STAT 641
Homework 4

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Problem 1

1. the likelihood function is:

$$L(\theta_1, \theta_2, \ldots, \theta_k; y) = \Pi_{i=1}^{n} f(y_i; \theta)$$

   So plugging in the poisson distribution and our data we have:

$$\begin{aligned}
L(\theta_1, \theta_2, \ldots, \theta_k; y) &= \Pi_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} \\
&= \frac{\lambda^0 e^{-\lambda}}{0!} \cdot \frac{\lambda^1 e^{-\lambda}}{1!} \cdot (\frac{\lambda^2 e^{-\lambda}}{2!})^2 \cdot \frac{\lambda^3 e^{-\lambda}}{3!} \cdot \frac{\lambda^4 e^{-\lambda}}{4!} \cdot \frac{\lambda^5 e^{-\lambda}}{5!} \\
&= e^{-\lambda} \cdot \lambda e^{-\lambda} \cdot \frac{\lambda^4 e^{-2\lambda}}{4} \cdot \frac{\lambda^3 e^{-\lambda}}{3!} \cdot \frac{\lambda^4 e^{-\lambda}}{4!} \cdot \frac{\lambda^5 e^{-\lambda}}{5!} \\
&= \frac{\lambda^{17} \cdot e^{-7\lambda}}{69120}
\end{aligned}$$

2. The log likelihood is simply taking the natural log of the likelihood function, so we have:

$$\begin{aligned}
\ln(L(\theta_1, \theta_2, \ldots, \theta_k; y)) &= \ln(\frac{\lambda^{17} \cdot e^{-7\lambda}}{69120}) \\
&= \ln(\lambda^{17}) + \ln(e^{-7\lambda}) - \ln(69120) \\
&= \ln(\lambda^{17}) - 7\lambda - \ln(69120)
\end{aligned}$$

3. The derivative of the log likelihood is as follows:

$$\begin{aligned}
\frac{d}{d\lambda}(\ln(\lambda^{17}) - 7\lambda - \ln(69120)) &= 17 \cdot \lambda^{16} \cdot \frac{1}{\lambda^{17}} - 7 \\
&= 17 \cdot \frac{1}{\lambda} - 7
\end{aligned}$$

solving for the maximum we have:

$$\begin{aligned} 17 \cdot \frac{1}{\lambda} - 7 &= 0 \\ 7 \cdot \lambda &= 17 \\ \lambda &= \frac{17}{7} \\ &\approx 2.4286 \end{aligned}$$

Problem 2

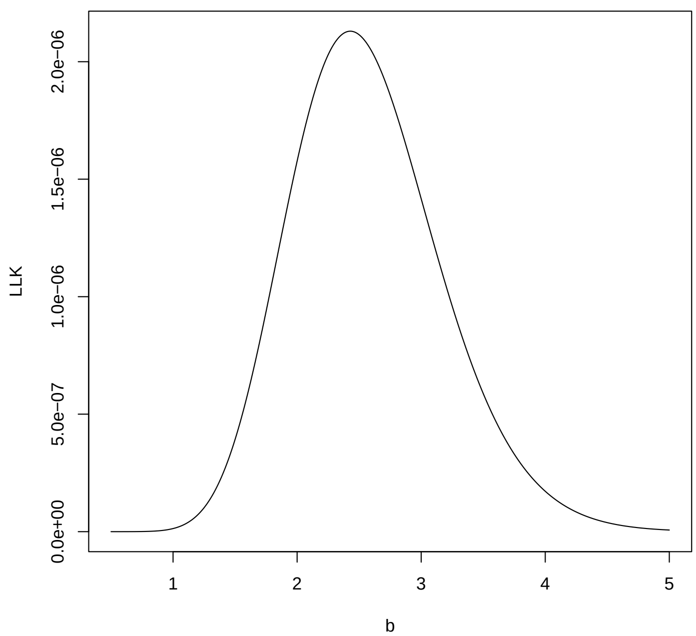
1. This is the R code for defining the log likelihood function:

```
b = seq(0, 10, .01)
LLK = b^(17) * exp(-7 * b) / 69120
```

2. This is the R code for plotting the log likelihood function:

```
b = seq(0.5, 5, .01)
LLK = b^(17) * exp(-7 * b) / 69120
plot(b, LLK, type="l")
```

This is the generated plot:


LLK
2.0e−06
1.5e−06
1.0e−06
5.0e−07
0.0e+00
1
2
3
4
5
b

3. This is the R code for finding the max value:

```
b = seq(0.5, 5, .01)
LLK = b^(17) * exp(-7 * b) / 69120
plot(b, LLK, type="l")
LK_max = max(LLK)
bmax = which(LLK == LK_max)
print(LK_max)
print(bmax)
MLE = b[bmax]
print(MLE)
```

4. The numerical output was 2.43

Problem 3

1. The survival function of the poisson distribution is:

$$S(x) = 1 - F(x)$$

where $F(x)$ is the cdf of the Poisson distribution and is defined as:

$$F(x) = \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}$$

The problem does not state that we need to find a closed form for $S(x)$, so we can just write:

$$S(x) = 1 - \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}$$

2. Plugging $x = 2$ and $\lambda = 2.4286$ into the above we get:

$$\begin{aligned} S(2) &= 1 - \left(\frac{2.4286^0 e^{-2.4286}}{0!} + \frac{2.4286^1 e^{-2.4286}}{1!} + \frac{2.4286^2 e^{-2.4286}}{2!}\right) \\ &\approx 0.4359 \end{aligned}$$

3. Instead of using the poisson distribution, we can use a non-parametric estimation where we use the edf:

$$F(x) = \frac{1}{n} \sum_{k=0}^{n-1} I(y_k \leq x)$$

where the survival function is:

$$S(x) = 1 - \frac{1}{n}\sum_{k=0}^{n-1} I(y_k \leq x)$$

so we end up with an estimation of:

$$S(2) = 1 - \frac{4}{7} \approx 0.4286$$

We can see that the estimations are surprisingly close, especially considering the sample size. This suggests that the Poisson might be a good estimation of the population distribution.