Advanced Counting Techniques

Chapter 8

Chapter Summary

- Applications of Recurrence Relations
- Solving Linear Recurrence Relations
  - Homogeneous Recurrence Relations
  - Nonhomogeneous Recurrence Relations
- Divide-and-Conquer Algorithms and Recurrence Relations
- Generating Functions
- Inclusion-Exclusion
- Applications of Inclusion-Exclusion

Applications of Recurrence Relations

Section 8.1

Recurrence Relations
(recalling definitions from Chapter 2)

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

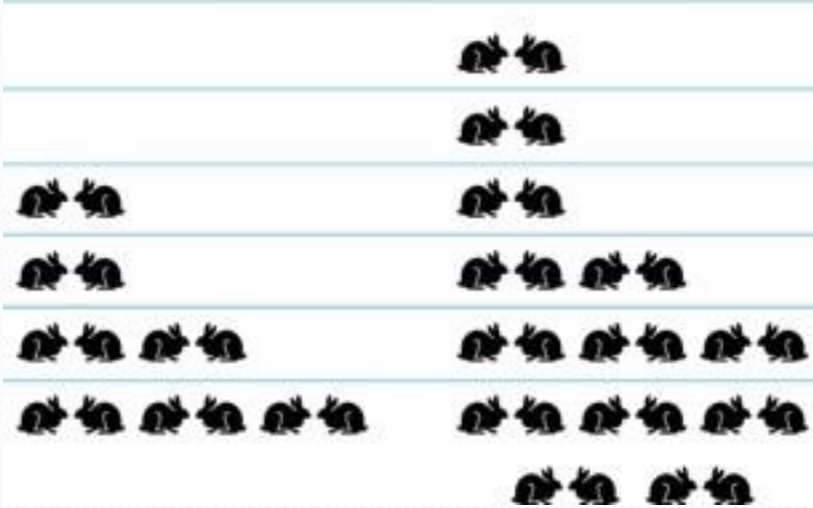
Rabbits and the Fiobonacci Numbers

Example: A young pair of rabbits (one of each gender) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month.

Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that rabbits never die.

This is the original problem considered by Leonardo Pisano (Fibonacci) in the thirteenth century.

Rabbits and the Fiobonacci Numbers (*cont.*)



Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

Modeling the Population Growth of Rabbits on an Island

Rabbits and the Fibonacci Numbers (*cont.*)

Solution: Let f_n be the the number of pairs of rabbits after n months.

- There are $f_1 = 1$ pairs of rabbits on the island at the end of the first month.
- We also have $f_2 = 1$ because the pair does not breed during the first month.
- To find the number of pairs on the island after n months, add the number on the island after the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least two months old.

Consequently the sequence $\{f_n\}$ satisfies the recurrence relation

$f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ with the initial conditions $f_1 = 1$ and $f_2 = 1$.
The number of pairs of rabbits on the island after n months is given by the nth Fibonacci number.

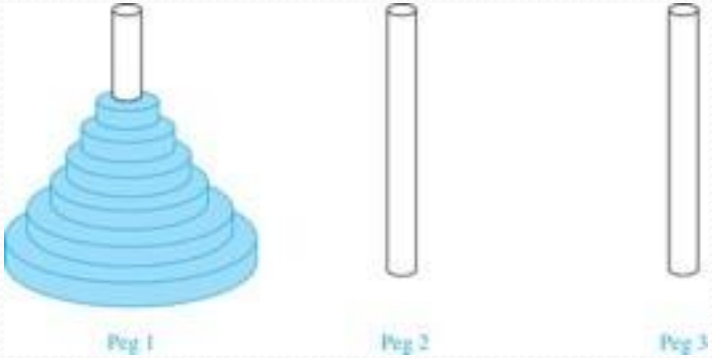
The Tower of Hanoi

In the late nineteenth century, the French mathematician Édouard Lucas invented a puzzle consisting of three pegs on a board with disks of different sizes. Initially all of the disks are on the first peg in order of size, with the largest on the bottom.

Rules: You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.

Goal: Using allowable moves, end up with all the disks on the second peg in order of size with largest on the bottom.

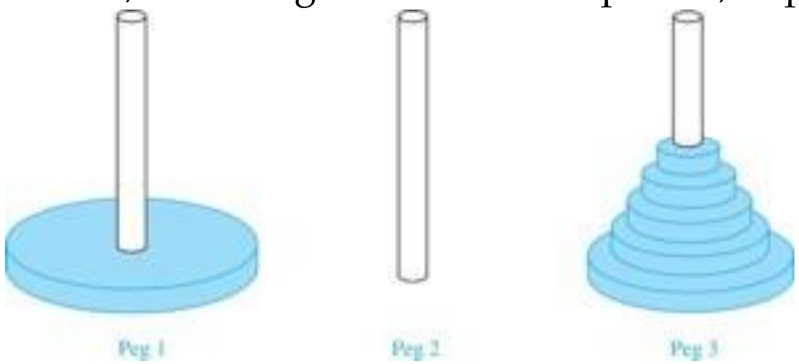
The Tower of Hanoi (*continued*)



The Initial Position in the Tower of Hanoi Puzzle

The Tower of Hanoi (*continued*)

Solution: Let $\{H_n\}$ denote the number of moves needed to solve the Tower of Hanoi Puzzle with n disks. Set up a recurrence relation for the sequence $\{H_n\}$. Begin with n disks on peg 1. We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves.



First, we use 1 move to transfer the largest disk to the second peg. Then we transfer the $n-1$ disks from peg 3 to peg 2 using H_{n-1} additional moves. This can not be done in fewer steps. Hence,

$$H_n = 2H_{n-1} + 1.$$

The initial condition is $H_1 = 1$ since a single disk can be transferred from peg 1 to peg 2 in one move.

The Tower of Hanoi (*continued*)

- We can use an iterative approach to solve this recurrence relation by repeatedly expressing H_n in terms of the previous terms of the sequence.

$H_n = 2H_{n-1} + 1$

$= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1$

$= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1$

$\vdots$

$= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1$

$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1$ because $H_1 = 1$

$= 2^n - 1$ using the formula for the sum of the terms of a geometric series

- There was a myth created with the puzzle. Monks in a tower in Hanoi are transferring 64 gold disks from one peg to another following the rules of the puzzle. They move one disk each day. When the puzzle is finished, the world will end.

Using this formula for the 64 gold disks of the myth,

$2^{64} - 1 = 18{,}446, 744{,}073, 709{,}551{,}615$

days are needed to solve the puzzle, which is more than 500 billion years.

- Reve's puzzle (proposed in 1907 by Henry Dudeney) is similar but has 4 pegs. There is a well-known unsettled conjecture for the the minimum number of moves needed to solve this puzzle. (*see Exercises* 38-45)

Counting Bit Strings

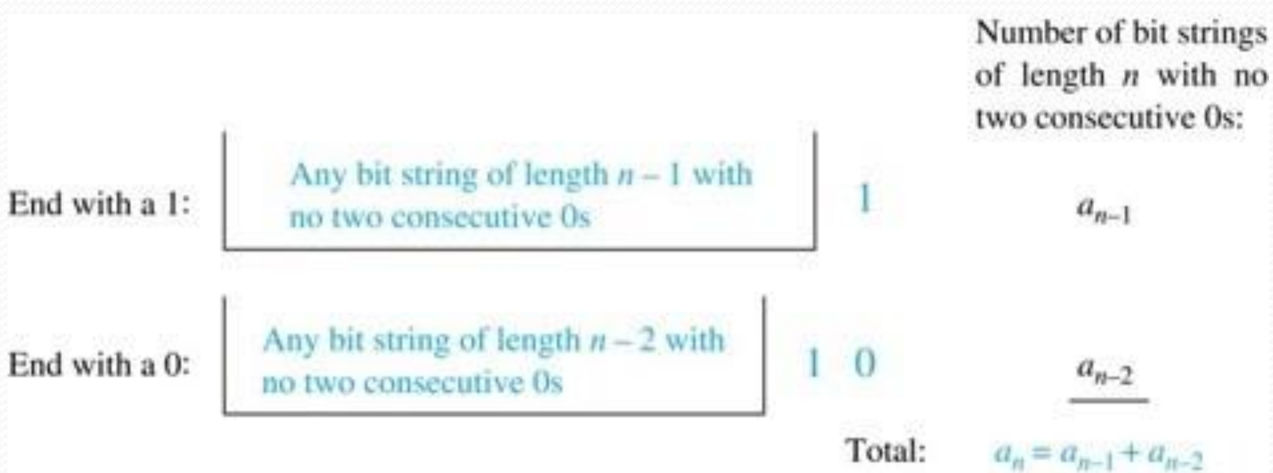
Example 3: Find a recurrence relation and give initial conditions for the number of bit strings of length n without two consecutive 0s. How many such bit strings are there of length five?

Solution: Let a_n denote the number of bit strings of length n without two consecutive 0s. To obtain a recurrence relation for $\{a_n\}$ note that the number of bit strings of length n that do not have two consecutive 0s is the number of bit strings ending with a 0 plus the number of such bit strings ending with a 1.

Now assume that $n \geq 3$.

- The bit strings of length n ending with 1 without two consecutive 0s are the bit strings of length n −1 with no two consecutive 0s with a 1 at the end. Hence, there are a_{n-1} such bit strings.
- The bit strings of length n ending with 0 without two consecutive 0s are the bit strings of length n −2 with no two consecutive 0s with 10 at the end. Hence, there are a_{n-2} such bit strings.

We conclude that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.



Bit Strings (*continued*)

The initial conditions are:

- $a_1 = 2$, since both the bit strings 0 and 1 do not have consecutive 0s.
- $a_2 = 3$, since the bit strings 01, 10, and 11 do not have consecutive 0s, while 00 does.

To obtain a_5, we use the recurrence relation three times to find that:

- $a_3 = a_2 + a_1 = 3 + 2 = 5$
- $a_4 = a_3 + a_2 = 5+ 3 = 8$ a_5
- $= a_4 + a_3 = 8+ 5 = 13$

Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Since $a_1 = f_3$ and $a_2 = f_4$, we conclude that $a_n = f_{n+2}$.

Counting the Ways to Parenthesize a Product

Example: Find a recurrence relation for C_n, the number of ways to parenthesize the product of $n + 1$ numbers, $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$, to specify the order of multiplication. For example, $C_3 = 5$, since all the possible ways to parenthesize 4 numbers are

$$((x_0 \cdot x_1) \cdot x_2) \cdot x_3, \quad (x_0 \cdot (x_1 \cdot x_2)) \cdot x_3, \quad (x_0 \cdot x_1) \cdot (x_2 \cdot x_3), \quad x_0 \cdot ((x_1 \cdot x_2) \cdot x_3), \quad x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$$

Solution: Note that however parentheses are inserted in $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$, one "$\cdot$" operator remains outside all parentheses. This final operator appears between two of the $n + 1$ numbers, say x_k and x_{k+1}. Since there are C_k ways to insert parentheses in the product $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_k$ and C_{n-k-1} ways to insert parentheses in the product $x_{k+1} \cdot x_{k+2} \cdot \cdots \cdot x_n$, we have

$$\begin{aligned} C_n &= C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-2}C_1 + C_{n-1}C_0 \\ &= \sum_{k=0}^{n-1} C_kC_{n-k-1} \end{aligned}$$

The initial conditions are $C_0 = 1$ and $C_1 = 1$.

The sequence $\{C_n\}$ is the sequence of **Catalan Numbers**. This recurrence relation can be solved using the method of generating functions; see Exercise 41 in Section 8.4.

Solving Linear Recurrence Relations

Section 8.2

Section Summary

- Linear Homogeneous Recurrence Relations
- Solving Linear Homogeneous Recurrence Relations with Constant Coefficients.
- Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.

Linear Homogeneous Recurrence Relations

Definition: A *linear homogeneous recurrence relation of degree* k *with constant coefficients* is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k}$, where $c_1, c_2, \ldots., c_k$ are real numbers, and $c_k \neq 0$

•it is *linear* because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n.
•it is *homogeneous* because no terms occur that are not multiples of the a_js. Each coefficient is a constant.
•the *degree* is k because a_n is expressed in terms of the previous k terms of the sequence.

By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions $a_0 = C_1$, $a_0 = C_1, \ldots , a_{k-1} = C_{k-1}$.

Examples of Linear Homogeneous Recurrence Relations

- $P_n = (1.11)P_{n-1}$ linear homogeneous recurrence relation of degree one
- $f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two
- $a_n = a_{n-1} + a_{n-2}^2$ not linear
- $H_n = 2H_{n-1} + 1$ not homogeneous
- $B_n = nB_{n-1}$ coefficients are not constants

Solving Linear Homogeneous Recurrence Relations

- The basic approach is to look for solutions of the form $a_n = r^n$, where r is a constant.
- Note that $a_n = r^n$ is a solution to the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$ if and only if

$$r^n = c_1r^{n-1} + c_2r^{n-2} + \cdots + c_k r^{n-k}.$$

- Algebraic manipulation yields the *characteristic equation*: $r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_{k-1}r - c_k = 0$
- The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution to the characteristic equation.
- The solutions to the characteristic equation are called the *characteristic roots* of the recurrence relation. The roots are used to give an explicit formula for all the solutions of the recurrence relation.

Solving Linear Homogeneous Recurrence Relations of Degree Two

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if

$$a_n = \alpha r_1^n + \alpha_2 r_2^n$$

for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

EXAMPLE 3 What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Extra Examples

Solution: Theorem 1 can be used to solve this problem. The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$. Its roots are $r = 2$ and $r = -1$. Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2(-1)^n,$$

for some constants α_1 and α_2. From the initial conditions, it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2,$$
$$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1).$$

Solving these two equations shows that $\alpha_1 = 3$ and $\alpha_2 = -1$. Hence, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$$a_n = 3 \cdot 2^n - (-1)^n.$$

An Explicit Formula for the Fibonacci Numbers

We can use Theorem 1 to find an explicit formula for the Fibonacci numbers. The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions: $f_0 = 0$ and $f_1 = 1$.

Solution: The roots of the characteristic equation $r^2 - r - 1 = 0$ are

$$r_1 = \frac{1+\sqrt{5}}{2}$$

$$r_2 = \frac{1-\sqrt{5}}{2}$$

Fibonacci Numbers (*continued*)

Therefore by Theorem 1

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants α_1 and α_2.

Using the initial conditions $f_0 = 0$ and $f_1 = 1$, we have

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

Solving, we obtain $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$.

Hence,

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

The Solution when there is a Repeated Root

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has one repeated root r_0. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha r_0^n + \alpha_2 n r_0^n$$

for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Using Theorem 2

Example: What is the solution to the recurrence relation
$a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The characteristic equation is $r^2 - 6r + 9 = 0$.
The only root is $r = 3$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 3^n + \alpha_2 n(3)^n$$

where α_1 and α_2 are constants.

To find the constants α_1 and α_2, note that

$a_0 = 1 = \alpha_1$ and $a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$.

Solving, we find that $\alpha_1 = 1$ and $\alpha_2 = 1$.
Hence,
$a_n = 3^n + n3^n$.

THEOREM 3 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

EXAMPLE 6 Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solution: The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6.$$

The characteristic roots are $r = 1$, $r = 2$, and $r = 3$, because $r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3)$. Hence, the solutions to this recurrence relation are of the form

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

To find the constants α_1, α_2, and α_3, use the initial conditions. This gives

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3,$$
$$a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3,$$
$$a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9.$$

When these three simultaneous equations are solved for α_1, α_2, and α_3, we find that $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$ ◀

The General Case with Repeated Roots Allowed

Theorem 4: Let c_1, c_2 ,…, c_k be real numbers. Suppose that the characteristic equation

$r^k - c_1 r^{k-1} - \dots - c_k = 0$

has t distinct roots r_1, r_2, …, r_t with multiplicities m_1, m_2, …, m_t, respectively so that $m_i \geq 1$ for $i = 0, 1, 2, \dots, t$ and $m_1 + m_2 + \dots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$

if and only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n$$
$$+ (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n$$
$$+ \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$$

for $n = 0, 1, 2, \dots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_{i-1}$.

EXAMPLE 7 Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, and 9 (that is, there are three roots, the root 2 with multiplicity three, the root 5 with multiplicity two, and the root 9 with multiplicity one). What is the form of the general solution?

Solution: By Theorem 4, the general form of the solution is

$$(\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)2^n + (\alpha_{2,0} + \alpha_{2,1}n)5^n + \alpha_{3,0}9^n.$$

We now illustrate the use of Theorem 4 to solve a linear homogeneous recurrence relation with constant coefficients when the characteristic equation has a root of multiplicity three.

EXAMPLE 8 Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

Solution: The characteristic equation of this recurrence relation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

Because $r^3 + 3r^2 + 3r + 1 = (r + 1)^3$, there is a single root $r = -1$ of multiplicity three of the characteristic equation. By Theorem 4 the solutions of this recurrence relation are of the form

$$a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1}n(-1)^n + \alpha_{1,2}n^2(-1)^n.$$

To find the constants $\alpha_{1,0}$, $\alpha_{1,1}$, and $\alpha_{1,2}$, use the initial conditions. This gives

$$\begin{aligned} a_0 &= 1 = \alpha_{1,0}, \\ a_1 &= -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2}, \\ a_2 &= -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}. \end{aligned}$$

The simultaneous solution of these three equations is $\alpha_{1,0} = 1$, $\alpha_{1,1} = 3$, and $\alpha_{1,2} = -2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Definition: A *linear nonhomogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k} + F(n) \, ,$$

where $c_1, c_2, \ldots., c_k$ are real numbers, and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k} \, ,$$

is called the associated homogeneous recurrence relation.

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*cont.*)

The following are linear nonhomogeneous recurrence relations with constant coefficients:

$a_n = a_{n-1} + 2^n$,

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$,

$a_n = 3a_{n-1} + n3^n$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

where the following are the associated linear homogeneous recurrence relations, respectively:

$a_n = a_{n-1}$,

$a_n = a_{n-1} + a_{n-2}$, $a_n = 3a_{n-1}$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Theorem 5: If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots\cdots + c_k a_{n-k} + F(n)\ ,$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where

$\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots\cdots + c_k a_{n-k}\ .$$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*continued*)

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.
What is the solution with $a_1 = 3$?

Solution: The associated linear homogeneous equation is $a_n = 3a_{n-1}$.

Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.

Because $F(n) = 2n$ is a polynomial in n of degree one, to find a particular solution we might try a linear function in n, say $p_n = cn + d$, where c and d are constants. Suppose that $p_n = cn + d$ is such a solution.

Then $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d) + 2n$.

Simplifying yields $(2 + 2c)n + (2d - 3c) = 0$. It follows that $cn + d$ is a solution if and only if $2 + 2c = 0$ and $2d - 3c =$
Therefore, $cn + d$ is a solution if and only if $c = -1$ and $d = -3/2$. Consequently, $a_n^{(p)} = -n - 3/2$ is a particular solution.

By Theorem 5, all solutions are of the form $a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha 3^n$, where α is a constant.

To find the solution with $a_1 = 3$, let $n = 1$ in the above formula for the general solution. Then $3 = -1 - 3/2 + 3\alpha$, and $\alpha = 11/$
Hence, the solution is $a_n = -n - 3/2 + (11/6)3^n$.

EXAMPLE 11 Find all solutions of the recurrence relation

Extra Examples

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

Solution: This is a linear nonhomogeneous recurrence relation. The solutions of its associated homogeneous recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

are $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$, where α_1 and α_2 are constants. Because $F(n) = 7^n$, a reasonable trial solution is $a_n^{(p)} = C \cdot 7^n$, where C is a constant. Substituting the terms of this sequence into the recurrence relation implies that $C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n$. Factoring out 7^{n-2}, this equation becomes $49C = 35C - 6C + 49$, which implies that $20C = 49$, or that $C = 49/20$. Hence, $a_n^{(p)} = (49/20)7^n$ is a particular solution. By Theorem 5, all solutions are of the form

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n.$$

◀

In Examples 10 and 11, we made an educated guess that there are solutions of a particular form. In both cases we were able to find particular solutions. This was not an accident. Whenever $F(n)$ is the product of a polynomial in n and the nth power of a constant, we know exactly what form a particular solution has, as stated in Theorem 6. We leave the proof of Theorem 6 as Exercise 52.

THEOREM 6 Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_tn^t + b_{t-1}n^{t-1} + \cdots + b_1n + b_0)s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_tn^t + p_{t-1}n^{t-1} + \cdots + p_1n + p_0)s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m(p_tn^t + p_{t-1}n^{t-1} + \cdots + p_1n + p_0)s^n.$$

EXAMPLE 12 What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2 + 1)3^n$?

Solution: The associated linear homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$. Its characteristic equation, $r^2 - 6r + 9 = (r - 3)^2 = 0$, has a single root, 3, of multiplicity two. To apply Theorem 6, with $F(n)$ of the form $P(n)s^n$, where $P(n)$ is a polynomial and s is a constant, we need to ask whether s is a root of this characteristic equation.

Because $s = 3$ is a root with multiplicity $m = 2$ but $s = 2$ is not a root, Theorem 6 tells us that a particular solution has the form $p_0 n^2 3^n$ if $F(n) = 3^n$, the form $n^2(p_1 n + p_0)3^n$ if $F(n) = n3^n$, the form $(p_2 n^2 + p_1 n + p_0)2^n$ if $F(n) = n^2 2^n$, and the form $n^2(p_2 n^2 + p_1 n + p_0)3^n$ if $F(n) = (n^2 + 1)3^n$. ◀

Care must be taken when $s = 1$ when solving recurrence relations of the type covered by Theorem 6. In particular, to apply this theorem with $F(n) = b_t n_t + b_{t-1} n_{t-1} + \cdots + b_1 n + b_0$, the parameter s takes the value $s = 1$ (even though the term 1^n does not explicitly appear). By the theorem, the form of the solution then depends on whether 1 is a root of the characteristic equation of the associated linear homogeneous recurrence relation. This is illustrated in Example 13, which shows how Theorem 6 can be used to find a formula for the sum of the first n positive integers.

EXAMPLE 13 Let a_n be the sum of the first n positive integers, so that

$$a_n = \sum_{k=1}^{n} k.$$

Note that a_n satisfies the linear nonhomogeneous recurrence relation

$$a_n = a_{n-1} + n.$$

(To obtain a_n, the sum of the first n positive integers, from a_{n-1}, the sum of the first $n-1$ positive integers, we add n.) Note that the initial condition is $a_1 = 1$.

The associated linear homogeneous recurrence relation for a_n is

$$a_n = a_{n-1}.$$

The solutions of this homogeneous recurrence relation are given by $a_n^{(h)} = c(1)^n = c$, where c is a constant. To find all solutions of $a_n = a_{n-1} + n$, we need find only a single particular solution. By Theorem 6, because $F(n) = n = n \cdot (1)^n$ and $s = 1$ is a root of degree one of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form $n(p_1 n + p_0) = p_1 n^2 + p_0 n$.

Inserting this into the recurrence relation gives $p_1 n^2 + p_0 n = p_1(n-1)^2 + p_0(n-1) + n$. Simplifying, we see that $n(2p_1 - 1) + (p_0 - p_1) = 0$, which means that $2p_1 - 1 = 0$ and $p_0 - p_1 = 0$, so $p_0 = p_1 = 1/2$. Hence,

$$a_n^{(p)} = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$$

Divide-and-Conquer Algorithms and Recurrence Relations

Section Summary

- Divide-and-Conquer Algorithms and Recurrence Relations
- Examples
  - Binary Search
  - Merge Sort
  - Fast Multiplication of Integers
- Master Theorem

Divide-and-Conquer Algorithmic Paradigm

Definition: A *divide-and-conquer algorithm* works by first *dividing* a problem into one or more instances of the same problem of smaller size and then *conquering* the problem using the solutions of the smaller problems to find a solution of the original problem.

Examples:

- Binary search: It works by comparing the element to be located to the middle element. The original list is then split into two lists and the search continues recursively in the appropriate sublist.

Merge sort: A list is split into two approximately equal sized sublists, each

- recursively sorted by merge sort. Sorting is done by successively merging pairs of lists.

Divide-and-Conquer Recurrence Relations

- Suppose that a recursive algorithm divides a problem of size n into a subproblems.
- Assume each subproblem is of size n/b.
- Suppose $g(n)$ extra operations are needed in the conquer step.
- Then $f(n)$ represents the number of operations to solve a problem of size n satisisfies the following recurrence relation:

$$f(n) = af(n/b) + g(n)$$

- This is called a *divide-and-conquer recurrence relation*.

Example: Binary Search

- Binary search reduces the search for an element in a sequence of size n to the search in a sequence of size $n/2$. Two comparisons are needed to implement this reduction;
  - one to decide whether to search the upper or lower half of the sequence and
  - the other to determine if the sequence has elements.
- Hence, if $f(n)$ is the number of comparisons required to search for an element in a sequence of size n, then

$$f(n) = f(n/2) + 2$$

when n is even.

Example: Merge Sort

- The merge sort algorithm splits a list of n (assuming n is even) items to be sorted into two lists with $n/2$ items. It uses fewer than n comparisons to merge the two sorted lists.
- Hence, the number of comparisons required to sort a sequence of size n, is no more than than $M(n)$ where

$$M(n) = 2M(n/2) + n.$$

Example: Fast Multiplication of Integers

An algorithm for the fast multiplication of two $2n$-bit integers (assuming n is even) first splits each of the $2n$-bit integers into two blocks, each of n bits.

- Suppose that a and b are integers with binary expansions of length $2n$. Let $a = (a_{2n-1}a_{2n-2} \ldots a_1a_0)_2$ and $b = (b_{2n-1}b_{2n-2} \ldots b_1b_0)_2$.
- Let $a = 2^nA_1 + A_0$, $b = 2^nB_1 + B_0$, where
- $A_1 = (a_{2n-1} \ldots a_{n+1}a_n)_2$, $A_0 = (a_{n-1} \ldots a_1a_0)_2$, $B_1 = (b_{2n-1} \ldots b_{n+1}b_n)_2$, $B_0 = (b_{n-1} \ldots b_1b_0)_2$.
- The algorithm is based on the fact that ab can be rewritten as:

$$ab = (2^{2n} + 2^n)A_1B_1 + 2^n(A_1 - A_0)(B_0 - B_1) + (2^n + 1)A_0B_0.$$

- This identity shows that the multiplication of two $2n$-bit integers can be carried out using three multiplications of n-bit integers, together with additions, subtractions, and shifts.

$$f(2n) = 3f(n) + Cn$$

Hence, if $f(n)$ is the total number of operations needed to multiply two n-bit integers, then

where Cn represents the total number of bit operations; the additions, subtractions and shifts that are a constant multiple of n-bit operations.

Estimating the Size of Divide-and-Conquer Functions

Theorem 1: Let f be an increasing function that satisfies the recurrence relation

$f(n) = af(n/b) + cn^d$

whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and c is a positive real number. Then

$$f(n) \text{ is } \begin{cases} O(n^{\log_b a}) & \text{if } \quad a > 1 \\ O(\log n) & \text{if } \quad a = 1. \end{cases}$$

Furthermore, when $n = b^k$ and $a \neq 1$, where k is a positive integer,

$$f(n) = C_1 n^{\log_b a} + C_2$$

where $C_1 = f(1) + c/(a-1)$ and $C_2 = -c/(a-1)$.

EXAMPLE 6 Let $f(n) = 5f(n/2) + 3$ and $f(1) = 7$. Find $f(2^k)$, where k is a positive integer. Also, estimate $f(n)$ if f is an increasing function.

Extra Examples

Solution: From the proof of Theorem 1, with $a = 5$, $b = 2$, and $c = 3$, we see that if $n = 2^k$, then

$$\begin{aligned} f(n) &= a^k[f(1) + c/(a-1)] + [-c/(a-1)] \\ &= 5^k[7 + (3/4)] - 3/4 \\ &= 5^k(31/4) - 3/4. \end{aligned}$$

Also, if $f(n)$ is increasing, Theorem 1 shows that $f(n)$ is $O(n^{\log_b a}) = O(n^{\log 5})$. ◀

We can use Theorem 1 to estimate the computational complexity of the binary search algorithm and the algorithm given in Example 2 for locating the minimum and maximum of a sequence.

Complexity of Binary Search

Binary Search Example: Give a big-O estimate for the number of comparisons used by a binary search.

Solution: Since the number of comparisons used by binary search is $f(n) = f(n/2) + 2$ where n is even, by Theorem 1, it follows that $f(n)$ is $O(\log n)$.

Estimating the Size of Divide-and-conquer Functions (*continued*)

Theorem 2. Master Theorem: Let f be an increasing function that satisfies the recurrence relation

$f(n) = af(n/b) + cn^d$

whenever $n = b^k$, where k is a positive integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d, \\ O(n^d \log n) & \text{if } a = b^d, \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

Complexity of Merge Sort

Merge Sort Example: Give a big-O estimate for the number of comparisons used by merge sort.

Solution: Since the number of comparisons used by mergesort to sort a list of n elements is less than

$M(n)$ where $M(n) = 2M(n/2) + n$, by the master theorem $M(n)$ is $O(n \log n)$.

Complexity of Fast Integer Multiplication Algorithm

Integer Multiplication Example: Give a big-O estimate for the number of bit operations used needed to multiply two n-bit integers using the fast multiplication algorithm.

Solution: We have shown that $f(n) = 3f(n/2) + Cn$, when n is even, where $f(n)$ is the number of bit operations needed to multiply two n-bit integers. Hence by the master theorem with $a = 3$, $b = 2$, $c = C$, and $d = 0$ (so that we have the case where $a > b^d$), it follows that $f(n)$ is $O(n^{\log 3})$.

Note that $\log 3 \approx 1.6$. Therefore the fast multiplication algorithm is a substantial improvement over the conventional algorithm that uses $O(n^2)$ bit operations.

Generating Functions

- Generating functions can be used to solve many types of counting problems.
- number of ways to select or distribute objects of different kinds, subject to a variety of constraints.
- number of ways to make change for a dollar using coins of different denominations.
- Generating functions can be used to solve recurrence relations by translating a recurrence relation for the terms of a sequence into an equation involving a generating function.
- This equation can then be solved to find a closed form for the generating function.
- From this closed form, the coefficients of the power series for the generating function can be found, solving the original recurrence relation.

Generating Functions

Definition: The *generating function for the sequence* $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1x + \cdots + a_kx^k + \cdots = \sum_{k=0}^{\infty} a_kx^k.$$

Examples:

- The sequence $\{a_k\}$ with $a_k = 3$ has the generating function $\sum_{k=0}^{\infty} 3x^k$.
- The sequence $\{a_k\}$ with $a_k = k + 1$ has the generating function has the generating function $\sum_{k=0}^{\infty} (k+1)x^k$.
- The sequence $\{a_k\}$ with $a_k = 2^k$ has the generating function $\sum_{k=0}^{\infty} 2^kx^k$.

Generating Functions for Finite Sequences

- Generating functions for finite sequences of real numbers can be defined by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting

$a_{n+1} = 0$, $a_{n+2} = 0$, and so on.

- The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n because no terms of the form $a_j x^j$ with $j > n$ occur, that is,

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

Generating Functions for Finite Sequences (continued)

Example: What is the generating function for the sequence 1,1,1,1,1,1?

Solution: The generating function of 1,1,1,1,1,1 is $1 + x + x^2 + x^3 + x^4 + x^5$.

we have $(x^6 - 1)/(x - 1) = 1 + x + x^2 + x^3 + x^4 + x^5$

when $x \neq 1$.

Consequently $G(x) = (x^6 - 1)/(x - 1)$ is the generating function of the sequence.

Useful Facts About Power Series

EXAMPLE 4

The function *$f(x) = 1/(1 - x)$ is the generating function of the sequence 1, 1, 1, 1, . . . , because*

$1/(1 - x) = 1 + x + x2 + \cdots$

for $|x| < 1$.

EXAMPLE 5

The function *$f(x) = 1/(1 - ax)$ is the generating function of the sequence 1, a, a2, a3, . . . ,*because

$1/(1 - ax) = 1 + ax + a2x2 + \cdots$

when $|ax| < 1$, *or equivalently, for* $|x| < 1/|a|$ *for* $a = 0$.

Useful Facts About Power Series

THEOREM 1

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k \quad \text{and} \quad f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k.$$

THEOREM 2

THE EXTENDED BINOMIAL THEOREM Let x be a real number with $|x| < 1$ and let u be a real number. Then

$$(1 + x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k.$$

EXAMPLE 6 Let $f(x) = 1/(1-x)^2$. Use Example 4 to find the coefficients $a_0, a_1, a_2, \ldots$ in the expansion $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

Solution: From Example 4 we see that

$$1/(1-x) = 1 + x + x^2 + x^3 + \cdots.$$

Hence, from Theorem 1, we have

$$1/(1-x)^2 = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} 1 \right) x^k = \sum_{k=0}^{\infty} (k+1) x^k.$$

DEFINITION 2 Let u be a real number and k a nonnegative integer. Then the *extended binomial coefficient* $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

EXAMPLE 7 Find the values of the extended binomial coefficients $\binom{-2}{3}$ and $\binom{1/2}{3}$.

Solution: Taking $u = -2$ and $k = 3$ in Definition 2 gives us

$$\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4.$$

Similarly, taking $u = 1/2$ and $k = 3$ gives us

$$\begin{aligned} \binom{1/2}{3} &= \frac{(1/2)(1/2-1)(1/2-2)}{3!} \\ &= (1/2)(-1/2)(-3/2)/6 \\ &= 1/16. \end{aligned}$$

EXAMPLE 8 When the top parameter is a negative integer, the extended binomial coefficient can be expressed in terms of an ordinary binomial coefficient. To see that this is the case, note that

$$\binom{-n}{r} = \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!} \quad \text{by definition of extended binomial coefficient}$$

$$= \frac{(-1)^r n(n+1)\cdots(n+r-1)}{r!} \quad \text{factoring out } -1 \text{ from each term in the numerator}$$

$$= \frac{(-1)^r (n+r-1)(n+r-2)\cdots n}{r!} \quad \text{by the commutative law for multiplication}$$

$$= \frac{(-1)^r (n+r-1)!}{r!(n-1)!} \quad \text{multiplying both the numerator and denominator by } (n-1)!$$

$$= (-1)^r \binom{n+r-1}{r} \quad \text{by the definition of binomial coefficients}$$

$$= (-1)^r C(n+r-1, r). \quad \text{using alternative notation for binomial coefficients}$$

◀

EXAMPLE 9 Find the generating functions for $(1+x)^{-n}$ and $(1-x)^{-n}$, where n is a positive integer, using the extended binomial theorem.

Solution: By the extended binomial theorem, it follows that

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k.$$

Using Example 8, which provides a simple formula for $\binom{-n}{k}$, we obtain

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k C(n+k-1, k) x^k.$$

Replacing x by $-x$, we find that

$$(1-x)^{-n} = \sum_{k=0}^{\infty} C(n+k-1, k) x^k.$$

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k

$\dfrac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\dfrac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\dfrac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$ $= 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\dfrac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k$ $= 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\dfrac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k$ $= 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k} x^k = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

Note: The series for the last two generating functions can be found in most calculus books when power series are discussed.

Counting Problems and Generating Functions

- Generating functions used to count the number of combinations of various types.
- Counting the solutions to equations of the form $e_1 + e_2 + \cdots + e_n = C$, where C *is a constant and each* e_i *is a nonnegative integer.*

Example: Find the number of solutions of $e_1 + e_2 + e_3 = 17$,

where e_1, e_2, and e_3 are nonnegative integers with
$2 \leq e_1 \leq 5$, $3 \leq e_2 \leq 6$, and $4 \leq e_3 \leq 7$.

Solution: The number of solutions is the coefficient of x^{17} in the expansion of
$(x^2 + x^3 + x^4 + x^5)\ (x^3 + x^4 + x^5 + x^6)\ (x^4 + x^5 + x^6 + x^7)$.

This follows because a term equal to is obtained in the product by picking a term in the first sum x^{e_1}, a term in the second sum x^{e_2}, and a term in the third sum x^{e_3}, where $e_1 + e_2 + e_3 = 17$.

There are three solutions since the coefficient of x^{17} in the product is 3.

EXAMPLE

In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?

Solution:

Because each child receives at least two but no more than four cookies, for each child there is a factor equal to $(x^2 + x^3 + x^4)$

in the generating function for the sequence $\{cn\}$*, where cn is the number of ways to distribute n* cookies.

Because there are three children, this generating function is

$(x^2 + x^3 + x^4)^3$.

We need the coefficient of x^8 *in this product.*

x^8 terms in the expansion correspond to the ways that three terms can be selected.

Computation shows that this coefficient equals 6

Counting Problems and Generating Functions (*continued*)

Example: Use generating functions to find the number of k-combinations of a set with n elements, i.e., $C(n,k)$.

Solution: Each of the n elements in the set contributes the term $(1 + x)$ to the generating function $f(x) = \sum_{k=0}^{n} a^k x^k$.

Hence $f(x) = (1 + x)^n$ where $f(x)$ is the generating function for $\{a^k\}$, where a^k represents the number of k-combinations of a set with n elements.

By the binomial theorem, we have

$$f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Hence,

$$C(n,k) = \frac{n!}{k!(n-k)!}.$$

Using Generating Functions to Solve Recurrence Relations

EXAMPLE 16 Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \ldots$ and initial condition $a_0 = 2$.

Extra Examples

Solution: Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is, $G(x) = \sum_{k=0}^{\infty} a_k x^k$. First note that

$$xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k.$$

Using the recurrence relation, we see that

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1} x^k \\ &= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k \\ &= 2, \end{aligned}$$

because $a_0 = 2$ and $a_k = 3a_{k-1}$. Thus,

$$G(x) - 3xG(x) = (1 - 3x)G(x) = 2.$$

Solving for $G(x)$ shows that $G(x) = 2/(1 - 3x)$. Using the identity $1/(1 - ax) = \sum_{k=0}^{\infty} a^k x^k$, from Table 1, we have

$$G(x) = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k.$$

Consequently, $a_k = 2 \cdot 3^k$. ◄

EXAMPLE 17 Suppose that a valid codeword is an n-digit number in decimal notation containing an even number of 0s. Let a_n denote the number of valid codewords of length n. In Example 4 of Section 8.1 we showed that the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1}$$

and the initial condition $a_1 = 9$. Use generating functions to find an explicit formula for a_n.

Solution: To make our work with generating functions simpler, we extend this sequence by setting $a_0 = 1$; when we assign this value to a_0 and use the recurrence relation, we have $a_1 = 8a_0 + 10^0 = 8 + 1 = 9$, which is consistent with our original initial condition. (It also makes sense because there is one code word of length 0—the empty string.)

We multiply both sides of the recurrence relation by x^n to obtain

$$a_n x^n = 8a_{n-1}x^n + 10^{n-1}x^n.$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $a_0, a_1, a_2, \ldots$. We sum both sides of the last equation starting with $n = 1$, to find that

$$\begin{aligned}
G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n) \\
&= 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n \\
&= 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1} \\
&= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n \\
&= 8xG(x) + x/(1-10x),
\end{aligned}$$

where we have used Example 5 to evaluate the second summation. Therefore, we have

$$G(x) - 1 = 8xG(x) + x/(1 - 10x).$$

Solving for $G(x)$ shows that

$$G(x) = \frac{1 - 9x}{(1 - 8x)(1 - 10x)}.$$

Expanding the right-hand side of this equation into partial fractions (as is done in the integration of rational functions studied in calculus) gives

$$G(x) = \frac{1}{2}\left(\frac{1}{1 - 8x} + \frac{1}{1 - 10x}\right).$$

Using Example 5 twice (once with $a = 8$ and once with $a = 10$) gives

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n.$$

Consequently, we have shown that

$$a_n = \frac{1}{2}(8^n + 10^n).$$

Inclusion-Exclusion

Principle of Inclusion-Exclusion

- In Section 2.2, we developed the following formula for the number of elements in the union of two finite sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

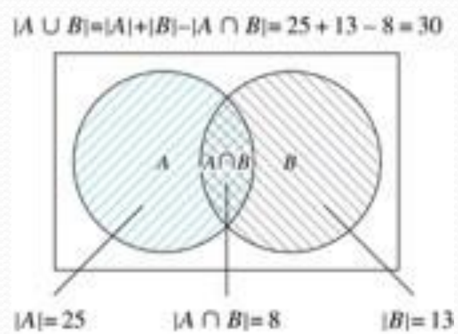
- We will generalize this formula to finite sets of any size.

Two Finite Sets

Example: In a discrete mathematics class every student is a major in computer science or mathematics or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in the class?

Solution: $|A \cup B| = |A| + |B| - |A \cap B|$

$= 25 + 13 - 8 = 30$

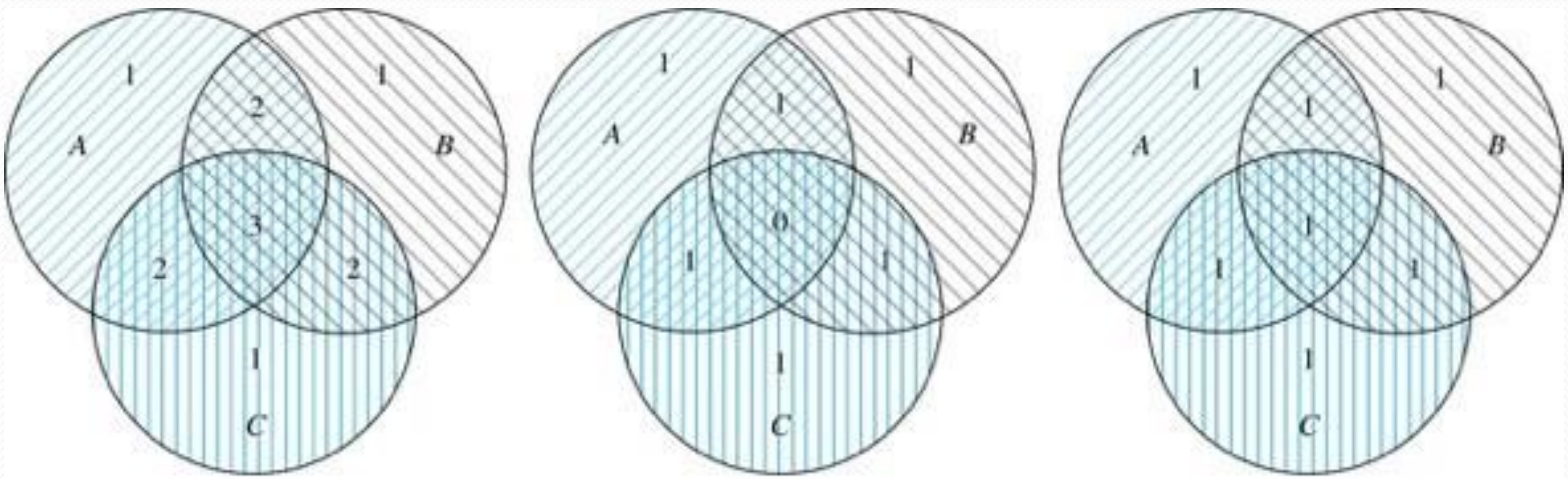


The Set of Students in a Discrete Mathematics Class.

Three Finite Sets

$$|A \cup B \cup C| =$$

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$

Three Finite Sets Continued

Example: A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken a course in at least one of Spanish French and Russian, how many students have taken a course in all 3 languages.

Solution: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian. Then, we have

$|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$, and $|S \cup F \cup R| = 23$.

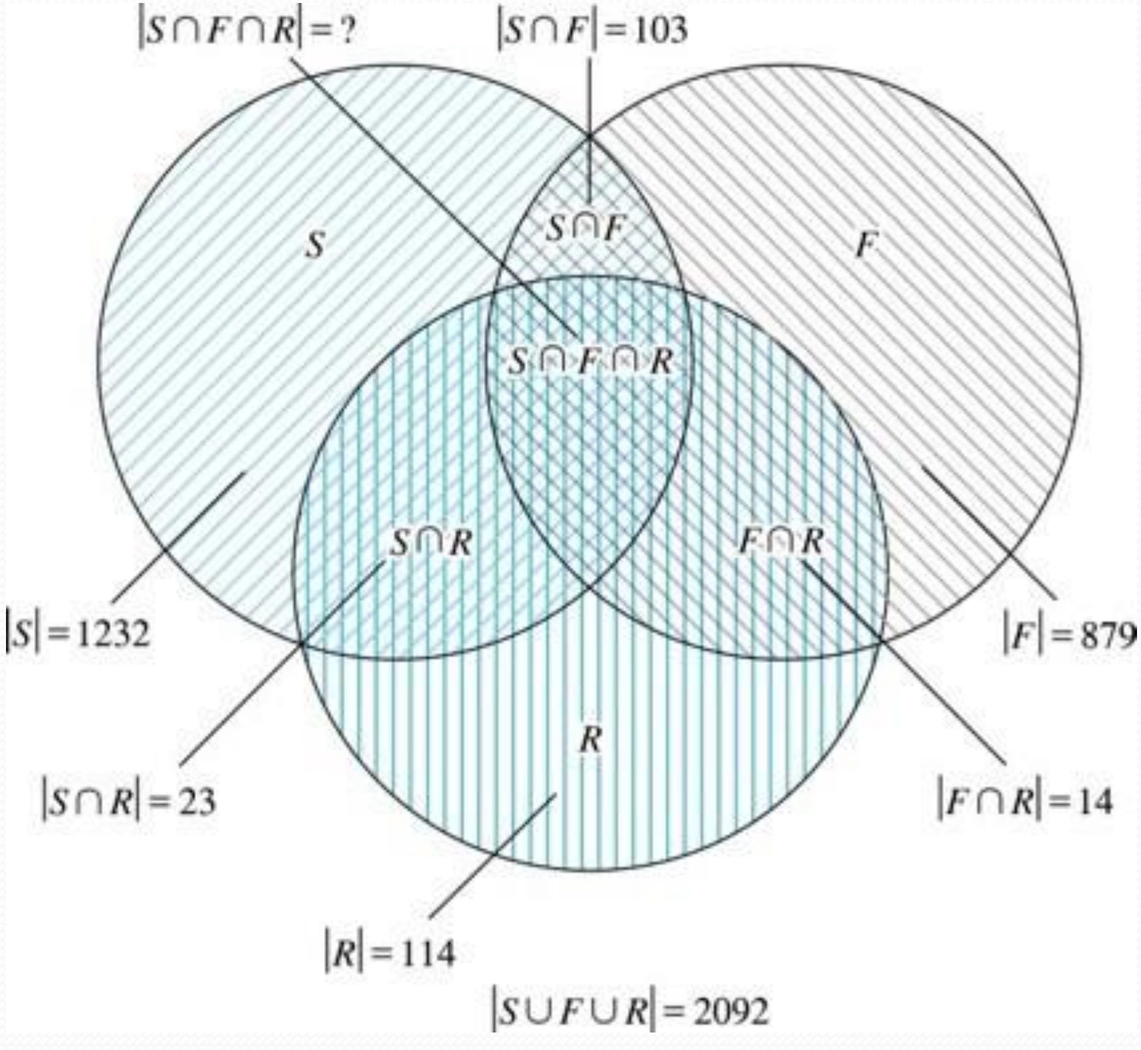
Using the equation

$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$, we obtain

$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$.

Solving for $|S \cap F \cap R|$ yields 7.

Illustration of Three Finite Set Example



The Principle of Inclusion-Exclusion

Theorem 1. **The Principle of Inclusion-Exclusion**:
Let A_1, A_2, …, A_n be finite sets. Then:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| =$$

$$\sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i \leq j \leq n} |A_i \cap A_j| +$$

$$\sum_{1 \leq i \leq j \leq k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

The Principle of Inclusion-Exclusion (*continued*)

Proof: An element in the union is counted exactly once in the right-hand side of the equation. Consider an element a that is a member of r of the sets $A_1, \ldots, A_n$ where $1 \leq r \leq n$.

- It is counted $C(r,1)$ times by $\sum |A_i|$
- It is counted $C(r,2)$ times by $\sum |A_i \cap A_j|$
- In general, it is counted $C(r,m)$ times by the summation of m of the sets A_i.

The Principle of Inclusion-Exclusion (cont)

- Thus the element is counted exactly

$C(r,1) - C(r,2) + C(r,3) - \ldots\cdots + (-1)^{r+1} C(r,r)$

times by the right hand side of the equation.

- By Corollary 2 of Section 6.4, we have

$C(r,0) - C(r,1) + C(r,2) - \ldots\cdots + (-1)^{r} C(r,r) = 0.$

- Hence,

$1 = \quad C(r,0) = C(r,1) - C(r,2) + \ldots\cdots + (-1)^{r+1} C(r,r).$

Applications of Inclusion-Exclusion

Section Summary

- The Sieve of Eratosthenes
- Counting Onto-Functions
- Derangements

The Sieve of Eratosthenes

- find all primes less than a specified positive integer n
- find the number of primes not exceeding 100
- Composite integers not exceeding 100 must have a prime factor not exceeding 10.
- primes not exceeding 10 are 2, 3, 5, and 7
- Thus, the number of primes not exceeding 100 is

$4 + N(P'_1 P'_2 P'_3 P'_4)$.

Because there are 99 positive integers greater than 1 and not exceeding 100, the principle of inclusion–exclusion shows that

$$
\begin{aligned}
N(P_1'P_2'P_3'P_4') = 99 &- N(P_1) - N(P_2) - N(P_3) - N(P_4)\\
&+ N(P_1P_2) + N(P_1P_3) + N(P_1P_4) + N(P_2P_3) + N(P_2P_4) + N(P_3P_4)\\
&- N(P_1P_2P_3) - N(P_1P_2P_4) - N(P_1P_3P_4) - N(P_2P_3P_4)\\
&+ N(P_1P_2P_3P_4).
\end{aligned}
$$

The number of integers not exceeding 100 (and greater than 1) that are divisible by all the primes in a subset of $\{2, 3, 5, 7\}$ is $\lfloor 100/N \rfloor$, where N is the product of the primes in this subset. (This follows because any two of these primes have no common factor.) Consequently,

$$
\begin{aligned}
N(P_1'P_2'P_3'P_4') = 99 &- \left\lfloor \frac{100}{2} \right\rfloor - \left\lfloor \frac{100}{3} \right\rfloor - \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{7} \right\rfloor\\
&+ \left\lfloor \frac{100}{2\cdot 3} \right\rfloor + \left\lfloor \frac{100}{2\cdot 5} \right\rfloor + \left\lfloor \frac{100}{2\cdot 7} \right\rfloor + \left\lfloor \frac{100}{3\cdot 5} \right\rfloor + \left\lfloor \frac{100}{3\cdot 7} \right\rfloor + \left\lfloor \frac{100}{5\cdot 7} \right\rfloor\\
&- \left\lfloor \frac{100}{2\cdot 3\cdot 5} \right\rfloor - \left\lfloor \frac{100}{2\cdot 3\cdot 7} \right\rfloor - \left\lfloor \frac{100}{2\cdot 5\cdot 7} \right\rfloor - \left\lfloor \frac{100}{3\cdot 5\cdot 7} \right\rfloor + \left\lfloor \frac{100}{2\cdot 3\cdot 5\cdot 7} \right\rfloor\\
&= 99 - 50 - 33 - 20 - 14 + 16 + 10 + 7 + 6 + 4 + 2 - 3 - 2 - 1 - 0 + 0\\
&= 21.
\end{aligned}
$$

Hence, there are $4 + 21 = 25$ primes not exceeding 100.

The Number of Onto Functions

Example: How many onto functions are there from a set with six elements to a set with three elements?

Solution: Suppose that the elements in the codomain are b_1, b_2, and b_3. Let P_1, P_2, and P_3 be the properties that b_1, b_2, and b_3 are not in the range of the function, respectively. The function is onto if none of the properties P_1, P_2, and P_3 hold.

By the inclusion-exclusion principle the number of onto functions from a set with six elements to a set with three elements is

$$N - [N(P_1) + N(P_2) + N(P_3)] + [N(P_1P_2) + N(P_1P_3) + N(P_2P_3)] - N(P_1P_2P_3)$$

- Here the total number of functions from a set with six elements to one with three elements is $N = 3^6$.
- The number of functions that do not have in the range is $N(P_1) = 2^6$. Similarly, $N(P_2) = N(3_1) = 2^6$.
- Note that $N(P_1P_2) = N(P_1P_3) = N(P_2P_3) = 1$ and $N(P_1P_2P_3)= 0$.

Hence, the number of onto functions from a set with six elements to a set with three

elements is: $3^6 - 3 \cdot 2^6 + 3 = 729 - 192 + 3 = 540$

The Number of Onto Functions (continued)

Theorem 1: Let m and n be positive integers with $m \geq$ n. Then there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1}C(n,n-1)\cdot 1^m$$

onto functions from a set with m elements to a set with n elements.

Derangements

Definition: A *derangement* is a permutation of objects that leaves no object in the original position.

- **Example**: The permutation of 21453 is a derangement of 12345 because no number is left in its original position. But 21543 is not a derangement of 12345, because 4 is in its original position.

Derangements (continued)

Theorem 2: The number of derangements of a set with *n* elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

Let Ai be the number of permutations that number i is in position i.

Number of dearrangements is n!-|A1 ∪ A2 ∪ A3 ∪ A4….. ∪ An|

Then |Ai|=(n-1)!,

|Ai ∩ Aj|=(n-2)!

|Ai1∩ Ai2∩ Ai3……∩ Aik|=(n-k)!

Derangements (continued)

The Hatcheck Problem: A new employee checks the hats of n people at restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat.

Solution: The answer is the number of ways the hats can be arranged so that there is no hat in its original position divided by $n!$, the number of permutations of n hats.

Remark: It can be shown that the probability of a derangement approaches $1/e$ as n grows without bound.

$$\frac{D_n}{n!} = \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

TABLE 1 The Probability of a Derangement.

n	2	3	4	5	6	7
$D_n/n!$	0.50000	0.33333	0.37500	0.36667	0.36806	0.36786

Relations

- In mathematics we study relationships such as those between a positive integer and one that it divides, an integer and one that it is congruent to modulo 5, a real number and one that is larger than it, a real number *x and the* value *f (x) where f is a function, and so on.*
- *Relationships such as that between a program and* a variable it uses, and that between a computer language and a valid statement in this language often arise in computer science.
- Relationships between elements of sets are represented using the structure called a relation, which is just a subset of the Cartesian product of the sets.
- Relations can be used to solve problems such as determining which pairs of cities are linked by airline flights in a network, finding a viable order for the different phases of a complicated project, or producing a useful way to store information in computer databases.

Outline

- 8.1 Relations and their properties
- 8.3 Representing Relations
- 8.4 Closures of Relations
- 8.5 Equivalence Relations
- 8.6 Partial Orderings

8.1 Relations and their properties.

※The most direct way to express a relationship between elements of two sets is to use ordered pairs.

For this reason, sets of ordered pairs are called **binary relations**.

Def 1

Let A and B be sets. A binary relation from A to B is a subset R of $A \times B = \{ (a, b) : a \in A, b \in B \}$.

Example 1.

A : the set of students in your school.

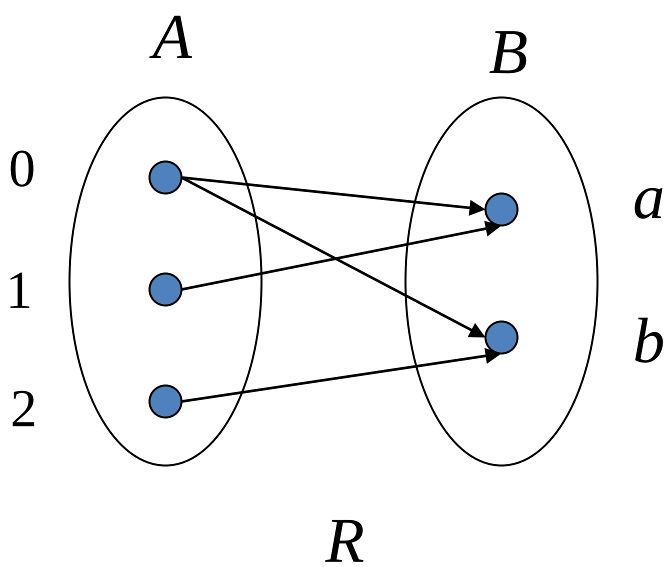
B : the set of courses.

$R = \{ (a, b) : a \in A, b \in B, a \text{ is enrolled in course } b \}$

Def 1′. We use the notation aRb to denote that $(a, b) \in R$, and $a \not R b$ to denote that $(a,b) \notin R$.

Moreover, a is said to be related to b by R if aRb.

Example 3. Let $A=\{0, 1, 2\}$ and $B=\{a, b\}$, then $\{(0,a),(0,b),(1,a),(2,b)\}$ is a relation R from A to B. This means, for instance, that $0Ra$, but that $1 \not R b$.



$$R \subseteq A \times B = \{ (0,a) , (0,b) , (1,a) , \underbrace{(1,b)}_{\notin R} , \underbrace{(2,a)}_{\notin R} , (2,b) \}$$

Note. Relations vs. Functions

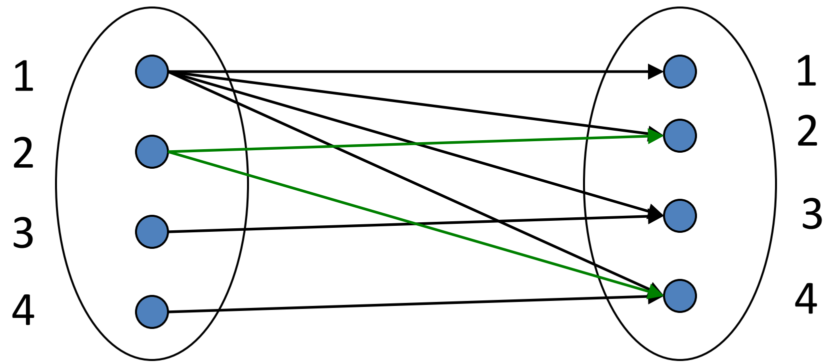
- A relation can be used to express a 1-to-many relationship between the elements of the sets A and B.
- Function represents a relation where exactly one element of *B is related to each element* of *A.*

Def 2. A relation on the set A is a subset of $A \times A$ (i.e., a relation from A to A).

Example 4.

Let $\boldsymbol{A}$ be the set $\{\mathbf{1, 2, 3, 4}\}$. Which ordered pairs are in the relation $\boldsymbol{R} = \{\ (\boldsymbol{a}, \boldsymbol{b}) |\ \boldsymbol{a} \text{ divides } \boldsymbol{b}\ \}$?

Sol :



$$\boldsymbol{R} = \{\ (1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\ \}$$

Example 5. Consider the following relations on $\mathbf{Z}$.

$R_1 = \{ (a, b) \mid a \le b \}$

$R_2 = \{ (a, b) \mid a > b \}$

$R_3 = \{ (a, b) \mid a = b \text{ or } a = -b \}$

$R_4 = \{ (a, b) \mid a = b \}$

$R_5 = \{ (a, b) \mid a = b+1 \}$

$R_6 = \{ (a, b) \mid a + b \le 3 \}$

Which of these relations contain each of the pairs (1,1), (1,2), (2,1), (1,–1), and (2,2)?

Sol :

	(1,1)	(1,2)	(2,1)	(1,–1)	(2,2)
R_1	●	●			●
R_2			●	●	
R_3	●			●	●
R_4	●				●
R_5			●		
R_6	●	●	●	●	

Example 6. How many relations are there on a set with $\boldsymbol{n}$ elements?

Sol :

A relation on a set $\boldsymbol{A}$ is a subset of $\boldsymbol{A}\times\boldsymbol{A}$.

$\Rightarrow \boldsymbol{A}\times\boldsymbol{A}$ has $\boldsymbol{n}^2$ elements.

$\Rightarrow$ a set with *n elements has* 2^n *subsets*.

$\Rightarrow \boldsymbol{A}\times\boldsymbol{A}$ has $2^{\boldsymbol{n}^2}$ subsets.

$\Rightarrow$ There are $2^{\boldsymbol{n}^2}$ relations.

Properties of Relations

Def 3. A relation R on a set A is called reflexive if $(a,a) \in R$ for every $a \in A$.

Example 7. Consider the following relations on $\{1, 2, 3, 4\}$:

$R_2 = \{ (1,1), (1,2), (2,1) \}$

$R_3 = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$

$R_4 = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

which of them are reflexive ?

Sol :

R_3

Example 8. Which of the relations from Example 5 are reflexive?

$$R_1 = \{ (a, b) \mid a \le b \}$$

$$R_2 = \{ (a, b) \mid a > b \}$$

$$R_3 = \{ (a, b) \mid a = b \text{ or } a = -b \}$$

$$R_4 = \{ (a, b) \mid a = b \}$$

$$R_5 = \{ (a, b) \mid a = b+1 \}$$

$$R_6 = \{ (a, b) \mid a + b \le 3 \}$$

Sol : R_1, R_3 and R_4

Example 9. Is the "divides" relation on the set of positive integers reflexive?

Sol : Yes.

Def 4.

(1) A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called symmetric

for all $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{A}$,

$$\textbf{if } (\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R} \;\Rightarrow\; (\boldsymbol{b}, \boldsymbol{a}) \in \boldsymbol{R}.$$

(2) A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called antisymmetric for *all* $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{A}$,

$$\textit{if } (\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R} \text{ and } (\boldsymbol{b}, \boldsymbol{a}) \in \boldsymbol{R} \;\Rightarrow\; \boldsymbol{a} = \boldsymbol{b}.$$

Example 10. Which of the relations from Example 7 are symmetric or antisymmetric ?

$R_2 = \{ (1,1), (1,2), (2,1) \}$

$R_3 = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$

$R_4 = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

Sol :

R_2, R_3 are symmetric

R_4 are antisymmetric.

Example 11. Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

Sol : It is not symmetric since $1|2$ but $2 \nmid 1$.

It is antisymmetric since $a|b$ and $b|a$ implies $a=b$.

Def 5. A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called

transitive

if $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R}$ and $(\boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{R} \Rightarrow (\boldsymbol{a}, \boldsymbol{c}) \in \boldsymbol{R}$.

for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{A}$

Example 15. Is the "divides" relation on the set of positive integers transitive?

Sol : Suppose $a|b$ and $b|c$

$\Rightarrow a|c$

$\Rightarrow$ transitive

Example 13. Which of the relations in Example 7 are transitive ?

$\boldsymbol{R_2} = \{ (1,1), (1,2), (2,1) \}$

$\boldsymbol{R_3} = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$

$\boldsymbol{R_4} = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

Sol :

$\boldsymbol{R_2}$ is not transitive since

$(2,1) \in \boldsymbol{R_2}$ and $(1,2) \in \boldsymbol{R_2}$ but $(2,2) \notin \boldsymbol{R_2}$.

$\boldsymbol{R_3}$ is not transitive since

$(2,1) \in \boldsymbol{R_3}$ and $(1,4) \in \boldsymbol{R_3}$ but $(2,4) \notin \boldsymbol{R_3}$.

$\boldsymbol{R_4}$ is transitive.

Combining Relations

Example 17. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$.

The relation $R_1 = \{(1,1), (2,2), (3,3)\}$ and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined to obtain

$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2), (3,3)\}$

$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

$R_1 \oplus R_2 = \{(2,2), (3,3), (1,2), (1,3), (1,4)\}$

symmetric difference, $(R_1 \cup R_2) - (R_1 \cap R_2)$

Def 6. Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a,c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. We denote the composite of R and S by $S \circ R$.

Example 20. What is the composite of relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Sol. $S \circ R$ is the relation from $\{1, 2, 3\}$ to $\{0, 1, 2\}$ with $S \circ R = \{(1, 0), (1,1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.

Def 7. Let R be a relation on the set A.
The powers R^n, $n = 1, 2, 3, \ldots,$ are defined recursively by
$R^1 = R$ and $R^{n+1} = R^n \circ R$.

Example 22. Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$.
Find the powers R^n, $n=2, 3, 4, \ldots$.

Sol. $R^2 = R \circ R = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$.

$R^3 = R^2 \circ R = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

$R^4 = R^3 \circ R = \{(1, 1), (2, 1), (3, 1), (4, 1)\} = R^3$.

Therefore $R^n = R^3$ for $n=4, 5, \ldots$.

Thm 1. The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

Combining Relations

Example 17. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$.

The relation $R_1 = \{(1,1), (2,2), (3,3)\}$

and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined to obtain

$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2), (3,3)\}$

$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

$R_1 \oplus R_2 = \{(2,2), (3,3), (1,2), (1,3), (1,4)\}$

symmetric difference, $(R_1 \cup R_2) - (R_1 \cap R_2)$

Example 18

Let *A and B be the set of all students and the set of all courses at a school, respectively*. **Suppose that** *R1 consists of all ordered pairs (a, b), where a is a student who has taken course b,***and***R2 consists of all ordered pairs (a, b), where a is a student who requires course b to graduate.***What are the relations** *R1 ∪R2, R1 ∩ R2, R1 ⊕R2, R1 − R2, and R2 − R1?*

Solution:

R1 ∪R2 consists of all ordered pairs (a, b), where a is a student who **either has taken course** *b or needs course b to graduate.*

R1 ∩ R2 is the set of all ordered **pairs** *(a, b), where a is a student who has taken course b and needs this course to graduate.*

R1 ⊕R2 consists of all ordered pairs (a, b), where student a has taken course b but does **not need it to graduate or needs course** *b to graduate but has not taken it.*

R1 − R2 is the set of **ordered pairs** *(a, b), where a has taken course b but does not need it to graduate; that is, b is* **an elective course that** *a has taken.*

R2 − R1 is the set of all ordered pairs (a, b), where b is a **course that** *a needs to graduate but has not taken.*

Example 19

Let *R1 be the "less than" relation on the set of real numbers and let R2 be the "greater than"*relation on the set of real numbers, that is, *$R1 = \{(x, y) \mid x < y\}$ and $R2 = \{(x, y) \mid x > y\}$*.What are *$R1 \cup R2$, $R1 \cap R2$, $R1 - R2$, $R2 - R1$, and $R1 \oplus R2$?*

Solution:

$(x, y) \in R1 \cup R2$ if and only if $(x, y) \in R1$ or $(x, y) \in R2$.

Hence, $(x, y) \in R1 \cup R2$ if and only if $x < y$ or $x > y$ i.e., $x! = y$,

$R1 \cup R2 = \{(x, y) \mid x! = y\}$.

$R1 \cap R2 = \emptyset$, $x< y$ and$x > y$

$R1 - R2 = R1$

$R2 - R1 = R2$

$R1 \oplus R2 = R1 \cup R2 - R1 \cap R2 = \{(x, y) \mid x! = y\}$.

Def 6. Let $\boldsymbol{R}$ be a relation from a set $\boldsymbol{A}$ to a set $\boldsymbol{B}$ and $\boldsymbol{S}$ a relation from $\boldsymbol{B}$ to a set $\boldsymbol{C}$. The composite of $\boldsymbol{R}$ and $\boldsymbol{S}$ is the relation consisting of ordered pairs $(\boldsymbol{a},\boldsymbol{c})$, where $\boldsymbol{a}\in\boldsymbol{A}$, $\boldsymbol{c}\in\boldsymbol{C}$, and for which there exists an element $\boldsymbol{b}\in\boldsymbol{B}$ such that $(\boldsymbol{a},\boldsymbol{b})\in\boldsymbol{R}$ and $(\boldsymbol{b},\boldsymbol{c})\in\boldsymbol{S}$. We denote the composite of $\boldsymbol{R}$ and $\boldsymbol{S}$ by $\boldsymbol{S}\circ\boldsymbol{R}$.

Example 20. What is the composite of relations $\boldsymbol{R}$ and $\boldsymbol{S}$, where $\boldsymbol{R}$ is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $\boldsymbol{R} = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and $\boldsymbol{S}$ is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $\boldsymbol{S} = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Sol. $\boldsymbol{S}\circ\boldsymbol{R}$ is the relation from $\{1, 2, 3\}$ to $\{0, 1, 2\}$ with
$\boldsymbol{S}\circ\boldsymbol{R} = \{(1, 0), (1,1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.

Def 7. Let R be a relation on the set A.
The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively by
$R^1 = R$ and $R^{n+1} = R^n \circ R$.

Example 22. Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$.
Find the powers R^n, $n=2, 3, 4, \ldots$.

Sol. $R^2 = R \circ R = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$.

$R^3 = R^2 \circ R = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

$R^4 = R^3 \circ R = \{(1, 1), (2, 1), (3, 1), (4, 1)\} = R^3$.

Therefore $R^n = R^3$ for $n=4, 5, \ldots$.

Thm 1. The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

n-ary Relations and Their Applications

- Relationship involving the name of a student, the student's major, and the student's grade point average.
- Relationship involving the airline, flight number, starting point, destination,departure time, and arrival time of a flight.
- relationships among elements from more than two sets are called ***n-ary relations.***
- These relations are used to represent computer databases.
- EX: Which flights land at O'Hare Airport between 3 a.m. and 4 a.m.?
- Which students at your school are sophomores majoring in mathematics or computer science and have greater than a 3.0 average?
- Which employees of a company have worked for the company less than 5 years and make more than $50,000?

n-ary Relations

DEFINITION 1

Let $A_1, A_2, \ldots, A_n$ *be sets.An n-ary relation on these sets is a subset of* $A_1 \times A_2 \times \cdot \cdot \cdot \times A_n$ The sets $A_1, A_2, \ldots, A_n$ *are called the domains of the relation, and n is called its degree.*

EXAMPLE 1

Let *R be the relation on* **N × N × N *consisting of triples (a, b, c), where a, b, and c are integers*** with $a < b < c$. *Then* $(1, 2, 3) \in R$, *but* $(2, 4, 3) \notin R$.

The degree of this relation is 3. Its domains are all equal to the set of natural numbers.

EXAMPLE 2

Let *R be the relation on* **Z × Z × Z *consisting of all triples of integers (a, b, c) in which*** *a, b, and c form an arithmetic progression.*

(a, b, c) $\in$ *R if and only if there is* an integer *k such that*

$b = a + k$ *and* $c = a + 2k$,

$b - a = k$ *and* $c - b = k$.

(1, 3, 5) $\in$ *R because* $3 = 1 + 2$ *and* $5 = 1 + 2 \cdot 2$,

but (2, 5, 9) !$\in$ *R because* $5 - 2 = 3$ while $9 - 5 = 4$.

This relation has degree 3 and its domains are all equal to the set of integers.

EXAMPLE 3

Let*R be the relation on***Z × Z × Z+ *consisting of triples (a, b, m), where a, b, and m are integers*** with $m \geq 1$ *and* $a \equiv b \pmod{m}$.

(8, 2, 3), (−1, 9, 5), and (14, 0, 7) ∈ R,

but (7, 2, 3), (−2,−8, 5), and (11, 0, 6) !∈ R

$8 \equiv 2 \pmod{3}, -1 \equiv 9 \pmod{5}$, and $14 \equiv 0 \pmod{7}$,

but $7 \equiv 2 \pmod{3}$, $-2 \equiv -8 \pmod{5}$, *and* $11 \equiv 0 \pmod{6}$.

This relation has degree 3 and its first two domains are the set of all integers and its third domain is the set of positive integers.

EXAMPLE 4

Let *R be the relation consisting of 5-tuples (A,N, S,D, T) representing airplane flights,* where *A is the airline, N is the flight number, S is the starting point, D is the destination, and T is* the departure time.

sol:

if Nadir Express Airlines has flight 963 from Newark to Bangor at 15:00, then (Nadir, 963, Newark, Bangor, 15:00) $\in R$.

The degree of this relation is 5, and its domains are the set of all airlines, the set of flight numbers, the set of cities, the set of cities (again), and the set of times.

Databases and Relations

- The time required to manipulate information in a database depends on how this information is stored.
- The operations of adding and deleting records, updating records, searching for records, and combining records from overlapping databases are performed millions of times each day in a large database.
- Because of the importance of these operations, various methods for representing databases have been developed.
- A database consists of **records, which are *n-tuples, made up of fields.***
- The fields are the entries of the *n-tuples*.

student records

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

are represented as 4-tuples of the form (*Student_name, ID_number, Major, GPA*). A sample database of six such records is

(Ackermann, 231455, Computer Science, 3.88)
(Adams, 888323, Physics, 3.45)
(Chou, 102147, Computer Science, 3.49)
(Goodfriend, 453876, Mathematics, 3.45)
(Rao, 678543, Mathematics, 3.90)
(Stevens, 786576, Psychology, 2.99).

Databases and Relations

- Relations used to represent databases are also called **tables.**
- Each column of the table corresponds to an *attribute of the database.*
- *For* instance, the same database of students is displayed in Table 1.
- The attributes of this database are Student Name, ID Number, Major, and GPA.
- A domain of an *n-ary relation is called a* ***primary key when the value of the n-tuple from*** this domain determines the *n-tuple.*
- Domain is a primary key when no two *n-tuples in* the relation have the same value from this domain.

Databases and Relations

- Records are often added to or deleted from databases.
- So,domain is a primary key is time-dependent.
- A primary key should be chosen that remains one whenever the database is changed.
- The current collection of *n-tuples in a relation* is called the **extension of the relation.**
- **The more permanent part of a database, including the** name and attributes of the database, is called its **intension.**
- **When selecting a primary key,** select a key that can serve as a primary key for all possible extensions of the database.
- To do this, it is necessary to examine the intension of the database to understand the set of possible *n-tuples that can occur in an extension.*

EXAMPLE 5

Which domains are primary keys for the *n-ary relation displayed in Table 1, assuming that no n-tuples will be added in the future?*

Solution:

domain of student names is a primary key.

ID numbers in this table are unique, so the domain of ID numbers is also a primary key.

domain of major fields of study is not a primary key, because more than one 4-tuple contains the same major field of study.

domain of grade point averages is also not a primary key, because there are two 4-tuples containing the same GPA.

composite key

Combinations of domains can also uniquely identify *n-tuples in an n-ary relation*.

When the values of a set of domains determine an *n-tuple in a relation, the Cartesian product of these* domains is called a **composite key.**

EXAMPLE 6

Is the Cartesian product of the domain of major fields of study and the domain of GPAs a composite key for the *n-ary relation from Table 1, assuming that no n-tuples are ever added?*

***Solution**:*

*Because no two 4-tuples from this table have both the same major and the same GPA,*this Cartesian product is a composite key.

Because primary and composite keys are used to identify records uniquely in a database, it is important that keys remain valid when new records are added to the database.

Combining Relations

Example 17. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$.

The relation $R_1 = \{(1,1), (2,2), (3,3)\}$

and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined to obtain

$$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$$

$$R_1 \cap R_2 = \{(1,1)\}$$

$$R_1 - R_2 = \{(2,2), (3,3)\}$$

$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$$

$$R_1 \oplus R_2 = \{(2,2), (3,3), (1,2), (1,3), (1,4)\}$$

symmetric difference, $(R_1 \cup R_2) - (R_1 \cap R_2)$

Example 18

Let *A and B be the set of all students and the set of all courses at a school, respectively*. **Suppose that** *R1 consists of all ordered pairs (a, b), where a is a student who has taken course b,***and***R2 consists of all ordered pairs (a, b), where a is a student who requires course b to graduate.***What are the relations** *R1 ∪ R2, R1 ∩ R2, R1 ⊕ R2, R1 − R2, and R2 − R1?*

Solution:

R1 ∪ R2 consists of all ordered pairs (a, b), where a is a student who **either has taken course** *b or needs course b to graduate.*

R1 ∩ R2 is the set of all ordered **pairs** *(a, b), where a is a student who has taken course b and needs this course to graduate.*

R1 ⊕ R2 consists of all ordered pairs (a, b), where student a has taken course b but does **not need it to graduate or needs course** *b to graduate but has not taken it.*

R1 − R2 is the set of **ordered pairs** *(a, b), where a has taken course b but does not need it to graduate; that is, b is* **an elective course that** *a has taken.*

R2 − R1 is the set of all ordered pairs (a, b), where b is a **course that** *a needs to graduate but has not taken.*

Example 19

Let ***R1 be the "less than" relation on the set of real numbers and let R2 be the "greater than"*****relation on the set of real numbers, that is,** $R1 = \{(x, y) \mid x < y\}$ ***and*** $R2 = \{(x, y) \mid x > y\}$**.What are** $R1 \cup R2$, $R1 \cap R2$, $R1 - R2$, $R2 - R1$, ***and*** $R1 \oplus R2$**?**

Solution:

$(x, y) \in R1 \cup R2$ *if and only if* $(x, y) \in R1$ *or* $(x, y) \in R2$.

Hence, $(x, y) \in R1 \cup R2$ *if and only if* $x < y$ *or* $x > y$ *i.e.,* $x != y$,

$R1 \cup R2 = \{(x, y) \mid x != y\}$.

$R1 \cap R2 = \emptyset$, $x < y$ *andx* $> y$

$R1 - R2 = R1$

$R2 - R1 = R2$

$R1 \oplus R2 = R1 \cup R2 - R1 \cap R2 = \{(x, y) \mid x != y\}$.

Def 6. Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a,c), where $a\in A$, $c\in C$, and for which there exists an element $b\in B$ such that $(a,b)\in R$ and $(b,c)\in S$. We denote the composite of R and S by $S\circ R$.

Example 20. What is the composite of relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Sol. $S\circ R$ is the relation from $\{1, 2, 3\}$ to $\{0, 1, 2\}$ with
$S\circ R = \{(1, 0), (1,1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.

Def 7. Let $\boldsymbol{R}$ be a relation on the set $\boldsymbol{A}$.
The powers $\boldsymbol{R^n}$, $\boldsymbol{n} = 1, 2, 3, \ldots$, are defined recursively by
$\boldsymbol{R}^1 = \boldsymbol{R}$ and $\boldsymbol{R}^{\boldsymbol{n}+1} = \boldsymbol{R^n} \circ \boldsymbol{R}$.

Example 22. Let $\boldsymbol{R} = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$.
Find the powers $\boldsymbol{R^n}$, $\boldsymbol{n}=2, 3, 4, \ldots$.

Sol. $\boldsymbol{R}^2 = \boldsymbol{R} \circ \boldsymbol{R} = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$.

$\boldsymbol{R}^3 = \boldsymbol{R}^2 \circ \boldsymbol{R} = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

$\boldsymbol{R}^4 = \boldsymbol{R}^3 \circ \boldsymbol{R} = \{(1, 1), (2, 1), (3, 1), (4, 1)\} = \boldsymbol{R}^3$.

Therefore $\boldsymbol{R^n} = \boldsymbol{R}^3$ for $\boldsymbol{n}=4, 5, \ldots$.

Thm 1. The relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is transitive if and only if $\boldsymbol{R^n} \subseteq \boldsymbol{R}$ for $\boldsymbol{n} = 1, 2, 3, \ldots$.

n-ary Relations and Their Applications

- Relationship involving the name of a student, the student's major, and the student's grade point average.
- Relationship involving the airline, flight number, starting point, destination,departure time, and arrival time of a flight.
- relationships among elements from more than two sets are called ***n-ary relations.***
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n-ary Relations

DEFINITION 1

Let $A_1, A_2, \ldots, A_n$ *be sets.An n-ary relation on these sets is a subset of* $A_1 \times A_2 \times \cdot \cdot \cdot \times A_n$ The sets $A_1, A_2, \ldots, A_n$ *are called the domains of the relation, and n is called its degree*.

EXAMPLE 1

Let *R be the relation on* **N × N × N** ***consisting of triples (a, b, c), where a, b, and c are integers*** with $a < b < c$. *Then* $(1, 2, 3) \in R$*, but* $(2, 4, 3) \notin R$.

The degree of this relation is 3. Its domains are all equal to the set of natural numbers.

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Let R *be the relation on* **Z × Z × Z** ***consisting of all triples of integers (a, b, c) in which*** *a, b, and c form an arithmetic progression.*

(a, b, c) $\in R$ *if and only if there is* an integer k *such that*

$b = a + k$ *and* $c = a + 2k$,

$b - a = k$ *and* $c - b = k$.

$(1, 3, 5) \in R$ *because* $3 = 1 + 2$ *and* $5 = 1 + 2 \cdot 2$,

but $(2, 5, 9)\ !\in R$ *because* $5 - 2 = 3$ while $9 - 5 = 4$.

This relation has degree 3 and its domains are all equal to the set of integers.

EXAMPLE 3

Let*R be the relation on***Z × Z × Z+ *consisting of triples (a, b, m), where a, b, and m are integers*** with *m* ≥ *1 and a* ≡ *b (mod m).*

(8, 2, 3), (−1, 9, 5), and (14, 0, 7) ∈ R,

but (7, 2, 3), (−2,−8, 5), and (11, 0, 6) !∈ R

8 ≡ 2 *(mod 3),−1* ≡ *9 (mod 5),* and 14 ≡ 0 *(mod 7),*

but 7 ≡ 2 *(mod 3), −2* ≡ *−8 (mod 5), and 11* ≡ *0 (mod 6).*

This relation has degree 3 and its first two domains are the set of all integers and its third domain is the set of positive integers.

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Let *R be the relation consisting of 5-tuples (A,N, S,D, T) representing airplane flights,* where *A is the airline, N is the flight number, S is the starting point, D is the destination, and T is* the departure time.

sol:

if Nadir Express Airlines has flight 963 from Newark to Bangor at 15:00, then (Nadir, 963, Newark, Bangor, 15:00) $\in R$.

The degree of this relation is 5, and its domains are the set of all airlines, the set of flight numbers, the set of cities, the set of cities (again), and the set of times.

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- A database consists of **records, which are *n-tuples, made up of fields.***
- The fields are the entries of the *n-tuples.*

student records

TABLE 1 **Students.**			
Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

are represented as 4-tuples of the form (*Student_name, ID_number, Major, GPA*). A sample database of six such records is

(Ackermann, 231455, Computer Science, 3.88)
(Adams, 888323, Physics, 3.45)
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(Rao, 678543, Mathematics, 3.90)
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Databases and Relations

- Relations used to represent databases are also called **tables.**
- Each column of the table corresponds to an *attribute of the database.*
- *For* instance, the same database of students is displayed in Table 1.
- The attributes of this database are Student Name, ID Number, Major, and GPA.
- A domain of an *n-ary relation is called a* ***primary key when the value of the n-tuple from*** this domain determines the *n-tuple.*
- Domain is a primary key when no two *n-tuples in* the relation have the same value from this domain.

Databases and Relations

- Records are often added to or deleted from databases.
- So,domain is a primary key is time-dependent.
- A primary key should be chosen that remains one whenever the database is changed.
- The current collection of *n-tuples in a relation* is called the **extension of the relation.**
- **The more permanent part of a database, including the** name and attributes of the database, is called its **intension.**
- **When selecting a primary key,** select a key that can serve as a primary key for all possible extensions of the database.
- To do this, it is necessary to examine the intension of the database to understand the set of possible *n-tuples that can occur in an extension*.

EXAMPLE 5

Which domains are primary keys for the *n-ary relation displayed in Table 1, assuming that no n-tuples will be added in the future?*

Solution:

domain of student names is a primary key.

ID numbers in this table are unique, so the domain of ID numbers is also a primary key.

domain of major fields of study is not a primary key, because more than one 4-tuple contains the same major field of study.

domain of grade point averages is also not a primary key, because there are two 4-tuples containing the same GPA.

composite key

Combinations of domains can also uniquely identify *n-tuples in an n-ary relation.*

When the values of a set of domains determine an *n-tuple in a relation, the Cartesian product of these* domains is called a **composite key.**

EXAMPLE 6

Is the Cartesian product of the domain of major fields of study and the domain of GPAs a composite key for the *n-ary relation from Table 1, assuming that no n-tuples are ever added?*

***Solution**:*

*Because no two 4-tuples from this table have both the same major and the same GPA,*this Cartesian product is a composite key.

Because primary and composite keys are used to identify records uniquely in a database, it is important that keys remain valid when new records are added to the database.

Operations on *n-ary Relations*

- Operations can answer queries on databases that ask for all *n-tuples that* satisfy certain conditions.

  find all the records of all computer science majors in a database of student records.

  find all students who have a grade point average above 3.5

  find the records of all computer science majors who have a grade point average above 3.5.

- **selection operator**

  Let *R be an n-ary relation and C a condition that elements in R may satisfy. Then the selection operator* s_C *maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy* the condition *C*.

Example

- To find the records of computer science majors in the *n-ary relation R shown in Table 1,*

  C1 is the condition Major=*"Computer Science."*

  The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Chou, 102147, Computer Science, 3.49).
- To find the records of students who have a grade point average above 3.5 in this database

  C2 is the condition GPA > *3.5.*

  The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Rao, 678543, Mathematics, 3.90).
- Finally, to find the records of computer science majors who have a GPA above 3.5, we use the operator *sC3 ,*
- *C3 is the condition (Major*=*"Computer Science"* $\wedge$ *GPA* > *3.5).*

  The result consists of the single 4-tuple (Ackermann, 231455, Computer Science, 3.88).

Projections

- Projections are used to form new *n-ary relations by deleting the same fields in every record* of the relation.
- The *projection P+where i1 < i2 < · · · < im, maps the n-tuple (a1, a2, . . . , an) to them-tuple (ai1 , ai2, . . . , aim), where m ≤ n.*

What relation results when the projection *P1,4 is applied to the relation in Table 1?*

TABLE 2 GPAs.	
Student_name	*GPA*
Ackermann	3.88
Adams	3.45
Chou	3.49
Goodfriend	3.45
Rao	3.90
Stevens	2.99

Example

What is the table obtained when the projection *P1,2 is applied to the relation in Table 3?*

TABLE 3 Enrollments.

Student	*Major*	*Course*
Glauser	Biology	BI 290
Glauser	Biology	MS 475
Glauser	Biology	PY 410
Marcus	Mathematics	MS 511
Marcus	Mathematics	MS 603
Marcus	Mathematics	CS 322
Miller	Computer Science	MS 575
Miller	Computer Science	CS 455

TABLE 4 Majors.

Student	*Major*
Glauser	Biology
Marcus	Mathematics
Miller	Computer Science

Join

- The **join operation is used to combine two tables into one when these tables share some** identical fields.
- For instance, a table containing fields for airline, flight number, and gate, and another table containing fields for flight number, gate, and departure time can be combined into a table containing fields for airline, flight number, gate, and departure time.
- Let R be a relation of degree m and S a relation of degree n. The join $J_p(R, S)$, where $p \le m$ and $p \le n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p)$-tuples $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$, where them-tuple $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p)$ belongs to R and the n-tuple $(c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$ belongs to S.

Example

What relation results when the join operator *J2 is used to combine the relation displayed in* Tables 5 and 6?

TABLE 5 Teaching_assignments.

Professor	*Department*	*Course_ number*
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

TABLE 6 Class_schedule.

Department	*Course_ number*	*Room*	*Time*
Computer Science	518	N521	2:00 P.M.
Mathematics	575	N502	3:00 P.M.
Mathematics	611	N521	4:00 P.M.
Physics	544	B505	4:00 P.M.
Psychology	501	A100	3:00 P.M.
Psychology	617	A110	11:00 A M.
Zoology	335	A100	9:00 A M.
Zoology	412	A100	8:00 A M.

TABLE 7 Teaching_schedule.

Professor	*Department*	*Course_number*	*Room*	*Time*
Cruz	Zoology	335	A100	9:00 A M.
Cruz	Zoology	412	A100	8:00 A M.
Farber	Psychology	501	A100	3:00 P M.
Farber	Psychology	617	A110	11:00 A M.
Grammer	Physics	544	B505	4:00 P M.
Rosen	Computer Science	518	N521	2:00 P M.
Rosen	Mathematics	575	N502	3:00 P M.

SQL

- The database query language SQL(Structured Query Language) can be used to carry out the operations(SELECTION,PROJECTION,JOIN).
- SELECT Departure_time FROM Flights WHERE Destination='Detroit'

TABLE 8 Flights.

Airline	*Flight_number*	*Gate*	*Destination*	*Departure_time*
Nadir	122	34	Detroit	08:10
Acme	221	22	Denver	08:17
Acme	122	33	Anchorage	08:22
Acme	323	34	Honolulu	08:30
Nadir	199	13	Detroit	08:47
Acme	222	22	Denver	09:10
Nadir	322	34	Detroit	09:44

Representing Relations.

Representing Relations Using Matrices

→ A relation between finite sets can be represented using a zero-one matrix.

R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots b_n\}$.

Relation R can be represented by matrix $M_R = [m_{ij}]$,

where $m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$

1) Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1, a_2 = 2, a_3 = 3$; $b_1 = 1, b_2 = 2$?

Solution: $R = \{(2,1), (3,1), (3,2)\}$

the matrix for R is $M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

The 1s in M_R show that the pairs (2,1), (3,1) & (3,2) belong to R.

0's show that no other pairs belong to R.

2) Let $A = \{a_1, a_2, a_3\}$ & $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by matrix.

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

Sol: R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$

So, $R = \{(a_1, b_2), (a_2, b_1), (a_3, b_3), (a_2, b_4),$
$(a_3, b_1), (a_3, b_3), (a_3, b_5)\}$.

→ The matrix of a relation on a set, which is a square matrix can be used to determine whether the relation has certain properties.

→ A relation R on A is reflexive if $(a,a) \in R$. Whenever $a \in A$.

∴ R is Reflexive if & only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots n$.

Here R is reflexive iff only if $m_{ii} = 1$ for $i = 1, 2, \ldots n$.

R is reflexive if all the elements on main diagonal of M_R are equal to 1.

$$\begin{bmatrix} 1 & & & 0\text{ or }1 \\ & 1 & & \\ & & \ddots & \\ 0\text{ or }1 & & & 1 \end{bmatrix}$$

Zero-one matrix for reflexive Relation.

Symmetric Relation R is Symmetric if $(a,b) \in R \rightarrow (b,a) \in R$.

the relation R on set $A = \{a_1, a_2 \dots a_n\}$ is Symmetric iff $(a_j, a_i) \in R$. whenever $(a_i, a_j) \in R$.

In terms of Matrix entries M_R.

R is Symmetric if & only if $m_{ji} = 1$. whenever $m_{ij} = 1$.

$m_{ji} = 0$ whenever $m_{ij} = 0$.

i.e., R is Symmetric iff $m_{ij} = m_{ji}$.

$i = 1, 2 \dots n$
$j = 1, 2, \dots n$.

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} \cdot & 1 & \cdot & 1 \\ 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix} \end{array} \qquad M_R = (M_R)^t.$$

→ Anti Symmetric

Relation R is antisymmetric if & only ~~if (a,b)∈R~~
if $(a,b) \in R$ & $(b,a) \in R \rightarrow$ that $a = b$.

The matrix of an anti Symmetric relation:

if $m_{ij} = 1$. with $i \neq j$ then $m_{ji} = 0$.

either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} \cdot & 1 & 0 \\ 0 & \cdot & \cdot \\ 0 & \cdot & \cdot \end{bmatrix} \end{array}$$

Ex: 3) Suppose that the Relation R on a set is represented by matrix $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Is R reflexive, Symmetric, antiSymmetric?

Sol: $\because$ diagonal elements are equal to 1, it is reflexive.

$$\therefore \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & & & \\ 1 & & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} & \\ 2 & & & \end{array}$$

R is Symmetric $(0,1) = (1,0) = 1$
$(1,2) = (2,1) = 1$.

R is not Antisymmetric. $\boxed{\text{by def if } a_{ij}=1, \text{ then } a_{ji}=0,\ i \neq j.}$

$\because$ if $(0,1) = 1$, then $(1,0) \neq 0$. here in this matrix So, it is not antisymmetric.

→ Boolean operations join and meet. used to find matrices representing the union & intersection of two relations.

→ R_1 & R_2 relations on a set A. represented by matrices MR_1 and M_{R_2} resp.

Union – Matrix representing the union of these relations has a 1 in positions where either M_{R_1} or M_{R_2} has a 1.

Intersection - Matrix representing intersection of these relations has a 1 in positions where both M_{R_1} & M_{R_2} has 1.

$$\therefore \quad M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}.$$

Ex. 4) Suppose that relations R_1 & R_2 on a Set A are represented by the matrices.

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \& \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ & $R_1 \cap R_2$?

Sol:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Composite of relations

Suppose that R is a relation from A to B. ~~& S is~~
S is a relation from B to C.

Suppose that A, B and C have m, n, p elements resp.

Matrices for $S \circ R$, R & S be $M_{S \circ R} = [t_{ij}]$

$M_R = [r_{ij}]$

$M_S = [s_{ij}]$.

ordered pair $(a_i, c_j) \in S \circ R$ iff there is an element b_k such that $(a_i, b_k) \in R$ & $(b_k, c_j) \in S$.

$t_{ij} = 1$ if & only if $r_{ik} = s_{kj} = 1$.

From definition of Boolean product.

i.e., $M_{S \circ R} = M_R \odot M_S$.

Ex 5 Find the matrix representing the relations $S \circ R$, where the matrices representing R & S are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \& \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Sol:

M_R ordered pairs are $(0,0)$, $(0,2)$, $(1,0)$ $(1,1)$.

M_S ordered pairs are $(0,1)$, $(1,2)$, $(2,0)$ $(2,2)$.

So $S \circ R = (0,1),\ (0,0),\ (0,2),\ (1,2),\ (1,2)$.

$$M_{S \circ R} = M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

→ The matrix representing the composite of two relations can be used to find matrix for M_{R^n}.

$$M_{R^n} = M_R^{[n]}.$$

Ex 6) Find the matrix representing the relation R^2 where the matrix representing R is

$$M_R = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}.$$

Sol: M_R ordered pairs $\cancel{(1,0)}$ $(0,1)$, $(1,1)$, $(1,2)$ $(2,0)$.

$(0,1)$ $(1,1)$ $(1,2)$ $(2,0)$.

$$R^2 = \cancel{(1,0)}\ (0,1),\ (1,1),\ (1,2),\ (1,0),\ (0,2),\ (2,1)$$

$$M_{R^2} = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Ex: 6. Find the matrix representing the relation R^2. Where the matrix representing R is

$$M_R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Sol: ordered pairs for R $(0,1)$, $(1,1)$ $(1,2)$ $(2,0)$

R $(0,1)$ $(1,1)$ $(1,2)$ $(2,0)$

$$R^2 = \{(0,1), (0,2), (1,1), (1,2), (1,0), (2,1)\}.$$

$$M_{R^2} = M_R^{[2]} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Representing Relations Using Digraphs

→ Representing Relation using a pictorial representation.

→ Each element of a set represented by a point.

→ Each ordered pair is represented using an arc. with its direction indicated by an arrow.

→ Pictorial representations of relations on a finite set as directed graphs, or digraphs.

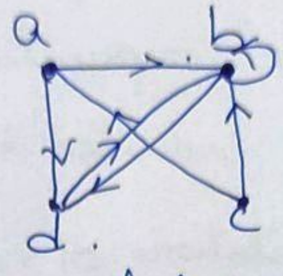
Def: A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs)

Vertex a is called the initial vertex of edge (a, b)

Vertex b is called the terminal vertex of this edge.

→ An edge of form (a, a) is represented using an arc from vertex a back to itself. It is called a loop.

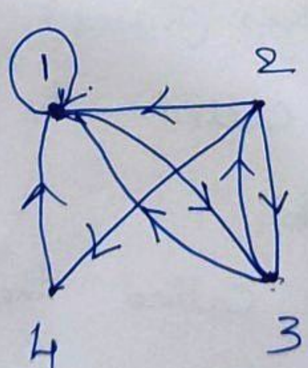
Ex: 7. The directed graph with vertices a, b, c, d & edges (a,b), (a, d), b,b), (b,d), (c,a), (c,b). & (d,b).



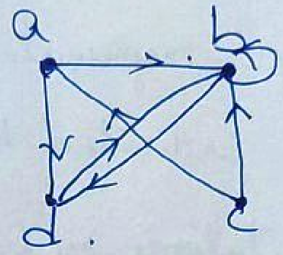
Directed graph.

Ex: 8 The directed graph of a relation.

$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$. on set $\{1,2,3,4\}$.



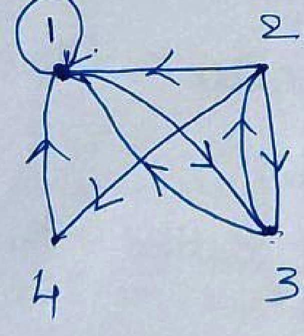
Ex: 7. The directed graph with vertices a, b, c, d & edges (a,b), (a,d), (b,b), (b,d), (c,a), (c,b). & (d,b).



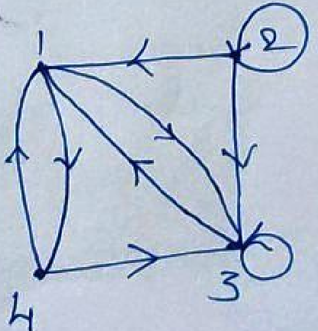
Directed graph.

Ex: 8 The directed graph of a relation.

$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$. on set $\{1,2,3,4\}$.



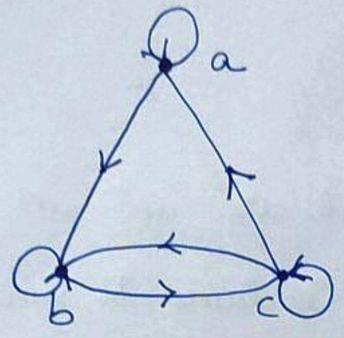
Ex: 9. What are the ordered pairs in the relation R represented by directed graph shown in Figure



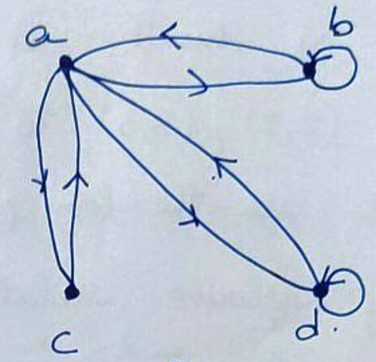
The ordered pairs (x, y) in the relation are

$R = \{(1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,3), (4,1), (4,3)\}$.

Ex: 10. Determine whether the relations for the directed graphs shown in below figures are reflexive, Symmetric, antisymmetric, & / or transitive.



Directed graph of R.



Directed graph of S.

R is Reflexive

R is not Symmetric, not AntiSymmetric, not Transitive

S is not reflexive

S is Symmetric

S is not antisymmetric

S is not Transitive

Closures of Relations

→ Let R be a relation on a set A. R may or may not have some property P, such as reflexivity, Symmetry or transitivity

→ If there is a relation S with property P containing R such that S is a subset of every relation with property P containing R, then S is called the closure of R with respect to P.

Closures

The relation $R = \{(1,1), (1,2), (2,1), (3,2)\}$ on the ~~set~~ set $A = \{1, 2, 3\}$ is not reflexive. How can we produce a reflexive relation containing R that is as small as possible?

→ Add $(2,2)$, $(3,3)$ to R.

these are the form (a,a) which are not in R.

→ Any Reflexive relation that contains R must also contain $(2,2)$ & $(3,3)$.

→ Bcoz this relation contains R, is reflexive and is contained within every reflexive relation that contains R, it is called the reflexive closure of R.

reflexive closure of $R = R \cup \Delta$

$\Delta = \{(a,a) \mid a \in A\}$ is ~~dig~~ diagonal relation on A.

Ex: 1 What is the reflexive closure of relation $R = \{(a,b) \mid a < b\}$ on the set of integers?

Sol: The reflexive closure of R is

$$R \cup \Delta = \{(a,b) \mid a<b\} \cup \{(a,a) \mid a \in Z\} = \{(a,b) \mid a \leq b\}$$

The Relation $\{(1,1), (1,2), (2,2), (2,3), (3,1), (3,2\}$ on $\{1,2,3\}$ is not Symmetric. How can we produce a Symmetric relation that is as small as possible and contains R?

→ For this we should add $(2,1)$ & $(1,3)$

→ It is of form (b,a) with $(a,b) \in R$ which is not in R.

→ This new relation is Symmetric & Contains R.

→ So, Further any Symmetric relation that Contains R must Contain this new relation, b'coz this new relation containing $(2,1), (1,3)$ is called Symmetric closure of R.

→ $R \cup R^{-1}$ is symmetric closure of R,
where $R^{-1} = \{(b,a) \mid (a,b) \in R\}$.

Ex:2 What is the Symmetric closure of the relation $R = \{(a,b) \mid a > b\}$ on set of positive integers?

Sol: The Symmetric closure of R is the relation

$$R \cup R^{-1} = \{(a,b) \mid a > b\} \cup \{(b,a) \mid a > b\} = \{(a,b) \mid a \neq b\}.$$

Transitive

Suppose a relation R is not transitive.

R = {(1,3), (1,4), (2,1), (3,2)} on set {1,2,3,4}.

This relation is not transitive b'coz it doesn't contain all pairs of form (a,c) where (a,b) & (b,c) are in R

→ So, (1,2), (2,3), (2,4), (3,1) are not in R.

→ By adding these pairs it doesnot produce a transitive relation, Because after adding these pairs we have (3,1), (1,4) but (3,4) is not there.

→ So, constructing the transitive closure of a relation is more complicated than reflexive or Symmetric closure.

→ Transitive closure of a relation can be found by adding new ordered pairs that must be present and then repeating this process until no new ordered pairs are needed.

Paths in Directed graphs

A path in a directed graph is obtained by traversing along edges.

Def: A path from a to b in directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where n is a nonnegative integer & $x_0 = a$ & $x_n = b$, i.e., a sequence of edges where the terminal vertex of an edge is same as the initial vertex in next edge in path.

→ This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ & has length n.

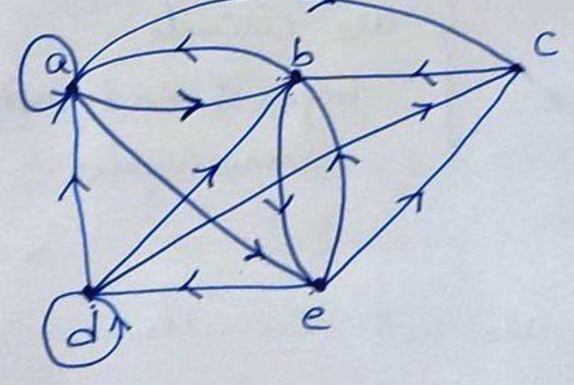
→ The empty set of edges as a path of length zero from a to a.

→ A path of length $n \geq 1$ that begins & ends at same vertex is called a circuit or cycle.

→ A path in a directed graph can pass through a vertex more than once.

→ An edge in a directed graph can occur more than once in a path.

<u>Ex:3.</u> Which of the following are paths in directed graph shown in below figure a, b, e, d; a, e, c, d, b; b, a, c, b, a, a, b; d, c; c, b, a; e, b, a, b, a, b, e? What are the lengths of those that are paths? Which of the paths in this list are circuits?



<u>Sol:</u> a, b, e, d.

(a, b), (b, e), (e, d) are edges from a to d.

So length of path is 3.

→ a, e, c, d, b.

here d → c is an edge but c → d is not an edge.

So, a, e, c, d, b is not a path.

→ b, a, c, b, a, a, b;

(b, a), (a, c), (c, b), (b, a), (a, a), (a, b).

length. 5

→ (d, c) is an edge. for d, c so length 1.

→ c, b, a.

(c, b), (b, a)

length 2.

→ e, b, a, b, a, b, e.

(e, b), (b, a), (a, b), (b, a), (a, b), (b, e)

So length - 6.

→ b, a, c, b, a, a, b & e, b, a, b, a, b, e } are circuits. begin & end with Same vertex..

a, b, e, d. c, b, a, d, c } are not circuits

Theorem 1: Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a,b) \in R^n$.

Proof: By Mathematical induction. we prove.

By definition, there is a path from a to b of length one if & only if $(a,b) \in R$.

So, true for $n=1$.

Inductive hypothesis: Assume it is true for positive integer n.

There is a path of length $n+1$ from a to b if & only if there is a element $c \in A$ such that there is a path of length one from a to c,

So, $(a,c) \in R$.

path of length n from c to b i.e., $(c,b) \in R^n$.

there is a path of length $n+1$ from a to b if & only if $(a,b) \in R^{n+1}$.

Transitive closures

→ Now finding the transitive closure of a relation is equivalent to determining which pair of vertices in the associated directed graph are connected by a path.

Def 2: Let R be a relation on a set A. The Connectivity relation R^* Consists of the pairs (a,b) such that there is a path of length at least one from a to b in R.

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

Ex:4. Let R be a relation on set of all people in the world that contains (a, b) if a has met b. What is R^n, where n is a positive integers greater than one? What is R^*?

Sol:- R^2 contains (a,b) if there is a person c such that $(a,c) \in R$ & $(c,b) \in R$.

i.e., if there is a person c such that a has met c and c has met b.

Illy R^n consists of those pairs (a,b) such that there are people $x_1, x_2 \cdots\cdots x_{n-1}$ such that a has met x_1, x_1 has met $x_2 \cdots$ & x_{n-1} has met b.

R^* contains (a,b) if sequence of people starts with a & ends with b, such that each person in sequence has met next person.

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i.e., if there is a person c such that a has met c and c has met b.

Illy R^n consists of those pairs (a,b) such that there are people $x_1, x_2 \ldots x_{n-1}$ such that a has met x_1, x_1 has met x_2 ... & x_{n-1} has met b.

R^* contains (a,b) if sequence of people starts with a & ends with b, such that each person in sequence has met next person.

Ex 5 Let R be the relation on set of all ~~sub~~ states in US that contains (a,b) if state a and state b have a common border. What is R^n, where n is a +ve integer? What is R^*?

Sol: R^n consists of pairs (a,b) i.e., it is possible to go from state 'a' to state b by crossing exactly n state borders

R^* consists of ordered pairs (a,b)

R^* – not there are those containing states that are not connected to continental United states.

Theorem: The transitive closure of a relation R equals the Connectivity relation R^*.

Proof: R^* Contains R by Definition

To show R^* is transitive closure of R.

we need to show that R^* is transitive &

$R^* \subseteq S$ whenever S is a transitive relation that Contains R.

-> 1st To show R^* is transitive.

If $(a,b) \in R^*$ and $(b,c) \in R^*$, then there are paths from a to b & from b to c in R.

we obtain a path from a to c, Hence $(a,c) \in R^*$.

So, R^* is transitive.

-> Suppose S is a transitive relation containing R.

$\because$ S is transitive, S^n is also transitive and.

$S^n \subseteq S$.

$$S^* = \bigcup_{k=1}^{\infty} S^k.$$

$S^k \subseteq S$.

So, $S^* \subseteq S$

if $R \subseteq S$, then $R^* \subseteq S^*$, b'coz any path in R is also a path in S.

$R^* \subseteq S^* \subseteq S$., So, any transitive relation that Contains R must also Contain R^*. $\therefore$ R^* is transitive closure of R.

LEMMA 1 Let A be a set with n elements, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. When $a \neq b$, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding $n-1$.

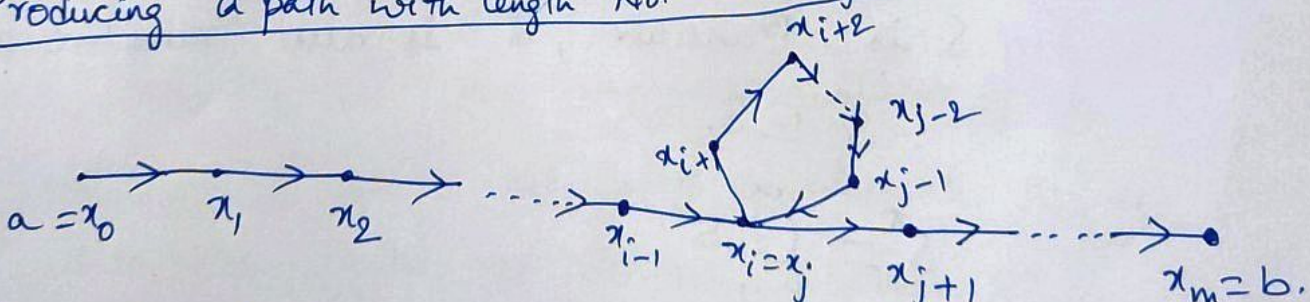
Proof: Suppose there is a path from a to b in R.

Let m be the length of the shortest. Such path.

Suppose $x_0, x_1, x_2 \dots x_{m-1}, x_m$. where $x_0 = a$ & $x_m = b$ is such a path.

Suppose that $a = b$ & $m > n$., so that $m \geq n+1$.

Producing a path with length Not exceeding n.



By Pigeonhole principle, b'coz there are n vertices in A among the m vertices $x_0, x_1, \dots x_{m-1}$, atleast two are equal.

Suppose $x_i = x_j$ with $0 \le i \le j \le m-1$.

Then the path contains a circuit from x_i to itself.

This circuit can be deleted from path a to b.

of shorter length. $x_0, x_1, \ldots, x_i, x_{j+1}, \ldots, x_{m-1}, x_m$.

Hence, the path of shortest length must have length less than or equal to n.

From lemma 1, we see that transitive closure of R is union of $R, R^2, R^3, \ldots,$ and R^n.

This follows because there is a path in R^* between two vertices if & only if there is a path between these vertices in R^i, for some +ve integer i with $i \le n$.

B, coz $R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$.

Theorem 3 Let M_R be ~~zero~~ zero-one matrix of the relation R on a set with n elements. Then the zero-one matrix of transitive closure R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]}.$$

Ex: 7 Find the zero - one matrix of transitive closure of relation R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Sol: By Theorem 3. zero - one matrix of R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}.$$

because $M_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $M_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$M_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Algorithm 1: Procedure for Computing the Transitive closure.

procedure transitive closure (M_R : zero- one $n \times n$ matrix).

$A := M_R$
$B := A$.
for $i := 2$ to n
$A := A \odot M_R$
$B := B \vee A$.
return B {B is the zero- one matrix for R^*}

Warshall's

To find no. of bit operations used by algorithm 1. to determine the transitive closure of a relation.

-> Computing the Boolean powers $M_R, M_R^{[2]}, \ldots, M_R^{[n]}$. requires that $n-1$ Boolean products of $n \times n$ zero-one matrices be found.

-> Each of these Boolean products can be found. using $n^2(2n-1)$ bit operations.

=> Hence, products can be computed using

$n^2(2n-1)(n-1)$ bit operations.

-> To find M_{R^*} from n Boolean powers of M_R. $n-1$ joins of 0-1 (zero-one) matrices need to be found.

-> To compute each of these joins uses n^2 bit operations.

-> $(n-1)n^2$ bit operations used for computations

-> $\therefore$ when Algorithm 1 is used, the matrix of transitive closure of a relation on a set with n elements can be found using

$n^2(2n-1)(n-1) + (n-1)(n)^2 = 2n^3(n-1)$.

i.e., $O(n^4)$ bit operations.

Warshall's Algorithm

→ Warshall's Algorithm, is described in 1960 by Stephen Warshall, is a efficient method for computing the transitive closure of a relation.

→ Algorithm1 uses $2n^3(n-1)$ bit operations where as Warshall's algorithm using only $2n^3$ bit operations ~~to~~ finds the transitive closure.

→ Warshall's Algorithm is sometimes called Roy-Warshall algorithm., b'coz in 1959, Benard Roy described it.

→ Suppose R is a relation on a set with n elements.

→ let $V_1, V_2, ----V_n$ be an arbitary listing of these n elements.

→ ~~con~~ concept of interior vertices is used here.

if . $a, x_1, x_2, ----, x_{m-1}, b$ is a path, its interior vertices are $x_1, x_2, ---, x_{m-1}$.

→ ~~inter~~ interior vertices of a path a, c, d, f, g, h, b, j in a directed graph are c, d, f, g, h, b.

→ 1st vertex in the path is not an interior vetex unless it is visited again by the path, except as the last vertex.

→ The last vertex in path is not an interior vertex unless it was visited previously by the path, except as first vertex.

→ Warshall's algorithm is based on construction of a Sequence of zero-one matrices.

→ These matrices are $W_0, W_1, \ldots W_n$, where $W_0 = M_R$. is the zero-one matrix of this relation & $W_K = [w_{ij}^{(K)}]$. where $w_{ij}^{(K)} = '1'$. if there is a path from V_i to V_j such that all the interior vertices of this path are in the set $\{V_1, V_2 \ldots V_K\}$. & is '0' otherwise.

→ $W_n = M_{R^*}$ because $(i,j)^{th}$ entry of M_{R^*} is 1 iff there is a path from V_i to V_j with all interior vertices in set $\{V_1, V_2, \ldots V_n\}$.

→ The last vertex in path is not an interior vertex unless it was visited previously by the path, except as first vertex.

→ Warshall's algorithm is based on construction of a Sequence of zero-one matrices.

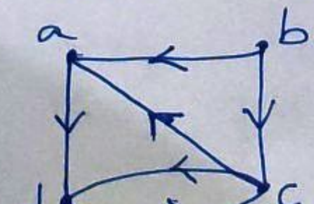
→ These matrices are $W_0, W_1, \ldots W_n$, where $W_0 = M_R$. is the zero-one matrix of this relation & $W_K = [W_{ij}^{(K)}]$. where $W_{ij}^{(K)} = 1$. if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2 \ldots v_K\}$. & is '0' otherwise.

→ $W_n = M_{R^*}$ because $(i,j)^{th}$ entry of M_{R^*} is 1 iff there is a path from v_i to v_j with all interior vertices in set $\{v_1, v_2, \ldots v_n\}$.

Ex: 8

Let R be the relation with directed graph below. Let a, b, c, d be a listing of the elements of the set. Find the matrices W_0, W_1, W_2, W_3 & [illegible] The matrix W_4 is transitive closure of R.

Sol:



Sol: Let $v_1 = a$, $v_2 = b$, $v_3 = c$, & $v_4 = d$. W_0 is matrix of relation.

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

W_1 has 1 as its $(i,j)^{th}$ entry if there is a path from v_i to v_j that has only $v_1 = a$ as an interior vertex.

path from b to d. i.e., b, a, d.

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

W_2 has 1 as its $(i,j)^{th}$ entry if there is a path. from v_i to v_j that has only $v_1 = a$ and/or $v_2 = b$. as its interior vertices, if any.

there are no edges with b as terminal vertex.. no new paths.

So $W_2 = W_1$.

W_3 has 1 as its $(i,j)^{th}$ entry if there is a path from v_i to v_j that has only $v_1 = a$, $v_2 = b$ &/or $v_3 = c$. as its interior vertices

So, we have d, c, a. & d, c, a, d.

$$\text{So, } W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Finally W_4 has 1 as its $(i,j)^{th}$ entry if there is a path from v_i to v_j that has $v_1 = a$, $v_2 = b$, $v_3 = c$. &/or $v_4 = d$. as interior vertices.

there are all vertices of graph this entry is 1 iff there is a path from v_i to v_j.

$$So, \quad W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

~~This is~~ W_4 is the matrix of transitive closure.

Warshall algorithm computes M_{R^*} efficiently by computing $W_0 = M_R$, $W_1, W_2 \ldots W_n = M_{R^*}$

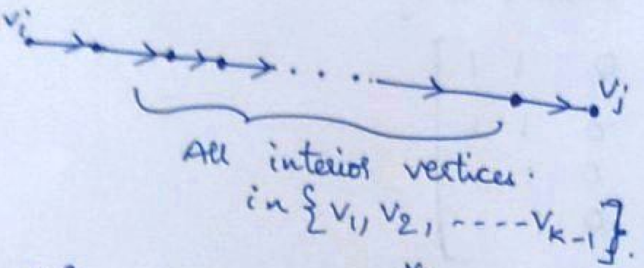
So, we can compute W_k directly from W_{k-1}

There is a path from v_i to v_j with no vertices other than $v_1, v_2, \ldots v_k$ as interior vertices iff either there is a path from v_i to v_j with its interior vertices among the first $k-1$ vertices in list or there are paths from v_i to v_k & from v_k to v_j that have interior vertices only among the 1^{st} $k-1$ vertices in list.

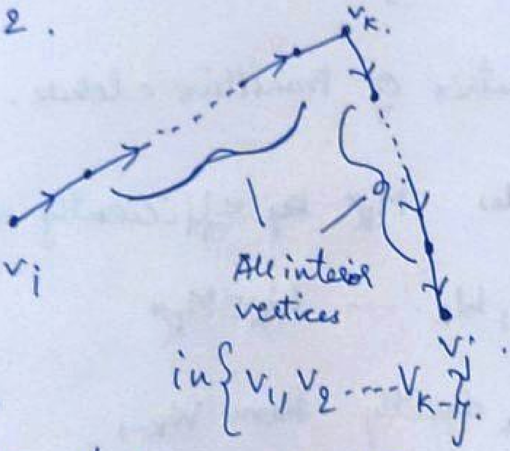
i.e., either a path from v_i to v_j already existed before v_k was permitted as an interior vertex, & allowing v_k as an interior vertex produces a path that goes from v_i to v_k & then from v_k to v_j.

The first type of path exists iff. $W_{ij}^{[k-1]} = 1$ &
the Second type of path exists iff $W_{ik}^{[k-1]}$ & $W_{kj}^{[k-1]}$ are 1.
So, $W_{ij}^{[k]} = 1$ iff either $W_{ij}^{[k-1]}$ is 1 or both $W_{ik}^{[k-1]}$ & $W_{kj}^{[k-1]}$ are '1'.

Case 1.



Case 2.



LEMMA.

~~Lemma 2~~ let $W_k = [W_{ij}^{[k]}]$ be zero-one matrix that has a 1 in its $(i,j)^{th}$ position iff there is a path from v_i to v_j with interior vertices from set $\{v_1, v_2 ---- v_k\}$ then.

$$W_{ij}^{[k]} = W_{ij}^{[k-1]} \vee \left(W_{ik}^{[k-1]} \wedge W_{kj}^{[k-1]}\right).$$

i, j, k are +ve integers not exceeding n.

~~Lemma~~ Algorith 2 Warshall Algorithm.

procedure Warshall (M_R: $n \times n$ zero-one Matrix).
$W := M_R$
for $k := 1$ to n.
for $i := 1$ to n
for $j := 1$ to n.
$w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$.
return W $\{W = [w_{ij}]$ is $M_R^*\}$

Total no. of bit operation used is $n \cdot 2n^2 = 2n^3$.

To find. $w_{ij}^{[k]}$ using lemma 2 requires 2 bit operations. To find all n^2 entries of W_k needs $2n^2$ bit operations.

Equivalence Relations

Def1: A relation on a set A is called an equivalence relation if it is reflexive, and transitive.

Def2: Two elements a and b that are related by an equivalence relation are called equivalent. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relations.

To make the sense notion of equivalent elements,

- Every element should be equivalent to itself, as the reflexive property guarantees for an equivalence relation.
- we can say a is related to b, by Symmetric property, b is related to a.
- B.coz an equivalence relation is transitive if a & b are equivalent & b & c are equivalent, so a & c are equivalent.

Ex:1 Let R be the relation on set of integers such that aRb if and only if $a=b$ or $a=-b$.

Ex: 2 Let R be the relation on set of real numbers such that aRb if & only if $a-b$ is an integer. IS R an equivalence relation?

Sol: $\because$ $a-a=0$ is an integer for all real numbers, a,

aRa for all real numbers a. So, R is reflexive.

→ Suppose aRb, then $a-b$ is an integer & $b-a$ is also an integer.

$\therefore$ bRa, R is Symmetric.

→ If aRb & bRc, then $a-b$ & $b-c$ are integers.

$\therefore$ $a-c=(a-b)+(b-c)$ is also an integer.

$\therefore$ aRc. So, R is transitive.

$\therefore$ R is an equivalence Relation.

Ex: 3 Congruence Modulo m: let m be an integer with $m>1$. show that relation $R=\{(a,b) \mid a \equiv b \pmod m\}$ is an equivalence relation on set of integers.

→ $a \equiv b \pmod m$ iff m divides $(a-b)$.

$a-a=0$ is divisible by m, $\therefore$ $a \equiv a \pmod m$, is reflexive.

→ $a \equiv b \pmod m$, $a-b$ is divisible by m, so

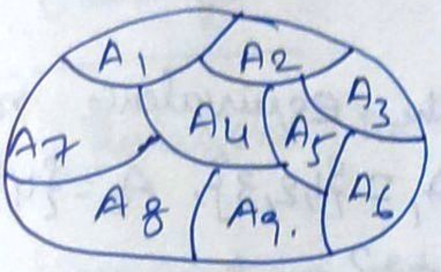
$a-b=Km$, where K is an integer.

$(b-a)=(-K)m$, So $b \equiv a \pmod m$. So it is Symmetric.

→ $a \equiv b \pmod m$ & $b \equiv c \pmod m$, Then m divides both $a-b$ & $b-c$. $a-b=Km$ & $b-c=lm$.

$a-c=(a-b)+(b-c)=Km+lm=(K+l)m$.

i.e., $a \equiv c \pmod m$. i.e. transitive.



Ex:12

Suppose that $S = \{1,2,3,4,5,6\}$. The collection of sets $A_1 = \{1,2,3\}$

$A_2 = \{4,5\}$

$A_3 = \{6\}$. forms a partition of S.

These sets are disjoint & their union is S.

Theorem 2.

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i / i \in I\}$ of the set S, there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes.

Examples:

List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1,2,3\}$, $A_2 = \{4,5\}$. $A_3 = \{6\}$ of $S = \{1,2,3,4,5,6\}$.

Sol: Subsets in partition are equivalence classes of R.

$(a,b) \in R$ iff a & b are in same subset of partition

So., $(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)$ belong to R. $\because A_1 = \{1,2,3\}$ is an equivalence class.

Illy $(4,4), (4,5), (5,4), (5,5) \in R$ $\because A_2 = \{4,5\}$ is an equivalence class.

Illy $(6,6) \in R$ $\because A_3 = \{6\}$ is an equivalence class.

Partial Orderings

→ we use Relations to order some or all of the elements of Sets.

→ we will put in order words using relation containing pairs of words (x, y) where x comes before y in dictionary.

Def: A relation R on a set S is called a partial ordering or partial order if it is reflexive, antisymmetric & transitive.

- A Set S together with a partial ordering R is called a partially ordered set or poset and is denoted by (S, R).
- Members of S are called elements of the poset.

Ex!1 Show that the "greater than or equal" relation $(\geq)$ is a partial ordering on the set of ~~strings~~ integers.

$\because$ $a \geq a$ for every integer a, $\geq$ is reflexive.

If $a \geq b$ & $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric

Finally, $\geq$ is transitive b'coz $a \geq b$ & ~~bc~~ $b \geq c$ imply that $a \geq c$. It follows that $\geq$ is a partial ordering.

on the Set of integers & $(\mathbb{Z}, \geq)$ is a poset.

Ex:3 Show that the inclusion relation $\subseteq$ is a partial ordering on power set of a set S.

Sol:- $\because$ $A \subseteq A$ when A is a Subset of S.

$\subseteq$ is reflexive.

if $A \subseteq B$ & $B \subseteq A \rightarrow A = B$., So it is antisymmetric.

if $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$., So transitive

$\therefore$ $\subseteq$ is a partial ordering on $P(S)$ &

$(P(S), \subseteq)$ is a poset.

Ex:4 let R be the relation on the set of people such that $x R y$ if x & y are people and x is older than y. Show that R is not a partial ordering.

$\rightarrow$ if a person x is older than a person y, then y is not older than x.

i.e., if $x R y$, then $y \not R x$.

- if person x is older than person y and y is older than person z, then x is older than z.

i.e, xRy, yRz, then xRz., So it is ~~antisymmetric~~. Transitive.

- But it is not reflexive. $\because$ no person is older than him self or herself. i.e., $x \not R x$.

$\therefore$ R is not a partial ordering.

→ $a \preccurlyeq b$ is used to denote that $(a,b) \in R$. in an arbitary poset (S, R).

Def:2 The elements a and b of a poset $(S, \preccurlyeq)$ are Called Comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called incomparable.

Ex In poset $(Z^+, |)$ are integers 3 & 9 are comparable? Are 5 & 7 comparable?

→ The integers 3 & 9 are comparable ∵ $9/3$.

∴ but 5 & 7 are incomparable ∵ $5 \nmid 7$ & $7 \nmid 5$.

→ Adjective 'Partial' is used to describe partial orderings because pairs of elements may be incomparable.

→ When every 2 elements in set are comparable, the relation is called a "<u>Total ordering</u>".

Def: 3 If $(S, \preccurlyeq)$ is a poset and every 2 elements of S are comparable, S is called a 'totally ordered' or linearly ordered set. & $\preccurlyeq$ is called a 'total order' or a 'linear order'

→ A totally ordered set is also called a chain.

Ex:6 poset $(Z, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a & b are integers.

Ex:7 poset $(Z^+, |)$ is not totally ordered.

$\because$ it contains elements that are incomparable like 5 & 7.

Def:4 $(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Ex: Set of ordered pairs of positive integers $Z^+ \times Z^+$ with $(a_1, a_2) \preccurlyeq (b_1, b_2)$ if $a_1 < b_1$ or if $a_1 = b_1$ & $a_2 \leq b_2$ (lexicographic ordering) is a well ordered set.

Theorem 1

Principle of Well-ordered Induction

Suppose that S is a well ordered set. Then $P(x)$ is true for all $x \in S$, if.

Inductive Step: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

Proof by Contradiction.

we assume. $P(x)$ is true for all $x \in S$ is not the case.

Then $y \in S$ and $P(y)$ is false.

Consequently

Set $A = \{x \in S \mid P(x) \text{ is false}\}$ is non empty

$\because$ S is well ordered, A has a least element a.

So, $P(x)$ is true for all $x \in S$ with $x \prec a$.

$\Rightarrow$ $P(a)$ is true.

Shows that $P(x)$ must be true for all $x \in S$.

Lexicographic Order

→ Special Case of an ordering of strings on a set. Constructed from a partial ordering on set.

How to Construct, a partial ordering on the Cartesian product of two posets, $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$.

Lexicographic ordering $\preccurlyeq$ on $A_1 \times A_2$.

$$(a_1, a_2) \prec (b_1, b_2).$$

either if $a_1 \prec_1 b_1$ or if both $a_1 = b_1$ & $a_2 \prec_2 b_2$.

We obtain a partial ordering $\preccurlyeq$ by adding equality to ordering $\prec$ on $A_1 \times A_2$.

Ex: Determine whether $(3,5) \prec (4,8)$, ~~whether~~ whether $(3,8) \prec (4,5)$.

whether $(4,9) \prec (4,11)$ in poset $(Z \times Z, \preccurlyeq)$.

where $\preccurlyeq$ is the lexicographic ordering constructed from the usual $\leq$ relation on Z.

→ $\because 3 < 4 \rightarrow (3,5) \prec (4,8)$ & $(3,8) \prec (4,5)$.

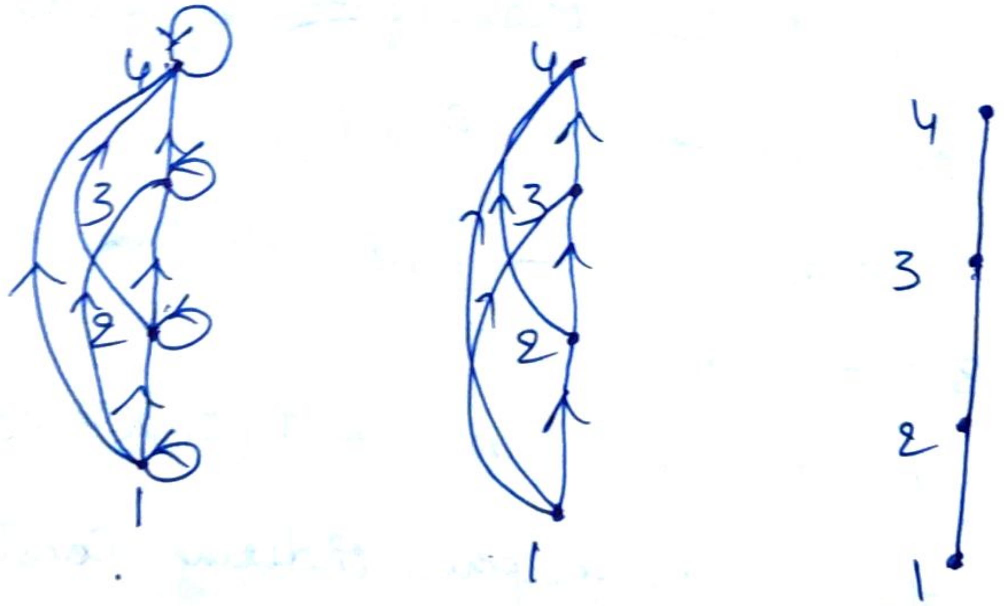
$(4,9) \prec (4,11)$. $\because 4 = 4$, $9 < 11$.

Hasse Diagrams

Consider the directed graph for partial orderings $\{(a,b) \mid a \le b\}$ on set $\{1,2,$

∵ the partial ordering is reflexive & its directed graph has loops at all vertices we do not have to show these loops. b'coz they must be present.

∵ the partial ordering is transitive we do not have to show those edges that must be present for transitivity.



If we assume that all edges are pointed "upward", we do not have to show the directions of edges.

we represent a finite poset $(S, \preceq)$ using the procedure:

-> Start with directed graph for the relation.

-> $\because$ the partial ordering is reflexive, a loop (a,a) is present at every vertex a, Remove these loops.

-> Next remove all edges that must be in partial ordering b'coz of presence of other edges and transitivity.

-> Remove all edges (x,y) for which there is an element $z \in S$ such that $x \prec z$ & $z \prec x$.

-> Finally arrange each edge so that its initial vertex is below its terminal vertex.

-> Remove all arrows on the directed edges, $\because$ all edges point "upward" toward their terminal vertex.

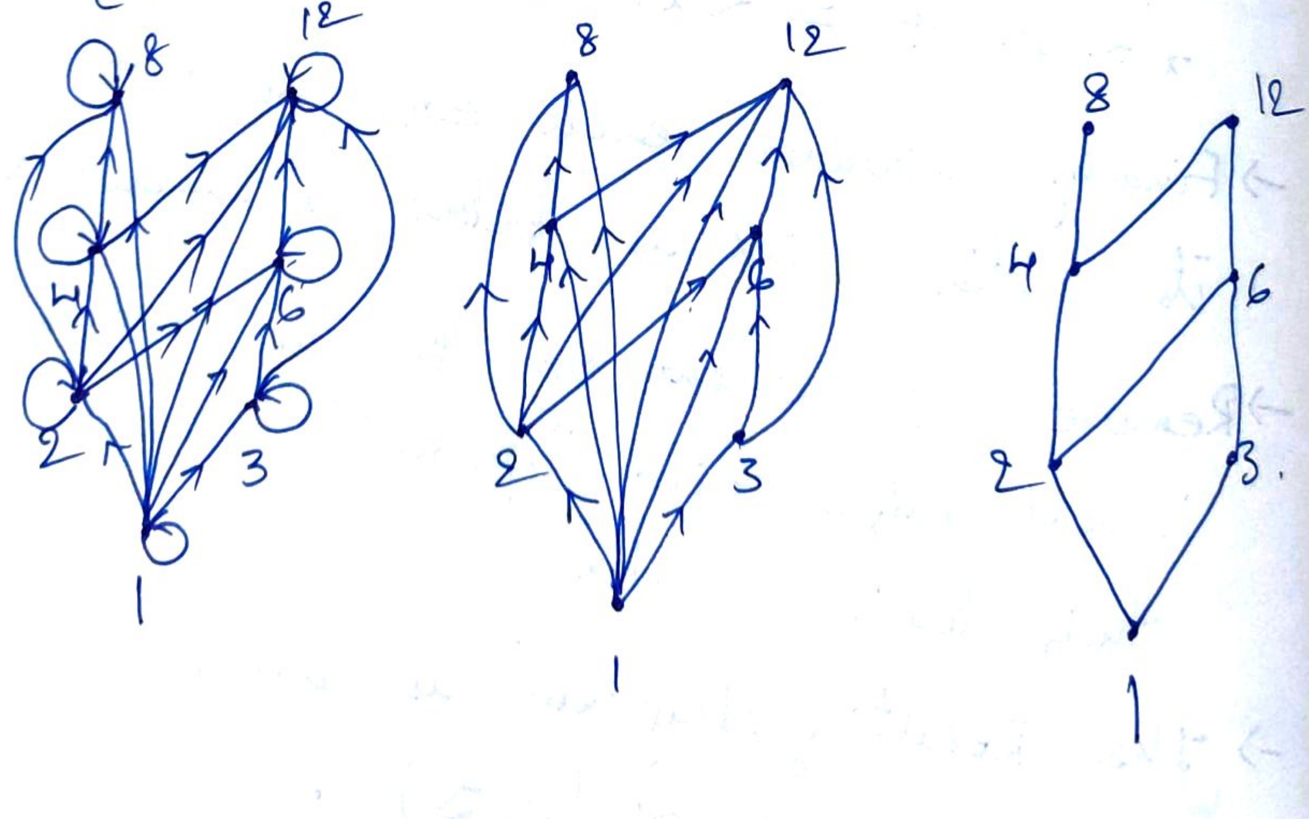
-> The Resulting diagram is called Hasse diagram of $(S, \preceq)$.

Let $(S, \leq)$ be a poset.

→ $y \in S$ covers an element $x \in S$ if $x < y$ & there is no element $z \in S$ such that $x < z < y$.

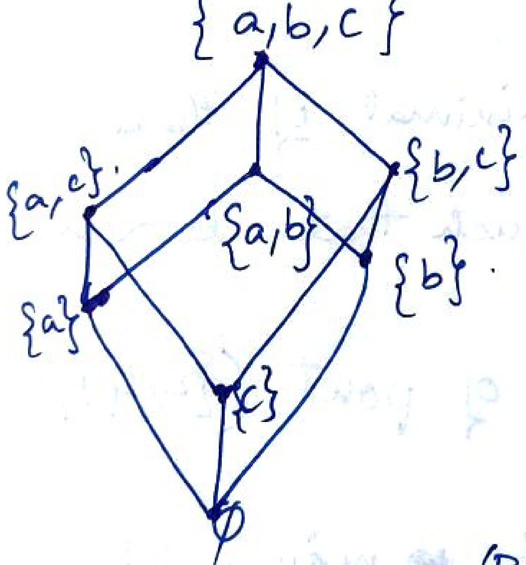
→ Set of pairs (x, y) such that y covers x is called covering relation of $(S, \leq)$.

Ex: Draw the Hasse Diagram representing the partial ordering $\{(a,b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.



Ex Draw the Hasse diagram for partial ordering $\{(A,B) \mid A \subseteq B\}$ on power set $P(S)$ where $S = \{a,b,c\}$.

→ Delete $(\phi, \{a,b\})$, $(\phi, \{a,c\})$,
$(\phi, \{b,c\})$, $(\phi, \{a,b,c\})$.
$(\{a\}, \{a,b,c\})$, $(\{b\}, \{a,b,c\})$.
$(\{c\}, \{a,b,c\})$.



Hasse diagram of $(P\{a,b,c\}, \subseteq)$

Maximal and Minimal elements

→ An element of a poset is called maximal if it is not less than any element of poset.

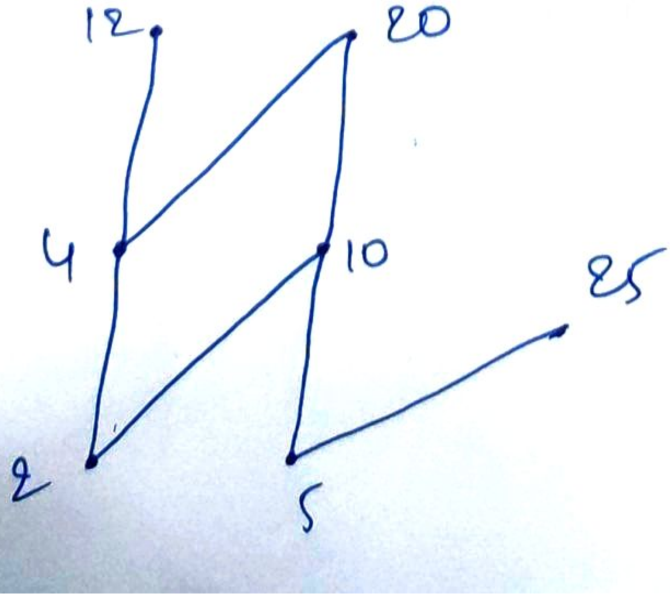
i.e., a is maximal in poset $(S, \preceq)$

if there is no $b \in S$ such that $a < b$.

→ An element of a poset is called minimal if it is not greater than any element of poset.

i.e., a is minimal if there is no element $b \in S$ such that $b < a$.

Ex:: Which elements of poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal & minimal?



The minimal elements are 2 & 5.
maximal elements are 12, 20, 25.

a poset can have more than one maximal & minimal element.

→ There is an element in a poset that is greater than every other element. Called the greatest element.

i.e., a is greatest element of poset $(S, \preceq)$.

if $b \preceq a$ for all $b \in S$.

→ Greatest element is unique when it exists.

→ An Element is Called least element if it is less than all other elements in poset.

i.e., a is least element of $(S, \preceq)$ if $a \preceq b$ for all $b \in S$.

Least element is unique when it exists.

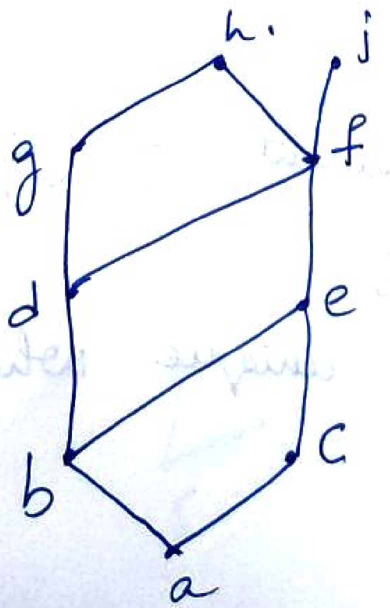
Least upper bound.

→ x is called least upper bound of subset A if x is an upper bound that is less than every other upper bound of A.

x is least upper bound of A if $a \preceq x$ whenever $a \in A$ & $x \preceq z$ whenever z is an upper bound of A. $lub(A)$

→ element y is called the greatest lower bound of A if y is a lower bound of A & $z \preceq y$ whenever z is a lower bound of A. $glb(A)$

Ex: Find the lower & upper bounds of subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with Hasse diagram below.



upper bounds of $\{a,b,c\}$ are e, f, j, h.

lower bound is a.

→ There is no upperbounds of $\{j, h\}$.

→ & lower bounds of $\{j, h\}$. a, b, c, d, e, f.

– upperbounds of $\{a, c, d, f\}$ are f, h, & j.

→ lower bound " is a.

Ex: Find the glb and lub of $\{b, d, g\}$ if they exist in above poset.

The upperbounds of $\{b, d, g\}$ are g & h.

$\therefore$ $g < h$. ~~glb~~ lub – g. lub – g.

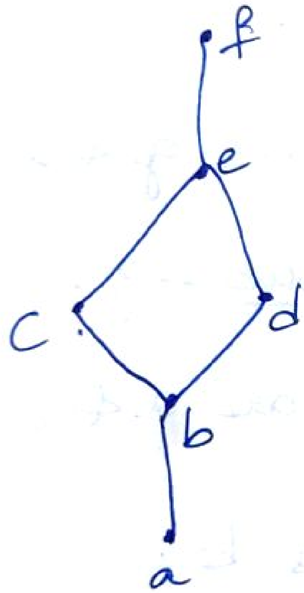
lower bounds of $\{b, d, g\}$ are a & b.

$\therefore$ $a < b$. glb is b.

Lattices

→ A partially ordered set in which every pair of elements has both a lub & a glb is called a lattice.

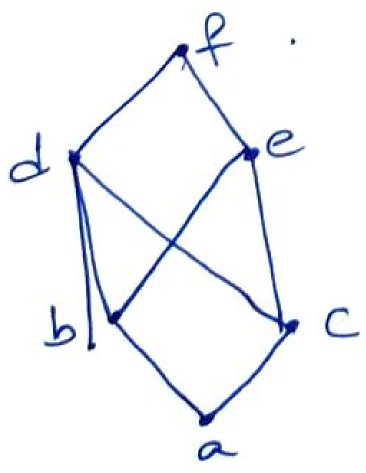
Ex:- Determine whether the posets represented by each of the Hasse diagram below are lattices.



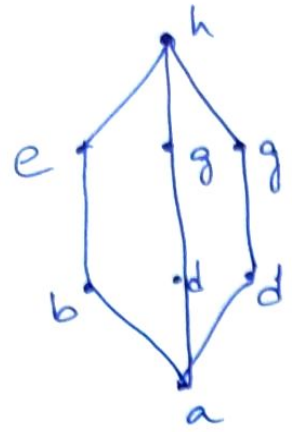
is lattice b'coz there is a lub & glb.

lub → ~~f~~e

glb → ~~a~~b.



is not a lattice there are not lub & g'b.



is a lattice.

Topological Sorting

A total ordering $\leq$ is said to be compatible with the partial ordering R if $a \leq b$. whenever $a R b$.

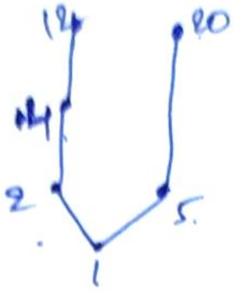
Constructing a compatible total ordering from a partial ordering is called "topological sorting"

LEMMA 1 Every finite nonempty poset $(S, \preceq)$ has at least one minimal element.

Ex: Find a compatible total ordering for poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

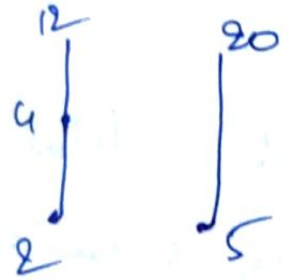
1* → choose a minimal element.

choose - 1



2 → choose a minimal element from $(\{2, 4, 5, 12, 20\}, |)$

2 & 5 are minimal elements.



we select 5.

So remaining elements $(\{2, 4, 12, 20\}, |)$

So the minimal element is 2.

Next select minimal element from {4, 12, 20}.

4 is minimal element.

Next. 12 & 20 are minimal elements of ({12, 20}, |

1^{st} 20 is choosen. 12 20

12 is the last element. 12.

$$\boxed{1 < 5 < 2 < 4 < 20 < 12}$$