1.Introduction

The problem of N particles interacting gravitationally is typically Newton's law plus, in case, an external potential field, it's a set of non-linear second order ordinary differential equations relating the acceleration with the position of all the particles in the system. The gravitational force equation presents a singularity when the distance of two particles approaches 0 and to avoid that a smoothing length called softening was injected within the equation. The computational complexity of the numerical solution of a N-body system for a fixed number of time steps scales as N^2.

2.Astrophysical domains and timescales

N-body simulations are applied to a wide range of different astrophysical problems, and depends on the timescale and collisionality of the problem we can identify four main astrophysical domains.

-Celestial mechanics: study of solar and extra solar planetary systems where one dominant body's gravity affects other objects. High accuracy is needed to correctly evaluate perturbations and avoid numerical errors.

-Dense stellar systems: such as open and globular clusters, require the study of multiple close encounters of stars and the correct description of short-range interactions on a relaxation timescale.

-The sphere of influence of a massive BH at the center of a stellar system resembles solar system dynamics but with frequent two-body encounters and the possibility of including post-Newtonian physics for high accuracy.

-Galaxy dynamics and cosmology: involve studying the mean field dynamics of a large number of particles, usually employing softening to avoid unphysical binary formation. Special attention is given to the dynamics of self-interacting dark matter particles which are described by the Collisional Boltzmann Equation.

3.Newtonian gravity: methods

The history of N-body simulation started in the 1940s with a 37-particle system calculated using light bulbs and galvanometers. Computer simulations began in the 1960s with up to 100 particles and improved in the 1980s with the development of efficient algorithms for collisionless systems. Today, simulations can handle up to 10^5 particles for direct integration and 10^{10} for collisionless dynamics/cosmology. Advancements in hardware and special purpose hardware, such as GRAPE, have also played a significant role in the development of N-body simulations.

Direct N-body methods achieve highest accuracy but have long computation time, using adaptive timesteps, Hermite integrator and treating close encounters and bound subsystems exactly. The number of particles that can be effectively followed is limited. Special purpose hardware like GRAPE can be used to achieve high performance. The tree code method is a fast, general integrator for collisionless systems that uses approximations that introduce some errors, but saves significant CPU time. The Fast Multipole Method (FMM) uses multipole expansion to compute forces and reduces complexity to O(N) but exact scaling is dependent on the implementation. It can guarantee exact conservation of momentum. Successful implementations include GyrfalcON code and PKDGRAV. The particle mesh method speeds up direct force evaluation for collisionless systems by representing the gravitational potential on a grid, calculated from the density field, where particles interact through a mean field. It softens interactions at small scales but sacrifices short-range accuracy and has a linear complexity in the number of particles and O(Ng log(Ng)) in the number of grid cells.

Another method is Adaptive Mesh Refinement (AMR), where the grid elements are concentrated in areas where higher resolution is needed. The Self-Consistent Field method, where the density and potential of the system are expanded in terms of a basis of orthogonal eigen functions. P3M and PM-Tree codes which increase the force resolution by combining a mean field description on large scales with a direct, softened treatment of the gravitational interactions on smaller scales. Celestial mechanics codes which are targeted at studying the dynamics of small number of bodies and require high precision due to the chaotic nature of the N-body problem.

4.Mean Field Methods

Dynamics of particle systems can be followed using time-dependent Boltzmann Equation and Poisson equation. There are different approaches for solving those Equations, which are used to model the dynamics of systems of particles in astrophysics. The first approach is a grid-based method that utilizes standard computational methods but requires a large amount of memory. The second approach is Fokker-Planck and Monte Carlo methods, which are less memory-intensive but are limited by the number of particles used in the simulation. The final approach discussed is the use of Post-Newtonian approximations in strong gravitational fields, such as those near black holes, as a full general relativity solution is too difficult to compute.