COMP582-HW1

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Problem - 1

1. f(n) = $\sqrt{n}$,
   $g(n) = 2\log n^2 = 2(2\log n) = 4\log n$
   We say f(n) is asymptotically equal to g(n) when $c_1 g(n) \leq f(n) \leq c_2 g(n)$ where c1,c2 are positive constants and n is sufficiently large.
   Lets take n = 2^{20} ,
   f(n)= $\sqrt{n} = 2^{20/2} = 2^{10}$
   g(n)= $4\log n = 4\log 2^{20} = 80$
   $2^{10} > 80$
   asymptotically f(n) is larger than g(n) for any large values of n .Hence , the given statement is **False**.

2. f(n)= $3n\log n + n$ = asymptotically equal to $n\log n$
   $g(n) = \frac{n^2-n}{2}$ = asymptotically equal to n^2
   And $n\log n$ is asymptotically smaller than n^2
   we can prove it by taking large values of n say 2^{16},
   f(n)= $3 * 2^{16}\log 2^{16} + 2^{16} \approx 2^{16}(16) = 2^{20}$
   g(n)= $\frac{2^{16}(2^{16}-1)}{2}$ =$2^{15}(2^{16}-1) \approx 2^{31}$
   $2^{31} > 2^{20}$
   g(n) is asymptotically larger than f(n) for any large values of n.Hence , the given statement is **True**.

3. Given statement is in the form , If p then q .
   lets take $f(n) = n$ and $g(n) = 2^n$, Since both n and 2^n are positive functions for any positive size of input n,thus satisfying the given conditions of f and g.
   Lets prove if statement p is true (i.e., $f(n) + g(n) = \Omega(f(n))$), for the above functions:
   In order for p to be true $n + 2^n$ should be greater than $c * n$, where $c > 0$.
   Being a exponential function the growth of 2^n is always faster than n for any high values of n.
   We can prove this by taking $n = 2^{20}$.
   $n + 2^n = 2^{20} + 2^{2^{20}} \approx 2^{\left(2^{20}\right)}$ which is greater than 2^{20}
   $n + 2^n = \Omega(n)$. Hence statement p is true,
   Now lets see if q is true or false for the same above functions.
   $g(n) = 2^n$, $f(n)^2 = n^2$
   for large values of n say $n = 2^{20}$,
   $2^n > n^2$ (i.e., $2^{\left(2^{20}\right)} > \left(2^{20}\right)^2$)
   g(n) is asymptotically larger than $f(n)^2$. Hence statement q is false.
   According to conditional logic if p then q , is false when p is true and q is false.
   Hence, the given statement is **False**.

Problem - 2

Base Case: Since $n \geq 1$, lets take the minimum value of n for the base case i.e., $n = 1$.
LHS :

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$$

RHS :

$$\frac{n}{(n+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$$

LHS=RHS . Hence , the statement holds for $n = 1$.
Inductive Hypothesis Case: Let us assume that the statement holds for some positive integer k, i.e., $n = k$.

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

Inductive Step: We need to prove that the statement holds for $n = k + 1$. i.e., we need to prove :

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$$

$$\begin{aligned}
LHS = \sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)((k+1)+1)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \text{(using the inductive hypothesis case)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

RHS:

$$\frac{k+1}{k+2}$$

LHS=RHS . Hence , the statement holds for $n = k + 1$.
By the principle of mathematical induction, the given statement holds for all the positive integers $n \geq 1$.

Problem - 3

1. $T(n) = 8T\left(\frac{n}{4}\right) + n$
   The above recurrence can be solved using masters theorem .
   a=8, b=4, f(n)=n.
   $n^{log_b a} = n^{log_4 8} = n^{\frac{log 8}{log 4}} = n^{3/2}$
   $f(n) < n^{log_b a}$
   $f(n) = O(n^{log_b a - \epsilon})$ (case-1 of masters theorem) . Hence, $T(n) = \theta(n^{3/2})$

2. $T(n) = 8T\left(\frac{n}{2}\right) + n^3$
   The above recurrence can be solved using masters theorem .
   a=8, b=2, f(n)=n^3.
   $n^{log_b a} = n^{log_2 8} = n^{\frac{log 8}{log 2}} = n^3$
   $f(n) = n^{log_b a}$
   $f(n) = \theta(n^{log_b a})$ (case-2 of masters theorem) . Hence, $T(n) = \theta(n^3 * logn)$

3. $T(n) = 2T\left(\frac{n}{3}\right) + \log n$
   $f(n) = \log n$, in order to apply masters theorem we have to have $f(n) > 0$ but $\log n$ is not always positive.
   So we cant use basic masters theorem , Lets use Substitution method.
   $T\left(\frac{n}{3}\right) = 2T\left(\frac{n}{3^2}\right) + \log\frac{n}{3}$
   $T\left(\frac{n}{3}\right) = 2T\left(\frac{n}{3^2}\right) + \log n - \log 3$
   $T(n) = 2 * \left(2T\left(\frac{n}{3^2}\right) + \log n - \log 3\right) + \log n$
   $T(n) = 4T\left(\frac{n}{3^2}\right) + 3\log n - 2\log 3$
   $T(n) = 4 * \left(2T\left(\frac{n}{3^3}\right) + \log\left(\frac{n}{3^2}\right)\right) + 3\log n - 2\log 3$
   $T(n) = 4 * \left(2T\left(\frac{n}{3^3}\right) + \log n - \log\left(3^2\right)\right) + 3\log n - 2\log 3$
   $T(n) = 8T\left(\frac{n}{3^3}\right) + 7\log n - 10\log 3$
   $\vdots$
   $T(n) = 2^k * T\left(\frac{n}{3^k}\right) + k_1\log n - k_2\log 3$
   Where k,k1,k2 are positive constants, since T(1)=1, lets say $n = 3^k$
   substituting $n = 3^k$ and by ignoring the constants,
   $T(n) = 2^k * T(1) + O(\log n)$
   We know that $k = \log_3 n$
   Multiply and divide by $\log 2$ to get 2^k
   $k = \frac{\log 2 * \log n}{\log 3 * \log 2}$
   $\log 2 * k = \frac{\log 2 * \log n}{\log 3}$
   $\log 2 * k = \log_3 2 * \log n$
   taking anti-log on both sides we get
   $2^k = n^{\log_3 2}$
   $T(n) = \theta(n^{\log_3 2}) + \theta(\log n)$
   Asymptotically $n^{\log_3 2}$ is greater than $\log n$ so $T(n) = \theta(n^{\log_3 2})$

4. $T(n) = 30T\left(\frac{n}{30}\right) + n$ The above recurrence can be solved using masters theorem .
   a=30, b=30, f(n)=n.
   $n^{log_b a} = n^{log_{30} 30} = n$
   $f(n) = \theta(n^{log_b a})$ (case-2 of masters theorem) . Hence, $T(n) = \theta(n * logn)$