LAPLACE TRANSFORM

1. $L[1] = \frac{1}{s}$

2. $L[e^{at}] = \frac{1}{s-a}$, $s > a$.

3. $L[\sin at] = \frac{a}{s^2+a^2}$.

4. $L[\cos at] = \frac{s}{s^2+a^2}$

5. $L[t^n] = \frac{\gamma_{n+1}}{s^{n+1}}$

6. $L[\sinh at] = \frac{a}{s^2-a^2}$ $s > |a|$

7. $L[\cosh at] = \frac{s}{s^2-a^2}$. $s > |a|$.

8. If $L[f(t)] = \bar{f}(s)$ then [FIRST SHIFTING PROPERTY]
$L[e^{at} f(t)] = \bar{f}(s-a)$.

9. If $L[f(t)] = \bar{f}(s)$ then.
   i) $L[\sinh at\ f(t)] = \frac{1}{2}\left[\bar{f}(s-a) - \bar{f}(s+a)\right]$
   ii) $L[\cosh at.\ f(t)] = \frac{1}{2}\left[\bar{f}(s-a) + \bar{f}(s+a)\right]$

10. If $L[f(t)] = \bar{f}(s)$ then
$L[f(at)] = \frac{1}{|a|}\bar{f}(s/a)$.

REMARK

$L[f(t)] = \bar{f}(s)$ exists then
i) $\underset{s\to\infty}{Lt}\ \bar{f}(s) = 0$ ii) $\underset{s\to\infty}{Lt}\ [s.\bar{f}(s)]$ is bounded.

11. $L[f'(t)] = s\bar{f}(s) - f(0)$

12. $L[f^n(t)] = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{n-1}(0)$.

13. If $L[f(t)] = \bar{f}(s)$ then
$L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}\bar{f}(s)$.

14. If $L[f(t)] = \bar{f}(s)$ then

$$L[f(t).t^n] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]. \quad n=1,2,3,..$$

15. If $L[f(t)] = \bar{f}(s)$ then

$$L[\tfrac{1}{t}f(t)] = \int_s^\infty \bar{f}(s)\,ds.$$ provided integral exists.

16. BESSEL FUNCTION

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots$$

$$J_1(x) = -J_0'(x)$$

$$L[J_0(x)] = \frac{1}{\sqrt{1+s^2}}$$

$$L[J_1(x)] = 1 - \frac{s}{\sqrt{1+s^2}}$$

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$
Bessel equations.

$$J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!}\left(\frac{t}{2}\right)^{n+2r}.$$

17. Periodic Function

$$f(t+T) = f(t)$$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)\,dt.$$

18. Error Function

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u(x,y) \rightarrow \text{Harmonic function}$$

$$\mathrm{erf}(\sqrt{x}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} e^{-t^2}\,dt$$

$$\mathrm{erfc}(\sqrt{x}) = 1 - \mathrm{erf}(\sqrt{x}).$$
Complementary function.

$$L[\mathrm{erf}(\sqrt{x})] = \frac{1}{s\sqrt{s+1}}.$$

** 19. Some questions require not to breakdown into parts.

eg. $L\left[\frac{1-e^{-t}}{t}\right]$, $L\left[\frac{\cos at - \cos bt}{t}\right]$

** 20 Solve some questions (integrals) using L.T. technique.

eg. $\int_0^\infty t.e^{-3t}.\sin t\,dt$, $\int_0^\infty \frac{\sin mt}{t}\,dt$ (Assume as $s \rightarrow 0$)

2. For any interval ... if ...

INVERSE LAPLACE TRANSFORM

$L[F(t)] = \bar{f}(s)$

$L^{-1}[\bar{f}(s)] = f(t)$

1. $L^{-1}\left[\frac{1}{s}\right] = 1$

2. $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$, $n \in N$

3. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

4. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

5. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

6. $L^{-1}\left[\frac{1}{(s-a)^n}\right] = e^{at}\cdot\frac{t^{n-1}}{(n-1)!}$, $n \in N$.

7. $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$

8. $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

9. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = e^{at}\cdot\frac{1}{b}\sin bt$

10. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cdot\cos bt$

11. $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}t\cdot\sin at$

12. $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}[\sin at - at\cos at]$

13. Use of partial fractions while finding I.L.T.

14. If $L^{-1}[\bar{f}(s)] = f(t)$ then
$L^{-1}[\bar{f}(s-a)] = e^{at}\cdot f(t)$

15. If $L^{-1}[\bar{f}(s)] = f(t)$, $F(0) = 0$
then $L^{-1}[s\bar{f}(s)] = \frac{d\,f(t)}{dt}$.

16. In general
$L^{-1}[s^n\cdot\bar{f}(s)] = \frac{d^n}{dt^n}f(t)$ provided $F(0) = f'(0) = \ldots = f^{n-1}(0) = 0$..

17. If $L^{-1}[\bar{f}(s)] = f(t)$,

then $L^{-1}\left[\frac{\bar{f}(s)}{s}\right] = \int_0^t f(t)\,dt$

18. $L^{-1}\left[\frac{\bar{f}(s)}{s^2}\right] = \int_0^t\left(\int_0^t f(t)\,dt\right)dt$

19. $L^{-1}\left[-\frac{d\bar{f}(s)}{ds}\right] = tf(t)$.

20. $L^{-1}\left[\int_s^\infty \bar{f}(s)\,ds\right] = \frac{1}{t}f(t)$.

21. Solving I.V.P. by taking laplace, solve and then inverse laplace.

22. $L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s+a)^3}\right] \Rightarrow L^{-1}\left[\frac{1}{s-a}\right] \to L^{-1}\left[\frac{1}{s}\,\frac{1}{s-a}\right] \cdots L^{-1}\left[\frac{1}{s^3}\cdot\frac{1}{s-a}\right]$ $\boxed{e^{-at}} L^{-1}\left(\frac{1}{(s+a)^3}\cdot s\right)$
Factor

23. Some require to manipulate
$tf(t) = L^{-1}\left[-\frac{d}{ds}f(s)\right]$
then find $f(t)$.

eg. $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$, $L^{-1}\left[\cot^{-1}\frac{s}{2}\right]$, $L^{-1}\left[\log\frac{s^2+1}{s(s+1)}\right]$.
$L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right]$

23. $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}t\sin at$.

24. $L^{-1}\left[s\cdot\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}(\sin at + at\cos at)$

25. $L^{-1}\left[\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$.

UNIT STEP FUNCTION

Heaviside's unit step function
$H(t-a)$; $u(t-a)$; $u_a(t)$

1. $u(t-a) = \begin{cases} 0 & ; t<a \\ 1 & ; t>a. \end{cases}$

2. $L[u(t-a)] = \frac{e^{-as}}{s}$.

3. $f(t).u(t-a) = \begin{cases} 0 & ; t<a \\ f(t) & ; t>a. \end{cases}$

4. Second Shifting Theorem
If $L[f(t)] = \bar{f}(s)$ then $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$.

5. Convert equations into $f(t-a)u(t-a)$ for easier calculation.

Eg. $f(t) = \begin{cases} t-1 & 1<t<2 \\ 3-t & 2<t<3 \end{cases}$

$f(t) = (t-1)[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$

$= (t-1)[u(t-1)] - 2(t-2)[u(t-2)] + (t-3)[u(t-3)]$

OTHER

1. In IVP if conditions are not defined at $t=0$, then assume that as some constant, solve it and find value of unknown constant.

Eg. $y'' + 9y = \cos 2t$. given $y(0) = 1, y(\frac{\pi}{2}) = -1$

Assume $y'(0) = \alpha$.

Solve for $y(t)$.

Get α value using $y(\frac{\pi}{2}) = 1$

2 IVP with shifted initial condition.

Eg. $y'' + y = 2t$; $y(\frac{\pi}{4}) = \frac{\pi}{2}$, $y'(\frac{\pi}{4}) = 2-\sqrt{2}$

set $t = \tilde{t} + \frac{\pi}{4}$

$y(t) = \tilde{y}(\tilde{t})$

& $y(\frac{\pi}{4}) = \frac{\pi}{2} = \tilde{y}(0)$

$y'(\frac{\pi}{4}) = 2-\sqrt{2} = \tilde{y}'(0)$

$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \frac{\pi}{4})$

solve for $\tilde{y}(\tilde{t})$. & replace $\tilde{t}$.

3. Solution of 2 differential equation.

Eg. $\frac{dx}{dt} + 5x - 2y = t$ $\quad x(0) = 0$

$\frac{dy}{dt} + 2x + y = 0$ $\quad y(0) = 0$.

Apply L.T.

& solve $X(s), Y(s)$

Find inverse & get $x(t), y(t)$.

DELTA FUNCTION

Dirac - Delta Function

1. $\delta(t-a) = 0, \ \forall\ t, t \neq a$.
2. For any interval $[c,d]$

$$\int_c^d \delta(t-a)\,dt = \begin{cases} 1 & \text{if interval } [c,d] \text{ contains } a, \text{ ie. } c \le a \le d. \\ 0 & \text{otherwise.} \end{cases}$$

3. $$\int_c^d \delta(t-a) f(t)\,dt = \begin{cases} f(a) & \text{if } c \le a \le d. \\ 0 & \text{otherwise} \end{cases}$$

4. $L[\delta(t-a)] = e^{-as}$.

5. $L[u'(t-a)] = L[\delta(t-a)]$

6. $L[\delta(t)] = 1$.

LAGUERRE POLYNOMIAL

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(e^{-t}.t^n)$$

solution of $x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0$

$n = 0, 1, 2, \ldots$

1. $L[L_n(t)] = \frac{(s-1)^n}{s^{n+1}}$

CONVOLUTION THEOREM

$L[f(t)] = \bar{f}(s)$

$L[g(t)] = \bar{g}(s)$

1. $f(t) * g(t) = \int_0^t f(u).g(t-u)du$

2. $f(t) * g(t) = g(t) * f(t)$.

4. $L[f(t) * g(t)] = \bar{f}(s).\bar{g}(s)$.

FOURIER SERIES

$$f(x) \simeq \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

$(T = 2\pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx.$$

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$(T = 2l)$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

DIRICHLET CONDITIONS.

1. $f(x)$ is periodic, single valued, finite.
2. $f(x)$ has finite number of discontinuities in any one period.
3. $f(x)$ has atmost a finite number of maxima or minima.