MATH 417 502 HW8

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November 4, 2024

Problem 1

Assuming we have:

$$f(t, y(t)) = \lambda \cdot y(t)$$

We can substitute into the original:

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{2} \cdot h \cdot \lambda \cdot (y_n + y_{n+1}) \\ y_{n+1} - \frac{1}{2} \cdot h \cdot \lambda \cdot y_{n+1} &= y_n + \frac{1}{2} \cdot h \cdot \lambda \cdot y_n \\ y_{n+1}(1 - \frac{1}{2} \cdot h \cdot \lambda) &= y_n(1 + \frac{1}{2} \cdot h \cdot \lambda) \\ y_{n+1} &= \frac{y_n(1 + \frac{1}{2} \cdot h \cdot \lambda)}{1 - \frac{1}{2} \cdot h \cdot \lambda} \end{aligned}$$

Thus we have the amplitude factor:

$$w(z) = \frac{1 + \frac{1}{2} \cdot z}{1 - \frac{1}{2} \cdot z}$$

We will call the real part of λ a, and the imaginary part of λ b.
We will call the real part of $w(z)$ $w_r(z)$, and imaginary $w_i(z)$:

$$w_r(z) = \frac{1 + \frac{1}{2}a}{1 - \frac{1}{2}a}$$

$$\begin{aligned} w_i(z) &= \frac{\frac{1}{2}b}{-\frac{1}{2}b} \\ &= -1 \end{aligned}$$

the rule is stable when $|w(z)| < 1$ so we have:

$$\begin{aligned}
|w(z)| &= \sqrt{w_r(z)^2 + w_i(z)^2} \\
\sqrt{w_r(z)^2 + w_i(z)^2} &< 1 \\
\sqrt{(\frac{1+\frac{1}{2}a}{1-\frac{1}{2}a})^2 + 1} &< 1 \\
(\frac{1+\frac{1}{2}a}{1-\frac{1}{2}a})^2 &< 0 \\
\frac{1+\frac{1}{2}a}{1-\frac{1}{2}a} &< 0 \\
1 + \frac{1}{2}a &< 0
\end{aligned}$$

This when the real part of λ is less than 0, the trapezoidal rule is stable.

Problem 2

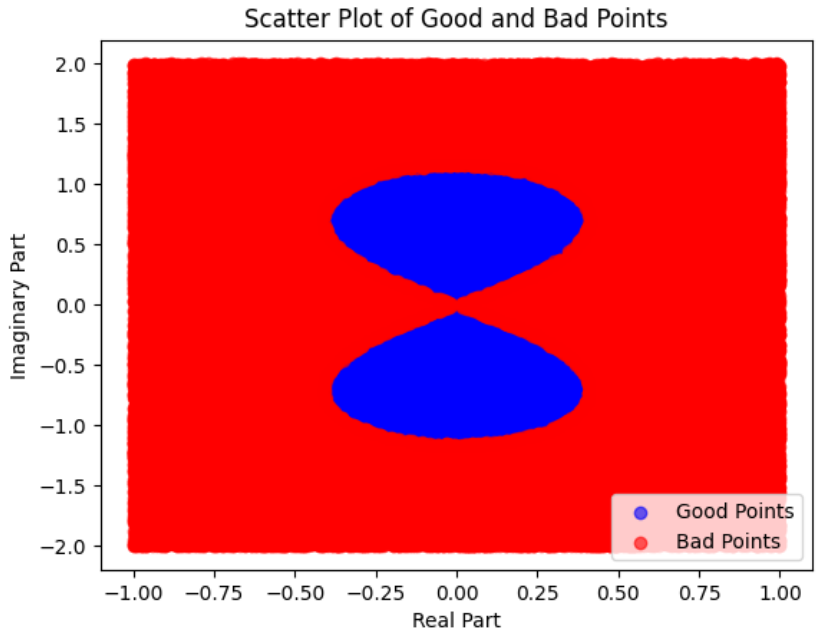
again assuming $f(t, y(t)) = \lambda \cdot y(t)$:

$$\begin{aligned}
y_{n+1} &= y_n + h \cdot f(t_n + \frac{1}{2}h, y_n + \frac{1}{2} \cdot h \cdot \lambda \cdot y_n) \\
y_{n+1} &= y_n + h \cdot \lambda \cdot (y_n + \frac{1}{2} \cdot h \cdot \lambda \cdot y_n) \\
y_{n+1} &= y_n + h \cdot \lambda \cdot y_n + h \cdot \lambda \cdot \frac{1}{2} \cdot h \cdot \lambda \cdot y_n \\
y_{n+1} &= y_n + h \cdot \lambda \cdot y_n + \frac{1}{2} \cdot h^2 \cdot \lambda^2 \cdot y_n \\
y_{n+1} &= y_n(1 + h \cdot \lambda + \frac{1}{2} \cdot h^2 \cdot \lambda^2)
\end{aligned}$$

thus we have:

$$w(z) = 1 + z + \frac{1}{2}z^2$$

Testing the points yields a Hyperboloid looking figure, where the function is stable within the hyperboloid:




```python
import random
import matplotlib.pyplot as plt
class Complex:
    def __init__(self, real, imaginary):
        self.real = real
        self.imaginary = imaginary
    def magnitude(self):
        return (self.real**2 + self.imaginary**2) ** .5
    def add(a, b):
        return Complex(a.real + b.real, a.imaginary + b.
            imaginary)
    def square(a):
        return Complex(a.real**2 - a.imaginary**2, 2 * a.
            real * a.imaginary)
def w(z):
    one = Complex(1, 0)
    z_square = Complex.square(z)
    z_square_term = Complex(1/2 * z_square.real, 1/2 *
        z_square.imaginary)
    result = Complex.add(one, Complex.add(z_square,
        z_square_term))
    return result
def magnitude(z):
    return w(z).magnitude()
```

```
21
22 def plot(bad_points, good_points):
23     plt.scatter(*zip(*good_points), color='blue', label='
           Good Points', alpha=0.6)
24     plt.scatter(*zip(*bad_points), color='red', label='Bad
           Points', alpha=0.6)
25     plt.xlabel("Real Part")
26     plt.ylabel("Imaginary Part")
27     plt.legend()
28     plt.title("Scatter Plot of Good and Bad Points")
29     plt.show()
30     return
31
32 def generate_points(real_interval, imag_interval, num_points
       ):
33     result = []
34     for i in range(0, num_points):
35         real = random.uniform(real_interval[0],
               real_interval[1])
36         imag = random.uniform(imag_interval[0],
               imag_interval[1])
37         result.append([real, imag])
38     return result
39
40 def main():
41     real_interval = [-1, 1]
42     imag_interval = [-2, 2]
43     num_points = 100000
44     points = generate_points(real_interval,imag_interval,
           num_points)
45     good_points = []
46     bad_points = []
47     for i in range(0, len(points)):
48         my_complex = Complex(points[i][0], points[i][1])
49         my_magnitude = magnitude(my_complex)
50         if(my_magnitude < 1):
51             good_points.append(points[i])
52         else:
53             bad_points.append(points[i])
54     plot(bad_points, good_points)
55
56 if __name__ == "__main__":
57     main()
```