1.1 Compound Propositions

1.1 复合命题

In English, we can modify, combine, and relate propositions with words such as "not", "and", "or", "implies", and "if-then". For example, we can combine three propositions into one like this:

If all humans are mortal **and** all Greeks are human, **then** all Greeks are mortal.

在英语中，我们可以用如下的词修改、组合、关联命题——“非”、“且”、“或”、“蕴涵”、“如果-那么”。例如，可以把三个命题组合成类似下面的这个：

如果所有人都终归一死**且**希腊人是人，**那么**希腊人终究会死。

For the next while, we won't be much concerned with the internals of propositions——whether they involve mathematics or Greek mortality—— but rather with how propositions are combined and related. So we'll frequently use variables such as P and Q in place of specific propositions such as "All humans are mortal" and "2 + 3 = 5". The understanding is that these variables, like propositions, can take on only the values T(true) and F(false). Such true/false variables are sometimes called Boolean variables after their inventor, George——you guessed it——Boole.

接下来的这段时间，我们不会太过关注命题的内在含义——无论它们涉及数学还是希腊人的必死性——相反，我们会更关注怎样组合关联命题。所以我们经常使用像P和Q一样的变量来代替像“人终有一死”和"2 + 3 = 5"一样的具体命题。我们这样做的原因是，这些变量像命题一样，可取的值只有两个：T(真)和F(假）。这种真/假变量有时叫做布尔变量——以其发明者的名字乔治·布尔（你猜对了）命名。

1.1.1 NOT, AND, and OR

1.1.1 非、与、或

We can precisely define these special words using truth tables. For example, if P denotes an arbitrary proposition, then the truth of the proposition "NOT(P)" is defined by the following truth table:

P	NOT(P)
T	F
F	T

可使用**真值表**精确定义这些关键词。例如，如果P代表某个任意命题，则命题“非P"的取值由以下真值表定义：

P	非P
T	F
F	T

The first row of the table indicates that when proposition P is true, the proposition "NOT(P)" is false. The second line indicates that when P is false, "NOT(P)" is true. This is probably what you would expect.

表格的第一行表明，命题P为真时，命题“非P”为假。第二行表明，P为假时，“非P”为真。这可能是你所期望的。

In general, a truth table indicates the true/false value of a proposition for each possible setting of the variables.

真值表通常反映了在每个可能的变量取值下，命题的真/假值。

For example, the truth table for the proposition "P AND Q" has four lines, since the two variables can be set in four different ways:

P	Q	P AND Q
T	T	T
T	F	F
F	T	F
F	F	F

例如，命题“P与Q”的真值表有四行，因为可以用四种方式来设定两个变量的值：

P	Q	P 与 Q
T	T	T
T	F	F
F	T	F
F	F	F

According to this table, the proposition "P AND Q" is true only when P and Q are both true. This is probably the way you think about the word "and".

根据这个表格，只有P和Q都为真时，命题“P与Q”才为真。也许你也是这样考虑单词“**与**”的。

There is a subtlety in the truth table for "P OR Q":

P	Q	P OR Q
T	T	T
T	F	T
F	T	T
F	F	F

"P或Q"的真值表中有个细微差别：

P	Q	P 或 Q
T	T	T
T	F	T
F	T	T
F	F	F

The second row of this table says that "P OR Q" is true even if both P and Q are true. This isn't always the intended meaning of "or" in everyday speech, but this is the standard definition in mathematical writing. So if a mathematician says, "You may have cake, or you may have ice cream," he means that you could have both.

该表格的第二行表明，即使P和Q都为真，“P或Q"也为真。这并非总是日常会话中”或“的预期含义，但却是数学著作中的标准定义。所以如果某个数学家说，”你可以吃蛋糕，**或**冰淇淋。“ 他的意思是两种都可以吃。

If you want to exclude the possibility of both having and eating, you should use "exclusive-or" (XOR):

P	Q	P XOR Q
T	T	F
T	F	T
F	T	T
F	F	F

如果你想要排除把两种都拿来吃的可能性，应使用”异或“（XOR):

P	Q	P 异或 Q
T	T	F
T	F	T
F	T	T
F	F	F

1.1.2 IMPLIES

1.1.2 蕴涵

The least intuitive connecting word is "implies". Here is its truth table, with the lines labeled so we can refer to them later.

P	Q	P IMPLIES Q
T	T	T (tt)
T	F	F (tf)
F	T	T (ft)
F	F	T (ff)

最不直观的连接词是“蕴涵”。下面是它的真值表，每行都加了标记，以便在之后提及。

P	Q	P 蕴涵 Q
T	T	T (tt)
T	F	F (tf)
F	T	T (ft)
F	F	T (ff)

Let's experiment with this definition. For example, is the following proposition true or false?

"If the Riemann Hypothesis is true, then $x^2 \geq 0$ for every real number x."

让我们试用下该定义。例如，以下命题是真还是假?

"如果黎曼假设为真，那么对于每个实数x,有$x^2 \geq 0$"。

The Riemann Hypothesis is a famous unresolved conjecture in mathematics —no one knows if it is true or false. But that doesn’t prevent you from answering the question! This proposition has the form P IMPLIES Q where the *hypothesis*, P , is “the Riemann Hypothesis is true” and the *conclusion*, Q, is “$x^2 \geq 0$ for every real number x”. Since the conclusion is definitely true, we’re on either line (tt) or line (ft) of the truth table. Either way, the proposition as a while is *true*!

黎曼假设是数学中一个尚未证实的著名猜想——没人知道它的真假。但这并不妨碍你回答该问题！这个命题有着“P蕴涵Q"的形式——假设P是”黎曼假设是真的“，而结论Q是”对于每个实数x, $x^2 \geq 0$“。因为结论肯定为真，所以该命题要么落在真值表的tt行，要么落在真值表的ft行。不管是哪种情况，该命题总是为真！

One of our original examples demonstrates an even stranger side of implications.

"If pigs can fly, then you can understand the Chebyshev bound."

我们最初的一个示例明显显露了**蕴涵**更为奇怪的一面。

”如果猪会飞，那么你就能理解契比雪夫不等式。“

Don't take this as an insult; we just need to figure out whether this proposition is true or false. Curiously, the answer has nothing to do with whether or not you can understand the Chebyshev bound. Pigs cannot fly, so we're on either line(ft) or line(ff) of the truth table. In both cases, the proposition is true!

别把它当侮辱；我们只需要搞清楚这一命题是真还是假。奇怪的是，答案和你是否能够理解契比雪夫不等式没有一点关系。猪不会飞，所以该命题落在真值表的ft行或ff行。在两种情形下，该命题都为真！

In contrast, here's an example of a false implication:

"If the moon shines white, then the moon is made of white cheddar."

相反，下面的示例是一个值为假的蕴涵：

"如果月亮发白光，那么它是用白色切达干酪做成的。"

Yes, the moon shines white. But, no, the moon is not made of white cheddar cheese. So we're on line(tf) of the truth table, and the proposition is false.

是的，月亮发白光。但是，月亮却不是用白色切达干酪做成的。所以该命题落在真值表的tf行，它的值为假。

The truth table for implications can be summarized in words as follows:

An implication is true exactly when the if-part is false or the then-part is true.

可以用如下的话来总结蕴涵的真值表：

当某蕴涵”**如果部分**“的值为**假**或”**那么部分**“的值为**真**时，该蕴涵的值正好为**真**。

This sentence is worth remembering; a large fraction of all mathematical statements are of the if-then form!

这句话值得牢记; 一大部分数学公式有着**如果-那么**的格式！

2 Patterns of Proof

证明的方法

2.1 The Axiomatic Method

2.1 公理化方法

The standard procedure for establishing truth in mathematics was invented by Eu- clid, a mathematician working in Alexandria, Egypt around 300 BC. His idea was to begin with five assumptions about geometry, which seemed undeniable based on direct experience.For example, one of the assumptions was "There is a straight line segment between every pair of points." Propositions like these that are simply accepted true are called axioms.

数学家欧几里得，生活于公元前300年左右的埃及亚历山大港，他发明了在数学中证实命题正确性的标准步骤。他的思想发端于几何学上的五个假设，它们基于直接经验，似乎毋庸置疑。例如，其中一个假设是“两点之间只有一个直线段。”像这样不证自明的命题称为公理。

Starting from these axioms, Euclid established the truth of many additional propositions by providing "proofs".A proof is a sequence of logical deductions **from** axioms and previously-proved statements that concludes with the proposition in question.

从这些公理开始，通过提供“证明”，欧几里得证实了许多附加命题的正确性。证明是一系列逻辑推论——从公理和之前证实过的陈述开始，以所讨论的命题作为结束。

There are several common terms for a proposition that has been proved.The different terms hint at the role of the proposition within a larger body of work.

- Important propositions are called *theorems*.
- A *lemma* is a preliminary proposition useful for proving later propositions.
- A *corollary* is a proposition that follows in just a few logical steps from a lemma or a theorem.

可用多个常见术语来指代已证明的命题。不同的术语暗示着该命题在更大工作体系中的地位。

- 重要的命题称作**定理**。
- **引理**是个可用于证明后续命题的初级命题。
- 从某个引理或定理开始，仅推导几步，就可得到叫做**推论**的命题。

Euclid's axiom-and-proof approach, now called the axiomatic method, is the foundation for mathematics today. In fact, just a handful of axioms, collectively called Zermelo-Frankel Set Theory with Choice(ZFC), together with a few logical deduction rules, appear to be sufficient to derive essentially all of mathematics.

欧几里德的公理&证明方法，现在称为公理化方法，是现代数学的基石。事实上，只需几个公理——它们统称为包括选择公理的策梅洛-弗兰克尔集合论——加上一些逻辑推论规则，似乎就足以推导出大体上所有的数学理论。

2.1.1 Our Axioms

2.1.1 我们的公理

The ZFC axioms are important in studying and justifying the foundations of mathematics, but for practical purposes, they are much too primitive.Proving theorems in ZFC is a little like writing programs in byte code instead of a full-fledged programming language——by one reckoning, a formal proof in ZFC that 2 + 2 = 4 requires more than 20,000 steps! So instead of starting with ZFC, we're going to take a huge set of axioms as our foundation: we'll accept all familiar facts from high school math!

在研究及证明数学基础的合理性方面，ZFC公理价值很大，但对实际应用来讲，它们却太过简陋。用ZFC证明定理有点象用字节码而非完备的编程语言来编写程序——据估计，用规范的ZFC来证明2 + 2 = 4需要超过20,000 个步骤！所以与其从ZFC开始，倒不如把超大的公理集当作我们的基础：我们将把高中数学中的常见事实都默认为公理！

This will give us a quick launch, but you may find this imprecise specification of the axioms troubling at times.For example, in the midst of a proof, you may find yourself wondering, "Must I prove this little fact or can I take it as an axiom?"Feel free to ask for guidance, but really there is no absolute answer. Just be up front about what you're assuming, and don't try to evade homework and exam problems by declaring everything an axiom!

这让我们得以快速开始，但有时你也会认为公理的这种不精确的规范会让人苦恼。例如，你可能在证明时陷入疑惑，“我是否必须证明这一小处事实，还是可以把它当公理？”不要羞于寻求指导，但的确不存在绝对正确的答案。只需直面你的职责，不要把一切都声明为公理来逃避作业和考试中的问题！

2.1.2 Logical Deductions

2.1.2 逻辑推论

Logical deductions or **inference rules** are used to prove new propositions using previously proved ones.

使用之前证实的命题来证明一个新命题时，会用到**逻辑推论**或**推理规则**。

A fundamental inference rule is *modus ponens*. This rule says that a proof of P together with a proof that P *IMPLIES* Q is a proof of Q.

假言推理是一个基本的**推理规则**。该规则称，**P**为真且**P蕴涵Q**为真，即可证明**Q**为真。

Inference rules are sometimes written in a funny notation. For example, *modus ponens* is written:

推理规则有时用古怪的符号标记。例如，假言推理标记如下：

Rule 2.1.1

$$\frac{P, P \;\; IMPLIES \;\; Q}{Q}$$

规则2.1.1

$$\frac{P, P \;\; \text{蕴 涵} \;\; Q}{Q}$$

When the statements above the line, called the antecedents, are proved, then we can consider the statement below the line, called the conclusion or **consequent**, to also be proved.

直线上面的陈述叫做**前件**，下面的陈述叫做**结论**或**后件**。证明了前件，就可以认为也证明了后件。

A key requirement of an **inference rule** is that it must be *sound*: any assignment of truth values that makes all the antecedents true must also make the consequent true. So if we start off with true axioms and apply sound inference rules, everything we prove will also be true.

一定得合理是推理规则的必要条件：只要指定真值，使所有前件为真，必然使后件也为真。所以如果我们从正确的公理着手，应用合理的推理规则，那么我们所证明的一切也都将为真。

You can see why modus ponens is a sound inference rule by checking the truth table of P *IMPLIES* Q. There is only one case where P and P IMPLIES Q are both true, and in that case Q is also true.

通过核查"P蕴涵Q"的真值表，你就会明白为什么要说假言推理是合理的推理规则。只在一种情形下**P**与**P蕴涵Q**都为真，同时**Q**也为真。

P	Q	$P \to Q$
F	F	T
F	T	T
T	F	F
T	T	T

There are many other natural, sound inference rules, for example:

也有很多自然合理的推理规则，如：

Rule 2.1.2

$$\frac{P \text{ } IMPLIES \text{ } Q, \;\; Q \;\; IMPLIES \;\; R}{P \text{ } IMPLIES \text{ } R}$$

规则 2.1.2

$$\frac{P\text{ 蕴 涵 } Q, Q\text{ 蕴 涵 } R}{P\text{ 蕴 涵 } R}$$

Rule 2.1.3

$$\frac{P \text{ } IMPLIES \text{ } Q, \;\; NOT(Q)}{NOT(P)}$$

规则 2.1.3

$$\frac{P\text{ 蕴 涵 } Q, \text{非 } Q}{\text{非 } P}$$

Rule 2.1.4

$$\frac{NOT(P) \text{ } IMPLIES \text{ } NOT(Q)}{Q \;\; IMPLIES \;\; P}$$

规则 2.1.4

$$\frac{\text{非}\,P\ \textbf{蕴 涵}\ \text{非}\,Q}{Q\ \textbf{蕴 涵}\ P}$$

On the other hand,

Non-Rule.

$$\frac{NOT(P)\ IMPLIES\ NOT(Q)}{P\ \ IMPLIES\ \ Q}$$

is not sound: if P is assigned **T** and Q is assigned **F**, then the antecedent is true and the consequent is not.

另一方面，

非规则

$$\frac{\text{非}\,P\ \textbf{蕴 涵}\ \text{非}\,Q}{P\ \textbf{蕴 涵}\ Q}$$

并不合理：如果指定P为**真**，Q为**假**，那么前件为真，但后件却不为真。

Note that a propositional inference rule is sound precisely when the conjunction(AND) of all its antecedents implies its consequent.

只有当某个**命题推理规则**所有前件的合取(AND)蕴涵着它的后件，才可以说**该规则**真正合理。

As with axioms, we will not be too formal about the set of legal inference rules. Each step in a proof should be clear and "logical"; in particular, you should state what previously proved facts are used to derive each new conclusion.

就像对待公理那样，我们不要求太过规范的**合法推理规则集**。证明中的每一步都应清楚、“合乎逻辑”；你尤其应该声明使用了之前证实过的哪些事实来推导每个新结论。

2.1.3 Proof Templates

2.1.3 证明模板

In principle, a proof can be any sequence of logical deductions from axioms and previously proved statements that concludes with the proposition in question. This freedom in constructing a proof can seem overwhelming at first. How do you even start a proof?

从理论上讲，证明可以是任意的逻辑推论序列——从公理与之前证实过的陈述开始，以所讨论的命题作为结束。**证明构造**中的这种自由乍看似乎让人无所适从。究竟怎样开始一段证明?

Here’s the good news: many proofs follow one of a handful of standard templates. Each proof has it own details, of course, but these templates at least provide you with an outline to fill in. In the remainder of this chapter, we’ll go through several of these standard patterns, pointing out the basic idea and common pitfalls and giving some examples. Many of these templates fit together; one may give you a top-level outline while others help you at the next level of detail.

And we’ll show you other, more sophisticated proof techniques in Chapter 3.

好消息是，许多证明都遵循少数几个标准模板中的某一个。当然，每个证明都有它自己的细节，但这些模板至少为你提供了一个可以增补的框架。本章的剩余部分，我们将通读几个这类标准方法，指出基本思想、常见陷阱，并提供一些示例。许多这类模板可以组合使用；某个模板可能为你提供顶层框架，而其他模板会在另一个细节层次上帮到你。在第3章中，我们还将向你介绍其他更高超的证明技巧。

The recipes that follow are very specific at times, telling you exactly which words to write down on your piece of paper. You’re certainly free to say things your own way instead; we’re just giving you something you *could* say so that you’re never at a complete loss.

下面列出的方法有时非常具体，精确地告诉你在纸上写下哪个字。与之相反，你当然可不受限制地用自己的方式来完成证明；我们只是给你一些建议，以免你全然不知所措。

2.2 Proof by Cases

2.2 分情况证明

Breaking a complicated proof into cases and proving each case separately is a useful and common proof strategy. In fact, we have already implicitly used this strategy when we used truth tables to show that certain propositions were true or valid. For example, in section 1.1.5, we showed that an implication P IMPLIES Q is equivalent to its contrapositive NOT(Q) IMPLIES NOT(P) by considering all