Relations

- In mathematics we study relationships such as those between a positive integer and one that it divides, an integer and one that it is congruent to modulo 5, a real number and one that is larger than it, a real number *x and the* value *f (x) where f is a function, and so on.*
- *Relationships such as that between a program and* a variable it uses, and that between a computer language and a valid statement in this language often arise in computer science.
- Relationships between elements of sets are represented using the structure called a relation, which is just a subset of the Cartesian product of the sets.
- Relations can be used to solve problems such as determining which pairs of cities are linked by airline flights in a network, finding a viable order for the different phases of a complicated project, or producing a useful way to store information in computer databases.

Outline

- 8.1 Relations and their properties
- 8.3 Representing Relations
- 8.4 Closures of Relations
- 8.5 Equivalence Relations
- 8.6 Partial Orderings

8.1 Relations and their properties.

※The most direct way to express a relationship between elements of two sets is to use ordered pairs.

For this reason, sets of ordered pairs are called **binary relations**.

Def 1

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be sets. A binary relation from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a subset $\boldsymbol{R}$ of $\boldsymbol{A} \times \boldsymbol{B} = \{\, (\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A}, \boldsymbol{b} \in \boldsymbol{B} \,\}$.

Example 1.

$\boldsymbol{A}$: the set of students in your school.

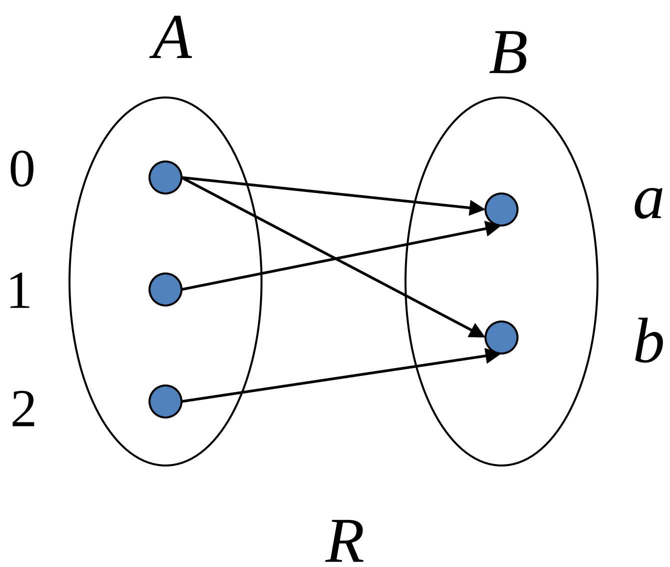
$\boldsymbol{B}$: the set of courses.

$\boldsymbol{R} = \{\, (\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A}, \boldsymbol{b} \in \boldsymbol{B}, \boldsymbol{a} \text{ is enrolled in course } \boldsymbol{b} \,\}$

Def 1′. We use the notation aRb to denote that $(a, b)\in R$, and $a\not{R}b$ to denote that $(a,b)\notin R$.

Moreover, a is said to be related to b by R if aRb.

Example 3. Let $A=\{0, 1, 2\}$ and $B=\{a, b\}$, then $\{(0,a),(0,b),(1,a),(2,b)\}$ is a relation R from A to B. This means, for instance, that $0Ra$, but that $1\not{R}b$.



$$R \subseteq A\times B = \{ (0,a) , (0,b) , (1,a) , \underbrace{(1,b)}_{\notin R} , \underbrace{(2,a)}_{\notin R} , (2,b)\}$$

Note. Relations vs. Functions

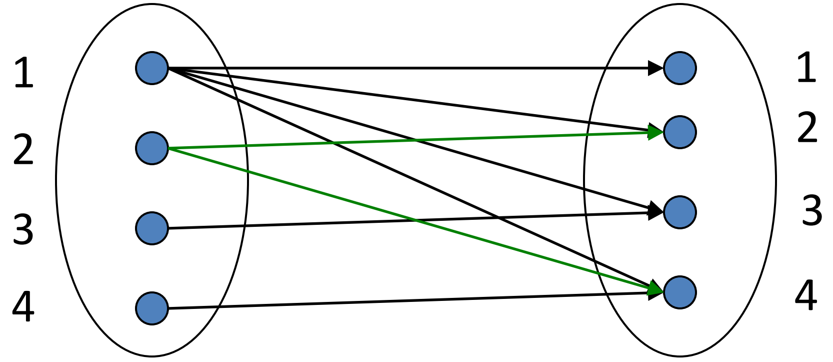
- A relation can be used to express a 1-to-many relationship between the elements of the sets $\boldsymbol{A}$ and $\boldsymbol{B}$.
- Function represents a relation where exactly one element of *B is related to each element* of *A.*

Def 2. A relation on the set $\boldsymbol{A}$ is a subset of $\boldsymbol{A} \times \boldsymbol{A}$ (i.e., a relation from $\boldsymbol{A}$ to $\boldsymbol{A}$).

Example 4.

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{ (a, b) \mid a \text{ divides } b \}$?

Sol :



$$R = \{ (1,1), (1,2), (1,3), (1,4),$$
$$(2,2), (2,4),$$
$$(3,3),$$
$$(4,4) \}$$

Example 5. Consider the following relations on $\mathbf{Z}$.

$R_1 = \{ (a, b) \mid a \le b \}$

$R_2 = \{ (a, b) \mid a > b \}$

$R_3 = \{ (a, b) \mid a = b \text{ or } a = -b \}$

$R_4 = \{ (a, b) \mid a = b \}$

$R_5 = \{ (a, b) \mid a = b+1 \}$

$R_6 = \{ (a, b) \mid a + b \le 3 \}$

Which of these relations contain each of the pairs (1,1), (1,2), (2,1), (1,–1), and (2,2)?

Sol :

	(1,1)	**(1,2)**	**(2,1)**	**(1,–1)**	**(2,2)**
R_1	●	●			●
R_2			●	●	
R_3	●			●	●
R_4	●				●
R_5			●		
R_6	●	●	●	●	

Example 6. How many relations are there on a set with $\boldsymbol{n}$ elements?

Sol :

A relation on a set $\boldsymbol{A}$ is a subset of $\boldsymbol{A}\times\boldsymbol{A}$.

$\Rightarrow \boldsymbol{A}\times\boldsymbol{A}$ has $\boldsymbol{n}^2$ elements.

$\Rightarrow$ a set with *n elements has* 2^n *subsets*.

$\Rightarrow \boldsymbol{A}\times\boldsymbol{A}$ has $2^{\boldsymbol{n}^2}$ subsets.

$\Rightarrow$ There are $2^{\boldsymbol{n}^2}$ relations.

Properties of Relations

Def 3. A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called reflexive if $(\boldsymbol{a},\boldsymbol{a})\in\boldsymbol{R}$ for every $\boldsymbol{a}\in\boldsymbol{A}$.

Example 7. Consider the following relations on $\{1, 2, 3, 4\}$:

$\boldsymbol{R_2} = \{ (1,1), (1,2), (2,1) \}$

$\boldsymbol{R_3} = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$

$\boldsymbol{R_4} = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

which of them are reflexive ?

Sol :

$\boldsymbol{R_3}$

Example 8. Which of the relations from Example 5 are reflexive?

$$R_1 = \{ (a, b) \mid a \le b \}$$
$$R_2 = \{ (a, b) \mid a > b \}$$
$$R_3 = \{ (a, b) \mid a = b \text{ or } a = -b \}$$
$$R_4 = \{ (a, b) \mid a = b \}$$
$$R_5 = \{ (a, b) \mid a = b+1 \}$$
$$R_6 = \{ (a, b) \mid a + b \le 3 \}$$

Sol : R_1, R_3 and R_4

Example 9. Is the "divides" relation on the set of positive integers reflexive?

Sol : Yes.

Def 4.

(1) A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called <u>symmetric</u>

for all $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{A}$,

$$\textbf{if } (\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R} \ \Rightarrow \ (\boldsymbol{b}, \boldsymbol{a}) \in \boldsymbol{R}.$$

(2) A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called <u>antisymmetric</u> for $\boldsymbol{all}$ $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{A}$,

$$\boldsymbol{if}\ (\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R} \text{ and } (\boldsymbol{b}, \boldsymbol{a}) \in \boldsymbol{R} \ \Rightarrow \ \boldsymbol{a} = \boldsymbol{b}.$$

Example 10. Which of the relations from Example 7 are symmetric or antisymmetric ?

$R_2 = \{ (1,1), (1,2), (2,1) \}$

$R_3 = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$

$R_4 = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

Sol :

R_2, R_3 are symmetric

R_4 are antisymmetric.

Example 11. Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

Sol : It is not symmetric since $1|2$ but $2 \nmid 1$.

It is antisymmetric since $a|b$ and $b|a$ implies $a=b$.

Def 5. A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called transitive

if $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R}$ and $(\boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{R} \Rightarrow (\boldsymbol{a}, \boldsymbol{c}) \in \boldsymbol{R}$.

for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{A}$

Example 15. Is the "divides" relation on the set of positive integers transitive?

Sol : Suppose $a|b$ and $b|c$

$\Rightarrow a|c$

$\Rightarrow$ transitive

Example 13. Which of the relations in Example 7 are transitive ?

$\boldsymbol{R_2} = \{ (1,1), (1,2), (2,1) \}$

$\boldsymbol{R_3} = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$

$\boldsymbol{R_4} = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

Sol :

$\boldsymbol{R_2}$ is not transitive since

$(2,1) \in \boldsymbol{R_2}$ and $(1,2) \in \boldsymbol{R_2}$ but $(2,2) \notin \boldsymbol{R_2}$.

$\boldsymbol{R_3}$ is not transitive since

$(2,1) \in \boldsymbol{R_3}$ and $(1,4) \in \boldsymbol{R_3}$ but $(2,4) \notin \boldsymbol{R_3}$.

$\boldsymbol{R_4}$ is transitive.

Combining Relations

Example 17. Let $\boldsymbol{A} = \{1, 2, 3\}$ and $\boldsymbol{B} = \{1, 2, 3, 4\}$.

The relation $\boldsymbol{R_1} = \{(1,1), (2,2), (3,3)\}$

and $\boldsymbol{R}_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined to obtain

$$\boldsymbol{R_1} \cup \boldsymbol{R_2} = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$$

$$\boldsymbol{R_1} \cap \boldsymbol{R_2} = \{(1,1)\}$$

$$\boldsymbol{R_1} - \boldsymbol{R_2} = \{(2,2), (3,3)\}$$

$$\boldsymbol{R_2} - \boldsymbol{R_1} = \{(1,2), (1,3), (1,4)\}$$

$$\boldsymbol{R_1} \oplus \boldsymbol{R_2} = \{(2,2), (3,3), (1,2), (1,3), (1,4)\}$$

symmetric difference, $(\boldsymbol{R_1} \cup \boldsymbol{R_2}) - (\boldsymbol{R_1} \cap \boldsymbol{R_2})$

Def 6. Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a,c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. We denote the composite of R and S by $S \circ R$.

Example 20. What is the composite of relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Sol. $S \circ R$ is the relation from $\{1, 2, 3\}$ to $\{0, 1, 2\}$ with $S \circ R = \{(1, 0), (1,1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.

Def 7. Let $\boldsymbol{R}$ be a relation on the set $\boldsymbol{A}$.
The powers $\boldsymbol{R^n}$, $\boldsymbol{n} = 1, 2, 3, \ldots$, are defined recursively by
$\boldsymbol{R}^1 = \boldsymbol{R}$ and $\boldsymbol{R}^{\boldsymbol{n}+1} = \boldsymbol{R^n} \circ \boldsymbol{R}$.

Example 22. Let $\boldsymbol{R} = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$.
Find the powers $\boldsymbol{R^n}$, $\boldsymbol{n}=2, 3, 4, \ldots$.

Sol. $\boldsymbol{R}^2 = \boldsymbol{R} \circ \boldsymbol{R} = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$.

$\boldsymbol{R}^3 = \boldsymbol{R}^2 \circ \boldsymbol{R} = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

$\boldsymbol{R}^4 = \boldsymbol{R}^3 \circ \boldsymbol{R} = \{(1, 1), (2, 1), (3, 1), (4, 1)\} = \boldsymbol{R}^3$.

Therefore $\boldsymbol{R^n} = \boldsymbol{R}^3$ for $\boldsymbol{n}=4, 5, \ldots$.

Thm 1. The relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is transitive if and only if $\boldsymbol{R^n} \subseteq \boldsymbol{R}$ for $\boldsymbol{n} = 1, 2, 3, \ldots$.

Combining Relations

Example 17. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$.

The relation $R_1 = \{(1,1), (2,2), (3,3)\}$

and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined to obtain

$$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$$

$$R_1 \cap R_2 = \{(1,1)\}$$

$$R_1 - R_2 = \{(2,2), (3,3)\}$$

$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$$

$$R_1 \oplus R_2 = \{(2,2), (3,3), (1,2), (1,3), (1,4)\}$$

symmetric difference, $(R_1 \cup R_2) - (R_1 \cap R_2)$

Example 18

Let *A and B be the set of all students and the set of all courses at a school, respectively***. Suppose that** *R1 consists of all ordered pairs (a, b), where a is a student who has taken course b,***and***R2 consists of all ordered pairs (a, b), where a is a student who requires course b to graduate.***What are the relations** *$R1 \cup R2$, $R1 \cap R2$, $R1 \oplus R2$, $R1 - R2$, and $R2 - R1$?*

Solution:

$R1 \cup R2$ consists of all ordered pairs (a, b), where a is a student who **either has taken course** *b or needs course b to graduate.*

$R1 \cap R2$ is the set of all ordered **pairs** *(a, b), where a is a student who has taken course b and needs this course to graduate.*

$R1 \oplus R2$ consists of all ordered pairs (a, b), where student a has taken course b but does **not need it to graduate or needs course** *b to graduate but has not taken it.*

$R1 - R2$ is the set of **ordered pairs** *(a, b), where a has taken course b but does not need it to graduate; that is, b is* **an elective course that** *a has taken.*

$R2 - R1$ is the set of all ordered pairs (a, b), where b is a **course that** *a needs to graduate but has not taken.*

Example 19

Let ***R1 be the "less than" relation on the set of real numbers and let R2 be the "greater than"*****relation on the set of real numbers, that is,** ***R1*** **=** ***{(x, y) | x*** **<** ***y} and R2*** **=** ***{(x, y) | x*** **>** ***y}*****.What are** ***R1 ∪R2, R1 ∩ R2, R1 − R2, R2 − R1, and R1 ⊕ R2?***

Solution:

(x, y) ∈R1 ∪R2 if and only if (x, y) ∈R1 or (x, y) ∈R2.

Hence, (x, y) ∈R1 ∪R2 if and only if x < y or x > y i.e., x! = y,

R1 ∪R2 = {(x, y) | x! = y}.

R1 ∩ R2 = Ø , x< y andx > y

R1 − R2 = R1

R2 − R1 = R2

R1 ⊕ R2 = R1 ∪R2 − R1 ∩ R2 = {(x, y) | x! = y}.

Def 6. Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a,c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. We denote the composite of R and S by $S \circ R$.

Example 20. What is the composite of relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Sol. $S \circ R$ is the relation from $\{1, 2, 3\}$ to $\{0, 1, 2\}$ with $S \circ R = \{(1, 0), (1,1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.

Def 7. Let R be a relation on the set A.
The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively by
$R^1 = R$ and $R^{n+1} = R^n \circ R$.

Example 22. Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$.
Find the powers R^n, $n=2, 3, 4, \ldots$.

Sol. $R^2 = R \circ R = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$.

$R^3 = R^2 \circ R = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

$R^4 = R^3 \circ R = \{(1, 1), (2, 1), (3, 1), (4, 1)\} = R^3$.

Therefore $R^n = R^3$ for $n=4, 5, \ldots$.

Thm 1. The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

n-ary Relations and Their Applications

- Relationship involving the name of a student, the student's major, and the student's grade point average.
- Relationship involving the airline, flight number, starting point, destination,departure time, and arrival time of a flight.
- relationships among elements from more than two sets are called ***n-ary relations.***
- These relations are used to represent computer databases.
- EX: Which flights land at O'Hare Airport between 3 a.m. and 4 a.m.?
- Which students at your school are sophomores majoring in mathematics or computer science and have greater than a 3.0 average?
- Which employees of a company have worked for the company less than 5 years and make more than $50,000?

n-ary Relations

DEFINITION 1

Let $A_1, A_2, \ldots, A_n$ *be sets.An n-ary relation on these sets is a subset of* $A_1 \times A_2 \times \cdot \cdot \cdot \times A_n$ The sets $A_1, A_2, \ldots, A_n$ *are called the domains of the relation, and n is called its degree.*

EXAMPLE 1

Let *R be the relation on* **N × N × N** ***consisting of triples (a, b, c), where a, b, and c are integers*** with $a < b < c$. *Then* $(1, 2, 3) \in R$, *but* $(2, 4, 3) \notin R$.

The degree of this relation is 3. Its domains are all equal to the set of natural numbers.

EXAMPLE 2

Let *R be the relation on* **Z × Z × Z *consisting of all triples of integers (a, b, c) in which*** *a, b, and c form an arithmetic progression.*

(a, b, c) $\in$ *R if and only if there is* an integer *k such that*

$b = a + k$ *and* $c = a + 2k$,

$b - a = k$ *and* $c - b = k$.

(1, 3, 5) $\in$ *R because* $3 = 1 + 2$ *and* $5 = 1 + 2 \cdot 2$,

but (2, 5, 9) !$\in$ *R because* $5 - 2 = 3$ while $9 - 5 = 4$.

This relation has degree 3 and its domains are all equal to the set of integers.

EXAMPLE 3

Let$R$ *be the relation on* **Z × Z × Z+** ***consisting of triples (a, b, m), where a, b, and m are integers*** with $m \geq 1$ *and* $a \equiv b \pmod{m}$.

$(8, 2, 3)$, $(-1, 9, 5)$, *and* $(14, 0, 7) \in R$,

but $(7, 2, 3)$, $(-2,-8, 5)$, *and* $(11, 0, 6)$ $!\in R$

$8 \equiv 2 \pmod{3}$,$-1 \equiv 9 \pmod{5}$, and $14 \equiv 0 \pmod{7}$,

but $7 \equiv 2 \pmod{3}$, $-2 \equiv -8 \pmod{5}$, *and* $11 \equiv 0 \pmod{6}$.

This relation has degree 3 and its first two domains are the set of all integers and its third domain is the set of positive integers.

EXAMPLE 4

Let *R be the relation consisting of 5-tuples (A,N, S,D, T) representing airplane flights,* where *A is the airline, N is the flight number, S is the starting point, D is the destination, and T is* the departure time.

sol:

if Nadir Express Airlines has flight 963 from Newark to Bangor at 15:00, then (Nadir, 963, Newark, Bangor, 15:00) $\in R$.

The degree of this relation is 5, and its domains are the set of all airlines, the set of flight numbers, the set of cities, the set of cities (again), and the set of times.

Databases and Relations

- The time required to manipulate information in a database depends on how this information is stored.
- The operations of adding and deleting records, updating records, searching for records, and combining records from overlapping databases are performed millions of times each day in a large database.
- Because of the importance of these operations, various methods for representing databases have been developed.
- A database consists of **records, which are *n-tuples, made up of fields.***
- The fields are the entries of the *n-tuples*.

student records

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

are represented as 4-tuples of the form (*Student_name, ID_number, Major, GPA*). A sample database of six such records is

(Ackermann, 231455, Computer Science, 3.88)
(Adams, 888323, Physics, 3.45)
(Chou, 102147, Computer Science, 3.49)
(Goodfriend, 453876, Mathematics, 3.45)
(Rao, 678543, Mathematics, 3.90)
(Stevens, 786576, Psychology, 2.99).

Databases and Relations

- Relations used to represent databases are also called **tables.**
- Each column of the table corresponds to an *attribute of the database.*
- *For* instance, the same database of students is displayed in Table 1.
- The attributes of this database are Student Name, ID Number, Major, and GPA.
- A domain of an *n-ary relation is called a **primary key when the value of the n-tuple from*** this domain determines the *n-tuple.*
- Domain is a primary key when no two *n-tuples in* the relation have the same value from this domain.

Databases and Relations

- Records are often added to or deleted from databases.
- So,domain is a primary key is time-dependent.
- A primary key should be chosen that remains one whenever the database is changed.
- The current collection of *n-tuples in a relation* is called the **extension of the relation.**
- **The more permanent part of a database, including the** name and attributes of the database, is called its **intension.**
- **When selecting a primary key,** select a key that can serve as a primary key for all possible extensions of the database.
- To do this, it is necessary to examine the intension of the database to understand the set of possible *n-tuples that can occur in an extension.*

EXAMPLE 5

Which domains are primary keys for the *n-ary relation displayed in Table 1, assuming that no n-tuples will be added in the future?*

Solution:

domain of student names is a primary key.

ID numbers in this table are unique, so the domain of ID numbers is also a primary key.

domain of major fields of study is not a primary key, because more than one 4-tuple contains the same major field of study.

domain of grade point averages is also not a primary key, because there are two 4-tuples containing the same GPA.

composite key

Combinations of domains can also uniquely identify *n-tuples in an n-ary relation*.

When the values of a set of domains determine an *n-tuple in a relation, the Cartesian product of these* domains is called a **composite key.**

EXAMPLE 6

Is the Cartesian product of the domain of major fields of study and the domain of GPAs a composite key for the *n-ary relation from Table 1, assuming that no n-tuples are ever added?*

Solution:

Because no two 4-tuples from this table have both the same major and the same GPA, this Cartesian product is a composite key.

Because primary and composite keys are used to identify records uniquely in a database, it is important that keys remain valid when new records are added to the database.

Combining Relations

Example 17. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$.

The relation $R_1 = \{(1,1), (2,2), (3,3)\}$

and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined to obtain

$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2), (3,3)\}$

$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

$R_1 \oplus R_2 = \{(2,2), (3,3), (1,2), (1,3), (1,4)\}$

symmetric difference, $(R_1 \cup R_2) - (R_1 \cap R_2)$

Example 18

Let *A and B be the set of all students and the set of all courses at a school, respectively*. **Suppose that** *R1 consists of all ordered pairs (a, b), where a is a student who has taken course b,***and***R2 consists of all ordered pairs (a, b), where a is a student who requires course b to graduate.***What are the relations** *R1 ∪R2, R1 ∩ R2, R1 ⊕R2, R1 − R2, and R2 − R1?*

Solution:

R1 ∪R2 consists of all ordered pairs (a, b), where a is a student who **either has taken course** *b or needs course b to graduate.*

R1 ∩ R2 is the set of all ordered **pairs** *(a, b), where a is a student who has taken course b and needs this course to graduate.*

R1 ⊕R2 consists of all ordered pairs (a, b), where student a has taken course b but does **not need it to graduate or needs course** *b to graduate but has not taken it.*

R1 − R2 is the set of **ordered pairs** *(a, b), where a has taken course b but does not need it to graduate; that is, b is* **an elective course that** *a has taken.*

R2 − R1 is the set of all ordered pairs (a, b), where b is a **course that** *a needs to graduate but has not taken.*

Example 19

Let ***R1 be the "less than" relation on the set of real numbers and let R2 be the "greater than"*****relation on the set of real numbers, that is,** $R1 = \{(x, y) \mid x < y\}$ *and* $R2 = \{(x, y) \mid x > y\}$**.What are** $R1 \cup R2$, $R1 \cap R2$, $R1 - R2$, $R2 - R1$, *and* $R1 \oplus R2$?

Solution:

$(x, y) \in R1 \cup R2$ *if and only if* $(x, y) \in R1$ *or* $(x, y) \in R2$.

Hence, $(x, y) \in R1 \cup R2$ *if and only if* $x < y$ *or* $x > y$ *i.e.,* $x != y$,

$R1 \cup R2 = \{(x, y) \mid x != y\}$.

$R1 \cap R2 = \emptyset$, $x< y$ *andx* $> y$

$R1 - R2 = R1$

$R2 - R1 = R2$

$R1 \oplus R2 = R1 \cup R2 - R1 \cap R2 = \{(x, y) \mid x != y\}$.

Def 6. Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a,c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. We denote the composite of R and S by $S \circ R$.

Example 20. What is the composite of relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Sol. $S \circ R$ is the relation from $\{1, 2, 3\}$ to $\{0, 1, 2\}$ with $S \circ R = \{(1, 0), (1,1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.

Def 7. Let $\boldsymbol{R}$ be a relation on the set $\boldsymbol{A}$.
The powers $\boldsymbol{R^n}$, $\boldsymbol{n} = 1, 2, 3, \ldots$, are defined recursively by
$\boldsymbol{R}^1 = \boldsymbol{R}$ and $\boldsymbol{R}^{\boldsymbol{n}+1} = \boldsymbol{R^n} \circ \boldsymbol{R}$.

Example 22. Let $\boldsymbol{R} = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$.
Find the powers $\boldsymbol{R^n}$, $\boldsymbol{n}=2, 3, 4, \ldots$.

Sol. $\boldsymbol{R}^2 = \boldsymbol{R} \circ \boldsymbol{R} = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$.

$\boldsymbol{R}^3 = \boldsymbol{R}^2 \circ \boldsymbol{R} = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

$\boldsymbol{R}^4 = \boldsymbol{R}^3 \circ \boldsymbol{R} = \{(1, 1), (2, 1), (3, 1), (4, 1)\} = \boldsymbol{R}^3$.

Therefore $\boldsymbol{R^n} = \boldsymbol{R}^3$ for $\boldsymbol{n}=4, 5, \ldots$.

Thm 1. The relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is transitive if and only if $\boldsymbol{R^n} \subseteq \boldsymbol{R}$ for $\boldsymbol{n} = 1, 2, 3, \ldots$.

n-ary Relations and Their Applications

- Relationship involving the name of a student, the student's major, and the student's grade point average.
- Relationship involving the airline, flight number, starting point, destination,departure time, and arrival time of a flight.
- relationships among elements from more than two sets are called ***n-ary relations.***
- These relations are used to represent computer databases.
- EX: Which flights land at O'Hare Airport between 3 a.m. and 4 a.m.?
- Which students at your school are sophomores majoring in mathematics or computer science and have greater than a 3.0 average?
- Which employees of a company have worked for the company less than 5 years and make more than $50,000?

n-ary Relations

DEFINITION 1

Let $A_1, A_2, \ldots, A_n$ *be sets.An n-ary relation on these sets is a subset of* $A_1 \times A_2 \times \cdot \cdot \cdot \times A_n$ The sets $A_1, A_2, \ldots, A_n$ *are called the domains of the relation, and n is called its degree.*

EXAMPLE 1

Let *R be the relation on* **N × N × N** ***consisting of triples (a, b, c), where a, b, and c are integers*** with $a < b < c$. *Then* $(1, 2, 3) \in R$, *but* $(2, 4, 3) \notin R$.
The degree of this relation is 3. Its domains are all equal to the set of natural numbers.

EXAMPLE 2

Let *R be the relation on* **Z × Z × Z *consisting of all triples of integers (a, b, c) in which*** *a, b, and c form an arithmetic progression.*

(a, b, c) ∈ R if and only if there is an integer *k such that*

$b = a + k$ *and* $c = a + 2k$,

$b - a = k$ *and* $c - b = k$.

(1, 3, 5) ∈ R because $3 = 1 + 2$ *and* $5 = 1 + 2 \cdot 2$,

but (2, 5, 9) !∈ R because $5 - 2 = 3$ while $9 - 5 = 4$.

This relation has degree 3 and its domains are all equal to the set of integers.

EXAMPLE 3

Let*R be the relation on***Z × Z × Z+ *consisting of triples* *(a, b, m), where a, b, and m are integers*** with $m \geq 1$ *and* $a \equiv b \pmod{m}$.

$(8, 2, 3)$, $(-1, 9, 5)$, *and* $(14, 0, 7) \in R$,

but $(7, 2, 3)$, $(-2, -8, 5)$, *and* $(11, 0, 6)$ $! \in R$

$8 \equiv 2 \pmod{3}$, $-1 \equiv 9 \pmod{5}$, and $14 \equiv 0 \pmod{7}$,

but $7 \equiv 2 \pmod{3}$, $-2 \equiv -8 \pmod{5}$, *and* $11 \equiv 0 \pmod{6}$.

This relation has degree 3 and its first two domains are the set of all integers and its third domain is the set of positive integers.

EXAMPLE 4

Let R *be the relation consisting of 5-tuples* (A,N,S,D,T) *representing airplane flights,* where A *is the airline,* N *is the flight number,* S *is the starting point,* D *is the destination, and* T *is* the departure time.

sol:

if Nadir Express Airlines has flight 963 from Newark to Bangor at 15:00, then (Nadir, 963, Newark, Bangor, 15:00) $\in R$.

The degree of this relation is 5, and its domains are the set of all airlines, the set of flight numbers, the set of cities, the set of cities (again), and the set of times.

Databases and Relations

- The time required to manipulate information in a database depends on how this information is stored.
- The operations of adding and deleting records, updating records, searching for records, and combining records from overlapping databases are performed millions of times each day in a large database.
- Because of the importance of these operations, various methods for representing databases have been developed.
- A database consists of **records, which are *n-tuples, made up of fields.***
- The fields are the entries of the *n-tuples*.

student records

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

are represented as 4-tuples of the form (*Student_name, ID_number, Major, GPA*). A sample database of six such records is

(Ackermann, 231455, Computer Science, 3.88)
(Adams, 888323, Physics, 3.45)
(Chou, 102147, Computer Science, 3.49)
(Goodfriend, 453876, Mathematics, 3.45)
(Rao, 678543, Mathematics, 3.90)
(Stevens, 786576, Psychology, 2.99).

Databases and Relations

- Relations used to represent databases are also called **tables.**
- Each column of the table corresponds to an *attribute of the database.*
- *For* instance, the same database of students is displayed in Table 1.
- The attributes of this database are Student Name, ID Number, Major, and GPA.
- A domain of an *n-ary relation is called a* ***primary key when the value of the n-tuple from*** this domain determines the *n-tuple.*
- Domain is a primary key when no two *n-tuples in* the relation have the same value from this domain.

Databases and Relations

- Records are often added to or deleted from databases.
- So,domain is a primary key is time-dependent.
- A primary key should be chosen that remains one whenever the database is changed.
- The current collection of *n-tuples in a relation* is called the **extension of the relation.**
- **The more permanent part of a database, including the** name and attributes of the database, is called its **intension.**
- **When selecting a primary key,** select a key that can serve as a primary key for all possible extensions of the database.
- To do this, it is necessary to examine the intension of the database to understand the set of possible *n-tuples that can occur in an extension*.

EXAMPLE 5

Which domains are primary keys for the *n-ary relation displayed in Table 1, assuming that no n-tuples will be added in the future?*

Solution:

domain of student names is a primary key.

ID numbers in this table are unique, so the domain of ID numbers is also a primary key.

domain of major fields of study is not a primary key, because more than one 4-tuple contains the same major field of study.

domain of grade point averages is also not a primary key, because there are two 4-tuples containing the same GPA.

composite key

Combinations of domains can also uniquely identify *n-tuples in an n-ary relation*.

When the values of a set of domains determine an *n-tuple in a relation, the Cartesian product of these* domains is called a **composite key.**

EXAMPLE 6

Is the Cartesian product of the domain of major fields of study and the domain of GPAs a composite key for the *n-ary relation from Table 1, assuming that no n-tuples are ever added?*

***Solution**:*

*Because no two 4-tuples from this table have both the same major and the same GPA,*this Cartesian product is a composite key.

Because primary and composite keys are used to identify records uniquely in a database, it is important that keys remain valid when new records are added to the database.

Operations on *n-ary Relations*

- Operations can answer queries on databases that ask for all *n-tuples that* satisfy certain conditions.

  find all the records of all computer science majors in a database of student records.

  find all students who have a grade point average above 3.5

  find the records of all computer science majors who have a grade point average above 3.5.

- **selection operator**

  Let *R be an n-ary relation and C a condition that elements in R may satisfy. Then the selection operator* s_C *maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy* the condition *C.*

Example

- To find the records of computer science majors in the *n-ary relation R shown in Table 1,*

  C1 is the condition Major="Computer Science."

  The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Chou, 102147, Computer Science, 3.49).
- To find the records of students who have a grade point average above 3.5 in this database

  C2 is the condition GPA > 3.5.

  The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Rao, 678543, Mathematics, 3.90).
- Finally, to find the records of computer science majors who have a GPA above 3.5, we use the operator *sC3 ,*
- *C3 is the condition (Major="Computer Science" ∧ GPA > 3.5).*

  The result consists of the single 4-tuple (Ackermann, 231455, Computer Science, 3.88).

Projections

- Projections are used to form new *n-ary relations by deleting the same fields in every record* of the relation.
- The *projection P+where i1 < i2 < · · · < im, maps the n-tuple (a1, a2, . . . , an) to them-tuple (ai1 , ai2, . . . , aim), where m ≤ n.*

What relation results when the projection *P1,4 is applied to the relation in Table 1?*

TABLE 2 GPAs.	
Student_name	*GPA*
Ackermann	3.88
Adams	3.45
Chou	3.49
Goodfriend	3.45
Rao	3.90
Stevens	2.99

Example

What is the table obtained when the projection *P1,2 is applied to the relation in Table 3?*

TABLE 3 Enrollments.

Student	*Major*	*Course*
Glauser	Biology	BI 290
Glauser	Biology	MS 475
Glauser	Biology	PY 410
Marcus	Mathematics	MS 511
Marcus	Mathematics	MS 603
Marcus	Mathematics	CS 322
Miller	Computer Science	MS 575
Miller	Computer Science	CS 455

TABLE 4 Majors.

Student	*Major*
Glauser	Biology
Marcus	Mathematics
Miller	Computer Science

Join

- The **join operation is used to combine two tables into one when these tables share some** identical fields.
- For instance, a table containing fields for airline, flight number, and gate, and another table containing fields for flight number, gate, and departure time can be combined into a table containing fields for airline, flight number, gate, and departure time.
- Let R be a relation of degree m and S a relation of degree n. The join $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p)$-tuples $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$, where them-tuple $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p)$ belongs to R and the n-tuple $(c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$ belongs to S.

Example

What relation results when the join operator *J*2 *is used to combine the relation displayed in* Tables 5 and 6?

TABLE 5 Teaching_assignments.

Professor	*Department*	*Course_ number*
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

TABLE 6 Class_schedule.

Department	*Course_ number*	*Room*	*Time*
Computer Science	518	N521	2:00 P.M.
Mathematics	575	N502	3:00 P.M.
Mathematics	611	N521	4:00 P.M.
Physics	544	B505	4:00 P.M.
Psychology	501	A100	3:00 P.M.
Psychology	617	A110	11:00 A M.
Zoology	335	A100	9:00 A M.
Zoology	412	A100	8:00 A M.

TABLE 7 Teaching_schedule.

Professor	*Department*	*Course_number*	*Room*	*Time*
Cruz	Zoology	335	A100	9:00 A M.
Cruz	Zoology	412	A100	8:00 A M.
Farber	Psychology	501	A100	3:00 P M.
Farber	Psychology	617	A110	11:00 A M.
Grammer	Physics	544	B505	4:00 P M.
Rosen	Computer Science	518	N521	2:00 P M.
Rosen	Mathematics	575	N502	3:00 P M.

SQL

- The database query language SQL(Structured Query Language) can be used to carry out the operations(SELECTION,PROJECTION,JOIN).
- SELECT Departure_time FROM Flights WHERE Destination='Detroit'

TABLE 8 Flights.

Airline	*Flight_number*	*Gate*	*Destination*	*Departure_time*
Nadir	122	34	Detroit	08:10
Acme	221	22	Denver	08:17
Acme	122	33	Anchorage	08:22
Acme	323	34	Honolulu	08:30
Nadir	199	13	Detroit	08:47
Acme	222	22	Denver	09:10
Nadir	322	34	Detroit	09:44