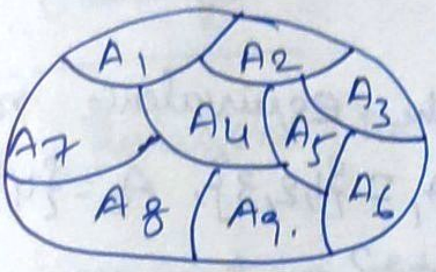

Ex:12

Suppose that $S = \{1,2,3,4,5,6\}$. The collection of sets $A_1 = \{1,2,3\}$

$A_2 = \{4,5\}$

$A_3 = \{6\}$. forms a partition of S.

These sets are disjoint & their union is S.

Theorem 2.

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i / i \in I\}$ of the set S, there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes.

Examples:

List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1,2,3\}$, $A_2 = \{4,5\}$, $A_3 = \{6\}$ of $S = \{1,2,3,4,5,6\}$.

Sol: Subsets in partition are equivalence classes of R.

$(a,b) \in R$ iff a & b are in same subset of partition

So., $(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)$ belong to R. $\because A_1 = \{1,2,3\}$ is an equivalence class.

Illy $(4,4), (4,5), (5,4), (5,5) \in R$ $\because A_2 = \{4,5\}$ is an equivalence class.

Illy $(6,6) \in R$ $\because A_3 = \{6\}$ is an equivalence class.

Partial Orderings

→ We use Relations to order some or all of the elements of Sets.

→ We will put in order words using relation containing pairs of words (x, y) where x comes before y in dictionary.

Def: A relation R on a set S is called a partial ordering or partial order if it is reflexive, antisymmetric & transitive.

- A Set S together with a partial ordering R is called a partially ordered set or poset and is denoted by (S, R).

- Members of S are called elements of the poset.

Ex:1) Show that the "greater than or equal" relation $(\geq)$ is a partial ordering on the set of ~~strings~~ integers.

$\because$ $a \geq a$ for every integer a, $\geq$ is reflexive.

If $a \geq b$ & $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric

Finally, $\geq$ is transitive b'coz $a \geq b$ & ~~b~~ $b \geq c$ imply that $a \geq c$. It follows that $\geq$ is a partial ordering.

on the Set of integers & $(\mathbb{Z}, \geq)$ is a poset.

Ex:3 Show that the inclusion relation $\subseteq$ is a partial ordering on power set of a set S.

Sol:- $\because A \subseteq A$ when A is a Subset of S.

$\subseteq$ is reflexive.

if $A \subseteq B$ & $B \subseteq A \rightarrow A = B$., So it is antisymmetric

if $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$., So transitive

$\therefore \subseteq$ is a partial ordering on $P(S)$ &

$(P(S), \subseteq)$ is a poset.

Ex:4 Let R be the relation on the set of people such that xRy if x & y are people and x is older than y. Show that R is not a partial ordering.

$\rightarrow$ if a person x is older than a person y, then y is not older than x.

i.e., if xRy, then $y \not R x$.

- if person x is older than person y and y is older than person z, then x is older than z.

i.e, xRy, yRz, then xRz., So it is ~~antisymmetric~~. Transitive.

- But it is not reflexive. ∵ no person is older than him self or herself. i.e., $x \not R x$.

∴ R is not a partial ordering.

→ $a \preccurlyeq b$ is used to denote that $(a,b) \in R$ in an arbitary poset (S, R).

Def:2 The elements a and b of a poset $(S, \preccurlyeq)$ are called Comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called incomparable.

Ex In poset $(Z^+, |)$ are integers 3 & 9 are comparable? Are 5 & 7 comparable?

→ The integers 3 & 9 are comparable $\because 9/3$.

but 5 & 7 are incomparable

$\because 5 \nmid 7$ & $7 \nmid 5$.

→ Adjective 'Partial' is used to describe partial orderings because pairs of elements may be incomparable.

→ When every 2 elements in set are comparable, the relation is called a "Total Ordering".

Def: 3 If $(S, \preccurlyeq)$ is a poset and every 2 elements of S are comparable, S is called a 'totally ordered' or linearly ordered set. & $\preccurlyeq$ is called a 'total order' or a 'linear order'

→ A totally ordered set is also called a chain.

Ex:6 poset $(Z, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a & b are integers.

Ex:7 poset $(Z^+, |)$ is not totally ordered.
∵ it contains elements that are incomparable like 5 & 7.

Def:4 $(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Ex: Set of ordered pairs of positive integers $Z^+ \times Z^+$ with $(a_1, a_2) \preccurlyeq (b_1, b_2)$ if $a_1 < b_1$ or if $a_1 = b_1$ & $a_2 \leq b_2$ (lexicographic ordering) is a well ordered set.

Theorem 1

Principle of Well-Ordered Induction

Suppose that S is a well ordered set. Then $P(x)$ is true for all $x \in S$, if.

Inductive Step: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

Proof by Contradiction.

we assume. $P(x)$ is true for all $x \in S$ is not the case.

Then $y \in S$ and $P(y)$ is false.

Consequently

Set $A = \{x \in S \mid P(x) \text{ is false}\}$ is nonempty

$\because$ S is well ordered, A has a least element a.

So, $P(x)$ is true for all $x \in S$ with $x \prec a$.

$\Rightarrow P(a)$ is true.

Shows that $P(x)$ must be True for all $x \in S$.

Lexicographic order

→ Special Case of an ordering of Strings on a Set. Constructed from a partial ordering on set.

How to Construct, a partial ordering on the Cartesian product of two posets, $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$.

Lexicographic ordering $\preccurlyeq$ on $A_1 \times A_2$.

$(a_1, a_2) \prec (b_1, b_2)$.

either if $a_1 \prec_1 b_1$ or if both $a_1 = b_1$ & $a_2 \prec_2 b_2$.

we obtain a partial ordering $\preccurlyeq$ by adding equality to ordering $\prec$ on $A_1 \times A_2$.

Ex: Determine whether $(3,5) \prec (4,8)$, ~~whether~~ whether $(3,8) \prec (4,5)$.

whether $(4,9) \prec (4,11)$ in poset $(Z \times Z, \preccurlyeq)$.

where $\preccurlyeq$ is the lexicographic ordering Constructed from the usual ~~<~~ $\leq$ relation on Z.

→ $\because 3 < 4 \rightarrow (3,5) \prec (4,8)$ & $(3,8) \prec (4,5)$.

$(4,9) \prec (4,11)$. $\because 4 = 4$, $9 < 11$.