STAT 641
Homework 6

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Problem 1

1. To create a reference plot, we must first make it such that the Weibull is a location-scale distribution. We can accomplish this (as outlined in Handout 8) by applying the transformation:

$$X = \log(Y)$$

so if X follows a log-Weibull distribution, then Y follows a Weibull distribution.
The cdf of the log weibull is:

$$\begin{aligned} P(\ln(Y) \leq x) &= P(Y \leq e^x) \\ &= 1 - e^{-(\frac{e^x}{\alpha})^\gamma} \\ &= 1 - e^{-e^{\frac{x-\theta_1}{\theta_2}}} \end{aligned}$$

We can then solve for the quantile function:

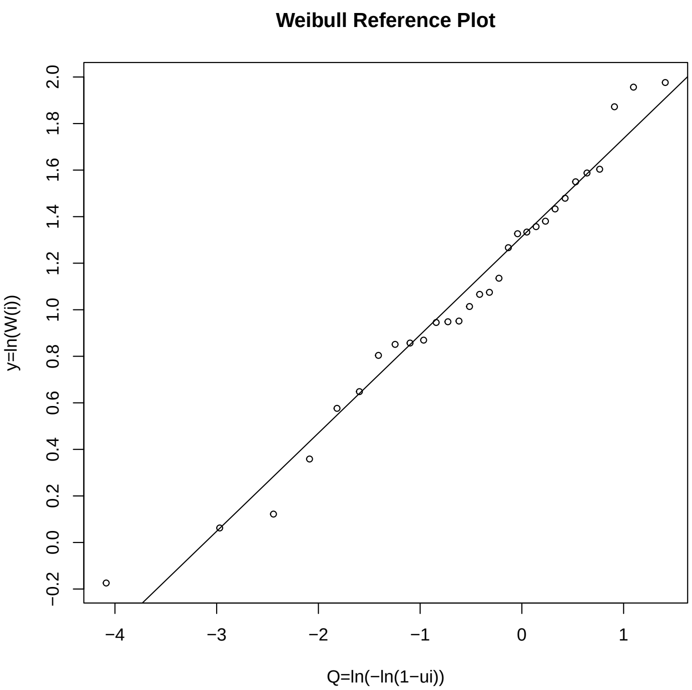
$$\begin{aligned} s &= 1 - e^{-e^{\frac{x-\theta_1}{\theta_2}}} \\ \ln(-s+1) &= -e^{\frac{x-\theta_1}{\theta_2}} \\ \ln(-\ln(-s+1)) &= \frac{x-\theta_1}{\theta_2} \\ \theta_2 \cdot \ln(-\ln(-s+1)) + \theta_1 &= x \end{aligned}$$

Thus we have:

$$Q_z(u) = \ln(-\ln(-u+1))$$

We can then make the reference plot with the following R code (continued from the previous R code):

```
\
x <- c(0.8402, 1.0644, 1.1298, 1.4314, 1.7795, 1.9121, 2.2343, 2.3424, 2.3559, 2.3855,
2.5734, 2.5815, 2.5893, 2.7562, 2.9040, 2.9295, 3.1124, 3.5490, 3.7684, 3.7953,
3.8846, 3.9766, 4.1918, 4.3887, 4.7106, 4.8918, 4.9716, 6.5018, 7.0740, 7.2158)
y = -log(x)
y = sort(y)
n = length(y)
weib= -y
weib= sort(weib)
i= 1:n
ui= (i-.5)/n
QW= log(-log(1-ui))
plot(QW,weib,abline(lm(weib~QW)),
main="Weibull Reference Plot",cex=.75,lab=c(7,11,7),
xlab="Q=ln(-ln(1-ui))",
ylab="y=ln(W(i))")
```



We will also compute the Anderson-Darling GOF test in R:

```
library(MASS)
mle <- fitdistr(x,"weibull")
shape = mle$estimate[1]
scale = mle$estimate[2]
a = -log(scale)
b = 1/shape
z = exp(-exp(-(y-a)/b))
A1i = (2*i-1)*log(z)
A2i = (2*n+1-2*i)*log(1-z)
s1 = sum(A1i)
s2 = sum(A2i)
AD = -n-(1/n)*(s1+s2)
ADM = AD*(1+.2/sqrt(n))
AD
ADM
```

This results in $A^2 = 0.3059062$. From the table in Handout 9, this corresponds with a p-value $> .25$. Combining this with the fact that the reference plot is fairly straight, I am inclined to say the Weibull distribution is a good fit for the data.

2. We can solve for the CI for the upper quantile using the following R code (adapted from handout 11):

```
x <- c(0.8402, 1.0644, 1.1298, 1.4314, 1.7795, 1.9121, 2.2343, 2.3424, 2.3559, 2.3855,
2.5734, 2.5815, 2.5893, 2.7562, 2.9040, 2.9295, 3.1124, 3.5490, 3.7684, 3.7953,
3.8846, 3.9766, 4.1918, 4.3887, 4.7106, 4.8918, 4.9716, 6.5018, 7.0740, 7.2158)
n=length(x)
L=.95
P=.75
s=ceiling(n*P)-1
r=floor(n*P)+1
cov=0
while(s<n-1 && r>1 && cov<L)
{s=s+1
cov=pbinom(s-1,n,P)-pbinom(r-1,n,P)
if(cov>=L) break;
r=r-1
cov=pbinom(s-1,n,P)-pbinom(r-1,n,P)
}
x[r]
x[s]
cov
```

this results in the 95% confidence interval $[3.549, 6.5018]$ with coverage probability .9678

3. We will assume the survival times of engines are independent. We will use power transformations (as outlined in handout 9) to transform the data to a somewhat normal distribution. We can find the power θ to be used with the following code:

```
y <- c(0.8402, 1.0644, 1.1298, 1.4314, 1.7795, 1.9121, 2.2343, 2.3424, 2.3559, 2.3855,
2.5734, 2.5815, 2.5893, 2.7562, 2.9040, 2.9295, 3.1124, 3.5490, 3.7684, 3.7953,
3.8846, 3.9766, 4.1918, 4.3887, 4.7106, 4.8918, 4.9716, 6.5018, 7.0740, 7.2158)
n = length(y)
yt0 = log(y)
s = sum(yt0)
varyt0 = var(yt0)
Lt0 = -1*s - .5*n*(log(2*pi*varyt0)+1)
th = 0
Lt = 0
t = -3.01
i = 0
while(t < 3)
{t = t+.001
i = i+1
th[i] = t
yt = (y^t -1)/t
varyt = var(yt)
Lt[i] = (t-1)*s - .5*n*(log(2*pi*varyt)+1)
if(abs(th[i])<1.0e-10)Lt[i]<-Lt0
if(abs(th[i])<1.0e-10)th[i]<-0
}
# The following outputs the values of the likelihood and theta and yields
# the value of theta where likelihood is a maximum
out = cbind(th,Lt)
Ltmax= max(Lt)
Ltmax
imax= which(Lt==max(Lt))
thmax= th[imax]
thmax
```

This yields $\theta = 0.325$. Thus the transformation is:

$$Y = \frac{y^{.325} - 1}{.325}$$

We can apply this transformation and find the 95% CI of the mean of the new distribution with R:

```
y = (y^.325 - 1) / .325
mean = mean(y)
std = sd(y)
error = qnorm(0.975)*std/sqrt(n)
lower = mean - error
upper = mean + error
lower
upper
```

we can then apply the inverse of the transformation function to get back the lower and upper bounds for the original distribution:

```
lower_transform = (lower * .325 + 1)^(1 / .325)
upper_transform = (upper * .325 + 1)^(1 / .325)
lower_transform
upper_transform
```

This results in the 95% CI of $[2.526999, 3.663964]$

4. For this, we will construct a two sided tolerance interval. First we will use a power transformation to normalize the data, then we will use the code from handout 11 for creating a two sided tolerance interval, and then we will back transform the interval endpoints:

```
y <- c(0.8402, 1.0644, 1.1298, 1.4314, 1.7795, 1.9121, 2.2343, 2.3424, 2.3559, 2.3855,
2.5734, 2.5815, 2.5893, 2.7562, 2.9040, 2.9295, 3.1124, 3.5490, 3.7684, 3.7953,
3.8846, 3.9766, 4.1918, 4.3887, 4.7106, 4.8918, 4.9716, 6.5018, 7.0740, 7.2158)
n = length(y)
yt0 = log(y)
s = sum(yt0)
varyt0 = var(yt0)
Lt0 = -1*s - .5*n*(log(2*pi*varyt0)+1)
th = 0
Lt = 0
t = -3.01
i = 0
while(t < 3)
{t = t+.001
i = i+1
th[i] = t
yt = (y^t -1)/t
varyt = var(yt)
Lt[i] = (t-1)*s - .5*n*(log(2*pi*varyt)+1)
if(abs(th[i])<1.0e-10)Lt[i]<-Lt0
```

```
if(abs(th[i])<1.0e-10)th[i]<-0
}
# The following outputs the values of the likelihood and theta and yields
# the value of theta where likelihood is a maximum
out = cbind(th,Lt)
Ltmax= max(Lt)
Ltmax
imax= which(Lt==max(Lt))
thmax= th[imax]
thmax

y = (y^thmax - 1) / thmax
G = .95
P = .95
Chi = qchisq(1-(1 - G) / 2,n-1)
z = qnorm((1+P)/2)
K2Side = sqrt(((n-1)*(n+1)*z^2)/(n*Chi))
lower = mean(y) - K2Side * sd(y)
upper = mean(y) + K2Side * sd(y)
lower_transform = (lower * thmax + 1)^(1 / thmax)
upper_transform = (upper * thmax + 1)^(1 / thmax)
lower_transform
upper_transform
```

The result is that we can be 95% confident that at least 95% of values will fall in the interval [1.174811, 6.343665]

Problem 2

1. (a) Adapting the code from handout 11:

```
y <- c(1.1, 2.6, 3.0, 3.7, 4.1, 6.5, 8.1, 9.1, 9.2, 11.7, 13.8, 14.3, 17.8, 19.3, 1
22.7, 22.9, 23.4, 23.9, 24.9, 26.9, 27.4, 27.5, 28.7, 31.4, 35.9, 41.7, 42.5, 43.1,
45.4, 46.2, 48.3, 54.2, 54.4, 54.8, 60.7, 61.0, 70.0, 70.1, 70.2, 75.4, 75.4, 75.7,
76.6, 76.9, 76.9, 78.7, 80.6, 83.6, 85.7, 86.0, 87.4, 90.0, 93.4,94.3, 96.7, 96.8,
100.1, 105.4, 105.9, 110.9, 112.8, 113.3, 114.3, 114.9, 119.4, 119.5, 120.6, 121.0,
127.7, 129.5, 129.8, 133.3, 133.6, 136.0, 137.5, 137.6, 138.5, 140.1, 158.4, 158.7,
165.9, 166.0, 166.9, 169.2, 174.4, 183.7, 188.0, 227.9, 248.0, 257.6, 271.4, 280.9,
283.5, 287.8, 303.2, 308.1, 326.5, 334.9, 520.4)
n= length(y)
thest = mean(y)
B = 9999
thestS = numeric(B)
thestS = rep(0,times =B)
for (i in 1:B)
```

```
thestS[i] = mean(sample(y,replace=T))
RS= sort(thestS-thest)
LRS = RS[250]
URS = RS[9750]
thL = thest-URS
thU = thest-LRS
thL
thU
```

This yields the interval $[86.876, 121.981]$

(b) Again adapting the code from handout 11:

```
n= length(y)
thest = mean(y)
V = thest**2/n
B = 9999
W = numeric(B)
W = rep(0,times =B)
for (i in 1:B)
W[i] = mean(sample(y,replace=T))
Z = sqrt(n)*(W-thest)/W
Z = sort(Z)
LZ = Z[250]
UZ = Z[9750]
thL = thest-UZ*sqrt(V)
thU = thest-LZ*sqrt(V)
thL
thU
```

Which results in the interval $[89.64146, 125.3428]$

(c) We again adapt the code from handout 11:

```
n= length(y)
thest = mean(y)
V = thest**2/n
B = 9999
#calculate mle for lambda
lambda_hat <- 1 / thest
for(i in seq_len(B)) {
  W[i] <- mean(rexp(n, rate = lambda_hat))
}
Z = sqrt(n)*(W-thest)/W
Z = sort(Z)
LZ = Z[250]
UZ = Z[9750]
```

```
thL <- thest - UZ * sqrt(V)
thU <- thest - LZ * sqrt(V)
thL
thU
```

This results in the interval $[86.94429, 129.0083]$

2. For basic bootstrap we have:

```
n = 100
B_sim = 50
true_mu = 100
count = 0
width_total = 0
for (i in 1:B_sim) {
  x_sim <- rexp(n, rate = 1/true_mu)
  thest = mean(x_sim)
  B = 9999
  thestS = numeric(B)
  thestS = rep(0,times =B)
  for (i in 1:B)
  thestS[i] = mean(sample(x_sim,replace=T))
  RS= sort(thestS-thest)
  LRS = RS[249] #handout was wrong, wasn't doing 0 indexing.... too lazy to fix it in m
  URS = RS[9749]
  thL = thest-URS
  thU = thest-LRS
  width = thU - thL
  width_total += width
  if (thL <= true_mu && true_mu <= thU) {
    count <- count + 1
  }
}

cov <- count / B_sim
cov
width_avg = width_total / B_sim
width_avg
```

which yields coverage .96 and average width 38.66296.
For studentized we have:

```
n = 100
B_sim = 50
true_mu = 100
```

```
count = 0
width_total = 0
for (i in 1:B_sim) {
    x_sim <- rexp(n, rate = 1/true_mu)
    thest = mean(x_sim)
    V = thest**2/n
    B = 9999
    W = numeric(B)
    W = rep(0,times =B)
    for (i in 1:B)
    W[i] = mean(sample(x_sim,replace=T))
    Z = sqrt(n)*(W-thest)/W
    Z = sort(Z)
    LZ = Z[250]
    UZ = Z[9750]
    thL = thest-UZ*sqrt(V)
    thU = thest-LZ*sqrt(V)
    width = thU - thL
    width_total = width_total + width
    if (thL <= true_mu && true_mu <= thU) {
        count <- count + 1
    }
}

cov <- count / B_sim
cov
width_avg = width_total / B_sim
width_avg
```

Which yields a coverage of .98 and an average width of 39.16417
For parametric bootstrap we have:

```
n = 100
B_sim = 50
true_mu = 100
count = 0
width_total = 0
for (i in 1:B_sim) {
    x_sim <- rexp(n, rate = 1/true_mu)
    thest = mean(x_sim)
    V = thest**2/n
    B = 9999
    W = numeric(B)
    W = rep(0,times =B)
    lambda_hat <- 1 / thest
```

```
    for(i in seq_len(B)) {
    W[i] <- mean(rexp(n, rate = lambda_hat))
    }
    Z = sqrt(n)*(W-thest)/W
    Z = sort(Z)
    LZ = Z[250]
    UZ = Z[9750]
    thL <- thest - UZ * sqrt(V)
    thU <- thest - LZ * sqrt(V)
    width = thU - thL
    width_total = width_total + width
    if (thL <= true_mu && true_mu <= thU) {
        count <- count + 1
    }
}
cov <- count / B_sim
cov
width_avg = width_total / B_sim
width_avg
```

Which yields a coverage of .98 and an average width of 40.04141

Problem 3

1. (a) from handout 11, the formula for Wald CI for a proportion is:

$$CI = \hat{p} \pm Z_{\frac{\alpha}{2}} \cdot \frac{\sqrt{\hat{p} \cdot (1-\hat{p})}}{\sqrt{n}}$$

From the data we can easily find that $\hat{p} = \frac{9}{24} \approx .375$, so for a 95% CI we have:

$$\begin{aligned} CI &= .375 \pm Z_{.025} \cdot \frac{\sqrt{.375 \cdot (1-.375)}}{\sqrt{24}} \\ &= .375 \pm -1.96 \cdot \frac{\sqrt{.375 \cdot (1-.375)}}{\sqrt{24}} \\ &= 0.18131049331468677, 0.5686895066853133 \end{aligned}$$

(b) from handout 11, the formula for Wilson CI for a proportion is:

$$CI = \frac{Y + \frac{1}{2} \cdot Z^2_{\frac{\alpha}{2}}}{n + Z^2_{\frac{\alpha}{2}}} \pm \frac{\sqrt{n} Z_{\frac{\alpha}{2}} \sqrt{\hat{p}(1-\hat{p}) + \frac{1}{4n} Z^2_{\frac{\alpha}{2}}}}{n + Z^2_{\frac{\alpha}{2}}}$$

We know $Y = 9$, plugging everything in we get:

$$\frac{Y + \frac{1}{2} \cdot Z^2_{\frac{\alpha}{2}}}{n + Z^2_{\frac{\alpha}{2}}} \pm \frac{\sqrt{n} Z_{\frac{\alpha}{2}} \sqrt{\hat{p}(1-\hat{p}) + \frac{1}{4n} Z^2_{\frac{\alpha}{2}}}}{n + Z^2_{\frac{\alpha}{2}}} = \frac{9 + \frac{1}{2} \cdot 1.96^2}{24 + 1.96^2} \pm \frac{\sqrt{24} \cdot -1.96 \sqrt{.375(1-.375) + \frac{1}{4 \cdot 24} 1.96^2}}{24 + 1.96^2}$$

$$= 0.21159133433631455, 0.3922475719786219$$

(c) Adapting the previous code we have:

```
y <- c(10.75, 10.87, 10.92, 11.23, 11.26, 11.30, 11.31, 11.32, 11.39, 11.41, 11.52,
11.58, 11.62, 11.62, 11.73, 11.77, 11.80, 11.88, 11.90, 11.93, 12.02, 12.10, 12.27)
n= length(y)
thest = 9 / 24
B = 9999
thestS = numeric(B)
thestS = rep(0,times =B)
for (i in 1:B)
thestS[i] = mean(sample(y,replace=T) > 11.7)
RS= sort(thestS-thest)
LRS = RS[250]
URS = RS[9750]
thL = thest-URS
thU = thest-LRS
thL
thU
```

which yields $[0.1666667, 0.5416667]$

2. We will use power transformation to transform the data to a normal distribution. The code is still the same:

```
  y <- c(10.75, 10.87, 10.92, 11.23, 11.26, 11.30, 11.31, 11.32, 11.39, 11.41, 11.52, 1
11.58, 11.62, 11.62, 11.73, 11.77, 11.80, 11.88, 11.90, 11.93, 12.02, 12.10, 12.27)
n = length(y)
yt0 = log(y)
s = sum(yt0)
varyt0 = var(yt0)
Lt0 = -1*s - .5*n*(log(2*pi*varyt0)+1)
th = 0
Lt = 0
t = -3.01
i = 0
while(t < 3)
{t = t+.001
i = i+1
```

```
th[i] = t
yt = (y^t -1)/t
varyt = var(yt)
Lt[i] = (t-1)*s - .5*n*(log(2*pi*varyt)+1)
if(abs(th[i])<1.0e-10)Lt[i]<-Lt0
if(abs(th[i])<1.0e-10)th[i]<-0
}
# The following outputs the values of the likelihood and theta and yields
# the value of theta where likelihood is a maximum
out = cbind(th,Lt)
Ltmax= max(Lt)
Ltmax
imax= which(Lt==max(Lt))
thmax= th[imax]
thmax
```

this yields the optimal $\theta \approx 3$. Thus we have the transformation:

```
y = (y^3 - 1) / 3
```

and we then have the calculation of the tolerance lower bound:

```
n = length(y)
G = .90
P = .95
za = qnorm(G)
zb = qnorm(P)
a = 1-za^2/(2*(n-1))
b = zb^2-za^2/n
K1Side = (zb+sqrt(zb^2-a*b))/a
lower = mean(y) - K1Side * sd(y)
lower
```

And its back transformation:

```
lower_transform = (lower * 3 + 1)^(1/3)
lower_transform
```

Which yields a lower bound of 10.67127. So we are 95% confident that at least 90% of bug report times are greater than 10.67127.

Problem 4

1. The pmf for the poisson is:

$$P(X = k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}$$

in our case $\lambda = 18.4$. This is the python code i wrote to find the expected number of occurences in each range:

```
  import math
def poisson(k, _lambda = 18.4):
    return _lambda**k * math.e**(-_lambda) / math.factorial(k)
def q4():
    ranges = [
        [0, 10],
        [11, 15],
        [16, 20],
        [21, 25],
        [26, 30],
        [31, 125]
    ]
    probabilities = [0] * len(ranges)
    index = 0
    for range in ranges:
        i = range[0]
        while(i <= range[1]):
            probabilities[index] += poisson(i)
            i+= 1
        index += 1
    print(probabilities)
    expected_values = [0] * len(ranges)
    index = 0
    for prob in probabilities:
        expected_values[index] = prob * 150
        index += 1
    print(expected_values)
def main():
    q4()

if __name__ == "__main__":
    main()
```

which yielded [3.7318574019243775, 34.71073099633768, 66.27724184984828, 37.056722125605354, 7.544586902599101, 0.6788607236853513] for the respective ranges.

2. I computed the Q stat in python as well:

```
observed = [5, 34, 64, 40, 5, 2]
q = 0
i = 0
while(i < len(observed)):
    q += (observed[i] - expected_values[i])**2 / expected_values[i]
    i += 1
print(q)
```

   which yields a q stat of 4.186811813146186. Plugging this into R:

```
k = 6
Q = 4.186811813146186
p = 1 - pchisq(Q, k - 1)
p
```

   we get a p value of 0.5228452 ¿ .25 which indicates the poisson model does in fact fit the distribution very well.

Problem 5

1. The handout recommends using A-C for when $n \geq 40$, so that's what we will use:

```
x <- c(5.8, 10.1, 12.0, 12.5, 16.0, 18.6, 18.9, 19.0, 19.2, 19.6, 21.5, 22.3, 23.2, 23.
23.7, 24.3, 24.8, 25.2, 25.5, 25.9, 26.2, 26.3, 26.7, 26.8, 27.0, 27.1, 27.1, 27.1,
27.2, 28.1, 28.1, 28.3, 28.4, 28.4, 28.6, 28.7, 29.5, 29.6, 29.7, 30.0, 30.2, 30.2,
30.6, 30.8, 31.2, 31.5, 31.6, 31.8, 32.3, 32.4, 32.8, 33.1, 34.4, 34.6, 35.5, 36.6,
36.7, 38.0, 38.9, 39.2)
n = length(x)
Y = sum(x < 30)
p = Y / n
p_hat = (Y + 2) / (n + 4)
n_hat = n + 4
lower = p_hat - 1.96 * sqrt(p_hat * (1 - p_hat)) / sqrt(n_hat)
upper = p_hat + 1.96 * sqrt(p_hat * (1 - p_hat)) / sqrt(n_hat)
lower
upper
```

   yields: [0.5230698, 0.7581802]

2. Using a very similar method to that used in problem 1 b, we have:

```
L=.95
P=.5
s=ceiling(n*P)-1
r=floor(n*P)+1
cov=0
while(s<n-1 && r>1 && cov<L)
{s=s+1
cov=pbinom(s-1,n,P)-pbinom(r-1,n,P)
if(cov>=L) break;
r=r-1
cov=pbinom(s-1,n,P)-pbinom(r-1,n,P)
}
x[r]
x[s]
cov
```

which yields 26.3, 29.6 as the 95% CI for the median.

3. For this we will transform to normal, use the code for finding two sided tolerance confidence interval, then back-transform the endpoints:

```
yt0 = log(y)
s = sum(yt0)
varyt0 = var(yt0)
Lt0 = -1*s - .5*n*(log(2*pi*varyt0)+1)
th = 0
Lt = 0
t = -3.01
i = 0
while(t < 3)
{t = t+.001
i = i+1
th[i] = t
yt = (y^t -1)/t
varyt = var(yt)
Lt[i] = (t-1)*s - .5*n*(log(2*pi*varyt)+1)
if(abs(th[i])<1.0e-10)Lt[i]<-Lt0
if(abs(th[i])<1.0e-10)th[i]<-0
}
# The following outputs the values of the likelihood and theta and yields
# the value of theta where likelihood is a maximum
out = cbind(th,Lt)
Ltmax= max(Lt)
Ltmax
```

```
imax= which(Lt==max(Lt))
thmax= th[imax]
thmax

y = (y^thmax - 1) / thmax
G = .95
P = .95
Chi = qchisq(1-(1 - G) / 2,n-1)
z = qnorm((1+P)/2)
K2Side = sqrt(((n-1)*(n+1)*z^2)/(n*Chi))
lower = mean(y) - K2Side * sd(y)
upper = mean(y) + K2Side * sd(y)
lower_transform = (lower * thmax + 1)^(1 / thmax)
upper_transform = (upper * thmax + 1)^(1 / thmax)
lower_transform
upper_transform
```

We can be 95% confident that at least 95% of values will fall in [14.78721, 36.91857]

Problem 6

1. B
2. A
3. B
4. B
5. C
6. D
7. A
8. A