One of the assigned exercises was:

Exercise For a weight function $w(x)$, denote

$$\mu_k = \int_{-1}^{1} x^k \, w(x)\, dx \quad , \qquad Q_n(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

Where Q_n's are the corresponding orthogonal polynomials. Write down a system of equations for the unknowns $a_0, a_1, \dots, a_{n-1}$. What condition must be satisfied for Q_n's to exist?

To solve this, let's consider the special case of $Q_3(z)$, the polynomial of degree 3. This polynomials satisfies:

$$\int_{-1}^{1} Q_3(z)\, dz = 0,$$

$$\int_{-1}^{1} Q_3(z)\, z\, dz = 0,$$

$$\int_{-1}^{1} Q_3(z)\, z^2\, dz = 0.$$

If we write $Q_3(z) = z^3 + a_2 z^2 + a_1 z + a_0$, we see that

$$\overbrace{\int z^3 w(z) dz}^{\mu_3} + a_2 \overbrace{\int z^2 w(z) dz}^{\mu_2} + a_1 \overbrace{\int z\, w(z) dz}^{\mu_1} + a_0 \overbrace{\int w(z) dz}^{\mu_0} = 0$$

$$\int z^4 w(z) dz + a_2 \int z^3 w(z) dz + a_1 \int z^2 w(z) dz + a_0 \int z\, w(z) dz = 0$$

$$\int z^5 w(z) dz + a_2 \int z^4 w(z) dz + a_1 \int z^3 w(z) dz + a_0 \int z^2 w(z) dz = 0$$

this yields

$$\mu_3 + a_2\mu_2 + a_1\mu_1 + a_0\mu_0 = 0$$

$$\mu_4 + a_2\mu_3 + a_1\mu_2 + a_0\mu_1 = 0$$

$$\mu_5 + a_2\mu_4 + a_1\mu_3 + a_0\mu_2 = 0$$

Written as a matrix, this yields:

$$\text{Hankel Matrix}\left\{\begin{pmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{pmatrix}\right. \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -\mu_3 \\ -\mu_4 \\ -\mu_5 \end{pmatrix}$$

In general, coefficients of Q_n solve an $n \times n$ matrix equation, and a polynomial of degree n exists if and only if the corresponding Hankel matrix is invertible. If this Hankel matrix is <u>not</u> invertible, that indicates that the polynomial Q_n <u>has degree less than n</u>.

Orthonormal Basis

One can view Q_n's as an orthonormal basis for

$$\mathbb{P}_n = \{ p(x) \mid p(x) \text{ polynomial}, \deg p \leq n \}$$

with respect to the inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x) g(x)\, w(x)\, dx \quad , \quad w(x) \geq 0$$

Exercise check that the above is indeed an inner product. What fails if $w(x)$ is not nonnegative?

In this way, we can start with another basis for $\mathbb{P}_n$, say

$$\{1, x, \dots, x^n\}$$

and, using Gram-Schmidt process to arrive at an orthogonal basis

$$\{Q_0(x), Q_1(x), \dots, Q_n(x)\}$$

Furthermore, we can look for an orthonormal basis,

$$\{q_0(x), q_1(x), \dots, q_n(x)\} \qquad \text{where} \qquad q_n(x) = \frac{1}{\underbrace{\sqrt{\langle Q_n, Q_n\rangle}}_{\text{leading coefficient}}} Q_n(x)$$

Exercise Using orthogonality, show that these polynomials satisfy

$$x q_n(x) = a_{n+1} q_{n+1}(x) + b_n q_n(x) + a_n q_{n-1}(x)$$

Exercise Write $q_n(x) = k_n x^n + \cdots$. By comparing coefficients, conclude that $a_{n+1} = \frac{k_n}{k_{n+1}}$. Use this to find a formula for β_n in terms of k_n's.

This recurrence relation offers yet another way to compute orthogonal polynomials.

Applications

Historically, orthogonal polynomials appeared in the study of continued fractions.
A continued fraction is a number of the form

$$\cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{\ddots}}}$$

From Euclid's algorithm, we can write a finite continued fraction for p/q:

$$p = a_0 q + r_0$$
$$q = a_1 r_0 + r_1$$
$$r_0 = a_1 r_1 + r_2$$
$$\vdots$$
$$r_{n-1} = a_n r_n$$

and so

$$\frac{p}{q} = a_0 + \frac{r_0}{q} = a_0 + \frac{1}{q/r_0} = a_0 + \cfrac{1}{a_1 + \cfrac{r_1}{r_0}}$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_1 + r_2/r_1}}$$

$$\vdots$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

Furthermore, for any real number x, we can write

$$x = [x] + \{x\}$$

where $[x]$ is the integer part of x, and $\{x\}$ is the fractional part. Then

$$x = [x] + \{x\} = [x] + \frac{1}{1/\{x\}}$$

where now, $1/\{x\} > 1$, and so can be written as

$$\frac{1}{\{x\}} = \left[\frac{1}{\{x\}}\right] + \left\{\frac{1}{\{x\}}\right\}$$

and this can be continued on. If the process terminates, then x must be rational, and otherwise x is rational.

These continued fractions are of interest since they offer a great way to approximate numbers!

The jump to functions

Inspired by this, we attempt to approximate functions using a similar trick!

Consider the function

$$f(z) = \int_{-1}^{1} \frac{w(x)}{z-x}\, dx$$

This is a function of z!

Exercise By taking out a factor $1/z$ and expanding $\frac{1}{1-\frac{x}{z}}$ as a geometric series, write

$$f(z) = \sum_{k=1}^{\infty} \frac{\mu_k}{z^k}.$$

Now, we play a similar trick and write

$$f(z) = \sum_{k=1}^{\infty} \frac{\mu_k}{z^k} = \cfrac{a_0}{z-b_0 + \cfrac{a_1}{z-b_1 + \cfrac{a_2}{z-b_2+\dots}}}$$

If we truncate the right hand side, we get a rational function

$$\cfrac{a_0}{z-b_0 + \cfrac{a_1}{z-b_1 + \cfrac{a_2}{z-b_2+\dots_{\ z-b_n}}}} = \frac{P_n(z)}{Q_n(z)}$$

and the denominator Q_n satisfies

$$\int_{-1}^{1} Q_n(x)\, x^k\, w(x)\, dx = 0 \qquad \text{for } k = 0, 1, \dots, n-1$$

The ratio

$$[n/n]_f(z) := \frac{P_n(z)}{Q_n(z)}$$

is known as the n^{th} diagonal Padé approximant.