Homework 1

ELEC 540: Advanced Wireless Communications

Problems are from *Fundamentals of Wireless Communications,* by David Tse and Pramod Viswanath. The equation number in the problems refer to the ones in Chapter 3 of the book.

1. (Book Problem 3.1, 20 points)
Verify (3.19) and the high SNR approximation (3.21).
Hint: Write the expression as a double integral and interchange the order of integration.

Solution: We have,

$$\begin{aligned}
P_e &= E_h[Q(\sqrt{2|h|^2\mathsf{SNR}})], \\
&= \int_0^\infty e^{-x} \int_{\sqrt{2x\mathsf{SNR}}}^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt dx, \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^{t^2/(2\mathsf{SNR})} e^{-t^2/2} e^{-x} dx dt, \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2}(1 - e^{-t^2/(2\mathsf{SNR})}) dt, \\
&= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2(1+1/\mathsf{SNR})/2} dt, \\
&= \frac{1}{2}\left(1 - \sqrt{\frac{\mathsf{SNR}}{1+\mathsf{SNR}}}\right),
\end{aligned}$$

where the third step follows from changing the order of integration. Now, for large SNR, we also have

$$\sqrt{\frac{\text{SNR}}{1+\text{SNR}}} \approx 1 - \frac{1}{2\text{SNR}}$$

which implies

$$P_e \approx \frac{1}{4\text{SNR}}$$

2. (Book Problem 3.7, 30 points)
In this exercise, we study the performance of the rotated QAM code in Section 3.2.2.

1. Give an explicit expression for the *exact* pairwise error probability $\mathbb{P}\{x_A \to x_B\}$ in (3.49). Hint: The techniques from Exercise 1 will be useful here.
2. This pairwise error probability was upper bounded in (3.54). Show that the product of SNR and the difference between the upper bound and the actual pairwise error probability goes to zero with increasing SNR. In other words, the upper bound in (3.54) is tight up to the leading term in 1/SNR.

Solution:

1. We have (for simplicity we denote the squared-distances as d_1 and d_2)

$$\begin{aligned}
\mathbb{P}[x_A \to x_B] &= E_{h_1,h_2}\left[Q\left(\sqrt{\frac{\mathsf{SNR}(|h_1|^2 d_1 + |h_2|^2 d_2)}{2}}\right)\right] \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty \int_0^\infty \int_{\sqrt{\frac{\mathsf{SNR}(xd_1+yd_2)}{2}}}^\infty e^{-t^2/2}e^{-x}e^{-y}dtdxdy \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty \int_0^{2t^2/(d_2\mathsf{SNR})} \int_0^{(2t^2/\mathsf{SNR}-d_2y)/d_1} e^{-x}dxe^{-y}dye^{-t^2/2}dt \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty \int_0^{2t^2/(d_2\mathsf{SNR})} (1-e^{-(2t^2/\mathsf{SNR}-d_2y)/d_1})e^{-y}dye^{-t^2/2}dt \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty \left(1-e^{-2t^2/(d_2\mathsf{SNR})} - \frac{e^{-2t^2/d_1\mathsf{SNR}}}{1-d_2/d_1}(1-e^{-(1/d_2-/d_1)2t^2/\mathsf{SNR}})\right)e^{-t^2/2}dt, \\
&= 0.5 + \frac{1}{\sqrt{2\pi}(d_2-d_1)}\int_0^\infty e^{-t^2/2}(d_1e^{-2t^2/(d_1\mathsf{SNR})} - d_2e^{-2t^2/(d_2\mathsf{SNR})})dt, \\
&= 0.5 + \frac{0.5}{d_2-d_1}\left(\frac{d_1}{\sqrt{1+4/(d_1\mathsf{SNR})}} - \frac{d_2}{\sqrt{1+4/(d_2\mathsf{SNR})}}\right)
\end{aligned}$$

2. For the high SNR scenario, we get (using Taylor series),

$$\frac{1}{\sqrt{1+4/(d_2\mathrm{SNR}}} = 1 - 2/(d_2\mathrm{SNR}) + 6/(d_2^2\mathrm{SNR}^2)$$

which implies,

$$\mathbb{P}[x_A \to x_B] = 3/(d_1 d_2 \mathrm{SNR}^2)$$

3. (Book Problem 3.10, 50 points)
In Section 3.2.2, we looked at the example of the rotation code to achieve time diversity (with the number of branches, L, equal to 2). Another coding scheme is the permutation code. Shown in Figure 1 are two 16-QAM constellations. Each codeword in the permutation code for $L = 2$ is obtained by picking a pair of points, one from each constellation, which are represented by the same icon. The codeword is transmitted over two (complex) symbol times.

1. Why do you think this is called a permutation code?
2. What is the data rate of this code?
3. Compute the diversity gain and the minimum product distance for this code.

Solution:

1. The code can be represented by a permutation of the 16-point QAM. What is transmitted on the second subchannel can be obtained as a simple permutation of what is transmitted on the first sub-channel.
2. Data rate = 2 bits/channel use (since all the information is contained in one of the QAMs itself).

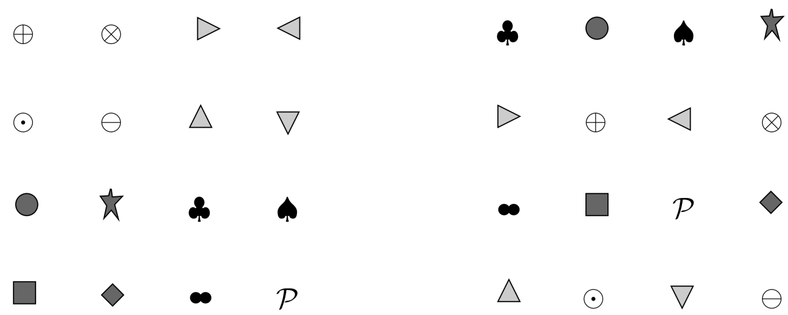

Figure 1: A permutation code.

3. Since the product distance is non-zero, the diversity gain is 2. The minimum product distance is given by by $64a^4$ where $2a$ is the minimum distance between the QAM symbols. Then, by normalizing the average receiver SNR to be 1 per time symbol, we get:

$$2 \times 2 \times (4 \times 20a^2)/16 = 1 \implies a^2 = 0.05$$

Therefore, the product distance is given by 0.32.