STAT 641
Homework 1

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Question Group 1

1.1

1. The cumulative distribution function $F(z)$ can be given by:

   $$F(z) = \int_{-\infty}^{z} f(x)dx$$

   Thus we have:

   $$\begin{aligned}
   F(z) &= \int_{-\infty}^{0} f(x)dx + \int_{0}^{z} f(x)dx \\
   F(z) &= 0 + \int_{0}^{z} \lambda \cdot e^{-\lambda \cdot x} dx \\
   F(z) &= (-e^{-\lambda \cdot x})_{0}^{z} \\
   F(z) &= -e^{-\lambda \cdot z} + 1
   \end{aligned}$$

2. Given $F(z)$ from the above:

   $$\begin{aligned}
   F(z) &= 0.5 \\
   -e^{-\lambda \cdot z} + 1 &= .5 \\
   -e^{-\lambda \cdot z} &= -.5 \\
   -\lambda \cdot z &= \ln(.5) \\
   z &= -\frac{\ln(.5)}{\lambda}
   \end{aligned}$$

1.2

1. The z stat for 60 is:

$$z = \frac{60 - 50}{10}$$
$$z = 1$$

   Using a calculator, the cdf of z is .8413

2. we are trying to find the value such that cdf(z) = .95 where z is the z statistic:

$$\begin{aligned} cdf(z) &= .95 \\ z &= icdf(.95) \\ \frac{x-\mu}{\sigma} &= icdf(.95) \\ x &= icdf(.95) \cdot \sigma + \mu \\ x &\approx 66.4485 \end{aligned}$$

Question Group 2

2.1

1. μ is a location variable if:

$$f_W(w) = f_Y(w + \mu)$$

   does not depend on μ where $W = Y - \mu$.
   For $\xi \neq 0$ we have:

$$\begin{aligned} f_W(w) &= f_Y(w+\mu) \\ &= \frac{1}{\sigma} \cdot (1 + \xi \cdot \frac{(z+\mu)-\mu}{\sigma})^{-1-\frac{1}{\xi}} \exp(-(1+\xi \cdot \frac{(z+\mu)-\mu}{\sigma})^{-\frac{1}{\xi}}) \\ &= \frac{1}{\sigma} \cdot (1 + \xi \cdot \frac{z}{\sigma})^{-1-\frac{1}{\xi}} \exp(-(1+\xi \cdot \frac{z}{\sigma})^{-\frac{1}{\xi}}) \end{aligned}$$

   The pdf of $f_W(w)$ does not depend on μ so μ is a location variable when $\xi \neq 0$

For $\xi = 0$ we have:

$$\begin{aligned} f_W(w) &= f_Y(w+\mu) \\ &= \frac{1}{\sigma} \cdot \exp(-\frac{(z+\mu)-\mu}{\sigma})\exp(\text{-}\exp(-\frac{(z+\mu)-\mu}{\sigma})) \\ &= \frac{1}{\sigma} \cdot \exp(-\frac{z}{\sigma})\exp(\text{-}\exp(-\frac{z}{\sigma})) \end{aligned}$$

again the pdf of $f_W(w)$ does not depend on μ so μ is a location variable when $\xi = 0$. Therefore μ is a location variable.
σ is a scaling variable if:

$$f_W(w) = \sigma \cdot f_Y(\sigma w)$$

does not depend on σ where $W = \frac{Y}{\sigma}$.
Since μ is a location variable, let $Y = Z - \mu$
For $\xi \neq 0$ we have:

$$\begin{aligned} f_W(w) &= \sigma \cdot f_Y(\sigma w) \\ &= \sigma \cdot \frac{1}{\sigma} \cdot (1+\xi \cdot \frac{y\cdot\sigma}{\sigma})^{-1-\frac{1}{\xi}} \exp(-(1+\xi\cdot\frac{y\cdot\sigma}{\sigma})^{-\frac{1}{\xi}}) \\ &= (1+\xi\cdot y)^{-1-\frac{1}{\xi}}\exp(-(1+\xi\cdot y)^{-\frac{1}{\xi}}) \end{aligned}$$

The final function does not depend on σ so σ is a scaling variable when $\xi \neq 0$
For $\xi = 0$ we have:

$$\begin{aligned} f_W(w) &= \sigma \cdot f_Y(\sigma w) \\ &= \sigma \cdot \frac{1}{\sigma} \cdot \exp(-\frac{y\cdot\sigma}{\sigma})\exp(\text{-}\exp(-\frac{y\cdot\sigma}{\sigma})) \\ &= \exp(-y)\exp(\text{-}\exp(-y)) \end{aligned}$$

The final function does not depend on σ so σ is a scaling variable for $\xi = 0$.
σ is a scaling variable both when $\xi = 0$ and $\xi \neq 0$ so σ is a scaling variable for the family.

2. The quantile function is simply the inverse of the cdf:

$$Q(p) = F^{-1}(p)$$

Thus we have:

$$
\begin{aligned}
p &= \exp(-(1+\xi(\frac{z-\mu}{\sigma}))^{-\frac{1}{\xi}}) \\
\ln(p) &= -(1+\xi(\frac{z-\mu}{\sigma}))^{-\frac{1}{\xi}} \\
(-\ln(p))^{-\xi} &= 1+\xi(\frac{z-\mu}{\sigma}) \\
(-\ln(p))^{-\xi} - 1 &= \xi(\frac{z-\mu}{\sigma}) \\
\frac{(-\ln(p))^{-\xi} - 1}{\xi} \cdot \sigma &= z-\mu \\
z &= \frac{(-\ln(p))^{-\xi} - 1}{\xi} \cdot \sigma + \mu
\end{aligned}
$$

Thus the quantile function for $\xi \neq 0$ is:

$$Q(p) = \frac{(-\ln(p))^{-\xi} - 1}{\xi} \cdot \sigma + \mu$$

for when $\xi = 0$:

$$
\begin{aligned}
p &= \exp(-\exp(-\frac{z-\mu}{\sigma})) \\
\ln(p) &= -\exp(-\frac{z-\mu}{\sigma}) \\
\ln(-\ln(p)) &= -\frac{z-\mu}{\sigma} \\
-\sigma \cdot \ln(-\ln(p)) + \mu &= z
\end{aligned}
$$

Thus the quantile function for $\xi = 0$ is:

$$Q(p) = -\sigma \cdot \ln(-\ln(p)) + \mu$$

3. The probability that a value is greater than x is given by:

$$P(X > x) = 1 - F(x)$$

Thus we have:

$$
\begin{aligned}
P(X > 12) &= 1 - F(12) \\
&= 1 - \exp(-(1 + \frac{1}{2} \cdot \frac{12-10}{2})^{-\frac{1}{\frac{1}{2}}}) \\
&\approx .3588
\end{aligned}
$$

4. We simply need to evaluate $Q(.25)$:

$$\begin{aligned} Q(.25) &= \frac{(-\ln(.25))^{-\frac{1}{2}} - 1}{\frac{1}{2}} \cdot 2 + 10 \\ &\approx 9.397 \end{aligned}$$

2.2

1. β acts as a scaling parameter if:

$$f_W(w) = \beta \cdot f_Y(\beta w)$$

does not depend on β where $W = \frac{Y}{\beta}$.
Thus we have:

$$\begin{aligned} f_W(w) &= \beta \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)}(y \cdot \beta)^{\alpha-1} \cdot e^{\frac{-\beta \cdot y}{\beta}} \\ f_W(w) &= \beta \cdot \frac{1}{\beta \cdot \Gamma(\alpha)} y^{\alpha-1} \cdot e^{-y} \\ f_W(w) &= \frac{1}{\Gamma(\alpha)} y^{\alpha-1} \cdot e^{-y} \end{aligned}$$

The final function doesn't rely on β so β is a scaling variable for $y \geq 0$. The case for $y < 0$ is trivial since the pdf is just 0. So β is a scaling variable for this family.

2. We now have:

$$f(y) = \frac{1}{\frac{1}{\theta}^\alpha \Gamma(\alpha)}(y)^{\alpha-1} \cdot e^{\frac{-y}{\frac{1}{\theta}}}$$

Instead of using $W = \frac{Y}{\theta}$, we will use $W = \frac{Y}{\theta^{-1}}$:

$$\begin{aligned} f_W(w) &= \frac{1}{\theta} \cdot \frac{1}{\frac{1}{\theta}^\alpha \Gamma(\alpha)}(y \cdot \theta^{-1})^{\alpha-1} \cdot e^{\frac{-(y \cdot \theta^{-1})}{\frac{1}{\theta}}} \\ f_W(w) &= \frac{1}{\theta} \cdot \frac{1}{(\theta^{-1})\Gamma(\alpha)}(y^{\alpha-1}) \cdot e^{-y} \\ f_W(w) &= \frac{1}{\Gamma(\alpha)}(y^{\alpha-1}) \cdot e^{-y} \end{aligned}$$

The final function doesn't depend on θ so θ is a sclaing variable (again $y < 0$ is trivial).

2.3

1. Assuming the call rate follows a Poisson distribution with $\lambda = 12$ then we have:

$$P(Y = y) = \frac{e^{-12} \cdot 12^y}{y!}$$

$$\begin{aligned}
P(Y \geq 3) &= 1 - P(Y \leq 2) \\
&= 1 - P(Y = 0) - P(Y = 1) - P(Y = 2) \\
&= 1 - e^{-12} - 12e^{-12} - 72e^{-12} \\
&= 1 - 85 \cdot e^{-12} \\
&= 0.9995
\end{aligned}$$

2. Since the rate of calls is constant, the distribution for the calls in a week also follows a poisson distribution with $\lambda = 12 \cdot 7 = 84$
Thus we have:

$$P(Y = y) = \frac{e^{-84} \cdot 84^y}{y!}$$

$$\begin{aligned}
P(Y \geq 3) &= 1 - P(Y \leq 2) \\
&= 1 - P(Y = 0) - P(Y = 1) - P(Y = 2) \\
&= 1 - e^{-84} - 84e^{-84} - 3528e^{-84} \\
&\approx 1.0000
\end{aligned}$$

2.4

1. The cdf for an exponential distribution with mean wait time of 10 minutes is given by:

$$F(t) = 1 - e^{-\frac{1}{10} \cdot t}$$

So we have:

$$\begin{aligned}
F(5) &= 1 - e^{-\frac{1}{10} \cdot 5} \\
&= 1 - e^{-\frac{1}{2}} \\
&\approx 0.3935
\end{aligned}$$

2. The probability that time is greater than 15 is $1 - F(15)$

$$\begin{aligned}
1 - F(15) &= 1 - (1 - e^{-\frac{1}{10} \cdot 15}) \\
&= 1 - (1 - e^{-\frac{3}{2}}) \\
&= e^{-\frac{3}{2}} \\
&\approx 0.2231
\end{aligned}$$

3. The probability that a call will occur in the next 5 minutes is 0.3935 since the amount of time between events does not depend on the amount of time that has passed.