MATH 417 502
Homework 2

Keegan Smith

September 4, 2024

Problem 1

a.) The first 20 iterations of the Newton method are shown below:

Iteration	Approximation	Residual
0	1.000000e+00	7.182818e-01
1	5.819767e-01	2.075957e-01
2	3.190550e-01	5.677201e-02
3	1.679962e-01	1.493591e-02
4	8.634887e-02	3.837726e-03
5	4.379570e-02	9.731870e-04
6	2.205769e-02	2.450693e-04
7	1.106939e-02	6.149235e-05
8	5.544905e-03	1.540144e-05
9	2.775014e-03	3.853917e-06
10	1.388149e-03	9.639248e-07
11	6.942351e-04	2.410369e-07
12	3.471577e-04	6.026621e-08
13	1.735889e-04	1.506742e-08
14	8.679696e-05	3.766965e-09
15	4.339911e-05	9.417547e-10
16	2.169971e-05	2.354406e-10
17	1.084989e-05	5.886025e-11
18	5.424958e-06	1.471512e-11
19	2.712481e-06	3.678613e-12
20	1.356302e-06	9.199308e-13

From the above, we can see that Newton's method probably converges, but not as quickly as we would like.
Requirements to converge quadratically:

$$g'(x*) = 0$$
$$g''(x) \leq M$$

In our case, from newton's method:

$$g(x) = x - \frac{e^x - x - 1}{e^x - 1}$$

Thus we have:

$$g'(x) = 1 - ((-e^x)(e^x - 1)^{-2}(e^x - x - 1) + (e^x - 1)^{-1}(e^x - 1))$$
$$g'(0) = 1 - ((-1)(1 - 1)^{-2}(1 - 0 - 1) + \frac{0}{0})$$

$\frac{0}{0}$ tells us nothing, NOTHING about a function. Thus we will take the limit as x approaches 0:

$$\lim_{x\to 0}(1-((-e^x)(e^x-1)^{-2}(e^x-x-1)+(e^x-1)^{-1}(e^x-1))) = \lim_{x\to 0}(1-(\frac{(-e^{2x} + xe^x + e^x)}{(e^x - 1)^2}+1))$$

$$\lim_{x\to 0}(1 - (\frac{(-e^{2x} + xe^x + e^x)}{(e^x - 1)^2} + 1)) = \lim_{x\to 0}(-\frac{(-e^{2x} + xe^x + e^x)}{(e^x - 1)^2})$$

Using L'Hopital's:

$$\lim_{x\to 0}(-\frac{(-e^{2x} + xe^x + e^x)}{(e^x - 1)^2}) = \lim_{x\to 0}(-\frac{(-2e^{2x} + e^x + xe^x + e^x)}{2e^x(e^x - 1)})$$

This is still $\frac{0}{0}$................use L'Hopital's again.

$$\lim_{x\to 0}(-\frac{(-2e^{2x} + 2e^x + xe^x)}{2e^x(e^x - 1)}) = \lim_{x\to 0}(-\frac{(-4e^{2x} + 2e^x + e^x + xe^x)}{2e^x(e^x - 1) + 2e^{2x}})$$

This finally gives us $\frac{1}{2}$, so we can safely say that since $g'(0) < 1$, g converges, but since $g'(0) \neq 0$, g does not converge quadratically.

b.) Start with $\mu(x)$ and $\mu'(x)$:

$$\mu(x) = \frac{f(x)}{f'(x)}$$

$$\begin{aligned}
\mu'(x) &= \frac{d}{dx}(f(x)(f'(x))^{-1} \\
&= \frac{f'(x)}{f'(x)} + (-1) \cdot f''(x) \cdot f'(x)^{-2} \cdot f(x) \\
&= \frac{f'(x)}{f'(x)} - \frac{f''(x) \cdot f(x)}{f'(x)^2}
\end{aligned}$$

Plugging into $g(x)$:

$$\begin{aligned}
g(x) &= x - \frac{\mu(x)}{\mu'(x)} \\
&= x - \frac{f(x)}{f'(x) \cdot (\frac{f'(x)}{f'(x)} - \frac{f''(x) \cdot f(x)}{f'(x)^2})} \\
&= x - \frac{f(x)}{\frac{f'(x)^2}{f'(x)} - \frac{f''(x) \cdot f(x)}{f'(x)}} \\
&= x - \frac{f(x)}{f'(x)^{-1}(f'(x)^2 - f''(x) \cdot f(x))} \\
&= x - \frac{f(x) \cdot f'(x)}{f'(x)^2 - f''(x) \cdot f(x)}
\end{aligned}$$

c.) We can see that the new $g(x)$ is approximately correct within the first 5 iterations:

Iteration	Approximation	Residual
0	1.000000e+00	7.182818e-01
1	-2.342106e-01	2.540578e-02
2	-8.458280e-03	3.567061e-05
3	-1.189018e-05	7.068790e-11
4	-4.218591e-11	0.000000e+00
5	-4.218591e-11	0.000000e+00

This $g(x)$ obviously converges much more quickly than the original, and thus it can be concluded that this $g(x)$ converges quadratically.

Problem 2

1. We know that if $|g'(x*)| < 1$ then $g(x)$ converges. So for a we have:

$$\begin{aligned}
g(x) &= \frac{1}{21}(20x + 21x^{-2}) \\
g'(x) &= \frac{1}{21}(20 - 42x^{-3}) \\
g'(21^{\frac{1}{3}} &= \frac{1}{21}(20 - 42(21^{\frac{1}{3}})^{-3}) \\
&= \frac{1}{21}(20 - 42(21^{-1}) \\
&= \frac{1}{21}(20 - 2) \\
&= \frac{1}{21}(18) \\
&= \frac{18}{21} < 1
\end{aligned}$$

Thus a likely converges, but does not converge quadratically since $g'(x) \neq 0$.
for b we have:

$$\begin{aligned}
g(x) &= x - \frac{1}{3}(x^3 - 21)x^{-2}, \\
g'(x) &= 1 - \frac{1}{3}((3x^2)x^{-2} + (x^3 - 21)(-2)x^{-3}), \\
g'(x) &= 1 - \frac{1}{3}(3 + (x^3 - 21)(-2)x^{-3}), \\
g'(21^{\frac{1}{3}}) &= 1 - \frac{1}{3}(3 + (21 - 21)(-2)(21^{-1})) \\
&= 1 - \frac{1}{3}(3) \\
&= 1 - 1 \\
&= 0
\end{aligned}$$

so b converges, and b also likely quadratically converges since $g'(x*) \equiv 0$
For c we have:

$$
\begin{aligned}
g(x) &= x - (x^4 - 21x)(x^2 - 21)^{-1} \\
g'(x) &= 1 - ((4x^3 - 21)(x^2 - 21)^{-1} + (x^4 - 21x)(-1)(2x)(x^2 - 21)^{-2} \\
g'(21^{\frac{1}{3}}) &= 1 - (4(21 - 21)(21^{\frac{2}{3}} - 21)^{-1} - (21^{\frac{4}{3}} - 21^{\frac{4}{3}})(2(21^{\frac{1}{3}}))(21^{\frac{2}{3}} - 21)^{-2}) \\
&= 1 - (0 - 0) \\
&= 1
\end{aligned}
$$

$g'(x*) \geq 1$ so c does not converge linearly or quadratically.

2. Table for a is below:

Iteration	Approximation	Error
0	1.000000e+00	1.758924e+00
1	1.952381e+00	8.065432e-01
2	2.121754e+00	6.371699e-01
3	2.242850e+00	5.160745e-01
4	2.334840e+00	4.240845e-01
5	2.407093e+00	3.518308e-01
6	2.465059e+00	2.938649e-01
7	2.512243e+00	2.466807e-01
8	2.551057e+00	2.078671e-01
9	2.583238e+00	1.756864e-01
10	2.610081e+00	1.488427e-01
11	2.632580e+00	1.263439e-01
12	2.651510e+00	1.074147e-01
13	2.667484e+00	9.143969e-02
14	2.681000e+00	7.792397e-02
15	2.692459e+00	6.646529e-02

Table for b is below:

Iteration	Approximation	Error
0	1.000000e+00	1.758924e+00
1	7.666667e+00	4.907742e+00
2	5.230204e+00	2.471280e+00
3	3.742697e+00	9.837727e-01
4	2.994854e+00	2.359294e-01
5	2.777022e+00	1.809805e-02
6	2.759042e+00	1.176900e-04
7	2.758924e+00	5.020131e-09
8	2.758924e+00	4.440892e-16
9	2.758924e+00	4.440892e-16
10	2.758924e+00	4.440892e-16
11	2.758924e+00	4.440892e-16
12	2.758924e+00	4.440892e-16
13	2.758924e+00	4.440892e-16
14	2.758924e+00	4.440892e-16
15	2.758924e+00	4.440892e-16

Table for c is below:

Iteration	Approximation	Error
0	1.000000e+00	1.758924e+00
1	0.000000e+00	2.758924e+00
2	0.000000e+00	2.758924e+00
3	0.000000e+00	2.758924e+00
4	0.000000e+00	2.758924e+00
5	0.000000e+00	2.758924e+00
6	0.000000e+00	2.758924e+00
7	0.000000e+00	2.758924e+00
8	0.000000e+00	2.758924e+00
9	0.000000e+00	2.758924e+00
10	0.000000e+00	2.758924e+00
11	0.000000e+00	2.758924e+00
12	0.000000e+00	2.758924e+00
13	0.000000e+00	2.758924e+00
14	0.000000e+00	2.758924e+00
15	0.000000e+00	2.758924e+00

We can see from the above that our hypotheses do hold since a does converge, but not quickly. b converges and does so much more quickly than a, and c does not converge to $21^{\frac{1}{3}}$. The python code for this is below:

```
import math;
NUM_ITER = 16
EXPECTED = 21**(1/3)
def f(x):
    return math.e**x - x - 1;
def f_prime(x):
```

```
    return math.e**x - 1;
def f_double_prime(x):
    return math.e**x;
def g1(x):
    return x - f(x) / f_prime(x);
def g2(x):
    return x - (f(x)*f_prime(x)) / (f_prime(x)**2 - f(x) * f_double_prime(x))
def a(x):
    return (20 * x + 21 / (x**2)) / 21
def b(x):
    return x - (x**3 - 21)/(3 * x**2)
def c(x):
    return x - (x**4 - 21 * x) / (x**2 - 21)
def compute(initial_x, function, f_function = None):
    result = "\\begin{tabular}{|c|c|c|c|}\n"
    result += "\\hline\n"
    result += "Iteration & Approximation & Error\\\\\n"
    result += "\\hline\n"
    curr_approx = initial_x
    error = abs(curr_approx - EXPECTED)
    for i in range(0, NUM_ITER):
        result += f"{i} & {curr_approx:.6e} & {error:.6e}\\\\\n"
        result += "\\hline\n"
        curr_approx = function(curr_approx)
        error = abs(curr_approx - EXPECTED)
    result += "\\end{tabular}\n"

    return result
    #return approx;
function_mapping = {
    "f" : f,
    "g1": g1,
    "g2": g2,
    "a" : a,
    "b" : b,
    "c" : c,
}
if __name__ == "__main__":
    initial_num = float(input("enter initial num: "))
    my_func = input("enter the name of the function: ")
    #my_func_2 = input("enter the name of the function (f): ")
    result = compute(initial_num, function_mapping[my_func])
    print(result)
```