Pinning Control of Networks: Choosing the Pinned Sites

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Abstract

In this master thesis we address the problem of optimal pin selection in four elementary topologies. The *augmented connectivity* of a graph is defined as an extension of the algebraic connectivity in a pinning control scenario, and its key role in the pinning control problem is illustrated. For each of the considered topologies several pinning configurations are examined and they are compared in terms of the control strength they require to yield a desired value for the augmented connectivity. For each of the examined configurations a direct expression is provided for the control strength as a function of the augmented connectivity.

Contents

Chapter 1

Introduction

In this chapter we introduce the subject of pinning control and its connection to graph theory. First we give a quick overview of the existing work on pinning control and graph theory, and we describe the contribution provided by this thesis. Then we review some well known concepts of elementary graph theory and we introduce some new definitions specifically designed to address the pinning control problem. We consider a particular pinning control problem to show how the introduced elements of graph theory can be effectively employed to study it. Finally we give a quick outline of the analysis procedure adopted throughout the report.

1.1 Previous Work

The problem of *pinning control* has recently been the subject of a great deal of interest from the automatic control community. In this problem a set of interacting agents must be driven onto a common reference trajectory known a priori. A control action is applied to a subset of the agents, which are said to be *pinned*, while convergence of the non-pinned agents to the reference trajectory must be achieved thanks to the interaction with the pinned agents. The success of the control task depends on several factors, such as the dynamics of the individual agents, the intensity of the interactions and of the control action, the topology of the network and the location of the pin nodes. When the agents' dynamics is given, the intensity of the control action is fixed and the topology and the intensity of the interactions are assigned, the selection of the pin nodes is left out as the key element of the control design.

Pinning control of nonlinear oscillators over a static topology is addressed in [1, 2, 3]. Adaptation of interaction intensity is studied for pinning control of nonlinear oscillators in [4]. In [5] the concept of *pinning controllability* is defined in terms of the spectral properties of the network topology, and the roles of the coupling and control gains are discussed as well. In [6] criteria for global pinning controllability of networks of nonlinear oscillators are provided in terms of the network topology, the oscillator dynamics and the feedback control law. Strategies for optimal pin selection are presented in [7, 8]. In [9] analytical tools are developed to study the controllability of a network and to identify the optimal subset of driver nodes. Decentralized adaptive pinning strategies are introduced in [10, 11]. In [12] pinning control over a time varying topology is investigated. Pinning control with nonlinear interaction protocol is studied in [13]. In [14] pinning controllability in networks with and without communication delay is investigated and a selective pinning criterion is proposed. In [15] local stochastic stability of networks under pinning control is studied, with stochastic perturbations to the interaction intensity. In [16] pinning control is applied to a network of non identical oscillators. Recently, an overview of the pinning control problem has been presented in [17].

Optimal pin selection can be studied with respect to the spectral properties of the graph associated to the network topology. As a consequence, algebraic graph theory is a good starting point to design some pin selection criteria.

Introductions to algebraic graph theory are provided in [18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. In particular, [19, 23] are focused on applications, while [27] is focused on algorithms defined on graphs. Spectral graph theory is addressed in a set of lessons available at [28]. Here a number of fundamental topologies are taken into account, and their spectral properties are derived analitically with some straightforward calculations.

Specific subjects of spectral graph theory have been addressed in papers such as [29, 30, 31, 32, 33, 34]. The concept of *algebraic connectivity* of a graph is defined and studied in [29]. The spectrum of the Laplacian matrix of a graph is studied in [30]. A survey about the Laplacian matrix of a graph is given in [31], with special emphasis on the second smallest eigenvalue. Spectral properties of graphs are addressed from an optimization point of view in [32, 33]. Finally, [34] focuses on the sum of the Laplacian eigenvalues of a tree graph.

1.2 Motivation

In this work we focus on a particular aspect of the pinning control problem, which is the selection of the pinned agents. This problem is often indicated as *leader selection* or *pin selection*. Our motivation was to provide an analytical framework to address the leader selection problem, which in the literature so far has been addressed mostly via numerical or stochastic approaches.

1.3 Contributions

The contribution of this thesis consists in providing an algebraic approach to the problem of leader selection, and applying this approach to a number of standard graph topologies.

First we introduce some mathematical formalisms that we later employ in our analysis. Specifically, we take inspiration from [6] to define the concepts of *augmented graph*, *augmented Laplacian* and *augmented connectivity* as pinning-based extensions of graph, graph Laplacian and algebraic connectivity respectively. Then we use the introduced formalisms to describe a network of interacting agents under pinning control and we show that the controllability of the network is strongly related to the value assumed by the augmented connectivity.

After that, four standard network topologies are considered and for all of them several pinning configurations are examined. For each of the examined configurations, we try to show how the augmented connectivity varies with respect to the intensity of the control action. In order to do this we use an algebraic procedure designed by ourselves, but largely inspired by [28]. Special emphasis is put on the upper bound that the connectivity may exhibit in each configuration.

For all the examined graphs, comparisons among different pinning strategies are proposed as well, in order to establish which one is more suitable for the graph itself.

1.4 Notations and properties

The operator $|\ |$ on a set shall indicate the cardinality of that set. The operator $||\ ||$ on a vector shall indicate the euclidean norm of that vector.

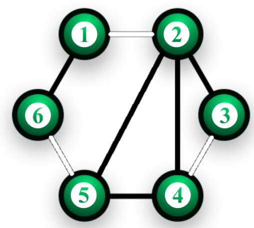
For a positive integer n we shall denote with $1_n \in \mathbb{R}_n$ the vector made up of n unitary components, and with I_n the identity matrix of order n.

The operator $\otimes$ between two matrices shall indicate the Kronecker product. We recall here some properties of the Kronecker product that we are going to use in the upcoming analysis.

Consider two square matrices $A \in \mathbb{R}_{N_a \times N_a}$ and $B \in \mathbb{R}_{N_b \times N_b}$. For $i = 1 \dots N_a$ we denote with a_i and α_i the i-th eigenvalue of matrix A and the corresponding eigenvector, while for $j = 1 \dots N_b$ we denote with b_j and β_j the j-th eigenvalue of matrix B and the corresponding eigenvector. Then the eigenvalues of matrix $A \otimes B$ are given by $\lambda_{ij} = a_i\, b_j$ while the corresponding eigenvectors are given by $v_{ij} = \alpha_i \otimes \beta_j$.

Given a vector $v \in \mathbb{R}_n$ we shall denote $v_{[N]} := 1_N \otimes v \in \mathbb{R}_{Nn}$.

Figure 1.1: Planar representation for a simple undirected graph with $N = 6$ nodes.



Consider four matrices A,B,C,D such that $A \cdot C$ and $B \cdot D$ are defined. Then it holds that $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$.

A function $f : \mathbb{R}_n \to \mathbb{R}_n$ is said to be *one-side Lipschitz* with a *Lipschitz constant* L_f if for any $x, y \in \mathbb{R}_n$ it holds that $(x - y)^T [f(x) - f(y)] \leq L_f ||x - y||^2$.

1.5 Elements of Graph Theory

Let us consider a set $\mathcal{V} = \{1, 2, \ldots, N\}$ and a set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The couple $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is called a *graph*. The elements of $\mathcal{V}$ are called *nodes* of the graph while the elements of $\mathcal{E}$ are called *edges* of the graph.

If $(i, j) \in \mathcal{E} \iff (j, i) \in \mathcal{E}$ the graph is said to be *undirected*, otherwise it is said to be *directed*. If $(i, i) \notin \mathcal{E}\ \forall i \in \mathcal{V}$ the graph is said to be *simple*. All the graphs considered in our work are simple and undirected. Therefore, all the definitions and properties given from now on are related to this particular kind of graphs.

Note that it is possible to represent a simple and undirected graph as a set of labeled points connected by lines on a plane. Each labeled point represents a node and a line connecting two points means that the corresponding couple of nodes appears in the edge set. Figure 1.1 shows such representation for a graph with $N = 6$ nodes.

In a simple undirected graph nodes i and j are said to be *neighbors* If $(i, j) \in \mathcal{E}$. For example, in the graph in Figure 1.1, nodes 1 and 6 are neighbors, as well as nodes 2 and 4.

The set of the neighbors of node i is denoted with $\mathcal{N}_i \subset \mathcal{V}$. For example, in the graph in Figure 1.1, the set of neighbors of node 4 is $\mathcal{N}_4 = \{2, 5\}$.

The number $d_i = |\mathcal{N}_i|$ of the neighbors of node i is called *degree* of node i. For example, in the graph in Figure 1.1 of node 4 is $d_4 = |\mathcal{N}_4| = 2$.

The diagonal matrix $D = diag\{d_1, d_2, \ldots, d_N\}$ is called *degree matrix* of the graph. The matrix $A = A^T = \{a_{ij}\} \in \mathbb{R}_{N \times N}$ such that

$$a_{ij} = \begin{cases} 1 & \text{if nodes } i \text{ and } j \text{ are neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

is called *adjacency matrix* of the graph. For example, the degree and adjacency matrices of the graph in Figure 1.1 are given respectively by

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.2}$$

The matrix $L = L^T = D - A$ is called *graph Laplacian*. As an example, the Laplacian of the graph in figure 1.1 is

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 3 & -1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.3}$$

It is a known result of graph theory that the Laplacian has zero row sum, and therefore it holds that $L \cdot 1_N = 0$. It is also possible to show that a Laplacian is positive semidefinite with at least one null eigenvalue. As a reference for these properties see for example [28].

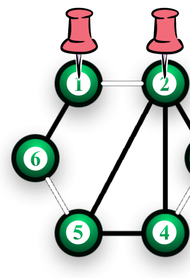
A set $\mathcal{C} \subseteq \mathcal{V}$ is called a *component* of the graph if its nodes have no neighbors outside of $\mathcal{C}$ itself. For example, the graph in Figure 1.1 contains three components, namely $\mathcal{C}_1 = \{1, 6\}$, $\mathcal{C}_2 = \{2, 3, 4, 5\}$, $\mathcal{C}_3 = \{1, 2, 3, 4, 5, 6\} = \mathcal{V}$.

When a component has no subset that is itself a component, it is said to be *connected*. Components $\mathcal{C}_1$ and $\mathcal{C}_2$ from the previous example are connected components.

A graph made up of only one connected component is said to be *connected* itself. As for the example graph in Figure 1.1, it is easy to see that it is not connected, since it contains two distinct connected components.

It is possible to show that the number of null eigenvalues of the Laplacian coincides with the number of connected components in the graph. In particular, the Laplacian of a connected graph has exactly one null eigenvalue. As for the example graph in Figure 1.1, it must have two null eigenvalues, since it contains two connected components. In fact, if we calculate its eigenvalues we obtain $eig(L) = \{0, 0, 1, 2, 3, 4\}$.

Figure 1.2: Augmented graph with $N = 6$ nodes and two pins.



The graph formalism provides an excellent starting point to model those control problems featuring a number of interacting systems. Nevertheless, by adding just a few more elements we can obtain a much more solid base to tackle the pinning control problem specifically. For the upcoming definitions of this section we have been largely inspired by [6].

Given a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, let us consider a set $\mathcal{P} \subseteq \mathcal{V}$. The nodes belonging to $\mathcal{P}$ are said to be *pinned* and the set itself is called *pin set*. Let us also consider two positive scalars $\gamma, \rho > 0$ which we shall call *coupling strength* and *pinning strength* respectively. We call the set $\tilde{\mathcal{G}} = \{\mathcal{V}, \mathcal{E}, \mathcal{P}, \gamma, \rho\}$ an *augmented graph*.

Note that the sets $\mathcal{V}$,$\mathcal{E}$ and $\mathcal{P}$ of an augmented graph can still be represented in a point-line drawing, as long as we denote the pinned nodes with a special label. Figure 1.2 illustrates an augmented version of the example graph in Figure 1.1, where we have pinned the nodes 1 and 2, that is to say $\mathcal{P} = \{1, 2\}$. The concept of augmented graph provides a formalism specifically designed to address the pinning control problem.

The diagonal matrix $P = diag\{p_1, p_2 \dots p_N\}$ such that

$$p_i = \begin{cases} 1 & \text{if node } i \text{ is pinned} \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

is called *pinning matrix* of the augmented graph. For example, the pinning matrix of the augmented graph in figure 1.2 is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.5}$$

With this elements we can define the *augmented Laplacian* of the augmented graph as

$$\tilde{L} = \gamma\, L + \rho\, P \tag{1.6}$$

It is worth pointing out that according to this definition the augmented Laplacian is itself a function of the coupling and the pinning strength, and so are all its eigenvalues.

For example, the augmented Laplacian of the augmented graph in Figure 1.2 is given by

$$\tilde{L} = \gamma \begin{bmatrix} \gamma+\rho & 0 & 0 & 0 & 0 & -\gamma \\ 0 & 3\gamma+\rho & -\gamma & -\gamma & -\gamma & 0 \\ 0 & -\gamma & \gamma & 0 & 0 & 0 \\ 0 & -\gamma & 0 & 2\gamma & -\gamma & 0 \\ 0 & -\gamma & 0 & -\gamma & 2\gamma & 0 \\ -\gamma & 0 & 0 & 0 & 0 & \gamma \end{bmatrix} \tag{1.7}$$

In the following we will denote with $\lambda_1, \lambda_2, \ldots, \lambda_N$ the eigenvalues of the augmented Laplacian of an augmented graph with N nodes.

It is possible to show that the augmented Laplacian is positive semidefinite and has as many null eigenvalues as the number of connected components in the augmented graph that do not contain pin nodes. Proof of this can be obtained as an extension of the known properties of the ordinary Laplacian.

An augmented graph in which every connected component contains at least one pin node is said to be *pinned*. For example, the augmented graph in figure 1.2 happens to be pinned, since it features two connected components and each of them contains one pin node. In a pinned graph there are no connected components that do not contain pin nodes, therefore the augmented Laplacian does not have any null eigenvalues. In fact, if we take our example graph in Figure 1.2 and we choose for example $\gamma = \rho = 1$ we get the following eigenvalues for the augmented Laplacian.

$$\begin{array}{lll} \lambda_1 = .21 & \lambda_2 = .38 & \lambda_3 = 1 \\ \lambda_4 = 2.6 & \lambda_5 = 3 & \lambda_6 = 4.8 \end{array} \tag{1.8}$$

Of course if we scale γ and ρ by the same positive factor, all the eigenvalues get scaled by the same factor.

We call the minimum eigenvalue of the augmented Laplacian *augmented connectivity*. As a consequence, we can say that the augmented connectivity of a pinned graph is strictly positive. As for the augmented graph in Figure 1.2, we have already calculated that for unitary values of γ and ρ we get $\lambda_1 = .21$.

The augmented connectivity plays a very important role in the pinning control problem. In a large number of formulations of the problem, convergence of the agents to the reference trajectory can be achieved if the augmented connectivity is large enough. Morevoer, larger values of the connectivity usually correspond to better levels of performance and robustness. As a consequence, design of the control system for the pinning control problem usually aims at making this eigenvalue as large as possible.

1.6 A Pinning Control Problem

In this section we would like to show the importance of the augmented connectivity in a common formulation of the pinning control problem.

Let us consider an augmented graph with N nodes and let us associate each node i of the graph to a nonlinear agent whose state is described by $x_i \in \mathbb{R}_n$ and whose individual dynamics is described by $\dot{x}_i = f(x_i) + u_i$, where $f : \mathbb{R}_n \to \mathbb{R}_n$ is one-side Lipschitz and $u_i \in \mathbb{R}_n$.

Let us assume that our goal is to make the agents synchronize onto a reference trajectory $s(t) \in \mathbb{R}_n$ whose dynamics is described by $\dot{s} = f(s)$. The convergence can be expressed as

$$\lim_{t \to +\infty} ||s - x_i|| = 0 \qquad i = 1 \dots N \tag{1.9}$$

In order to drive the agents onto the reference trajectory we adopt the following expression for the control signals

$$u_i = \gamma \sum_{j=1}^{N} a_{ij}\ (x_j - x_i) + \rho\ p_i(s - x_i) \tag{1.10}$$

where $A = A^T = \{a_{ij}\}$ and $P = diag\{p_1 \dots p_N\}$ are the adjacency and the pinning matrix of the augmented graph respectively, and γ, ρ are its coupling and pinning strength respectively. If we introduce the state stack vector $x = [x_1^T \dots x_N^T]^T$, and we denote $F(x) = [f(x_1)^T \dots f(x_N)^T]^T$ it is possible to express the dynamics of the state as

$$\dot{x} = F(x) - \gamma(L \otimes I_n)x + \rho(P \otimes I_n)(s_{[N]} - x) \tag{1.11}$$

where L is the Laplacian of the graph.

If for each node i we introduce the error trajectory $e_i = s - x_i$ we can also introduce the error stack vector $e = [e_1^T \dots e_N^T]^T = x - s_{[N]}$. It is easy to see that the synchronization condition (1.9) corresponds to convergence of the error stack to zero. Therefore our goal can be restated as finding a sufficient condition to drive the error stack to zero.

The dynamics of the error stack can be expressed as

$$\dot{e} = f(s)_{[N]} - F(x) + \gamma(L \otimes I_n)x - \rho(P \otimes I_n)e \tag{1.12}$$

It makes no difference to subtract $0 = (L \otimes I_n)s_{[N]}$ from the right member, so we can also rewrite

$$\dot{e} = f(s)_{[N]} - F(x) - (\tilde{L} \otimes I_n)e \tag{1.13}$$

where $\tilde{L}$ is the augmented Laplacian of the augmented graph. Now let us consider a Lyapunov candidate function $V(e) = \frac{1}{2}e^T e$, so that we can write

$$\dot{V}(e) = e^T \dot{e} = e^T[f(s)_{[N]} - F(x)] - e^T(\tilde{L} \otimes I_n)e \tag{1.14}$$

Thanks to Lipschitzianity of function f, we can say that $e^T[f(s)_{[N]} - F(x)] \leq L_f||e||^2$, while if we denote with $\lambda_1 \geq 0$ the minimum eigenvalue of the augmented Laplacian we can say that $e^T(\tilde{L} \otimes I_n)e \geq \lambda_1||e||^2$. Therefore we can upper bound the time derivative of the candidate Lyapunov function with

$$\dot{V}(e) \leq (L_f - \lambda_1)||e||^2 \tag{1.15}$$

Hence, a sufficient condition for the function's time derivative to be negative definite for any $||e|| \neq 0$ is that $\lambda_1 > L_f$. Thanks to Lyapunov's theorem we can say that when this condition is satisfied the error stack converges to zero asymptotically. Moreover, it is appeerent that with a bigger value of λ_1 we get a faster convergence to zero of the error norm.

1.7 Work Outline

In Section 1.6 we show that a pinning control problem can be seen as the problem of maximizing the smallest eigenvalue of the augmented Laplacian of the graph associated to the network under control. Note that such eigenvalue is influenced by the ratio between the coupling and the pinning strength, but also, for fixed values of the coupling and pinning strengths, by the number and location of the pin nodes.

In the following chapters we consider some standard graph topologies, and for each of them we try to identify the optimal pinning configuration in a number of different circumstances. Different pinning configurations are compared in terms of the total pinning strength they require to obtain a certain value of the augmented connectivity. The configuration that requires the lower total pinning strength is considered the better. For each pinning configuration the total pinning strength is calculated as the value of the pinning strength ρ multiplied by the number of pinned nodes. Of course if two configurations feature the same number of pinned nodes the comparison is based on the pinning strength straight away.

Without loss of generality, we work with a normalized coupling strength $\gamma = 1$, so that the augmented Laplacian is given by $\tilde{L} = L + \rho\, P$. In order to find the relationship between the pinning strength and the eigenvalues of the augmented Laplacian we use an ordinary eigenequation, that is

$$\tilde{L}\, x = \lambda\, x \tag{1.16}$$

where x is a generic vector of $\mathbb{R}_N$. Note that when there are no pin nodes we have $\tilde{L} = L$, and (1.16) can be used to obtain the eigenvalues of the Laplacian. The expressions of the eigenvalues of the Laplacian of standard graphs are known in the literature. Nevertheless, in this report we always present the resolution of (1.16) also in absence of pin nodes, since the procedure provides the framework to address the pinned configurations.

When solving (1.16) for the complete graph or the star garph we often denote with s the sum of the components of the generic vector x, that is to say $s = x_1 + x_2 + \ldots + x_N$. In fact such quantity appears rather frequently in the consequential calculations.

Note that the pinning strength is a control parameter in the pinning control problem. This means that we can tune it in order to obtain a desired value for the augmented connectivity. In other words, the pinning stregnth can be regarded as an input for the network, while the resulting value of the augemnted connectivity can be regarded as the network's response to such input. Therefore it would be maybe more natural to solve (1.16) for λ as a function of ρ. Nevertheless we prefer to adopt a reversed point of view, and use (1.16) to obtain an expression of the pinning strength as a function of the augmented connectivity. Such expression contains as much information, and has the advantage of being achievable with much more manageable calculations. The reason for this is that degree of (1.16) as an equation in the unknown λ is upper-bounded by the number N of nodes in the graph, while its degree as an equation in the unkown ρ is upper-bounded by the number m of pinned nodes, which is usually a small fraction of N.

The rest of this report is organized as follows.

- In Chapter 2 we analyse and compare several pinning configurations on the complete graph.
- In Chapter 3 we analyse and compare several pinning configurations on the star graph.
- In Chapter 4 we analyse a second-order recursion which appears constantly in the study of the upcoming topologies.
- In Chapter 5 we analyse and compare several pinning configurations on the path graph.
- In Chapter 6 we analyse and compare several pinning configurations on the ring graph.

- In Chapter 7 we point out the limitations of our work and give some possible inspiration for future developments.

Chapter 2

Complete Graph

In this chapter we address the study of the augmented connectivity of a *complete graph*. A complete graph is characterized by every node being connected to every other node.

Figure 2.1 shows an example of a complete graph with $N = 6$ nodes. It is easy to see that the Laplacian of a complete graph is given by

$$L = \begin{bmatrix} N-1 & -1 & \dots & -1 \\ -1 & N-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & N-1 \end{bmatrix} \tag{2.1}$$

For the sake of completeness before studying any pinning startegy for the complete graph let us calculate the eigenvalues of the Laplacian itself. This is a known result of graph theory, and can be found for example in [28]. Here we propose our version f the proof as a testbed for our analysis procedure.

Theorem 1. *The Laplacian of a complete graph has eigenvalues*

Figure 2.1: Complete graph with $N = 6$.

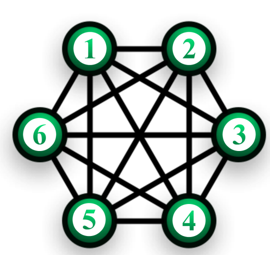
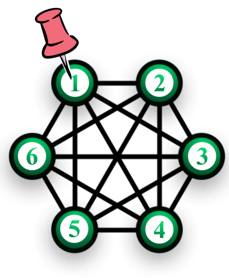


Figure 2.2: Complete Graph with $N = 6$ nodes and one pin node.



- $\lambda = 0$, *with multiplicity* 1;
- $\lambda = N$, *with multiplicity* $N - 1$.

Proof. Given the expression (2.1) for the Laplacian, it is possible to rewrite (1.16) as

$$Nx_i - s = \lambda x_i \quad i = 1, \dots, N \tag{2.2}$$

where $s = x_1 + \ldots + x_N$. With this formulation, it is easy to notice that $\lambda = N$ solves the equation for any x such that $s = 0$. Therefore $\lambda = N$ must be a $(N - 1)$-multiplicity eigenvalue. Instead, $\lambda = 0$ solves the equation with $s = Nx_i$ for all $i = 1, \dots, N$. Therefore $\lambda = 1$ must be a simple eigenvalue. □

2.1 Single-Node Pinning

In this section we consider the case in which one of the nodes of the complete graph is pinned. This configuration is represented in Figure 2.2 Of course there is no difference in pinning any of the nodes, since the graph is completely symmetrical.

Theorem 2. *The augemnted connectivity λ_1 of a complete graph with one pin is bounded by $0 < \lambda_1 < 1$. Moreover, for any admissible value of λ_1, the pinning strength is given by*

$$\rho = \frac{\lambda_1(N - \lambda_1)}{1 - \lambda_1} \tag{2.3}$$

Proof. Without loss of generality, let us assume that the first node is pinned. In this case it is easy to see that the augmented Laplacian can be rewritten as

$$\tilde{L} = L + \rho P = \begin{bmatrix} N-1+\rho & -1 & \dots & -1 \\ -1 & N-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & N-1 \end{bmatrix} \tag{2.4}$$

Therefore equation (1.16) yields

$$\begin{cases} (N+\rho)x_1 - s = \lambda x_1 & \quad (2.5a) \\ N x_i - s = \lambda x_i \qquad i = 2, \dots, N & \quad (2.5b) \end{cases}$$

where $s = x_1 + x_2 + \ldots + x_N$. This time $\lambda = N$ solves the equations for $s = 0$ and $x_1 = 0$, so it must be a $(N-2)$-multiplicity eigenvalue. Conversely, $\lambda = N + \rho$ only solves the equations for $x = 0_N$, so it cannot be an eigenvalue.

In order to find the two missing eigenvalues let us observe that, for any $\lambda \neq N, N+\rho$, we can rewrite the previous system as

$$\begin{cases} x_1 = \dfrac{s}{N+\rho-\lambda} & \quad (2.6a) \\ x_i = \dfrac{s}{N-\lambda} \qquad i = 2, \dots, N & \quad (2.6b) \end{cases}$$

If we substitute these expressions in the definition of s we get

$$s = \frac{s}{N+\rho-\lambda} + (N-1)\frac{s}{N-\lambda} \tag{2.7}$$

Since $s = 0$ leads to $x = 0_N$, this can be rewritten as

$$1 = \frac{1}{N+\rho-\lambda} + (N-1)\frac{1}{N-\lambda} \tag{2.8}$$

which after simple manipulation yields

$$(1-\lambda)\rho = \lambda(N-\lambda) \tag{2.9}$$

For $\lambda = 1$ the previous equation yields $N = 1$, which does not make sense in our scenario. Instead for any $\lambda \neq 1$, we can solve this equation for ρ, obtaining

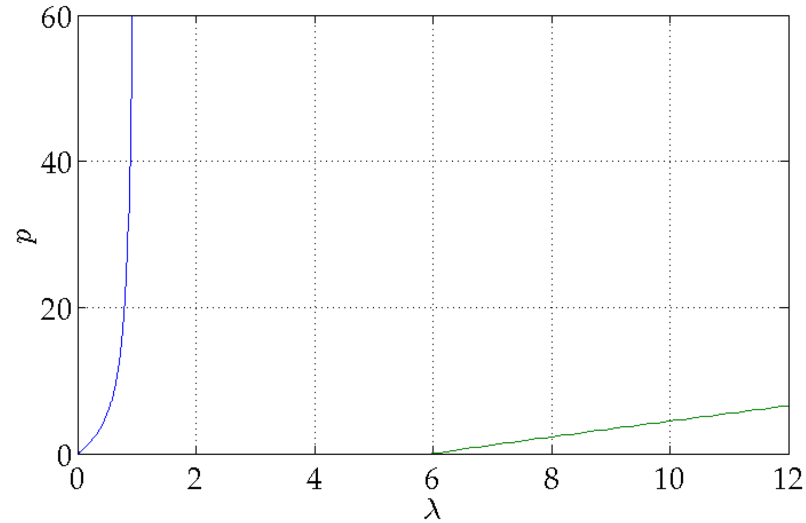
$$\rho = \frac{\lambda(N-\lambda)}{1-\lambda} \tag{2.10}$$

For $\rho > 0$ function (2.10) has two branches, the former for any $0 < \lambda < 1$ and the latter for any $\lambda > N$. The minimum eigenvalue must correspond to the first branch. □

Figure 2.3 shows the trend of the function (2.10) for $N = 6$. From the plot we can also guess how the remaining eigenvalue varies with respect to the pinning strength.

Figure 2.3: Trend of function (2.10) $N = 6$ when pinning one node. Note that the first branch is bounded by $\lambda < 1$.

(a) pinning strength



(b) upper bound

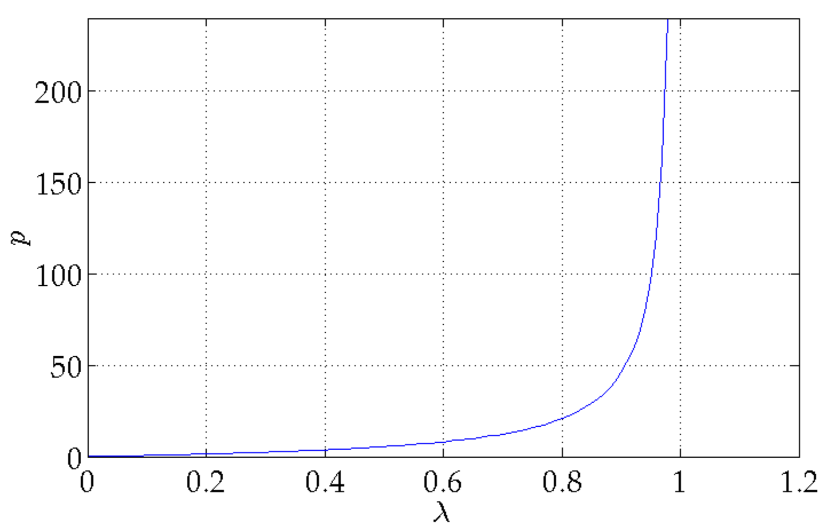
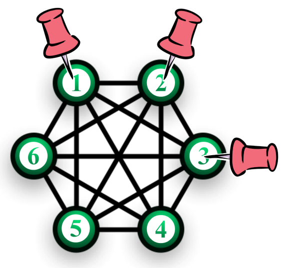


Figure 2.4: Complete Graph with $N = 6$ nodes and $m = 3$ pin nodes.



2.2 Multiple-Node Pinning

In this section we would like to generalize the result that we have presented in section 2.1 to the case in which $m < N$ nodes out of N are pinned in the complete graph. This configuration is represented in Figure 2.4 for $N = 6$ and $m = 3$. Of course, given the symmetry of the graph, there is no difference at all in pinning some nodes instead of some others.

Theorem 3. *The augmented connectivity λ_1 of a complete graph with $m < N$ pins is bounded by $0 < \lambda_1 < m$. Moreover, for any admissible value of λ_1, the pinning strength is given by*

$$\rho = \frac{\lambda_1(N - \lambda_1)}{m - \lambda_1} \tag{2.11}$$

Proof. Without loss of generality let us assume that the first m nodes are pinned. In this case it is possible to rewrite (1.16) as

$$\begin{cases} (N+\rho)x_i - s = \lambda x_i & i = 1, \dots, m \qquad (2.12a) \\ Nx_j - s = \lambda x_j & j = m+1, \dots, N \qquad (2.12b) \end{cases}$$

where we denote $s = x_1 + x_2 + \ldots + x_N$ as usual. This time $\lambda = N$ solves the equation if $s = 0$ and $x_i = 0$ for all $i = 1, \dots, m$, therefore it is a $(N-m-1)$-multiplicity eigenvalue. Instead, $\lambda = N + \rho$ solves the equation for $s = 0$ and $x_j = 0$ for all $j = m+1, \dots, N$, therefore it is an eigenvalue of multiplicity $N - (N-m) - 1 = m - 1$. We still have to find $N - (N-m-1) - (m-1) = 2$ eigenvalues.

In order to calculate the missing eigenvalues, let us note that, for any $\lambda \neq N, N + \rho$, we can rewrite the previous system as

$$\begin{cases} x_i = \dfrac{s}{N+\rho-\lambda} & i = 1,\dots,m \quad \text{(2.13a)} \\ x_j = \dfrac{s}{N-\lambda} & j = m+1,\dots,N \quad \text{(2.13b)} \end{cases}$$

Like in the previous case, we substitute these expressions in the definition of s, and we observe that it must be $s \neq 0$ we get

$$1 = \frac{m}{N+\rho-\lambda} + \frac{N-m}{N-\lambda} \tag{2.14}$$

which after a few passages leads to

$$(m-\lambda)\rho = \lambda(N-\lambda) \tag{2.15}$$

For $\lambda = m$ the previous inequality yields either $m = 0$, which does not make sense in our scenario, or $\lambda = N$, which has already been ruled out.

Instead for any $\lambda \neq m$, we can solve this equation for ρ, obtaining

$$\rho = \frac{\lambda(N-\lambda)}{m-\lambda} \tag{2.16}$$

For $\rho > 0$ function (2.16) has two branches, the former for any $0 < \lambda < m$ and the latter for any $\lambda > N$. Therefore the minimum eigenvalue must correspond to the first branch. □

Figure 2.5 shows the trend of function (2.16) for $N = 10$ and $m = 3$.

2.3 All-Node Pinning

In this section we would like to analyse the case when $m = N$ - that is to say - when we pin all the nodes in the graph. This configuration is represented in Figure 2.6 for $N = m = 6$.

Theorem 4. *The augmented connectivity λ_1 of a complete graph where all the nodes are pinned is equal to the pinning strength ρ.*

Proof. In this case equation (1.16) can be written as

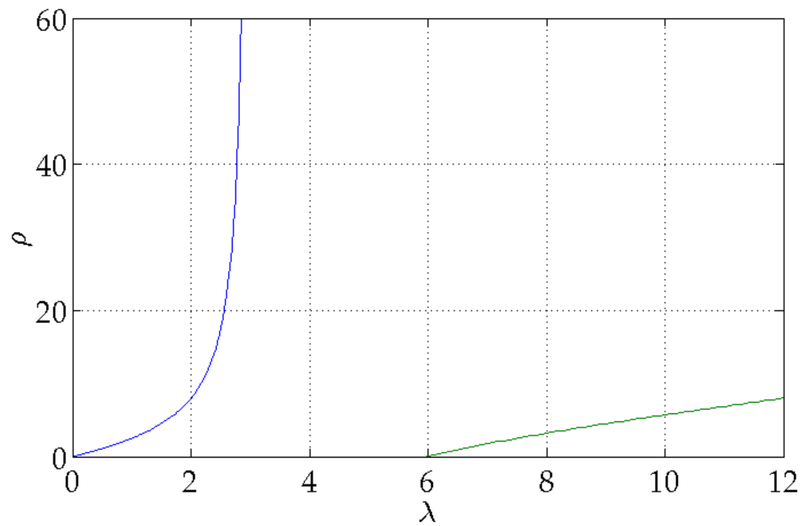
$$(N+\rho)x_i - s = \lambda x_i \quad i = 1,\dots,N \tag{2.17}$$

It is easy to see that this time $\lambda = N + \rho$ solves the equation for $s = 0$, so it is a $(N-1)$-multiplicity eigenvalue. In order to find the missing eigenvalue, it is sufficient to observe that, for any $\lambda \neq N + \rho$, we can write

$$x_i = \frac{s}{N+\rho-\lambda} \qquad i = 1,\dots,N \tag{2.18}$$

Figure 2.5: Trend of function (2.16), with $N = 6$ and $m = 3$. Note that the first branch is bounded by $\lambda < m = 3$.

(a) pinning strength



(b) upper bound

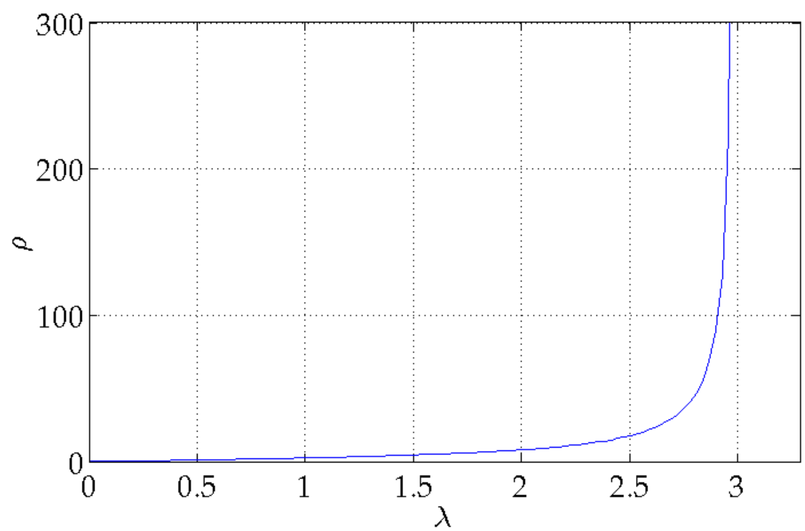
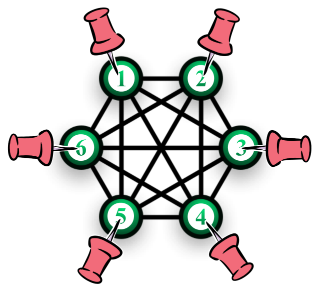


Figure 2.6: Complete Graph with $N = 6$ nodes all of which are pinned.



As usual, we substitute this expression in the definition of s and we use that $s \neq 0$, obtaining

$$1 = \frac{N}{N + \rho - \lambda} \tag{2.19}$$

which leads immediately to $\rho = \lambda$. □

2.4 Pinning Strategies Comparison

In this section we would like to compare the different possible pinning strategies for the complete graph. Since it makes no difference at all to pin a certain subset of nodes instead of a certain other with the same cardinality, the only meaningful comparisons are those among strategies that feature a different number of pin nodes.

To this regard, let us observe that when we pin $m < N$ nodes, if we want to get a certain value $0 < \lambda_1 < m$ for the augmented connectivity, we have to apply to each of the m nodes a pinning strength given by (2.16). Therefore, we have to apply a total pinning strength given by

$$\rho_m = m\frac{\lambda_1(N - \lambda_1)}{m - \lambda_1} \tag{2.20}$$

Since this is valid for any $1 \leq m < N$, we can say that when we pin $m+1 < N$ nodes, if we want to get a certain value $0 < \lambda_1 < m+1$ for the augmented connectivity, we have to apply a total pinning strength given by

$$\rho_{m+1} = (m+1)\frac{\lambda_1(N - \lambda_1)}{m + 1 - \lambda_1} \tag{2.21}$$

Therefore it is easy to see that, for any $0 < \lambda_1 < m$, we have

$$\frac{\rho_{m+1}}{\rho_m} = \frac{(m+1)(m - \lambda_1)}{m(m + 1 - \lambda_1)} = \frac{m^2 + (1 - \lambda_1)m - \lambda_1}{m^2 + (1 - \lambda_1)m} < 1 \tag{2.22}$$

Of course this reasoning can be iterated for any value of $m < N$ to show that, for any $m < n < N$, the inequality $\rho_n < \rho_m$ holds for any $0 < \lambda_1 < m$.

In the special case when we pin all the nodes, if we want to get a certain value $0 < \lambda_1 < N$ for the augmented connectivity, we have to apply a total pinning strength given by $\rho_N = N\lambda_1$. If we compare this with the total pinning strength that we have to apply when pinning $m < N$ nodes, for any $0 < \lambda_1 < m$ we get

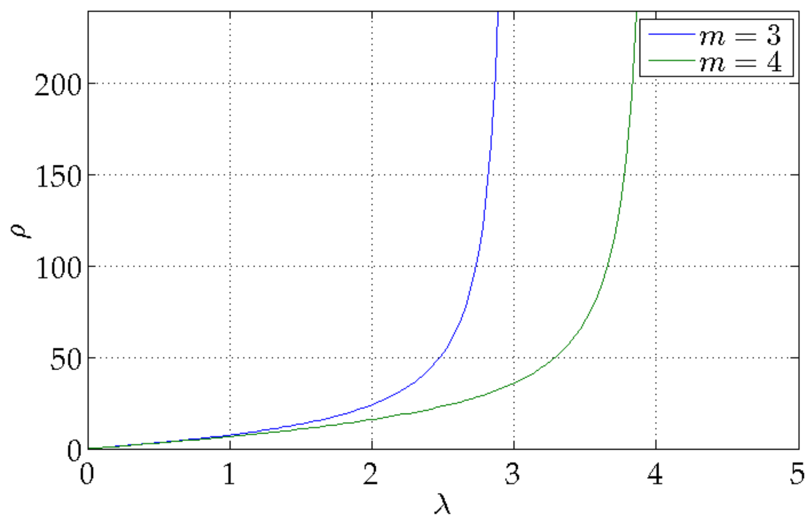
$$\frac{\rho_N}{\rho_m} = \frac{N(m-\lambda_1)}{m(N-\lambda_1)} = \frac{Nm - N\lambda_1}{Nm - m\lambda_1} < 1 \tag{2.23}$$

Therefore we can say that in all cases, if we distribute the pinning strength among a larger number of pin nodes we get both a better upper bound for the augmented connectivity and, for any admissible value fo the connectivity, a better trend of the pinning strength.

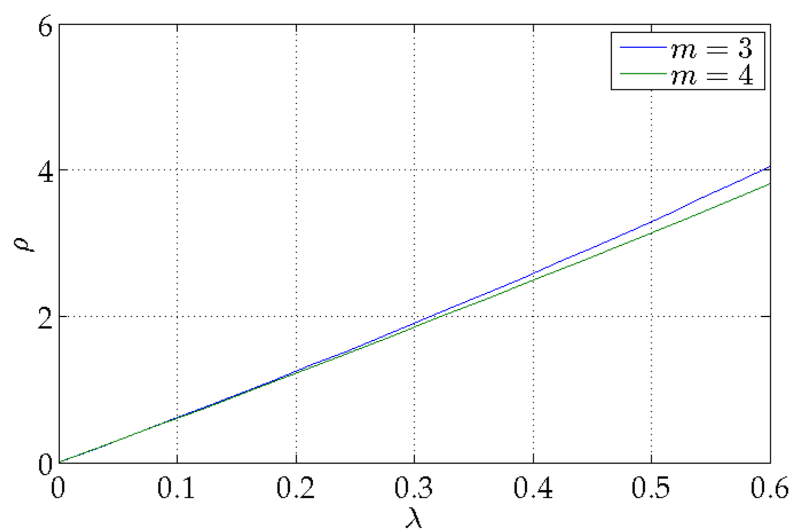
As a confirmation of our analytical results, Figure 2.7 compares the trend of functions ρ_m for $m = 3$ and $m = 4$ when $N = 6$.

Figure 2.7: functions ρ_m for $m = 3$ (blue) and $m = 4$ (green) with $N = 6$.

(a) comparison



(b) zoom



Chapter 3

Star Graph

In this chapter we address the study of the augmented connectivity of a *star graph*. In a star graph there is one node connected to all the other nodes, which have no further connections.

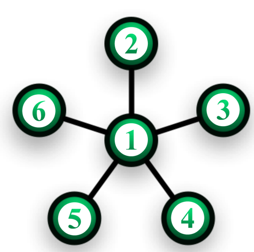
Figure 3.1 shows a star graph with $N = 6$ nodes.

Without loss of generality, let us assume that the central node is the first node. In this case it is easy to see that the Laplacian of a star graph can be written as

$$L = \begin{bmatrix} N-1 & -1 & \dots & -1 \\ -1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & 1 \end{bmatrix} \tag{3.1}$$

Before addressing any pinning configuration for the star graph, let us calculate the eigenvalues of the Laplacian of the graph itself. This is a known result in graph theory, and can be found for example in [28]. Nevertheless we use this

Figure 3.1: Star graph with $N = 6$ nodes



proof as a testbed for our analysis procedure, before applying it to the pinning configurations.

Theorem 5. *The Laplacian of a star graph with no pins has eigenvalues*

- $\lambda = N$, *with multiplicity* 1*;*
- $\lambda = 1$, *with multiplicity* $N-2$*;*
- $\lambda = 0$, *with multiplicity* 1*.*

Proof. Using expression (3.1) for the Laplacian, we can easily write equation (1.16) as

$$\begin{cases} Nx_1 - s = \lambda x_1 & \quad (3.2a) \\ -x_1 + x_j = \lambda x_j \qquad j = 2, \dots, N & \quad (3.2b) \end{cases}$$

It is easy to see that $\lambda = 1$ solves the equation for $x_1 = 0$ and $s = 0$, so it must be a $(N-2)$-multiplicity eigenvalue. In order to find the two missing eigenvalues let us rewrite the previous system as

$$\begin{cases} x_1 = (1-\lambda)x_j \qquad j = 2, \dots, N & \quad (3.3a) \\ s = (N-\lambda)x_1 & \quad (3.3b) \end{cases}$$

Now if we observe that

$$s = x_1 + (N-1)x_j \tag{3.4}$$

and we substitute (3.3a), (3.3b) into it, we get easily

$$(N-1)(1-\lambda)x_j = (1-\lambda)x_j + (N-1)x_j \tag{3.5}$$

We can exclude $x_j = 0$, since it leads to $x = 0_N$. Therefore, after simple manipulation we get from (3.5) that

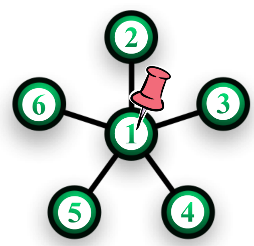
$$\lambda(N-\lambda) = 0 \tag{3.6}$$

Therefore the two missing eigenvalues must be $\lambda = 0$ and $\lambda = N$. □

3.1 Central-Node Pinning

In this section we study the case when the central node is pinned in a star graph with N nodes. This configuration is represented in Figure 3.2 for $N = 6$. Since we assume that the central node is the first one in the node set, the augmented Laplacian is given by

Figure 3.2: Star Graph with $N = 6$ nodes, the central node being pinned.



$$\tilde{L} = \begin{bmatrix} N-1+\rho & -1 & \dots & -1 \\ -1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & 1 \end{bmatrix} \tag{3.7}$$

Theorem 6. *In a star graph where the central node is pinned, the augmented connectivity λ_1 is bounded by $0 < \lambda_1 < 1$. Moreover, for all the admissible values of λ_1 the pinning strength is given by*

$$\rho = \frac{\lambda_1(N - \lambda_1)}{1 - \lambda_1} \tag{3.8}$$

Proof. Accounting for expression (3.7) of the augmented Laplacian, it is possible to write equation (1.16) as

$$\begin{cases} (N+\rho)x_1 - s = \lambda x_1 & \quad (3.9a) \\ -x_1 + x_j = \lambda x_j \qquad j = 2, \dots, N & \quad (3.9b) \end{cases}$$

Let us observe that $\lambda = 1$ solves the equations for $x_1 = 0$ and $s = 0$. Therefore it must be a $(N-2)$-multiplicity eigenvalue. In order to find the two missing eigenvalues, let us rewrite the previous system as

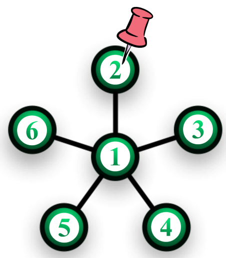
$$\begin{cases} s = (N + \rho - \lambda) & \quad (3.10a) \\ x_1 = (1-\lambda)x_j \qquad j = 2, \dots, N & \quad (3.10b) \end{cases}$$

Now if we observe that

$$s = x_1 + (N-1)x_j \tag{3.11}$$

and we substitute (3.10a), (3.10b) into it, we get easily

Figure 3.3: Star graph with $N = 6$ nodes and one peripheral node pinned.



$$(N + \rho - 1)(1 - \lambda)x_j = (1 - \lambda)x_j + (N - 1)x_j \tag{3.12}$$

We can exclude $x_j = 0$, since it leads to $x = 0_N$. Therefore, after simple manipulation we get from (3.12)

$$(1 - \lambda)\rho = \lambda(N - \lambda) \tag{3.13}$$

Since $\lambda = 1$ has been already ruled out, we can conclude that

$$\rho = \frac{\lambda(N - \lambda)}{1 - \lambda} \tag{3.14}$$

□

Let us observe that the expression obtained for the pinning strength as a function of the connectivity is identical to the one obtained when pinning one node in the complete graph. This means that when we pin an agent which is connected to all the other ones, the presence of additional connections does not affect the augmented connectivity at all. On the other hand, additional connections affect the values of the other eigenvalues of the augmented Laplacian. Since function (3.14) is identical to function (2.10), its trend has already been plotted in Figure 2.2.

3.2 Single-Peripheral-Node Pinning

In this section we study the case of one peripheral node being pinned in the star graph. This configuration is presented in Figure 3.3 for $N = 6$.

Theorem 7. *In a star graph where one peripheral node is pinned, the augmented connectivity λ_1 is bounded by*

$$0 < \lambda_1 < \frac{N - \sqrt{N^2 - 4}}{2} \tag{3.15}$$

Moreover, for any admissible value of λ_1, the pinning strength is given by

$$\rho = \frac{\lambda_1(1-\lambda_1)(N-\lambda_1)}{\lambda_1^2 - N\lambda_1 + 1} \tag{3.16}$$

Proof. Of course, given the symmetrical structure of the graph, there is no difference in pinning any of the peripheral nodes. Therefore, without loss of generality, let us say that the second node is pinned. In this case equation (1.16) yields

$$\begin{cases} Nx_1 - s = \lambda x_1 & \text{(3.17a)} \\ -x_1 + (1+\rho)x_2 = \lambda x_2 & \text{(3.17b)} \\ -x_1 + x_j = \lambda x_j \qquad j = 3, \dots, N & \text{(3.17c)} \end{cases}$$

This time $\lambda = 1$ solves the equation for $x_1 = 0$, $x_2 = 0$ and $s = 0$, so it must be a $(N-3)$-multiplicity eigenvalue. Let us also observe that $\lambda = 1 + \rho$ leads to $x = 0_N$, so it is not an eigenvalue. In order to find the three missing eigenvalues, let us rewrite the previous system as

$$\begin{cases} s = N - \lambda x_1 & \text{(3.18a)} \\ x_2 = \dfrac{1}{1+\rho-\lambda} x_1 & \text{(3.18b)} \\ x_j = \dfrac{1}{1-\lambda} x_1 \qquad j = 3, \dots, N & \text{(3.18c)} \end{cases}$$

Now let us observe that

$$s = x_1 + x_2 + (N-2)x_j \tag{3.19}$$

and let us substitute equations (3.18a), (3.18b), (3.18c) into it, obtaining

$$(N-\lambda)x_1 = x_1 + \frac{1}{1+\rho-\lambda}x_1 + (N-2)\frac{1}{1-\lambda}x_1 \tag{3.20}$$

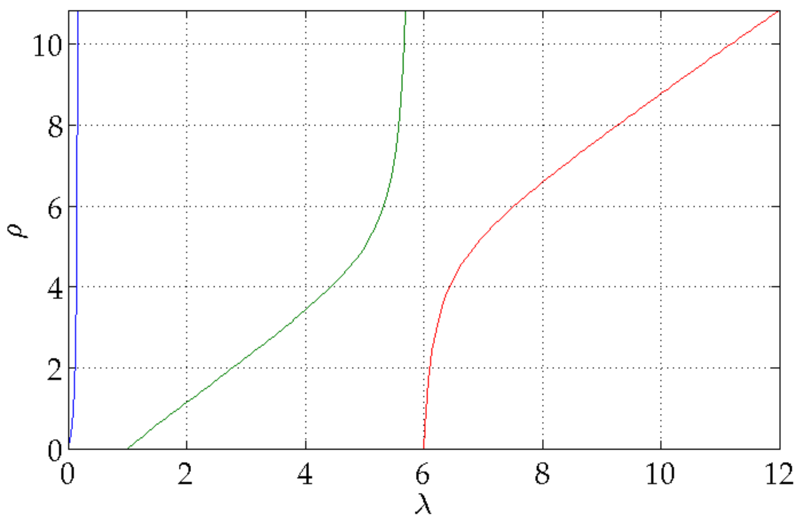
For $\lambda^2 - N\lambda + 1 = 0$ equation (3.20) yields $\lambda(1-\lambda)(N-\lambda) = 0$, meaning that $\lambda = 0, 1, N$. But none of these values is a root of $\lambda^2 - N\lambda + 1 = 0$ in the first place. Therefore we can state that $\lambda^2 - N\lambda + 1 \neq 0$, and rewrite equation (3.20) as

$$\rho = \frac{\lambda(1-\lambda)(N-\lambda)}{\lambda^2 - N\lambda + 1} \tag{3.21}$$

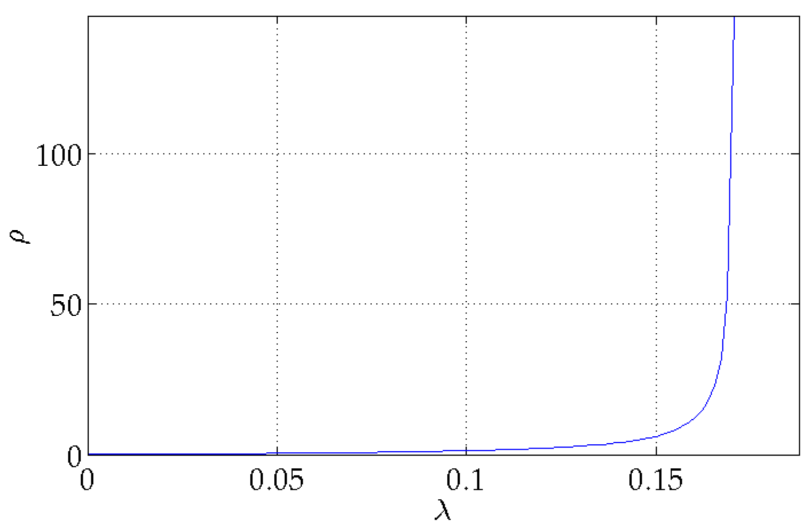
For $\rho > 0$ this expression has three branches, the first one with $0 < \lambda < \frac{N-\sqrt{N^2-4}}{2} < 1$, the second one with $1 < \lambda < \frac{N+\sqrt{N^2-4}}{2} < N$ and the third one with $\lambda > N$. Therefore the minimum eigenvalue must correspond to the first branch. □

Figure 3.4: Trend of function (3.21) with $N = 6$. Note that the first branch is bounded by $\lambda < \frac{N-\sqrt{N^2-4}}{2} \simeq \frac{1}{N} \simeq .17$.

(a) pinning strength



(b) upper bound



Let us observe that for growing values of N the upper bound tends to $\frac{1}{N}$.

Figure 3.4 shows the trend of function (3.21) for $N = 6$. In this case the upper bound for the minimum eigenvalue is $\frac{N-\sqrt{N^2-4}}{2} \simeq \frac{1}{N} \simeq .17$, and it is easy to see that the first branch of the function lays indeed before this value.

3.3 Single-Pin Strategies Comparison

In this section we would like to compare the two pinning strategies that we have so far introduced for the star graph. To this aim, let us denote with ρ_c the pinning strength used when pinning the central node and ρ_p the pinning strength used when pinning one peripheral node. For a generical value λ_1 of the augemnted connectivity we have

$$\rho_c = \frac{\lambda_1(N-\lambda_1)}{1-\lambda_1} \qquad \rho_p = \frac{\lambda_1(1-\lambda_1)(N-\lambda_1)}{\lambda_1^2 - N\lambda_1 + 1} \tag{3.22}$$

Therefore, for any $0 < \lambda_1 < \frac{N-\sqrt{N^2-4}}{2}$, we can calculate

$$\frac{\rho_c}{\rho_p} = \frac{\lambda_1^2 - N\lambda_1 + 1}{(1-\lambda_1)^2} = \frac{\lambda_1^2 - N\lambda_1 + 1}{\lambda_1^2 - 2\lambda_1 + 1} < 1 \tag{3.23}$$

This means that, if we pin a peripheral node, not only we get a smaller upper bound for the augmented connectivity, but also, for all the admissible values of λ_1, we need to apply a higher pinning strength.

Figure 3.5 compares the first branch of functions ρ_c and ρ_p for $N = 6$. Such figure confirms that, for the relevant values of λ_1, function ρ_c is always below function ρ_p.

3.4 Multiple-Peripheral-Node Pinning

In this section we would like to study the case when $m < N - 1$ out of N peripheral nodes are pinned in the star graph. This configuration is represented in Figure 3.6 for $N = 6$ and $m = 3$.

Theorem 8. *In a star graph where $m < N - 1$ peripheral nodes are pinned out of N, the augmented connectivity λ_1 is bounded by*

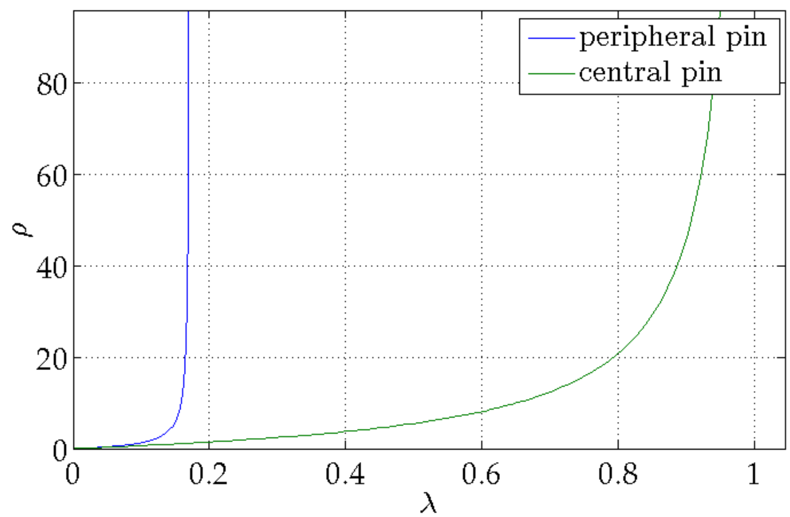
$$0 < \lambda_1 < \frac{N - \sqrt{N^2 - 4m}}{2} < 1 \tag{3.24}$$

Moreover, for any admissible value of λ_1, the pinning strength is given by

$$\rho = \frac{\lambda_1(1-\lambda_1)(N-\lambda_1)}{\lambda_1^2 - N\lambda_1 + m} \tag{3.25}$$

Figure 3.5: First branch of functions ρ_p (blue) and ρ_c (green) for $N = 6$. Note that function ρ_c is always below function ρ_p.

(a) comparison



(b) zoom

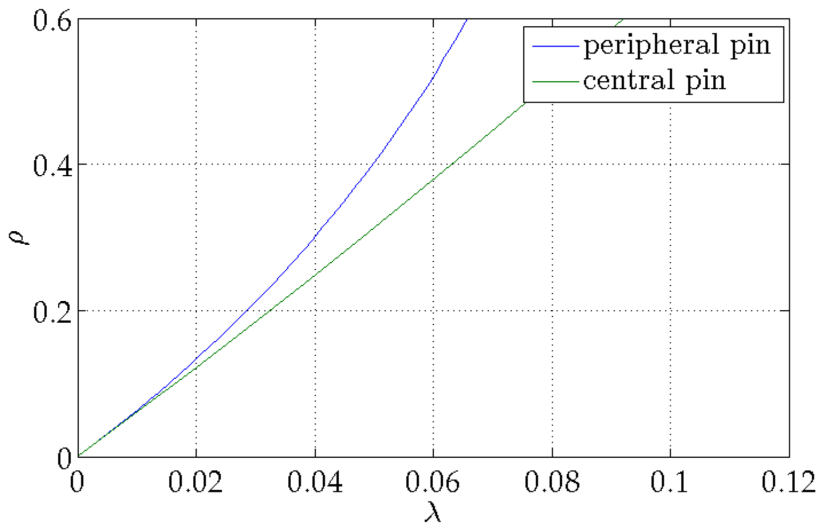
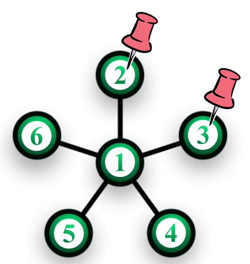


Figure 3.6: Star graph with $N = 6$ nodes and $m = 3$ peripheral pins



Proof. Without loss of generality let us say that nodes from 2 to $m+1$ are pinned. In this case equation (1.16) yields

$$\begin{cases} Nx_1 - s = \lambda x_1 & \quad (3.26a)\\ -x_1 + (1+\rho)x_i = \lambda x_i \qquad i = 2,\dots,m+1 & \quad (3.26b)\\ -x_1 + x_j = \lambda x_j \qquad j = m+2,\dots,N & \quad (3.26c)\end{cases}$$

This time $\lambda = 1$ solves the equations for $x_1 = 0$, $x_i = 0$ with $i = 2,\dots,m+1$ and $s = 0$, so it must be an eigenvalue with multiplicity equal to $N-1-m-1 = N-m-2$.

Instead, $\lambda = 1+\rho$ solves the equations for $x_1 = 0$, $x_j = 0$ with $j = m+2,\dots,N$ and $s = 0$, so it must be an eigenvalue with multiplicity equal to $N-1-(N-m-1)-1 = m-1$.

Therefore we are missing $N-(N-m-2)-(m-1) = 3$ eigenvalues. In order to find the three missing eigenvalues we reason as in the previous section and we observe that for any $\lambda \neq 1, 1+\rho$ we can rewrite the system as

$$\begin{cases} s = (N-\lambda)x_1 & \quad (3.27a)\\ x_i = \dfrac{1}{1+\rho-\lambda}x_1 \qquad i = 2,\dots,m+1 & \quad (3.27b)\\ x_j = \dfrac{1}{1-\lambda}x_1 \qquad j = m+2,\dots,N & \quad (3.27c)\end{cases}$$

As usual, if we observe that

$$s = x_1 + m x_i + (N-m-1)x_j \tag{3.28}$$

and we substitute equations (3.27a), (3.27b), (3.27c) into it we get

$$(N-\lambda)x_1 = x_1 + m\frac{1}{1+\rho-\lambda}x_1 + (N-m-1)\frac{1}{N-\lambda}x_1 \tag{3.29}$$

We exclude $x_1 = 0$ since it leads to $x = 0_N$ and we perform simple manipulations to solve the equation for ρ. In this case we obtain

$$(\lambda^2 - N\lambda + m)\,\rho = \lambda(1-\lambda)(N-\lambda) \tag{3.30}$$

For $\lambda^2 - N\lambda + m = 0$ equation (3.30) yields $\lambda(1-\lambda)(N-\lambda) = 0$, meaning that $\lambda = 0, 1, N$. But none of these values is a root of $\lambda^2 - N\lambda + m = 0$ in the first place. Therefore we can state that $\lambda^2 - N\lambda + m \neq 0$, and rewrite equation (3.30) as

$$\rho = \frac{\lambda(1-\lambda)(N-\lambda)}{\lambda^2 - N\lambda + m} \tag{3.31}$$

For $\rho > 0$ this expression has three branches, the first one with $0 < \lambda < \frac{N-\sqrt{N^2-4m}}{2} < 1$, the second one with $1 < \lambda < \frac{N+\sqrt{N^2-4m}}{2} < N$ and the third one with $\lambda > N$. Therefore the minimum eigenvalue must correspond to the first branch. □

Let us note that for a large value of N the upper bound for the augmented connectivity tends to $\frac{m}{N}$.

Figure 3.7 shows the trend of function (3.31) for $N = 6$ and $m = 3$. In this case the upper bound for the minimum eigenvalue is $\frac{N-\sqrt{N^2-4m}}{2} \simeq \frac{m}{N} \simeq .5$, and it is easy to see that the first branch lays indeed before this value.

3.5 Multiple-Peripheral-Pin Strategies Comparison

In this section we would like to compare the total pinning strength required when pinning different numbers of peripheral nodes. From theorem 8 we can easily see that distributing the control strength among a larger number of nodes always yields a higher upper bound for the augmented connectivity. In fact, when pinning m peripheral nodes the bound is given by (3.24) which grows larger for larger values of m.

Let us now focus on the trend of the total pinning strength. From Theorem 8 it is immediate to deduce that when we pin m peripheral nodes the total pinning strength is given by

$$\rho_m = m\frac{\lambda(1-\lambda)(N-\lambda)}{\lambda^2 - N\lambda + m} \tag{3.32}$$

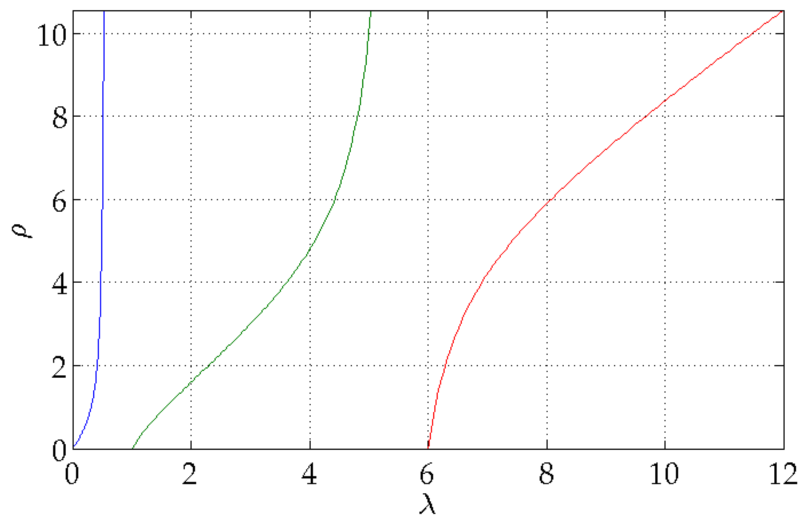
while when we pin $m+1$ peripheral nodes we have a total pinning strength

$$\rho_{m+1} = (m+1)\frac{\lambda(1-\lambda)(N-\lambda)}{\lambda^2 - N\lambda + m + 1} \tag{3.33}$$

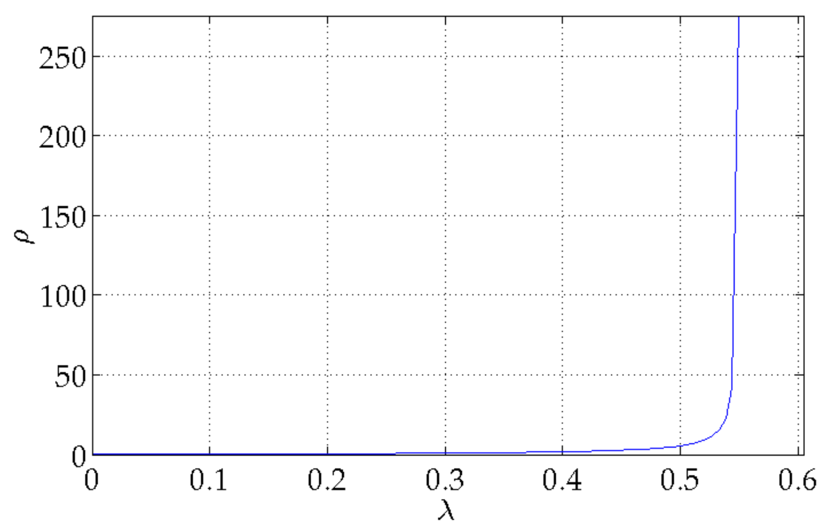
So the ratio between this two quantities is given by

Figure 3.7: Trend of function (3.31) with $N = 6$ and $m = 3$. Note that the first branch is bounded by $\lambda < \frac{N-\sqrt{N^2-4m}}{2} \simeq \frac{m}{N} \simeq .5$.

(a) pinning strength



(b) upper bound



$$\frac{\rho_{m+1}}{\rho_m} = \frac{(m+1)(\lambda^2 - N\lambda + m)}{m(\lambda^2 - N\lambda + m + 1)} = \frac{m(\lambda^2 - N\lambda + m) + \lambda^2 - N\lambda + m}{m(\lambda^2 - N\lambda + m) + m} \tag{3.34}$$

hence we can conclude that $\rho_{m+1} < \rho_m \iff \lambda(N - \lambda) < 0$. Since the last inequality is always satisfied by the augmented connectivity, we can conclude that distributing the pinning strength among a larger number of peripheral nodes always yields a better trend of the total pinning strength as a function of the augmented connectivity.

Figure 3.8 compares the first branch of functions ρ_m with $m = 3$ and $m = 4$ for $N = 6$. The picture confirms that the trend corresponding to the higher distribution is the better.

3.6 All-Peripheral-Node Pinning

In this section we would like to study the case when all the peripheral nodes are pinned in a star graph. This configuration is represented in Figure 3.9 for a star graph with $N = 6$ nodes.

Theorem 9. *In a star graph where all the peripheral nodes are pinned the augmented connectivity λ_1 is bounded by*

$$0 < \lambda_1 < N - 1 \tag{3.35}$$

Moreover, for any admissible value of λ_1, the pinning strength is given by

$$\rho = \frac{\lambda_1(N - \lambda_1)}{N - 1 - \lambda_1} \tag{3.36}$$

Proof. In this case equation (1.16) yields

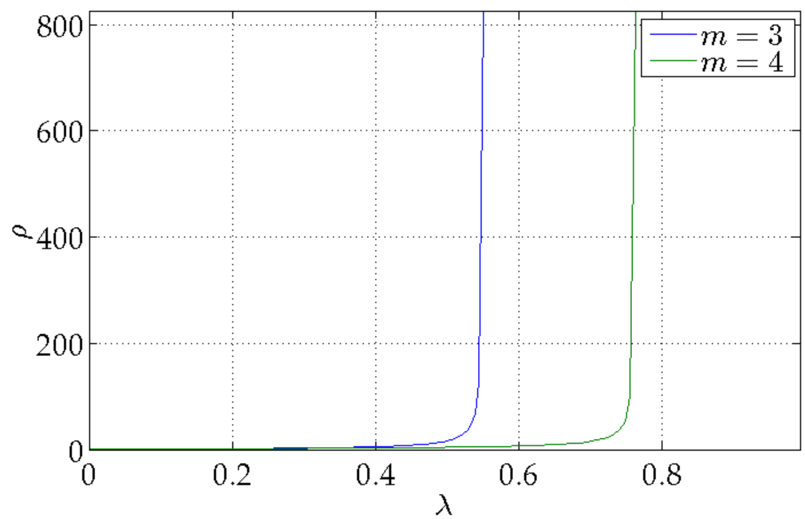
$$\begin{cases} Nx_1 - s = \lambda x_1 & \text{(3.37a)} \\ -x_1 + (1+\rho)x_i = \lambda x_i \qquad i = 2, \dots, N & \text{(3.37b)} \end{cases}$$

We observe that $\lambda = 1+\rho$ solves the equation for $x_1 = 0$ and $s = 0$, meaning that it must be an eigenvalue with multiplicity equal to $N - 2$. In order to find the two missing eigenvalues, let us note that, for $\lambda \neq 1 + \rho$, we can rewrite the system as

$$\begin{cases} s = (N - \lambda)x_1 & \text{(3.38a)} \\ x_i = \dfrac{x_1}{1 + \rho - \lambda} \qquad i = 2, \dots, N & \text{(3.38b)} \end{cases}$$

Figure 3.8: First branch of functions ρ_m with $m = 3$ (blue) and $m = 4$ (green) for $N = 6$. Note that function ρ_4 is always below function ρ_3.

(a) comparison



(b) zoom

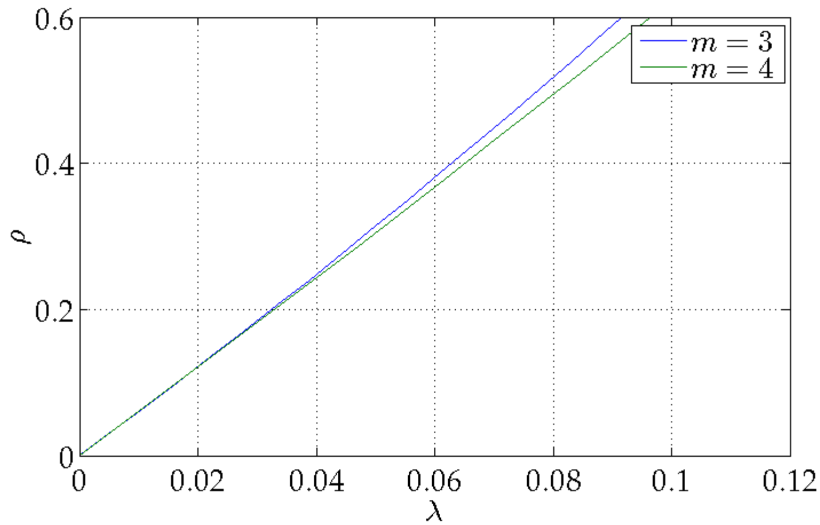
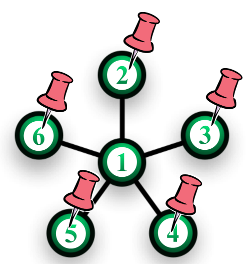


Figure 3.9: Star Graph with $N = 6$ nodes and all peripheral nodes pinned.



Now if we observe that

$$s = x_1 + (N-1)x_i \tag{3.39}$$

and we substitute (3.38a), (3.38b) into it, we obtain

$$(N-\lambda)x_1 = x_1 + \frac{N-1}{1+\rho-\lambda}x_1 \tag{3.40}$$

As usual, we exclude $x_1 = 0$, which leads to $x = 0_N$, and we perform simple manipulation, obtaining

$$(N-1-\lambda)\rho = \lambda(N-\lambda) \tag{3.41}$$

We can exclude $\lambda = N-1$ which leads to $N = 1$, which does not make sense in our scenario. Therefore we can solve equation (3.41) for ρ, obtaining

$$\rho = \frac{\lambda(N-\lambda)}{N-1-\lambda} \tag{3.42}$$

For $\rho > 0$ this expression has two branches, the former with $0 < \lambda < N-1$, and the latter with $\lambda > N$.

Now we have to understand whether the minimum eigenvalue is given by $\lambda = 1+\rho$ or it corresponds to the first branch of (3.42). This can be done very easily if we adopt a reversed point of view and we compare

$$\rho_a = \lambda - 1, \qquad \rho_b = \frac{\lambda(N-\lambda)}{N-1-\lambda} \tag{3.43}$$

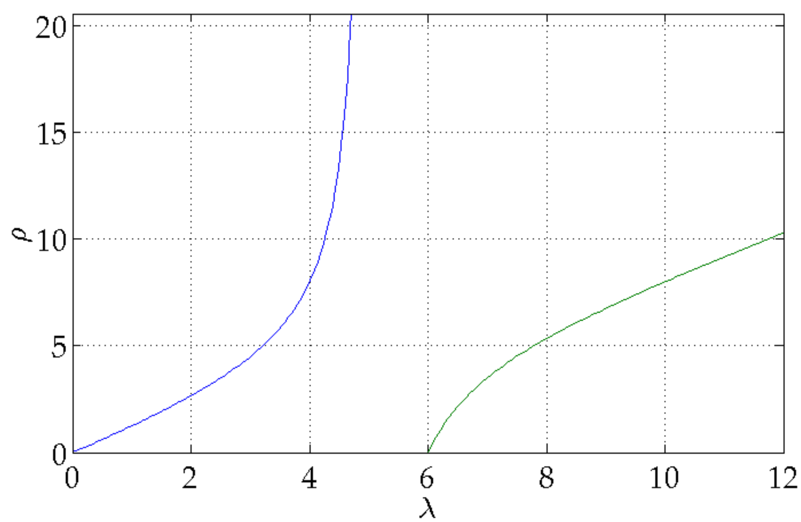
For $\lambda < N-1$ it is immediate to write

$$\rho_a < \rho_b \iff (\lambda-1)(N-1-\lambda) < \lambda(N-\lambda) \iff 1 < N \tag{3.44}$$

from which we can conclude that the minimum eigenvalue corresponds to p_b for $\lambda < N-1$, or equivalently, to the first branch of (3.42). □

Figure 3.10: Trend of function (3.42) for $N = 6$. Note that the first branch is bounded by $\lambda < N - 1 = 5$.

(a) pinning strength



(b) upper bound

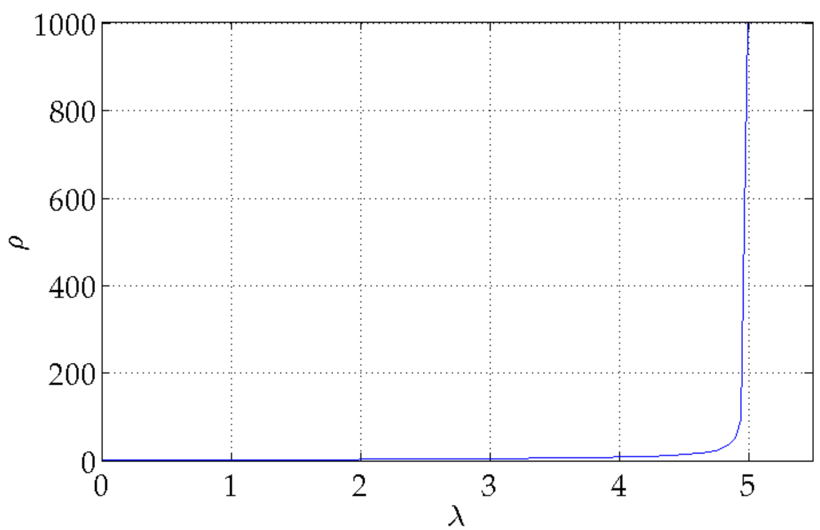


Figure 3.10 shows the trend of function (3.42) for $N = 6$. As we expected, the upper bound for the minimum eigenvalue is given by $N - 1 = 5$.

3.7 Multiple-Peripheral-Pin Strategies Comparison

In this section we would like to compare the total pinning strength required when pinning all the peripheral nodes and when pinning only a fraction of them. From Theorem 8 we know that when we pin $m < N-1$ perpheral nodes the total pinning strength is given by (3.32) and the connectivity is bounded by (3.24). From Theorem 9 we know that when pinning all the peripheral pins the total pinning strength is given by

$$\rho_{N-1} = (N-1)\frac{\lambda(N-\lambda)}{N-\lambda-1} \tag{3.45}$$

and the connectivity is bounded by $\lambda_1 < N - 1$. It is immediate to see that the last is a better upper bound.

As for the trend of the pinning strength, from the expressions of ρ_m and ρ_{N-1} it only takes a few passages to write

$$\frac{\rho_{N-1}}{\rho_m} = \frac{(N-1)\lambda(\lambda-N) + m(N-1)}{m\lambda(\lambda-N) + m(N-1)} \tag{3.46}$$

from which it is easy to see that $\rho_{N-1} < \rho_m \iff \lambda < N$, which is satisfied by any admissible value of λ. Hence we can state that distributing the pinning strength among all the peripheral nodes is always better than pin only a subset of them.

Figure 3.11 compares functions ρ_m and ρ_{N-1} when $N = 6$. The picture confirms that having all the peripheral nodes pinned always yields a lower total pinning strength than pinning only a subset of them.

3.8 Multiple-Central-Node Pinning

In this section we would like to address the case when $m < N$ nodes are pinned in the star graph, including the central one. This configuration is represented in Figure 3.12 for $N = 6$ and $m = 3$.

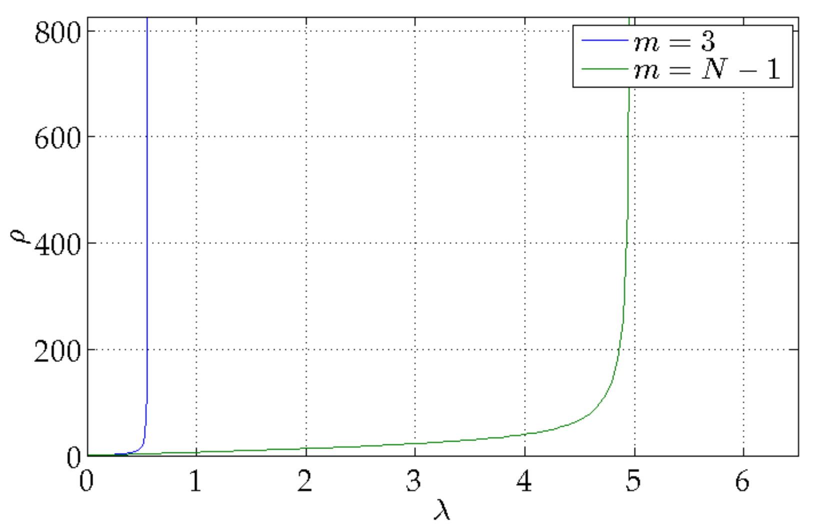
Theorem 10. *In a star graph where $m < N$ multiple nodes are pinned, including the central one, the augmented connectivity λ_1 is bounded by*

$$0 < \lambda_1 < 1 \tag{3.47}$$

Moreover, for any admissible value of λ_1, the pinning strength is given by the positive roots of

Figure 3.11: Comparison between the first branch of functions ρ_m (blue) and ρ_{N-1} (green) with $m = 3$ and $N = 6$. Note that function ρ_{N-1} lays always below function ρ_m.

(a) comparison



(b) zoom

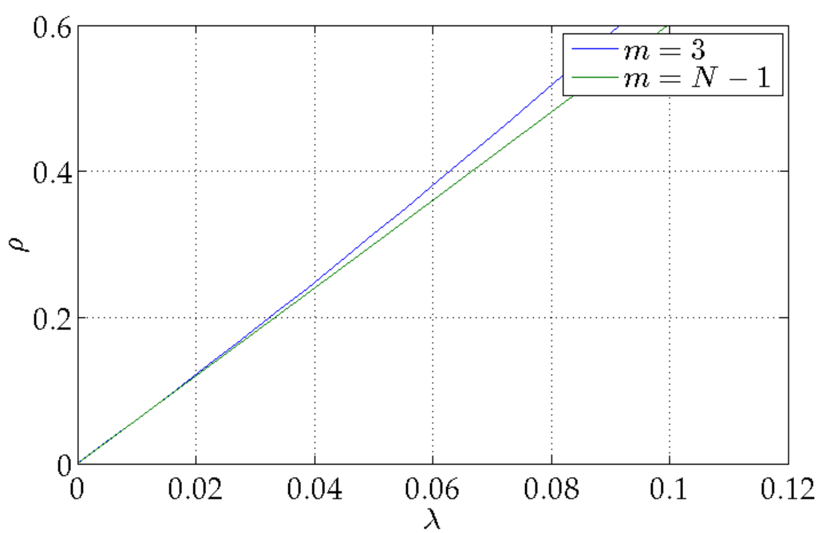
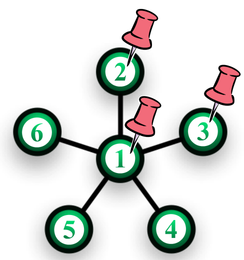


Figure 3.12: Star Graph with $N = 6$ nodes and $m = 3$ pin nodes, including the central node.



$$(1-\lambda_1)\rho^2 + (2\lambda_1^2 - (N+2)\lambda_1 + m)\rho - \lambda_1(1-\lambda_1)(N-\lambda_1) = 0 \tag{3.48}$$

Proof. Without loss of generality, let us say that the first m nodes are pinned. In this case it is easy to see that equation (1.16) yields

$$\begin{cases} (N+\rho)x_1 - s = \lambda x_1 & & (3.49a)\\ -x_1 + (1+\rho)x_i = \lambda x_i & i = 2,\dots,m & (3.49b)\\ -x_1 + x_j = \lambda x_j & j = m+1,\dots,N & (3.49c) \end{cases}$$

This time we can see that $\lambda = 1$ solves the equations for $x_1 = 0$, $x_i = 0$ with $i = 2,\dots,m$ and $s = 0$. Therefore it must be an eigenvalue of multiplicity $N-1-(m-1)-1 = N-m-1$.

Instead $\lambda = 1+\rho$ solves the equations for $x_1 = 0$, $x_j = 0$ with $j = m+1,\dots,N$ and $s = 0$. Therefore it must be an eigenvalue of multiplicity $N-1-(N-m)-1 = m-2$.

This means that we are missing $N-(N-m-1)-(m-2) = 3$ eigenvalues. In order to calculate them, let us observe that, for any $\lambda \neq 1, 1+\rho$, we can rewrite the system as

$$\begin{cases} s = (N+\rho-\lambda)x_1 & & (3.50a)\\ x_i = \dfrac{x_1}{1-\lambda+\rho} & i = 2,\dots,m & (3.50b)\\ x_j = \dfrac{x_1}{1-\lambda} & j = m+1,\dots,N & (3.50c) \end{cases}$$

As usual, we observe that

$$s = x_1 + (m-1)x_i + (N-m-1)x_j \tag{3.51}$$

and we substitute expressions (3.50a),(3.50b),(3.50c) into it, obtaining

$$(N+\rho-\lambda)x_1 = x_1 + \frac{m-1}{1+\rho-\lambda} + \frac{N-m-1}{1-\lambda}x_1 \tag{3.52}$$

We exclude $x_1 = 0$ since it leads to $x = 0_N$ and we perform simple algebraic passages to obtain

$$(1-\lambda)\rho^2 + (2\lambda^2 - (N+2)\lambda + m)\rho - \lambda(1-\lambda)(N-\lambda) = 0 \tag{3.53}$$

For $\lambda = 1$ equation (3.53) yields $m = N$, which has been excluded in our scenario.

For $\lambda \neq 1$ the equation can be solved for ρ. The function corresponding to the positive solutions for ρ has three branches, one for $0 < \lambda < 1$, one for $\lambda > 1$ and one for $\lambda > N$. The minimum eigenvalue must correspond to the first branch, therefore it is bounded by $0 < \lambda_1 < 1$. □

Figure 3.13 shows the trend of function (3.53) for $N = 6$ and $m = 3$. It is easy to see that the first branch of the function is upper-bounded by $\lambda < 1$.

3.9 Multiple-Pin Strategies Comparison

In this section we would like to compare the case when $m < N$ peripheral nodes are pinned and the case when $m < N$ nodes, including the central one, are pinned.

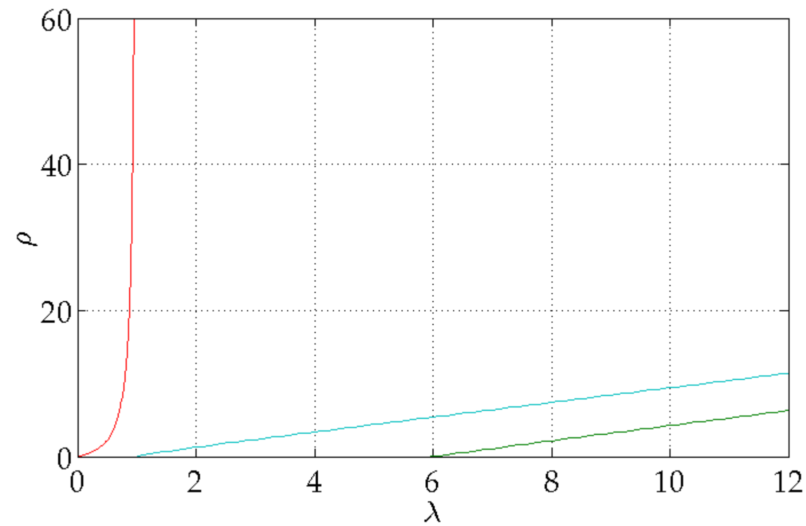
Let us first focus on the case in which $m < N-1$. We already know that if the central node is one of the pins the augmented connectivity is bounded by $0 < \lambda_1 < 1$, regardless of the value of m. Instead, if we only pin peripheral nodes, the augmented connectivity is bounded by $0 < \lambda_1 < \frac{N-\sqrt{N^2-4m}}{2} < 1$. Therefore, in terms of upper bounds for the augmented connectivity, the former strategy is always better. Now we would like also to compare the pinning strengths required in the two configurations to get a desired admissible value for λ_1.

Let us denote with ρ_p the pinning strength required in the scenario in which $m < N-1$ peripheral nodes are pinned and with ρ_c the pinning strength required in the scenario in which $m < N-1$ nodes, including the central one, are pinned. We have shown that for admissible values of λ_1, the two following equalities must hold

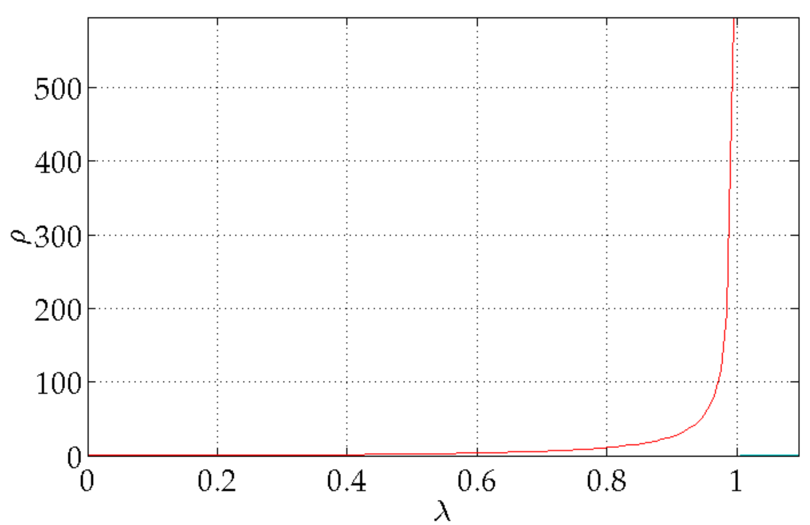
$$(1-\lambda_1)\rho_c^2 + (2\lambda_1^2 - (N+2)\lambda_1 + m)\rho_c - \lambda_1(1-\lambda_1)(N-\lambda_1) = 0 \tag{3.54}$$

Figure 3.13: Trend of the positive solution of (3.53) for $N = 6$ and $m = 3$. Note that the first branch is bounded by $\lambda < 1$.

(a) pinning strength



(b) upper bound



$$(\lambda_1^2 - N\lambda_1 + m)\rho_p = \lambda_1(1-\lambda_1)(N-\lambda_1) > 0 \tag{3.55}$$

Replacing (3.54) in (3.55), we get

$$(\lambda_1^2 - N\lambda_1 + m)\rho_p = (1-\lambda_1)\rho_c^2 + (2\lambda_1^2 - (N+2)\lambda_1 + m)\rho_c > 0 \tag{3.56}$$

Therefore, if pinning only peripheral nodes were more convenient, meaning that $\rho_p \leq \rho_c$, it should hold that

$$(\lambda_1^2 - N\lambda_1 + m)\rho_c \geq (1-\lambda_1)\rho_c^2 + (2\lambda_1^2 - (N+2)\lambda_1 + m)\rho_c \tag{3.57}$$

which after simple manipulation becomes

$$\rho_c \leq \frac{\lambda_1(2-\lambda_1)}{1-\lambda_1} \tag{3.58}$$

Using this inequality in equation (3.54), we obtain

$$\frac{\lambda_1^2(2-\lambda_1)^2}{1-\lambda_1} + (2\lambda_1^2 - (N+2)\lambda_1 + m)\,\frac{\lambda_1(2-\lambda_1)}{1-\lambda_1} +$$

$$-\lambda_1(1-\lambda_1)(N-\lambda_1) \geq 0 \tag{3.59}$$

which after simple manipulation leads to

$$\lambda_1 \leq \frac{2m-N}{m-1} \tag{3.60}$$

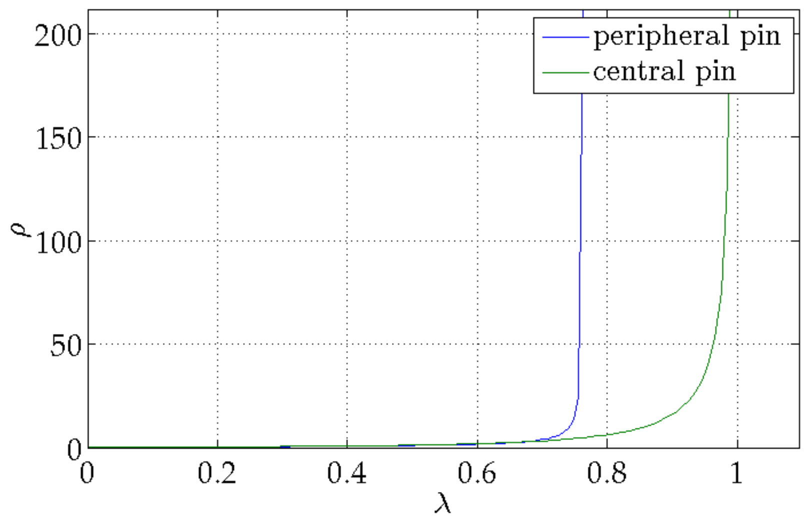
This means that pinning only peripheral nodes may actually be more convenient in terms of the pinning strength, but only if the desired value of the augmented connectivity respects inequality (3.60). For higher values of the augmented connectivity, it is more convenient, conversely, to pin also the central node. Note that for $2m \leq N$, inequality (3.60) cannot be respected, since the augmented connectivity must be strictly positive. Therefore, if we pin less than half the nodes of the graph, pinning also the central node will always be more convenient than pinning only peripheral nodes.

Figure 3.14 shows the trend of functions ρ_c and ρ_p for $N = 6$ and $m = 4$. It is easy to see that the two curves intersect at $\lambda = \frac{2m-N}{m-1} = \frac{2}{3}$.

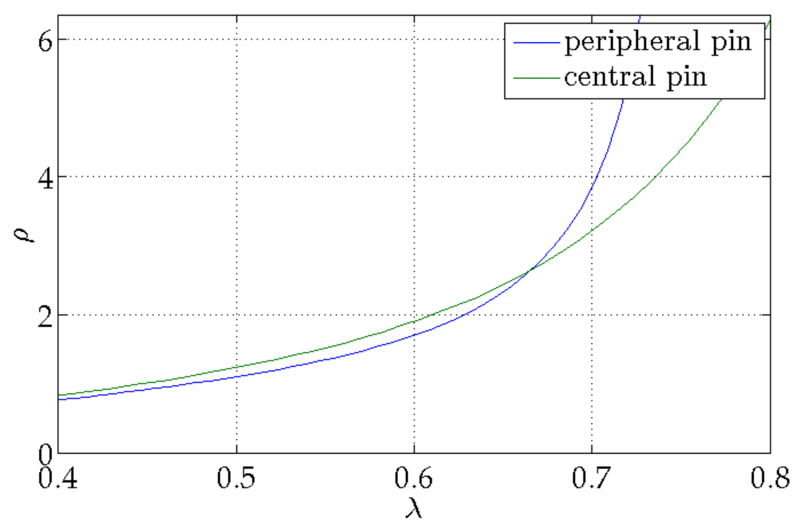
Let us now focus on the case in which $m = N-1$. In this case, if we pin also the central node, the upper bound for the augmented connectivity is still given by $\lambda_1 < 1$, while if we pin only peripheral nodes the upper bound is given by $\lambda_1 < N-1$, which is necessarily larger. Moreover, let us observe that for $\lambda_1 \neq 1$ the pinning strength ρ_p does still obey (3.55). Therefore our whole reasoning can be repeated, leading to equation (3.60), which for $m = N-1$ becomes

Figure 3.14: Trend of the first branches of functions ρ_p (blue) and ρ_c (green) for $N = 6$ and $m = 4$. Note that the two curves intersect at $\lambda = \frac{2m-N}{m-1} = \frac{2}{3}$.

(a) comparison



(b) zoom



$\lambda_1 \leq 1$, which is always true. This means that, for any admissible value of λ_1, we have $\rho_p \leq \rho_c$. Hence, for $m = N-1$, if the central node is one of the pins, not only we get a smaller upper bound for the augmented connectivity, but also, for all the admissible values of λ_1, we need to apply a higher pinning strength.

Figure 3.15 shows the trend of functions ρ_c and ρ_p for $N = 6$ and $m = 5$.

3.10 All-Node Pinning

In this section we would like to address the case in which all the nodes of the star graph are pinned.

Theorem 11. *In a star graph where all the nodes are pinned the augmented connectivity λ_1 is equal to the pinning strength ρ.*

Proof. In this case it is possible to rewrite equation (1.16) as

$$\begin{cases} (N+\rho)x_1 - s = \lambda x_1 & \quad (3.61a) \\ -x_1 + (1+\rho)x_i = \lambda x_i \qquad i = 2, \dots, N & \quad (3.61b) \end{cases}$$

We note that $\lambda = 1+\rho$ solves the equation for $x_1 = 0$ and $s = 0$, so it must be a $(N-2)$-multiplicity eigenvalue. In order to find the two missing eigenvalues it is sufficient to observe that, for $\lambda \neq 1+\rho$, we can rewrite the system as

$$\begin{cases} s = (N+\rho-\lambda)x_1 & \quad (3.62a) \\ x_i = \dfrac{1}{1+\rho-\lambda}x_1 \qquad i = 2, \dots, N & \quad (3.62b) \end{cases}$$

If we substitute (3.62a), (3.62b) in the definition of s, that is $s = x_1 + (N-1)x_i$, we obtain

$$(N+\rho-\lambda)x_1 = x_1 + \frac{N-1}{1+\rho-\lambda}x_1 \tag{3.63}$$

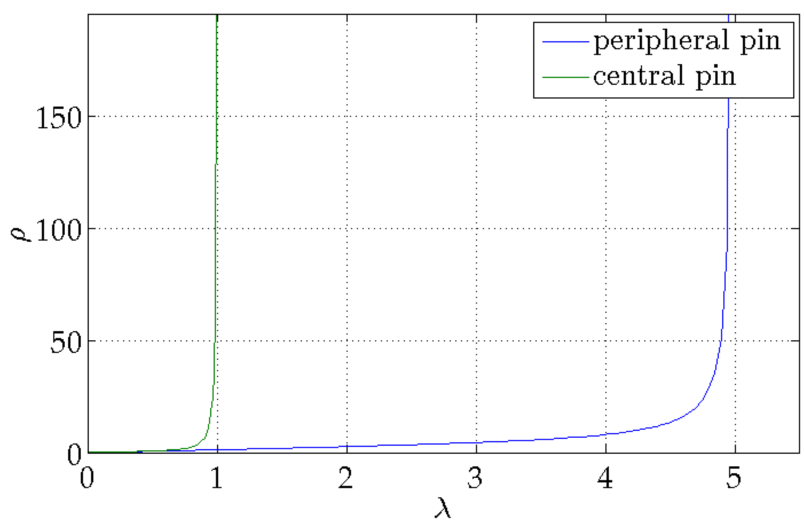
Excluding $x_1 = 0$, which leads to $x = 0_N$ we obtain after a few passages

$$(\rho-\lambda)(N+\rho-\lambda) = 0 \tag{3.64}$$

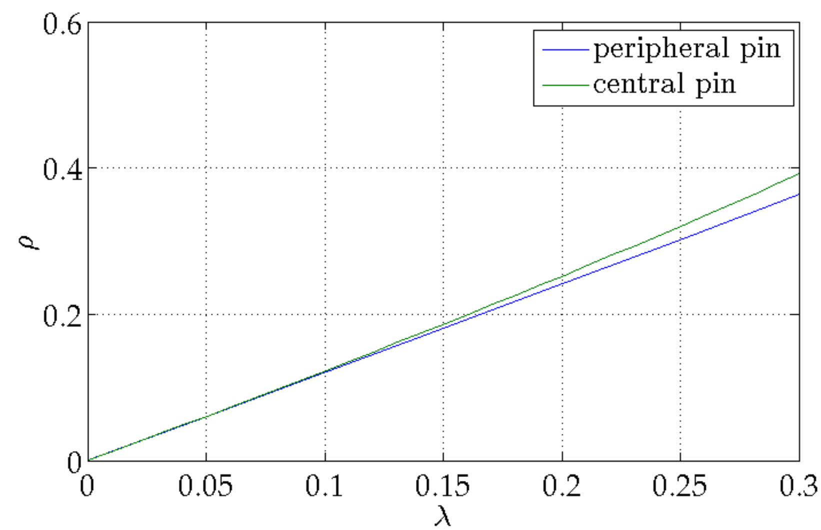
which means that the two missing eigenvalues are $\lambda = \rho$ and $\lambda = N+\rho$. Therefore the minimum eigenvalue must be $\lambda = \rho$. □

Figure 3.15: Trend of the first branches of functions ρ_c (green) and ρ_p (blue) for $N = 6$ and $m = 5$. Note that this time function ρ_p is always below function ρ_c.

(a) comparison



(b) zoom



3.11 Multiple-Pin Strategies Comparison

In this section we would like to compare the two possible strategies in which all the peripheral nodes are pinned. In terms of upper bound for the augmented connectivity, it is convenient to include the central node, which leads to an unbounded connectivity.

As for the trend of the total pinning strength, from Theorem 9 we know that when pinning all the peripheral nodes we need a total pinning strength given by (3.45), while from Theorem 11 we know that when pinning all the nodes, including the central one, we need a total pinning strength of $\rho_N = N\lambda$. The ratio between this two quantities is given by

$$\frac{\rho_N}{\rho_{N-1}} = \frac{N^2 - N - N\lambda}{N^2 - N\lambda - N + \lambda} < 1 \tag{3.65}$$

This proves that including the central node results in a better trend for the total pinning strength as well.

Chapter 4

A Second-Order Recursion

Algebraic analysis of the spectral properties of the complete graph and of the star graph is made possible by the high degree of simmetry exhibited by these topologies. As for the path graph and the ring graph, their structure could be described as recursive more than symmetric. In particular, one specific second-order recursion is often found in the calculations regarding these two topologies. In this chapter we would like to highlight some properties exhibited by this recursion, which we will then use extensively in our analysis of the path graph and the ring graph.

The recursion we would like to study is expressed by

$$- x_{k-1} + ax_k - x_{k+1} = 0 \tag{4.1}$$

It is easy to see that the same recursion can be equivalently expressed by

$$\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix} \begin{bmatrix} x_{k-1} \\ x_k \end{bmatrix} \tag{4.2}$$

Therefore, if we define

$$A = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix} \qquad \xi_k = \begin{bmatrix} x_{k-1} \\ x_k \end{bmatrix} \tag{4.3}$$

the recursion is simply given by

$$\xi_{k+1} = A\xi_k \tag{4.4}$$

Of course, enforcing the recursion m times we get

$$\xi_{k+m} = A^m \xi_k \tag{4.5}$$

Let us now study the properties of the generic power A^m of the matrix A for $m \in \mathbb{N}$. First of all we would like to prove the following result.

Lemma 1. *There exists a sequence $\{Q_m(a)\}$ of polynomials such that, for any $m > 0$, the generic power A^m of matrix A can be written as*

$$A^m = \begin{bmatrix} -Q_{m-2} & Q_{m-1} \\ -Q_{m-1} & Q_m \end{bmatrix} \tag{4.6}$$

where

$$Q_m = -Q_{m-2} + a\, Q_{m-1} \tag{4.7}$$

has degree m for any $m > 0$, and conventionally we define $Q_0 = 1$, $Q_1 = a$ and $Q_l = 0$ for any $l < 0$.

Proof. We prove this lemma by the induction principle. Let us first observe that the lemma is true for $m = 1$ since

$$A = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix} = \begin{bmatrix} Q_{-1} & Q_0 \\ -Q_0 & Q_1 \end{bmatrix} \tag{4.8}$$

$$Q_1 = a = 0 + a \cdot 1 = -Q_{-1} + a\, Q_0 \tag{4.9}$$

Now let us assume that the lemma holds for a particular value of m. Therefore A^m can be written as in (4.6). Now let us calculate

$$A^{m+1} = A^m\, A = \begin{bmatrix} -Q_{m-2} & Q_{m-1} \\ -Q_{m-1} & Q_m \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix} =$$

$$= \begin{bmatrix} -Q_{m-1} & -Q_{m-2} + a\, Q_{m-1} \\ -Q_m & -Q_{m-1} + a\, Q_m \end{bmatrix} = \begin{bmatrix} -Q_{m-1} & Q_m \\ -Q_m & Q_{m+1} \end{bmatrix} \tag{4.10}$$

which means that the lemma holds also for the value $m + 1$. Therefore the induction principle guarantees that the lemma holds for any value of $m > 0$. □

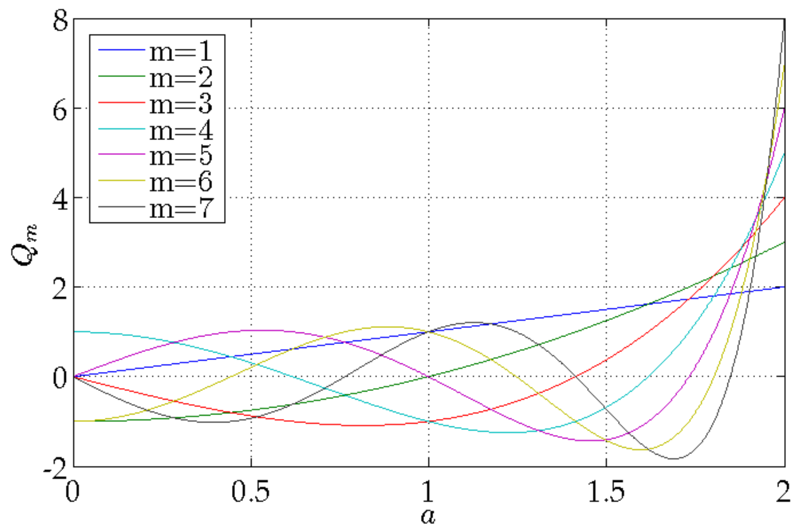
Lemma 1 gives a recursive rule to build the polynomials Q_m for any $m \geq 2$. In the following sections we will make large use of equation (4.7) in order to simplify the expression of some relevant quantities.

Figure 4.1 shows the trend of polynomials $Q_m(a)$ with $m = 1, \ldots, 7$ for $a \in [0, 2]$.

Corollary 1. *Given the polynomial $R_m(a) = Q_m(a) - Q_{m-1}(a)$ for any $m > 0$, the following equality holds*

$$R_m(a) = -R_{m-2}(a) + aR_{m-1}(a) \tag{4.11}$$

Figure 4.1: Trend of the polynomials Q_m for $m = 1, \ldots, 7$.



Proof. Taking advantage of Lemma 1, we can write

$$\begin{aligned}
-R_{m-2} + a\,R_{m-1} &= -(Q_{m-2} - Q_{m-3}) + a(Q_{m-1} - Q_{m-2}) = \\
&= -(a\,Q_{m-1} - Q_m) + Q_{m-3} + a\,Q_{m-1} - a\,Q_{m-2} = \\
&= Q_m + Q_{m-3} - a\,Q_{m-2} = \\
&= Q_m - (-Q_{m-3} + a\,Q_{m-2}) = Q_m - Q_{m-1} = R_m
\end{aligned} \tag{4.12}$$

which proves the corollary. □

Figure 4.2 shows the trend of polynomials $R_m(a)$ with $m = 1, \ldots, 7$ for $a \in [0, 2]$.

Lemma 2. *For any $m \geq 0$ the following equality holds*

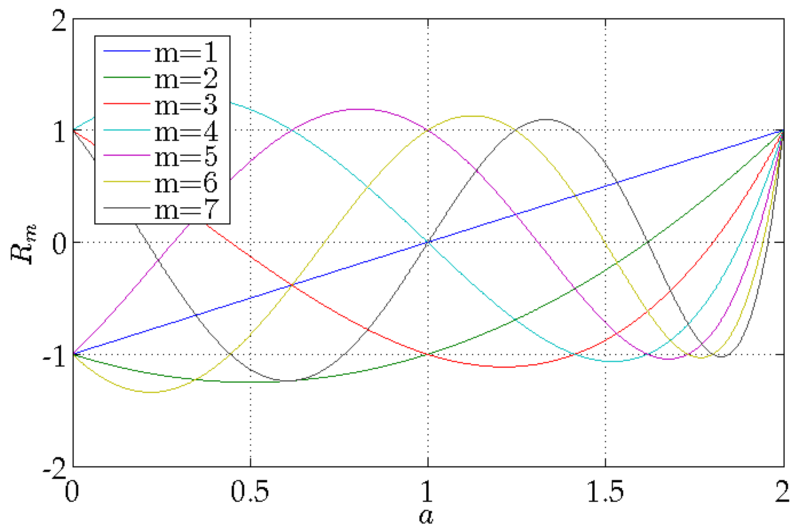
$$Q_m(2) = m + 1 \tag{4.13}$$

Proof. We prove this lemma by using the induction principle.

It is immediate to see that the lemma holds for $m = 0$ and $m = 1$. Now let us assume that the lemma holds for two values m and $m + 1$. Using equation (4.7) we can calculate

$$Q_{m+2}(2) = -Q_m(2) + 2\,Q_{m+1}(2) = -(m + 1) + 2(m + 2) = m + 3 \tag{4.14}$$

Figure 4.2: Trend of the polynomials R_m for $m = 1, \ldots, 7$.



which means that the lemma holds for the value $m + 2$. Therefore, by the induction principle, the lemma must hold for every value of m. □

Corollary 2. *For any $m > 0$ the following equality holds for the polynomial $R_m(a) = Q_m(a) - Q_{m-1}(a)$*

$$R_m(2) = 1 \tag{4.15}$$

Proof. It is sufficient to observe that for any $m > 0$ we have

$$R_m(2) = Q_m(2) - Q_{m-1}(2) = m + 1 - m = 1 \tag{4.16}$$

□

Figure 4.2 gives numerical validation for Corollary 2.

Lemma 3. *For any $m > 0$ the following equality holds*

$$Q_{2m} = (Q_m - Q_{m-1})(Q_m + Q_{m-1}) \tag{4.17}$$

Proof. Using expression (4.6) for A^m, we can write

$$\begin{bmatrix} * & * \\ * & Q_{2m} \end{bmatrix} = A^{2m} = A^m \, A^m =$$

$$= \begin{bmatrix} -Q_{m-2} & Q_{m-1} \\ -Q_{m-1} & Q_m \end{bmatrix} \begin{bmatrix} -Q_{m-2} & Q_{m-1} \\ -Q_{m-1} & Q_m \end{bmatrix} = \begin{bmatrix} * & * \\ * & Q_m^2 - Q_{m-1}^2 \end{bmatrix} \tag{4.18}$$

Therefore

$$Q_{2m} = Q_m^2 - Q_{m-1}^2 = (Q_m - Q_{m-1})(Q_m + Q_{m-1}) \tag{4.19}$$

□

Corollary 3. *For any $m > 0$ the highest root of the polynomial Q_{2m} coincides with the highest root of the polynomial $R_m = Q_m - Q_{m-1}$.*

$$a_{2m}^{(q)} = a_m^{(r)} \tag{4.20}$$

Proof. Thanks to Lemma 3, we know that equality (4.17) holds. Moreover, thanks to recursion (4.7) we can see that both the polynomials Q_m and Q_{m-1} must go to infinity when a goes to infinity as well. As a consequence, if there exists a value of a in which Q_m and Q_{m-1} assume opposite values, there must be a larger value of a in which they assume an equal value instead. This means that the highest root $a_{2m}^{(q)}$ of the polynomial Q_{2m} coincides with the highest root $a_m^{(r)}$ of the polynomial $R_m = Q_m - Q_{m-1}$. □

Lemma 4. *Let us denote with $a_m^{(q)}$ the highest root of the polynomial $Q_m(a)$. For any $m > 0$ the following inequality holds*

$$0 < a_m^{(q)} < a_{m+1}^{(q)} < 2 \tag{4.21}$$

Moreover

$$\lim_{m \to +\infty} a_m^{(q)} = 2 \tag{4.22}$$

The proof of this lemma relies on the study of the path graph, therefore it is postponed to the corresponing chapter.

Figure 4.3 shows the values of $a_m^{(q)}$ for $m = 1, \ldots, 7$.

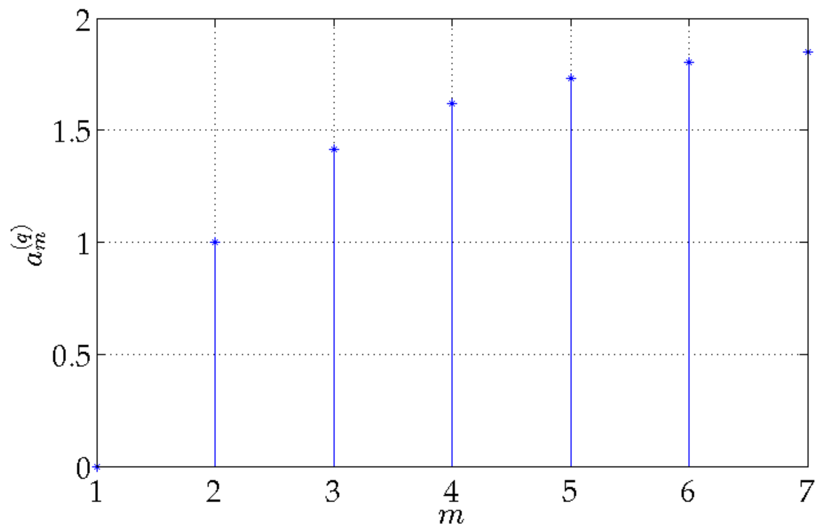
Corollary 4. *Let us denote with $a_m^{(r)}$ the highest root of the polynomial $R_m(a) = Q_m(a) - Q_{m-1}(a)$. For any $m > 0$ the following inequality holds*

$$0 < a_m^{(r)} < a_{m+1}^{(r)} < 2 \tag{4.23}$$

Moreover

$$\lim_{m \to +\infty} a_m^{(r)} = 2 \tag{4.24}$$

Figure 4.3: Highest root of the polynomials Q_m for $m = 1, \dots, 7$.



Proof. From Corollary 3 we know that $a_m^{(r)} = a_{2m}^{(q)}$ and that $a_{m+1}^{(r)} = a_{2(m+1)}^{(q)}$ for any $m > 0$. Therefore we can apply Lemma 4 to write

$$0 < a_m^{(r)} = a_{2m}^{(q)} < a_{2m+1}^{(q)} < a_{2(m+1)}^{(q)} = a_{m+1}^{(r)} < 2 \quad \forall m > 0 \tag{4.25}$$

which proves inequality (4.23). Moreover we can write that

$$\lim_{m \to +\infty} a_m^{(r)} = \lim_{m \to +\infty} a_{2m}^{(q)} = 2 \tag{4.26}$$

which proves (4.24). □

Figure 4.4 shows the values of $a_m^{(r)}$ for $m = 1, \dots, 20$.

Lemma 5. *For any $0 < m \leq n$, the following inequality holds*

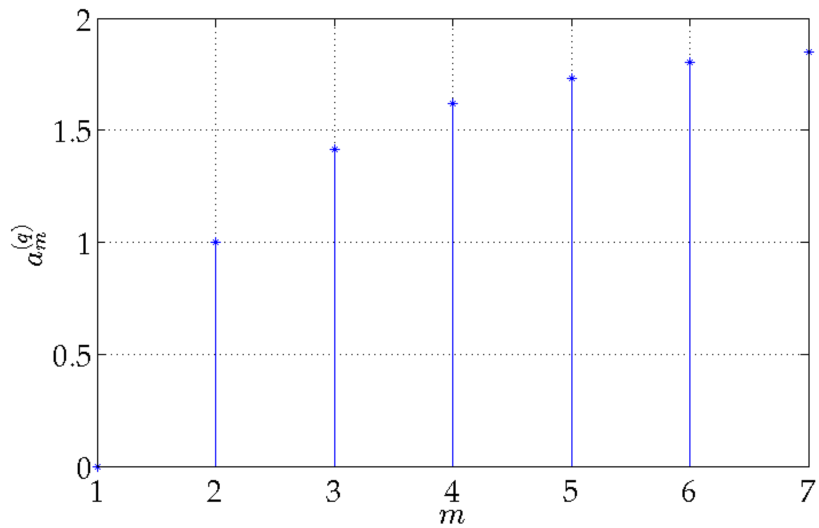
$$Q_m(a) > 0 \qquad \forall a > a_n^{(q)} \tag{4.27}$$

Proof. By definition, the polynomial $Q_m(a)$ is going to infinity for increasing values of a, therefore we know that it is positive for any $a > a_m^{(q)}$. From Lemma 4 we know that $a_n^{(q)} \geq a_m^{(q)}$, which leads to (4.27). □

Corollary 5. *For any $0 < m \leq n$, the following inequality holds*

$$R_m(a) > 0 \qquad \forall a > a_n^{(r)} \tag{4.28}$$

Figure 4.4: Highest root of the polynomials R_m for $m = 1, \ldots, 7$.



Proof. We can reason in the same way as done for polynomial $Q_m(a)$ in Lemma 5, using Corollary 4 instead of Lemma 4. □

Lemma 6. *The polynomials $Q_m(a)$ and $Q_{m-1}(a)$ have no roots in common for any $m > 0$.*

Proof. Let us suppose by contradiction that one of the roots of the polynomial Q_m, say $\bar{a}$, is also a root of the polynomial Q_{m-1}. Taking advantage of (4.7), we can write

$$Q_m(\bar{a}) = -Q_{m-2}(\bar{a}) + \bar{a}\, Q_{m-1}(\bar{a}) \tag{4.29}$$

which yields $Q_{m-2}(\bar{a}) = 0$, meaning that $\bar{a}$ must be also a root of the polynomial Q_{m-2}. If we repeat such reasoning for decreasing values of m, we get that $\bar{a}$ must be also a root of the polynomial Q_0, but this is absurd since $Q_0(a) = 1$. This proves that the polynomials $Q_m(a)$ and $Q_{m-1}(a)$ have no roots in common for any $m > 0$. □

Corollary 6. *The polynomials $Q_m(a)$ and $R_m(a)$ have no roots in common for any $m > 0$. The same goes for $Q_{m-1}(a)$ and $R_m(a)$.*

Proof. Let us suppose by contradiction that one of the roots of the polynomial Q_m, say $\bar{a}$, is also a root of the polynomial R_m. This means that

$$0 = R_m(\bar{a}) = Q_m(\bar{a}) - Q_{m-1}(\bar{a}) = -Q_{m-1}(\bar{a}) \tag{4.30}$$

Therefore $\bar{a}$ must be also a root of the polynomial Q_{m-1}, but this is in contrast with Lemma 6. This proves that the polynomials $Q_m(a)$ and $R_m(a)$ have no roots in common for any $m > 0$.

Same reasoning can be carried out for the polynomials Q_{m-1} and $R_m(a)$. □

Corollary 7. *The polynomials $R_m(a)$ and $R_{m+1}(a)$ have no roots in common for any $m > 0$.*

Proof. Let us suppose by contradiction that one of the roots of the polynomial R_m, say $\bar{a}$, is also a root of the polynomial R_{m+1}. So, using also (4.7), we can write

$$\begin{gathered} 0 = R_{m+1}(\bar{a}) = Q_{m+1}(\bar{a}) - Q_m(\bar{a}) = \\ = -Q_{m-1}(\bar{a}) + \bar{a}\, Q_m(\bar{a}) - Q_m(\bar{a}) = \\ = -Q_{m-1}(\bar{a}) + Q_m(\bar{a}) - Q_m(\bar{a}) + \bar{a}\, Q_m(\bar{a}) - Q_m(\bar{a}) = \\ = R_m(\bar{a}) + (\bar{a} - 2)Q_m(\bar{a}) = (\bar{a} - 2)Q_m(\bar{a}) \end{gathered} \tag{4.31}$$

or equivalently

$$(\bar{a} - 2)Q_m(\bar{a}) = 0 \tag{4.32}$$

But from Corollaries 4 and 6 we can easily state that equality (4.32) cannot be satisfied, therefore this proves that the polynomials $R_m(a)$ and $R_{m+1}(a)$ have no roots in common for any $m > 0$. □

Chapter 5

Path Graph

In this chapter we address the study of the augmented connectivity of a *path graph*. A path graph is a graph in which consecutive nodes are connected to each other.

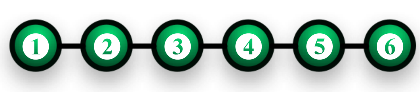
Figure 5.1 shows a path graph with $N = 6$ nodes.

It is easy to see that the expression of the Laplacian of the path graph is

$$L = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{bmatrix} \tag{5.1}$$

Before studying some possible pinning configuration for the path graph, let us examine the case in which the graph is not pinned. For the Laplacian of the path graph it is possible to find simple goniometrical expressions of eigenvalues and eigenvectors, exploiting the symmetrical structure of the graph itself.

Figure 5.1: path graph with $N = 6$



This result is illustrated in [28]. However, such symmetry is lost when one or more pins are introduced. Therefore we would like to follow a different approach, which leads to results that are less explicit, but more easily extendable to the study of the augmented Laplacian.

Theorem 12. *The non-null eigevalues λ of the Laplacian of a path graph with no pins correspond to the roots of the polynomial $Q_{N-1}(a)$, with $a = 2 - \lambda$.*

Proof. Looking at the expression of the Laplacian, we can rewrite (1.16) as

$$\begin{cases} x_1 - x_2 = \lambda x_1 & (5.2a) \\ -x_{j-1} + 2x_j - x_{j+1} = \lambda x_j \qquad j = 2, \ldots, N-1 & (5.2b) \\ -x_{N-1} + x_N = \lambda x_N & (5.2c) \end{cases}$$

If we define $a = 2 - \lambda$, we can rewrite the central equation as equation (4.1). We also know that such recursion can be written as

$$\begin{bmatrix} x_j \\ x_{j+1} \end{bmatrix} = A \begin{bmatrix} x_{j-1} \\ x_j \end{bmatrix} \tag{5.3}$$

which holds for any $j = 2, \ldots, N-1$. If we enforce the recursion for $N-2$ times we get

$$\begin{bmatrix} x_{N-1} \\ x_N \end{bmatrix} = A^{N-2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -Q_{N-4}(a) & Q_{N-3}(a) \\ -Q_{N-3}(a) & Q_{N-2}(a) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5.4}$$

Therefore we end up with the following system of four linear equations in four unknowns.

$$\begin{cases} (a-1)x_1 - x_2 = 0 & (5.5a) \\ x_{N-1} = -Q_{N-4}(a)\, x_1 + Q_{N-3}(a)\, x_2 & (5.5b) \\ x_N = -Q_{N-3}(a)\, x_1 + Q_{N-2}(a)\, x_2 & (5.5c) \\ -x_{N-1} + (a-1)x_N = 0 & (5.5d) \end{cases}$$

We can rewrite (5.5a) and (5.5d) as

$$\begin{aligned} x_2 &= (a-1)\, x_1 \\ x_{N-1} &= (a-1)\, x_N \end{aligned} \tag{5.6}$$

and we can substitute these expressions into (5.5b), (5.5c), obtaining

$$\begin{cases} (a-1)\, x_N = -q_4\, x_1 + q_3\, (a-1)\, x_1 & (5.7a) \\ x_N = -q_3\, x_1 + q_2\, (a-1)\, x_1 & (5.7b) \end{cases}$$

where $q_i = Q_{N-i}(a)$.

Substitution of (5.7b) into (5.7a) yields

$$(a-1)(-q_3 + q_2\ (a-1)) = -q_4 + q_3\ (a-1) \tag{5.8}$$

Using recursion (4.7) for q_4, we can substitute $q_4 = -q_2 + a\ q_3$ so that we obtain

$$(a-1)(-q_3 + q_2\ (a-1)) = q_2 - a\ q_3 + q_3\ (a-1) \tag{5.9}$$

After simple manipulation we get

$$(2-a)\ (q_3 - a\ q_2) = 0 \tag{5.10}$$

Using recursion (4.7) for q_1, we can substitute $q_1 = -q_3 + a\ q_2$ so that we finally obtain $(2-a)\ q_1 = 0$, or equivalently

$$(2-a)\ Q_{N-1}(a) = 0 \tag{5.11}$$

which one may solve for a, from which $\lambda = 2 - a$ can be obtained.

Equation (5.11) accounts for the null eigenvalue of the Laplacian, which corresponds to the solution $a = 2$, while the other $N-1$ eigenvalues correspond to the roots of the polynomial Q_{N-1}. □

Thanks to this result, we can now prove Lemma 4.

Proof (of Lemma 4). From [28] we know that the second smallest eigenvalue of the Laplacian of the path graph with N nodes is

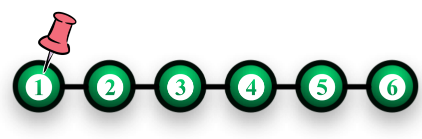
$$\lambda_2 = 2 - 2\cos\left(\frac{\pi}{N}\right) \tag{5.12}$$

Thanks to equation (5.11), we can see that this eigenvalue must correspond to the highest root of the polynomial $Q_{N-1}(a)$, with $a = 2 - \lambda$. Therefore we can write

$$a_{N-1}^{(q)} = 2 - \lambda_2 = 2\cos\left(\frac{\pi}{N}\right) \tag{5.13}$$

Therefore both inequality (4.21) and equation (4.22) are known goniometrical properties. □

Figure 5.2: Path graph with $N = 6$ nodes and one peripheral pin.



5.1 Single Peripheral Pin

In this section we would like to study the trend of the augmented connectivity when one peripheral node - that is to say, either the first or the last node - is pinned in the path graph. Figure 5.2 presents this configuration for $N = 6$.

Theorem 13. *In a path graph where either of the peripheral nodes is pinned, the augmented connectivity λ_1 is bounded by*

$$0 < \lambda_1 < 2 - a_{N-1}^{(r)} \tag{5.14}$$

Moreover, for any admissible value of λ_1, the pinning strength is given by

$$\rho = \frac{R_{N-2}(a)}{R_{N-1}(a)} + 1 - a \tag{5.15}$$

with $a = 2 - \lambda_1$.

Proof. Of course there is no difference in pinning the first or the last node. Without loss of generality, let us say that the first node is pinned. In this case, equation (1.16) becomes

$$\begin{cases} (1+\rho)\, x_1 - x_2 = \lambda x_1 & (5.16a) \\ -x_{j-1} + 2x_j - x_{j+1} = \lambda x_j \qquad j = 2, \ldots, N-1 & (5.16b) \\ -x_{N-1} + x_N = \lambda x_N & (5.16c) \end{cases}$$

The recursion can be written again as in equation (5.3), and again it can be enforced $N-2$ times to get equation (5.4). Therefore, using also $a = 2 - \lambda$, this time we obtain the following linear system.

$$\begin{cases} (a-1+\rho)\,x_1 - x_2 = 0 & \text{(5.17a)} \\ x_{N-1} = -Q_{N-4}(a)\,x_1 + Q_{N-3}(a)\,x_2 & \text{(5.17b)} \\ x_N = -Q_{N-3}(a)\,x_1 + Q_{N-2}(a)\,x_2 & \text{(5.17c)} \\ -x_{N-1} + (a-1)\,x_N = 0 & \text{(5.17d)} \end{cases}$$

Now we rewrite (5.17a) and (5.17d) as

$$\begin{aligned} x_2 &= (a-1+\rho)\,x_1 \\ x_{N-1} &= (a-1)\,x_N \end{aligned} \tag{5.18}$$

then we substitute these expressions into (5.17b) and (5.17c), obtaining

$$\begin{cases} (a-1)\,x_N = -q_4\,x_1 + q_3\,(a-1+\rho)\,x_1 & \text{(5.19a)} \\ x_N = -q_3\,x_1 + q_2\,(a-1+\rho)\,x_1 & \text{(5.19b)} \end{cases}$$

where $q_i = Q_{N-i}(a)$. Finally, we substitute (5.19b) into (5.19a) and we obtain

$$(a-1)(-q_3 + q_2\,(a-1+\rho)) = -q_4 + q_3\,(a-1+\rho) \tag{5.20}$$

which after some simple passages becomes

$$((a-1)q_2 - q_3)(a-1+\rho) = (a-1)q_3 - q_4 \tag{5.21}$$

Using recursion (4.7), we can substitute $q_3 = -q_1 + a\,q_2$ for the first member and $q_4 = -q_2 + a\,q_3$ for the second member, obtaining

$$(q_1 - q_2)(a-1+\rho) = q_2 - q_3 \tag{5.22}$$

or equivalently

$$R_{N-1}(a)\,(a-1+\rho) = R_{N-2}(a) \tag{5.23}$$

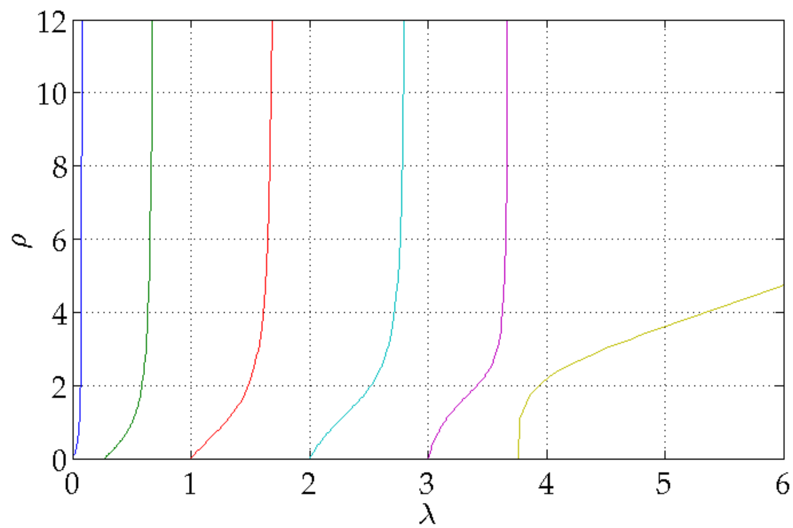
For $R_{N-1}(a) = 0$ equation (5.23) leads to $R_{N-2}(a) = 0$, which can be excluded thanks to Corollary 7. Therefore we can state that, for all the values of a that satisfy equation (5.23), it holds that $R_{N-1}(a) \neq 0$, meaning that we can rewrite equation (5.23) as

$$\rho = \frac{R_{N-2}(a)}{R_{N-1}(a)} + 1 - a \tag{5.24}$$

Thanks to Corollary 5 we know that in a right neighborhood of $a = a_{N-1}^{(r)}$ this function is going to infinity. Therefore $\lambda = 2 - a_{N-1}^{(r)}$ is the first vertical asymptote for ρ as a function of λ. This means that the minimum eigenvalue λ_1 is bounded by $0 < \lambda_1 < 2 - a_{N-1}^{(r)}$. □

Figure 5.3: Trend of function (5.24) for $N = 6$. The first branch is bounded by $\lambda < 2 - a^{(r)}_{N-1} \simeq .081$.

(a) pinning strength



(b) upper bound

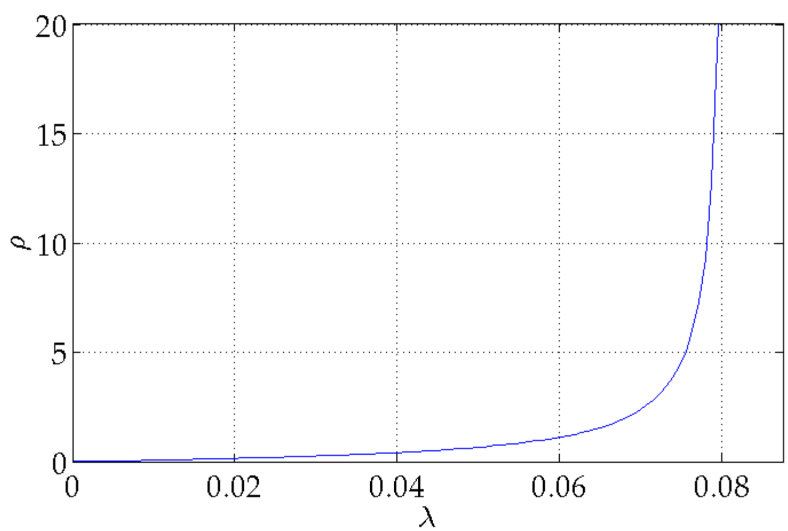


Figure 5.4: Path graph with $N = 6$ nodes and one semiperipheral pin.

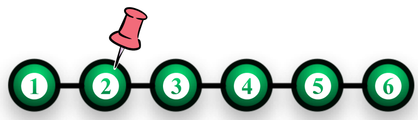


Figure 5.3 shows the trend of function (5.24) for $N = 6$. In this case we have $2 - a_{N-1}^{(r)} \simeq .081$, and it is easy to see from the figure that the first branch of the function is indeed upper-bounded by this value of λ.

5.2 Single Semiperipheral Pin

In this section we would like to address the case in which a semiperipheral node - that is to say, either the second or the second-last node - is pinned in a path graph. This configuration is represented in Figure 5.4 for $N = 6$.

Theorem 14. *In a path graph where the second or the second-last node is pinned the augmented connectivity λ_1 is bounded by*

$$0 < \lambda_1 < 2 - a_{N-2}^{(r)} \tag{5.25}$$

Moreover, for any admissible value of λ_1, the pinning strength is given by

$$\rho = \frac{(a-1)\, R_{N-3}(a) + R_{N-2}(a)}{(a-1)\, R_{N-2}(a)} - a \tag{5.26}$$

with $a = 2 - \lambda_1$.

Proof. Of course the symmetry of the path graph guarantees that as for the spectral properties of the augmented Laplacian there is no difference in pinning the second or the second last node. Therefore, without loss of generality, let us assume that the second node is pinned. In this case equation (1.16) becomes

$$\begin{cases} x_1 - x_2 = \lambda x_1 & \text{(5.27a)} \\ -x_1 + (2+\rho)x_2 - x_3 = \lambda x_2 & \text{(5.27b)} \\ -x_{j-1} + 2x_j - x_{j+1} = \lambda x_j \qquad j = 3, \ldots, N-1 & \text{(5.27c)} \\ -x_{N-1} + x_N = \lambda x_N & \text{(5.27d)} \end{cases}$$

Therefore, recursion (5.3) can be enforced $N-3$ times, yielding

$$\begin{bmatrix} x_{N-1} \\ x_N \end{bmatrix} = A^{N-3} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -Q_{N-5}(a) & Q_{N-4}(a) \\ -Q_{N-4}(a) & Q_{N-3}(a) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5.28}$$

Substituting these expressions in the system, and accounting also for $a = 2-\lambda$, we obtain a system of five linear equations in five unknowns.

$$\begin{cases} (a-1)\, x_1 - x_2 = 0 & \text{(5.29a)} \\ -x_1 + (a+\rho)\, x_2 - x_3 = 0 & \text{(5.29b)} \\ x_{N-1} = -Q_{N-5}(a)\, x_2 + Q_{N-4}(a)\, x_3 & \text{(5.29c)} \\ x_N = -Q_{N-4}(a)\, x_2 + Q_{N-3}(a)\, x_3 & \text{(5.29d)} \\ -x_{N-1} + (a-1)\, x_N = 0 & \text{(5.29e)} \end{cases}$$

Equations (5.29a), (5.29b), (5.29e) can be reqritten as

$$\begin{gathered} x_2 = (a-1)\, x_1 \\ x_3 = -x_1 + (a+\rho)\, x_2 = (-1 + (a+\rho)(a-1))\, x_1 \\ x_{N-1} = (a-1) x_N \end{gathered} \tag{5.30}$$

Substitution of the previous expressions in (5.29b), (5.29c)

$$\begin{cases} (a-1)\, x_N = -q_5\, (a-1)\, x_1 + q_4\, (-x_1 + (a+\rho)(a-1)\, x_1) & \text{(5.31a)} \\ x_N = -q_4\, (a-1)\, x_1 + q_3\, (-x_1 + (a+\rho)(a-1)\, x_1) & \text{(5.31b)} \end{cases}$$

where $q_i = Q_{N-i}(a)$. Upon substitution of (5.31b) into (5.31a) we get

$$\begin{gathered} (a-1)[-q_4\, (a-1) + q_3\, (-1 + (a+\rho)(a-1))] = \\ = -q_5\, (a-1) + q_4\, (-1 + (a+\rho)(a-1)) \end{gathered} \tag{5.32}$$

Isolating the term $(-1 + (a+\rho)(a-1))$, we get

$$\begin{gathered} [(a-1)q_3 - q_4]\, (-1 + (a+\rho)(a-1)) = \\ = (a-1)[q_4\, (a-1) - q_5] \end{gathered} \tag{5.33}$$

Using recursion (4.7), we substitute $q_4 = -q_2 + a\, q_3$ for the first member and $q_5 = -q_3 + a\, q_4$ for the second member, obtaining

$$(q_2 - q_3)\, (-1 + (a+\rho)(a-1)) = (a-1)\, (q_3 - q_4) \tag{5.34}$$

Isolating the term $(a+\rho)(a-1)$, we simply get

$$(q_2 - q_3)\, (a+\rho)\, (a-1) = (a-1)\, (q_3 - q_4) + (q_2 - q_3) \tag{5.35}$$

or equivalently

$$(a+\rho)\ (a-1)\ R_{N-2}(a) = (a-1)\ R_{N-3}(a) + R_{N-2}(a) \tag{5.36}$$

For any value of a such that $R_{N-2}(a)\ (a-1) \neq 0$, equation (5.36) can be solved for ρ yielding

$$\rho = \frac{(a-1)\ R_{N-3}(a) + R_{N-2}(a)}{(a-1)\ R_{N-2}(a)} - a \tag{5.37}$$

Thanks to Corollaries 4 and 5, we know that this function has its first vertical asymptote in $\lambda = 2 - a^{(r)}_{N-2}$, which therefore must be an upper bound for the minimum eigenvalue. □

Figure 5.5 shows the trend of function (5.37) for $N = 6$. In this case we have $2 - a^{(r)}_{N-2} \simeq .12$, and it is easy to see from the figure that the first branch of the function is indeed upper-bounded by this value of λ.

5.3 Single Nonperipheral Pin

In this section we would like to consider the case in which a central node - that is to say - neither peripheral nor semiperipheral - is pinned in the path graph. This configuration is represented in Figure 5.6 for a graph with $N = 6$ nodes.

Theorem 15. *In a path graph where a node n is pinned, with $2 < n < N-1$, the augmented connectivity λ_1 is bounded by*

$$0 < \lambda_1 < 2 - \max\{a^{(r)}_{n-1}, a^{(r)}_{N-n}\} \tag{5.38}$$

Moreover, for any admissible value of λ_1, the pinning strength is given by

$$\rho = \frac{R_{N-n-1}(a)\ R_{n-1}(a) - R_n(a)\ R_{N-n}(a)}{R_{n-1}(a)\ R_{N-n}(a)} \tag{5.39}$$

with $a = 2 - \lambda_1$.

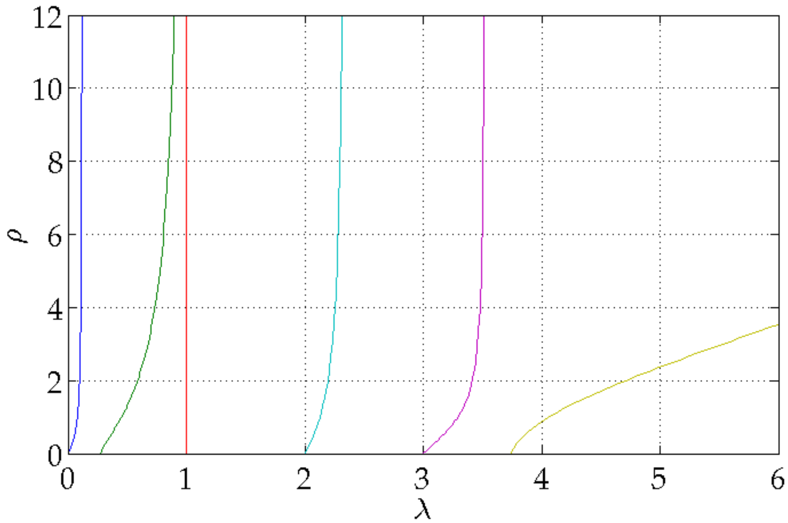
Proof. In this case equation (1.16) becomes

$$\begin{cases} x_1 - x_2 = \lambda x_1 & (5.40a) \\ -x_{j-1} + 2x_j - x_{j+1} = \lambda x_j \qquad j = 2, \ldots, n-1 & (5.40b) \\ -x_{n-1} + (2+\rho)x_n - x_{n+1} = \lambda x_n & (5.40c) \\ -x_{j-1} + 2x_j - x_{j+1} = \lambda x_j \qquad j = n+1, \ldots, N-1 & (5.40d) \\ -x_{N-1} + x_N = \lambda x_N & (5.40e) \end{cases}$$

Therefore we can write recursion (5.3) two times, the first one for $j = 2, \ldots, n-1$ and the second one for $j = n+1, \ldots, N-1$. Enforcing the two recursions, we obtain the following two equations respectively.

Figure 5.5: Trend of function (5.37) for $N = 6$. Note that the first branch is bounded by $\lambda < 2 - a^{(r)}_{N-2} \simeq .12$.

(a) pinning strength



(b) upper bound

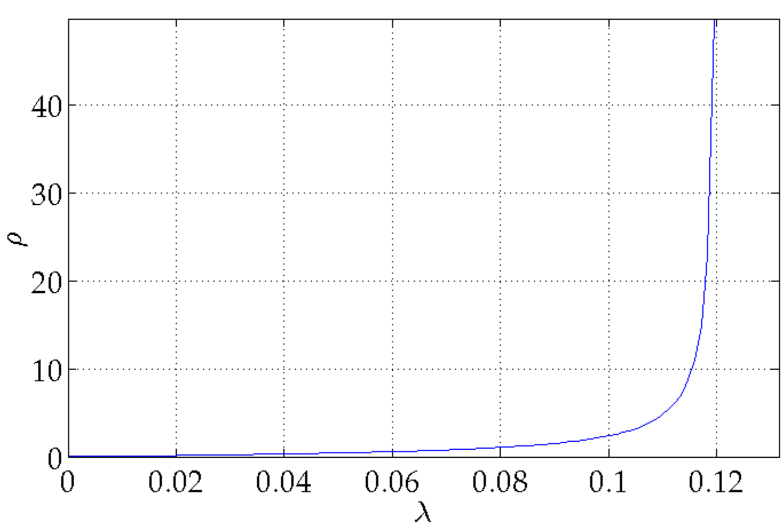
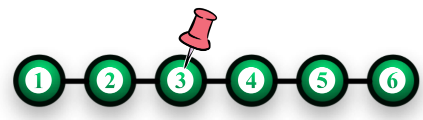


Figure 5.6: Path graph with $N = 6$ nodes and one nonperipheral pin.



$$\begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} = A^{n-2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -Q_{n-4}(a) & Q_{n-3}(a) \\ -Q_{n-3}(a) & Q_{n-2}(a) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5.41}$$

$$\begin{bmatrix} x_{N-1} \\ x_N \end{bmatrix} = A^{N-n-1} \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} -Q_{N-n-3}(a) & Q_{N-n-2}(a) \\ -Q_{N-n-2}(a) & Q_{N-n-1}(a) \end{bmatrix} \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} \tag{5.42}$$

Substituting these expressions in the system, and using also $a = 2 - \lambda$, we obtain the following system of seven linear equations in seven unknowns.

$$\begin{cases} (a-1)\, x_1 - x_2 = 0 & \text{(5.43a)} \\ x_{n-1} = -Q_{n-4}(a)\, x_1 + Q_{n-3}(a)\, x_2 & \text{(5.43b)} \\ x_n = -Q_{n-3}(a)\, x_1 + Q_{n-2}(a)\, x_2 & \text{(5.43c)} \\ -x_{n-1} + (a+\rho)\, x_n - x_{n+1} = 0 & \text{(5.43d)} \\ x_{N-1} = -Q_{N-n-3}(a)\, x_n + Q_{N-n-2}(a)\, x_{n+1} & \text{(5.43e)} \\ x_N = -Q_{N-n-2}(a)\, x_n + Q_{N-n-1}(a)\, x_{n+1} & \text{(5.43f)} \\ -x_{N-1} + (a-1)\, x_N = 0 & \text{(5.43g)} \end{cases}$$

Equations (5.43a), (5.43d) and (5.43g) can be rewritten as

$$\begin{aligned} x_2 &= (a-1)\, x_1 \\ x_{n-1} &= -x_{n+1} + (a+\rho)\, x_n \\ x_{N-1} &= (a-1)\, x_N \end{aligned} \tag{5.44}$$

Substituting these expressions into the remaining equations of the system we obtain

$$\begin{cases} (a+\rho)\, x_n - x_{n+1} = -q_4\, x_1 + q_3(a-1)\, x_1 & \text{(5.45a)} \\ x_n = -q_3\, x_1 + q_2(a-1)\, x_1 & \text{(5.45b)} \\ (a-1)\, x_N = -\tilde{q}_3\, x_n + \tilde{q}_2\, x_{n+1} & \text{(5.45c)} \\ x_N = -\tilde{q}_2\, x_n + \tilde{q}_1\, x_{n+1} & \text{(5.45d)} \end{cases}$$

where $q_i = Q_{N-i}(a)$ and $\tilde{q}_i = Q_{N-n-i}(a)$.

Substituting the second and the fourth equation of the system into the first and the third one, we get

$$\begin{cases} (a+\rho)(-q_3+q_2(a-1))\,x_1 - x_{n+1} = (-q_4+q_3(a-1))\,x_1 & \text{(5.46a)} \\ (a-1)(-\tilde{q}_2(-q_3+q_2(a-1))\,x_1 + \tilde{q}_1\,x_{n+1}) = \\ \quad = -\tilde{q}_3(-q_3+q_2(a-1))\,x_1 + \tilde{q}_2\,x_{n+1} & \text{(5.46b)} \end{cases}$$

Using recursion (4.7), we substitute $q_3 = -q_1 + a\,q_2$ for the first members and $q_4 = -q_2 + a\,q_3$ for the second members of (**??**), obtaining

$$\begin{cases} (a+\rho)\,r_1\,x_1 - x_{n+1} = r_2\,x_1 \\ (a-1)(-\tilde{q}_2\,r_1\,x_1 + \tilde{q}_1\,x_{n+1}) = -\tilde{q}_3\,r_1\,x_1 + \tilde{q}_2\,x_{n+1} \end{cases} \tag{5.47}$$

where $r_i = q_i - q_{i+1}$.

From the first equation of (5.47) we can obtain an expression for x_{n+1} that can be substituted in the second equation. This substituion yields

$$\begin{aligned} (a-1)[-\tilde{q}_2\,r_1 + \tilde{q}_1((a+\rho)\,r_1 - r_2)] = \\ = -\tilde{q}_3\,r_1 + \tilde{q}_2[(a+\rho)\,r_1 - r_2] \end{aligned} \tag{5.48}$$

Isolating the term $(a+\rho)\,r_1 - r_2$, we get

$$[(a-1)\,\tilde{q}_1 - \tilde{q}_2][(a+\rho)\,r_1 - r_2] = [(a-1)\,\tilde{q}_2 - \tilde{q}_3]\,r_1 \tag{5.49}$$

Using recursion (4.7), we substitute $\tilde{q}_2 = -\tilde{q}_0 + a\,\tilde{q}_1$ for the first member and $\tilde{q}_3 = -\tilde{q}_1 + a\,\tilde{q}_2$ for the second member of the previous equation, obtaining

$$\tilde{r}_0[(a+\rho)\,r_1 - r_2] = \tilde{r}_1\,r_1 \tag{5.50}$$

where $\tilde{r}_i = \tilde{q}_i - \tilde{q}_{i+i}$.

Isolating the term $\tilde{r}_0\,r_1\,(a+\rho)$, we get

$$\tilde{r}_0\,r_1\,(a+\rho) = \tilde{r}_1\,r_1 + \tilde{r}_0\,r_2 \tag{5.51}$$

or equivalently

$$\begin{aligned} R_{N-n}(a)\,R_{n-1}(a)\,(a+\rho) = \\ = R_{N-n-1}(a)\,R_{n-1}(a) + R_{N-n}(a)\,R_{n-2}(a) \end{aligned} \tag{5.52}$$

For any value of a such that $R_{N-n}(a)\,R_{n-1}(a) \neq 0$ we can rewrite equation (5.52) as

$$\rho = \frac{R_{N-n-1}(a)}{R_{N-n}(a)} + \frac{R_{n-2}(a)}{R_{n-1}(a)} - a \tag{5.53}$$

Suppose that $n-1 \geq N-n$. Thanks to Corollaries 4 and 5 we can say that

$$\lim_{a \to a_{n-1}^{(r)\,+}} \rho = +\infty \tag{5.54}$$

meaning that function (5.53) has its first vertical asymptote in $\lambda = 2 - a_{n-1}^{(r)}$. Same reasoning can be carried out in the case in which $n-1 \leq N-n$, leading to $\lambda = 2 - a_{N-n}^{(r)}$ being the first vertical asymptote of function (5.53).

Therefore in general we can say that the first vertical asymptote of function (5.53) is in $\lambda = 2 - \max\{a_{n-1}^{(r)}, a_{N-n}^{(r)}\}$. As a consequence such value must be an upper bound for the minimum eigenvalue. □

Figure 5.7 shows the trend of function (5.53) for $N = 6$ and $n = 3$. In this case we have $n-1 = 2$ and $N-n = 3$, therefore the first branch of the function must be bounded by $\lambda < 2 - a_3^{(r)} \simeq .20$.

5.4 Single-Pin Strategies Comparison

In this section we would like to compare the possible single-pin strategies for the path graph.

First of all let us point out that:

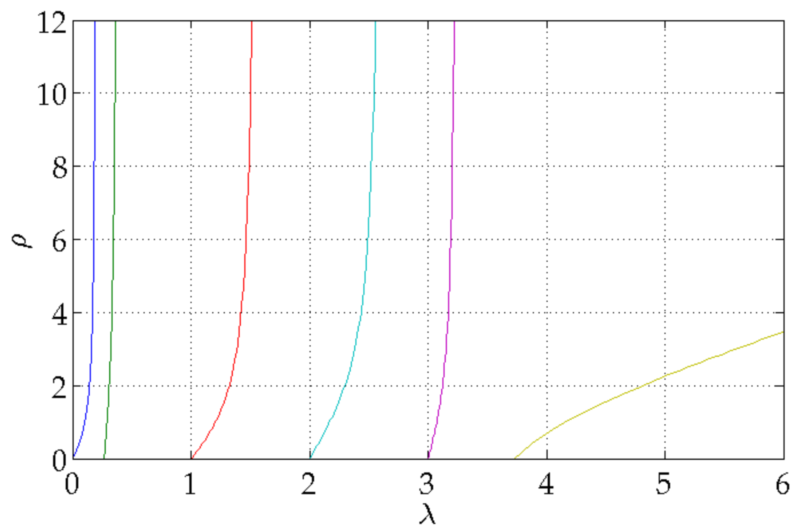
- if we pin a peripheral node the upper bound for the augmented connectivity is set by $a_{N-1}^{(r)}$;
- if we pin the second or the second-last node the upper bound is set by $a_{N-2}^{(r)}$;
- otheriwise, denoting with n the cardinality of the pinned node, we have that the upper bound is set by $a = \max\{a_{n-1}^{(r)}, a_{N-n}^{(r)}\}$.

From this information only, we can say that, in terms of upper bound for the augmented connectivity, the best choice is to pin a node as far from the periphery as possible. In fact, when the pinned node n is $2 < n < N-1$, the upper bound is set by the root of a polynomial whose degree is either $n-1$ or $N-n$, which is always lower than $N-2$.

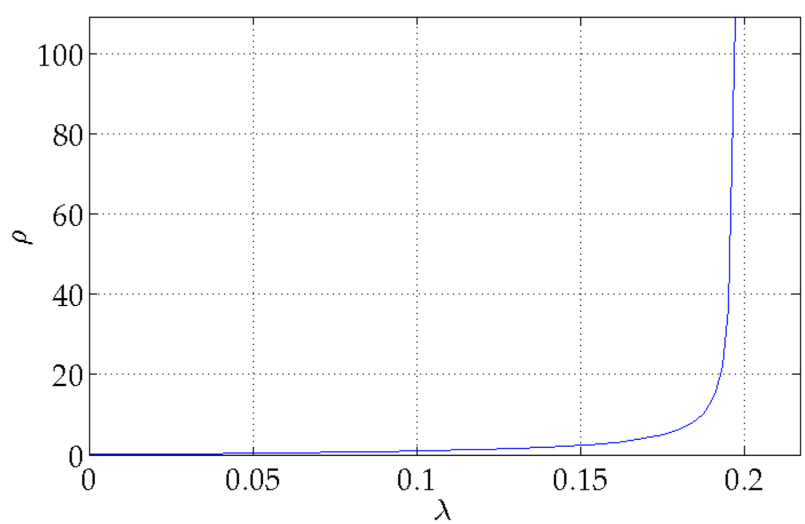
Of course this analysis is only relative to the upper bounds for the augmented connectivity, and does not give any information on the trend of the eigenvalue itself for finite values of the pinning strengths. A more complete analysis should be based on the comparison of the trend of the pinning strength, seen as a function of a, in the three scenarios. To this regard we have plotted the expressions of ρ in a large number of scenarios, and in all cases the curve happens to be better for pins that are closer to the center of the path. Therefore,

Figure 5.7: Trend of function (5.53) with $N = 6$ and $n = 3$. Since we have $n - 1 = 2$ and $N - n = 3$, the first branch of the function is bounded by $\lambda < 2 - a_3^{(r)} \simeq .20$

(a) pinning strength



(b) upper bound



a better upper bound for the augmented connectivity seems to correspond always to a better trend for the pinning strength. So far we have not proved analytically that this result holds for any value of N.

As an example of our numerical experiments, Figure 5.8 shows the trend of the pinning strength, as a function of the augmented connectivity, in a path graph with $N = 6$. As expected, the pinning strength for the central pin is always below the pinning strength required for a semiperipheral pin, which in turn is below the pinning strength required for a peripheral pin.

5.5 Double Peripheral Pin

In this section we would like to examine the case in which both the peripheral nodes are pinned in the path graph. This configuration is represented in Figure 5.9 for a graph with $N = 6$ nodes.

Theorem 16. *In a path graph where both the peripheral nodes are pinned the augmented connectivity λ_1 is bounded by*

$$0 < \lambda_1 < 2 - a^{(q)}_{N-2} \tag{5.55}$$

Moreover, for any admissible value of λ_1, the pinning strength is given by the positive solutions of

$$Q_{N-2}(a)\,\rho^2 + 2\,R_{N-1}(a)\,\rho - (2-a)\,Q_{N-1}(a) = 0 \tag{5.56}$$

with $a = 2 - \lambda_1$.

Proof. In this case equation (1.16) becomes

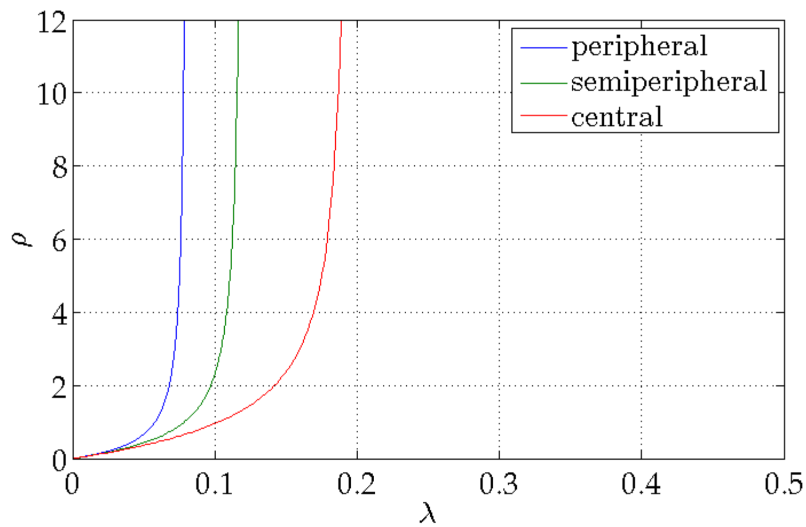
$$\begin{cases} (1+\rho)\,x_1 - x_2 = \lambda x_1 & (5.57a) \\ -x_{j-1} + 2x_j - x_{j+1} = \lambda x_j \qquad j = 2, \dots, N-1 & (5.57b) \\ -x_{N-1} + (1+\rho)\,x_N = \lambda x_N & (5.57c) \end{cases}$$

Therefore we can enforce recursion (5.3) $N-2$ times, for $j = 2, \dots, N-1$, obtaining equation (5.4). Substituting equation (5.4) and $a = 2 - \lambda$ in the system, we obtain

$$\begin{cases} (a-1+\rho)\,x_1 - x_2 = 0 & (5.58a) \\ x_{N-1} = -Q_{N-4}(a)\,x_1 + Q_{N-3}(a)\,x_2 & (5.58b) \\ x_N = -Q_{N-3}(a)\,x_1 + Q_{N-2}(a)\,x_2 & (5.58c) \\ -x_{N-1} + (a-1+\rho)\,x_N = 0 & (5.58d) \end{cases}$$

Figure 5.8: Comparison of the trends of the pinning strength in a path graph with $N = 6$, when pinning a peripheral node(blue), when pinning a semiperipheral node (green) and when pinning a central node (red). Note that the trend get better as the pin is drawn closer to the center of the graph.

(a) comparison



(b) zoom

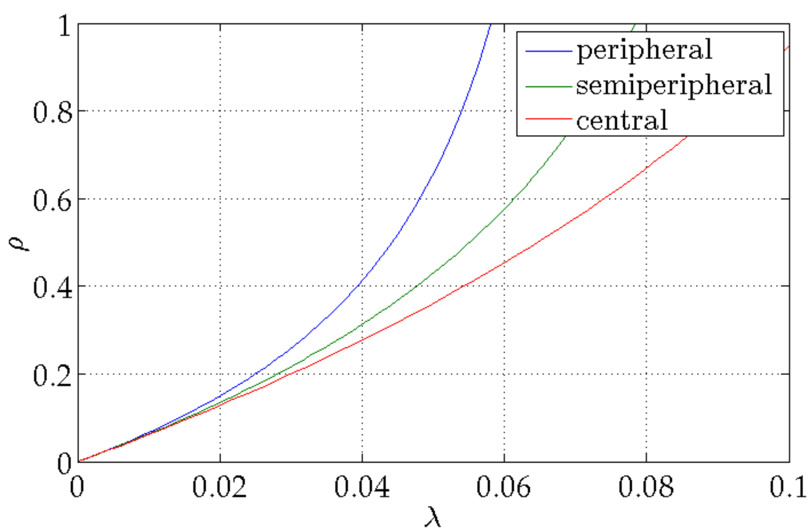
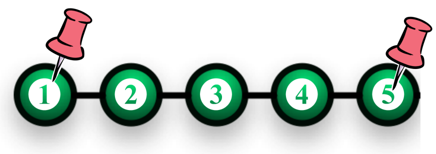


Figure 5.9: Path graph with $N = 5$ nodes and both the peripheral nodes pinned.



which is a system of four linear equations in four unknowns. The first and the last equation can be rewritten as

$$\begin{aligned} x_2 &= (a-1+\rho)\, x_1 \\ x_{N-1} &= (a-1+\rho)\, x_N \end{aligned} \tag{5.59}$$

After substituting these expressions into the second and the third equation of the system, we obtain

$$\begin{cases} (a-1+\rho)\, x_N = -q_4\, x_1 + q_3\, (a-1+\rho)\, x_1 & (5.60a) \\ x_N = -q_3\, x_1 + q_2\, (a-1+\rho)\, x_1 & (5.60b) \end{cases}$$

where $q_i = Q_{N-i}(a)$.

Substituting the second equation into the first one, we easily get

$$(a-1+\rho)(-q_3 + q_2(a-1+\rho)) = -q_4 + q_3(a-1+\rho) \tag{5.61}$$

If we isolate the pinning strength ρ, the previous equation can be rewritten as

$$\begin{gathered} (a-1)(-q_3 + q_2(a-1)) + q_2(a-1)\, \rho + \\ +(-q_3 + q_2(a-1))\, \rho + q_2\, \rho^2 - q_3\, \rho + q_4 - q_3(a-1) = 0 \end{gathered} \tag{5.62}$$

Using recursion (4.7) for q_4, we can substitute $q_4 = -q_2 + a\, q_3$ so that (5.62) becomes

$$\begin{gathered} q_2\, \rho^2 + 2(-q_3 + q_2(a-1))\, \rho + \\ +(a-1)(-q_3 + q_2(a-1)) - (q_2 - q_3) = 0 \end{gathered} \tag{5.63}$$

Again, using recursion (4.7) we can substitute $a\, q_2 = q_1 + q_3$ so that (5.63) can be rewritten as

$$q_2\, \rho^2 + 2\, r_1\, \rho + ((a-1)\, r_1 - r_2) = 0 \tag{5.64}$$

where $r_i = q_i - q_{i+1}$. Furthermore, using Corollary 1 we can substitute $r_2 = -r + a\, r_1$, so that we obtain

$$q_2\, \rho^2 + 2\, r_1\, \rho + (q - 2\, q_1 + q_2) = 0 \tag{5.65}$$

Using recursion (4.7) for q_2, we can substitute $q_2 = -q + a\, q_1$ so that (5.65) becomes

$$q_2\, \rho^2 + 2\, r_1\, \rho - (2 - a)\, q_1 = 0 \tag{5.66}$$

or equivalently

$$Q_{N-2}(a)\, \rho^2 + 2\, R_{N-1}(a)\, \rho - (2 - a)\, Q_{N-1}(a) = 0 \tag{5.67}$$

Let us note that for $Q_{N-2} = 0$, since $Q_{N-1} \neq 0$ thanks to Lemma 6, equation (**??**) is satisfied by

$$0 < \rho = 1 - \frac{a}{2} < +\infty \tag{5.68}$$

Instead in a right neighborhood of $a = a^{(q)}_{N-2}$, remembering that $R_{N-1}(a) = Q_{N-1}(a) - Q_{N-2}(a)$, equation (**??**) can be rewritten as

$$\rho^2 + 2\left(\frac{Q_{N-1}(a)}{Q_{N-2}(a)} - 1\right)\, \rho = (2 - a)\frac{Q_{N-1}(a)}{Q_{N-2}(a)} \tag{5.69}$$

Using recursion (4.7) for $Q_{N-1}(a)$, we can substitute $Q_{N-1}(a) = -Q_{N-3}(a) + a\, Q_{N-2}(a)$ so that (5.69) becomes

$$\begin{aligned} \rho^2 + 2\left(-\frac{Q_{N-3}(a)}{Q_{N-2}(a)} + a - 1\right)\, \rho = (2 - a)\left(-\frac{Q_{N-3}(a)}{Q_{N-2}(a)} + a\right) = \\ = (2 - a)\left(-\frac{Q_{N-3}(a)}{Q_{N-2}(a)} + a - 1\right) + (2 - a) \end{aligned} \tag{5.70}$$

which leads to leads to

$$\rho^2 + (2p + a - 2)\left(-\frac{Q_{N-3}(a)}{Q_{N-2}(a)} + a - 1\right) = 2 - a \tag{5.71}$$

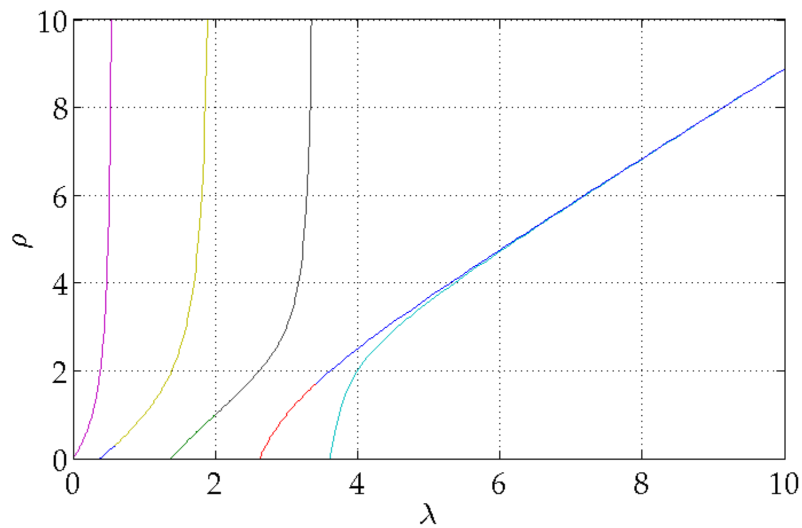
Thanks to Lemma 4 we know that $2 - a > 0$, while thanks to Lemma 5 we know that $\frac{Q_{N-3}(a)}{Q_{N-2}(a)} \gg 0$ in a right neighborhood of $a = a^{(q)}_{N-2}$. Therefore it is possible to state that (5.71) cannot be satisfied by negative or finite values of ρ, meaning that

$$\lim_{a \to a^{(q)\,+}_{N-2}} \rho = +\infty \tag{5.72}$$

This means that $\lambda = 2 - a^{(q)}_{N-2}$ must be an upper bound for the minimum eigenvalue. □

Figure 5.10: Trend of the positive solution of (5.67) for $N = 5$. Note that the first branch is bounded by $\lambda < 2 - a^{(q)}_{N-2} \simeq .60$.

(a) pinning strength



(b) upper bound

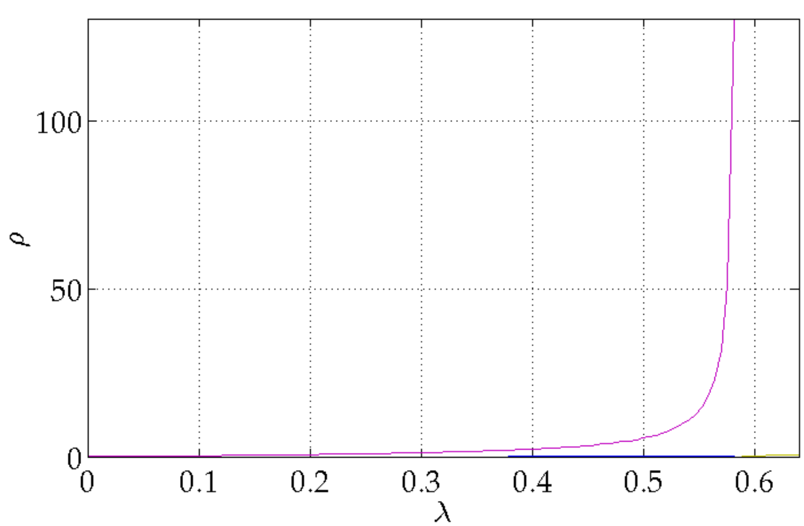


Figure 5.11 shows the trend of the positive solutions of (5.67) with $\rho > 0$, for $N = 5$. It is easy to see that the first branch of the function is always upper-bounded by $\lambda = 2 - a^{(q)}_{N-2} \simeq 0.60$.

5.6 Double-Pin Strategies Comparison

In this section we would like to compare two different pinning approaches for the path graph. Assuming that the graph contains an odd number of nodes, in one case we pin the two peripheral nodes, while in the other case we concentrate the pinning strength in the central node.

Let us denote with ρ_p the pinning strength required when pinning the two peripheral nodes. We know that for a generic value of $a = 2 - \lambda$, the value of ρ_p must obey equation (5.67) and therefore the highest root $a^{(q)}_{N-2}$ of Q_{N-2} sets an upper bound for the minimum eigenvalue.

Instead, let us denote with ρ_c the pinning strength required when pinning the central node. We know that for a generic value of $a = 2 - \lambda$, the value of ρ_c must obey (5.53), with $n = \frac{N+1}{2}$. For this value of n we have

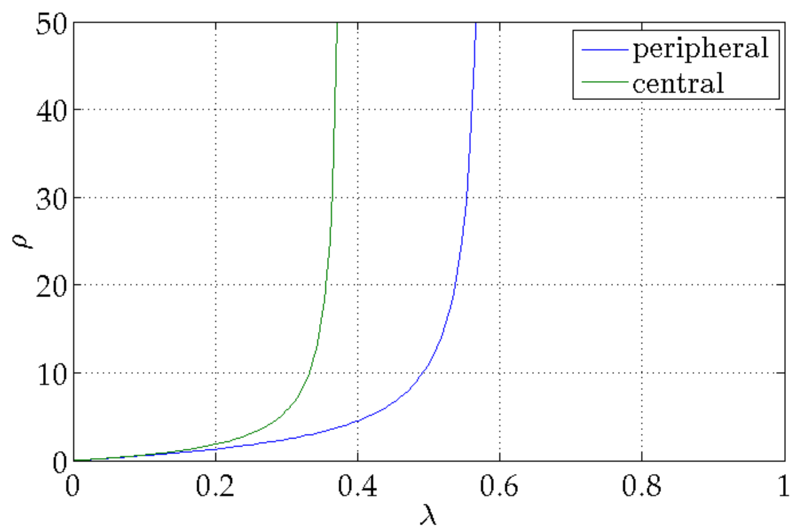
$$n - 1 = N - n = \frac{N-1}{2} \tag{5.73}$$

Therefore in this case the upper bound for the minimum eigenvalue is set by the highest root of $R_{\frac{N-1}{2}}$, which, thanks to Corollary 3, we know to be equal to the highest root $a^{(q)}_{N-1}$ of Q_{N-1}. Since $a^{(q)}_{N-2} < a^{(q)}_{N-1}$ thanks to Lemma 4, if we only care about the upper bound for the minimum eigenvalue, pinning the two peripheral nodes is always a better strategy than pinning only the central node. Of course this analysis is only relative to the upper bounds for the minimum eigenvalue, and does not give any information on the trend of the eigenvalue itself for finite values of the pinning strengths. A more complete analysis should be based on the comparison of the trend of the overall pinning strengths seen as functions of a. In the first case the overall pinning strength is given by $2\rho_p$, while in the second case it is given by ρ_c. To this regard we have plotted the expressions of $2\rho_p$ and ρ_c in a large number of scenarios, and in all cases the whole curve of $2\rho_p$ is always below the whole curve of ρ_c. Therefore, a better upper bound for the minimum eigenvalue seems to always correspond to a better trend for the pinning strength overall. So far we have not proved analytically that this result holds for any value of N.

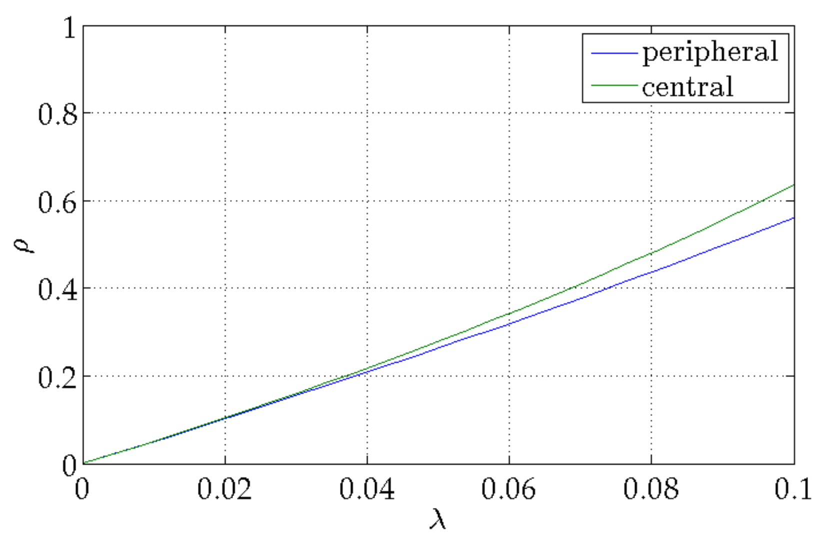
As an example of our numerical experiments, Figure 5.11 shows the trend of $2\rho_p$ and ρ_c, as a function of the augmented connectivity, in a path graph with $N = 5$. As expected, the curve for $2\rho_p$ is always below the curve for ρ_c.

Figure 5.11: trend of the total pinning strength in a path graph with $N = 5$, when pinning the two peripheral nodes (blue) and when pinning the central node (green)

(a) comparison



(b) zoom



Chapter 6

Ring Graph

In this section we address the study of the augmented connectivity of a *ring graph*. A ring graph is a graph in which consecutive nodes are connected, and also the first and the last node are connected.

Figure 6.1 shows a ring graph with $N = 6$ nodes.

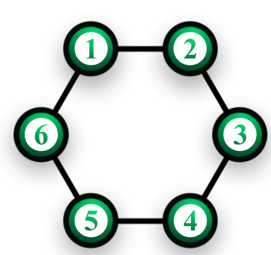
It is easy to see that the expression of the Laplacian of a ring graph is

$$L = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{bmatrix} \tag{6.1}$$

Before studying the augmented Laplacian of the ring graph, let us calculate the eigenvalues of the Laplacian when the ring graph contains no pin nodes.

For the Laplacian of a ring graph it is possible to find simple goniometrical expressions of eigenvalues and eigenvectors, exploiting the symmetrical

Figure 6.1: ring graph with $N = 6$



structure of the graph itself. This result is illustrated in [28]. However, such symmetry is lost when one or more pins are introduced. Therefore we would like to follow a different approach, which leads to results that are less explicit, but more easily extendable to the study of the augmented Laplacian.

First of all let us define

$$x_0 = x_N \qquad x_{N+1} = x_1 \tag{6.2}$$

Theorem 17. *The eigevalues of the Laplacian of a ring graph with no pins satisfy the following equation.*

$$(1 + Q_{N-2}(a))(Q_N(a) - 1) = Q_{N-1}^2(a) \tag{6.3}$$

with $a = 2 - \lambda$.

Proof. Thanks to (6.2), it is possible to see that equation (1.16) for the ring graph can be rewritten as simply as

$$-x_{j-1} + 2x_j - x_{j+1} = \lambda x_j \quad j = 1, \ldots, N \tag{6.4}$$

Therefore, if we define $a = 2 - \lambda$, equation (5.3) can be written again for $j = 1, \ldots, N$. Enforcing the recursion N times, we get

$$\begin{bmatrix} x_N \\ x_{N+1} \end{bmatrix} = A^N \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \tag{6.5}$$

Substituting the expression of A^N in (6.5), and using also (6.2), we obtain

$$\begin{cases} x_N = -Q_{N-2}(a)\, x_N + Q_{N-1}(a)\, x_1 & \text{(6.6a)} \\ x_1 = -Q_{N-1}(a)\, x_N + Q_N(a)\, x_1 & \text{(6.6b)} \end{cases}$$

or equivalently

$$\begin{cases} (1 + Q_{N-2}(a))\, x_N = Q_{N-1}(a)\, x_1 & \text{(6.7a)} \\ Q_{N-1}(a)\, x_N = (Q_N(a) - 1)\, x_1 & \text{(6.7b)} \end{cases}$$

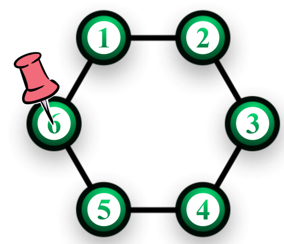
which is a system of two linear equations in two unknowns. Solving the system by Cramer's rule, we easily get the following equation

$$(1 + Q_{N-2}(a))(Q_N(a) - 1) = Q_{N-1}^2(a) \tag{6.8}$$

□

In particular we can notice that equation (6.8) accounts for the null eigenvalue since, thanks to Lemma 2, for $a = 2$ it becomes the identity $N^2 = N^2$.

Figure 6.2: Ring Graph with $N = 6$ nodes and one pin.



6.1 Single-Node Pinning

In this section we would like to address the case in which one node of the ring graph is pinned. This configuration is represented in Figure 6.2 for a graph with $N = 6$ nodes.

Theorem 18. *In a ring graph where only one node is pinned, the augmented connectivity λ_1 is bounded by*

$$0 < \lambda_1 < 2 - a_{N-1}^{(q)} \tag{6.9}$$

Moreover, for any admissible value of λ_1, the pinning strength is given by

$$\rho = \frac{(1 + Q_{N-2}(a))^2}{Q_{N-1}(a)} - (a + Q_{N-3}(a)) \tag{6.10}$$

with $a = 2 - \lambda_1$.

Proof. Of course, due to the symmetry of the graph itself, there is no difference in pinning one node or one other. Therefore, without loss of generality, let us say that we pin the last node. In this case it is possible to rewrite equation (1.16) as

$$\begin{cases} -x_{j-1} + 2x_j - x_{j+1} = \lambda x_j \quad j = 1, \ldots, N-1 & \text{(6.11a)} \\ -x_{N-1} + (2 + \rho)x_N - x_1 = \lambda x_N & \text{(6.11b)} \end{cases}$$

Therefore in this case, if we define $a = 2 - \lambda$, recursion (5.3) can be enforced $N - 1$ times for $j = 1, \ldots, N-1$, leading to

$$\begin{bmatrix} x_{N-1} \\ x_N \end{bmatrix} = A^{N-1} \begin{bmatrix} x_N \\ x_1 \end{bmatrix} = \begin{bmatrix} -Q_{N-3}(a) & Q_{N-2}(a) \\ -Q_{N-2}(a) & Q_{N-1}(a) \end{bmatrix} \begin{bmatrix} x_N \\ x_1 \end{bmatrix} \tag{6.12}$$

Therefore we can rewrite the system as as

$$\begin{cases} x_{N-1} = -Q_{N-3}(a)\, x_N + Q_{N-2}(a)\, x_1 & \text{(6.13a)} \\ x_N = -Q_{N-2}(a)\, x_N + Q_{N-1}(a)\, x_1 & \text{(6.13b)} \\ -x_{N-1} + (a+\rho)x_N - x_1 = 0 & \text{(6.13c)} \end{cases}$$

which is a system of three linear equations in three unknowns.
Equation (6.13c) can be rewritten as

$$x_{N-1} = (a+\rho)x_N - x_1 \tag{6.14}$$

After substituting this expression in (6.13a) we obtain

$$\begin{cases} (a+\rho+q_3)\, x_N = (1+q_2)\, x_1 & \text{(6.15a)} \\ (1+q_2)\, x_N = q_1\, x_1 & \text{(6.15b)} \end{cases}$$

where $q_i = Q_{N-i}(a)$.
Solving the system by Cramer's rule, we easily get the following equation

$$q_1\,(a+\rho+q_3) = (1+q_2)^2 \tag{6.16}$$

In a right neighborhood of $a = a^{(q)}_{N-1}$ we can solve the previous equation for ρ, obtaining

$$\rho = \frac{(1+q_2)^2}{q_1} - (a+q_3) \tag{6.17}$$

or equivalently

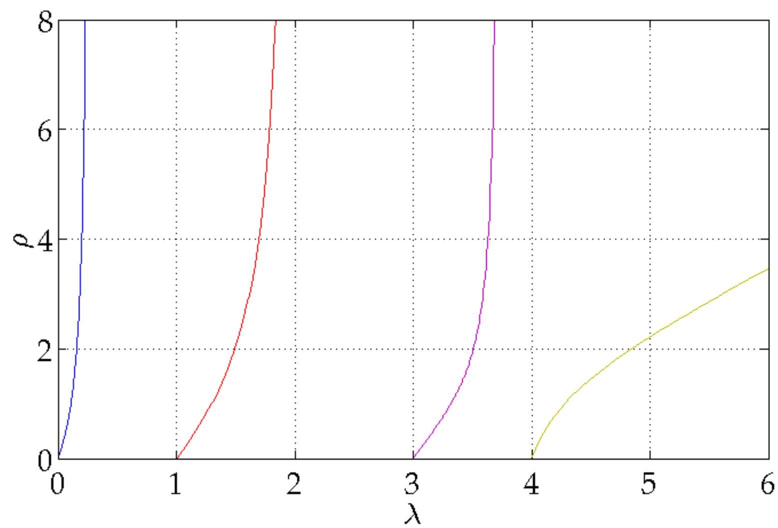
$$\rho = \frac{(1+Q_{N-2}(a))^2}{Q_{N-1}(a)} - (a+Q_{N-3}(a)) \tag{6.18}$$

Thanks to Lemma 5 we know that $Q_{N-2}(a^{(q)}_{N-1}) > 0$, meaning that function (6.18) is going to infinity in a right neighborhood of $a = a^{(q)}_{N-1}$. Therefore $\lambda = 2 - a^{(q)}_{N-1}$ is the first vertical asymptote for ρ as a function of λ. This means that the minimum eigenvalue λ_1 is bounded by $0 < \lambda_1 < 2 - a^{(q)}_{N-1}$. □

Figure 6.2 shows the trend of function (6.18) for $N = 6$. In this case we have $2 - a^{(q)}_{N-1} \simeq .26$, and it is possible to see from the figure that the first branch of the function is indeed upper-bounded by this value of λ.

Figure 6.3: Trend of function 6.18 with $N = 6$. Note that the first branch is bounded by $\lambda < 2 - a^{(q)}_{N-1} \simeq .26$

(a) pinning strength



(b) upper bound

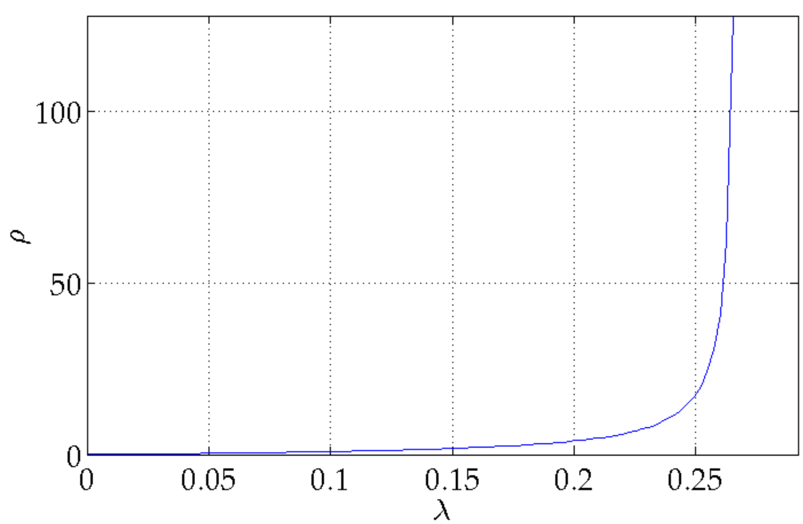
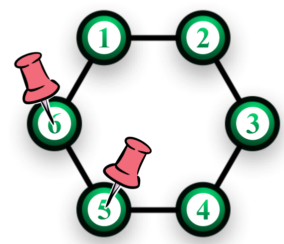


Figure 6.4: Ring graph with $N = 6$ nodes and two consecutive nodes pinned.



6.2 Double-Consecutive-Node Pinning

In this section we would like to address the case in which two consecutive nodes are pinned in a ring graph. This configuration is represented in Figure 6.4 for a graph with $N = 6$ nodes.

Theorem 19. *In a ring graph where two consecutive nodes are pinned, the augmented connectivity λ_1 is bounded by*

$$0 < \lambda_1 < 2 - a^{(q)}_{N-2} \tag{6.19}$$

Moreover, for any admissible value of λ_1, the pinning strength is given by the positive solution of

$$\begin{aligned} Q_{N-2}(a)\,(a+\rho)^2 - 2\,Q_{N-3}(a)\,(a+\rho) &= \\ = Q^2_{N-3}(a) - (Q_{N-2}(a)+1)\,(a\,Q_{N-3}(a) - Q_{N-2}(a)) \end{aligned} \tag{6.20}$$

with $a = 2 - \lambda_1$.

Proof. Of course, given the simmetrical structure of the graph there is no difference in pinning any of all the possible couples of consecutive nodes. Therefore let us say, without loss of generality, that nodes $N-1$ and N are pinned. In this case, after defining $a = 2-\lambda$, recursion (5.3) can be written for $j = 1, \ldots, N-2$. Enforcing the recursion $N-2$ times we obtain

$$\begin{bmatrix} x_{N-2} \\ x_{N-1} \end{bmatrix} = A^{N-2} \begin{bmatrix} x_N \\ x_1 \end{bmatrix} = \begin{bmatrix} -Q_{N-4}(a) & Q_{N-3}(a) \\ -Q_{N-3}(a) & Q_{N-2}(a) \end{bmatrix} \begin{bmatrix} x_N \\ x_1 \end{bmatrix} \tag{6.21}$$

Instead, for the pinned nodes we have the equations

$$\begin{aligned} -x_{N-2} + (2+\rho)x_{N-1} - x_N &= \lambda x_{N-1} \\ -x_{N-1} + (2+\rho)x_N - x_1 &= \lambda x_N \end{aligned} \tag{6.22}$$

Therefore, remembering the expression of A^{N-2} and substituting $a = 2-\lambda$, we can write the following system of four linear equations in four unknowns

$$\begin{cases} x_{N-2} = -Q_{N-4}(a)\, x_N + Q_{N-3}(a)\, x_1 & (6.23a) \\ x_{N-1} = -Q_{N-3}(a)\, x_N + Q_{N-2}(a)\, x_1 & (6.23b) \\ -x_{N-2} + (a+\rho)x_{N-1} - x_N = 0 & (6.23c) \\ -x_{N-1} + (a+\rho)x_N - x_1 = 0 & (6.23d) \end{cases}$$

The third and fourth equation of the system can be rewritten as

$$\begin{aligned} x_{N-2} &= (a+\rho)\, x_{N-1} - x_N \\ x_{N-1} &= (a+\rho)\, x_N - x_1 \end{aligned} \tag{6.24}$$

Substituting these expressions into the first and the second equation we obtain

$$\begin{cases} (a+\rho)\, ((a+\rho)\, x_N - x_1) - x_N = -q_4\, x_N + q_3\, x_1 & (6.25a) \\ (a+\rho)\, x_N - x_1 = -q_3\, x_N + q_2\, x_1 & (6.25b) \end{cases}$$

where $q_i = Q_{N-i}(a)$.
From this system we easily obtain

$$\begin{cases} ((a+\rho)^2 + q_4)\, x_N = ((a+\rho) + q_3)\, x_1 & (6.26a) \\ ((a+\rho) + q_3)\, x_N = (1 + q_2)\, x_1 & (6.26b) \end{cases}$$

Solving the system by Cramer's rule, we easily get the following equatio

$$(1+q_2)((a+\rho)^2 + q_4) = ((a+\rho) + q_3)^2 \tag{6.27}$$

which can be rewritten as a second-order equation in the unknown $a+\rho$. After few simple passages, we get

$$q_2\, (a+\rho)^2 - 2\, q_3\, (a+\rho) = q_3^2 - (q_2+1)\, q_4 \tag{6.28}$$

Using recursion (4.7) for q_4, we can substitute $q_4 = a\, q_3 - q_2$ so that (6.28) becomes

$$q_2\, (a+\rho)^2 - 2\, q_3\, (a+\rho) = q_3^2 - (q_2+1)(a\, q_3 - q_2) \tag{6.29}$$

which is equivalent to (6.20). Thanks to Lemma 5 and Descartes's rule, we can say that equation (6.29) has at least one positive solution in the unknown $a+\rho$. In a right neighborhood of $a = a^{(q)}_{N-2}$, we can rewrite (6.29) as

$$(a+\rho)^2 = 2\, \frac{q_3}{q_2}\, (a+\rho) + \frac{q_3^2}{q_2} - \left(1 + \frac{1}{q_2}\right)(a\, q_3 - q_2) \tag{6.30}$$

which after a few passages yields

$$(a+\rho)^2 = [2\,(a+\rho) + q_3 - a]\,\frac{q_3}{q_2} - (a\,q_3 - q_2 - 1) \tag{6.31}$$

Using Lemma 5 again, it is possible to state that positive solutions of equation (6.31) in the unknown $a+\rho$ cannot be finite. Therefore we can conclude that

$$\lim_{a \to a_{N-2}^{(q)\,+}} \rho = +\infty \tag{6.32}$$

This means that $\lambda = 2 - a_{N-2}^{(q)}$ must be an upper bound for the minimum eigenvalue. □

Figure 6.5 shows the trend of the positive solutions of (6.29) with $\rho > 0$, for $N = 6$. It is easy to see that the first branch of the function is upper-bounded by $\lambda = 2 - a_{N-2}^{(q)} \simeq 0.38$.

6.3 Double-Nonconsecutive-Node Pinning

In this section we would like to address the case in which two non consecutive nodes are pinned in the ring graph. This configuration is represented in Figure 6.6 for a graph with $N = 6$ nodes.

Theorem 20. *In a ring graph where two non consecutive nodes are pinned, the augmented connectivity λ_1 is bounded by*

$$0 < \lambda_1 < 2 - \max\{a_{n-1}^{(q)},\ a_{N-n-1}^{(q)}\} \tag{6.33}$$

Moreover, for any admissible value of λ_1, the pinning strength is given by the positive solution of

$$\begin{gathered}\tilde{q}_1\,q_1(a+\rho)^2 - 2(\tilde{q}_1\,q_2 + \tilde{q}_2\,q_1)(a+\rho) + \\ +\tilde{q}_2^2\,q_3\,q_1 + \tilde{q}_3\,q_2^2\,\tilde{q}_1 - \tilde{q}_3\,q_1\,\tilde{q}_1\,q_3 + \\ +\tilde{q}_3\,q_1 + \tilde{q}_1\,q_3 - (1 - \tilde{q}_2\,q_2)^2 = 0\end{gathered} \tag{6.34}$$

with $a = 2 - \lambda_1$, where we define

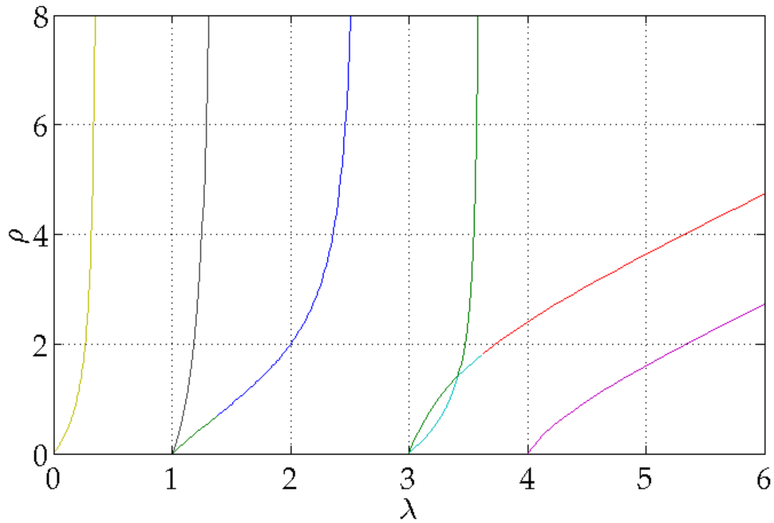
$$q_i = Q_{n-i}(a), \quad \tilde{q}_i = Q_{N-n-i}(a) \qquad i = 1,2,3 \tag{6.35}$$

Proof. Without loss of generality, let us say that the n-th and the N-th nodes are pinned. In this case, after defining $a = 2 - \lambda$, recursion (5.3) can be written $n-1$ times for $j = 1, \ldots, n-1$ and $N-n-1$ times for $j = n+1, \ldots, N-1$.

Enforcing the first recursion we get

Figure 6.5: Trend of the positive solutions of (6.29) for $N = 6$. Note that the first branch is bounded by $\lambda < 2 - a_{N-2}^{(q)} \simeq .38$.

(a) pinning strength



(b) upper bound

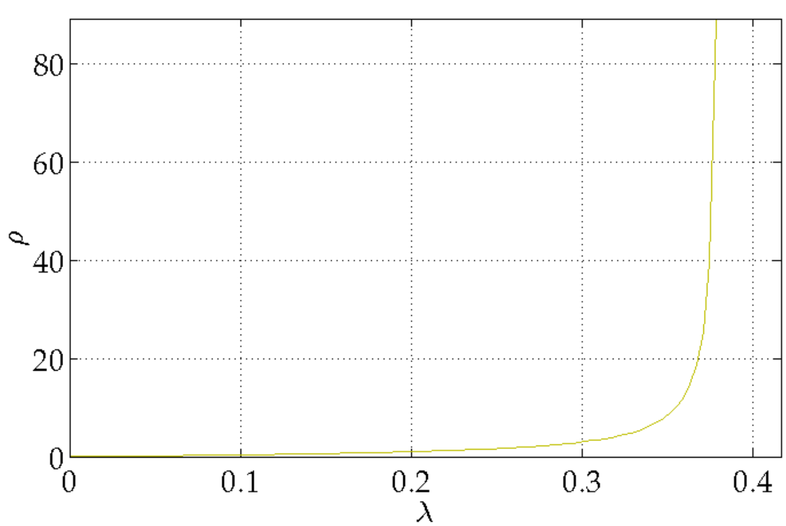
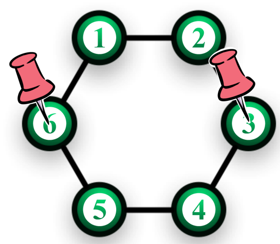


Figure 6.6: Ring Graph with $N = 6$ nodes and two nonconsecutive pin nodes.



$$\begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} = A^{n-1} \begin{bmatrix} x_N \\ x_1 \end{bmatrix} = \begin{bmatrix} -Q_{n-3}(a) & Q_{n-2}(a) \\ -Q_{n-2}(a) & Q_{n-1}(a) \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} \tag{6.36}$$

while enforcing the second recursion we get

$$\begin{bmatrix} x_{N-1} \\ x_N \end{bmatrix} = A^{N-n-1} \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} -Q_{N-n-3}(a) & Q_{N-n-2}(a) \\ -Q_{N-n-2}(a) & Q_{N-n-1}(a) \end{bmatrix} \begin{bmatrix} x_{N-1} \\ x_N \end{bmatrix} \tag{6.37}$$

Finally, for the pinned nodes we have the following equations.

$$\begin{aligned} -x_{n-1} + (2+\rho)x_n - x_{n+1} &= \lambda x_n \\ -x_{N-1} + (2+\rho)x_N - x_1 &= \lambda x_N \end{aligned} \tag{6.38}$$

We can so obtain a system of six linear equations in six unknowns, which can be written as follows.

$$\begin{cases} x_{n-1} = -Q_{n-3}(a)\, x_N + Q_{n-2}(a)\, x_1 & \text{(6.39a)} \\ x_n = -Q_{n-2}(a)\, x_N + Q_{n-1}(a)\, x_1 & \text{(6.39b)} \\ -x_{n-1} + (a+\rho)x_n - x_{n+1} = 0 & \text{(6.39c)} \\ x_{N-1} = -Q_{N-n-3}(a)\, x_n + Q_{N-n-2}(a)\, x_{n+1} & \text{(6.39d)} \\ x_N = -Q_{N-n-2}(a)\, x_n + Q_{N-n-1}(a)\, x_{n+1} & \text{(6.39e)} \\ -x_{N-1} + (a+\rho)x_N - x_1 = 0 & \text{(6.39f)} \end{cases}$$

The third and the last equation of the system can be rewritten as

$$\begin{aligned} x_{n-1} &= (a+\rho)x_n - x_{n+1} \\ x_{N-1} &= (a+\rho)x_N - x_1 \end{aligned} \tag{6.40}$$

Substituting these expressions into the remaining equations of the system we obtain

$$\begin{cases} (a+\rho)\,x_n - x_{n+1} = -q_3\,x_N + q_2\,x_1 & \text{(6.41a)} \\ x_n = q_2\,x_N + q_1\,x_1 & \text{(6.41b)} \\ (a+\rho)\,x_N - x_1 = -\tilde{q}_3\,x_n + \tilde{q}_2\,x_{n+1} & \text{(6.41c)} \\ x_N = -\tilde{q}_2\,x_n + \tilde{q}_1\,x_{n+1} & \text{(6.41d)} \end{cases}$$

which can be rewritten as

$$\begin{bmatrix} a+\rho & -1 & q_3 & -q_2 \\ 1 & 0 & q_2 & -q_1 \\ \tilde{q}_3 & -\tilde{q}_2 & a+\rho & -1 \\ \tilde{q}_2 & -\tilde{q}_1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n+1} \\ x_N \\ x_1 \end{bmatrix} = 0_4 \tag{6.42}$$

Solving (6.42) with Cramer, we get

$$\begin{aligned} &\tilde{q}_1\,q_1(a+\rho)^2 - 2(\tilde{q}_1\,q_2 + \tilde{q}_2\,q_1)(a+\rho)+ \\ &\quad +\tilde{q}_2^2\,q_3\,q_1 + \tilde{q}_3\,q_2^2\,\tilde{q}_1 - \tilde{q}_3\,q_1\,\tilde{q}_1\,q_3+ \\ &\quad +\tilde{q}_3\,q_1 + \tilde{q}_1\,q_3 - (1-\tilde{q}_2\,q_2)^2 = 0 \end{aligned} \tag{6.43}$$

which corresponds to (6.34).

Let us suppose that $n > N-n$. Thanks to Lemma 5, we know that in a right neighborhood of $a = a_{n-1}^{(q)}$ it holds that $\tilde{q}_1 > 0$. This means that we can rewrite equation (6.43) as

$$\begin{aligned} &(a+\rho)^2 - 2\,\frac{q_2}{q_1}(a+\rho) = \\ &= \left[(2-a)\frac{q_2}{q_1}(1+q_2\,\tilde{q}_2) + \left(\frac{q_2(\tilde{q}_1+\tilde{q}_2)-1}{\tilde{q}_1\,q_1}\right)^2\right] \end{aligned} \tag{6.44}$$

where the terms without q_1 as a denominator have been neglected when summed with terms whose denominator contains q_1 instead.

If we consider again Lemma 5 together with Descartes's rule, we know that equation (6.44) must have one positive and one negative solution in the unknown $a+\rho$. It is easy to see that the positive solution cannot be finite in the mentioned neighborhood, meaning that

$$\lim_{a \to a_{n-1}^{(q)}{}^+} \rho = +\infty \tag{6.45}$$

Therefore we can conclude that $\lambda = 2 - a_{n-1}^{(q)}$ must be an upper bound for the minimum eigenvalue.

A similar reasoning can be carried out when $n < N-n$, leading to $\lambda = 2 - a_{N-n-1}^{(q)}$ being an upper bound for the minimum eigenvalue.

Finally, if $n = N - n$ we have that

$$q_i = \tilde{q}_i \qquad \forall i = 1, 2, 3 \tag{6.46}$$

so that equation (6.43) can be rewritten as

$$\begin{gathered} q_1^2(a+\rho)^2 - 4\, q_1\, q_2(a+\rho) + \\ +2\, q_1\, q_2^2\, q_3 - q_1^2\, q_3^2 + 2\, q_1\, q_3 - (1-q_2^2)^2 = 0 \end{gathered} \tag{6.47}$$

It is easy to see that, in a right neighborhood of $a = a_{n-1}^{(q)}$, we can rewrite the previous equation as

$$q_1^2(a+\rho)^2 - 4\, q_1\, q_2(a+\rho) - (1-q_2^2)^2 = 0 \tag{6.48}$$

Again, if we consider Lemma 5 together with Descartes's rule, we can draw the same conclusions as in the two previous cases, meaning that

$$\lim_{a \to a_{n-1}^{(q)\,+}} \rho = \lim_{a \to a_{N-n-1}^{(q)\,+}} \rho = +\infty \tag{6.49}$$

Therefore, in general, it is possible to say that the upper bound for the minimum eigenvalue is given by

$$\lambda = 2 - \max\{a_{n-1}^{(q)},\ a_{N-n-1}^{(q)}\} \tag{6.50}$$

□

Figure 6.7 shows the trend of the positive solutions of (6.43) with $\rho > 0$, for $N = 6$ and $n = 3$. In this case we have $n-1 = N-n-1 = 2$, therefore the first branch of the function is upper-bounded by $\lambda = 2 - a_2^{(q)} = 1$.

6.4 Pinning Strategies Comparison

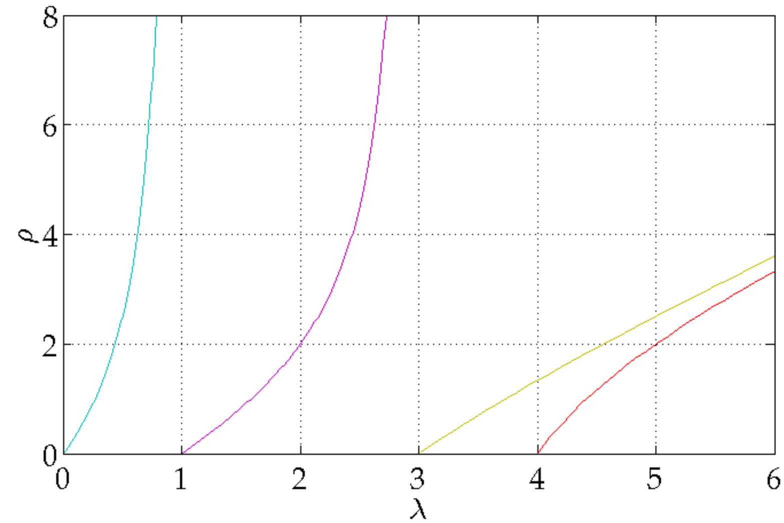
In this section we would like to compare different pinning scenarios for the ring graph. We will consider the three following cases:

- one node is pinned with pinning strength ρ_1, so that the total pinning strength is still ρ_1;
- two consecutive nodes are pinned with pinning strength ρ_c, so that the total pinning strength is equal to $2\rho_c$;
- two non consecutive nodes are pinned with pinning strength ρ_s, so that the total pinning strength is equal to $2\rho_s$.

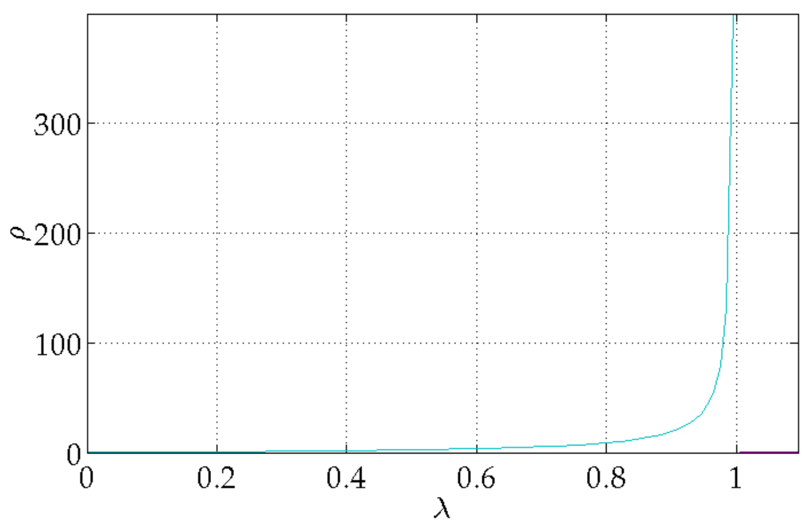
It is worth remembering that:

Figure 6.7: Trend of the positive solutions of (6.43) with $N = 6$ and $n = 3$. Note that the first branch is bounded by $\lambda < 2 - a_2^{(q)} = 1$

(a) pinning strength



(b) upper bound



- if we pin one node, the upper bound for the augmented connectivity is given by $2 - a^{(q)}_{N-1}$;
- if we pin two consecutive nodes, the upper bound for the augmented connectivity is given by $2 - a^{(q)}_{N-2}$;
- if we pin two non consecutive nodes, the upper bound for the augmented connectivity is given by $2 - \max\{a^{(q)}_{n-1},\ a^{(q)}_{N-n-1}\}$.

We note that when nodes n and N are not consecutive, meaning that $1 < n < N-1$, both subscripts $n-1$ and $N-n-1$ are strictly lower than $N-2$. Thanks to Lemma 4 we can so conclude that, at least in term of upper bounds for the augmented connectivity, pinning two non consecutive nodes is always better that pinning two consecutive nodes, and of course than pinning just one node with double strength.

In particular, the best choice is to pin two nodes that are as distant as possible. For example, if the number of nodes is even, and node N is pinned, the best choice for the second pin will be the $\frac{N}{2}$-th node. In fact, in this case both subscripts $n-1$ and $N-n-1$ will be equal to $\frac{N}{2}-1$.

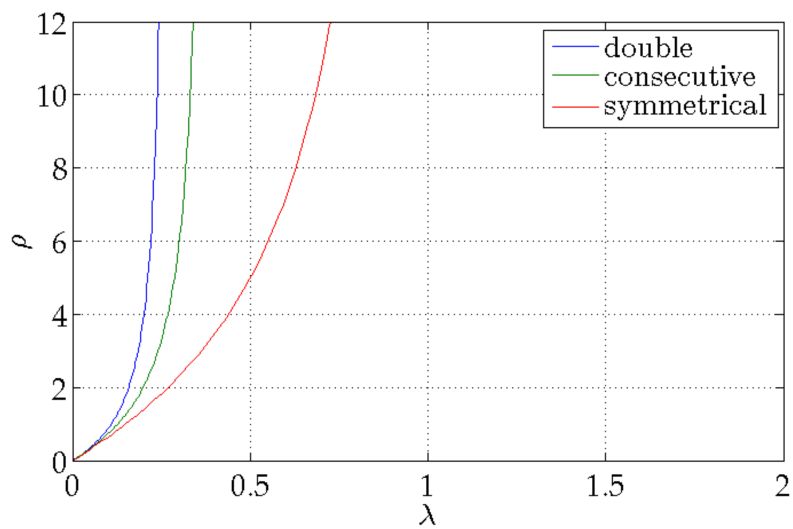
Of course this analysis is valid only with respect to the upper bound for the augmented connectivity, and gives no information about the trend of the eigenvalue itself for finite values of the pinning strength.

As usual, in order to make a more exhaustive comparison, we have plotted the trend of the overall pinning strengths in a number of scenarios. In all the cases we have observed that the whole curve of $2\rho_s$ stays below the whole curve of $2\rho_c$, which in turn stays below the whole curve of ρ_1. Therefore a configuration that gives a better upper bound for the augmented connectivity, appears to give also a better trend for the overall pinning strength. So far we have not proved analytically that this result holds for any value of N.

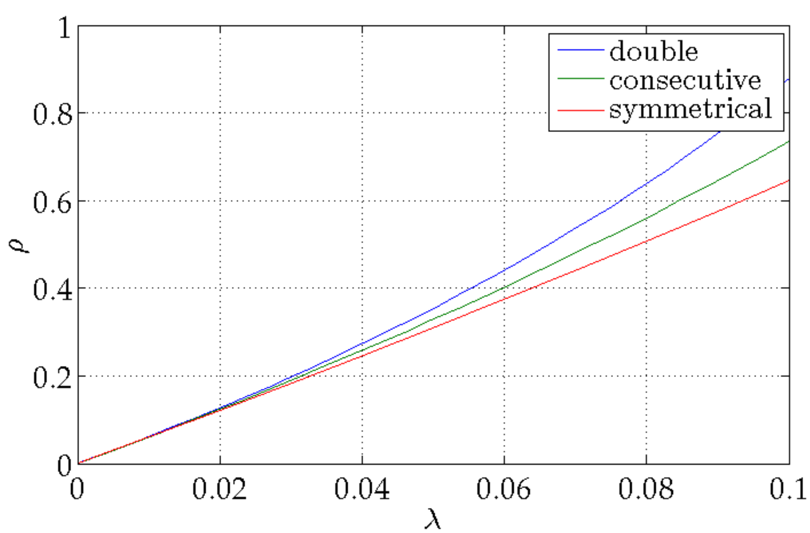
As an example of our numerical results, Figure 6.8 shows the trend of ρ_1, $2\rho_c$ and $2\rho_s$ in a ring graph with $N = 6$.

Figure 6.8: Comparison among the first branches of functions ρ_1 (blue), $2\rho_c$ (green) and $2\rho_s$ (red) with $N = 6$. Note that the trend gets better as the distance between the two pinned nodes increases.

(a) comparison



(b) zoom



Chapter 7

Conclusions

This chapter concludes the report with a short summary of our results and an overview of possible extensions and developments.

7.1 Main Results Summary

In this report the problem of optimal pin selection was addressed as part of the control design in the pinning control problem. The augmented graph formalism was introduced, and we showed its effectiveness in the mathematical formulation of a pinning control problem. Specifically, we showed that pinning control of a network of nonlinear agents is possible if the augmented connectivity of the underlying graph is large enough.

Four fundamental topologies were considered, and for each of them several pinning configurations were examined and compared. All the comparisons were based on the total pinning strength required to obtain a certain value for the augmented connectivity. Without loss of generality, a normalized value for the coupling strength has been used. Our results can be summarized as follows.

- In a complete graph it is always better to distribute the pinning strength among as much nodes as possible. A larger number of nodes results always in a larger upper bound for the augmented connectivity. Moreover, for any admissible value of the augmented connectivity, a smaller value of the total pinning strength is required when a larger number of nodes are pinned. Pinning all the nodes is the only configuration that does not place any finite upper bound to the augmented connectivity.
- In the star graph, pinning the central node is always better than pinning one peripheral node, both in terms of bounds for the augmented connec-

tivity and in terms of total pinning strength required. Similiarly, when pinning less than half of the nodes, it is always better to include the central node among the pins, both in terms of bounds for the augmented connectivity and in terms of the total pinning strength required.

- When more than half of the nodes are pinned in the star graph, it is better not to have the central node pinned if the required value for the augmented connectivity is below a certain threshold. Conversely, if a larger value for the augmented connectivity is desired, it is better to include the central node in the pins. In this case having the central node pinned gives also a better bound for the augmented connectivity.

- In the star graph, pinning all the peripheral nodes is better than all the aforementioned strategies. Pinning all the nodes is still better and it is also the only configuration that does not place any finite upper bound on the augmented connectivity.

- When one node is pinned in the path graph, it is better to have it as far from the periphery as possible. However, pinning the two peripheral nodes is better than concentrating the pinning strength in the central node.

- In the ring graph, pinning two different nodes is always better than concentrating the pinning strength in one node. Moreover, when pinning two different nodes, a higher distance between them always correponds to a better upper bound for the augmented connectivity and a better trend of the pinning strength required.

For the complete graph and star graph, all the results were proved analytically. For the path graph and the ring graph, analytical proof was given for all the results but for the comparison of the trend of the pinning strength in different configurations.

Of course the majority of the reported results are intuitive or at least can be easily verified via numerical simulations. However, to the best of the authors' knowledge, analytical proof of such properties has not been addressed yet in the literature.

7.2 Future Developments

Pinning control is a relatively newborn branch of automation and a pinning control theory is in process of formation. Optimal pin selection is addressed in modern research papers, and increasingly complex scenarios are examined.

A large set of open questions and possibilities for future developments can be outlined from the presented work. Here we list some points which could be interesting to address.

- The pin selection criteria obtained in this work could be compared to the output of some adaptive and/or decentralized pinning strategies, such as those proposed in [10, 11]. Conversely, the presented results may be used as a simple testbed for novel adaptive pinning strategies.
- In this work we use a normalized value for the coupling strength and focus our attention on the variations of the pinning strength. Strictly speaking this causes no loss of generality, but it would be interesting to adopt a reversed point of view and focus the attention on the effects of the variation of the coupling strength when the pinning strength is fixed.
- The possibility of having a different pinning strength for different nodes may be considered.
- Our analysis is limited to four kinds of graphs, but other topologies could be studied with the same approach. Graphs whose topologies are obtained joining two or more of the graphs examined in this work could are a feasible example. Pinning strategies presented here may be tested on the joint graphs to see if optimality is obtained with the same pin allocation. Then more complex topologies may be examined with an incremental approach.
- Weighted or directed graphs may be examined as well.

All the configurations examined in this work are static. However, it is intuitive that periodical variations of the pin allocation may have beneficial effect on the synchronization speed. For simple topologies it could be manageable to study such effects analitically. Variation of the edge allocation has been recently studied in synchronization problems under the name of *blinking*. Interesting results about blinking networks can be found in [35, 36, 37, 38, 39, 40]. Some of this results have been extended to the pinning control problem, letting the blinking dynamics affect the pin selection other than the interactions among the agents. This scenario has been considered in [41], where some sufficient conditions are given for local pinning controllability of a complex oscillator network with cyclical pin selection. Periodically intermittent pinning control is also studied in [42], here on a network of delayed oscillators. Interesting developments in this direction might involve the following.

- Sufficient conditions for global pinning controllability could be researched, maybe drawing inspiration from [6].

- Success of the control task may be analytically related to the frequency of the pin switchings.

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