Proof Terminology

- A ***proof*** is a valid argument that establishes the truth of a mathematical statement
- ***Axiom*** (or ***postulate***): a statement that is assumed to be true
- ***Theorem***
  - A statement that has been proven to be true
- ***Hypothesis***, ***premise***
  - An assumption (often unproven) defining the structures about which we are reasoning

More Proof Terminology

- ***Lemma***
  - A minor theorem used as a stepping-stone to proving a major theorem.
- ***Corollary***
  - A minor theorem proved as an easy consequence of a major theorem.
- ***Conjecture***
  - A statement whose truth value has not been proven. (A conjecture may be widely believed to be true, regardless.)

Proof Methods

- For proving a statement p alone
  - ***Proof by Contradiction*** (indirect proof): Assume $\neg p$, and prove $\neg p \rightarrow \mathbf{F}$.

Proof Methods

- For proving implications $p \to q$, we have:
  - ***Trivial* proof:** Prove q by itself.
  - ***Direct* proof:** Assume p is true, and prove q.
  - ***Indirect* proof:**
    - ***Proof by Contraposition*** ($\neg q \to \neg p$): Assume $\neg q$, and prove $\neg p$.
    - ***Proof by Contradiction***: Assume $p \wedge \neg q$, and show this leads to a contradiction. (i.e. prove ($p \wedge \neg q$) $\to$ **F)**
  - ***Vacuous* proof:** Prove $\neg p$ by itself.

Direct Proof Example

- **Definition:** An integer n is called *odd* iff $n=2k+1$ for some integer k; n is *even* iff $n=2k$ for some k.
- **Theorem:** Every integer is either odd or even, but not both.
  - This can be proven from even simpler axioms.
- **Theorem:**
  (For all integers n) If n is odd, then n^2 is odd.

  Proof:

  If n is odd, then $n = 2k + 1$ for some integer k.

  Thus, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

  Therefore n^2 is of the form $2j + 1$ (with j the integer $2k^2 + 2k$), thus n^2 is odd. ■

Indirect Proof Example: Proof by Contraposition

- **Theorem:** (For all integers n)
  If $3n + 2$ is odd, then n is odd.

- **Proof:**

  (Contrapositive: If n is even, then $3n + 2$ is even)

  Suppose that the conclusion is false, *i.e.*, that n is even.

  Then $n = 2k$ for some integer k.

  Then $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$.

  Thus $3n + 2$ is even, because it equals $2j$ for an integer $j = 3k + 1$. So $3n + 2$ is not odd.

  We have shown that $\neg(n \text{ is odd}) \rightarrow \neg(3n + 2 \text{ is odd})$, thus its contrapositive $(3n + 2 \text{ is odd}) \rightarrow (n \text{ is odd})$ is also true. ■

Vacuous Proof Example

- Show $\neg p$ (i.e. p is false) to prove $p \rightarrow q$ is true.

- **Theorem:** (For all n) If n is both odd and even, then $n^2 = n + n$.
- **Proof:**

  The statement "n is both odd and even" is necessarily false, since no number can be both odd and even. So, the theorem is vacuously true. ■

Trivial Proof Example

- Show q (i.e. q is true) to prove $p \to q$ is true.

- **Theorem:** (For integers n) If n is the sum of two prime numbers, then either n is odd or n is even.
- **Proof:**

  Any integer n is either odd or even. So the conclusion of the implication is true regardless of the truth of the hypothesis. Thus the implication is true trivially. ■

Proof by Contradiction

- A method for proving p.
  - Assume $\neg p$, and prove both q and $\neg q$ for some proposition q. (Can be anything!)
  - Thus $\neg p \rightarrow (q \wedge \neg q)$
  - $(q \wedge \neg q)$ is a trivial contradiction, equal to **F**
  - Thus $\neg p \rightarrow$ **F**, which is only true if $\neg p =$ **F**
  - Thus p is true

Rational Number

- Definition:

  The real number r is *rational* if there exist integers p and q with $q \neq 0$ such that $r = p/q$. A real number that is not rational is called *irrational*.

Proof by Contradiction Example

- **Theorem:** $\sqrt{2}$ is irrational.
  - **Proof:**
  - Assume that $\sqrt{2}$ is rational. This means there are integers x and y ($y \neq 0$) with no common divisors such that $\sqrt{2} = x/y$.

    Squaring both sides, $2 = x^2/y^2$, so $2y^2 = x^2$. So x^2 is even; thus x is even (see earlier).

    Let $x = 2k$. So $2y^2 = (2k)^2 = 4k^2$. Dividing both sides by 2, $y^2 = 2k^2$. Thus y^2 is even, so y is even.

    But then x and y have a common divisor, namely 2, so we have a contradiction.

    Therefore, $\sqrt{2}$ is irrational. ∎

Proof by Contradiction

- Proving implication $p \rightarrow q$ by contradiction
  - Assume $\neg q$, and use the premise p to arrive at a contradiction, i.e. $(\neg q \wedge p) \rightarrow \mathbf{F}$ $(p \rightarrow q \equiv (\neg q \wedge p) \rightarrow \mathbf{F})$
  - How does this relate to the proof by contraposition?
  - ***Proof by Contraposition*** $(\neg q \rightarrow \neg p)$: Assume $\neg q$, and prove $\neg p$.

Proof by Contradiction Example: Implication

- **Theorem:** (For all integers n)

  If $3n + 2$ is odd, then n is odd.

- **Proof:**

  Assume that the conclusion is false, *i.e.*, that n is even, and that $3n + 2$ is odd.

  Then $n = 2k$ for some integer k and $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. Thus $3n + 2$ is even, because it equals $2j$ for an integer $j = 3k + 1$.

  This contradicts the assumption "$3n + 2$ is odd".

  This completes the proof by contradiction, proving that if $3n + 2$ is odd, then n is odd. ■

Circular Reasoning

- The fallacy of (explicitly or implicitly) assuming the very statement you are trying to prove in the course of its proof. Example:
- Prove that an integer n is even, if n^2 is even.
- **Attempted proof:**

Assume n^2 is even. Then $n^2 = 2k$ for some integer k.

Dividing both sides by n gives $n = (2k)/n = 2(k/n)$.

So there is an integer j (namely k/n) such that $n = 2j$. Therefore n is even.

  - Circular reasoning is used in this proof. Where?

*Begs the question: How do you show that $j = k/n = n/2$ is an integer, without **first** assumig that n is even?*

The Identity Function

- For any domain A, the ***identity function*** $I: A \rightarrow A$ (also written as I_A, $\mathbf{1}$, $\mathbf{1}_A$) is the unique function such that $\forall a \in A$: $I(a) = a$.
- Note that the identity function is always both one-to-one and onto (i.e., bijective).
- For a bijection $f : A \rightarrow B$ and its inverse function f^{-1}: $B \rightarrow A$,

$$f^{-1} \circ f = I_A$$

- Some identity functions you've seen:
  - $+$ 0, $\times$ 1, $\wedge$ **T**, $\vee$ **F**, $\cup$ $\varnothing$, $\cap$ U.

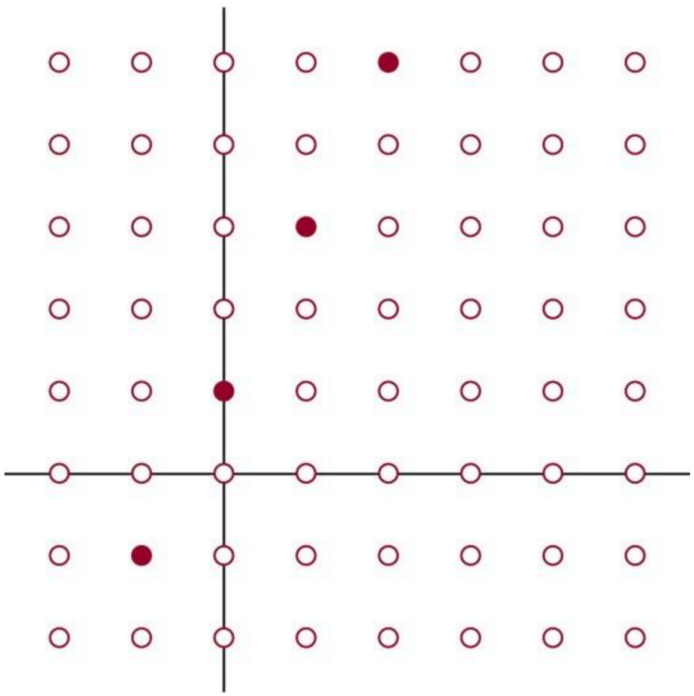
Graphs of Functions

- We can represent a function $f : A \rightarrow B$ as a set of ordered pairs $\{(a, f(a)) \mid a \in A\}$. ← The function's *graph*.

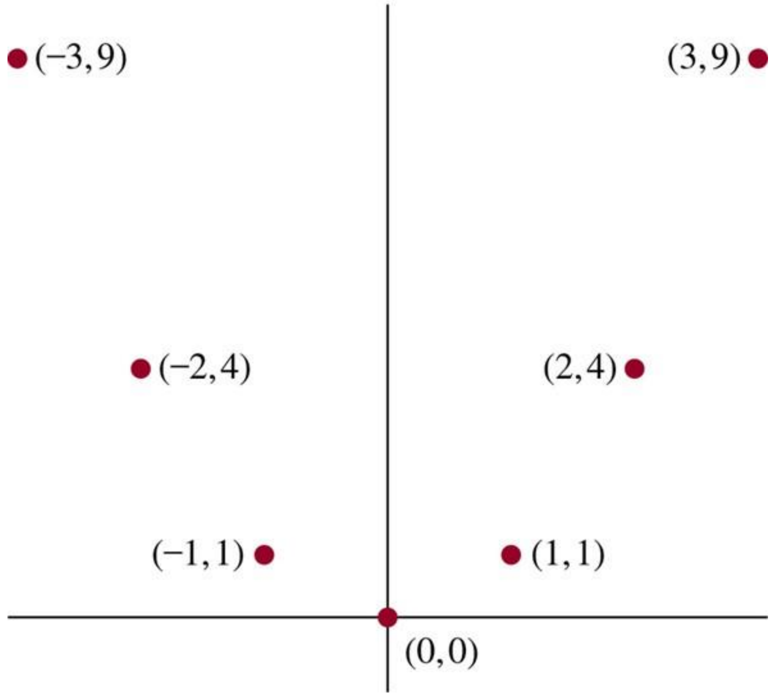
- Note that $\forall a \in A$, there is only 1 pair (a, b).

- For functions over numbers, we can represent an ordered pair (x, y) as a point on a plane.
- A function is then drawn as a curve (set of points), with only one y for each x.

Graphs of Functions: Examples



The graph of $f(n) = 2n + 1$ from **Z** to **Z**



The graph of $f(x) = x^2$ from **Z** to **Z**

Floor&Ceiling Functions

In discrete math, we frequently use the following two functions over real numbers:

- The ***floor function*** $\lfloor \cdot \rfloor$***: R → Z, where*** $\lfloor x \rfloor$ (“**floor** of *x*”) *means the* ***largest integer*** **≤ *x*,** i.e., $\lfloor x \rfloor = \max(\{i \in \mathbf{Z} \mid i \le x\})$.

E.g. $\lfloor 2.3 \rfloor = 2$, $\lfloor 5 \rfloor = 5$, $\lfloor -1.2 \rfloor = -2$

- The ***ceiling function*** $\lceil \cdot \rceil$***: R → Z, where*** $\lceil x \rceil$ (“ceiling of *x*”) *means the* ***smallest integer*** **≥ *x*,** i.e., $\lceil x \rceil = \min(\{i \in \mathbf{Z} \mid i \ge x\})$

E.g. $\lceil 2.3 \rceil = 3$, $\lceil 5 \rceil = 5$, $\lceil -1.2 \rceil = -$

Visualizing Floor & Ceiling

- Real numbers “fall to their floor” or “rise to their ceiling.”

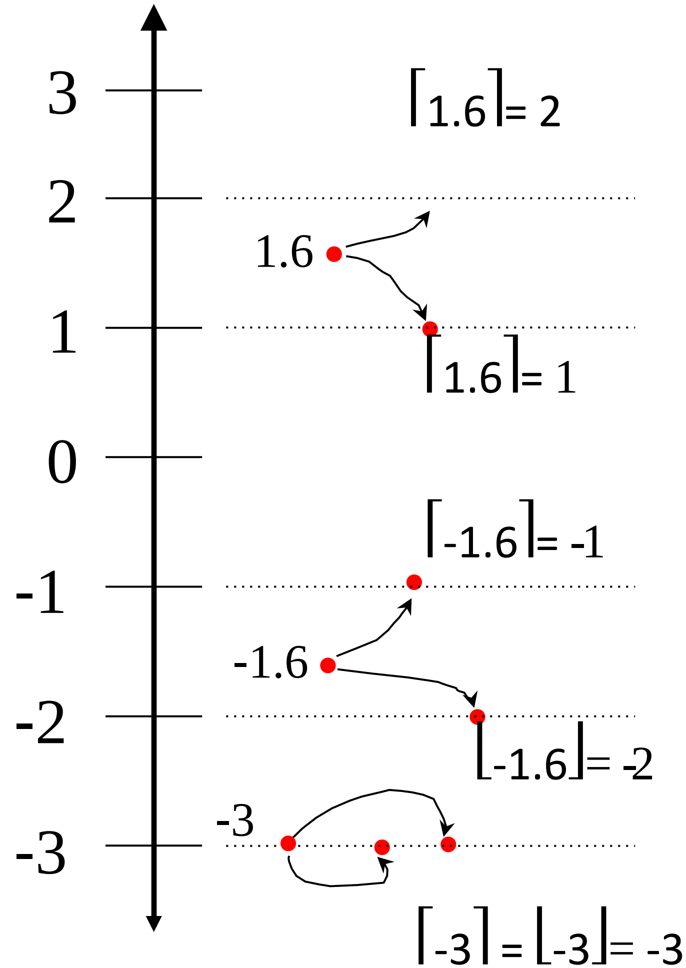
- Note that if $x \notin \mathbf{Z}$,

$$\lfloor -x \rfloor \neq -\lfloor x \rfloor \;\&$$

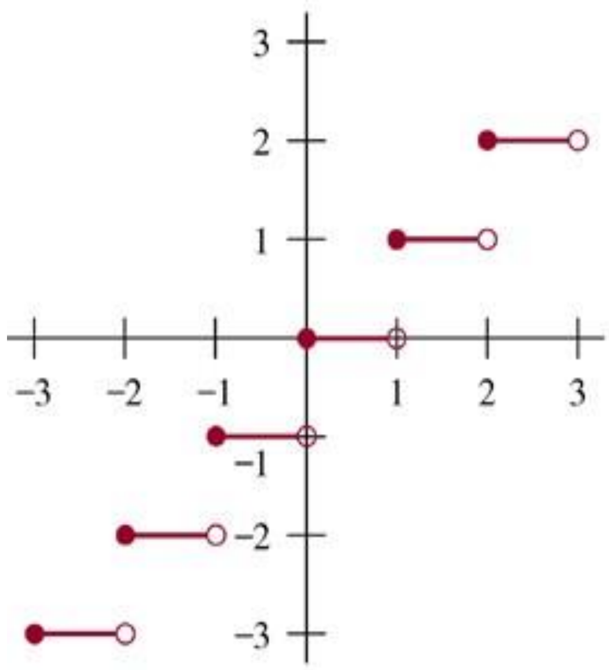
$$\lceil -x \rceil \neq -\lceil x \rceil$$

- Note that if $x \in \mathbf{Z}$,

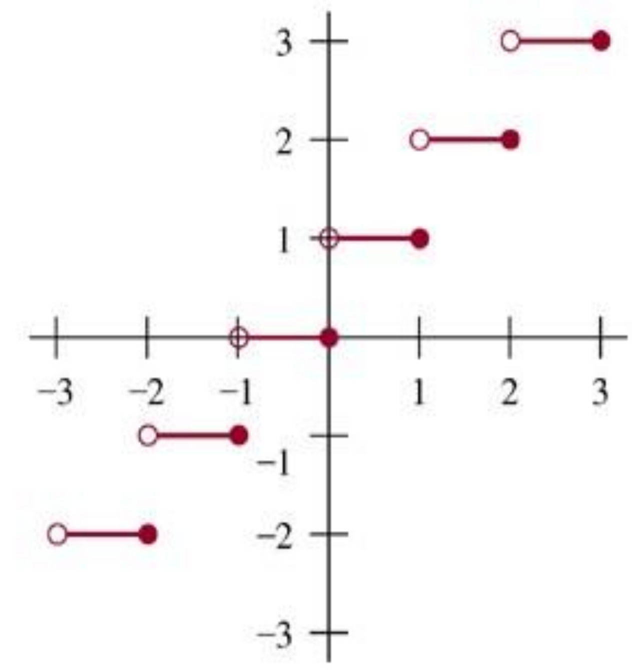
$$\lfloor x \rfloor = \lceil x \rceil = x.$$



Plots with Floor/Ceiling: Example



$y = \lfloor x \rfloor$



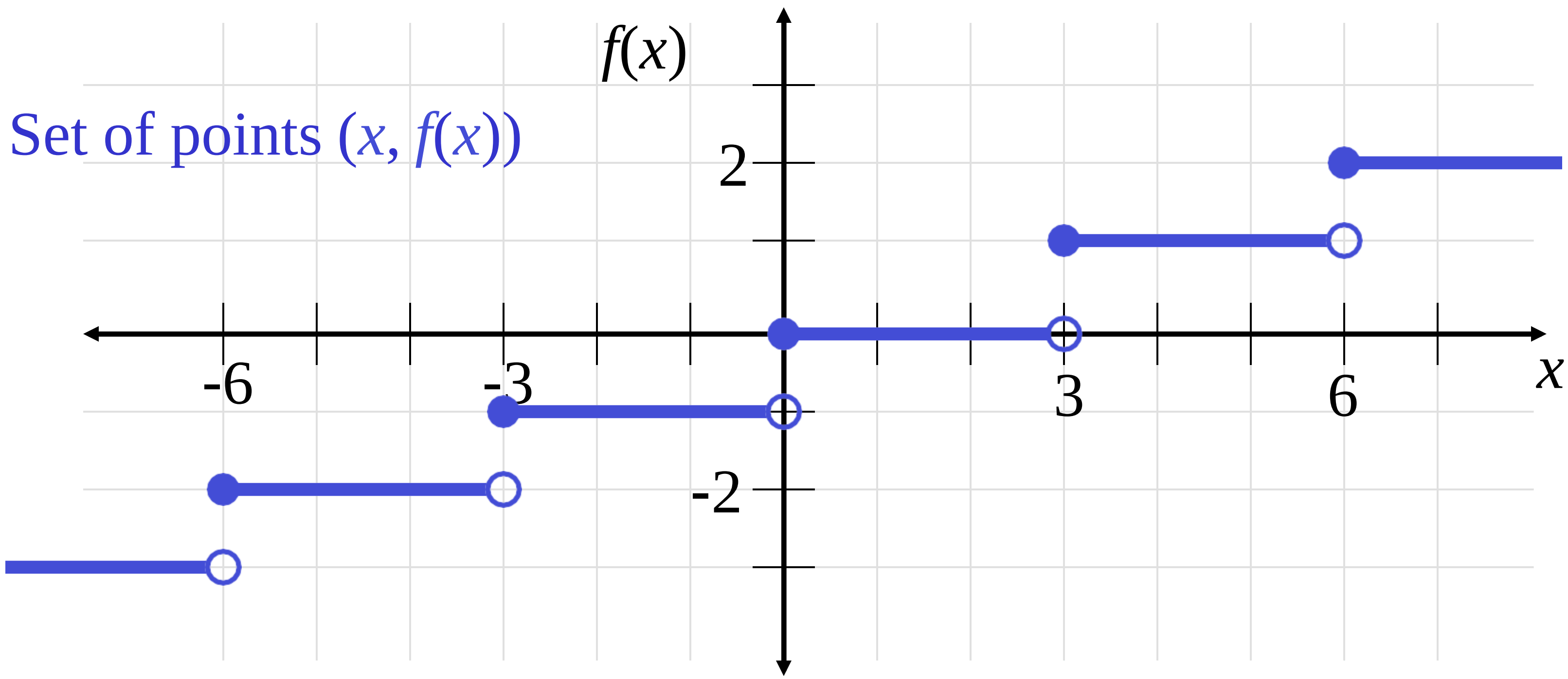
$y = \lceil x \rceil$

Plots with Floor/Ceiling

- Note that for $f(x) = \leq\lfloor x f \rfloor$, the graph of f includes the point (a, 0) for all values of a such that
$0 \leq a < 1$, but not for the value $a = 1$.
- We say that the set of points (a, 0) that is in f
does not include its *limit* or *boundary* point (a,1).
- Sets that do not include all of their limit points are called *open sets*.
- In a plot, we draw a limit point of a curve using an open dot (circle) if the limit point is not on the curve, and with a closed (solid) dot if it is on the curve.

Plots with Floor/Ceiling: Another Example

- Plot of graph of function $f(x)$ = ≤⌊x/3ƒ ⌋:



2.4 Sequences and Summations

- Sequences are ordered lists of elements, represents solutions to certain counting problems.
- A sequence is a discrete structure used to represent an ordered list.
- A *sequence is a function from a subset of the set of integers.*
- *Notation* a_n *to denote the image of the integer n.* a_n *a term of the sequence.*
- A ***summation*** is a compact notation for the sum of the terms in a (possibly infinite) sequence.

Sequences

- A ***sequence*** or ***series*** $\{a_n\}$ is identified with a ***generating function*** $f : I \rightarrow S$ for some subset $I \subseteq \mathbf{N}$ and for some set S.
- Often we have $I = \mathbf{N}$ or $I = \mathbf{Z}^+ = \mathbf{N} - \{0\}$.
- If f is a generating function for a sequence $\{a_n\}$, then for $n \in I$, the symbol a_n denotes $f(n)$, also called ***term*** *n* of the sequence.
- The ***index*** of a_n is n. (Or, often i is used.)
- A sequence is sometimes denoted by listing its first and/or last few elements, and using ellipsis (…) notation.
- *E.g.*, "$\{a_n\} = 0, 1, 4, 9, 16, 25, \ldots$" is taken to mean $\forall n \in \mathbf{N}, a_n = n^2$.

Sequence Examples

- Some authors write "the sequence $a_1, a_2, \ldots$" instead of $\{a_n\}$, to ensure that the set of indices is clear.
- Be careful: Our book often leaves the indices ambiguous.
- An example of an infinite sequence:
- Consider the sequence $\{a_n\} = a_1, a_2, \ldots,$ where $(\forall n \geq 1)\ a_n = f(n) = 1/n$.
- Then, we have $\{a_n\} = 1, 1/2, 1/3, \ldots$
- Called "harmonic series"

Example with Repetitions

- Like tuples, but unlike sets, a sequence may contain *repeated* instances of an element.
- Consider the sequence $\{b_n\} = b_0, b_1, \ldots$ (note that 0 is an index) where $b_n = (-1)^n$.
- Thus, $\{b_n\} = 1, -1, 1, -1, \ldots$
- Note repetitions!
- This $\{b_n\}$ denotes an infinite sequence of 1's and -1's, <u>*not*</u> the 2-element set {1, -1}.

Geometric Progression

- A geometric progression is a sequence of the form

***a*, *ar*, *ar*^{2}, …, *ar*^{n},…**

where the ***initial term a*** and the ***common ratio r*** are real numbers.

- A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$
- Examples

Assuming $n = 0, 1, 2,\ldots$

  - $\{b_n\}$ with $b_n = (-1)^n$ — initial term 1, common ratio –1
  - $\{c_n\}$ with $c_n = 2\cdot5^n$ — initial term 2, common ratio 5
  - $\{d_n\}$ with $d_n = 6\cdot(1/3)^n$ — initial term 6, common ratio 1/3

Arithmetic Progression

- An arithmetic progression is a sequence of the form

***a*, *a*+*d*, *a*+2*d*, …, *a*+*nd*,…**

where the ***initial term a*** and the ***common difference d*** are real numbers.

- An arithmetic progression is a discrete analogue of the linear function $f(x) = a + dx$

- Examples

Assuming $n = 0, 1, 2, \ldots$

  - $\{s_n\}$ with $s_n = -1 + 4n$ initial term –1, common diff. 4
  - $\{t_n\}$ with $t_n = 7 - 3n$ initial term 7, common diff. –3

Recognizing Sequences (I)

- Sometimes, you're given the first few terms of a sequence,
- and you are asked to find the sequence's generating function,
- or a procedure to enumerate the sequence.

- Examples: What's the next number?
  - 1, 2, 3, 4,… 5 (the 5th smallest number > 0)
  - 1, 3, 5, 7, 9,… 11 (the 6th smallest odd number > 0)
  - 2, 3, 5, 7, 11,... 13 (the 6th smallest prime number)

Recognizing Sequences (II)

- General problems
  - Given a sequence, find a formula or a general rule that produced it
- Examples: How can we produce the terms of a sequence if the first 10 terms are
  - 1, 2, 2, 3, 3, 3, 4, 4, 4, 4?

Possible match: next five terms would all be 5, the following six terms would all be 6, and so on.

  - 5, 11, 17, 23, 29, 35, 41, 47, 53, 59?

Possible match: nth term is $5 + 6(n - 1) = 6n - 1$ (assuming $n = 1, 2, 3, \ldots$)

Special Integer Sequences

- A useful technique for finding a rule for generating the terms of a sequence is to compare the terms of a sequence of interest with the terms of a well-known integer sequences (e.g. arithmetic/geometric progressions, perfect squares, perfect cubes, etc.)

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, … .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, … .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, … .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, … .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, … .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, … .

Coding: Fibbonaci Series

- Series $\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

- Generating function (recursive definition!):
  - $a_0 = a_1 = 1$ and
  - $a_n = a_{n-1} + a_{n-2}$ for all $n > 1$

- Now let's find the entire series $\{a_n\}$:

```
int [] a = new int [n];
  a[0] = 1;
a[1] =    1;
  for (int i = 2; i < n; i++) {
    a[i] = a[i-1] + a[i-2];
  }
  return a;
```

Coding: Factorial Series

- Factorial series $\{a_n\} = \{1, 2, 6, 24, 120, \ldots\}$

- Generating function:
- $a_n = n! = 1 \times 2 \times 3 \times \ldots \times n$

- This time, let's just find the term a_n:

```
int an = 1;
for (int     i       = 1; i <= n; i++) {
an = an * i;
}
return an;
```

Summation Notation

- Given a sequence $\{a_n\}$, an integer *lower bound* (or *limit*) $j \geq 0$, and an integer *upper bound* $k \geq j$, then the *summation of* $\{a_n\}$ *from* a_j *to* a_k is written and defined as follows:

$$\boxed{\sum_{i=j}^{k} a_i = a_j + a_{j+1} + \ldots + a_k}$$

- Here, i is called the *index of summation*.

$$\sum_{i=j}^{k} a_i = \sum_{m=j}^{k} a_m = \sum_{l=j}^{k} a_l$$

Generalized Summations

- For an infinite sequence, we write:

$$\sum_{i=j}^{\infty} a_i = a_j + a_{j+1} + \cdots$$

- To sum a function over all members of a set $X = \{x_1, x_2, \ldots\}$:

$$\sum_{x \in X} f(x) = f(x_1) + f(x_2) + \cdots$$

- Or, if $X = \{x \mid P(x)\}$, we may just write:

$$\sum_{P(x)} f(x) = f(x_1) + f(x_2) + \cdots$$

Simple Summation Example

- $$\sum_{i=2}^{4}\left(i^2+1\right)=$$

- $$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\cdots+\frac{1}{100}=\sum_{i=1}^{100}\frac{1}{i}$$

More Summation Examples

- An infinite sequence with a finite sum:

$$\sum_{i=0}^{\infty} 2^{-i} = 2^0 + 2^{-1} + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$$

- Using a predicate to define a set of elements to sum over:

$$\sum_{\substack{(x \text{ is prime}) \wedge \\ x<10}} x^2 = 2^2 + 3^2 + 5^2 + 7^2$$

$$= 4 + 9 + 25 + 49 = 87$$

Summation Manipulations

- Some handy identities for summations:
  - Summing constant value

$$\sum_{n=i}^{j} c = (j - i + 1) \cdot c$$

Number of terms in the summation

$$\sum_{n=1}^{3} 2 =$$

$$= 6$$

$$\sum_{n=-1}^{2} 2i$$

$$= 4 \oplus (2i) = 8i$$

Summation Manipulations

- Distributive law

$$\sum_{n=i}^{j} cf(n) = c\sum_{n=i}^{j} f(n)$$

$$\begin{aligned}\sum_{n=1}^{3}\left(4\cdot n^2\right) &= 4\cdot 1^2 + 4\cdot 2^2 + 4\cdot 3^2 \\ &= 4\cdot\left(1^2 + 2^2 + 3^2\right) \\ &= 4\sum_{n=1}^{3} n^2\end{aligned}$$

Summation Manipulations

- An application of commutativity

$$\sum_{n=i}^{j}\left(f(n)+g(n)\right)=\sum_{n=i}^{j} f(n)+\sum_{n=i}^{j} g(n)$$

$$\sum_{n=2}^{4}(n+2n)=(2+2\cdot 2)+(3+2\cdot 3)+(4+2\cdot 4)$$

$$=(2+3+4)+(2\cdot 2+2\cdot 3+2\cdot 4)$$

$$=\sum_{n=2}^{4} n+\sum_{n=2}^{4} 2n$$

Index Shifting

$$\sum_{i=j}^{m} f(i) = \sum_{k=j+n}^{m+n} f(k-n)$$

$$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2$$

- Let $k = i + 2$, then $i = k - 2$

$$\sum_{k=1+2}^{4+2} (k-2)^2 = \sum_{k=3}^{6} (k-2)^2$$

$$= (3-2)^2 + (4-2)^2 + (5-2)^2 + (6-2)^2$$

More Summation Manipulations

- Sequence splitting

$$\sum_{i=j}^{k} f(i) = \sum_{i=j}^{m} f(i) + \sum_{i=m+1}^{k} f(i) \quad \text{if } j \le m < k$$

$$\sum_{i=0}^{4} i^3 = 0^3 + 1^3 + 2^3 + 3^3 + 4^3$$

$$= (0^3 + 1^3 + 2^3) + (3^3 + 4^3)$$

$$= \sum_{i=0}^{2} i^3 + \sum_{i=3}^{4} i^3$$

More Summation Manipulations

- Order reversal

$$\sum_{i=0}^{k} f(i) = \sum_{i=0}^{k} f(k-i)$$

$$\sum_{i=0}^{3} i^3 = 0^3 + 1^3 + 2^3 + 3^3$$

$$= (3-0)^3 + (3-1)^3 + (3-2)^3 + (3-3)^3$$

$$= \sum_{i=0}^{3} (3-i)^3$$

Example: Geometric Progression

- A *geometric progression* is a sequence of the form $a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$ where $a, r \in \mathbf{R}$.
- The sum of such a sequence is given by:

$$\boxed{S = \sum_{i=0}^{n} ar^{i}}$$

- We can reduce this to *closed form* via clever manipulation of summations...

THEOREM 1

If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$

Proof: Let

$$S_n = \sum^{n} ar^j.$$

To compute S, first multiply both sides of the equality by r and then manipulate the resulti
sum as follows:

$$
\begin{aligned}
r S_n &= r \sum_{j=0}^{n} ar^j && \text{substituting summation formula for } S \\
&= \sum_{j=0}^{n} ar^{j+1} && \text{by the distributive property} \\
&= \sum_{k=1}^{n+1} ar^k && \text{shifting the index of summation, with } k = j+1 \\
&= \left(\sum_{k=0}^{n} ar^k \right) + (ar^{n+1} - a) && \text{removing } k = n+1 \text{ term and adding } k = 0 \text{ term} \\
&= S_n + (ar^{n+1} - a) && \text{substituting } S \text{ for summation formula}
\end{aligned}
$$

From these equalities, we see that

$$rS_n = S_n + (ar^{n+1} - a).$$

Solving for S_n shows that if $r \neq 1$, then

$$S_n = \frac{ar^{n+1} - a}{r - 1}.$$

If $r = 1$, then the $S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n+1)a$.

Gauss' Trick, Illustrated

- Consider the sum:

$$1+2+\ldots+(n/2)+((n/2)+1)+\ldots+(n-1)+n$$

$n+1$

$\vdots$

$n+1 \quad n+1$

- We have $n/2$ pairs of elements, each pair summing to $n+1$, for a total of $(n/2)(n+1)$.

Some Shortcut Expressions

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*	
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\dfrac{ar^{n+1} - a}{r - 1}, r \neq 1$	Geometric sequence
$\sum_{k=1}^{n} k$	$\dfrac{n(n+1)}{2}$	Gauss' trick
$\sum_{k=1}^{n} k^2$	$\dfrac{n(n+1)(2n+1)}{6}$	Quadratic series
$\sum_{k=1}^{n} k^3$	$\dfrac{n^2(n+1)^2}{4}$	Cubic series
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\dfrac{1}{1 - x}$	
$\sum_{k=1}^{\infty}, kx^{k-1}, \lvert x \rvert < 1$	$\dfrac{1}{(1 - x)^2}$	

Using the Shortcuts

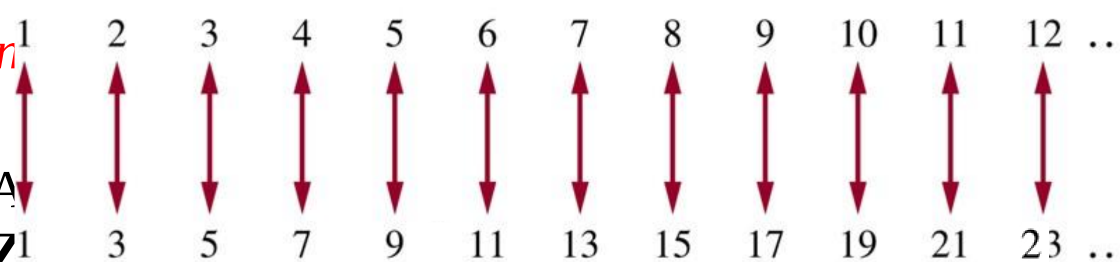
- Example: Evaluate $\sum_{k=50}^{100} k^2$
- Use series splitting. $$\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2$$
- Solve for desired summation. $$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$$
- Apply quadratic series rule. $$= \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6}$$
- Evaluate. $$= 338{,}350 - 40{,}425$$ $$= 297{,}925.$$

Cardinality

- The sets *A* and *B* have the same ***cardinality*** if and only if there is a one-to-one correspondence from *A* to *B*.
- A set that is either finite or has the same cardinality as the set of positive integers is called **countable**.
- A set that is not countable is called **uncountable**.
- Example: Show that the set of odd positive integers is a countable set.

Consider the function *f*(*n*
integers

A
between **Z**



Summation Manipulations

- Useful identities:

$$\sum_{i=j}^{k} f(i) = \sum_{i=j}^{m} f(i) + \sum_{i=m+1}^{k} f(i) \quad \text{if } j \le m < k$$

(Sequence splitting.)

$$\sum_{i=0}^{k} f(i) = \sum_{i=0}^{k} f(k-i)$$

(Order reversal.)

$$\sum_{i=1}^{2k} f(i) = \sum_{i=1}^{k} \big(f(2i-1) + f(2i) \big)$$

(Grouping.)