→ The last vertex in path is not an interior vertex unless it was visited previously by the path, except as first vertex.

→ Warshall's algorithm is based on construction of a Sequence of zero-one matrices.

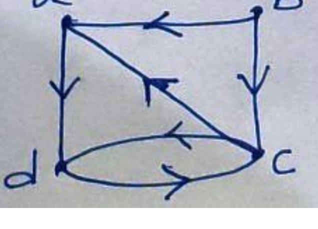
→ These matrices are $W_0, W_1, \ldots W_n$, where $W_0 = M_R$. is the zero-one matrix of this relation & $W_K = [w_{ij}^{(K)}]$. where $w_{ij}^{(K)} = $ '1'. if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2 \ldots v_K\}$. & is '0' otherwise.

→ $W_n = M_{R^*}$ because $(i,j)^{th}$ entry of M_{R^*} is $\underline{1}$ iff there is a path from v_i to v_j with all interior vertices in set $\{v_1, v_2, \ldots V_n\}$.

<u>Ex: 8</u>

Let R be the relation with directed graph. below. Let a,b,c,d be a listing of the elements of the set. Find the matrices W_0, W_1, W_2, W_3 & W_4. The matrix W_4 is transitive closure of R.

Sol:



Sol: Let $v_1 = a$, $v_2 = b$, $v_3 = c$, & $v_4 = d$. W_0 is matrix of relation.

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

W_1 has 1 as its $(i,j)^{th}$ entry if there is a path from v_i to v_j that has only $v_1 = a$ as an interior vertex.

path from b to d. i.e., b,a,d.

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

W_2 has 1 as its $(i,j)^{th}$ entry if there is a path. from v_i to v_j that has only $v_1 = a$ and/or $V_2 = b$. as its interior vertices, if any.

there are no edges with b as terminal vertex.. no new paths.

So $W_2 = W_1$.

W_3 has 1 as its $(i,j)^{th}$ entry if there is a path from v_i to v_j that has only $v_1 = a$, $v_2 = b$ &/or $v_3 = c$. as its interior vertices

So, we have d,c,a. & d,c,a,d.

$$\text{So, } W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Finally W_4 has 1 as its $(i,j)^{th}$ entry if there is a path from v_i to v_j that has $v_1 = a, v_2 = b, v_3 = c$. &/or $v_4 = d$. as interior vertices.

there are all vertices of graph this entry is 1 iff there is a path from v_i to v_j.

$$So,\ W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

~~This is~~ W_4 is the matrix of transitive closure.

Warshall algorithm computes M_{R^*} efficiently by computing $W_0 = M_R, W_1, W_2 \ldots W_n = M_{R^*}$

So, we can compute W_k directly from W_{k-1}

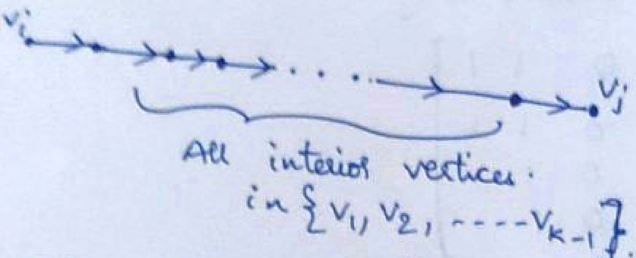
There is a path from v_i to v_j with no vertices other than $v_1, v_2 \ldots v_k$ as interior vertices iff either there is a path from v_i to v_j with its interior vertices among the first $k-1$ vertices in list or there are paths from v_i to v_k & from v_k to v_j that have interior vertices only among the 1^{st} $k-1$ vertices in list.

i.e., either a path from v_i to v_j already existed before v_k was permitted as an interior vertex, & allowing v_k as an interior vertex produces a path that goes from v_i to v_k & then from v_k to v_j.

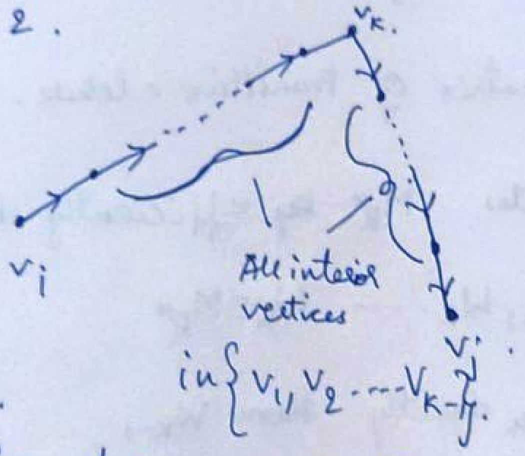
The first type of path exists iff. $w_{ij}^{[k-1]} = 1$ & the Second type of path exists iff $w_{ik}^{[k-1]}$ & $w_{kj}^{[k-1]}$ are 1.

So, $w_{ij}^{[k]} = 1$ iff either $w_{ij}^{[k-1]}$ is 1 or both $w_{ik}^{[k-1]}$ & $w_{kj}^{[k-1]}$ are '1'.

Case 1.



Case 2.



LEMMA 2.

~~Lemma 2~~ let $W_k = [w_{ij}^{[k]}]$ be zero-one matrix that has a 1 in its $(i,j)^{th}$ position iff there is a path from v_i to v_j with interior vertices from Set $\{v_1, v_2 ---- v_k\}$ then.

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee \left(w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]} \right).$$

i, j, k are +ve integers not exceeding n.

~~Lemma~~ Algorithm 2 Warshall Algorithm.

```
procedure Warshall (M_R: nxn zero-one Matrix).
  W: = M_R
  for k: = 1 to n.
     for i: = 1 to n
        for j: = 1 to n.
          w_ij := w_ij v (w_ik ^ w_kj).
return W {W = [w_ij] is M_R*}
```

Total no. of bit operations used is $n \cdot 2n^2 = 2n^3$.

To find. $w_{ij}^{[k]}$ using lemma 2 requires 2 bit operations. To find all n^2 entries of W_k needs $2n^2$ bit operations.