1

The statement is **false**.

If $m \geq 3$, then for any $n \in \mathcal{N}$(since $n \geq 1$), we have $3m + 5n \geq 9 + 5 \geq 14 > 12$. Thus, if we can find such m and n, it must have $m < 3$, i.e., $m \in \{1, 2\}$. But

- $m = 1$, it does not exist a $n \in \mathcal{N}$ such that $5n = 9$;
- $m = 2$, it does not exist a $n \in \mathcal{N}$ such that $5n = 6$;

Hence, we can not find $m \in \mathcal{N}$ and $n \in \mathcal{N}$ satisfying $3m + 5n = 12$, which is saying $(\exists m \in \mathcal{N})(\exists n \in \mathcal{N})(3m + 5n = 12)$ is false.

2

The statement is **true**.

We assume the consecutive five numbers are $n-2$, $n-1$, n, $n+1$, $n+2$, where n is an integer. Sum those together, we get $(n-2)+(n-1)+(n)+(n+1)+(n+2) = 5n$, which can be divisible by 5. Hence the statement is true.

3

The statement is **true**.

We can make polynomial transformation on $n^2 + n + 1$ to $n(n+1) + 1$

- if n is odd, then $n + 1$ is even, then $n(n+1)$ is even, then $n(n+1) + 1$ is odd;
- if n is even, then $n(n+1)$ is even, then $n(n+1) + 1$ is odd

Thus, no matter n is even or odd, $n(n+1) + 1$ is odd, which is saying for any integer n the number $n^2 + n + 1$ is always odd.

4

According to remainder theorem, every natural number n can be written as $n = 4q + r$, where q and r are also natural numbers, comes with $0 \leq r < 4$. That is saying every natural number n can be written as

- $n = 4q + 0$, which is even;
- $n = 4q + 1$, which is odd;
- $n = 4q + 2$, which is even;
- $n = 4q + 3$, which is odd.

So, every odd natural number can be written as $n = 4q + 1$ or $n = 4q + 3$, which proves the statement.

5

According to remainder theorem, every natural number n can be written as $n = 3q + r$, where q and r are also natural numbers, comes with $0 \leq r < 3$. That is saying every natural number n can be written as

- when $n = 3q + 0$, n is divisible by 3;
- when $n = 3q + 1$, $n + 2$ is divisible by 3;
- when $n = 3q + 2$, $n + 4$ is divisible by 3.

Which proves the statement.

6

Remind what we proved in Question-5

> At least one of the integers n, $n + 2$, $n + 4$ is divisible by 3.

We assume we can find another prime triple $(p, p + 2, p + 4)$ except $(3, 5, 7)$. According to the statement we proved in Question-5, at least one of $(p, p + 2, p + 4)$ is divisible by 3. We know that the only prime number can be divisible by 3 is 3 itself. So 3 is one of $(p, p + 2, p + 4)$. But p is a prime number, so $p \geq 2$, so the only option 3 in $(p, p + 2, p + 4)$ is $p = 3$. In which case, $(p, p + 2, p + 4)$ is $(3, 5, 7)$, which is in contradiction to our assumption. Hence, the only prime triple is $(3, 5, 7)$, which proves the statement.

7

Let's denote:

$$S = 2^1 + 2^2 + 2^3 + \cdots + 2^n$$

Multiply both sides by 2:

$$2S = 2^2 + 2^3 + 2^5 + \cdots + 2^{n+1}$$

Substract S from 2S:

$$S = 2S - S = (2^2 + 2^3 + 2^5 + \cdots + 2^{n+1}) - (2^1 + 2^2 + 2^3 + \cdots + 2^n) = 2^{n+1} - 2$$

Which proves the statement.

8

By definition of limit, the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L means:

$$(\forall \varepsilon > 0)(\exists n \in \mathcal{N})(\forall m \geq n)[|a_m - L| < \varepsilon]$$

Let $\varepsilon > 0$ be given. Choose N large enough so that $|a_m - L| < \dfrac{\varepsilon}{M}$, where M is any fixed number and $M > 0$.

Then for $n \geq N$,

$$|Ma_m - ML| = M|a_m - L| < M\frac{\varepsilon}{M} = \varepsilon$$

By the definition of limit, this proves the statement.

9

Let $A_n = (0, \dfrac{1}{n})$.

Then, $A_n = \{x \mid 0 < x < \dfrac{1}{n}\}$, $A_{n+1} = \{x \mid 0 < x < \dfrac{1}{n+1}\}$. Because $\dfrac{1}{n} > \dfrac{1}{n+1}$, so every element in A_{n+1} also exists in A_n, by definition, A_{n+1} is a subset of A_n, i.e, $A_{n+1} \subset A_n$.

By the definition:

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid (\forall n)(x \in A_n)\}$$

Assume there is an element x in $\bigcap_{n=1}^{\infty} A_n$, we can always find an natural number n such that $nx > 1$, i.e, $x > \dfrac{1}{n}$, which is saying x is not in A_n, which is a contradiction to our assumption. Hence, there's no element in $\bigcap_{n=1}^{\infty} A_n$, i.e, $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

which proves all the stated properties.

10

Let $A_n = [0, \frac{1}{n})$.

Then, $A_n = \{x \mid 0 \leq x < \frac{1}{n}\}$, $A_{n+1} = \{x \mid 0 \leq x < \frac{1}{n+1}\}$. Because $\frac{1}{n} > \frac{1}{n+1}$, so every element in A_{n+1} also exists in A_n, by definition, A_{n+1} is a subset of A_n, i.e, $A_{n+1} \subset A_n$.

But:

$$A_n = [0, \frac{1}{n}) = \{0\} \cup (0, \frac{1}{n})$$

Let's denote $A'_n = (0, \frac{1}{n})$, then, $A_n = \{0\} \cup A'_n$.

Thus, $\bigcap_{n=1}^{\infty} A_n$ defined as

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid (\forall n)(x \in A_n)\}$$

can also be defined as:

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid (\forall n)(x \in \{0\} \vee x \in A'_n)\} = \{0\} \cup \bigcap_{n=1}^{\infty} A'_n$$

According to what we proved in Question-9, $\bigcap_{n=1}^{\infty} A'_n = \emptyset$, hence:

$$\bigcap_{n=1}^{\infty} A_n = \{0\} \cup \emptyset = \{0\}$$

which is saying $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number (0 here), which proves all the stated properties.