UNIT IV : Dynamic Programming

Syllabus:

- Dynamic programming methodology
- Applications
- 0/1 knapsack problem
- Optimal binary search trees
- All pairs shortest path problem
- Travelling sales person problem

❖ Dynamic Programming:

Dynamic programming is an algorithm design method that can be used when the solution to problem can be viewed as the result of sequence of decisions.

Dynamic programming is applicable when the sub-problems are dependent.

For generating decision sequence we use the principle of optimality i.e., output of stage-1 will be given as input stage-2, output of stage-2 will be given as input for stage-3 and so on.

Initial conditions are given as input for stage-1.

In greedy method only one decision sequence is generated but in dynamic programming many decision sequences may be generated.

❖ Applications:

1) All pairs shortest path
2) 0/1 knapsack problem
3) Optimal binary search tree
4) Travelling sales person problem
5) Matrix chain multiplication

❖ 0/1 KNAPSACK PROBLEM

Solution to the knapsack problem can be obtained by making a sequence of decisions on variables $x_1,x_2,x_3.......x_n$ i.e., the decision determine which of the values 0/1 is assigned to the variables.

It may be in one of the two possible states, if an object is directly placed into a knapsack then x=1 otherwise x=0.

PROCEDURE:-

Step1:- Let $f_i(m)$ represents the optimum solution.

Step2:- S[i] is a pair of (profit, weight) by using S^i we calculate next values i.e., S^{i+1} .

Step3:- Initially $s^i=\{0,0\}$. To obtain the optimal solution we use $\mathbf{f_n(m)=max\{ f_{n-1}(m),f_{n-1}(m-w_n)+f_n \}}$ we need to compute the ordered set

$S^i=\{ f((y_i),y_j) ¥ 1 \leq j \leq k \}$ to represent $f_i(y)$.

Step4:- Each member of S^i is a pair of (p,w) where $p=f_i(y_i)$, $w=y_j$.

Step5:- We can compute S^{i+1} from S^i by first computing s^i_1 .If contains two pairs (p_j ,w_j) ,(p_k ,w_k) with the property $p_j \leq p_k$, $w_j \geq w_k$.then the pair (p_j , w_j) can be discarded.This is known as rule and sometimes it is known as purging rule.

EXAMPLE:- Let us consider a knapsack instance

p_i	w_i
1	2
2	3
5	4
6	5

Where n=4 and m=8.

<u>Sol</u>:- We have $S^0=(0,0)$

$S_1^0=(0+1,0+2)=(1,2)$

$S^1=\{(0,0)(1,2)\}$ (Apply purging rule)

$S_1^1=\{(0+2,0+3)(1+2,2+3)\}=\{(2,3)(3,5)\}$

$S^2=\{(0,0)(1,2)(2,3)(3,5)\}$

$S_1^2=\{(0+5,0+4)(1+5,2+4)(2+5,3+4)(3+5,5+4)\}$

$=\{(5,4)(6,6)(7,7)(8,9)\}$

$S^3=\{(0,0)(1,2)(2,3)(3,5)(5,4)(6,6)(7,7)(8,9)\}$

(after applying purging rule (3,5) will be discarded)

$=\{(0,0)(1,2)(2,3)(5,4)(6,6)(7,7)(8,9)\}$

$S_1^3=\{(0+6,0+5)(1+6,2+5)(2+6,3+5)(5+6,4+5)(6+6,6+5),$
$(7+6,7+5)(8+6,9+5)\}$

$=\{(6,5)(7,7)(8,8)(11,9)(12,11)(13,12)(14,14)\}$

$S^4=\{(0,0)(1,2)(2,3)(5,4)(6,6)(7,7)(8,9)(6,5)(8,8)(11,9)$
$(12,11)(13,12)(14,14)\}$

Now let us take the following assumptions.

1. If (p_1,w_1) is a last tuple in S^n, a set of 0 or 1 values

for the x such that $\sum p_i x_i = p_1$ and $\sum w_i x_i = w$ can be determine carrying out a search through S^i s

2.→ If $(p_1,w_1) \in S^{n-1}$ then $x_n=0$.

→ If (p_1,w_1) S^{n-1} , $(p_1-p_n , w_1-w_n) \in S^{n-1}$ then $x_n=1$.

PROCEDURE:

→Take $(p , w) \in S^n$ and $w \le m$.

→ If $(p,w) \in S^N$,(p,w) does not belongs to S^{n-1},$x_n=1$.

→Take (p_n ,w_n) and substract from (p,w).

*Consider the previous example (p,w)=(8,8).

Note:- (p,w)=(8,8) because 8≤8and comparing all the pairs of weight≤m the max profit is 8.

*(p,w)=(8,8)

8≤8(w≤m)

*$(8,8) \in S^4$,(8,8)does not belongs to S^3

then $x_4=1$

*$(p_4,w_4)=(6,5)$

$(p-p_4,w-w_4)=(8-6,8-5)=(2,3)$

(p,w)=(2,3)

*(p,w)=(2,3)

$(2,3) \in S^3$ and $(2,3) \in S^2$

So $X_3 = 0$

$(2,3) \in S^2$ and $(2,3)$ does not belongs to S^1

So $x_2 = 1$

*$(p_2,w_2)=(2,3) \Rightarrow (p-p_2,w-w_2)=(2-2,3-3) = (0,0)$

*$(p,w) = (0,0)$

$(0,0) \in S^1$ and $(0,0) \in S^0$

So $x_1=0$

Therefore $(x_1\ x_2\ x_3\ x_4) = (0\ 1\ 0\ 1)$

$\sum p_i x_i = p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4$

$=1(0)+2(1)+5(0)+6(1)$
$= 0+2+0+6$

$= 8$

Therefore profit is 8.

ALGORITHM:-

Algorithm DKP (p,w,n,m){

$S^0=\{(0,0)\};$

for i:=1 to n-1 do{

$S_1^0 = S_1^{i-1} := \{(p,w)/(p-p_i\ ,\ w-w_i) \in S^{i-1}$ and $w \leq m$;

S^i:= merge purge(S^{i-1}, S^{i-1}); $\rightarrow (S^0 + S_1^1 = S^1)$

$S^1 = (S^0, S^i)$

}

$(p_{X,} w_X)$: = last pair in S^{n-1}

$(p_{Y,} w_{Y)}$:= (p^1+p_n, w^1, w_n) where w^1 is large w pair in S^{n-1} such

that $w+w_n \leq m$;

//True back for $x_n, x_{n-1}, \ldots\ldots. x_1$

If $(p_x > p_y)$ then $x_n = 0$;

else

$x_n := 1$

True back for$(x_{n-1} \ldots\ldots. x_1)$;

}

Time Complexity is - $O(2^{n/2})$

❖ OPTIMAL BINARY STRUCTURE EXAMPLE PROBLEM

*Construct the optimal binary search tree for identifiers (a_1,a_2,a_3,a_4) = (do,if,int,while) , P(1:4) = (3,3,1,1) , Q(0:4) = (2,3,1,1) , $0 \leq i < 4$

Solution:

Possibilities	j - i = 0		j - i = 1		j - i = 2		j - i = 3		j - i = 4	
	i	j	i	j	i	j	i	j	i	j
	0	0	0	1	0	2	0	3	0	4
	1	1	1	2	1	3	1	4		
	3	3	2	3	2	4				
	4	4								

⇨ For j - i = 0 the possibilities are (i , j) => (0,0)
(1,1)
(2,2)
(3,3)
(4,4)

(i) $W_{00} = q_0 = 2$
$C_{00} = r_{00} = 0$

(ii) $W_{11} = q_1 = 3$
$C_{11} = r_{11} = 0$

(iii) $W_{22} = q_2 = 1$
$C_{22} = r_{22} = 0$

(iv) $W_{33} = q_3 = 1$
$C_{33} = r_{33} = 0$

(v) $W_{44} = q_4 = 1$
$C_{44} = r_{44} = 0$

⇨ For $j - i = 1$ the possibilities are $(i , j) \Rightarrow (0,0)$

$(1,2)$

$(2,3)$

$(3,4)$

$W_{ij} = W_{ij-1} + P[j] + Q[j]$

$$C_{ij} = \min_{i < k \leq j} \{ C(i , k-1) + C(k , j) \} + W[i , j]$$

$(0,1) \Rightarrow W_{01} = W_{00} + P[1] + Q[1]$

$= 2 + 3 + 3$

$W_{01} = 8$

$$C_{01} = \min_{0< k \leq 1} \{ C(0 , 0) + C(1 , 1) \} + W[0 , 1]$$

$$= \min_{0 < k \leq 1} \{ 0 + 0 \} + W[0,1]$$

$= \min \{0\} + 8$

$C_{01} = 8$

$r_{01} = k = 1$

$(1,2) \Rightarrow W_{12} = W_{11} + P[2] + Q[2]$

$= 3 + 3 + 1$

$W_{12} = 7$

$C_{12} = \min \{ C(1 , 1) + C(2 , 2) \} + W[1 , 2]$

$1 < k \leq sss2$

$= \min \{ 0 + 0 \} + W[1 , 2]$

$1 < k \leq 2$

$= \min \{0\} + 7$

$C_{01} = 7$

$r_{12} = k = 2$

$(2,3) \Rightarrow W_{23} = W_{22} + P[3] + Q[3]$

$= 1 + 1 + 1$

$W_{23} = 3$

$C_{23} = \min \{ C(2 , 2) + C(3 , 3) \} + W[2 , 3]$

$2 < k \leq 3$

$= \min \{ 0 + 0 \} + W[2,3]$

$2 < k \leq 3$

$= \min \{0\} + 3$

$C_{23} = 3$

$r_{23} = k = 3$

$(3,4) \Rightarrow W_{34} = W_{33} + P[4] + Q[4]$

$= 1 + 1 + 1$

$W_{34} = 3$

$C_{34} = \min \{ C(3 , 3) + C(4 , 4) \} + W[3 , 4]$

$3 < k \leq 4$

$= \min \{ 0 + 0 \} + W[3,4]$

$3 < k \leq 4$

$= \min \{0\} + 3$

$C_{34} = 3$

$r_{34} = k = 4$

⇨ For j - i = 2 the possibilities are (i , j) => (0,2)
(1,3)
(2,4)

$W_{ij} = W_{ij-1} + P[j] + Q[j]$

$C_{ij} = \min \{ C(i , k-1) + C(k , j) \} + W[i , j]$

$i < k \leq j$

(0,2) => $W_{02} = W_{01} + P[2] + Q[2]$

$= 8 + 3 + 1$

$W_{02} = 12$

$C_{02} = \min \{ C(0 , 0) + C(1 , 2), C(0 , 1) + C(2 , 2) \} + W[0 , 2]$

$0 < k \leq 2$

$= \min \{ 0 + 7 , 8 + 0 \} + W[0,2]$

$0 < k \leq 2$

$= 7 + 12$

$C_{02} = 19$

$r_{02} = k = 1$

(1,3) => $W_{13} = W_{12} + P[3] + Q[3]$

$= 7 + 1 + 1$

$W_{13} = 9$

$C_{13} = \min \{ C(1,1) + C(2,3), C(1,2) + C(3,3) \} + W[1,3]$

$1 < k \leq 3$

$= \min \{ 0 + 3, 7 + 0 \} + W[1, 3]$

$1 < k \leq 3$

$= 3 + 9$

$C_{13} = 12$

$r_{13} = k = 2$

$(2,4) \Rightarrow W_{24} = W_{23} + P[4] + Q[4]$

$= 3 + 1 + 1$

$W_{24} = 5$

$C_{24} = \min \{ C(2,2) + C(3,4), C(2,3) + C(4,4) \} + W[0,3]$

$2 < k \leq 4$

$= \min \{ 0 + 3, 3 + 0 \} + W[0,3]$

$2 < k \leq 4$

$= 3 + 5$

$C_{24} = 8$

$r_{24} = k = 3$

⇨ For j - i = 3 the possibilities are (i , j) => (0,3)
(1,4)

$W_{ij} = W_{ij-1} + P[j] + Q[j]$

$C_{ij} = \min_{i < k \leq j} \{ C(i , k-1) + C(k , j) \} + W[i , j]$

$(0,3) \Rightarrow W_{03} = W_{02} + P[3] + Q[3]$

$= 12 + 1 + 1$

$W_{03} = 14$

$C_{03} = \min_{0 < k \leq 3} \{ C(0,0)+C(1,3) , C(0,1)+C(2,3) , C(0,2)+C(3,3) \} + W[0 , 3]$

$= \min_{0 < k \leq 3} \{ 0 + 12 , 8 + 3 , 19 + 10 \} + W[0,3]$

$= 11 + 14$

$C_{03} = 25$

$r_{03} = k = 2$

$(1 , 4) \Rightarrow W_{14} = W_{13} + P[4] + Q[4]$

$= 9 + 1 + 1$

$W_{14} = 12$

$C_{14} = \min_{1 < k \leq 4} \{ C(1,1)+C(2,4) , C(1,2)+C(3,4) , C(1,3)+C(4,4) \} + W[1 , 4]$

$= \min_{1 < k \leq 4} \{ 0 + 8 , 7 + 3 , 12 + 10 \} + W[1 , 4]$

$= 8 + 11$

$C_{14} = 19$

$r_{14} = k = 2$

⇨ For $j - i = 4$ the possibility is $(0 , 4)$

$(0 , 4) \Rightarrow W_{04} = W_{03} + P[4] + Q[4]$

$= 14 + 1 + 1$

$W_{04} = 16$

$C_{04} = \min\{C(0,0)+C(1,4) , C(0,1)+C(2,4) , C(0,2)+C(3,4) , C(0,3)+C(4,4)\} + W[0 , 4]$

$0 < k \leq 4$

$= \min \{ 0 + 19 , 8 + 8 , 19 + 13 , 25 + 0 \} + W[0 , 4]$

$0 < k \leq 4$

$= 16 + 16$

$= 32$

$C_{04} = 32$

$r_{04} = k = 2$

❖ All Pair Shortest Path:

Let G=(V,E) be a directed graph with n vertices and each edge contains same cost. When the cost of adjacency matrix for G.

Cost(i,j) = 0.

Cost(i , j) = length of the edge <i , j> if <i , j> belongs to E(G).

Cost(i,j) = ∞ if i ≠ j, <i , j>doesnot belongs to E(G).

All pair shortest path is to dertemine matrix A such that A(i , j) is the length of the shortest path from i to j.

Procedure:

1. By using the given graph construct the adjacent matrix with intial weights.
   i) A(i , j) = 0
   ii) A(i ,j) = cost(i ,j)
   iii) A(i ,j) = ∞ if there is no direct edge between i to j.

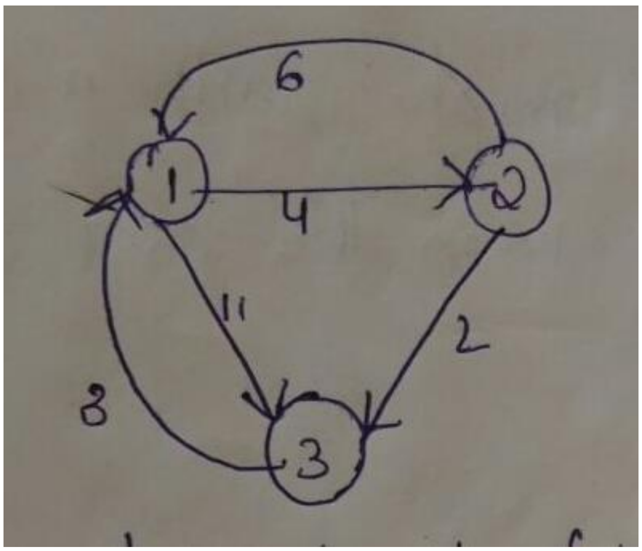
By using the following rules there is no direct edge between i to j.

2. By using the intial matrix A^0 construct all pair shortest path for remaining vertices using the following recursive relations.

$A^k(i, j) = \min\{ A^{k-1}(i, j), A^{k-1}(i, k) + A^{k-1}(k, j) \}$ where $k = 1,2,3,\ldots\ldots,n$.

Example:

Construct all pair shortest path for the following graph.



Adajacency matrix for the following graph: $A^0 = \begin{bmatrix} \mathbf{0} & \mathbf{4} & \mathbf{11} \\ \mathbf{6} & \mathbf{0} & \mathbf{2} \\ \mathbf{3} & \infty & \mathbf{0} \end{bmatrix}$

- Let us compute k=1

  $A^1(1,1) = 0$

  $A^1(1,2) = \min\{ A^{1-1}(1,2), A^{1-1}(1,2) + A^{i-1}(1,2) \}$

  $= \min\{ A^0(1,2), A^0(1,2) + A^0(1,2) \}$

  $= \min\{4, 0, 4)$

$= \min \{4 , 4\}$

$= 4$

$A^1(1,3)=\min \{ A^0(1,3) , A^0(1,1) + A^0(1,3) \}$

$= \min \{ 11 , 0+11 \}$

$= \min \{ 11 , 11 \}$

$= 11$

$A^1(2,1)= \min \{ A^0(2,1) , A^0(2,1) + A^0(1,1) \}$

$= \min \{ 6 , 0+6 \}$

$= \min \{ 6, 6 \}$

$= 6$

$A^1(2,2)= \min \{ A^0(2,3) , A^0(2,1) + A^0(1,2) \}$

$= \min (0 , 6+4 \}$

$= \min \{ 0 , 10)$

$= 0$

$A^1(2,3)= \min \{ A^0(2,3) , A^0(2,1) + A^0(1,3) \}$

$= \min \{ 2 , 6+11 \}$

$= \min \{ 2, 17 \}$

$= 2$

$A^1(3,1)= \min \{ A^0(3,1) , A^0(3,1) + A^0(1,1) \}$

-

$= \min \{ 3 , 3 \}$

$= 3$

$A^1(3,2)= \min \{ A^0(3,2) , A^0(3,1) + A^0(1,2) \}$

$= \min \{ \infty , 3+4 \}$

$= \min \{ \infty , 7 \} = 7$

$A^1(3,3) = \min \{ A^0(3,3) , A^0(3,1) + A^0(1,3) \}$

$= \min \{ 0 , 3+11\}$

$= \min \{ 0 , 14 \}$

$= 0$

$$A^1 = \begin{bmatrix} 0 & 4 & 11 \\ 6 & 0 & 2 \\ 3 & 7 & 0 \end{bmatrix}$$

- Let us compute k = 2

$A^2(1,1) = 0$

$A^2(1,2) = \min \{ A^1(1,2) , A^1(1,2) + A^1(2,2) \}$

$= \min \{ 4 , 4+0 \}$

$= \min \{ 4 , 4 \}$

$= 4$

$A^2(1,3) = \min \{ A^1(1,3) , A^1(1,2) + A^1(2,3) \}$

$= \min \{ 11 , 4+2)$

$= \min \{ 11 , 6 \}$

$= 6$

$A^2(2,1) = \min \{ A^1(2,1) , A^1(2,2) + A^1(2,1) \}$

$= \min \{ 6 , 0+6 \}$

$= \min \{ 6 , 6 \}$

$= 6$

$A^2(2,2) = \min \{ A^1(2,2) , A^1(2,2) + A^1(2,2) \}$

$= \min \{ 0 , 0 \}$

$= 0$

$A^2(2,3) = \min \{ A^1(2,3) , A^1(2,2) + A^1(2,3) \}$

$= \min \{ 2 , 0+2 \}$

$= \min \{ 2 , 2 \}$

$= 2$

$A^2(3,1) = \min \{ A^1(3,1) , A^1(3,2) + A^1(2,1) \}$

$= \min \{ 3 , 7+6 \}$

$= \min \{ 3 , 13 \}$

$= 3$

$A^2(3,2) = \min \{ A^1(3,2) , A^1(3,2) + A^1(2,2) \}$

$= \min \{ 7 , 7+0 \}$

$= \min \{ 7 , 7 \}$

$= 7$

$A^2(3,3) = \min \{ A^1(3,3) , A^1(3,2) + A^1(2,3) \}$

$= \min \{ 0 , 7+2 \}$

$= \min \{ 0 , 9 \}$

$= 0$

$$A^2 = \begin{bmatrix} 0 & 4 & 6 \\ 6 & 0 & 2 \\ 3 & 7 & 0 \end{bmatrix}$$

- Let us compute k = 3

  $A^3(1,1) = 0$

  $A^3(1,2) = \min \{ A^2(1,2) , A^2(1,3) + A^2(3,2) \}$

  $= \min \{ 4 , 6+7 \}$

$= \min\{4, 13\}$

$= 4$

$A^3(1,3) = \min\{A^2(1,3), A^2(1,1) + A^2(3,3)\}$

$= \min\{6, 6+0\}$

$= \min\{6, 6\}$

$= 6$

$A^3(2,1) = \min\{A^2(2,1), A^2(2,3) + A^2(3,1)\}$

$= \min\{6, 2+3\}$

$= \min\{6, 5\}$

$= 5$

$A^3(2,2) = \min\{A^2(2,2), A^2(2,3) + A^2(3,2)\}$

$= \min\{0, 2+7\}$

$= \min\{0, 9\}$

$= 9$

$A^3(2,3) = \min\{A^2(2,3), A^2(2,3) + A^2(3,3)\}$

$= \min\{2, 2+0\}$

$= \min\{2, 2\}$

$= 2$

$A^3(3,1) = \min\{A^2(2,3), A^2(2,3) + A^2(3,3)\}$

$= \min\{3, 0+3\}$

$= \min\{3, 3\}$

$= 3$

$A^3(3,2) = \min\{A^2(3,2), A^2(3,3) + A^2(3,2)\}$

$= \min\{7, 0+7\}$

$= \min\{7, 7\}$

$= 7$

$A^3(3,3) = \min \{ A^2(3,3) , A^2(3,3) + A^2(3,3) \}$

$= \min \{ 0 , 0 \}$

$= 0$

$$A^3 \begin{bmatrix} 0 & 4 & 6 \\ 5 & 0 & 2 \\ 3 & 7 & 0 \end{bmatrix}$$

Algorithm:

```
  Algorithm all pair ( cost , n ,A )
// n is number of  nodes , cost is an adjacent matrix that.
// contains distance between every pair of vertices.
// A is adjacency matrix contains shortest path for every pair.
//of nodes , cost[i,i] = 0.0
{
for i:=1 to n
for j:=1 to n do
A[i,j]:=cost[i,j];
for k:=1 to n do
for i:=1 to n do
for j:=1 to n do
                A[i,j] := min( A[i,j] ,A[i,k]+A[k,j])
}
```

- Time Complexity of All Pair Shortest Path is = $O(n^3)$.

Travelling Sales Persons Problem:

- Let the ‘G’ be a directed graph with n vertices and edges each edge contains same cost.
- A tour of G is simple cyclic that includes every vertex in ‘G’.
  The cost of the tour is the sum of the edges on the tour.
- The main objective of travelling sales persons problem is to find the tour of the minimum cost.

Example:

1. Suppose we have to route a postal van to pickup mails from boxex located at n different sites.
2. And n+1 vertex graph may be used to represents the situation.
3. One vertex represents the post office from which postal van starts and to which it return.
4. The route taken by a van is atour.
5. We need to find the tour with minimum length.
   - Every tour consists of an edge <i,k> for some k belongs to v – { 1 }.
     And path from k to 1
   - A path from vertex k to vertex 1 goes through each vertex in v – {1,k} exactly once.
   - Let g(2,5) be the length of shortest path starting a vertex I going through all vertices in s and terminated at vertex i.

- The function g(1,v – {1}) is the length of the an optimal sales person four then.

  $$g(1,v-\{1\}) = \min_{2\leq k \leq n} \{ c_{1k} + g(k, v-\{1,k\}) \}$$

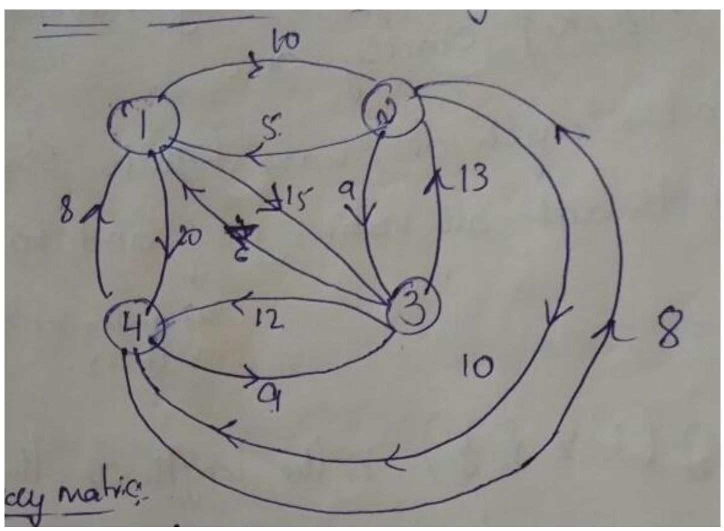
  Genaralizing the above we get,

  $$v(i,s) = \min_{j\in s} \{ c_{ij} + g(j , s-\{j\}) \}$$

- For calculating distance from intial vertex to remaining vertices.

  $g(i,\emptyset) = c_{i1}, 1\leq i\leq n$

  Example: consider the following graph



Adajacency matrix for the above graph is : $\begin{bmatrix} 0 & 10 & 15 & 20 \\ 5 & 0 & 9 & 10 \\ 6 & 13 & 0 & 12 \\ 8 & 8 & 9 & 0 \end{bmatrix}$

We know that, $g(i,\emptyset) = c_{i1}$, 1<i<n

$g(1,\emptyset) = c_{11} = 0$

$g(2,\emptyset) = c_{21} = 5$

$g(3,\emptyset) = c_{31} = 6$

$g(4,\emptyset) = c_{41} = 8$

Let us assume that mod s = 1 means set contains only one element. Before solving this problem we make an assumption that sales person starts at vertex 1 from that we can move to vertex 1.

g(I,s) =g(2,3) , g(2,4) ,g(3,2) ,g(3,4) , g(4,2) ,g(4,3) ,.

Now we determine the above six possibilities using

$g(i,s) = \min \{ c_{ij} + g(j , s - \{ j \}) \}$

$j \in s$

$g(2,3) = \min \{ c_{23} + g(3 , 3 - \{ 3 \}) \}$

$= \min \{ 9 + g(3,\emptyset)\}$

$= \min \{ 9 + 6)$

$= \min \{15\} = 15$

$g(2,4) = \min \{ c_{24} + g(4 , 4 - \{ 4 \}) \}$

$= \min \{10 + g(4,\emptyset)\}$

$= \min \{ 10 + 8)$

$= \min \{18\} = 18$

$g(3,2) = \min \{ c_{32} + g(2 , 2 – \{ 4 \}) \}$

$= \min \{ 13 + g(2,\emptyset)\}$

$= \min \{ 13 + 5)$

$= \min \{18\} = 18$

$g(3,4) = \min \{ c_{34} + g(4 , 4 – \{ 4 \}) \}$

$= \min \{12 + g(4,\emptyset)\}$

$= \min \{ 12 + 8)$

$= \min \{20\} = 20$

$g(4,2) = \min \{ c_{42} + g(2 , 2 – \{ 2\}) \}$

$= \min \{ 8 + g(2,\emptyset)\}$

$= \min \{ 8 + 5)$

$= \min \{13\} = 13$

$g(4,3) = \min \{ c_{43} + g(3 , 3 – \{ 3 \}) \}$

$= \min \{ 9 + g(3,\emptyset)\}$

$= \min \{ 9 + 6)$

= min {15} = 15

- Let us consider mod s = 2
  g(2 , {3,4}) = 25
  g(3 , {2,4}) = 25
  g(4 , {2,3}) = 33

g(2 , {3,4}) =

= min { c_{23} + g(3 , {3,4} – 3) , c_{24} + g(4 , {3,4} – 4) }

= min { 9 + g(3,4) , 10 + g(4,3) }

= min { 9 + 20 ,10 + 15}

= min { 29 ,25 } = 25

g(3 , {2,4}) =

= min { c_{32} + g(2 , {2,4} – 2) , c_{34} + g(4 , {2,4} – 4) }

= min { 13 + g(2,4) , 12 + g(4,3) }

= min { 13 + 18 ,12 + 13 }

= min{ 31 ,25 } = 25

g(4 , {23}) =

= min { c_{42} + g(2 , {2,3} – 2) , c_{43} + g(3 , {2,3} – 3) }

= min { 8 + g(2,3) , 10 + g(3,2) }

= min { 8 + 15 ,9 + 18}

= min{ 23 ,27 } = 23

- Let us consider mod s = 3

g(1 , {2 , 3 , 4})

= min { c_{12} + g(2 , {2,3,4} – 2) , c_{13} + g(3 , {2,3,4} – 3), c_{14} + g(4 , {2,3,4} – 4) }

= min { c_{12} + g(2 , {3,4}) , c_{13} + g(3 , {2,4}), c_{14} + g(4 , {2,3}) }

= min { 10+25 , 15+25 , 20+23 }

=min {35 ,40,43}

35

The shortest path for visiting all the vertices is **1-2-4-3-1**

Analysis:-Time complexity is O(n^2 ,2^n)

Space complexity is O(n,2^n)