ELEC 533
Homework 2

PROBLEM 1

Consider X a uniform r.v. in $[0,1]$. Then,

$$\int_0^1 x dF_X = \lim_{d\to 0} \sum_{i=1}^{n-1} y_i (F_X(x_{i+1}) - F_X(x_i))$$

where $d = max_i \, |x_{i+1} - x_i|, y_i \in [x_i, x_{i+1}]$ and $\{x_i\}_{i=1}^n$ is a partition of $[0,1]$ such that $x_1 = 0$ and $x_n = 1$. For any interval $[x_i, x_{i+1}]$, F_X is differentiable. Then, by applying the mean value theorem,

$$\int_0^1 x dF_X = \lim_{d\to 0} \sum_{i=1}^{n-1} y_i F_X'(y_i)(x_{i+1} - x_i) = \int_0^1 x f_X(x) dx.$$

and since $dF_X = 0 = f_X(x)$, $\forall x \in \mathcal{R} - [0,1]$, it is immediately concluded that $\int_{-\infty}^{\infty} x dF_X = \int_{-\infty}^{\infty} x f_X(x) dx$.

Note: Some people considered an interval $[a,b]$ with $a < 0$ and $b > 1$ and proceeded as above. In that case however, F_X is not differentiable at $x = 0$ and $x = 1$ and the mean value theorem cannot be applied.

PROBLEM 2

Let

$$Y^+ = \begin{cases} 0 & n \text{ even} \\ \frac{2^n}{n} & n \text{ odd} \end{cases} \quad \text{and} \quad Y^- = \begin{cases} \frac{2^n}{n} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Then, $Y = Y^+ - Y^-$. But,

$$E[Y^+] = \sum_{\substack{n=1 \\ n \text{ odd}}} \frac{2^n}{n} \frac{1}{2^n} = \infty = \sum_{\substack{n=1 \\ n \text{ even}}} \frac{2^n}{n} \frac{1}{2^n} = E[Y^-].$$

Thus, $E[Y] = E[Y^+] - E[Y^-]$ does not exist.

PROBLEM 3

a) X is a binomial r.v..

- mean

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} \theta^k (1-\theta)^{n-k} = \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} \theta^k (1-\theta)^{n-k} =$$

$$n\theta \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} \theta^{k-1} (1-\theta)^{n-k} = n\theta \sum_{k=1}^{n} \binom{n-1}{k-1} \theta^{k-1} (1-\theta)^{n-k} = n\theta.$$

- variance

$\mathrm{var}(X) = E[X^2] - (E[X])^2$

$$E[X^2] = \sum_{k=0}^{n} k^2 \binom{n}{k} \theta^k (1-\theta)^{n-k} = n\theta \sum_{k=1}^{n} k \frac{(n-1)!}{(k-1)!(n-k)!} \theta^{k-1} (1-\theta)^{n-k} =$$

$$n\theta \sum_{k=1}^{n} (k-1) \frac{(n-1)!}{(k-1)!(n-k)!} \theta^{k-1} (1-\theta)^{n-k} + n\theta =$$

$$n(n-1)\theta^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} \theta^{k-2} (1-\theta)^{n-k} + n\theta =$$

$$n(n-1)\theta^2 \sum_{k=2}^{n} \binom{n-2}{k-2} \theta^{k-2} (1-\theta)^{n-k} + n\theta = n(n-1)\theta^2 + n\theta$$

Thus, $\mathrm{var}(X) = n\theta(1-\theta)$

b) X is a Poisson r.v..

- mean

$$E[X] = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} = \lambda$$

- variance

$$E[X^2] = E[X(X-1)] + E[X] = \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k}{k!} e^{-\lambda} + \lambda = \lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} e^{-\lambda} + \lambda = \lambda^2 + \lambda$$

Thus, $\mathrm{var}(X) = \lambda$.

c) X is a constant c.

$$E[X] = \sum_{k} x_k P_X(x_k) = c \cdot 1 = c$$

PROBLEM 4

a) $X \sim n(\mu, \sigma^2)$, $\quad f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-(x-\mu)^2/2\sigma^2)$.

- mean

$$E[X] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} x \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx. \text{ Set } \frac{x-\mu}{\sigma} = y.$$

Then,

$$E[X] = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-y^2/2} dy + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = 0 + \frac{\mu}{\sqrt{2\pi}} \sqrt{2\pi} = \mu$$

- variance

$$E[X^2] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} x^2 \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx. \text{ Again, set } \frac{x-\mu}{\sigma} = y.$$

Then,

$$E[X^2] = \frac{1}{\sqrt{2\pi}} \sigma^2 \int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy + 2\sigma\mu 0 + \mu^2 = \sigma^2 + \mu^2.$$

Thus, $\mathrm{var}(X) = \sigma^2$.

b)For simplicity, assume that the unit on the $x-$axis is one. It is obviously one on the $y-$axis. Notice that $F_X(x)$ can be expressed as $F_X(x) = F_d(x) + F_c(x)$ where,

$$F_d(x) = \begin{cases} 0 & x < 2 \\ 1/3 & x \geq 2 \end{cases}$$

Then, $F_c(x)$ is continuous and $f_c(x)$ exists. Evidently,

$$f_c(x) = \begin{cases} 2/3 & 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 dF_X = \int_{-\infty}^{\infty} x^2 dF_d + \int_{-\infty}^{\infty} x^2 dF_c =$$

$$\sum x^2 P_d(x) + \int_{-\infty}^{\infty} x^2 f_c(x) dx = 2^2 \frac{1}{3} + \frac{2}{3} \frac{x^3}{3} \Big|_2^3 = \frac{4}{3} + \frac{38}{9} = \frac{50}{9}.$$