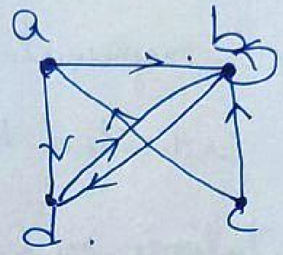
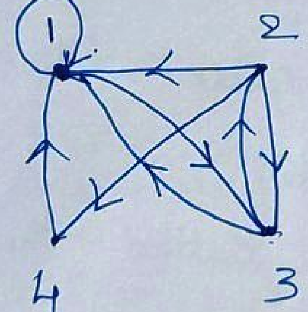
Ex: 7. The directed graph with vertices a, b, c, d & edges (a,b), (a, d), (b, b), (b, d), (c,a), (c,b). & (d, b).



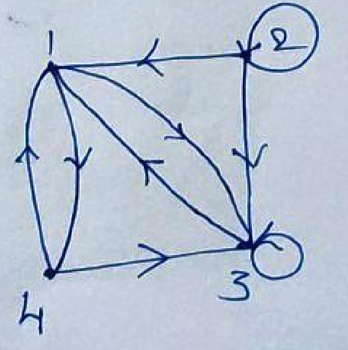
Directed graph.

Ex: 8 The directed graph of a relation.

$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$. on set $\{1,2,3,4\}$.



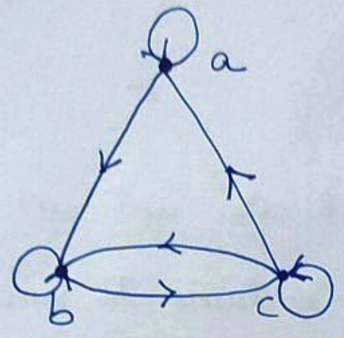
Ex: 9. What are the ordered pairs in the relation R represented by directed graph shown in Figure



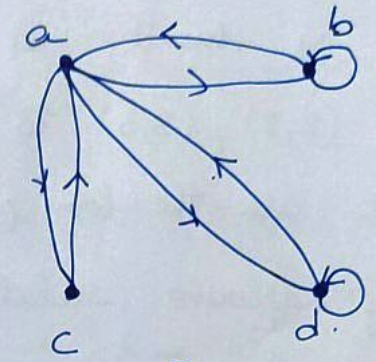
The ordered pairs (x, y) in the relation are

$R = \{(1,3), (1,4), (2,1), (2,2), (2,3)$
$(3,1)\ (3,3), (4,1), (4,3)\}$.

Ex: 10. Determine whether the relations for the directed graphs shown in below figures are reflexive, Symmetric, antisymmetric, & /or transitive.



Directed graph of R.



Directed graph of S.

R is Reflexive

R is
- not Symmetric
- not AntiSymmetric
- not Transitive

S is not reflexive

S is Symmetric

S is not antisymmetric

S is not Transitive

Closures of Relations

→ Let R be a relation on a set A. R may or may not have some property P, such as reflexivity, symmetry or transitivity

→ If there is a relation S with property P containing R such that S is a subset of every relation with property P containing R, then S is called the closure of R with respect to P.

Closures

The relation $R = \{(1,1), (1,2), (2,1), (3,2)\}$ on the ~~set~~ set $A = \{1, 2, 3\}$ is not reflexive. How can we produce a reflexive relation containing R that is as small as possible?

→ Add $(2,2)$, $(3,3)$ to R.

these are the form (a,a) which are not in R.

→ Any Reflexive relation that contains R must also contain $(2,2)$ & $(3,3)$.

→ Bcoz this relation contains R, is reflexive and is contained within every reflexive relation that contains R, it is called the reflexive closure of R.

reflexive closure of $R = R \cup \Delta$

$\Delta = \{(a,a) \mid a \in A\}$ is ~~dig~~ diagonal relation on A.

Ex: 1 What is the reflexive closure of relation $R = \{(a,b) \mid a < b\}$ on the set of integers?

Sol: The reflexive closure of R is

$$R \cup \Delta = \{(a,b) \mid a < b\} \cup \{(a,a) \mid a \in Z\} = \{(a,b) \mid a \leq b\}$$

The Relation $\{(1,1),(1,2),(2,2),(2,3),(3,1),(3,2)\}$ on $\{1,2,3\}$ is not Symmetric. How can we produce a Symmetric relation that is as small as possible and contains R?

→ For this we should add $(2,1)$ & $(1,3)$

→ It is of form (b,a) with $(a,b) \in R$ which is not in R.

→ This new relation is Symmetric & Contains R.

→ So, Further any Symmetric relation that contains R must contain this new relation, b'coz this new relation containing $(2,1),(1,3)$ is called Symmetric closure of R.

→ $R \cup R^{-1}$ is Symmetric closure of R, where $R^{-1} = \{(b,a) \mid (a,b) \in R\}$.

Ex:2 What is the Symmetric closure of the relation $R = \{(a,b) \mid a > b\}$ on set of positive integers?

Sol: The Symmetric closure of R is the relation

$$R \cup R^{-1} = \{(a,b) \mid a > b\} \cup \{(b,a) \mid a > b\} = \{(a,b) \mid a \neq b\}.$$

Transitive

Suppose a relation R is not transitive.

$R = \{(1,3), (1,4), (2,1), (3,2)\}$ on set $\{1,2,3,4\}$.

This relation is not transitive b'coz it doesn't contain all pairs of form (a,c) where (a,b) & (b,c) are in R

→ So, $(1,2), (2,3), (2,4), (3,1)$ are not in R.

→ By adding these pairs it doesnot produce a transitive relation, Because after adding these pairs we have $(3,1), (1,4)$ but $(3,4)$ is not there.

→ So, constructing the transitive closure of a relation is more complicated than reflexive & Symmetric closure.

→ Transitive closure of a relation can be found by adding new ordered pairs that must be present and then repeating this process until no new ordered pairs are needed.

Paths in Directed graphs

A path in a directed graph is obtained by traversing along edges.

Def: A path from a to b in directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where n is a nonnegative integer & $x_0 = a$ & $x_n = b$, i.e., a sequence of edges where the terminal vertex of an edge is same as the initial vertex in next edge in path.

→ This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ & has length n.

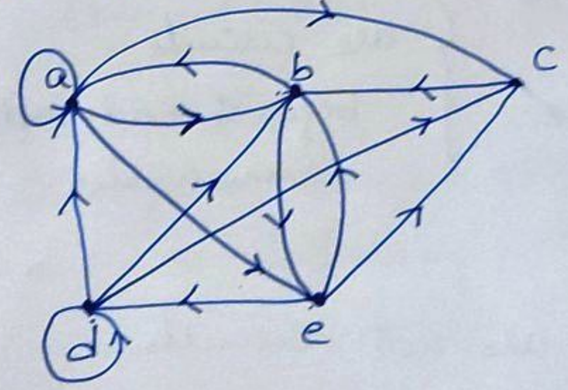
→ The empty set of edges as a path of length zero from a to a.

→ A path of length $n \geq 1$ that begins & ends at same vertex is called a circuit or cycle.

→ A path in a directed graph can pass through a vertex more than once.

→ An edge in a directed graph can occur more than once in a path.

<u>Ex:3</u>. Which of the following are paths in directed graph shown in below figure a, b, e, d; a, e, c, d, b; b, a, c, b, a, a, b; d, c; c, b, a; e, b, a, b, a, b, e? What are the lengths of those that are paths? Which of the paths in this list are circuits?



<u>Sol:</u> a, b, e, d.

(a, b), (b, e), (e, d) are edges from a to d.

So length of path is 3.

→ a, e, c, d, b.

here d → c is an edge but . c → d is not an edge.

So, a, e, c, d, b is not a path.

→ b, a, c, b, a, a, b;

(b, a), (a, c), (c, b), (b, a), (a, a), (a, b).

length. 5

→ (d, c) is an edge. for d, c so length 1.

→ c, b, a.

(c, b), (b, a)

length 2.

→ e, b, a, b, a, b, e.

(e, b), (b, a), (a, b), (b, a), (a, b), (b, e)

So length - 6.

→ b, a, c, b, a, a, b & e, b, a, b, a, b, e } are circuits. begin & end with Same vertex..

a, b, e, d. c, b, a d, c } are not circuits

Theorem 1: let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a,b) \in R^n$.

Proof: By Mathematical induction we prove.

By definition, there is a path from a to b of length one if & only if $(a,b) \in R$.

So, true for $n=1$.

Inductive hypothesis:
Assume it is true for positive integer n.

There is a path of length $n+1$ from a to b if & only if there is a element $c \in A$ such that there is a path of length one from a to c,

So, $(a,c) \in R$.

path of length n from c to b i.e., $(c,b) \in R^n$.
there is a path of length $n+1$ from a to b if & only if $(a,b) \in R^{n+1}$.

Transitive closures

→ Now finding the transitive closure of a relation is equivalent to determining which pair of vertices in the associated directed graph are connected by a path.

Def 2: Let R be a relation on a set A. The Connectivity relation R^* consists of the pairs (a,b) such that there is a path of length at least one from a to b in R.

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

Ex:4. Let R be a relation on set of all people in the world that contains. (a, b) if a has met b What is R^n, where n is a positive integers greater than one ? what is R^*?

Sol:- R^2 contains (a, b) if there is a person c such that $(a, c) \in R$ & $(c, b) \in R$.

i.e., if there is a person c such that a has met c and c has met b.

Illy R^n consists of those pairs (a, b) such that there are people $x_1, x_2 \ldots\ldots x_{n-1}$ such that a has met x_1, x_1 has met x_2 .... & x_{n-1} has met b.

R^* contains (a, b) if sequence of people starts with a & ends with b, such that each person in sequence has met next person.