ELEC 533 - Solution Homework #5

Problem 1.

(a) The characteristic function for X_{kn} is given as

$$\Phi_{X_{kn}} = E[e^{iuX_{kn}}] = 1 \cdot (1 - \frac{\lambda}{n}) + e^{iu}\frac{\lambda}{n} = 1 + \frac{\lambda}{n}(e^{iu} - 1).$$

Since $X_{1n}, \ldots, X_{nn}$ are i.i.d. random variables

$$\Phi_{Y_n}(u) = [1 + \frac{\lambda}{n}(e^{iu} - 1)]^n.$$

(b) Since $\lim_{n\to\infty}(1 + \frac{a}{n})^n = e^a$ then $\lim_{n\to\infty} \Phi_{Y_n}(u) = e^{\lambda(e^{iu}-1)}$.

This is the characteristic function of a r.v. (say Y) with Poisson distribution and parameter λ, i.e.,

$$P_Y(k) = \begin{cases} 0 & \text{for } k < 0 \\ \frac{\lambda^k e^{-\lambda}}{k!} & \text{for } k \geq 0 \end{cases}$$

Problem 2.

a)

$$P(|X| > b) = \int_{-\infty}^{-b} dF_X + \int_b^{\infty} dF_X \geq \int_{-\infty}^{-b} g(x)dF_X + \int_b^{\infty} g(x)dF_X \qquad (\text{since } g(x) \leq 1 \ \forall x)$$

$$= \int_{-\infty}^{\infty} g(x)dF_X - \int_{-b}^{b} g(x)dF_X \geq E[g(x)] - g(b)\int_{-b}^{b} dF_X \qquad (\text{since } g(x) \text{ is increasing})$$

$$\geq E[g(x)] - g(b) \qquad (\text{since } \int_{-b}^{b} dF_X \leq 1)$$

(b) Let $A_i = [\omega \in \Omega \,||\, X_i(\omega) - X(\omega)| \geq \epsilon]$.

"$\Rightarrow$"

$$X_i \xrightarrow{\text{i.p.}} X \Rightarrow 0 = \lim_{i\to\infty} P(|X_i - X| \geq \epsilon) \geq \lim_{i\to\infty} E[g(|X_i - X|)] - g(\epsilon) \qquad \text{(from part (a))}$$

$$\Rightarrow 0 \geq \lim_{i\to\infty} E[g(|X_i - X|)] \leq g(\epsilon).$$

Let $\epsilon \to 0$. With $g(0) = 0$ and $g(\cdot)$ continuous and increasing, it is obvious that $\lim_{i\to\infty} E[g(|X_i - X|)] = 0$.

"$\Leftarrow$"

$$\lim_{i\to\infty} E[g(|X_i - X|)] = 0 = \lim_{i\to\infty} \int_{A_i} g(|x_i - x|)dP + \lim_{i\to\infty} \int_{A_i^c} g(|x_i - x|)dP \geq g(\epsilon) \lim_{i\to\infty} \int_{A_i} dP \geq 0$$

since $\inf_{A_i} |X_i - X| = \epsilon$ and $\inf_{A_i^c} |X_i - X| = 0$. With $g(\epsilon) > 0 \ \forall \epsilon > 0$ we get that $\lim_{i\to\infty} \int_{A_i} dP = 0$ or $X_i \xrightarrow{\text{i.p.}} X$. Therefore

$$X_i \xrightarrow{\text{i.p.}} X \Leftrightarrow E[g(|X_i - X|)] = 0$$

Problem 3.

(a) The characteristic function of $\underline{X}$ is

$$\Phi_{\underline{X}}(\underline{u}) = \exp[i\underline{\mu}^T \underline{u} - \frac{1}{2}\underline{u}^T \Sigma \underline{u}]$$

Denoting by $\nabla_{\underline{u}}$ the gradient with respect to $\underline{u}$

$$\nabla_{\underline{u}} \Phi_{\underline{X}}(\underline{u}) = (i\underline{\mu} - \Sigma \underline{u}) \Phi_{\underline{X}}(\underline{u})$$

and

$$\nabla_{\underline{u}}(\nabla_{\underline{u}} \Phi_{\underline{X}}(\underline{u})) = [-\Sigma + (i\underline{\mu} - \Sigma\underline{u})(i\underline{\mu} - \Sigma\underline{u})^T] \Phi_{\underline{X}}(\underline{u})$$

Now,

$$E[\underline{X}] = -i \nabla_{\underline{u}} \Phi_{\underline{X}}(\underline{u})|_{\underline{u}=\underline{0}} = \underline{\mu}$$

and

$$Cov(\underline{X}, \underline{X}) = E[\underline{X}\underline{X}^T] - E[\underline{X}]\, E[\underline{X}]^T = -1 \cdot [\nabla_{\underline{u}}(\nabla_{\underline{u}} \Phi_{\underline{X}}(\underline{u}))]|_{\underline{u}=\underline{0}} - \underline{\mu}\,\underline{\mu}^T = \Sigma.$$

(b) Let $\underline{Y} = A\underline{X} + \underline{b}$. Then,

$$\Phi_{\underline{Y}}(\underline{u}) = E[e^{i\underline{u}^T\underline{y}}] = E[e^{i\underline{u}^T A\underline{x}}]e^{i\underline{u}^T\underline{b}} = \Phi_{\underline{X}}(A^T\underline{u})e^{i\underline{u}^T\underline{b}} = \exp[i\underline{u}^T(A\underline{\mu}+\underline{b}) - \frac{1}{2}\underline{u}^T(A\Sigma A^T)\underline{u}].$$

Thus, $Y \sim N(A\underline{\mu} + \underline{b}, A\Sigma A^T)$.

(c) Compare with (b) and set $A = C^{-1}$, $\underline{b} = -C^{-1}\underline{\mu}$. Then,

$$Z = C^{-1}(\underline{X} - \underline{\mu}) \sim N(C^{-1}\underline{\mu} - C^{-1}\underline{\mu}, C^{-1}\Sigma(C^{-1})^T) = N(\underline{0}, I).$$

Problem 4.

(a) Notice that

$$F_{\frac{X_i}{n}}(x) = Pr(\frac{X_i}{n} \leq x) = Pr(X_i \leq nx) = F_{X_i}(nx)$$

Therefore,

$$f_{\frac{X_i}{n}}(x) = nf_{X_i}(nx) = \frac{1}{\pi}\frac{na}{a^2 + n^2x^2} = \frac{1}{\pi}\frac{a/n}{(a/n)^2 + x^2}$$

The previous p.d.f. is Cauchy with parameter $\frac{a}{n}$ and consequently

$$\Phi_{\frac{X_i}{n}}(u) = e^{-\frac{a|u|}{n}} \Rightarrow \Phi_{\overline{X}}(u) = \prod_{i=1}^{n}\Phi_{\frac{X_i}{n}}(u) = e^{-a|u|}$$

Therefore $\overline{X}$ is Cauchy with parameter a!

(b) The CLT does not apply because $E[X_i]$ does not exist, since each X_i is Cauchy. Alternatively, the CLT does not apply because var(X_i) is infinite.

Must compute E(x) explicitly using the probability density function of Cauchy, and show that the integral doesn't exist.