LOG(M) PROJECT ZERO

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1. Introduction

In general, orthogonal polynomials are polynomials that are perpendicular to other polynomials as defined by a specific inner product. Our research will focus on the specific inner product on polynomials P and Q_n defined as following.

$$\langle P, Q_n \rangle = \int_{-1}^{1} P(x)Q_n(x)w(x)dx$$

Where $w(x)$ is a predefined weight function around which we are studying, and P is any polynomial of degree less than or equal to $n-1$.

In order to find and calculate orthogonal polynomials, we must find Q_n such that the above integral evaluates to 0 for any polynomial P of degree less than n. Linearity of integrals implies that it suffices to find Q_n that satisfy the following.

$$\int_{-1}^{1} x^k Q_n(x)w(x)dx = 0 \quad (k = 0, 1, ..., n-1)$$

Below are some examples of the Orthogonal Polynomials

1.1. **Legendre Polynomials** $(w(x) \equiv 1)$.

$$\int_{-1}^{1} x^k L_n(x)dx = 0 \quad (k = 0, 1, ..., n-1)$$

1.2. **Jacobi Polynomials** $(w(x) = (1-x)^{\alpha}(1+x)^{\beta})$.

$$\int_{-1}^{1} x^k J_n(x)w_j(x)dx = 0 \quad (k = 0, 1, ..., n-1)$$

1.3. **Chebyshev Polynomials** $(w(x) = 1/\sqrt{(1-x^2)})$.

$$\int_{-1}^{1} x^k C_n(x)w_c(x)dx = 0 \quad (k = 0, 1, ..., n-1)$$

2. Background

Theorem 2.1. *Given $w(x) \geq 0$ for $x \in [a,b]$, all polynomials Q_n satisfying*

$$\int_a^b x^k Q_n(x) w(x) dx = 0 \quad (k = 0, 1, ..., n-1),$$

it follows that Q_n has n simple zeros lying in [a,b].

Proof. Assume Q_n has $k(< n)$ numbers of zeroes lying in [a,b]. So For n x_is which satisfy $Q(x_i) = 0$, let $x_i \in [a,b]$ for $i = 1, 2, ..., k$ and $x_i \notin [a,b]$ for $i = k+1, ..., n$.

Let $Q_n(x) = (x - x_1)...(x - x_k)(x - x_{k+1})...(x - x_n)$. Also, since $(x - x_1)...(x - x_k)$ is k degree polynomial, we can say that

$$\int_a^b (x - x_1)...(x - x_k) Q_n(x) w(x) dx = 0$$

is true by the given condition $\int_a^b x^k Q_n(x) w(x) dx = 0$ $(k = 0, 1, ..., n-1)$.

However, for

$$\int_a^b w(x) Q_n(x)(x - x_1)...(x - x_k) dx$$

$$= \int_a^b w(x)(x - x_1)^2 ...(x - x_k)^2 * (x - x_{k+1})...(x - x_n) dx$$

$w(x)$, $(x - x_1)^2 ...(x - x_k)^2$ is always positive. Also, $(x - x_{k+1}), ..., (x - x_n)$ does not change its sign in the range [a,b] since $x_i \notin [a,b]$ for $i = k+1, ..., n$. So, the total product inside the integral does not change its sign in the range [a,b]. Then integral cannot be 0, which contradicts to our previous assumption

$$\int_a^b (x - x_1)...(x - x_k) Q_n(x) w(x) dx = 0$$

So, Q_n has n simple zeroes lying in [a,b]. □