

STAT 641

Homework 1

Keegan Smith

June 9, 2025

Question Group 1

1.1

1. The cumulative distribution function $F(z)$ can be given by:

$$F(z) = \int_{-\infty}^z f(x)dx$$

Thus we have:

$$F(z) = \int_{-\infty}^0 f(x)dx + \int_0^z f(x)dx$$

$$F(z) = 0 + \int_0^z \lambda \cdot e^{-\lambda \cdot x} dx$$

$$F(z) = (-e^{-\lambda \cdot x})_0^z$$

$$F(z) = -e^{-\lambda \cdot z} + 1$$

2. Given $F(z)$ from the above:

$$F(z) = 0.5$$

$$-e^{-\lambda \cdot z} + 1 = .5$$

$$-e^{-\lambda \cdot z} = -.5$$

$$-\lambda \cdot z = \ln(.5)$$

$$z = -\frac{\ln(.5)}{\lambda}$$

1.2

1. The z stat for 60 is:

$$z = \frac{60 - 50}{10}$$
$$z = 1$$

Using a calculator, the cdf of z is .8413

2. we are trying to find the value such that $\text{cdf}(z) = .95$ where z is the z statistic:

$$\text{cdf}(z) = .95$$
$$z = \text{icdf}(.95)$$
$$\frac{x - \mu}{\sigma} = \text{icdf}(.95)$$
$$x = \text{icdf}(.95) \cdot \sigma + \mu$$
$$x \approx 66.4485$$

Question Group 2

2.1

1. μ is a location variable if:

$$f_W(w) = f_Y(w + \mu)$$

does not depend on μ where $W = Y - \mu$.

For $\xi \neq 0$ we have:

$$f_W(w) = f_Y(w + \mu)$$
$$= \frac{1}{\sigma} \cdot \left(1 + \xi \cdot \frac{(z + \mu) - \mu}{\sigma}\right)^{-1 - \frac{1}{\xi}} \exp\left(-\left(1 + \xi \cdot \frac{(z + \mu) - \mu}{\sigma}\right)^{-\frac{1}{\xi}}\right)$$
$$= \frac{1}{\sigma} \cdot \left(1 + \xi \cdot \frac{z}{\sigma}\right)^{-1 - \frac{1}{\xi}} \exp\left(-\left(1 + \xi \cdot \frac{z}{\sigma}\right)^{-\frac{1}{\xi}}\right)$$

The pdf of $f_W(w)$ does not depend on μ so μ is a location variable when $\xi \neq 0$

For $\xi = 0$ we have:

$$\begin{aligned} f_W(w) &= f_Y(w + \mu) \\ &= \frac{1}{\sigma} \cdot \exp\left(-\frac{(z + \mu) - \mu}{\sigma}\right) \exp(-\exp(-\frac{(z + \mu) - \mu}{\sigma})) \\ &= \frac{1}{\sigma} \cdot \exp\left(-\frac{z}{\sigma}\right) \exp(-\exp(-\frac{z}{\sigma})) \end{aligned}$$

again the pdf of $f_W(w)$ does not depend on μ so μ is a location variable when $\xi = 0$. Therefore μ is a location variable.
 σ is a scaling variable if:

$$f_W(w) = \sigma \cdot f_Y(\sigma w)$$

does not depend on σ where $W = \frac{Y}{\sigma}$.
Since μ is a location variable, let $Y = Z - \mu$
For $\xi \neq 0$ we have:

$$\begin{aligned} f_W(w) &= \sigma \cdot f_Y(\sigma w) \\ &= \sigma \cdot \frac{1}{\sigma} \cdot (1 + \xi \cdot \frac{y \cdot \sigma}{\sigma})^{-1-\frac{1}{\xi}} \exp(-(1 + \xi \cdot \frac{y \cdot \sigma}{\sigma})^{-\frac{1}{\xi}}) \\ &= (1 + \xi \cdot y)^{-1-\frac{1}{\xi}} \exp(-(1 + \xi \cdot y)^{-\frac{1}{\xi}}) \end{aligned}$$

The final function does not depend on σ so σ is a scaling variable when $\xi \neq 0$
For $\xi = 0$ we have:

$$\begin{aligned} f_W(w) &= \sigma \cdot f_Y(\sigma w) \\ &= \sigma \cdot \frac{1}{\sigma} \cdot \exp\left(-\frac{y \cdot \sigma}{\sigma}\right) \exp(-\exp(-\frac{y \cdot \sigma}{\sigma})) \\ &= \exp(-y) \exp(-\exp(-y)) \end{aligned}$$

The final function does not depend on σ so σ is a scaling variable for $\xi = 0$.
 σ is a scaling variable both when $\xi = 0$ and $\xi \neq 0$ so σ is a scaling variable for the family.

2. The quantile function is simply the inverse of the cdf:

$$Q(p) = F^{-1}(p)$$

Thus we have:

$$\begin{aligned}
 p &= \exp(-(1 + \xi(\frac{z - \mu}{\sigma}))^{-\frac{1}{\xi}}) \\
 \ln(p) &= -(1 + \xi(\frac{z - \mu}{\sigma}))^{-\frac{1}{\xi}} \\
 (-\ln(p))^{-\xi} &= 1 + \xi(\frac{z - \mu}{\sigma}) \\
 (-\ln(p))^{-\xi} - 1 &= \xi(\frac{z - \mu}{\sigma}) \\
 \frac{(-\ln(p))^{-\xi} - 1}{\xi} \cdot \sigma &= z - \mu \\
 z &= \frac{(-\ln(p))^{-\xi} - 1}{\xi} \cdot \sigma + \mu
 \end{aligned}$$

Thus the quantile function for $\xi \neq 0$ is:

$$Q(p) = \frac{(-\ln(p))^{-\xi} - 1}{\xi} \cdot \sigma + \mu$$

for when $\xi = 0$:

$$\begin{aligned}
 p &= \exp(-\exp(-\frac{z - \mu}{\sigma})) \\
 \ln(p) &= -\exp(-\frac{z - \mu}{\sigma}) \\
 \ln(-\ln(p)) &= -\frac{z - \mu}{\sigma} \\
 -\sigma \cdot \ln(-\ln(p)) + \mu &= z
 \end{aligned}$$

Thus the quantile function for $\xi = 0$ is:

$$Q(p) = -\sigma \cdot \ln(-\ln(p)) + \mu$$

3. The probability that a value is greater than x is given by:

$$P(X > x) = 1 - F(x)$$

Thus we have:

$$\begin{aligned}
 P(X > 12) &= 1 - F(12) \\
 &= 1 - \exp(-(1 + \frac{1}{2} \cdot \frac{12 - 10}{2})^{-\frac{1}{\frac{1}{2}}}) \\
 &\approx .3588
 \end{aligned}$$

4. We simply need to evaluate $Q(.25)$:

$$Q(.25) = \frac{(-\ln(.25))^{-\frac{1}{2}} - 1}{\frac{1}{2}} \cdot 2 + 10$$

$$\approx 9.397$$

2.2

1. β acts as a scaling parameter if:

$$f_W(w) = \beta \cdot f_Y(\beta w)$$

does not depend on β where $W = \frac{Y}{\beta}$.

Thus we have:

$$f_W(w) = \beta \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} (y \cdot \beta)^{\alpha-1} \cdot e^{-\frac{\beta \cdot y}{\beta}}$$

$$f_W(w) = \beta \cdot \frac{1}{\beta \cdot \Gamma(\alpha)} y^{\alpha-1} \cdot e^{-y}$$

$$f_W(w) = \frac{1}{\Gamma(\alpha)} y^{\alpha-1} \cdot e^{-y}$$

The final function doesn't rely on β so β is a scaling variable for $y \geq 0$. The case for $y < 0$ is trivial since the pdf is just 0. So β is a scaling variable for this family.

2. We now have:

$$f(y) = \frac{1}{\frac{1}{\theta} \Gamma(\alpha)} (y)^{\alpha-1} \cdot e^{-\frac{y}{\theta}}$$

Instead of using $W = \frac{Y}{\theta}$, we will use $W = \frac{Y}{\theta-1}$:

$$f_W(w) = \frac{1}{\theta} \cdot \frac{1}{\frac{1}{\theta} \Gamma(\alpha)} (y \cdot \theta^{-1})^{\alpha-1} \cdot e^{-\frac{(y \cdot \theta^{-1})}{\theta}}$$

$$f_W(w) = \frac{1}{\theta} \cdot \frac{1}{(\theta^{-1}) \Gamma(\alpha)} (y^{\alpha-1}) \cdot e^{-y}$$

$$f_W(w) = \frac{1}{\Gamma(\alpha)} (y^{\alpha-1}) \cdot e^{-y}$$

The final function doesn't depend on θ so θ is a scaling variable (again $y < 0$ is trivial).

2.3

1. Assuming the call rate follows a Poisson distribution with $\lambda = 12$ then we have:

$$P(Y = y) = \frac{e^{-12} \cdot 12^y}{y!}$$

$$\begin{aligned} P(Y \geq 3) &= 1 - P(Y \leq 2) \\ &= 1 - P(Y = 0) - P(Y = 1) - P(Y = 2) \\ &= 1 - e^{-12} - 12e^{-12} - 72e^{-12} \\ &= 1 - 85 \cdot e^{-12} \\ &= 0.9995 \end{aligned}$$

2. Since the rate of calls is constant, the distribution for the calls in a week also follows a poisson distribution with $\lambda = 12 \cdot 7 = 84$
Thus we have:

$$P(Y = y) = \frac{e^{-84} \cdot 84^y}{y!}$$

$$\begin{aligned} P(Y \geq 3) &= 1 - P(Y \leq 2) \\ &= 1 - P(Y = 0) - P(Y = 1) - P(Y = 2) \\ &= 1 - e^{-84} - 84e^{-84} - 3528e^{-84} \\ &\approx 1.0000 \end{aligned}$$