Orthogonal Polynomials

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Key Technical Terms: Orthogonal Polynomials

- Orthogonal Polynomials is a family of polynomials in the sequence are orthogonal to each other under some inner product.
- We can define an inner product of polynomial P and Q_n defined as following.

$$\langle P, Q_n \rangle = \int_{-1}^{1} P(x)Q_n(x)w(x)dx$$

Where w(x) is a given function on [-1,1]. (w(x) could be on any interval in the real line as well)

Compute Orthogonal Polynomials: Gram-Schmidt Algorithm

- Given an basis 1,x,x^2, ... ,x_n, for an inner product space V, the Gram-Schmidt algorithm constructs an orthogonal basis $P_0(x)$,$P_1(x)$, $P_2(x)$,...,$P_n(x)$ for V:
- The coefficients of the orthogonal polynomials further depend on our $w(x)$ and interval $[a, b]$

Continued to Orthogonal Polynomials

$$\langle P, Q_n \rangle = \int_{-1}^{1} P(x) Q_n(x) w(x) dx$$

where $P(x)$ is any polynomial of degree $n-1$

$$\langle x^k, Q_n \rangle = \int_{-1}^{1} x^k Q_n(x) w(x) dx$$

for $k = 0, 1, ..., n-1$

where w(x) is a given positive function on [-1,1]

Some Families of Orthogonal Polynomials

- Legendre Polynomials ($w(x) = 1$)

$$\int_{-1}^{1} x^k L_n(x) dx = 0 \quad (k = 0, 1, ..., n-1)$$

- Jacobi Polynomials ($w(x) = (1-x)^{\alpha}(1+x)^{\beta}$)

$$\int_{-1}^{1} x^k J_n(x) w_j(x) dx = 0 \quad (k = 0, 1, ..., n-1)$$

- Chebyshev Polynomials ($w(x) = 1/\sqrt{(1-x^2)}$)

$$\int_{-1}^{1} x^k C_n(x) w_c(x) dx = 0 \quad (k = 0, 1, ..., n-1)$$

Hankel Matrix(1)

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- Denote
$$\mu_k = \int_{-1}^{1} x^k w(x)dx$$
$$for\, k = 0, 1, ..., n-1$$
- Let
$$Q_n = x^n + a_{n-1}x^{n-1} + ... + a_0$$
- By definition of Orthogonal Polynomials
$$\int_{-1}^{1} x^k Q_n(x)w(x)dx = 0$$

for all $k = 0, \cdots, n-1$

Hankel Matrix(2)

- We can then write the previous equations into a linear system of equations using μ_k as follows:

$$\mu_{2n-1} + a_{n-1}\mu_{2n-2} + \cdots a_0\mu_{=}0$$

$$\vdots$$

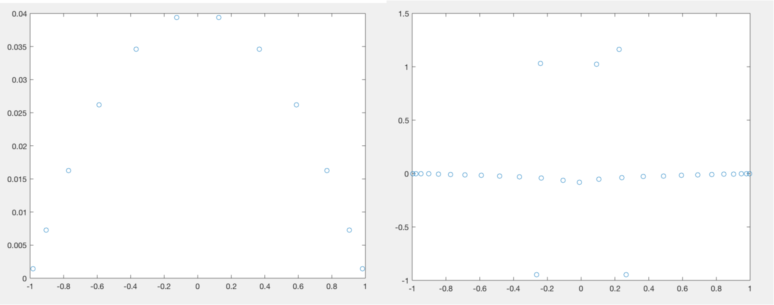
$$\mu_n + a_{n-1}\mu_{n-1} + \cdots a_0\mu_0 = 0$$

  We can convert this into a matrix equation as shown below:

- $$\begin{bmatrix} \mu_{2n-2} & \mu_{2n-3} & \cdots & \mu_{n-1} \\ \mu_{2n-3} & \mu_{2n-4} & \cdots & \mu_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_{n-2} & \cdots & \mu_0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \\ \vdots \\ a_0 \end{bmatrix} = \begin{bmatrix} -\mu_{2n-1} \\ -\mu_{2n-2} \\ \vdots \\ -\mu_n \end{bmatrix}$$

What is the Goal?

- We want to visualize the zeroes of Orthogonal Polynomials.
  - We want a high degree of accuracy as our n increases.
- Code will be primarily written in Matlab/R/Python



These images were both individually generated in MatLab

Preliminary Knowledge

- We have been exploring the stuff that has been well documented in order to familiarize ourselves with the concepts.
- Starts with proving some useful results, and using them in the code.
- The following theorem, for example, proves to be extremely helpful:

Theorem (Szego)

Given $w(x) \geq 0$ for $x \in [a, b]$, and polynomials Q_n satisfying

$$\int_a^b x^k Q_n(x) w(x) dx = 0 \quad (k = 0, 1, ..., n-1)$$

It follows that Q_n has n simple zeros lying in $[a, b]$.

What's next?

- After getting the preliminaries done, the goal is to then explore the complex plane.
- We try to see and predict the zeroes of the Orthogonal Polynomials as we vary $w(x)$.

Applications & Connection to other Math

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- Orthogonal polynomials can be used in many Mathematics fields
  - Continued Fraction
  - Gaussian Quadrature Rule
  - Others

Continued Fraction

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ddots}}}}$$

Continued Fraction

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- We want to estimate $\int_{-1}^{1} \frac{w(x)}{z-x} dx$
- Using Calculus,

$$\int_{-1}^{1} \frac{w(x)}{z-x} dx = \Sigma_{k=1}^{\infty} \frac{\mu_k}{z^k}$$

where $\mu_k = \int_{-1}^{1} w(x) x^{k-1} dx$

Continued Fraction

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- We want to estimate $\sum_{k=1}^{\infty} \frac{\mu_k}{z^k}$. How?
- Write $\sum_{k=1}^{\infty} \frac{\mu_k}{z^k}$ as a continued fraction form

$$f(z) = \sum_{k=1}^{\infty} \frac{\mu_k}{z^k} = \cfrac{a_0}{z - b_0 + \cfrac{a_1}{z - b_1 + \cfrac{a_2}{z - b_2 + \dots}}}$$

- ...And we cut this continued fraction at n

$$\cfrac{a_0}{z - b_0 + \cfrac{a_1}{z - b_1 + \cfrac{a_2}{\dots + \cfrac{a_n}{z - b_n}}}} = \frac{P_n(z)}{Q_n(z)}$$

Where $Q_n(z)$ and $P_n(z)$ are some n-degree polynomial. (we call $\frac{P_n(z)}{Q_n(z)}$ n^{th} diagonal Pade approximant)

Continued Fraction

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- Amazing Fact!

$$Q_n(z)\ satisfies \int_{-1}^{1} Q_n(z)\, x^k w(x) dx = 0, for\ k = 0,1,\dots,n-1$$

- So if we want to estimate $\int_{-1}^{1} \frac{w(x)}{z-x} dx$, we can treat $Q_n(z)$, the Orthogonal Polynomial of $w(x)$, as a denominator and corresponding $P_n(z)$ as a numerator of n^{th} diagonal Pade approximant.

Gaussian Quadrature Rule

- Now we want to estimate $\int_a^b f(x)w(x)dx$
- Let

$$\int_a^b f(x)w(x)dx = a_0 f(x_0) + a_1 f(x_1) + \ ... + a_n f(x_n) \ \ (x_i \epsilon (a,b), a_i \epsilon \ \Re)$$

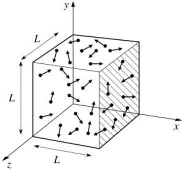
- Similar to Riemann sum

Gaussian Quadrature Rule

- What are the best x_i values to estimate $\int_a^b f(x)w(x)dx$?
- The zeros of the $Q_{n+1}(z)$
  - $\int_a^b Q_{n+1}(z)x^k w(x)dx = 0$, for $k = 0, 1, ..., n$
- So we have to learn Orthogonal Polynomial to use Gaussian Quadrature Rule to estimate $\int_a^b f(x)w(x)dx$

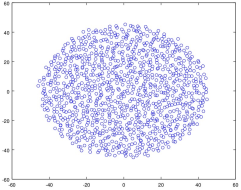
Others

- We can use Orthogonal Polynomials in more diverse place
  - Statistical Mechanisms



[1]

  - Random Matrix



[2]

[1] https://platosrealm.blog/2018/12/15/stop-using-thermodynamics/

[2] https://math.stackexchange.com/questions/1348679/random-matrices-eigenvalue-distribution

References

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[1] Gabor Szegö. *Orthogonal polynomials by Gabor Szegö*. Published 1937.

[2] Lab of Geometry at Michigan. LoG(M) Beamer Template. *University of Michigan Department of Mathematics*. 2018.

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