ELEC 533: Homework 7

Problem 1

Suppose $\{X_t; t \in R\}$ is a wss, zero mean Gaussian random process with autocorrelation function $R_x(\tau); \tau \in \mathbb{R}$ and power spectral density $S_x(\omega); \omega \in R$. Define the random process $\{Y_t; t \in R\}$ by $Y_t = (X_t)^2$.

a) Find the mean function Y_t.

$$\mu_y(t) = E[Y_t] = E[(X_t)^2] = E[X_t X_{t+0}] = R_x(0)$$

$\boxed{\mu_y(t) = R_x(0)}$

b) Find the autocorrelation function of Y_t.

$$\begin{aligned} R_y(t,s) &= E[Y_t Y_s^*] \\ &= E[X_t^2 X_s^2] \\ &= E[X_t^2]E[X_s^2] + 2E[X_t X_s]^2 \\ R_y(\tau) &= R_x(0)^2 + 2R_x(\tau)^2 \end{aligned}$$

$\boxed{R_y(\tau) = R_x(0)^2 + 2R_x(\tau)^2}$

Step 2 to Step 3 can be easily made by properties of multivariate normal distributions where we know

$$E[x_i^2 x_j^2] = \Sigma_{ii}\Sigma_{jj} + 2(\Sigma_{ij})^2$$

c) Does the power spectral density exist for Y_t? What if we were to use the delta function?

The delta function must be used to find the power spectral density as below.

$\boxed{S_y(\omega) = \delta(\omega)R_x(0)^2 + 2(S_X(\omega) * S_X(\omega))}$

Problem 2

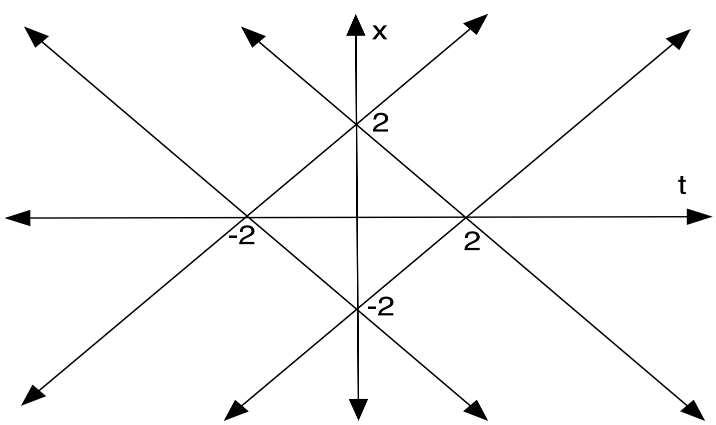
Show that the autocorrelation function of a WSS process $\{X_t; t \in \mathbb{R}\}$ is continuous for all $\tau \in \mathbb{R}$ if it is contionuous at $\tau = 0$.

$$
\begin{aligned}
&\lim_{\tau \to \tau_0} |R_x(\tau) - R_x(\tau_0)| \\
&= \lim_{\tau \to \tau_0} |E[X_0 X_\tau^*] - E[X_0 X_{\tau_0}^*]| \\
&= \lim_{\tau \to \tau_0} |E[X_0 (X_\tau^* - X_{\tau_0}^*)]| \\
&\leq \lim_{\tau \to \tau_0} [E[X_0^2] E[(X_\tau^* - X_{\tau_0}^*)^2]]^{1/2} \\
&\leq \lim_{\tau \to \tau_0} [R_x(0)(E[(X_\tau^*)^2] - 2E[X_\tau^2 X_{\tau_0}^*] + E[(X_{\tau_0}^*)^2])]^{1/2} \\
&\leq \lim_{\tau \to \tau_0} [R_x(0)(R_x(0) - 2R_x(\tau - \tau_0) + R_x(0))]^{1/2} \\
&\leq \lim_{\tau \to \tau_0} [2R_x(0)(R_x(0) - R_x(\tau - \tau_0))]^{1/2} \\
&\leq 0
\end{aligned}
$$

Problem 3

Let A and B be independent with $\Pr(A = 1) = \Pr(A = -1) = \Pr(B = 1) = \Pr(B = -1) = \frac{1}{2}$. Define the random process $\{X_t; t \in \mathbb{R}\}$ by $X_t = 2A + Bt$.

a) Sketch a possible sample path.



b) Find $\Pr(X_t \geq 0)$ for all $t \in \mathbb{R}$.

$$
Pr(X_t \geq 0) = \begin{cases} 3/4 & |t| = 2 \\ 1/2 & |t| \neq 2 \end{cases}
$$

c) Find $\Pr(X_t \geq 0 \text{ for all } t \in \mathbb{R})$.

By inspection $\boxed{Pr(X_t \geq 0 \;\; \forall t \in \mathbb{R}) = 0}$

Problem 4

Suppose $\{X_n; n \in \mathbb{Z}\}$ and $\{Z_n; n \in \mathbb{Z}\}$ are mutually independent, i.i.d. zero mean Gaussian random process with autocorrelations
$R_x(k) = \sigma_x^2 \delta_k$ and $R_z(k) = \sigma_z^2 \delta_k$ where $\delta_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$
These process are used to construct new processes as follows
$Y_n = Z_n + rY_{n-1}$
$U_n = X_n + Y_n$
$W_n = U_n - rU_{n-1}$
Find the covariances and power spectral densities of $\{U_n\}$ and $\{W_n\}$. Find $E[(X_n - W_n)^2]$

First, we write the linear system of Y_n as a time invariant finite impulse response.

$$Y_n = Z_n + rY_{n-1} = \sum_{l=-\infty}^{k} r^{k-l} Z_l$$

and

$$h_k = \begin{cases} \alpha^k & k \geq 0 \\ 0 & k < 0 \end{cases}$$

Thus

$$H(z) = \frac{1}{1 - rz^{-1}} \qquad \text{and} \qquad H(\frac{1}{z}) = \frac{1}{1 - rz}$$

Thus the spectral density of Y_n

$$S_y(z) = \frac{\sigma_z^2}{(1 - rz^{-1})(1 - rz)} = \frac{\sigma_z^2}{1 - 2r\cos(w) + r^2}$$

Thus the autocorrelation function of Y_n is

$$R_y(k) = \frac{\sigma_z^2 r^{|k|}}{1 - r^2}, k \in \mathbb{Z}$$

We can calculate the mean of Y_n as

$$E[Y_n] = E[\sum_{l=-\infty}^{k} r^{k-l} Z_l] = \sum_{l=-\infty}^{k} r^{k-l} E[Z_l] = 0$$

due to the fact $E[Z_n] = 0$.

$$\begin{aligned}
\operatorname{cov}(U_n) &= E[U_n U_n^*] - E[U_n]E[U_n^*] \\
&= E[(X_n + Y_n)(X_n^* + Y_n^*)] - E[X_n + Y_n]E[X_n + Y_n]^* \\
&= R_x(0) + 2E[X_n Y_n] + R_y(0) - E[X_n]^2 - 2E[X_n]E[Y_n] - E[Y_n]^2 \\
&= R_x(0) + 2E[X_n]E[Y_n] + R_y(0) - E[X_n]^2 - 2E[X_n]E[Y_n] - E[Y_n]^2 \\
&= R_x(0) + R_y(0) = \sigma_x^2 + \sigma_z^2 \frac{1}{(1 - r^2)}
\end{aligned}$$

$$\boxed{\mathrm{cov}(U_n) = \sigma_x^2 + \sigma_z^2 \frac{1}{(1-r^2)}}$$

$$\begin{aligned}
S_u(\omega) &= \mathfrak{F}\{E[U_n U_{n+k}^*]\} \\
&= \mathfrak{F}\{E[X_n X_{n+k}] + E[X_n Y_{n+k}] + E[X_{n+k} Y_n] + E[Y_n Y_{n+k}]\} \\
&= \mathfrak{F}\{E[X_n X_{n+k}] + E[X_n]E[Y_{n+k}] + E[X_{n+k}]E[Y_n] + E[Y_n Y_{n+k}]\} \\
&= \mathfrak{F}\{R_x(k) + R_y(k)\}
\end{aligned}$$

$$\boxed{S_u(\omega) = \sigma_x^2 + \frac{\sigma_z^2}{1 - 2r\cos(\omega) + r^2}}$$

$$\begin{aligned}
\mathrm{cov}(W_n) &= E[W_n W_n^*] - E[W_n]E[W_n^*] \\
&= E[U_n^2] - 2E[rU_n U_{n-1}] + E[(rU_{n-1})^2] \\
&= R_x(0) + R_y(0) - 2r(R_x(-1) + R_y(-1)) + r^2(R_x(0) + R_y(0)) \\
&= \frac{\sigma_x^2(1-r^4) + \sigma_z^2(1-r^2)}{1-r^2}
\end{aligned}$$

$$\boxed{\mathrm{cov}(W_n) = \sigma_x^2(1+r^2) + \sigma_z^2}$$

$$\begin{aligned}
S_w(\omega) &= \mathfrak{F}\{E[W_n W_{n+k}^*]\} \\
&= \mathfrak{F}\{E[(U_n - rU_{n-1}])(U_{n+k} - rU_{n-1+k})] \\
&= \mathfrak{F}\{E[U_n U_{n+k}] - E[U_n r U_{n-1+k}] - E[rU_{n-1} U_{n+k}] + r^2 E[U_{n-1} U_{n-1+k}]\} \\
&= S_x(\omega) + S_y(\omega) - r[e^{-j\omega}(S_x(\omega) + S_y(\omega)) + e^{j\omega}(S_x(\omega) + S_y(\omega))] + r^2[S_x(\omega) + S_y(\omega)] \\
&= \sigma_x^2(r^2 - 2r\cos(\omega) + 1) + \sigma_z^2
\end{aligned}$$

$$\boxed{S_w(\omega) = \sigma_x^2(r^2 - 2r\cos(\omega) + 1) + \sigma_z^2}$$

$$\begin{aligned}
E[(X_n - W_n)^2] &= E[X_n^2] - 2E[X_n W_n] + E[W_n^2] \\
&= R_x(0) - 2E[X_n U_n] - 2E[X_n r U_{n-1}] + R_w(0) \\
&= \sigma_x^2 - 2R_x(0) - 2rR_x(-1) + R_w(0) \\
&= -\sigma_x^2 + R_w(0) \\
&= \sigma_z^2 + \sigma_x^2 r^2
\end{aligned}$$

$$\boxed{E[(X_n - W_n)^2] = \sigma_z^2 + \sigma_x^2 r^2}$$

where

$$
\begin{aligned}
R_w(0) &= E[W_n W_n^*] \\
&= E[(U_n - rU_{n-1})(U_n - rU_{n-1})^*] \\
&= E[U_n U_n^*] - 2rE[U_n U_{n-1}^*] + r^2 E[U_{n-1} U_{n-1}^*] \\
&= R_u(0) - 2rR_u(-1) + r^2 R_u(0) \\
&= R_x(0) + R_y(0) - 2rR_y(-1) + r^2 R_x(0) + r^2 R_y(0) \\
&= \sigma_x^2(1 + r^2) + \sigma_z^2
\end{aligned}
$$