Post-Newtonian Moreschi Supermomentum

By definition, the Moreschi supermomentum is

$$\Psi^{\mathrm{M}} = \Psi_2 + \sigma\dot{\bar{\sigma}} + \eth^2\bar{\sigma}. \tag{1}$$

But, by utilizing the Bianchi identities, namely

$$\dot{\Psi}_2 = -\eth^2\dot{\bar{\sigma}} - \sigma\ddot{\bar{\sigma}} = -\eth^2\bar{\sigma} - \sigma\dot{\bar{\sigma}} + \int_{-\infty}^{u} |\dot{\sigma}|^2 - M_{\mathrm{ADM}}, \tag{2}$$

we can rewrite this as

$$\Psi^{\mathrm{M}} = \int_{-\infty}^{u} |\dot{\sigma}|^2 - M_{\mathrm{ADM}}, \tag{3}$$

In principle, we should be able to take the post-Newtonian (PN) expression for σ and then compute this integral to obtain a PN expression for the $\ell \geq 2$ components of Ψ^{M}, since M_{ADM} is only a function of time, but not angle on S^2. Note that the $\ell \geq 2$ components of Ψ^{M} are

$$\begin{aligned}
\Psi^{M}_{\ell m} &= \int_{-\infty}^{u} |\dot{\sigma}(u)|^2_{\ell m} \\
&= \sum_{\ell_1 m_1}\sum_{\ell_2 m_2}\int_{-\infty}^{u} \dot{\sigma}_{\ell_1 m_1}(u)\dot{\bar{\sigma}}_{\ell_2 m_2}(u)(-1)^{m+m_2}\int Y_{\ell,-m\ +2}Y_{\ell_1,+m_1\ -2}Y_{\ell_2,-m_2}\,d\Omega \\
&= \sum_{\ell_1 m_1}\sum_{\ell_2 m_2}\int_{-\infty}^{u} \dot{\sigma}_{\ell_1 m_1}(u)\dot{\bar{\sigma}}_{\ell_2 m_2}(u)(-1)^{m+m_2} \\
&\qquad \sqrt{\frac{(2\ell+1)(2\ell_1+2)(2\ell_2+1)}{4\pi}}\begin{pmatrix}\ell & \ell_1 & \ell_2 \\ -m & +m_1 & -m_2\end{pmatrix}\begin{pmatrix}\ell & \ell_1 & \ell_2 \\ 0 & -2 & +2\end{pmatrix}.
\end{aligned} \tag{4}$$

According to PN theory,

$$h_{\ell m} = 2\sqrt{\frac{16\pi}{5}}\frac{GM\nu x}{Rc^2}\hat{H}_{\ell m}e^{-im\psi}, \tag{5}$$

where

$$M = m_1 + m_2, \quad \nu = \frac{m_1 m_2}{(m_1+m_2)^2}, \quad x = (\frac{GM\omega}{c^3})^{\frac{2}{3}}, \quad \text{and} \quad \psi = \phi - 3x^{\frac{3}{2}}\left(1 - \frac{\nu}{2}x\right)\ln\left(\frac{x}{x_0}\right). \tag{6}$$

Therefore

$$\sigma_{\ell m} = \sqrt{\frac{16\pi}{5}}\frac{GM\nu x}{Rc^2}\hat{H}^{*}_{\ell m}e^{+im\psi} \tag{7}$$

and

$$\dot{\sigma}_{\ell m} = \sqrt{\frac{16\pi}{5}}\frac{GM\nu}{Rc^2}e^{+im\psi(u)}\left[\left(\frac{d}{du}\hat{H}^{*}_{\ell m}\right)x(u) + \hat{H}^{*}_{\ell m}\left(\dot{x}(u) + imx(u)\dot{\psi}(u)\right)\right]. \tag{8}$$

Currently, we just have PN expressions for σ up to 3PN order (i.e., $x^{\leq 3}$). If we focus on the time integral,

$$\begin{aligned}
I &= \int_{-\infty}^{u} \dot{\sigma}_{\ell_1 m_1}(u)\dot{\bar{\sigma}}_{\ell_2 m_2}(u)\,du \\
&= \left(\sqrt{\frac{16\pi}{5}}\frac{GM\nu}{Rc^2}\right)^2\int_{-\infty}^{u} e^{i(m_1-m_2)\psi(u)}\left[\left(\frac{d}{du}\hat{H}^{*}_{\ell_1 m_1}\right)x(u) + \hat{H}^{*}_{\ell_1 m_1}\left(\dot{x}(u) + im_1 x(u)\dot{\psi}(u)\right)\right]du \\
&\qquad \left[\left(\frac{d}{du}\hat{H}_{\ell_2 m_2}\right)x(u) + \hat{H}_{\ell_2 m_2}\left(\dot{x}(u) - im_2 x(u)\dot{\psi}(u)\right)\right]du.
\end{aligned} \tag{9}$$

At this point, however, we should convert the integration variable to x by making use of the fact that

$$du = (\frac{dx}{du})^{-1} dx = \dot{x}^{-1} dx \quad \text{and} \quad \frac{d}{du} = \frac{dx}{du}\frac{d}{dx} = \dot{x}\frac{d}{dx}. \tag{10}$$

Doing so yields

$$\begin{aligned} I &= \left(\sqrt{\frac{16\pi}{5}}\frac{GM\nu}{Rc^2}\right)^2 \int_0^x e^{i(m_1-m_2)\psi(x)} \left[\left(\dot{x}\frac{d}{dx}\hat{H}^*_{\ell_1 m_1}\right)x + \hat{H}^*_{\ell_1 m_1}\left(\dot{x} + i m_1 x\dot{\psi}(x)\right)\right] \\ &\qquad \left[\left(\dot{x}\frac{d}{dx}\hat{H}_{\ell_2 m_2}\right)x + \hat{H}_{\ell_2 m_2}\left(\dot{x} - i m_2 x\dot{\psi}(x)\right)\right]\dot{x}^{-1}\, dx \\ &= \left(\sqrt{\frac{16\pi}{5}}\frac{GM\nu}{Rc^2}\right)^2 \int_0^x \Bigg(e^{i(m_1-m_2)\psi(x)} \left[\left(\frac{d}{dx}\hat{H}^*_{\ell_1 m_1}\right)x + \hat{H}^*_{\ell_1 m_1}\left(1 + i m_1 x\left(\dot{\psi}(x)/\dot{x}\right)\right)\right] \\ &\qquad \left[\left(\frac{d}{dx}\hat{H}_{\ell_2 m_2}\right)x + \hat{H}_{\ell_2 m_2}\left(1 - i m_2 x\left(\dot{\psi}(x)/\dot{x}\right)\right)\right]\dot{x}^2\Bigg)\dot{x}^{-1}\, dx, \end{aligned} \tag{11}$$

where the $[\cdots]$ terms only depend on x and not $\dot{x}$. Then, using the fact that

$$\begin{aligned} \dot{x} &= \frac{64\nu}{5M}x^5\Bigg\{1 + x\left(-\frac{743}{336} - \frac{11}{4}\nu\right) + 4\pi x^{3/2} + x^2\left(\frac{34103}{18144} + \frac{13661}{2016}\nu + \frac{59}{18}\nu^2\right) + \pi x^{5/2}\left(-\frac{4159}{672} - \frac{189}{8}\nu\right) \\ &\quad + x^3\left[\frac{16447322263}{139708800} + \frac{16}{3}\pi^2 - \frac{856}{105}\left(2\gamma_E + \ln(16x)\right) + \left(-\frac{56198689}{217728} + \frac{451}{48}\pi^2\right)\nu + \frac{541}{896}\nu^2 - \frac{5605}{2592}\nu^3\right] \\ &\quad + \pi x^{\frac{7}{2}}\left(-\frac{4415}{4032} + \frac{358675}{6048}\nu + \frac{91495}{1512}\nu^2\right) + \mathcal{O}(8)\Bigg\}, \end{aligned} \tag{12}$$

$$\begin{aligned} \psi(x) &= -\frac{1}{32\nu}x^{-\frac{5}{2}}\Bigg\{1 + x\left(\frac{3715}{1008} + \frac{55}{12}\nu\right) - 10\pi x^{\frac{3}{2}} + x^2\left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2\right) \\ &\quad + \pi x^{\frac{5}{2}}\ln(\frac{x}{x_0})\left(\frac{38645}{1344} - \frac{65}{16}\nu\right) + x^3\Big[\frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{856}{21}\left(2\gamma_E + \ln(16x)\right) \\ &\quad + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3\Big] \\ &\quad + \pi x^{\frac{7}{2}}\left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right) + \mathcal{O}(8)\Bigg\}. \end{aligned} \tag{13}$$

and the following integral:

$$\int_{-\infty}^u x^n e^{-im\psi}\, du = \int_0^x x^n e^{-im\psi}\dot{x}^{-1}\, dx = \begin{cases} i\frac{M}{m}x^{n-\frac{3}{2}}e^{-im\psi} & m \neq 0, \\ \frac{5M}{64\nu}\left(n-4\right)x^{n-4} & m = 0. \end{cases} \tag{14}$$

we find,

$$I = \tag{15}$$