Representing Relations.

Representing Relations Using Matrices

→ A relation between finite sets can be represented using a zero-one matrix.

R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots b_n\}$.

Relation R can be represented by matrix $M_R = [m_{ij}]$,

where $m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$

1) Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R. be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1, a_2 = 2, a_3 = 3$; $b_1 = 1, b_2 = 2$?

Solution: $R = \{(2,1), (3,1), (3,2)\}$

the matrix for R is $M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

The 1s in M_R show that the pairs $(2,1), (3,1)$ & $(3,2)$ belong to R.

0's show that no other pairs belong to R.

2) Let $A = \{a_1, a_2, a_3\}$ & $B = \{b_1, b_2, b_3, b_4, b_5\}$.
Which ordered pairs are in the relation R represented by matrix.

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

Sol: R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$

So, ~~8~~ $R = \{(a_1, b_2), (a_2, b_1), (a_3, b_3), (a_2, b_4),$
$(a_3, b_1), (a_3, b_3), (a_3, b_5)\}$.

→ The matrix of a relation on a set, which is a square matrix can be used to determine whether the relation has certain properties.

→ A relation R on A is reflexive if $(a,a) \in R$. Whenever $a \in A$.

$\therefore$ R is Reflexive if & only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots n$.

Here R is reflexive if only if $m_{ii} = 1$ for $i = 1, 2, \ldots n$.

R is reflexive if all the elements on main diagonal of M_R are equal to 1.

$$\begin{bmatrix} 1 & & & 0 \text{ or } 1 \\ & 1 & & \\ & & \ddots & \\ 0 \text{ or } 1 & & & 1 \end{bmatrix}$$

Zero-one matrix for reflexive Relation.

Symmetric Relation

R is Symmetric if $(a,b) \in R \rightarrow (b,a) \in R$.

the relation R on set $A = \{a_1, a_2 \dots a_n\}$ is Symmetric iff $(a_j, a_i) \in R$. whenever $(a_i, a_j) \in R$.

In terms of Matrix entries M_R.

R is Symmetric if & only if $m_{ji} = 1$. whenever $m_{ij} = 1$.

$m_{ji} = 0$ whenever $m_{ij} = 0$.

i.e., R is Symmetric iff $m_{ij} = m_{ji}$.

$i = 1, 2, \dots n$
$j = 1, 2, \dots n$.

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} \cdot & 1 & \cdot & 1 \\ 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix} \end{array}$$

$$M_R = (M_R)^t.$$

→ Anti Symmetric

Relation R is antisymmetric if & only ~~if (a,b)∈R~~

if $(a,b) \in R$ & $(b,a) \in R$ → that $a = b$.

The matrix of an anti Symmetric relation.

if $m_{ij} = 1$. with $i \neq j$ then $m_{ji} = 0$.

either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} \cdot & 1 & 0 \\ 0 & \cdot & \cdot \\ 0 & \cdot & \cdot \end{bmatrix} \end{array}$$

Ex: 3) Suppose that the Relation R on a set is represented by matrix $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Is R reflexive, Symmetric, antiSymmetric?

Sol: $\because$ diagonal elements are equal to 1, it is reflexive.

$$\therefore \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{array}$$

R is Symmetric $(0,1) = (1,0) = 1$
$(1,2) = (2,1) = 1$.

R is not Antisymmetric. $\boxed{\text{by def if } a_{ij} = 1, \text{ then } a_{ji} = 0 \quad i \neq j.}$

$\because$ if $(0,1) = 1$, then $(1,0) \neq 0$. here in this matrix So, it is not antisymmetric.

→ Boolean operations join and meet. used to find matrices representing the union & intersection of two relations.

→ R_1 & R_2 relations on a set A. represented by matrices MR_1 and M_{R_2} resp.

<u>Union</u> – Matrix representing the union of these relations has a 1 in positions where either M_{R_1} or M_{R_2} has a 1.

③

Intersection - Matrix representing intersection of these relations has a 1 in positions where both M_{R_1} & M_{R_2} has 1.

$\therefore$ $M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$

$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$.

Ex. 4) Suppose that relations R_1 & R_2 on a Set A are represented by the matrices.

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \& \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ & $R_1 \cap R_2$?

Sol:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Composite of relations

Suppose that R is a relation from A to B. ~~[illegible]~~

S is a relation from B to C.

Suppose that A, B and C have m, n, p elements resp.

Matrices for SoR, R & S be $M_{SoR} = [t_{ij}]$

$M_R = [r_{ij}]$

$M_S = [s_{ij}]$.

Ordered pair $(a_i, c_j) \in SoR$ iff there is an element b_k such that $(a_i, b_k) \in R$ & $(b_k, c_j) \in S$.

$t_{ij} = 1$ if & only if $r_{ik} = s_{kj} = 1$.

From definition of Boolean product.

i.e., $M_{SoR} = M_R \odot M_S$.

Ex 5 Find the matrix representing the relations $S \circ R$, where the matrices representing R & S are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \& \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Sol:

M_R ordered pairs are $(0,0)$, $(0,2)$, $(1,0)$ $(1,1)$.

M_S ordered pairs are $(0,1)$, $(1,2)$, $(2,0)$ $(2,2)$.

So $S \circ R = (0,1)$, $(0,0)$, $(0,2)$, $(1,2)$, $(1,2)$.

$$M_{S \circ R} = M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

→ The matrix representing the composite of two relations can be used to find matrix for M_{R^n}.

$$M_{R^n} = M_R^{[n]}.$$

Ex 6) Find the matrix representing the relation R^2 where the matrix representing R is

$$M_R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}.$$

Sol: M_R ordered pairs $(0,1)$, $(1,1)$, $(1,2)$ $(2,0)$.

$(0,1)$ $(1,1)$ $(1,2)$ $(2,0)$.

$R^2 = (0,1)$, $(1,1)$, $(1,2)$, $(1,0)$, $(0,2)$, $(2,1)$

$$M_{R^2} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Ex: 6. Find the matrix representing the relation R^2. Where the matrix representing R is

$$M_R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Sol: ordered pairs for R - (0,1), (1,1) (1,2) (2,0)

R (0,1) (1,1) (1,2) (2,0)

$$R^2 = \{(0,1), (0,2), (1,1), (1,2), \cancel{(\ )}, (1,0), \cancel{(2,0)}, (2,1)\}.$$

$$M_{R^2} = M_R^{[2]} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

⑤

Representing Relations Using Digraphs

→ Representing Relation using a pictorial representation.

→ Each element of a set represented by a point.

→ Each ordered pair is represented using an arc. with its direction indicated by an arrow.

→ Pictorial representations of relations on a finite set as directed graphs, or digraphs.

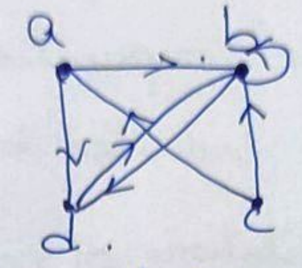
Def: A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs)

Vertex a is called the initial vertex of edge (a,b)

Vertex b is called the terminal vertex of this edge.

→ An edge of form (a,a) is represented using an arc from vertex a back to itself. It is called a loop.

Ex: 7. The directed graph with vertices a, b, c, d & edges (a,b), (a, d), b, b), (b,d), (c,a), (c,b). & (d,b).



Directed graph.

Ex: 8 The directed graph of a relation.

$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$. on set $\{1,2,3,4\}$.