16-8-21

Gates & Circuits

Can a classical logic circuit be represented / simulated by (or) circuit be implemented by quantum gates ?

Classical - Irreversible gates (except NOT)

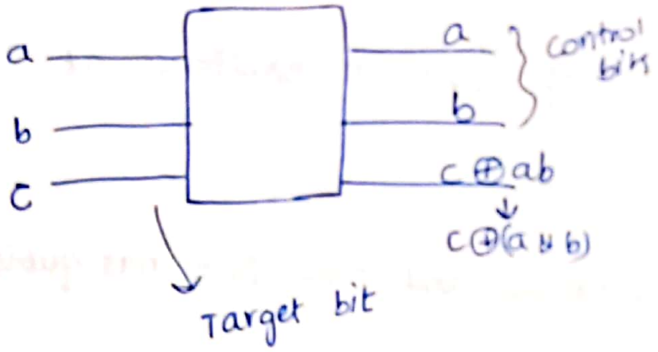
XOR - causes loss of information

Quantum - Only reversible gates & computations

$$UU^\dagger = I$$

classical gate

Toffoli gate - Reversible classical gate



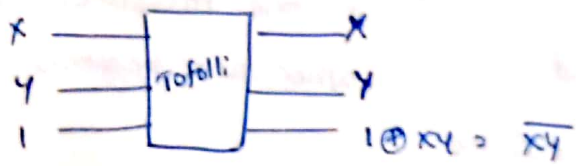
~~C inverted if~~

$\sim c$ if $ab = 1$ $(a \& b = 1)$

c if $ab \neq 1$ $(a \& b \neq 1)$

Reversible → Inverse is itself

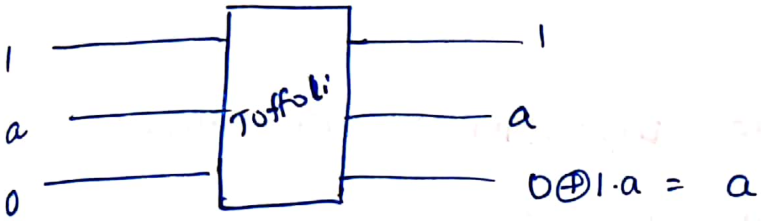
Q Can it simulate other classical circuits ? i.e. Can it can act as NAND ?



NAND Gate

Fan-out

(Get input a in 2 places of output)



With NAND and Fan-out every classical circuit can be obtained with Toffoli Gate

Q. Can we use quantum circuits to implement classical logic circuits ?

yes, we use Toffoli gate

Toffoli - Valid Quantum Gate

Matrix for Toffoli Gate

$$
U_T = \begin{array}{c|c}
{}_{a,b,c}\backslash^{\text{output}} & \begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array} \\
\hline
\begin{array}{c} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

} - order changed here when ab = 1

Vector Spaces

The space of all n tuples of complex numbers - $\mathbb{C}^n$

n tuples - $(z_1, z_2 \ldots z_n)$

Column Matrix Representation

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

1) Addition of complex numbers

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} z_1' \\ \vdots \\ z_n' \end{bmatrix} = \begin{bmatrix} z_1 + z_1' \\ \vdots \\ z_n + z_n' \end{bmatrix}$$

Addition of vector

2) Multiplication by scalar

$$z \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} zz_1 \\ \vdots \\ zz_n \end{bmatrix}$$

multiplying two complex numbers.

Dirac Notation

$|\psi\rangle$ — Ket (says its a vector) $|P\rangle$ — P is vector

Zero vector $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ denoted by 0 not $|0\rangle$

reserved for computational basis state

Notations

z^* → Complex conjugate of scalar z
↓
(complex no.)

$|\psi\rangle$ → vector

$\langle\psi|$ → dual vector of $|\psi\rangle$

$\langle\phi|\psi\rangle$ → Inner Product of 2 vectors $|\phi\rangle$ & $|\psi\rangle$
↙
Results in scalar

$|\phi\rangle \otimes |\psi\rangle$ - Tensor product of $|\phi\rangle$ and $|\psi\rangle$

* Abbreviated as $\underline{|\phi\rangle|\psi\rangle}$
↓ vector product

A^* - complex conjugate of A matrix

A^T - Transpose of A matrix

$A^\dagger$ - Hermitian Conjugate or Adjoint

$= (A^T)^*$

$\langle\phi|A|\psi\rangle$ = Inner Product of ϕ with $A|\psi\rangle$
(or)
Inner product between $A^\dagger|\phi\rangle$ and $|\psi\rangle$

18-8-21

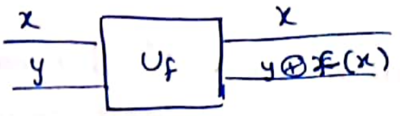
I. Quantum Parallelism

II. Linear Algebra

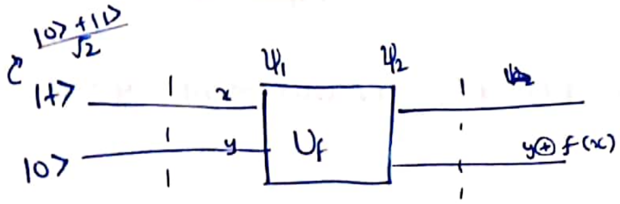
Quantum Parallelism

$f(x) : \{0,1\} \rightarrow \{0,1\}$

Quantum



(Uf - circuit
(Can be verified reversible



ψ_1 - Combined input state

ψ_2 - Combined output state

$$|\psi_2\rangle = \frac{|0, f(0)\rangle + |1, f(1)\rangle}{\sqrt{2}}$$

So circuit computes both $F(0)$, $F(1)$ simultaneously

If x is 0, they output on other will be $F(0)$

If x is 1, then output on other will be $F(1)$

Hadamard Transform (also called Walsh Hadamard Transform)

Suppose 2 qubits — 2 Uf gates (can be generalized to n qubits)

Initial qubits - $\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$

All 4 states are there

For n qubits

- n U_f gates
- n qubits prep in $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$

Output $= \frac{1}{\sqrt{2^n}} \sum_x |x\rangle \longrightarrow$ input states

n qubits $\longrightarrow$ 2^n different input combinations

Though 2^n combinations are present, measurement yields for 1 of these states

** Deutsch Algorithm

Linear Algebra

Vector Space

(i) Spanning set

For a vector space is a set of vectors $(|v_1\rangle, |v_2\rangle, \ldots |v_n\rangle)$ if any vector can be represented as linear combination of these vectors

$$|v\rangle = \sum_i a_i |v_i\rangle \qquad (a_i \in \mathbb{C})$$

Then $(|v_1\rangle, |v_2\rangle \ldots, |v_n\rangle)$ set is called span of V

$\mathbb{C}^2$ is our vector space

$|v_1\rangle$, $|v_2\rangle$

Eg: $|v_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad |v_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$|v\rangle = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + z_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = z_1 v_1 + z_2 v_2$$

Another spanning set for $\mathbb{C}^2$

$$|v_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad |v_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|v\rangle = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + z_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

in terms of $|v_1'\rangle$, $|v_2'\rangle$?

$$|v\rangle = \frac{z_1 + z_2}{\sqrt{2}} |v_1'\rangle + \frac{z_1 - z_2}{\sqrt{2}} |v_2'\rangle$$

"Any vector can be represented using $|+\rangle$ and $|-\rangle$"

<u>Linear Independence</u>

A set of vectors $|v_1\rangle, |v_2\rangle, \ldots\ldots |v_n\rangle$ are linearly dependent if there exists a set of complex numbers $c_1, c_2 \ldots c_n$ such that if $c_i \neq 0$ for atleast one i such that ~~~~

$$c_1|v_1\rangle + c_2|v_2\rangle + \ldots c_n|v_n\rangle = 0$$

A set of vectors $|v_1\rangle, |v_2\rangle, \ldots |v_n\rangle$ are linearly independent if they are not linearly dependent.

(*) Any 2 sets of linearly independent vectors which span a vector space V contain same ~~of~~ number of elements

Proof : Let $A = \{a_1, a_2 \dots a_n\}$ $B = \{b_1, b_2 \dots b_m\}$ be two L.I sets of vectors and $a_i, b_i \in V$ $(m<n)$ Both A & B span V

$B' = \{a_1, b_1, \dots b_m\}$ will be linearly dependent

So $a_1 = \alpha_1 b_1 + \alpha_2 b_2 + \dots \alpha_m b_m$ where atleast one $\alpha_i \neq 0$

This type of spanning set is called "Basis". It can be proved that such basis state always exists.

* Number of vectors in the basis is called dimension

→ Mostly finite dimensional spaces are discussed

Q. Show $(1,-1), (1,2), (2,1)$ is linearly independent.

$\alpha(1,-1) + \beta(1,2) + \gamma(2,1) = (0,0)$

$$\Rightarrow \left.\begin{aligned} \alpha + \beta + 2\gamma &= 0 \\ -\alpha + 2\beta + \gamma &= 0 \end{aligned}\right\} \quad \begin{aligned} \alpha &= 0 \\ \beta + \gamma &= 0 \end{aligned}$$

~~So if α =~~

$$\underset{\substack{(+1) \\ \alpha_1}}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} + \underset{\substack{(+1) \\ \alpha_2}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} - \underset{\substack{(-1) \\ \alpha_3}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

27-08-21

Linear Operator on a vector space

→ Dirac representation

→ Matrix representation.

1) Linear operator between vector spaces V & W is - defined to be function $A: V \longrightarrow W$ such that

$$A\left(\sum_i a_i |v_i\rangle\right) = \sum_i a_i A |v_i\rangle$$

V $\quad\quad$ W

$\sum_i v_i$ $\quad\quad$ $\sum_j w_j$

basis for this vector space

$$|x\rangle \in V$$

$$|x\rangle = \sum_i a_i v_i \longrightarrow \sum_i a_i A |v_i\rangle$$

Operator

$$A|x\rangle = \sum_j b_j w_j \quad = \sum_j b_j |w_j\rangle$$

same

(Applying operator to whole vector x and individually to each $|v_i\rangle$ results same output)

* If the action of operator on all basis vectors is defined then mappings on all vectors is defined

(Because every vector can be spanned by basis)

Composition

V, W, X are vector spaces

$A: V \to W \qquad B: W \to X$

BA : denotes composition of B with A

$$(BA)|v\rangle = B(A|v\rangle) = BA|v\rangle$$

Matrix Representations

$$\begin{array}{ccc} A: & V & \to & W \\ & \downarrow & & \downarrow \\ & \mathbb{C}^n & & \mathbb{C}^m \end{array}$$

$\begin{pmatrix} n \text{ basis} \\ \text{vectors} \end{pmatrix} \qquad \begin{pmatrix} m \text{ basis} \\ \text{vectors} \end{pmatrix}$

coefficients of matrix

$$A|v_i\rangle = \sum_{j=1}^{m} a_{ij} |b_j\rangle$$

A is a $m \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x'_1 \\ \vdots \\ x'_m \end{bmatrix} \longrightarrow \text{vector space in } W$$

Ex: Suppose V is vector space with basis $|0\rangle$ & $|1\rangle$ and A is linear op. from V to V such that

$$A|0\rangle = |1\rangle$$
$$A|1\rangle = |0\rangle$$

Give matrix representation for A with resp to input basis $|0\rangle\, |1\rangle$ and output basis $|0\rangle\, |1\rangle$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

ii) For same question, find input & output basis which gives rise to different matrix representation of A

eg:. Input basis – $|0\rangle, |1\rangle$ $\quad$ $A|v_j\rangle = \sum A_{ij} |w_i\rangle$

Output basis – $|1\rangle, |0\rangle$ $\quad$ (Think)

$$A|0\rangle = A_{11}|1\rangle + A_{21}|0\rangle = |1\rangle$$

$$A_{11} = 1, \; A_{21} = 0$$

$$A|1\rangle = A_{12}|1\rangle + A_{22}|0\rangle = |0\rangle$$

$$A_{12} = 0, \; A_{22} = 1$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

iii) What is matrix representation of Identity operator in vector space V ?

Ans: Identity Matrix (I_n)

Pauli Matrices

→ 2×2 Matrices

→ Mapped from $V \to V$

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I|v\rangle = |v\rangle$$

$$\sigma_x = \sigma_1 \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad X|0\rangle = |1\rangle, \; X|1\rangle = |0\rangle$$

$$\sigma_y = \sigma_2 \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Y|0\rangle = i|1\rangle, \; Y|1\rangle = -i|0\rangle$$

$$\sigma_z = \sigma_3 \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Inner Product

Function which takes two vectors from vector space V and maps to a complex number

Inner Product : $V \times V \rightarrow C$

$\downarrow$

$(|v\rangle, |w\rangle)$

Also represented as

$\langle v|w\rangle$

$\downarrow$ Inner product of $|v\rangle$ and $|w\rangle$

(*) A function from $V \times V$ to C is an inner product if it satisfies -

1) Linear in the second argument

$$(|v\rangle, \sum_i \lambda_i |w_i\rangle)$$

$$= \sum_i \lambda_i (|v\rangle, |w_i\rangle)$$

2) $(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$ → Complex conjugate

3) $(|v\rangle, |w\rangle) \geqslant 0$ with equality only if $|v\rangle$ or $|w\rangle = 0$

30-8-21

Recap

- Linear Operators on vector space
  - Matrix rep
  - Dirac rep

↓

n to m vector space

↓

m×n operator

Pauli matrices $\mathbb{C} \to \mathbb{C}$

σ_0 σ_1 σ_2 σ_3

I X Y Z

Inner Products

$$F : V \times V \longrightarrow C$$

Representation: $(|v\rangle, |w\rangle)$ (or) $\langle v|w\rangle$

vectors in V

$\langle v|$ — used for "dual vector" of $|v\rangle$ in dirac notation

* dual is a linear operator from V to C

$$\langle v|(|w\rangle) \equiv \langle v|w\rangle = \underbrace{(|v\rangle, |w\rangle)}_{\text{Inner Product}}$$

* Dual vector has to be row matrix

What characterestics should a function have to be called as "inner product"

1) $(\cdot,\cdot)$ is linear in the second argument

$$(|v\rangle, \sum_i \lambda_i |w_i\rangle) = \sum_i \lambda_i (|v\rangle, |w_i\rangle$$

2) $(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$ i.e the complex conjugate of the inner product in reverse order

3) $(|v\rangle, |v\rangle) \geq 0$ and is only 0 if $|v\rangle = 0$ (zero vector)

Any function $V \times V \to \mathbb{C}$ that satisfies the above three properties can be called inner product on vector space V.

→ our inner product of interest (right now)

$$((y_1, y_2 \cdots y_n), (z_1, z_2 \cdots z_n)) \equiv \sum_i y_i^* z_i = [y_1^* \; y_2^* \cdots y_n^*] \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

Q verify that inner product defined satisfies above 3 properties? (Assignment)

Prop 1 :

* A vector $|v\rangle$ is normalized if $\| |v\rangle \| = 1$

Hence any vector can be normalized by dividing with its norm.

Eg: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \longrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = |-\rangle$

Orthonormal vectors

A set of vectors $|i\rangle$ is orthonormal if

① Each vector is unit vector

② Distinct vectors in the set are orthogonal

i.e. for $i \neq j$ their inner product is zero

$\langle v_i | v_j \rangle = 0$

* If we have a set of vectors forms a basis set, then we can convert it to an "orthonormal set" by Gram-schmidt procedure

(i.e. every vector can be represented as linear sum of these basis vectors)

Gram- Schmidt Procedure

→ Used to produce an orthonormal basis set from $(|w_1\rangle, \ldots\ldots |w_d\rangle)$

→ ① Define first vector as $|v_1\rangle = \frac{|w_1\rangle}{\| |w_1\rangle \|}$ (normalized)

② $1 \leq k \leq d-1$

$$|v_{k+1}\rangle = \frac{|w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle |v_i\rangle}{\| |w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle |v_i\rangle \|}$$

→ Finally produces $|v_1\rangle, |v_2\rangle \ldots |v_d\rangle$

Exercise : ~~Given~~ $|w_1\rangle = \begin{bmatrix}1\\1\end{bmatrix}$, $|w_2\rangle = \begin{bmatrix}1\\-1\end{bmatrix}$ $\Big\}$ Basis set

$$|\psi\rangle = \alpha\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\end{bmatrix} = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi\rangle = \underbrace{\left(\frac{\alpha+\beta}{2}\right)}^{x}\begin{bmatrix}1\\1\end{bmatrix} + \overbrace{\left(\frac{\alpha-\beta}{2}\right)}^{y}\begin{bmatrix}1\\-1\end{bmatrix}$$

Convert it to orthonormal basis using Gram Schmidt procedure

$$|v_1\rangle = \frac{|w_1\rangle}{\| |w_1\rangle \|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$|v_2\rangle = \frac{\begin{bmatrix}1\\-1\end{bmatrix} - \sum_{i=1}^{1}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}}{\left\| \begin{bmatrix}1\\-1\end{bmatrix} \right\|}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$|v_1\rangle, |v_2\rangle$ are orthonormal vectors.

31-08-21

overview

Quantum Computation

1. Deutsch Algorithm (2 bit version) → Deutsch Josza Algorithm (n-bit version)
2. Outer Product, completeness Relation

Problem :

Function : $\{0,1\} \rightarrow \{0,1\}$

$f(0) = f(1) = 0$
$f(1) = f(0) = 1$ } Single value

$f(0) \neq f(1)$
$f(0), f(1) = (0,1)$
$(1,0)$
different value

Suppose if we want to determine if f is single-valued (or) different valued function

x → [F] → F(x)

How many lookups are needed for the same ?

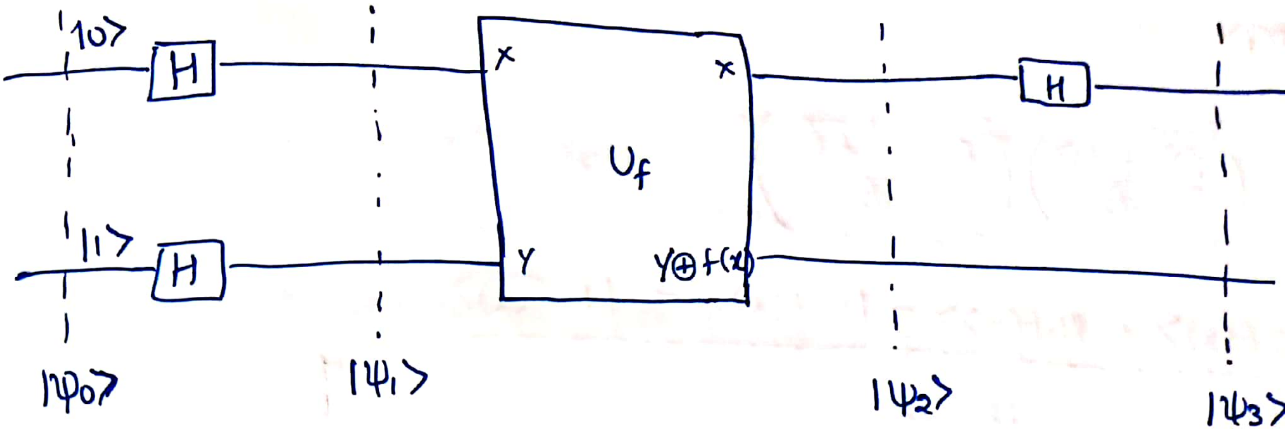
Ans 2. Check for f(0) and f(1) by looking up for $x=0$ and $x=1$ and see if they are same.

So 2 bits are to be examined

Now this can be done better by Quantum circuit

1 qubit can be examined to answer the question if F is single valued (or) different valued

Deutsch Algorithm



$$|\psi_0\rangle = |01\rangle$$

$$|\psi_1\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$|\psi_2\rangle = |x\rangle\left(\frac{|f(x)\rangle - |\overline{f(x)}\rangle}{\sqrt{2}}\right)$$

$$|\psi_2\rangle = (-1)^{f(x)}|x\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$|\psi_2\rangle = (-1)^{f(x)}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

As $(-1)^{f(x)}\,\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ came from $\frac{|f(x)\rangle - |\overline{f(x)}\rangle}{\sqrt{2}}$

replace it with that

$$|\psi_2\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|f(x)\rangle - |\overline{f(x)}\rangle}{\sqrt{2}}\right)$$

If $f(0) = f(1)$,

$$|\psi_2\rangle = \pm\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

If $f(0) \neq f(1)$

$$|\psi_2\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{f(x) - \overline{f(x)}}{\sqrt{2}}\right)$$

$$= \frac{|0 f(x)\rangle + |1 \cdot f(x)\rangle - |0 \cdot \overline{f(x)}\rangle - |1 \cdot \overline{f(x)}\rangle}{\sqrt{2} \cdot \sqrt{2}}$$

$$= \frac{\pm(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)}{\sqrt{2} \cdot \sqrt{2}}$$

(Just check the cases when $f(0) = 0$, $f(1) = 1$ and $f(1) = 0$, $f(0) = 1$)

Thus

$$|\psi_2\rangle = \begin{cases} \pm\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0) = f(1) \\ \pm\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0) \neq f(1) \end{cases}$$

$$|\psi_3\rangle = \begin{cases} \pm \underbrace{\boxed{|0\rangle}}_{\text{first bit}}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0) = f(1) \\ \pm \underbrace{\boxed{|1\rangle}}_{\text{first bit}}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0) \neq f(1) \end{cases}$$

Measuring the first bit, if first bit is

$|0\rangle \longrightarrow f(0) = f(1) \Rightarrow$ single valued function

$|1\rangle \longrightarrow f(0) \neq f(1) \rightarrow$ different valued function

Matrix Representation for inner product on Hilbert Space

$|w\rangle = \sum_i w_i |i\rangle$ basis vectors $\qquad |v\rangle = \sum_j w_j |j\rangle$ basis vector

$$\langle i|j\rangle = \left.\begin{matrix} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{matrix}\right\} = \delta_{ij}$$

δ_{ij} ↓ Kronecker constant ↓

1 if $i=j$

0 if $i \neq j$

$$\langle v|w\rangle = \left(\sum_i v_i |i\rangle , \sum_j w_j |j\rangle\right)$$

$$= \sum_{ij} v_i^* w_j \, \delta_{ij}$$

$$= \sum_j v_i^* w_i$$

[$\because$ δ_{ij} becomes 0 if $i \neq j$]

$$= [v_1^* \dots v_n^*] \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

(provided they are expressed in same basis) → $|v\rangle, |w\rangle$

i.e. for example if

$$|\psi_1\rangle = \alpha|+\rangle + \beta|-\rangle$$

$$|\psi_2\rangle = x|0\rangle + y|1\rangle$$

then $\langle\psi_1|\psi_2\rangle$ cannot be expressed as $[\alpha^* \ \beta^*]\begin{bmatrix} x \\ y \end{bmatrix}$

due to different basis (obv reasons)

Its expressed as $\left[\left(\frac{\alpha+\beta}{\sqrt{2}}\right)^* \left(\frac{\alpha-\beta}{\sqrt{2}}\right)^*\right]\begin{bmatrix} x \\ y \end{bmatrix}$

because $|\psi_1\rangle = \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$

→ Convinient rep $\langle v|w\rangle$

$$\begin{tikzcd} |v\rangle \arrow[r, "\text{dual}"] \arrow[d] & \langle v| \arrow[d] \\ \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} & [v_1^* \dots v_n^*] \end{tikzcd}$$

Outer Product

Suppose $|v\rangle$ is a vector ~~space~~ in Inner Product Space V

and $|w\rangle$ is a vector in Inner Product space W

Operator - $(|w\rangle\langle v|)$

$$\underbrace{(|w\rangle\langle v|)}_{op} \underbrace{|v'\rangle}_{\in V} = |w\rangle\langle v|v'\rangle$$

$$= \langle v|v'\rangle|w\rangle$$