ASSIGNMENT - 1

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Advanced Numerical Techniques

1) Find the Truncation Error and check for consistency for discretization of the linear BVP by the central difference scheme.

Soln: Consider the linear BVP:

$$y'' + A(x)\cdot y' + B(x)\cdot y = C(x) \quad \text{in } 0<x<a \quad \cdots\cdots (i)$$

with Boundary conditions: $\alpha_0 y(0) + \beta_0 y'(0) = \gamma_0$, and

$$\alpha_a y(a) + \beta_a y'(a) = \gamma_a$$

where ~~not~~ α_0, β_0 or α_a, β_a are not all zero.

We discretize (i) by Central Difference Scheme as follows:

$$\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + A_i \times \left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + B_i \times y_i = C_i \quad ; \text{ where } A_i = A(x_i), \; B_i = B(x_i), \; C_i = C(x_i)$$

$$\Rightarrow \left(\frac{1}{h^2} - \frac{A_i}{2h}\right) y_{i-1} + \left(B_i - \frac{2}{h^2}\right) y_i + \left(\frac{1}{h^2} + \frac{A_i}{2h}\right) y_{i+1} = C_i \quad \cdots\cdots (ii)$$

Suppose y_i is the value of y at $x = x_i$ that we get from this discretized central difference scheme, whereas, let Y_i denote the actual value of the soln. of (i) at $x = x_i$.

Then, Y_i satisfies the original ODE (i) at $x = x_i$

So, $Y_i'' + A_i \times Y_i' + B_i Y_i = C_i$

$$\Rightarrow Y_i'' + A_i Y_i' + B_i Y_i - C_i = 0 \quad \cdots\cdots (iii)$$

Now, Truncation Error for the scheme (ii) is given by:

$$T_i = \left(\frac{1}{h^2} - \frac{A_i}{2h}\right) Y_{i-1} + \left(B_i - \frac{2}{h^2}\right) Y_i + \left(\frac{1}{h^2} + \frac{A_i}{2h}\right) Y_{i+1} - C_i$$

at $x = x_i$ (Local Truncation Error)

$$= \frac{1}{h^2} \times (Y_{i-1} + Y_{i+1}) + \frac{A_i}{2h} \times (Y_{i+1} - Y_{i-1}) + \left(B_i - \frac{2}{h^2}\right) Y_i - C_i$$

$$= \frac{1}{h^2} \times \left[\left\{Y_i - hY_i' + \frac{h^2}{2!} Y_i'' - \frac{h^3}{3!} Y_i''' + \cdots\right\} + \left\{Y_i + hY_i' + \frac{h^2}{2!} Y_i'' + \frac{h^3}{3!} Y_i''' + \cdots\right\}\right]$$

$$+ \frac{A_i}{2h} \times \left[\left\{Y_i + hY_i' + \frac{h^2}{2!} Y_i'' + \cdots\right\} - \left\{Y_i - hY_i' + \frac{h^2}{2!} Y_i'' - \cdots\right\}\right] + \left(B_i - \frac{2}{h^2}\right) Y_i - C_i$$

$$\Rightarrow T_i = \frac{1}{h^2} \times \left[2Y_i + \frac{2h^2}{2!}Y_i'' + \frac{2h^4}{4!}Y_i^{(4)} + \dots\right] + \frac{A_i}{2h} \times \left[2hY_i' + \frac{2h^3}{3!}Y_i''' + \frac{2h^5}{5!}Y_i^{(5)} + \dots\right]$$

$$+ \left(B_i - \frac{2}{h^2}\right) Y_i - C_i$$

$$\Rightarrow T_i = \frac{1}{h^2} \times (2Y_i - 2Y_i) + \frac{1}{h} \times (0) + (Y_i'' + A_i Y_i' + B_i Y_i - C_i) + h \times (0)$$

$$+ h^2 \times \left(\frac{2}{4!} \times Y_i^{(4)} + \frac{2}{3!} \times Y_i''' \times \frac{A_i}{2}\right) + \underbrace{\dots}_{\text{higher powers of } h}$$

[Writing the expression in increasing powers of h]

$$\Rightarrow T_i = \frac{1}{h^2} \times (0) + \frac{1}{h} \times (0) + (0) + h \times (0) + h^2 \times \left(\frac{1}{12} Y_i^{(4)} + \frac{A_i}{6} Y_i'''\right) + \dots \text{ higher powers of } h$$

[$\because$ Y_i satisfies the ODE (i).
So, $Y_i'' + A_i Y_i' + B_i Y_i - C_i = 0$]

And, coeff. of h^2 is non-zero. Hence, $T_i = O(h^2)$

And, all terms have powers of h (with exponent ≥ 2) associated with them, so clearly, as $h \to 0$, $T_i \to 0$.

Hence, for the discretization of the linear BVP by the central Difference Scheme, the truncation error is of $O(h^2)$, and this scheme is consistent.

2) Solve the BVP : $x^2 y'' + xy' = 1$; $y(1) = 0$; $y(1.4) = 0.0566$

for step size $h = 0.1$.

Soln: From discretization of the given ODE by central difference scheme, we get:

$$x^2 y'' + xy' = 1 \Rightarrow y'' + \frac{1}{x} y' = \frac{1}{x^2}$$

$$\Rightarrow \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{1}{x_i} \cdot \left(\frac{y_{i+1} - y_{i-1}}{2h}\right) = \frac{1}{x_i^2}$$

$$\Rightarrow \left(\frac{1}{h^2} - \frac{1}{2hx_i}\right) y_{i-1} - \frac{2}{h^2} y_i + \left(\frac{1}{h^2} + \frac{1}{2hx_i}\right) y_{i+1} = \frac{1}{x_i^2} \quad \dots \text{(i)}$$

If we take $h = 0.1$, then $n = 4$, and $x_0 = 1$, $x_n = 1.4$ and $y_0 = 0$, $y_n = 0.0566$

In (i), putting $i = 1, 2, 3$, we get the system of equations:

$$\begin{bmatrix} -\frac{2}{h^2} & \left(\frac{1}{h^2}+\frac{1}{2hx_1}\right) & 0 \\ \left(\frac{1}{h^2}-\frac{1}{2hx_2}\right) & -\frac{2}{h^2} & \left(\frac{1}{h^2}+\frac{1}{2hx_2}\right) \\ 0 & \left(\frac{1}{h^2}-\frac{1}{2hx_3}\right) & -\frac{2}{h^2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{x_1^2} - 0 \\ \frac{1}{x_2^2} \\ \frac{1}{x_3^2} - \left(\frac{1}{h^2}+\frac{1}{2hx_3}\right) \times y_4 \end{bmatrix}$$

Putting $h = 0.1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$, $y_4 = 0.0566$, we get:

$$\begin{bmatrix} -200 & \frac{1150}{11} & 0 \\ \frac{575}{6} & -200 & \frac{625}{6} \\ 0 & \frac{1250}{13} & -200 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{100}{121} \\ \frac{25}{36} \\ \frac{-89333}{16900} \end{bmatrix}$$

On solving: $y_1 = y(1.1) = 0.0045742$, $y_2 = y(1.2) = 0.01665575$,

$y_3 = y(1.3) = 0.0344375$

3) Solve the BVP: $y'' - 2xy' - 2y = -4x$; $y(0) = y'(0)$; $2y(1) - y'(1) = 1$

taking step size $h = 0.25$. [And $h \leq 0.1$ for lab]

Soln: From discretization of the given ODE by Central difference scheme,

we get: [For $h = 0.25$, $n = 4$]

$$\left(\frac{1}{h^2} + \frac{x_i}{h}\right) y_{i-1} - \left(2 + \frac{2}{h^2}\right) y_i + \left(\frac{1}{h^2} - \frac{x_i}{h}\right) y_{i+1} = -4x_i \quad \cdots \text{(i)}$$

→ for $i = 1, 2, \ldots, n-1$

we get $(n-1)$ equations.

But, we have $(n+1)$ unknowns. From the Boundary Conditions, we have:

$y(0) = y'(0)$. So, $y_0 = y_0' = \dfrac{-3y_0 + 4y_1 - y_2}{2h}$ [Using forward difference approximation of y_0' of 2nd order]

$$\Rightarrow (2h+3)y_0 - 4y_1 + y_2 = 0 \quad \cdots \text{(ii)}$$

And, $2y(1) - 1 = y'(1)$ $\Rightarrow 2y_n - 1 = \dfrac{3y_n - 4y_{n-1} + y_{n-2}}{2h}$ [Using backward difference approx. of 2nd order]

$$\Rightarrow -y_{n-2} + 4y_{n-1} + (4h-3)y_n = 2h \quad \cdots \text{(iii)}$$

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From (i), (ii) and (iii), we get the system of equations:

$$\begin{bmatrix} 3.5 & -4 & 1 & 0 & 0 \\ 17 & -34 & 15 & 0 & 0 \\ 0 & 18 & -34 & 14 & 0 \\ 0 & 0 & 19 & -34 & 13 \\ 0 & 0 & -1 & 4 & -2 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -2 \\ -3 \\ 0.5 \end{bmatrix}$$

On solving, we get: $y_0 = y(0) = -3.448484$, $y_1 = y(0.25) = -4.7469697$,

$y_2 = y(0.5) = -6.918182$, $y_3 = y(0.75) = -10.8409091$, $y_4 = y(1) = -18.4727273$

[Note: However, from the Lab Task, to solve it for step sizes $h \leq 0.1$,

we observe that the results obtained here for $h = 0.25$ diverge

from the actual solution by a great amount, though for

$h \leq 0.1$, the values obtained are convergent.

The values at the above points, obtained with $h = 0.005$ in the

Lab Task are as follows:

$y(0) = 1.0004693$, $y(0.25) = 1.3151157$, $y(0.5) = 1.7849026$,

$y(0.75) = 2.5063879$, $y(1) = 3.72048848$]

4) Show that the discretization in Control Volume Method reduces to the tridiagonal central difference scheme if $p(x), q(x), r(x)$ are continuous.

Soln: We have the given ODE:

$$\frac{d}{dx}\left[p(x)\cdot\frac{dy}{dx}\right] + q(x)\cdot y = r(x) \quad ; \quad 0 < x < a \quad \text{in Sturm-Liouville form} \quad \ldots (i)$$

$$\Rightarrow p(x)\cdot y'' + p'(x)\cdot y' + q(x)\cdot y = r(x)$$

$$\Rightarrow y'' + \frac{p'(x)}{p(x)}\cdot y' + \frac{q(x)}{p(x)} y = \frac{r(x)}{p(x)} \quad \ldots (ii)$$

Applying Central difference discretization scheme in (ii), we get:

$$\left(\frac{1}{h^2} - \frac{p_i'}{2h p_i}\right) y_{i-1} + \left(\frac{q_i}{p_i} - \frac{2}{h^2}\right) y_i + \left(\frac{1}{h^2} + \frac{p_i'}{2h p_i}\right) y_{i+1} = \frac{r_i}{p_i} \quad \ldots (iii)$$

And, discretizing (i) by Control Volume Method, we had reached upto:

$$p_{i+\frac{1}{2}} \times \left(\frac{y_{i+1}-y_i}{\delta x_i}\right) - p_{i-\frac{1}{2}} \times \left(\frac{y_i - y_{i-1}}{\delta x_{i-1}}\right) + y_i \times \left[\frac{q_{i_-} \times \delta x_{i-1} + q_{i_+} \times \delta x_i}{2}\right]$$

$$= \frac{r_{i_-} \times \delta x_{i-1} + r_{i_+} \times \delta x_i}{2}$$

Now, given $p(x)$, $q(x)$, $r(x)$ are continuous. So, $q_{i_-} = q_{i_+} = q_i$ and $r_{i_-} = r_{i_+} = r_i$.

Also, we take uniform step size. So, let $\delta x_i = \delta x_{i-1} = h$.

Then, $p_{i+\frac{1}{2}} \times \left(\frac{y_{i+1}-y_i}{h}\right) - p_{i-\frac{1}{2}} \times \left(\frac{y_i - y_{i-1}}{h}\right) + y_i \times (q_i h) = r_i h$

$$\Rightarrow \left(\frac{p_{i-\frac{1}{2}}}{h^2}\right) y_{i-1} + \left(q_i - \frac{p_{i+\frac{1}{2}}}{h^2} - \frac{p_{i-\frac{1}{2}}}{h^2}\right) y_i + \left(\frac{p_{i+\frac{1}{2}}}{h^2}\right) y_{i+1} = r_i$$

Now, $p_{i-\frac{1}{2}} \simeq \frac{p_{i-1}+p_i}{2}$ and $p_{i+\frac{1}{2}} \simeq \frac{p_i + p_{i+1}}{2}$.

Putting these, we get:

$$\left(\frac{p_{i-1}+p_i}{2h^2}\right) y_{i-1} + \left(q_i - \frac{p_i + p_{i+1}}{2h^2} - \frac{p_{i-1}+p_i}{2h^2}\right) y_i + \left(\frac{p_i + p_{i+1}}{2h^2}\right) y_{i+1} = r_i$$

Dividing by p_i throughout, we get:

$$\left(\frac{\frac{p_{i-1}}{p_i} + 1}{2h^2}\right) y_{i-1} + \left(\frac{q_i}{p_i} - \frac{1}{h^2} - \frac{p_{i+1}+p_{i-1}}{2h^2 p_i}\right) y_i + \left(\frac{1 + \frac{p_{i+1}}{p_i}}{2h^2}\right) y_{i+1} = \frac{r_i}{p_i}$$

$$\Rightarrow \left(\frac{\frac{p_{i-1}}{p_i} - 1 + 2}{2h^2}\right) y_{i-1} + \left(\frac{q_i}{p_i} - \frac{1}{h^2} - \frac{\frac{p_{i+1}+p_{i-1}}{2}}{h^2 p_i}\right) y_i + \left(\frac{2 + \frac{p_{i+1}}{p_i} - 1}{2h^2}\right) y_{i+1}$$

$$= \frac{r_i}{p_i}$$

$$\Rightarrow \left(\frac{1}{h^2} - \frac{p_i - p_{i-1}}{2h^2 p_i}\right) y_{i-1} + \left(\frac{q_i}{p_i} - \frac{1}{h^2} - \frac{\frac{p_{i+1}+p_{i-1}}{2}}{h^2 p_i}\right) y_i + \left(\frac{1}{h^2} + \frac{p_{i+1}-p_i}{2h^2 p_i}\right) y_{i+1}$$

$$= \frac{r_i}{p_i}$$

Now, from forward and backward difference approximation,

$$\frac{p_i - p_{i-1}}{h} \simeq p_i' \quad \text{and} \quad \frac{p_{i+1} - p_i}{h} \simeq p_i'$$

Also, $\frac{p_{i+1} + p_{i-1}}{2} \simeq p_i$. Using these approximations, we get :

$$\left(\frac{1}{h^2} - \frac{p_i'}{2hp_i}\right) y_{i-1} + \left(\frac{q_i}{p_i} - \frac{1}{h^2} - \frac{1}{h^2}\right) y_i + \left(\frac{1}{h^2} + \frac{p_i'}{2hp_i}\right) y_{i+1} = \frac{r_i}{p_i}$$

$$\Rightarrow \left(\frac{1}{h^2} - \frac{p_i'}{2hp_i}\right) y_{i-1} + \left(\frac{q_i}{p_i} - \frac{2}{h^2}\right) y_i + \left(\frac{1}{h^2} + \frac{p_i'}{2hp_i}\right) y_{i+1} = \frac{r_i}{p_i} \quad \ldots \text{(iv)}$$

Clearly, (iii) and (iv) are identical. This shows that the discretization in Control Volume method reduced to the tridiagonal central difference scheme if $p(x), q(x), r(x)$ are continuous. [Proved.]

5) Show that : $\left.\frac{dy}{dx}\right|_{x_{i+1/2}} = \frac{y_{i+1} - y_i}{\delta x_i} + O(\delta x_i^2)$.

We have : $y_{i+1} = y(x_{i+1}) = y(x_i + \delta x_i) = y\left(x_{i+1/2} + \frac{\delta x_i}{2}\right)$

$$\Rightarrow y_{i+1} = y_{i+1/2} + \frac{\delta x_i}{2} \times y'_{i+1/2} + \frac{\left(\frac{\delta x_i}{2}\right)^2}{2!} \times y''_{i+1/2} + \frac{\left(\frac{\delta x_i}{2}\right)^3}{3!} \times y'''_{i+1/2} + \cdots \quad \left[\text{Taylor expansion about } x_{i+1/2}\right] \quad \ldots \text{(i)}$$

And, $y_i = y(x_i) = y\left(x_{i+1/2} - \frac{\delta x_i}{2}\right)$

$$\Rightarrow y_i = y_{i+1/2} - \frac{\delta x_i}{2} \times y'_{i+1/2} + \frac{\left(\frac{\delta x_i}{2}\right)^2}{2!} \times y''_{i+1/2} - \frac{\left(\frac{\delta x_i}{2}\right)^3}{3!} \times y'''_{i+1/2} + \cdots \quad \ldots \text{(ii)}$$

From (i) - (ii), we get :-

$$y_{i+1} - y_i = \delta x_i \times y'_{i+1/2} + \frac{2 \times \left(\frac{\delta x_i}{2}\right)^3}{3!} \times y'''_{i+1/2} + \cdots$$

$$\Rightarrow \frac{y_{i+1} - y_i}{\delta x_i} = y'_{i+1/2} + \frac{1}{24} \times (\delta x_i)^2 \times y'''_{i+1/2} + \cdots$$

$$\Rightarrow \frac{y_{i+1} - y_i}{\delta x_i} = \left.\frac{dy}{dx}\right|_{x_{i+1/2}} + O(\delta x_i^2)$$

[Proved.] Hence this is $O(\delta x_i^2)$ central diff. approximation.

6) Solve the BVP : $y'' - 2y = 0$; $y(0) = 1$, $y'(1) = 0$

for step size $h = 0.2$ with the method of defining fictitious points for derivative boundary conditions.

Soln : Central difference scheme discretization gives :

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - 2y_i = 0 \quad \Rightarrow \left(\frac{1}{h^2}\right)y_{i-1} - \left(2 + \frac{2}{h^2}\right)y_i + \left(\frac{1}{h^2}\right)y_{i+1} = 0 \quad \ldots \text{(i)}$$

Here, $y_0 = 1$, but y_n is unknown. So there are n unknowns →

$y_1, y_2, \ldots, y_n$

If we introduce a fictitious point x_{n+1} and correspondingly, y_{n+1},

then, $y'(1) = 0 \Rightarrow y_n' = 0 \Rightarrow \dfrac{y_{n+1} - y_{n-1}}{2h} = 0 \Rightarrow y_{n+1} = y_{n-1} \quad \ldots \text{(ii)}$

So, put $i = 1, 2, 3, \ldots, n$ in (i) and we get n equations,

however, for $i = n$, replace y_{n+1} by y_{n-1}.

For $h = 0.2$, we get the system of equations : $(h = 0.2 \Rightarrow n = 5)$

$$\begin{bmatrix} -52 & 25 & 0 & 0 & 0 \\ 25 & -52 & 25 & 0 & 0 \\ 0 & 25 & -52 & 25 & 0 \\ 0 & 0 & 25 & -52 & 25 \\ 0 & 0 & 0 & 50 & -52 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} -25 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

as $h = \frac{1}{5} \Rightarrow \frac{1}{h^2} = 25$,

$\frac{1}{h} = 5$

On solving, we get :

$y_1 = y(0.2) = 0.7865088$, $y_2 = y(0.4) = 0.6359383$, $y_3 = y(0.6) = 0.536243$,

$y_4 = y(0.8) = 0.4794468$, $y_5 = y(1) = 0.46100658$

7) Solve the BVP : $y'' + 2xy' + 2y = 4x$; $y(0) = 1$, $y(0.5) = 1.279$

for $h = 0.1$.

Soln: Discretization of this ODE by Central difference scheme gives :

$$\left(\frac{1}{h^2} - \frac{2x_i}{2h}\right) y_{i-1} + \left(2 - \frac{2}{h^2}\right) y_i + \left(\frac{1}{h^2} + \frac{2x_i}{2h}\right) y_{i+1} = 4x_i$$

$$\Rightarrow \left(\frac{1}{h^2} - \frac{x_i}{h}\right) y_{i-1} + \left(2 - \frac{2}{h^2}\right) y_i + \left(\frac{1}{h^2} + \frac{x_i}{h}\right) y_{i+1} = 4x_i \quad \cdots (i)$$

For $h = 0.1$, $x_0 = 0$, $x_n = 0.5$, we have $n = 5$.

In (i), putting $i = 1, 2, 3, 4$. we get the following system of eqns :

$$\begin{bmatrix} -198 & 101 & 0 & 0 \\ 98 & -198 & 102 & 0 \\ 0 & 97 & -198 & 103 \\ 0 & 0 & 96 & -198 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -493/5 \\ 4/5 \\ 6/5 \\ -131.416 \end{bmatrix}$$

On solving, we get :

$y_1 = y(0.1) = 1.0902945$, $y_2 = y(0.2) = 1.16117143$, $y_3 = y(0.3) = 1.214344$,

$y_4 = y(0.4) = 1.25249$

8) Solve the BVP : $y'' - 2y = 0$; $y(0) = 1$, $y'(1) = 0$ for step size $h = 0.2$ with 2nd order backward difference approximation for derivative boundary conditions.

Soln: Central difference scheme discretization gives :

$$\left(\frac{1}{h^2}\right) y_{i-1} - \left(2 + \frac{2}{h^2}\right) y_i + \left(\frac{1}{h^2}\right) y_{i+1} = 0 \quad \cdots (i)$$

Here, $y_0 = 1$, but y_n is unknown . So there are n unknowns →

$y_1, y_2, \ldots, y_n$.

Using 2nd order backward difference approximation for $y'(1)$,

we get : $y_n' = y'(1) = 0$ And, $y_n' = \dfrac{3y_n - 4y_{n-1} + y_{n-2}}{2h}$ (Backward diff. approx.)

So, $y_n' = 0 \Rightarrow \dfrac{3y_n - 4y_{n-1} + y_{n-2}}{2h} = 0 \Rightarrow 3y_n - 4y_{n-1} + y_{n-2} = 0$ ...(ii)

Now, putting $i = 1, 2, \ldots, n-1$ in (i) and taking (ii), we get n equations.

For $h = 0.2$, we get the system of equations : $(h = 0.2 \Rightarrow n = 5)$

$$\begin{bmatrix} -52 & 25 & 0 & 0 & 0 \\ 25 & -52 & 25 & 0 & 0 \\ 0 & 25 & -52 & 25 & 0 \\ 0 & 0 & 25 & -52 & 25 \\ 0 & 0 & 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} -25 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving, we get :

$y_1 = y(0.2) = 0.7861564$, $y_2 = y(0.4) = 0.63520533$, $y_3 = y(0.6) = 0.5350707$,

$y_4 = y(0.8) = 0.4777417$, $y_5 = y(1) = 0.4586320$