Homework 3

ELEC 540: Advanced Wireless Communications

5 Problems, *** points

Problems are taken from *Fundamentals of Wireless Communications,* by David Tse and Pramod Viswanath.

1. (Book Problem 5.13, 30 points)
Consider a system with 1 transmit antenna and L receive antennas. Independent $\mathcal{CN}(0, N_0)$ noise corrupts the signal at each of the receive antennas. The transmit signal has a power constraint of P.

1. Suppose the gain between the transmit antenna and each of the receive antennas is constant, equal to 1. What is the capacity of the channel? What is the performance gain compared to a single receive antenna system? What is the nature of the performance gain?
2. Suppose now the signal to each of the receive antenna is subject to independent Rayleigh fading. Compute the capacity of the (fast) fading channel with channel information only at the receiver. What is the nature of the performance gain compared to a single receive antenna system? What happens when $L \to \infty$?
3. Give an expression for the capacity of the fading channel in part 2. with CSI at both the transmitter and the receiver. At low SNR, do you think the benefit of having CSI at the transmitter is more or less significant when there are multiple receive antennas? How about when the operating SNR is high?
4. Now consider the slow fading scenario when the channel is random but constant. Compute the outage probability and quantify the performance gain of having multiple receive antennas.

Solution:

1. Let $\mathbf{1} = [1,\ 1,\cdots,\ 1]^T \in \mathbb{R}^L$. Then the channel equation is: $y = \mathbf{1}x + z$, where $z \sim CN(0, N_0 I_L)$ and x must satisfy the power constraint $E[x] \leq P$. We note that we can project the received signal onto the direction of $\mathbf{1}$ obtaining the sufficient statistic:

$$r = \frac{\mathbf{1}}{\sqrt{L}} y = \sqrt{L}x + \tilde{z}$$

   where, $\tilde{z} \sim CN(0, N_0)$. Defining $\tilde{x} = \sqrt{L}x$ we see that we have an AWGN channel with power constraint LP and noise variance N_0. Therefore $C = log(1 + LP/N_0)$.
   We see that there is a power gain of L with respect to the single receive antenna system.

2. Let $\mathbf{h} = [h_1, h_2, \cdots, h_L]^T \in \mathbb{C}^L$. Then the channel equation is: $y = \mathbf{h}x + z$, where $z \sim CN(0, N_0 I_L)$, $\mathbf{h}$ is known at the receiver and x must satisfy the power constraint

$E[x] \leq P$. Since the receiver knows the channel, it can project the received signal onto the direction of $\mathbf{h}$ obtaining the sufficient statistic:

$$r = \frac{\mathbf{h}}{\|\mathbf{h}\|} y = \|\mathbf{h}\| x + \tilde{z}$$

where $\tilde{z} \sim CN(0, N_0)$. Then the problem reduces to computing the capacity of a scalar fading channel, with fading coefficient given by $\|\mathbf{h}\|$. It follows that:

$$C = E\left[\log\left(1 + \frac{\|\mathbf{h}\|^2 P}{N_0}\right)\right] = E\left[\log\left(1 + \frac{LP}{N_0}\frac{\|\mathbf{h}\|^2}{L}\right)\right]$$

In contrast, the single receive antenna system has a capacity $C = E\left[\log(1 + |h|^2 P/N_0)\right]$. The capacity is increased by having multiple receive antennas for two reasons: first there is a power gain L, and second $\|\mathbf{h}\|^2/L$ has the same mean but less variance than $|h|^2$, and we get a diversity gain. Note that $Var[\|\mathbf{h}\|^2/L] = 1/L$ whereas $Var[|h|^2] = 1$. As $L \to \infty$, $\|\mathbf{h}\|^2/L \to 1$ almost surely, so it follows that $C \approx \log(1 + LP/N_0)$ for large L.

3. With full CSI, the transmitter knows the channel, and for a given realization of the fading process $\{\mathbf{h}[n]\}_{n=1}^{N}$ the channel supports a rate:

$$R = \frac{1}{N}\sum_{n=1}^{N} \log\left(1 + \frac{\|\mathbf{h}[n]\|^2 P[n]}{N_0}\right)$$

and the problem becomes that of finding the optimal power allocation strategy. We note that the problem is the same as the one corresponding to the case of a single receive antenna, replacing $|h[n]|^2$ by $\|h[n]\|^2$. It follows that the optimal solution is also obtained by waterfilling:

$$P^*(\|\mathbf{h}\|^2) = \left(\frac{1}{\lambda} - \frac{N_0}{\|\mathbf{h}\|^2}\right)^+$$

where λ is chosen so that the power constraint is satisfied, i.e. $E[P(\|h\|^2)] = P$. The resulting capacity is:

$$C = E\left[\log\left(1 + \frac{P^*\|h\|^2}{N_0}\right)\right]$$

At low SNR, when the system is power limited, the benefit of having CSI at the transmitter comes from the fact that we can transmit only when the channel is good, saving power (which is the limiting resource) when the channel is bad. The larger the fluctuation in the channel gain, the larger the benefit. If the channel gain is constant, then the waterfilling strategy reduces to transmitting with constant power, and there is no benefit in having CSI at the transmitter. When there are multiple receive antennas, there is diversity and $\|h\|^2/L$ does not fluctuate much. In the limit as $L \to \infty$ we have seen that this random variable converges to a constant with probability one. Then, as L increases, the benefit of having CSI at the transmitter is reduced.

4.

$$P_{out} = Pr\left[\log\left(1 + \frac{\|\mathbf{h}\|^2 P}{N_0}\right) < R\right] = Pr\left[\|\mathbf{h}\|^2 < (2^R - 1)\frac{N_0}{P}\right]$$

We know that we can approximate the pdf of $\|\mathbf{h}\|_2$ around 0 by: $f(x) \approx x^{L-1}/(L-1)!$, where Rayleigh fading was assumed, and hence the distribution function of $\|\mathbf{h}\|^2$ evaluated at x is approximately given by: $F(x) \approx x^L/L!$ for x small. Thus, for large SNR we get the following approximation for the outage probability: $P_{out} \approx 1/L! \left[(2^R-1)N_0/P\right]^L$.
We see that having multiple antennas reduces the outage probability by a factor of $(2^R - 1)^L/L!$ and also increases the exponent of SNR^{-1} by a factor of L.

2. (Book Problem 5.14, 20 points)
Consider a MISO slow fading channel.

1. Verify that the Alamouti scheme radiates energy in an isotropic manner.
2. Show that a transmit diversity scheme radiates energy in an isotropic manner if and only if the signals transmitted from the antennas have the same power and are uncorrelated

Solution:

1.

The Alamouti scheme transmits two independent symbols u_1, u_2 over the two antennas in two channel uses as follows:

$$\mathbf{X} = \begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix}.$$

To show that the scheme radiates energy in an isotropic manner, we need to show that the energy in the projection of this codeword in any direction $d \in C^2$ depends only on the magnitude of d and not its direction. Let $E[u_1u_2] = 0$ and $E[|u_1|^2] = E[|u_2|^2] = P/2$. We then have:

$$\mathbf{d}^\dagger \mathbb{E}[\mathbf{X}\mathbf{X}^\dagger]\mathbf{d} = [d_1^* \ d_2^*] \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = P\|\mathbf{d}\|^2.$$

2.

Suppose that the transmitted vector $\mathbf{x} = [x_1 \ x_2]^T$ is such that $\mathbb{E}[x_1x_2^*] = 0$ and $\mathbb{E}[|x_1|^2] = \mathbb{E}[|x_2|^2] = P$. Then, for any $\mathbf{d} = [d_1 \ d_2]^T$,

$$\mathbf{d}^\dagger \mathbb{E}[\mathbf{x}\mathbf{x}^\dagger]\mathbf{d} = \mathbf{d}^\dagger \mathbb{E} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1^* & x_2^* \end{bmatrix} \mathbf{d} = \mathbf{d}^\dagger P \mathbf{I} \mathbf{d} = P\|\mathbf{d}\|^2,$$

hence the scheme radiates energy isotropically.

To prove the converse, assume that the scheme $\mathbf{x} = [x_1 \ x_2]^T$ is isotropic, i.e., for any two vectors $\mathbf{d}_a$ and $\mathbf{d}_b$ such that $\|\mathbf{d}_a\|^2 = \|\mathbf{d}_b\|^2 = 1$, we have that

$$\mathbf{d}_a^\dagger \mathbb{E}[\mathbf{x}\mathbf{x}^\dagger]\mathbf{d}_a = \mathbf{d}_b^\dagger \mathbb{E}[\mathbf{x}\mathbf{x}^\dagger]\mathbf{d}_b. \tag{5.19}$$

Then we must prove that $E[x_1x_2] = 0$ and $E[|x_1|^2] = E[|x_2|^2]$. To see that this must be so, first choose $d_a = [10]^T$ and $d_b = [01]^T$. Substituting this into (5.19) we obtain that $E[|x_1|^2] = E[|x_2|^2]$.

Now, choose $\mathbf{d}_a = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, and $\mathbf{d}_b = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$. Then, (5.19) yields

$$\mathbb{E}[|x_1|^2 + x_1^* x_2 + x_1 x_2^* + |x_2|^2] = \mathbb{E}[|x_1|^2 - x_1^* x_2 - x_1 x_2^* + |x_2|^2],$$

Hence we get that $\mathbb{E}[x_1^* x_2 + x_1 x_2^*] = 0$ which implies that $\text{Real}(\mathbb{E}[x_1^* x_2]) = 0$.

Now, choose $\mathbf{d}_a = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{j\sqrt{2}} \end{bmatrix}$, and $\mathbf{d}_b = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{j\sqrt{2}} \end{bmatrix}$. Then, (5.19) yields

$$\mathbb{E}\left[|x_1|^2 + \frac{x_1^* x_2}{j} - \frac{x_1 x_2^*}{j} + |x_2|^2\right] = \mathbb{E}\left[|x_1|^2 - \frac{x_1^* x_2}{j} + \frac{x_1 x_2^*}{j} + |x_2|^2\right].$$

Hence we get that $\mathbb{E}\left[\frac{x_1^* x_2}{j} - \frac{x_1 x_2^*}{j}\right] = 0$, which implies that $\text{Imag}(\mathbb{E}[x_1^* x_2]) = 0$.
Thus we conclude that $\mathbb{E}[x_1^* x_2] = 0$ and we have established the converse.

3. (Book Problem 5.15, 30 points)
Consider the MISO channel with L transmit antennas and channel gain vector $\mathbf{h} = [h_1, \cdots, h_L]^t$. The noise variance is N_0 per symbol and the total power constraint across the transmit antennas is P.

1. First, think of the channel gains as fixed. Suppose someone uses a transmission strategy for which the input symbols at any time is zero mean and has a covariance matrix $\mathbf{K}_x$. Argue that the maximum achievable reliable rate of communication under this strategy is no larger than $\log\left(1 + \frac{\mathbf{h}^t \mathbf{K}_x \mathbf{h}}{N_0}\right)$ bits/symbol.
2. Now suppose we are in a slow fading scenario and $\mathbf{h}$ is random and i.i.d. Rayleigh. The outage probability of the scheme in Part 1 is given by $P_{out}(R) = \mathbb{P}\left\{\left(1 + \frac{\mathbf{h}^t \mathbf{K}_x \mathbf{h}}{N_0}\right) < R\right\}$.

   Show that correlation never improves the outage probability: i.e., given a total power constraint P , one can do no worse by choosing $\mathbf{K}_x$ to be diagonal.
   Hint: Observe that the covariance matrix $\mathbf{K}_x$ admits a decomposition of the form $\mathbf{U}diag\{P_1, \cdots, P_L\}\mathbf{U}$.

Solution:

1. A MISO channel is given by the following input-output relation:
$$y[m] = \mathbf{h}^\dagger \mathbf{x}[m] + z[m],$$
   with the total power constraint $\mathbb{E}[\|\mathbf{x}\|^2] \le P$. The received SNR is given by
$$\mathsf{SNR} = \frac{\mathbb{E}[|\mathbf{h}^\dagger \mathbf{x}|^2]}{N_0} = \frac{\mathbb{E}[\mathbf{h}^\dagger \mathbf{x}\mathbf{x}^\dagger \mathbf{h}]}{N_0} = \frac{\mathbf{h}^\dagger \mathbf{K}_x \mathbf{h}}{N_0}.$$
   Thus this channel is equivalent to a scalar channel with the same received SNR.
   Hence, the maximal rate of reliable communication on this channel is given by
$$C = \log(1 + \mathsf{SNR}) = \log\left(1 + \frac{\mathbf{h}^\dagger \mathbf{K}_x \mathbf{h}}{N_0}\right).$$

2. Since the covariance matrix K_x is positive semi-definite, it admits the decomposition $K_x = U\Sigma U^T$, where Σ is a diagonal matrix and U is a unitary matrix. Since the channel is

i.i.d. Rayleigh, the vector h is isotropically distributed, i.e., $h^T U$ has the same distribution as h^T. Thus the quadratic form $hTK_x h = h^T U\Sigma U^T h = (h^T U)\Sigma(h^T U)^T$ has the same distribution as $h^T \Sigma h$. Therefore, we can restrict K_x to be diagonal without sacrificing outage performance.

4. (Book Problem 5.21, 20 points)
In Chapter 3, we have seen that one way to communicate over the MISO channel is to convert it into a parallel channel by sending symbols over the different transmit antennas one at at time.

1. Consider first the case when the channel is fixed (known to both the transmitter and the receiver). Evaluate the capacity loss of using this strategy at high and low SNR. In which regime is this transmission scheme a good idea?
2. Now consider the slow fading MISO channel. Evaluate the loss in performance of using this scheme in terms of (i) the outage probability $p_{out}(R)$ at high SNR; (ii) the ϵ-outage capacity C_ϵ at low SNR.

Solutions:

1.

Using only one antenna at a time, we convert the MISO channel into a parallel channel. The maximal rate achievable with this strategy is given by:

$$C^{\text{parallel}} = \frac{1}{L}\sum_{l=1}^{L} \log(1 + |h_l|^2 \mathsf{SNR}),$$

compared by the capacity of this MISO channel (observe that the channel gain is constant and known to both the transmitter and receiver):

$$C^{\text{MISO}} = \log(1 + \|\mathbf{h}\|^2 \mathsf{SNR}).$$

At high-SNR, we can approximate the two rates as follows:

$$C^{\text{parallel}} \approx \log \mathsf{SNR} + \frac{1}{L}\sum_{l=1}^{L} \log |h_l|^2,$$
$$C^{\text{MISO}} \approx \log \mathsf{SNR} + \log \|\mathbf{h}\|^2.$$

Hence, at high-SNR, the ratio of the two rates goes to 1.

At low-SNR, we can make the following approximations:

$$\begin{aligned} C^{\text{parallel}} &\approx \frac{1}{L}\sum_{l=1}^{L} |h_l|^2 \mathsf{SNR} \log_2 e = \frac{1}{L}\|\mathbf{h}\|^2 \mathsf{SNR} \log_2 e, \\ C^{\text{MISO}} &\approx \|\mathbf{h}\|^2 \mathsf{SNR} \log_2 e. \end{aligned}$$

Thus, the loss from capacity goes to $\frac{1}{L}$ as $\mathsf{SNR} \to 0$.

The parallelization scheme is degree-of-freedom efficient so at high-SNR its performance is close to the optimal performance on the MISO channel due to the fact that the AWGN MISO channel is degree-of-freedom limited at high-SNR. However, the optimal strategy is for the transmitter to do beamforming (having knowledge of the channel) and hence harness the power gain afforded in this way. The parallelization scheme does not perform beamforming and hence suffers a loss from capacity in the SNR-limited low-SNR regime.

2.

The outage probability expressions of the MISO channel and the scheme which turns it into a parallel channel are given by:

$$\begin{aligned} P_{\text{out}}^{\text{parallel}} &:= \mathbb{P}\left(\sum_{l=1}^{L} \log(1+|h_l|^2\mathsf{SNR}) < LR\right), \\ P_{\text{out}}^{\text{MISO}} &:= \mathbb{P}\left(\log(1+\|\mathbf{h}\|^2\mathsf{SNR}) < R\right). \end{aligned}$$

Assuming i.i.d. Rayleigh fading, we can use the result of Exercise 5.18 to obtain the high-SNR approximations:

$$\begin{aligned} P_{\text{out}}^{\text{parallel}} &\approx \left(\frac{2^R-1}{\mathsf{SNR}}\right)^L, \\ P_{\text{out}}^{\text{MISO}} &\approx \frac{1}{L!}\left(\frac{2^R-1}{\mathsf{SNR}}\right)^L, \end{aligned}$$

hence, the outage probability of the scheme which converts the MISO channel to a parallel channel is $L!$ times larger than the actual outage probability of the MISO channel at high-SNR.

At low-SNR, we have

$$\begin{aligned} C_\epsilon^{\text{MISO}} &\approx F^{-1}(1-\epsilon)\mathsf{SNR}\log_2 e, \\ C_\epsilon^{\text{parallel}} &\approx \frac{1}{L}F^{-1}(1-\epsilon)\mathsf{SNR}\log_2 e, \end{aligned}$$

hence, the outage capacity of the scheme is L times smaller than the outage capacity of the MISO channel at low-SNR.

5. (Book Problem 5.24, 20 points)
(The price of channel inversion)

1. Consider a narrowband Rayleigh flat fading SISO channel. Show that the average power (averaged over the channel fading) to implement the channel inversion scheme is infinite for any positive target rate.

2. Suppose now there are $L > 1$ receive antennas. Show that the average power for channel inversion is now finite.

3. Compute numerically and plot the average power as a function of the target rate for different L to get a sense of the amount of gain from having multiple receive antennas. Qualitatively describe the nature of the performance gain.

Solutions:

1. The channel model is: $y[m] = h[m]x[m] + w[m]$, and the rate it can support when channel state is h[m] is,

$$R = \log\left(1 + \frac{|h[m]|^2 P(h[m])}{N_0}\right).$$

Using channel inversion to keep a constant rate R, we need

$$P(h[m]) = \frac{(2^R-1)N_0}{|h[m]|^2}.$$

Thus the average power needed is

$$
\begin{aligned}
\mathbb{E}[P] &= (2^R - 1)N_0 \mathbb{E}\left(\frac{1}{|h[m]|^2}\right) \\
&= (2^R - 1)N_0 \int_0^\infty \frac{1}{x} e^{-x} dx \\
&> (2^R - 1)N_0 \int_0^M \frac{1}{x} e^{-x} dx \\
&> (2^R - 1)N_0 e^{-M} \int_0^M \frac{1}{x} dx = \infty
\end{aligned}
$$

2. The Channel model is, $y_l[m] = h_l[m]x[m] + w_l[m], l = 1, \cdots, L$

   and the rate it can support when channel state is $\mathbf{h}[m] = (h_1[m], \ldots, h_L[m])$ is

$$
R = \frac{1}{2}\log\left(1 + \frac{|\mathbf{h}[m]|^2 P(\mathbf{h}[m])}{N_0}\right).
$$

   Using channel inversion to keep a constant rate R, we need

$$
P(\mathbf{h}[m]) = \frac{(2^R - 1)N_0}{|\mathbf{h}[m]|^2}.
$$

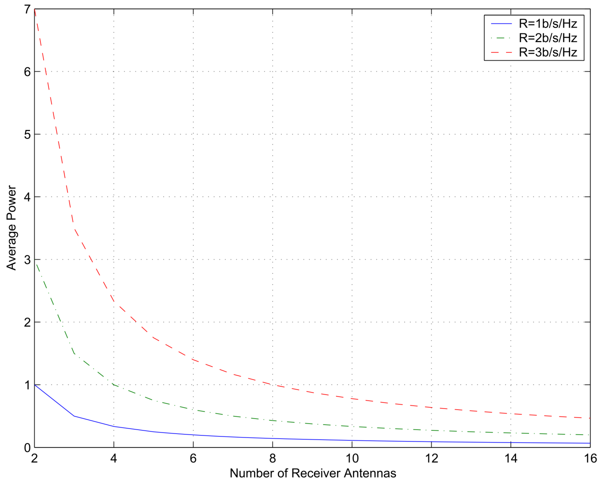
   Since $|\mathbf{h}[m]|^2$ is a χ^2 distribution with pdf

$$
f(x) = \frac{x^{L-1}}{(L-1)!} e^{-x},
$$

   the average power needed is

$$
\begin{aligned}
\mathbb{E}[P] &= (2^R - 1)N_0 \mathbb{E}\left(\frac{1}{|\mathbf{h}[m]|^2}\right) \\
&= (2^R - 1)N_0 \int_0^\infty \frac{1}{x}\frac{x^{L-1}}{(L-1)!} e^{-x} dx \\
&= \frac{(2^R - 1)N_0}{L-1}.
\end{aligned}
$$

3. Assume the noise $w \sim CN(0,1)$, for different target rate and L, the average power is plotted in the Figure 1. We can see that the power needed is decreasing with increasing number of receiver antennas (actually inversely proportional to $L-1$).



Requared rate(kb/s)	Optimal SINR threshold(dB)	SINR threshold using IS-856(dB)
38.4	-16.7	-11.5
76.8	-13.6	-9.2
153.6	-10.5	-6.5
307.2	-7.3	-3.5
614.4	-3.9	-0.5
921.6	-1.8	2.2
1228.8	-0.1	3.9
1843.2	2.5	8.0
2457.6	4.6	10.3

Figure 1: Figure for Problem 5.3