MATH 417 502
Homework 2

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Problem 1

a.) The first 20 iterations of the Newton method are shown below:

Iteration	Approximation	Residual
0	1.000000e+00	7.182818e-01
1	5.819767e-01	2.075957e-01
2	3.190550e-01	5.677201e-02
3	1.679962e-01	1.493591e-02
4	8.634887e-02	3.837726e-03
5	4.379570e-02	9.731870e-04
6	2.205769e-02	2.450693e-04
7	1.106939e-02	6.149235e-05
8	5.544905e-03	1.540144e-05
9	2.775014e-03	3.853917e-06
10	1.388149e-03	9.639248e-07
11	6.942351e-04	2.410369e-07
12	3.471577e-04	6.026621e-08
13	1.735889e-04	1.506742e-08
14	8.679696e-05	3.766965e-09
15	4.339911e-05	9.417547e-10
16	2.169971e-05	2.354406e-10
17	1.084989e-05	5.886025e-11
18	5.424958e-06	1.471512e-11
19	2.712481e-06	3.678613e-12
20	1.356302e-06	9.199308e-13

From the above, we can see that Newton's method converges, but not as quickly as we would like.
Requirements to converge quadratically:

$$g'(x*) = 0$$
$$g''(x) \leq M$$

In our case, from newton's method:

$$g(x) = x - \frac{e^x - x - 1}{e^x - 1}$$

Thus we have:

$$g'(x) = 1 - ((-e^x)(e^x - 1)^{-2}(e^x - x - 1) + (e^x - 1)^{-1}(e^x - 1))$$
$$g'(0) = 1 - ((-1)(1 - 1)^{-2}(1 - 0 - 1) + \frac{0}{0})$$

So $g'(0)$ is undefined, and thus does not satisfy the condition for quadratic convergence.

b.) Start with $\mu(x)$ and $\mu'(x)$:

$$\mu(x) = \frac{f(x)}{f'(x)}$$

$$\begin{aligned}
\mu'(x) &= \frac{d}{dx}(f(x)(f'(x))^{-1} \\
&= \frac{f'(x)}{f'(x)} + (-1) \cdot f''(x) \cdot f'(x)^{-2} \cdot f(x) \\
&= \frac{f'(x)}{f'(x)} - \frac{f''(x) \cdot f(x)}{f'(x)^2}
\end{aligned}$$

Plugging into $g(x)$:

$$\begin{aligned}
g(x) &= x - \frac{\mu(x)}{\mu'(x)} \\
&= x - \frac{f(x)}{f'(x) \cdot (\frac{f'(x)}{f'(x)} - \frac{f''(x) \cdot f(x)}{f'(x)^2})} \\
&= x - \frac{f(x)}{\frac{f'(x)^2}{f'(x)} - \frac{f''(x) \cdot f(x)}{f'(x)}} \\
&= x - \frac{f(x)}{f'(x)^{-1}(f'(x)^2 - f''(x) \cdot f(x))} \\
&= x - \frac{f(x) \cdot f'(x)}{f'(x)^2 - f''(x) \cdot f(x)}
\end{aligned}$$

c.) We can see that the new $g(x)$ is approximately correct within the first 5 iterations:

Iteration	Approximation	Residual
0	1.000000e+00	7.182818e-01
1	-2.342106e-01	2.540578e-02
2	-8.458280e-03	3.567061e-05
3	-1.189018e-05	7.068790e-11
4	-4.218591e-11	0.000000e+00
5	-4.218591e-11	0.000000e+00

This $g(x)$ obviously converges much more quickly than the original, and thus it can be concluded that this $g(x)$ converges quadratically.

Problem 2

1. We know that if $|g'(x*)| < 1$ then $g(x)$ converges. So for a we have:

$$\begin{aligned} g(x) &= \frac{1}{21}(20x + 21x^{-2}) \\ g'(x) &= \frac{1}{21}(20 - 42x^{-3}) \\ g'(21^{\frac{1}{3}} &= \frac{1}{21}(20 - 42(21^{\frac{1}{3}})^{-3}) \\ &= \frac{1}{21}(20 - 42(21^{-1}) \\ &= \frac{1}{21}(20 - 2) \\ &= \frac{1}{21}(18) \\ &= \frac{18}{21} < 1 \end{aligned}$$

Thus a likely converges, but does not converge quadratically since $g'(x) \neq 0$.
for b we have:

$$
\begin{aligned}
g(x) &= x - \frac{1}{3}(x^3 - 21)x^{-2}, \\
g'(x) &= 1 - \frac{1}{3}((3x^2)x^{-2} + (x^3 - 21)(-2)x^{-3}), \\
g'(x) &= 1 - \frac{1}{3}(3 + (x^3 - 21)(-2)x^{-3}), \\
g'(21^{\frac{1}{3}}) &= 1 - \frac{1}{3}(3 + (21 - 21)(-2)(21^{-1})) \\
&= 1 - \frac{1}{3}(3) \\
&= 1 - 1 \\
&= 0
\end{aligned}
$$

so b converges, and b also likely quadratically converges since $g'(x*) \equiv 0$
For c we have:

$$
\begin{aligned}
g(x) &= x - (x^4 - 21x)(x^2 - 21)^{-1} \\
g'(x) &= 1 - ((4x^3 - 21)(x^2 - 21)^{-1} + (x^4 - 21x)(-1)(2x)(x^2 - 21)^{-2} \\
g'(21^{\frac{1}{3}}) &= 1 - (4(21 - 21)(21^{\frac{2}{3}} - 21)^{-1} - (21^{\frac{4}{3}} - 21^{\frac{4}{3}})(2(21^{\frac{1}{3}}))(21^{\frac{2}{3}} - 21)^{-2}) \\
&= 1 - (0 - 0) \\
&= 1
\end{aligned}
$$

$g'(x*) \geq 1$ so c does not converge linearly or quadratically.