MATH 417 502
Homework 4

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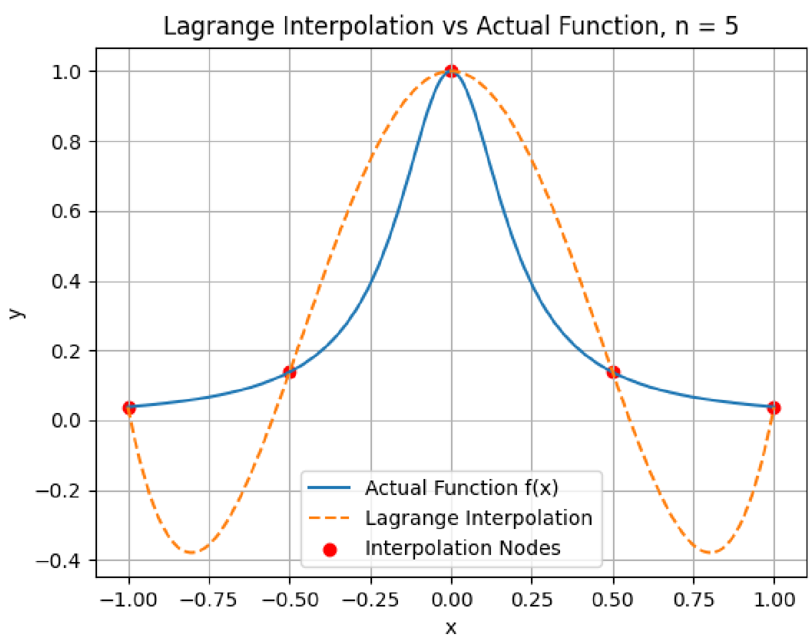
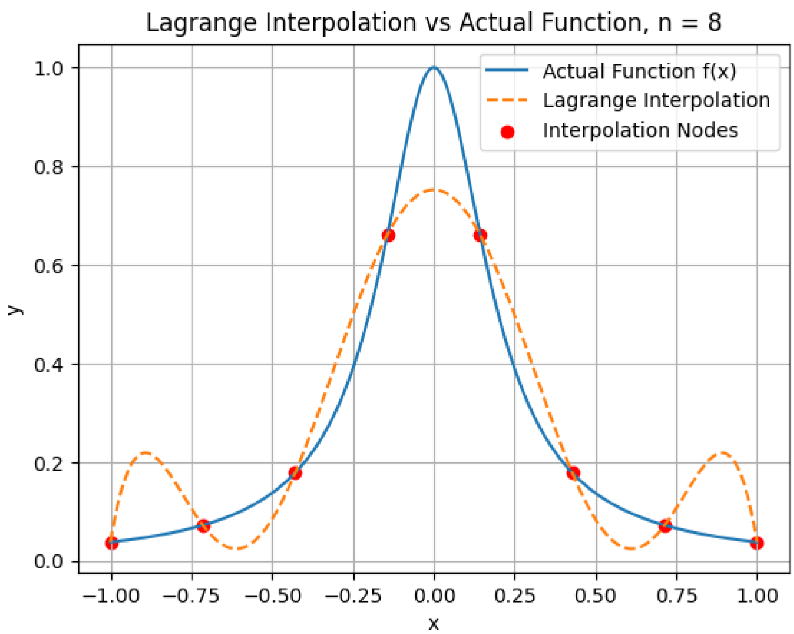
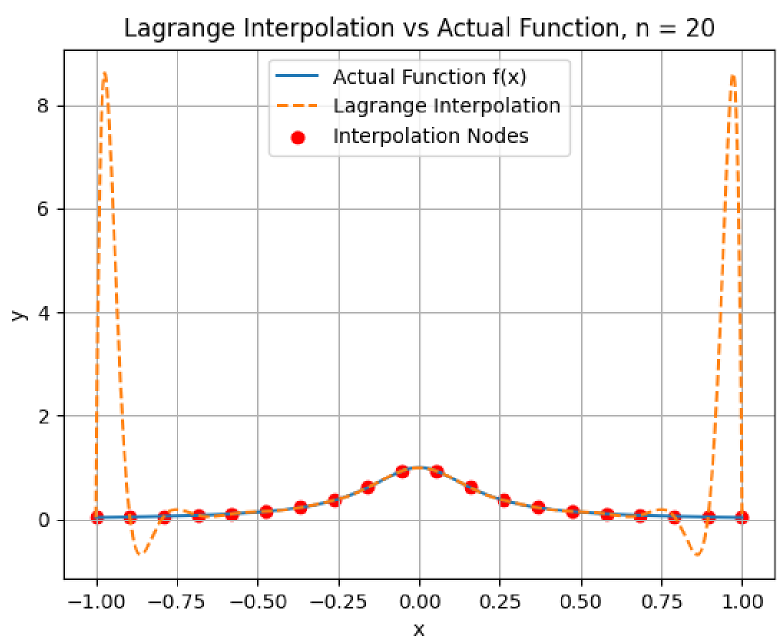
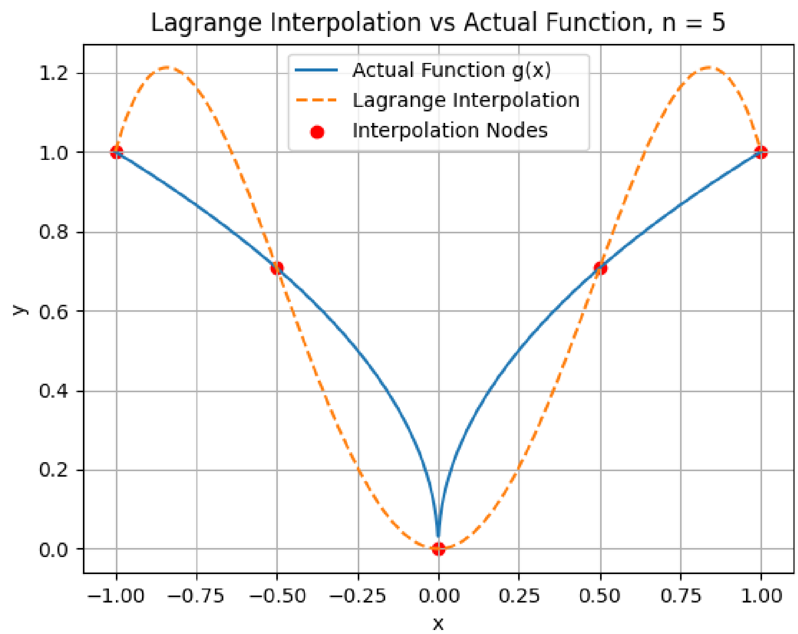
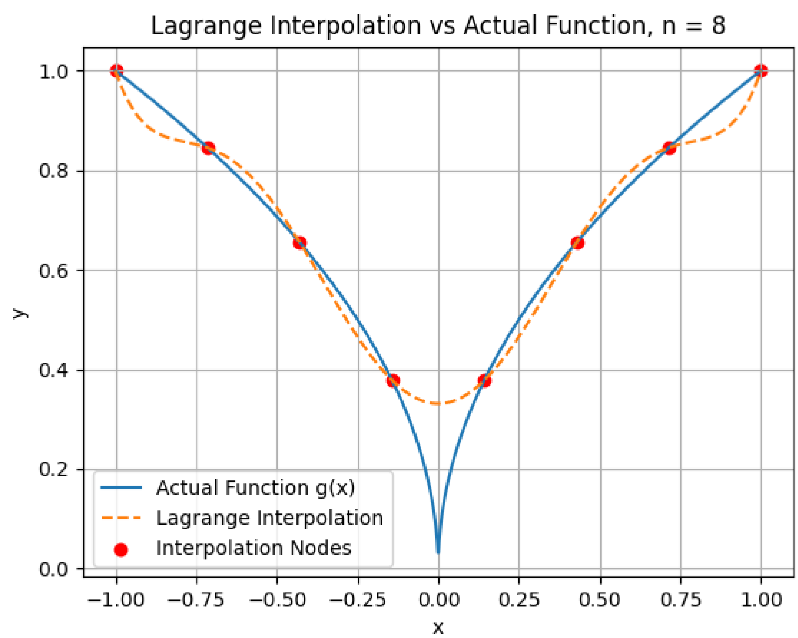
Problem 1

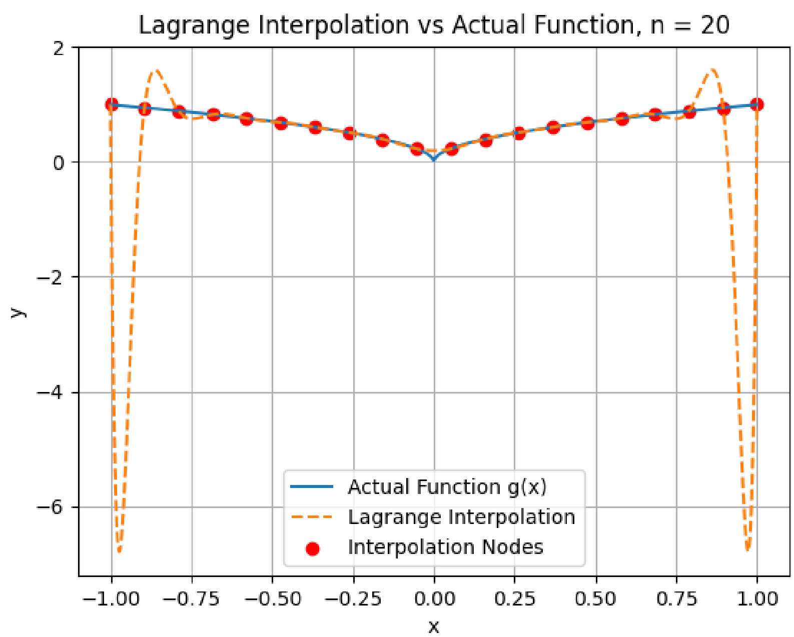



Lagrange Interpolation vs Actual Function, n = 8
Actual Function f(x)
Lagrange Interpolation
Interpolation Nodes
1.0
0.8
0.6
0.4
0.2
0.0
y
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x


Lagrange Interpolation vs Actual Function, n = 20
Actual Function f(x)
Lagrange Interpolation
Interpolation Nodes
8
6
4
2
0
y
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x


Lagrange Interpolation vs Actual Function, n = 5
Actual Function g(x)
Lagrange Interpolation
Interpolation Nodes
y
x
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00


Lagrange Interpolation vs Actual Function, n = 8
Actual Function g(x)
Lagrange Interpolation
Interpolation Nodes
y
x
1.0
0.8
0.6
0.4
0.2
0.0
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00



code:

```
import numpy as np
import matplotlib.pyplot as plt
def lagrange(x, points):
    lagrange_results = [];
    for i in range(0, len(points)):
        numerator = 1;
        denominator = 1;
        for j in range(0, len(points)):
            if(i == j):
                continue;
            numerator *= (x - points[j][0]);
            denominator *= (points[i][0] - points[j][0]);
        lagrange_results.append(numerator / denominator);
    result = 0;
    for i in range(0, len(points)):
        result += points[i][1] * lagrange_results[i]
    return result;

def f(x):
    return 1 / (1 + 25 * x**2);
def g(x):
    return (abs(x)) ** (1/2);
```

```

def get_x_coords(interval, num_points):
    start = interval[0];
    orig_start = start
    end = interval[1];
    result = [];
    result.append(start);
    for i in range(0, num_points - 1):
        start += (end - orig_start) / (num_points - 1)
        result.append(start)
    return result;
def do_the_thing(my_function, n):
    x_coords = get_x_coords([-1, 1], n);
    actual_function_values = []
    for x in x_coords:
        actual_function_values.append([x, my_function(x)]);

    x_plot = np.linspace(-1, 1, 1000)
    y_actual = []
    y_interp = []
    for i in range(0, len(x_plot)):
        y_actual.append(my_function(x_plot[i]));
        y_interp.append(lagrange(x_plot[i], actual_function_values));

    plt.plot(x_plot, y_actual, label=f'Actual Function {my_function.__name__}(x)')
    plt.plot(x_plot, y_interp, '--', label='Lagrange Interpolation')
    plt.scatter(x_coords, [my_function(x) for x in x_coords], color='red', label='Interpolation Nodes')
    plt.title(f'Lagrange Interpolation vs Actual Function, n = {n}')
    plt.xlabel('x')
    plt.ylabel('y')
    plt.legend()
    plt.grid(True)
    plt.show()
def main():
    functions = [f, g]
    nums = [5, 8, 20]
    for function in functions:
        for num in nums:
            do_the_thing(function, num)
if __name__ == "__main__":
    main();
```

Problem 2

1. Recall the max error is bounded by:

$$|f(x) - p(x)| \leq \frac{(b-a)^{n+1}}{(n+1)!}(\max|f^{n+1}(x)|)$$

So our error for $f(x) = e^{\lambda x}$ is bounded by:

$$\lim_{n\to\infty} \frac{(b-a)^{n+1}}{(n+1)!}(\max|f^{n+1}(x)|) = \lim_{n\to\infty} \frac{(b-a)^{n+1}}{(n+1)!}(\max|\lambda^{n+1}e^{\lambda x}|)$$

if $\lambda < 0$ then $\max|\lambda^{n+1}e^{\lambda x}| = \lambda^{n+1}e^{\lambda a}$. If $\lambda > 0$ then $max|\lambda^{n+1}e^{\lambda x}| = \lambda^{n+1}e^{\lambda b}$. Either way, the max is λ^{n+1} multiplied by some constant C. Thus we have:

$$\lim_{n\to\infty} \frac{(b-a)^{n+1}}{(n+1)!}(\max|\lambda^{n+1}e^{\lambda x}|) = \lim_{n\to\infty} \frac{(b-a)^{n+1}}{(n+1)!}\lambda^{n+1} \cdot C$$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{(b-a)\lambda(b-a)\lambda(b-a)\lambda \cdot ... \cdot (b-a)\lambda}{(n+1)(n)(n-1) \cdot ... \cdot (1)} \cdot C \\
&= \lim_{n\to\infty} \frac{(b-a)\lambda}{n+1} \cdot \frac{(b-a)\lambda}{n} \cdot \frac{(b-a)\lambda}{n-1} \cdot ... \cdot \frac{(b-a)\lambda}{1} \cdot C \\
&= 0 \cdot 0 \cdot 0 \cdot ... \cdot \frac{(b-a)\lambda}{1} \cdot C \\
&= 0
\end{aligned}$$

2. The same does not hold true for $f(x) = 4(1+x^2)^{-1}$:

$$\begin{aligned}
f'(x) &= -4(2)x(1+x^2)^{-2} \\
f''(x) &= -8(1+x^2)^{-2} + -4(2)x(-2)(2x)(1+x^2)^{-3} \\
|f'''(x)| &\geq 4(2x)(2x)(2x)(1)(2)(3)(1+x^2)^{-4} \\
|f^{n+1}(x)| &\geq 4(2^{n+1}x^{n+1})n! \cdot (1+x^2)^{-(n+1)}
\end{aligned}$$

We will say that $f^{n+1}(x)$ is it's maximum at $f^{n+1}(C)$ (C is some constant on the interval $[a,b]$).
Plugging this in we get:

$$\begin{aligned}
\lim_{n\to\infty}\frac{(b-a)^{n+1}}{(n+1)!}(\max|f^{n+1}(x)|) &= \lim_{n\to\infty}\frac{(b-a)^{n+1}}{(n+1)!}(4(2^{n+1}C^{n+1})n!\cdot(1+C^2)^{-(n+1)}) \\
&= \lim_{n\to\infty}\frac{(b-a)^{n+1}}{(n+1)!}4(2^{n+1})n!\cdot\frac{C^{n+1}}{(1+C^2)^{n+1}} \\
&= \lim_{n\to\infty}\frac{(b-a)^{n+1}}{(n+1)!}4(2^{n+1})n!\cdot(\frac{C}{1+C^2})^{n+1} \\
&= \lim_{n\to\infty}\frac{n!}{(n+1)!}4(2^{n+1})(b-a)^{n+1}\cdot(\frac{C}{1+C^2})^{n+1} \\
&= \lim_{n\to\infty}\frac{1}{(n+1)}4(2^{n+1})(b-a)^{n+1}\cdot(\frac{C}{1+C^2})^{n+1} \\
&= \lim_{n\to\infty}\frac{4(2(b-a))^{n+1}}{(n+1)}\cdot(\frac{C}{1+C^2})^{n+1} \\
&= \lim_{n\to\infty}\frac{4}{(n+1)}\cdot(\frac{C\cdot 2(b-a)}{1+C^2})^{n+1}
\end{aligned}$$

We can see that if $(\frac{C\cdot 2(b-a)}{1+C^2}) > 1$, this limit approaches infinity. Thus the limit does not necessarily converge to 0.

Problem 3

$$\begin{aligned}
P_{0,0} &= 5.3 \\
P_{1,1} &= 2 \\
P_{2,2} &= 3.19 \\
P_{3,3} &= 1
\end{aligned}$$

$$\begin{aligned}
P_{0,1} &= \frac{(x-(-.1))(2)-(x-0)(5.3)}{0-(-.1)} \\
&= \frac{(x+.1)(2)-(x)(5.3)}{.1} \\
&= \frac{2x+.2-5.3x}{.1} \\
&= 20x+2-53x \\
&= -33x+2
\end{aligned}$$

$$\begin{aligned}
P_{1,2} &= \frac{(x-0)(3.19)-(x-.2)(2)}{.2-0}\\
&= \frac{3.19x-2x+.4}{.2}\\
&= \frac{1.19x+.4}{.2}\\
&= 5.95x+2
\end{aligned}$$

$$\begin{aligned}
P_{2,3} &= \frac{(x-.2)(1)-(x-.3)(3.19)}{.3-.2}\\
&= \frac{x-.2-3.19x+0.957}{.1}\\
&= \frac{-2.19x+0.757}{.1}\\
&= -21.9x+7.57
\end{aligned}$$

$$\begin{aligned}
P_{0,2} &= \frac{(x-(-.1))(5.95x+2)-(x-.2)(-33x+2)}{.2-(-.1)}\\
&= \frac{(x+.1)(5.95x+2)-(x-.2)(-33x+2)}{.3}\\
&= \frac{5.95x^2+2x+.595x+.2-(-33x^2+2x+6.6x-.4)}{.3}\\
&= \frac{5.95x^2+2x+.595x+.2+33x^2-2x-6.6x+.4)}{.3}\\
&= \frac{38.95x^2-6.005x+.6)}{.3}
\end{aligned}$$

$$\begin{aligned}
P_{1,3} &= \frac{(x-0)(-21.9x+7.57)-(x-.3)(5.95x+2)}{.3}\\
&= \frac{(x)(-21.9x+7.57)-(x-.3)(5.95x+2)}{.3}\\
&= \frac{-21.9x^2+7.57x-(5.95x^2+2x-1.785x-.6)}{.3}\\
&= \frac{-21.9x^2+7.57x-5.95x^2-2x+1.785x+.6)}{.3}\\
&= \frac{-27.85x^2+7.355x+.6)}{.3}
\end{aligned}$$

Final polynomial is:

$$P_{0,3} = \frac{(x-(-.1))(\frac{-27.85x^2+7.355x+.6)}{.3}) - (x-.3)(\frac{38.95x^2-6.005x+.6)}{.3})}{.3-(-.1)}$$