8-9-21

Eigen Vectors / Eigen Values

property of op. A

For operator A, its eigen vector (on vector space V) is non-zero vector $|v\rangle$ such that

$$A|v\rangle = v|v\rangle$$

$\downarrow$ complex scalar

(*) v is called eigen-value of A corresponding to eigen vector $|v\rangle$

→ How do we find eigen vectors ? (Assume you have matrix rep. of operator)

A: Using characteristic equation

characteristic function : $c(\lambda)$

$$c(\lambda) = \det|A - \lambda I|$$

$c(\lambda) = 0$ → used to find eigen-values

solutions of characteristic eqn $c(\lambda) = 0$ are eigen-values of operator A

(*) By fundamental theorem of algebra, we say every polynomial has atleast one complex root, so every operator has atleast one eigen-value and a corresponding eigen vector

$(\lambda_1, \lambda_2, \lambda_3 \cdots\cdots) \rightarrow$ eigen values

$$\downarrow \quad \downarrow \quad \downarrow$$

$|\lambda_1\rangle \; |\lambda_2\rangle \; |\lambda_3\rangle \cdots\cdots \rightarrow$ corresponding eigen vectors

Eigen space : Corresponding to eigen value v, there could be set of vectors which have Eigen vector v. It is a subspace of the vector space V

Q Why is eigen space a subspace ?

For V' to be subspace

i) V' is not empty → eigenspace is not empty as it contains atleast one eigen vector

ii) If $u, v \in V'$ then $\alpha u + \beta v \in V'$
where $\alpha, \beta \in \mathbb{C}$

$$A(\alpha|u\rangle + \beta|v\rangle) = A(\alpha|u\rangle) + A(\beta|v\rangle)$$
$$= \alpha A|u\rangle + \beta|v\rangle$$
$$= \lambda(\alpha|u\rangle + \beta|v\rangle) \in V'$$

Thus eigenspace is a subspace

Diagonal Representation (for an operator)

For an operator A on vector space V,

$$A = \sum_i \lambda_i |i\rangle\langle i| \rightarrow \text{Diagonal Representation}$$

where vectors $|i\rangle$ form orthonormal ~~basis~~ set of eigen vectors for A with corresponding eigenvalues λ_i

$$A|q\rangle = \sum_i \lambda_i |i\rangle\langle i| (|q\rangle)$$
$$= \sum_i \lambda_i |i\rangle\langle i|q\rangle$$
$$= \sum_i \lambda_i \langle i|q\rangle |i\rangle$$

Operators — diagonalizable
Operators — some are not

diagonalizable: if it has a diagonal representation

Q. Find eigenvectors, eigen values & diagonal representations for X, Y, Z. (Pauli Matrices)

Sol :

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det(X - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 = 1 \Rightarrow \boxed{\lambda = \pm 1}$$

① $\lambda = 1$

$$(X - \lambda I)|v\rangle = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_1 = c_2$$

$$|v_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

② $\lambda = -1$

$$(X - \lambda I)|v\rangle = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0 \Rightarrow \quad c_1 = -c_2$$

$$|v_{-1}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

By diagonalising X,

$$X = \sum \lambda_i |i\rangle\langle i|$$

$$X = \lambda_1 |v_1\rangle\langle v_1| + \lambda_{-1} |v_{-1}\rangle\langle v_{-1}|$$

$$= \frac{1}{2} \left| |0\rangle + |1\rangle \right\rangle \left\langle |0\rangle + |1\rangle \right| - \frac{1}{2} \left| |0\rangle - |1\rangle \right\rangle \left\langle |0\rangle - |1\rangle \right|$$

$$\boxed{X = |+\rangle\langle +| - |-\rangle\langle -|}$$

ii) $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\det(Y - \lambda I) = 0$

$\Rightarrow \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

① $\lambda = 1$

$Y|\lambda_1\rangle = 1 \cdot |\lambda_1\rangle$

$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$-ic_2 = c_1 \qquad ic_1 = c_2$

$c_1 = 1, \quad c_2 = i$

$|\lambda_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$

② $\lambda = -1$

$Y|\lambda_{-1}\rangle = -1 \cdot |\lambda_{-1}\rangle$

$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}$

$+ic_2 = +c_1 \qquad ic_1 = -c_2$

$c_1 = 1, \quad c_2 = -i$

$|\lambda_{-1}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$

By diagonalising Y,

$$Y = \lambda_1 |\lambda_1\rangle\langle\lambda_1| + \lambda_{-1}|\lambda_{-1}\rangle\langle\lambda_{-1}|$$

$$\boxed{Y = 1\cdot\frac{1}{2}(|0\rangle + i|1\rangle)(\langle 0| + i\langle 1|) - \frac{1}{2}(|0\rangle - i|1\rangle)(\langle 0| - i\langle 1|)}$$

iii) $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\det(Z - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(1-\lambda)(1+\lambda) = 0$$

$$\Rightarrow \lambda = \pm 1$$

① $\lambda = 1$

$$Z|\lambda_1\rangle = 1\cdot|\lambda_1\rangle$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$c_2 = 0, \; c_1 = 1$$

$$|\lambda_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

② $\lambda = -1$

$$Z|\lambda_{-1}\rangle = -1\,|\lambda_{-1}\rangle$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}$$

$$c_1 = 0, \; c_2 = 1$$

$$|\lambda_{-1}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

By diagonalising Z,

$$Z = \lambda_1|\lambda_1\rangle\langle\lambda_1| + \lambda_{-1}|\lambda_{-1}\rangle\langle\lambda_{-1}|$$

$$= 1\cdot|0\rangle\langle 0| + (-1)\,|1\rangle\langle 1|$$

$$\boxed{Z = |0\rangle\langle 0| - |1\rangle\langle 1|}$$

When an eigenspace has more than one dimension, we call it "degenerate"

For same eigen vector

Ex:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C(\lambda) = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$(2-\lambda)(2-\lambda)(-\lambda) = 0$$

$$\Rightarrow (2-\lambda)^2(-\lambda) = 0$$

$$\Rightarrow \lambda = 0, \quad \lambda = 2$$

Eigen Vectors

$$\lambda = 2$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0-2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\gamma = 0$$

0-9-21

Adjoints & Hermitian Operators

→ Dirac Representation

Adjoint → conjugate of transpose of matrix
↓
all elements in matrix converted to their complex conjugates

Hermitian Conjugate

(A) Suppose A is an linear operator on Hilbert space V, then there exists a unique linear operator $A^\dagger$ on V such that for all vector $|v\rangle, |w\rangle \in V$

$$(|v\rangle, A|w\rangle) = (A^\dagger|v\rangle, |w\rangle)$$

$A^\dagger$ is called Adjoint / Hermitian conjugate of A

① $(AB)^\dagger = B^\dagger A^\dagger$

Proof:

$$(|v\rangle, (AB)^\dagger|w\rangle) = (AB|v\rangle, |w\rangle)$$

$$= (B|v\rangle, A^\dagger|w\rangle)$$

$$= (|v\rangle, B^\dagger A^\dagger|w\rangle)$$

$$= (|v\rangle, B^\dagger A^\dagger|w\rangle)$$

By usual convention,

$$\boxed{|v\rangle^\dagger = \langle v|}$$

$$* \boxed{(A|v\rangle)^\dagger = \langle v|A^\dagger}$$

Proof ↗ → [Correct Proof in Next class]

we know $(|w\rangle, A|v\rangle) = \langle w|A|v\rangle$

$$(|w\rangle, (A|v\rangle)^\dagger) = \langle w|(A|v\rangle)^\dagger\rangle$$

$$= |(A|v\rangle)^\dagger\rangle\langle w|$$

$(A|v\rangle)^\dagger \rightarrow$ dual vector of $(A|v\rangle)$

$$= \langle v|A^\dagger$$

Ex: Show that for any two vectors $|v\rangle, |w\rangle$,

$$(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$$

Sol: for arbitrary vectors $|x\rangle, |y\rangle$

$$(|x\rangle, (|w\rangle\langle v|)^\dagger |y\rangle) = ((|w\rangle\langle v|)|x\rangle, |y\rangle) \quad \text{[defn of hermitian conjugate]}$$

$$= (\langle v|x\rangle |w\rangle, |y\rangle)$$

$$= (\langle v|x\rangle)^\dagger \langle w|y\rangle$$

$$= \langle x|v\rangle\langle w|y\rangle$$

$$= (\langle x|)(|v\rangle\langle w|)|y\rangle$$

$$= (|x\rangle, \underline{\underline{|v\rangle\langle w|}} y\rangle)$$

So $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$

Scanned with CamScanner

Anti-linearity of Adjoint

$$\left(\sum_i a_i A_i\right)^\dagger = \sum_i a_i^* (A_i)^\dagger$$

Proof:

$$\left(\left(\sum_i a_i A_i\right)^\dagger |v\rangle, |w\rangle\right) = \left(|v\rangle, \left(\sum_i a_i A_i\right)|w\rangle\right)$$

$$= \sum_i a_i \left(|v\rangle, A_i|w\rangle\right) \quad \left(\text{Property of Inner Product}\right)$$

$$= \sum_i a_i \left(A_i^\dagger |v\rangle, |w\rangle\right)$$

$$= \sum_i \left(a_i^* A_i^\dagger |v\rangle, |w\rangle\right)$$

$$= \left(\left(\sum_i a_i^* A_i^\dagger\right)|v\rangle, |w\rangle\right)$$

Thus $\boxed{\left(\sum_i a_i A_i\right)^\dagger = \sum_i a_i^* A_i^\dagger}$

Q: Prove $(A^\dagger)^\dagger = A$

A: So we need to prove $\left((A^\dagger)^\dagger |v\rangle, |w\rangle\right) = \left(A|v\rangle, |w\rangle\right)$

$$\left((A^\dagger)^\dagger |v\rangle, |w\rangle\right) = \left(|v\rangle, A^\dagger |w\rangle\right)$$

$$= \left(A^\dagger|w\rangle, |v\rangle\right)^*$$

$$= \left(|w\rangle, A|v\rangle\right)^*$$

$$= \left(A|v\rangle, |w\rangle\right)$$

Implication of Hermitian conjugate in matrix notation

$$A^\dagger = (A^*)^T$$

eg: $A = \begin{bmatrix} i & -i \\ 2+i & 3-i \end{bmatrix}$

$$A^\dagger = \begin{bmatrix} -i & -(i+2) \\ i & i-3 \end{bmatrix}$$

Hermitian Operator (also called as self Adjoint Operator)

$\downarrow$

definition: $A^\dagger = A$

where A is an operator

Q. Are Pauli Matrices self adjoint ?

σ_1	I	$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$I^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
σ_2	X	$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$X^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
σ_3	Y	$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$	$Y^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
σ_4	Z	$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$Z^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\therefore$ All Pauli Matrices are self-adjoint and hence hermitian operators

→ There's an important class of ~~operators~~ Hermitian operators called <u>Projectors</u>

<u>Projectors</u>

Suppose W is a K-dim vector subspace within d-dimensional vector space V. We can construct using Gram Schmidt procedure, an orthonormal basis set $|1\rangle \dots |d\rangle$ for V

$\sum^{d} |i\rangle$ → orthonormal basis for V ($|1\rangle, |2\rangle$ ~~and~~ $\dots |d\rangle$)

and

$\sum^{K} |i\rangle$ → Orthonormal basis for W ($|1\rangle, \dots |K\rangle$)

then

$$P = \sum_{i=1}^{K} |i\rangle\langle i|$$

P is a projector onto vector subspace W

(A) consider $|v\rangle \in V$ then $|v\rangle = \sum_{i=1}^{d} \alpha_i |i\rangle$

$$P(|v\rangle) = \sum_{i=1}^{K} |i\rangle\langle i| \left(\sum_{j=1}^{d} \alpha_j |j\rangle \right)$$

$$= \sum_{i=1}^{K} |i\rangle \alpha_i$$

(i.e. when $i \neq j$ ~~then~~ $\langle i|j\rangle = 0$ so those terms are eliminated)

↓

⇒ we get a vector that belongs to subspace W

So many vectors $|v\rangle$ in V can get mapped to some $|w\rangle$ in W (~~as~~ like many-one relation)

13-9-21

Recap

Hermitian conjugate of operator

$(A|v\rangle)^\dagger = \langle v|A^\dagger$ $\qquad (|v\rangle^\dagger = \langle v|)$

~~Proof~~ Clarification in Proof

Proof: $(A|v\rangle)^\dagger$ is dual vector of $A|v\rangle$

$(A|v\rangle, |v\rangle) = \langle (A|v\rangle)^\dagger |v\rangle$ — ①

↓ Also

$(A|v\rangle, |v\rangle) = (|v\rangle, A^\dagger|v\rangle)$

$= \langle v|A^\dagger|v\rangle$ — ②

As ① & ② are same

$\langle (A|v\rangle)^\dagger |v\rangle = \langle v|A^\dagger|v\rangle$

$\boxed{(A|v\rangle)^\dagger = \langle v|A^\dagger}$

Hermitian / Self-Adjoint Operator $(A^\dagger = A)$

→ Special Class – Projector

↓

Vector subspace W of vectorspace V

↓ K dim $\qquad$ ↓ d dim

orthonormal basis $|i\rangle$ $\qquad \{1, \ldots K\}$ $\qquad \{1, \ldots K, K+1, \ldots d\}$

$P = \sum_{i=1}^{K} |i\rangle\langle i|$ is a projector for V

↓ it projects onto subspace W

Usually also called subspace projected by P

Q: Prove P is Hermitian

① we know $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$

If $|w\rangle = |v\rangle$

$$= (|v\rangle\langle v|)^\dagger = |v\rangle\langle v|$$

So $|v\rangle\langle v|$ is Hermitian

~~Hence $\sum_{i=1}^{n} |i\rangle\langle i|$ is also hermitian.~~

~~proof~~

$$\boxed{\sum_i (a_i A_i)^\dagger = \sum_i a_i^* A_i^\dagger}$$

Result

Proof :

$$P = \sum_{i=1}^{K} |i\rangle\langle i|$$

$$P^\dagger = \left(\sum_{i=1}^{K} |i\rangle\langle i|\right)^\dagger$$

$$P^\dagger = (1)^* \sum_{i=1}^{K} (|i\rangle\langle i|)^\dagger \quad \text{(Using above result)}$$

$$P^\dagger = \sum_{i=1}^{K} |i\rangle\langle i|$$

$\boxed{P^\dagger = P}$ Thus P is Hermitian.

Orthogonal Component of a Projector

$Q = I - P$ is called orthogonal complement of P

$$I = \sum_{i=1}^{d} |i\rangle\langle i|$$

$$Q = \sum_{i=1}^{d} |i\rangle\langle i| - \sum_{j=1}^{K} |j\rangle\langle j|$$

$$= \sum_{i=K+1}^{d} |i\rangle\langle i| .$$

$\downarrow$ projects onto

This vector space also called as Orthogonal Complement of vector space of P $\leftarrow$ vector space spanned by $\{K+1, \ldots, d\}$

Q. Prove $P^2 = P$

Sol: $P = \sum_i |i\rangle\langle i|$

$$P^2 = P \cdot P = \left(\sum_i |i\rangle\langle i|\right)\left(\sum_j |j\rangle\langle j|\right)$$

$$= \sum_{i,j} |i\rangle \underbrace{\langle i|j\rangle}_{\delta_{ij}} \langle j|$$

$$= \sum_{i,j} |i\rangle\langle j| \delta_{ij} \qquad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$= \sum_i |i\rangle\langle i|$$

$$= P$$

Thus $\boxed{P^2 = P}$

Let $P(v_1|1\rangle + v_2|2\rangle \ldots + v_d|d\rangle)$
where $P = \sum_{i=1}^{K} |i\rangle\langle i|$

we get only those vectors $\{1, 2, \ldots K\}$

i.e. $v_1|1\rangle + \ldots v_K|K\rangle$

Now $P(v_1|1\rangle + \ldots v_K|K\rangle)$

will give same vector back

Normal Operator

An operator A is said to be normal if $AA^{\dagger} = A^{\dagger}A$

Ex: Prove that an operator that is hermitian is also normal

Sol: As A is hermitian, $A^{\dagger} = A$

so $AA^{\dagger} = A^{\dagger}A = A^2$, Hence its normal

Ex: Is ~~conve~~ reverse true ? Does A is normal means A is hermitian ?

Ans: False, it need not be,

Consider $A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ $\quad A \neq A^{\dagger}$

$A^{\dagger} = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$

$AA^{\dagger} = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$A^{\dagger}A = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$AA^{\dagger} = A^{\dagger}A$ but $A \neq A^{\dagger}$

Spectral Decomposition

Theorem

Any normal operator M on vector space V is diagonal with respect to some orthonormal basis for V.

Conversely every diagonalizable operator is normal

$M = \sum_i \lambda_i |i\rangle\langle i| \rightarrow$ λ_i's are eigen values

$|i\rangle$ form orthonormal basis formed with eigen vectors corresponding to eigen values

Q. **Prove** - A normal matrix is Hermitian if & only if it has real eigen values

Sol.

Part-I - We prove if an operator is normal & hermitian then it has real eigen values

Sol - Let A be normal & hermitian

Let $|\lambda\rangle$ be an eigen vector and λ is corresponding eigen value

$$A|\lambda\rangle = \lambda|\lambda\rangle$$

$$A^{\dagger}A|\lambda\rangle = A^{\dagger}(\lambda|\lambda\rangle)$$

$$= \lambda^{*}A^{\dagger}|\lambda\rangle$$

$$= \lambda^{*}A|\lambda\rangle \quad (\because A \text{ is hermitian})$$

$$= \lambda^{*}\lambda|\lambda\rangle \quad -①$$

On other hand,

$$AA^{\dagger}|\lambda\rangle = AA|\lambda\rangle$$

$$= A\lambda|\lambda\rangle$$

$$= \lambda A|\lambda\rangle$$

$$= \lambda(\lambda|\lambda\rangle)$$

$$= \lambda^{2}|\lambda\rangle \quad -②$$

From ① & ②

So $\lambda^{*}\lambda = \lambda^{2}$

$\Rightarrow \lambda = \lambda^{*}$

So $\boxed{\lambda \text{ is Real}}$

Part-II - If eigen values are real, then the normal matrix is also hermitian

Sol. Using Spectral Theorem,

$$A = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

Eigen vectors that form orthonormal basis

eigen values (given λ_i's are real)

$$A^\dagger = \sum_i \lambda_i^* (|\lambda_i\rangle\langle\lambda_i|)^\dagger$$

$$= \sum_i \lambda_i^* (|\lambda_i\rangle\langle\lambda_i|) \quad [\because (|w\rangle\langle v|)^\dagger = |v\rangle\langle w|]$$

$$= \sum_i \lambda_i (|\lambda_i\rangle\langle\lambda_i|)$$

$$= A$$

So $\boxed{A^\dagger = A} \Rightarrow A$ is Hermitian

Unitary Matrices

$$U^\dagger U = UU^\dagger = I$$

Operator is unitary $\Rightarrow$ Each of its matrix representations is also unitary

depends on chosen basis

As $UU^\dagger = U^\dagger U$, U is also normal and it has spectral decomposition

Unitary $\rightarrow$ They preserve inner product of two vectors i.e. Lets consider $|v\rangle, |w\rangle$ be two vectors, U is unitary operator

$$(U|v\rangle, U|w\rangle) = \langle (U|v\rangle)^{\dagger} | U|w\rangle$$

$$= \langle v|U^{\dagger} U|w\rangle \qquad [\because (A|v\rangle)^{\dagger} = \langle v|A^{\dagger}]$$

$$= \langle v|(U^{\dagger}U)|w\rangle$$

$$= \langle v|I|w\rangle \qquad [\because UU^{\dagger} = I]$$

$$= \langle v|w\rangle$$

15-9-21

Recap

- Hermitian Operators
- Normal Operators
- Spectral Decomposition Theorem
- Unitary Operator

Unitary Operators

$UU^{\dagger} = U^{\dagger}U = I \rightarrow$ normal

$\downarrow$

has spectral decomposition

$\downarrow$

They preserve Inner Products

$$(U|v\rangle, U|w\rangle) = \langle v|\underbrace{U^{\dagger}|U}_{I}|w\rangle$$

$$= \langle v|I|w\rangle$$

$$= \langle v|w\rangle = (|v\rangle, |w\rangle)$$

Scanned with CamScanner

(*) $|v\rangle, |w\rangle \in$ Orthonormal basis

And inner product is preserved even after apply U operator

So Operator by U on orthonormal basis set gives rise to another orthonormal basis set

(*) We can write,

$$U = \sum |w_i\rangle\langle v_i|$$

Let $|v_i\rangle$ be orthonormal basis

$U|v_i\rangle$ be the $\underbrace{\text{set of vectors } |w_i\rangle}_{\text{another orthonormal basis}}$

$\underline{U = \sum_i |w_i\rangle\langle v_i|}$ → It is true as if U operates on $x \in V$ $\langle v_i|x\rangle$ is scalar an we get U as linear sum of multiples of $|w_i\rangle$

↓

is unitary & orthonormal basis conversion

① $\sum_i |w_i\rangle\langle v_i|$ is unitary (Can be proved by $UU^\dagger = U^\dagger U = I$ <u>Left as Exercise</u>

(*) Eigen values of unitary matrix have modulus 1 (i.e can be written as $e^{i\theta}$ for some θ)

<u>Proof</u> Suppose $|v\rangle$ is a eigen vector with λ as eigen value

$$U|v\rangle = \lambda|v\rangle \quad - ①$$

$$\langle v|U^\dagger = \lambda^* \langle v| \quad - ②$$

$(① \times ②) \Rightarrow$ $\quad \langle v|U^\dagger|U|v\rangle = (\lambda^*\langle v|)(\lambda|v\rangle)$

↓
Inner Product

$$\langle v|I|v\rangle = \lambda^*\lambda\langle v|v\rangle \qquad \boxed{\lambda = 1 \cdot e^{i\theta}}$$

$$\langle v|v\rangle = \lambda^*\lambda\langle v|v\rangle$$

$$\Rightarrow \lambda^*\lambda = 1 \Rightarrow |\lambda|^2 = 1 \Rightarrow |\lambda| = 1$$

Basis Changes

Suppose A' and A'' are matrix representations of an operator A on vector space V with respect to two different orthonormal bases, $|v_i\rangle$ and $|w_i\rangle$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\searrow A' \quad\quad\quad \searrow A''$

Elements of A' – $A'_{ij} = \langle v_i|A|v_j\rangle$

Elements of A'' – $A''_{ij} = \langle w_i|A|w_j\rangle$

Q. Characterize the relationship between A' and A'' (i.e. knowing A'', how you find A')

Sol:

U is an operator from $|w_i\rangle$ to $|v_i\rangle$

orthonormal basis

$$UU^\dagger = U^\dagger U = I$$

$$U = \sum_i |w_i\rangle\langle v_i| \Rightarrow U^\dagger = \sum_i |v_i\rangle\langle w_i|$$

$$\begin{aligned}
A'_{ij} &= \langle v_i|A|v_j\rangle \\
&= \langle v_i|IAI|v_j\rangle \\
&= \langle v_i|UU^\dagger A UU^\dagger|v_j\rangle \\
&= \langle v_i|\Big(\sum_p |w_p\rangle\langle v_p|\Big)\Big(\sum_q |v_q\rangle\langle w_q|\Big) A \Big(\sum_r |w_r\rangle\langle v_r|\Big) \\
&\quad\quad \Big(\sum_s |v_s\rangle\langle w_s|\Big)|v_j\rangle \\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle\langle v_p|v_q\rangle\langle w_q|A|w_r\rangle\langle v_r|v_s\rangle\langle w_s|v_j\rangle
\end{aligned}$$

As all $|v_i\rangle$ and $|w_i\rangle$ are part of orthonormal basis

$$\langle v_i | v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} = \cancel{\delta_{pq}}\ \delta_{ij}$$

so

$$A'_{ij} = \sum_{p,q,r,s} \langle v_i | w_p \rangle \delta_{pq} \langle w_q | A | w_r \rangle \delta_{rs} \langle w_s | v_j \rangle$$

$$= \sum_{p,r} \langle v_i | w_p \rangle \underbrace{\langle w_p | A | w_r \rangle}_{A''_{qr}} \langle w_r | v_j \rangle \qquad \left(\begin{array}{l} \text{As } p = q \to \delta_{pq} = 1 \\ \text{and } r = s \to \delta_{rs} = 1 \end{array} \right)$$

$$= \sum_{p,r} \langle v_i | w_p \rangle A''_{pr} \langle w_r | v_j \rangle$$

$$\boxed{A'_{ij} = \sum_{p,r} \langle v_i | w_p \rangle \langle w_r | v_j \rangle A''_{pr}}$$

⊕ Special sub class of Hermitian → "Positive Operators"

<u>Positive Operator</u>

An operator A, such that for any vector $|v\rangle$, $(|v\rangle, A|v\rangle)$ is real & non-negative.

<u>Positive Definite Operator</u>

A positive operator A where $(|v\rangle, A|v\rangle) > 0$ (strictly greater than 0) for all $|v\rangle \neq 0$

Scanned with CamScanner

Q. Prove all positive operators are Hermitian

Proof Let A be a positive operator

$$(|v\rangle, A|v\rangle) \geq 0 \Rightarrow \langle v|A|v\rangle \geq 0$$

$$A = \frac{A+A^{\dagger}}{2} + i\,\frac{A-A^{\dagger}}{2i}$$

$$= B + iC \quad \text{where } B = \frac{A+A^{\dagger}}{2},\; C = \frac{A-A^{\dagger}}{2i}$$

Clearly, as we can see B and C are hermitian

$$B^{\dagger} = \left(\frac{A+A^{\dagger}}{2}\right)^{\dagger} = \frac{A^{\dagger}+(A^{\dagger})^{\dagger}}{2} = \frac{A+A^{\dagger}}{2} = \underline{\underline{B}}$$

$$C^{\dagger} = \left(\frac{A-A^{\dagger}}{2i}\right)^{\dagger} = \frac{A^{\dagger}-(A^{\dagger})^{\dagger}}{-2i} = \frac{A-A^{\dagger}}{2i} = C$$

We know,

$$\langle v|A|v\rangle = \langle v|(B+iC)|v\rangle$$

$$= \langle v|B|v\rangle + \langle v|iC|v\rangle$$

$$= \langle v|B|v\rangle + i\langle v|C|v\rangle$$

$$= \alpha + i\beta \quad \text{where } \alpha = \langle v|B|v\rangle,\; \beta = \langle v|C|v\rangle$$

As $\langle v|A|v\rangle \geq 0 \Rightarrow \beta = 0$ for all $|v\rangle$

$$\Rightarrow \langle v|C|v\rangle = 0 \text{ for all } |v\rangle$$

$$\Rightarrow C = 0$$

$$\Rightarrow \frac{A-A^{\dagger}}{2i} = 0$$

$$\Rightarrow \boxed{A = A^{\dagger}}$$

So, A is Hermitian

7-9-21

Spectral Decomposition Theorem

Theorem : Any normal operator M on vector space V is diagonal with respect to some orthonormal basis for V. Conversely any diagonalizable operator is normal.

$M = \sum a_i |i\rangle\langle i|$ → ~~St~~ Proving Converse part is simple, we prove $MM^\dagger = M^\dagger M$ by expanding sum (trivial)

↓

we prove $(MM^\dagger)|v\rangle = (M^\dagger M)|v\rangle$ by expanding $|v\rangle = \sum_i v_i|i\rangle$

M is diagonalizable (Proof)

① If V has only 1 dimension (M has only one entry)

eg: if the dim is 1 → orth : $|0\rangle$ (only basis vector)

then $M|v\rangle = \cancel{\ \ }$

$= \underbrace{\boxed{}^{\text{scalar}}|0\rangle\langle 0|}_{M}\ |v\rangle$ → simple expression (only one term) [M will be just a complex no. so $M^\dagger$ is also complex no.] ↓ [And as complex no's are commutative under *, $MM^\dagger = M^\dagger M$]

② What if d > 1 i.e. $M = \phi_0|0\rangle\langle 0| + \phi_1|1\rangle\langle 1|$

↓

So we need to show M is of this form when M is normal

M is normal on V and $d > 1$

Let λ be eigen value of M

↓ create

eigen space → of λ

(set of eigen vectors with same eigen value)

Take the projector P onto the subspace which is subspace of λ

↓

i.e it contains those set of orthormal vector that project onto eigenspace of λ.

so it may have basis vectors k $(<d)$

(i.e k orthonormal vectors as basis which are also a part of orthonormal basis of V)

Also consider Q → orthogonal complement of P

→ contains basis vectors $\{k+1, \dots d\}$

Then,

$$M = (P+Q)\cdot M \cdot (P+Q) \qquad [\because P+Q = I]$$

$$= \left(\sum_{i=1}^{k} |i\rangle\langle i| + \sum_{k+1}^{d} |i\rangle\langle i|\right)(M)\left(\sum_{j=1}^{k} |j\rangle\langle j| + \sum_{k+1}^{d} |j\rangle\langle j|\right)$$

$$= \left(\sum_{i=1}^{d} |i\rangle\langle i|\right) M \sum_{j=1}^{d} |j\rangle\langle j|$$

$$M = PMP + QMP + PMQ + QMQ \quad - (1)$$

(1) PMP (P is projector onto λ subspace)

[Continued in next Page] →

① PMP (P is projector onto λ eigen subspace)

Let $|\psi\rangle$ be any vector on M , then

$P|\psi\rangle \in$ eigen subspace of λ

(i.e $P|\psi\rangle$ is a eigen vector of λ)

So by ~~property~~ defn of eigen vectors

For $|v\rangle \rightarrow$ eigen vector of λ on M vectorspace

$$M|v\rangle = \lambda|v\rangle$$

similarly,

$$M(P|\psi\rangle) = \lambda P|\psi\rangle$$

$$MP|\psi\rangle = \lambda P|\psi\rangle$$

In short we can say $\boxed{MP = \lambda P}$

Scanned with CamScanner

$MP = \lambda P$

$\Rightarrow \quad P(MP) = P(\lambda P)$

$\Rightarrow \quad PMP = \lambda P^2$

$\Rightarrow \quad \boxed{PMP = \lambda P}$ $\quad$ [$\because$ we proved already that for projector P, $P^2 = P$]

② $\underline{QMP}$

Q only contains orthogonal bases not in P

As already seen, $MP = \lambda P$

$QMP = Q(\lambda P)$

$= \lambda(QP)$

$= 0$

proof that $QP = 0$

Sol: we know $Q = I - P$

$QP = (I-P)P$

$= IP - P^2$

$= P - P \quad (\because P^2 = P)$

$= 0$

③ PMQ

To prove this term is 0, we cant use the same argument used for QMP as we don't know if there are any eigen vectors in orthogonal space.

Lets consider an eigen vector $|v\rangle \in$ subspace of P

$MM^\dagger|v\rangle = M^\dagger M|v\rangle \quad$ (as M is normal $\Rightarrow MM^\dagger = M^\dagger M$)

$\underbrace{MM^\dagger|v\rangle = \lambda M^\dagger|v\rangle}$ $\quad$ (since $|v\rangle \in$ eigenspace of λ)

$\downarrow$

M operating on $M^\dagger|v\rangle$ gives $\lambda . M^\dagger|v\rangle$

$\Rightarrow$ It implies that $M^\dagger|v\rangle \in$ eigenspace of λ (i.e. it belongs to subspace P)

As $M^\dagger|v\rangle \in$ subspace of P

So, $QM^\dagger P = Q(\underbrace{M^\dagger P}_{\in \text{ subspace of } P})$

$$QM^\dagger P = 0$$

Taking its adjoint,

$$(QM^\dagger P)^\dagger = 0^\dagger$$

[As P, Q are projectors and hermitian, $P^\dagger = P$, $Q^\dagger = Q$ which is proved earlier]

$$\Rightarrow P^\dagger M Q^\dagger = 0$$

~~As P~~

$$\Rightarrow \boxed{PMQ = 0}$$

Hence, from ①, we have

$$M = PMP + \cancelto{0}{QMP} + \cancelto{0}{PMQ} + QMQ$$

$$M = PMP + QMQ$$

we already know PMP is diagonalizable,

Because $PMP = \lambda P$

and P is diagonalizable (by defn)

So λP is also diagonalisable.

We now need to show QMQ is normal.

To prove this, we need to show

$$QMQ(QMQ)^\dagger = (QMQ)^\dagger(QMQ)$$

Consider,

$$QM = QM \cdot I$$

$$= QM(P+Q) \qquad [\because P+Q = I]$$

$$= \cancel{QMP}^{\to 0} + QMQ \qquad [\text{Already proved } QMP = 0]$$

$$\boxed{QM = QMQ}$$

$$QM^\dagger = QM^\dagger \cdot I$$

$$= QM^\dagger (P+Q)$$

$$= \cancel{QM^\dagger P}^{\to 0} + QM^\dagger Q \qquad [\text{Already proved } QM^\dagger P = 0]$$

$$\boxed{QM^\dagger = QM^\dagger Q}$$

$$Q^2 = Q \cdot Q$$

$$= (I-P)(I-P)$$

$$= I - P - P + P^2$$

$$= I - P \cancel{- P + P} \qquad [\because P^2 = P]$$

$$= I - P$$

$$\boxed{Q^2 = Q}$$

Other way to prove $Q^2 = Q$

As Q is normal, hermitian

$$QQ^\dagger = Q^\dagger Q$$

and $Q = Q^\dagger$

<u>Proof</u> : QMQ is normal

Consider $QMQ(QMQ)^\dagger = QMQ(Q^\dagger M^\dagger Q)$

$$= QMQ(QM^\dagger Q) \qquad \left[\begin{array}{l}\because Q^\dagger = Q \\ Q \to \text{hermitian}\end{array}\right.$$

$$= QMQQM^\dagger Q$$

$$= QMQ^2M^\dagger Q$$

$$= QMQM^\dagger Q \qquad [\because Q^2 = Q]$$

$$= (QMQ)M^{\dagger}Q$$

$$= QMM^{\dagger}Q \qquad [\because QMQ = QM]$$

$$= QM^{\dagger}MQ \qquad [\because M \text{ is normal}]$$

$$= (QM^{\dagger})MQ$$

$$= QM^{\dagger}QMQ \qquad [\because QM^{\dagger} = QM^{\dagger}Q]$$

$$= (QM^{\dagger}Q)\cdot(QMQ) \qquad [\because Q = Q^{2}]$$

$$= (Q^{\dagger}M^{\dagger}Q^{\dagger})(QMQ) \qquad [\because Q = Q^{\dagger} \text{ as } Q \text{ is hermitian}]$$

$$= (QMQ)^{\dagger}(QMQ)$$

Thus QMQ is normal

Use of proving QMQ is normal

If QMQ is normal, then by induction we can claim QMQ is diagonal with respect to some orthogonal basis for subspace because its dimension is definitely less than d as rest of dimensions is contained in P subspace

In terms of outer product representation,

$$M = \sum_i \lambda_i |i\rangle\langle i|$$ where λ_i are eigen values of M

$|i\rangle$ is orthonormal basis of V

and each $|i\rangle$ is eigen vector of M with eigen value λ

In terms of Projectors,

$$M = \sum \lambda_i P_i$$ where λ_i are eigenvalues of M

P_i is projector onto λ_i eigenspace of M

* These projectors satisfy completeness relation

$$\sum_i P_i = I$$

* It also satisfies orthonormality relation

$$P_i P_j = \delta_{ij} P_i$$

01-09-21

Recap

~~Quantum D~~

Spectral Decomposition

Any normal operator M on vector space V is diagonal with respect to some orthonormal basis for V.

(Diagonal ⇒ normal too)

$P, Q \rightarrow \{k+1, \cdots d\}$

$\downarrow$

$\{1, 2 \cdots k\}$

$M = \underbrace{PMP + QMQ}_{\text{normal}}$

⊛ M can be written as $M = \sum_i \lambda_i |i\rangle\langle i|$

where λ_i is eigen value

$|i\rangle$ are the eigen vector corres to λ_i which forms an orthonormal basis

$M = \sum_i \lambda_i P_i \longrightarrow$ where P_i is proj onto λ_i eigenspace

$\sum_i P_i = I$ ↘ completeness

⊛ It also satisfies orthonormality relation

i.e. $P_i P_j = \delta_{ij} P_i$

Tensor Products

Ex: $|01\rangle, |11\rangle \ldots$

→ Just a way of putting vector spaces together to form larger vector space.

* Mainly used in multiparticle systems.

Defn: Suppose V & W are vector spaces and have dimensions m, n respectively. Also we suppose V, W are Hilbert spaces. Then $V \otimes W$ (read as V tensor W) is called Tensor Product, forms an 'mn' dimensional vector space.

Linear Combinations of "tensor products" → also falls into same vector space.

→ In particular $|i\rangle$ - orthonormal basis for V and $|j\rangle$ is orthonormal basis for W, then

$|i\rangle \otimes |j\rangle$ → Orthonormal basis for $V \otimes W$

Abbreviated as -

$$\left.\begin{array}{l} |v\rangle|w\rangle \\ |v,w\rangle \\ |vw\rangle \end{array}\right\} \Rightarrow |v\rangle \otimes |w\rangle$$

Eg: $\{|0\rangle, |1\rangle\}$ → orthonormal basis for V

$\{|0\rangle, |1\rangle\}$ → orthonormal basis of W

→ These four are orthonormal basis for $V \otimes W$

$|0\rangle \otimes |0\rangle \Rightarrow |00\rangle$ $\qquad$ $|1\rangle \otimes |0\rangle \Rightarrow |10\rangle$

$|0\rangle \otimes |1\rangle \Rightarrow |01\rangle$ $\qquad$ $|1\rangle \otimes |1\rangle \Rightarrow |11\rangle$

Properties of Tensor Products

① for any arbitrary scalar z and elements $|v\rangle \in V$ and $|w\rangle \in W$,

$$z(|v\rangle \otimes |w\rangle) = (z|v\rangle) \otimes (|w\rangle) = |v\rangle \otimes (z|w\rangle)$$

Eg: $p = \underset{\in V}{\alpha_p|0\rangle} + \beta_p|1\rangle$

$q = \underset{\in W}{\alpha_q|0\rangle} + \beta_q|1\rangle$

then $z(p \otimes q) = z|p\rangle \otimes |q\rangle$

$= |p\rangle \otimes z|q\rangle$

② Let $|v_1\rangle, |v_2\rangle \in V$ and $|w\rangle \in W$ be arbitrary vectors,

$$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$$

③ Let $|v\rangle \in V$ and $|w_1\rangle$ and $|w_2\rangle \in W$ be arbitrary vectors,

$$|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$$

④ What sort of linear operators act on the space $|v\rangle \otimes |w\rangle$?

✓ Answered

Operators on the Product Space

Let A, B be linear operators on V & W respectively

Linear Operator $A \otimes B \longrightarrow$ operates on $V \otimes W$

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle$$

Eg: $|v\rangle = \alpha|0\rangle + \beta|1\rangle$

$|w\rangle = \gamma|0\rangle + \delta|1\rangle$

$A = X$
$B = I$ $\gg (X \otimes I)(|v\rangle \otimes |w\rangle)$

$= X|v\rangle \otimes I|w\rangle$

* Thus $A \otimes B$ is operator that can be extended to all elements of $V \otimes W$

* This defn of $A \otimes B$ is also linear, i.e.

$$A \otimes B \left(\sum_i a_i |v_i\rangle \otimes |w_i\rangle \right) = \sum_i a_i A|v_i\rangle \otimes B|w_i\rangle$$

Eg: $A = X$, $B = I$ $\Rightarrow (X \otimes I)(a_1|v_1\rangle \otimes |w_1\rangle + a_2|v_2\rangle \otimes |w_2\rangle)$

$$= a_1 X|v_1\rangle \otimes I|w_1\rangle + a_2 X|v_2\rangle \otimes I|w_2\rangle$$

* It also can be shown that $A \otimes B$ defined as above is well-defined linear operator on $V \otimes W$

Operators mapping to different basis

Consider $\left.\begin{array}{l} A : V \rightarrow V' \\ B : W \rightarrow W' \end{array}\right\}$ one vector space to another (different)

An arbitrary linear operator $C : V \otimes W \rightarrow V' \otimes W'$

$\downarrow$

expressed as linear combination of tensor products of operators A, B

$$C = \sum_i c_i A_i \otimes B_i$$

Eg: $C(|v\rangle \otimes |w\rangle) = \sum_i c_i (A_i \otimes B_i)(|v\rangle \otimes |w\rangle)$

$$= \sum_i c_i A_i|v\rangle \otimes B_i|w\rangle$$

Inner Products on $V \otimes W$

Inner Product space $V \otimes W$ vector space.
(notation of inner products on individual vector spaces are used)

$$\underbrace{\left(\sum a_i |v\rangle \otimes |w_i\rangle , \sum b_j |v_j'\rangle \otimes |w_j'\rangle\right)}_{\text{Inner Product of two elements in } V\otimes W} = \underbrace{\sum_{i,j} a_i^* b_j \langle v_i | v_j' \rangle \langle w_i | w_j' \rangle}_{\text{satisfies properties of inner product}}$$

Q: Try to prove these in above cases of tensor products.

1. satisfies inner product properties
2. Adjoint operator
3. Unitary Operation
4. Normality
5. Hermitian Operator.

Matrix Representation of Tensor Product

- Known as "Kronecker Product"

Let A be $m \times n$ matrix
B be $p \times q$ matrix

$$A \otimes B = \left.\begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix}\right\} mp$$

(the columns are marked nq above the matrix)

$A_{ij}B \rightarrow$ matrix B multiplied by constant A_{ij}

Eg: $A = \begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$

$2 \times 2 = 4$ col

$$A \otimes B = \begin{bmatrix} 5\times3 & 5\times2 & 6\times3 & 6\times2 \\ 5\times4 & 5\times5 & 6\times4 & 6\times5 \\ 3\times3 & 3\times2 & 2\times3 & 2\times2 \\ 3\times4 & 3\times5 & 2\times4 & 2\times5 \end{bmatrix}$$

$2 \times 2 = 4$ rows

Eg ②: Tensor product of operators X, Y

$$X \otimes Y = \begin{bmatrix} 0 \cdot Y & 1 \cdot Y \\ 1 \cdot Y & 0 \cdot Y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

..-09-21

Recap

Tensor Products

- defn
- operator $A \otimes B$
- Inner Product

V, W are inner product space

$V \otimes W$:

$$\left(\sum_i a_i |v_i\rangle \otimes |w_i\rangle, \sum_j b_j |v_j'\rangle \otimes |w_j'\rangle\right)$$

$$= \sum_{i,j} a_i^* b_j \langle v_i | v_j' \rangle \langle w_i | w_j' \rangle$$

→ Kronecker product representation

$$\underset{(m \times n)}{A} \otimes \underset{(p \times q)}{B} = \underbrace{\begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}}_{nq} \Big\} mp$$

Q Is $A \otimes B$ commutative ? A: No, $X \otimes Y$ and $Y \otimes X$ aren't same where X, Y are pauli matrices

Useful Notation in Tensor Product

$|\psi\rangle^{\otimes k}$ → means $|\psi\rangle$ tensored with itself k times

Q. $|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$. Write $|\psi\rangle^{\otimes 2}$, $|\psi\rangle^{\otimes 3}$ in tensor products and as well as Kronecker product.

a) $|\psi\rangle^{\otimes 2}$

$$= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \rightarrow \text{Tensor Product}$$

<u>Kronecker product</u>

$$|\psi\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$|\psi\rangle^{\otimes 2} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \otimes \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \\ 1/\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \end{bmatrix}$$

$$|\psi\rangle^{\otimes 2} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

b) $|\psi\rangle^{\otimes 3}$

$$= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{2\sqrt{2}}\left(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\right)$$

$$|\psi\rangle^{\otimes 3} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \text{Kronecker Product}$$

Q: Tensor Product on Pauli Operators

(a) $X \otimes Z$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Q. Show conjugate ~~transpose~~ distributes over tensor product

i.e. $(A \otimes B)^* = A^* \otimes B^*$

Ans:

$$(A \otimes B)^* = \begin{bmatrix} A_{11}B & \cdots & A_{1m}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^* = \begin{bmatrix} A_{11}^* B^* & \cdots & A_{1n}^* B^* \\ \vdots & \ddots & \vdots \\ A_{m1}^* B^* & \cdots & A_{mn}^* B^* \end{bmatrix}$$

$$= A^* \otimes B^*$$

Q. Show transpose distributes over tensor product

i.e. $(A \otimes B)^T = A^T \otimes B^T$

Sol:

$$(A \otimes B)^T = \begin{bmatrix} A_{11}B & \cdots & A_{1m}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^T$$

$$= \begin{bmatrix} A_{11}B^T & \cdots & A_{m1}B^T \\ \vdots & \ddots & \vdots \\ A_{1n}B^T & \cdots & A_{mn}B^T \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B^T & \cdots & A_{1m}B^T \\ \vdots & \ddots & \vdots \\ A_{n1}B^T & \cdots & A_{nm}B^T \end{bmatrix}$$

$$= A^T \otimes B^T$$

Q ④ Show that adjoint operator distributes over tensor product.

i.e. $(A \otimes B)^{\dagger} = A^{\dagger} \otimes B^{\dagger}$

Sol:

$$(A \otimes B)^{\dagger} = \left((A \otimes B)^{T}\right)^{*}$$

$$= \left(A^{T} \otimes B^{T}\right)^{*}$$

$$= (A^{T})^{*} \otimes (B^{T})^{*}$$

$$= A^{\dagger} \otimes B^{\dagger}$$

⑤. Show tensor product of two unitary operators is unitary.

Sol:

<u>Prop of Unitary</u> — $UU^{\dagger} = I$

Let U_1 and U_2 be unitary operators

$$U_1 U_1^{\dagger} = I \qquad U_2 U_2^{\dagger} = I$$

Let $U = U_1 \otimes U_2$

$$UU^{\dagger} = (U_1 \otimes U_2)(U_1 \otimes U_2)^{\dagger} = (U_1 \otimes U_2)(U_1^{\dagger} \otimes U_2^{\dagger})$$

$$= U_1 U_1^{\dagger} \otimes U_2 U_2^{\dagger}$$

$$= I \otimes I$$

$$\cancel{= I}$$

$$U^{\dagger}U = (U_1 \otimes U_2)^{\dagger}(U_1 \otimes U_2) = I \otimes I$$

Operator Function

Function of Operator

$f :$ complex $\longrightarrow$ complex

→ Use it to define

Let $A = \sum_a a |a\rangle\langle a|$ → Spectral decomposition

↓

normal

Define - $f(A) = \sum_a f(a) |a\rangle\langle a|$

↓

function f of operator A

28-9-21

Recap

- → Tensor Products
  - Operators
- → Matrices - Kronecker Product
  - various proofs

Operator Functions

f : complex $\longrightarrow$ Complex

Let $A = \sum_a a |a\rangle\langle a|$ (A ↓ normal)

a is eigen value & $|a\rangle$ is corresponding eigen vector

then $f(A) = \sum_a f(a) |a\rangle\langle a|$

$f(A)$ ↘ function on operator A

Eg: sq.root of operator ($\sqrt{X}, \sqrt{Y}, \ldots$ etc)

Eg② $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Eigen values = 1 & -1

Eigenvectors = $|0\rangle$ $|1\rangle$

$Z = 1|0\rangle\langle 0| + (-1)|1\rangle\langle 1|$

$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

Let $f(x) = e^{\theta x}$ then $f(Z) = e^{\theta Z}$

$f(Z) = e^{\theta \cdot 1}|0\rangle\langle 0| + e^{-\theta \cdot 1}|1\rangle\langle 1|$

$= \begin{bmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{bmatrix}$

Q. Find the square root of $\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$

A. $\det(A - \lambda I) = 0$

$(4-\lambda)^2 = 9$

$4 - \lambda = \pm 3$

$\boxed{\lambda = 1, +7}$

$A|\lambda\rangle = \lambda|\lambda\rangle$

$\lambda = 1$

$\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 1\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$4c_1 + 3c_2 = c_1 \Rightarrow c_1 + c_2 = 0$ so $c_1 = \frac{1}{\sqrt{2}}$

$3c_1 + 4c_2 = c_2 \Rightarrow c_1 + c_2 = 0$ $\quad c_2 = \frac{-1}{\sqrt{2}}$

$|\lambda = 1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 7$

$\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 7\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\left.\begin{matrix} 4c_1 + 3c_2 = 7c_1 \\ 3c_1 + 4c_2 = 7c_2 \end{matrix}\right\} \Rightarrow c_1 = c_2$ so $c_1 = c_2 = \frac{1}{\sqrt{2}}$

$|\lambda = 7\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A = 1 \cdot |-\rangle\langle -| + 7\,|+\rangle\langle +|$

$\sqrt{A} = 1 \cdot |-\rangle\langle -| + \sqrt{7}\,|+\rangle\langle +|$

$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\sqrt{7}}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\sqrt{7} & -1+\sqrt{7} \\ -1+\sqrt{7} & 1+\sqrt{7} \end{bmatrix}$

(Q) Find logarithm of matrix $\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$

Eigen values = 1, 7

$$1 \rightarrow |-\rangle, \quad 7 \rightarrow |+\rangle$$

$$\log(A) = \log(1)\,|-\rangle\langle -| + \log(7)\,|+\rangle\langle +|$$

$$= \frac{\log(7)}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

<u>Trace of Matrix</u>

$$tr(A) = \sum_i A_{ii}$$

<u>Properties</u> :

1) Trace is cyclic

$$tr(AB) = tr(BA) \quad (\text{To Prove})$$

<u>proof</u> $tr(AB) = \sum_i \langle i|AB|i\rangle$

$$= \sum_i \langle i|AIB|i\rangle$$

$$= \sum_i \langle i|A\left(\sum_j |j\rangle\langle j|\right)B|i\rangle \qquad \left[\because I = \sum_j |j\rangle\langle j|\right]$$

$$= \sum_{i,j} \langle i|A|j\rangle\langle j|B|i\rangle$$

scalars (are commutative)

$$= \sum_{i,j} \langle j|B|i\rangle\langle i|A|j\rangle$$

$$= \sum_j \langle j|B\left(\sum_i |i\rangle\langle i|\right)A|j\rangle$$

$$= \sum_j \langle j|BIA|j\rangle = \sum_j \langle j|BA|j\rangle = tr(BA)$$

2) Trace is linear

$tr(A+B) = tr(A) + tr(B)$

Proof: $tr(A+B) = \sum_i \langle i|(A+B)|i\rangle$

$= \sum_i \langle i|A|i\rangle + \langle i|B|i\rangle$ $\quad \boxed{\because (A+B)|i\rangle = A|i\rangle + B|i\rangle}$

$= tr(A) + tr(B)$

3) $tr(zA) = z\, tr(A)$

Proof: $tr(zA) = \sum_i \langle i|zA|i\rangle$

$= \cancel{z} \sum_i z\langle i|A|i\rangle$

$= z \sum_i \langle i|A|i\rangle$

$= z\, tr(A)$

Scanned with CamScanner

27-09-21

Recap

→ operator functions

→ Trace of Matrix

$$Tr(A) = \sum_i A_{ii}$$

→ Cyclic → $tr(AB) = tr(BA)$

→ Linear → $tr(A+B) = tr(A) + tr(B)$

→ $tr(zA) = z\,tr(A)$

⊛ Trace is invariant under unitary similarity transformation

$$A \xrightarrow{\sim} UAU^{\dagger}$$

Transformation

adjoint

Unitary Matrix

$$tr(UAU^{\dagger}) = tr(U^{\dagger}UA) \qquad [\because \text{ Cyclic Property}]$$

$$= tr(A) \qquad [\because U^{\dagger}U = I]$$

Thus, trace is invariant under this transformation

⊛ If an operator A is transformed to ~~another~~ $UAU^{\dagger}$, to represent it in different basis, then trace would still remain the same.

⊛ As in, the trace is property of operator but doesn't depend on its representation. It is well-defined

Trace of Operator

Trace of any matrix representation of A

* Suppose $|\psi\rangle$ is a unit vector, A is an arbitrary operator

Evaluate $tr(A|\psi\rangle\langle\psi|)$

→ Start with $|\psi\rangle$ & use the Gram-Schmidt procedure to construct orthonormal basis $|i\rangle$

→ Let its first element be $|\psi\rangle$

$$tr(A|\psi\rangle\langle\psi|) = \sum_i \langle i|A|\psi\rangle\langle\psi|i\rangle$$

$$= \sum_i \langle\psi|i\rangle\langle i|A|\psi\rangle$$

$$= \langle\psi|\left(\sum_i |i\rangle\langle i|\right)A|\psi\rangle$$

$$= \langle\psi|IA|\psi\rangle$$

$$= \langle\psi|A|\psi\rangle$$

$$\boxed{tr(A|\psi\rangle\langle\psi|) = \langle\psi|A|\psi\rangle}$$

Q. Show that all Pauli Matrices except I have trace 0.

$$Tr(X) = Tr\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$$

$$Tr(Y) = Tr\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 0$$

$$Tr(Z) = Tr\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0$$

$$Tr(I) = Tr\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

Hilbert - Schmidt Inner Product on Operators

Consider the set L_V of linear operators on vector space V

This set L_V follows certain properties -

1. Sum of 2 linear operators is linear
2. If z is a complex number, A is ~~an~~ linear operator, then $zA \in L_V$
3. There is a zero element O in L_V $\qquad OA = AO = O$

$$A = O \quad \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & \cdots & & & 0 \end{bmatrix} \in L_V$$

$AO|v\rangle =$ zero vector

$OA|v\rangle =$ zero vector

L_V — $A+B \in L_V$, $zA \in L_V$, $O \in L_V$ $\Big\}$ $\rightarrow$ L_V is a vector space

Q. What would make L_V a hilbert space?

- Needs an "inner product" definition

$\downarrow$

Hilbert - Schmidt Inner Product

(or)

Trace Inner Product

Definition -

The function $(\cdot,\cdot)$ on $L_V \times L_V$, $F: L_V \times L_V \to \mathbb{C}$

$$(A,B) = tr(A^{\dagger}B) \qquad \text{where } A,B \in L_V$$

It is an inner product.

Proof:

To prove F above i.e. $(A,B) = tr(A^{\dagger}B)$ is inner product, it should satisfy -

i) Linear in the second argument

$$A \in L_V, \quad B = \sum_i z_i B_i$$

$$(A,B) = (A, \sum_i z_i B_i) = \sum_i z_i (A, B_i)$$

Proof -

$$tr(A^{\dagger} \sum_i z_i B_i) = \sum_i z_i \, tr(A^{\dagger} B_i)$$

$$tr(A^{\dagger} \sum_i z_i B_i) = \sum_i tr(A^{\dagger} z_i B_i) \qquad [\because \text{trace is linear}]$$

$$= \sum_i z_i \, tr(A^{\dagger} B_i) \qquad [\because tr(zA) = z\, tr(A)]$$

ii) $(A,B)^* = (B,A)$

Proof -

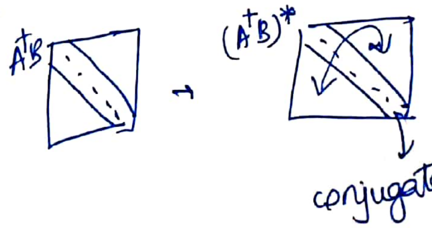
$$(A,B)^* = (tr(A^{\dagger}B))^*$$

$$= (tr(A^{\dagger}B)^{\dagger})$$

$$= tr(B^{\dagger}A)$$

$$= (B,A)$$

$[\because tr(A^{\dagger}B)^{\dagger} = (tr(A^{\dagger}B))^*]$



iii) $(A, A) = tr(A^{+}A)$

$$= \sum_{i} |A_{ii}|^2$$

$$\geq 0$$

Thus L_V is a Hilbert space

Commutators

The commutator between two operators A, B is defined as -

$$[A, B] = AB - BA$$

⊛ If $AB - BA = 0$, then $AB = BA$, then we say that A COMMUTES WITH B

Anti-Commutators

The anti-commutator of two operators A, B is defined as -

$$\{A, B\} = AB + BA$$

⊛ If $\{A, B\} = 0$, then $AB = -BA$ and we say that A ANTI-COMMUTES WITH B

→ Simultaneous Diagonalization - is a .

Simultaneous Diagonalization Theorem

Suppose A, B are hermitian operators (i.e. $A = A^\dagger$, $B = B^\dagger$), then $[A, B] = 0$ if and only if there exists an orthonormal basis such that both A and B are simultaneously diagonalizable in that basis

← Check [illegible] if simul. diagonalizable

56 Check if X, Y are simultaneously diagonalizable?

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$[X, Y] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$= \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \neq 0$$

So, NO they are not simultaneously diagonalizable

1-10-21

Recap

→ Commutator - $[A, B] \equiv AB - BA$

→ Anti-commutator - $\{A, B\} \equiv AB + BA$

→ Simultaneous Diagonalization theorem

Simultaneous Diagonalization Theorem

Let A, B be Hermitian operators then $[A, B] = 0$ if and only if A, B are diagonal with respect to a same orthonormal basis (i.e. simultaneously diagonalizable)

Part-I :

If A, B are simultaneously diagonalizable, then $[A, B] = 0$ (Trivial)

Part-II :

If $[A,B] = 0$ then A, B are simultaneously diagonalizable.

Proof :

Given A, B are hermitian

Let V_a be the eigenspace for A with Eigen Value a.

$|a,j\rangle$ be orthonormal basis for eigenspace V_a

↓

indexing the degeneracies in case of nondegenerate vectors (just a notation not any operation)

$AB|a,j\rangle = BA|a,j\rangle$ [$\because [A,B]=0$ i.e. $AB = BA$]

↓

AB operating on orthonormal basis vector

$= Ba|a,j\rangle$ [$\because |a,j\rangle$ is eigen vector in eigenspace for a]

$= aB|a,j\rangle$

$\Rightarrow B|a,j\rangle \in V_a$ because $AB|a,j\rangle = aB|a,j\rangle$

Let P_a be a projector onto the space V_a.

Define new operator B_a as

$$B_a = P_a B P_a$$

→ B_a is hermitian on V_a

$$B_a = P_a B P_a$$

$$B_a = (P_a B P_a)^{+}$$

$$= P_a^{+} B^{+} P_a^{+}$$

$$= P_a B P_a$$ [$\because P_a$ - Projector so hermitian and B is given hermitian]

Thus it has a spectral decomposition in terms of orthonormal set of eigen vectors that span the space V_a

Lets call these eigen vectors which form the orthonormal basis for spectral decomposition of B_a

$|a,b,k\rangle \rightarrow$ a - Eigen Value 'a' of A

b - Eigen Value of B_a

k - Denotes degeneracy
(i.e. extra index to allow for possibility of degenerate B_a)

We can see that,

$B|a,b,k\rangle$ belongs to vector space V_a

[$\because$ we already proved $B|a,j\rangle \in V_a$ thus $B|a,b,k\rangle \in V_a$ too]

$$B|a,b,k\rangle \in V_a$$

$$B|a,b,k\rangle = P_a B|a,b,k\rangle$$

$$P_a|a,b,k\rangle = |a,b,k\rangle \qquad [\because |a,b,k\rangle \in V_a]$$

$$B|a,b,k\rangle = P_a B\underbrace{|a,b,k\rangle}$$

$$= P_a B \underbrace{P_a|a,b,k\rangle}$$

$$= b|a,b,k\rangle$$

($\because$ It belongs to eigenspace of B with eigen value b)

$|a,b,k\rangle \rightarrow$ eigenvector of B with eigen value b

Thus $|a,b,k\rangle$ is orthonormal set of eigen vectors of both A, B, spanning entire vector space on A, B

So A, B are simultaneously diagonalizable

(8) If A, B are hermitian? Is $i[A,B]$ hermitian? Prove.

Sol: $A = A^{\dagger}$, $B = B^{\dagger}$ (Given)

$$\left(i[A,B]\right)^{\dagger} = (-i)\left([A,B]\right)^{\dagger}$$

$$= (-i)\left(AB - BA\right)^{\dagger}$$

$$= -i\left((AB)^{\dagger} - (BA)^{\dagger}\right)$$

$$= -i\left(B^{\dagger}A^{\dagger} - A^{\dagger}B^{\dagger}\right)$$

$$= i\left(A^{\dagger}B^{\dagger} - B^{\dagger}A^{\dagger}\right)$$

$$= i\left(AB - BA\right)$$

$$= i[A,B]$$

Hence $i[A,B]$ is hermitian

4-10-21

Polar & Single Value Decompositions

Linear operator — generic - properties are not easily visualizable

→ Break them up into simpler products

→ Into products of unitary & positive operators
$\downarrow$
$(|v\rangle, A|v\rangle) \geq 0$

→ Polar and singular value decompositions allow us to represent operators in that way.

Polar Decomposition Theorem

Let A be linear operator on vector space V, then there exists Unitary Operator U and positive operators J, K such that

$$A = UJ = KU$$

where unique positive operators J, K satisfy

$$J = \sqrt{A^\dagger A} \qquad K = \sqrt{AA^\dagger}$$

Moreover, if A is invertible then U is unique

⊛ $A = UJ$ → Left Polar Decomposition of A

⊛ $A = KU$ → Right Polar Decomposition of A

Proof :

Given A, linear operator

Let J be $\sqrt{A^\dagger A}$

($\because$ why does sqrt of $A^\dagger A$ be defined ?)

(i.e function of operator)

→ what is $\sqrt{A^\dagger A}$

→ Does it guarantee sqrt is defined ? How ?

Properties of $A^\dagger A$ -

- Hermitian $\left((A^\dagger A)^\dagger = A^\dagger A \right)$

- Normal

↳ Spectral Decomposition in terms of eigen vectors

→ Eigen values - real & positive

coz of hermitian operator

coz $A^\dagger A$ is positive

So square root is defined

Thus J is a valid operator

Show that J is positive

$J = \sqrt{A^\dagger A}$ → $(|v\rangle, J|v\rangle) \geq 0$

As $A^\dagger A$ has spectral decomposition,

$$J = \sum_a \sqrt{a}\, |a\rangle\langle a|$$ → Eigen vectors of $A^\dagger A$

$$|v\rangle = \sum_{a'} v_{a'} |a'\rangle$$

$$J|v\rangle = \sum_a \sqrt{a}\, |a\rangle\langle a| \left(\sum_{a'} v_{a'} |a'\rangle \right)$$

$$= \sum_a \sqrt{a}\, v_a |a\rangle$$

(In inner sum & outer sum only for $a = a'$ we get $\langle a|a'\rangle = 1$ or else it will be 0)

Scanned with CamScanner

$(|v\rangle)^\dagger = \sum_a v_a^* \langle a|$

$(|v\rangle, J|v\rangle) = \sum_a \sqrt{a}\, \boxed{v_a^* v_a}$ → real and positive ($\because v_a^* v_a = \|v_a\|^2 \geq 0$)

real & positive (Given $A^\dagger A$ has real & positive eigen ~~vecto~~ values)

so $(|v\rangle, J|v\rangle) \geq 0$

Thus J is a positive operator

→ As J is positive, it has spectral decomposition

$$J = \sum_i \lambda_i |i\rangle\langle i| \qquad (\lambda_i \geq 0)$$

Defining some new vectors → eigen vectors of J

$$|\psi_i\rangle = A|i\rangle$$

We claim that $\langle\psi_i|\psi_i\rangle = \lambda_i^2$ →

Proof :

$\langle\psi_i| = (A|i\rangle)^\dagger$

$= \langle i|A^\dagger$

$\langle i|A^\dagger A|i\rangle \Rightarrow \langle\psi_i|\psi_i\rangle$

$= \langle i|J^2|i\rangle$

$= \langle i|J\, J|i\rangle$

$= \langle i|J\, \lambda_i|i\rangle$

$= \lambda_i \langle i|J|i\rangle$

$= \lambda_i^2 \langle i|i\rangle$

$= \lambda_i^2$