ELEC 533: Homework 3 Solutions

Problem 1

a) As we will prove in Problem 2(b),

$$\begin{aligned} F_X(a) &= \lim_{b\to\infty} F_{XY}(a,b) \\ &= \int_{-\infty}^{a}\int_{-\infty}^{\infty} f_{XY}(x,y)dydx \end{aligned}$$

Using Leibniz' rule, we get

$$f_X(a) = \frac{\partial F_X(a)}{\partial a} = \frac{\partial}{\partial a}\int_{-\infty}^{a}\int_{-\infty}^{\infty} f_{XY}(x,y)dydx = \int_{-\infty}^{\infty} f_{XY}(x,y)dy.$$

b) No, they are not jointly continuous. The simplest counterexample is to choose X uniformly distributed on $(0,1)$ and to choose $Y = X$. Then, the joint CDF is

$$\begin{aligned} F_{XY}(x,y) &= Pr(X \le x, Y \le y) \\ &= Pr(X \le x, X \le y) \\ &= Pr(X \le \min(x,y)) \\ &= \min(x,y), 0 \le x,y \le 1. \end{aligned}$$

It's easy to show that the second partial derivative of F_{XY} does not exist, so neither does the density f_{XY}, thus X and Y are not jointly continuous.

Problem 2

a) We need to show that $F_{XY}(a,b) \le F_{XY}(c,d)$ whenever $a \le b$ and $c \le d$. So, for such a,b,c,d, we have

$$\begin{aligned} F_{XY}(c,d) &= Pr(X \le c \cup Y \le d) \\ &= Pr((X \le a \cap a \le X \le c) \cup (Y \le b \cap b \le Y \le d)) \\ &= Pr(X \le a \cup (Y \le b \cap b \le Y \le d)) + Pr(a \le X \le c \cup (Y \le b \cap b \le Y \le d)) \\ &= Pr(X \le a \cup Y \le b) + Pr(X \le a \cup b \le Y \le d) + Pr(a \le X \le c \cup (Y \le b \cap b \le Y \le d)) \\ &\ge Pr(X \le a \cup Y \le b) \\ &= F_{XY}(a,b), \end{aligned}$$

where we have used the fact that probabilities of disjoint events add and that probabilities are always non-negative.

b) We can write the limit as

$$\begin{aligned}
\lim_{a\to\infty} F_{XY}(a,b) &= \lim_{a\to\infty} Pr(X \le a \cup Y \le b) \\
&= Pr(X \le \infty \cup Y \le b) \\
&= Pr(Y \le b) \\
&= F_Y(b).
\end{aligned}$$

c) ($\Rightarrow$): if X and Y are independent, then

$$F_{XY}(x,y) = Pr(X \le x \cup Y \le y) = Pr(X \le x)P(Y \le y) = F_X(x)F_Y(y).$$

Taking derivatives:

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(xy) = \frac{\partial}{\partial x} F_X(x) \frac{\partial}{\partial y} F_Y(y) = f_X(x) f_Y(y).$$

($\Leftarrow$): If the joint density factors, then

$$F_{XY}(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_X(x) f_Y(y) dy dx = \int_{-\infty}^{a} f_X(x) dx \int_{-\infty}^{b} f_Y(y) dy = F_X(a) F_Y(b),$$

and X and Y are therefore independent.

Problem 3

a) Using the (Cauchy-)Schwartz inequality,

$$E[(X-\mu_x)(Y-\mu_y)]^2 \le \sigma_x^2 \sigma_y^2.$$

Therefore,

$$\rho^2 = \frac{E[(X-\mu_x)(Y-\mu_y)]^2}{\sigma_x^2 \sigma_y^2} \le 1,$$

thus $|\rho| \le 1$.

b) ($\Rightarrow$): If X and Y are independent, then

$$E[(X-\mu_x)(Y-\mu_y)] = E[(X-\mu_x)]E[(Y-\mu_y)] = 0,$$

so $\rho = 0$.

($\Leftarrow$): If $\rho = 0$, then the joint density of X and Y is

$$\begin{aligned}
f_{XY}(x,y) &= \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[\frac{-(x-\mu_x^2)}{2\sigma_x^2} + \frac{-(y-\mu_y^2)}{2\sigma_y^2}\right] \\
&= \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[\frac{-(x-\mu_x^2)}{2\sigma_x^2}\right] \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left[\frac{-(y-\mu_y^2)}{2\sigma_y^2}\right] \\
&= f_X(x) f_Y(y),
\end{aligned}$$

and X and Y are independent.

Problem 4

We'll show that Z is not uniformly distributed by finding its density. We'll start by finding the CDF:

$$\begin{aligned}
F_Z(z) &= Pr(XY \leq z) \\
&= Pr(X \leq z/Y) \\
&= \int_0^1 \int_0^{\min(z/y,1)} dxdy \\
&= \int_0^1 \min(z/y, 1)dy \\
&= \int_0^z dy + \int_z^1 z/ydy, 0 \leq z \leq 1 \\
&= z - z\log(z), 0 \leq z \leq 1.
\end{aligned}$$

Differentiating $F_Z(z)$ gives us

$$f_Z(z) = -\log(z), 0 \leq z \leq 1.$$