STAT 641
Homework 7

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Problem 1

1. The rejection region is defined as:

$$R = X : T(X) > C$$

where $T(X)$ is the distribution of the test statistic X.
Since Y is normally distributed we can use the test statistic:

$$T = \frac{\sqrt{n}(\bar{Y} - \mu_o)}{\sigma}$$

Substituting our values we get:

$$T = \frac{\sqrt{36}(\hat{\mu} - 10)}{\hat{\sigma}}$$

And we have the critical value (obtained from the table):

$$t_{.025,35} = 2.0315$$

Thus our rejection region is:

$$R = (\hat{\mu}, \hat{\sigma}) : \frac{\sqrt{36}(\hat{\mu} - 10)}{\hat{\sigma}} > 2.0315$$

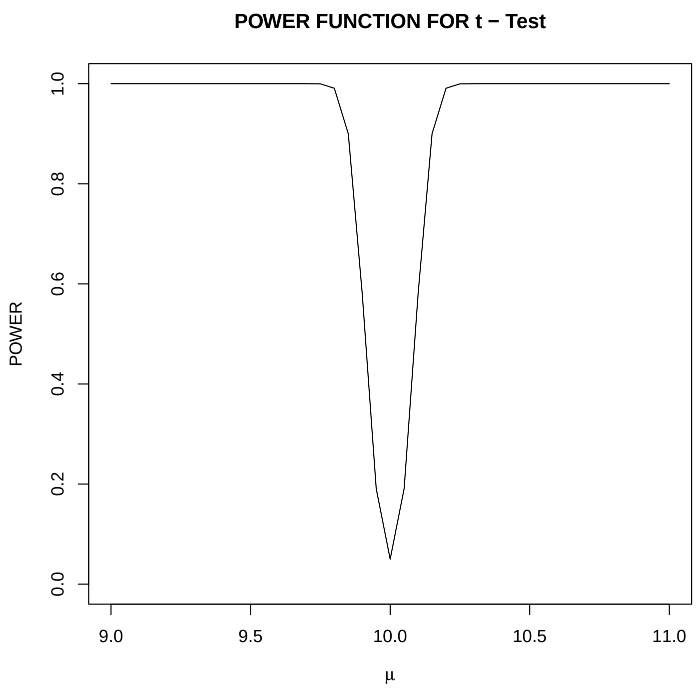
2. The power function is:

$$\gamma(\mu) = G(-t_{.025}) + 1 - G(t_{t.025})$$

Where G is the cdf of the students T distribution centered at:

$$T = \frac{\sqrt{36}(\mu - 10)}{.27}$$

with $df = 35$.
Graphing this function in R:


POWER FUNCTION FOR t − Test
POWER
1.0
0.8
0.6
0.4
0.2
0.0
9.0
9.5
10.0
10.5
11.0
μ

The R code is:

```
n = 36
df = n-1
muo = 10
sigma = .27
mu = seq(9,11,.05)
delta = sqrt(n)*(mu-muo)/sigma
power = 1-pt(qt(.975,df),df,delta) + pt(qt(.025, df), df, delta)
#par(lab=c(15,20,4))
plot(mu,power,type="l",ylim=c(0,1),xlab=expression(mu),
ylab="POWER")
title("POWER FUNCTION FOR t - Test")
out = cbind(mu,delta,power)
```

The sample mean is 10, so our T statistic is 0, thus G is centered at 0, so:

$$\begin{aligned}\gamma(\mu) &= G(-t_{.025}) + 1 - G(t_{.025}) \\ &= .025 + 1 - .975 \\ &= .05\end{aligned}$$

3. To determine the sample size, we have $\alpha = .05$, $\beta = .2$, and $\phi = \frac{.2}{.27} = 0.7407$. From the table $A11$, this gives us a sample size of 13.

Problem 2

1. The null hypothesis is $\mu \geq 5$, with the alternative hypothesis being $\mu < 5$. σ is unkown, so we will be using t testing.
   We know that: $n = 20$, $\hat{\mu} = 4.2$, $S = 1.2$, and $\alpha = .05$, so we have the T statistic:
   $$T = \frac{\sqrt{20}(4.2 - 5)}{1.2} = -2.98142396999972$$
   the t critical value is:
   $$t_{.05} = -1.729$$
   (from the table).
   Since $T < t_{.05}$, we reject the null hypothesis. We also have the p-value for T which is approximately $.005 < .05$ from the table.

2. The probability that the test will be able to detect that the real mean is 4.1 or less can be given by the power function [ ]

$$\gamma(\mu) = G(-t_{.05})$$

where G is the cumulative t distribution centered at $\delta = \frac{\sqrt{20}(4.1-5)}{1.2}$ with $df = 19$. This can be solved with R:

```
result = pt(-qt(.05, 19), 19, 0.3727)
```

which yields a power of 0.9006489

3. For sample size estimation we know: $\alpha = .05$, $\beta = .1$, and $\phi = \frac{|4.4-5|}{1.2} = .5$. From the table this gives us 44

Problem 3

1. We will assume the population distribution is normal, since this is a manufacturing process.
The null hypothesis is that $\sigma \geq 1$, the alternative hypothesis is that $\sigma < 1$. We can calculate the test statistic:

$$\begin{aligned} TS &= \frac{(n-1) * S^2}{\sigma_o} \\ &= \frac{19 * .8458^2}{1} \\ &= 13.5922 \end{aligned}$$

we can determine the critical value with:

$$\begin{aligned} t_{.05} &= \chi^2_{19,.95} \\ &= 10.117 \end{aligned}$$

The test statistic is not less than the critical value, thus we fail to reject the null hypothesis.

2. We have the power function:

$$\begin{aligned} power &= G(\frac{\sigma_o^2}{\sigma_1^2}\chi^2_{n-1,1-\alpha}) \\ &= G(10.117) \\ &= .05 \end{aligned}$$

so the type II error rate for $\sigma_1 = 1$ is $1 - .05 = .95$. For $\sigma_1 = .9$:

$$\begin{aligned} power &= G(\frac{\sigma_o^2}{\sigma_1^2}\chi^2_{n-1,1-\alpha}) \\ &= G(\frac{1^2}{.9^2} \cdot 10.117) \\ &= G(12.4901) \\ &= 0.1363781 \end{aligned}$$

so the type II error is 0.8636219. For $\sigma_1 = .8$:

$$\begin{aligned} power &= G(\frac{\sigma_o^2}{\sigma_1^2}\chi^2_{n-1,1-\alpha}) \\ &= G(\frac{1^2}{.8^2} \cdot 10.117) \\ &= G(15.81) \\ &= 0.32994 \end{aligned}$$

so the type II error is 0.67006. For $\sigma_1 = .7$:

$$\begin{aligned} power &= G(\frac{\sigma_o^2}{\sigma_1^2}\chi^2_{n-1,1-\alpha}) \\ &= G(\frac{1^2}{.7^2} \cdot 10.117) \\ &= G(20.6469) \\ &= 0.6433 \end{aligned}$$

so the type II error is .3567. For $\sigma_1 = .6$:

$$\begin{aligned} power &= G(\frac{\sigma_o^2}{\sigma_1^2}\chi^2_{n-1,1-\alpha}) \\ &= G(\frac{1^2}{.6^2} \cdot 10.117) \\ &= G(28.1027) \\ &= 0.918528 \end{aligned}$$

so the type II error is .08147.

3. The bound for the upper 90% CI is given by:

$$
\begin{aligned}
CI &= \frac{\sqrt{n-1}\cdot S}{\sqrt{X^2_{1-\alpha,n-1}}} \\
&= \frac{\sqrt{19}\cdot .8458}{\sqrt{X^2_{.9,19}}} \\
&= \frac{\sqrt{19}\cdot .8458}{\sqrt{11.651}} \\
&= 1.0801
\end{aligned}
$$

This aligns with my results in part 1 since 1 is included the in the CI.

Problem 4

1. $S_+ = 5$, so we have:

   $$p = G(5, 20, .5) = .021$$

   where G is the cdf of the binomial distribution with $p = .5$, $n = 20$, and $x = 5$.
   .021 ¡ .05, so we reject the null hypothesis.

2. Following the procedure in handout 12, we find that $W_+ = 68.5$ and $W_- = 141.5$.
   We find from psignrank(68.5, 20, TRUE) that $p = 0.09467411 < .05$ so we reject the null hypothesis.

3. We can invert the wilcox test to find an upper bound. We need to find the first value where we would reject the null hypothesis if it were the sample median. We will do a binary search:
   Trying $H_0 = 10.07$:
   $W_+ = 100, W_- = 90$
   psignrank(100, 20, TRUE) = 0.4347439 ¿ .1, so fail to reject.
   Trying $H_0 = 10.93$:
   $W_+ = 20, W_- = 170$
   psignrank(20, 20, TRUE) = 0.0003538132, so reject.
   Trying $H_0 = 10.15$:
   $W_+ = 79.5$
   psignrank(79.5, 20, TRUE) = 0.1841383 ¿ .1 so fail to reject.
   Trying $H_0 = 10.24$:
   $W_+ = 78$
   psignrank(78, 20, TRUE) = 0.1649914 ¿ .1 so fail to reject.
   So the 90% upper bound is 10.24.

Problem 5

1. idk, just use wald CI I guess:

$$\begin{aligned} CI &= \hat{p} \pm Z_{.025} \frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}} \\ &= .65 \pm 1.96 \frac{\sqrt{.65(1-.65)}}{\sqrt{200}} \\ &= 0.583895385940163, 0.716104614059837 \end{aligned}$$

2. we have the test statistic:

$$\begin{aligned} TS &= \frac{.65 - .6}{\sqrt{\frac{.65 \cdot .35}{200}}} \\ &= 1.4824986333222026 \end{aligned}$$

   the p value is:

$$p = 1 - pnorm(1.48) = 0.06943662$$

   greater than α so fail to reject.

3. so we have the power function:

$$power = 1 - pnorm(Z_\alpha \cdot \sqrt{\frac{p_o(1-p_o)}{p_1(1-p_1)} + \frac{\sqrt{n}(p_o - p_1)}{\sqrt{p_1(1-p_1)}}})$$

   This is the power function implemented in R:

```
power_from_formula <- function(n, p0, p1, alpha = 0.05) {
z_alpha <- qnorm(1 - alpha)
inside_sqrt <-
  p0*(1 - p0) / (p1*(1 - p1)) +
  ( sqrt(n) * (p0 - p1) ) / sqrt(p1*(1 - p1))
1 - pnorm( z_alpha * sqrt(inside_sqrt) )
}
```

   When plugging in our values, we get:
   0.001178664
   0.05
   0.4180287
   0.9078522

0.9988094
for .55, .6, .65, .7, .75 respectively.

section*Problem 6

1. b

2. c

3. d

4. a

5. b

6. c