Theorem 1: let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a,b) \in R^n$.

Proof: By Mathematical induction. we prove.

By definition, there is a path from a to b of length one if & only if $(a,b) \in R$.

So, true for $n=1$.

Inductive hypothesis:
Assume it is true for positive integer n.

There is a path of length $n+1$ from a to b if & only if there is a element $c \in A$ such that there is a path of length one from a to c,

So, $(a,c) \in R$.

path of length n from c to b i.e., $(c,b) \in R^n$.
there is a path of length $n+1$ from a to b if & only if $(a,b) \in R^{n+1}$.

Transitive closures

→ Now finding the transitive closure of a relation is equivalent to determining which pair of vertices in the associated directed graph are connected by a path.

Def 2: Let R be a relation on a set A. The Connectivity relation R^* Consists of the pairs (a,b) such that there is a path of length at least one from a to b in R.

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

Ex:4. Let R be a relation on set of all people in the world that contains. (a, b) if a has met b. What is R^n, where n is a positive integer greater than one? What is R^*?

Sol:- R^2 contains (a, b) if there is a person c such that $(a, c) \in R$ & $(c, b) \in R$.

i.e., if there is a person c such that a has met c and c has met b.

Illy R^n consists of those pairs (a, b) such that there are people $x_1, x_2 \ldots\ldots x_{n-1}$ such that a has met x_1, x_1 has met x_2 .... & x_{n-1} has met b.

R^* contains (a, b) if sequence of people starts with a & ends with b, such that each person in sequence has met next person.

Ex 5 Let R be the relation on set of all ~~sub~~ states in US that contains (a, b) if state a and state b have a common border. What is R^n, where n is a +ve integer? What is R^*?

Sol: R^n consists of pairs (a, b) i.e., it is possible to go from state 'a' to state b by crossing exactly n state borders

R^* consists of ordered pairs (a, b)

R^* - not there are those containing states that are not connected to continental United states.

<u>Theorem 2</u> The transitive closure of a relation R equals the Connectivity relation R^*.

Proof: R^* Contains R by Definition

To show R^* is transitive closure of R.

we need to show that R^* is transitive &

$R^* \subseteq S$ whenever S is a transitive relation that Contains R.

-> 1st To show R^* is transitive.

If $(a,b) \in R^*$ and $(b,c) \in R^*$, then there are paths from a to b & from b to c in R.

we obtain a path from a to c, Hence $(a,c) \in R^*$.

So, R^* is transitive.

-> Suppose S is a transitive relation containing R.

$\because$ S is transitive, S^n is also transitive and.

$S^n \subseteq S$.

$$S^* = \bigcup_{k=1}^{\infty} S^k.$$

$S^k \subseteq S$.

So, $S^* \subseteq S$

if $R \subseteq S$, then $R^* \subseteq S^*$, b'coz any path in R is also a path in S.

$R^* \subseteq S^* \subseteq S$., So, any transitive relation that Contains R must also Contain R^*. $\therefore$ R^* is transitive closure of R.

LEMMA 1 Let A be a set with n elements, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. When $a \neq b$, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding $n-1$.

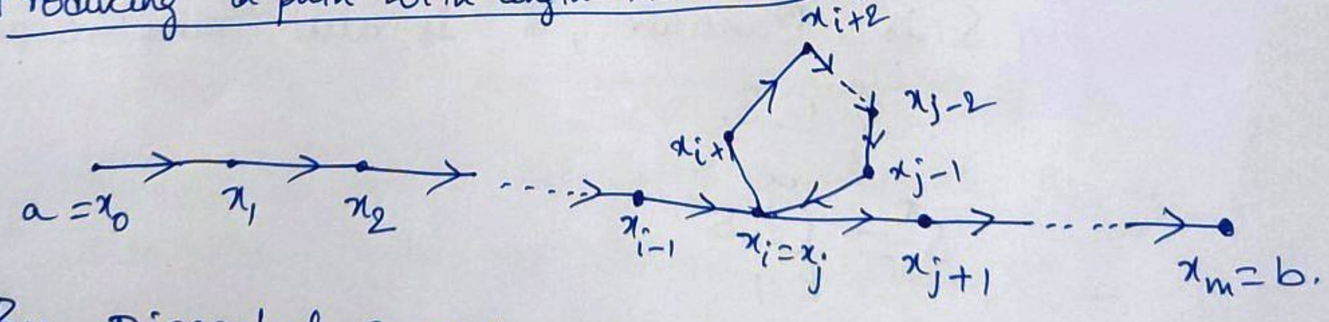
Proof: Suppose there is a path from a to b in R.

Let m be the length of the shortest. Such path.

Suppose $x_0, x_1, x_2 \cdots x_{m-1}, x_m$. where $x_0 = a$ & $x_m = b$ is such a path.

Suppose that $a = b$ & $m > n$., so that $m \geq n+1$.

Producing a path with length Not exceeding n.



By Pigeonhole principle, b'coz there are n vertices in A among the m vertices $x_0, x_1, \cdots x_{m-1}$, atleast two are equal.

Suppose $x_i = x_j$ with $0 \le i \le j \le m-1$.

Then the path contains a circuit from x_i to itself.

This circuit can be deleted from path a to b.

of shorter length. $x_0, x_1, \ldots x_i, x_{j+1}, \ldots, x_{m-1}, x_m$.

Hence, the path of shortest length must have length less than or equal to n.

From lemma 1, we see that transitive closure of R is union of $R, R^2, R^3, \ldots,$ and R^n.

This follows because there is a path in R^* between two vertices if & only if there is a path between these vertices in R^i, for some +ve integer i with $i \le n$.

By coz $R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$.

Theorem 3 Let M_R be ~~zero~~ zero-one matrix of the relation R on a set with n elements. Then the zero-one matrix of transitive closure R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \ldots \vee M_R^{[n]}.$$

Ex: 7 Find the zero-one matrix of transitive closure of relation R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Sol: By Theorem 3. zero-one matrix of R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}.$$

Because $M_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $M_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$M_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Algorithm 1: Procedure for Computing the Transitive closure.

procedure transitive closure (M_R : zero-one $n \times n$ matrix).

$A := M_R$
$B := A$.
for $i := 2$ to n
  $A := A \odot M_R$
  $B := B \vee A$.
return B {B is the zero-one matrix for R^*}

Warshall's

To find no. of bit operations used by algorithm 1. to determine the transitive closure of a relation.

-> Computing the Boolean powers $M_R, M_R^{[2]}, \ldots, M_R^{[n]}$. requires that $n-1$ Boolean products of $n \times n$ zero-one matrices be found.

-> Each of these Boolean products can be found using $n^2(2n-1)$ bit operations.

=> Hence, products can be computed using

$n^2(2n-1)(n-1)$ bit operations.

-> To find M_{R^*} from n Boolean powers of M_R. $n-1$ joins of zero-one 0-1 matrices need to be found.

-> To compute each of these joins uses n^2 bit operations.

-> $(n-1)n^2$ bit operations used for computations

-> $\therefore$ when Algorithm1 is used, the matrix of transitive closure of a relation on a set with n elements can be found using

$n^2(2n-1)(n-1) + (n-1)(n)^2 = 2n^3(n-1)$.

i.e., $O(n^4)$ bit operations.

Warshall's Algorithm

→ Warshall's Algorithm, is described in 1960 by Stephen Warshall, is a efficient method for computing the transitive closure of a relation.

→ Algorithm1 uses $2n^3(n-1)$ bit operations where as Warshall's algorithm using only $2n^3$ bit operations ~~to~~ finds the transitive closure.

→ Warshall's Algorithm is sometimes called Roy-Warshall algorithm., b'coz in 1959, Benard Roy described it.

→ Suppose R is a relation on a set with n elements.

→ let $v_1, v_2, ---- v_n$ be an arbitary listing of these n elements.

→ ~~con~~ Concept of interior vertices is used here.

if. $a, x_1, x_2, ----, x_{m-1}, b$ is a path, its interior vertices are $x_1, x_2, ---, x_{m-1}$.

→ interior vertices of a path a, c, d, f, g, h, b, j in a directed graph are c, d, f, g, h, b.

→ 1^{st} vertex in the path is not an interior vetex unless it is visited again by the path, except as the last vertex.

→ The last vertex in path is not an interior vertex unless it was visited previously by the path, except as first vertex.

→ Warshall's algorithm is based on construction of a sequence of zero-one matrices.

→ These matrices are $W_0, W_1, \ldots W_n$, where $W_0 = M_R$ is the zero-one matrix of this relation & $W_K = [w_{ij}^{(K)}]$. where $w_{ij}^{(K)} = '1'$ if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2 \ldots v_K\}$. & is '0' otherwise.

→ $W_n = M_{R^*}$ because $(i,j)^{th}$ entry of M_{R^*} is 1 iff there is a path from v_i to v_j with all interior vertices in set $\{v_1, v_2, \ldots v_n\}$.