Representing Relations.

Representing Relations Using Matrices

→ A relation between finite sets can be represented using a zero-one matrix.

R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$

to

$B = \{b_1, b_2, \ldots b_n\}$.

Relation R can be represented by matrix $M_R = [m_{ij}]$,

where $m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$

1) Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, $a_3 = 3$; $b_1 = 1$, $b_2 = 2$?

Solution: $R = \{(2,1), (3,1), (3,2)\}$

the matrix for R is $M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

The 1s in M_R show that the pairs $(2,1), (3,1)$ & $(3,2)$ belong to R.

0's show that no other pairs belong to R.

2) Let $A = \{a_1, a_2, a_3\}$ & $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by matrix.

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \text{ ?}$$

Sol: R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$

So, $R = \{(a_1, b_2), (a_2, b_1), (a_3, b_3), (a_2, b_4),$
$(a_3, b_1), (a_3, b_3), (a_3, b_5)\}$.

→ The matrix of a relation on a set, which is a square matrix can be used to determine whether the relation has certain properties.

→ A relation R on A is reflexive if $(a,a) \in R$. whenever $a \in A$.

$\therefore$ R is Reflexive if & only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots n$.

Here R is reflexive if only if $m_{ii} = 1$ for $i = 1, 2, \ldots n$.

R is reflexive if all the elements on main diagonal of M_R are equal to 1.

$$\begin{bmatrix} 1 & & & 0 \text{ or } 1 \\ & 1 & & \\ & & \ddots & \\ 0 \text{ or } 1 & & & 1 \end{bmatrix}$$

Zero-one matrix for reflexive Relation.

2

<u>Symmetric</u>

Relation R is Symmetric if $(a,b) \in R \rightarrow (b,a) \in R$.

the relation R on set $A = \{a_1, a_2 \dots a_n\}$ is Symmetric iff $(a_j, a_i) \in R$. whenever $(a_i, a_j) \in R$.

In terms of Matrix entries M_R.

R is Symmetric if & only if $m_{ji} = 1$. whenever $m_{ij} = 1$.

$m_{ji} = 0$ whenever $m_{ij} = 0$.

i.e., R is Symmetric iff $m_{ij} = m_{ji}$.

$i = 1, 2 \dots n$
$j = 1, 2, \dots n$.

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \cdot & 1 & \cdot & 1 \\ 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix} \end{array} \qquad M_R = (M_R)^t.$$

→ Anti Symmetric

Relation R is antisymmetric if & only ~~if (a,b)∈R~~ if $(a,b) \in R$ & $(b,a) \in R \rightarrow$ that $a = b$.

The matrix of an anti Symmetric relation:

if $m_{ij} = 1$. with $i \neq j$ then $m_{ji} = 0$.

either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \cdot & 1 & 0 \\ 0 & \cdot & \cdot \\ 0 & \cdot & \cdot \end{bmatrix} \end{array}$$

Ex: 3) Suppose that the Relation R on a set is represented by matrix $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Is R reflexive, Symmetric, antiSymmetric?

Sol: $\because$ diagonal elements are equal to 1, it is reflexive.

$$\therefore \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{array}$$

R is Symmetric $(0,1) = (1,0) = 1$

$(1,2) = (2,1) = 1$.

R is not Antisymmetric. $\boxed{\text{by def if } a_{ij} = 1, \text{ then } a_{ji} = 0,\ i \neq j.}$

$\because$ if $(0,1) = 1$, then $(1,0) \neq 0$. here in this matrix So, it is not antisymmetric.

→ Boolean operations join and meet. used to find matrices representing the union & intersection of two relations.

→ R_1 & R_2 relations on a set A. represented by matrices MR_1 and M_{R_2} resp.

<u>Union</u> – Matrix representing the union of these relations has a 1 in positions where either M_{R_1} or M_{R_2} has a 1.

Intersection - Matrix representing intersection of these relations has a 1 in positions where both M_{R_1} & M_{R_2} has 1.

$$\therefore \quad M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}.$$

Ex. 4) Suppose that relations R_1 & R_2 on a Set A are represented by the matrices.

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \& \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ & $R_1 \cap R_2$?

Sol:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Composite of relations

Suppose that R is a relation from A to B. ~~[illegible]~~

S is a relation from B to C.

Suppose that A, B and C. have m, n, p elements. resp.

Matrices for SoR, R & S be $M_{SoR} = [t_{ij}]$

$$M_R = [r_{ij}]$$

$$M_S = [s_{ij}].$$

Ordered pair $(a_i, c_j) \in SoR$. iff. there is an element b_k such that $(a_i, b_k) \in R$ & $(b_k, c_j) \in S$.

$t_{ij} = 1$ if & only if $r_{ik} = s_{kj} = 1$.

From definition of Boolean product.

i.e., $M_{SoR} = M_R \odot M_S$.

Ex 5 Find the matrix representing the relations $S \circ R$, where the matrices representing R & S are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \& \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Sol:

M_R ordered pairs are $(0,0)$, $(0,2)$, $(1,0)$ $(1,1)$.

M_S ordered pairs are $(0,1)$, $(1,2)$, $(2,0)$ $(2,2)$.

So $S \circ R = (0,1),\ (0,0),\ (0,2),\ (1,2),\ (1,2)$.

$$M_{S \circ R} = M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

→ The Matrix representing the composite of two relations can be used to find matrix for M_{R^n}.

$$M_{R^n} = M_R^{[n]}.$$

Ex 6) Find the matrix representing the relation R^2 where the matrix representing R is

$$M_R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}.$$

Sol: M_R ordered pairs $\overset{(0,1)}{\cancel{(1,0)}}$, $(1,1)$, $(1,2)$ $(2,0)$.

$(0,1)$ $(1,1)$ $(1,2)$ $(2,0)$.

$R^2 = \overset{(0,1)}{\cancel{(1,0)}}, (1,1), (1,2), (1,0), (0,2), (2,1)$

$$M_{R^2} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Ex: 6. Find the matrix representing the relation R^2. Where the matrix representing R is

$$M_R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Sol: ordered pairs for R $(0,1), (1,1)\ (1,2)\ (2,0)$

$R\quad (0,1)\ (1,1)\ (1,2)\ (2,0)$

$$R^2 = \{(0,1),\ (0,2),\ (1,1),\ (1,2),\ (1,0),\ (2,1)\}.$$

$$M_{R^2} = M_R^{[2]} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

⑤

Representing Relations Using Digraphs

-> Representing Relation using a pictorial representation.

-> Each element of a set represented by a point.

-> Each ordered pair is represented using an arc. with its direction indicated by an arrow.

-> Pictorial representations of relations on a finite set as directed graphs, or digraphs.

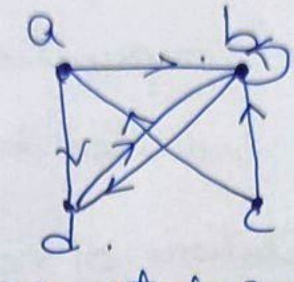
Def: A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs)

Vertex a is called the initial vertex of edge (a, b)

Vertex b is called the terminal vertex of this edge.

-> An edge of form (a, a) is represented using an arc from vertex a back to itself. It is called a loop.

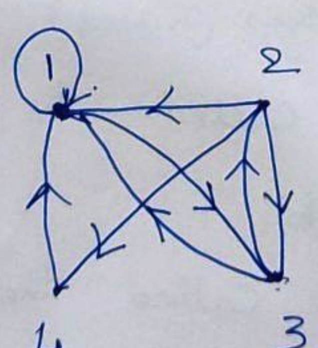
Ex: 7. The directed graph with vertices a, b, c, d & edges (a,b), (a, d), b, b) , (b, d) , (c,a), (c,b). & (d,b).



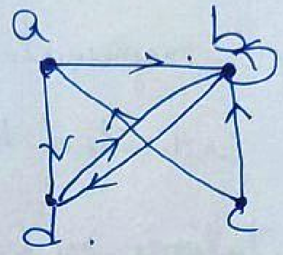
Directed graph.

Ex: 8 The directed graph of a relation.

$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$. on set $\{1,2,3,4\}$.



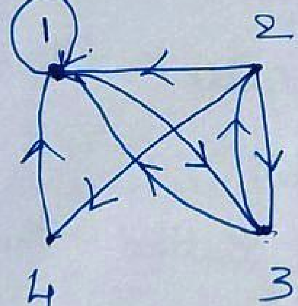
Ex: 7. The directed graph with vertices a, b, c, d & edges (a,b), (a,d), b,b), (b,d), (c,a), (c,b). & (d,b).



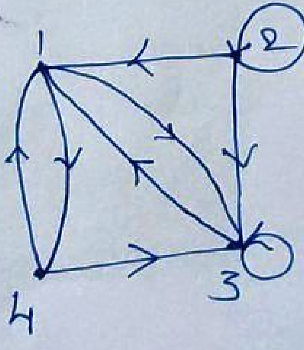
Directed graph.

Ex: 8 The directed graph of a relation.

$$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$$ on set $\{1,2,3,4\}$.



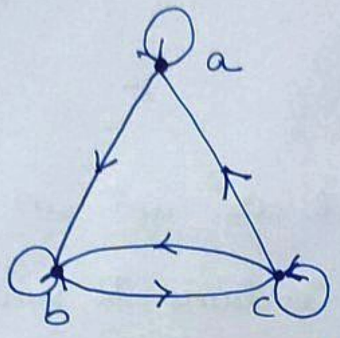
Ex: 89. What are the ordered pairs in the relation R represented by directed graph shown in Figure



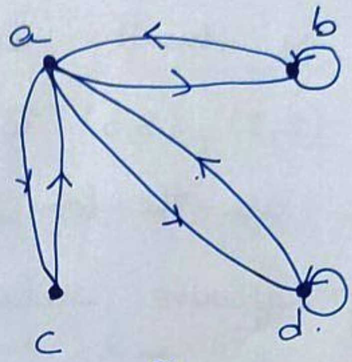
The ordered pairs (x, y) in the relation are

$$R = \{(1,3), (1,4), (2,1), (2,2)(2,3) \; (3,1) \; (3,3), (4,1), (4,3)\}.$$

Ex: 10. Determine whether the relations for the directed graphs shown in below figures are reflexive, Symmetric, antisymmetric, & / or transitive.



Directed graph of R.



Directed graph of S.

R is Reflexive

R is not Symmetric, not AntiSymmetric, not Transitive

S is not reflexive

S is Symmetric

S is not antisymmetric

S is not Transitive

Closures of Relations

→ Let R be a relation on a set A. R may or may not have some property P, such as reflexivity, Symmetry or transitivity

→ If there is a relation S with property P containing R such that S is a subset of every relation with property P containing R, then S is called the closure of R with respect to P.

Closures

The relation $R = \{(1,1), (1,2), (2,1), (3,2)\}$ on the ~~set~~ set $A = \{1, 2, 3\}$ is not reflexive. How can we produce a reflexive relation containing R that is as small as possible?

→ Add $(2,2)$, $(3,3)$ to R.

these are the form (a,a) which are not in R.

→ Any Reflexive relation that contains R must also contain $(2,2)$ & $(3,3)$.

→ Bcoz this relation contains R, is reflexive and is contained within every reflexive relation that contains R, it is called the reflexive closure of R.

reflexive closure of $R = R \cup \Delta$

$\Delta = \{(a,a) \mid a \in A\}$ is ~~dig~~ diagonal relation on A.

Ex: 1 What is the reflexive closure of relation $R = \{(a,b) \mid a < b\}$ on the set of integers?

Sol: The reflexive closure of R is

$$R \cup \Delta = \{(a,b) \mid a<b\} \cup \{(a,a) \mid a \in Z\} = \{(a,b) \mid a \le b\}$$

The Relation $\{(1,1),(1,2),(2,2),(2,3),(3,1),(3,2)\}$ on $\{1,2,3\}$ is not Symmetric. How can we produce a Symmetric relation that is as small as possible and contains R?

→ For this we should add $(2,1)$ & $(1,3)$

→ It is of form (b,a) with $(a,b) \in R$ which is not in R.

→ This new relation is Symmetric & Contains R.

→ So, Further any Symmetric relation that contains R must contain this new relation, b'coz this new relation containing $(2,1),(1,3)$ is called Symmetric closure of R.

→ $R \cup R^{-1}$ is symmetric closure of R, where $R^{-1} = \{(b,a) \mid (a,b) \in R\}$.

Ex:2 What is the Symmetric closure of the relation $R = \{(a,b) \mid a>b\}$ on set of positive integers?

Sol: The Symmetric closure of R is the relation

$$R \cup R^{-1} = \{(a,b) \mid a>b\} \cup \{(b,a) \mid a>b\} = \{(a,b) \mid a \neq b\}.$$

Transitive

Suppose a relation R is not transitive.

$R = \{(1,3), (1,4), (2,1), (3,2)\}$ on set $\{1,2,3,4\}$.

This relation is not transitive b'coz it doesn't contain all pairs of form (a,c) where (a,b) & (b,c) are in R

→ So, $(1,2), (2,3), (2,4), (3,1)$ are not in R.

→ By adding these pairs it doesnot produce a transitive relation, Because after adding these pairs we have $(3,1), (1,4)$ but $(3,4)$ is not there.

→ So, constructing the transitive closure of a relation is more complicated than reflexive or Symmetric closure.

→ Transitive closure of a relation can be found by adding new ordered pairs that must be present and then repeating this process until no new ordered pairs are needed.

Paths in Directed graphs

A path in a directed graph is obtained by traversing along edges.

Def: A path from a to b in directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where n is a nonnegative integer & $x_0 = a$ & $x_n = b$, i.e., a sequence of edges where the terminal vertex of an edge is same as the initial vertex in next edge in path.

→ This path is denoted by $x_0, x_1, x_2, \dots, x_{n-1}, x_n$ & has length n.

→ The empty set of edges as a path of length zero from a to a.

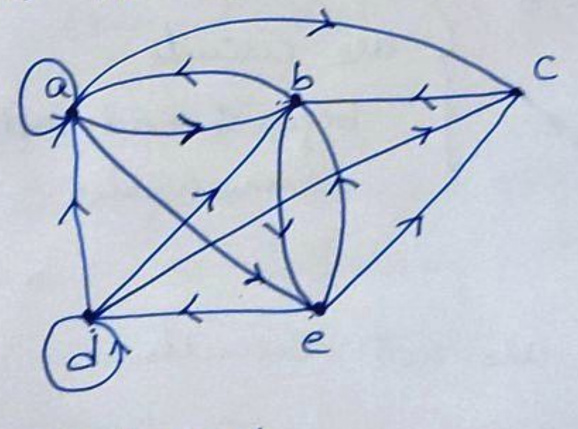
→ A path of length $n \geq 1$ that begins & ends at same vertex is called a circuit or cycle.

→ A path in a directed graph can pass through a vertex more than once.

→ An edge in a directed graph can occur more than once in a path.

Ex: 3. Which of the following are paths in directed graph shown in below figure a, b, e, d; a, e, c, d, b; b, a, c, b, a, a, b; d, c; c, b, a; e, b, a, b, a, b, e? What are the lengths of those that are paths? Which of the paths in this list are circuits?

Sol:



Sol: a, b, e, d.

(a, b), (b, e), (e, d) are edges from a to d.

So length of path is 3.

→ a, e, c, d, b.

here d → c is an edge but. c → d is not an edge.

So, a, e, c, d, b is not a path.

→ b, a, c, b, a, a, b;

(b, a), (a, c), (c, b), (b, a), (a, a), (a, b).

length. 5

→ (d, c) is an edge. for d, c so length 1.

→ c, b, a.

(c, b), (b, a)

length 2.

→ e, b, a, b, a, b, e.

(e, b), (b, a), (a, b), (b, a), (a, b), (b, e)

So length - 6.

→ b, a, c, b, a, a, b & e, b, a, b, a, b, e } are circuits. begin & end with Same vertex..

a, b, e, d. c, b, a d, c } are not circuits

Theorem 1: Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a,b) \in R^n$.

Proof: By Mathematical induction. we prove.

By definition, there is a path from a to b of length one if & only if $(a,b) \in R$.

So, true for $n=1$.

<u>Inductive hypothesis:</u>
Assume it is true for positive integer n.

There is a path of length $n+1$ from a to b if & only if there is a element $c \in A$ such that there is a path of length one from a to c,

So, $(a,c) \in R$.

path of length n from c to b i.e., $(c,b) \in R^n$.
there is a path of length $n+1$ from a to b if & only if $(a,b) \in R^{n+1}$.

<u>Transitive closures</u>

→ Now finding the transitive closure of a relation is equivalent to determining which pair of vertices in the associated directed graph are connected by a path.

Def 2: Let R be a relation on a set A. The Connectivity relation R^* consists of the pairs (a,b) such that there is a path of length at least one from a to b in R.

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

Ex:4. Let R be a relation on set of all people in the world that contains. (a, b) if a has met b. What is R^n, where n is a positive integers greater than one? what is R^*?

Sol:- R^2 contains (a, b) if there is a person c such that $(a, c) \in R$ & $(c, b) \in R$.

i.e., if there is a person c such that a has met c and c has met b.

Illy R^n consists of those pairs (a, b) such that there are people $x_1, x_2 \dots x_{n-1}$ such that a has met x_1, x_1 has met x_2 ... & x_{n-1} has met b.

R^* contains (a, b) if sequence of people starts with a & ends with b, such that each person in sequence has met next person.

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R^* contains (a, b) if sequence of people starts with a & ends with b, such that each person in sequence has met next person.

Ex 5 Let R be the relation on set of all ~~sub~~ states in US that contains (a, b) if state a and state b have a common border. What is R^n, where n is a +ve integer? What is R^*?

Sol: R^n consists of pairs (a, b) i.e., it is possible to go from state 'a' to state b by crossing exactly n state borders

R^* consists of ordered pairs (a, b)

R^* – not there are those containing states that are not connected to continental United states.

Theorem: The transitive closure of a relation R equals the Connectivity relation R^*.

Proof: R^* Contains R by Definition

To show R^* is transitive closure of R.

we need to show that R^* is transitive &

$R^* \subseteq S$ whenever S is a transitive relation that Contains R.

→ 1st To show R^* is transitive.

If $(a,b) \in R^*$ and $(b,c) \in R^*$, then there are paths from a to b & from b to c in R.

we obtain a path from a to c, Hence $(a,c) \in R^*$.

So, R^* is transitive.

→ Suppose S is a transitive relation containing R.

∵ S is transitive, S^n is also transitive and.

$S^n \subseteq S$.

$$S^* = \bigcup_{k=1}^{\infty} S^k.$$

$S^k \subseteq S$.

So, $S^* \subseteq S$

if $R \subseteq S$, then $R^* \subseteq S^*$, b'coz any path in R is also a path in S.

$R^* \subseteq S^* \subseteq S$., So, any transitive relation that Contains R must also Contain R^*. ∴ R^* is transitive closure of R.

LEMMA 1 Let A be a set with n elements, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. When $a \neq b$, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding $n-1$.

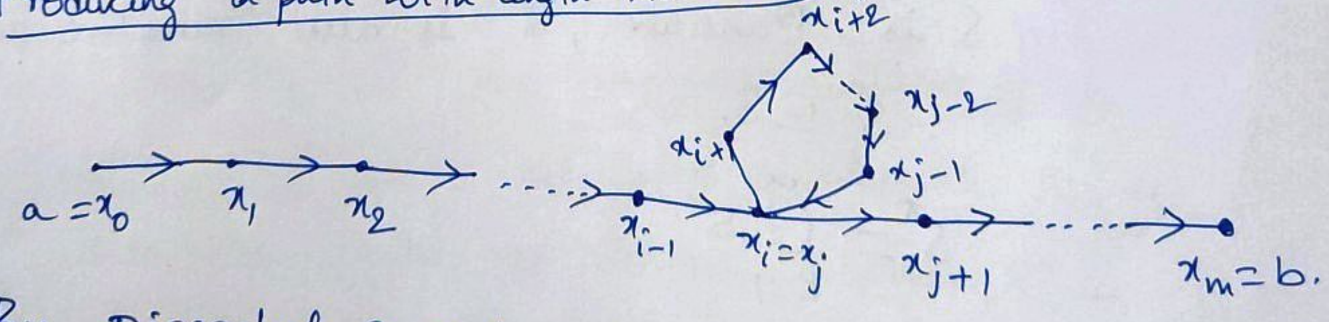
Proof: Suppose there is a path from a to b in R.

Let m be the length of the shortest such path.

Suppose $x_0, x_1, x_2 \ldots x_{m-1}, x_m$. where $x_0 = a$ & $x_m = b$ is such a path.

Suppose that $a = b$ & $m > n$, so that $m \geq n+1$.

Producing a path with length Not exceeding n.



By Pigeonhole principle, b'coz there are n vertices in A among the m vertices $x_0, x_1, \ldots x_{m-1}$, atleast two are equal.

Suppose $x_i = x_j$ with $0 \le i \le j \le m-1$.

Then the path Contains a circuit from x_i to itself.

This circuit can be deleted from path a to b.

of shorter length. $x_0, x_1, \ldots x_i, x_{j+1}, \ldots, x_{m-1}, x_m$.

Hence, the path of shortest length must have length less than or equal to n.

From lemma 1, we see that transitive closure of R is union of $R, R^2, R^3, \ldots,$ and R^n.

This follows because there is a path in R^* between two vertices if & only if there is a path between these vertices in R^i, for some +ve integer i with $i \le n$.

B, coz $R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$.

Theorem 3 Let M_R be ~~zero~~ zero-one matrix of the relation R on a set with n elements. Then the zero-one matrix of transitive closure R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]}.$$

Ex: 7 Find the zero-one matrix of transitive closure of relation R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Sol: By Theorem 3. zero-one matrix of R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}.$$

because $M_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $M_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$M_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Algorithm 1: Procedure for Computing the Transitive closure.

procedure transitive closure (M_R : zero-one $n \times n$ matrix).

$A := M_R$
$B := A$.
for $i := 2$ to n
  $A := A \odot M_R$
  $B := B \vee A$.
return B {B is the zero-one matrix for R^*}

Warshall's

To find no. of bit operations used by algorithm 1. to determine the transitive closure of a relation.

-> Computing the Boolean powers $M_R, M_R^{[2]}, \ldots, M_R^{[n]}$. requires that $n-1$ Boolean products of $n \times n$ zero-one matrices be found.

-> Each of these Boolean products can be found. using $n^2(2n-1)$ bit operations.

-> Hence, products can be computed using $n^2(2n-1)(n-1)$ bit operations.

-> To find M_{R^*} from n Boolean powers of M_R. $n-1$ joins of 0-1 (zero-one) matrices need to be found.

-> To compute each of these joins uses n^2 bit operations.

-> $(n-1)n^2$ bit operations used for computations

-> ∴ When Algorithm 1 is used, the matrix of transitive closure of a relation on a set with n elements can be found using

$$n^2(2n-1)(n-1) + (n-1)(n^2) = 2n^3(n-1).$$

i.e., $O(n^4)$ bit operations.

Warshall's Algorithm

-> Warshall's Algorithm, is described in 1960 by Stephen Warshall, is a efficient method for computing the transitive closure of a relation.

-> Algorithm1 uses $2n^3(n-1)$ bit operations where as Warshall's algorithm using only $2n^3$ bit operations ~~to~~ finds the transitive closure.

-> Warshall's Algorithm is sometimes called Roy-Warshall algorithm., b'coz in 1959, Benard Roy described it.

-> Suppose R is a relation on a set with n elements.

-> let $V_1, V_2, ----V_n$ be an arbitary listing of these n elements.

-> ~~con~~ Concept of interior vertices is used here.

if . $a, x_1, x_2, ----, x_{m-1}, b$ is a path, its interior vertices are $x_1, x_2, ---, x_{m-1}$.

-> interior vertices of a path a, c, d, f, g, h, b, j in a directed graph are c, d, f, g, h, b.

-> 1st vertex in the path is not an interior vetex unless it is visited again by the path, except as the last vertex.

→ The last vertex in path is not an interior vertex unless it was visited previously by the path, except as first vertex.

→ Warshall's algorithm is based on construction of a Sequence of zero-one matrices.

→ These matrices are $W_0, W_1, \ldots W_n$, where $W_0 = M_R$. is the zero-one matrix of this relation & $W_K = [w_{ij}^{(K)}]$. where $w_{ij}^{(K)} = '1'$. if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2 \ldots v_K\}$. & is '0' otherwise.

→ $W_n = M_{R^*}$ because $(i,j)^{th}$ entry of M_{R^*} is 1 iff there is a path from v_i to v_j with all interior vertices in set $\{v_1, v_2, \ldots v_n\}$.

→ The last vertex in path is not an interior vertex unless it was visited previously by the path, except as first vertex.

→ Warshall's algorithm is based on construction of a Sequence of zero-one matrices.

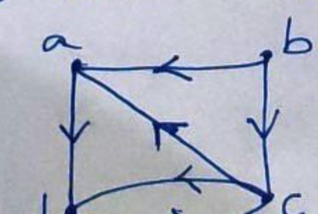
→ These matrices are $W_0, W_1, \ldots W_n$, where $W_0 = M_R$. is the zero-one matrix of this relation & $W_K = [w_{ij}^{(K)}]$. where $w_{ij}^{(K)} = '1'$. if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2 \ldots v_K\}$. & is '0' otherwise.

→ $W_n = M_{R^*}$ because $(i, j)^{th}$ entry of M_{R^*} is 1 iff there is a path from v_i to v_j with all interior vertices in set $\{v_1, v_2, \ldots v_n\}$.

Ex: 8

Let R be the relation with directed graph below. Let a, b, c, d be a listing of the elements of the set. Find the matrices W_0, W_1, W_2, W_3 & W_4. The matrix W_4 is transitive closure of R.

Sol:



Sol: Let $v_1 = a$, $v_2 = b$, $v_3 = c$, & $v_4 = d$. W_0 is matrix of relation.

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

W_1 has 1 as its $(i,j)^{th}$ entry if there is a path from v_i to v_j that has only $v_1 = a$ as an interior vertex.

path from b to d. i.e., b,a,d.

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

W_2 has 1 as its $(i,j)^{th}$ entry if there is a path. from v_i to v_j that has only $v_1 = a$ and/or $V_2 = b$. as its interior vertices, if any.

there are no edges with b as terminal vertex..
no new paths.
So $W_2 = W_1$.

W_3 has 1 as its $(i,j)^{th}$ entry if there is a path from v_i to v_j that has only $v_1 = a$, $v_2 = b$ &/or $v_3 = c$. as its interior vertices

So, we have d,c,a. & d,c,a,d.

$$\text{So, } W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Finally W_4 has 1 as its $(i,j)^{th}$ entry if there is a path from v_i to v_j that has $v_1=a, v_2=b, v_3=c$. &/or $v_4=d$. as interior vertices.

these are all vertices of graph this entry is 1 iff there is a path from v_i to v_j.

$$\text{So, } W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

~~This las~~ W_4 is the matrix of transitive closure.

Warshall algorithm computes M_{R^*} ~~by~~ efficiently by computing $W_0 = M_R, W_1, W_2 \ldots W_n = M_{R^*}$

So, we can compute W_k directly from W_{k-1}

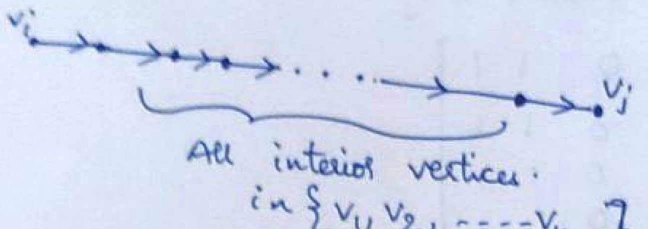
There is a path from v_i to v_j with no vertices other than $v_1, v_2 \ldots v_k$ as interior vertices iff either there is a path from v_i to v_j with its interior vertices among the first $k-1$ vertices in list or there are paths from v_i to v_k & from v_k to v_j that have interior vertices only among the 1^{st} $k-1$ vertices in list.

i.e., either a path from v_i to v_j already existed before v_k was permitted as an interior vertex, & allowing v_k as an interior vertex produces a path that goes from v_i to v_k & then from v_k to v_j.

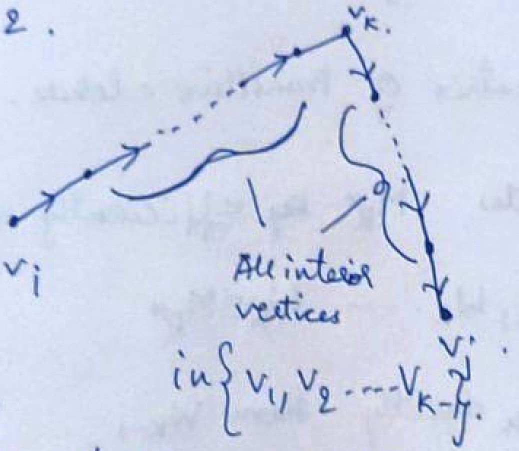
The first type of path exists iff $w_{ij}^{[k-1]}=1$ & the second type of path exists iff $w_{ik}^{[k-1]}$ & $w_{kj}^{[k-1]}$ are 1.

So, $w_{ij}^{[k]}=1$ iff either $w_{ij}^{[k-1]}$ is 1 or both $w_{ik}^{[k-1]}$ & $w_{kj}^{[k-1]}$ are '1'.

Case 1.



Case 2.



LEMMA 2.

~~Lemma 2~~ Let $W_k = [w_{ij}^{[k]}]$ be zero-one matrix that has a 1 in its $(i,j)^{th}$ position iff there is a path from v_i to v_j with interior vertices from set $\{v_1, v_2 ----v_k\}$ then.

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee \left(w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]}\right).$$

i, j, k are +ve integers not exceeding n.

~~Lemma~~ **Algorithm 2 Warshall Algorithm.**

```
procedure Warshall (M_R: nxn zero-one matrix).
  W: = M_R
  for k: = 1 to n.
      for i: = 1 to n
          for j: = 1 to n.
              w_ij := w_ij v (w_ik ^ w_kj).
return W {W = [w_ij] is M_R*}
```

Total no. of bit operations used is $n \cdot 2n^2 = 2n^3$.

To find. $w_{ij}^{[k]}$ using lemma 2 requires 2 bit operations.
To find all n^2 entries of W_k needs $2n^2$ bit operations.

Equivalence Relations

Def1: A relation on a set A is called an equivalence relation if it is reflexive, and transitive.

Def2: Two elements a and b that are related by an equivalence relation are called equivalent. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relations.

To make sense the notion of equivalent elements,

- Every element should be equivalent to itself, as the reflexive property guarantees for an equivalence relation.
- we can say a is related to b, by Symmetric property, b is related to a.
- B.coz an equivalence relation is transitive if a & b are equivalent & b & c are equivalent, so a & c are equivalent.

Ex:1 Let R be the relation on set of integers such that aRb if and only if $a=b$ or $a=-b$.

Ex:2 Let R be the relation on set of real numbers such that aRb if & only if $a-b$ is an integer. IS R an equivalence relation?

Sol: $\because$ $a-a=0$ is an integer for all real numbers, a,

aRa for all real numbers a. So, R is reflexive.

→ Suppose aRb, then $a-b$ is an integer & $b-a$ is also an integer.

$\therefore$ bRa, R is Symmetric.

→ If aRb & bRc then $a-b$ & $b-c$ are integers.

$\therefore$ $a-c=(a-b)+(b-c)$ is also an integer.

$\therefore$ aRc. So, R is transitive.

$\therefore$ R is an equivalence Relation.

Ex:3 Congruence Modulo m Let m be an integer with $m>1$. show that relation $R=\{(a,b) \mid a\equiv b \pmod{m}\}$ is an equivalence relation on set of integers.

→ $a\equiv b \pmod{m}$ iff m divides $(a-b)$.

$a-a=0$ is divisible by m, $\therefore$ $a\equiv a \pmod{m}$, is reflexive.

→ $a\equiv b \pmod{m}$ $a-b$ is divisible by m, so

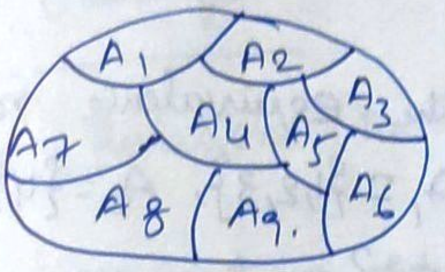
$a-b=Km$, where K is an integer.

$(b-a)=(-K)m$, so $b\equiv a \pmod{m}$. So it is Symmetric.

→ $a\equiv b \pmod{m}$ & $b\equiv c \pmod{m}$, Then m divides both $a-b$ & $b-c$. $a-b=Km$ & $b-c=lm$.

$a-c=(a-b)+(b-c)=Km+lm=(K+l)m$.

i.e., $a\equiv c \pmod{m}$. i.e. transitive.



Ex:12

Suppose that $S = \{1,2,3,4,5,6\}$. The collection of sets $A_1 = \{1,2,3\}$

$A_2 = \{4,5\}$

$A_3 = \{6\}$. forms a partition of S.

These sets are disjoint & their union is S.

Theorem 2:

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i / i \in I\}$ of the set S, there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes.

Examples:

List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1,2,3\}$, $A_2 = \{4,5\}$, $A_3 = \{6\}$ of $S = \{1,2,3,4,5,6\}$.

Sol: Subsets in partition are equivalence classes of R.

$(a,b) \in R$ iff a & b are in same subset of partition.

So., $(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)$ belong to R. $\because A_1 = \{1,2,3\}$ is an equivalence class.

Illy $(4,4), (4,5), (5,4), (5,5) \in R$ $\because A_2 = \{4,5\}$ is an equivalence class.

Illy $(6,6) \in R$ $\because A_3 = \{6\}$ is an equivalence class.

Partial Orderings

→ We use Relations to order some or all of the elements of Sets.

→ We will put in order words using relation containing pairs of words (x, y) where x comes before y in dictionary.

Def: A relation R on a set S is called a partial ordering or partial order if it is reflexive, antisymmetric & transitive.

- A set S together with a partial ordering R is called a partially ordered set or poset and is denoted by (S, R).
- Members of S are called elements of the poset.

Ex:1) Show that the "greater than or equal" relation $(\geq)$ is a partial ordering on the set of ~~strings~~ integers.

∴ $a \geq a$ for every integer a, $\geq$ is reflexive.

If $a \geq b$ & $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric.

Finally, $\geq$ is transitive b'coz $a \geq b$ & ~~b≥~~ $b \geq c$ imply that $a \geq c$. It follows that $\geq$ is a partial ordering.

on the Set of integers & $(\mathbb{Z}, \geq)$ is a poset.

Ex: 3 Show that the inclusion relation $\subseteq$ is a partial ordering on power set of a set S.

Sol:- $\because A \subseteq A$ when A is a Subset of S.

$\subseteq$ is reflexive.

if $A \subseteq B$ & $B \subseteq A \rightarrow A = B$., So it is antisymmetric

if $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$., So transitive

$\therefore \subseteq$ is a partial ordering on P(S) &

$(P(S), \subseteq)$ is a poset.

Ex: 4 Let R be the relation on the Set of people such that xRy if x & y are people and x is older than y. Show that R is not a partial ordering.

$\rightarrow$ if a person x is older than a person y, then y is not older than x.

i.e., if xRy, then $y \not R x$.

- if person x is older than person y and y is older than person z, then x is older than z.

i.e, xRy, yRz, then xRz., So it is ~~antisymmetric~~. Transitive.

- But it is not reflexive. $\because$ no person is older than himself or herself. i.e., $x \not R x$.

$\therefore$ R is not a partial ordering.

→ $a \preccurlyeq b$ is used to denote that $(a,b) \in R$. in an arbitary poset (S, R).

Def:2 The elements a and b of a poset $(S, \preccurlyeq)$ are called comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called incomparable.

Ex In poset $(Z^+, |)$ are integers 3 & 9 are comparable? Are 5 & 7 comparable?

→ The integers 3 & 9 are comparable ∵ $9/3$.

but 5 & 7 are incomparable ∵ $5 \nmid 7$ & $7 \nmid 5$.

→ Adjective 'Partial' is used to describe partial orderings because pairs of elements may be incomparable.

→ When every 2 elements in set are comparable, the relation is called a "Total Ordering".

Def: 3 If $(S, \preccurlyeq)$ is a poset and every 2 elements of S are comparable, S is called a 'totally ordered' or linearly ordered set. & $\preccurlyeq$ is called a 'total order' or a 'linear order'

→ A totally ordered set is also called a chain.

Ex:6 poset $(Z, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a & b are integers.

Ex:7 poset $(Z^+, |)$ is not totally ordered.

∵ it contains elements that are incomparable like 5 & 7.

Def:4 $(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Ex: Set of ordered pairs of positive integers $Z^+ \times Z^+$ with $(a_1, a_2) \preccurlyeq (b_1, b_2)$ if $a_1 < b_1$ or if $a_1 = b_1$ & $a_2 \leq b_2$ (lexicographic ordering) is a well ordered set.

Theorem 1

Principle of Well-ordered Induction

Suppose that S is a well ordered set. Then P(x) is true for all $x \in S$, if.

Inductive Step: For every $y \in S$, if P(x) is true for all $x \in S$ with $x \prec y$, then P(y) is true.

Proof by Contradiction.

we assume. P(x) is true for all $x \in S$ is not the case.

Then $y \in S$ and P(y) is false.

Consequently

Set $A = \{x \in S \mid P(x) \text{ is false}\}$ is non empty

$\because$ S is well ordered, A has a least element a.

So, P(x) is true for all $x \in S$ with $x \prec a$.

$\overset{\circ}{\Rightarrow}$ P(a) is true.

Shows that P(x) must be true for all $x \in S$.

Lexicographic order

→ Special case of an ordering of strings on a set. Constructed from a partial ordering on set.

How to construct, a partial ordering on the Cartesian product of two posets, $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$.

Lexicographic ordering $\preccurlyeq$ on $A_1 \times A_2$.

$$(a_1, a_2) \prec (b_1, b_2).$$

either if $a_1 \prec_1 b_1$ or if both $a_1 = b_1$ & $a_2 \prec_2 b_2$.

We obtain a partial ordering $\preccurlyeq$ by adding equality to ordering $\prec$ on $A_1 \times A_2$.

Ex: Determine whether $(3,5) \prec (4,8)$, ~~whether~~ whether $(3,8) \prec (4,5)$.

whether $(4,9) \prec (4,11)$ in poset $(Z \times Z, \preccurlyeq)$.

where $\preccurlyeq$ is the lexicographic ordering constructed from the usual $\leq$ relation on Z.

→ $\because 3 < 4 \rightarrow (3,5) \prec (4,8)$ & $(3,8) \prec (4,5)$.

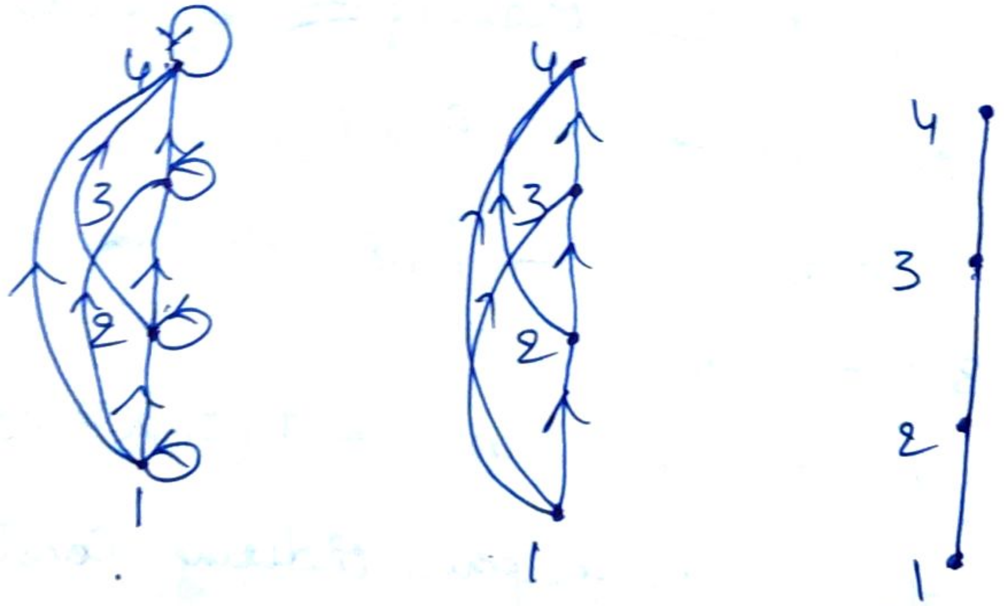
$(4,9) \prec (4,11)$. $\because 4 = 4$, $9 < 11$.

Hasse Diagrams

Consider the directed graph for partial orderings $\{(a,b) \mid a \leq b\}$ on set $\{1,2,$

∵ the partial ordering is reflexive & its directed graph has loops at all vertices we do not have to show these loops. b'coz they must be present.

∵ the partial ordering is transitive we do not have to show those edges that must be present for transitivity.



If we assume that all edges are pointed "upward", we do not have to show the directions of edges.

we represent a finite poset $(S, \preceq)$ using the procedure:

-> Start with directed graph for the relation.

-> $\because$ the partial ordering is reflexive, a loop (a,a) is present at every vertex a, Remove these loops.

-> Next remove all edges that must be in partial ordering b'coz of presence of other edges and transitivity.

-> Remove all edges (x,y) for which there is an element $z \in S$ such that $x \prec z$ & $z \prec x$.

-> Finally arrange each edge so that its initial vertex is below its terminal vertex.

-> Remove all arrows on the directed edges, $\because$ all edges point "upward" toward their terminal vertex.

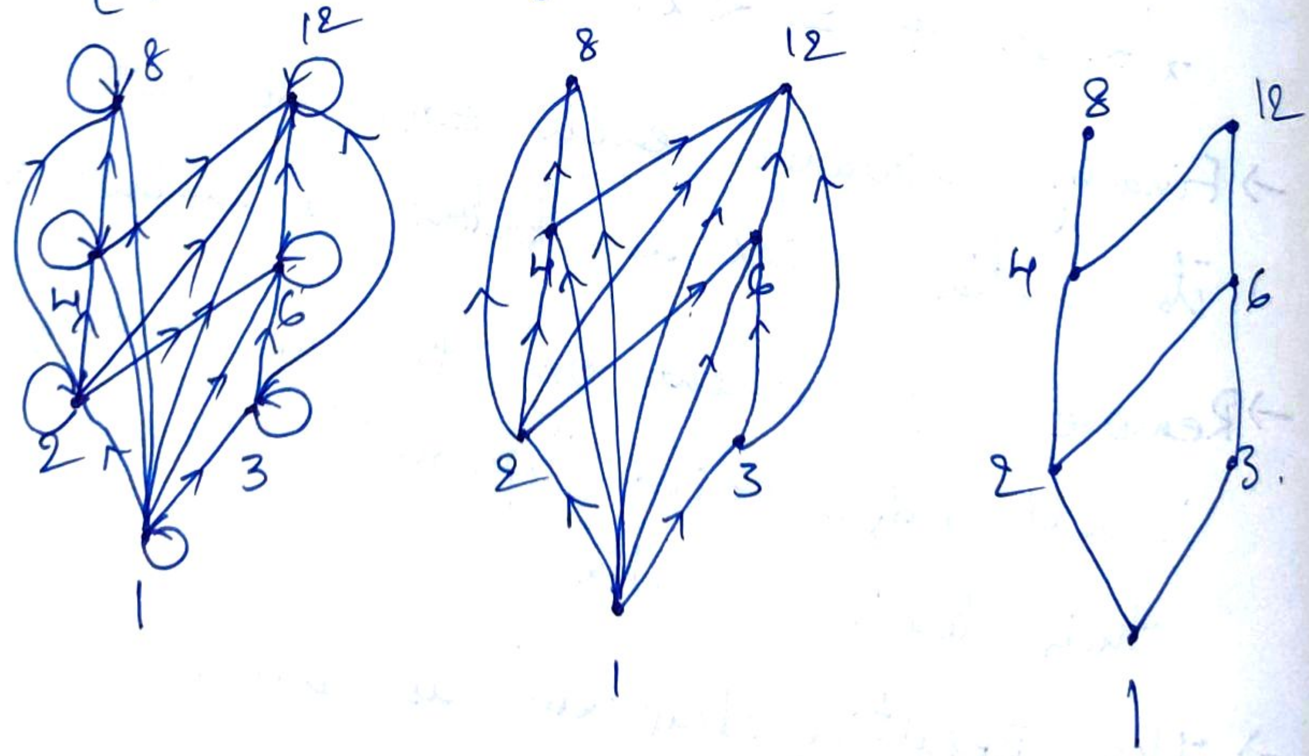
-> The Resulting diagram is called Hasse diagram of $(S, \preceq)$.

Let $(S, \preceq)$ be a poset.

→ $y \in S$ covers an element $x \in S$ if $x \prec y$ & there is no element $z \in S$ such that $x \prec z \prec y$.

→ Set of pairs (x, y) such that y covers x is called covering relation of $(S, \preceq)$.

<u>Ex:</u> Draw the Hasse Diagram representing the partial ordering $\{(a,b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.



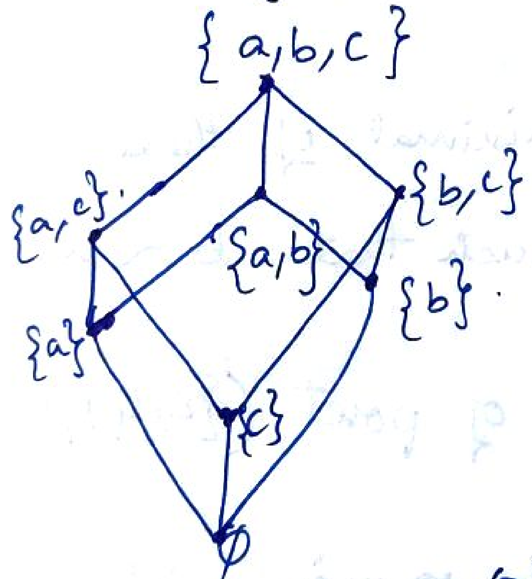
Ex Draw the Hasse diagram for partial ordering $\{(A, B) \mid A \subseteq B\}$ on power set $P(S)$ where $S = \{a, b, c\}$.

→ Delete $(\phi, \{a, b\}), (\phi, \{a, c\}),$

$(\phi, \{b, c\}), (\phi, \{a, b, c\}).$

$(\{a\}, \{a, b, c\}), (\{b\}, \{a, b, c\}).$

$(\{c\}, \{a, b, c\}).$



Hasse diagram of $(P\{a, b, c\}, \subseteq)$

Maximal and Minimal elements

→ An element of a poset is called maximal if it is not less than any element of poset.

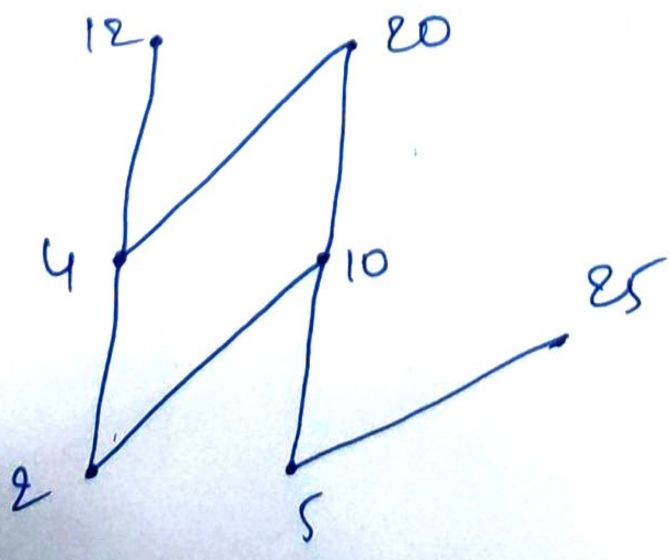
i.e., a is maximal in poset $(S, \preceq)$

if there is no $b \in S$ such that $a < b$.

→ An element of a poset is called minimal if it is not greater than any element of poset.

i.e., a is minimal if there is no element $b \in S$ such that $b < a$.

Ex:: Which elements of poset $(\{2,4,5,10,12,20,25\}, |)$ are maximal & minimal?



The minimal elements are 2 & 5.
maximal elements are 12, 20, 25.

a poset can have more than one maximal & minimal element.

→ There is an element in a poset that is greater than every other element. Called the greatest element.

i.e., a is greatest element of poset $(S, \preceq)$.
if $b \preceq a$ for all $b \in S$.

→ Greatest element is unique when it exists.

→ An Element is Called least element if it is less then all other elements in poset.

i.e., a is least element of $(S, \preceq)$ if $a \preceq b$ for all $b \in S$.

Least element is unique when its exists.

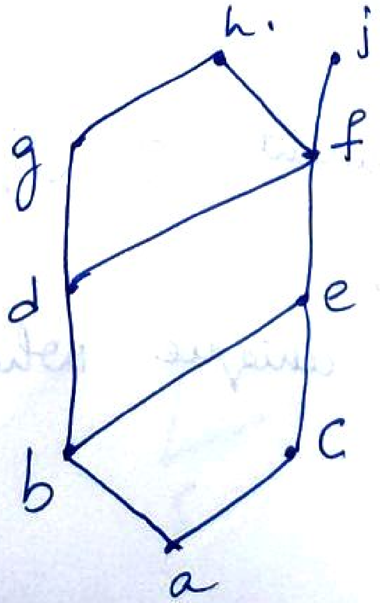
Least upper bound.

→ x is called least upper bound of subset A if x is an upper bound that is less than every other upper bound of A.

x is least upper bound of A if $a \preceq x$ whenever $a \in A$ & $x \preceq z$ whenever z is an upper bound of A. lub(A)

→ element y is called the greatest lower bound of A if y is a lower bound of A & $z \preceq y$ whenever z is a lower bound of A. glb(A)

Ex: Find the lower & upper bounds of subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with Hasse diagram below.



upper bounds of $\{a, b, c\}$ are e, f, j, h.

lower bound is a.

→ There is no upperbounds of $\{j, h\}$.

→ & lower bounds of $\{j, h\}$. a, b, c, d, e, f.

– upperbounds of $\{a, c, d, f\}$ are f, h, & j.

→ lower bound " is a.

Ex: Find the glb and lub of $\{b, d, g\}$ if they exist in above poset.

The upperbounds of $\{b, d, g\}$ are g & h.

$\because g < h$. ~~glb~~ lub – g. lub – g.

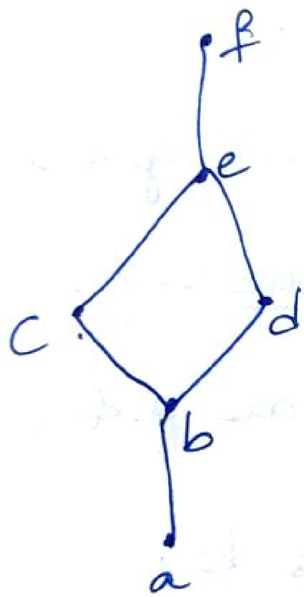
lower bounds of $\{b, d, g\}$ are a & b.

$\because a < b$. glb is b.

Lattices

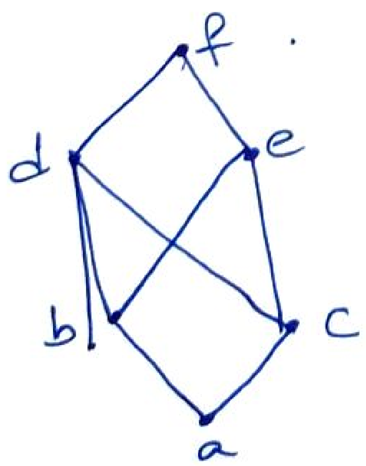
→ A partially ordered set in which every pair of elements has both a lub & a glb is called a lattice.

Ex:- Determine whether the posets represented by each of the Hasse diagram below are lattices.

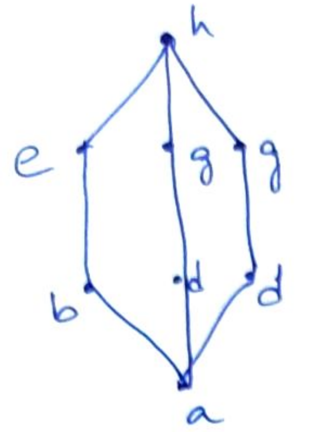


is lattice b'coz there is a lub & glb.

lub → ~~f~~e   glb → ~~a~~b.



is not a lattice there are not lub & glb.

 is a lattice.

Topological Sorting

A total ordering $\leq$ is said to be compatible with the partial ordering R if $a \leq b$. whenever $a R b$.

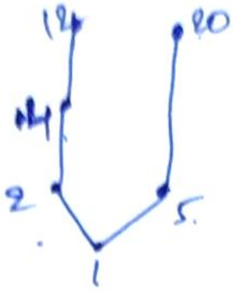
Constructing a compatible total ordering from a partial ordering is called "topological sorting"

LEMMA 1 Every finite nonempty poset $(S, \preceq)$ has at least one minimal element.

Ex: Find a compatible total ordering for poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

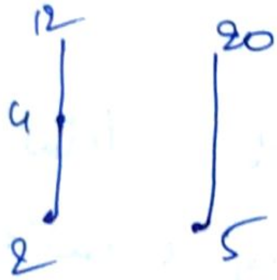
1 → choose a minimal element.

choose - 1



2 → choose a minimal element from $(\{2, 4, 5, 12, 20\}, |)$

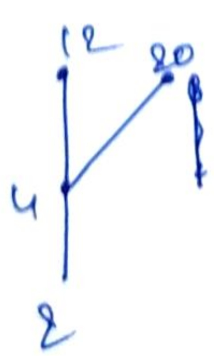
2 & 5 are minimal elements.



we select 5.

So remaining elements $(\{2, 4, 12, 20\}, |)$

So the minimal element is 2.



Next select minimal element from $\{4, 12, 20\}$.

4 is minimal element.

Next. 12 & 20 are minimal elements of $(\{12, 20\}, |$

1^{st} 20 is choosen. 12 20

12 is the last element. 12.

$$\boxed{1 < 5 < 2 < 4 < 20 < 12}$$