07-09-21

Recap

$|w\rangle\langle v|\,(|v'\rangle) = |w\rangle\langle v|v'\rangle$

$= \langle v|v'\rangle\,|w\rangle$

This is called outer product

Linearity of Outer Product $\rightarrow$ $\sum_i a_i |w_i\rangle\langle v_i|$.

$$\left(\sum_i a_i |w_i\rangle\langle v_i|\right)|v'\rangle = \sum_i a_i |w_i\rangle\langle v_i|v'\rangle$$

This concept can be used to another result i.e. Completeness relation for orthonormal vectors

Let $|i\rangle$ be an orthonormal basis on vector space V

$|v\rangle \in V$ can be written as $|v\rangle = \sum_i \underbrace{v_i}_{\text{complex no.}} |i\rangle$

so $\langle i|v\rangle = v_i$

$$\left(\sum_i |i\rangle\langle i|\right)(|v\rangle) = \sum_i |i\rangle\langle i|v\rangle$$

$$= \sum_i |i\rangle v_i$$

$$= \sum_i v_i |i\rangle$$

$$= |v\rangle$$

$$= \underset{\text{Identity}}{I}|v\rangle$$

Thus,

$$\sum_i |i\rangle\langle i| = I$$

Completeness Relation

So, ~~for~~ for orthonormal basis vectors $|i\rangle$ on vector space V

$$\sum_i |i\rangle\langle i| = I$$

* Outer Product is used to give vector representation to operators

$\downarrow$

outer product representation

$\downarrow$

represented as matrices

Suppose $A: V \to W$ is a linear operator, $|v_i\rangle$ orthonormal basis for V and $|w_j\rangle$ for W

$$\begin{array}{c} \\ j \text{ rows} \end{array} \left\{ \overset{i \text{ col} \quad A}{\begin{bmatrix} & & \\ & & \end{bmatrix}} \right. \overset{v}{\begin{bmatrix} \vdots \\ \vdots \end{bmatrix}} \Big\} \overset{\vdots}{\text{i row}} = \overset{w}{\begin{bmatrix} \vdots \\ \vdots \end{bmatrix}} \Big\} \overset{\vdots}{\text{j row}}$$

$$\underset{\text{op}}{\overset{\downarrow}{A}} = \underbrace{I_W \, A \, I_V}_{\text{Identity}}$$

$$A = \left(\sum_j |w_j\rangle\langle w_j|\right) A \sum_i |v_i\rangle\langle v_i|$$

$$= \sum_{i,j} |w_j\rangle\langle w_j| A |v_i\rangle\langle v_i|$$

over all pairs of i,j

$$= \sum_{i,j} \underbrace{\langle w_j|A|v_i\rangle}_{\downarrow} \langle w_j|\, v_i| = \sum_{i,j} a_{ji} \langle w_j|v_i\rangle$$

corresponds to A_{ji} (i^{th} column j^{th} row)

Reason :

$A|v_i\rangle$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ | & & & \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ v_i \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}$$

$\langle w_j|A|v_i\rangle$

$$= [0\ 0 \cdots w_j \cdots 0] \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix} = a_{ji}$$

Cauchy - Schwarz Inequality

For two vectors $|v\rangle$, $|w\rangle$

$$|\langle v|w\rangle|^2 \leq \langle v|v\rangle \langle w|w\rangle$$

Proof:

We use Gram-schmidt procedure to construct an orthonormal basis vector for vector space. Let it be $|i\rangle$

$|i\rangle \rightarrow$ orthonormal vector basis

Construct first vector in $|i\rangle$ as $\dfrac{|w\rangle}{\sqrt{\langle w|w\rangle}} \rightarrow$ call it $|0\rangle$

Using completeness relation, $\sum_i |i\rangle\langle i| = I$

$$\langle v|v\rangle \langle w|w\rangle$$

$$= \sum_i \langle v|i\rangle\langle i|v\rangle \langle w|w\rangle$$

$$\geq \langle v|0\rangle \langle 0|v\rangle \langle w|w\rangle$$

$$\geq \frac{\langle v|w\rangle\langle w|v\rangle}{\sqrt{\langle w|w\rangle}\sqrt{\langle w|w\rangle}} \langle w|w\rangle$$

$$\geq \langle v|w\rangle \langle w|v\rangle$$

$$\geq |\langle v|w\rangle|^2$$

Thus $\langle v|v\rangle \langle w|w\rangle \geq |\langle v|w\rangle|^2$

When $|v\rangle = k|w\rangle$, the equality occurs in above eqn

8-9-21

Eigen Vectors / Eigen Values

For operator A, its eigen vector (on vector space V) is non-zero vector $|v\rangle$ such that

property of op. A

$$A|v\rangle = \vartheta |v\rangle$$

ϑ → complex scalar

(*) ϑ is called eigen-value of A corresponding to eigen vector $|v\rangle$

→ How do we find eigen vectors ? (Assume you have matrix rep. of operator)

A: Using characteristic equation

Characteristic function : $c(\lambda)$

$$c(\lambda) = \det|A - \lambda I|$$

$c(\lambda) = 0$ → used to find eigen-values

solutions of characteristic eqn $c(\lambda) = 0$ are eigen-values of operator A.

Eigen space : Corresponding to eigen value v, there could be set of vectors which have Eigen vector v. It is a subspace of the vector space V

(*) Why is eigen space a subspace ?

For V' to be subspace

V' is not empty $\rightarrow$ eigenspace is not empty as it contains atleast one eigen vector

If $u, v \in V'$ then $\alpha u + \beta v \in V'$
where $\alpha, \beta \in \mathbb{C}$

$$A(\alpha|u\rangle + \beta|v\rangle) = A(\alpha|u\rangle) + A(\beta|v\rangle)$$
$$= \alpha A|u\rangle + \beta|v\rangle$$
$$= \lambda(\alpha|u\rangle + \beta|v\rangle) \in V'$$

Thus eigenspace is a subspace

Diagonal Representation (for an operator)

For an operator A on vector space V,

$$A = \sum_i \lambda_i |i\rangle\langle i| \quad \rightarrow \text{Diagonal Representation}$$

where vectors $|i\rangle$ form orthonormal ~~basis~~ set of eigen vectors for A with corresponding eigenvalues λ_i

$$A|q\rangle = \sum_i \lambda_i |i\rangle\langle i| (|q\rangle)$$
$$= \sum_i \lambda_i |i\rangle\langle i|q\rangle$$
$$= \sum_i \lambda_i \langle i|q\rangle |i\rangle$$

When an eigenspace has more than one dimension, we call it "degenerate"

For same eigen vector

Ex:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$c(\lambda) = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$(2-\lambda)(2-\lambda)(-\lambda) = 0$$

$$\Rightarrow (2-\lambda)^2(-\lambda) = 0$$

$$\Rightarrow \lambda = 0, \quad \lambda = 2$$

Eigen vectors

$$\lambda = 2$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0-2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\gamma = 0$$

Q: Prove that $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is not diagonalisable

$$\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 = 0 \Rightarrow \lambda = 1 \Rightarrow \boxed{|\lambda\rangle = |1\rangle}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_1 = 0, \quad c_2 = 1$$

$$\boxed{|\lambda\rangle = |1\rangle}$$

Only 1 eigen vector

$$|1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

There exists no such c

10-9-21

Adjoints & Hermitian Operators

→ Dirac Representation

Adjoint → conjugate of transpose of matrix
↓
all elements in matrix converted to their complex conjugates

Hermitian Conjugate

* Suppose A is an linear operator on Hilbert space V, then there exists a unique linear operator $A^{\dagger}$ on V such that for all vector $|v\rangle, |w\rangle \in V$

$$(|v\rangle, A|w\rangle) = (A^{\dagger}|v\rangle, |w\rangle)$$

$A^{\dagger}$ is called Adjoint / Hermitian conjugate of A

① $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$

Proof :

$$\begin{aligned}(|v\rangle, (AB)^{\dagger}|w\rangle) &= (AB|v\rangle, |w\rangle) \\ &= (B|v\rangle, A^{\dagger}|w\rangle) \\ &= (|v\rangle, B^{\dagger}A^{\dagger}|w\rangle)) \\ &= (|v\rangle, B^{\dagger}A^{\dagger}|w\rangle)\end{aligned}$$

By usual convention,

$$\boxed{|v\rangle^\dagger = \langle v|}$$

$$* \boxed{(A|v\rangle)^\dagger = \langle v|A^\dagger}$$

Proof → [Correct Proof in Next class]

we know $(|w\rangle, A|v\rangle) = \langle w|A|v\rangle$

$$(|w\rangle, (A|v\rangle)^\dagger) = \langle w|(A|v\rangle)^\dagger$$

$$= |(A|v\rangle)^\dagger\rangle \langle w|$$

$(A|v\rangle)^\dagger \to$ dual vector of $(A|v\rangle)$

$$= \langle v|A^\dagger$$

Ex: Show that for any two vectors $|v\rangle, |w\rangle$,

$$(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$$

Sol: for arbitrary vectors $|x\rangle, |y\rangle$

$$(|x\rangle, (|w\rangle\langle v|)^\dagger |y\rangle) = ((|w\rangle\langle v|)|x\rangle, |y\rangle)$$ [defn of hermitian conjugate]

$$= (\langle v|x\rangle |w\rangle, |y\rangle)$$

$$= (\langle v|x\rangle)^\dagger \langle w|y\rangle$$

$$= \langle x|v\rangle\langle w|y\rangle$$

$$= (\langle x|)(|v\rangle\langle w|)|y\rangle$$

$$= (|x\rangle, \underline{|v\rangle\langle w|} y\rangle)$$

so $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$

Anti-linearity of Adjoint

$$\left(\sum_i a_i A_i\right)^\dagger = \sum_i a_i^* (A_i)^\dagger$$

Proof:

$$\left(\left(\sum_i a_i A_i\right)^\dagger |v\rangle, |w\rangle\right) = \left(|v\rangle, \left(\sum_i a_i A_i\right) |w\rangle\right)$$

$$= \sum_i a_i \left(|v\rangle, A_i |w\rangle\right) \quad \text{(property of Inner Product)}$$

$$= \sum_i a_i \left(A_i^\dagger |v\rangle, |w\rangle\right)$$

$$= \sum_i \left(a_i^* A_i^\dagger |v\rangle, |w\rangle\right)$$

$$= \left(\left(\sum_i a_i^* A_i^\dagger\right) |v\rangle, |w\rangle\right)$$

Thus $\boxed{\left(\sum_i a_i A_i\right)^\dagger = \sum_i a_i^* A_i^\dagger}$

Q: Prove $(A^\dagger)^\dagger = A$

A: So we need to prove $\left((A^\dagger)^\dagger |v\rangle, |w\rangle\right) = \left(A|v\rangle, |w\rangle\right)$

$$\left((A^\dagger)^\dagger |v\rangle, |w\rangle\right) = \left(|v\rangle, A^\dagger |w\rangle\right)$$

$$= \left(A^\dagger |w\rangle, |v\rangle\right)^*$$

$$= \left(|w\rangle, A|v\rangle\right)^*$$

$$= \left(A|v\rangle, |w\rangle\right)$$

Implication of Hermitian conjugate in matrix notation

$$A^\dagger = (A^*)^T$$

Eg :

$$A = \begin{bmatrix} i & -i \\ 2+i & 3-i \end{bmatrix}$$

$$A^\dagger = \begin{bmatrix} -i & -(i+2) \\ i & i-3 \end{bmatrix}$$

Hermitian Operator (also called as self Adjoint Operator)

↓

definition : $A^\dagger = A$

where A is an operator

Q. Are Pauli Matrices self adjoint ?

σ_1 · I $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

σ_2 X $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $X^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

σ_3 Y $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $Y^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

σ_4 Z $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $Z^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

∴ All pauli Matrices are self-adjoint and hence hermitian operators

→ There's an important class of ~~operators~~ Hermitian operators called Projectors

Projectors

Suppose W is a k-dim vector subspace within d-dimensional vector space V. We can construct using Gram Schmidt procedure, an orthonormal basis set $|1\rangle \dots |d\rangle$ for V

$$\sum^{d} |i\rangle \rightarrow \text{orthonormal basis for } V\ (|1\rangle, |2\rangle, \dots |d\rangle)$$

and

$$\sum^{k} |i\rangle \rightarrow \text{Orthonormal basis for } W\ (|1\rangle, \dots |k\rangle)$$

then

$$P = \sum_{i=1}^{k} |i\rangle\langle i|$$

P is a projector onto vector subspace W

(*) consider $|v\rangle \in V$ then $|v\rangle = \sum_{i=1}^{d} \alpha_i |i\rangle$

$$P(|v\rangle) = \sum_{i=1}^{k} |i\rangle\langle i| \left(\sum_{j=1}^{d} \alpha_j |j\rangle \right)$$

$$= \sum_{i=1}^{k} |i\rangle \alpha_i$$

(i.e. when $i \neq j$ ~~thus~~ $\langle i|j\rangle = 0$ so those terms are eliminated)

↓

⇒ we get a vector that belongs to subspace W

So many vectors $|v\rangle$ in V can get mapped to some $|w\rangle$ in W (~~one~~ like many-one relation)

13-9-21

Recap

Hermitian conjugate of operator

$(A|v\rangle)^\dagger = \langle v|A^\dagger$ $\qquad\qquad (|v\rangle^\dagger = \langle v|)$

~~Proof~~ Clarification in Proof

Proof:

$(A|v\rangle)^\dagger$ is dual vector of $A|v\rangle$

$(A|v\rangle, |v\rangle) = \langle (A|v\rangle)^\dagger |v\rangle$ — ①

↓ Also

$(A|v\rangle, |v\rangle) = (|v\rangle, A^\dagger|v\rangle)$

$= \langle v|A^\dagger|v\rangle$ — ②

As ① & ② are same

$\langle (A|v\rangle)^\dagger |v\rangle = \langle v|A^\dagger|v\rangle$

$$\boxed{(A|v\rangle)^\dagger = \langle v|A^\dagger}$$

Hermitian/ Self-Adjoint Operator $(A^\dagger = A)$

→ Special Class – Projector

↓

Vector subspace W of vectorspace V

W ↓ K dim $\qquad$ V ↓ d dim

orthonormed basis $|i\rangle$: $\{1, \dots K\}$ $\qquad$ $\{1, \dots K, K+1, \dots d\}$

$$P = \sum_{i=1}^{K} |i\rangle\langle i|$$ is a Projector for V

usually also called subspace projected by P

↓

It projects onto subspace W

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Orthogonal Component of a Projector

$Q = I - P$ is called orthogonal complement of P

$$I = \sum_{i=1}^{d} |i\rangle\langle i|$$

$$Q = \sum_{i=1}^{d} |i\rangle\langle i| - \sum_{j=1}^{k} |j\rangle\langle j|$$

$$= \sum_{i=k+1}^{d} |i\rangle\langle i| .$$

↓ projects onto

This vector space also Called as Orthogonal Complement of vector space of P ← vector space spanned by $\{k+1, \dots, d\}$

Q. Prove $P^2 = P$

Sol:

$$P = \sum_i |i\rangle\langle i|$$

$$P^2 = P \cdot P = \left(\sum_i |i\rangle\langle i|\right)\left(\sum_j |j\rangle\langle j|\right)$$

$$= \sum_{i,j} |i\rangle \underbrace{\langle i|j\rangle}_{\delta_{ij}} \langle j|$$

$$= \sum_{i,j} |i\rangle\langle j| \delta_{ij} \qquad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$= \sum_i |i\rangle\langle i|$$

$$= P$$

Thus $\boxed{P^2 = P}$

Let $P(v_1|1\rangle + v_2|2\rangle \dots + v_d|d\rangle)$

where $P = \sum_{i=1}^{k} |i\rangle\langle i|$

we get only those vectors $\{1, 2 \dots k$

i.e. $v_1|1\rangle + \dots v_k|k\rangle$

Now $P(v_1|1\rangle + \dots v_k|k\rangle)$

will give same vector back

$= v_1|1\rangle + \dots v_k|k\rangle$

Normal Operator

An operator A is said to be normal if $AA^\dagger = A^\dagger A$

Ex: Prove that an operator that is hermitian is also normal

Sol: As A is hermitian, $A^\dagger = A$

so $AA^\dagger = A^\dagger A = A^2$, Hence its normal

Ex: Is ~~conve~~ reverse true? Does A is normal means A is hermitian?

Ans: False, it need not be.

Consider $A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ $\quad A \neq A^\dagger$

$A^\dagger = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$

$$AA^\dagger = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A^\dagger A = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$AA^\dagger = A^\dagger A$ but $A \neq A^\dagger$

Spectral Decomposition

Theorem

Any normal operator M on vector space V is diagonal with respect to some orthonormal basis for V.

Conversely every diagonalizable operator is normal

$$M = \sum_i \lambda_i |i\rangle\langle i| \rightarrow$$

λ_i's are eigen values

$|i\rangle$ form orthonormal bais formed with eigen vectors corresponding to eigen values

Part-II - If eigen values are real, then the normal matrix is also hermitian

Sol. Using Spectral Theorem,

$$A = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

↓ eigen values (given λ_i's are real)

→ Eigen vectors that form orthonormal basis

$$A^\dagger = \sum_i \lambda_i^* (|\lambda_i\rangle\langle\lambda_i|)^\dagger$$

$$= \sum_i \lambda_i^* (|\lambda_i\rangle\langle\lambda_i|) \qquad [\because (|w\rangle\langle v|)^\dagger = |v\rangle\langle w|]$$

$$= \sum_i \lambda_i (|\lambda_i\rangle\langle\lambda_i|)$$

$$= A$$

So $\boxed{A^\dagger = A}$ ⇒ A is Hermitian

Unitary Matrices

$$U^\dagger U = UU^\dagger = I$$

Operator is unitary ⇒ Each of its matrix representations is also unitary

depends on chosen basis

As $UU^\dagger = U^\dagger U$, U is also normal and it has spectral decomposition

Unitary → They preserve inner product of two vectors i.e. Lets consider $|v\rangle, |w\rangle$ be two vectors, U is unitary operator

$$(U|v\rangle, U|w\rangle) = \langle (U|v\rangle)^{\dagger} | U|w\rangle$$

$$= \langle v|U^{\dagger} U|w\rangle \qquad [\because (A|v\rangle)^{\dagger} = \langle v|A$$

$$= \langle v|(U^{\dagger}U)|w\rangle$$

$$= \langle v|I|w\rangle \qquad [\because UU^{\dagger} = I]$$

$$= \langle v|w\rangle$$

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