How Phasors Work

Keegan Green

Oct 25, 2025

CONTENTS

Note

The latest HTML version of this book is available at:

keeganmjgreen.github.io/how_phasors_work

The latest PDF version is available at:

github.com/keeganmjgreen/how_phasors_work/blob/main/how_phasors_work.pdf

CHAPTER

ONE

UNDERSTANDING AC POWER

AC power is driven by AC voltage. AC means that the voltage (or power) oscillates with time at a fixed frequency f. In the vast majority of cases, this oscillation follows a sine wave $V\sin(2\pi f t)$ because rotary generators produce a sinusoidal voltage, and because voltages that follow a sine wave are most efficiently converted by transformers.

In a pure DC circuit, current flows according to Ohm's law,

$$I = \frac{V}{R},$$

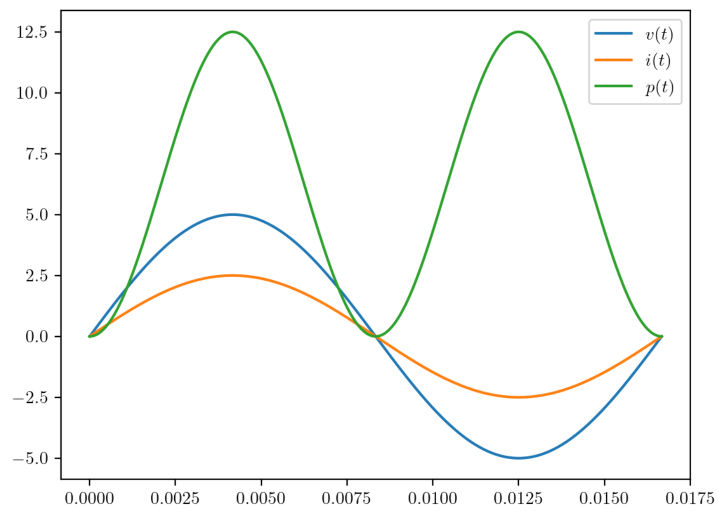
meaning that the power dissipated by the resistor—the rate at which electrical energy is converted to thermal energy—is:

$$P = VI = \frac{V^2}{R}.$$

In a purely resistive AC circuit, Ohm's law still applies:

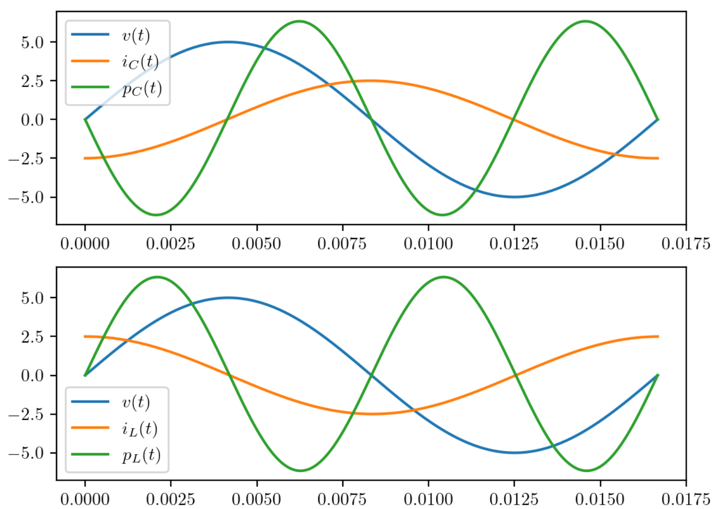
$$i(t) = \frac{v(t)}{R}, \quad p(t) = v(t)\,i(t) = \frac{v^2(t)}{R}.$$

This means that as the voltage sine wave $v(t)$ reaches a peak, so does the current sine wave $i(t)$—they are *in phase*. $p(t)$ is also a sinusoid that is in-phase with the voltage and current. This is because $v(t)$ can be expressed as $V\sin(x)$, and $\sin^2(x)$ is itself also a sinusoid through the trigonometric identity $\sin x = (1 - \cos 2x)/2$.



However, in an AC circuit with inductive and capacitive components, the voltage, current, and power are not in-phase. This is because inductors and capacitors *store electrical energy* in magnetic and electric fields, respectively. They do so by converting electrical energy to magnetic and electric potential energy. These components also release their energy by converting it back to electrical energy. The rate at which the component stores and releases energy is its power.

When an AC voltage is applied to these components, they store and release energy in a cycle. In fact, their power is a sine wave whose frequency f matches that of the AC voltage. However, the times at which these components store and release energy—and the power at which they do so—does not align with the peaks and valleys of the AC voltage. Omitting the mathematical derivations, the sine wave of an inductor's and capacitor's power leads and lags behind that of the AC voltage by exactly an eighth of a cycle, respectively. In a related manner, a capacitor's current lags behind the AC voltage by exactly a quarter cycle (making it a cosine wave), and an inductor's leads behind it by exactly a quarter cycle (making it a negative cosine wave):



CHAPTER TWO

WHAT IS A PHASOR?

2.1 Phasors: A notation and way to make calculations straightforward

A phasor is analogous to a negative number, in the sense that both negative numbers and phasors are mathematical representations, notations, and ways to make calculations straightforward.

A negative number is a *mathematical representation* of the opposite of a quantity. It is distinct from the number zero which represents the absence or lack of a quantity. You can have no fewer than zero apples. Similarly, you cannot have a negative quantity of dollar bills. However, if the quantity of dollars refers not to physical legal tender but to an accounting of how many units of currency you have at your disposal, then a negative quantity represents the opposite of having dollars at your disposal—that is, owing dollars. In addition to this representation, negative numbers offer:

- A *notation*. By prefixing a regular number (x) with a minus sign $(-)$, it becomes negative $(-x)$.
- A *way to make calculations straightforward*. A negative number is like a different kind of quantity on which arithmetic operations can be done by treating them accordingly; to subtract a negative quantity you add its positive counterpart, to multiply you multiply by the positive counterpart and negate the product, and so on.

Similarly, a phasor is a *mathematical representation* of a sine wave, in terms of its amplitude A and phase θ. A sine wave already has a mathematical representation in the time domain with the notation $A\sin(\omega t+\theta)$. However, while this representation makes calculations possible, it does not make them straightforward and does not have a concise notation. Representing sine waves in the phasor domain offers:

- A *notation*. Phasors can be written concisely, such as in polar form $A\angle\theta$. The "ωt" is implicit.
- A *way to make calculations straightforward*. As we will see in *Life With and Without Phasors*, working in the phasor domain allows you to write and solve AC circuit analysis equations more quickly and easily than working in the time domain. Working in the time domain requires writing and solving differential equations from scratch for each circuit, even though the differential equations and their solutions are similar from one circuit to another. It turns out that the calculations that can be done with phasors are the bare minimum calculations that must be done in order to solve an AC circuit.

2.2 Back to basics: Sine waves are vector-like

Consider the right triangle XYZ in Figure 2.1(a). This triangle (like every triangle) has a ratio between length of side YZ (opposite to ϕ_X) and the length of side XY (the hypotenuse). The sine function represents the relationship between this ratio and angle ϕ_X:

$$\sin \phi_X = \frac{YZ}{XY}$$

As ϕ_X ranges from 0 to 2π radians (or 0 to 360 degrees), point Y draws a circle centered around X whose radius A is the hypotenuse's length, as in Figure 2.1(b).

If we were to plot the length of side YZ as a function of $x = \phi_X$, we would have a sine wave of amplitude $A = XY$, as in Figure 2.1(c), that repeats every multiple of 2π radians. This sine wave may represent AC voltage or current.

$$y = A \sin \phi_X$$

If we wanted the drawing of our circle to depend on time t, and we wanted f circles to be drawn per second, we would replace the independent variable $x = \phi_X$ with $x = t$:

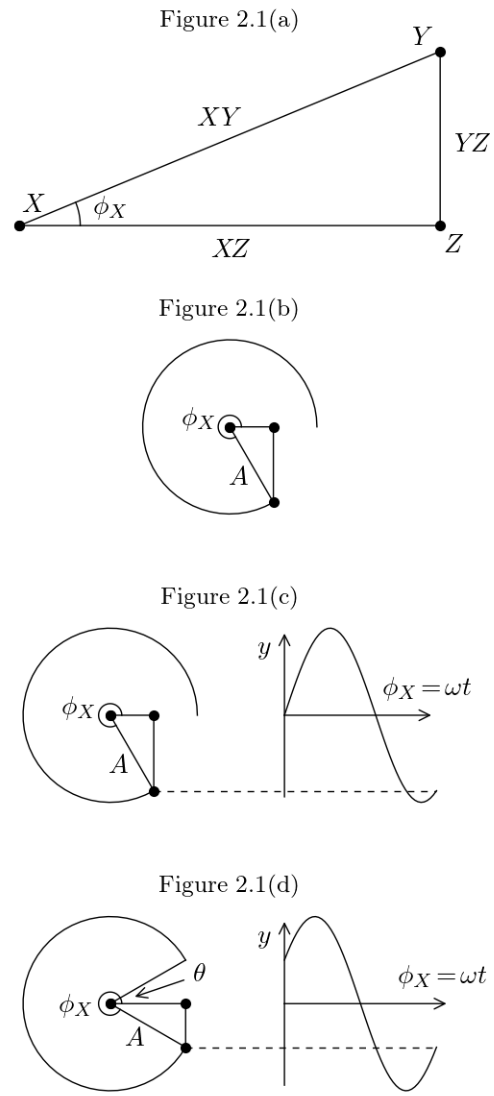
$$y = A \sin 2\pi f t$$

where f is frequency in units of cycles per second (or *Hertz*, Hz). For conciseness, we can replace $2\pi f$ with ω, the *natural frequency*, which is in units of radians per second:

$$y = A \sin \omega t$$

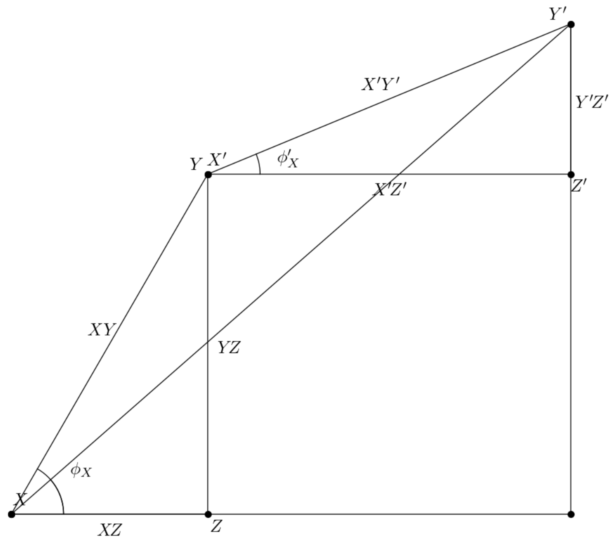
If we wanted to shift the drawing of our circle in time, we could add or subtract a constant from t. However, the mathematically equivalent convention is to add or subtract a constant θ, called the *phase*, from ωt. Now we're drawing our circle with a head start of θ, from θ to $\theta + 2\pi$ radians, as in Figure 2.1(d). This corresponds to a phase-shifted sine wave:

$$y = A \sin(\omega t + \theta)$$



What if we wanted to add sine waves? If the sine wave represents AC voltage or current, being able to do so is important for AC circuit analysis. We can prove that the sum of two phase-shifted sine waves is another phase-shifted sine wave, and apply trigonometric identities to calculate it. However, the geometric approach is more intuitive. Consider that point X of the first wave's triangle is at the origin of its two-dimensional space, and that the second sine wave is similarly formed at any moment in time by a second triangle $X'Y'Z'$. If the two waves were added, then the second triangle would be translated such that its point X' would be situated at the first triangle's point Y. Thus, the second triangle's point Y' would have the coordinates $(XZ + X'Z', YZ + Y'Z')$, representing the sum of the two waves. Similarly, if we were instead subtracting the second wave from the first, the point Y' would have the coordinates $(XZ - X'Z', YZ - Y'Z')$. Clearly, the addition and subtraction of the triangles by which sine waves are formed satisfies the properties of vector addition and subtraction.

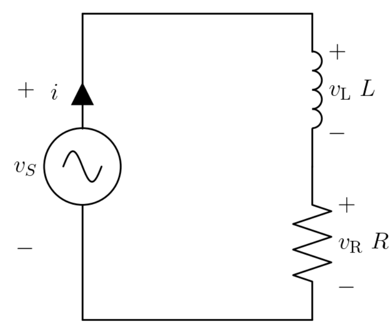
Figure 2.2



Phasors satisfy properties of vectors—addition, subtraction, multiplication by a scalar, and so on. However, phasors are not vectors. Vectors exist in n-dimensional euclidean space. As the geometric analogy suggested, phasors exist in two-dimensional space. As we will learn in the next section, the two-dimensional space in which phasors exist is not even euclidean, and phasors must satisfy additional properties that vectors do not have.

2.3 Reverse engineering the concept of phasors

Let's reverse-engineer the concept of phasors. Consider the following series RL circuit consisting of a voltage source S, a resistor R, and an inductor L:



The voltage source generates a sine wave $v_S(t) = V_S \sin(\omega t + \theta_S)$. If using the geometric analogy to represent the voltage source's sine wave, it would have coordinates $(v_{Sx}, v_{Sy}) = (V_S \cos\theta_S, V_S \sin\theta_S)$. Say that we want to solve for the steady-state current $i(t)$ through the circuit. We would start by writing the integro-differential equation:

$$\begin{aligned} i(t) &= \frac{1}{L}\int_0^t v_L(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{L}\int_0^t (v_S(\tau) - v_R(\tau))\,\mathrm{d}\tau \\ &= \frac{1}{L}\int_0^t (v_S(\tau) - i(\tau)R)\,\mathrm{d}\tau \end{aligned} \tag{2.1}$$

The steady-state solution to this integro-differential equation is as follows. Note that setting up and solving AC circuits' integro-differential equations will be discussed further in *Life With and Without Phasors*.

$$i(t) = \frac{V_S}{\sqrt{R^2 + (\omega L)^2}} \sin\Big(\omega t + \theta_S - \arctan \frac{\omega L}{R}\Big)$$

Using the geometric analogy, we can represent the sine wave of $i(t)$ using coordinates (i_x, i_y). i_x can be written as:

$$\begin{aligned}
i_x &= \frac{V_S}{\sqrt{R^2 + (\omega L)^2}} \cos\Big(\theta_S - \arctan \frac{\omega L}{R}\Big) \\
&= \frac{V_S}{\sqrt{R^2 + (\omega L)^2}} \Big(\cos\theta_S \cdot \cos\Big(\arctan \frac{\omega L}{R}\Big) + \sin\theta_S \cdot \sin\Big(\arctan \frac{\omega L}{R}\Big)\Big) \\
&= \frac{V_S}{\sqrt{R^2 + (\omega L)^2}} \left(\cos\theta_S \cdot \frac{R}{\sqrt{R^2 + (\omega L)^2}} + \sin\theta_S \cdot \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}}\right) \\
&= \frac{V_S \cos\theta_S \cdot R + V_S \sin\theta_S \cdot \omega L}{R^2 + (\omega L)^2}
\end{aligned} \tag{2.2}$$

And, similarly, i_y can be written as follows.

$$\begin{aligned}
i_y &= \frac{V_S}{\sqrt{R^2 + (\omega L)^2}} \sin\Big(\theta_S - \arctan \frac{\omega L}{R}\Big) \\
&= \frac{V_S \sin\theta_S \cdot R - V_S \cos\theta_S \cdot \omega L}{R^2 + (\omega L)^2}
\end{aligned} \tag{2.3}$$

Let's substitute v_{Sx} for $V_S \cos\theta_S$ and v_{Sy} for $V_S \sin\theta_S$. Let's also replace R and ωL with new quantities Z_x and Z_y, whose significance will become apparent shortly.

$$(i_x, i_y) = \left(\frac{v_{Sx} Z_x + v_{Sy} Z_y}{Z_x^2 + Z_y^2}, \frac{v_{Sy} Z_x - v_{Sx} Z_y}{Z_x^2 + Z_y^2}\right)$$

Making use of the properties of complex numbers (in the box below), we discover what is important about being able to write (i_x, i_y) in this way. The complex number $i_x + j\, i_y$ (where j is the imaginary unit) is the result of the following complex number division:

$$i_x + j\, i_y = \frac{v_{Sx} + j\, v_{Sy}}{Z_x + j\, Z_y} \tag{2.4}$$

Properties of complex numbers

Addition and subtraction:

$$(a + j\, b) \pm (c + j\, d) = (a \pm c) + j\, (b \pm d)$$

Multiplication:

$$(a + j\, b)(c + j\, d) = ac + j\, da + j\, bc + (j\, b)(j\, d) = (ac - bd) + j\, (bc + ad)$$

Division:

$$\begin{aligned}
\frac{a + j\, b}{c + j\, d} &= \frac{a + j\, b}{c + j\, d} \frac{c - j\, d}{c - j\, d} \quad \text{(Multiply both numerator and denominator by denominator's conjugate.)} \\
&= \frac{ac - j\, da + j\, bc - (j\, b)(j\, d)}{c^2 - j\, dc + j\, dc - (j\, d)(j\, d)} \\
&= \frac{(ac + bd) + j\, (bc - ad)}{c^2 + d^2}
\end{aligned}$$

Knowing that $Z_x = R$, Equation Equation (2.4) feels like Ohm's law. If we were to remove the inductor L from the circuit and thus set $Z_y = \omega L$ to zero, the equation indeed becomes Ohm's law for a purely resistive circuit:

$$\begin{aligned} i_x + j\, i_y &= \frac{v_x + j\, v_y}{R} = \frac{V_S \cos\theta_S}{R} + j\,\frac{V_S \sin\theta_S}{R} \\ \implies (i_x, i_y) &= \left(\frac{V_S \cos\theta_S}{R}, \frac{V_S \sin\theta_S}{R}\right) \\ \implies i(t) &= \frac{v_S(t)}{R} \end{aligned}$$

In fact, Equation (2.4) is the generalization of Ohm's law for an AC circuit with inductive and capacitive components, and $Z_x + j\, Z_y$, called *impedance*, is the generalization of electrical resistance R for such a circuit. Just as we could write $i(t) = v_S(t)/R$ for a purely resistive AC circuit, we can write $i_x + j\, i_y = (v_{Sx} + j\, v_{Sy})/(Z_x + j\, Z_y)$ for an AC circuit with inductive and capacitive components.

This indicates to us that the coordinates we've been working with in our geometric analogy are actually complex numbers. They satisfy all the properties of complex numbers, including addition, subtraction, multiplication by a scalar, multiplication, and division (as we've just seen). A complex number, as used to represent a sinusoidal voltage or current, is what we call a phasor. Furthermore, a complex number, as used to represent the ratio between a voltage phasor and a current phasor, is what we call impedance. Impedance, however, is not itself a phasor because it does not represent a sine wave.

The most important feature of phasors is that they allow us to do calculations using complex numbers rather than solving differential equations. For example, rather than solving differential equation (2.1), we can quickly convert $v_S(t)$ to a phasor, solve a regular equation in the phasor domain, and convert the solution to $i(t)$ back in the time domain.

You may be thinking that complex numbers are an unintuitive or downright far-fetched way to represent solving differential equations. In answer to this, consider that the behavior of multiplying a complex number of unit magnitude by j is identical to the behavior of differentiating a sine wave of unit amplitude with respect to time, as shown in the following diagram. Similarly, by reversing the diagram, the behavior of dividing a complex number of unit magnitude by j is identical to the behavior of integrating a sine wave of unit amplitude with respect to time. So, instead of writing derivatives and integrals to model capacitors and inductors, we can simply multiply and divide by j.

$$\begin{array}{rccl} & 1 & \sin t & \\ \text{Multiply by } j & \downarrow & \downarrow & \text{Differentiate w.r.t. } t \\ & j & \cos t & \\ \text{Multiply by } j & \downarrow & \downarrow & \text{Differentiate w.r.t. } t \\ & -1 & -\sin t & \\ \text{Multiply by } j & \downarrow & \downarrow & \text{Differentiate w.r.t. } t \\ & -j & -\cos t & \\ \text{Multiply by } j & \downarrow & \downarrow & \text{Differentiate w.r.t. } t \\ & 1 & \sin t & \end{array}$$

2.4 Notation for phasors

Previously, we treated phasors as coordinates in two-dimensional space, with x- and y components. For example, we had:

$$(v_{Sx}, v_{Sy}) \qquad (i_x, i_y)$$

However, now that we know that phasors (and impedances) are just complex numbers, we can replace this notation with complex number notation, in terms of a real and imaginary component. For example:

$$\mathbf{v}_S = \mathrm{Re}(\mathbf{v}_S) + j\,\mathrm{Im}(\mathbf{v}_S) \qquad \mathbf{i} = \mathrm{Re}(\mathbf{i}) + j\,\mathrm{Im}(\mathbf{i}) \qquad Z = \mathrm{Re}(Z) + j\,\mathrm{Im}(Z)$$

2.4.1 Polar form

Let's introduce a polar form of complex number notation, which is used to write a phasor (or impedance) in terms of amplitude and phase, rather than in terms of its real and imaginary components. For example:

$$\begin{aligned}\mathbf{v}_S &= V_S \cos\theta_S + j\,V_S \sin\theta_S \\ &= \boxed{V_S \angle \theta_S}\end{aligned}$$

We can apply this polar form to equations (2.2) and (2.3):

$$\begin{aligned}\mathbf{i} &= \frac{V_S}{\sqrt{R^2+(\omega L)^2}} \cos\left(\theta_S - \arctan\frac{\omega L}{R}\right) + j\,\frac{V_S}{\sqrt{R^2+(\omega L)^2}} \sin\left(\theta_S - \arctan\frac{\omega L}{R}\right) \\ &= \boxed{\left(\frac{V_S}{\sqrt{R^2+(\omega L)^2}}\right) \angle \left(\theta_S - \arctan\frac{\omega L}{R}\right)}\end{aligned} \tag{2.5}$$

Previously, equations (2.2) and (2.3) led us to see a way that $\mathbf{i}$ can be calculated from $\mathbf{v}_S$ by dividing by impedance Z using division of complex numbers. If we take a close look at Equation (2.5), we can find an additional, equivalent way in which $\mathbf{i}$ can be calculated from $\mathbf{v}_S$. Notice how $\mathbf{v}_S$ can be converted to $\mathbf{i}$ by dividing its amplitude V_S by $\sqrt{R^2+(\omega L)^2}$ and by subtracting $\arctan(\omega L/R)$ from θ_S. This demonstrates a useful property of phasors and complex numbers in general in polar form—two complex numbers can easily be divided by dividing their amplitudes and subtracting their phases:

$$\frac{A_1\angle\theta_1}{A_2\angle\theta_2} = \frac{A_1}{A_2}\angle(\theta_1 - \theta_2)$$

Of course, all properties of complex numbers can be written in polar form, but multiplying and dividing complex numbers is easiest when they are in polar form. On the other hand, adding and subtracting complex numbers is easiest when they are in the non-polar form.

Properties of complex numbers (polar form)

Addition and subtraction:

$$A_1\angle\theta_1 \pm A_2\angle\theta_2 = \sqrt{(A_1\cos\theta_1 \pm A_2\cos\theta_2)^2 + (A_1\sin\theta_1 \pm A_2\sin\theta_2)^2}\angle\left(\arctan\frac{A_1\sin\theta_1 \pm A_2\sin\theta_2}{A_1\cos\theta_1 \pm A_2\cos\theta_2}\right)$$

Multiplication:

$$A_1\angle\theta_1 \cdot A_2\angle\theta_2 = A_1A_2\angle(\theta_1+\theta_2)$$

Division:

$$\frac{A_1\angle\theta_1}{A_2\angle\theta_2} = \frac{A_1}{A_2}\angle(\theta_1 - \theta_2)$$

CHAPTER

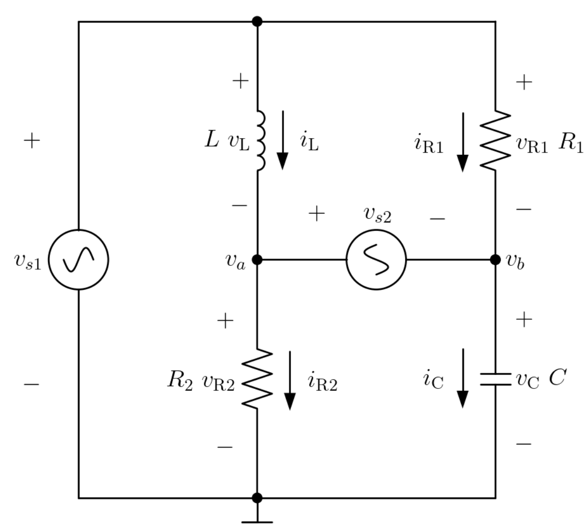
THREE

LIFE WITH AND WITHOUT PHASORS

One of the main reasons for many people being confused about phasors is that phasors are abstract, but they don't understand why the abstraction exists. In other words, they may know what phasors *represent*, but they don't know what they *replace*—they don't know of any alternative approach for analyzing AC circuits that can be used to put phasors in perspective. This is understandable as many electrical engineering textbooks poorly explain why we use phasors.

With the aid of an example AC circuit, this section goes over what's hidden behind the phasor abstraction. By first trying to solve the circuit without phasors, we will better understand how AC circuit analysis works at its core. And by then solving the circuit with phasors, we will see that it is not only a more concise notation, but also a simpler mathematical representation and a faster method for analyzing AC circuits.

The following AC circuit is our example circuit. Note that it has multiple AC sources and an bridge-type topology that does not allow us to consolidate impedances using formulae for impedances in series or in parallel.



Say that we want to solve for voltages $v_a(t)$ and $v_b(t)$. Doing so would allow us to solve for the voltage across—and therefore the current through—any one component. Just as for a purely resistive DC network, Kirchhoff's Current Law (KCL) applies and we can do nodal analysis to determine the voltages at nodes a and b. In particular we must use the supernode technique with a single node ab because the current through voltage source v_{s2} cannot be put in terms of $v_a(t)$ or $v_b(t)$. This loses us an equation, but we gain one in its place with the following relationship:

$$v_{s2}(t) = v_a(t) - v_b(t) \tag{3.1}$$

Let's try solving the circuit using node voltage analysis, with and without phasors.

> **Note**
>
> While the parameters V_{s1}, ω, R_1, etc. can take on any numerical values, at certain points in the chapter we will substitute the following very arbitrary values to simplify or be able to plot our equations.
>
> $$\begin{gathered} V_{s1} = 1\ \text{V} \qquad V_{s2} = 2\ \text{V} \qquad \omega = 3\ \text{rad/s} \\ R_1 = 4\ \Omega \qquad R_2 = 5\ \Omega \qquad L = 6\ \text{H} \qquad C = 7\ \text{F} \end{gathered} \tag{3.2}$$

3.1 Solving the circuit without phasors

Using KCL, the sum of currents flowing out of supernode ab must sum to zero. We can therefore write:

$$-i_{\text{L}}(t) + i_{\text{R2}}(t) - i_{\text{R1}}(t) + i_{\text{C}}(t) = 0$$

Substituting in the components' characteristic equations gives us:

$$-\frac{1}{L}\int_0^t v_{\text{L}}(\tau)\,\mathrm{d}\tau + \frac{v_{\text{R2}}(t)}{R_2} - \frac{v_{\text{R1}}(t)}{R_1} + C\frac{\mathrm{d}v_{\text{C}}(t)}{\mathrm{d}t} = 0$$

Putting everything in terms of the node voltages yields:

$$-\frac{1}{L}\int_0^t (v_{s1}(\tau) - v_a(\tau))\,\mathrm{d}\tau + \frac{v_a(t)}{R_2} - \frac{v_{s1}(t) - v_b(t)}{R_1} + C\frac{\mathrm{d}v_b(t)}{\mathrm{d}t} = 0 \tag{3.3}$$

Paired with Equation (3.1), this represents a system of two integro-differential equations. By differentiating, we put this in the more familiar form of a system of differential equations (in which the highest-order derivative is two):

$$\begin{aligned} &-\frac{1}{L}(v_{s1}(t) - v_a(t)) + \frac{1}{R_2}\frac{\mathrm{d}v_a(t)}{\mathrm{d}t} - \frac{1}{R_1}\frac{\mathrm{d}}{\mathrm{d}t}(v_{s1}(t) - v_b(t)) + C\frac{\mathrm{d}^2 v_b(t)}{\mathrm{d}t^2} = 0 \\ &v_{s2}(t) = v_a(t) - v_b(t) \end{aligned}$$

We put this into a canonical form of a system of first-order differential equations by introducing an auxiliary variable $v_b'(t)$ and by using Equation (3.1) to replace the need for variable $v_a(t)$ with $v_b'(t)$:

$$\begin{aligned} \frac{\mathrm{d}v_b(t)}{\mathrm{d}t} &= v_b'(t) \\ \frac{\mathrm{d}v_b'(t)}{\mathrm{d}t} &= \frac{1}{C}\left[\frac{v_{s1}(t) - (v_{s2}(t) + v_b(t))}{L} - \frac{v_{s2}'(t) + v_b'(t)}{R_2} + \frac{v_{s1}'(t) - v_b'(t)}{R_1}\right] \end{aligned}$$

We specify that $v_{si}(t) = V_{si}\sin\omega t$ (and $v_{si}'(t) = \omega V_{si}\cos\omega t$):

$$\begin{aligned} \frac{\mathrm{d}v_b(t)}{\mathrm{d}t} &= v_b'(t) \\ \frac{\mathrm{d}v_b'(t)}{\mathrm{d}t} &= \frac{1}{C}\left[\frac{(V_{s1} - V_{s2})\sin\omega t - v_b(t)}{L} - \frac{\omega V_{s2}\cos\omega t + v_b'(t)}{R_2} + \frac{\omega V_{s1}\cos\omega t - v_b'(t)}{R_1}\right] \end{aligned}$$

Our system is now in terms of two unknowns—$v_b(t)$ and $v_b'(t)$—two functions to solve for. We start to solve our system of differential equations by assuming a solution of the form:

$$\begin{aligned} v_b(t) &= V_b\cos(\omega t + \theta_b) \\ v_b'(t) &= -\omega V_b\sin(\omega t + \theta_b) \end{aligned}$$

By substituting into our system of differential equations, we convert it to a regular system of equations—or, in this case, just one equation:

$$-\omega^2 V_b \cos(\omega t + \theta_b) = \frac{1}{C}\left[\begin{aligned}&\frac{(V_{s1} - V_{s2})\sin\omega t - V_b\cos(\omega t + \theta_b)}{L} \\ &+ \omega\left[\left(\frac{V_{s1}}{R_1} - \frac{V_{s2}}{R_2}\right)\cos\omega t + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_b\sin(\omega t + \theta_b)\right]\end{aligned}\right]$$

This equation is in terms of two unknowns—V_b and θ_b. While having only one equation may appear to be insufficient to solve for two unknowns, the equation is parameterized by t and must be satisfied for all values of t. So, in this sense, we actually have an infinite number of equations at our disposal. We can pick two 'easy' values of t that result in two relatively simple equations:

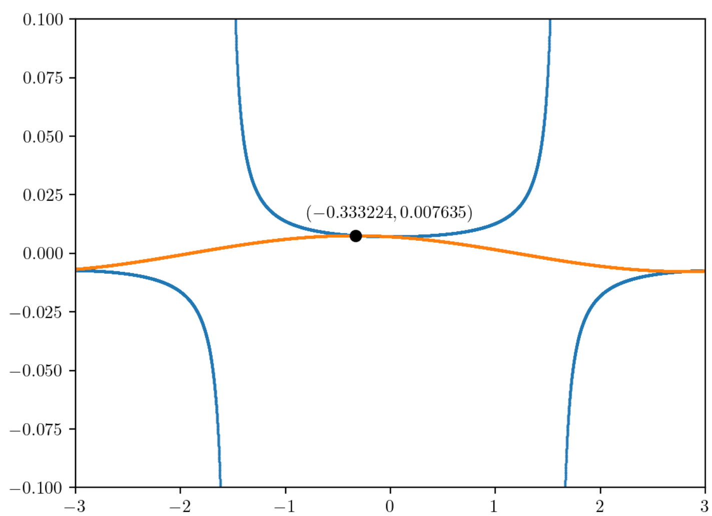
$t = 0$:

$$-\omega^2 C V_b \cos\theta_b = \frac{-V_b\cos\theta_b}{L} + \omega\left[\left(\frac{V_{s1}}{R_1} - \frac{V_{s2}}{R_2}\right) + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_b\sin\theta_b\right]$$

$t = -\theta_b/\omega$:

$$-\omega^2 C V_b = \frac{-(V_{s1} - V_{s2})\sin\theta_b - V_b}{L} + \omega\left(\frac{V_{s1}}{R_1} - \frac{V_{s2}}{R_2}\right)\cos\theta_b$$

This nonlinear system is inconvenient to solve. We can plot these two equations as follows and solve the system numerically. However, numerical methods are less efficient than being able to directly calculate the solution. We should be able to solve AC circuits very quickly.



Note

There is technically an infinite number of solutions for θ_b, because any multiple of 2π can be added to (or subtracted from) θ_b, resulting in an identical sinusoid with a different equation.

Furthermore, as we have seen, the process to derive the system of equations is long and notation-heavy despite being consistent between different AC circuits.

Before we see how the circuit can be solved with phasors, let's see how we can solve its system of integro-differential equations with the aid of the Laplace transform.

3.2 Solving the circuit with the Laplace transform

The Laplace transform $\mathcal{L}$ converts a function $f(t)$ (in the *time domain*) into a function $F(s)$ (in the *complex frequency domain*) through the following integration:

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t)\, \mathrm{e}^{-st}\, \mathrm{d}t \tag{3.4}$$

The Laplace transform is closely related to the Fourier transform. Both transforms convert to the frequency domain. However, the Laplace transform converts to a frequency domain whose variable s is a *complex number*, which can simplify calculations. This domain is known as the *s-domain* or *Laplace domain*.

The Laplace transform has a number of favorable properties. One such property is that differentiating with respect to t in the time domain becomes multiplication by s in the Laplace domain. Similarly, integrating with respect to t in the time domain becomes division by s in the Laplace domain. Because of this, the Laplace transform is often used to convert a system of differential equations into a regular system of equations which can be solved more easily.

Furthermore, many useful functions in the time domain become simple, rational functions in the Laplace domain, for example:

Sine wave:	$\mathcal{L}\{\sin(\omega t)\, u(t)\} = \dfrac{\omega}{s^2 + \omega^2}$
Phase-shifted cosine wave:	$\mathcal{L}\{\cos(\omega t + \theta)\, u(t)\} = \dfrac{s\cos\theta - \omega\sin\theta}{s^2 + \omega^2}$
Exponential decay:	$\mathcal{L}\{\mathrm{e}^{-\alpha t}\, u(t)\} = \dfrac{1}{s + \alpha}$
Exponentially decaying cosine wave:	$\mathcal{L}\{\mathrm{e}^{-\alpha t}\cos(\omega t)\, u(t)\} = \dfrac{s + \alpha}{(s + \alpha)^2 + \omega^2}$

Listings of the Laplace transforms of common functions, such as those above, are called Laplace tables.

$u(t)$ is the Heaviside step function, which is 1 for $t \geq 0$ and 0 otherwise. The time-domain functions are multiplied by $u(t)$—that is, only nonzero for $t \geq 0$, for the same reason that the lower bound of integration in Equation (3.4) is 0. The Laplace transform only looks at $t \geq 0$, and is thus useful for solving differential equations involving functions that are only nonzero for $t \geq 0$. This aspect allows us to answer questions about a system—particularly an electric circuit—that is modeled by differential equations, such as: "what happens when the system is turned on?", "what happens when the value of this component suddenly changes?", and "what happens when the circuit's topology changes in this way?" To consider functions that are nonzero for $t < 0$, and thus model no sudden change at $t = 0$, the *two-sided* or *bilateral* Laplace transform can be used.

We want to solve the system of two integro-differential equations (3.1) and (3.3) that we derived in the previous section. That system was in terms of $v_a(t)$ and $v_b(t)$, in the time domain. First, we will apply the Laplace transform to convert it to a regular system of equations in terms of $V_a(s)$ and $V_b(s)$, in the Laplace domain. Then, we will be able to solve for $V_a(s)$ and $V_b(s)$ very easily. Finally, we will convert our solution back to the time domain using the inverse Laplace transform.

Applying the Laplace transform to equations (3.1) and (3.3) gives us:

$$\begin{aligned} &-\frac{V_{s1}(s) - V_a(s)}{Ls} + \frac{V_a(s)}{R_2} - \frac{V_{s1}(s) - V_b(s)}{R_1} + CsV_b(s) = 0 \\ &V_{s2}(s) = V_a(s) - V_b(s) \end{aligned}$$

Here, $V_{s1}(s)$ and $V_{s2}(s)$ are the Laplace transforms of $v_{s1}(t)$ and $v_{s2}(t)$—not to be confused with the amplitudes V_{s1} and V_{s2} of $v_{s1}(t)$ and $v_{s2}(t)$.

Solving for $V_a(s)$ in the second equation and substituting it into the first equation yields:

$$-\frac{V_{s1}(s) - (V_{s2}(s) + V_b(s))}{Ls} + \frac{V_{s2}(s) + V_b(s)}{R_2} - \frac{V_{s1}(s) - V_b(s)}{R_1} + CsV_b(s) = 0$$

Solving for $V_b(s)$ gives us:

$$V_b = \frac{R_2(R_1 + Ls)V_{s1}(s) - R_1(R_2 + Ls)V_{s2}(s)}{R_1R_2 + (R_1 + R_2)Ls + R_1R_2LCs^2}$$

Substituting in $V_{si}(s) = \mathcal{L}\{V_{si}\sin(\omega t)\,u(t)\} = V_{si}\,\omega/(s^2+\omega^2)$ yields the following. Note that, because the voltage sources are multiplied by $u(t)$, we are essentially answering the question of how the circuit responds when turned on at $t = 0$.

$$V_b = \frac{R_2(R_1 + Ls)V_{s1}\omega - R_1(R_2 + Ls)V_{s2}\omega}{(s^2+\omega^2)(R_1R_2 + (R_1 + R_2)Ls + R_1R_2LCs^2)}$$

At this point, the algebra will be significantly easier if we substitute in known values for our parameters. Let's introduce numerical values (3.2), after which we will omit units for brevity. However, $V_b(s)$ is still in units of volts and s is still, like ω, in units of radians per second.

$$\begin{aligned} V_b(s) &= \frac{(5\ \Omega)(4\ \Omega + (6\ \text{H})s)(1\ \text{V})(3\ \text{rad/s}) - (4\ \Omega)(5\ \Omega - (6\ \text{H})s)(2\ \text{V})(3\ \text{rad/s})}{(3\ \text{rad/s})(-(4\ \Omega)(5\ \Omega) + (4\ \Omega - 5\ \Omega)(6\ \text{H})s + (4\ \Omega)(5\ \Omega)(6\ \text{H})(7\ \text{F})s^2)} \\ &= \frac{-54s - 60}{(s^2+9)(840s^2 + 54s + 20)} \end{aligned}$$

We need to convert this solution for $V_b(s)$ back to the time domain. Converting a function from the Laplace domain to the time domain can be done by looking it up in a Laplace table. However, $V_b(s)$ is too complicated in its current form to find a matching function in a Laplace table—in particular, because the order of the polynomial in the denominator is four. This is higher than the highest-order denominator of two that can be found in typical Laplace tables. We can handle this by rewriting our solution as a linear combination of simpler functions that we will be able to find in the Laplace table. To determine such simpler functions, we factor the denominator then apply *partial fraction decomposition*. The denominator needs to be split into factors whose order is at most two. This is already the case. Next, we assume the following modified format of our V_b solution and find the values A_1 through A_4 that make it true:

$$\begin{aligned} & V_b(s) = \frac{A_1 s + A_2}{s^2+9} + \frac{A_3 s + A_4}{840s^2 + 54s + 20} \\ & \implies \frac{-54s - 60}{(s^2+9)(840s^2 + 54s + 20)} = \frac{A_1 s + A_2}{s^2+9} + \frac{A_3 s + A_4}{840s^2 + 54s + 20} \\ & \implies -54s - 60 = (A_1 s + A_2)(840s^2 + 54s + 20) + (A_3 s + A_4)(s^2 + 9) \\ & \implies (-840A_1 - A_3)s^3 \\ & \quad + (-54A_1 - 840A_2 - A_4)s^2 \\ & \quad + (-20A_1 - 54A_2 - 9A_3 - 54)s \\ & \quad + (-20A_2 - 9A_4 - 60) \\ & \quad = 0 \\ & \implies \begin{cases} -840A_1 - A_3 = 0 \\ -54A_1 - 840A_2 - A_4 = 0 \\ -20A_1 - 54A_2 - 9A_3 - 54 = 0 \\ -20A_2 - 9A_4 - 60 = 0 \end{cases} \\ & \implies \begin{cases} A_1 = 102600/14219461 \\ A_2 = 106539/14219461 \\ A_3 = -86184000/14219461 \\ A_4 = -95033160/14219461 \end{cases} \end{aligned}$$

Substituting A_1 through A_4 gives us:

$$V_b(s) = \underbrace{\frac{\frac{102600}{14219461}s + \frac{106539}{14219461}}{s^2+9}}_{\text{First term}} + \underbrace{\frac{-\frac{86184000}{14219461}s - \frac{95033160}{14219461}}{840s^2+54s+20}}_{\text{Second term}}$$

Inverse Laplace transform of the first term. The first term matches the following Laplace table function:

$$\mathcal{L}\{\cos(\omega t+\theta)\,u(t)\} = \frac{s\cos\theta + \omega\sin\theta}{s+\omega^2}$$

Therefore:

$$\text{First term} = \frac{\frac{102600}{14219461}s + \frac{106539}{14219461}}{s^2+9} = B_1\frac{s\cos\theta_1 - \omega_1\sin\theta_1}{s+\omega_1^2}$$

The ω_1 here is (not coincidentally) the same as our natural frequency $\omega = 3$ rad/s. Now, we need to solve for θ_1 and B_1, which we can do by setting up the following system of equations:

$$\begin{aligned} B_1\cos\theta_1 &= 102600/14219461 \\ -3B_1\sin\theta_1 &= 106539/14219461 \end{aligned}$$

We can solve for θ_1 by dividing the second equation by the first:

$$\frac{106539/14219461}{102600/14219461} = -3\tan\theta_1 \implies \theta_1 = \arctan\left(-\frac{35513}{102600}\right) = -19.092^\circ$$

And we can solve for B_1 by dividing the second equation by -3 and adding its square to the square of the first:

$$\begin{aligned} &\left(\frac{102600}{14219461}\right)^2 + \left(\frac{106539}{-3\cdot 14219461}\right)^2 = B_1^2\cos^2\theta_1 + B_1^2\sin^2\theta_1 = B_1^2 \\ &\implies B_1 = \sqrt{\left(\frac{102600}{14219461}\right)^2 + \left(\frac{35513}{14219461}\right)^2} = 0.007635 \end{aligned}$$

Therefore, the inverse Laplace transform of the first term is:

$$\mathcal{L}^{-1}\left\{\frac{\frac{102600}{14219461}s + \frac{106539}{14219461}}{s^2+9}\right\} = 0.007635\cos(3t - 19.092^\circ)\,u(t)$$

Derivation: Laplace transform of an exponentially decaying, phase-shifted cosine wave

$$\begin{aligned} \mathcal{L}\{e^{-\alpha t}\cos(\omega t+\theta)\,u(t)\} &= \mathcal{L}\{e^{-\alpha t}\left[\cos\omega t\cos\theta - \sin\omega t\sin\theta\right]u(t)\} \\ &= \cos\theta\,\mathcal{L}\{e^{-\alpha t}\cos\omega t\,u(t)\} - \sin\theta\,\mathcal{L}\{e^{-\alpha t}\sin\omega t\,u(t)\} \\ &= \cos\theta\frac{s+\alpha}{(s+\alpha)^2+\omega^2} - \sin\theta\frac{\omega}{(s+\alpha)^2+\omega^2} \\ &= \frac{(s+\alpha)\cos\theta - \omega\sin\theta}{(s+\alpha)^2+\omega^2} \end{aligned} \tag{3.5}$$

Inverse Laplace transform of the second term. The second term matches the following function from (3.5):

$$\mathcal{L}\{e^{-\alpha t}\cos(\omega t+\theta)\,u(t)\} = \frac{(s+\alpha)\cos\theta - \omega\sin\theta}{(s+\alpha)^2+\omega^2}$$

Therefore:

$$\text{Second term} = \frac{-\frac{86184000}{14219461}s - B_2\frac{95033160}{14219461}}{840s^2+54s+20} = \frac{(s+\alpha_2)\cos\theta_2 - \omega_2\sin\theta_2}{(s+\alpha_2)^2+\omega_2^2}$$
$$\implies \frac{-\frac{102600}{14219461}s - \frac{791943}{99536227}}{s^2+\frac{9}{140}s+\frac{1}{42}} = B_2\frac{s\cos\theta_2 + (\alpha_2\cos\theta_2 - \omega_2\sin\theta_2)}{s^2+2\alpha_2 s+(\alpha_2^2+\omega_2^2)}$$

We need to solve for α_2, ω_2, θ_2, and B_2:

$$9/140 = 2\alpha_2 \implies \alpha_2 = 9/280 = 0.03214$$
$$\alpha_2^2+\omega_2^2 = \frac{1}{42} \implies \omega_2 = \sqrt{\frac{1}{42} - \left(\frac{9}{280}\right)^2} = 0.15092 \text{ rad/s}$$

We can solve for θ_2 and B_2 by setting up and solving the following system of equations:

$$\left.\begin{aligned} -102600/14219461 &= B_2\cos\theta_2 \\ -791943/99536227 &= B_2\left(0.03214\cos\theta_2 - 0.15092\sin\theta_2\right) \end{aligned}\right\} \implies \begin{cases} \theta_2 = -81.9756^\circ \\ B_2 = -0.05169 \end{cases}$$

Therefore, the inverse Laplace transform of the second term is:

$$\mathcal{L}^{-1}\left\{\frac{-\frac{86184000}{14219461}s - \frac{95033160}{14219461}}{840s^2+54s+20}\right\} = -0.05169\,\mathrm{e}^{-0.03214t}\cos(0.15092\,t - 81.9756^\circ)\,u(t)$$

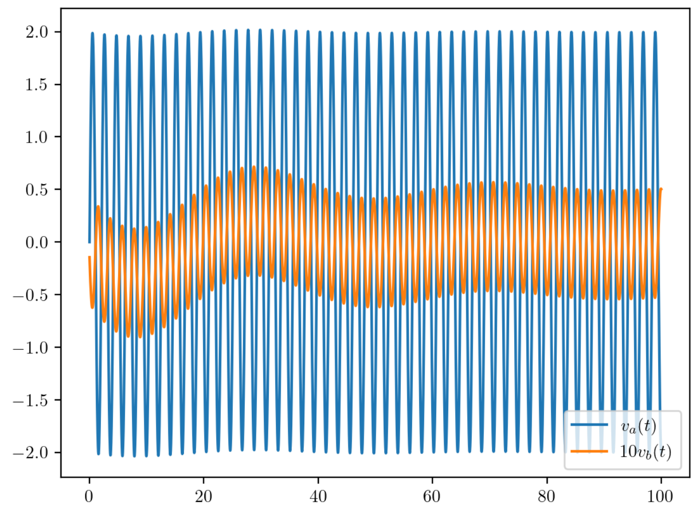
And finally, substituting in the inverse Laplace transforms of the first and second terms yields the following full solution for $v_b(t)$ in the time domain:

$$v_b(t) = (\underbrace{0.007635\cos(3t-19.092^\circ)}_{\text{Steady-state response, } v_b^{SS}(t)} - \underbrace{0.05169\,\mathrm{e}^{-0.03214t}\cos(0.15092\,t-81.9756^\circ)}_{\text{Transient response}})\,u(t)$$

From this we can calculate $v_a(t)$ via trigonometric identities. Its transient response is the same as for $v_b(t)$, and its steady-state response is:

$$\begin{aligned} v_a^{SS}(t) &= v_{s2}(t) + v_b^{SS}(t) \\ &= 0.007635\cos(3t-19.092) + 2\sin(3t) \\ &= 1.99752\cos(3t - 89.7930^\circ) \end{aligned}$$

The solution for $v_b(t)$ (and $v_a(t)$) has two distinct terms. It is a linear combination of cosine waves, each with their own natural frequency and phase, but one of the cosine waves is exponentially decaying. The exponentially decaying cosine models how the circuit responds when turned on at $t = 0$—its *transient response*. The non-decaying cosine is how the circuit responds in the longer term—its *steady-state response* as t tends to infinity (but practically much sooner as the transient quickly dies out).



3.3 Solving the circuit with phasors

Using KCL, the sum of currents flowing out of supernode ab must sum to zero. We can therefore write:

$$-\mathbf{i}_{\mathrm{L}} + \mathbf{i}_{\mathrm{R2}} - \mathbf{i}_{\mathrm{R1}} + \mathbf{i}_{\mathrm{C}} = 0$$

$$-\frac{\mathbf{v}_{\mathrm{L}}}{Z_{\mathrm{L}}} + \frac{\mathbf{v}_{\mathrm{R2}}}{Z_{\mathrm{R2}}} - \frac{\mathbf{v}_{\mathrm{R1}}}{Z_{\mathrm{R1}}} + \frac{\mathbf{v}_{\mathrm{C}}}{Z_{\mathrm{C}}} = 0$$

Substituting in the components' impedances and putting everything in terms of the node voltages gives us:

$$-\frac{\mathbf{v}_{s1} - \mathbf{v}_a}{j\,\omega L} + \frac{\mathbf{v}_a}{R_2} - \frac{\mathbf{v}_{s1} - \mathbf{v}_b}{R_1} + \frac{\mathbf{v}_b}{j\frac{-1}{\omega C}} = 0$$

Paired with Equation (3.1), $\mathbf{v}_a - \mathbf{v}_b = \mathbf{v}_{s2}$, this represents a system of two linear equations in two unknowns $\mathbf{v}_a$ and $\mathbf{v}_b$. Substituting Equation (3.1) into the above yields:

$$j\frac{\mathbf{v}_{s1} - (\mathbf{v}_{s2} + \mathbf{v}_b)}{\omega L} + \frac{\mathbf{v}_{s2} + \mathbf{v}_b}{R_2} - \frac{\mathbf{v}_{s1} - \mathbf{v}_b}{R_1} + j\,\omega C\mathbf{v}_b = 0$$

Solving for $\mathbf{v}_b$ yields the following, and we can solve for $\mathbf{v}_a$ from this using Equation (3.1).

$$\mathbf{v}_b = \frac{\omega L(R_2\mathbf{v}_{s1} - R_1\mathbf{v}_{s2}) + j\,R_1R_2(\mathbf{v}_{s2} - \mathbf{v}_{s1})}{\omega L(R_1 + R_2) + j\,(\omega^2 CL - 1)R_1R_2}$$

Getting here required some algebra involving complex numbers. However, when solving a real circuit whose parameters have known values, this process would have been aided significantly by being able to evaluate/simplify expressions at each step. Let's introduce numerical values (3.2) to see how $\mathbf{v}_b$ and $\mathbf{v}_a$ can be evaluated.

$$\begin{aligned}
\mathbf{v}_b &= \frac{(3\text{ rad/s})(6\text{ H})((5\,\Omega)(j\,1\text{ V}) - (4\,\Omega)(j\,2\text{ V})) + j\,(4\,\Omega)(5\,\Omega)((j\,2\text{ V}) - (j\,1\text{ V}))}{(3\text{ rad/s})(6\text{ H})((4\,\Omega) + (5\,\Omega)) + j\,((3\text{ rad/s})^2(7\text{ F})(6\text{ H}) - 1)(4\,\Omega)(5\,\Omega)} \\
&= \frac{-20 - j\,54}{162 + j\,7540}\text{ V} \\
&= \frac{-20 - j\,54}{162 + j\,7540}\,\frac{162 - j\,7540}{162 - j\,7540}\text{ V} \\
&= \frac{(-20 - j\,54)(162 - j\,7540)}{162^2 + 7540^2}\text{ V} \\
&= \frac{-3240 + j\,150800 - j\,8748 - 407160}{26244 + 56851600}\text{ V} \\
&= \frac{-410400 + j\,142052}{56877844}\text{ V} \\
&= \boxed{(-0.007215 + j\,0.002497)\text{ V}}
\end{aligned}$$

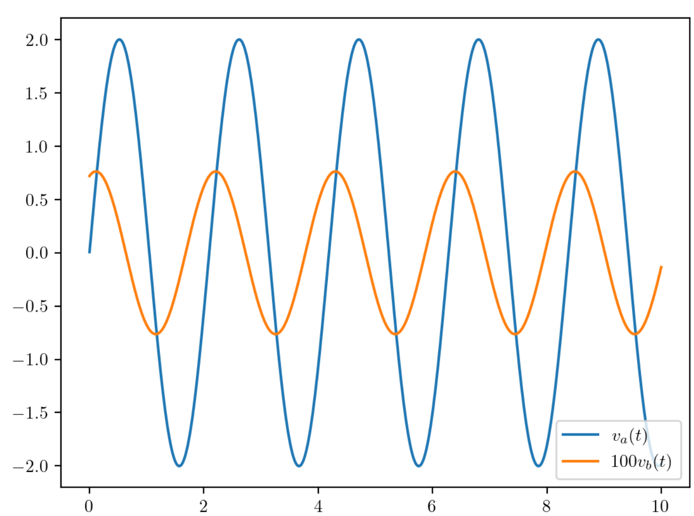
$$\begin{aligned}
\mathbf{v}_a &= \mathbf{v}_b + \mathbf{v}_{s2} \\
&= (-0.007215 + j\,0.002497)\text{ V} + j\,2\text{ V} \\
&= \boxed{(-0.007215 + j\,2.002497)\text{ V}}
\end{aligned}$$

By calculating the amplitude A and phase θ, we can convert from complex number phasor notation $a + j\,b$ to either polar phasor notation $A\angle\theta$ or back into the time domain as $A\cos(\omega t + \theta)$. The amplitude and phase are calculated using:

$$A = \sqrt{a^2 + b^2}, \qquad \theta = \arctan\frac{b}{a}$$

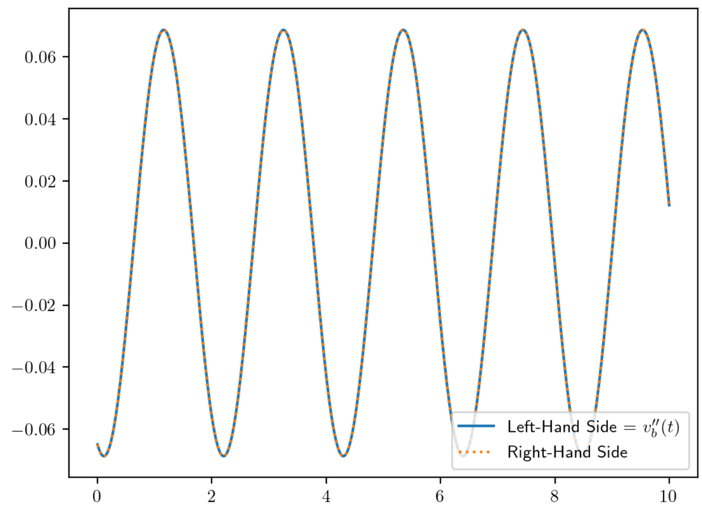
$$\begin{aligned}
\mathbf{v}_b &= 0.007635\,\angle{-19.092^\circ} \\
\mathbf{v}_a &= 2.002510\,\angle{-89.794^\circ}
\end{aligned}$$

With the amplitude and phase calculated, it becomes very easy to plot the solution for $v_a(t)$ and $v_b(t)$:



3.4 Checking the phasor solution

At this point, we have derived the equations representing the circuit with and without phasors, and we have solved the circuit with phasors. To check our work, we can plug the solution obtained using phasors into the differential equation obtained without using phasors, and verify that the equation holds true (that the left-hand side equals the right-hand side). This can be done conveniently by plotting both the left- and right-hand sides:



As can be seen, the left- and right-hand sides are equal.