GRADUATE MACROECONOMICS

UC Irvine

2015-2016

(Partial and Incomplete)

"A barbarian is not aware that he is a barbarian."

– Jack Vance, *Big Planet*

Quarter I i - 1

1.1 The Solow Growth Model . . . i - 1
- 1.1.1 Solow's Model . . . i - 1
- 1.1.2 Firm Optimization in the Solow Model . . . i - 2
- 1.1.3 The Production Function in Intensive Form . . . i - 2
- 1.1.4 The Dynamics of the Solow Model . . . i - 3
- 1.1.5 The Steady–State Equilibrium . . . i - 4
- 1.1.6 The Comparative Statics of the Solow Model . . . i - 4
- 1.1.7 The Golden–Rule of Capital Accumulation . . . i - 5
- 1.1.8 The Transitional Dynamics of the Solow Model . . . i - 6
- 1.1.9 The Dynamics with the Cobb-Douglas Production Function . . . i - 7
- 1.1.10 The Discrete–Time Solow Growth Model . . . i - 8

1.2 Linear, First–Order Differential Equations . . . i - 9
- 1.2.1 Autonomous Equations . . . i - 9
- 1.2.2 The Solution Method . . . i - 9
- 1.2.3 The Homogeneous Solution . . . i - 9
- 1.2.4 The Particular Solution . . . i - 10
- 1.2.5 The Initial Value . . . i - 11
- 1.2.6 Convergence . . . i - 11

1.3 Nonlinear, First–Order Differential Equations . . . i - 12
- 1.3.1 Qualitative Analysis . . . i - 12
- 1.3.2 Phase Diagrams . . . i - 12
- 1.3.3 Stability Analysis . . . i - 13
- 1.3.4 Linearizing a Differential Equation . . . i - 13
- 1.3.5 Interpretation . . . i - 14
- 1.3.6 The Neoclassical Model of Economic Growth . . . i - 15

1.4 Growth in a Overlapping Generations Economy . . . i - 16
- 1.4.1 The OLG Model . . . i - 16
- 1.4.2 The Steady–State and Dynamics of the OLG Economy . . . i - 18
- 1.4.3 Dynamic Inefficiency in the OLG Economy . . . i - 19
- 1.4.4 The Samuelson Paradox . . . i - 20
- 1.4.5 Impure Altruism in the OLG Economy . . . i - 21

1.5 Fiat Money in the Overlapping Generations Economy . . . i - 23
- 1.5.1 The OLG Barter Economy . . . i - 23
- 1.5.2 The OLG Monetary Economy . . . i - 24
- 1.5.3 The Role of Money in the OLG Economy . . . i - 25
- 1.5.4 Money and Inflation in the OLG Economy . . . i - 26

1.5.5 Dynamics of the OLG Economy . . . . i - 27
1.6 Introduction to Optimal Control Theory . . . . i - 30
1.6.1 A Simple Optimal Control Problem . . . . i - 30
1.6.2 Discrete Time Optimal Control . . . . i - 30
1.6.3 Continuous Time Optimal Control . . . . i - 32
1.6.4 The Maximum Principle . . . . i - 33
1.6.5 Sufficient Conditions for Optimal Control . . . . i - 33
1.6.6 Economic Interpretation of Optimal Control . . . . i - 34
1.6.7 Costate Variables . . . . i - 35
1.6.8 Infinite Time Horizon Optimal Control . . . . i - 35
1.6.9 Generalization of the Optimal Control Problem . . . . i - 37
1.7 The Ramsey Model of Optimal Growth . . . . i - 39
1.7.1 The Ramsey Model . . . . i - 39
1.7.2 The Steady–State of the Ramsey Model . . . . i - 41
1.7.3 The Dynamics of the Ramsey Model . . . . i - 42
1.8 Phase Diagrams . . . . i - 44
1.8.1 Construction of a Phase Diagram . . . . i - 44
1.8.2 Vector Fields . . . . i - 44
1.8.3 Finding a Saddle Path . . . . i - 46
1.9 The Neoclassical Growth Model in Continuous Time . . . . i - 49
1.9.1 Ramsey's Neoclassical Growth Model . . . . i - 49
1.9.2 The Steady–State and Dynamics of the Ramsey Model . . . . . . i - 52
1.9.3 The Neoclassical Growth Model with Government . . . . i - 52
1.10 A Model of Endogenous Growth . . . . i - 54
1.10.1 The Endogenous Growth Model . . . . i - 54
1.10.2 The Balanced Growth Equilibrium . . . . i - 55
1.10.3 Endogenous Growth with Physical and Human Capital . . . . . . i - 56
1.10.4 Endogenous Growth with Knowledge Spillovers . . . . i - 57
1.11 Dynamic Programming Applications . . . . i - 59
1.11.1 Optimal Unemployment . . . . i - 59
1.11.2 Search Unemployment . . . . i - 62

Quarter II ii - 1
2.1 Introduction to Stochastic Processes . . . . ii - 1
2.1.1 Stochastic Processes . . . . ii - 1
2.1.2 Time Series . . . . ii - 1
2.1.3 Rational Expectations . . . . ii - 3
2.2 Real Business Cycle Models . . . . ii - 4

2.2.1 Solving the RBC Bellman Equation . . . . ii - 4
2.2.2 Productivity Shocks . . . . ii - 5
2.2.3 Calibration . . . . ii - 6
2.3 Asset Pricing and Financial Markets . . . . ii - 7
2.3.1 Complete Markets . . . . ii - 7
2.3.2 The Lucas Asset Pricing Model . . . . ii - 7
2.3.3 The Martingale Theory of Stock–Prices . . . . ii - 8
2.3.4 Term Structure of Interest Rates . . . . ii - 8
2.3.5 The Equity Premium Puzzle 9

Quarter III iii - 1
3.1 Reduced-Form and Structural Vector Autoregressions . . . . iii - 1
3.1.1 Reduced–Form Vector Autoregressions . . . . iii - 1
3.1.2 Structural Vector Autoregressions . . . . iii - 2
3.2 Methods of Identifying Structural VARs . . . . iii - 3
3.2.1 Recursive Identification . . . . iii - 3
3.2.2 Long-Run Restrictions . . . . iii - 3
3.2.3 Sign Restrictions . . . . iii - 3
3.2.4 High-Frequency Identification . . . . iii - 3
3.2.5 External Instruments . . . . iii - 3
3.3 The Lucas Critique: Reduced-form vs. Structural Models . . . . . . iii - 3
3.4 Heterogeneity . . . . iii - 3
3.4.1 The Existence of a Representative Firm . . . . iii - 3
3.4.2 The Existence of a Representative Agent . . . . iii - 6
3.4.3 Gorman Aggregation . . . . iii - 6
3.5 Balanced Growth and Real Business Cycles . . . . iii - 7
3.5.1 Balanced Growth . . . . iii - 7
3.6 Monopolistic Competition and Nominal Rigidities . . . . iii - 7
3.7 The New Keynesian DSGE Model . . . . iii - 7
3.7.1 Applications of NK DSGE Models . . . . iii - 8
3.8 Expectations and Time-Inconsistency . . . . iii - 8
3.9 Sticky Wages and Prices . . . . iii - 8
3.10 Macroeconomic vs. Microeconomic Elasticities 9

Quarter I

"We need no chieftain; such folk eat more than their share."
– Jack Vance, *Rhialto the Marvellous*

This section will introduce you to some benchmark models of the macroeconomy and to methods and tools that are commonly used in modern macroeconomics. It will cover models of economic growth that can also be used to study economic fluctuations, consumption, and investment. Most of the models will be dynamic and will include rational optimizing agents. The basic model of economic growth comes from Solow. While the model has no microfoundations, it is dynamic and it can serve as a building block for the neoclassical growth model. You will be presented with different extensions of this model that make the saving rate endogenous. You will first go over a model from Samuelson and Diamond where individuals live for a finite number of periods (for simplicity, two) and generations overlap. You will show that this simple model is very tractable and delivers important insights for welfare theorems in economies with an infinite time horizon. As a by-product, the model can also be used to study the role of fiat money and social security schemes. The second type of models you will be presented with has infinitely lived individuals. You will consider the problem of optimal economic growth from Ramsey. You will decentralize the Ramsey economy in order to obtain the neoclassical growth model with markets, households, and firms. Finally, you will go over models of endogenous growth. In terms of the methods, you will study dynamic systems (e.g., difference and differential equations, phase diagrams. . .) and dynamic optimization (optimal control theory and dynamic programming).

1.1. The Solow Growth Model

How can we explain the huge income differences across countries? A major paradigm began with Robert Solow and his contributions to the study of economic growth. The Solow model is a building block of modern macroeconomics and looks at the determinants of economic growth and the standard of living in the long run.

1.1.1. Solow's Model

- 4 variables: output (Y), capital (K), labor (L), and knowledge (A).
- A production function $F[\cdot]$ to link inputs and output.
- An equation for saving, investment, and capital accumulation.

Definition: The Neoclassical Production Function

$$Y(t) = F[K(t), A(t)L(t)]$$

- $Y(t)$ is the flow of output; 1 good that can be consumed or invested.
- $A(t)$ is labor augmenting technological progress that depends on time.
- $A(t)L(t)$ are efficiency units of labor.

Properties:

- Diminishing marginal products with respect to each input

$$\frac{\partial^2 F}{\partial K^2} < 0 \quad \text{and} \quad \frac{\partial^2 F}{\partial L^2} < 0.$$

- Constant returns to scale (C.R.S.) in its two arguments

$$F[cK, cAL] = cF[K, AL] \ \text{ for all } \ c \geq 0.$$

- The production function satisfies the Inada conditions

$$\lim_{K\to 0} F_K = \lim_{K\to 0} F_L = \infty$$
$$\lim_{K\to \infty} F_K = \lim_{K\to \infty} F_L = 0$$

1.1.2. Firm Optimization in the Solow Model

The firm's profit maximization problem is

$$\max_{\{K,L\}} F[K(t), A(t)L(t) - R(t)K(t) - w(t)L(t)].$$

- The rental price of capital is $R(t)$.
- The real wage is $w(t)$.

The first order conditions are

$$\begin{aligned} w(t) &= A(t)F_L[K(t), A(t)L(t)] \\ R(t) &= F_K[K(t), A(t)L(t)]. \end{aligned}$$

Theorem: Euler's Theorem
If $F(x, y)$ has constant returns to scale—it is homogenous of degree 1—then

$$F(x,y) = \frac{\partial F(x,y)}{\partial x}x + \frac{\partial F(x,y)}{\partial y}y.$$

So,

$$\begin{aligned} F[K(t), A(t)L(t)] &= F_K K(t) + A(t)F_L L(t) \\ F[K(t), A(t)L(t)] &= R(t)K(t) + w(t)L(t). \end{aligned}$$

Note that real profits are

$$\begin{aligned} \pi &= F[K(t), A(t)L(t)] - R(t)K(t) - w(t)L(t) \\ \pi &= R(t)K(t) + w(t)L(t) - R(t)K(t) - w(t)L(t) = 0 \end{aligned}$$

Thus, payment of input factors exhausts profits.

1.1.3. The Production Function in Intensive Form

To write the production function in intensive form, normalize all the variables by the efficiency of labor supply $A(t)L(t)$. The production of one elementary unit of effective labor is

$$y(t) = F\left(\frac{K(t)}{A(t)L(t)}, 1\right) \equiv f\big(k(t)\big)$$

with effective capital

$$k \equiv \frac{K(t)}{A(t)L(t)}.$$

Properties:

- Monotonic: $f'(k) > 0$
- Concave: $f''(k) < 0$
- $\lim_{k\to 0}$ $f'(k) = +\infty$
- $\lim_{k\to +\infty}$ $f'(k) = 0$

The rental price of capital is

$$R(t) = f'[k(t)].$$

From Euler's theorem

$$f[k(t)] = k(t)R(t) + \frac{w(t)}{A(t)}.$$

Therefore, the real wage is

$$\frac{w(t)}{A(t)} = f[k(t)] - k(t)f'[k(t)].$$

Example: The Cobb–Douglas Specification

$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha} \quad , \quad 0 < \alpha < 1$$

Written in intensive form

$$f(k) = F(k, 1)$$

$$f(k) = k^{\alpha}$$

This production function satisfies the 3 properties of a neoclassical production function; diminishing marginal returns, constant returns to scale, and the Inada conditions.

1.1.4. The Dynamics of the Solow Model

Labor grows at the rate n

$$\dot{L}(t) = nL(t) \Rightarrow L(t) = L(0)\mathrm{e}^{nt}.$$

Knowledge grows at the rate g

$$\dot{A}(t) = gA(t) \Rightarrow A(t) = A(0)\mathrm{e}^{gt}.$$

Agents save (invest) a fraction s of their income, while capital depreciates at rate δ

$$\dot{K}(t) = sY(t) - \delta K(t).$$

Definition: The saving rate, $s(\cdot)$, is the fraction of output that is saved.

The rate should depend on preferences for curent and future consumption, the level of wealth, the interest rate, etc. For simplification, s is assumed to be constant. This assumption matters for short–run dynamics and welfare results.

Take the log of $k = K/AL$

$$\ln k = \ln K - \ln A - \ln L$$

differentiate with respect to time

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - g - n$$

replace $\dot{K}/K$ to obtain

$$\frac{\dot{k}}{k} = \frac{sY}{K} - \delta - g - n.$$

Thus, the transitional equation for capital is

$$\dot{k} = sy - (\delta + g + n)k.$$

1.1.5. The Steady–State Equilibrium

Definition: An equilibrium of the Solow model is a function $k(t)$ that satisfies

$$\dot{k} = sf(k) - (\delta + g + n)k$$

with the initial condition $k(0) = k_0$.

Definition: The Balanced Growth Path
Note that $sf(k)$ is actual investment per unit of effective labor and $(\delta + g + n)k$ is breakeven investment. When actual investment equals breakeven investment, $\dot{k} = 0$, there is a steady state such that

$$sf(k) = (\delta + g + n)k.$$

- Since k is constant, K is growing at rate $n + g$.
- $Y = f(k)AL$ is also growing at rate $n + g$.
- Capital per worker $K/L = Ak$, output per worker $f(k)A$, and consumption per worker are growing at rate g.

Example: The Cobb–Douglas Specification
Production per efficient unit of labor is

$$f(k) = k^{\alpha}.$$

Steady–state capital per efficient unit of labor is

$$k^* = \left(\frac{s}{\delta + g + n}\right)^{\frac{1}{1-\alpha}}.$$

The conclusion is that countries that have high savings rates will tend to be richer and countries that have high population growth will tend to be poorer.

1.1.6. The Comparative Statics of the Solow Model

The effect from a change in the saving rate, s, on capital, k^*, is positive, found by differentiating $sf(k) = (\delta + g + n)$, it is

$$\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{n + g + \delta - sf'(k^*)} = \frac{f(k^*)k^*}{s]f(k^*) - k^* f'(k^*)]} > 0.$$

The long run effect of a change in the saving rate, s, on output, y^*, is positive

$$\frac{\partial y^*}{\partial s} = f'(k^*)\frac{\partial k^*}{\partial s} = \frac{f'(k^*)f(k^*)k^*)}{s[f(k^*) - k^* f'(k^*)]} > 0.$$

The elasticity of output, y^*, with respect to the saving rate, s, is

$$\frac{s}{y}\frac{\partial y^*}{\partial s} = \frac{f'(k^*)k^*}{f(k^*) - f'(k^*)k^*}.$$

The effect from a change in a growth rate, n or g, on capital, k^*, is negative

$$\frac{\partial k^*}{\partial n} = \frac{\partial k^*}{\partial g} = \frac{-k^*}{n + g + \delta - sf'(k^*)} = \frac{-(k^*)^2}{s[f(k^*) - k^* f'(k^*)]} < 0.$$

Therefore,

$$\frac{s}{y}\frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1 - \alpha_K(k^*)} \quad \text{with} \quad \alpha + K(k^*) = \frac{k^* f'(k^*)}{f(k^*)}.$$

Example: Explaining Income Difference
Consider two countries, A and B. Let output per worker be $Af(k)$ and assume that

$$y_A = 10y_B \Rightarrow \ln y_A - \ln y_B = \ln 10.$$

Using the fact that

$$\alpha_K = \frac{\Delta \ln y}{\Delta \ln k}$$

then

$$\ln k_A - \ln k_B = \frac{\ln 10}{\alpha_K}.$$

If $\alpha = 1/3$, then the difference in k should be

$$\frac{k_A}{k_B} = 10^{\frac{1}{\alpha_K}} \simeq 10^3,$$

and there is no evidence of such differences in capital per worker, thus, differences in k cannot account for large differences in y.

Alternatively, consider the rate of return on capital (without depreciation)

$$r_K = f'(k).$$

Under a Cobb-Douglas specification

$$f(k) = k^\alpha.$$

So,

$$r_K = \alpha k^{\alpha-1} = \alpha y^{\frac{\alpha-1}{\alpha}}.$$

If $y_A = 10y_B$ and $\alpha = 1/3$, then $r_K \approx 1/3y^{-2}$ and

$$r_{K,B} = 100r_{K,A}.$$

There would be large incentives to invest in poor countries. Another way to explain differences in y is from differences in A. However, the growth of A is exogenous; A represents everything that we do not *know*.

1.1.7. The Golden–Rule of Capital Accumulation

Consumption per unit of effective labor is

$$c = f(k) - sf(k) = (1-s)f(k).$$

On the balanced growth path

$$c^* = f(k^*) - (n+g+\delta)k^*.$$

Thus

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - (n+g+\delta)]\frac{\partial k^*}{\partial s}.$$

Consumption is maximized when

$$\frac{\partial c^*}{\partial s} = 0.$$

This occurs when

$$f'(k^*) = n+g+\delta.$$

Definition: The golden–rule saving rate, s_{gold}, is the consumption–maximizing rate.

Definition: An economy is dynamically inefficient if $s > s_{\text{gold}}$. A reduction of the saving rate from s to s_{gold} would provide more consumption during the transition toward the new steady state, and more consumption at the steady state (i.e. the economy is oversaving: consumption could be raised at all points in time).

Suppose that $s < s_{\text{gold}}$. An increase of the saving rate would provide less consumption during the transition toward the new steady state, but more consumption at the steady state. Overall, the effect is positive if households do not care too much about current consumption. In the Solow growth model, there is nothing to guarantee that k will not be larger than the golden–rule level of capital. The saving decisions do not reflect intertemporal trade–offs.

1.1.8. The Transitional Dynamics of the Solow Model

How does the economy converge to its steady–state? Define

$$\gamma_k \equiv \frac{\dot{k}}{k} = s\frac{f(k)}{k} - (\delta + g + n).$$

Note that

$$\left[\frac{f(k)}{k}\right]' = \frac{f'(k)k - f(k)}{k^2} < 0.$$

Using l'Hopital's rule it can be shown

$$\lim_{k \to 0} \frac{f(k)}{k} = \lim_{k \to 0} f'(k) = \infty$$
$$\lim_{k \to \infty} \frac{f(k)}{k} = \lim_{k \to \infty} f'(k) = 0$$

If $k < k^*$, then $\gamma_k > 0$. The rate of growth of capital converges to 0 asymptotically. This implies that there are diminishing returns to capital. Similarly, an increase in s generates a positive effect on the growth rate of capital and output, but this effect is transitory. It can be shown

$$\frac{\partial \gamma_k}{\partial k} = s\frac{f'(k)k - f(k)}{k^2} < 0.$$

So

$$\frac{\partial \gamma_k}{\partial k} < 0.$$

Definition: Absolute convergence is that, other things equal, counties with a low capital stock per capita grow faster.

This concept suggests poor countries tend to *catch up*. For this to be true, countries must have the same s, n, g, δ, and $f(\cdot)$.

Definition: Conditional convergence is that an economy grows faster the further it is from its own steady–state value of capital.

Conditional convergence allows for heterogeneity across countries (i.e. countries may have different steady–states). Use

$$s = \frac{(\delta + g + n)k^*}{f(k^*)}$$

to rewrite γ_k as

$$\gamma_k = (\delta + g + n)\left[\frac{f(k)/k}{f(k^*)/k^*} - 1\right].$$

To determine how rapidly k approaches k^*, use a 1st order Taylor–series approximation

$$\dot{k} = [sf'(k^*) - (\delta + g + n)](k - k^*).$$

Let λ be defined as

$$\lambda = -[sf'(k^*) - (\delta + g + n)] > 0.$$

The solution to the 1st order linear differential equation is

$$k(t) - k^* = [k(0) - k^*]\exp(-\lambda t).$$

The speed of convergence depends on λ

$$\begin{aligned}
\lambda &= -[sf'(k^*) - (\delta + g + n)] \\
\lambda &= -\left[\frac{(\delta + g + n)k^*}{f(k^*)} f'(k^*) - (\delta + g + n)\right] \\
\lambda &= (\delta + g + n)[1 - \alpha_K(k^*)].
\end{aligned}$$

Let τ_{half} be the time required to be half–way between the initial capital stock and its steady–state value. Then

$$\frac{k(0) - k^*}{2} = [k(0) - k^*]\exp(-\lambda\tau_{\text{half}}) \Rightarrow \tau_{\text{half}} = \frac{\ln 2}{\lambda}.$$

1.1.9. The Dynamics with the Cobb-Douglas Production Function

- The production function is $f(k) = k^\alpha$.
- The law of motion for capital is

$$\dot{k} = sk^\alpha - (\delta + g + n)k.$$

- Define $x = k^{1-\alpha}$. The law of motion for $x(t)$ is

$$\dot{x} = (1 - \alpha)s - (1 - \alpha)(\delta + g + n)x.$$

  The solution to the linear differential equation is

$$x(t) = \frac{s}{\delta + g + n} + \left(x(0) - \frac{s}{\delta + g + n}\right)\mathrm{e}^{-(1-\alpha)(\delta+g+n)t}.$$

  Therefore, the path for capital accumulation is

$$k(t) = \left[\frac{s}{\delta + g + n} + \left(k_0^{1-\alpha} - \frac{s}{\delta + g + n}\right)\mathrm{e}^{-(1-\alpha)(\delta+g+n)t}\right]^{\frac{1}{1-\alpha}}.$$

  The path for output accumulation is

$$y(t) = \left[\frac{s}{\delta + g + n} + \left(k_0^{1-\alpha} - \frac{s}{\delta + g + n}\right)\mathrm{e}^{-(1-\alpha)(\delta+g+n)t}\right]^{\frac{\alpha}{1-\alpha}}.$$

Definition: The rate of adjustment is $(1 - \alpha)(\delta + g + n)$.

A higher α means that there is less diminishing returns to capital, and hence a lower rate of adjustment. Similarly, a lower depreciation rate, δ, or a lower rate of technological progress, g, slows down the adjustment toward steady state.

The effects of an increase in the saving rate s are

- At any point in time, the capital stock and output increase.
- The path for consumption per efficient unit of labor is

$$c(t) = (1-2)\left[\frac{s}{\delta+g+n} + \left(k_0^{1-\alpha} - \frac{s}{\delta+g+n}\right)e^{-(1-\alpha)(\delta+g+n)t}\right]^{\frac{\alpha}{1-\alpha}}.$$

- $c(t)$ decreases for low t.
- $c(t)$ increases for high t if $s < \alpha$.

1.1.10. The Discrete–Time Solow Growth Model

Many macro models are written in discrete time. Suppose there is no population growth ($n = 0$) and no technological progress ($g = 0$). The law of motion for the capital stock is

$$\begin{aligned} k_{t+1} &= k_t + i_t - \delta k_t \\ k_{t+1} &= (1-\delta)k_t + s y_t \\ k_{t+1} &= (1-\delta)k_t + s f(k_t) \end{aligned}$$

and is a first–order, nonlinear, difference equation.

There is an equilibrium (steady–state) such that

$$k_{t+1} = k_t = k^*.$$

The solution is

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}.$$

Since $f(k)/(k)$ is decreasing in k, there is a unique solution. Note that this is the same expression as in the continuous–time model.

All equilibria converge to the positive steady–state. To see this note

$$\frac{k_{t+1}-k_t}{k_t} = s\left[\frac{f(k_t)}{k_t} - \frac{\delta}{s}\right] = \left[\frac{f(k_t)}{k_t} - \frac{f(k^*)}{k^*}\right].$$

- If $k_t < k^*$, then $k_{t+1} > k_t$.
- If $k_t > k^*$, then $k_{t+1} < k_t$.
- Moreover,

$$k_{t+1} - k^* = [(1-\delta)k_t + s f(k_t)] - [(1-\delta)k^* + s f(k^*)]$$

  has the same sign as $k_t - k^*$.

Reading: Introduction and Section 1 of the paper titled "A Contribution to the Empirics of Economic Growth" by Mankiw, Romer and Weil (1992)

Reading: Chapter 2 and 3 from the book "Barriers to Riches" by Parente and Prescott (2000).

1.2. Linear, First–Order Differential Equations

1.2.1. Autonomous Equations

The general form of the linear, autonomous, first–order differential equation is

$$\dot{y} + ay = b$$

where

$$\dot{y} = \frac{\partial y(t)}{\partial t}$$

and a, b are known constants.

- Implicitly, y is a function of time t.
- Time is continuous ($t \in \mathbb{R}^+$).

Let $z(t)$ be a particular solution to this differential equation

$$\dot{z} + az = b.$$

Take the difference with the general equation

$$(\dot{y} - \dot{z}) + a(y - z) = 0$$

$$\dot{\widetilde{y}} + a\widetilde{y} = 0.$$

1.2.2. The Solution Method

- The first step is to find a solution $\widetilde{y}$ to the homogeneous form $\dot{\widetilde{y}} + a\widetilde{y} = 0$.
- The second step is to find a particular solution to the complete equation z.

The general solution is

$$y = \widetilde{y} + z.$$

1.2.3. The Homogeneous Solution

The homogeneous solution form is

$$\dot{y} + ay = 0$$

where $a \neq 0$.

$$\frac{\dot{y}}{y} = -a$$

$$\int \frac{\dot{y}}{y}\,\mathrm{d}t = -at + c_1.$$

Recall that

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln\big(f(x)\big) + c,$$

if we assume that $f(x) > 0$ for all x. Therefore,

$$\int \frac{\dot{y}}{y}\,\mathrm{d}t = -at + c_1 \Leftrightarrow \ln(y) + c_2 = -at + c_1$$

$$y = \exp[-at + (c_1 - c_2)]$$

$$y = C\exp[-at],$$

where $C = e^{c_1 - c_2}$.

Theorem: The general solution to the *homogeneous* form of the linear, autonomous, first–order differential equation is

$$\widetilde{y}(t) = C\mathrm{e}^{-at}.$$

Example:

$$\dot{y} - 3y = 0$$
$$\frac{\dot{y}}{y} = 3.$$

Integrate both sides

$$\ln y + c_2 = 3t + c_1.$$

Simplify and take the exponential

$$y(t) = C\mathrm{e}^{3t}.$$

1.2.4. The Particular Solution

When b is constant, a particular solution is the steady–state equilibrium value of y. A steady–state value $\overline{y}$ of a differential equation is defined by $\dot{y} = 0$.

$$\dot{y} + ay = b \underset{\dot{y}=0}{\Rightarrow} \overline{y} = \frac{b}{a}.$$

Theorem: The general solution to the *complete*, autonomous, linear, first–order differential equation is

$$y(t) = C\mathrm{e}^{-at} + \frac{b}{a}.$$

Example:

$$\dot{y} + 2y = 8.$$

In homogeneous form

$$\dot{\widetilde{y}} + 2\widetilde{y} = 0$$
$$\widetilde{y} = C\mathrm{e}^{-2t}.$$

A particular solution is

$$\dot{y} = 0 \Rightarrow \overline{y} = 4.$$

The general solution is

$$y(t) = \widetilde{y}(t) + \overline{y} = C\mathrm{e}^{-2t} + 4.$$

Example: Let $K(t)$ be the capital stock at time t. Capital depreciates at the rate $\boldsymbol{\Omega}$. Investment per unit of time is $\overline{I}$. The differential equation for capital is

$$\dot{K} = \overline{I} - \delta K.$$

The homogeneous form is

$$\dot{\widetilde{K}} = -\delta \widetilde{K}$$
$$\widetilde{K}(t) = C\mathrm{e}^{-\delta t}.$$

The steady–state solution is

$$\dot{K} = 0 \Rightarrow \overline{K} = \frac{\overline{I}}{\delta}.$$

The general solution is

$$K(t) = \widetilde{K}(t) + \overline{K} = C\mathrm{e}^{-\delta t} + \overline{K}$$

1.2.5. The Initial Value

In order to determine the constant C of the general solution, you need to know the value of y at some arbitrary time t_0. Assume

$$y(t_0) = y_0.$$

From the general solution

$$\begin{aligned} y_0 &= C\mathrm{e}^{-at_0} + \frac{b}{a} \\ C &= \mathrm{e}^{at_0}\left(y_0 - \frac{b}{a}\right). \end{aligned}$$

The solution for the differential equation becomes

$$\begin{aligned} y(t) &= \mathrm{e}^{at_0}\left(y_0 - \frac{b}{a}\right)\mathrm{e}^{-at} + \frac{b}{a} \\ y(t) &= \left(y_0 - \frac{b}{a}\right)\mathrm{e}^{-a(t-t_0)} + \frac{b}{a}. \end{aligned}$$

Example: Consider the example about capital accumulation and assume $K(0) = K_0$. From the general solution

$$\begin{aligned} K_0 \quad &= C + \overline{K} \Rightarrow C = K_0 - \overline{K} \\ K(t) &= (K_0 - \overline{K})\mathrm{e}^{-\delta t} + \overline{K}. \end{aligned}$$

1.2.6. Convergence

Does $y(t)$ converge to its steady–state value? Assume that $y(0) = y_0$. It follows that

$$y(t) = (y_0 - \overline{y})\mathrm{e}^{-at} + \overline{y}.$$

and

$$\lim_{t\to+\infty} y(t) = \overline{y} \Leftrightarrow \lim_{t\to+\infty} \mathrm{e}^{-at} = 0 \Leftrightarrow a > 0.$$

Theorem: The solution to a linear, autonomous, first–order differential equation, $y(t)$, converges to the steady–state equilibrium $\overline{y} = \frac{b}{a}$, no matter what the initial value, y_0, if and only if the coefficient in the differential equation is positive, $a > 0$.

1.3. Nonlinear, First–Order Differential Equations

1.3.1. Qualitative Analysis

Under which conditions can a solution to a nonlinear differential equation exist? Even if a solution exists, it is in general difficult to find. Usually qualitative analysis (e.g. phase diagrams) are useful.

Definition: The initial–value problem for an autonomous, nonlinear, first–order differential equation is expressed as

$$\begin{aligned} \dot{y} &= g(y) \\ y(t_0) &= y_0. \end{aligned}$$

Theorem: If the function g and its partial derivative $\frac{\partial g}{\partial y}$ are *continuous* in some closed rectangle containing the point (t_0, y_0), then in a neighborhood around t_0 contained in the rectangle, there is a unique solution, $y = \xi(t)$, satisfying

$$\begin{aligned} \dot{y} &= g(y) \\ y(t_0) &= y_0. \end{aligned}$$

Sometimes, you may not have an initial condition, but have a terminal condition or a transversality condition (e.g. the dynamics of the price of an asset).

Example: Consider the following nonlinear differential equation

$$\dot{y} = y - y^2 = y(1 - y).$$

The steady–state values are

$$\dot{y} = 0 \Rightarrow y = 0 \quad \text{or} \quad y = 1.$$

The function $g(y) = y - y^2$ is concave and reaches a maximum for $y = 0.5$.

1.3.2. Phase Diagrams

Definition: A phase diagram shows $\dot{y}$ as a function of y.

It is useful to find the following.

- The range of y values of which y is *increasing* over time.
- The range of y values over which y is *decreasing* over time.
- Introduce arrows of motion to indicate the direction of motion of the variable y in the different regions.

Remember that the phase diagram for a difference equation plots y_{t+1} as a function of y_t. Steady–states are at the intersection of the phase line and the 45°degree line. The phase diagram for a differential equation plots the change in y, $\dot{y}$, as a function of y. Steady–states are at the intersection of the phase line and the horizontal axis.

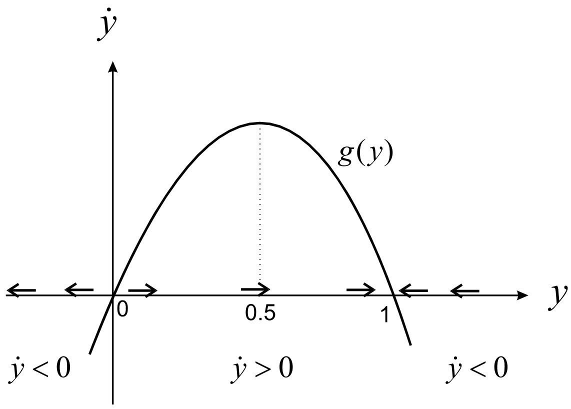


Figure I.1: Phase Diagram of a Differential Equation

1.3.3. Stability Analysis

To determine stability, you need to know the arrows of motion around the steady–state values. In Figure I.1, $y = 1$ is a stable equilibrium. The arrows of motion point toward the stable equilibrium and away from the unstable one.

1.3.4. Linearizing a Differential Equation

First, linearize $\dot{y} = g(y)$ in the neighborhood of the steady–state $\overline{y}$

$$g(y) = g(\overline{y}) + (y - \overline{y})g'(\overline{y})$$

where $g(\overline{y}) = 0$. The differential equation can be approximated by

$$\dot{y} = (y - \overline{y})g'(\overline{y}).$$

Thus

$$y(t) = C\mathrm{e}^{g'(\overline{y})t} + \overline{y}$$

implies convergence if $g'(\overline{y}) < 0$.

Theorem: A steady–state equilibrium point of a nonlinear, first–order differential equation is stable if the derivative $\frac{\mathrm{d}\dot{y}}{\mathrm{d}y}$ is negative at that point and unstable if the derivative is positive at that point.

Example: Consider again $\dot{y} = g(y) = y - y^2$.

$$\frac{\mathrm{d}\dot{y}}{\mathrm{d}y} = g'(y) = 1 - 2y.$$

At the steady–state value, $y = 0$,

$$\left.\frac{\mathrm{d}\dot{y}}{\mathrm{d}y}\right|_{y=0} = g'(0) = 1 > 0$$

and the equation is unstable.
At the steady–state value, $y = 1$,

$$\left.\frac{\mathrm{d}\dot{y}}{\mathrm{d}y}\right|_{y=1} = g'(1) = -1 < 0$$

and the equation is stable.

1.3.5. Interpretation

Assume the system is at its steady–state. It is pushed away from the equilibrium point by an amount $\mathrm{d}y$.

- If $\frac{\mathrm{d}\dot{y}}{\mathrm{d}y} < 0$, then the system will move backward and return to the equilibrium point.
- If $\frac{\mathrm{d}\dot{y}}{\mathrm{d}y} > 0$, then the system moves further away from equilibrium.

Example:

$$\dot{y} = 3y^2 - 2y = y(3y - 2)$$

The steady–state points, $\dot{y} = 0$, are

$$y = 0 \quad \text{or} \quad y = \frac{2}{3}.$$

The differential

$$\frac{\mathrm{d}\dot{y}}{\mathrm{d}y} = 6y - 2.$$

If $y = 0$, then

$$\left.\frac{\mathrm{d}\dot{y}}{\mathrm{d}y}\right|_{y=0} = -2 < 0$$

and the equation is stable.
If $y = \frac{2}{3}$, then

$$\left.\frac{\mathrm{d}\dot{y}}{\mathrm{d}y}\right|_{y=2/3} = 2 > 0$$

and the equation is unstable.

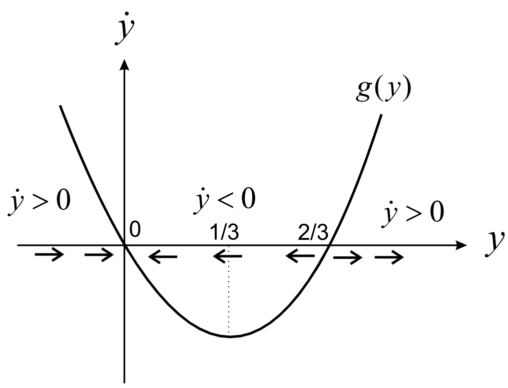


Figure I.2: Phase Diagram of a Differential Equation

1.3.6. The Neoclassical Model of Economic Growth

The production function, $Y = F(K, L)$, exhibits constant returns to scale. Output per person is

$$y \equiv \frac{Y}{L} = F\left(\frac{K}{L}, 1\right) \equiv f(k),$$

where $f(\cdot)$ is a concave function. The law of motion of the capital stock is

$$\dot{K} = sY.$$

The change in the capital-labor ratio is

$$\dot{k} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{K}{L}\right)$$
$$\dot{k} = \frac{\dot{K}}{L} - \frac{K\dot{L}}{L^2}$$
$$\dot{k} = \frac{\dot{K}}{L} - k\frac{\dot{L}}{L}.$$

The labor force grows at the constant rate, $\frac{\dot{L}}{L} = n$. It follows that

$$\dot{k} = \frac{sY}{L} - kn$$
$$\dot{k} = sf(k) - kn.$$

This nonlinear differential equation describes the growth of the economy.
The steady–state points, such that $\dot{k} = 0$, that occurs where

$$sf(k) = kn,$$

$$\text{are} \quad k = 0 \quad \text{and} \quad k^* > 0.$$

Furthermore,

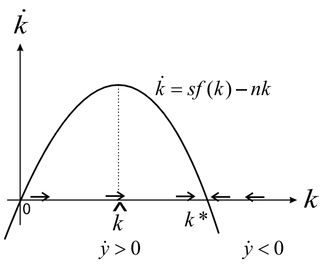
$$\frac{\mathrm{d}\dot{k}}{\mathrm{d}k} = sf'(k) - n$$
$$\frac{\mathrm{d}\dot{k}}{\mathrm{d}k} = 0 \Rightarrow sf'(k) = n$$

and this point is $\hat{k}$. If

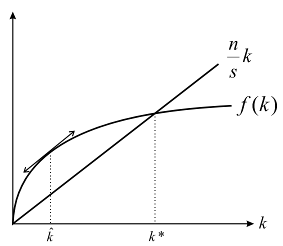
$$\frac{\mathrm{d}^2\dot{k}}{\mathrm{d}k^2} = sf''(k) < 0,$$

then $\dot{k}$ is maximized at $k = \hat{k}$.

Phase diagram



Steady-states



1.4. Growth in a Overlapping Generations Economy

The objective is to endogenize the savings rate in the Solow growth model by introducing a natural heterogeneity across individuals at a point in time. This is an example of an economy in which the competitive equilibrium may not be Pareto optimal. You can study the aggregate implications of life–cycle saving by individuals.

1.4.1. The OLG Model

Assumptions:

- Time is discrete ($t = 1, 2, \dots$)
- Each individual lives for two periods (the simplest case where generations overlap).
- L_t individuals are born in period t

$$L_t = (1+n)L_{t-1}$$

- At time 1, there is a generation who only lives for one period—the initial old—who own the initial capital stock.
- At any time, the economy is composed of 2 generations; the young and the old.
- Each individual supplies 1 unit of labor when young.
- Individuals are not productive when old.
- Capital saved in one period is a input in the production process of the following period.
- There is no depreciation of capital stock.

Households consume part of their first period income and save the rest to finance their second period retirement consumption. The capital stock is generated by individuals who save during their working lives. The timing of events is as follows

- 1^{st} period of life: an individual is born, works, consumes, and saves capital.
- 2^{nd} period of life: an individual spends revenue from capital, consumes, and dies.

The constant–relative–risk–aversion utility is

$$U_t = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2,t+1}^{1-\theta}}{1-\theta},$$

where $\theta > 0$, $\rho > -1$.

- $C_{1,t}$ is consumption in period t of young individuals.
- $C_{2,t+1}$ is consumption in period $t+1$ of old individuals.

$$C_{2,t+1} = (1 + r_{t+1})(w_t - C_{1,t}),$$

and the lifetime budget constraint is

$$C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}} = w_t.$$

The Lagrangian for the individual's maximization problem is

$$\max_{\{C_{1,t}, C_{2,t+1}\}} \mathscr{L} = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2,t+1}^{1-\theta}}{1-\theta} + \lambda\left[w_t - \left(C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}}\right)\right].$$

The first–order conditions are

$$\frac{\partial \mathscr{L}}{\partial C_{1,t}} = C_{1,t}^{-\theta} - \lambda = 0$$

and

$$\frac{\partial \mathscr{L}}{\partial C_{2,t+1}} = \frac{1}{1+\rho} C_{2,t+1}^{-\theta} - \frac{1}{1+r_{t+1}} \lambda = 0$$

So

$$C_{1,t}^{-\theta} = \lambda,$$

and the Euler equation is

$$C_{1,t}^{-\theta} = \frac{1+r_{t+1}}{1+\rho} C_{2,t+1}^{-\theta}.$$

You can note that

$$\frac{C_{1,t}}{C_{2,t+1}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}} \approx \frac{r_{t+1}-\rho}{\theta} - 1.$$

The interpretation of the Euler equation is

$$\frac{C_{2,t+1} - C_{1,t}}{C_{1,t}} \approx \frac{1}{\theta} \times (r_{t+1} - \rho)$$

"Rate of growth of consumption"

"Preference for consumption smoothing"

"Incentives to postpone consumption"

From the Euler equation and the budget constraint

$$C_{1,t} + C_{1,t} \frac{(1+r_{t+1})^{\frac{1}{\theta}-1}}{(1+\rho)^{\frac{1}{\theta}}} = w_t$$

and it follows that

$$C_{1,t} = [1 - s(r_{t+1})] w_t$$

with

$$s(r) = \frac{(1+r)^{\frac{1}{\theta}-1}}{(1+r)^{\frac{1}{\theta}-1} + (1+\rho)^{\frac{1}{\theta}}}.$$

- The saving rate, $s(r)$, is increasing in r if and only if $(1+r)^{\frac{1}{\theta}-1}$ is increasing in r. This occurs when $\theta < 1$.
- A rise in r has a negative substitution effect on current consumption.
- A rise in r has a positive income effect on current consumption (because the young agents are lenders of capital).
- If θ is low, then there is a high intertemporal elasticity of substitution and the substitution effect dominates.
- If $\theta = 1$, then it is the logarithmic case and the substitution effect and the income effect perfectly offset each other.

The capital stock in period $t+1$ is the amount saved by young individuals in period t

$$K_{t+1} = L_t s(r_{t+1}) w_t,$$

where L_t are the young in period t and s is the saving rate. If you divide by L_{t+1}

$$k_{t+1} = \frac{s(r_{t+1})}{(1+n)} w_t.$$

There are many firms, each with production function

$$Y_t = F(K_t, L_t).$$

Markets are competitive, so, labor and capital earn their marginal products

$$r_t = f'(k_t)$$
$$w_t = f(k_t) - k_t f'(k_t)$$

where $k_t \equiv \frac{K_t}{L_t}$. The initial capital stock K_0 is owned equally by all old individuals.

An equilibrium of the OLG model is a triple of sequences, $\{w_t\}$, $\{r_t\}$, $\{k_t\}$, that satisfy

$$r_t = f'(k_t)$$
$$w_t = f(k_t) - k_t f'(k_t)$$
$$k_{t+1} = \frac{s(r_{t+1})}{(1+n)} w_t$$

where k_0 is given.

1.4.2. The Steady–State and Dynamics of the OLG Economy

For examination of the dynamics for the capital stock, substitute r_{t+1} and w_t by their expressions as functions of the capital stock k_t

$$k_{t+1} = \frac{s[f'(k_{t+1})]}{(1+n)} [f(k_t) - k_t f'(k_t)].$$

The dynamics depends crucially on the saving rate function.

Next, with logarithmic utility and Cobb–Douglas production, $\theta = 1$, $f(k) = k^\alpha$, and $s(r) = \frac{1}{2+\rho}$, the saving rate is constant and independent of r, then

$$k_{t+1} = \frac{(1-\alpha)(k_t)^\alpha}{(2+\rho)(1+n)}.$$

The steady–state for k is

$$k_{t+1} = k_t = k^*$$
$$k^* = \left[\frac{(1-\alpha)}{(2+\rho)(1+n)}\right]^{\frac{1}{1-\alpha}}.$$

The steady–state capital stock is decreasing with ρ and n.

For examination of the speed of convergence, linearize the dynamic system around the balanced growth path

$$k_{t+1} \simeq k^* + \left.\frac{\mathrm{d}k_{t+1}}{\mathrm{d}k_t}\right|_{k_t=k^*} (k_t - k^*).$$

Then compute

$$\left.\frac{\mathrm{d}k_{t+1}}{\mathrm{d}k_t}\right|_{k_t=k^*} = \frac{\alpha(1-\alpha)(k^*)^{\alpha-1}}{(2+\rho)(1+n)} = \alpha.$$

The dynamic of the system can be approximated by

$$k_t - k^* = \alpha^t (k_0 - k^*).$$

The median lag satisfies

$$\frac{k_0 - k^*}{2} = \alpha^{\tau_{\text{half}}}(k_0 - k^*).$$

If $\alpha = 1/3$, then the median lag satisfies

$$\tau_{\text{half}} = \frac{\ln 2}{\ln 3}.$$

1.4.3. Dynamic Inefficiency in the OLG Economy

Next, we consider heterogenous agents and the questions of how to measure welfare and how to asses the efficiency of the equilibrium. A robust criterion is Pareto efficiency. Is it possible to raise consumption for all agents in all periods? If so, then the economy is dynamically inefficient. First, a feasible allocation satisfies

$$L_t C_{1,t} + L_{t-1} C_{2,t} + L_{t+1} k_{t+1} = L_t k_t + L_t f(k_t).$$

Now divide by L_t to yield

$$C_{1,t} + \frac{C_{2,t}}{1+n} + (1+n)k_{t+1} = k_t + f(k_t).$$

If the economy is in steady state, then

$$C_1 + \frac{C_2}{1+n} = f(k) - nk.$$

Definition: The golden–rule level of capital stock is where aggregate steady–state consumption is maximized.

Here, that is when

$$f'(k) = n.$$

If the steady–state equilibrium capital stock is larger than k_{GR}, then the economy is dynamically inefficient. That is, agents can "eat" the capital stock above k_{GR} and still increase aggregate consumption in subsequent periods. Consider a social planner and assume that $k^* > k_{\text{GR}}$. If the social planner does not change the capital stock, then the output available for consumption is $f(k^*) - nk^*$. The social planner can increase consumption in the current period and maintain the golden–rule level for the capital stock in subsequent periods. Consumption in the initial period is

$$f(k^*) - nk_{\text{GR}} + (k^* - k_{\text{GR}}),$$

and consumption in subsequent periods is

$$f(k_{\text{GR}}) - nk_{\text{GR}}.$$

The steady–state equilibrium capital stock is given by

$$k^* = \left[\frac{(1-\alpha)}{(2+\rho)(1+n)}\right]^{\frac{1}{1-\alpha}}.$$

The marginal product of capital is

$$f'(k^*) = \alpha {k^*}^{\alpha-1} = \frac{\alpha(2+\rho)(1+n)}{(1-\alpha)},$$

which may be greater than or less than $f'(k_{\text{GR}}) = n$. For sufficiently small α, then $f'(k^*) < f'(k_{\text{GR}})$. Equivalently, if the saving rate, $s = \frac{1}{2+\rho}$, is too large, then the steady–state capital stock exceeds the golden–rule level and the equilibrium is Pareto inefficient.

1.4.4. The Samuelson Paradox

Note that the 1st Welfare Theorem states that competitive equilibria are always efficient. So, why does the 1st Welfare Theorem fail to hold in the OLG model? This is because the 1st Welfare Theorem assumes that there are no externalities, competitive markets, and no *missing* markets. In an economy with births and deaths, all agents cannot meet in a single market. A infinite number of dated commodities and a infinite number of agents, that is two infinite quantities, explains the Samuelson paradox.

Proof. Given infinite goods, g, and households, h, let $\{p_g\}$ be the competitive prices and $\{c_{hg}\}$ be the competitive allocation. The proof of the 1st Welfare Theorem is sketched out as follows. Let $\{c'_{hg}\}$ be an allocation that Pareto–dominates the competitive allocation. It follows that

$$\sum_g p_g(c'_{hg} - c_{hg}) \geq 0,$$

for every h, with strict inequality for some h. Adding over over households yields

$$\sum_h \sum_g p_g(c'_{hg} - c_{hg}) > 0.$$

Interchanging the summations yields

$$\sum_g p_g \sum_h (c'_{hg} - c_{hg}) > 0,$$

which implies

$$\sum_h (c'_{hg} - c_{hg}) > 0,$$

for some good g. This violates feasibility; people are consuming more than their combined endowment of good g. However, this proof cannot be used in OLG models, because the sets of goods and agents are infinite. The proof of the 1st Welfare Theorem requires the double summation to be finite. Thus, the 1st Welfare Theorem does not hold in OLG economies. □

Example: Assume that there is no production, the endowment of the young is $e_1 = 1$, the endowment of the old $e_0 = 0$, and agents have linear utility

$$U_t = C_{1,t} + C_{2,t+1}.$$

The competitive equilibria are such that

$$(C_{1,t}, C_{2,t+1}) = (1, 0).$$

An allocation that generates a Pareto–improvement is

$$(C_{1,t}, C_{2,t+1}) = (0, 1),$$

because the old at time 1 are strictly better–off whereas the following generations are indifferent. Note that this is because generations are infinite, the young will always give their endowment to the old and receive an endowment when they are old. Everyone is just as well off, but the first generation of old is strictly better–off because they can consume an endowment.

1.4.5. Impure Altruism in the OLG Economy

So far, agents do not care about the utility of future generations. However, altruism might be empirically relevant, so, it is important to explain bequests. Perhaps, parents have warm glow preferences and derive utility from their bequests. We will examine how such altruism affects the dynamics of the OLG economy.

Assume that there are a continuum of individuals with measure normalized to 1 and that the population is constant at 1. Each individual lives for 2 periods; childhood and adulthood. In adulthood, the second period, each individual receives one child and an endowment of 1 unit of labor. Capital fully depreciates after use. Agents do not enjoy consumption in childhood, the first period. The preferences of agent i at time t are

$$U_i(t, c, b) = \log[c_i(t)] + \beta \log[b_i(t)],$$

where $c_i(t)$ is consumption when an adult and $b_i(t)$ is a bequest to the individual's offspring. The maximization problem is

$$\max_{c_i(t), b_i(t)} \log[c_i(t)] + \beta \log[b_i(t)]$$

$$\text{s. t. } c_i(t) + b_i(t) = w(t) + R(t) b_i(t-1),$$

where $R(t)$ is the rental price of capital and $w(t)$ is the real wage rate. Assuming competitve prices

$$\begin{aligned} R(t) &= f'[k(t)] \\ w(t) &= f[k(t)] - k(t) f'[k(t)]. \end{aligned}$$

The solution to this problem is

$$\begin{aligned} c_i(t) &= \frac{y_i(t)}{1+\beta} \\ b_i(t) &= \frac{\beta y_i(t)}{1+\beta}. \end{aligned}$$

The result is a distribution of wealth that evolves endogenously over time. The capital–labor ratio at time $t+1$ is given by aggregating the bequests of all adults at time t

$$\begin{aligned} k(t+1) &= \int_0^1 b_i(t) \, \mathrm{d}i \\ k(t+1) &= \int_0^1 \frac{\beta}{1+\beta} [w(t) + R(t) b_i(t-1)] \, \mathrm{d}i \\ k(t+1) &= \frac{\beta}{1+\beta} [w(t) + R(t) k(t)] \\ k(t+1) &= \frac{\beta}{1+\beta} f[k(t)]. \end{aligned}$$

This equation represents the aggregate equilibrium dynamics, and it is similar to the baseline Solow growth model. There is a unique positive steady–state, where capital stock increases with β,

$$k^* = \frac{\beta}{1+\beta} f(k^*).$$

At the steady–state, individual bequest dynamics are given by

$$b_i(t) = \frac{\beta}{1+\beta}[w^* + R^* b_i(t-1)],$$

and it can be checked that $\frac{R^*\beta}{1+\beta} < 1$. Thus, the distribution of bequests converges to full equality

$$b_i(t) \to b^* = \frac{\beta w^*}{1+\beta(1-R^*)}.$$

1.5. Fiat Money in the Overlapping Generations Economy

Definition: Fiat money is inconvertible, there is no promise it can be converted into anything else, and it is intrinsically useless, it cannot be used in the utility function nor in the production function.

Thus, fiat money is an efficient form of money; it can be produced at no cost. The Hahn problem is the question of how can an intrinsically useless object can have a positive value in exchange? This is a puzzle in monetary theory. The OLG model offers a “deep” model of money that can offer help to solve this puzzle by adding an inter-generational friction to motivate a meaningful role for money.

1.5.1. The OLG Barter Economy

First, look at the OLG barter economy.

Assumptions:

- Time is discrete; $t = 0, 1, 2, \ldots$
- There are L_t individuals are born at time t
$$L_t = (1+n)^t.$$
- Individuals live for 2 periods.
- All agents have perfect foresight.
- Each agent is endowed with 1 unit of good when young.
- The good can be exchanged, consumed, or stored.
- Each unit saved at time t yields $1+r$ units at time $t+1$ (storage technology).
- The lifetime utility function of an individual born at time t is
$$u(c_{y,t}, c_{o,t+1}) = \ln(c_{y,t}) + \beta \ln(c_{o,t+1}),$$
where $c_{y,t}$ is consumption when young and $c_{o,t+1}$ is consumption when old.

Let $\{(c_{y,t}, c_{o,t}), t = 0, 1, 2, \ldots\}$ be an allocation of the consumption of the young and old agents in each period. When good are perishable, $r = -1$, feasible allocations satisfy

$$L_t c_{y,t} + L_{t-1} c_{o,t} \leq L_t$$
$$c_{y,t} + \frac{c_{o,t}}{1+n} \leq 1,$$

for all t. If agents face a similar budget constraint, then their program is

$$\max_{c_{y,t}, c_{o,t+1}} \ln(c_{y,t}) + \beta \ln(c_{o,t+1})$$
$$\text{s. t. } c_{y,t} + \frac{c_{o,t+1}}{1+n} = 1.$$

The solution (Euler equation) for the maximization problem is

$$\frac{c_{o,t+1}}{c_{y,t}} = \beta(1+n).$$

From the budget constraint, you can find that

$$c_{y,t} = \frac{1}{1+\beta}$$
$$c_{o,t+1} = \frac{\beta(1+n)}{1+\beta}$$

Even though the allocation $\{(\frac{1}{1+\beta}, \frac{\beta(1+n)}{1+\beta})\}$ is feasible, it is not attainable through bilateral trade. The young would like to exchange goods in this period against goods in the next period with the future young, but they can only trade with the current old. Therefore, no trade can take place and the decentralized outcome is $c_{y,t} = 1$ and $c_{o,t+1} = 0$. The decentralized equilibrium is not Pareto optimal and is therefore dynamically inefficient. If the young agents each transfer $\frac{\beta}{1+\beta}$ to the old generation and if each old agent receives $\frac{\beta(1+n)}{1+\beta}$, then there is a Pareto improvement and everyone is strictly better-off.

1.5.2. The OLG Monetary Economy

At time 0, the government gives to the old H divisible units of a fiat object called money. Suppose that at time t the price of goods in terms of this fiat object is P_t. Money is valued $P_t < +\infty$ or $\frac{1}{P_t} > 0$. The maximization problem of the agent is then

$$\max_{c_{y,t}, c_{o,t+1}} \ln(c_{y,t}) + \beta \ln(c_{o,t+1})$$

$$\begin{aligned} \text{s. t. } P_t c_{y,t} + m_t^d &= P_t \\ P_{t+1} c_{o,t+1} &= m_t^d, \end{aligned}$$

where m_t^d is the household's money balance. Let z_t be the real money balances

$$z_t = \frac{m_t^d}{P_t}.$$

The constraints become

$$\begin{aligned} c_{y,t} &= 1 - z_t \\ c_{o,t+1} &= \tfrac{P_t}{P_{t+1}} z_t, \end{aligned}$$

and the program can be rewritten

$$\max_{z_t} \ln(1 - z_t) + \beta \ln\left(\frac{P_t}{P_{t+1}} z_t\right),$$

with first order condition

$$\frac{1}{1 - z_t} = \frac{\beta}{z_t}.$$

The solution to this program is

$$\begin{aligned} c_{y,t} &= \frac{1}{1+\beta} \\ z_t &= \frac{\beta}{1+\beta} \\ c_{o,t+1} &= \frac{P_t}{P_{t+1}} \left(\frac{\beta}{1+\beta}\right). \end{aligned}$$

The equilibrium of the money market is

$$L_t P_t \left(\frac{\beta}{1+\beta}\right) = H,$$

where $L_t P_t(\frac{\beta}{1+\beta})$ is the *demand for money* and the H is the *money supply*. It follows that there is deflation (i.e. $P_t > P_{t+1}$) at the rate n

$$\frac{L_t P_t}{L_{t+1} P_{t+1}} = 1$$

$$\frac{P_t}{P_{t+1}} = 1 + n.$$

The allocation at the steady–state monetary equilibrium is

$$c_{y,t} = \frac{1}{1+\beta}$$

$$c_{o,t+1} = (1+n)\frac{\beta}{1+\beta}.$$

Thus, the introduction of money leads to a Pareto optimal allocation of resources across generations.

The assumption that the economy goes on forever is a necessary condition for money to be valued. If the economy ended at time T, the young at time T would not want to buy money. Proceeding backward, no one would ever want to buy money. Furthermore, even if a monetary equilibrium exists, there is also a barter equilibrium where fiat money is not valued. An implication of the incovertibility and intrinsic uselessness of fiat money is that equilibria in which fiat money is valued are tenuous.

1.5.3. The Role of Money in the OLG Economy

Now, suppose that there is fiat money in an economy with storage (i.e. $r > -1$). The amount of goods that is stored is k_t. The program of the agent is

$$\max_{c_{y,t}, c_{o,t+1}} \ln(c_{y,t}) + \beta \ln(c_{o,t+1})$$

$$\text{s. t. } c_{y,t} = 1 - k_t - z_t$$

$$c_{o,t+1} = (1+r)k_t + \frac{P_t}{P_{t+1}} z_t$$

The program can be rewritten as

$$\max_{k_t, z_t} \ln(1 - k_t - z_t) + \beta \ln\left((1+r)k_t + \frac{P_t}{P_{t+1}} z_t\right),$$

with first order conditions

$$-\frac{1}{c_{y,t}} + \beta\frac{(1+r)}{c_{o,t+1}} \begin{cases} \leq 0 \\ = 0 \quad \text{if } k_t > 0. \end{cases}$$

$$-\frac{1}{c_{y,t}} + \beta\frac{P_t/P_{t+1}}{c_{o,t+1}} \begin{cases} \leq 0 \\ = 0 \quad \text{if } z_t > 0. \end{cases}$$

From the Inada conditions, $c_{o,t+1} > 0$, which implies $k_t > 0$ or $z_t = \frac{m_t^d}{P_t} > 0$. From the FOC,

$$\text{If } \frac{P_t}{P_{t+1}} < 1 + r, \text{ then } k_t > 0 \text{ and } \frac{m_t^d}{P_t} = 0$$

$$\text{If } \frac{P_t}{P_{t+1}} > 1 + r, \text{ then } k_t = 0 \text{ and } \frac{m_t^d}{P_t} > 0.$$

If agents are willing to hold money, then

$$\frac{P_t}{P_{t+1}} = 1 + n.$$

The result is as follows.

- If $r < n$ then money can be valued and has a rate of return equal to n. The monetary economy achieves a Pareto optimum and storage is not used.
- If $r > n$, then the barter equilibrium is a Pareto optimum. There cannot be a monetary equilibrium with a constant money stock.

In conclusion, there can be a monetary equilibrium only if the barter equilibrium is not a Pareto optimum. In this case, there is a monetary equilibrium that is Pareto optimal. That is, if the economy is dynamically inefficient (i.e. $r < n$), then the introduction of money can make everybody better–off. Furthermore, money is valued only when it is not dominated in rate of return by any other asset.

1.5.4. Money and Inflation in the OLG Economy

Next, assume that the nominal money stock grows at rate σ

$$H_{t+1} = (1 + \sigma)H_t.$$

New money is introduced through lump–sum transfers to the old. Let T_t be the amount of the monetary transfer received by the old at time t. The program of an agent is

$$\max_{c_{y,t}, c_{o,t+1}} \ln(c_{y,t}) + \beta \ln(c_{o,t+1})$$

$$\begin{aligned} \text{s. t. } c_{y,t} &= 1 - k_t - z_t \\ c_{o,t+1} &= (1 + r)k_t + \frac{P_t}{P_{t+1}} z_t + \frac{T_{t+1}}{P_{t+1}}. \end{aligned}$$

The equilibrium condition in the money market is

$$L_t m_t^d = H_t,$$

or in real money balances it is

$$L_t z_t = \frac{H_t}{P_t}.$$

In a steady–state, per–capita real balances are constant

$$\frac{H_t}{L_t P_t} = \frac{H_{t+1}}{L_{t+1} P_{t+1}}.$$

This implies that

$$\frac{P_{t+1}}{P_t} = \frac{L_t H_{t+1}}{L_{t+1} H_t} = \frac{1 + \sigma}{1 + n}$$

where

$$\frac{1 + \sigma}{1 + n} \simeq 1 + \sigma - n.$$

The rate of return of money is

$$\frac{P_t}{P_{t+1}} - 1 = \frac{1 + n}{1 + \sigma} - 1 \simeq n - \sigma.$$

Assume that $\frac{1+n}{1+\sigma} > 1 + r$. This implies that agents prefer to hold money rather than to store goods. Let z_t be the demand for real balances. The program of the agent is then

$$\max_{z_t} \ln(1 - z_t) + \beta \ln\left(\frac{P_t}{P_{t+1}} z_t + \frac{T_{t+1}}{P_{t+1}}\right),$$

with first order condition

$$\frac{1}{1 - z_t} = \frac{\beta}{z_t + \frac{T_{t+1}}{P_t}}.$$

The additional money is used to finance the transfer to the old

$$T_{t+1} = \frac{\sigma H_t}{L_t} = \sigma m_t^d,$$

or in real money balances

$$\frac{T_{t+1}}{P_t} = \sigma z_t.$$

It can then be deduced that

$$\begin{aligned} c_{y,t} &= \frac{1+\sigma}{1+\beta+\sigma} \\ z_t &= \frac{\beta}{1+\beta+\sigma} \\ c_{o,t+1} &= (1+n)\frac{\beta}{1+\beta+\sigma}. \end{aligned}$$

In conclusion, when real money balances are constant, then the price level is proportional to the money supply, that is, money is neutral. Money *does affect* the allocation of resources, that is, money is not superneutral. The monetary equilibrium with inflation is no longer a Pareto optimum. If $\frac{1+n}{1+\sigma} < 1 + r$, then there is no monetary equilibrium (i.e. the storage technology outperforms money as a store of value). Also, note that money growth cannot be too large, otherwise the economy resorts to a barter economy.

1.5.5. Dynamics of the OLG Economy

Assume that goods are perishable (i.e. $r = -1$), the endowments are 1 when young and $\alpha < 1$ when old, and that the population is constant, $L_t = 1$ and $H = 1$. Consider the following utility function with no discounting

$$u(c_{y,t}, c_{o,t+1}) = \frac{(c_{y,t})^{1-\gamma_1}}{1-\gamma_1} + \frac{(c_{o,t+1})^{1-\gamma_2}}{1-\gamma_2},$$

where there are different coefficients for RRA across consumption, γ_1, $\gamma_2 > 0$. The program of the agent is

$$\begin{aligned} \max_{c_{y,t}, c_{o,t+1}} \ & \frac{(c_{y,t})^{1-\gamma_1}}{1-\gamma_1} + \frac{(c_{o,t+1})^{1-\gamma_2}}{1-\gamma_2} \\ \text{s. t. } \ & c_{y,t} + z_t = 1 \\ & c_{o,t+1} = \frac{P_t}{P_{t+1}} z_t + \alpha. \end{aligned}$$

The FOC implies that

$$\frac{(c_{o,t+1})^{\gamma_2}}{(c_{y,t})^{\gamma_1}} = \frac{P_t}{P_{t+1}}.$$

By definition

$$z_{t+1} = z_t \frac{P_t}{P_{t+1}},$$

so, the budget constraints of the agent born at time t are

$$\begin{aligned} z_t &= 1 - c_{y,t} \\ \alpha + z_{t+1} &= c_{o,t+1} \end{aligned}$$

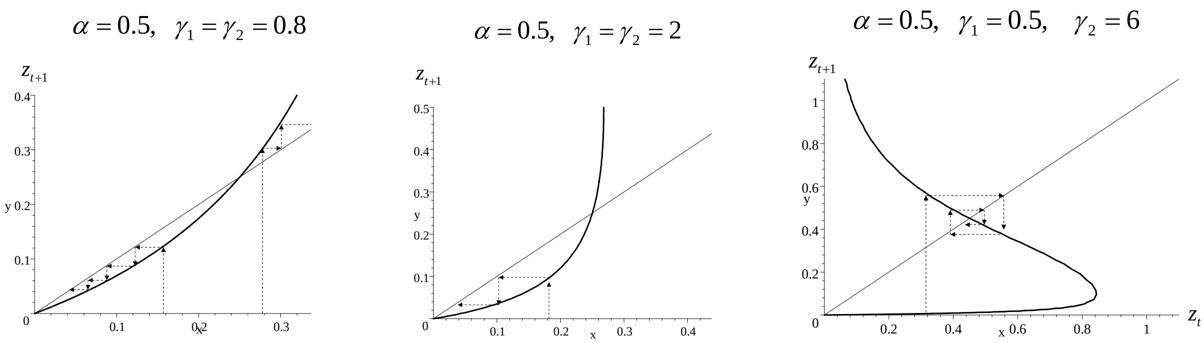
As a consequence

$$z_{t+1} = z_t \frac{(\alpha + z_{t+1})^{\gamma_2}}{(1 - z_t)^{\gamma_1}},$$

and it follows that

$$\frac{z_t}{(1 - z_t)^{\gamma_1}} = \frac{z_{t+1}}{(\alpha + z_{t+1})^{\gamma_2}}.$$

This represents a phase line. The left–hand side is increasing in z_t, and the right–hand side, $(\alpha(z_{t+1}^{-1/\gamma_2} + (z_{t+1})^{1-1/\gamma_2})^{-\gamma_2}$, is increasing in $z_t + 1$ if $\gamma < 1$. Otherwise, if $\gamma_2 > 1$, then the right–hand side is non-monotonic and the phase line may be backward bending.



The steady–states of this first–order difference equation are $\overline{z} = 0$ or $\overline{z}$ such that

$$(\alpha + \overline{z})^{\gamma_2} = (1 - \overline{z})^{\gamma_1},$$

where $\alpha < 1$ is required for a steady–state monetary equilibrium to exist.

When $\alpha = 0.5$ and $\gamma_1 = \gamma_2 = 0.8$, then there are two steady stats; $\overline{z} = 0$ and $\overline{z} = 0.25$. The monetary equilibrium is unstable. Paths that start to the right transition such that z increases and becomes larger than the initial endowment, 1, which is impossible. Path starting to the left transition such that z decreases asymptotically to 0. If you impose that the price level will not explode, then the monetary equilibrium is unique and the price level is uniquely determined. The monetary equilibrium will be locally stable. The economy converges to the monetary steady–state starting from any z in the neighborhood of $\overline{z}$. This implies that there are a multiplicity of convergent solutions, which in turn implies that the price level is indeterminate.

The reasons why the phase line can be backward bending are that the supply of goods when young depends on the rate of return of money, and if $\frac{P_t}{P_{t+1}}$ increases then there are two affects; a substitution effect, agents want to save more, and an income effect, agents want to consume more. The substitution effect may dominate for low values of $\frac{P_t}{P_{t+1}}$ whereas the income effect may dominate for large values of $\frac{P_t}{P_{t+1}}$.

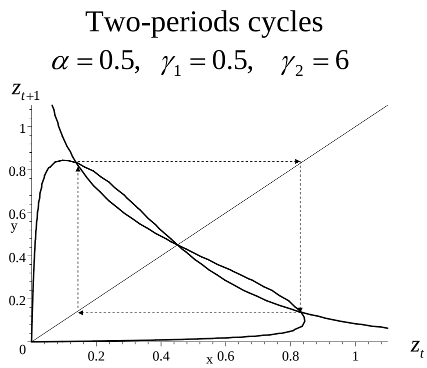
In order to find a cyclical solution, let the phase line by given by

$$z_{t+1} = \psi(z_t).$$

A 2–periods cycle is

$$z_2 = \psi(z_1) \text{ and } z_1 = \psi(z_2).$$

In order to find such a solution, you can draw the mirror image of the phase line, $z_t = \psi(z_{t+1})$, and check if it intersects the original phase line.



1.6. Introduction to Optimal Control Theory

1.6.1. A Simple Optimal Control Problem

There is one good and one agent (or social planner). The good can be either consumed or used as capital. The agent maximizes her lifetime discounted utility over the time horizon $[0, T]$. At time 0, there are $k_0 > 0$ units of capital. The terminal condition is that $k(T) = k_T$. The production function is

$$y = f(k)$$

where $f' > 0$, $f'' < 0$, $f'(0) = +\infty$, and $f'(+\infty) = 0$. Capital depreciates at rate $\delta > 0$. The law of motion for the capital stock is then

$$\dot{k} = f(k) - c - \delta k.$$

The utility of the agent is

$$U(\{c(t)\}) = \int_0^T \mathrm{e}^{-\rho t} u[c(t)] \,\mathrm{d}t,$$

where $\rho > 0$. $u(\cdot)$ is increasing and strictly concave; $u' > 0$, $u'' < 0$, $u'(0) = +\infty$, and $u'(+\infty) = 0$.

Definition: A state variable determines the position of the (economic) system at each point of time.

In this context, k is a state variable; $\{k(t), t \in [0, T]\}$ gives the trajectory of the system.

Definition: A control varialbe is the choice variable of the agent and affects her current utility and the path of the state variable(s).

In this context, c is a control variable.

The $\{c(t), t \in [0, T]\}$, initial condition $k(0) = k_0$, and the ordinary differential equation (ODE) for k define a unique trajectory for the system. A pair of functions, $c(t)$ and $k(t)$, that satisfy the law of motion for capital, the initial and terminal conditions, and non–negativity constraints is an admissible pair of functions.

The program of the agent is

$$\max_{\{c(t), k(t)\}} U(\{c(t)\}) = \int_0^T \mathrm{e}^{-\rho t} u[c(t)] \,\mathrm{d}t$$

$$\text{s. t.} \quad \dot{k} = f(k) - c - \delta k, \quad k(0) = k_0, \ \text{and} \quad k(T) = k_T.$$

1.6.2. Discrete Time Optimal Control

Consider first a finite dimensional problem; a discrete problem where the number of control variables is finite. Divide the interval interval of time $[0, T]$ into N periods. The length of each period is $\Delta \equiv \frac{T}{N}$. The Agent's utility can be rewritten as

$$U(\{c(t)\}) = \sum_{\tau=0}^{N-1} \int_{\tau\Delta}^{(\tau+1)\Delta} \mathrm{e}^{-\rho t} u[c(t)] \,\mathrm{d}t.$$

At time $\tau\Delta$, with $\tau \in \{0, 1, \ldots, N-1\}$, the agent chooses $c_{\tau\Delta}$ given their utility over $[\tau\Delta, (\tau+1)\Delta)$

$$U(\{c(t)\}) = \int_{\tau\Delta}^{(\tau+1)\Delta} \mathrm{e}^{-\rho t} u[c_{\tau\Delta}] \,\mathrm{d}t = u[c_{\tau\Delta}] \mathrm{e}^{-\rho\tau\Delta} \frac{1 - \mathrm{e}^{-\rho\Delta}}{\rho}.$$

If $\Delta \approx 0$, then $\mathrm{e}^{-\rho\Delta} \approx 1 - \rho\Delta$ and

$$\int_{\tau\Delta}^{(\tau+1)\Delta} \mathrm{e}^{-\rho t} u[c_{\tau\Delta}]\,\mathrm{d}t \approx \mathrm{e}^{-\rho\tau\Delta} u[c_{\tau\Delta}]\Delta.$$

Capital accumulation, over $[\tau\Delta, (\tau+1)\Delta]$, follows the law of motion for k

$$k_{(\tau+1)\Delta} - k_{\tau\Delta} = \int_{\tau\Delta}^{(\tau+1)\Delta} [f(k_t) - c_t - \delta k_t]\,\mathrm{d}t.$$

If Δ is small, the production flow can be approximated by

$$\int_{\tau\Delta}^{(\tau+1)\Delta} f(k_t)\,\mathrm{d}t \approx f[k_{\tau\Delta}]\Delta.$$

Depreciation of capital is approximated by

$$\int_{\tau\Delta}^{(\tau+1)\Delta} \delta k_t\,\mathrm{d}t \approx k_{\tau\Delta}\delta\Delta.$$

Therefore, the law of motion of k can be rewritten as

$$k_{(\tau+1)\Delta} = k_{\tau\Delta} + f(k_{\tau\Delta})\Delta - c_{\tau\Delta} - \delta\Delta k_{\tau\Delta}.$$

The agent's problem can be rewritten as

$$\max_{\{c_{\tau\Delta}, k_{(\tau+1)\Delta}\}} U(\{c(t)\}) = \sum_{\tau=0}^{N-1} \mathrm{e}^{-\rho\Delta\tau} u(c_{\tau\Delta})\Delta$$

$$\text{s. t. } \; k_{(\tau+1)\Delta} = k_{\tau\Delta} + f(k_{\tau\Delta}) - c_{\tau\Delta} - \delta\Delta k_{\tau\Delta}, \quad k(0) = k_0, \text{ and } \; k(T) = k_T.$$

Example: A Two–Period Optimal Control Problem
Suppose $N = 2$ and $\Delta = 1$. The program is

$$\begin{aligned} &\max_{\{c_0, c_1\}} \{u(c_0) + \mathrm{e}^{-\rho} u(c_1)\} \\ \text{s. t. } \; & k_1 = k_0 + f(k_0) - c_0 - \delta k_0 \\ & k_2 = k_1 + f(k_1) - c_1 - \delta k_1 \end{aligned}$$

To solve, substitute c_1 and c_0 by their expressions given by the budget constraints. Now, the problem is

$$\max_{\{k1, k2\}} \{u[k_0 + f(k_0) - \delta k_0 - k_1] + \mathrm{e}^{-\rho} u[k_1 + f(k_1) - \delta k_1 - k_2]\}.$$

If k_2 is free, then $k_2 = 0$. Otherwise, k_2 is the terminal value. The first–order condition with respect to k_1 yields the Euler equation

$$u'(c_0) = [1 + f'(k_1) - \delta]\mathrm{e}^{-\rho} u'(c_1).$$

Example: A N-Period Optimal Control Problem
Let $\mu_{\tau\Delta}$ be the Lagrange multiplier associated with the law of motion k. Its economic interpretation is the shadow price of capital at time $\tau\Delta$. The Lagrangian is

$$\mathscr{L} = \sum_{\tau=0}^{N-1} \left\{ \mathrm{e}^{-\rho\Delta\tau} u(c_{\tau\Delta})\Delta + \mu_{\tau\Delta}\left[k_{\tau\Delta} + f(k_{\tau\Delta})\Delta - c_{\tau\Delta}\Delta - \delta\Delta k_{\tau\Delta} - k_{(\tau+1)\Delta}\right] \right\}.$$

Let $\lambda_{\tau\Delta} \equiv \mathrm{e}^{\rho\Delta\tau}\mu_{\tau\Delta}$ denote the current–value multiplier; the Lagrangian can be rewritten

$$\mathscr{L} = \sum_{\tau=0}^{N-1} \mathrm{e}^{-\rho\Delta\tau}\Big\{u(c_{\tau\Delta})\Delta + \lambda_{\tau\Delta}\big[k_{\tau\Delta} + f(k_{\tau\Delta})\Delta - c_{\tau\Delta}\Delta - \delta\Delta k_{\tau\Delta} - k_{(\tau+1)\Delta}\big]\Big\}.$$

The first–order conditions with respect to $c_{\tau\Delta}$ are

$$u'(c_{\tau\Delta}) = \lambda_{\tau\Delta} \quad \text{for all} \quad \tau = 0, \ldots, N-1.$$

The first–order conditions with respect to $k_{\tau\Delta}$ are

$$\lambda_{(\tau-1)\Delta}\mathrm{e}^{\rho\Delta} = \lambda_{\tau\Delta} + \lambda_{\tau\Delta}[f'(k_{\tau\Delta})\Delta - \delta\Delta] \quad \text{for all} \quad \tau = 1, \ldots, N.$$

Definition: The Hamiltonian function is defined as

$$H(c_{\tau\Delta}, k_{\tau\Delta}, \lambda_{\tau\Delta}) = u(c_{\tau\Delta}) + \lambda_{\tau\Delta}[f(k_{\tau\Delta}) - c_{\tau\Delta} - \delta k_{\tau\Delta}]$$

Definition: The multiplier of the Hamiltonian function is called a costate variable.

The first–order conditions with respect to $c_{\tau\Delta}$ are

$$\frac{\partial H}{\partial c_{\tau\Delta}} = 0 \quad \text{for all} \quad \tau = 0, \ldots, N-1.$$

The shadow price of capital obeys

$$\lambda_{(\tau-1)\Delta} = \mathrm{e}^{-\rho\Delta}\left(\frac{\partial H}{\partial k_{\tau\Delta}}\Delta + \lambda_{\tau\Delta}\right),$$

and the equation for the costate variable can be rewritten as

$$\left(\frac{\mathrm{e}^{\rho\Delta} - 1}{\Delta}\right)\lambda_{(\tau-1)\Delta} = \frac{\partial H}{\partial k_{\tau\Delta}} + \left(\frac{\lambda_{\tau\Delta} - \lambda_{(\tau-1)\Delta}}{\Delta}\right).$$

The solution is a system of two first–order difference equations. From the first first–order condition, you can express c as a function of λ. Therefore, there are two unknowns; the state variable k and the costate variable λ. An initial condition, $k(0) = k_0$, and a terminal condition, $k(T) = k_T$, are needed to pin down the trajectory of the system.

1.6.3. Continuous Time Optimal Control

For continuous time, take the limit as N goes to infinity (i.e. $\Delta \to 0$). Let Δ go to 0 while τ goes to infinity such as to maintain $\tau\Delta$ equal to t. The conditions

$$\frac{\partial H(c(t), k(t), \lambda(t))}{\partial c(t)} = 0 \quad \text{for all} \quad t \in [0, T]$$

and $k(T) = k_t$ remain unchanged. The difference equation for λ is a differential equation

$$\lim_{\Delta\to 0} \frac{\mathrm{e}^{\rho\Delta} - 1}{\Delta} = \rho$$

$$\lim_{\Delta\to 0} \frac{\lambda(t) - \lambda(t-\Delta)}{\Delta} = \lambda'(t).$$

The equation for the costate variable becomes

$$\rho\lambda = \frac{\partial H}{\partial k} + \frac{\partial \lambda}{\partial t}.$$

The candidates for an optimum satisfy the two first–order differential equations above.

It is assumed that the end–point is chosen freely by the agent. The boundary condition must hold

$$\lambda(T)k(T) = 0.$$

- If $k(T) > 0$, then the price of the state variable must be zero, $\lambda(T) = 0$, that is, the capital stock should be used until its marginal contribution is 0 at T.
- If $\lambda(T) > 0$, then $k(T) = 0$, that is, if capital is valuable, then the agent will die with no capital stock.

1.6.4. The Maximum Principle

Theorem: The Maximum Principle
The Maximum Principle states that an optimal solution to

$$\max_{\{c(t),k(t)\}} U(\{c(t)\}) = \int_0^T \mathrm{e}^{-\rho t} u[c(t)]\,\mathrm{d}t$$

$$\text{s. t.} \quad \dot{k} = f(k) - c - \delta k, \quad k(0) = k_0, \text{ and } \quad k(T) = k_T.$$

is a triplet $\{c(t), k(t), \lambda(t)\}$ that must satisfy the following conditions.

- There is optimal control, $c(t)$ maximizes $H\big(c(t), k(t), \lambda(t)\big)$ for any $t \in [0, T]$.
- The costate variable obeys the differential equation

$$\rho\lambda = \frac{\partial H}{\partial k} + \frac{\partial \lambda}{\partial t}.$$

- The terminal condition, $k(T) = k_T$ or $\lambda(T)k(T) = 0$ if the end–point is free, holds.

1.6.5. Sufficient Conditions for Optimal Control

The Maximum Principle only give the necessary conditions for an optimum. The first–order conditions may yield a local maximum, a minimum, or neither (i.e. a saddle–point). Some additional requirements are needed to isolate the trajectory(ies) that maximizes the welfare criterion.

Assumption: Assume that the Hamiltonian, $H(c, k, \lambda^*)$, is jointly concave in (c, k) where λ^* is generated by the Maximum Principle. The necessary conditions given by the Maximum Principle are sufficient conditions for a maximum. For the problem above, the Hamiltonian is

$$H(c, k, \lambda^*) = u(c) + \lambda^*[f(k) - c - \delta k],$$

where $\lambda^* \geq 0$. Thus, both the utility function and the production function must be strictly concave.

Proof. Consider a candidate solution $\{c^*(t), k^*(t), \lambda^*(t)\}$ that satisfies the first–order conditions. Now consider another admissible path $\{c(t), k(t)\}$. Denote $k^*(t)$ the trajectory under $c^*(t)$ and $k(t)$ if $c(t)$. Let Δ be defined as

$$\Delta = \int_0^T \mathrm{e}^{-\rho t} u[c^*(t)]\,\mathrm{d}t - \int_o^T \mathrm{e}^{-\rho t} u[c(t)]\,\mathrm{d}t.$$

Note that

$$u(c) = H(c, k, \lambda^*) - \lambda^* \dot{k}.$$

Substitute $u(c)$ by its expression given by H to get

$$\Delta = \int_0^T e^{-\rho t}[H(c^*, k^*, \lambda^*) - H(c, k, \lambda^*)]\, dt - \int_0^T e^{-\rho t} \lambda^* [\dot{k}^* - \dot{k}]\, dt.$$

Use the concavity of the Hamiltonian function in (c, k) to find

$$H(c, k, \lambda^*) \leq H(c^*, k^*, \lambda^*) + H_c(c^*, k^*, \lambda^*)(c - c^*) + H_k(c^*, k^*, \lambda^*)(k - k^*),$$

where H_c and H_k are the partial derivatives of the Hamiltonian with respect to c and k respectively. From the first order conditions

$$H_c(c^*, k^*, \lambda^*) = 0$$
$$H_k(c^*, k^*, \lambda^*) = \rho\lambda^* - \dot{\lambda}^*$$

you can obtain the following inequality

$$\Delta \geq \int_0^T e^{-\rho t}[\rho\lambda^* - \dot{\lambda}^*](k^* - k)\, dt - \int_0^T e^{-\rho t}\lambda^*[\dot{k}^* - \dot{k}]\, dt$$
$$\Delta \geq \int_0^T \frac{d[-e^{-\rho t}\lambda^*(k^* - k)]}{dt}\, dt$$
$$\Delta \geq \lambda^*(0)[k^*(0) - k(0)] - e^{-\rho T}\lambda^*(T)[k^*(T) - k(T)].$$

Note that $k^*(0) = k(0) = k_0$ and $k^*(T) = k(T) = k_t$. Consequently,

$$\Delta \geq 0.$$

This inequality is strict if the Hamiltonian is strictly concave in (c, k). □

Theorem: The Mangasarian Sufficiency Theorem
If $(c^*(t), k^*(t))$ is a solution of the conditions provided by the Maximum Principle and if $H(c, k, \lambda^*)$ is concave in (c, k) with the costate variable, λ^*, supplied by the maximum principle, then $(c^*(t), k^*(t))$ solves the optimal control problem. If $H(c, k, \lambda^*)$ is strictly concave in (c, k), then $(c^*(t), k^*(t))$ is the unique solution to the problem.

1.6.6. Economic Interpretation of Optimal Control

The Hamiltonian of an agent is

$$H(c, k, \lambda) = u(c) + \lambda\dot{k}.$$

If the agent decides to modify her control variable, c, there are two consequences.

- First, the choice modifies the current utility of the agent.
- Second, the choice will affect the state variable, k, in future periods.

The question is, how to value this effect on $\dot{k}$? You can use a shadow price, λ, analogous to the Lagrange multiplier. The consequences of the current choice for the future are summarized by $\lambda\dot{k}$. The first first–order condition for the Hamiltonian is

$$\frac{\partial H(c, k, \lambda^*)}{\partial c} = 0$$

and as in a static problem, the agent chooses the control to maximize the objective. The second first–order condition for the Hamiltonian is

$$\rho\lambda = \frac{\partial H}{\partial k} + \frac{\partial \lambda}{\partial t}$$

and is like an asset–pricing equation. The left–hand side, $\rho\lambda$, can be interpreted as an opportunity cost. The first term on the right–hand side, $\frac{\partial H}{\partial k}$, is the dividend of the asset. The last term on the right–hand side, $\frac{\partial \lambda}{\partial t}$ is the capital gain or loss.

1.6.7. Costate Variables

The utility of the agent when the optimal control has been chosen is

$$U^* = \int_0^T u(c^*)\mathrm{e}^{-\rho t}\,\mathrm{d}t.$$

For any function, λ, you have

$$\lambda[f(k^*) - c^* - \delta k^*] = \lambda\dot{k}^*.$$

Consequently,

$$U^* = \int_0^T \mathrm{e}^{-\rho t}\{u(c^*) + \lambda[f(k^*) - c^* - \delta k^*] - \lambda\dot{k}^*\}\,\mathrm{d}t.$$

Integration by parts yields

$$\int_0^T \mathrm{e}^{-\rho t}\lambda k^*\,\mathrm{d}t = [\mathrm{e}^{-\rho t}\lambda k^*]_0^T - \int_0^T \mathrm{e}^{-\rho t}(-\rho\lambda + \dot{\lambda})k^*\,\mathrm{d}t$$
$$\int_0^T \mathrm{e}^{-\rho t}\lambda k^*\,\mathrm{d}t = \mathrm{e}^{-\rho T}\lambda(T)k^*(T) - \lambda(0)k^*(0) - \int_0^T \mathrm{e}^{-\rho t}(-\rho\lambda + \dot{\lambda})k^*\,\mathrm{d}t.$$

Substitute into the equation for U^* to obtain

$$U^* = \int_0^T \mathrm{e}^{-\rho t}\{H(c^*, k^*, \lambda) + (-\rho\lambda + \dot{\lambda})k^*\}\,\mathrm{d}t + \lambda(0)k^*(0) - \mathrm{e}^{-\rho T}\lambda(T)k^*(T)$$

Differentiate with respect to $k(0)$ to obtain

$$\frac{\partial U^*}{\partial k_0} = \int_0^T \mathrm{e}^{-\rho t}\left\{H_c(c^*, k^*, \lambda)\frac{\partial c^*}{\partial k_0} + [H_k(c^*, k^*, \lambda) - \rho\lambda + \dot{\lambda}]\frac{\partial k^*}{\partial k_0}\right\}\mathrm{d}t + \lambda(0) - \mathrm{e}^{-\rho T}\lambda(T)\frac{\partial k^*(T)}{\partial k_0}.$$

Note that the shadow price, λ, is an arbitrary function of time. A change in the initial condition, $k(0)$, does not affect the costate variable, λ, or its derivative. Furthermore, $\frac{\partial k^*(T)}{\partial k_0} = 0$, because the terminal point is exogenously specified. Suppose that you select the optimal path $\lambda^*(t)$ from the Maximum Principle. Then

$$\frac{\partial U^*}{\partial k_0} - \lambda^*(0).$$

Note that $\lambda^*(0)$ measures the impact of a change in the initial capital stock on the utility of the agent. The effect of a change in k_T on the agent's utility is

$$\frac{\partial U^*}{\partial k_T} = -\mathrm{e}^{-\rho T}\lambda^*(T).$$

1.6.8. Infinite Time Horizon Optimal Control

When the time horizon is infinite, the method is similar. The Hamiltonian is

$$H(c, k, \lambda) = u(c) + \lambda[f(k) - c - \delta k].$$

The optimal control satisfies

$$\frac{\partial H}{\partial c} = 0.$$

The dynamic equation for the costate variable satisfies

$$\rho\lambda = \frac{\partial H}{\partial k} + \dot{\lambda}.$$

There is an initial condition $k(0) = k_0$. However, there is no terminal value for the capital stock. Instead, there is a transversality condition.

Definition: The Transversality Condition
Assume that $H(c, k, \lambda)$ is jointly concave in c and k. Let c^* and k^* be the optimal paths for consumption and the capital stock, and let λ^* be the associated costate variable. Consider another path, c and k, that satisfies

$$\begin{aligned} \dot{k} &= f(k) - c - \delta k \\ k(0) &= k_0. \end{aligned}$$

The path (c^*, k^*) is an optimum if and only if

$$\Delta = \int_0^{+\infty} \mathrm{e}^{-\rho t} u[c^*(t)\,\mathrm{d}t - \int_0^{+\infty} \mathrm{e}^{-\rho t} u[c(t)]\,\mathrm{d}t \geq 0.$$

Note that

$$\begin{aligned} u(c^*) &= H(c^*, k^*, \lambda^*) - \lambda^* \frac{\partial k^*}{\partial t} \\ u(c) &= H(c, k, \lambda^*) - \lambda^* \frac{\partial k}{\partial t} \end{aligned}$$

From the concavity of the Hamiltonian, you have

$$H(c^*, k^*, \lambda^*) - H(c, k, \lambda^*) \geq H_k(c^*, k^*, \lambda^*)(k^* - k) + H_c(c^*, k^*, \lambda^*)(c^* - c).$$

From the first–order conditions

$$\begin{aligned} H_c(c^*, k^*, \lambda^*)(c^* - c) &= 0 \\ H_k(c^*, k^*, \lambda^*)(k^* - k) &= (\rho\lambda^* - \dot{\lambda}^*)(k^* - k) \end{aligned}$$

It follows that

$$\begin{aligned} \int_0^{+\infty} \mathrm{e}^{-\rho t}[u[c^*(t)] - u[c(t)]]\,\mathrm{d}t &\geq \int_0^{+\infty} \mathrm{e}^{-\rho t}(\rho\lambda^* - \dot{\lambda}^*)(k^* - k)\,\mathrm{d}t - \int_0^{+\infty} \mathrm{e}^{-\rho t}\lambda^*\left(\frac{\partial k^*}{\partial t} - \frac{\partial k}{\partial t}\right)\mathrm{d}t \\ &\geq \int_o^{+\infty} -\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{-\rho t}\lambda^*(k^* - k)]\,\mathrm{d}t \\ &\geq \lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda^*(k - k^*). \end{aligned}$$

A sufficient condition for (c^*, k^*) to be a maximum is that

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda^*(k - k^*) \geq 0 \text{ for all } k.$$

In this problem, the shadow price of capital will always be positive and the capital stock cannot be negative. Consequently, a sufficient condition for an optimum is that

$$\boxed{\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda^* k^* = 0}$$

Theorem: The Mangasarian Sufficiency Theorem
Let (c^*, k^*, λ^*) be a triplet generated by the Maximum Principle. If $H(c, k, \lambda^*)$ is jointly concave in k and c, and if $lim_{t\to\infty}\mathrm{e}^{-\rho t}\lambda^*(k - k^*) \geq 0$ for all possible paths, $k(t)$, then (c^*, k^*) is optimal.

1.6.9. Generalization of the Optimal Control Problem

For a more general problem, $x(t)$ is the state variable and $y(t)$ is the control variable. The optimal control problem is

$$\max_{\{x(t),y(t)\}} \int_0^\infty u[t, x(t), y(t)]\,\mathrm{d}t$$

subject to

$$\dot{x}(t) = g[t, x(t), y(t)],$$

an initial condition, $x(t) = x_0$, and $\lim_{t\to\infty} b(t)x(t) \geq x_1$ with $\lim_{t\to\infty} b(t) < \infty$.

In the previous example, $x(t) = k(t)$, $y(t) = c(t)$, $u(t, x, y) = \mathrm{e}^{-\rho t}u(c)$, and $g(t, x, y) = f(x) - \delta x - y$. However, with an infinite horizon, then $\lim_{t\to\infty} b(t)x(t) \geq x_1$ is a terminal value constraint. In many applications, $b(t) = 1$.

Definition: Value Function
The optimal value of the dynamic maximization problem starting at time t_0 with state variable $x(t_0)$ is given by the value function

$$V[t_0, x(t_0)] = \max_{\{x(t),y(t)\}} \int_{t_0}^\infty u[t, x(t), y(t)]\,\mathrm{d}t$$

$$\text{s. t.} \quad \dot{x}(t) = g[t, x(t), y(t)], \quad x(t) = x_0, \quad \text{and} \quad \lim_{t\to\infty} b(t)x(t) \geq x_1.$$

Theorem: The Principle of Optimality
Suppose that $(x^*(t), y^*(t))$ is a solution to the dynamic optimization problem. Then

$$V[t_0, x(t_0)] = \int_{t_0}^{t_1} u[t, x^*(t), y^*(t)]\,\mathrm{d}t + V[t_1, x^*(t1)],$$

for all $t_1 \geq t_0$.

Proof. Assuming that $(x^*(t), y^*(t))$ is a solution to the optimization problem implies

$$V[t_0, x(t_0)] = \int_{t_0}^{\infty} u[t, x^*(t), y^*(t)]\,\mathrm{d}t$$

$$V[t_0, x(t_0)] = \int_{t_0}^{t_1} u[t, x^*(t), y^*(t)]\,\mathrm{d}t + \int_{t_1}^{\infty} u[t, x^*(t), y^*(t)]\,\mathrm{d}t.$$

By definition of the value function

$$V[t_1, x^*(t_1)] \geq \int_{t_1}^{\infty} u[t, x^*(t), y^*(t)]\,\mathrm{d}t.$$

Thus, it is clear that the inequality cannot be strict, otherwise there would be a profitable deviation after t_1. □

Theorem: The Infinite–Horizon Maximum Principle
Suppose that the dynamic maximization problem has a solution $(x^*(t), y^*(t))$. Define the present value Hamiltonian as

$$H(t, x, y, \lambda) = u(t, x, y) + \lambda g(t, x, y).$$

Then

$$y^*(t) \in \arg\max_{\{y\}} H[t, x^*(t), y, \lambda(t)] \quad \text{for all } t.$$

$$\dot{\lambda}(t) = -H_x[t, x^*(t), y^*(t), \lambda(t)].$$

Example: Optimal Growth
The Hamiltonian is

$$H(t, k, c, \lambda) = \mathrm{e}^{-\rho t} u(c) + \lambda[f(k) - \delta k - c].$$

The maximization of H with respect to c yields

$$u'(c(t)) = \mathrm{e}^{\rho t} \lambda(t).$$

The differential equation for the costate variable is

$$\dot{\lambda}(t) = -\lambda(t)[f'(k) - \delta].$$

Denote $\mu(t) \equiv \mathrm{e}^{\rho t} \lambda(t)$. Then,

$$\rho \mu(t) = \mu(t)[f' - \delta] + \dot{\mu}(t).$$

1.7. The Ramsey Model of Optimal Growth

This section will introduce you to a growth model from Ramsey 1928. The question is, how much should a nation save? A framework for studying the optimal intertemporal allocation of resources is introduced. The model begins with microfoundations, wherein the optimizing behavior of agents is explicit, and the result is that the saving rate is endogenous.

1.7.1. The Ramsey Model

Assumptions:

- There is a household composed of L agents who have to decide how much to consume and how much to save (invest).
- Output is produced according to a Neoclassical production function

$$Y = F(K, L).$$

- There is initial capital, k_0.
- Capital depreciates at rate δ.

For simplicity, you can normalize L to 1, and work in per capita terms. Let

$$f(k) \equiv F(k, 1).$$

The function $f(\cdot)$ is strictly concave and satisfies the Inada conditions;

$$f(0) = 0, \quad f'(0) = \infty, \quad f'(\infty) = 0.$$

The household's utility function is

$$U = \int_0^{+\infty} \mathrm{e}^{-\rho t} u\big(c(t)\big)\,\mathrm{d}t,$$

where ρ is the rate of time preference. The instantaneous utility function is

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \quad \text{where } \theta > 0$$

and

$$u(c) = \ln c \quad \text{if } \theta = 1.$$

Definition: The Coefficient of Relative Risk Aversion

$$\text{RRA} \equiv \frac{-cu''(c)}{u'(c)} = \theta.$$

Definition: The Intertemporal Elasticity of Substitution

$$\eta = \frac{1}{\theta}.$$

Proof. The utility can be rewritten

$$U = u[c(t)] + \mathrm{e}^{-\rho(s-t)} u[c(s)].$$

Thus the marginal rate of substitution is

$$-\left.\frac{\mathrm{d}c(t)}{\mathrm{d}c(s)}\right|_{U=cste} = \frac{u'[c(s)]\mathrm{e}^{-\rho(s-t)}}{u'[c(t)]}$$

$$\text{MRS} = \left(\frac{c(t)}{c(s)}\right)^{\theta} \mathrm{e}^{-\rho(s-t)}.$$

and the intertemporal elasticity of substitution is

$$\eta_{(c(t)/c(s))/\text{MRS}} = \frac{\partial \ln\left(\frac{c(t)}{c(s)}\right)}{\partial \ln(\text{MRS})} = \frac{1}{\theta}.$$

□

The program of the household is

$$\max_{\{c(t),k(t)\}} U = \int_0^{+\infty} \mathrm{e}^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta}\, \mathrm{d}t$$
$$\text{s. t.} \quad c + \dot{k} = f(k) - \delta k$$
$$k(0) = k_0.$$

The current–value Hamiltonian is a technique to transform a dynamic problem into a static one, where k is a state variable and λ is a costate variable (the shadow price of capital). The Hamiltonian is the instantaneous utility plus the change in capital stock valued according to λ

$$H(c, k, \lambda) = \frac{c^{1-\theta}}{1-\theta} + \lambda[f(k) - \delta k - c].$$

The first–order conditions are found from the Maximum Principle. Maximizing the Hamiltonian with respect to the control variable yields the optimal control

$$c(t)^{-\theta} = \lambda(t).$$

The first–order condition with respect to the costate variable yields the dynamic equation

$$\rho\lambda = [f'(k) - \delta]\lambda + \dot{\lambda}.$$

From the first–order conditions

$$\frac{\dot{\lambda}}{\lambda} = \rho + \delta - f'(k),$$

$$-\theta \ln c = \ln \lambda.$$

By taking a time derivative on both sides

$$-\theta \frac{\dot{c}}{c} = \frac{\dot{\lambda}}{\lambda},$$

and solving yields the Keynes-Ramsey rule for the growth of consumption

$$\boxed{\frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho}{\theta}}$$

The Mangasarian sufficiency conditions are

- If $\lambda > 0$, then the Hamiltonian $H(c, k, \lambda)$ is strictly jointly concave in (c, k).
- The transversality condition holds

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda(t)k(t) = 0.$$

Note that this is similar to a complementary slackness condition.

Definition: Equilibrium of the Ramsey Model
An equilibrium of the Ramsey model is a pair of functions $\big(c(t), k(t)\big)$ satisfying

$$\frac{\dot{c}}{c} = \frac{f'(k) - \delta - \rho}{\theta} \tag{1}$$

$$\dot{k} = f(k) - \delta k - c \tag{2}$$

$$\lim_{s\to+\infty} \mathrm{e}^{-\rho s}\frac{k(s)}{[c(s)]^{\theta}} = 0 \tag{3}$$

$$k(0) = k_0 \tag{4}$$

1.7.2. The Steady–State of the Ramsey Model

Next, a steady–state is a pair (k, c) such that $\dot{k} = 0$ and $\dot{c} = 0$. This respectively implies

$$f'(k) = \rho + \delta$$
$$f(k) - \delta k = c$$

Notice that the steady–state capital stock decreases with δ and ρ, and the capital stock increases if the production technology, $f(k)$, becomes more efficient. Also notice that the agents' willingness to smooth consumption across time does not influence the steady–state capital stock.

Notice that steady–state investment is $f(k) - c = \delta k$. Thus, the saving rate at the steady–state is

$$s = \frac{\delta k^*}{f(k^*)}.$$

The steady–state saving rate is increasing with k^*, and has a negative relationship with ρ (e.g. a decrease in ρ raises the saving rate).

Example: The Ramsey Model with Cobb-Douglas Production
Given a Cobb-Douglas production function

$$f(k) = Ak^{\alpha},$$

the steady–state, where $\dot{k} = 0$, occurs where

$$A\alpha k^{\alpha-1} = \rho + \delta.$$

Thus, steady–state capital stock is

$$k^* = \left(\frac{A\alpha}{\rho + \delta}\right)^{\frac{1}{1-\alpha}},$$

and the saving rate is

$$s = \frac{\delta k}{Ak^{\alpha}}$$

$$s = k^{1-\alpha}\frac{\delta}{A}$$

$$s = \left(\frac{A\alpha}{\rho+\delta}\right)^{\frac{1-\alpha}{1-\alpha}}\frac{\delta}{A}$$

$$s = \alpha\frac{\delta}{\rho+\delta}.$$

The capital stock that maximizes steady–state consumption is the golden–rule capital stock, k_{GR}, and satisfies

$$f'(k_{\mathrm{GR}}) = \delta.$$

The capital stock in equilibrium satisfies $f'(k^*) = \delta + \rho$. Therefore, $k^* < k_{\mathrm{GR}}$ and the economy is dynamically efficient.

1.7.3. The Dynamics of the Ramsey Model

Next, to study the properties of the dynamic system, linearize the transition equations in the neighborhood of the steady state

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \frac{f''(k^*)c^*}{\theta} \\ -1 & \rho \end{pmatrix}}_{J} \begin{pmatrix} c - c^* \\ k - k^* \end{pmatrix}.$$

Notice that

$$\det J = \frac{f''(k^*)c^*}{\theta} < 0.$$

This implies that the steady–state is a saddle–point. The transversality condition is satisfied on the saddle–path. Indeed,

$$\lim_{s\to\infty} \lambda(s)k(s) = \lambda^* k^*,$$

and

$$\lim_{s\to\infty} \mathrm{e}^{-\rho s}\lambda(s)k(s) = 0.$$

From the strict concavity of the Hamiltonian and the Mangasarian sufficiency condition, the saddle path is the unique solution to the Ramsey problem.

Proof. Let $\mathbf{q}_1$ and $\mathbf{q}_2$ be the two eigenvectors and $\lambda_1 < 0$ and $\lambda_2 > 0$ be the two eigenvalues associated with the Jacobian matrix. It follows that

$$\begin{pmatrix} c - c^* \\ k - k^* \end{pmatrix} = C_1\mathbf{q}_1\mathrm{e}^{\lambda_1 t} + C_2\mathbf{q}_2\mathrm{e}^{\lambda_2 t}.$$

From the transversality condition

$$\lim_{t\to\infty} \begin{pmatrix} c - c^* \\ k - k^* \end{pmatrix} = 0,$$

which implies that $C_2 = 0$. Therefore,

$$c(t) = c^* + \mathrm{e}^{\lambda_1 t}[c(0) - c^*]$$
$$k(t) = k^* + \mathrm{e}^{\lambda_1 t}[k(0) - k^*].$$

$\square$

The speed of adjustment to the steady–state is given by $|\lambda_1|$, where

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 - 4\frac{f''(k^*)c^*}{\theta}}}{2}.$$

The higher the elasticity of substitution, the faster capital accumulates. This is because people are more willing to accept low consumption early on in their life in exchange for higher consumption later.

A fall in ρ is the closest analogue to a rise in the saving rate in the Solow model. Suppose that the change is unexpected, that is, at some date households suddenly discover that they now discount utility at a lower rate. In the phase diagram, only the c-locus is affected which will lead to an increase in k^*. Note that k is a predetermined variable and cannot change discontinuously. In contrast, c can jump to a new value at any time.

1.8. Phase Diagrams

Phase diagrams are useful when conducting a qualitative analysis of a system of two differential equations, studying systems of nonlinear differential equations, or illustrating the different types of steady–state equilibria.

1.8.1. Construction of a Phase Diagram

Definition: A phase plane consists of a horizontal axis, y_1, and a vertical axis, y_2.

Definition: An isocline for y_i is the locus of points for which $\dot{y}_i = 0$

Definition: The isocline divides the phase plane into two isosectors. One where $\dot{y}_i$ is negative and the other where $\dot{y}_i$ is positive.

The two isoclines, for y_1 and y_2, intersect where both $\dot{y}_1$ and $\dot{y}_2$ equal zero. These are the steady–state points. From the two isoclines, the four quadrants can be deduced. In each quadrant, it is customary to draw arrows of motion to indicate how the system evolves.

- $\leftarrow$ and $\downarrow$ indicates that $\dot{y}_1 < 0$ and $\dot{y}_2 < 0$.
- $\leftarrow$ and $\uparrow$ indicates that $\dot{y}_1 < 0$ and $\dot{y}_2 > 0$.
- $\leftarrow$ and $\downarrow$ indicates that $\dot{y}_1 > 0$ and $\dot{y}_2 < 0$.
- $\rightarrow$ and $\uparrow$ indicates that $\dot{y}_1 > 0$ and $\dot{y}_2 > 0$.,

1.8.2. Vector Fields

The system of differential equations is interpreted as the equations of motion of a particle in the plane, with velocity vector $(\dot{y}_1, \dot{y}_2)$.

Definition: A vector field is a family of vectors where, for each point in the phase plane, you draw the velocity vector, $(\dot{y}_1, \dot{y}_2)$, with its tail at the point and pointing in the direction of the particle's motion.

Example: Phase Diagram with a Stable Node
Consider the following differential equation system

$$\dot{y}_1 = -2y_1 + 2$$
$$\dot{y}_2 = -3y_2 + 6$$

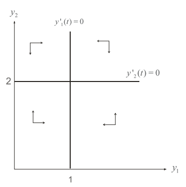
Written as $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{b}$, then

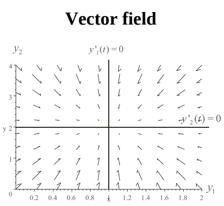
$$\mathbf{A} = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}.$$

The tr$\mathbf{A} = -5$, $\det \mathbf{A} = 6$, so $(\text{tr}\mathbf{A})^2 - 4\det\mathbf{A} = 25 - 24 = 1$. Thus, there is a stable node. The isoclines for y_1 and y_2 are as follows.

- If $\dot{y}_1 = 0$, then $y_1 = 2$.
- If $\dot{y}_1 > 0$, then $y_1 < 1$.
- If $\dot{y}_1 < 0$, then $y_1 > 1$.

- If $\dot{y}_2 = 0$, then $y_2 = 2$.
- If $\dot{y}_2 > 0$, then $y_2 < 2$.
- If $\dot{y}_2 < 0$, then $y_2 > 2$.





Example: Phase diagram with an Unstable Node
Consider the following system of differntial equations

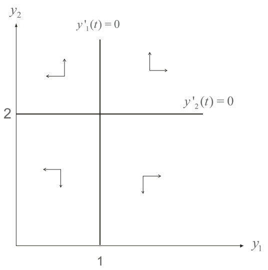
$$\dot{y}_1 = 2y_1 - 2$$
$$\dot{y}_2 = 3y_2 - 6$$

Written as $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{b}$, then

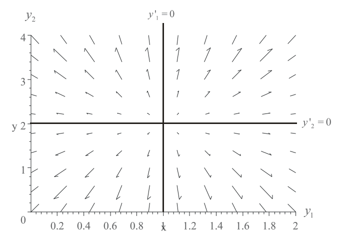
$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

The $\text{tr}\mathbf{A} = 5$, $\det \mathbf{A} = 6$, so $(\text{tr}\mathbf{A})^2 - 4\det \mathbf{A} = 25 - 24 = 1$. Thus, there is an unstable node. The isoclines for y_1 and y_2 are as follows.

- If $\dot{y}_1 = 0$, then $y_1 = 1$.
- If $\dot{y}_1 > 0$, then $y_1 > 1$.
- If $\dot{y}_1 < 0$, then $y_1 < 1$.
- If $\dot{y}_2 = 0$, then $y_2 = 2$.
- If $\dot{y}_2 > 0$, then $y_2 > 2$.
- If $\dot{y}_2 < 0$, then $y_2 < 2$.



Vector field



Example: Phase Diagram with a Saddle Point
Consider the following differential equation system

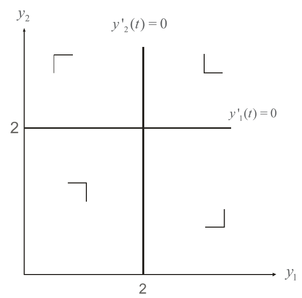
$$\dot{y}_1 = y_2 - 2$$
$$\dot{y}_2 = \frac{y_1}{4} - \frac{1}{2}$$

Written as $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{b}$, then

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{pmatrix}.$$

The $\text{tr}\mathbf{A} = 0$, $\det \mathbf{A} = -\frac{1}{4}$, so $(\text{tr}\mathbf{A})^2 - 4\det \mathbf{A} = 0 - 4(-\frac{1}{4}) = 1$. Thus, there is a saddle point. The isoclines for y_1 and y_2 are as follows.

- If $\dot{y}_1 = 0$, then $y_2 = 2$.
- If $\dot{y}_1 > 0$, then $y_2 > 2$.
- If $\dot{y}_1 < 0$, then $y_2 < 2$.
- If $\dot{y}_2 = 0$, then $y_1 = 2$.
- If $\dot{y}_2 > 0$, then $y_1 > 2$.
- If $\dot{y}_2 < 0$, then $y_1 < 2$.



1.8.3. Finding a Saddle Path

In order to find a saddle path, first diagonalize the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}.$$

Then utilize the change of variables technique

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The system becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ -\frac{1}{2} \end{pmatrix}.$$

Notice that x_1 is unstable whereas x_2 is stable. The saddle path is such that x_1 is equal to its steady–state value so that it does not diverge

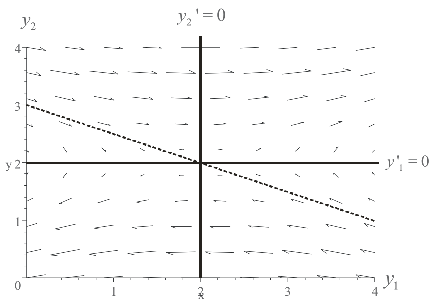
$$\dot{x}_1 = \frac{1}{2}x_1 - 3 = 0$$

$$x_1 = \overline{x}_1 = 6.$$

Using the fact that $x_1 = y_1 + 2y_2$, the equation of the saddle path is

$$y_1 + 2y_2 = 6.$$

Vector field



Example: A Phase Diagram with a Stable Focus
Consider the following differential equation system

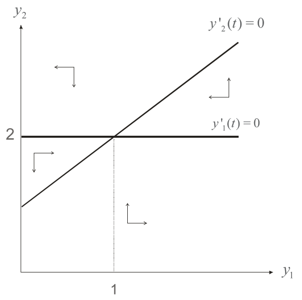
$$\dot{y}_1 = -y_2 + 2$$
$$\dot{y}_2 = y_1 - y_2 + 1$$

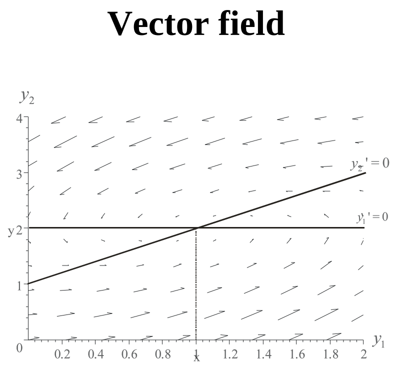
Written as $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{b}$, then

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

The tr$\mathbf{A} = -1$, $\det \mathbf{A} = 1$, so $(\text{tr}\mathbf{A})^2 - 4\det \mathbf{A} = 2 - 4(1) = -2 < 0$. Thus, there is a stable focus. The isoclines for y_1 and y_2 are as follows.

- If $\dot{y}_1 = 0$, then $y_2 = 2$.
- If $\dot{y}_1 > 0$, then $y_2 < 2$.
- If $\dot{y}_1 < 0$, then $y_2 > 2$.
- If $\dot{y}_2 = 0$, then $y_1 - y_2 = -1$.
- If $\dot{y}_2 > 0$, then $y_1 - y_2 > -1$.
- If $\dot{y}_2 < 0$, then $y_1 - y_2 < -1$.





There is a steady–state, where $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$, such that $y_1 = 1$ and $y_2 = 2$.

Example: The Dornbusch Model of Exchange–Rate Overshooting

A dynamic version of the Mundell–Fleming model helps analyze how exchange rates respond to a change in the money supply in an economy where the goods market does not clear instantaneously. The real demand for money is

$$m^D = -ar + b\overline{y}.$$

The equilibrium of the money market is

$$m - p = -ar + b\overline{y}.$$

There is perfect–foresight expectations of the depreciation of the national currency, $\dot{e}$, resulting in the interest rates parity

$$r = r^* + \dot{e},$$

where e is the exchange rate defined as the domestic price of foreign currency. From the two last equations

$$\dot{e} = -r + \frac{b\overline{y} - m + p}{a}.$$

There is sluggish adjustment of prices

$$\overline{p} = \alpha(y^D - \overline{y}),$$

where $\alpha > 0$ and aggregate demand is given by

$$y^D = u + v(e - p).$$

The differential equation for prices is then

$$\dot{p} = \alpha(u + ve - vp - \overline{y}).$$

The system of differential equations can be written in matrix form

$$\begin{pmatrix} \dot{p} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} -\alpha v & \alpha v \\ \frac{1}{\alpha} & 0 \end{pmatrix} \begin{pmatrix} p \\ e \end{pmatrix} + \begin{pmatrix} \alpha(u - \overline{y}) \\ \frac{b\overline{y} - m}{a} - r^* \end{pmatrix}.$$

Denote $\mathbf{A}$ the matrix of coefficients. Then

$$\det \mathbf{A} = -\frac{\alpha v}{a} < 0,$$

and all the roots are real valued and of opposite sign. Thus, the steady–state is a saddle–point. At the steady–state

$$\begin{pmatrix} \dot{p} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\alpha v & \alpha v \\ \frac{1}{\alpha} & 0 \end{pmatrix} \begin{pmatrix} p \\ e \end{pmatrix} = \begin{pmatrix} \alpha(u-\overline{y}) \\ \frac{b\overline{y}-m}{a} - r^* \end{pmatrix}$$

$$\begin{pmatrix} p \\ e \end{pmatrix} = \frac{a}{\alpha v} \begin{pmatrix} 0 & -\alpha v \\ -\frac{1}{a} & -\alpha v \end{pmatrix} \begin{pmatrix} \alpha(u-\overline{y}) \\ \frac{b\overline{y}-m}{a} - r^* \end{pmatrix}$$

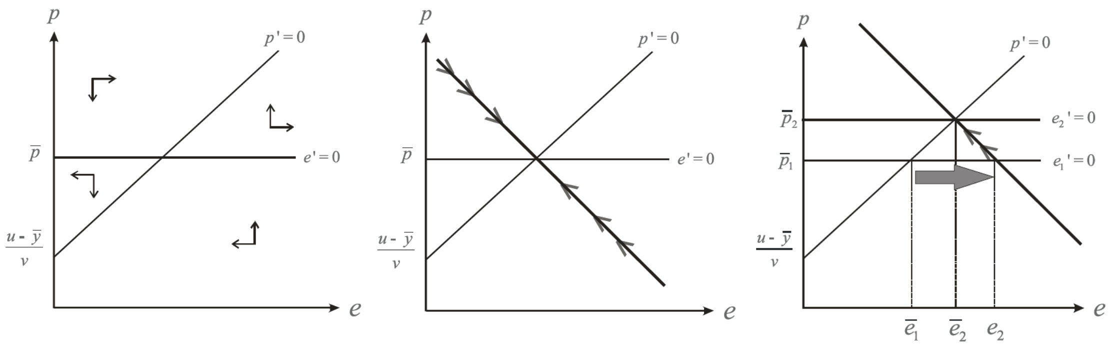
$$\begin{pmatrix} p \\ e \end{pmatrix} = \begin{pmatrix} -b\overline{y} + m + ar^* \\ -\frac{u-\overline{y}}{v} - b\overline{y} + m + ar^* \end{pmatrix}.$$

The p isocline in the phase diagram is

$$\dot{p} = 0$$
$$p = \frac{u-\overline{y}}{v} + e.$$

The e isocline in the phase diagram is

$$\dot{e} = 0$$
$$p = ar^* - b\overline{y} + m = \overline{p}.$$



The domestic price, p, changes sluggishly, because of the initial condition, p_0. The nominal exchange rate, e, can adjust instantly, because there is no initial value for e. To determine the trajectory of the economy, a condition imposed by the assumption of perfect foresight is that agents only anticipate trajectories that converge to the steady state

$$\lim_{t\to\infty} e(t) = \overline{e}.$$

Assume there is an initial point, steady–state $(\overline{p}_1, \overline{e}_1)$, and an increase in the money supply, m. Then, the p isocline is not affected, the e isocline moves upward, and in the long run, both p and e increase, while the real exchange rate rate remains unchanged. The increase in m triggers a jump of the exchange rate

$$\overline{e}_1 \to e_2.$$

Following the jump, the nominal exchange rate is larger than its new steady-state value, initially overshooting the new steady-state exchange rate

$$e_2 > \overline{e}_2.$$

1.9. The Neoclassical Growth Model in Continuous Time

The Ramsey problem is that an agent wish to maximize her lifetime utility subject to a technological constraint. This is equivalent to the program of a social planner. In the decentralized economy, assume that there are households who consume and supply labor services, and firms who rent capital and labor services and produce output. The objective is to discover if the decentralized equilibrium efficient.

1.9.1. Ramsey's Neoclassical Growth Model

First, time is continuous and infinite. there are a large number of identical firms with CRS production function

$$Y = F(K, L).$$

The firms hire workers and rent capital. The factor markets and output markets are competitive. The firms are owned by the households, where there are a large number, H, of identical households. Each household supplies 1 unit of labor at every point in time and rents its capital to the firms, where its initial capital is

$$\frac{K(0)}{H},$$

and capital depreciates at rate δ. There is a debt market in which households can borrow and lend. Loans and capital pay the same real rate of return, $r(t)$, and are thus perfect substitutes. The rental rate of capital is

$$p_k(t) = r(t) + \delta.$$

Both households and firms are price–takers and have perfect foresight expectations. This implies that current and future values of $r(t)$ and $w(t)$ are known

$$r(t) = p_k K, w(t) = wL.$$

The firm's behavior is equivalent to the static program

$$\max_{K,L} F(K, L) - p_k K - wL,$$

or equivalently in intensive form

$$\max_{k} k(k) - p_k k - w.$$

The first order conditions imply that

$$\begin{aligned} p_k &= f'(k), \\ w &= f(k) - p_k k. \end{aligned}$$

Note that

$$w = f(k) - f'(k)k.$$

Next, the household's utility function

$$U = \int_0^{+\infty} \mathrm{e}^{-\rho t} u(c(t))\, \mathrm{d}t,$$

takes the form of an instantaneous utility function

$$u(C) = \begin{cases} \frac{C^{1-\theta}}{1-\theta} & \text{if } \theta > 0, \\ \ln C & \text{if } \theta = 1, \end{cases},$$

where $\frac{1}{\theta}$ is the intertemporal elasticity of substitution. Let $a(t)$ denote the net value of the household's assets. All assets guarantee the same rate of return, r. The law of motion for the household's assets is

$$c + \dot{a} = w + ra$$

By the method of the integrating factor

$$\lim_{s \to +\infty} \mathrm{e}^{-R(s)} a(s) = a(0) + \int_0^{+\infty} \mathrm{e}^{-R(t)} [w(t) - c(t)]\, \mathrm{d}t,$$

where

$$R(t) = \int_0^t r(\tau)\, \mathrm{d}\tau.$$

There is a no–Ponzi game condition that is imposed

$$\lim_{s \to +\infty} \mathrm{e}^{-R(s)} a(s) \geq 0.$$

The present value of the household's asset holdings cannot be negative in the limit. So, someone cannot issue debt and roll it over forever

$$a(0) + \int_o^{+\infty} \mathrm{e}^{-R(t)} [w(t) - c(t)]\, \mathrm{d}t \geq 0.$$

The program of the household is then

$$\begin{aligned} \max_{\{c\}} U = \int_0^{+\infty} \mathrm{e}^{-\rho t} \frac{C^{1-\theta}}{1-\theta}\, \mathrm{d}t \\ \text{s. t. } c + \dot{a} = w + ra \\ \lim_{t \to +\infty} \mathrm{e}^{-R(t)} a(t) \geq 0 \\ a(0) = a_0. \end{aligned}$$

The current-value Hamiltonian is

The first–order condition corresponding to the optimal control is

$$c(t)^{-\theta} = \lambda,$$

where the costate variable, λ, is the shadow price of capital. The equation for the costate variable is

$$\rho\lambda = \lambda r + \dot{\lambda}.$$

If you take a time derivative of the first–order condition,

$$-\theta \dot{c} c^{-\theta-1} = \dot{\lambda},$$

and rearrange the costate equation, so that

$$\frac{\dot{\lambda}}{\lambda} = \rho - r.$$

Thus, optimal consumption growth is given by

$$\frac{\dot{c}}{c} = \frac{r-\rho}{\theta}.$$

The Mangasarian sufficient conditions must hold.

- The Hamiltonian $H(c, k, \lambda^*)$, is jointly concave in (c, k) where λ^* is generated by the maximum principle.
- The transversality condition holds

  $$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda^*(t)[a(t) - a^*(t)] \geq 0,$$

  where $a^*(t)$ is the candidate for a maximum and $a(t)$ is an alternative admissible trajectory.

The dynamic equation for λ implies

$$\lambda^* = \overline{\lambda}\mathrm{e}^{\rho t - R(t)}$$
$$\mathrm{e}^{-\rho t}\lambda^* = \overline{\lambda}\mathrm{e}^{-R(t)}.$$

The no–Ponzi game condition can then be rewritten as

$$\lim_{t\to+\infty} \mathrm{e}^{-\rho t}\lambda^*(t)a(t) \geq 0.$$

From the previous transversality condition, a sufficient condition for a maximum is that

$$\lim_{t\to+\infty} \mathrm{e}^{-\rho t}\lambda^*(t)a(t) = 0.$$

In equilibrium, The labor and the capital markets clear

$$L = H,$$
$$a = k.$$

Therefore, the law of motion for the household is

$$c + \dot{k} = f(k) - \delta k.$$

Definition: An equilibrium of the decentralized Ramsey model is a 4–tuple

$$\{c(t), k(t), r(t), w(t)\},$$

that satisfies

$$\begin{aligned}
\frac{\dot{c}}{c} &= \left(\frac{f'(k) - \delta - \rho}{\theta}\right), \\
\dot{k} &= f(k) - \delta k - c, \\
r(t) &= f'(k(t)) - \delta \\
w(t) &= f(k(t)) - f'(k(t))k(t) \\
\lim_{s\to+\infty} \mathrm{e}^{-\rho s}\frac{k(s)}{c(s)^{\theta}} &= 0 \\
k(0) &= k_0.
\end{aligned}$$

1.9.2. The Steady–State and Dynamics of the Ramsey Model

The decentralized Ramsey model is equivalent to the centralized Ramsey model. The models share the same phase diagram, steady–state and dynamics. In particular, the steady–state, where $k = k^*$, is

$$\begin{aligned} f'(k) &= \delta + \rho, \\ c &= f(k) - \delta k, \\ r^* &= \rho. \end{aligned}$$

There are the same first–order conditions as the ones of the competitive equilibrium. Thus, the decentralized equilibrium is Pareto efficient and satisfies the 1st Welfare Theorem.

Note that from a an unanticipated fall in the discount rate, there is a new steady–state and a new saddle path. Initially, consumption falls, then capital converges asymptotically to its new steady–state value which is higher, resulting in higher long–run consumption.

1.9.3. The Neoclassical Growth Model with Government

Suppose that there is a government who buys output at rate $g(t)$ per unit of labor per unit time. Government purchases affect neither the utility of private agents nor the production technology. There is a balanced budget, financed by lump–sum taxes of amount $g(t)$. The law of motion for capital is

$$\dot{k} = f(k) - \delta k - c - g(t).$$

Notice that the k–locus is affected. The c-locus given by the Euler equation is unaffected by the presence of the government. The no–Ponzi game condition still holds

$$k(0) + \int_0^{+\infty} \mathrm{e}^{-R(t)} [w(t) - g(t) - c(t)] \, \mathrm{d}t \geq 0.$$

Example: A Non–Anticipated Increase in the Government's Spending
Suppose that there is a non–anticipated increase in the government's spending. The trajectory for government's expenditure is

$$g(t) = \begin{cases} g_0 & \text{for all } t < t_1, \\ g_1 > g_0 & \text{for all } t \geq t_1. \end{cases}$$

At $t = t_1$, the system jumps to its new steady–state, consumption falls by the exact amount of the increase in g, and capital stock is unchanged.

Example: An Anticipated Increase in the Government's Spending
Now, suppose that there is an anticipated increase in government's spending. At $t = t_0 < t_1$ the government announces the future increase in government's spending. The households want to smooth consumption, this implies that consumption starts to fall before taxes have increased. The capital stock increases and then decreases, and in the long run is unchanged.

Example: Ricardian Equivalence
Suppose that the path for government's spending, $g(t)$, is exogenous. The government's purchases can be financed through taxation or by issuing bonds. The initial debt of the

government per unit of labor is D_0. The budget constraint for the government is an identity

$$g(t) + rD = \dot{D} + T(t),$$

and there is a no–Ponzi game condition

$$\lim_{s \to +\infty} \mathrm{e}^{-R(s)} D(s) = 0.$$

The budget constraint of the government can be rewritten as

$$D_0 + \int_0^{\infty} \mathrm{e}^{-R(t)} g(t)\, \mathrm{d}t = \int_0^{\infty} \mathrm{e}^{-R(t)} T(t)\, \mathrm{d}t.$$

The budget constraint of households is

$$a(0) + \int_0^{+\infty} \mathrm{e}^{-R(t)} [w(t) - T(t)]\, \mathrm{d}t = \int_0^{\infty} \mathrm{e}^{-R(t)} c(t)\, \mathrm{d}t.$$

If you combine the two, then

$$a(0) - D_0 + \mathrm{e}^{-R(t)} [w(t) - g(t)]\, \mathrm{d}t = \int_0^{\infty} \mathrm{e}^{-R(t)} c(t)\, \mathrm{d}t.$$

The result is Ricardian equivalence, that is, the choice of $T(t)$ has no real consequences. Agents know that a reduction in taxes today has to be matched by an increase in taxes in the future. Following a decrease in taxes, households keep their consumption constant and raise their savings.

1.10. A Model of Endogenous Growth

In the Solow and Ramsey models, the rate of growth of the economy is exogenous, output per–capita grows at the rate of technological progress. The saving rate does influence transitional dynamics, but not long–run growth. To extend the Ramsey model to allow for endogenous growth, you can drop the assumption of diminishing returns to capital accumulation (Paul Romer, 1986). This is the AK approach of endogenous growth.

1.10.1. The Endogenous Growth Model

Assume that there is a household (or one agent) who is infinitely-lived. She chooses how much to consume and to save (invest) in order to maximize her lifetime utility. There is initial capital, $k(0)$, and no capital depreciation. There is a non–neoclassical production function. There is a linear production function

$$f(k) = Ak,$$

where the marginal product of capital, A, is not diminishing. Thus, the Inada conditions are violated

$$\lim_{k\to\infty} f'(k) = A \neq 0.$$

This is the key element that underlies endogenous growth.

The program of the household is

$$\begin{aligned} \max_{\{c\}} U &= \int_0^{+\infty} \mathrm{e}^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta}\,\mathrm{d}t \\ \text{s. t. } c + \dot{k} &= Ak \\ k(0) &= k_0, \end{aligned}$$

where it is assumed that $(1-\theta)A < \rho < A$. The current–value Hamiltonian is

$$(H, c, k, \lambda) = \frac{c(t)^{1-\theta}}{1-\theta} + \lambda[f(k) - c].$$

The first–order condition results in the optimal control

$$c(t)^{-\theta} = \lambda(t).$$

The equation for the costate variable is

$$\rho\lambda = A\lambda + \dot{\lambda}.$$

From the two previous equations it follows that

$$\begin{aligned} \dot{\lambda} &= -\theta \dot{c} c^{-\theta-1} \\ \frac{\dot{\lambda}}{\lambda} &= \rho - A. \end{aligned}$$

Thus,

$$\frac{\dot{c}}{c} = \frac{A-\rho}{\theta} > 0,$$

consumption growth does not depend on the stock of capital. Furthermore, from the first–order condition and the law of motion of capital, then

$$c + \dot{k} = Ak \geq 0,$$

and there is no steady–state with positive consumption.

1.10.2. The Balanced Growth Equilibrium

There is no steady state for k in the endogenous growth model. However, you can look for an equilibrium where k and c grow at constant rate and then the ratio (c/k) would be constant.

Definition: An equilibrium of the endogenous growth model is a pair of functions, $\{k(t), c(t)\}$, that satisfy

$$\frac{\dot{c}}{c} = \frac{A-\rho}{\theta} \qquad \text{and} \qquad \dot{k} = Ak - c,$$

$$\text{s. t.} \lim_{t\to\infty} \mathrm{e}^{-\rho t}\frac{k(t)}{c(t)^{\theta}} = 0 \qquad \text{and} \qquad k(0) = k_0.$$

First, define $z \equiv (c/k)$. To write a differential equation for z, you can take the log of z and differentiate with respect to time

$$\frac{\dot{z}}{z} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k}$$
$$\frac{\dot{z}}{z} = \frac{A-\rho}{\theta} - A + z$$
$$\frac{\dot{z}}{z} = \frac{A(1-\theta)-\rho}{\theta} + z.$$

Thus, the steady–state for $z = \frac{c}{k}$ is

$$z = \begin{cases} 0, \\ \frac{\rho - A(1-\theta)}{\theta}. \end{cases}$$

The transversality condition is

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda(t)k(t) = 0.$$

You then have

$$\lambda(t)k(t) = (\tfrac{k}{c})[c(t)]^{1-\theta}$$
$$\lambda(t)k(t) = (\tfrac{k}{c})[c(0)]^{1-\theta}\mathrm{e}^{\frac{A(1-\theta)-\rho}{\theta}t}.$$

At the steady-state value of $z = \frac{c}{k}$, then

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda(t)k(t) = \lim_{t\to\infty}(\tfrac{k}{c})[c(0)]^{1-\theta}\mathrm{e}^{\frac{A(1-\theta)-\rho}{\theta}t} = 0.$$

Under the balanced growth path, the ratio $\frac{c}{k}$ is constant and equal to

$$\left(\frac{c}{k}\right)^{*} = \frac{\rho - A(1-\theta)}{\theta},$$

and consumption, capital, and output are all growing at the same rate.

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{y}}{y} = \frac{A-\rho}{\theta}.$$

The ratio (c/k) is a jump variable, that is, it adjusts instantly to its steady–state value. The rates of growth of c, k, and y are constant and do not depend on any endogenous variables. There are no transitional dynamics.

The saving rate is given by

$$\begin{aligned}
s &= 1 - \frac{c}{Ak} \\
s &= 1 - \frac{\rho - A(1-\theta)}{A\theta} \\
s &= \frac{A - \rho}{A\theta}.
\end{aligned}$$

There is a relationship between the rate of growth and the saving rate

$$As = \frac{\dot{y}}{y}.$$

In the long–run, growth rate depends on the willingness to save and the productivity of capital. Lower values of ρ and θ imply a higher willingness to save, and this will result in a higher growth rate. An improvement in the level of technology will also lead to a higher growth rate. These are conclusions that are very different from those of the Ramsey model.

1.10.3. Endogenous Growth with Physical and Human Capital

A shortcoming of the previous model is that the share of capital in national income is equal to one. Consider the introduction of human capital, h, with a production function

$$y = kf\left(\frac{h}{k}\right),$$

that has CRS with respect to h and k, and $f(\cdot)$ is strictly concave. Let investment in physical capital be i. The program of the agent is

$$\begin{aligned}
\max_{\{c\}} U &= \int_0^{+\infty} \mathrm{e}^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta} \, \mathrm{d}t \\
\text{s. t. } \dot{h} &= kf(\tfrac{h}{k}) - c - i \\
\dot{k} &= i \\
k(0) &= k_0.
\end{aligned}$$

The Hamiltonian of the agent is

$$H(k, h, c, i, \lambda_k, \lambda_h) = \frac{c(t)^{1-\theta}}{1-\theta} + \lambda_h \left[kf\left(\frac{h}{k}\right) - c - i \right] + \lambda_k i.$$

The first–order conditions are

$$\begin{aligned}
c^{-\theta} &= \lambda_h \\
\lambda_h &= \lambda_k.
\end{aligned}$$

The shadow value of physical capital is

$$\rho\lambda_k = \lambda_h[f(\tfrac{h}{k}) - f'(\tfrac{h}{k})\tfrac{h}{k}] + \dot{\lambda}_k,$$

and the shadow value of human capital

$$\rho\lambda_h = \lambda_h f'(\tfrac{h}{k}) + \dot{\lambda}_h,$$

The condition $\lambda_h = \lambda_k$ gives

$$f\left(\frac{h}{k}\right) - f'\left(\frac{h}{k}\right)\frac{h}{k} = f'\left(\frac{h}{k}\right).$$

Thus, both types of capital have the same rate of return.

Let $\kappa = \frac{h}{k}$ denote the solution to

$$f(\kappa) - f'(\kappa)\kappa = f'(\kappa),$$

and let $A = f'(\kappa)$. The model reduces to the simple AK model described earlier. The rate of growth of consumption and output is given by

$$\frac{\dot{c}}{c} = \frac{\dot{y}}{y} = \frac{f'(\frac{h}{k}) - \rho}{\theta}.$$

1.10.4. Endogenous Growth with Knowledge Spillovers

Assume that each agent's knowledge is a public good, that is, knowledge of one agent spills over across the whole economy. The knowledge agents depends on the aggregate stock of capital through a learning–by–doing effect. The production for agent can then be written as

$$y_i = f(k_i, \overline{k}),$$

where $f(\cdot)$ is a CRS production function and $\overline{k}$ is aggregate capital stock. The program of the household

$$\begin{aligned} \max_{\{c\}} U &= \int_0^{+\infty} \mathrm{e}^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta}\,\mathrm{d}t \\ \text{s. t. } c + \dot{k} &= f(k, \overline{k}) \\ k(0) &= k_0. \end{aligned}$$

This is the standard Ramsey model with diminishing private returns of capital accumulation. The solution to the program is

$$\frac{\dot{c}}{c} = \frac{f_k(k, \overline{k}) - \rho}{\theta}.$$

All the agents choose the same capital stock, $k = \overline{k}$. The rate of growth of consumption and output in equilibrium is then

$$\frac{\dot{c}}{c} = \frac{\dot{y}}{y} = \frac{f_k(\overline{k}, \overline{k}) - \rho}{\theta}.$$

Example: Knowledge Spillovers with Cobb–Douglas Production
Given a Cobb–Douglas production function

$$f(k_i, \overline{k}) = A k_i^{\alpha} \overline{k}^{1-\alpha},$$

you can find the first order condition

$$f_k(k_i, \overline{k}) = A\alpha \left(\frac{\overline{k}}{k_i}\right)^{1-\alpha}.$$

Thus, the growth of consumption and output in equilibrium given Cobb–Douglas technology is

$$\frac{\dot{c}}{c} = \frac{\dot{y}}{y} = \frac{A\alpha - \rho}{\theta}.$$

Example: Knowledge Spillovers with a Social Planner
Consider a social planner who internalizes the spillovers of knowledge across the firms. The program of the social planner is

$$\begin{aligned}\max_{\{c\}} U &= \int_0^{+\infty} \mathrm{e}^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta}\,\mathrm{d}t \\ \text{s. t. } c + \dot{k} &= f(k, k) \\ k(0) &= k_0.\end{aligned}$$

The solution to the program is

$$\frac{\dot{c}}{c} = \frac{f_k(k, k) + f_{\overline{k}}(k, k) - \rho}{\theta}.$$

Thus, the rate of growth in the planned economy is greater than in the decentralized economy. Note that under the Cobb–Douglas specification, that the rate of growth should be

$$\frac{A - \rho}{\theta}.$$

The agents in the decentralized economy do not internalize the positive externality of capital accumulation from knowledge spillovers.

1.11. Dynamic Programming Applications

1.11.1. Optimal Unemployment

The labor market differs from other markets in that labor is not homogeneous. It takes time for a worker to find a job, and for a vacancy to find a worker. There are large flows of workers and jobs between activity and inactivity, and large and persistent stocks of vacancies and unemployed workers. This section looks at the trade sector of the labor market and its frictions. A modeling device, similar to the aggregate production function introduced by Pissarides (1985, 1990), takes the form

$$M = m(U, V),$$

where U is the number of unemployed workers, V is the number of vacancies, and M is the number of match creations (hires).

Properties:

- The matching function is increasing with respect to its two arguments

$$\begin{aligned} m_U &> 0, \\ m_V &> 0. \end{aligned}$$

- There cannot be match creation without agents to be matched on both sides of the market

$$m(0, V) = m(U, 0) = 0.$$

- The matching function is strictly concave with respect to each of its arguments

$$\begin{aligned} m_{UU} &< 0, \\ m_{VV} &< 0. \end{aligned}$$

- The matching function is homogenous of degree 1 with respect to U and V.

Consider two economies which differ only with respect to their size. Under CRS the two economies have the same unemployment rate. The matching technology can be rewritten as

$$M = m_U U + m_V V,$$

where

$$\begin{aligned} m_U &= \frac{\partial m}{\partial U}, \\ m_v &= \frac{\partial m}{\partial V}. \end{aligned}$$

The job finding rate of an unemployed worker is

$$p = \frac{m(U, V)}{U} = m(1, \theta),$$

where $\theta = V/U$ indicates market tightness.

Example: Cobb–Douglas Matching Technology
The Cobb–Douglas specification of the matching function is

$$m(U, V) = AU^{\alpha} V^{1-\alpha},$$

where the efficiency of the matching process $A > 0$, and $0 < \alpha < 1$. This specification is reasonably successful in empirical studies (Blanchard and Diamond, 1990).

Let the labor force be normalized to one. The measure of employed (employment rate) is $e(t)$ and the unemployment rate is $u(t) = 1 - e(t)$. There are $v(t)$ vacancies and a separation rate s. The law of motion for the employment rate is

$$\dot{e} = m(u, v) - se.$$

All agents are risk neutral and discount future utility at rate r. The income of an unemployed agent is b. The productivity of a match is y and there is a flow cost of opening a vacancy, γ. A benevolent planner who maximizes society's net output faces the optimization problem

$$\begin{aligned} \max_{\{e(t),v(t)\}} \int_o^\infty & \mathrm{e}^{-rt}[e(t)y + (1-e(t))b - v(t)\gamma]\,\mathrm{d}t \\ \text{s. t. } & \dot{e}(t) = m[1-e(t), v(t)] - se(t), \\ & e(0) = e_0. \end{aligned}$$

The planner is subject to the matching frictions as described by m. The current-value Hamiltonian is

$$H(e, v, \lambda) = e(y - b) + b - v\gamma + \lambda[m(1-e, v) - se],$$

where λ, the shadow value of a job, is the co–state variable. The optimal control is

$$\gamma = \lambda m_v(1-e, v).$$

The cost of opening a vacancy is equal to the shadow value of a job times the marginal contribution of a vacancy to the matching process. Equivalently,

$$\gamma = \overbrace{\frac{m(1-e,v)}{v}}^{\text{Vacancy Filling Rate}} \overbrace{\frac{m_v(1-e,v)v}{m(1-e,v)}}^{\text{Vacancy Share}} \lambda.$$

The equation for the co–state variable is

$$r\lambda = y - b - m_u(1-e, v)\lambda - s\lambda + \dot{\lambda},$$

and has the usual interpretation as an asset pricing equation. The term $b + m_u(1-e, v)\lambda)$ can be interpreted as the flow value of an unemployed worker. It can be rewritten as

$$b + \overbrace{\frac{m(1-e,v)}{u}}^{\text{Job Finding Rate}} \overbrace{\frac{m_u(1-e,v)u}{m(1-e,v)}}^{\text{Unemployed Share}} \overbrace{\lambda}^{\text{Match Value}}.$$

Example: Cobb–Douglas Matching Technology (Continued)
For simplicity, let

$$m(u, v) = A\sqrt{u}\sqrt{v}.$$

The optimal control implies that

$$\sqrt{\frac{v}{1-e}} = \frac{\lambda A}{2\gamma}.$$

The number of vacancies per unemployed, $\frac{v}{1-e}$ is called market tightness. The equation for the co–state becomes

$$r\lambda = y - b - \frac{\lambda A}{2}\sqrt{\frac{v}{1-e}} - s\lambda + \dot{\lambda}$$
$$r\lambda = y - b - \frac{1}{\gamma}\left(\frac{\lambda A}{2}\right)^2 - s\lambda + \dot{\lambda}.$$

An equilibrium can be reduced to a pair of functions, $e(t)$ and $\lambda(t)$, that satisfy the following system of differential equations

$$\dot{\lambda} = (r+s)\lambda + \frac{1}{\gamma}\left(\frac{\lambda A}{2}\right)^2 + b - y$$
$$\dot{e} = A^2\frac{\lambda}{2\gamma}(1-e) - se,$$

where the initial condition, $e(0) = e_0$, and the Mangasarian sufficiency condition hold from the the fact that the state variable is non negative

$$\lim_{t\to\infty} \mathrm{e}^{-rt}\lambda(t)e(t) = 0.$$

To reach the stationary solution, consider the solution such that

$$\dot{\lambda} = 0,$$
$$\dot{e} = 0.$$

Then, the shadow value of a job solves

$$\frac{1}{\gamma}\left(\frac{A}{2}\right)^2\lambda^2 + (r+s)\lambda - (y-b) = 0.$$

The positive root is

$$\lambda^* = \frac{2\gamma}{A^2}\left[\sqrt{(r+s)^2 + \frac{A^2}{\gamma}(y-b)} - (r+s)\right].$$

The employment rate is

$$e^* = \frac{A^2\lambda^*}{2\gamma s + A^2\lambda^*},$$

where $e^* < 1$. Thus, it is optimal to have some unemployment.

Next, you can linearize the first–order equations around their steady states

$$\begin{pmatrix} \dot{\lambda} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} (r+s) + \frac{2\lambda^*}{\gamma}(\frac{A}{2})^2 & 0 \\ A^2\frac{(1-e^*)}{2\gamma} & -\frac{A^2\lambda^*}{2\gamma} - s \end{pmatrix}\begin{pmatrix} \lambda - \lambda^* \\ e - e^* \end{pmatrix}.$$

The steady–state is a saddle point and the saddle path is such that $\lambda(t) = \lambda^*$ and

$$\dot{e} = e^* + (e_0 - e^*)\exp\left[-\left(A^2\frac{\lambda^*}{2\gamma} + s\right)t\right].$$

Graphically, the saddle path coincides with the λ–isocline, and in the space (e, λ) it is horizontal.

The model provides a rich set of comparative statics.

	r	s	y	b	A	γ
λ^*	$-$	$-$	$+$	$-$	$-$	$+$
e^*	$-$	$-$	$+$	$-$	$+$	$-$

A higher separation rate, s means a lower value of a match and higher unemployment. Higher productivity, y, means a higher value of a match and lower unemployment. Higher match efficiency, A, means lower value of a match and lower unemployment.

The speed at which employment converges to its steady–state value is

$$A^2 \frac{\lambda^*}{2\gamma} + s = \sqrt{(r+s)^2 + \frac{A^2(y-b)}{\gamma}} - r.$$

Notice that the transition to the steady–state is faster if y is high and b is low. If the cost to open vacancies, γ, is high (e.g., because if credit market frictions) then the speed of convergence is low.

If there is an unanticipated shock that raises workers' productivity, y, then the shadow value of a job, λ^*, increases. Note that λ jumps instantly to its new steady–state value, and that the same is true for market tightness. Employment and unemployment converge gradually to their steady–state values.

1.11.2. Search Unemployment

Consider a Walrasian frictionless market where workers can find a job instantly. The only decision of a worker is whether or not to participate in the market. Search theory helps describe the worker's optimal search strategy, that may be affected by the distribution of job offers, the job destruction rate, and search costs. The first model in the economics literature, by Stigler (1961, 1962), regards choosing the optimal size of a sample. Next, search models where applied to the labor market by McCall (1970) who suggested that searching is sequential. For a review, see Mortensen (1986).

Assume that time is discrete and represented by $t \in \mathbb{N}^*$. The lifetime utility of a worker is

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t U(c_t)\right],$$

where c_t is the consumption at time t, and $\beta \in (0,1)$. It is assumed that

$$\begin{aligned} U' &> 0 \\ U'' &< 0 \\ U(0) &= 0 \\ U'(0) &< \infty. \end{aligned}$$

The worker cannot borrow or lend, her consumption is equal to her earnings. The worker begins each period with a current wage offer, w. She then has two alternatives; she can can work at that wage, or she can search for a new wage offer. All wage offers lie in $[0, \overline{w}]$. Let f be the density of wages, w, on that interval and F the c.d.f. If the worker chooses to work during the current period, then with probability $1-s$ the same wage is available to her next period. With probability s she will lose her job at the beginning of the next period and begin the next period with a wage of 0.

For a recursive formulation of the problem, the state variable is the current wage, w, and the control variable is the search variable $y \in \{0,1\}$, where

$$y = \begin{cases} 0 & \text{if the worker searches,} \\ 1 & \text{if the workerworks at her current job.} \end{cases}$$

The value function is $v(w)$. If the worker chooses to work, $y = 1$, her expected present discounted value of utility is

$$v(y = 1) = U(w) + \beta[(1 - s)v(w) + sv(0)].$$

If the worker chooses to search instead, her expected utility is

$$v(y = 0) = 0 + \beta \int_0^{\overline{w}} v(x)f(x)\,\mathrm{d}x.$$

The value function solves the following Bellman equation

$$v(w) = \max_{y\in\{0,1\}} \left\{U(w) + \beta[(1 - s)v(w) + sv(0)], 0 + \beta \int_0^{\overline{w}} v(x)f(x)\,\mathrm{d}x\right\}.$$

Note that $v(w)$ is bounded above by $U(\overline{w})/(1 - \beta)$ and below by $U(0) = 0$. Therefore, you can work with the space of bounded functions $B([0, \overline{w}])$ with the supremum metric. Also, since $U(w)$ is continuous, you can work with the subspace of continuous bounded functions $C([0, \overline{w}])$. You can then check that the Blackwell sufficient conditions are satisfied. First, the value function is monotonic. If $h \leq g$, then

$$\beta[(1 - s)h(w) + sh(0)] \leq \beta[(1 - s)g(w) + sg(0)]$$
$$\int_0^{\overline{w}} h(x)f(x)\,\mathrm{d}x \leq \int_0^{\overline{w}} g(x)f(x)\,\mathrm{d}x.$$

Therefore, $Th \leq Tg$. Second, the value function satisfies discounting

$$T(h + a) = Th + \beta a.$$

Theorem: The Banach Fixed–Point Theorem
The Banach fixed–point theorem states that given $C([0, \overline{w}])$ where the supremum metric is a complete metric space, then the mapping given by the Bellman equation is a contraction mapping.

Therefore, from Banach fixed–point theorem, there is a unique v that solves the Bellman equation. This is because of the properties of the value function. Note that $U(w)$ is increasing and you can work with a closed subspace of weakly increasing functions. Furthermore, $v(w)$ is weakly increasing. Let

$$A = \beta \int_0^{\overline{w}} v(x)f(x)\,\mathrm{d}x.$$

From the Bellman equation

$$v(0) = \max\left\{\beta v(0), \beta \int_0^{\overline{w}} v(x)f(x)\,\mathrm{d}x\right\} = A$$
$$v(\overline{w}) > \beta \int_0^{\overline{w}} v(x)f(x)\,\mathrm{d}x = A.$$

The term $U(w)+\beta[(1-s)v(w)+sv(0)]$ is strictly increasing in w. That is, at at $w = 0$ it is less than A, and at $w = \overline{w}$ it is greater than A. Therefore there is a unique reservation wage, $w^* \in (0, \overline{w})$, such that

$$v(w^*) = U(w^*) + \beta[(1 - s)v(w^*) + sv(0)] = A.$$

Solving for $U(w^*)$ yields

$$U(w^*) = (1 - \beta)A.$$

Quarter II

"My fees are not too high. Your wage scale may simply be too low."
– Jack Vance, *Showboat World*

The objective of this section is to introduce you to the theories and methods of dynamic, stochastic macroeconomics. The hallmarks of dynamic macroeconomic models are intertemporal decision making and stochastic equilibrium processes. In order to study these two elements, this section is structured to emphasize computational tools and dynamic economics. Quarter II begins with some of the basic tools of decision-making in uncertain environments. All modern macroeconomic theories build on the workhorse growth models that are covered in Quarter I. You can use the insights from those frameworks to build up, step by step, to the Dynamic Stochastic General Equilibrium (DSGE) Model. This section then concludes with applications of the DSGE model to business cycles, asset pricing, and monetary/fiscal policy.

2.1. Introduction to Stochastic Processes

"An economic model is a probability distribution over a sequence."
– Thomas J. Sargent

2.1.1. Stochastic Processes

Let x_t be a realization form a random variable X_t.

Definition: A stochastic process is a sequence $\{x_t\}$ defined on a probability space.

Definition: An auto–covariance function is defined as

$$\gamma_x(r,s) \equiv \operatorname{cov}(x_r, x_s) = \mathbb{E}(x_r - \mathbb{E}\, x_r)(x_s - \mathbb{E}\, x_s).$$

Definition: The stochastic process $\{x_t\}_{t=-\infty}^{t=\infty}$ is covariance stationary provided

- $\mathbb{E}\, x_t = m$ for all t,
- $\gamma_x(r,s) = \gamma_x(r+t, s+t)$ for all t.

For simpler notation, set $t = -s$ and it follows that

$$\gamma_x(r,s) = \gamma_x(r-s, 0).$$

Now, let h be the order of the autocovariance in

$$\gamma_x(h) = \gamma_x(h, 0).$$

- When $h = 0$, then $\gamma_x(0)$ is the variance.
- When $h = 1$, then $\gamma_x(1)$ is the 1^{st} order auto–covariance.

2.1.2. Time Series

It is helpful to introduced the autoregressive moving average (ARMA) representation.

Definition: An autoregressive moving average (ARMA) model is a linear representation of a stochastic process with constant coefficients.

An ARMA(p,q) representation of x_t is

$$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + z_t + \theta_1 z_{t-1} + \cdots + \theta_q z_{t-q},$$

or

$$\phi(L)x_t = \theta(L)z_t,$$

where L is a lag operator such that

$$\begin{aligned}\phi(L) &= 1 - \phi_1(L) - \cdots - \phi_p L^p \\ \theta(L) &= 1 + \theta_1 L + \cdots + \theta_q L^q.\end{aligned}$$

An AR(p) is simply an ARMA($p, 0$), and likewise a MA(q) is simply an ARMA($0, q$).

Claim: An AR(1) is stationary provided $|\phi_1| < 1$.

Proof.

$$x_t = \phi_1 x_{t-1} + z_t,$$

and can be rewritten as

$$x_t = \phi_1^{k+1}(x_{t-(k+1)}) + \sum_{j=0}^{k} \phi_1^j z_{t-j}.$$

Note that

$$\lim_{k \to 0} \phi_1^{k+1}(x_{t-(k+1)}) = 0.$$

It follows that

$$x_t = \sum_{j=0}^{\infty} \phi_1^j z_{t-j},$$

and

$$\mathbb{E}\, x_t x_{t-1} = \mathbb{E}(z_t + \phi_1 z_{t-1} + \phi_1^2 z_{t-2} + \dots)(z_{t-1} + \phi_1 z_{t-2} + \phi_1^2 z_{t-3} + \dots).$$

Suppose that $|\phi_1| > 1$. Then

$$x_t = \phi_1 x_{t-1} + z_t,$$

can be rewritten as

$$x_t = \phi_1^{-1} x_{t+1} - \phi_1^{-1} z_{t+1}.$$

Continuing forward

$$x_t = -\sum_{j=1}^{\infty} \phi_1^j z_{t+j}.$$

$\square$

Definition: White noise is a process $\{z_t\} \overset{\text{i.i.d.}}{\sim} \mathcal{WN}(0, \sigma^2)$ if and only if

- $\mathbb{E}\, z_t = 0$ for all t,
- $\gamma_z(h) = \begin{cases} \sigma^2 & \text{if } h = 0, \\ 0 & \text{otherwise.} \end{cases}$

2.1.3. Rational Expectations

An expectational difference equation is a process

$$y_t = \alpha + \beta\,\mathbb{E}_t\,y_{t+1} + \gamma z_t, \tag{1}$$

where $z_t = e z_{t-1} + \varepsilon_t$ and $-1 < e < 1$.

Definition: A rational expectations equilibrium is a sequence $\{y_t\}$ that is a non–explosive solution to the expectational difference equation (1).

Finding solutions may lead to multiple equilibria, in which case refinement is needed. One such refinement is the Minimal State Variable (MSV) method. The MSV method finds a linear process for y_t with constant coefficients that depends on a minimal number of variables. The guess and verify method is as follows.

- First, guess

  $$y_t = a + b z_t,$$

  take future expectations

  $$\mathbb{E}_t\,y_{t+1} = a + b\,\mathbb{E}_t\,z_{t+1},$$

  substitute for $\mathbb{E}_t\,z_{t+1}$

  $$\mathbb{E}_t\,y_{t+1} = a + b e z_t,$$

  then substitute into the difference equation (1)

  $$y_t = \alpha + \beta a + \beta b e z_t + \gamma z_t.$$

  According to the guess, it is implied that

  $$\begin{aligned} a &= \alpha + \beta a, \\ b &= \beta b e + \gamma. \end{aligned}$$

  Solving yields

  $$\begin{aligned} \bar{a} &= \frac{\alpha}{1-\beta}, \\ \bar{b} &= \frac{\gamma}{1-\beta e}. \end{aligned}$$

  The MSV is

  $$y_t = \frac{\alpha}{1-\beta} + \frac{\gamma}{1-\beta e} z_t.$$

- Next, verify

  $$\mathbb{E}_t\,y_{t+1} = \frac{\alpha}{1-\beta} + \frac{\gamma}{1-\beta e} e z_t,$$

  and

  $$y_t = \alpha + \frac{\alpha\beta}{1-\beta} + \frac{\gamma\beta}{1-\beta e} e z_t + \gamma z_t.$$

  This is the rational expectations equilibrium.

2.2. Real Business Cycle Models

Begin with the social planner's problem

$$\max_{\{c_t,x_t,l_t,n_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$\begin{aligned} \text{s. t. } c_t + x_t &= A_t F(k_t, n_t) \\ k_{t+1} &= (1-\delta)k_t + x_t \\ n_t + l_t &= 1. \end{aligned}$$

Assume that $F(k_t, n_t)$ has constant returns to scale.

The intertemporal condition (Euler equation) is

$$u'(c_t, 1-n_t) = \beta\, \mathbb{E}_t\, u'(c_{t+1}, 1-n_{t+1})[A_{t+1}F_k(k_{t+1}, n_{t+1}) + (1-\delta)]. \tag{1}$$

$$[A_{t+1}F_k(k_{t+1},n_{t+1})+(1-\delta)]\equiv 1+r_{t+1}$$

The intratemporal condition (labor/leisure choice) is then found

$$-u_l(c_t, 1-n_t) + \lambda A_t F_n(k_t, n_t) = 0$$

$$A_tF_n(k_t,n_t)\equiv \text{MPL}=w_t$$

$$\lambda = u_c(c, 1-n)$$

$$\frac{u_l(c_t, 1-n_t)}{u_c(c_t, 1-n_t)} = A_t F_n(k_t, n_t) \equiv w_t. \tag{2}$$

Also, note that the resource constraint is

$$c_t + x_t = A_t F(k_t, n_t) + (1-\delta)k_t. \tag{3}$$

The solution is the sequence $\{c_t, x_t, n_t\}_{t=0}^{t=\infty}$ that solves the intertemporal condition (1), the intratemproal condition (2), and satisfies the resource constraint (3). A standard technique is to log–linearize the equations, (1), (2), and (3), around their steady–state.

Notice that

$$\mathbb{E}_t\, u_c(c_{t+1}, 1-n_{t+1})(1+r_{t+1}) = \mathbb{E}_t\, u_c(c_{t+1}, 1-n_{t+1})\, \mathbb{E}_t(1+r_{t+1}) + \text{cov}_t\big(u_c(c_{t+1}, 1-n_{t+1}), (1+r_{t+1})\big).$$

- If $\text{cov}_t\big(u_c, (1+r_{t+1})\big) < 0$, then when r_{t+1} is high, consumption is also high.
- If $\text{cov}_t\big(u_c, (1+r_{t+1})\big) > 0$, then when r_{t_1} is high, consumption is low.

2.2.1. Solving the RBC Bellman Equation

The social planner's problem can be setup using a Bellman equation and solved using various methods. Take note that the state variables are k and A, and that the control variables are c, n, and k'. The value function is

$$V(k, A) = \max_{\{c,n,k'\}} u(c, 1-n) + \beta\, \mathbb{E}_{A'|A}\, V(k', A')$$

$$\text{s. t. } c + k' = AF(k, n,) + (1-\delta)k.$$

Using the Envelope Theorem

$$V_k(k, A) = u_c(c, 1-n)\frac{\mathrm{d}c}{\mathrm{d}k} = u_c(c, 1-n)[AF_k(k, n) + (1-\delta)].$$

The resulting policy functions are

$$\begin{aligned} c_t &= c(k_t, A_t) \\ n_t &= n(k_t, A_t) \\ k_{t+1} &= k(k_t, A_t). \end{aligned}$$

The corresponding Lagrangian is

$$\mathscr{L} = u(c, 1-n) + \beta\,\mathbb{E}_{A'|A}\,V(k', A') - \lambda(c + k' - AF(k, n) - (1-\delta)k).$$

The first–order conditions are

$$\begin{aligned} u_c(c, 1-n) - \lambda &= 0 \\ -u_l(c, 1-n) + \lambda AF_n(k, n) &= 0 \\ -\lambda + \beta\,\mathbb{E}_{A'|A}\,V_k(k', A') &= 0. \end{aligned}$$

The Euler (intertemporal) equation is

$$u_c(c, 1-n) = \beta\,\mathbb{E}_{A'|A}\,u_c(c', 1-n')[A'F_k(k', n') + (1-\delta)].$$

"The opportunity cost of another unit of capital."

"The return to savings (MPK + undeprecitated capital) valued at next period's MU."

2.2.2. Productivity Shocks

If A_t increases temporarily there are important effects.

Definition: A wealth effect occurs when $A_tF(k_t, n_t)$ increases and causes c_t to increase[1].

Definition: An income effect occurs when the MPL increases and causes an increase in w_t, $\frac{u_l}{u_c}$, and l_t (a decrease in n_t).

Definition: A substitution effect occurs when the MPL increases and causes an increase in w_t, that in turn causes n_t to increase[2].

Definition: An interest rate effect occurs when the MPK increases and causes an increase in r_{t+1}, that in turn causes more capital to be accumulated, which will lead to an increase in future output and consumption.

If A_t increases for one period, then short–run wages are greater than long–run wages and labor, n_t, will increase.

If A_t increases permanently, then long–run wages, w, increase and labor, n_t, is substituted for leisure, l_t, in the short–run, that is, n_t will decrease.

[1] With consumption smoothing, then $\Delta c_t < \Delta y_t$.
[2] Empirically, the substitution effect usually dominates the income effect in most applications.

2.2.3. Calibration

Two main goals for calibration of the RBC model are choosing the parameters for the utility and production functions. There are concerns that parameters of the utility or production function may not satisfy the balanced growth path restrictions. Typically, parameters are chosen from microeconomic studies. Estimating the production function is often of key importance.

Example: Estimation of a Cobb-Douglas Production Function
Given Cobb-Douglas technology

$$Y_t = A_t k_t^{\alpha} n_t^{1-\alpha},$$

a researcher could attempt to estimate

$$\log\left(\frac{y_t}{n_t}\right) = \alpha \log\left(\frac{k_t}{n_t}\right) + \log(A_t).$$

Definition: The Solow residual is $\log(A_t)$.

The Solow residual can be estimated by an AR(1) process

$$\log(A_t) = \rho \log(A_{t-1}) + \varepsilon_t.$$

Reading: See Prescott "Theory Ahead of Business–Cycle Measurement".

Reading: Robert King and Sergio Rebelo (1999). "Resuscitating Real Business Cycles," *Handbook of Macroeconomics*.

2.3. Asset Pricing and Financial Markets

The macroeconomic approach is based on consumption and savings models. Here, you will be introduced to complete markets of financial assets.

2.3.1. Complete Markets

In a complete market, there are prices for assets that pay off in a given state (i.e. a price for 1 unit of consumption at time t in state s_t). There is an introduction of a redundant asset, a 'risk–free bond'. Households make consumption decisions for the future. The future is uncertain (i.e. there are individual and aggregate shocks). Financial markets allow for smoothing consumption across dates and states. This is a core component of modern macroeconomic models.

2.3.2. The Lucas Asset Pricing Model

Now, consider the Lucas Asset Pricing Model where there is a representative household. There is one durable good–exogenous endowments of the 'Lucas tree', N_t. Trees yield a stochastic flow of nondurable goods–dividends or 'fruit', y_t. Consuming fruit yields utility, $u(c_t)$. Furthermore, households can purchase claims to trees, N_t, that are traded at price P_t. Household's can also buy or sell risk–free bonds with a face value of 1 unit of consumption. Let L_t be the holdings of bonds with bond price

$$R_t^{-1} = \frac{1}{1+r}.$$

The household's optimization problem is

$$\max_{\{C_t, N_t, L_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t)$$
$$\text{s. t. } C_t + P_t N_t + R_t^{-1} L_t = (P_t + y_t) N_{t-1} + L_{t-1}.$$

The first–order conditions are

$$\begin{aligned} u'(C_t) - \lambda_t &= 0 \\ -P_t \lambda_t + \mathbb{E}_t (P_{t+1} + y_{t+1}) \lambda_{t+1} &= 0 \\ -R_t^{-1} \lambda_t + \mathbb{E}_t \, \lambda_{t+1} &= 0. \end{aligned}$$

The Euler equations are

$$\begin{aligned} R_t^{-1} &= \beta \, \mathbb{E}_t \left[\frac{u'(C_{t+1})}{u'(C_t)} \right] \\ P_t &= \beta \, \mathbb{E}_t \left[\frac{u'(C_{t+1})}{u'(C_t)} \right] (P_{t+1} + y_{t+1}). \end{aligned}$$

The result is complete market prices, $\{P_t, R_t^{-1}\}$, for the consumption good and bonds. There is an equilibrium if and only if the following conditions hold.

- There is a sequence $\{C_t, N_t, L_t\}$ that solves the household's optimization problem given complete market prices $\{P_t, R_t^{-1}\}$.
- All markets clear,

$$\begin{aligned} C_t &= y_t \\ N_t &= N \\ L_t &= 0, \end{aligned}$$

  including the bond market.

2.3.3. The Martingale Theory of Stock–Prices

Definition: A martingale is a process x_t such that $\mathbb{E}[x_t + 1] = x_t$.

Note that in the Lucas Asset Pricing Model

$$P_t = \beta\,\mathbb{E}_t\left[\frac{u'(y_{t+1})}{u'(y_t)}\right]\mathbb{E}_t[P_{t+1} + y_{t+1}] + \beta\text{cov}\left[\frac{u'(y_{t+1})}{u'(y_t)}, P_{t+1} + y_{t+1}\right].$$

If

- the ratio $\left[\frac{u'(y_{t+1})}{u'(y_t)}\right]$ is constant
- and $\text{cov}\left[\frac{u'(y_{t+1})}{u'(y_t)}, P_{t+1} + y_{t+1}\right] = 0$,

then the household is risk neutral and

$$P_t = \beta\,\mathbb{E}_t[P_{t+1} + y_{t+1}].$$

If this is correct, then the future price for discounted dividends is the best forecast of future prices, P_t. Alternatively, P_t contains all useful information about future payoffs. This result is the Efficient Market Hypothesis.

2.3.4. Term Structure of Interest Rates

Now, consider a term structure of interest rates in the Lucas Asset Pricing Model. For simplicity, assume that there are 1 and 2 period bonds, the analysis can be readily extended with the addition of longer period bonds. There are two strategies for the household.

- Strategy 1: buy $\frac{1}{P_{1,t}}$ and receive payoff $\frac{1}{P_{1,t}}$.
- Strategy 2: buy $\frac{1}{P_{2,t}}$ and receive payoff $\frac{1}{P_{2,t}}P_{1,t+1}$.

It is assumed that there is no arbitrage

$$\frac{1}{P_{1,t}} = \frac{1}{P_{2,t}}P_{1,t+1}.$$

It follows that

$$\begin{aligned} P_{2,t} &= P_{1,t}P_{1,t+1} \\ R_{2,t}^{-1} &= R_{1,t}^{-1}R_{1,t+1}^{-1} \\ R_{2,t} &= R_{1,t}\,\mathbb{E}_t[R_{1,t+1}]. \end{aligned}$$

The household's optimization problem is

$$\max_{\{C_t,N_t,L_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t)$$

$$\text{s. t. } C_t + P_tN_t + R_{1,t}^{-1}L_{1,t} + R_{2,t}^{-1}L_{2,t} = (P_t + y_t)N_{t-1} + L_{1,t-1} + R_{1,t}^{-1}L_{2,t-1}.$$

The first order conditions are

$$\begin{aligned} u'(C_t) - \lambda_t &= 0 \\ -P_t\lambda_t + \beta\,\mathbb{E}_t\,\lambda_{t+1}(P_{t+1} + y_{t+1}) &= 0 \\ -R_{1,t}^{-1}\lambda_t + \beta\,\mathbb{E}_t\,\lambda_{t+1} &= 0 \\ -R_{2,t}^{-1}\lambda_t + \beta\,\mathbb{E}_t\,R_{1,t+1}^{-1}\lambda_{t+1} &= 0. \end{aligned}$$

The Euler equations is

$$R_{2,t}^{-1} = \beta\,\mathbb{E}_t\, R_{1,t}^{-1}\left[\frac{u'(y_{t+1})}{u'(y_t)}\right]$$

or

$$R_{2,t}^{-1} = \beta\,\mathbb{E}_t\, R_{1,t}^{-1}\,\mathbb{E}_t\left[\frac{u'(y_{t+1})}{u'(y_t)}\right] + \beta\mathrm{cov}_t\left[R_{1,t+1}^{-1}, \frac{u'(y_{t+1})}{u'(y_t)}\right].$$

Note that the price of a 1–period bond is

$$R_{1,t}^{-1} = \beta\,\mathbb{E}_t\left[\frac{u'(y_{t+1})}{u'(y_t)}\right].$$

Thus the price of a 2–period bond is,

$$R_{2,t}^{-1} = R_{1,t}^{-1}\,\underset{\substack{\uparrow\\ \text{Expectations hypothesis.}}}{\mathbb{E}_t}\, R_{1,t+1}^{-1} + \beta\mathrm{cov}_t\left[\underset{\substack{\uparrow\\ \text{Risk Premium.}}}{R_{1,t+1}^{-1}}, \frac{u'(y_{t+1})}{u'(y_t)}\right]$$

If the $\mathrm{cov}_t < 0$, then the price of a 1–period bond is positively correlated with consumption growth. This pushes price down on a 2–year bond and increases the return, because of the bond–price risk.

2.3.5. The Equity Premium Puzzle

In the United States, as a rough estimate, the average rate of return on a bond is

$$r_{t+1} \approx 6\%.$$

Let

$$1 + r_{t+1}^i \equiv R_{t+1}^i,$$

with Euler equations

$$1 = \beta\,\mathbb{E}_t\left[(1 + r_{t+1}^i)\frac{u'(c_{t+1})}{u'(c_t)}\right],$$

for $i = s, b$.

Quarter III

"It is useless, after all, to complain against inexorable reality."
– Jack Vance, *Cugel's Saga*

The goal of this section is to cover topics in Macroeconomics that every economics graduate student should know. The two main topics covered are are vector autoregressions (VARs) and New Keynesian Dynamic Stochastic General Equilibrium models (NK DSGE models).

3.1. Reduced-Form and Structural Vector Autoregressions

Reading: Stock, James, and Mark Watson (2001). "Vector Autoregressions," *Journal of Economic Perspectives* 15, 101-115.

3.1.1. Reduced–Form Vector Autoregressions

Assume a system of linear equations for output, y, inflation, π, and the interest rate, i,

$$\begin{aligned} y_t &= \alpha_y + \cdots + \beta_j y_{t-j} + \cdots + \gamma_j \pi_{t-j} + \cdots + \delta_j i_{t-j} + \mu_t^y \\ \pi_t &= \alpha_\pi + \cdots + \theta_j y_{t-j} + \cdots + \phi_j \pi_{t-j} + \cdots + \lambda_j i_{t-j} + \mu_t^\pi \\ i_t &= \alpha_i + \cdots + \psi_j y_{t-j} + \cdots + \kappa_j \pi_{t-j} + \cdots + \rho_j i_{t-j} + \mu_t^i, \end{aligned}$$

that can be written in matrix form as

$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = \begin{pmatrix} \alpha_y \\ \alpha_\pi \\ \alpha_i \end{pmatrix} + A_1 \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + \cdots + A_j \begin{pmatrix} y_{t-j} \\ \pi_{t-j} \\ i_{t-j} \end{pmatrix} + \begin{pmatrix} \mu_t^y \\ \mu_t^\pi \\ \mu_t^i \end{pmatrix}.$$

Let

$$A(L) = A_0 + A_1 L + A_2 L^2 + \cdots + A_k L^k,$$

where

$$L^j x_t \equiv x_{t-j}.$$

Then a reduced–form vector autoregression (VAR) is written

$$\vec{x}_t = \vec{\alpha} + A(L)\vec{x}_{t-1} + \vec{\mu}_t,$$

where A is a $n \times k$ matrix of scalar lag polynomials of any number of lags k.

You can estimate

$$y_t = x_t \vec{\beta} + \mu_t,$$

where

$$\begin{aligned} x_t &= [1 \ldots y_{t-k} \ldots \pi_{t-k} \ldots i_{t-k}] \\ \vec{\beta}\,' &= [\alpha_y \beta_1 \ldots \beta_k \ \ldots \alpha_z \lambda_1 \ldots \lambda_k]. \end{aligned}$$

by OLS

$$\hat{\beta} = (x'x)^{-1} x'y.$$

The estimates are consistent, but biased because μ_t is not independent of x_t. The variance of μ_t is

$$\mathbb{V}[\mu_t] = \Omega.$$

Typically,

$$\text{corr}(\mu_t^i, \mu_t^y) \neq 0.$$

Example: Forecasting with a Reduced–Form Vector Autoregression
Consider the reduced–form model VAR(1) model of output, y, inflation, π, and the interest rate, i,

$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = \vec{\alpha} + A(L) \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + \vec{\mu}_t.$$

Then

$$\mathbb{E}_t \begin{pmatrix} y_{t+1} \\ \pi_{t+1} \\ i_{t+1} \end{pmatrix} = \vec{\alpha} + A(L) \begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix}.$$

The best forecast leaves only the exogenous shock unexplained, so $\mathbb{E}[\vec{\mu}_t] = 0$. Furthermore, this forecast are the best explanatory variables for the next forecast. This process can be repeated.

3.1.2. Structural Vector Autoregressions

Now, consider a structural vector autoregression representation

$$\vec{x}_t = \vec{\alpha} + B(L)\vec{x}_{t-1} + S\vec{\varepsilon}_t.$$

For the system of linear equations for output, y, inflation, π, and the interest rate, i, the structural VAR is

$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = \vec{\alpha} + B(L) \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + \begin{pmatrix} S \end{pmatrix} \begin{pmatrix} \varepsilon_t^y \\ \varepsilon_t^\pi \\ \varepsilon_t^i \end{pmatrix}.$$

If

$$\mathbb{V}\left(\vec{\varepsilon}_t \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$S \cdot \vec{\varepsilon}_t = \vec{\mu}_t.$$

Note that

$$\begin{aligned} \mathbb{V}[\vec{\mu}]_t &= S \cdot \mathbb{V}[\vec{\varepsilon}_t] \cdot S' \\ \Omega_\mu &= S\Omega_\varepsilon S' \\ \Omega_\mu &= SS', \end{aligned}$$

and the reduced–form VAR does not identify S. If S has 9 elements, then Ω_μ needs 6 restrictions.

Let $S_\circ$ satisfy $S_\circ S_\circ' = \Omega_\mu$ and let U be any 3×3 orthogonal matrix (rotation matrix). Then

$$\begin{aligned} (S_\circ U') \cdot (S_\circ U')' &= S_\circ U \cdot U \cdot S_\circ' \\ (S_\circ U') \cdot (S_\circ U')' &= S_\circ S_\circ' \\ (S_\circ U') \cdot (S_\circ U')' &= \Omega_\mu \end{aligned}$$

3.2. Methods of Identifying Structural VARs

Reading: Christiano, Lawrence, Martin Eichenbaum, and Charles Evans (1999). “Monetary Policy Shocks: What Have We Learned and to What End?” *Handbook of Macroeconomics* 1, 65-148.

3.2.1. Recursive Identification

3.2.2. Long-Run Restrictions

Reading: Blanchard, Olivier, and Danny Quah (1989). “The Dynamic Effects of Aggregate Demand and Supply Disturbances,” *American Economic Review* 79, 655-673.

Reading: Galí , Jordi (1999). “Technology, Employment, and the Business Cycle: Do Technology Shocks Explain Aggregate Fluctuations?” *American Economic Review* 89, 249-271.

3.2.3. Sign Restrictions

Reading: Uhlig, Harald (2005). “What Are the Effects of Monetary Policy on Output? Results from an Agnostic Identification Procedure.” *Journal of Monetary Economics* 52, 381–419.

Reading: Baumeister, Christiane, and James Hamilton (2015). “Sign Restrictons, Strucutral Vector Autoregressions, and Useful Prior Information,” *Econometrica* 83, 1963-1999.

3.2.4. High-Frequency Identification

Reading: Cochrane, John, and Monika Piazzesi (2002). “The Fed and Interest Rates: A High-Frequency Identification,” *American Economic Review* 92, 90-95.

Reading: Fisher, Jonas, and Ryan Peters (2010). “Using Stock Returns to Identify Government Spending Shocks,” *Economic Journal* 120, 414-436.

Reading: Gertler, Mark, and Peter Karadi (2015). “Monetary Policy Surprises, Credit Costs, and Economic Activity,” *American Economic Journal: Macroeconomics* 7, 44-76.

3.2.5. External Instruments

Reading: Stock, James, and Mark Watson (2012). “Disentangling the Channels of the 2007-09 Recession,” *Brookings Papers on Economic Activity*, Spring, 81-135.

3.3. The Lucas Critique: Reduced-form vs. Structural Models

Reading: Lucas, Robert (1976). ”Econometric Policy Evaluation: A Critique,” *Carnegie-Rochester Conference Series on Public Policy* 1, 19-46.

3.4. Heterogeneity

Reading: Krusell, Per, and Anthony Smith (1998). “Income and Wealth Heterogeneity and the Macroeconomy,” *Journal of Political Economy* 106, 867-896.

Reading: Gourinchas, Pierre-Oliver, and Jonathan Parker (2002). “Consumption over the Life Cycle,” *Econometrica* 70, 47-89.

3.4.1. The Existence of a Representative Firm

Suppose that there are $n = 1, \ldots, N$ heterogeneous firms.

Definition: Aggregate Production Function
Define an aggregate production function as

$$Y = F(K, L),$$

where total capital, K, total labor, L, and total output, Y, are given by

$$K = \sum_{n=1}^{N} k_n \qquad L = \sum_{n=1}^{N} l_n \qquad Y = \sum_{n=1}^{N} y_n,$$

for firms $n = 1, \ldots, N$ with capital $k_1, \ldots, k_n$, labor $l_1, \ldots, l_n$, and output $y_1, \ldots, y_n$ respectively. There does not exist an aggregate production function

$$F(K, L) = \sum_{n=1}^{N} y_n,$$

unless all $f_n(k, l)$ are linear with the same slope.

Proof. Suppose that

$$\frac{\partial f_n}{\partial k} \neq \frac{\partial f_j}{\partial k},$$

for some $j \neq n$. Also, let

$$\frac{\partial f_n}{\partial k} > \frac{\partial f_j}{\partial k}.$$

Then it is possible to reallocate dk from j to n without changing aggregate capital, K, but increasing aggregate output, Y. Continue until

$$\frac{\partial f_n}{\partial k} = \frac{\partial f_j}{\partial k},$$

for all $j, n \in N$ and $k \in K$. Then it is also possible to reallocate dl from j to n until

$$\frac{\partial f_n}{\partial l} = \frac{\partial f_j}{\partial l},$$

for all $j, n \in N$ and $l \in L$. Now, let

$$\beta = \frac{\partial f_n}{\partial k},$$
$$\gamma = \frac{\partial f_n}{\partial l}.$$

Then it must be that

$$f_n(k, l) = \alpha_n + \beta k_n + \gamma l_n.$$

□

Let each firm n have production function $f_n(k, l)$ and profit functions $\pi_n(p, r, w)$, where p is the price, r is the rental price of capital, and w is the wage rate of labor. If the profit functions of the firms are homogeneous of degree 1, convex, monotonic in price, p, and strictly decreasing in rental price, r, and wage rate, w, then there is an aggregate profit function.

Definition: Aggregate Profit Function
Define an aggregate profit function as

$$\pi(p,r,w) = \sum_{i=1}^{N} \pi_n(p,r,w).$$

Proof. From Hotelling's Lemma, note that

$$\begin{aligned} \frac{\partial \pi_n}{\partial p} &= \quad y_n, \\ \frac{\partial \pi_n}{\partial r} &= -k_n, \\ \frac{\partial \pi_n}{\partial w} &= -l_n. \end{aligned}$$

You can check that

$$\begin{aligned} \frac{\partial \pi}{\partial p} &= \quad \sum_{i=1}^{N} \frac{\partial \pi_n}{\partial p} = \sum_{i=1}^{N} y_n = Y, \\ -\frac{\partial \pi}{\partial r} &= -\sum_{i=1}^{N} \frac{\partial \pi_n}{\partial r} = \sum_{i=1}^{N} k_n = K, \\ -\frac{\partial \pi}{\partial w} &= -\sum_{i=1}^{N} \frac{\partial \pi_n}{\partial w} = \sum_{i=1}^{N} l_n = L. \end{aligned}$$

Therefore, $\pi(p,r,w)$ is an aggregate production function. □

Theorem: If and only if you have perfectly competitive markets, then you can have perfect aggregation.

The conclusion is that the economy acts as if it is a single firm with profit function $\pi(p,r,w)$. You can then convert $\pi(p,r,w)$ to the production function $Y = F(k,L)$ through duality.

Given competitive markets and profit maximization, you can show that the economy behaves as if there is an aggregate production function with an aggregate production possibilities set

$$Y = \left\{ \sum_{i=1}^{N} y_n : y_n \in Y_n \right\},$$

but this does not prove the existence of an aggregate production function.

3.4.2. The Existence of a Representative Agent

Assume that there are commodities, $n = 1, \dots, N$, and households, $h = 1, \dots, H$, with utility, $U_n(\vec{c})$. The households each demand $\vec{c}_h(\vec{p}, m)$, where the houshold's income is m, commodities demanded are $\vec{c} \in \mathbb{R}^n$ and the respective prices are $\vec{p} \in \mathbb{R}^N$.

Definition: Aggregate Consumption Function
Define an aggregate consumption function as

$$\vec{C}(\vec{p}, \{m_h\}_{h=1}^H) = \sum_{h=1}^H \vec{c}_h(\vec{p}, m_h).$$

If you wish to restrict the distribution of $\{m_h\}_{h=1}^H$ so that you can represent aggregate consumption as $\vec{C}(\vec{p}, M)$, where $m = \sum_{h=1}^H m_h$, then it must be that reallocations of household income, m, do not affect consumption. It must be that

$$\frac{\partial \vec{c}_h(\vec{p}, m)}{\partial m} = \frac{\partial \vec{c}_j}{\partial m},$$

for all households, $h, j \in H$, and income levels, m. Thus, it must be that

$$\frac{\partial \vec{c}_h(\vec{p}, m)}{\partial m} = \vec{f}(\vec{p}),$$

where the marginal propensity to consume, $\vec{f}(\vec{p}$, is independent from household, h, and income level, m. So, it must be that consumption of every good is linear in m and has a linear Engel curve

$$\vec{c}_h(\vec{p}, m) = \vec{g}(\vec{p}) + \vec{f}(\vec{p})m.$$

Roy's Identity implies that

$$\vec{c}_h(\vec{p}, m) = -\frac{\partial V_h(\vec{p}, m)/\partial \vec{p}}{\partial V_h(\vec{p}, m)/\partial m},$$

where $V_h(\vec{p}, m)$ is the indirect utility function of household h. This implies that for an aggregate consumption function to exist, you need

$$-\frac{\partial V_h(\vec{p}, m)/\partial \vec{p}}{\partial V_h(\vec{p}, m)/\partial m} = \vec{g}(\vec{p}) + \vec{f}(\vec{p})m.$$

3.4.3. Gorman Aggregation

Definition: Representative Household
Heterogeneous agents act as if the economy is representative.

Theorem: The Gorman Theorem
Let

$$V_h(\vec{p}, m) = a_h(\vec{p}) + b(\vec{p})m.$$

Then $\vec{c}_h(\vec{p}, m)$ is linear in income, m, and the economy behaves as if there is a representative agent.

Proof. □

3.5. Balanced Growth and Real Business Cycles

Reading: King, Robert, Charles Plosser, and Sergio Rebelo (2002). "Production, Growth, and Business Cycles: Technical Appendix," *Computational Economics* 20, 87-116.

3.5.1. Balanced Growth

You may notice that the trend in a countries GDP is pretty constant. An economy is on a balanced growth path if output, Y, capital, K, labor, L, investment, I, and consumption, C, grow at constant rates

3.6. Monopolistic Competition and Nominal Rigidities

Reading: Mankiw, N. Gregory (1985). "Small Menu Costs and Large Business Cycles: A Macroeconomic Model of Monopoly," *Quarterly Journal of Economics* 100, 529-537.

Reading: Blanchard, Olivier, and Nobuhiro Kiyotaki (1987). "Monopolistic Competition and the Effects of Aggregate Demand," *American Economic Review* 77, 647-666.

3.7. The New Keynesian DSGE Model

First, we start with the household's problem

$$\max_{\{C_t,N_t\}_{t=0}^{\infty}} U = \sum_{t=0}^{\infty} \beta^t \Big[\frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{N_t^{1-\theta}}{1-\theta}\Big] Z_t$$

$$\text{s.t.} \quad P_tC_t + Q_tB_t = B_{t-1} + w_tN_t + D_t,$$

where B_t is the household's stock of bonds, D_t are dividends, N_t is the household's labor supply, and w_t is the wage rate in period t. The stochastic exogenous shocks $z_t = \log Z_t$ follow an AR(1) process $z_t = \rho_z + \epsilon_t^z$. The household's aggregate consumption and price indexes are

$$C_t \equiv \Big[\int_0^1 c_t(i)^{\epsilon-1/\epsilon}\,\mathrm{d}i\Big]^{\epsilon/\epsilon-1},$$

$$P_t \equiv \Big[\int_0^1 p_t(i)^{1-\epsilon}\Big]^{1/1-\epsilon}.$$

Notice that if we let $Q = \frac{1}{1+i_t}$, with interest rate i_t, then the budget can be written as

$$B_t = (1+i_t)(B_{t-1} + w_tN_t + D_t - P_tC_t).$$

The Lagrangian for the household's problem is

$$\max_{\{C_t,N_t\}_{t=0}^{\infty}} \mathscr{L} = \sum_{t=0}^{\infty} \beta^t \Big[\frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{N_t^{1-\theta}}{1-\theta}\Big] Z_t - \sum_{t=0}^{\infty} \lambda_t[B_t-(1+i_t)(B_{t-1}+w_tN_t+D_t-P_tC_t)].$$

The first order conditions are

$$\lambda_t = \frac{\beta^t C_t^{-\sigma} Z_t}{(1+i_t)P_t},$$

$$\lambda_t = \frac{\beta^t N_t^{\theta} Z_t}{(1+i_t)w_t},$$

$$\lambda_t = (1+i_{t+1})\lambda_{t+1}.$$

The intratemporal optimality condition is:

$$\frac{N_t^{\theta}}{C_t^{-\sigma}} = \frac{w_t}{P_t}.$$

The intertemporal optimality condition is:

$$\frac{C_t^{-\sigma} Z_t}{(1+i_t)P_t} = \beta(1+i_{t+1})\frac{C_{t+1}^{-\sigma} Z_{t+1}}{(1+i_{t+1})P_{t+1}}.$$

The Euler equation is:

$$C_t^{-\sigma} = \beta\,\mathbb{E}_t\Big[\frac{(1+i_t)}{(1+\pi_{t+1})}C_{t+1}^{-\sigma}\Big(\frac{Z_{t+1}}{Z_t}\Big)\Big].$$

Inflation is defined as $\Pi_{t+1} \equiv 1+\pi_{t+1} \equiv \frac{P_{t+1}}{P_t}$. Notice that if there is zero inflation, $\pi = 0\ \forall\ t$, then

$$C^{-\sigma} = \beta\frac{1+i}{1}C^{-\sigma}.$$

This implies that $1+i = 1/\beta$.

The next step is to log-linearize the equation. So,

$$\mathbb{E}_t\,\mathrm{d}z_{t+1} = \rho_z\,\mathrm{d}z_t,$$

where $r_{t+1} \equiv i_t - \pi_{t+1}$. Note that $Y_t = C_t$, because there is no government spending and the economy is closed. Note that $\lim_{j\to\infty}\mathbb{E}_t\,\hat{Y}_{t+j} = 0$, because $\mathbb{E}_t\,\mathrm{d}z_{t+j} = \rho_z^j\,\mathrm{d}z_t$. We can now see that output today is negatively related to interest rates in the future. Thus any deviation from steady-state output is related to long-term interest rates[3].

$$\hat{Y}_t = -\frac{1}{\sigma}\,\mathbb{E}_t\sum_{j=1}^{\infty}\mathrm{d}r_{t+j} + \frac{1}{\sigma}\,\mathrm{d}z_t.$$

3.7.1. Applications of NK DSGE Models

Reading: Woodford, Michael (2011). "Simple Analytics of the Government Expenditure Multiplier," *American Economic Journal: Macroeconomics* 3, 1-35.

3.8. Expectations and Time-Inconsistency

Reading: Barro, Robert, and David Gordon (1983). "Rules, Discretion, and Reputation in a Model of Monetary Policy," *Journal of Monetary Economics* 12, 101-121.

3.9. Sticky Wages and Prices

Reading: Erceg, Christopher, Dale Henderson, and Andrew Levin (2000). "Optimal Monetary Policy with Staggered Wage and Price Contracts," *Journal of Monetary Economics* 46, 281-313.

Reading: Christiano, Lawrence, Martin Eichenbaum, and Charles Evans (2005). "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *Journal of Political Economy* 113, 1-45.

Reading: Smets, Frank, and Raf Wouters (2007). "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach," *American Economic Review* 97, 586-606.

[3] Note that in the case of zero inflation, then $\mathrm{d}r_{t+j} = r_{t+j} - r = r_{t+j} - i$, so, $i = \frac{1}{R} - 1$

3.10. Macroeconomic vs. Microeconomic Elasticities

Reading: Altig, David, Lawrence Christiano, Martin Eichenbaum, and Jesper Lind'e (2010). "Firm-Specific Capital, Nominal Rigidities, and the Business Cycle," *Review of Economic Dynamics* 14, 225-247.