ME 547 Homework 1
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Due 1/26/2024

Question 1

A)

A1

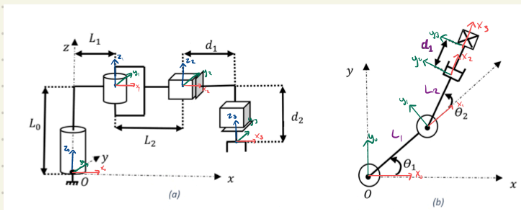

Figure 1. (a) 3D view (b) 2D top view of manipulator.

Since this robot's revolute joints are both in the same axis, it's linear & revolute displacement are fairly easy to consider seperately.

Rotationally:

There are 2 revolute joints in the z axis, so

$$\Theta_3 = \Theta_2 = \begin{array}{cc} \begin{bmatrix} 0 \\ 0 \\ \Theta_1 + \Theta_2 \end{bmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \end{array}$$

No revolute joints about x or y, so no rotation about x or y axis.

Forward Model for x, y & z

$$x_1 = x_0 + L_1 \cos\theta_1$$

$$x_2 = x_1 + L_2 \cos(\theta_1 + \theta_2)$$

$$x_3 = x_2 + d_1 \cos(\theta_1 + \theta_2)$$

Back-substitute:

$$x_3 = x_0 + L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + d_1 \cos(\theta_1 + \theta_2)$$

$$x_3 = x_0 + L_1 \cos\theta_1 + (L_2 + d_1) \cos(\theta_1 + \theta_2)$$

$$y_1 = y_0 + L_1 \sin\theta_1$$

$$y_2 = y_1 + L_2 \sin(\theta_1 + \theta_2)$$

$$y_3 = y_2 + d_1 \sin(\theta_1 + \theta_2)$$

Back-substitute:

$$y_3 = y_0 + L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + d_1 \sin(\theta_1 + \theta_2)$$

$$y_3 = y_0 + L_1 \sin\theta_1 + (L_2 + d_1) \sin(\theta_1 + \theta_2)$$

$$z_1 = z_0 + L_0$$

$$z_2 = z_1$$

$$z_3 = z_2 - d_2$$

$$z_3 = z_0 + L_0 - d_2$$

Combined into vector

$$\vec{q}_3 = \begin{bmatrix} x_0 + L_1 \cos\theta_1 + (L_2 + d_1) \cos(\theta_2 + \theta_1) \\ y_0 + L_1 \sin\theta_1 + (L_2 + d_1) \sin(\theta_2 + \theta_1) \\ z_0 + L_0 - d_2 \end{bmatrix}$$

$$L_0 = 1m, \quad L_1 = L_2 = 0.5m$$

$$\vec{q}_3 = \begin{bmatrix} x_0 + \cos\theta_1 + (0.5 + d_1) \cos(\theta_2 + \theta_1) \\ y_0 + \sin\theta_1 + (0.5 + d_1) \sin(\theta_2 + \theta_1) \\ z_0 + 1 - d_2 \end{bmatrix}$$

If q_0 can be assumed to be same as world frame, x_0, y_0, $z_0 = 0$, and can be removed.

B)

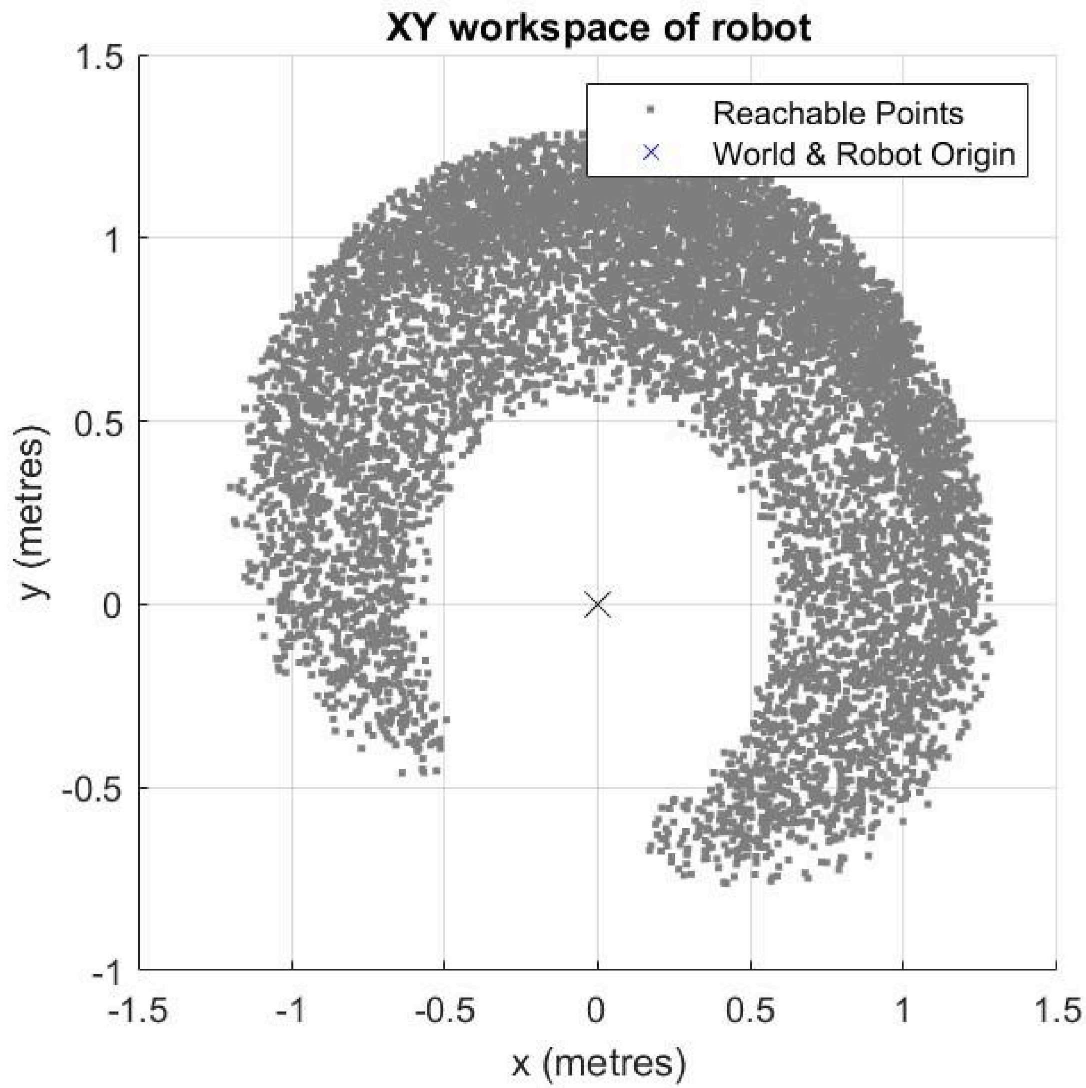

Figure 1: The XY workspace of the given robot, without any translation or rotation from the world frame at (0, 0)

C)

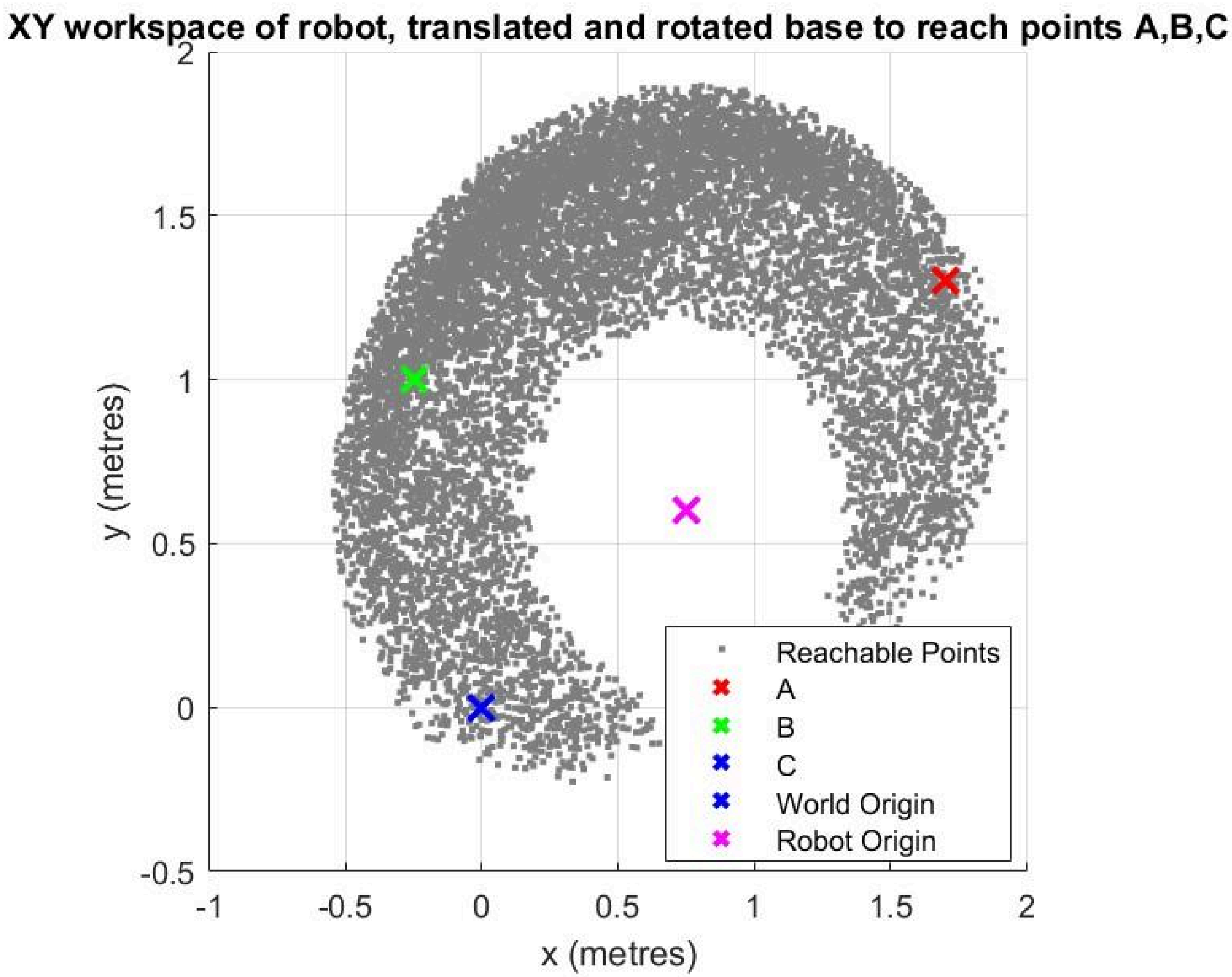

Figure 2: XY workspace when the robot base is translated 0.75 meters right, 0.6 meters up, and rotated 40 degrees counterclockwise. End effector destination points A, B and C marked in colour.
Note - Points A, B, and C all have a Z value of 0.8 meters, which is attainable by setting d_2 to 0.2 meters.

Question 2

A)

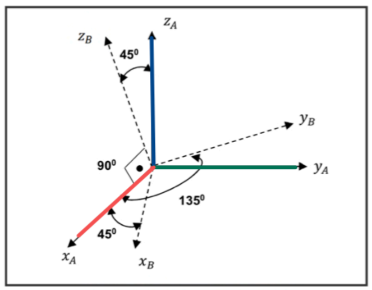

Figure 2. Coordinate frames {A} and {B}

Assuming all vectors are unit length.

If z_B & z_A are both $\perp$ to x_A, then x_A is normal to the plane defined by z_B & z_A.

ie, $z_B \times z_A = x_A$ (need to validate sign)

Also known:

$$z_B \cdot z_A = \cos 45^\circ$$
$$x_A \cdot y_B = \cos 135^\circ$$
$$x_A \cdot x_B = \cos 45^\circ$$
$$(z_B \times z_A) \cdot x_A = 0$$

If it is known that z_B & z_A are 45° apart, and are both orthogonal to x_A, then that 45° rotation must come from a rotation about x_A, as a rotation about z_A would not change the angle between z_A & z_B, and a rotation about y_A would cause z_B & z_A to not be $\perp$ with x_A.

So, First rotation:

$$R_{x_A}(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_{x_A}(45^\circ) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

So frame $A' = R_{x_A}(45^\circ)A$. Taking A as base reference frame, $A = I$, so $A' = R_{x_A}(45^\circ) I = R_{x_A}(45^\circ)$.

The next rotation needs to satisfy

$$x_A \cdot y_B = \cos 135^\circ = -\frac{1}{\sqrt{2}}$$
$$x_A \cdot x_B = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

without changing the position of z_B, since $z_B \cdot z_A = \cos 45^\circ$ & $(z_B \times z_A) \cdot x_A = 0$ is currently satisfied.

lets see if a rotation about the new z_B can achieve this.

$$B = R_x(45^\circ) R_z(\gamma) = \begin{array}{c c} & \begin{array}{ccc} x_B & y_B & z_B \end{array} \\ \begin{array}{c} x_A \\ y_A \\ z_A \end{array} & \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ ? & ? & ? \\ ? & ? & \frac{1}{\sqrt{2}} \end{bmatrix} \end{array}$$

Question marks are interesting, as no constraints were given for those relations. Maybe ignorable?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ ? & ? & ? \\ ? & ? & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$1 \cos\alpha = \frac{1}{\sqrt{2}}$
$\Rightarrow \alpha = 45^\circ$ or -45°

$-1 \sin\alpha = -\frac{1}{\sqrt{2}}$
$\sin\alpha = \frac{1}{\sqrt{2}}$
$\Rightarrow \alpha = 45^\circ$ or 135°

$0 = 0$

$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\alpha = 45^\circ$

Okay lets make the rotation matrix & see if it works.

For cascade of rotations about new axis, apply "first" rotations last, & "last" first, so:

$$R_A^B = R_x(45^\circ)R_z(45^\circ) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45^\circ & -\sin 45^\circ \\ 0 & \sin 45^\circ & \cos 45^\circ \end{bmatrix}\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{array}{c c} & \begin{array}{ccc} x_B & y_B & z_B \end{array} \\ \begin{array}{c} x_A \\ y_A \\ z_A \end{array} & \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \end{array}$$

Each row & column is of unit length, that's a good sign.

Double check!

$y_B \cdot x_A = -\frac{1}{\sqrt{2}} = \cos(135^\circ)$ ✓

$x_A \cdot x_B = \frac{1}{\sqrt{2}} = \cos(45^\circ)$ ✓

$z_B \cdot z_A = \frac{1}{\sqrt{2}} = \cos(45^\circ)$ ✓

$(z_B \times z_A) \cdot x_A = 0 \quad \cos(90^\circ)$ ✓

I'm gonna try transforming the unit vectors in A as well, to be sure transform B in correct way ($A \to B$, not $B \to A$)

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

unit length ✓

visually that checks out with the given diagram & found rotations. All components > 0. ✓

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

unit length ✓

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$ ✓

unit length ✓

so, $$\boxed{R_A^B = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}}$$

$$R^T R = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ✓

$$\det R = \frac{1}{\sqrt{2}}\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right) - -\frac{1}{\sqrt{2}}\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right)$$
$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)$$
$$= 1$$ ✓

R is indeed a rotation matrix

B)

B)

R is a rotation about some u. Since R is a rotation matrix, in 3D one of it's eigenvalues is 1, which corresponds to u, about which there is rotation & no scaling.

Also, since no B axis is aligned with it's A axis, it is known that u is not aligned with any A axis, hence $u_x, u_y, u_z \neq 0$

Since u is going to have magnitude of 1, we are really only interested in the sizes of u_x, u_y & u_z relative to eachother. To avoid having to do row-reduction, let $u_x = 1$

$$R^B_A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} u_x - \frac{1}{\sqrt{2}} u_y = u_x$$
$$u_y = \sqrt{2}\left(\frac{1}{\sqrt{2}} u_x - u_x\right)$$
$$u_y = \sqrt{2}\left(\frac{1}{\sqrt{2}} - 1\right)$$
$$u_y \approx -0.414$$

$$\frac{1}{2} u_x + \frac{1}{2} u_y - \frac{1}{\sqrt{2}} u_z = u_y$$
$$u_z = \sqrt{2}\left(-u_y + \frac{1}{2} u_y + \frac{1}{2} u_x\right)$$
$$u_z = \sqrt{2}(0.414 - 0.207 + 0.5)$$

$$\text{so } u = \begin{bmatrix} 1 \\ -0.414 \\ 1 \end{bmatrix}, \quad \|u\| = 1.47$$

Double check third eq:

$$\frac{1}{2} u_x + \frac{1}{2} u_y + \frac{1}{\sqrt{2}} u_z = u_z$$
$$\frac{1}{2} \cdot 1 + \frac{1}{2}(-0.414) + \frac{1}{\sqrt{2}} 1 = 1$$
$$1 = 1 \quad \text{nice}$$

$$\hat{u} = \begin{bmatrix} 0.679 \\ -0.282 \\ 0.679 \end{bmatrix}$$

Still need ϕ.

$$\mathrm{Trace}(R) = 1 + 2\cos\phi$$

$$\frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{\sqrt{2}} = 1 + 2\cos\phi$$
$$1.914 = 1 + 2\cos\phi$$
$$\frac{0.914}{2} = \cos\phi$$
$$\phi = \pm 62.8°$$

Expected pos, lets use $R_u(\phi)$ to find out elements (2,1)

$$u_x u_y (1 - \cos\phi) + u_z \sin\phi = \frac{1}{2}$$
$$-0.282 \cdot 0.679 (1 - \cos 62.8°) + 0.679 \sin\phi = 0.5$$
$$-0.104 + 0.679 \sin\phi = 0.5$$
$$\sin\phi = 0.889$$
$$\phi = 62.8, 117.2$$

$$\phi = 62.8°$$

Double check w/ element (1,3)

$$u_x u_z (1 - c\phi) + u_y s\phi = 0$$
$$0.679 \cdot 0.679 (1 - c_{62.8°}) - 0.282 s_{62.8°} = 0$$
$$0.250 - 0.250 = 0 \checkmark$$