How Phasors Work

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Oct 19, 2025

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UNDERSTANDING AC POWER

AC power is driven by AC voltage. AC means that the voltage (or power) oscillates with time at a fixed frequency f. In the vast majority of cases, this oscillation follows a sine wave $V\sin(2\pi ft)$ because rotary generators produce a sinusoidal voltage, and because voltages that follow a sine wave are most efficiently converted by transformers.

In a pure DC circuit, current flows according to Ohm's law,

$$I = \frac{V}{R},$$

meaning that the power dissipated by the resistor—the rate at which electrical energy is converted to thermal energy—is:

$$P = VI = \frac{V^2}{R}.$$

In a purely resistive AC circuit, Ohm's law still applies:

$$i(t) = \frac{v(t)}{R}, \quad p(t) = v(t)\,i(t) = \frac{v^2(t)}{R}.$$

This means that as the voltage sine wave $v(t)$ reaches a peak, so does the current sine wave $i(t)$—they are *in phase*. $p(t)$ is also a sinusoid that is in-phase with the voltage and current. This is because $v(t)$ can be expressed as $V\sin(x)$, and $\sin^2(x)$ is itself also a sinusoid through the trigonometric identity $\sin x = (1 - \cos 2x)/2$.
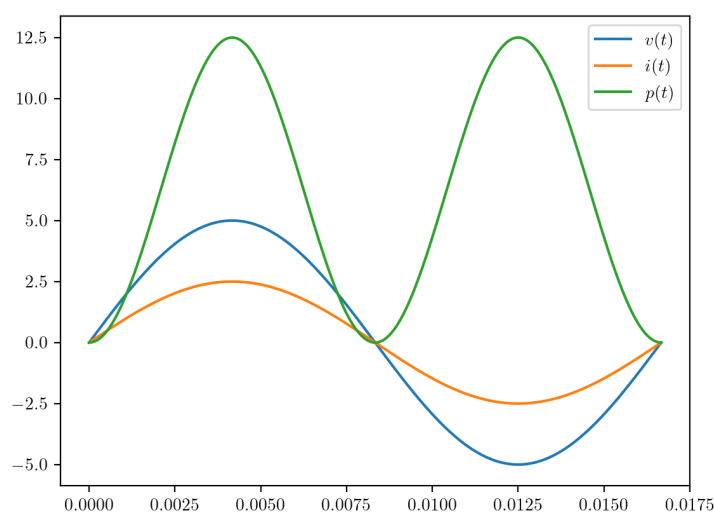
However, in an AC circuit with inductive and capacitive components, the voltage, current, and power are not in-phase. This is because inductors and capacitors *store electrical energy* in magnetic and electric fields, respectively. They do so

by converting electrical energy to magnetic and electric potential energy. These components also release their energy by converting it back to electrical energy. The rate at which the component stores and releases energy is its power.

When an AC voltage is applied to these components, they store and release energy in a cycle. In fact, their power is a sine wave whose frequency f matches that of the AC voltage. However, the times at which these components store and release energy—and the power at which they do so—does not align with the peaks and valleys of the AC voltage. Omitting the mathematical derivations, the sine wave of an inductor's and capacitor's power leads and lags behind that of the AC voltage by exactly an eighth of a cycle, respectively. In a related manner, a capacitor's current lags behind the AC voltage by exactly a quarter cycle (making it a cosine wave), and an inductor's leads behind it by exactly a quarter cycle (making it a negative cosine wave):

TWO
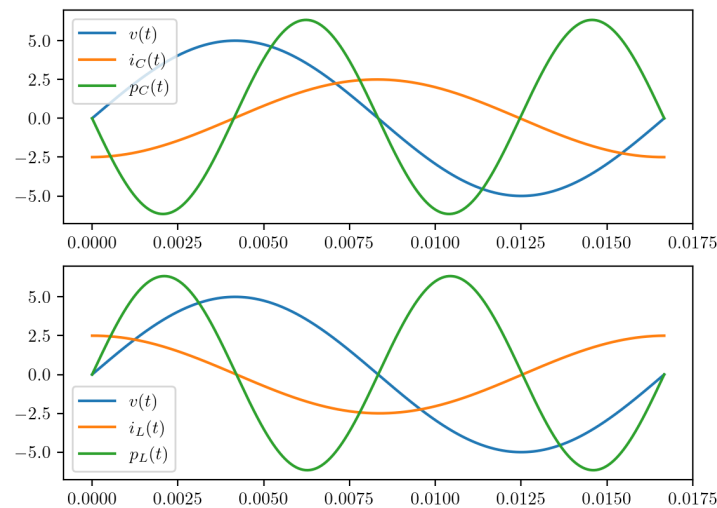
WHAT IS A PHASOR?

A phasor is analogous to a negative number. If that seems like an oversimplification, or that seems to not pay enough tribute to such a seemingly complex electrical engineering concept, then read on.

Both negative numbers and phasors are mathematical representations, notations, and ways to make calculations straightforward.

A negative number is a *mathematical representation* of the opposite of a quantity. It is distinct from the number zero which represents the absence or lack of a quantity. You can have no fewer than zero apples. Similarly, you cannot have a negative quantity of dollar bills. However, if the quantity of dollars refers not to physical legal tender but to an accounting of how many units of currency you have at your disposal, then a negative quantity represents the opposite of having dollars at your disposal—that is, owing dollars. In addition to this representation, negative numbers offer:

- A *notation*. By prefixing a regular number (x) with a minus sign ($-$), it becomes negative ($-x$).

- A *way to make calculations straightforward*. A negative number is like a different kind of quantity on which arithmetic operations can be done by treating them accordingly; to subtract a negative quantity you add its positive counterpart, to multiply you multiply by the positive counterpart and negate the product, and so on.

Similarly, a phasor is a *mathematical representation* of a sinusoid, in terms of its amplitude A and phase θ. A sinusoid already has a mathematical representation in the time domain with the notation $A\cos(\omega t + \theta)$. However, while this representation makes calculations possible, it does not make them straightforward and does not have a concise notation. Representing sinusoids in the phasor domain offers:

- A *notation*. Phasors can be written concisely in polar phasor notation $A\angle\theta$ or complex phasor notation $a + \mathbf{j}\,b$. The "ωt" is implicit.

- A *way to make calculations straightforward*. As we will see in *Life With and Without Phasors*, working in the phasor domain allows you to write and solve AC circuit analysis equations more quickly and easily than working in the time domain. Working in the time domain requires writing and solving differential equations from scratch for each circuit, even though the differential equations and their solutions are similar from one circuit to another. It turns out that the calculations that can be done with phasors are the bare minimum calculations that must be done in order to solve an AC circuit.

LIFE WITH AND WITHOUT PHASORS

One of the main reasons for many people being confused about phasors is that phasors are abstract, but they don't understand why the abstraction exists. In other words, they may know what phasors *represent*, but they don't know what they *replace*—they don't know of any alternative approach for analyzing AC circuits that can be used to put phasors in perspective. This is understandable as many electrical engineering textbooks poorly explain why we use phasors.

With the aid of an example AC circuit, this section goes over what's hidden behind the phasor abstraction. By first trying to solve the circuit without phasors, we will better understand how AC circuit analysis works at its core. And by then solving the circuit with phasors, we will see that it is not only a more concise notation, but also a simpler mathematical representation and a faster method for analyzing AC circuits.

The following AC circuit is our example circuit. Note that it has multiple AC sources and an bridge-type topology that does not allow us to consolidate impedances using formulae for impedances in series or in parallel.
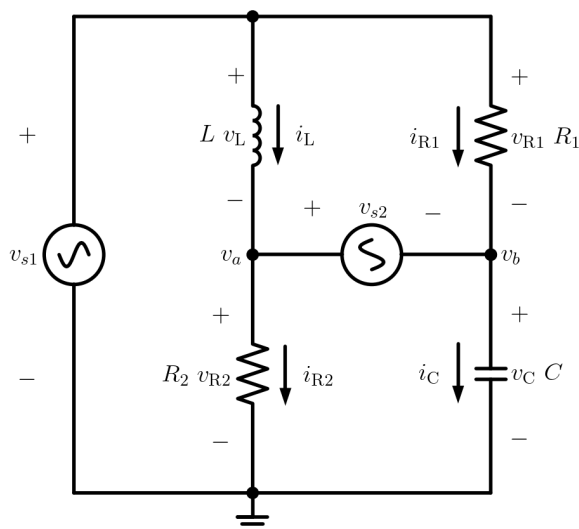
Say that we want to solve for voltages $v_a(t)$ and $v_b(t)$. Doing so would allow us to solve for the voltage across—and therefore the current through—any one component. Just as for a purely resistive DC network, Kirchhoff's Current Law (KCL) applies and we can do nodal analysis to determine the voltages at nodes a and b. In particular we must use the supernode technique with a single node ab because the current through voltage source v_{s2} cannot be put in terms of $v_a(t)$ or $v_b(t)$. This loses us an equation, but we gain one in its place with the following relationship:

$$v_{s2}(t) = v_a(t) - v_b(t) \tag{3.1}$$

Let's try solving the circuit using node voltage analysis, with and without phasors.

3.1 Solving the circuit without phasors

Using KCL, the sum of currents flowing out of supernode ab must sum to zero. We can therefore write:

$$-i_\mathrm{L}(t) + i_\mathrm{R2}(t) - i_\mathrm{R1}(t) + i_\mathrm{C}(t) = 0$$

Substituting in the components' characteristic equations gives us:

$$-\frac{1}{L}\int_0^t v_\mathrm{L}(\tau)\,\mathrm{d}\tau + \frac{v_\mathrm{R2}(t)}{R_2} - \frac{v_\mathrm{R1}(t)}{R_1} + C\frac{\mathrm{d}v_\mathrm{C}(t)}{\mathrm{d}t} = 0$$

Putting everything in terms of the node voltages yields:

$$-\frac{1}{L}\int_0^t (v_{s1}(\tau) - v_a(\tau))\,\mathrm{d}\tau + \frac{v_a(t)}{R_2} - \frac{v_{s1}(t) - v_b(t)}{R_1} + C\frac{\mathrm{d}v_b(t)}{\mathrm{d}t} = 0 \tag{3.2}$$

Paired with Equation (3.1), this represents a system of two integro-differential equations. By differentiating, we put this in the more familiar form of a system of differential equations (in which the highest-order derivative is two):

$$-\frac{1}{L}(v_{s1}(t) - v_a(t)) + \frac{1}{R_2}\frac{\mathrm{d}v_a(t)}{\mathrm{d}t} - \frac{1}{R_1}\frac{\mathrm{d}}{\mathrm{d}t}(v_{s1}(t) - v_b(t)) + C\frac{\mathrm{d}^2 v_b(t)}{\mathrm{d}t^2} = 0$$

$$v_{s2}(t) = v_a(t) - v_b(t)$$

We put this into a canonical form of a system of first-order differential equations by introducing an auxiliary variable $v_b'(t)$ and by using Equation (3.1) to replace the need for variable $v_a(t)$ with $v_b'(t)$:

$$\frac{\mathrm{d}v_b(t)}{\mathrm{d}t} = v_b'(t)$$

$$\frac{\mathrm{d}v_b'(t)}{\mathrm{d}t} = \frac{1}{C}\left[\frac{v_{s1}(t) - (v_{s2}(t) + v_b(t))}{L} - \frac{v_{s2}'(t) + v_b'(t)}{R_2} + \frac{v_{s1}'(t) - v_b'(t)}{R_1}\right]$$

We specify that $v_{si}(t) = V_{si}\sin\omega t$ (and $v_{si}'(t) = \omega V_{si}\cos\omega t$):

$$\frac{\mathrm{d}v_b(t)}{\mathrm{d}t} = v_b'(t)$$

$$\frac{\mathrm{d}v_b'(t)}{\mathrm{d}t} = \frac{1}{C}\left[\frac{(V_{s1} - V_{s2})\sin\omega t - v_b(t)}{L} - \frac{\omega V_{s2}\cos\omega t + v_b'(t)}{R_2} + \frac{\omega V_{s1}\cos\omega t - v_b'(t)}{R_1}\right]$$

Our system is now in terms of two unknowns—$v_b(t)$ and $v_b'(t)$—two functions to solve for. We start to solve our system of differential equations by assuming a solution of the form:

$$v_b(t) = V_b\cos(\omega t + \theta_b)$$

$$v_b'(t) = -\omega V_b\sin(\omega t + \theta_b)$$

By substituting into our system of differential equations, we convert it to a regular system of equations—or, in this case, just one equation:

$$-\omega^2 V_b\cos(\omega t + \theta_b) = \frac{1}{C}\left[\begin{array}{c} \dfrac{(V_{s1} - V_{s2})\sin\omega t - V_b\cos(\omega t + \theta_b)}{L} \\ + \omega\left[\left(\dfrac{V_{s1}}{R_1} - \dfrac{V_{s2}}{R_2}\right)\cos\omega t + \left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)V_b\sin(\omega t + \theta_b)\right] \end{array}\right]$$

This equation is in terms of two unknowns—V_b and θ_b. While having only one equation may appear to be insufficient to solve for two unknowns, the equation is parameterized by t and must be satisfied for all values of t. So, in this sense, we actually have an infinite number of equations at our disposal. We can pick two 'easy' values of t that result in two relatively simple equations:

$t = 0$:

$$-\omega^2 C V_b \cos\theta_b = \frac{-V_b \cos\theta_b}{L} + \omega\left[\left(\frac{V_{s1}}{R_1} - \frac{V_{s2}}{R_2}\right) + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_b \sin\theta_b\right]$$

$t = -\theta_b/\omega$:

$$-\omega^2 C V_b = \frac{-(V_{s1} - V_{s2})\sin\theta_b - V_b}{L} + \omega\left(\frac{V_{s1}}{R_1} - \frac{V_{s2}}{R_2}\right)\cos\theta_b$$

> ℹ **Note**
>
> There is technically an infinite number of solutions for θ_b, because any multiple of 2π can be added to (or subtracted from) θ_b, resulting in an identical sinusoid with a different equation.

This is still a difficult nonlinear system to solve. Furthermore, as we have seen, the process to derive the system of equations is long and notation-heavy despite being consistent between different AC circuits.

~~Now that we are familiar with the non-phasor approach to solving the AC circuit, let's solve the circuit with phasors to see how it compares.~~

3.2 Solving the circuit with the Laplace transform

The Laplace transform $\mathcal{L}$ converts a function $f(t)$ (in the *time domain*) into a function $F(s)$ (in the *complex frequency domain*) through the following integration:

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t$$

The Laplace transform is closely related to the Fourier transform. Both transforms convert to the frequency domain. However, the Laplace transform converts to a frequency domain whose variable s is a *complex number*, which can simplify calculations. This domain is known as the *s-domain* or *Laplace domain*.

The Laplace transform has a number of favorable properties. One such property is that differentiating with respect to t in the time domain becomes multiplication by s in the Laplace domain. Similarly, integrating with respect to t in the time domain becomes division by s in the Laplace domain. Because of this, the Laplace transform is often used to convert a system of differential equations into a regular system of equations which can be solved more easily.

Furthermore, many useful functions in the time domain become simple, rational functions in the Laplace domain, for example:

Sine wave: $\qquad\qquad \mathcal{L}\{\sin(\omega t)\,u(t)\} = \dfrac{\omega}{s^2 + \omega^2}$

Phase-shifted cosine wave: $\qquad \mathcal{L}\{\cos(\omega t + \theta)\,u(t)\} = \dfrac{s\cos\theta - \omega\sin\theta}{s^2 + \omega^2}$

Exponential decay: $\qquad\qquad \mathcal{L}\{\mathrm{e}^{-\alpha t}\,u(t)\} = \dfrac{1}{s + \alpha}$

Exponentially decaying cosine wave: $\quad \mathcal{L}\{\mathrm{e}^{-\alpha t}\cos(\omega t)\,u(t)\} = \dfrac{s + \alpha}{(s + \alpha)^2 + \omega^2}$

Listings of the Laplace transforms of common functions are called Laplace tables.

We want to solve the system of two integro-differential equations (3.1) and (3.2) that we derived in the previous section. That system was in terms of $v_a(t)$ and $v_b(t)$, in the time domain. First, we will apply the Laplace transform to convert

it to a regular system of equations in terms of $V_a(s)$ and $V_b(s)$, in the Laplace domain. Then, we will be able to solve for $V_a(s)$ and $V_b(s)$ very easily. Finally, we will convert our solution back to the time domain using the inverse Laplace transform.

Applying the Laplace transform to equations (3.1) and (3.2) gives us:

$$-\frac{V_{s1}(s) - V_a(s)}{Ls} + \frac{V_a(s)}{R_2} - \frac{V_{s1}(s) - V_b(s)}{R_1} + CsV_b(s) = 0$$
$$V_{s2}(s) = V_a(s) - V_b(s)$$

Here, $V_{s1}(s)$ and $V_{s2}(s)$ are the Laplace transforms of $v_{s1}(t)$ and $v_{s2}(t)$—not to be confused with the amplitudes V_{s1} and V_{s2} of $v_{s1}(t)$ and $v_{s2}(t)$.

Solving for $V_a(s)$ in the second equation and substituting it into the first equation yields:

$$-\frac{V_{s1}(s) - (V_{s2}(s) + V_b(s))}{Ls} + \frac{V_{s2}(s) + V_b(s)}{R_2} - \frac{V_{s1}(s) - V_b(s)}{R_1} + CsV_b(s) = 0$$

Solving for $V_b(s)$ gives us:

$$V_b = \frac{R_2(R_1 + Ls)V_{s1}(s) - R_1(R_2 + Ls)V_{s2}(s)}{R_1R_2 + (R_1 + R_2)Ls + R_1R_2LCs^2}$$

Substituting in $V_{si}(s) = \mathcal{L}\{V_{si}\sin(\omega t)\,u(t)\} = V_{si}\dfrac{\omega}{s^2 + \omega^2}$ yields:

$$V_b = \frac{R_2(R_1 + Ls)V_{s1}\omega - R_1(R_2 + Ls)V_{s2}\omega}{(s^2 + \omega^2)(R_1R_2 + (R_1 + R_2)Ls + R_1R_2LCs^2)}$$

At this point, the algebra will be significantly easier if we substitute in known values for our parameters. Let's introduce some arbitrary values, after which we will omit units for brevity. However, $V_b(s)$ is still in units of volts and s is still, like ω, in units of radians per second.

$$V_b(s) = \frac{(5\ \Omega)(4\ \Omega + (6\ \text{H})s)(1\ \text{V})(3\ \text{rad/s}) - (4\ \Omega)(5\ \Omega - (6\ \text{H})s)(2\ \text{V})(3\ \text{rad/s})}{(3\ \text{rad/s})(-(4\ \Omega)(5\ \Omega) + (4\ \Omega - 5\ \Omega)(6\ \text{H})s + (4\ \Omega)(5\ \Omega)(6\ \text{H})(7\ \text{F})s^2)}$$
$$= \frac{-54s - 60}{(s^2 + 9)(840s^2 + 54s + 20)}$$

We need to convert this solution for $V_b(s)$ back to the time domain. Converting a function from the Laplace domain to the time domain can be done by looking it up in a Laplace table. However, $V_b(s)$ is too complicated in its current form to find a matching function in a Laplace table—in particular, because the order of the polynomial in the denominator is four. This is higher than the highest-order denominator of two that can be found in typical Laplace tables. We can handle this by rewriting our solution as a linear combination of simpler functions that we will be able to find in the Laplace table. To determine such simpler functions, we factor the denominator then apply *partial fraction decomposition*. The denominator needs to be split into factors whose order is at most two. This is already the case. Next, we assume the following modified format of our V_b solution and find the values A_1 through A_4 that make it true:

$$V_b(s) = \frac{A_1 s + A_2}{s^2 + 9} + \frac{A_3 s + A_4}{840s^2 + 54s + 20}$$

$$\implies \frac{-54s - 60}{(s^2 + 9)(840s^2 + 54s + 20)} = \frac{A_1 s + A_2}{s^2 + 9} + \frac{A_3 s + A_4}{840s^2 + 54s + 20}$$

$$\implies -54s - 60 = (A_1 s + A_2)(840s^2 + 54s + 20) + (A_3 s + A_4)(s^2 + 9)$$

$$\implies (-840A_1 - A_3)s^3$$
$$+ (-54A_1 - 840A_2 - A_4)s^2$$
$$+ (-20A_1 - 54A_2 - 9A_3 - 54)s$$
$$+ (-20A_2 - 9A_4 - 60)$$
$$= 0$$

$$\implies \begin{cases} -840A_1 - A_3 = 0 \\ -54A_1 - 840A_2 - A_4 = 0 \\ -20A_1 - 54A_2 - 9A_3 - 54 = 0 \\ -20A_2 - 9A_4 - 60 = 0 \end{cases}$$

$$\implies \begin{cases} A_1 = & 102600/14219461 \\ A_2 = & 106539/14219461 \\ A_3 = & -86184000/14219461 \\ A_4 = & -95033160/14219461 \end{cases}$$

Substituting A_1 through A_4 gives us:

$$V_b(s) = \underbrace{\frac{\frac{102600}{14219461}s + \frac{106539}{14219461}}{s^2 + 9}}_{\text{First term}} + \underbrace{\frac{-\frac{86184000}{14219461}s - \frac{95033160}{14219461}}{840s^2 + 54s + 20}}_{\text{Second term}}$$

Inverse Laplace transform of the first term. The first term matches the following Laplace table function:

$$\mathcal{L}\{\cos(\omega t + \theta)\, u(t)\} = \frac{s\cos\theta + \omega\sin\theta}{s + \omega^2}$$

Therefore:

$$\text{First term} = \frac{\frac{102600}{14219461}s + \frac{106539}{14219461}}{s^2 + 9} = B_1 \frac{s\cos\theta_1 - \omega_1\sin\theta_1}{s + \omega_1^2}$$

The ω_1 here is not coincidentally the same as our natural frequency $\omega = 3$ rad/s. We need to solve for θ_1 and B_1, which we can do by setting up the following system of equations:

$$B_1\cos\theta_1 = 102600/14219461$$
$$-3B_1\sin\theta_1 = 106539/14219461$$

We can solve for θ_1 by dividing the second equation by the first:

$$\frac{106539/14219461}{102600/14219461} = -3\tan\theta_1 \implies \theta_1 = \arctan\left(-\frac{35513}{102600}\right) = -19.092°$$

And we can solve for B_1 by dividing the second equation by -3 and adding its square to the square of the first:

$$\left(\frac{102600}{14219461}\right)^2 + \left(\frac{106539}{-3\cdot 14219461}\right)^2 = B_1^2\cos^2\theta_1 + B_1^2\sin^2\theta_1 = B_1^2$$

$$\implies B_1 = \sqrt{\left(\frac{102600}{14219461}\right)^2 + \left(\frac{35513}{14219461}\right)^2} = 0.007635$$

Therefore, the inverse Laplace transform of the first term is:

$$\mathcal{L}^{-1}\left\{\frac{\frac{102600}{14219461}s + \frac{106539}{14219461}}{s^2 + 9}\right\} = 0.007635\cos(3t - 19.092°)\,u(t)$$

Inverse Laplace transform of the second term. The second term matches the following function from (3.3):

$$\mathcal{L}\{\mathrm{e}^{-\alpha t}\cos(\omega t + \theta)\,u(t)\} = \frac{(s + \alpha)\cos\theta - \omega\sin\theta}{(s + \alpha)^2 + \omega^2}$$

Therefore:

$$\text{Second term} = \frac{-\frac{86184000}{14219461}s - B_2\frac{95033160}{14219461}}{840s^2 + 54s + 20} = \frac{(s + \alpha_2)\cos\theta_2 - \omega_2\sin\theta_2}{(s + \alpha_2)^2 + \omega_2^2}$$

$$\implies \frac{-\frac{102600}{14219461}s - \frac{791943}{99536227}}{s^2 + \frac{9}{140}s + \frac{1}{42}} = B_2\frac{s\cos\theta_2 + (\alpha_2\cos\theta_2 - \omega_2\sin\theta_2)}{s^2 + 2\alpha_2 s + (\alpha_2^2 + \omega_2^2)}$$

We need to solve for α_2, ω_2, θ_2, and B_2:

$$9/140 = 2\alpha_2 \implies \alpha_2 = 9/280 = 0.03214$$

$$\alpha_2^2 + \omega_2^2 = \frac{1}{42} \implies \omega_2 = \sqrt{\frac{1}{42} - \left(\frac{9}{280}\right)^2} = 0.15092 \text{ rad/s}$$

We can solve for θ_2 and B_2 by setting up and solving the following system of equations:

$$\left.\begin{array}{l} -102600/14219461 = B_2\cos\theta_2 \\ -791943/99536227 = B_2\,(0.03214\cos\theta_2 - 0.15092\sin\theta_2) \end{array}\right\} \implies \left\{\begin{array}{l} \theta_2 = -81.9756° \\ B_2 = -0.05169 \end{array}\right.$$

Therefore, the inverse Laplace transform of the second term is:

$$\mathcal{L}^{-1}\left\{\frac{-\frac{86184000}{14219461}s - \frac{95033160}{14219461}}{840s^2 + 54s + 20}\right\} = -0.05169\,\mathrm{e}^{-0.03214t}\cos(0.15092\,t - 81.9756°)\,u(t)$$

And finally, $v_b(t)$ is:

$$v_b(t) = (0.007635\cos(3t - 19.092°) - 0.05169\,\mathrm{e}^{-0.03214t}\cos(0.15092\,t - 81.9756°))\,u(t)$$

> **ⓘ Derivation: Laplace transform of an exponentially decaying, phase-shifted cosine wave**
>
> $$\begin{aligned} \mathcal{L}\{\mathrm{e}^{-\alpha t}\cos(\omega t + \theta)\,u(t)\} &= \mathcal{L}\{\mathrm{e}^{-\alpha t}\,[\cos\omega t\cos\theta - \sin\omega t\sin\theta]\,u(t)\} \\ &= \cos\theta\,\mathcal{L}\{\mathrm{e}^{-\alpha t}\cos\omega t\,u(t)\} - \sin\theta\,\mathcal{L}\{\mathrm{e}^{-\alpha t}\sin\omega t\,u(t)\} \\ &= \cos\theta\frac{s + \alpha}{(s + \alpha)^2 + \omega^2} - \sin\theta\frac{\omega}{(s + \alpha)^2 + \omega^2} \qquad (3.3) \\ &= \frac{(s + \alpha)\cos\theta - \omega\sin\theta}{(s + \alpha)^2 + \omega^2} \end{aligned}$$

3.3 Solving the circuit with phasors

Using KCL, the sum of currents flowing out of supernode ab must sum to zero. We can therefore write:

$$-\mathbf{i}_\mathrm{L} + \mathbf{i}_\mathrm{R2} - \mathbf{i}_\mathrm{R1} + \mathbf{i}_\mathrm{C} = 0$$

$$-\frac{\mathbf{v}_\mathrm{L}}{Z_\mathrm{L}} + \frac{\mathbf{v}_\mathrm{R2}}{Z_\mathrm{R2}} - \frac{\mathbf{v}_\mathrm{R1}}{Z_\mathrm{R1}} + \frac{\mathbf{v}_\mathrm{C}}{Z_\mathrm{C}} = 0$$

Substituting in the components' impedances and putting everything in terms of the node voltages gives us:

$$-\frac{\mathbf{v}_{s1} - \mathbf{v}_a}{\mathbf{j}\,\omega L} + \frac{\mathbf{v}_a}{R_2} - \frac{\mathbf{v}_{s1} - \mathbf{v}_b}{R_1} + \frac{\mathbf{v}_b}{\mathbf{j}\frac{-1}{\omega C}} = 0$$

Paired with Equation (3.1), $\mathbf{v}_a - \mathbf{v}_b = \mathbf{v}_{s2}$, this represents a system of two linear equations in two unknowns $\mathbf{v}_a$ and $\mathbf{v}_b$. Substituting Equation (3.1) into the above yields:

$$\mathbf{j}\frac{\mathbf{v}_{s1} - (\mathbf{v}_{s2} + \mathbf{v}_b)}{\omega L} + \frac{\mathbf{v}_{s2} + \mathbf{v}_b}{R_2} - \frac{\mathbf{v}_{s1} - \mathbf{v}_b}{R_1} + \mathbf{j}\omega C\mathbf{v}_b = 0$$

Solving for $\mathbf{v}_b$ yields the following, and we can solve for $\mathbf{v}_a$ from this using Equation (3.1).

$$\mathbf{v}_b = \frac{\omega L(R_2\mathbf{v}_{s1} - R_1\mathbf{v}_{s2}) + \mathbf{j}\,R_1 R_2(\mathbf{v}_{s2} - \mathbf{v}_{s1})}{\omega L(R_1 + R_2) + \mathbf{j}\,(\omega^2 CL - 1)R_1 R_2}$$

Getting here required some algebra involving complex numbers. However, when solving a real circuit whose parameters have known values, this process would have been aided significantly by being able to evaluate/simplify expressions at each step. Let's introduce some very arbitrary values to see how $\mathbf{v}_b$ and $\mathbf{v}_a$ can be evaluated.

$$
\begin{aligned}
\mathbf{v}_b &= \frac{(3\text{ rad/s})(6\text{ H})((5\,\Omega)(\mathbf{j}\,1\text{ V}) - (4\,\Omega)(\mathbf{j}\,2\text{ V})) + \mathbf{j}\,(4\,\Omega)(5\,\Omega)((\mathbf{j}\,2\text{ V}) - (\mathbf{j}\,1\text{ V}))}{(3\text{ rad/s})(6\text{ H})((4\,\Omega) + (5\,\Omega)) + \mathbf{j}\,((3\text{ rad/s})^2(7\text{ F})(6\text{ H}) - 1)(4\,\Omega)(5\,\Omega)} \\
&= \frac{-20 - \mathbf{j}\,54}{162 + \mathbf{j}\,7540}\text{ V} \\
&= \frac{-20 - \mathbf{j}\,54}{162 + \mathbf{j}\,7540}\frac{162 - \mathbf{j}\,7540}{162 - \mathbf{j}\,7540}\text{ V} \\
&= \frac{(-20 - \mathbf{j}\,54)(162 - \mathbf{j}\,7540)}{162^2 + 7540^2}\text{ V} \\
&= \frac{-3240 + \mathbf{j}\,150800 - \mathbf{j}\,8748 - 407160}{26244 + 56851600}\text{ V} \\
&= \frac{-410400 + \mathbf{j}\,142052}{56877844}\text{ V} \\
&= \boxed{(-0.007215 + \mathbf{j}\,0.002497)\text{ V}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v}_a &= \mathbf{v}_b + \mathbf{v}_{s2} \\
&= (-0.007215 + \mathbf{j}\,0.002497)\text{ V} + \mathbf{j}\,2\text{ V} \\
&= \boxed{(-0.007215 + \mathbf{j}\,2.002497)\text{ V}}
\end{aligned}
$$

By calculating the amplitude A and phase θ, we can convert from complex number phasor notation $a + \mathbf{j}\,b$ to either polar phasor notation $A\angle\theta$ or back into the time domain as $A\cos(\omega t + \theta)$. The amplitude and phase are calculated using:

$$A = \sqrt{a^2 + b^2}, \qquad \theta = \arctan\frac{b}{a}$$

$$\mathbf{v}_b = 0.007635 \angle -19.092°$$
$$\mathbf{v}_a = 2.002510 \angle -89.794°$$

With the amplitude and phase calculated, it becomes very easy to plot the solution for $v_a(t)$ and $v_b(t)$:
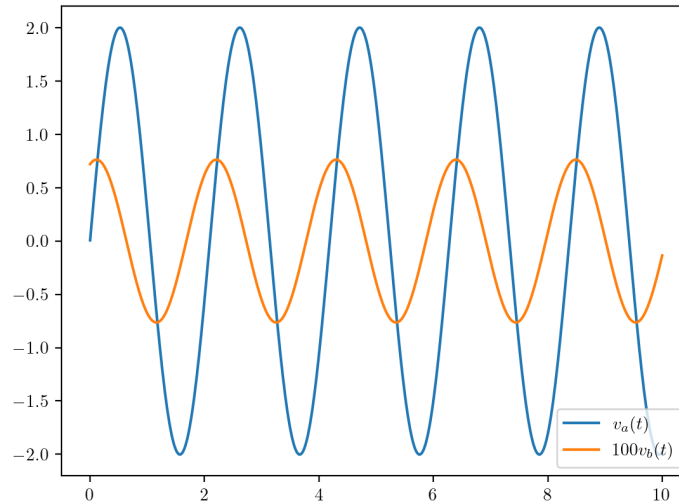
3.4 Checking the phasor solution

At this point, we have derived the equations representing the circuit with and without phasors, and we have solved the circuit with phasors. To check our work, we can plug the solution obtained using phasors into the differential equation obtained without using phasors, and verify that the equation holds true (that the left-hand side equals the right-hand side). This can be done conveniently by plotting both the left- and right-hand sides:
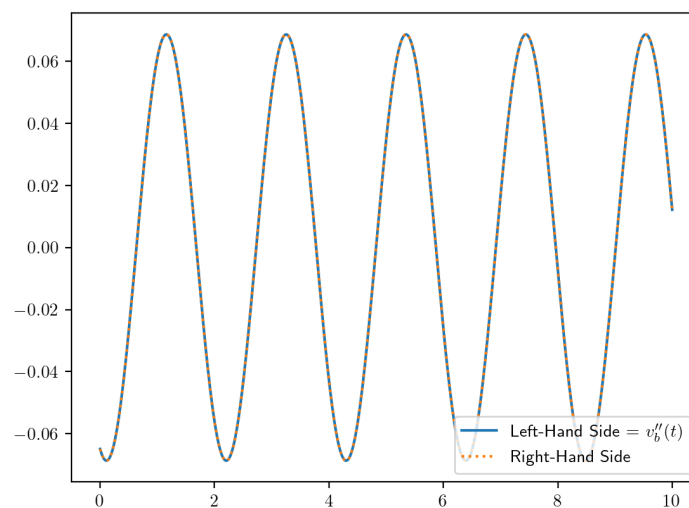
As can be seen, the left- and right-hand sides are equal.