Statistical Signal Processing
Exercise 2

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1 KL Divergence

Wiki: In probability theory and information theory, the **Kullback-Leibler divergence** is a non-symmetric measure of the difference between two probability distribution P and Q. Specifically, the KL-divergence of Q from P is a measure of the information lost when Q is used to approximate P: The KL divergence measure the expected number of extra bits required to code [Huffman code] samples from P when using a code based on Q, rather than using the the true code based on P. Typically P represents the "true" distribution of data, observations, or a precisely calculated theoretical distribution. The measure Q typically represents a theory, mode, description or approximation of P.

In this question we're finding the KL divergence of f_2 from f_1.

Using the Law of the Unconscious Statistician:

$$\mathcal{D}_{KL}\left(f_1 || f_2\right) = \mathbb{E}_{f_1}\left[\log\left(\frac{f_1(y)}{f_2(y)}\right)\right] = \int_{-\infty}^{\infty} \log\left(\frac{f_1(y)}{f_2(y)}\right) f_1(y)\,\mathrm{d}y = \int_{-\infty}^{\infty} -\log\left(\frac{f_2(y)}{f_1(y)}\right) f_1(y)\,\mathrm{d}y$$

Known inequality: $\forall x \geq 0 : \log(x) \leq x - 1$. f_1 and f_2 are probability distributions, hence $\dfrac{f_2(y)}{f_1(y)} \geq 0$. Therefore:

$$\mathcal{D}_{KL}\left(f_1 || f_2\right) = \int_{-\infty}^{\infty} -\log\left(\frac{f_2(y)}{f_1(y)}\right) f_1(y)\,\mathrm{d}y \geq \int_{-\infty}^{\infty} \left(-\frac{f_2(y)}{f_1(y)} + 1\right) f_1(y)\,\mathrm{d}y = \int_{-\infty}^{\infty} -f_2(y)\,\mathrm{d}y + \int_{-\infty}^{\infty} f_1(y)\,\mathrm{d}y$$

$$\mathcal{D}_{KL}\left(f_1 || f_2\right) \geq -1 + 1 = 0.$$

$\blacksquare$