Homework 2

ELEC 540: Advanced Wireless Communications

Problems are from *Fundamentals of Wireless Communications,* by David Tse and Pramod Viswanath. The equation number in the problems refer to the ones in Chapter 3 of the book.

1. (Book Problem 3.13, 15 points)
The optimal coherent receiver for repetition coding is L branches of diversity is a maximal ratio combiner. For implementation reasons, a simple receiver one often builds is a *selection combiner*. It does detection based on received signal along the branch with the strongest gain only, and ignores the rest. For the i.i.d. Rayleigh fading model, analyze the high SNR performance of this scheme. How much inherent diversity gain can this scheme get? Quantify the performance loss from optimal combining.

Solution: The channel equation is $y = hx + w$. Let $l = \arg\max_i |h[i]|$. The selection combiner bases its decision on the l^{th} branch only discarding the rest so the decision is based on $y[l] = h[l]x + w[l]$.

Let $\{x_i\}_{i=1}^{L}$ be i.i.d. $Exp(1)$ random variables, and $x = max_i x_i$. Then the pdf of x, $f(\cdot)$, for $x \to 0$ is given by:

$$\begin{aligned} f(x) &= L(1-e^{-x})^{L-1}e^{-x} = L[1-(1-x+o(x))]^{L-1}[1-x+o(x)] \\ &= Lx^{L-1} + o(x^{L-1}) \end{aligned}$$

Noting that $|h[l]|^2$ has the above pdf, we can use the derivation of Ex. 3.2, part 3) replacing β with L:

$$\lim_{\rho\to\infty} P_e \rho^L = \frac{L}{4^L}\frac{(2L-1)!}{L!} = \frac{(2L-1)!}{4^L(L-1)!}$$

We observe that this scheme still achieves a diversity gain of L but the error performance degrades by the factor $L/\beta = L!/(\Pi_{i=1}^{L} g_i[0]) = L!$ with respect to that of optimal combining.

2. (Book Problem 3.15, 25 points)
An $L \times 1$ MISO channel can be converted into a time-diversity channel with L diversity branches by simply transmitting over one antenna at a time.

1. In this way, any code designed for a time-diversity channel with L diversity branches can be used for a MISO (multi-input single-output) channel with L transmit antennas. If the code achieves k-fold diversity in the time-diversity channel, how much diversity can it obtain in the MISO channel? What is the relationship between the minimum product distance metric of the code when viewed as a time-diversity code and its minimum determinant metric when viewed as a transmit-diversity code?

2. Using this transformation, the rotation code can be used as a transmit diversity scheme. Compare the performance of this code and the Alamouti scheme in a 2×1 Rayleigh fading channel, using BPSK symbols. Which one is better? How about using QPSK symbols?

3. Use the permutation code (c.f. Figure 3.26) from Exercise 3.10 on the 2×1 Rayleigh fading channel and compare (via a numerical simulation) its performance with the Alamouti scheme using QPSK symbols (so the rate is the same in both the schemes).

Solution:

1. We obtain the same diversity gain over the MISO channel (assuming the same statistical characterization of the fading gains) as we have operationally converted the MISO channel into a parallel channel. Also, since the determinant of a diagonal matrix is the product of its diagonal elements, the determinant metric for the MISO channel is same as the product distance metric of the time diversity code.

2. For the rotation code the worst case pairwise error probability is $\frac{16SNR^{-2}}{min\ \delta_{ij}}$, where the optimal δ_{ij} is given by 16/5, which gives $5/\text{SNR}^2$ as the upper bound for the rotation code.

   On the other hand, for Alamouti scheme, the worst case probability of error is $\frac{16SNR^{-2}}{det(X_A - X_B)^2}$. If u_1 and u_2 are the BPSK (+/ a) symbols used for the Alamouti scheme, then the average power per time symbol is given by $2a^2$. The determinant is given by $u_1^2 + u_2^2$, thus the worst case determinant is given by $4a^2$. Thus, after normalizing we get worst case probability of error is given by $4/SNR^2$. Thus, the Alamouti scheme is better.

   For QPSK symbols, we are just using an additional degree of freedom which will change the power used for both the schemes, but the relative difference in the worst case error performance (a factor of 1.25) remains the same.

3. See Figure 1

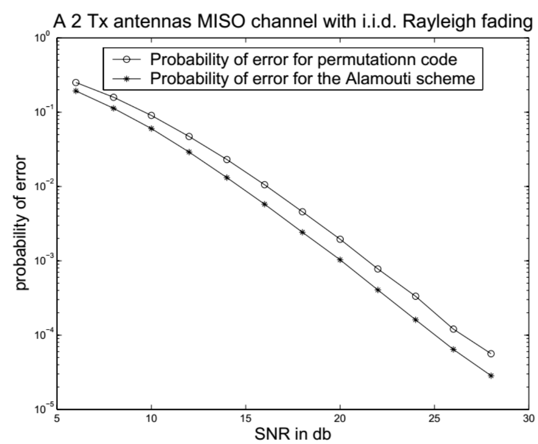

Figure 1: The error probability of uncoded QAM with the Alamouti scheme and that of a permutation code over one antenna at a time for the Rayleigh fading MISO channel with two transmit antennas: the permutation code is only about 1.5 dB worse than the Alamouti scheme over the plotted error probability range.

3. (Book Problem 3.19, 35 points)
In this exercise we study the performance of space time codes (the subject of Section 3.3.2) in the presence of multiple receive antennas.

1. Derive, as an extension of (3.83), the pairwise error probability for space time codes with n_r receive antennas.

2. Assuming that the channel matrix has i.i.d. Rayleigh components derive, as an extension of (3.86), a simple upper bound for the pairwise error probability.

3. Conclude that the code design criterion remains unchanged with multiple receive antennas.

Solution:

1. Let H be the fading matrix for the MIMO channel. Then the channel model can be written as, $Y = HX + W$. Now, this channel model can be rewritten as a MISO channel with block-length $n_r N$. Let $\tilde{X}$ and h be

$$\begin{aligned} \tilde{\mathbf{X}} &= \begin{bmatrix} \mathbf{X} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbf{X} \end{bmatrix}, \\ \mathbf{h} &= \left[\begin{array}{cccccccc} H(1,1) & \cdots & H(1,L) & H(2,1) & \cdots & \cdots & H(n_r,L) \end{array} \right]. \end{aligned}$$

Then the received signal can be rewritten as $y = h\tilde{X} + w$, with y and w appropriately defined in term of Y and W. Then the probability of pairwise error can be written as:

$$E\left[Q\left(\sqrt{\frac{\mathsf{SNR}\mathbf{h}(\tilde{\mathbf{X}}_A - \tilde{\mathbf{X}}_B)(\tilde{\mathbf{X}}_A - \tilde{\mathbf{X}}_B)^*\mathbf{h}^*}{2}}\right)\right]$$

2. and 3. Since we have reduced the MIMO problem with i.i.d. Rayleigh fading to a MISO problem with i.i.d. Rayleigh fading, probability of pairwise error can be upper bounded as:

$$\begin{aligned} \mathbb{P}(\mathbf{X}_A \to \mathbf{X}_B) &\leq \frac{4^{Ln_r}}{\mathsf{SNR}^{Ln_r} \det\left((\tilde{\mathbf{X}}_A - \tilde{\mathbf{X}}_B)(\tilde{\mathbf{X}}_A - \tilde{\mathbf{X}}_B)^*\right)}, \\ &= \left(\frac{4^L}{\mathsf{SNR}^L \det\left((\mathbf{X}_A - \mathbf{X}_B)(\mathbf{X}_A - \mathbf{X}_B)^*\right)}\right)^{n_r}, \end{aligned}$$

where the last step follows from the diagonal structure of $\tilde{X}_A$ and $\tilde{X}_B$. Thus, the code design criterion of maximizing the minimum determinant remains unchanged.

4. (Book Problem 3.20, 25 points)
We have studied the performance of the Alamouti scheme in a channel with two transmit and one receive antenna. Suppose now we have an additional receive antenna. Derive the ML detector for the symbols based on the received signals at both receive antennas. Show that the scheme effectively provides two independent scalar channels. What is the gain of each of the channels?

Solution: Using the same notation as in the text, and using a subindex to denote the receive antenna (either 1 or 2) we have:

$$\begin{bmatrix} y_1[1] \\ y_1[2]^* \\ y_2[1] \\ y_2[2]^* \end{bmatrix} = \begin{bmatrix} h_{11} & h_{21} \\ h_{21}^* & -h_{11}^* \\ h_{12} & h_{22} \\ h_{22}^* & -h_{12}^* \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w_1[1] \\ w_1[2]^* \\ w_2[1] \\ w_2[2]^* \end{bmatrix}$$

where h_{ij} is the complex channel gain from transmit antenna i to receive antenna j. In vector notation we can rewrite the above equation as $y = Hu + w$, where $w \sim CN(0, N_0 I_4)$. Noting that the columns of H (which we call h_i) are orthogonal, we can project the (slightly modified) received vector y onto the normalized columns of H to obtain the 2 sufficient statistics:

$$r_i = \frac{\mathbf{h}_i^*}{\|\mathbf{h}_i\|}\mathbf{y} = \|h\| u_i + \tilde{w}_i$$

for $i = 1, 2$, where $\tilde{w}_i \sim CN(0, N_0)$ independent across i, and $\|h\| = \|h_1\| = \|h_2\|$ is the effective gain.