STAT 641
Homework 2

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June 10, 2025

Question Group 1

1.1

1. The cumulative distribution function $F(z)$ can be given by:

$$F(z) = \int_{-\infty}^{z} f(x)dx$$

Thus we have:

$$\begin{aligned}
F(z) &= \int_{-\infty}^{0} f(x)dx + \int_{0}^{z} f(x)dx \\
F(z) &= 0 + \int_{0}^{z} \lambda \cdot e^{-\lambda \cdot x} dx \\
F(z) &= (-e^{-\lambda \cdot x})_{0}^{z} \\
F(z) &= -e^{-\lambda \cdot z} + 1
\end{aligned}$$

2. Given $F(z)$ from the above:

$$\begin{aligned}
F(z) &= 0.5 \\
-e^{-\lambda \cdot z} + 1 &= .5 \\
-e^{-\lambda \cdot z} &= -.5 \\
-\lambda \cdot z &= \ln(.5) \\
z &= -\frac{\ln(.5)}{\lambda}
\end{aligned}$$

1.2

1. The z stat for 60 is:

$$
\begin{aligned}
z &= \frac{60-50}{10} \\
z &= 1
\end{aligned}
$$

   Using a calculator, the cdf of z is .8413

2. we are trying to find the value such that cdf(z) = .95 where z is the z statistic:

$$
\begin{aligned}
cdf(z) &= .95 \\
z &= icdf(.95) \\
\frac{x-\mu}{\sigma} &= icdf(.95) \\
x &= icdf(.95) \cdot \sigma + \mu \\
x &\approx 66.4485
\end{aligned}
$$

Question Group 2

2.1

1. μ is a location variable if:

$$f_W(w) = f_Y(w+\mu)$$

   does not depend on μ where $W = Y - \mu$.
   For $\xi \neq 0$ we have:

$$
\begin{aligned}
f_W(w) &= f_Y(w+\mu) \\
&= \frac{1}{\sigma} \cdot (1 + \xi \cdot \frac{(z+\mu)-\mu}{\sigma})^{-1-\frac{1}{\xi}} \exp(-(1+\xi \cdot \frac{(z+\mu)-\mu}{\sigma})^{-\frac{1}{\xi}}) \\
&= \frac{1}{\sigma} \cdot (1 + \xi \cdot \frac{z}{\sigma})^{-1-\frac{1}{\xi}} \exp(-(1+\xi \cdot \frac{z}{\sigma})^{-\frac{1}{\xi}})
\end{aligned}
$$

   The pdf of $f_W(w)$ does not depend on μ so μ is a location variable when $\xi \neq 0$

For $\xi = 0$ we have:

$$\begin{aligned}
f_W(w) &= f_Y(w+\mu) \\
&= \frac{1}{\sigma} \cdot \exp(-\frac{(z+\mu)-\mu}{\sigma})\exp(\text{-}\exp(-\frac{(z+\mu)-\mu}{\sigma})) \\
&= \frac{1}{\sigma} \cdot \exp(-\frac{z}{\sigma})\exp(\text{-}\exp(-\frac{z}{\sigma}))
\end{aligned}$$

again the pdf of $f_W(w)$ does not depend on μ so μ is a location variable when $\xi = 0$. Therefore μ is a location variable.
σ is a scaling variable if:

$$f_W(w) = \sigma \cdot f_Y(\sigma w)$$

does not depend on σ where $W = \frac{Y}{\sigma}$.
Since μ is a location variable, let $Y = Z - \mu$
For $\xi \neq 0$ we have:

$$\begin{aligned}
f_W(w) &= \sigma \cdot f_Y(\sigma w) \\
&= \sigma \cdot \frac{1}{\sigma} \cdot (1 + \xi \cdot \frac{y \cdot \sigma}{\sigma})^{-1-\frac{1}{\xi}} \exp(-(1+\xi \cdot \frac{y \cdot \sigma}{\sigma})^{-\frac{1}{\xi}}) \\
&= (1+\xi \cdot y)^{-1-\frac{1}{\xi}} \exp(-(1+\xi \cdot y)^{-\frac{1}{\xi}})
\end{aligned}$$

The final function does not depend on σ so σ is a scaling variable when $\xi \neq 0$
For $\xi = 0$ we have:

$$\begin{aligned}
f_W(w) &= \sigma \cdot f_Y(\sigma w) \\
&= \sigma \cdot \frac{1}{\sigma} \cdot \exp(-\frac{y \cdot \sigma}{\sigma})\exp(\text{-}\exp(-\frac{y \cdot \sigma}{\sigma})) \\
&= \exp(-y)\exp(\text{-}\exp(-y))
\end{aligned}$$

The final function does not depend on σ so σ is a scaling variable for $\xi = 0$.
σ is a scaling variable both when $\xi = 0$ and $\xi \neq 0$ so σ is a scaling variable for the family.

2. The quantile function is simply the inverse of the cdf:

$$Q(p) = F^{-1}(p)$$

Thus we have:

$$
\begin{aligned}
p &= \exp(-(1+\xi(\frac{z-\mu}{\sigma}))^{-\frac{1}{\xi}}) \\
\ln(p) &= -(1+\xi(\frac{z-\mu}{\sigma}))^{-\frac{1}{\xi}} \\
(-\ln(p))^{-\xi} &= 1+\xi(\frac{z-\mu}{\sigma}) \\
(-\ln(p))^{-\xi} - 1 &= \xi(\frac{z-\mu}{\sigma}) \\
\frac{(-\ln(p))^{-\xi} - 1}{\xi} \cdot \sigma &= z-\mu \\
z &= \frac{(-\ln(p))^{-\xi} - 1}{\xi} \cdot \sigma + \mu
\end{aligned}
$$

Thus the quantile function for $\xi \neq 0$ is:

$$Q(p) = \frac{(-\ln(p))^{-\xi} - 1}{\xi} \cdot \sigma + \mu$$

for when $\xi = 0$:

$$
\begin{aligned}
p &= \exp(-\exp(-\frac{z-\mu}{\sigma})) \\
\ln(p) &= -\exp(-\frac{z-\mu}{\sigma}) \\
\ln(-\ln(p)) &= -\frac{z-\mu}{\sigma} \\
-\sigma \cdot \ln(-\ln(p)) + \mu &= z
\end{aligned}
$$

Thus the quantile function for $\xi = 0$ is:

$$Q(p) = -\sigma \cdot \ln(-\ln(p)) + \mu$$

3. The probability that a value is greater than x is given by:

$$P(X > x) = 1 - F(x)$$

Thus we have:

$$
\begin{aligned}
P(X > 12) &= 1 - F(12) \\
&= 1 - \exp(-(1 + \frac{1}{2} \cdot \frac{12-10}{2})^{-\frac{1}{\frac{1}{2}}}) \\
&\approx .3588
\end{aligned}
$$

4. We simply need to evaluate $Q(.25)$:

$$\begin{aligned} Q(.25) &= \frac{(-\ln(.25))^{-\frac{1}{2}} - 1}{\frac{1}{2}} \cdot 2 + 10 \\ &\approx 9.397 \end{aligned}$$

2.2

1. β acts as a scaling parameter if:

$$f_W(w) = \beta \cdot f_Y(\beta w)$$

does not depend on β where $W = \frac{Y}{\beta}$.
Thus we have:

$$\begin{aligned} f_W(w) &= \beta \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} (y \cdot \beta)^{\alpha-1} \cdot e^{\frac{-\beta \cdot y}{\beta}} \\ f_W(w) &= \beta \cdot \frac{1}{\beta \cdot \Gamma(\alpha)} y^{\alpha-1} \cdot e^{-y} \\ f_W(w) &= \frac{1}{\Gamma(\alpha)} y^{\alpha-1} \cdot e^{-y} \end{aligned}$$

The final function doesn't rely on β so β is a scaling variable for $y \geq 0$. The case for $y < 0$ is trivial since the pdf is just 0. So β is a scaling variable for this family.

2. We now have:

$$f(y) = \frac{1}{\frac{1}{\theta}^\alpha \Gamma(\alpha)} (y)^{\alpha-1} \cdot e^{\frac{-y}{\frac{1}{\theta}}}$$

Instead of using $W = \frac{Y}{\theta}$, we will use $W = \frac{Y}{\theta^{-1}}$:

$$\begin{aligned} f_W(w) &= \frac{1}{\theta} \cdot \frac{1}{\frac{1}{\theta}^\alpha \Gamma(\alpha)} (y \cdot \theta^{-1})^{\alpha-1} \cdot e^{\frac{-(y \cdot \theta^{-1})}{\frac{1}{\theta}}} \\ f_W(w) &= \frac{1}{\theta} \cdot \frac{1}{(\theta^{-1})\Gamma(\alpha)} (y^{\alpha-1}) \cdot e^{-y} \\ f_W(w) &= \frac{1}{\Gamma(\alpha)} (y^{\alpha-1}) \cdot e^{-y} \end{aligned}$$

The final function doesn't depend on θ so θ is a sclaing variable (again $y < 0$ is trivial).

2.3

1. Assuming the call rate follows a Poisson distribution with $\lambda = 12$ then we have:

$$P(Y = y) = \frac{e^{-12} \cdot 12^y}{y!}$$

$$\begin{aligned} P(Y \geq 3) &= 1 - P(Y \leq 2) \\ &= 1 - P(Y = 0) - P(Y = 1) - P(Y = 2) \\ &= 1 - e^{-12} - 12e^{-12} - 72e^{-12} \\ &= 1 - 85 \cdot e^{-12} \\ &= 0.9995 \end{aligned}$$

2. Since the rate of calls is constant, the distribution for the calls in a week also follows a poisson distribution with $\lambda = 12 \cdot 7 = 84$
Thus we have:

$$P(Y = y) = \frac{e^{-84} \cdot 84^y}{y!}$$

$$\begin{aligned} P(Y \geq 3) &= 1 - P(Y \leq 2) \\ &= 1 - P(Y = 0) - P(Y = 1) - P(Y = 2) \\ &= 1 - e^{-84} - 84e^{-84} - 3528e^{-84} \\ &\approx 1.0000 \end{aligned}$$

2.4

1. The cdf for an exponential distribution with mean wait time of 10 minutes is given by:

$$F(t) = 1 - e^{-\frac{1}{10} \cdot t}$$

So we have:

$$\begin{aligned} F(5) &= 1 - e^{-\frac{1}{10} \cdot 5} \\ &= 1 - e^{-\frac{1}{2}} \\ &\approx 0.3935 \end{aligned}$$

2. The probability that time is greater than 15 is $1 - F(15)$

$$\begin{aligned}1 - F(15) &= 1 - (1 - e^{-\frac{1}{10}\cdot 15}) \\ &= 1 - (1 - e^{-\frac{3}{2}}) \\ &= e^{-\frac{3}{2}} \\ &\approx 0.2231\end{aligned}$$

3. The probability that a call will occur in the next 5 minutes is 0.3935 since the amount of time between events does not depend on the amount of time that has passed.

Problem 3

3.1

1. This is an exponential (β) distribution

2. This is a gamma distribution with parameters $(\alpha_2 + \alpha_3 + \alpha_4,\, \beta)$

3. This is a gamma distribution with parameters $(\alpha_5,\, 5 \cdot \beta_5)$

4. This is a chi-squared distribution with parameter $v = 2 \cdot \alpha_6$

5. This is a maxwell distribution

3.2

1. the exponential cdf is:

$$1 - e^{-\lambda \cdot y}$$

In this case $\lambda = 2$.
Thus we have:

$$\begin{aligned}
.48 &= 1 - e^{-\lambda \cdot y} \\
.52 &= e^{-\lambda \cdot y} \\
\ln(.52) &= -\lambda \cdot y \\
y &= \frac{-\ln(.52)}{2} \\
&= 0.3269
\end{aligned}$$

2. The cdf for geometric distribution is:

$$1 - (1-p)^i$$

where $p = .5$ in our case:

$$\begin{aligned}
.48 &= 1 - (1-p)^i \\
.52 &= (1-p)^i \\
\log_{.5} .52 &= i \\
i &= 0.9434
\end{aligned}$$

3. I refuse to do this by hand. Here is the python code using scipy:

```
import numpy as np
from scipy.stats import binom
value = .48
n = 15
p = .6
i = 0
total_prob = 0
while(True):
    total_prob += binom.pmf(i, n, p)
    if total_prob >= value:
        print("result: ", i)
        break;
    i += 1
```

The result is 9

4. Again:

```
total_prob = 0
i = 0
while(True):
  total_prob += poisson.pmf(i, 4)
  if(total_prob >= value):
      print("result: ", i)
      break;
  i += 1
```

The result is 4.

5. For uniform (5, 10) This should simply be $5 + 5 \cdot .48 = 7.4$

question group 4

1. binomial
2. exponential
3. poisson
4. geometric
5. binomial
6. gamma
7. exponential
8. lognormal
9. poisson
10. lognormal
11. normal

12. hypergeometric

13. weibull

14. lognormal

15. gamma

16. poisson