# ELEC 533: Homework 8

Due on : Please check online

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## Problem 1

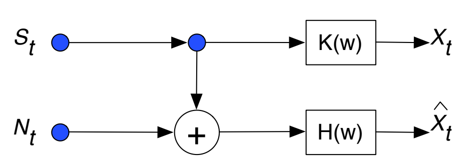
a) E $[(X_{t+\tau} - X_t)^2] = R_X(0) + R_X(0) - 2R_X(\tau) = 2[R_X(0) - R_X(\tau)]$
Chebysheff Inequality: $\mathcal{P}(|X_{t+\tau} - X_t| > a) \leq \frac{\mathrm{E}[(X_{t+\tau}-X_t)^2]}{a^2} = 2\frac{R_X(0)-R_X(\tau)}{a^2}$

b) $\mathrm{cov}(X_t, X_t') = R_{XX'}(t,t) - \mathrm{E}[X_t]\mathrm{E}[X_t']$

$$\begin{aligned} R_{XX'}(t,t) &= \mathrm{E}[X_t \lim_{t'\to t} \frac{X_{t'} - X_t}{t' - t}] = \lim_{t'\to t} \mathrm{E}\left[\frac{X_t X_{t'} - X_t X_t}{t' - t}\right] = \\ &= \frac{d}{d\tau} R_X(\tau)\Big|_{\tau=0} = 0 \text{ since } R_X(\tau) \text{ is even and differentiable} \end{aligned}$$

$E[X_t'] = 0 \Rightarrow cov(X_t, X_t') = 0$
$X_t$ and its derivative $X_t'$ are uncorrelated !

## Problem 2




$X_t - \hat{X}_t = \int\limits_{-\infty}^{\infty} (k(\tau) - h(\tau))S_{t-\tau}d\tau - \int\limits_{-\infty}^{\infty} h(\tau)N_{t-\tau}d\tau = (k(t) - h(t)) * S_t - h(t) * N_t$
$R_{X-\hat{X}}(t) = (k(t) - h(t)) * (k(-t) - h(-t)) * R_S(t) - h(t) * h(-t) * R_N(t)$, because $R_{SN}(t) = 0$.
$S_{X-\hat{X}}(\omega) = |K(\omega) - H(\omega)|^2 S_S(\omega) + |H(\omega)|^2 S_N(\omega)$

$$\mathrm{E}[(X_t - \hat{X}_t)^2] = R_{X-\hat{X}}(0) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} S_{X-\hat{X}}(\omega)d\omega = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \Big(|K(\omega) - H(\omega)|^2 S_S(\omega) + |H(\omega)|^2 S_N(\omega)\Big)\,d\omega$$

## Problem 3

We know that $|R_{\widehat{X}\widehat{Y}}(\tau)|^2 \leq R_{\widehat{X}}(0)R_{\widehat{Y}}(0),\ \forall\tau$ for two processes $\{\widehat{X}_t;\ t\in\mathbb{R}\}$, $\{\widehat{Y}_t;\ t\in\mathbb{R}\}$ which gives us for $\tau = 0$, the following:

$$|R_{\widehat{X}\widehat{Y}}(0)|^2 \leq R_{\widehat{X}}(0)R_{\widehat{Y}}(0)$$

Consider the following filter : $H(\omega) = \begin{cases} 1 & \omega \in [\omega_1, \omega_2] \\ 0 & \omega \notin [\omega_1, \omega_2] \end{cases}$

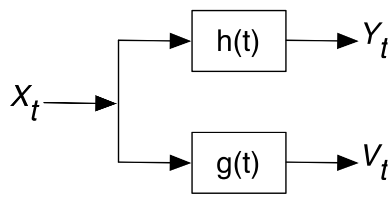
Pass $X_t$ and $Y_t$ through this filter to obtain $\widehat{X}_t$ and $\widehat{Y}_t$, respectively.
Then: $S_{\widehat{X}}(\omega) = |H(\omega)|^2 S_X(\omega)$ and $S_{\widehat{Y}}(\omega) = |H(\omega)|^2 S_Y(\omega)$.
Furthermore, $S_{\widehat{X}\widehat{Y}}(\omega) = |H(\omega)|^2 S_{XY}(\omega)$

Now we can rewrite the autocorrelations as: $R_{\widehat{X}\widehat{Y}}(0) = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} S_{XY}(\omega)d\omega$, $R_{\widehat{X}}(0) = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} S_X(\omega)d\omega$ and respectively $R_{\widehat{Y}}(0) = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} S_Y(\omega)d\omega$.

Having all of the above we can rewrite the inequality we had for $\tau = 0$:

$$4\pi^2 |R_{\widehat{X}\widehat{Y}}(0)|^2 = \left| \int_{\omega_1}^{\omega_2} S_{XY}(\omega)d\omega \right|^2 \leq \int_{\omega_1}^{\omega_2} S_X(\omega)d\omega \int_{\omega_1}^{\omega_2} S_Y(\omega)d\omega = 4\pi^2 R_{\widehat{X}}(0) R_{\widehat{Y}}(0)$$

# Problem 4




a)

$$\begin{aligned}
\mathrm{E}[Y_t V_s] &= \mathrm{E}[\int_{-\infty}^{\infty} h(t-\alpha)X_\alpha d\alpha \int_{-\infty}^{\infty} g(s-\beta)X_\beta d\beta] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t-\alpha)g(s-\beta)R_X(\alpha-\beta)d\alpha d\beta \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t-\alpha)g(s-\beta)\delta(\alpha-\beta)d\alpha d\beta \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty} h(\gamma)g(\gamma-(t-s))d\gamma
\end{aligned}$$

$$\Rightarrow R_{YV}(\tau) = \frac{N_0}{2}\int_{-\infty}^{\infty} h(\gamma)g(\gamma-(t-s))d\gamma = \frac{N_0}{2}(h * \widehat{g})(\tau)$$

where $\widehat{g}$ is obtained from $g$ by time reversal.

b) $X_t$ is a Gaussian r.p. $\Rightarrow$ $Y_t$ and $V_t$ are jointly Gaussian r.p.'s. Thus, $Y_t$ and $V_t$ are independent when they are uncorrelated: $R_{YV}(\tau) = 0, \forall\, \tau \in \mathbb{R} \Rightarrow S_{YV}(\omega) = \frac{N_0}{2}H(\omega)G^*(\omega) = 0\ \forall \omega \in \mathbb{R}$.
If $H(\omega)G^*(\omega) = 0\ \forall\omega \in \mathbb{R}$ $Y_t$ and $V_t$ are independent.

# Problem 5

$$Y_t' + \alpha Y_t = X_t \quad \text{m.s.} \quad \text{with} \quad Y_0 = 0$$

a) take expectation of the differential equation:

$$\begin{aligned} \mathrm{E}[Y_t'] + \alpha \mathrm{E}[Y_t] &= \mathrm{E}[X_t] \quad \text{m.s.} \quad \text{with} \quad Y_0 = 0 \\ \frac{d}{dt}\mu_Y(t) + \alpha\mu_Y(t) &= \mu_X(t) = 0 \end{aligned}$$

$$\text{the initial condition} \quad Y_0 = 0 \Rightarrow \mathrm{E}[Y_0] = \mu_y(0) = 0$$

b) multiply the differential equation with the conjugate of $X_t$:

$$\begin{aligned} Y_t' X_s^* + \alpha Y_t X_s^* &= X_t X_s^* \\ \text{take expectation}: \quad \mathrm{E}[Y_t' X_s^*] + \mathrm{E}[\alpha Y_t X_s^*] &= \mathrm{E}[X_t X_s^*] \\ R_{Y'X}(t,s) + \alpha R_{YX}(t,s) &= R_{XX}(t,s) \\ \frac{d}{dt} R_{YX}(t,s) + \alpha R_{YX}(t,s) &= R_{XX}(t,s) = a^2\delta(t-s) \end{aligned}$$

$$\text{the initial condition} \quad Y_0 = 0 \Rightarrow R_{YX}(0,s) = \mathrm{E}[Y_0 X_s] = \mathrm{E}[0 X_s] = 0 \Rightarrow R_{YX}(0,s) = 0 \ \forall \ s \in \mathbb{R}$$

c) multiply the differential equation by $Y_s^*$

$$\begin{aligned} Y_t' Y_s^* + \alpha Y_t Y_s^* &= X_t Y_s^* \\ \text{take expectation}: \quad \mathrm{E}[Y_t' Y_s^*] + \mathrm{E}[\alpha Y_t Y_s^*] &= \mathrm{E}[X_t Y_s^*] \\ R_{Y'Y}(t,s) + \alpha R_{YY}(t,s) &= R_{XY}(t,s) \\ \frac{d}{dt} R_{YY}(t,s) + \alpha R_{YY}(t,s) &= R_{XY}(t,s) \end{aligned}$$

$$\text{the initial condition} \quad Y_0 = 0 \Rightarrow R_{YY}(0,s) = \mathrm{E}[Y_0 Y_s] = \mathrm{E}[0 Y_s] = 0 \Rightarrow R_{YY}(0,s) = 0 \ \forall \ s \in \mathbb{R}$$

d) $Y_t$ is WSS since its mean is constant and the autocorrelation depends only on $(t-s)$. $Y_t$ is Gaussian, derivation of a Gaussian gives a Gaussian (because of the exponential function), linear transformation of a Gaussian gives a Gaussian.