1. Ring, Ideal

(1) Describe the group of units in $\mathbb{R}[x]$, $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}[i]$.

(2) An element a of a ring R is called nilpotent if $a^n = 0$ for some $n \in \mathbb{N}$. Show that if u is a unit and a is nilpotent in R then $u + a$ is a unit.

(3) Let E be the set of all integer sequences $a = (a_1, a_2, a_3, \ldots)$. We add sequences componentwise. Let R be the set of all mappings $f : E \longrightarrow E$ such that $f(a+b) = f(a) + f(b)$ for all $a, b \in E$. Let $T : E \longrightarrow E$ be the shift operator $T(a_1, a_2, a_3, ...) = (0, a_1, a_2, a_3, ...)$. Show that T has a left inverse but not a right inverse.

(4) Let R be a ring and $R[[t]]$ denote the set of all formal power series in an indeterminate t. A formal power series is a formal expression of the form $f(t) = \sum_{i=0}^{\infty} a_i t^i$ where $a_i \in R$ for all i. We add and multiply formal power series as we add and multiply polynomials. Under these operations $R[[t]]$ is a ring. Show that $f(t)$ is a unit if and only if a_0 is a unit. Show that if $f(t)$ is nilpotent then all a_i are so. Is the converse true ?

(5) Let $\mathbb{Q}[\alpha, \beta]$ denote the smallest subring of $\mathbb{C}$ containing $\alpha = \sqrt{2}$ and $\beta = \sqrt{3}$. Show that $\mathbb{Q}[\alpha, \beta] = \mathbb{Q}[\gamma]$ where $\gamma = \alpha + \beta$.

(6) Prove that every non zero ideal in $\mathbb{Z}[i]$ contains a non zero integer.

(7) Describe the kernels of the homomorphisms $\phi : \mathbb{R}[x] \longrightarrow \mathbb{C}$ given by $\phi(f(x)) = f(2+i)$.

(8) Show that nilpotent elements of a ring R form an ideal. This ideal, denoted by $nil(R)$, is called the nilradical of R. Determine the nilradical of $\mathbb{Z}/12\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{R}[x]$.

(9) Show that all ideals of the power series ring $R[[x]]$ are principal.

(10) Let I and J be ideals of a ring R. The sum $I + J$ of I and J is defined by $I + J = \{x + y | x \in I, y \in J\}$. Show that $I + J$ is an ideal of R.

(11) The product IJ of I and J is defined to be the set

$$IJ = \{\sum_i x_i y_i | x_i \in I, y_i \in J \text{ for all } i\}$$

Show that IJ is an ideal and $I \cap J \subseteq IJ$. Show by an example that IJ need not be equal to $I \cap J$

(12) An isomorphism of a ring R is called an automorphism of R. Determine all automorphisms of $\mathbb{Z}[x]$ and $\mathbb{R}$..