ELEC 533
Homework 1

PROBLEM 1

i) Let $A_1, A_2, \ldots$ be sets that belong to the σ-field $\mathcal{F}$. We can prove this part of the problem by either following a direct approach or by using mathematical induction.

The direct approach is based on the fact that $\cup_{i=1}^{\infty} A_i$ can be expressed as the union of disjoint sets as follows:

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup (A_1^c A_2) \cup (A_1^c A_2^c A_3) \cup \ldots$$

Since $A_1^c A_2^c \ldots A_{n-1}^c A_n \subseteq A_n$, the desired inequality $P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$ follows immediately.

If induction is to be employed, then we must show first that $P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$. The set $A_1 \cup A_2$ can be decomposed into three disjoint sets as follows:

$$A_1 \cup A_2 = A_1 A_2^c \cup A_1^c A_2 \cup A_1 A_2.$$

It is easy to show that $P(A_1 A_2^c) = P(A_1) - P(A_1 A_2)$ and similarly $P(A_2 A_1^c) = P(A_2) - P(A_1 A_2)$. Therefore

$$P(A_1 \cup A_2) = P(A_1 A_2^c) + P(A_2 A_1^c) + P(A_1 A_2) = P(A_1) + P(A_2) - P(A_1 A_2) \leq P(A_1) + P(A_2).$$

We are now ready to apply induction.
For $n = 1$ we have $P(A_1) \leq P(A_1)$ which is true. Suppose that $P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$. We will prove that $P(\bigcup_{i=1}^{n+1} A_i) \leq \sum_{i=1}^{n+1} P(A_i)$.
But

$$P(\bigcup_{i=1}^{n+1} A_i) = P(\bigcup_{i=1}^{n} A_i \cup A_{n+1}) \leq P(\bigcup_{i=1}^{n} A_i) + P(A_{n+1}) \leq \sum_{i=1}^{n} P(A_i) + P(A_{n+1}) = \sum_{i=1}^{n+1} P(A_i).$$

The first inequality comes from the relation $P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$, while the second comes from our assumption that $P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$.

ii) Since $A_n \nearrow A$ is a monotone increasing sequence we can use the identity
$A = A_1 \cup (A_2 - A_1) \cup (A_3 - A_2) \cup \ldots \cup (A_n - A_{n-1}) \cup \ldots$.

Since $A_i \subset A_{i+1}$, $A_i A_{i+1} = A_i$. Also, $A_{i+1} - A_i = A_{i+1} A_i^c$, and (see part i)) $P(A_{i+1} A_i^c) = P(A_{i+1}) - P(A_{i+1} A_i)$. Hence, $P(A_{i+i} - A_i) = P(A_{i+1}) - P(A_i)$ and finally

$$P(A) = P(A_1) + P(A_2) - P(A_1) + P(A_3) - P(A_2) + \ldots + P(A_i) - P(A_{i-1}) + \ldots = \lim_{n \to \infty} P(A_n).$$

iii) As in part i) we can prove this part either directly or by induction. A direct proof is presented here.

Since $A_1, A_2, \ldots \in \mathcal{F}$ and $A_iA_j = \emptyset$ $\forall i,j$ with $i \neq j$ we have that $A_1B, A_2B, \ldots \in \mathcal{F}$ and $A_iBA_jB = A_iA_jB = \emptyset$ $\forall i,j$ with $i \neq j$. Consequently,

$$P(\bigcup_{i=1}^{\infty} A_iB) = \sum_{i=1}^{\infty} P(A_iB)$$

Therefore,

$$P(\bigcup_{i=1}^{\infty} A_i \mid B) = \frac{P(\bigcup_{i=1}^{\infty} A_iB)}{P(B)} = \frac{\sum_{i=1}^{\infty} P(A_iB)}{P(B)} = \sum_{i=1}^{\infty} P(A_i \mid B)$$

PROBLEM 2

We need to show that $P_X(a) \geq 0$ $\forall a$ and $\sum_{a=-\infty}^{\infty} P_X(a) = 1$.

i) Obviously, $P_X(a) \geq 0$ $\forall a$. Also notice that $(b+c)^n = \sum_{k=0}^{n} \binom{n}{k} b^k c^{n-k}$. Therefore,

$$\sum_{a=-\infty}^{\infty} P_X(a) = \sum_{a=0}^{n} \binom{n}{a} \theta^a (1-\theta)^{n-a} = (\theta + (1-\theta))^n = 1$$

ii) Again, $P_X(a) \geq 0$ $\forall a$
and $\sum_{a=-\infty}^{\infty} P_X(a) = \sum_{a=0}^{\infty} e^{-\lambda} \frac{\lambda^a}{a!} = e^{-\lambda} e^{\lambda} = 1$.

PROBLEM 3

i) Consider two points a, b in $\mathcal{R}$ with $a \leq b$. Since $(-\infty, a] \subseteq (-\infty, b]$ we have that

$$P((-\infty, a]) \leq P((-\infty, b]) \Rightarrow F_X(a) \leq F_X(b)$$

Thus F_X is nondecreasing.

ii) Consider a monotone increasing sequence of sets $(-\infty, a_n] \nearrow (-\infty, \infty)$. In problem 1, we showed that if $A_n \nearrow A$, then $P(A_n) \to P(A)$. Therefore

$$\lim_{a_n \to \infty} F_X(a_n) = \lim_{a_n \to \infty} P_X((-\infty, a_n]) = P_X((-\infty, \infty)) = 1.$$

iii) Similar to ii).

iv) Consider a positive monotone decreasing sequence of numbers a_i such that $\lim_{i\to\infty} a_i = 0$. From the continuity of the probability measure (problem 1) we have that

$$\lim_{i\to\infty} F_X(a+a_i) - F_X(a) = \lim_{i\to\infty}[F_X(a+a_i) - F_X(a)] = \lim_{i\to\infty} P_X(x \in (a, a+a_i]) = P_X[\lim_{i\to\infty}(x \in (a, a+a_i])].$$

But,

$$\lim_{i\to\infty}(a, a+a_i] = \bigcap_{i=1}^{\infty}(a, a+a_i] = \emptyset.$$

Thus,

$$\lim_{i\to\infty} F_X(a+a_i) - F_X(a) = P_X(\emptyset) = 0$$

v) $P_X(a < x \leq b) = P_X((-\infty, b]) - P_X((-\infty, a]) = F_X(b) - F_X(a)$.

vi) As in part (iv), consider a positive monotone decreasing sequence of numbers a_i such that $\lim_{i\to\infty} a_i = 0$. From the continuity of the probability measure (problem 1) we have that

$$F_X(a) - \lim_{i\to\infty} F_X(a-a_i) = \lim_{i\to\infty}[F_X(a) - F_X(a-a_i)] =$$

$$\lim_{i\to\infty} P_X(x \in (a-a_i, a]) = P_X(\lim_{i\to\infty} x \in (a-a_i, a]).$$

But,

$$\lim_{i\to\infty}(a-a_i, a] = \bigcap_{i=1}^{\infty}(a-a_i, a] = a.$$

Thus,

$$F_X(a) - \lim_{i\to\infty} F_X(a-a_i) = P_X(x=a) \Rightarrow P_X(x=a) = F_X(a) - \lim_{\substack{b\to a\\ b<a}} F_X(b).$$

vii) Using the previous parts of the problem, we can easily obtain that

- $P_X(a \leq x \leq b) = F_X(b) - F_X(a) + P_X(x=a)$
- $P_X(a \leq x < b) = P_X(x=a) + F_X(b) - F_X(a) - P_X(x=b)$
- $P_X(a < x < b) = F_X(b) - F_X(a) - P_X(x=b)$.