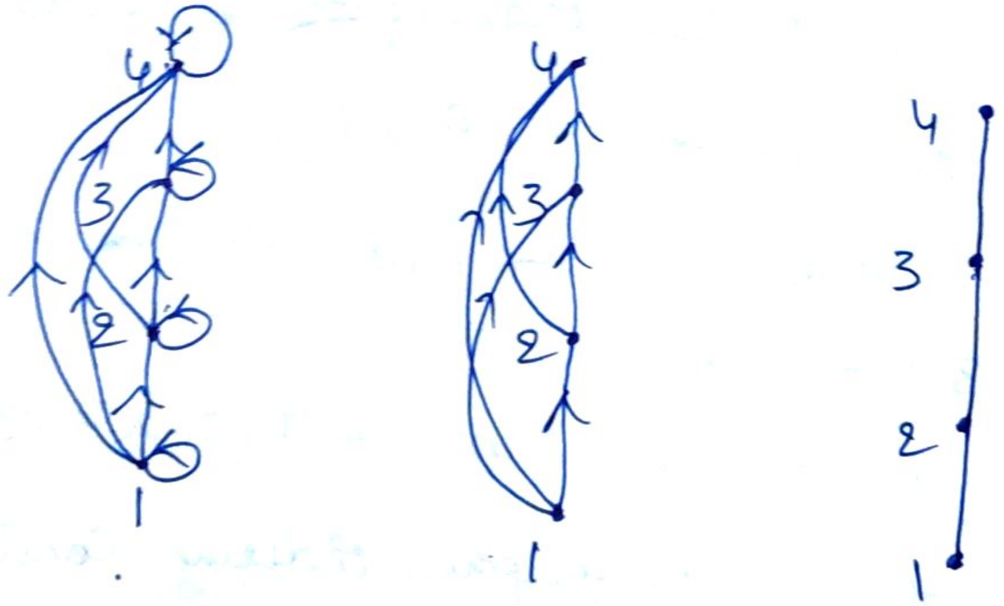
Hasse Diagrams

Consider the directed graph for partial orderings $\{(a,b) \mid a \leq b\}$ on set $\{1,2,\ldots$

∵ the partial ordering is reflexive & its directed graph has loops at all vertices we do not have to show these loops. b'coz they must be present.

∵ the partial ordering is transitive we do not have to show those edges that must be present for transitivity.



If we assume that all edges are pointed "upward", we do not have to show the directions of edges.

we represent a finite poset $(S, \preceq)$ using the procedure:

-> Start with directed graph for the relation.

-> $\because$ the partial ordering is reflexive, a loop (a,a) is present at every vertex a, Remove these loops.

-> Next remove all edges that must be in partial ordering b'coz of presence of other edges and transitivity.

-> Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ & $z \prec x$.

-> Finally arrange each edge so that its initial vertex is below its terminal vertex.

-> Remove all arrows on the directed edges, $\because$ all edges point "upward" toward their terminal vertex.

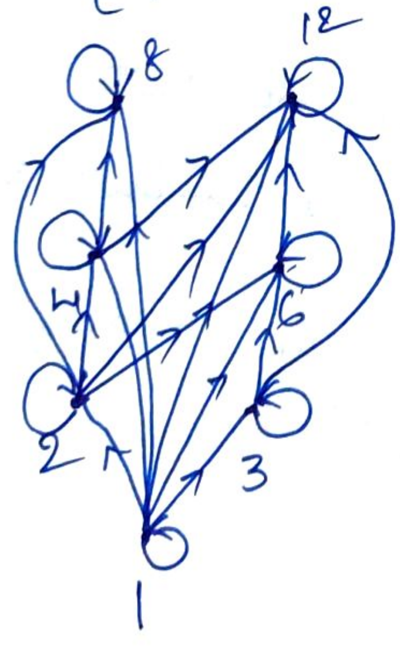
-> The Resulting diagram is called Hasse diagram of $(S, \preceq)$.

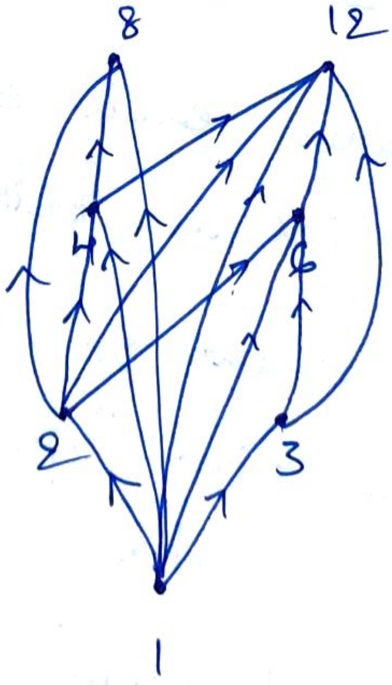
Let $(S, \leq)$ be a poset.

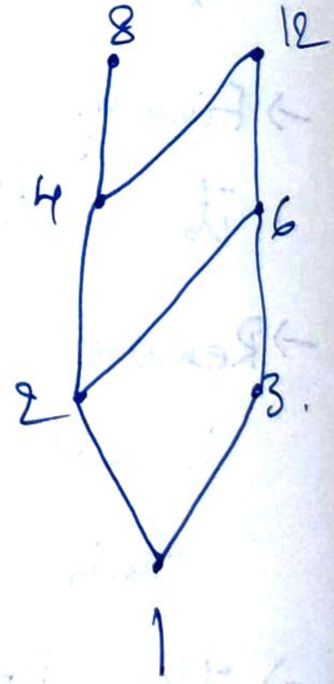
→ $y \in S$ covers an element $x \in S$ if $x < y$ & there is no element $z \in S$ such that $x < z < y$.

→ Set of pairs (x, y) such that y covers x is called covering relation of $(S, \leq)$.

Ex: Draw the Hasse Diagram representing the partial ordering $\{(a,b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.

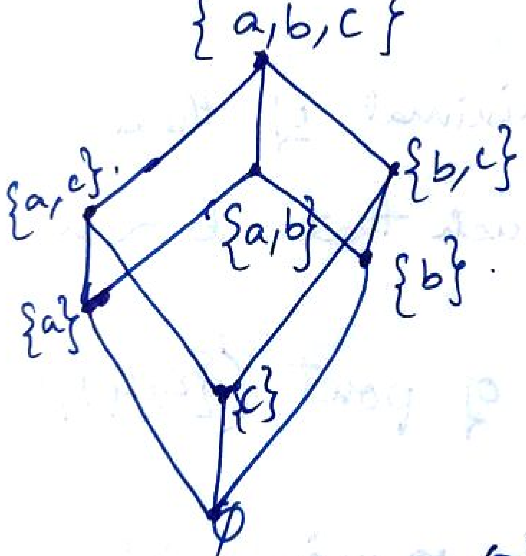






Ex Draw the Hasse diagram for partial ordering $\{(A,B) \mid A \subseteq B\}$ on power set $P(S)$ where $S = \{a,b,c\}$.

→ Delete $(\phi, \{a,b\}), (\phi, \{a,c\}),$
$(\phi, \{b,c\}), (\phi, \{a,b,c\}).$
$(\{a\}, \{a,b,c\}), (\{b\}, \{a,b,c\}).$
$(\{c\}, \{a,b,c\}).$



Hasse diagram of $(P\{a,b,c\}, \subseteq)$

Maximal and Minimal elements

→ An element of a poset is called maximal if it is not less than any element of poset.

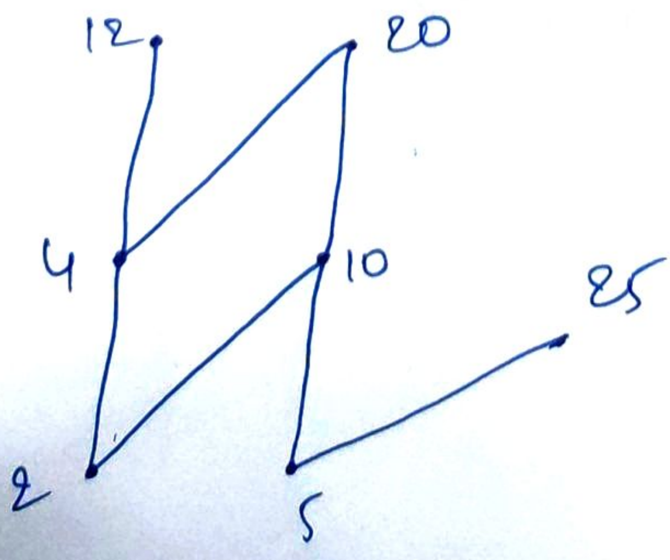
i.e., a is maximal in poset $(S, \preccurlyeq)$

if there is no $b \in S$ such that $a \prec b$.

→ An element of a poset is called minimal if it is not greater than any element of poset.

i.e., a is minimal if there is no element $b \in S$ such that $b \prec a$.

Ex:: Which elements of poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal & minimal?



The minimal elements are 2 & 5.

maximal elements are 12, 20, 25.

a poset can have more than one maximal & minimal element.

→ There is an element in a poset that is greater than every other element. Called the greatest element.

i.e., a is greatest element of poset $(S, \preceq)$ if $b \preceq a$ for all $b \in S$.

→ Greatest element is unique when it exists.

→ An Element is called least element if it is less than all other elements in poset.

i.e., a is least element of $(S, \preceq)$ if $a \preceq b$ for all $b \in S$.

Least element is unique when it exists.

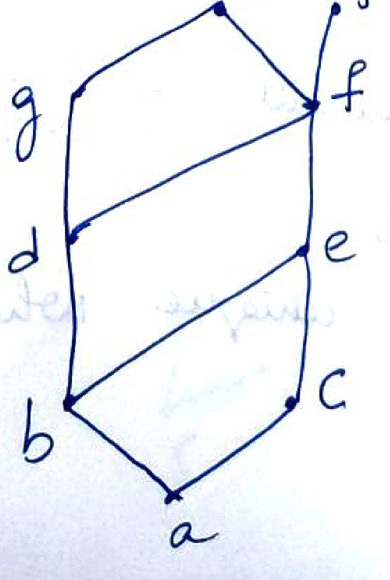
Least upper bound.

→ x is called least upper bound of subset A if x is an upper bound that is less than every other upper bound of A.

x is least upper bound of A if $a \preceq x$ whenever $a \in A$ & $x \preceq z$ whenever z is an upper bound of A. lub(A)

→ element y is called the greatest lower bound of A if y is a lower bound of A & $z \preceq y$ whenever z is a lower bound of A. glb(A)

Ex: Find the lower & upper bounds of subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with Hasse diagram below.



upper bounds of $\{a, b, c\}$ are e, f, j, h.

lower bound is a.

→ There is no upperbounds of $\{j, h\}$.

→ & lower bounds of $\{j, h\}$. a, b, c, d, e, f.

– upperbounds of $\{a, c, d, f\}$ are f, h, & j.

→ lower bound " is a.

Ex: Find the glb and lub of $\{b, d, g\}$ if they exist in above poset.

The upperbounds of $\{b, d, g\}$ are g & h.

$\therefore$ $g < h$. ~~glb~~ lub – g. lub – g.

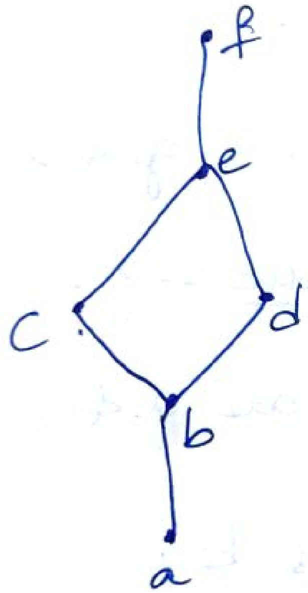
lower bounds of $\{b, d, g\}$ are a & b.

$\therefore$ $a < b$. glb is b.

Lattices

→ A partially ordered set in which every pair of elements has both a lub & a glb is called a lattice.

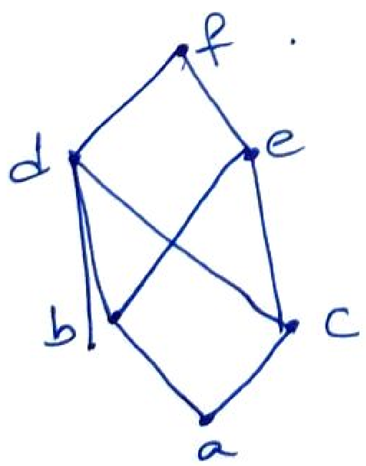
Ex:- Determine whether the posets represented by each of the Hasse diagram below are lattices.



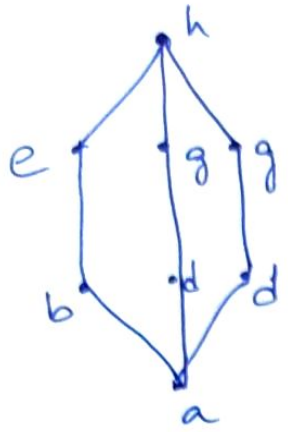
is lattice b'coz there is a lub & glb.

lub → ~~f~~e

glb → ~~a~~b.



is not a lattice there are not lub & glb.



is a lattice.

Topological Sorting

A total ordering $\leq$ is said to be compatible with the partial ordering R if $a \leq b$. whenever $a R b$.

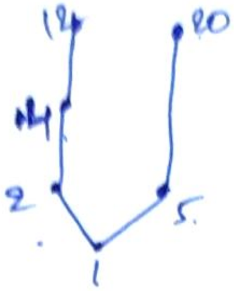
Constructing a compatible total ordering from a partial ordering is called "topological sorting"

LEMMA 1 Every finite nonempty poset $(S, \preccurlyeq)$ has at least one minimal element.

Ex: Find a compatible total ordering for poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

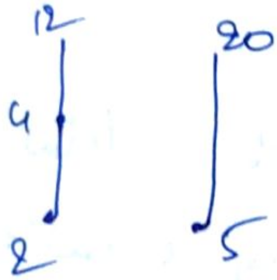
1 → choose a minimal element.

choose - 1



2 → choose a minimal element from $(\{2, 4, 5, 12, 20\}, |)$

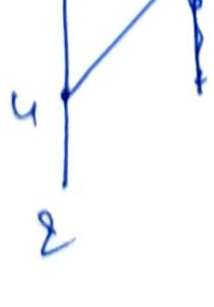
2 & 5 are minimal elements.



we select 5.

So remaining elements $(\{2, 4, 12, 20\}, |)$

So the minimal element is 2.



Next select minimal element from $\{4, 12, 20\}$.

4 is minimal element.

Next. 12 & 20 are minimal elements of $(\{12, 20\}, |$

1^{st} 20 is choosen. 12 20

12 is the last element. 12.

$$\boxed{1 < 5 < 2 < 4 < 20 < 12.}$$