

# Product Form of Inverse( PFI)of a Basis Matrix and Revised Simplex Method (RSM)

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We wish to compute the inverse of a basis matrix,  $B_c$ , that is differ by one column from the basis matrix,  $B$ , whose inverse is known. The product form of the inverse allows us to determine this new inverse in an efficient manner. We want to find  $B_c^{-1}$ .

First, let us consider the following definitions:

$B$  is the original basis matrix of size  $m \times m$ .

$B_c$  is the new basis matrix, which is identical to  $B$  except for the column  $r$ .

$c$  is the  $r$ th column of matrix  $B_c$ , the only column different from those in  $B$ .

$$e = B^{-1}c = (e_1, e_2, \dots, e_m)^T \quad (1)$$

$$\eta = \left( -\frac{e_1}{e_r}, \dots, -\frac{e_{r-1}}{e_r}, \frac{1}{e_r}, -\frac{e_{r+1}}{e_r}, \dots, -\frac{e_m}{e_r} \right)^T, e_r \neq 0 \quad (2)$$

where  $e_r$  is the  $r$ -th component of  $\mathbf{e}$  as computed in (1) and  $m$  is the total number of elements of the column vector  $\mathbf{e}$ . Thus,

$$B_c^{-1} = E_r B^{-1} \quad (3)$$

where  $B_c^{-1}$  = inverse of  $B_c$

$B^{-1}$  = inverse of the previous matrix

$E_r$  = an identity matrix with its  $r$ -th column replaced by  $\eta$ .

We now use (3) to illustrate the computation of the inverse of a basis matrix that differs by only a single column from another basis matrix, whose inverse is known.

**Example 1:**

Consider the two matrices shown below. Both are non-singular and differ by only one column, the first. The inverse of  $B$  is given and we wish to find the inverse of  $B_c$ .

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_c = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \quad B^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

We first compute  $\mathbf{e}$  from (1), where

$$\mathbf{c}_1 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

(i.e., the first column in  $B_c$ )

$$\mathbf{e} = B^{-1}c_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

Next, from (2) we establish  $\eta$ :

$$\eta = \begin{pmatrix} \frac{1}{2} \\ -1 \\ -2 \end{pmatrix}$$

Thus,

$$E_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$B_c^{-1} = E_1 B^{-1}$$

$$B_c^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

### Example 2:

Consider the two matrices shown below. Both are non singular and differ by only one column, the second. The inverse of  $B$  is given and we wish to find the inverse of  $B_c$ .

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_c = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix} \quad B_c^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

We first compute  $\mathbf{e}$  from (1), where

$$\mathbf{c}_2 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

(i.e., the second column in  $B_c$ )

$$\mathbf{e} = B^{-1}c_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

Next, from (2) we establish  $\eta$ :

$$\eta = \begin{pmatrix} -1 \\ \frac{1}{2} \\ -2 \end{pmatrix}$$

Thus,

$$E_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -2 & 1 \end{pmatrix}$$



$$B_c^{-1} = E_2 B^{-1}$$

$$B_c^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

### Example 3:

Consider the two matrices shown below. Both are non singular and differ by only one column, the third. The inverse of  $B$  is given and we wish to find the inverse of  $B_c$ .

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_c = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix} \quad B_c^{-1} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

We first compute  $\mathbf{e}$  from (1), where

$$\mathbf{c}_3 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

(i.e., the third column in  $B_c$ )

$$\mathbf{e} = B^{-1}c_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

Next, from (2) we establish  $\eta$ :

$$\eta = \begin{pmatrix} -1/2 \\ -1/2 \\ 1/4 \end{pmatrix}$$

Thus,

$$E_3 = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$B_c^{-1} = E_3 B^{-1}$$

$$B_c^{-1} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Let  $B$  be a basis matrix of size  $m \times m$ .

Let  $B = I_{m \times m}$  (an identity matrix of size  $m \times m$ )

Then  $B = B^{-1} = I_{m \times m}$ .

Let  $B_1, B_2, \dots, B_m$  are  $m$  non-singular matrices of size  $m \times m$ .

$B$  and  $B_1$  are differ by first column.

$B_1$  and  $B_2$  are differ by second column.

$B_2$  and  $B_3$  are differ by third column.

$B_3$  and  $B_4$  are differ by fourth column.

$\vdots$

$B_{m-1}$  and  $B_m$  are differ by  $m$ -th column.

Now  $B_1^{-1} = E_1 B^{-1} = E_1 I_{m \times m} = E_1$

Then  $B_2^{-1} = E_2 B_1^{-1} = E_2 E_1$

$B_3^{-1} = E_3 B_2^{-1} = E_3 E_2 E_1$

$B_4^{-1} = E_4 B_3^{-1} = E_4 E_3 E_2 E_1$

$B_m^{-1} = E_m B_{m-1}^{-1} = E_m E_{m-1} \dots E_1$

where  $E_r, (r = 1, 2, \dots, m)$  is defined in equation (3).

### Example 4:

Consider four different matrices shown below. All are non-singular matrices and differ by only one column. The inverse of the matrices are computed as follows:

$$B = B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_2 = \begin{pmatrix} 2 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 6 & 1 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 2 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 6 & 5 \end{pmatrix} = B_{new}$$

We first compute  $\mathbf{e}$  from (1), where

$$c_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

From  $B$  and  $B_1$  we find  $c_1$ .

(i.e., the first column in  $B_1$ )

$$\mathbf{e} = B^{-1}\mathbf{c}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

Next, from (2) we establish  $\eta$ :

$$\eta = \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}$$

Thus,

$$E_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B_1^{-1} = E_1 B^{-1}$$

$$B_1^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Then we compute  $c_2$ .

$$c_2 = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$$

From  $B_1$  and  $B_2$  we find  $c_2$ .

(i.e., the second column in  $B_2$ )

$$\mathbf{e} = B^{-1} \mathbf{c}_2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$$

Next, from (2) we establish  $\eta$ :

$$\eta = \begin{pmatrix} -1 \\ 1/2 \\ -3 \end{pmatrix}$$

Thus,

$$E_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

$$B_2^{-1} = E_2 B_1^{-1} = E_2 E_1$$

$$B_2^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

Then we compute  $c_3$ .

$$c_3 = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$$

(i.e., the third column in  $B_3$ )

$$\mathbf{e} = B_2^{-1} \mathbf{c}_3 = \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$$

Next, from (2) we establish  $\eta$ :

$$\eta = \begin{pmatrix} 0 \\ 0 \\ 1/5 \end{pmatrix}$$

Thus,

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{pmatrix}$$

$$B_3^{-1} = E_3 B_2^{-1} = E_3 E_2 E_1 = B_{new}^{-1}.$$

$$B_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{pmatrix} \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3/5 & 1/5 \end{pmatrix}$$

$$\text{Hence } B_{new}^{-1} = B_3^{-1} = \begin{pmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3/5 & 1/5 \end{pmatrix} = E_3 E_2 E_1$$

## Revised Simplex Method

Original simplex method calculates and stores all numbers in the simplex Tableau. Many are not needed.

Revised Simplex Method (more efficient for computing):

It is used in all commercial packages (e.g. IBM MPSX, CDC APEX III).

$$LPP \quad \max : \quad Z = c^T x$$

Subject to

$$Ax \leq b, \quad b \geq 0$$

$$x \geq 0.$$

Initially constraints becomes (standard form):

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ x_s \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

$x_s$  = slack variables

**Basis matrix:** Column relating to basic variables.

$$B = \begin{pmatrix} B_{11} & \dots & \dots & \dots & B_{1m} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ B_{m1} & \dots & \dots & \dots & B_{mm} \end{pmatrix}_{m \times m}$$

Initially  $B = I_{m \times m}$ ,  $B^{-1} = I_{m \times m}$ .



**Basic variable values:**  $X_B = \begin{pmatrix} X_{B1} \\ \dots \\ \dots \\ \dots \\ X_{Bm} \end{pmatrix}$

At any iteration all the non-basic variables are zero.

$$BX_B = b$$

Therefore  $X_B = B^{-1}b$  where  $B^{-1}$ , inverse basis matrix.

At any iteration, given the original  $b$  vector and the inverse matrix  $B^{-1}$ ,  $X_B$  can be calculated.

$Z = c_B^T X_B$ , where  $c_B$  = objective coefficients of basic variables.

## Steps in the Revised Simplex Method

**Step 1.** Determine the entering variable,  $x_j$ , with associated vector  $P_j$ .

–compute  $Y = c_B^T B^{-1}$

–compute  $z_j - c_j = Y P_j - c_j$  for all non-basic variables.

Select the largest negative value (For Max type LPP) among all  $z_j - c_j$ .

Break the ties arbitrarily. If all the  $z_j - c_j \geq 0$ , optimal solution is reached.

$$X_B = B^{-1}b$$

$$Z = c_B^T X_B$$

Otherwise go to Step 2.

**Step 2.** Determine leaving variable,  $x_r$ , with associated vector  $P_r$ .

–compute the current basic variable  $X_B = B^{-1}b$

– compute constraint coefficients of entering variables for  $P_j$ :

$$\alpha^j = B^{-1}P_j$$

Leaving variable  $x_r$  must be associated with

$$\theta = \min_k \left\{ \frac{(B^{-1}b)_k}{\alpha_k^j}, \alpha_k^j > 0 \right\}.$$

using minimum ratio rule.

If  $\alpha_k^j \leq 0, \forall k$ , then the problem is unbounded.

**Step 3.** Determination of the next basis matrix and  $B_{next}^{-1}$

For the given  $B^{-1}$  the  $B_{next}^{-1}$  is computed by

$B_{next}^{-1} = E_r B^{-1}$ , where  $r$  is the column number of the entering vector

Set  $B^{-1} = B_{next}^{-1}$

Go to step 1. Note  $E$  is computed using equation (3).

(See the next slide for the numerical example) Note: If you detect any typo please inform me.