

Chapter 2 Measures

The last section of the previous chapter discusses several deficiencies of Riemann integration. To remedy those deficiencies, in this chapter we extend the notion of the length of an interval to a larger collection of subsets of **R**. This leads us to measures and then in the next chapter to integration with respect to measures.

We begin this chapter by investigating outer measure, which looks promising but fails to have a crucial property. That failure leads us to σ -algebras and measurable spaces. Then we define measures in an abstract context that can be applied to settings more general than \mathbf{R} . Next, we construct Lebesgue measure on \mathbf{R} as our desired extension of the notion of the length of an interval.



Fifth-century AD Roman ceiling mosaic in what is now a UNESCO World Heritage site in Ravenna, Italy. Giuseppe Vitali, who in 1905 proved result 2.18 in this chapter, was born and grew up in Ravenna, where perhaps he saw this mosaic. Could the memory of the translation-invariant feature of this mosaic have suggested to Vitali the translation invariance that is the heart of his proof of 2.18?

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2A Outer Measure on R

Motivation and Definition of Outer Measure

The Riemann integral arises from approximating the area under the graph of a function by sums of the areas of approximating rectangles. These rectangles have heights that approximate the values of the function on subintervals of the function's domain. The width of each approximating rectangle is the length of the corresponding subinterval. This length is the term $x_j - x_{j-1}$ in the definitions of the lower and upper Riemann sums (see 1.3).

To extend integration to a larger class of functions than the Riemann integrable functions, we will write the domain of a function as the union of subsets more complicated than the subintervals used in Riemann integration. We will need to assign a size to each of those subsets, where the size is an extension of the length of intervals.

For example, we expect the size of the set $(1,3) \cup (7,10)$ to be 5 (because the first interval has length 2, the second interval has length 3, and 2 + 3 = 5).

Assigning a size to subsets of \mathbf{R} that are more complicated than unions of open intervals becomes a nontrivial task. This chapter focuses on that task and its extension to other contexts. In the next chapter, we will see how to use the ideas developed in this chapter to create a rich theory of integration.

We begin by giving the expected definition of the length of an open interval, along with a notation for that length.

2.1 **Definition** *length of open interval;* $\ell(I)$

The *length* $\ell(I)$ of an open interval I is defined by

$$\ell(I) = \begin{cases} b - a & \text{if } I = (a, b) \text{ for some } a, b \in \mathbf{R} \text{ with } a < b, \\ 0 & \text{if } I = \emptyset, \\ \infty & \text{if } I = (-\infty, a) \text{ or } I = (a, \infty) \text{ for some } a \in \mathbf{R}, \\ \infty & \text{if } I = (-\infty, \infty). \end{cases}$$

Suppose $A \subset \mathbf{R}$. The size of A should be at most the sum of the lengths of a sequence of open intervals whose union contains A. Taking the infimum of all such sums gives a reasonable definition of the size of A, denoted |A| and called the outer measure of A.

2.2 **Definition** outer measure; |A|

The *outer measure* |A| of a set $A \subset \mathbf{R}$ is defined by

$$|A|=\inf\Bigl\{\sum_{k=1}^\infty\ell(I_k):I_1,I_2,\ldots$$
 are open intervals such that $A\subset\bigcup_{k=1}^\infty I_k\Bigr\}.$

The definition of outer measure involves an infinite sum. The infinite sum $\sum_{k=1}^{\infty} t_k$ of a sequence t_1, t_2, \ldots of elements of $[0, \infty]$ is defined to be ∞ if some $t_k = \infty$. Otherwise, $\sum_{k=1}^{\infty} t_k$ is defined to be the limit (possibly ∞) of the increasing sequence $t_1, t_1 + t_2, t_1 + t_2 + t_3, \ldots$ of partial sums; thus

$$\sum_{k=1}^{\infty} t_k = \lim_{n \to \infty} \sum_{k=1}^{n} t_k.$$

2.3 Example finite sets have outer measure 0

Suppose $A = \{a_1, \dots, a_n\}$ is a finite set of real numbers. Suppose $\varepsilon > 0$. Define a sequence I_1, I_2, \dots of open intervals by

$$I_k = \begin{cases} (a_k - \varepsilon, a_k + \varepsilon) & \text{if } k \leq n, \\ \emptyset & \text{if } k > n. \end{cases}$$

Then I_1, I_2, \ldots is a sequence of open intervals whose union contains A. Clearly $\sum_{k=1}^{\infty} \ell(I_k) = 2\varepsilon n$. Hence $|A| \leq 2\varepsilon n$. Because ε is an arbitrary positive number, this implies that |A| = 0.

Good Properties of Outer Measure

Outer measure has several nice properties that are discussed in this subsection. We begin with a result that improves upon the example above.

2.4 countable sets have outer measure 0

Every countable subset of \mathbf{R} has outer measure 0.

Proof Suppose $A = \{a_1, a_2, ...\}$ is a countable subset of **R**. Let $\varepsilon > 0$. For $k \in \mathbf{Z}^+$, let

$$I_k = \left(a_k - \frac{\varepsilon}{2^k}, a_k + \frac{\varepsilon}{2^k}\right).$$

Then I_1, I_2, \ldots is a sequence of open intervals whose union contains A. Because

$$\sum_{k=1}^{\infty} \ell(I_k) = 2\varepsilon,$$

we have $|A| \le 2\varepsilon$. Because ε is an arbitrary positive number, this implies that |A| = 0.

The result above, along with the result that the set \mathbf{Q} of rational numbers is countable, implies that \mathbf{Q} has outer measure 0. We will soon show that there are far fewer rational numbers than real numbers (see 2.17). Thus the equation $|\mathbf{Q}| = 0$ indicates that outer measure has a good property that we want any reasonable notion of size to possess.

The next result shows that outer measure does the right thing with respect to set inclusion.

2.5 outer measure preserves order

Suppose A and B are subsets of **R** with $A \subset B$. Then $|A| \leq |B|$.

Proof Suppose $I_1, I_2, ...$ is a sequence of open intervals whose union contains B. Then the union of this sequence of open intervals also contains A. Hence

$$|A| \leq \sum_{k=1}^{\infty} \ell(I_k).$$

Taking the infimum over all sequences of open intervals whose union contains B, we have $|A| \le |B|$.

We expect that the size of a subset of \mathbf{R} should not change if the set is shifted to the right or to the left. The next definition allows us to be more precise.

2.6 **Definition** *translation*; t + A

If $t \in \mathbf{R}$ and $A \subset \mathbf{R}$, then the *translation* t + A is defined by

$$t + A = \{t + a : a \in A\}.$$

If t > 0, then t + A is obtained by moving the set A to the right t units on the real line; if t < 0, then t + A is obtained by moving the set A to the left |t| units.

Translation does not change the length of an open interval. Specifically, if $t \in \mathbf{R}$ and $a,b \in [-\infty,\infty]$, then t+(a,b)=(t+a,t+b) and thus $\ell(t+(a,b))=\ell((a,b))$. Here we are using the standard convention that $t+(-\infty)=-\infty$ and $t+\infty=\infty$.

The next result states that translation invariance carries over to outer measure.

2.7 outer measure is translation invariant

Suppose $t \in \mathbf{R}$ and $A \subset \mathbf{R}$. Then |t + A| = |A|.

Proof Suppose $I_1, I_2,...$ is a sequence of open intervals whose union contains A. Then $t + I_1, t + I_2,...$ is a sequence of open intervals whose union contains t + A. Thus

$$|t + A| \le \sum_{k=1}^{\infty} \ell(t + I_k) = \sum_{k=1}^{\infty} \ell(I_k).$$

Taking the infimum of the last term over all sequences $I_1, I_2, ...$ of open intervals whose union contains A, we have $|t + A| \le |A|$.

To get the inequality in the other direction, note that A = -t + (t + A). Thus applying the inequality from the previous paragraph, with A replaced by t + A and t replaced by -t, we have $|A| = |-t + (t + A)| \le |t + A|$. Hence |t + A| = |A|.

The union of the intervals (1,4) and (3,5) is the interval (1,5). Thus

$$\ell((1,4) \cup (3,5)) < \ell((1,4)) + \ell((3,5))$$

because the left side of the inequality above equals 4 and the right side equals 5. The direction of the inequality above is explained by noting that the interval (3,4), which is the intersection of (1,4) and (3,5), has its length counted twice on the right side of the inequality above.

The example of the paragraph above should provide intuition for the direction of the inequality in the next result. The property of satisfying the inequality in the result below is called *countable subadditivity* because it applies to sequences of subsets.

2.8 countable subadditivity of outer measure

Suppose A_1, A_2, \ldots is a sequence of subsets of **R**. Then

$$\left| \bigcup_{k=1}^{\infty} A_k \right| \le \sum_{k=1}^{\infty} |A_k|.$$

Proof If $|A_k| = \infty$ for some $k \in \mathbb{Z}^+$, then the inequality above clearly holds. Thus assume $|A_k| < \infty$ for all $k \in \mathbb{Z}^+$.

Let $\varepsilon > 0$. For each $k \in \mathbf{Z}^+$, let $I_{1,k}$, $I_{2,k}$, ... be a sequence of open intervals whose union contains A_k such that

$$\sum_{j=1}^{\infty} \ell(I_{j,k}) \le \frac{\varepsilon}{2^k} + |A_k|.$$

Thus

$$\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{j,k}) \le \varepsilon + \sum_{k=1}^{\infty} |A_k|.$$

The doubly indexed collection of open intervals $\{I_{j,k}: j,k\in \mathbb{Z}^+\}$ can be rearranged into a sequence of open intervals whose union contains $\bigcup_{k=1}^{\infty} A_k$ as follows, where in step k (start with k=2, then $k=3,4,5,\ldots$) we adjoin the k-1 intervals whose indices add up to k:

$$\underbrace{I_{1,1}}_{2}$$
, $\underbrace{I_{1,2}, I_{2,1}}_{3}$, $\underbrace{I_{1,3}, I_{2,2}, I_{3,1}}_{4}$, $\underbrace{I_{1,4}, I_{2,3}, I_{3,2}, I_{4,1}}_{5}$, $\underbrace{I_{1,5}, I_{2,4}, I_{3,3}, I_{4,2}, I_{5,1}}_{\text{sum of the two indices is 6}}$...

Inequality 2.9 shows that the sum of the lengths of the intervals listed above is less than or equal to $\varepsilon + \sum_{k=1}^{\infty} |A_k|$. Thus $\left|\bigcup_{k=1}^{\infty} A_k\right| \leq \varepsilon + \sum_{k=1}^{\infty} |A_k|$. Because ε is an arbitrary positive number, this implies that $\left|\bigcup_{k=1}^{\infty} A_k\right| \leq \sum_{k=1}^{\infty} |A_k|$.

Countable subadditivity implies finite subadditivity, meaning that

$$|A_1 \cup \dots \cup A_n| \le |A_1| + \dots + |A_n|$$

for all $A_1, \ldots, A_n \subset \mathbf{R}$, because we can take $A_k = \emptyset$ for k > n in 2.8.

The countable subadditivity of outer measure, as proved above, adds to our list of nice properties enjoyed by outer measure.

Outer Measure of Closed Bounded Interval

One more good property of outer measure that we should prove is that if a < b, then the outer measure of the closed interval [a,b] is b-a. Indeed, if $\varepsilon > 0$, then $(a-\varepsilon,b+\varepsilon),\emptyset,\emptyset,\ldots$ is a sequence of open intervals whose union contains [a,b]. Thus $|[a,b]| \le b-a+2\varepsilon$. Because this inequality holds for all $\varepsilon > 0$, we conclude that

$$|[a,b]| \le b - a.$$

Is the inequality in the other direction obviously true to you? If so, think again, because a proof of the inequality in the other direction requires that the completeness of **R** is used in some form. For example, suppose **R** was a countable set (which is not true, as we will soon see, but the uncountability of **R** is not obvious). Then we would have |[a,b]| = 0 (by 2.4). Thus something deeper than you might suspect is going on with the ingredients needed to prove that $|[a,b]| \ge b-a$.

The following definition will be useful when we prove that $|[a, b]| \ge b - a$.

2.10 **Definition** open cover

Suppose $A \subset \mathbf{R}$.

- A collection C of open subsets of R is called an open cover of A if A is contained in the union of all the sets in C.
- An open cover C of A is said to have a *finite subcover* if A is contained in the union of some finite list of sets in C.

2.11 Example open covers and finite subcovers

- The collection $\{(k, k+2) : k \in \mathbb{Z}^+\}$ is an open cover of [2,5] because $[2,5] \subset \bigcup_{k=1}^{\infty} (k, k+2)$. This open cover has a finite subcover because $[2,5] \subset (1,3) \cup (2,4) \cup (3,5) \cup (4,6)$.
- The collection $\{(k, k+2) : k \in \mathbf{Z}^+\}$ is an open cover of $[2, \infty)$ because $[2, \infty] \subset \bigcup_{k=1}^{\infty} (k, k+2)$. This open cover does not have a finite subcover because there do not exist finitely many sets of the form (k, k+2) whose union contains $[2, \infty)$.
- The collection $\{(0,2-\frac{1}{k}): k \in \mathbf{Z}^+\}$ is an open cover of (1,2) because $(1,2) \subset \bigcup_{k=1}^{\infty} (0,2-\frac{1}{k})$. This open cover does not have a finite subcover because there do not exist finitely many sets of the form $(0,2-\frac{1}{k})$ whose union contains (1,2).

The next result will be our major tool in the proof that $|[a,b]| \ge b-a$. Although we need only the result as stated, be sure to see Exercise 4 in this section, which when combined with the next result gives a characterization of the closed bounded subsets of **R**. Note that the following proof uses the completeness property of the real numbers (by asserting that the supremum of a certain nonempty bounded set exists).

2.12 Heine-Borel Theorem

Every open cover of a closed bounded subset of **R** has a finite subcover.

Proof Suppose F is a closed bounded subset of \mathbf{R} and C is an open cover of F.

First consider the case where F = [a, b] for some $a, b \in \mathbf{R}$ with a < b. Thus C is an open cover of [a, b]. Let

To provide visual clues, we usually denote closed sets by F and open sets by G.

$$D = \{d \in [a, b] : [a, d] \text{ has a finite subcover from } C\}.$$

Note that $a \in D$ (because $a \in G$ for some $G \in C$). Thus D is not the empty set. Let

$$s = \sup D$$
.

Thus $s \in [a,b]$. Hence there exists an open set $G \in \mathcal{C}$ such that $s \in G$. Let $\delta > 0$ be such that $(s - \delta, s + \delta) \subset G$. Because $s = \sup D$, there exist $d \in (s - \delta, s]$ and $n \in \mathbf{Z}^+$ and $G_1, \ldots, G_n \in \mathcal{C}$ such that

$$[a,d] \subset G_1 \cup \cdots \cup G_n$$
.

Now

$$[a,d'] \subset G \cup G_1 \cup \cdots \cup G_n$$

for all $d' \in [s, s + \delta)$. Thus $d' \in D$ for all $d' \in [s, s + \delta) \cap [a, b]$. This implies that s = b. Furthermore, 2.13 with d' = b shows that [a, b] has a finite subcover from C, completing the proof in the case where F = [a, b].

Now suppose F is an arbitrary closed bounded subset of \mathbf{R} and that \mathcal{C} is an open cover of F. Let $a, b \in \mathbf{R}$ be such that $F \subset [a, b]$. Now $\mathcal{C} \cup \{\mathbf{R} \setminus F\}$ is an open cover of \mathbf{R} and hence is an open cover of [a, b] (here $\mathbf{R} \setminus F$ denotes the set complement of F in \mathbf{R}). By our first case, there exist $G_1, \ldots, G_n \in \mathcal{C}$ such that

$$[a,b]\subset G_1\cup\cdots\cup G_n\cup (\mathbf{R}\setminus F).$$

Thus

$$F \subset G_1 \cup \cdots \cup G_n$$
,

completing the proof.



Saint-Affrique, the small town in southern France where Émile Borel (1871–1956) was born. Borel first stated and proved what we call the Heine–Borel Theorem in 1895. Earlier, Eduard Heine (1821–1881) and others had used similar results.

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Now we can prove that closed intervals have the expected outer measure.

2.14 outer measure of a closed interval

Suppose $a, b \in \mathbf{R}$, with a < b. Then |[a, b]| = b - a.

Proof See the first paragraph of this subsection for the proof that $|[a,b]| \le b-a$. To prove the inequality in the other direction, suppose I_1, I_2, \ldots is a sequence of open intervals such that $[a,b] \subset \bigcup_{k=1}^{\infty} I_k$. By the Heine–Borel Theorem (2.12), there exists $n \in \mathbb{Z}^+$ such that

$$[a,b] \subset I_1 \cup \cdots \cup I_n.$$

We will now prove by induction on n that the inclusion above implies that

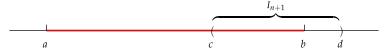
$$\sum_{k=1}^{n} \ell(I_k) \ge b - a.$$

This will then imply that $\sum_{k=1}^{\infty} \ell(I_k) \ge \sum_{k=1}^{n} \ell(I_k) \ge b - a$, completing the proof that $|[a,b]| \ge b - a$.

To get started with our induction, note that 2.15 clearly implies 2.16 if n = 1. Now for the induction step: Suppose n > 1 and 2.15 implies 2.16 for all choices of $a, b \in \mathbf{R}$ with a < b. Suppose $I_1, \ldots, I_n, I_{n+1}$ are open intervals such that

$$[a,b] \subset I_1 \cup \cdots \cup I_n \cup I_{n+1}.$$

Thus b is in at least one of the intervals $I_1, \ldots, I_n, I_{n+1}$. By relabeling, we can assume that $b \in I_{n+1}$. Suppose $I_{n+1} = (c,d)$. If $c \le a$, then $\ell(I_{n+1}) \ge b-a$ and there is nothing further to prove; thus we can assume that a < c < b < d, as shown in the figure below.



Hence

$$[a,c]\subset I_1\cup\cdots\cup I_n.$$

By our induction hypothesis, we have $\sum_{k=1}^{n} \ell(I_k) \ge c - a$. Thus

$$\sum_{k=1}^{n+1} \ell(I_k) \ge (c - a) + \ell(I_{n+1})$$

$$= (c - a) + (d - c)$$

$$= d - a$$

$$\ge b - a,$$

completing the proof.

Alice was beginning to get very tired of sitting by her sister on the bank, and of having nothing to do: once or twice she had peeped into the book her sister was reading, but it had no pictures or conversations in it, "and what is the use of a book," thought Alice, "without pictures or conversation?"

 opening paragraph of Alice's Adventures in Wonderland, by Lewis Carroll

The result above easily implies that the outer measure of each open interval equals its length (see Exercise 6).

The previous result has the following important corollary. You may be familiar with Georg Cantor's (1845–1918) original proof of the next result. The proof using outer measure that is presented here gives an interesting alternative to Cantor's proof.

2.17 nontrivial intervals are uncountable

Every interval in **R** that contains at least two distinct elements is uncountable.

Proof Suppose *I* is an interval that contains $a, b \in \mathbf{R}$ with a < b. Then

$$|I| \ge |[a,b]| = b - a > 0,$$

where the first inequality above holds because outer measure preserves order (see 2.5) and the equality above comes from 2.14. Because every countable subset of \mathbf{R} has outer measure 0 (see 2.4), we can conclude that I is uncountable.

Outer Measure is Not Additive

We have had several results giving nice properties of outer measure. Now we come to an unpleasant property of outer measure.

If outer measure were a perfect way to assign a size as an extension of the lengths of intervals, then the outer measure of the union of two disjoint sets would equal the sum of the outer measures of the two se

Outer measure led to the proof above that **R** is uncountable. This application of outer measure to prove a result that seems unconnected with outer measure is an indication that outer measure has serious mathematical value.

sum of the outer measures of the two sets. Sadly, the next result states that outer measure does not have this property.

In the next section, we begin the process of getting around the next result, which will lead us to measure theory.

2.18 nonadditivity of outer measure

There exist disjoint subsets A and B of \mathbf{R} such that

$$|A \cup B| \neq |A| + |B|.$$

Proof For $a \in [-1,1]$, let \tilde{a} be the set of numbers in [-1,1] that differ from a by a rational number. In other words,

$$\tilde{a} = \{c \in [-1, 1] : a - c \in \mathbf{Q}\}.$$

If $a, b \in [-1, 1]$ and $\tilde{a} \cap \tilde{b} \neq \emptyset$, then $\tilde{a} = \tilde{b}$. (Proof: Suppose there exists $d \in \tilde{a} \cap \tilde{b}$. Then a - d and b - d are rational numbers; subtracting, we conclude that a - b is a rational number. The equation

Think of \tilde{a} as the equivalence class of a under the equivalence relation that declares $a, c \in [-1, 1]$ to be equivalent if $a - c \in \mathbf{Q}$.

a-c=(a-b)+(b-c) now implies that if $c\in[-1,1]$, then a-c is a rational number if and only if b-c is a rational number. In other words, $\tilde{a}=\tilde{b}$.)

Clearly
$$a \in \tilde{a}$$
 for each $a \in [-1,1]$. Thus $[-1,1] = \bigcup_{a \in [-1,1]} \tilde{a}$.

Let V be a set that contains exactly one element in each of the distinct sets in

$$\{\tilde{a}: a \in [-1,1]\}.$$

In other words, for every $a \in [-1, 1]$, the set $V \cap \tilde{a}$ has exactly one element.

This step involves the Axiom of Choice, as discussed after this proof. The set V arises by choosing one element from each equivalence class.

Let r_1, r_2, \ldots be a sequence of distinct rational numbers such that

$$[-2,2] \cap \mathbf{Q} = \{r_1, r_2, \ldots\}.$$

Then

$$[-1,1]\subset\bigcup_{k=1}^{\infty}(r_k+V),$$

where the set inclusion above holds because if $a \in [-1,1]$, then letting v be the unique element of $V \cap \tilde{a}$, we have $a - v \in \mathbf{Q}$, which implies that $a = r_k + v \in r_k + V$ for some $k \in \mathbf{Z}^+$.

The set inclusion above, the order-preserving property of outer measure (2.5), and the countable subadditivity of outer measure (2.8) imply

$$|[-1,1]| \leq \sum_{k=1}^{\infty} |r_k + V|.$$

We know that |[-1,1]| = 2 (from 2.14). The translation invariance of outer measure (2.7) thus allows us to rewrite the inequality above as

$$2 \le \sum_{k=1}^{\infty} |V|.$$

Thus |V| > 0.

Note that the sets r_1+V , r_2+V ,... are disjoint. (Proof: Suppose there exists $t\in (r_j+V)\cap (r_k+V)$. Then $t=r_j+v_1=r_k+v_2$ for some $v_1,v_2\in V$, which implies that $v_1-v_2=r_k-r_j\in \mathbf{Q}$. Our construction of V now implies that $v_1=v_2$, which implies that $r_j=r_k$, which implies that j=k.)

Let $n \in \mathbf{Z}^+$. Clearly

$$\bigcup_{k=1}^{n} (r_k + V) \subset [-3,3]$$

because $V \subset [-1,1]$ and each $r_k \in [-2,2]$. The set inclusion above implies that

$$\left|\bigcup_{k=1}^{n} (r_k + V)\right| \le 6.$$

However

2.20
$$\sum_{k=1}^{n} |r_k + V| = \sum_{k=1}^{n} |V| = n |V|.$$

Now 2.19 and 2.20 suggest that we choose $n \in \mathbb{Z}^+$ such that n|V| > 6. Thus

$$\left| \bigcup_{k=1}^{n} (r_k + V) \right| < \sum_{k=1}^{n} |r_k + V|.$$

If we had $|A \cup B| = |A| + |B|$ for all disjoint subsets A, B of \mathbf{R} , then by induction on n we would have $\Big|\bigcup_{k=1}^n A_k\Big| = \sum_{k=1}^n |A_k|$ for all disjoint subsets A_1, \ldots, A_n of \mathbf{R} . However, 2.21 tells us that no such result holds. Thus there exist disjoint subsets A, B of \mathbf{R} such that $|A \cup B| \neq |A| + |B|$.

The Axiom of Choice, which belongs to set theory, states that if \mathcal{E} is a set whose elements are disjoint nonempty sets, then there exists a set D that contains exactly one element in each set that is an element of \mathcal{E} . We used the Axiom of Choice to construct the set D that was used in the last proof.

A small minority of mathematicians objects to the use of the Axiom of Choice. Thus we will keep track of where we need to use it. Even if you do not like to use the Axiom of Choice, the previous result warns us away from trying to prove that outer measure is additive (any such proof would need to contradict the Axiom of Choice, which is consistent with the standard axioms of set theory).

EXERCISES 2A

- 1 Prove that if A and B are subsets of **R** and |B| = 0, then $|A \cup B| = |A|$.
- **2** Suppose $A \subset \mathbf{R}$ and $t \in \mathbf{R}$. Let $tA = \{ta : a \in A\}$. Prove that |tA| = |t| |A|. [Assume that $0 \cdot \infty$ is defined to be 0.]
- 3 Prove that if A, B ⊂ \mathbf{R} and $|A| < \infty$, then $|B \setminus A| \ge |B| |A|$.
- 4 Suppose *F* is a subset of **R** with the property that every open cover of *F* has a finite subcover. Prove that *F* is closed and bounded.
- 5 Suppose \mathcal{A} is a set of closed subsets of \mathbf{R} such that $\bigcap_{F \in \mathcal{A}} F = \emptyset$. Prove that if \mathcal{A} contains at least one bounded set, then there exist $n \in \mathbf{Z}^+$ and $F_1, \ldots, F_n \in \mathcal{A}$ such that $F_1 \cap \cdots \cap F_n = \emptyset$.
- **6** Prove that if $a, b \in \mathbf{R}$ and a < b, then

$$|(a,b)| = |[a,b)| = |(a,b]| = b - a.$$

7 Suppose a, b, c, d are real numbers with a < b and c < d. Prove that

$$|(a,b)\cup(c,d)|=(b-a)+(d-c)$$
 if and only if $(a,b)\cap(c,d)=\emptyset$.

- **8** Prove that if $A \subset \mathbf{R}$ and t > 0, then $|A| = |A \cap (-t, t)| + |A \cap (\mathbf{R} \setminus (-t, t))|$.
- 9 Prove that $|A| = \lim_{t \to \infty} |A \cap (-t, t)|$ for all $A \subset \mathbf{R}$.

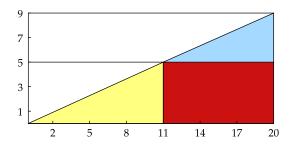
- **10** Prove that $|[0,1] \setminus \mathbf{Q}| = 1$.
- 11 Prove that if I_1, I_2, \ldots is a disjoint sequence of open intervals, then

$$\left|\bigcup_{k=1}^{\infty}I_{k}\right|=\sum_{k=1}^{\infty}\ell(I_{k}).$$

12 Suppose r_1, r_2, \ldots is a sequence that contains every rational number. Let

$$F = \mathbf{R} \setminus \bigcup_{k=1}^{\infty} \left(r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k} \right).$$

- (a) Show that F is a closed subset of \mathbf{R} .
- (b) Prove that if *I* is an interval contained in *F*, then *I* contains at most one element.
- (c) Prove that $|F| = \infty$.
- Suppose $\varepsilon > 0$. Prove that there exists a subset F of [0,1] such that F is closed, every element of F is an irrational number, and $|F| > 1 \varepsilon$.
- 14 Consider the following figure, which is drawn accurately to scale.



(a) Show that the right triangle whose vertices are (0,0), (20,0), and (20,9) has area 90.

[We have not defined area yet, but just use the elementary formulas for the areas of triangles and rectangles that you learned long ago.]

- (b) Show that the yellow (lower) right triangle has area 27.5.
- (c) Show that the red rectangle has area 45.
- (d) Show that the blue (upper) right triangle has area 18.
- (e) Add the results of parts (b), (c), and (d), showing that the area of the colored region is 90.5.
- (f) Seeing the figure above, most people expect parts (a) and (e) to have the same result. Yet in part (a) we found area 90, and in part (e) we found area 90.5. Explain why these results differ.

[You may be tempted to think that what we have here is a two-dimensional example similar to the result about the nonadditivity of outer measure (2.18). However, genuine examples of nonadditivity require much more complicated sets than in this example.]

2B Measurable Spaces and Functions

The last result in the previous section showed that outer measure is not additive. Could this disappointing result perhaps be fixed by using some notion other than outer measure for the size of a subset of \mathbf{R} ? The next result answers this question by showing that there does not exist a notion of size, called the Greek letter mu (μ) in the result below, that has all the desirable properties.

Property (c) in the result below is called *countable additivity*. Countable additivity is a highly desirable property because we want to be able to prove theorems about limits (the heart of analysis!), which requires countable additivity.

2.22 nonexistence of extension of length to all subsets of R

There does not exist a function μ with all the following properties:

- (a) μ is a function from the set of subsets of **R** to $[0, \infty]$.
- (b) $\mu(I) = \ell(I)$ for every open interval I of \mathbf{R} .
- (c) $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$ for every disjoint sequence A_1, A_2, \ldots of subsets of **R**.
- (d) $\mu(t+A) = \mu(A)$ for every $A \subset \mathbf{R}$ and every $t \in \mathbf{R}$.

Proof Suppose there exists a function μ with all the properties listed in the statement of this result.

Observe that $\mu(\emptyset) = 0$, as follows from (b) because the empty set is an open interval with length 0.

We will show that μ has all the properties of outer measure that were used in the proof of 2.18.

If $A \subset B \subset \mathbf{R}$, then $\mu(A) \leq \mu(B)$, as follows from (c) because we can write B as the union of the disjoint sequence $A, B \setminus A, \emptyset, \emptyset, \dots$; thus

$$\mu(B) = \mu(A) + \mu(B \setminus A) + 0 + 0 + \cdots = \mu(A) + \mu(B \setminus A) \ge \mu(A).$$

If $a, b \in \mathbf{R}$ with a < b, then $(a, b) \subset [a, b] \subset (a - \varepsilon, b + \varepsilon)$ for every $\varepsilon > 0$. Thus $b - a \le \mu([a, b]) \le b - a + 2\varepsilon$ for every $\varepsilon > 0$. Hence $\mu([a, b]) = b - a$.

If $A_1, A_2,...$ is a sequence of subsets of **R**, then $A_1, A_2 \setminus A_1, A_3 \setminus (A_1 \cup A_2),...$ is a disjoint sequence of subsets of **R** whose union is $\bigcup_{k=1}^{\infty} A_k$. Thus

$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \mu\Big(A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus (A_1 \cup A_2)) \cup \cdots\Big)$$
$$= \mu(A_1) + \mu(A_2 \setminus A_1) + \mu\Big(A_3 \setminus (A_1 \cup A_2)\Big) + \cdots$$
$$\leq \sum_{k=1}^{\infty} \mu(A_k),$$

where the second equality follows from the countable additivity of μ .

We have shown that μ has all the properties of outer measure that were used in the proof of 2.18. Repeating the proof of 2.18, we see that there exist disjoint subsets A, B of \mathbf{R} such that $\mu(A \cup B) \neq \mu(A) + \mu(B)$. Thus the disjoint sequence $A, B, \emptyset, \emptyset, \ldots$ does not satisfy the countable additivity property required by (c). This contradiction completes the proof.

σ -Algebras

The last result shows that we need to give up one of the desirable properties in our goal of extending the notion of size from intervals to more general subsets of **R**. We cannot give up 2.22(b) because the size of an interval needs to be its length. We cannot give up 2.22(c) because countable additivity is needed to prove theorems about limits. We cannot give up 2.22(d) because a size that is not translation invariant does not satisfy our intuitive notion of size as a generalization of length.

Thus we are forced to relax the requirement in 2.22(a) that the size is defined for all subsets of **R**. Experience shows that to have a viable theory that allows for taking limits, the collection of subsets for which the size is defined should be closed under complementation and closed under countable unions. Thus we make the following definition.

2.23 **Definition** σ -algebra

Suppose X is a set and S is a set of subsets of X. Then S is called a σ -algebra on X if the following three conditions are satisfied:

- $\emptyset \in \mathcal{S}$;
- if $E \in \mathcal{S}$, then $X \setminus E \in \mathcal{S}$;
- if $E_1, E_2, ...$ is a sequence of elements of S, then $\bigcup_{k=1}^{S} E_k \in S$.

Make sure you verify that the examples in all three bullet points below are indeed σ -algebras. The verification is obvious for the first two bullet points. For the third bullet point, you need to use the result that the countable union of countable sets is countable (see the proof of 2.8 for an example of how a doubly indexed list can be converted to a singly indexed sequence). The exercises contain some additional examples of σ -algebras.

2.24 Example σ -algebras

- Suppose X is a set. Then clearly $\{\emptyset, X\}$ is a σ -algebra on X.
- Suppose X is a set. Then clearly the set of all subsets of X is a σ -algebra on X.
- Suppose X is a set. Then the set of all subsets E of X such that E is countable or X \ E is countable is a σ-algebra on X.

Now we come to some easy but important properties of σ -algebras.

2.25 σ -algebras are closed under countable intersection

Suppose S is a σ -algebra on a set X. Then

- (a) $X \in \mathcal{S}$;
- (b) if $D, E \in \mathcal{S}$, then $D \cup E \in \mathcal{S}$ and $D \cap E \in \mathcal{S}$ and $D \setminus E \in \mathcal{S}$;
- (c) if $E_1, E_2, ...$ is a sequence of elements of S, then $\bigcap_{k=1}^{\infty} E_k \in S$.

Proof Because $\emptyset \in \mathcal{S}$ and $X = X \setminus \emptyset$, the first two bullet points in the definition of σ -algebra (2.23) imply that $X \in \mathcal{S}$, proving (a).

Suppose $D, E \in \mathcal{S}$. Then $D \cup E$ is the union of the sequence $D, E, \emptyset, \emptyset, \ldots$ of elements of \mathcal{S} . Thus the third bullet point in the definition of σ -algebra (2.23) implies that $D \cup E \in \mathcal{S}$.

De Morgan's Laws tell us that

$$X \setminus (D \cap E) = (X \setminus D) \cup (X \setminus E).$$

If $D, E \in \mathcal{S}$, then the right side of the equation above is in \mathcal{S} ; hence $X \setminus (D \cap E) \in \mathcal{S}$; thus the complement in X of $X \setminus (D \cap E)$ is in \mathcal{S} ; in other words, $D \cap E \in \mathcal{S}$.

Because $D \setminus E = D \cap (X \setminus E)$, we see that if $D, E \in \mathcal{S}$, then $D \setminus E \in \mathcal{S}$, completing the proof of (b).

Finally, suppose $E_1, E_2, ...$ is a sequence of elements of S. De Morgan's Laws tell us that

$$X\setminus\bigcap_{k=1}^{\infty}E_k=\bigcup_{k=1}^{\infty}(X\setminus E_k).$$

The right side of the equation above is in S. Hence the left side is in S, which implies that $X \setminus (X \setminus \bigcap_{k=1}^{\infty} E_k) \in S$. In other words, $\bigcap_{k=1}^{\infty} E_k \in S$, proving (c).

The word *measurable* is used in the terminology below because in the next section we introduce a size function, called a measure, defined on measurable sets.

2.26 **Definition** measurable space; measurable set

- A *measurable space* is an ordered pair (X, S), where X is a set and S is a σ -algebra on X.
- An element of S is called an S-measurable set, or just a measurable set if S is clear from the context.

For example, if $X = \mathbf{R}$ and S is the set of all subsets of \mathbf{R} that are countable or have a countable complement, then the set of rational numbers is S-measurable but the set of positive real numbers is not S-measurable.

Borel Subsets of R

The next result guarantees that there is a smallest σ -algebra on a set X containing a given set A of subsets of X.

2.27 smallest σ -algebra containing a collection of subsets

Suppose X is a set and A is a set of subsets of X. Then the intersection of all σ -algebras on X that contain A is a σ -algebra on X.

Proof There is at least one σ -algebra on X that contains \mathcal{A} because the σ -algebra consisting of all subsets of X contains \mathcal{A} .

Let S be the intersection of all σ -algebras on X that contain A. Then $\emptyset \in S$ because \emptyset is an element of each σ -algebra on X that contains A.

Suppose $E \in \mathcal{S}$. Thus E is in every σ -algebra on X that contains \mathcal{A} . Thus $X \setminus E$ is in every σ -algebra on X that contains \mathcal{A} . Hence $X \setminus E \in \mathcal{S}$.

Suppose E_1, E_2, \ldots is a sequence of elements of \mathcal{S} . Thus each E_k is in every σ -algebra on X that contains \mathcal{A} . Thus $\bigcup_{k=1}^{\infty} E_k$ is in every σ -algebra on X that contains \mathcal{A} . Hence $\bigcup_{k=1}^{\infty} E_k \in \mathcal{S}$, which completes the proof that \mathcal{S} is a σ -algebra on X.

Using the terminology *smallest* for the intersection of all σ -algebras that contain a set \mathcal{A} of subsets of X makes sense because the intersection of those σ -algebras is contained in every σ -algebra that contains \mathcal{A} .

2.28 Example *smallest* σ -algebra

 Suppose X is a set and A is the set of subsets of X that consist of exactly one element:

$$\mathcal{A} = \big\{ \{x\} : x \in X \big\}.$$

Then the smallest σ -algebra on X containing A is the set of all subsets E of X such that E is countable or $X \setminus E$ is countable, as you should verify.

• Suppose $\mathcal{A} = \{(0,1),(0,\infty)\}$. Then the smallest σ -algebra on \mathbf{R} containing \mathcal{A} is $\{\emptyset,(0,1),(0,\infty),(-\infty,0]\cup[1,\infty),(-\infty,0],[1,\infty),(-\infty,1),\mathbf{R}\}$, as you should verify.

Now we come to a crucial definition.

2.29 Definition Borel set

The smallest σ -algebra on **R** containing all open subsets of **R** is called the collection of *Borel subsets* of **R**. An element of this σ -algebra is called a *Borel set*.

We have defined the collection of Borel subsets of \mathbf{R} to be the smallest σ -algebra on \mathbf{R} containing all the open subsets of \mathbf{R} . We could have defined the collection of Borel subsets of \mathbf{R} to be the smallest σ -algebra on \mathbf{R} containing all the open intervals (because every open subset of \mathbf{R} is the union of a sequence of open intervals).

2.30 Example Borel sets

- Every closed subset of **R** is a Borel set because every closed subset of **R** is the complement of an open subset of **R**.
- Every countable subset of **R** is a Borel set because if $B = \{x_1, x_2, ...\}$, then $B = \bigcup_{k=1}^{\infty} \{x_k\}$, which is a Borel set because each $\{x_k\}$ is a closed set.
- Every half-open interval [a, b] (where $a, b \in \mathbb{R}$) is a Borel set because $[a, b] = \bigcap_{k=1}^{\infty} (a \frac{1}{k}, b)$.
- If f: R → R is a function, then the set of points at which f is continuous is the
 intersection of a sequence of open sets (see Exercise 12 in this section) and thus
 is a Borel set.

The intersection of every sequence of open subsets of \mathbf{R} is a Borel set. However, the set of all such intersections is not the set of Borel sets (because it is not closed under countable unions). The set of all countable unions of countable intersections of open subsets of \mathbf{R} is also not the set of Borel sets (because it is not closed under countable intersections). And so on *ad infinitum*—there is no finite procedure involving countable unions, countable intersections, and complements for constructing the collection of Borel sets.

We will see later that there exist subsets of \mathbf{R} that are not Borel sets. However, any subset of \mathbf{R} that you can write down in a concrete fashion is a Borel set.

Inverse Images

The next definition is used frequently in the rest of this chapter.

2.31 **Definition** *inverse image*; $f^{-1}(A)$

If $f: X \to Y$ is a function and $A \subset Y$, then the set $f^{-1}(A)$ is defined by

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

2.32 Example inverse images

Suppose $f: [0,4\pi] \to \mathbf{R}$ is defined by $f(x) = \sin x$. Then

$$f^{-1}((0,\infty)) = (0,\pi) \cup (2\pi,3\pi),$$

$$f^{-1}([0,1]) = [0,\pi] \cup [2\pi,3\pi] \cup \{4\pi\},$$

$$f^{-1}(\{-1\}) = \{\frac{3\pi}{2}, \frac{7\pi}{2}\},$$

$$f^{-1}((2,3)) = \emptyset,$$

as you should verify.

Inverse images have good algebraic properties, as is shown in the next two results.

2.33 algebra of inverse images

Suppose $f: X \to Y$ is a function. Then

- (a) $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ for every $A \subset Y$; (b) $f^{-1}(\bigcup_{A \in \mathcal{A}} A) = \bigcup_{A \in \mathcal{A}} f^{-1}(A)$ for every set \mathcal{A} of subsets of Y; (c) $f^{-1}(\bigcap_{A \in \mathcal{A}} A) = \bigcap_{A \in \mathcal{A}} f^{-1}(A)$ for every set \mathcal{A} of subsets of Y.

Proof Suppose $A \subset Y$. For $x \in X$ we have

$$x \in f^{-1}(Y \setminus A) \iff f(x) \in Y \setminus A$$

$$\iff f(x) \notin A$$

$$\iff x \notin f^{-1}(A)$$

$$\iff x \in X \setminus f^{-1}(A).$$

Thus $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$, which proves (a).

To prove (b), suppose A is a set of subsets of Y. Then

$$x \in f^{-1}(\bigcup_{A \in \mathcal{A}} A) \iff f(x) \in \bigcup_{A \in \mathcal{A}} A$$

$$\iff f(x) \in A \text{ for some } A \in \mathcal{A}$$

$$\iff x \in f^{-1}(A) \text{ for some } A \in \mathcal{A}$$

$$\iff x \in \bigcup_{A \in \mathcal{A}} f^{-1}(A).$$

Thus $f^{-1}(\bigcup_{A\in\mathcal{A}}A)=\bigcup_{A\in\mathcal{A}}f^{-1}(A)$, which proves (b).

Part (c) is proved in the same fashion as (b), with unions replaced by intersections and for some replaced by for every.

inverse image of a composition 2.34

Suppose $f: X \to Y$ and $g: Y \to W$ are functions. Then

$$(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A))$$

for every $A \subset W$.

Proof Suppose $A \subset W$. For $x \in X$ we have

$$x \in (g \circ f)^{-1}(A) \iff (g \circ f)(x) \in A \iff g(f(x)) \in A$$

 $\iff f(x) \in g^{-1}(A)$
 $\iff x \in f^{-1}(g^{-1}(A)).$

Thus $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)).$

Measurable Functions

The next definition tells us which real-valued functions behave reasonably with respect to a σ -algebra on their domain.

2.35 **Definition** measurable function

Suppose (X, S) is a measurable space. A function $f: X \to \mathbf{R}$ is called S-measurable (or just measurable if S is clear from the context) if

$$f^{-1}(B) \in \mathcal{S}$$

for every Borel set $B \subset \mathbf{R}$.

2.36 Example *measurable functions*

- If $S = \{\emptyset, X\}$, then the only S-measurable functions from X to **R** are the constant functions.
- If S is the set of all subsets of X, then every function from X to \mathbf{R} is S-measurable.
- If $S = \{\emptyset, (-\infty, 0), [0, \infty), \mathbf{R}\}$ (which is a σ -algebra on \mathbf{R}), then a function $f : \mathbf{R} \to \mathbf{R}$ is S-measurable if and only if f is constant on $(-\infty, 0)$ and f is constant on $[0, \infty)$.

Another class of examples comes from characteristic functions, which are defined below. The Greek letter chi (χ) is traditionally used to denote a characteristic function.

2.37 **Definition** *characteristic function*; χ_{F}

Suppose *E* is a subset of a set *X*. The *characteristic function* of *E* is the function $\chi_E \colon X \to \mathbf{R}$ defined by

 $\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$

The set X that contains E is not explicitly included in the notation χ_E because X will always be clear from the context.

2.38 Example *inverse image with respect to a characteristic function* Suppose (X, S) is a measurable space, $E \subset X$, and $B \subset \mathbf{R}$. Then

$$\chi_E^{-1}(B) = \begin{cases} E & \text{if } 0 \not\in B \text{ and } 1 \in B, \\ X \setminus E & \text{if } 0 \in B \text{ and } 1 \not\in B, \\ X & \text{if } 0 \in B \text{ and } 1 \in B, \\ \varnothing & \text{if } 0 \not\in B \text{ and } 1 \not\in B. \end{cases}$$

Thus we see that χ_E is an S-measurable function if and only if $E \in S$.

The definition of an S-measurable function requires the inverse image of every Borel subset of \mathbf{R} to be in S. The next result shows that to verify that a function is S-measurable, we can check the inverse images of a much smaller collection of subsets of \mathbf{R} .

Note that if $f: X \to \mathbf{R}$ is a function and $a \in \mathbf{R}$, then

$$f^{-1}\bigl((a,\infty)\bigr)=\{x\in X:f(x)>a\}.$$

2.39 condition for measurable function

Suppose (X, S) is a measurable space and $f: X \to \mathbf{R}$ is a function such that

$$f^{-1}((a,\infty)) \in \mathcal{S}$$

for all $a \in \mathbf{R}$. Then f is an S-measurable function.

Proof Let

$$\mathcal{T} = \{ A \subset \mathbf{R} : f^{-1}(A) \in \mathcal{S} \}.$$

We want to show that every Borel subset of **R** is in \mathcal{T} . To do this, we will first show that \mathcal{T} is a σ -algebra on **R**.

Certainly $\emptyset \in \mathcal{T}$, because $f^{-1}(\emptyset) = \emptyset \in \mathcal{S}$.

If $A \in \mathcal{T}$, then $f^{-1}(A) \in \mathcal{S}$; hence

$$f^{-1}(\mathbf{R} \setminus A) = X \setminus f^{-1}(A) \in \mathcal{S}$$

by 2.33(a), and thus $\mathbf{R} \setminus \mathcal{A} \in \mathcal{T}$. In other words, \mathcal{T} is closed under complementation. If $A_1, A_2, \ldots \in \mathcal{T}$, then $f^{-1}(A_1), f^{-1}(A_2), \ldots \in \mathcal{S}$; hence

$$f^{-1}\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \bigcup_{k=1}^{\infty} f^{-1}(A_k) \in \mathcal{S}$$

by 2.33(b), and thus $\bigcup_{k=1}^{\infty} A_k \in \mathcal{T}$. In other words, \mathcal{T} is closed under countable unions. Thus \mathcal{T} is a σ -algebra on \mathbf{R} .

By hypothesis, \mathcal{T} contains $\{(a, \infty) : a \in \mathbf{R}\}$. Because \mathcal{T} is closed under complementation, \mathcal{T} also contains $\{(-\infty, b] : b \in \mathbf{R}\}$. Because the σ -algebra \mathcal{T} is closed under finite intersections (by 2.25), we see that \mathcal{T} contains $\{(a, b] : a, b \in \mathbf{R}\}$. Because $(a, b) = \bigcup_{k=1}^{\infty} (a, b - \frac{1}{k}]$ and $(-\infty, b) = \bigcup_{k=1}^{\infty} (-k, b - \frac{1}{k}]$ and \mathcal{T} is closed under countable unions, we can conclude that \mathcal{T} contains every open subset of \mathbf{R} .

Thus the σ -algebra \mathcal{T} contains the smallest σ -algebra on \mathbf{R} that contains all open subsets of \mathbf{R} . In other words, \mathcal{T} contains every Borel subset of \mathbf{R} . Thus f is an \mathcal{S} -measurable function.

In the result above, we could replace the collection of sets $\{(a, \infty) : a \in \mathbf{R}\}$ by any collection of subsets of \mathbf{R} such that the smallest σ -algebra containing that collection contains the Borel subsets of \mathbf{R} . For specific examples of such collections of subsets of \mathbf{R} , see Exercises 3–6.

We have been dealing with S-measurable functions from X to \mathbf{R} in the context of an arbitrary set X and a σ -algebra S on X. An important special case of this setup is when X is a Borel subset of \mathbf{R} and S is the set of Borel subsets of \mathbf{R} that are contained in X (see Exercise 11 for another way of thinking about this σ -algebra). In this special case, the S-measurable functions are called Borel measurable.

2.40 Definition Borel measurable function

Suppose $X \subset \mathbf{R}$. A function $f \colon X \to \mathbf{R}$ is called *Borel measurable* if $f^{-1}(B)$ is a Borel set for every Borel set $B \subset \mathbf{R}$.

If $X \subset \mathbf{R}$ and there exists a Borel measurable function $f: X \to \mathbf{R}$, then X must be a Borel set [because $X = f^{-1}(\mathbf{R})$].

If $X \subset \mathbf{R}$ and $f: X \to \mathbf{R}$ is a function, then f is a Borel measurable function if and only if $f^{-1}((a, \infty))$ is a Borel set for every $a \in \mathbf{R}$ (use 2.39).

Suppose X is a set and $f: X \to \mathbf{R}$ is a function. The measurability of f depends upon the choice of a σ -algebra on X. If the σ -algebra is called \mathcal{S} , then we can discuss whether f is an \mathcal{S} -measurable function. If X is a Borel subset of \mathbf{R} , then \mathcal{S} might be the set of Borel sets contained in X, in which case the phrase *Borel measurable* means the same as \mathcal{S} -measurable. However, whether or not \mathcal{S} is a collection of Borel sets, we consider inverse images of Borel subsets of \mathbf{R} when determining whether a function is \mathcal{S} -measurable.

The next result states that continuity interacts well with the notion of Borel measurability.

2.41 every continuous function is Borel measurable

Every continuous real-valued function defined on a Borel subset of ${\bf R}$ is a Borel measurable function.

Proof Suppose $X \subset \mathbf{R}$ is a Borel set and $f: X \to \mathbf{R}$ is continuous. To prove that f is Borel measurable, fix $a \in \mathbf{R}$.

If $x \in X$ and f(x) > a, then (by the continuity of f) there exists $\delta_x > 0$ such that f(y) > a for all $y \in (x - \delta_x, x + \delta_x) \cap X$. Thus

$$f^{-1}((a,\infty)) = \left(\bigcup_{x \in f^{-1}((a,\infty))} (x - \delta_x, x + \delta_x)\right) \cap X.$$

The union inside the large parentheses above is an open subset of **R**; hence its intersection with *X* is a Borel set. Thus we can conclude that $f^{-1}((a, \infty))$ is a Borel set.

Now 2.39 implies that f is a Borel measurable function.

Next we come to another class of Borel measurable functions. A similar definition could be made for decreasing functions, with a corresponding similar result.

2.42 **Definition** increasing function

Suppose $X \subset \mathbf{R}$ and $f \colon X \to \mathbf{R}$ is a function.

- f is called *increasing* if $f(x) \le f(y)$ for all $x, y \in X$ with x < y.
- f is called *strictly increasing* if f(x) < f(y) for all $x, y \in X$ with x < y.

2.43 every increasing function is Borel measurable

Every increasing function defined on a Borel subset of ${\bf R}$ is a Borel measurable function.

Proof Suppose $X \subset \mathbf{R}$ is a Borel set and $f \colon X \to \mathbf{R}$ is increasing. To prove that f is Borel measurable, fix $a \in \mathbf{R}$.

Let $b = \inf f^{-1}((a, \infty))$. Then it is easy to see that

$$f^{-1}((a,\infty)) = (b,\infty) \cap X$$
 or $f^{-1}((a,\infty)) = [b,\infty) \cap X$.

Either way, we can conclude that $f^{-1}((a, \infty))$ is a Borel set.

Now 2.39 implies that f is a Borel measurable function.

The next result shows that measurability interacts well with composition.

2.44 composition of measurable functions

Suppose (X, \mathcal{S}) is a measurable space and $f: X \to \mathbf{R}$ is an \mathcal{S} -measurable function. Suppose g is a real-valued Borel measurable function defined on a subset of \mathbf{R} that includes the range of f. Then $g \circ f: X \to \mathbf{R}$ is an \mathcal{S} -measurable function.

Proof Suppose $B \subset \mathbf{R}$ is a Borel set. Then (see 2.34)

$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B)).$$

Because g is a Borel measurable function, $g^{-1}(B)$ is a Borel subset of \mathbf{R} . Because f is an \mathcal{S} -measurable function, $f^{-1}(g^{-1}(B)) \in \mathcal{S}$. Thus the equation above implies that $(g \circ f)^{-1}(B) \in \mathcal{S}$. Thus $g \circ f$ is an \mathcal{S} -measurable function.

2.45 Example if f is measurable, then so are -f, $\frac{1}{2}f$, |f|, f^2

Suppose (X, S) is a measurable space and $f: X \to \mathbf{R}$ is S-measurable. Then 2.44 implies that the functions $-f, \frac{1}{2}f, |f|, f^2$ are all S-measurable functions because each of these functions can be written as the composition of f with a continuous (and thus Borel measurable) function g.

Specifically, take g(x) = -x, then $g(x) = \frac{1}{2}x$, then g(x) = |x|, and then $g(x) = x^2$.

Measurability also interacts well with algebraic operations, as shown in the next result.

algebraic operations with measurable functions

Suppose (X, S) is a measurable space and $f, g: X \to \mathbf{R}$ are S-measurable. Then (a) f + g, f - g, and fg are S-measurable functions;

- (b) if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is an S-measurable function.

Suppose $a \in \mathbf{R}$. We will show that

$$(f+g)^{-1}((a,\infty)) = \bigcup_{r \in \mathbf{Q}} \left(f^{-1}((r,\infty)) \cap g^{-1}((a-r,\infty)) \right),$$

which implies that $(f+g)^{-1}((a,\infty)) \in \mathcal{S}$.

To prove 2.47, first suppose

$$x \in (f+g)^{-1}\big((a,\infty)\big).$$

Thus a < f(x) + g(x). Hence the open interval (a - g(x), f(x)) is nonempty, and thus it contains some rational number r. This implies that r < f(x), which means that $x \in f^{-1}((r, \infty))$, and a - g(x) < r, which implies that $x \in g^{-1}((a - r, \infty))$. Thus x is an element of the right side of 2.47, completing the proof that the left side of 2.47 is contained in the right side.

The proof of the inclusion in the other direction is easier. Specifically, suppose $x \in f^{-1}((r,\infty)) \cap g^{-1}((a-r,\infty))$ for some $r \in \mathbb{Q}$. Thus

$$r < f(x)$$
 and $a - r < g(x)$.

Adding these two inequalities, we see that a < f(x) + g(x). Thus x is an element of the left side of 2.47, completing the proof of 2.47. Hence f + g is an S-measurable function.

Example 2.45 tells us that -g is an S-measurable function. Thus f - g, which equals f + (-g) is an S-measurable function.

The easiest way to prove that fg is an S-measurable function uses the equation

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2}.$$

The operation of squaring an S-measurable function produces an S-measurable function (see Example 2.45), as does the operation of multiplication by $\frac{1}{2}$ (again, see Example 2.45). Thus the equation above implies that fg is an S-measurable function, completing the proof of (a).

Suppose $g(x) \neq 0$ for all $x \in X$. The function defined on $\mathbb{R} \setminus \{0\}$ (a Borel subset of **R**) that takes x to $\frac{1}{x}$ is continuous and thus is a Borel measurable function (by 2.41). Now 2.44 implies that $\frac{1}{g}$ is an S-measurable function. Combining this result with what we have already proved about the product of S-measurable functions, we conclude that $\frac{f}{\sigma}$ is an S-measurable function, proving (b).

The next result shows that the pointwise limit of a sequence of S-measurable functions is S-measurable. This is a highly desirable property (recall that the set of Riemann integrable functions on some interval is not closed under taking pointwise limits; see Example 1.17).

2.48 limit of S-measurable functions

Suppose (X, S) is a measurable space and $f_1, f_2, ...$ is a sequence of S-measurable functions from X to \mathbf{R} . Suppose $\lim_{k\to\infty} f_k(x)$ exists for each $x\in X$. Define $f\colon X\to \mathbf{R}$ by

$$f(x) = \lim_{k \to \infty} f_k(x).$$

Then f is an S-measurable function.

Proof Suppose $a \in \mathbf{R}$. We will show that

$$(a,\infty) = \bigcup_{j=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} f_k^{-1} \left(\left(a + \frac{1}{j}, \infty \right) \right),$$

which implies that $f^{-1}((a, \infty)) \in \mathcal{S}$.

To prove 2.49, first suppose $x \in f^{-1}((a, \infty))$. Thus there exists $j \in \mathbf{Z}^+$ such that $f(x) > a + \frac{1}{j}$. The definition of limit now implies that there exists $m \in \mathbf{Z}^+$ such that $f_k(x) > a + \frac{1}{j}$ for all $k \ge m$. Thus x is in the right side of 2.49, proving that the left side of 2.49 is contained in the right side.

To prove the inclusion in the other direction, suppose x is in the right side of 2.49. Thus there exist $j, m \in \mathbb{Z}^+$ such that $f_k(x) > a + \frac{1}{j}$ for all $k \ge m$. Taking the limit as $k \to \infty$, we see that $f(x) \ge a + \frac{1}{j} > a$. Thus x is in the left side of 2.49, completing the proof of 2.49. Thus f is an S-measurable function.

Occasionally we need to consider functions that take values in $[-\infty, \infty]$. For example, even if we start with a sequence of real-valued functions in 2.53, we might end up with functions with values in $[-\infty, \infty]$. Thus we extend the notion of Borel sets to subsets of $[-\infty, \infty]$, as follows.

2.50 **Definition** *Borel subsets of* $[-\infty, \infty]$

A subset of $[-\infty, \infty]$ is called a *Borel set* if its intersection with **R** is a Borel set.

In other words, a set $C \subset [-\infty, \infty]$ is a Borel set if and only if there exists a Borel set $B \subset \mathbf{R}$ such that C = B or $C = B \cup \{\infty\}$ or $C = B \cup \{-\infty\}$ or $C = B \cup \{\infty, -\infty\}$.

You should verify that with the definition above, the set of Borel subsets of $[-\infty, \infty]$ is a σ -algebra on $[-\infty, \infty]$.

Next, we extend the definition of S-measurable functions to functions taking values in $[-\infty, \infty]$.

2.51 **Definition** measurable function

Suppose (X, S) is a measurable space. A function $f: X \to [-\infty, \infty]$ is called *S-measurable* if

$$f^{-1}(B) \in \mathcal{S}$$

for every Borel set $B \subset [-\infty, \infty]$.

The next result, which is analogous to 2.39, states that we need not consider all Borel subsets of $[-\infty, \infty]$ when taking inverse images to determine whether or not a function with values in $[-\infty, \infty]$ is S-measurable.

2.52 condition for measurable function

Suppose (X, S) is a measurable space and $f: X \to [-\infty, \infty]$ is a function such that

$$f^{-1}((a,\infty]) \in \mathcal{S}$$

for all $a \in \mathbf{R}$. Then f is an S-measurable function.

The proof of the result above is left to the reader (also see Exercise 27 in this section).

We end this section by showing that the pointwise infimum and pointwise supremum of a sequence of S-measurable functions is S-measurable.

2.53 infimum and supremum of a sequence of S-measurable functions

Suppose (X, S) is a measurable space and $f_1, f_2, ...$ is a sequence of S-measurable functions from X to $[-\infty, \infty]$. Define $g, h: X \to [-\infty, \infty]$ by

$$g(x) = \inf\{f_k(x) : k \in \mathbf{Z}^+\} \text{ and } h(x) = \sup\{f_k(x) : k \in \mathbf{Z}^+\}.$$

Then g and h are S-measurable functions.

Proof Let $a \in \mathbf{R}$. The definition of the supremum implies that

$$h^{-1}\big((a,\infty]\big) = \bigcup_{k=1}^{\infty} f_k^{-1}\big((a,\infty]\big),\,$$

as you should verify. The equation above, along with 2.52, implies that h is an S-measurable function.

Note that

$$g(x) = -\sup\{-f_k(x) : k \in \mathbf{Z}^+\}$$

for all $x \in X$. Thus the result about the supremum implies that g is an S-measurable function.

EXERCISES 2B

- 1 Show that $S = \{ \bigcup_{n \in K} (n, n+1] : K \subset \mathbb{Z} \}$ is a σ -algebra on \mathbb{R} .
- 2 Verify both bullet points in Example 2.28.
- 3 Suppose S is the smallest σ -algebra on \mathbf{R} containing $\{(r,s]: r,s \in \mathbf{Q}\}$. Prove that S is the collection of Borel subsets of \mathbf{R} .
- **4** Suppose S is the smallest σ -algebra on \mathbf{R} containing $\{(r, n] : r \in \mathbf{Q}, n \in \mathbf{Z}\}$. Prove that S is the collection of Borel subsets of \mathbf{R} .
- 5 Suppose S is the smallest σ -algebra on \mathbf{R} containing $\{(r, r+1) : r \in \mathbf{Q}\}$. Prove that S is the collection of Borel subsets of \mathbf{R} .
- 6 Suppose S is the smallest σ-algebra on **R** containing $\{[r, ∞) : r ∈ \mathbf{Q}\}$. Prove that S is the collection of Borel subsets of **R**.
- 7 Prove that the collection of Borel subsets of **R** is translation invariant. More precisely, prove that if $B \subset \mathbf{R}$ is a Borel set and $t \in \mathbf{R}$, then t + B is a Borel set.
- 8 Prove that the collection of Borel subsets of **R** is dilation invariant. More precisely, prove that if $B \subset \mathbf{R}$ is a Borel set and $t \in \mathbf{R}$, then tB (which is defined to be $\{tb : b \in B\}$) is a Borel set.
- 9 Give an example of a measurable space (X, S) and a function $f: X \to \mathbf{R}$ such that |f| is S-measurable but f is not S-measurable.
- 10 Show that the set of real numbers that have a decimal expansion with the digit 5 appearing infinitely often is a Borel set.
- 11 Suppose \mathcal{T} is a σ -algebra on a set Y and $X \in \mathcal{T}$. Let $\mathcal{S} = \{E \in \mathcal{T} : E \subset X\}$.
 - (a) Show that $S = \{F \cap X : F \in \mathcal{T}\}.$
 - (b) Show that S is a σ -algebra on X.
- **12** Suppose $f: \mathbf{R} \to \mathbf{R}$ is a function.
 - (a) For $k \in \mathbf{Z}^+$, let

$$G_k = \{a \in \mathbf{R} : \text{there exists } \delta > 0 \text{ such that } |f(b) - f(c)| < \frac{1}{k} \}$$
 for all $b, c \in (a - \delta, a + \delta)$.

Prove that G_k is an open subset of **R** for each $k \in \mathbf{Z}^+$.

- (b) Prove that the set of points at which f is continuous equals $\bigcap_{k=1}^{\infty} G_k$.
- (c) Conclude that the set of points at which f is continuous is a Borel set.
- 13 Suppose (X, S) is a measurable space, E_1, \ldots, E_n are disjoint subsets of X, and c_1, \ldots, c_n are distinct nonzero real numbers. Prove that $c_1 \chi_{E_1} + \cdots + c_n \chi_{E_n}$ is an S-measurable function if and only if $E_1, \ldots, E_n \in S$.

14 (a) Suppose $f_1, f_2, ...$ is a sequence of functions from a set X to \mathbb{R} . Explain why

$$\{x \in X : \text{the sequence } f_1(x), f_2(x), \dots \text{ has a limit in } \mathbf{R}\}$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} (f_j - f_k)^{-1} \left(\left(-\frac{1}{n}, \frac{1}{n} \right) \right).$$

(b) Suppose (X, S) is a measurable space and $f_1, f_2, ...$ is a sequence of S-measurable functions from X to \mathbb{R} . Prove that

$$\{x \in X : \text{the sequence } f_1(x), f_2(x), \dots \text{ has a limit in } \mathbb{R} \}$$

is an S-measurable subset of X.

- Suppose *X* is a set and $E_1, E_2, ...$ is a disjoint sequence of subsets of *X* such that $\bigcup_{k=1}^{\infty} E_k = X$. Let $S = \{\bigcup_{k \in K} E_k : K \subset \mathbf{Z}^+\}$.
 - (a) Show that S is a σ -algebra on X.
 - (b) Prove that a function from X to \mathbf{R} is S-measurable if and only if the function is constant on E_k for every $k \in \mathbf{Z}^+$.
- **16** Suppose S is a σ -algebra on a set X and $A \subset X$. Let

$$S_A = \{ E \in S : A \subset E \text{ or } A \cap E = \emptyset \}.$$

- (a) Prove that S_A is a σ -algebra on X.
- (b) Suppose $f: X \to \mathbf{R}$ is a function. Prove that f is measurable with respect to \mathcal{S}_A if and only if f is measurable with respect to \mathcal{S} and f is constant on A.
- 17 Suppose X is a Borel subset of \mathbf{R} and $f: X \to \mathbf{R}$ is a function such that $\{x \in X : f \text{ is not continuous at } x\}$ is a countable set. Prove f is a Borel measurable function.
- Suppose $f: \mathbf{R} \to \mathbf{R}$ is differentiable at every element of \mathbf{R} . Prove that f' is a Borel measurable function from \mathbf{R} to \mathbf{R} .
- Suppose X is a nonempty set and S is the σ -algebra on X consisting of all subsets of X that are either countable or have a countable complement in X. Give a characterization of the S-measurable real-valued functions on X.
- **20** Suppose (X, S) is a measurable space and $f, g: X \to \mathbf{R}$ are S-measurable functions. Prove that if f(x) > 0 for all $x \in X$, then f^g (which is the function whose value at $x \in X$ equals $f(x)^{g(x)}$) is an S-measurable function.
- 21 Prove 2.52.
- Suppose $B \subset \mathbf{R}$ and $f \colon B \to \mathbf{R}$ is an increasing function. Prove that f is continuous at every element of B except for a countable subset of B.

23 Suppose $f: \mathbf{R} \to \mathbf{R}$ is a strictly increasing function. Prove that the inverse function $f^{-1}: f(\mathbf{R}) \to \mathbf{R}$ is a continuous function. [*Note that this exercise does not have as a hypothesis that f is continuous*.]

24 Suppose $B \subset \mathbf{R}$ is a Borel set and $f \colon B \to \mathbf{R}$ is an increasing function. Prove that f(B) is a Borel set.

25 Suppose $B \subset \mathbf{R}$ and $f : B \to \mathbf{R}$ is an increasing function. Prove that there exists a sequence f_1, f_2, \ldots of strictly increasing functions from B to \mathbf{R} such that

$$f(x) = \lim_{k \to \infty} f_k(x)$$

for every $x \in B$.

- **26** Suppose $B \subset \mathbf{R}$ and $f \colon B \to \mathbf{R}$ is a bounded increasing function. Prove that there exists an increasing function $g \colon \mathbf{R} \to \mathbf{R}$ such that g(x) = f(x) for all $x \in B$.
- 27 Prove or give a counterexample: If (X, S) is a measurable space and

$$f: X \to [-\infty, \infty]$$

is a function such that $f^{-1}((a,\infty)) \in \mathcal{S}$ for every $a \in \mathbb{R}$, then f is an \mathcal{S} -measurable function.

28 Suppose $f: B \to \mathbf{R}$ is a Borel measurable function. Define $g: \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in B, \\ 0 & \text{if } x \in \mathbf{R} \setminus B. \end{cases}$$

Prove that *g* is a Borel measurable function.

29 Give an example of a measurable space (X, S) and a family $\{f_t\}_{t \in \mathbb{R}}$ such that each f_t is an S-measurable function from X to [0,1], but the function $f \colon X \to [0,1]$ defined by

$$f(x) = \sup\{f_t(x) : t \in \mathbf{R}\}\$$

is not S-measurable.

[Compare this exercise to 2.53, where the index set is \mathbf{Z}^+ rather than \mathbf{R} .]

30 Show that

$$\lim_{j \to \infty} \left(\lim_{k \to \infty} \left(\cos(j!\pi x) \right)^{2k} \right) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

for every $x \in \mathbf{R}$.

[This example is due to Henri Lebesgue.]

2C Measures and Their Properties

Definition and Examples of Measures

The original motivation for the next definition came from trying to extend the notion of the length of an interval. However, the definition below allows us to discuss size in many more contexts. For example, we will see later that the area of a set in the plane or the volume of a set in higher dimensions fits into this structure. The word measure allows us to use a single word instead of repeating theorems for length, area, and volume.

2.54 **Definition** measure

Suppose X is a set and S is a σ -algebra on X. A *measure* on (X, S) is a function $\mu \colon S \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \sum_{k=1}^{\infty} \mu(E_k)$$

for every disjoint sequence E_1, E_2, \ldots of sets in S.

The countable additivity that forms the key part of the definition above allows us to prove good limit theorems. Note that countable additivity implies finite additivity: if μ is a measure on (X, S) and E_1, \ldots, E_n are disjoint sets in S, then

$$\mu(E_1 \cup \cdots \cup E_n) = \mu(E_1) + \cdots + \mu(E_n),$$

as follows from applying the equation $\mu(\emptyset) = 0$ and countable additivity to the disjoint sequence $E_1, \ldots, E_n, \emptyset, \emptyset, \ldots$ of sets in S.

In the mathematical literature, sometimes a measure on (X, S) is just called a measure on X if the σ -algebra S is clear from the context.

The concept of a measure, as defined here, is sometimes called a positive measure (although the phrase nonnegative measure would be more accurate).

2.55 Example measures

- If X is a set, then *counting measure* is the measure μ defined on the σ -algebra of all subsets of X by setting $\mu(E) = n$ if E is a finite set containing exactly n elements and $\mu(E) = \infty$ if E is not a finite set.
- Suppose *X* is a set, *S* is a σ -algebra on *X*, and $c \in X$. Define the *Dirac* measure δ_c on (X, S) by

$$\delta_c(E) = \begin{cases} 1 & \text{if } c \in E, \\ 0 & \text{if } c \notin E. \end{cases}$$

This measure is named in honor of mathematician and physicist Paul Dirac (1902–1984), who won the Nobel Prize for Physics in 1933 for his work combining relativity and quantum mechanics at the atomic level.

• Suppose X is a set, S is a σ -algebra on X, and $w: X \to [0, \infty]$ is a function. Define a measure μ on (X, S) by

$$\mu(E) = \sum_{x \in E} w(x)$$

for $E \in \mathcal{S}$. [Here the sum is defined as the supremum of all finite subsums $\sum_{x \in D} w(x)$ as D ranges over all finite subsets of E.]

• Suppose X is a set and S is the σ -algebra on X consisting of all subsets of X that are either countable or have a countable complement in X. Define a measure μ on (X, S) by

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable,} \\ 3 & \text{if } E \text{ is uncountable.} \end{cases}$$

- Suppose S is the σ -algebra on \mathbf{R} consisting of all subsets of \mathbf{R} . Then the function that takes a set $E \subset \mathbf{R}$ to |E| (the outer measure of E) is not a measure because it is not finitely additive (see 2.18).
- Suppose \mathcal{B} is the σ -algebra on \mathbf{R} consisting of all Borel subsets of \mathbf{R} . We will see in the next section that outer measure is a measure on $(\mathbf{R}, \mathcal{B})$.

The following terminology is frequently useful.

2.56 **Definition** *measure space*

A *measure space* is an ordered triple (X, \mathcal{S}, μ) , where X is a set, \mathcal{S} is a σ -algebra on X, and μ is a measure on (X, \mathcal{S}) .

Properties of Measures

The hypothesis that $\mu(D) < \infty$ is needed in part (b) of the next result to avoid undefined expressions of the form $\infty - \infty$.

2.57 measure preserves order; measure of a set difference

Suppose (X, S, μ) is a measure space and $D, E \in S$ are such that $D \subset E$. Then

- (a) $\mu(D) \leq \mu(E)$;
- (b) $\mu(E \setminus D) = \mu(E) \mu(D)$ provided that $\mu(D) < \infty$.

Proof Because $E = D \cup (E \setminus D)$ and this is a disjoint union, we have

$$\mu(E) = \mu(D) + \mu(E \setminus D) \ge \mu(D),$$

which proves (a). If $\mu(D) < \infty$, then subtracting $\mu(D)$ from both sides of the equation above proves (b).

The countable additivity property of measures applies to disjoint countable unions. The following countable subadditivity property applies to countable unions that may not be disjoint unions.

2.58 countable subadditivity

Suppose (X, \mathcal{S}, μ) is a measure space and $E_1, E_2, \ldots \in \mathcal{S}$. Then

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) \le \sum_{k=1}^{\infty} \mu(E_k).$$

Proof Let $D_1 = \emptyset$ and $D_k = E_1 \cup \cdots \cup E_{k-1}$ for $k \ge 2$. Then

$$E_1 \setminus D_1, E_2 \setminus D_2, E_3 \setminus D_3, \dots$$

is a disjoint sequence of subsets of X whose union equals $\bigcup_{k=1}^{\infty} E_k$. Thus

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \mu\Big(\bigcup_{k=1}^{\infty} (E_k \setminus D_k)\Big)$$
$$= \sum_{k=1}^{\infty} \mu(E_k \setminus D_k)$$
$$\leq \sum_{k=1}^{\infty} \mu(E_k),$$

where the second line above follows from the countable additivity of μ and the last line above follows from 2.57(a).

Note that countable subadditivity implies finite subadditivity: if μ is a measure on (X, S) and E_1, \ldots, E_n are sets in S, then

$$\mu(E_1 \cup \cdots \cup E_n) \leq \mu(E_1) + \cdots + \mu(E_n),$$

as follows from applying the equation $\mu(\emptyset) = 0$ and countable subadditivity to the sequence $E_1, \ldots, E_n, \emptyset, \emptyset, \ldots$ of sets in S.

The next result shows that measures behave well with increasing unions.

2.59 measure of an increasing union

Suppose (X, \mathcal{S}, μ) is a measure space and $E_1 \subset E_2 \subset \cdots$ is an increasing sequence of sets in \mathcal{S} . Then

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \lim_{k \to \infty} \mu(E_k).$$

Proof If $\mu(E_k) = \infty$ for some $k \in \mathbf{Z}^+$, then the equation above holds because both sides equal ∞ . Hence we can consider only the case where $\mu(E_k) < \infty$ for all $k \in \mathbf{Z}^+$.

For convenience, let $E_0 = \emptyset$. Then

$$\bigcup_{k=1}^{\infty} E_k = \bigcup_{j=1}^{\infty} (E_j \setminus E_{j-1}),$$

where the union on the right side is a disjoint union. Thus

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{j=1}^{\infty} \mu(E_j \setminus E_{j-1})$$

$$= \lim_{k \to \infty} \sum_{j=1}^{k} \mu(E_j \setminus E_{j-1})$$

$$= \lim_{k \to \infty} \sum_{j=1}^{k} \left(\mu(E_j) - \mu(E_{j-1})\right)$$

$$= \lim_{k \to \infty} \mu(E_k),$$



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as desired.

Measures also behave well with respect to decreasing intersections (but see Exercise 10, which shows that the hypothesis $\mu(E_1) < \infty$ below cannot be deleted).

2.60 measure of a decreasing intersection

Suppose (X, \mathcal{S}, μ) is a measure space and $E_1 \supset E_2 \supset \cdots$ is a decreasing sequence of sets in \mathcal{S} , with $\mu(E_1) < \infty$. Then

$$\mu\Big(\bigcap_{k=1}^{\infty} E_k\Big) = \lim_{k \to \infty} \mu(E_k).$$

Proof One of De Morgan's Laws tells us that

$$E_1 \setminus \bigcap_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} (E_1 \setminus E_k).$$

Now $E_1 \setminus E_1 \subset E_1 \setminus E_2 \subset E_1 \setminus E_3 \subset \cdots$ is an increasing sequence of sets in S. Thus 2.59, applied to the equation above, implies that

$$\mu\Big(E_1\setminus\bigcap_{k=1}^{\infty}E_k\Big)=\lim_{k\to\infty}\mu(E_1\setminus E_k).$$

Use 2.57(b) to rewrite the equation above as

$$\mu(E_1) - \mu\Big(\bigcap_{k=1}^{\infty} E_k\Big) = \mu(E_1) - \lim_{k \to \infty} \mu(E_k),$$

which implies our desired result.

The next result is intuitively plausible—we expect that the measure of the union of two sets equals the measure of the first set plus the measure of the second set minus the measure of the set that has been counted twice.

2.61 measure of a union

Suppose (X, \mathcal{S}, μ) is a measure space and $D, E \in \mathcal{S}$, with $\mu(D \cap E) < \infty$. Then

$$\mu(D \cup E) = \mu(D) + \mu(E) - \mu(D \cap E).$$

Proof We have

$$D \cup E = (D \setminus (D \cap E)) \cup (E \setminus (D \cap E)) \cup (D \cap E).$$

The right side of the equation above is a disjoint union. Thus

$$\mu(D \cup E) = \mu(D \setminus (D \cap E)) + \mu(E \setminus (D \cap E)) + \mu(D \cap E)$$

= $(\mu(D) - \mu(D \cap E)) + (\mu(E) - \mu(D \cap E)) + \mu(D \cap E)$
= $\mu(D) + \mu(E) - \mu(D \cap E)$,

as desired.

EXERCISES 2C

1 Explain why there does not exist a measure space (X, \mathcal{S}, μ) with the property that $\{\mu(E) : E \in \mathcal{S}\} = [0, 1)$.

Let 2^{Z^+} denote the σ -algebra on Z^+ consisting of all subsets of Z^+ .

2 Suppose μ is a measure on $(\mathbf{Z}^+, 2^{\mathbf{Z}^+})$. Prove that there is a sequence w_1, w_2, \ldots in $[0, \infty]$ such that

$$\mu(E) = \sum_{k \in F} w_k$$

for every set $E \subset \mathbf{Z}^+$.

3 Give an example of a measure μ on $(\mathbf{Z}^+, 2^{\mathbf{Z}^+})$ such that

$$\{\mu(E) : E \subset \mathbf{Z}^+\} = [0,1].$$

4 Give an example of a measure space (X, \mathcal{S}, μ) such that

$$\{\mu(E) : E \in \mathcal{S}\} = \{\infty\} \cup \bigcup_{k=0}^{\infty} [3k, 3k+1].$$

- 5 Suppose (X, \mathcal{S}, μ) is a measure space such that $\mu(X) < \infty$. Prove that if \mathcal{A} is a set of disjoint sets in \mathcal{S} such that $\mu(A) > 0$ for every $A \in \mathcal{A}$, then \mathcal{A} is a countable set.
- **6** Find all *c* ∈ $[3, \infty)$ such that there exists a measure space (X, \mathcal{S}, μ) with

$$\{\mu(E): E \in \mathcal{S}\} = [0,1] \cup [3,c].$$

7 Give an example of a measure space (X, \mathcal{S}, μ) such that

$$\{\mu(E): E \in \mathcal{S}\} = [0,1] \cup [3,\infty].$$

- **8** Give an example of a set X, a σ -algebra $\mathcal S$ of subsets of X, a set $\mathcal A$ of subsets of X such that the smallest σ -algebra on X containing $\mathcal A$ is $\mathcal S$, and two measures μ and ν on $(X,\mathcal S)$ such that $\mu(A)=\nu(A)$ for all $A\in\mathcal A$ and $\mu(X)=\nu(X)<\infty$, but $\mu\neq\nu$.
- 9 Suppose μ and ν are measures on a measurable space (X, S). Prove that $\mu + \nu$ is a measure on (X, S). [Here $\mu + \nu$ is the usual sum of two functions: if $E \in S$, then $(\mu + \nu)(E) = \mu(E) + \nu(E)$.]
- 10 Give an example of a measure space (X, S, μ) and a decreasing sequence $E_1 \supset E_2 \supset \cdots$ of sets in S such that

$$\mu\Big(\bigcap_{k=1}^{\infty} E_k\Big) \neq \lim_{k\to\infty} \mu(E_k).$$

11 Suppose (X, \mathcal{S}, μ) is a measure space and $C, D, E \in \mathcal{S}$ are such that

$$\mu(C \cap D) < \infty, \mu(C \cap E) < \infty$$
, and $\mu(D \cap E) < \infty$.

Find and prove a formula for $\mu(C \cup D \cup E)$ in terms of $\mu(C)$, $\mu(D)$, $\mu(E)$, $\mu(C \cap D)$, $\mu(C \cap E)$, $\mu(D \cap E)$, and $\mu(C \cap D \cap E)$.

Suppose X is a set and S is the σ -algebra of all subsets E of X such that E is countable or $X \setminus E$ is countable. Give a complete description of the set of all measures on (X, S).

2D Lebesgue Measure

Additivity of Outer Measure on Borel Sets

Recall that there exist disjoint sets $A, B \in \mathbf{R}$ such that $|A \cup B| \neq |A| + |B|$ (see 2.18). Thus outer measure, despite its name, is not a measure on the σ -algebra of all subsets of \mathbf{R} .

Our main goal in this section is to prove that outer measure, when restricted to the Borel subsets of \mathbf{R} , is a measure. Throughout this section, be careful about trying to simplify proofs by applying properties of measures to outer measure, even if those properties seem intuitively plausible. For example, there are subsets $A \subset B \subset \mathbf{R}$ with $|A| < \infty$ but $|B \setminus A| \neq |B| - |A|$ [compare to 2.57(b)].

The next result is our first step toward the goal of proving that outer measure restricted to the Borel sets is a measure.

2.62 additivity of outer measure if one of the sets is open

Suppose A and G are disjoint subsets of \mathbf{R} and G is open. Then

$$|A \cup G| = |A| + |G|.$$

Proof We can assume that $|G| < \infty$ because otherwise both $|A \cup G|$ and |A| + |G| equal ∞ .

Subadditivity (see 2.8) implies that $|A \cup G| \le |A| + |G|$. Thus we need to prove the inequality only in the other direction.

First consider the case where G = (a, b) for some $a, b \in \mathbf{R}$ with a < b. We can assume that $a, b \notin A$ (because changing a set by at most two points does not change its outer measure). Let I_1, I_2, \ldots be a sequence of open intervals whose union contains $A \cup G$. For each $n \in \mathbf{Z}^+$, let

$$J_n = I_n \cap (-\infty, a), \quad K_n = I_n \cap (a, b), \quad L_n = I_n \cap (b, \infty).$$

Then

$$\ell(I_n) = \ell(J_n) + \ell(K_n) + \ell(L_n).$$

Now $J_1, L_1, J_2, L_2, ...$ is a sequence of open intervals whose union contains A and $K_1, K_2, ...$ is a sequence of open intervals whose union contains G. Thus

$$\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \left(\ell(J_n) + \ell(L_n) \right) + \sum_{n=1}^{\infty} \ell(K_n)$$
$$\geq |A| + |G|.$$

The inequality above implies that $|A \cup G| \ge |A| + |G|$, completing the proof that $|A \cup G| = |A| + |G|$ in this special case.

Using induction on m, we can now conclude that if $m \in \mathbb{Z}^+$ and G is a union of m disjoint open intervals that are all disjoint from A, then $|A \cup G| = |A| + |G|$.

Now suppose G is an arbitrary open subset of \mathbf{R} that is disjoint from A. Then $G = \bigcup_{n=1}^{\infty} I_n$ for some sequence of disjoint open intervals I_1, I_2, \ldots , each of which is disjoint from A. Now for each $m \in \mathbf{Z}^+$ we have

$$|A \cup G| \ge |A \cup (\bigcup_{n=1}^{m} I_n)|$$
$$= |A| + \sum_{n=1}^{m} \ell(I_n).$$

Thus

$$|A \cup G| \ge |A| + \sum_{n=1}^{\infty} \ell(I_n)$$

$$\ge |A| + |G|,$$

completing the proof that $|A \cup G| = |A| + |G|$.

The next result shows that the outer measure of the disjoint union of two sets is what we expect if at least one of the two sets is closed.

2.63 additivity of outer measure if one of the sets is closed

Suppose A and F are disjoint subsets of **R** and F is closed. Then

$$|A \cup F| = |A| + |F|.$$

Proof Suppose I_1, I_2, \ldots is a sequence of open intervals whose union contains $A \cup F$. Let $G = \bigcup_{k=1}^{\infty} I_k$. Thus G is an open set with $A \cup F \subset G$. Hence $A \subset G \setminus F$, which implies that

$$2.64 |A| < |G \setminus F|.$$

Because $G \setminus F = G \cap (\mathbf{R} \setminus F)$, we know that $G \setminus F$ is an open set. Hence we can apply 2.62 to the disjoint union $G = F \cup (G \setminus F)$, getting

$$|G| = |F| + |G \setminus F|.$$

Adding |F| to both sides of 2.64 and then using the equation above gives

$$|A| + |F| \le |G|$$

$$\le \sum_{k=1}^{\infty} \ell(I_k).$$

Thus
$$|A| + |F| \le |A \cup F|$$
, which implies that $|A| + |F| = |A \cup F|$.

Recall that the collection of Borel sets is the smallest σ -algebra on **R** that contains all open subsets of **R**. The next result provides an extremely useful tool for approximating a Borel set by a closed set.

2.65 approximation of Borel sets from below by closed sets

Suppose $B \subset \mathbf{R}$ is a Borel set. Then for every $\varepsilon > 0$, there exists a closed set $F \subset B$ such that $|B \setminus F| < \varepsilon$.

Proof Let

$$\mathcal{L} = \{ D \subset \mathbf{R} : \text{for every } \varepsilon > 0, \text{ there exists a closed set } F \subset D \text{ such that } |D \setminus F| < \varepsilon \}.$$

The strategy of the proof is to show that \mathcal{L} is a σ -algebra. Then because \mathcal{L} contains every closed subset of \mathbf{R} (if $D \subset \mathbf{R}$ is closed, take F = D in the definition of \mathcal{L}), by taking complements we can conclude that \mathcal{L} contains every open subset of \mathbf{R} and thus every Borel subset of \mathbf{R} .

To get started with proving that \mathcal{L} is a σ -algebra, we want to prove that \mathcal{L} is closed under countable intersections. Thus suppose D_1, D_2, \ldots is a sequence in \mathcal{L} . Let $\varepsilon > 0$. For each $k \in \mathbf{Z}^+$, there exists a closed set F_k such that

$$F_k \subset D_k$$
 and $|D_k \setminus F_k| < \frac{\varepsilon}{2^k}$.

Thus $\bigcap_{k=1}^{\infty} F_k$ is a closed set and

$$\bigcap_{k=1}^{\infty} F_k \subset \bigcap_{k=1}^{\infty} D_k \quad \text{and} \quad \left(\bigcap_{k=1}^{\infty} D_k\right) \setminus \left(\bigcap_{k=1}^{\infty} F_k\right) \subset \bigcup_{k=1}^{\infty} (D_k \setminus F_k).$$

The last set inclusion and the countable subadditivity of outer measure (see 2.8) imply that

$$\left|\left(\bigcap_{k=1}^{\infty}D_{k}\right)\setminus\left(\bigcap_{k=1}^{\infty}F_{k}\right)\right|<\varepsilon.$$

Thus $\bigcap_{k=1}^{\infty} D_k \in \mathcal{L}$, proving that \mathcal{L} is closed under countable intersections.

Now we want to prove that \mathcal{L} is closed under complementation. Suppose $D \in \mathcal{L}$ and $\varepsilon > 0$. We want to show that there is a closed subset of $\mathbf{R} \setminus D$ whose set difference with $\mathbf{R} \setminus D$ has outer measure less than ε , which will allow us to conclude that $\mathbf{R} \setminus D \in \mathcal{L}$.

First we consider the case where $|D| < \infty$. Let $F \subset D$ be a closed set such that $|D \setminus F| < \frac{\varepsilon}{2}$. The definition of outer measure implies that there exists an open set G such that $D \subset G$ and $|G| < |D| + \frac{\varepsilon}{2}$. Now $\mathbf{R} \setminus G$ is a closed set and $\mathbf{R} \setminus G \subset \mathbf{R} \setminus D$. Also, we have

$$(\mathbf{R} \setminus D) \setminus (\mathbf{R} \setminus G) = G \setminus D$$
$$\subset G \setminus F.$$

Thus

$$|(\mathbf{R} \setminus D) \setminus (\mathbf{R} \setminus G)| \le |G \setminus F|$$

$$= |G| - |F|$$

$$= (|G| - |D|) + (|D| - |F|)$$

$$< \frac{\varepsilon}{2} + |D \setminus F|$$

$$< \varepsilon,$$

where the equality in the second line above comes from applying 2.63 to the disjoint union $G = (G \setminus F) \cup F$, and the fourth line above uses subadditivity applied to the union $D = (D \setminus F) \cup F$. The last inequality above shows that $\mathbf{R} \setminus D \in \mathcal{L}$, as desired.

Now, still assuming that $D \in \mathcal{L}$ and $\varepsilon > 0$, we consider the case where $|D| = \infty$. For $k \in \mathbb{Z}^+$, let $D_k = D \cap [-k, k]$. Because $D_k \in \mathcal{L}$ and $|D_k| < \infty$, the previous case implies that $\mathbb{R} \setminus D_k \in \mathcal{L}$. Clearly $D = \bigcup_{k=1}^{\infty} D_k$. Thus

$$\mathbf{R}\setminus D=\bigcap_{k=1}^{\infty}(\mathbf{R}\setminus D_k).$$

Because \mathcal{L} is closed under countable intersections, the equation above implies that $\mathbf{R} \setminus D \in \mathcal{L}$, which completes the proof that \mathcal{L} is a σ -algebra.

Now we can prove that the outer measure of the disjoint union of two sets is what we expect if at least one of the two sets is a Borel set.

2.66 additivity of outer measure if one of the sets is a Borel set

Suppose A and B are disjoint subsets of **R** and B is a Borel set. Then

$$|A \cup B| = |A| + |B|.$$

Proof Let $\varepsilon > 0$. Let F be a closed set such that $F \subset B$ and $|B \setminus F| < \varepsilon$ (see 2.65). Thus

$$|A \cup B| \ge |A \cup F|$$

$$= |A| + |F|$$

$$= |A| + |B| - |B \setminus F|$$

$$> |A| + |B| - \varepsilon,$$

where the second and third lines above follow from 2.63 [use $B = (B \setminus F) \cup F$ for the third line].

Because the inequality above holds for all $\varepsilon > 0$, we have $|A \cup B| \ge |A| + |B|$, which implies that $|A \cup B| = |A| + |B|$.

You have probably long suspected that not every subset of ${\bf R}$ is a Borel set. Now we can prove this suspicion.

2.67 existence of a subset of R that is not a Borel set

There exists a set $B \subset \mathbf{R}$ such that $|B| < \infty$ and B is not a Borel set.

Proof In the proof of 2.18, we showed that there exist disjoint sets $A, B \subset \mathbf{R}$ such that $|A \cup B| \neq |A| + |B|$. For any such sets, we must have $|B| < \infty$ because otherwise both $|A \cup B|$ and |A| + |B| equal ∞ (as follows from the inequality $|B| \leq |A \cup B|$). Now 2.66 implies that B is not a Borel set.

The tools we have constructed now allow us to prove that outer measure, when restricted to the Borel sets, is a measure.

2.68 outer measure is a measure on Borel sets

Outer measure is a measure on $(\mathbf{R}, \mathcal{B})$, where \mathcal{B} is the σ -algebra of Borel subsets of \mathbf{R} .

Proof Suppose $B_1, B_2, ...$ is a disjoint sequence of Borel subsets of **R**. Then for each $n \in \mathbb{Z}^+$ we have ∞ n

$$\left| \bigcup_{k=1}^{\infty} B_k \right| \ge \left| \bigcup_{k=1}^{n} B_k \right|$$
$$= \sum_{k=1}^{n} |B_k|,$$

where the first line above follows from 2.5 and the last line follows from 2.66 (and induction on n). Taking the limit as $n \to \infty$, we have $\left|\bigcup_{k=1}^{\infty} B_k\right| \ge \sum_{k=1}^{\infty} |B_k|$. The inequality in the other directions follows from countable subadditivity of outer measure (2.8). Hence

$$\left|\bigcup_{k=1}^{\infty} B_k\right| = \sum_{k=1}^{\infty} |B_k|.$$

Thus outer measure is a measure on the σ -algebra of Borel subsets of **R**.

The result above implies that the next definition makes sense.

2.69 **Definition** Lebesgue measure

Lebesgue measure is the measure on $(\mathbf{R}, \mathcal{B})$, where \mathcal{B} is the σ -algebra of Borel subsets of \mathbf{R} , that assigns to each Borel set its outer measure.

In other words, the Lebesgue measure of a set is the same as its outer measure, except that the term *Lebesgue measure* should not be applied to arbitrary sets but only to Borel sets (and also to what are called Lebesgue measurable sets, as we will soon see). Unlike outer measure, Lebesgue measure is actually a measure, as shown in 2.68. Lebesgue measure is named in honor of its inventor, Henri Lebesgue.



The cathedral in Beauvais, the French city where Henri Lebesgue (1875–1941) was born. Much of what we call Lebesgue measure and Lebesgue integration was developed by Lebesgue in his 1902 PhD thesis. Émile Borel was Lebesgue's PhD thesis advisor. CC-BY-SA James Mitchell

Lebesgue Measurable Sets

We have accomplished the major goal of this section, which was to show that outer measure restricted to Borel sets is a measure. As we will see in this subsection, outer measure is actually a measure on a somewhat larger class of sets called the Lebesgue measurable sets.

The mathematics literature contains many different definitions of a Lebesgue measurable set. These definitions are all equivalent—the definition of a Lebesgue measurable set in one approach becomes a theorem in another approach. The approach chosen here has the advantage of emphasizing that a Lebesgue measurable set differs from a Borel set by a set with outer measure 0. The attitude here is that sets with outer measure 0 should be considered small sets that do not matter much.

2.70 Definition Lebesgue measurable set

A set $A \subset \mathbf{R}$ is called *Lebesgue measurable* if there exists a Borel set $B \subset A$ such that $|A \setminus B| = 0$.

Every Borel set is Lebesgue measurable because if $A \subset \mathbf{R}$ is a Borel set, then we can take B = A in the definition above.

The result below gives several equivalent conditions for being Lebesgue measurable. The equivalence of (a) and (d) is just our definition and thus is not discussed in the proof.

Although there exist Lebesgue measurable sets that are not Borel sets, you are unlikely to encounter one. The most important application of the result below is that if $A \subset \mathbf{R}$ is a Borel set, then A satisfies conditions (b), (c), (e), and (f). Condition (c) implies that every Borel set is almost a countable union of closed sets, and condition (f) implies that every Borel set is almost a countable intersection of open sets.

2.71 equivalences for being a Lebesgue measurable set

Suppose $A \subset \mathbf{R}$. Then the following are equivalent:

- (a) A is Lebesgue measurable.
- (b) For each $\varepsilon > 0$, there exists a closed set $F \subset A$ with $|A \setminus F| < \varepsilon$.
- (c) There exist closed sets F_1, F_2, \ldots contained in A such that $\left| A \setminus \bigcup_{k=1}^{\infty} F_k \right| = 0$.
- (d) There exists a Borel set $B \subset A$ such that $|A \setminus B| = 0$.
- (e) For each $\varepsilon > 0$, there exists an open set $G \supset A$ such that $|G \setminus A| < \varepsilon$.
- (f) There exist open sets G_1, G_2, \ldots containing A such that $\left| \left(\bigcap_{k=1}^{\infty} G_k \right) \setminus A \right| = 0$.
- (g) There exists a Borel set $B \supset A$ such that $|B \setminus A| = 0$.

Proof Let \mathcal{L} denote the collection of sets $A \subset \mathbf{R}$ that satisfy (b). We have already proved that every Borel set is in \mathcal{L} (see 2.65). As a key part of that proof, which we will freely use in this proof, we showed that \mathcal{L} is a σ -algebra on \mathbf{R} (see the proof of 2.65). In addition to containing the Borel sets, \mathcal{L} contains every set with outer measure 0 [because if |A| = 0, we can take $F = \emptyset$ in (b)].

(b) \Longrightarrow **(c):** Suppose (b) holds. Thus for each $n \in \mathbb{Z}^+$, there exists a closed set $F_n \subset A$ such that $|A \setminus F_n| < \frac{1}{n}$. Now

$$A\setminus\bigcup_{k=1}^{\infty}F_k\subset A\setminus F_n$$

for each $n \in \mathbf{Z}^+$. Thus $|A \setminus \bigcup_{k=1}^{\infty} F_k| \leq |A \setminus F_n| \leq \frac{1}{n}$ for each $n \in \mathbf{Z}^+$. Hence $|A \setminus \bigcup_{k=1}^{\infty} F_k| = 0$, completing the proof that (b) implies (c).

- (c) \implies (d): Because every countable union of closed sets is a Borel set, we see that (c) implies (d).
- (d) \Longrightarrow (b): Suppose (d) holds. Thus there exists a Borel set $B \subset A$ such that $|A \setminus B| = 0$. Now

$$A = B \cup (A \setminus B)$$
.

We know that $B \in \mathcal{L}$ (because B is a Borel set) and $A \setminus B \in \mathcal{L}$ (because $A \setminus B$ has outer measure 0). Because \mathcal{L} is a σ -algebra, the displayed equation above implies that $A \in \mathcal{L}$. In other words, (b) holds, completing the proof that (d) implies (b).

At this stage of the proof, we now know that (b) \iff (c) \iff (d).

(b) \Longrightarrow (e): Suppose (b) holds. Thus $A \in \mathcal{L}$. Let $\varepsilon > 0$. Then because $\mathbf{R} \setminus A \in \mathcal{L}$ (which holds because \mathcal{L} is closed under complementation), there exists a closed set $F \subset \mathbf{R} \setminus A$ such that

$$|(\mathbf{R}\setminus A)\setminus F|<\varepsilon.$$

Now $\mathbf{R} \setminus F$ is an open set with $\mathbf{R} \setminus F \supset A$. Because $(\mathbf{R} \setminus F) \setminus A = (\mathbf{R} \setminus A) \setminus F$, the inequality above implies that $|(\mathbf{R} \setminus F) \setminus A| < \varepsilon$. Thus (e) holds, completing the proof that (b) implies (e).

(e) \Longrightarrow (f): Suppose (e) holds. Thus for each $n \in \mathbb{Z}^+$, there exists an open set $G_n \supset A$ such that $|G_n \setminus A| < \frac{1}{n}$. Now

$$\left(\bigcap_{k=1}^{\infty}G_{k}\right)\setminus A\subset G_{n}\setminus A$$

for each $n \in \mathbf{Z}^+$. Thus $|(\bigcap_{k=1}^{\infty} G_k) \setminus A| \leq |G_n \setminus A| \leq \frac{1}{n}$ for each $n \in \mathbf{Z}^+$. Hence $|(\bigcap_{k=1}^{\infty} G_k) \setminus A| = 0$, completing the proof that (e) implies (f).

- (f) \Longrightarrow (g): Because every countable intersection of open sets is a Borel set, we see that (f) implies (g).
- (g) \Longrightarrow (b): Suppose (g) holds. Thus there exists a Borel set $B \supset A$ such that $|B \setminus A| = 0$. Now

$$A = B \cap (\mathbf{R} \setminus (B \setminus A)).$$

We know that $B \in \mathcal{L}$ (because B is a Borel set) and $\mathbf{R} \setminus (B \setminus A) \in \mathcal{L}$ (because this set is the complement of a set with outer measure 0). Because \mathcal{L} is a σ -algebra, the displayed equation above implies that $A \in \mathcal{L}$. In other words, (b) holds, completing the proof that (g) implies (b).

Our chain of implications now shows that (b) through (g) are all equivalent.

In addition to the equivalences in the previous result, see Exercise 13 in this section for another condition that is equivalent to being Lebesgue measurable. Also see Exercise 6, which shows that a set with finite outer measure is Lebesgue mea-

In practice, the most useful part of Exercise 6 is the result that every Borel set with finite measure is almost a finite disjoint union of bounded open intervals.

surable if and only if it is almost a finite disjoint union of bounded open intervals.

Now we can show that outer measure is a measure on the Lebesgue measurable sets.

2.72 outer measure is a measure on Lebesgue measurable sets

- (a) The set \mathcal{L} of Lebesgue measurable subsets of **R** is a σ -algebra on **R**.
- (b) Outer measure is a measure on $(\mathbf{R}, \mathcal{L})$.

Proof Because (a) and (b) are equivalent in 2.71, the set \mathcal{L} of Lebesgue measurable subsets of **R** is the collection of sets satisfying (b) in 2.71. As noted in the first paragraph of the proof of 2.71, this set is a σ -algebra on **R**, proving (a).

To prove the second bullet point, suppose $A_1, A_2, ...$ is a disjoint sequence of Lebesgue measurable sets. By the definition of Lebesgue measurable set (2.70), for each $k \in \mathbb{Z}^+$ there exists a Borel set $B_k \subset A_k$ such that $|A_k \setminus B_k| = 0$. Now

$$\left| \bigcup_{k=1}^{\infty} A_k \right| \ge \left| \bigcup_{k=1}^{\infty} B_k \right|$$

$$= \sum_{k=1}^{\infty} |B_k|$$

$$= \sum_{k=1}^{\infty} |A_k|,$$

where the second line above holds because $B_1, B_2, ...$ is a disjoint sequence of Borel sets and outer measure is a measure on the Borel sets (see 2.68); the last line above holds because $B_k \subset A_k$ and by subadditivity of outer measure (see 2.8) we have $|A_k| = |B_k \cup (A_k \setminus B_k)| \le |B_k| + |A_k \setminus B_k| = |B_k|$.

The inequality above, combined with countable subadditivity of outer measure (see 2.8), implies that $\left|\bigcup_{k=1}^{\infty} A_k\right| = \sum_{k=1}^{\infty} |A_k|$, completing the proof of (b).

If A is a set with outer measure 0, then A is Lebesgue measurable (because we can take $B=\emptyset$ in the definition 2.70). Our definition of the Lebesgue measurable sets thus implies that the set of Lebesgue measurable sets is the smallest σ -algebra on \mathbf{R} containing the Borel sets and the sets with outer measure 0. Thus the set of Lebesgue measurable sets is also the smallest σ -algebra on \mathbf{R} containing the open sets and the sets with outer measure 0.

Because outer measure is not even finitely additive (see 2.18), 2.72(b) implies that there exist subsets of **R** that are not Lebesgue measurable.

We previously defined Lebesgue measure as outer measure restricted to the Borel sets (see 2.69). The term *Lebesgue measure* is sometimes used in mathematical literature with the meaning as we previously defined it and is sometimes used with the following meaning.

2.73 Definition Lebesgue measure

Lebesgue measure is the measure on $(\mathbf{R}, \mathcal{L})$, where \mathcal{L} is the σ -algebra of Lebesgue measurable subsets of \mathbf{R} , that assigns to each Lebesgue measurable set its outer measure.

The two definitions of *Lebesgue measure* disagree only on the domain of the measure—is the σ -algebra the Borel sets or the Lebesgue measurable sets? You may be able to tell which is intended from the context. In this book, the domain is specified unless it is irrelevant.

If you are reading a mathematics paper and the domain for Lebesgue measure is not specified, then it probably does not matter whether you use the Borel sets or the Lebesgue measurable sets (because every Lebesgue measurable set differs from a Borel set by a set with outer measure 0, and when dealing with measures, what happens on a set with measure 0 usually does not matter). Because all sets that arise from the usual operations of analysis are Borel sets, you may want to assume that *Lebesgue measure* means outer measure on the Borel sets, unless what you are reading explicitly states otherwise.

A mathematics paper may also refer to a *measurable* subset of \mathbf{R} , without further explanation. Unless some other σ -algebra is clear from the context, the author probably means the Borel sets or the Lebesgue measurable sets. Again, the choice probably does not matter, but using the Borel sets can be cleaner and simpler.

The emphasis in some textbooks on Lebesgue measurable sets instead of Borel sets probably stems from the historical development of the subject, rather than from any common use of Lebesgue measurable sets that are not Borel sets.

Lebesgue measure on the Lebesgue measurable sets does have one small advantage over Lebesgue measure on the Borel sets: every subset of a set with (outer) measure 0 is Lebesgue measurable but is not necessarily a Borel set. However, any natural process that produces a subset of $\bf R$ will produce a Borel set. Thus this small advantage does not often come up in practice.

Cantor Set and Cantor Function

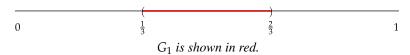
Every countable set has outer measure 0 (see 2.4). A reasonable question arises about whether the converse holds. In other words, is every set with outer measure 0 countable? The Cantor set, which is introduced in this subsection, provides the answer to this question.

The Cantor set also gives counterexamples to other reasonable conjectures. For example, Exercise 17 in this section shows that the sum of two sets with Lebesgue measure 0 can have positive Lebesgue measure.

2.74 Definition Cantor set

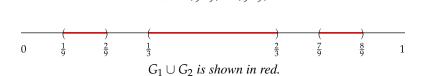
The *Cantor set C* is $[0,1] \setminus (\bigcup_{n=1}^{\infty} G_n)$, where $G_1 = (\frac{1}{3}, \frac{2}{3})$ and G_n for n > 1 is the union of the middle-third open intervals in the intervals of $[0,1] \setminus (\bigcup_{j=1}^{n-1} G_j)$.

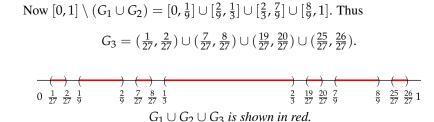
One way to envision the Cantor set C is to start with the interval [0,1] and then consider the process that removes at each step the middle-third open intervals of all intervals left from the previous step. At the first step, we remove $G_1 = (\frac{1}{3}, \frac{2}{3})$.



After that first step, we have $[0,1] \setminus G_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$. Thus we take the middle-third open intervals of $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$. In other words, we have

 $G_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}).$





Base 3 representations provide a useful way to think about the Cantor set. Just as $\frac{1}{10} = 0.1 = 0.09999...$ in the decimal representation, base 3 representations are not unique for fractions whose denominator is a power of 3. For example, $\frac{1}{3} = 0.1_3 = 0.02222..._3$, where the subscript 3 denotes a base 3 representations.

Notice that G_1 is the set of numbers in [0,1] whose base 3 representations have 1 in the first digit after the decimal point (for those numbers that have two base 3 representations, this means both such representations must have 1 in the first digit). Also, $G_1 \cup G_2$ is the set of numbers in [0,1] whose base 3 representations have 1 in the first digit or the second digit after the decimal point. And so on. Hence $\bigcup_{n=1}^{\infty} G_n$ is the set of numbers in [0,1] whose base 3 representations have a 1 somewhere.

Thus we have the following description of the Cantor set. In the following result, the phrase *a base* 3 *representation* indicates that if a number has two base 3 representations, then it is in the Cantor set if and only if at least one of them contains no 1s. For example, both $\frac{1}{3}$ (which equals $0.02222..._3$ and equals 0.1_3) and $\frac{2}{3}$ (which equals 0.2_3 and equals $0.12222..._3$) are in the Cantor set.

2.75 base 3 description of the Cantor set

The Cantor set C is the set of numbers in [0, 1] that have a base 3 representation containing only 0s and 2s.

The two endpoints of each interval in each G_n are in the Cantor set. However, many elements of the Cantor set are not endpoints of any interval in any G_n . For example, Exercise 14 asks you to show that $\frac{1}{4}$ and $\frac{9}{13}$ are in the Cantor set; neither

It is unknown whether or not every number in the Cantor set is either rational or transcendental (meaning not the root of a polynomial with integer coefficients).

of those numbers is an endpoint of any interval in any G_n . An example of an irrational number in the Cantor set is $\sum_{n=1}^{\infty} \frac{2}{3^{n!}}$.

The next result gives some elementary properties of the Cantor set.

2.76 C is closed, has measure 0, and contains no nontrivial intervals

- (a) The Cantor set is a closed subset of **R**.
- (b) The Cantor set has Lebesgue measure 0.
- (c) The Cantor set contains no interval with more than one element.

Proof Each set G_n used in the definition of the Cantor set is a union of open intervals. Thus each G_n is open. Thus $\bigcup_{n=1}^{\infty} G_n$ is open, and hence its complement is closed. The Cantor set equals $[0,1] \cap (\mathbb{R} \setminus \bigcup_{n=1}^{\infty} G_n)$, which is the intersection of two closed sets. Thus the Cantor set is closed, completing the proof of (a).

By induction on n, each G_n is the union of 2^{n-1} disjoint open intervals, each of which has length $\frac{1}{3^n}$. Thus $|G_n| = \frac{2^{n-1}}{3^n}$. The sets G_1, G_2, \ldots are disjoint. Hence

$$\left| \bigcup_{n=1}^{\infty} G_n \right| = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots$$

$$= \frac{1}{3} \left(1 + \frac{2}{3} + \frac{4}{9} + \cdots \right)$$

$$= \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}}$$

$$= 1.$$

Thus the Cantor set, which equals $[0,1] \setminus \bigcup_{n=1}^{\infty} G_n$, has Lebesgue measure 1-1 [by 2.57(b)]. In other words, the Cantor set has Lebesgue measure 0, completing the proof of (b).

A set with Lebesgue measure 0 cannot contain an interval that has more than one element. Thus (b) implies (c).

Now we can define an amazing function.

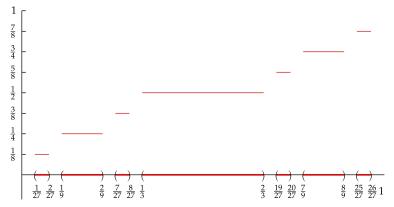
2.77 Definition Cantor function

The Cantor function $\Lambda: [0,1] \to [0,1]$ is defined by converting base 3 representations into base 2 representations as follows:

- If x ∈ C, then Λ(x) is computed from the unique base 3 representation of x containing only 0s and 2s by replacing each 2 by 1 and interpreting the resulting string as a base 2 number.
- If $x \in [0, 1] \setminus C$, then $\Lambda(x)$ is computed from a base 3 representation of x by truncating after the first 1, replacing each 2 before the first 1 by 1, and interpreting the resulting string as a base 2 number.

2.78 Example values of the Cantor function

- $\Lambda(0.0202_3) = 0.0101_2$; in other words, $\Lambda(\frac{20}{81}) = \frac{5}{16}$.
- $\Lambda(0.220121_3) = 0.1101_2$; in other words $\Lambda(\frac{664}{729}) = \frac{13}{16}$.
- Suppose $x \in \left(\frac{1}{3}, \frac{2}{3}\right)$. Then $x \notin C$ because x was removed in the first step of the definition of the Cantor set. Each base 3 representation of x begins with 0.1. Thus we truncate and interpret 0.1 as a base 2 number, getting $\frac{1}{2}$. Hence the Cantor function Λ has the constant value $\frac{1}{2}$ on the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$, as shown on the graph below.
- Suppose $x \in \left(\frac{7}{9}, \frac{8}{9}\right)$. Then $x \notin C$ because x was removed in the second step of the definition of the Cantor set. Each base 3 representation of x begins with 0.21. Thus we truncate, replace the 2 by 1, and interpret 0.11 as a base 2 number, getting $\frac{3}{4}$. Hence the Cantor function Λ has the constant value $\frac{3}{4}$ on the interval $\left(\frac{7}{9}, \frac{8}{9}\right)$, as shown on the graph below.



Graph of the Cantor function on the intervals from first three steps.

As shown in the next result, in some mysterious fashion the Cantor function manages to map [0,1] *onto* [0,1] even though the Cantor function is constant on each open interval in the complement of the Cantor set—see the graph in Example 2.78.

2.79 Cantor function

The Cantor function Λ is a continuous, increasing function from [0,1] onto [0,1]. Furthermore, $\Lambda(C) = [0,1]$.

Proof We begin by showing that $\Lambda(C) = [0,1]$. To do this, suppose $y \in [0,1]$. In the base 2 representation of y, replace each 1 by 2 and interpret the resulting string in base 3, getting a number $x \in [0,1]$. Because x has a base 3 representation consisting only of 0s and 2s, the number x is in the Cantor set C. The definition of the Cantor function shows that $\Lambda(x) = y$. Thus $y \in \Lambda(C)$. Hence $\Lambda(C) = [0,1]$, as desired.

Some careful thinking about the meaning of base 3 and base 2 representations and the definition of the Cantor function shows that Λ is an increasing function. This step is left to the reader.

If $x \in [0,1] \setminus C$, then the Cantor function Λ is constant on an open interval containing x and thus Λ is continuous at x. If $x \in C$, then again some careful thinking about base 3 and base 2 representations shows that Λ is continuous at x.

Alternatively, you can skip the paragraph above and note that an increasing function on [0,1] whose range equals [0,1] is automatically continuous (although you should think about why that holds).

Now we can use the Cantor function to show that the Cantor set is uncountable even though it is a closed set with outer measure 0.

2.80 C is uncountable

The Cantor set is uncountable.

Proof If *C* were countable, then $\Lambda(C)$ would be countable. However, 2.79 shows that $\Lambda(C)$ is uncountable.

As we see in the next result, the Cantor function shows that even a continuous function can map a set with Lebesgue measure 0 to nonmeasurable sets.

2.81 continuous image of a Lebesgue measurable set can be nonmeasurable

There exists a Lebesgue measurable set $A\subset [0,1]$ such that |A|=0 and $\Lambda(A)$ is not a Lebesgue measurable set.

Proof Let E be a subset of [0,1] that is not Lebesgue measurable (the existence of such a set follows from the discussion after 2.72). Let $A = C \cap \Lambda^{-1}(E)$. Then |A| = 0 because $A \subset C$ and |C| = 0 (by 2.76). Thus A is Lebesgue measurable because every subset of \mathbf{R} with Lebesgue measure 0 is Lebesgue measurable.

Because Λ maps C onto [0,1] (see 2.79), we have $\Lambda(A) = E$.

EXERCISES 2D

- 1 (a) Show that the set consisting of those numbers in (0,1) that have a decimal expansion containing one hundred consecutive 4s is a Borel subset of **R**.
 - (b) What is the Lebesgue measure of the set in part (a)?
- **2** Prove that there exists a bounded set $A \subset \mathbf{R}$ such that $|F| \leq |A| 1$ for every closed set $F \subset A$.
- 3 Prove that there exists a set $A \subset \mathbf{R}$ such that $|G \setminus A| = \infty$ for every open set G that contains A.
- 4 The phrase *nontrivial interval* is used to denote an interval of **R** that contains more than one element. Recall that an interval might be open, closed, or neither.
 - (a) Prove that the union of each collection of nontrivial intervals of \mathbf{R} is the union of a countable subset of that collection.
 - (b) Prove that the union of each collection of nontrivial intervals of R is a Borel set.
 - (c) Prove that there exists a collection of closed intervals of R whose union is not a Borel set.
- **5** Prove that if $A \subset \mathbf{R}$ is Lebesgue measurable, then there exists an increasing sequence $F_1 \subset F_2 \subset \cdots$ of closed sets contained in A such that

$$\left|A\setminus\bigcup_{k=1}^{\infty}F_{k}\right|=0.$$

- 6 Suppose *A* ⊂ **R** and |A| < ∞. Prove that *A* is Lebesgue measurable if and only if for every ε > 0 there exists a set *G* that is the union of finitely many disjoint bounded open intervals such that $|A \setminus G| + |G \setminus A| < ε$.
- 7 Prove that if $A \subset \mathbf{R}$ is Lebesgue measurable, then there exists a decreasing sequence $G_1 \supset G_2 \supset \cdots$ of open sets containing A such that

$$\left|\left(\bigcap_{k=1}^{\infty}G_{k}\right)\setminus A\right|=0.$$

- 8 Prove that the collection of Lebesgue measurable subsets of \mathbf{R} is translation invariant. More precisely, prove that if $A \subset \mathbf{R}$ is Lebesgue measurable and $t \in \mathbf{R}$, then t + A is Lebesgue measurable.
- 9 Prove that the collection of Lebesgue measurable subsets of **R** is dilation invariant. More precisely, prove that if $A \subset \mathbf{R}$ is Lebesgue measurable and $t \in \mathbf{R}$, then tA (which is defined to be $\{ta : a \in A\}$) is Lebesgue measurable.
- Prove that if A and B are disjoint subsets of **R** and B is Lebesgue measurable, then $|A \cup B| = |A| + |B|$.

- 11 Prove that if $A \subset \mathbf{R}$ and |A| > 0, then there exists a subset of A that is not Lebesgue measurable.
- Suppose b < c and $A \subset (b,c)$. Prove that A is Lebesgue measurable if and only if $|A| + |(b,c) \setminus A| = c b$.
- 13 Suppose $A \subset \mathbf{R}$. Prove that A is Lebesgue measurable if and only if

$$|(-n,n) \cap A| + |(-n,n) \setminus A| = 2n$$

for every $n \in \mathbf{Z}^+$.

- 14 Show that $\frac{1}{4}$ and $\frac{9}{13}$ are both in the Cantor set.
- 15 Show that $\frac{13}{17}$ is not in the Cantor set.
- 16 List the eight open intervals whose union is G_4 in the definition of the Cantor set (2.74).
- 17 Let C denote the Cantor set. Prove that $\frac{1}{2}C + \frac{1}{2}C = [0, 1]$.
- 18 Prove that every open interval of **R** contains either infinitely many or no elements in the Cantor set.
- 19 Evaluate $\int_0^1 \Lambda$, where Λ is the Cantor function.
- **20** Evaluate each of the following:
 - (a) $\Lambda\left(\frac{9}{13}\right)$;
 - (b) $\Lambda(0.93)$.
- 21 Find each of the following sets:
 - (a) $\Lambda^{-1}(\{\frac{1}{3}\});$
 - (b) $\Lambda^{-1}(\{\frac{5}{16}\}).$
- 22 (a) Suppose x is a rational number in [0,1]. Explain why $\Lambda(x)$ is rational.
 - (b) Suppose $x \in C$ is such that $\Lambda(x)$ is rational. Explain why x is rational.
- 23 Show that there exists a function $f \colon \mathbf{R} \to \mathbf{R}$ such that the image under f of every nonempty open interval is \mathbf{R} .
- **24** For $A \subset \mathbf{R}$, the quantity

$$\sup\{|F|: F \text{ is a closed bounded subset of } \mathbf{R} \text{ and } F \subset A\}$$

is called the *inner measure* of A.

- (a) Show that if A is a Lebesgue measurable subset of \mathbf{R} , then the inner measure of A equals the outer measure of A.
- (b) Show that inner measure is not a measure on the σ -algebra of all subsets of **R**.

2E Convergence of Measurable Functions

Recall that a measurable space is a pair (X, S), where X is a set and S is a σ -algebra on X. We defined a function $f: X \to \mathbf{R}$ to be S-measurable if $f^{-1}(B) \in S$ for every Borel set $B \subset \mathbf{R}$. In Section 2B we proved some results about S-measurable functions; this was before we had introduced the notion of a measure.

In this section, we return to study measurable functions, but now with an emphasis on results that depend upon measures. The highlights of this section are the proofs of Egorov's Theorem and Luzin's Theorem.

Pointwise and Uniform Convergence

We begin this section with some definitions that you probably saw in an earlier course.

2.82 **Definition** pointwise convergence; uniform convergence

Suppose X is a set, $f_1, f_2, ...$ is a sequence of functions from X to **R**, and f is a function from X to **R**.

• The sequence f_1, f_2, \dots converges pointwise on X to f if

$$\lim_{k \to \infty} f_k(x) = f(x)$$

for each $x \in X$.

In other words, f_1, f_2, \ldots converges pointwise on X to f if for each $x \in X$ and every $\varepsilon > 0$, there exists $n \in \mathbb{Z}^+$ such that $|f_k(x) - f(x)| < \varepsilon$ for all integers $k \ge n$.

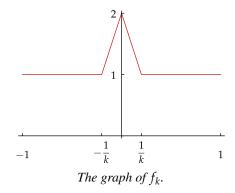
• The sequence f_1, f_2, \ldots converges uniformly on X to f if for every $\varepsilon > 0$, there exists $n \in \mathbb{Z}^+$ such that $|f_k(x) - f(x)| < \varepsilon$ for all integers $k \ge n$ and all $x \in X$.

2.83 Example a sequence converging pointwise but not uniformly

Suppose $f_k \colon [-1,1] \to \mathbf{R}$ is the function whose graph is shown here and $f \colon [-1,1] \to \mathbf{R}$ is the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 2 & \text{if } x = 0. \end{cases}$$

Then $f_1, f_2,...$ converges pointwise on [-1,1] to f but $f_1, f_2,...$ does not converge uniformly on [-1,1] to f, as you should verify.



Like the difference between continuity and uniform continuity, the difference between pointwise convergence and uniform convergence lies in the order of the quantifiers. Take a moment to examine the definitions carefully. If a sequence of functions converges uniformly on some set, then it also converges pointwise on the same set; however, the converse is not true, as shown by Example 2.83.

Example 2.83 also shows that the pointwise limit of continuous functions need not be continuous. However, the next result tells us that the uniform limit of continuous functions is continuous.

2.84 uniform limit of continuous functions is continuous

Suppose $B \subset \mathbf{R}$ and f_1, f_2, \ldots is a sequence of functions from B to \mathbf{R} that converges uniformly on B to a function $f \colon B \to \mathbf{R}$. Suppose $b \in B$ and f_k is continuous at b for each $k \in \mathbf{Z}^+$. Then f is continuous at b.

Proof Suppose $\varepsilon > 0$. Let $n \in \mathbf{Z}^+$ be such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in B$. Because f_n is continuous at b, there exists $\delta > 0$ such that $|f_n(x) - f_n(b)| < \frac{\varepsilon}{3}$ for all $x \in (b - \delta, b + \delta) \cap B$.

Now suppose $x \in (b - \delta, b + \delta) \cap B$. Then

$$|f(x) - f(b)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(b)| + |f_n(b) - f(b)| < \varepsilon.$$

Thus f is continuous at b.

Egorov's Theorem

A sequence of functions that converges pointwise need not converge uniformly. However, the next result says that a pointwise convergent sequence of functions on a measure space with finite total measure

Dmitri Egorov (1869–1931) proved the theorem below in 1911. You may encounter some books that spell his last name as Egoroff.

almost converges uniformly, in the sense that it converges uniformly except on a set that can have arbitrarily small measure.

As an example of the next result, consider Lebesgue measure λ on the interval [-1,1] and the sequence of functions f_1,f_2,\ldots in Example 2.83 that converges pointwise but not uniformly on [-1,1]. Suppose $\varepsilon>0$. Then taking $E=[-1,-\frac{\varepsilon}{4}]\cup[\frac{\varepsilon}{4},1]$, we have $\lambda([-1,1]\setminus E)<\varepsilon$ and f_1,f_2,\ldots converges uniformly on E, as in the conclusion of the next result.

2.85 Egorov's Theorem

Suppose (X, S, μ) is a measure space with $\mu(X) < \infty$. Suppose f_1, f_2, \ldots is a sequence of S-measurable functions from X to \mathbf{R} that converges pointwise on X to a function $f: X \to \mathbf{R}$. Then for every $\varepsilon > 0$, there exists a set $E \in S$ such that $\mu(X \setminus E) < \varepsilon$ and f_1, f_2, \ldots converges uniformly to f on E.

Proof Suppose $\varepsilon > 0$. Temporarily fix $n \in \mathbb{Z}^+$. The definition of pointwise convergence implies that

2.86
$$\bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \{x \in X : |f_k(x) - f(x)| < \frac{1}{n}\} = X.$$

For $m \in \mathbf{Z}^+$, let

$$A_{m,n} = \bigcap_{k=m}^{\infty} \{ x \in X : |f_k(x) - f(x)| < \frac{1}{n} \}.$$

Then clearly $A_{1,n} \subset A_{2,n} \subset \cdots$ is an increasing sequence of sets and 2.86 can be rewritten as

$$\bigcup_{m=1}^{\infty} A_{m,n} = X.$$

The equation above implies (by 2.59) that $\lim_{m\to\infty} \mu(A_{m,n}) = \mu(X)$. Thus there exists $m_n \in \mathbf{Z}^+$ such that

Now let

$$E=\bigcap_{n=1}^{\infty}A_{m_n,n}.$$

Then

$$\mu(X \setminus E) = \mu\left(X \setminus \bigcap_{n=1}^{\infty} A_{m_n,n}\right)$$

$$= \mu\left(\bigcup_{n=1}^{\infty} (X \setminus A_{m_n,n})\right)$$

$$\leq \sum_{n=1}^{\infty} \mu(X \setminus A_{m_n,n})$$

$$\leq \varepsilon.$$

where the last inequality follows from 2.87.

To complete the proof, we must verify that f_1, f_2, \ldots converges uniformly to f on E. To do this, suppose $\varepsilon' > 0$. Let $n \in \mathbf{Z}^+$ be such that $\frac{1}{n} < \varepsilon'$. Then $E \subset A_{m_n,n}$, which implies that

$$|f_k(x) - f(x)| < \frac{1}{n} < \varepsilon'$$

for all $k \ge m_n$ and all $x \in E$. Hence f_1, f_2, \ldots does indeed converge uniformly to f on E.

Approximation by Simple Functions

2.88 **Definition** *simple function*

A function is called *simple* if it takes on only finitely many values.

Suppose (X, S) is a measurable space, $f: X \to \mathbf{R}$ is a simple function, and c_1, \ldots, c_n are the distinct nonzero values of f. Then

$$f = c_1 \chi_{E_1} + \cdots + c_n \chi_{E_n},$$

where $E_k = f^{-1}(\{c_k\})$. Thus this function f is an S-measurable function if and only if $E_1, \ldots, E_n \in \mathcal{S}$ (as you should verify).

2.89 approximation by simple functions

Suppose (X, S) is a measure space and $f: X \to [-\infty, \infty]$ is S-measurable. Then there exists a sequence f_1, f_2, \ldots of functions from X to **R** such that

- (a) each f_k is a simple $\mathcal S$ -measurable function; (b) $|f_k(x)| \leq |f_{k+1}(x)| \leq |f(x)|$ for all $k \in \mathbf Z^+$ and all $x \in X$; (c) $\lim_{k \to \infty} f_k(x) = f(x)$ for every $x \in X$;
- (d) f_1, f_2, \ldots converges uniformly on X to f if f is bounded.

Proof The idea of the proof is that for each $k \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$, the interval [n, n+1) is divided into 2^k equally sized half-open subintervals. If $f(x) \in [0, k)$, we define $f_k(x)$ to be the left endpoint of the subinterval into which f(x) falls; if $f(x) \in (-k,0)$, we define $f_k(x)$ to be the right endpoint of the subinterval into which f(x) falls; and if $|f(x)| \ge k$, we define $f_k(x)$ to be $\pm k$. Specifically, let

$$f_k(x) \in (-k,0), \text{ we define } f_k(x) \text{ to be the right endpoint of the subinterval into which } f(x) \text{ falls; and if } |f(x)| \ge k, \text{ we define } f_k(x) \text{ to be } \pm k. \text{ Specifically, let}$$

$$f_k(x) = \begin{cases} \frac{m}{2^k} & \text{if } 0 \le f(x) < k \text{ and } m \in \mathbf{Z} \text{ is such that } f(x) \in \left[\frac{m}{2^k}, \frac{m+1}{2^k}\right], \\ \frac{m+1}{2^k} & \text{if } -k < f(x) < 0 \text{ and } m \in \mathbf{Z} \text{ is such that } f(x) \in \left[\frac{m}{2^k}, \frac{m+1}{2^k}\right], \\ k & \text{if } f(x) \ge k, \\ -k & \text{if } f(x) \le -k. \end{cases}$$

Each $f^{-1}([\frac{m}{2^k}, \frac{m+1}{2^k})) \in \mathcal{S}$ because f is an \mathcal{S} -measurable function. Thus each f_k is an S-measurable simple function; in other words, (a) holds.

Also, (b) holds because of how we have defined f_k .

The definition of f_k implies that

 $|f_k(x) - f(x)| \le \frac{1}{2^k}$ for all $x \in X$ such that $f(x) \in [-k, k]$. 2.90

Thus we see that (c) holds.

Finally, 2.90 shows that (d) holds.

Luzin's Theorem

Our next result is surprising. It says that an arbitrary Borel measurable function is almost continuous, in the sense that its restriction to a large closed set is continuous. Here, the phrase *large closed set* means that we can take the complement of the closed set to have arbitrarily small measure.

Nikolai Luzin (1883–1950) proved the theorem below in 1912. Most mathematics literature in English refers to the result below as Lusin's Theorem. However, Luzin is the correct transliteration from Russian into English; Lusin is the transliteration into German.

Be careful about the interpretation of the conclusion of Luzin's Theorem that $f|_B$ is a continuous function on B. This is not the same as saying that f (on its original domain) is continuous at each point of B. For example, $\chi_{\mathbf{Q}}$ is discontinuous at every point of \mathbf{R} . However, $\chi_{\mathbf{Q}}|_{\mathbf{R}\setminus\mathbf{Q}}$ is a continuous function on $\mathbf{R}\setminus\mathbf{Q}$ (because this function is identically 0 on its domain).

2.91 Luzin's Theorem

Suppose $g: \mathbf{R} \to \mathbf{R}$ is a Borel measurable function. Then for every $\varepsilon > 0$, there exists a closed set $F \subset \mathbf{R}$ such that $|\mathbf{R} \setminus F| < \varepsilon$ and $g|_F$ is a continuous function on F.

Proof First consider the special case where $g = d_1 \chi_{D_1} + \cdots + d_n \chi_{D_n}$ for some distinct nonzero $d_1, \ldots, d_n \in \mathbf{R}$ and some disjoint Borel sets $D_1, \ldots, D_n \subset \mathbf{R}$. Suppose $\varepsilon > 0$. For each $k \in \{1, \ldots, n\}$, there exist (by 2.71) a closed set $F_k \subset D_k$ and an open set $G_k \supset D_k$ such that

$$|G_k \setminus D_k| < \frac{\varepsilon}{2n}$$
 and $|D_k \setminus F_k| < \frac{\varepsilon}{2n}$.

Because $G_k \setminus F_k = (G_k \setminus D_k) \cup (D_k \setminus F_k)$, we have $|G_k \setminus F_k| < \frac{\varepsilon}{n}$ for each $k \in \{1, \ldots, n\}$.

Let

$$F = \left(\bigcup_{k=1}^n F_k\right) \cup \bigcap_{k=1}^n (\mathbf{R} \setminus G_k).$$

Then *F* is a closed subset of **R** and **R** \ $F = \bigcup_{k=1}^{n} (G_k \setminus F_k)$. Thus $|\mathbf{R} \setminus F| < \varepsilon$.

Because $F_k \subset D_k$, we see that g is identically d_k on F_k . Thus $g|_{F_k}$ is continuous for each $k \in \{1, ..., n\}$. Because

$$\bigcap_{k=1}^{n} (\mathbf{R} \setminus G_k) \subset \bigcap_{k=1}^{n} (\mathbf{R} \setminus D_k),$$

we see that g is identically 0 on $\bigcap_{k=1}^{n} (\mathbf{R} \setminus G_k)$. Thus $g|_{\bigcap_{k=1}^{n} (\mathbf{R} \setminus G_k)}$ is continuous. Putting all this together, we conclude that $g|_F$ is continuous (use Exercise 9 in this section), completing the proof in this special case.

Now consider an arbitrary Borel measurable function $g: \mathbb{R} \to \mathbb{R}$. By 2.89, there exists a sequence g_1, g_2, \ldots of functions from \mathbb{R} to \mathbb{R} that converges pointwise on \mathbb{R} to g, where each g_k is a simple Borel measurable function.

Suppose $\varepsilon > 0$. By the special case already proved, for each $k \in \mathbb{Z}^+$, there exists a closed set $C_k \subset \mathbb{R}$ such that $|\mathbb{R} \setminus C_k| < \frac{\varepsilon}{2k+1}$ and $g_k|_{C_k}$ is continuous. Let

$$C = \bigcap_{k=1}^{\infty} C_k.$$

Thus C is a closed set and $g_k|_C$ is continuous for every $k \in \mathbb{Z}^+$. Note that

$$\mathbf{R}\setminus C=\bigcup_{k=1}^{\infty}(\mathbf{R}\setminus C_k);$$

thus $|\mathbf{R} \setminus C| < \frac{\varepsilon}{2}$.

For each $m \in \mathbb{Z}$, the sequence $g_1|_{(m,m+1)}$, $g_2|_{(m,m+1)}$, ... converges pointwise on (m,m+1) to $g|_{(m,m+1)}$. Thus by Egorov's Theorem (2.85), for each $m \in \mathbb{Z}$, there is a Borel set $E_m \subset (m,m+1)$ such that g_1,g_2,\ldots converges uniformly to g on E_m and

$$|(m,m+1)\setminus E_m|<\frac{\varepsilon}{2^{|m|+3}}.$$

Thus g_1, g_2, \ldots converges uniformly to g on $C \cap E_m$ for each $m \in \mathbf{Z}$. Because each $g_k|_C$ is continuous, we conclude (using 2.84) that $g|_{C \cap E_m}$ is continuous for each $m \in \mathbf{Z}$. Thus $g|_D$ is continuous, where

$$D=\bigcup_{m\in\mathbf{Z}}(C\cap E_m).$$

Because

$$\mathbf{R} \setminus D \subset \mathbf{Z} \cup \left(\bigcup_{m \in \mathbf{Z}} ((m, m+1) \setminus E_m)\right) \cup (\mathbf{R} \setminus C),$$

we have $|\mathbf{R} \setminus D| < \varepsilon$.

There exists a closed set $F \subset D$ such that $|D \setminus F| < \varepsilon - |\mathbf{R} \setminus D|$ (by 2.65). Now

$$|\mathbf{R} \setminus F| = |(\mathbf{R} \setminus D) \cup (D \setminus F)| \le |\mathbf{R} \setminus D| + |D \setminus F| < \varepsilon.$$

Because the restriction of a continuous function to a smaller domain is also continuous, $g|_F$ is continuous, completing the proof.

We need the following result to get another version of Luzin's Theorem.

2.92 continuous extensions of continuous functions

- Every continuous function on a closed subset of R can be extended to a continuous function on all of R.
- More precisely, if F ⊂ R is closed and g: F → R is continuous, then there exists a continuous function h: R → R such that h|_F = g.

Proof Suppose $F \subset \mathbf{R}$ is closed and $g \colon F \to \mathbf{R}$ is continuous. Thus $\mathbf{R} \setminus F$ is the union of a collection of disjoint open intervals $\{I_k\}$. For each such interval of the form (a, ∞) or of the form $(-\infty, a)$, define h(x) = g(a) for all x in the interval.

For each interval I_k of the form (b, c) with b < c and $b, c \in \mathbf{R}$, define h on [b, c] to be the linear function such that h(b) = g(b) and h(c) = g(c).

Define h(x) = g(x) for all $x \in \mathbf{R}$ for which h(x) has not been defined by the previous two paragraphs. Then $h \colon \mathbf{R} \to \mathbf{R}$ is continuous and $h|_F = g$.

The next result gives a slightly modified way to state Luzin's Theorem. You can think of this version as saying that the value of a Borel measurable function can be changed on a set with small Lebesgue measure to produce a continuous function.

2.93 Luzin's Theorem, second version

Suppose $E \subset \mathbf{R}$ and $g \colon E \to \mathbf{R}$ is a Borel measurable function. Then for every $\varepsilon > 0$, there exists a closed set $F \subset E$ and a continuous function $h \colon \mathbf{R} \to \mathbf{R}$ such that $|E \setminus F| < \varepsilon$ and $h|_F = g|_F$.

Proof Suppose $\varepsilon > 0$. Extend g to a function $\tilde{g} \colon \mathbf{R} \to \mathbf{R}$ by defining

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in E, \\ 0 & \text{if } x \in \mathbf{R} \setminus E. \end{cases}$$

By the first version of Luzin's Theorem (2.91), there is a closed set $C \subset \mathbf{R}$ such that $|\mathbf{R} \setminus C| < \varepsilon$ and $\tilde{g}|_C$ is a continuous function on C. There exists a closed set $F \subset C \cap E$ such that $|(C \cap E) \setminus F| < \varepsilon - |\mathbf{R} \setminus C|$ (by 2.65). Thus

$$|E \setminus F| \le |((C \cap E) \setminus F) \cup (\mathbf{R} \setminus C)| \le |(C \cap E) \setminus F| + |\mathbf{R} \setminus C| < \varepsilon.$$

Now $\tilde{g}|_F$ is a continuous function on F. Also, $\tilde{g}|_F = g|_F$ (because $F \subset E$). Use 2.92 to extend $\tilde{g}|_F$ to a continuous function $h \colon \mathbf{R} \to \mathbf{R}$.



The building at Moscow State University where the mathematics seminar organized by Egorov and Luzin met. Both Egorov and Luzin had been students at Moscow State University and then later became faculty members at the same institution. Luzin's PhD thesis advisor was Egorov.

Lebesgue Measurable Functions

2.94 Definition Lebesgue measurable function

A function $f: A \to \mathbb{R}$, where $A \subset \mathbb{R}$, is called *Lebesgue measurable* if $f^{-1}(B)$ is a Lebesgue measurable set for every Borel set $B \subset \mathbb{R}$.

If $f: A \to \mathbf{R}$ is a Lebesgue measurable function, then A is a Lebesgue measurable subset of \mathbf{R} [because $A = f^{-1}(\mathbf{R})$]. If A is a Lebesgue measurable subset of \mathbf{R} , then the definition above is the standard definition of an S-measurable function, where S is a σ -algebra of all Lebesgue measurable subsets of A.

The following list summarizes and reviews some crucial definitions and results:

- A Borel set is an element of the smallest σ -algebra on **R** that contains all the open subsets of **R**.
- A Lebesgue measurable set is an element of the smallest σ -algebra on \mathbf{R} that contains all the open subsets of \mathbf{R} and all the subsets of \mathbf{R} with outer measure 0.
- The terminology *Lebesgue set* would make good sense in parallel to the terminology *Borel set*. However, *Lebesgue set* has another meaning, so we need to use *Lebesgue measurable set*.
- Every Lebesgue measurable set differs from a Borel set by a set with outer measure 0. The Borel set can be taken either to be contained in the Lebesgue measurable set or to contain the Lebesgue measurable set.
- ullet Outer measure restricted to the σ -algebra of Borel sets is called Lebesgue measure.
- Outer measure restricted to the σ -algebra of Lebesgue measurable sets is also called Lebesgue measure.
- Outer measure is not a measure on the σ -algebra of all subsets of **R**.
- A function $f: A \to \mathbb{R}$, where $A \subset \mathbb{R}$, is called Borel measurable if $f^{-1}(B)$ is a Borel set for every Borel set $B \subset \mathbb{R}$.
- A function $f: A \to \mathbb{R}$, where $A \subset \mathbb{R}$, is called Lebesgue measurable if $f^{-1}(B)$ is a Lebesgue measurable set for every Borel set $B \subset \mathbb{R}$.

Although there exist Lebesgue measurable sets that are not Borel sets, you are unlikely to encounter one. Similarly, a Lebesgue measurable function that is not Borel measurable is unlikely to arise in anything you do. A great way to simplify the potential confusion about Lebesgue measurable functions being defined by inverse images of Borel sets is to consider only Borel measurable functions.

"Passing from Borel to Lebesgue measurable functions is the work of the devil. Don't even consider it!"

-Barry Simon (winner of the American Mathematical Society Steele Prize for Lifetime Achievement), in his five-volume series A Comprehensive Course in Analysis

The next result states that if we adopt the philosophy that what happens on a set of outer measure 0 does not matter much, then we might as well restrict our attention to Borel measurable functions.

'He professes to have received no sinister measure."

Measure for Measure, by William Shakespeare

2.95 every Lebesgue measurable function is almost Borel measurable

Suppose $f \colon \mathbf{R} \to \mathbf{R}$ is a Lebesgue measurable function. Then there exists a Borel measurable function $g \colon \mathbf{R} \to \mathbf{R}$ such that

$$|\{x \in \mathbf{R} : g(x) \neq f(x)\}| = 0.$$

Proof There exists a sequence f_1, f_2, \ldots of Lebesgue measurable simple functions from **R** to **R** converging pointwise on **R** to f (by 2.89). Suppose $k \in \mathbb{Z}^+$. Then there exist $c_1, \ldots, c_n \in \mathbb{R}$ and disjoint Lebesgue measurable sets $A_1, \ldots, A_n \subset \mathbb{R}$ such that

$$f_k = c_1 \chi_{A_1} + \dots + c_n \chi_{A_n}.$$

For each $j \in \{1, ..., n\}$, there exists a Borel set $B_j \subset A_j$ such that $|A_j \setminus B_j| = 0$ [by the equivalence of (a) and (d) in 2.71]. Let

$$g_k = c_1 \chi_{B_1} + \cdots + c_n \chi_{B_n}$$

Then g_k is a Borel measurable function and $|\{x \in \mathbf{R} : g_k(x) \neq f_k(x)\}| = 0$. If $x \notin \bigcup_{k=1}^{\infty} \{x \in \mathbf{R} : g_k(x) \neq f_k(x)\}$, then $g_k(x) = f_k(x)$ for all $k \in \mathbf{Z}^+$ and hence $\lim_{k \to \infty} g_k(x) = f(x)$. Let

$$E = \{x \in \mathbf{R} : \lim_{k \to \infty} g_k(x) \text{ exists in } \mathbf{R}\}.$$

Then E is a Borel subset of **R** [by Exercise 14(b) in Section 2B]. Also,

$$\mathbf{R} \setminus E \subset \bigcup_{k=1}^{\infty} \{x \in \mathbf{R} : g_k(x) \neq f_k(x)\}$$

and thus $|\mathbf{R} \setminus E| = 0$. For $x \in \mathbf{R}$, let

$$g(x) = \lim_{k \to \infty} (\chi_E g_k)(x).$$

If $x \in E$, then the limit above exists by the definition of E; if $x \in \mathbf{R} \setminus E$, then the limit above exists because $(\chi_E g_k)(x) = 0$ for all $k \in \mathbf{Z}^+$.

For each $k \in \mathbb{Z}^+$, the function $\chi_E g_k$ is Borel measurable. Thus 2.96 implies that g is a Borel measurable function (by 2.48). Because

$$\{x \in \mathbf{R} : g(x) \neq f(x)\} \subset \bigcup_{k=1}^{\infty} \{x \in \mathbf{R} : g_k(x) \neq f_k(x)\},$$

we have $|\{x \in \mathbf{R} : g(x) \neq f(x)\}| = 0$, completing the proof.

EXERCISES 2E

- 1 Suppose X is a finite set. Explain why a sequence of functions from X to \mathbf{R} that converges pointwise on X also converges uniformly on X.
- 2 Give an example of a sequence of functions from \mathbf{Z}^+ to \mathbf{R} that converges pointwise on \mathbf{Z}^+ but does not converge uniformly on \mathbf{Z}^+ .
- 3 Give an example of a sequence of continuous functions $f_1, f_2, ...$ from [0, 1] to **R** that converge pointwise to a function $f: [0, 1] \to \mathbf{R}$ that is not a bounded function.
- 4 Prove or give a counterexample: If $A \subset \mathbf{R}$ and f_1, f_2, \ldots is a sequence of uniformly continuous functions from A to \mathbf{R} that converge uniformly to a function $f: A \to \mathbf{R}$, then f is uniformly continuous on A.
- 5 Give an example to show that Egorov's Theorem can fail without the hypothesis that $\mu(X)$ < ∞.
- 6 Suppose (X, S, μ) is a measure space with $\mu(X) < \infty$. Suppose f_1, f_2, \ldots is a sequence of S-measurable functions from X to \mathbf{R} such that $\lim_{k \to \infty} f_k(x) = \infty$ for each $x \in X$. Prove that for every $\varepsilon > 0$, there exists a set $E \in S$ such that $\mu(X \setminus E) < \varepsilon$ and f_1, f_2, \ldots converges uniformly to ∞ on E (meaning that for every t > 0, there exists $n \in \mathbf{Z}^+$ such that $f_k(x) > t$ for all integers $k \ge n$ and all $x \in E$).

[The exercise above is an Egorov-type theorem for sequences of functions that converge pointwise to ∞ .]

7 Suppose F is a closed bounded subset of \mathbf{R} and g_1, g_2, \ldots is an increasing sequence of continuous real-valued functions on F (thus $g_1(x) \leq g_2(x) \leq \cdots$ for all $x \in F$) such that $\sup\{g_1(x), g_2(x), \ldots\} < \infty$ for each $x \in F$. Define a real-valued function g on F by

$$g(x) = \lim_{k \to \infty} g_k(x).$$

Prove that g is continuous on F if and only if g_1, g_2, \ldots converges uniformly on F to g.

[The result above is called Dini's Theorem.]

8 Suppose μ is the measure on $(\mathbf{Z}^+, 2^{\mathbf{Z}^+})$ defined by

$$\mu(E) = \sum_{n \in F} \frac{1}{2^n}.$$

Prove that for every $\varepsilon > 0$, there exists a set $E \subset \mathbf{Z}^+$ with $\mu(\mathbf{Z}^+ \setminus E) < \varepsilon$ such that f_1, f_2, \ldots converges uniformly on E for every sequence of functions f_1, f_2, \ldots from \mathbf{Z}^+ to \mathbf{R} that converges pointwise on \mathbf{Z}^+ .

[This result does not follow from Egorov's Theorem because here we are asking for E to depend only on ε . In Egorov's Theorem, E depends on ε and on the sequence f_1, f_2, \ldots]

9 Suppose F_1, \ldots, F_n are disjoint closed subsets of **R**. Prove that if

$$g: F_1 \cup \cdots \cup F_n \to \mathbf{R}$$

is a function such that $g|_{F_k}$ is a continuous function for each $k \in \{1, ..., n\}$, then g is a continuous function.

- 10 Suppose $F \subset \mathbf{R}$ is such that every continuous function from F to \mathbf{R} can be extended to a continuous function from \mathbf{R} to \mathbf{R} . Prove that F is a closed subset of \mathbf{R} .
- Prove or give a counterexample: If $F \subset \mathbf{R}$ is such that every bounded continuous function from F to \mathbf{R} can be extended to a continuous function from \mathbf{R} to \mathbf{R} , then F is a closed subset of \mathbf{R} .
- Give an example of a Borel measurable function f from \mathbf{R} to \mathbf{R} such that there does not exist a set $B \subset \mathbf{R}$ such that $|\mathbf{R} \setminus B| = 0$ and $f|_B$ is a continuous function on B.
- Prove or give a counterexample: If $f_t : \mathbf{R} \to \mathbf{R}$ is a Borel measurable function for each $t \in \mathbf{R}$ and $f : \mathbf{R} \to (-\infty, \infty]$ is defined by

$$f(x) = \sup\{f_t(x) : t \in \mathbf{R}\},\$$

then f is a Borel measurable function.

14 Suppose b_1, b_2, \ldots is a sequence of real numbers. Define $f: \mathbf{R} \to [0, \infty]$ by

$$f(x) = \begin{cases} \sum_{k=1}^{\infty} \frac{1}{4^k |x - b_k|} & \text{if } x \notin \{b_1, b_2, \dots\}, \\ \infty & \text{if } x \in \{b_1, b_2, \dots\}. \end{cases}$$

Prove that $|\{x \in \mathbf{R} : f(x) < 1\}| = \infty$.

[This exercise is a variation of a problem originally considered by Borel. If b_1, b_2, \ldots contains all the rational numbers, then it is not even obvious that $\{x \in \mathbf{R} : f(x) < \infty\} \neq \emptyset$.]

Suppose *B* is a Borel set and $f: B \to \mathbf{R}$ is a Lebesgue measurable function. Show that there exists a Borel measurable function $g: B \to \mathbf{R}$ such that

$$|\{x \in B : g(x) \neq f(x)\}| = 0.$$

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