Name: Keerthi
ID: cometfwc045
Date: 09/02/2026

Twenty-fourth International Olympiad, 1983

1983/1. Find all functions f defined on the set of positive real numbers which take positive real values and satisfy the conditions:

(i) $f(xf(y)) = yf(x)$ for all positive x, y,

(ii) $f(x) \to 0$ as $x \to \infty$.

1983/2. Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2, respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1, and M_2 be the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

1983/3. Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$, where x, y and z are non-negative integers.

1983/4. Let ABC be an equilateral triangle and E the set of all points contained in the three segments AB, BC and CA (including A, B and C). Determine whether, for every partition of E into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle. Justify your answer.

1983/5. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression? Justify your answer.

1983/6. Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Determine when equality occurs.

Twenty-fifth International Olympiad, 1984

1984/1. Prove that $0 \leq yz + zx + xy - 2xyz \leq 7/27$, where x, y and z are non-negative real numbers for which $x + y + z = 1$.

1984/2. Find one pair of positive integers a and b such that:

(i) $ab(a + b)$ is not divisible by 7;

(ii) $(a + b)^7 - a^7 - b^7$ is divisible by 7^7.

Justify your answer.

1984/3. In the plane two different points O and A are given. For each point X of the plane, other than O, denote by $a(X)$ the measure of the angle between OA and OX in radians, counterclockwise from OA $(0 \leq a(X) < 2\pi)$. Let $C(X)$ be the circle with center O and radius of length $OX + a(X)/OX$. Each point of the plane is colored by one of a finite number of colors. Prove that there exists a point Y for which $a(Y) > 0$ such that its color appears on the circumference of the circle $C(Y)$.

1984/4. Let $ABCD$ be a convex quadrilateral such that the line CD is a tangent to the circle on AB as diameter. Prove that the line AB is a tangent to the circle on CD as diameter if and only if the lines BC and AD are parallel.

1984/5. Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices $(n > 3)$, and let p be its perimeter. Prove that

$$n - 3 < \frac{2d}{p} < \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] - 2,$$

where $[x]$ denotes the greatest integer not exceeding x.

1984/6. Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers k and m, then $a = 1$.

Twenty-sixth International Olympiad, 1985

1985/1. A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

1985/2. Let n and k be given relatively prime natural numbers, $k < n$. Each number in the set $M = \{1, 2, \dots, n-1\}$ is colored either blue or white. It is given that

(i) for each $i \in M$, both i and $n - i$ have the same color;

(ii) for each $i \in M$, $i \neq k$, both i and $|i - k|$ have the same color.

Prove that all numbers in M must have the same color.

1985/3. For any polynomial

$$P(x) = a_0 + a_1 x + \cdots + a_k x^k$$

with integer coefficients, the number of coefficients which are odd is denoted by $w(P)$. For $i = 0, 1, \dots$, let $Q_i(x) = (1+x)^i$. Prove that if $i_1, i_2, \dots, i_n$ are integers such that $0 \leq i_1 < i_2 < \cdots < i_n$, then

$$w(Q_{i_1} + Q_{i_2} + \cdots + Q_{i_n}) \geq w(Q_{i_1}).$$

1985/4. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer.

1985/5. A circle with center O passes through the vertices A and C of triangle ABC and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangles ABC and EBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle.

1985/6. For every real number x_1, construct the sequence $x_1, x_2, \dots$ by setting

$$x_{n+1} = x_n \left(x_n + \frac{1}{n} \right) \quad \text{for each } n \geq 1.$$

Prove that there exists exactly one value of x_1 for which

$$0 < x_n < x_{n+1} < 1$$

for every n.