Orthogonal Polynomials

Object

Consider polynomials Q_n satisfying

$$\int_{-1}^{1} x^k Q_n(x) \, w(x) dx = 0$$

for $k=0, 1, \ldots, n-1$, where $w(x)$ is a given positive function on $[-1,1]$

Exercise Show that the above conditions imply that for any polynomial P with degree $\leq n-1$,

$$\int_{-1}^{1} P(x) \cdot Q_n(x) \; w(x) \, dx = 0 .$$

Examples

Legendre Polynomials $(w(x) \equiv 1)$:

$$\int_{-1}^{1} x^k \, L_n(x) \, dx = 0 , \quad k=0,1,\ldots,n-1$$

Exercise Find the first three monic Legendre polynomials

Jacobi Polynomials $\left(w_j(x) = (1-x)^\alpha (1+x)^\beta \text{ for } \alpha, \beta > -1\right)$

$$\int_{-1}^{1} x^k \, J_n(x) \; w_j(x) dx = 0, \quad k=0,1,\ldots,n-1$$

Chebyshev Polynomials $\left(w_c(x) = \dfrac{1}{\sqrt{1-x^2}} \right)$

$$\int_{-1}^{1} x^k \, C_n(x) \, w_c(x) dx = 0 \quad , k=0,1,\ldots,n-1$$

Exercise Compute the first three monic Chebyshev polynomials.

Exercise For a weight function $w(x)$, denote

$$\mu_k = \int_{-1}^{1} x^k w(x)\, dx \quad , \qquad Q_n(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

Where Q_n's are the corresponding orthogonal polynomials. Write down a system of equations for the unknowns $a_0, a_1, \dots, a_{n-1}$. What condition must be satisfied for Q_n's to exist?

Zeros

From the above discussion, you may have guessed that the zeros of the orthogonal polynomials seem to stay within $[-1,1]$.

Theorem Given $w(x) \geq 0$ for $x \in [a,b]$, and polynomials Q_n satisfying

$$\int_a^b x^k Q_n(x)\, w(x)\, dx = 0 \qquad \text{for } k = 0, 1, \dots, n-1$$

it follows that Q_n has n simple zeros lying in $[a,b]$

Exercise Prove the above theorem.

Losing Positivity

What happens if $w(x)$ is allowed to be negative?

Bad example

Let

$$w(x) = \sin(2\pi \log(x)) \exp(-\log^2(x)) \quad , \quad x \in [0, \infty)$$

Exercise Using the substitution

$$u = \log x - \frac{n+1}{2}$$

$Q_1(z) = 1$

$Q_2(t) = 1$

$\vdots$

show that

$$\int_0^\infty x^k w(x)\, dx = 0 \qquad \text{for } k = 0, 1, 2, \dots$$

What conclusion can you make about the orthogonal polynomials? Can you write them down?

Proposition Let Q_n be polynomials satisfying

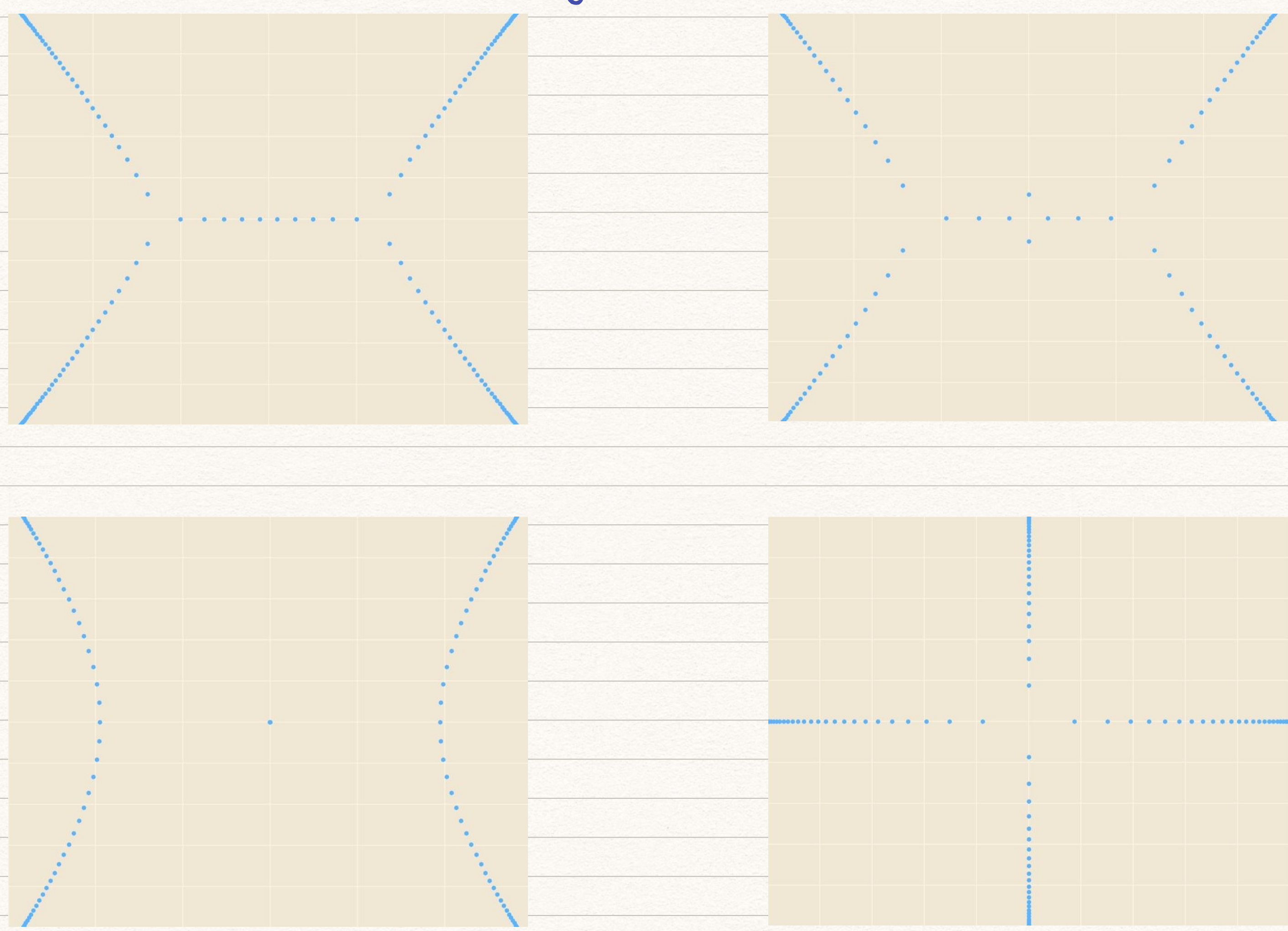
$$\int_a^b x^k \, Q_n(x) \, w(x) \, dx \qquad \text{for} \qquad k = 0, 1, \dots, n-1$$

where $w(x)$ is a smooth function on $[a,b]$ (not necessarily positive or even real!)

Then, while $\deg Q_n \leq n$, the minimal-degree monic polynomials satisfying the above conditions are unique.

Exercise Prove this proposition

These zeros can arrange themselves in extremely interesting ways! We will be interested in how many zeros polynomial Q_n has!



Goal of the project

The general theme will be to consider several different families of orthogonal polynomials and compute their degrees to look for possible degeneration patterns, all while trying to optimize how to compute/visualize this information.

Some Sources

* "Orthogonal Polynomials" by Gabor Szegő

* "Orthogonal Polynomials : Computation and Approximation" by Walter Gautschi

* "An Introduction to Orthogonal Polynomials" by Theodore Chihara