MATH 417 502 HW6

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Problem 1

To do this we will need the first 3 legendre polynomials. Recall:

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_{n+1}(x) &= \frac{2n+1}{n+1} P_n(x) \cdot x - \frac{n}{n+1} P_{n-1}(x)
\end{aligned}$$

Thus we have:

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$\begin{aligned}
P_3(x) &= \frac{5}{3}(\frac{3}{2}x^2 - \frac{1}{2}) \cdot x - \frac{2}{3}x \\
&= \frac{5}{2}x^3 - \frac{5}{6}x - \frac{2}{3}x \\
&= \frac{5}{2}x^3 - \frac{3}{2}x
\end{aligned}$$

So for 2 point quadrature we will find x_0, x_1 such that:

$$\begin{aligned}
P_2(x) &= 0 \\
\frac{3}{2}x^2 - \frac{1}{2} &= 0 \\
x^2 &= \frac{1}{3} \\
x &= \pm\frac{\sqrt{3}}{3}
\end{aligned}$$

So we have $x_0 = -\frac{\sqrt{3}}{3}, x_1 = \frac{\sqrt{3}}{3}$. Now we will find the weights according to newton cotes:

$$
\begin{aligned}
w_0 &= \int_{-1}^{1} L_0(x)dx \\
&= \int_{-1}^{1} \frac{x - \frac{\sqrt{3}}{3}}{-\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3}} \\
&= \frac{1}{-\frac{2\sqrt{3}}{3}} \int_{-1}^{1} (x - \frac{\sqrt{3}}{3})dx \\
&= -\frac{3}{2\sqrt{3}}(\frac{1}{2}(1-1) - (\frac{\sqrt{3}}{3} - (-\frac{\sqrt{3}}{3}))) \\
&= -\frac{\sqrt{3}}{2}(-\frac{2\sqrt{3}}{3}) \\
&= 1
\end{aligned}
$$

Weights in Newton Cotes are symmetrical so $w_1 = w_0$. Thus we have:

$$\int_{-1}^{1} f(x) \approx f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3})$$

for 3 point quadrature:

$$
\begin{aligned}
P_3(x) &= 0 \\
\frac{5}{2}x^3 - \frac{3}{2}x &= 0 \\
x(\frac{5}{2}x^2 - \frac{3}{2}) &= 0 \\
x(\frac{5x^2 - 3}{2}) &= 0
\end{aligned}
$$

So the roots are $x = 0, -\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}$
The weights are:

$$\begin{aligned}
w_0 &= \int_{-1}^{1} L_0(x)dx \\
&= \int_{-1}^{1} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}dx \\
&= \frac{1}{(x_0-x_1)(x_0-x_2)} \int_{-1}^{1} (x-x_1)(x-x_2)dx \\
&= \frac{1}{(-\sqrt{\frac{3}{5}}-0)(-\sqrt{\frac{3}{5}}-\sqrt{\frac{3}{5}})} \cdot \int_{-1}^{1} (x-0)(x-\sqrt{\frac{3}{5}})dx \\
&= \frac{1}{-\sqrt{\frac{3}{5}} \cdot -2 \cdot \sqrt{\frac{3}{5}}} \cdot \int_{-1}^{1} (x^2 - x \cdot \sqrt{\frac{3}{5}})dx \\
&= \frac{1}{2 \cdot \frac{3}{5}} \cdot (\frac{1}{3}(1-(-1)) - \frac{1}{2} \cdot \sqrt{\frac{3}{5}}(1-1)) \\
&= \frac{5}{6} \cdot \frac{2}{3} \\
&= \frac{5}{9}
\end{aligned}$$

$$\begin{aligned}
w_1 &= \int_{-1}^{1} L_1(x)dx \\
&= \int_{-1}^{1} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}dx \\
&= \frac{1}{(x_1-x_0)(x_1-x_2)} \int_{-1}^{1} (x-x_0)(x-x_2)dx \\
&= \frac{1}{(\sqrt{\frac{3}{5}})(-\sqrt{\frac{3}{5}})} \int_{-1}^{1} (x+\sqrt{\frac{3}{5}})(x-\sqrt{\frac{3}{5}})dx \\
&= \frac{1}{-\frac{3}{5}} \int_{-1}^{1} (x^2 - \frac{3}{5})dx \\
&= -\frac{5}{3}(\frac{1}{3}(2) - \frac{3}{5}(2)) \\
&= -\frac{5}{3}(\frac{2}{3} - \frac{6}{5}) \\
&= -\frac{10}{9} + \frac{18}{9} \\
&= \frac{8}{9}
\end{aligned}$$

The weights are symmetrical so $w_0 = w_2$, thus we have all of the weights $w_0 = \frac{5}{9}, w_1 = \frac{8}{9}, w_2 = \frac{5}{9}$. Our final answer is:

$$\int_{-1}^{1} f(x)dx \approx \frac{5}{9}f(-\sqrt{\frac{3}{5}}) + \frac{8}{9}f(0) + \frac{5}{9}f(\sqrt{\frac{3}{5}})$$

Problem 2

To transform the bounds of the integral [-1, 1], we will need to substitute x for a variable u such that when $x = -1$, $u = 7$ and when $x = 1$, $u = 20$. We will use a linear transformation to do so:

$$\begin{aligned}
x &= au + b - 1 & &= 7 \cdot a + b \\
b &= -1 - 7 \cdot a \\
1 &= 20 \cdot a + b \\
1 &= 20 \cdot a - 1 - 7 \cdot a \\
1 &= 13 \cdot a - 1 \\
a &= \frac{2}{13} \\
b &= -1 - 7 \cdot \frac{2}{13} \\
b &= -\frac{27}{13}
\end{aligned}$$

Thus we will have the transformation $x = \frac{2}{13}u - \frac{27}{13}$, and the other way $u = (x + \frac{27}{13}) \cdot \frac{13}{2}$:

$$\int_{-1}^{1} f(x)dx = \int_{7}^{20} f(\frac{2}{13}u - \frac{27}{13})du$$

Thus the Gaussian quadrature for the interval [7, 20] is:

$$\begin{aligned}
\int_{7}^{20} f(x)dx &\approx \frac{13}{2} \cdot \frac{5}{9}f((-\sqrt{\frac{3}{5}} + \frac{27}{13}) \cdot \frac{13}{2}) + \frac{13}{2} \cdot \frac{8}{9}f((0 + \frac{27}{13}) \cdot \frac{13}{2}) + \frac{13}{2} \cdot \frac{5}{9}f((\sqrt{\frac{3}{5}} + \frac{27}{13}) \cdot \frac{13}{2}) \\
&\approx \frac{13}{2} \cdot \frac{5}{9}f(\frac{13}{2} \cdot -\sqrt{\frac{3}{5}} + \frac{27}{2}) + \frac{13}{2} \cdot \frac{8}{9}f(\frac{27}{2}) + \frac{13}{2} \cdot \frac{5}{9}f(\frac{13}{2} \cdot \sqrt{\frac{3}{5}} + \frac{27}{2})
\end{aligned}$$

Problem 3

Definition: A quadrature rule has degree of precision k if it is exact for all polynomials of degree at most k. A gaussian quadrature rule should have degree of

precision $2n+1$. So the given quadrature rule should have degree of precision 3. We will verify this:
For the interpolating function given, it ouputs the following for each function:

$$\begin{aligned} f(x) &= 1 \\ I &= 1 + 1 = 2 \end{aligned}$$

which is correct ($\int_{-1}^{1} 1 = 2$). So I has degree at least 0.

$$\begin{aligned} f(x) &= x \\ I &= -\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} = 0 \end{aligned}$$

which is correct ($f(x)$ is odd so it must be 0). So I has degree at least 1.

$$\begin{aligned} f(x) &= x^2 \\ I &= \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \end{aligned}$$

which is correct. So I has degree at least 2.

$$\begin{aligned} f(x) &= x^3 \\ I &= 0 \end{aligned}$$

which is correct. So I has degree at least 3.

$$\begin{aligned} f(x) &= x^4 \\ I &= \frac{1}{9} + \frac{1}{9} = \frac{2}{9} \end{aligned}$$

which is incorrect. $\int_{-1}^{1} x^4 dx = \frac{2}{5}$.
So the degree of precision is 3.