1. UFD, PID, ED and irreducibility of polynomials

(1) Show that $\mathbb{R}[x,y]$ and $\mathbb{Z}[x]$ are not PIDs.
(2) Prove that $2, 3, 1 \pm \sqrt{-5}$ are irreducible elements in $\mathbb{Z}[\sqrt{-5}]$.
(3) Show that $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[w]$ are EDs, where $w = \exp^{2\pi i/3}$.
(4) Let F be a subfield of $\mathbb{C}$. Show that an irreducible polynomial in $F[x]$ has no multiple roots.
(5) Show that an integer prime p is a prime element of $\mathbb{Z}[\sqrt{3}]$ if and only if $x^2 - 3$ is irreducible in $\mathbb{F}_p[x]$.
(6) Factor $30, 10, 1 - 3i$ into product of Gaussian primes.
(7) Prove that every Gaussian prime divides exactly one integer prime.
(8) Prove that $f, g \in \mathbb{Z}[x]$ are relatively prime in $\mathbb{Q}[x]$ if and only if $(f,g)\mathbb{Z}[x] \cap \mathbb{Z} \neq (0)$.
(9) Prove that polynomials $f(x) = x^2 + 26x + 213$, $g(x) = 8x^3 - 6x + 1$ are irreducible over $\mathbb{Q}[x]$.
(10) Factor $x^5 + 5x + 5$ into irreducible factors in $\mathbb{Q}[x]$ and $\mathbb{F}_2[x]$.
(11) Factor $x^3 + x + 1$ in $\mathbb{F}_p[x]$ where $p = 2, 3, 5$.
(12) Let $f(x)$ be a monic integer polynomial of positive degree having a rational root r. Show that r is an integer.
(13) Let $R = \mathbb{Z}[w]$, where $w = \exp^{2\pi i/3}$ (a) Decompose 3 into irreducible factors in R.
(b) Let $p \neq 3$ be a rational prime. Show that if $x^2 + x + 1$ has a root in $\mathbb{F}_p$, then $p \equiv 1 (\mathrm{mod}\ 3)$.
(c) Show that (p) is a prime ideal of R if and only if $p \neq a^2 - ab + b^2$ for some integers a and b.
(This question is optional).