ELEC 533: Homework 4 Solutions

Problem 1

Supose X and Y are jointly continuous

i) Show that $F_{Y|X}(b|x) = \int\limits_{-\infty}^{b} \frac{f_{xy}(x,y)}{f_x(x)} dy$ and thus $f_{Y|X}(y|x) = \frac{f_{xy}(x,y)}{f_x(x)}$.

By definition,

$$F_{Y|X}(b|x) = E\{I_{(-\infty,b]}(Y)|X = x\}$$

That is, $F_{Y|X}(b|x)$ is any function satisfying

$$\int_B F_{Y|X}(b|x) P_X(dx) = \int_{\{X\in B\}} I_{(-\infty,b]}(Y) dP \ \ \forall B$$

or equivalently,

$$\int_B F_{Y|X}(b|x) f_X(x) dx = \int_\Omega I_B(X) I_{(-\infty,b]}(Y) dP = P(X \in B, Y \in (-\infty, b]).$$

Also,

$$P(X \in B, Y \in (-\infty, b]) = \int_B \int\limits_{-\infty}^{b} f_{XY}(x,y) dx dy = \int_B \int\limits_{-\infty}^{b} \frac{f_{XY}(x,y)}{f_X(x)} dy f_X(x) dx.$$

Therefore, $F_{Y|X}(b|x) = \int_{-\infty}^{b} \frac{f_{XY}(x,y)}{f_X(x)} dy$ and the result follows.

ii) Suppose $\int\limits_{-\infty}^{\infty} |y| f_{Y|X}(y|x) dy < +\infty \ \ \forall x$. Show that $E[Y|X = x] = \int\limits_{-\infty}^{\infty} y f_{Y|X}(y|x) dy \ \ \forall x$.

Let $x \in A$ and note that

$$\int_A E[Y|X = x] f_X(x) dx = \int_{\{x\in A\}} Y dP \ \ \forall A.$$

In addition,

$$\int_{\{x\in A\}} Y dP = \int_{\{(x,y):x\in A\}} \int y f_{XY}(x,y) dx dy = \int_{\{(x,y):x\in A\}} \int y f_X(x) f_{Y|X}(y|x) dx dy,$$

because in part (i) we showed that $f_{XY}(x,y) = f_X(x) f_{Y|X}(y|x)$. Since $|y f_{Y|X}(y|x)| = |y| f_{Y|X}(y|x)$ and $\int_{-\infty}^{\infty} |y| f_{Y|X}(y|x) dy < \infty$, we can apply Fubini's theorem to get that

$$\int_{\{x\in A\}} Y dP = \int_A f_X(x) \left[\int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy \right] dx.$$

Therefor, $E[Y|X = x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy$.

Problem 2

Suppose X and Y are jointly continuous random variables with joint pdf given by
$f_{XY}(x,y) = \begin{cases} e^{-y} & \text{if } x > 0 \text{ and } y > x \\ 0 & \text{otherwise} \end{cases}$

i) Show that f_{XY} is a legitimate joint pdf.

$f_{XY}(x,y)$ is obviously nonnegative. Also,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}dxdy = \int_{0}^{\infty}\int_{x}^{\infty} e^{-y}dydx = \int_{0}^{\infty} e^{-x}dx = 1.$$

Thus, $f_{XY}(x,y)$ is a legitimate joint pdf.

ii) Find the Marginal pdf's X and Y.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \begin{cases} 0 & \text{for } y \leq 0 \\ ye^{-y} & \text{for } y > 0 \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \begin{cases} 0 & \text{for } x \leq 0 \\ e^{-x} & \text{for } x > 0 \end{cases}$$

iii) Find $E[Y|X=x]$ for $x > 0$.

$$E[Y|X=x] = \int_{-\infty}^{\infty} y\frac{f_{XY}(x,y)}{f_X(x)}dy = x+1, \quad x > 0$$

Problem 3

Suppose $X_n \xrightarrow{i.p.} X$ and that there is a constant c such that $|X_n| \leq c$ for all n. Show that $X_n \xrightarrow{m.s.} X$.

$$|X_n| \leq c \;\; \forall n \Rightarrow F_{X_n} = \begin{cases} 1 & \text{for } x \geq c \\ ? & \text{for } -c < x < c \\ 0 & \text{for } x < -c \end{cases}$$

Therefore

$$F_X(x) = F_{X_n}(x) \quad \text{for } |x| > c$$

Now, given $\epsilon > 0$, let $A_n = \{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \epsilon\}$. If we use the indicator function I_{A_n} and for $I_{A_n^c}$, for each n we have:

$$\mathrm{E}[|X_n - X|^2] = \mathrm{E}[(I_{A_n} + I_{A_n^c})|X_n - X|^2] = \mathrm{E}[I_{A_n}|X_n - X|^2] + \mathrm{E}[I_{A_n^c}|X_n - X|^2] \tag{1}$$

Using triangle inequality, it can be proven that $|X_n - X| < 2c \Rightarrow |X_n - X|^2 < (2c)^2$. Using this result, we have the first part of equation (**??**) as:

$$\mathrm{E}[I_{A_n}|X_n - X|^2] \leq 4c^2\mathrm{P}(|X_n - X| > \epsilon), \forall \epsilon > 0$$

For the second part of equation (**??**) we have

$$\mathrm{E}[I_{A_n^c}|X_n - X|^2] = \int_{-\infty}^{+\infty} I_{A_n^c}|X_n - X|^2 \mathrm{d}F_{|X_n - X|} \leq \epsilon^2 \int_{-\infty}^{+\infty} \mathrm{d}F_{|X_n - X|} = \epsilon^2, \forall \epsilon > 0$$

Which means that : $\mathrm{E}[I_{A_n^c}|X_n - X|^2] = 0$ (ϵ small).

Thus, we have:

$$\lim_{n\to+\infty} E[|X_n - X|^2] = \lim_{n\to+\infty} \mathrm{E}[I_{A_n}|X_n - X|^2] + \lim_{n\to+\infty} \mathrm{E}[I_{A_n^c}|X_n - X|^2] = 4c^2 \lim_{n\to+\infty} \mathrm{P}(|X_n - X| > \epsilon) + 0$$

And we can conclude that $\lim_{n\to+\infty} \mathrm{E}[|X_n - X|^2] = 0$.

Problem 4

Show that if $X_n \xrightarrow{D} X$ and $P(X = c) = 1$ for some constant c then $X_n \xrightarrow{i.p.} X$.

$$\lim_{n\to+\infty} F_{X_n} = \begin{cases} 0 & \text{if } x < c \\ 1 & \text{if } x \geq c \end{cases}$$

For $\epsilon > 0$ and $n \geq 1$ we have

$$\begin{aligned} P(|X_n - c| \geq \epsilon) = P(X_n - c \geq \epsilon) + P(X_n - c < -\epsilon) = P(X_n \geq \epsilon + c) + P(X_n < c - \epsilon) = \\ 1 - F_{X_n}(c + \epsilon) + F_{X_n}(c - \epsilon) + P(X_n = c + \epsilon) \end{aligned} \tag{2}$$

Now

$$\lim_{n\to+\infty} P(|X_n - c| \geq \epsilon) = 1 - 1 + 0 + 0 = 0 \Rightarrow X_n \xrightarrow{i.p.} c.$$

Problem 5

We investigate the case of $|X_n| < Y,\ \forall n$.

Using triangle inequality we have: $|X| = |X - X_n + X_n| \leq |X - X_n| + |X_n|$, and thus

$$P(|X| > Y + \epsilon) \leq P(|X - X_n| + |X_n| - Y > \epsilon) \leq P(|X - X_n| > \epsilon)$$

Applying the limit it becomes:

$$\lim_{n\to+\infty} P(|X| > Y + \epsilon) \leq \lim_{n\to+\infty} P(|X - X_n| > \epsilon)$$

Because $X_n \xrightarrow{i.p.} X$, $\lim_{n\to+\infty} P(|X - X_n| > \epsilon) = 0$, and we have no dependence on n on the left hand side so:

$$P(|X| > Y + \epsilon) \leq 0 \Rightarrow P(X > Y + \epsilon) = 0$$

So, now we can use exploit the fact that $|X| \leq Y$:

$$|X_n| \leq |X_n| + |X| \leq 2Y \Rightarrow |X_n - X|^2 \leq 2Y^2.$$

With the notation that $\mathrm{E}[Y^2] = c^2$, the proof is now similar to the one done in Problem 3.