# ASSIGNMENT - 2

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## Advanced Numerical Techniques

1) For the following Higher order linear BVP, find the coupled system of equations in matrix form :

$$y''' + a(x)y'' + b(x)y' + c(x)y = d(x) \quad ; \quad 0 < x < a$$

with BCs : $y(0) = y_0$ , $y'(0) = y_0'$ , $y'(a) = y_a'$

Soln: Set $p = \frac{dy}{dx}$ ....(i) Then, given BVP becomes :

$$p'' + a(x)p' + b(x)p + c(x)y = d(x) \quad \text{....(ii)}$$

with BCs : $y(0) = y_0$ , $p(0) = y_0'$ , $p(a) = y_a'$

Now, to discretize (i) at the grid point $x_i$, integrate (i) between $x_{i-1}$ to $x_i$ using Trapezoidal rule :

$$\int_{x_{i-1}}^{x_i} dy = \int_{x_{i-1}}^{x_i} p.dx \Rightarrow y_i - y_{i-1} = \frac{h}{2} \times [p_i + p_{i-1}] + O(h^2)$$

$$\Rightarrow y_i - y_{i-1} - \frac{h}{2} \times (p_i + p_{i-1}) = 0 \quad \text{....(iii)}$$

And, applying the Central diff. scheme in (ii), we get :

$$\frac{p_{i+1} - 2p_i + p_{i-1}}{h^2} + a_i \times \frac{p_{i+1} - p_{i-1}}{2h} + b_i p_i + c_i y_i = d_i \quad \text{....(iv)}$$

$i = 1, 2, \ldots, n-1$

Now these coupled system of can be combined to a block matrix form as :

$$\underbrace{\begin{bmatrix} -1 & -\frac{h}{2} \\ 0 & \left(\frac{1}{h^2} - \frac{a_i}{2h}\right) \end{bmatrix}}_{A_i} \begin{bmatrix} y_{i-1} \\ p_{i-1} \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & -\frac{h}{2} \\ c_i & \left(b_i - \frac{2}{h^2}\right) \end{bmatrix}}_{B_i} \begin{bmatrix} y_i \\ p_i \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{1}{h^2} + \frac{a_i}{2h}\right) \end{bmatrix}}^{C_i} \begin{bmatrix} y_{i+1} \\ p_{i+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ d_i \end{bmatrix}}_{D_i} \quad \text{....(v)}$$

Let $X_i = \begin{bmatrix} y_i \\ p_i \end{bmatrix}$. Then the system in (v) can be written as:

$$A_i X_{i-1} + B_i X_i + C_i X_{i+1} = D_i \quad \ldots \text{(vi)}$$

where $X_0$ is known [$\because$ $y_0$ and $p_0$ are known]

Home Task was to find these block matrices $A_i$, $B_i$, $C_i$, $D_i$.

These are: $A_i = \begin{bmatrix} -1 & \frac{-h}{2} \\ 0 & \left(\frac{1}{h^2} - \frac{a_i}{2h}\right) \end{bmatrix}$, $B_i = \begin{bmatrix} 1 & \frac{-h}{2} \\ c_i & \left(b_i - \frac{2}{h^2}\right) \end{bmatrix}$,

$C_i = \begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{1}{h^2} + \frac{a_i}{2h}\right) \end{bmatrix}$ and $D_i = \begin{bmatrix} 0 \\ d_i \end{bmatrix}$. [Ans.]

2) Solve: $y^{(4)} + 81y = 81x^2$ with BCs: $y(0) = y(1) = y''(0) = y''(1) = 0$ for step size $h = 0.25$.

Soln: Let $z = y''$. Then the BVP becomes:

$$z'' + 81y = 81x^2 \quad \text{and} \quad y'' - z = 0$$

with BCs: $y(0) = y(1) = 0$, $z(0) = z(1) = 0$

Discretizing both using Central difference scheme:

$$\frac{z_{i+1} - 2z_i + z_{i-1}}{h^2} + 81y_i = 81x_i^2 \quad \text{and} \quad \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - z_i = 0$$

In block matrix form, we get: Taking $X_i = \begin{bmatrix} y_i \\ z_i \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 0 & \frac{1}{h^2} \\ \frac{1}{h^2} & 0 \end{bmatrix}}_{A_i} X_{i-1} + \underbrace{\begin{bmatrix} 81 & \frac{-2}{h^2} \\ \frac{-2}{h^2} & -1 \end{bmatrix}}_{B_i} X_i + \underbrace{\begin{bmatrix} 0 & \frac{1}{h^2} \\ \frac{1}{h^2} & 0 \end{bmatrix}}_{C_i} X_{i+1} = \underbrace{\begin{bmatrix} 81x_i^2 \\ 0 \end{bmatrix}}_{D_i}$$

So, $A_i X_{i-1} + B_i X_i + C_i X_{i+1} = D_i$. In matrix form, we get:

Taking $h = 0.25 \Rightarrow h = \frac{1}{4}$.

$$\begin{bmatrix} 81 & -32 & 0 & 16 & 0 & 0 \\ -32 & -1 & 16 & 0 & 0 & 0 \\ 0 & 16 & 81 & -32 & 0 & 16 \\ 16 & 0 & -32 & -1 & 16 & 0 \\ 0 & 0 & 0 & 16 & 81 & -32 \\ 0 & 0 & 16 & 0 & -32 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ z_1 \\ y_2 \\ z_2 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 5.0625 \\ 0 \\ 20.25 \\ 0 \\ 45.5625 \\ 0 \end{bmatrix}$$

On solving, we get: $y_1 \approx y(0.25) \simeq 0.1008276455$ [Ans.]

$y_2 \simeq y(0.5) \simeq 0.163436141$

$y_3 \simeq y(0.75) \simeq 0.13747923$

3) Solve: $y^{(4)} - y''' + y = x^2$ ; $y(0) = y'(0) = 0$ , $y(1) = 2$ , $y'(1) = 0$

for step size $h = \frac{1}{3}$.

Soln: Discretizing the BVP using Central difference scheme:

$$\frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4} - \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2h^3} + y_i = x_i^2$$

$$\Rightarrow \left(\frac{1}{h^4} + \frac{1}{2h^3}\right) y_{i-2} + \left(\frac{-4}{h^4} - \frac{1}{h^3}\right) y_{i-1} + \left(\frac{6}{h^4} + 1\right) y_i + \left(\frac{-4}{h^4} + \frac{1}{h^3}\right) y_{i+1} + \left(\frac{1}{h^4} - \frac{1}{2h^3}\right) y_{i+2} = x_i^2$$

Putting $h = \frac{1}{3}$, this becomes:

$$94.5\, y_{i-2} - 351\, y_{i-1} + 487 y_i - 297\, y_{i+1} + 67.5\, y_{i+2} = x_i^2 \quad \ldots (i) \quad \text{for } i = 1,2$$

Now, from the derivative BCs, we get:

$y_0' = 0 \Rightarrow \frac{y_1 - y_{-1}}{2h} = 0 \Rightarrow y_{-1} = y_1$ And, $y_3' = 0 \Rightarrow \frac{y_4 - y_2}{2h} = 0 \Rightarrow y_4 = y_2$

And, we have: $y(0) = y_0 = 0$ and $y(1) = y_3 = 2$.

Putting $i=1$ in (i), $94.5\, y_{-1} - 351 y_0 + 487 y_1 - 297 y_2 + 67.5\, y_3 = x_1^2$

$\Rightarrow 94.5 y_1 + 0 + 487 y_1 - 297 y_2 + 135 = \frac{1}{9}$

$\Rightarrow 581.5\, y_1 - 297\, y_2 = -134.888$ ...... (ii)

Now, putting $i=2$ in ①, $94.5y_0 - 351y_1 + 487y_2 - 297y_3 + 67.5y_4 = x_2^2$

$$\Rightarrow 0 - 351y_1 + 487y_2 - 297\times 2 + 67.5y_2 = \frac{4}{9}$$

$$\Rightarrow -351y_1 + 554.5y_2 = 594.444 \quad \ldots\ldots \text{(iii)}$$

On solving (ii) and (iii), we get :

$$y_1 \simeq y\left(\tfrac{1}{3}\right) \simeq 0.466347 \quad , \quad y_2 \simeq y\left(\tfrac{2}{3}\right) \simeq 1.367235 \qquad [\text{Ans.}]$$

4) Let $P_k(x)$ be a polynomial which interpolates $y=f(x)$ in $[x_k, x_{k+1}]$ using Cubic Spline Interpolation.

Given that, $P_k(x) = \frac{M_k}{h} \times \frac{(x_{k+1}-x)^3}{6} + \frac{M_{k+1}}{h} \times \frac{(x-x_k)^3}{6} + C_k(x-x_k) + D_k(x_{k+1}-x)$

where $f''(x_k) = M_k$ and $h = x_{i+1} - x_i \quad \forall i = 0,1,2,\ldots,n-1$

Find $C_k$, $D_k$, using $P_k(x_k) = f_k = P_{k+1}(x_k)$.

Soln: We have, $P_k(x) = \frac{M_k}{h} \times \frac{(x_{k+1}-x)^3}{6} + \frac{M_{k+1}}{h} \times \frac{(x-x_k)^3}{6} + C_k(x-x_k) + D_k(x_{k+1}-x)$

Using $P_k(x_k) = f_k$, we get :

$$\frac{M_k}{h} \times \frac{(x_{k+1}-x_k)^3}{6} + \frac{M_{k+1}}{h} \times 0 + C_k \times 0 + D_k \times (x_{k+1}-x_k) = f_k$$

$$\Rightarrow \frac{M_k}{6} \times h^2 + D_k \times h = f_k \qquad [\because x_{k+1} - x_k = h]$$

$$\Rightarrow D_k = \frac{1}{h} \times \left(f_k - \frac{M_k}{6} h^2\right)$$

Now, using $P_k(x_{k+1}) = f_{k+1}$, we get :

$$\frac{M_k}{h} \times 0 + \frac{M_{k+1}}{h} \times \frac{(x_{k+1}-x_k)^3}{6} + C_k \times (x_{k+1}-x_k) + D_k \times 0 = f_{k+1}$$

$$\Rightarrow \frac{M_{k+1}}{6} h^2 + C_k h = f_{k+1} \quad \Rightarrow C_k = \frac{1}{h} \times \left(f_{k+1} - \frac{M_{k+1}}{6} h^2\right)$$

$\therefore$ Using Cubic Spline interpolation, the polynomial which interpolates $f(x)$ in the interval $[x_k, x_{k+1}]$ is given by :

$$P_k(x) = \frac{M_k}{h} \times \frac{(x_{k+1}-x)^3}{6} + \frac{M_{k+1}}{h} \times \frac{(x-x_k)^3}{6} + \frac{1}{h}\left(f_{k+1} - \frac{M_{k+1}}{6}h^2\right)(x-x_k)$$

$$+ \frac{1}{h}\left(f_k - \frac{M_k}{6}h^2\right)(x_{k+1}-x)$$

[Ans.]

5)

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$ | 1.5 | 2.2 | 3.1 | 4.3 |

Let there be free end conditions. Find $y(1.2)$ and $y'(1)$ by Spline interpolation technique.

Find all the cubic polynomials for all intervals, and find the overall Interpolation polynomial $S(x)$.

Soln: We have $h = 1$ and $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$

Due to free end conditions, $M_0 = M_3 = 0$.

And we have the equations :

$$M_0 + 4M_1 + M_2 = 6 \times (y_2 - 2y_1 + y_0)$$

$$\Rightarrow 0 + 4M_1 + M_2 = 6 \times (3.1 - 2 \times 2.2 + 1.5) \Rightarrow 4M_1 + M_2 = 6 \times 0.2$$

$$\Rightarrow 4M_1 + M_2 = 1.2 \quad \dots (i)$$

And, $M_1 + 4M_2 + M_3 = 6 \times (y_3 - 2y_2 + y_1)$

$$\Rightarrow M_1 + 4M_2 = 6 \times (4.3 - 2 \times 3.1 + 2.2) \Rightarrow M_1 + 4M_2 = 1.8 \quad \dots (ii)$$

Solving (i) and (ii), we get : $M_1 = 0.2$, $M_2 = 0.4$.

Now, $P_0(x) = 0 + \frac{0.2}{6} \times \left[(x-1)^3 - (x-1)\right] + 1.5 \times (2-x) + 2.2 \times (x-1)$ ; $x \in [1,2]$

$$= \frac{1}{30} \times \left[x^3 - 3x^2 + 3x - 1 - x + 1\right] + 3 - \frac{3}{2}x + \frac{22}{10}x - 2.2$$

$$= \frac{1}{30}x^3 - \frac{1}{10}x^2 + \frac{23}{30}x + \frac{8}{10} \quad ; \quad x \in [1,2] \quad \dots (iii)$$

And, $P_1(x) = \frac{0.2}{6} \times \left[(3-x)^3 - (3-x)\right] + \frac{0.4}{6} \times \left[(x-2)^3 - (x-2)\right] + 2.2 \times (3-x)$
$+ 3.1 \times (x-2)$

$= \frac{1}{30} \times \left[27 - 27x + 9x^2 - x^3 - 3 + x\right] + \frac{2}{30} \times \left[x^3 - 6x^2 + 12x - 8 - x + 2\right]$

$+ 6.6 - 2.2x + 3.1x - 6.2 \quad ; \quad x \in [2,3]$

$= \frac{1}{30}x^3 - \frac{1}{10}x^2 + \frac{23}{30}x + \frac{8}{10} \quad \ldots\ldots \text{(iv)} \quad ; x \in [2,3]$

$P_2(x) = \frac{0.4}{6} \times \left[(4-x)^3 - (4-x)\right] + 3.1 \times (4-x) + 4.3 \times (x-3) \quad ; \quad x \in [3,4]$

$= \frac{2}{30} \times \left[64 - 48x + 12x^2 - x^3 - 4 + x\right] + 12.4 - 3.1x + 4.3x - 12.9$

$= \frac{-1}{15}x^3 + \frac{8}{10}x^2 - \frac{29}{15}x + \frac{7}{2} \quad ; \quad x \in [3,4] \quad \ldots\ldots \text{(v)}$

From (iii), (iv), (v), we get the final interpolating curve as:

$$S(x) = \begin{cases} \frac{1}{30}x^3 - \frac{1}{10}x^2 + \frac{23}{30}x + \frac{8}{10} & ; \quad x \in [1,3] \\ \frac{-1}{15}x^3 + \frac{8}{10}x^2 - \frac{29}{15}x + \frac{7}{2} & ; \quad x \in [3,4] \end{cases}$$ [Ans.]

Then, $y(1.2) = P_0(1.2) = \frac{1021}{625} = \underline{1.6336}$ [Ans.]

And, $y'(1) = P_0'(1) = \frac{2}{3} = \underline{0.667}$ [Ans.]

6) Solve by spline interpolation:
$y'' - y = 0 \quad ; \quad y(0) = y(1) = 1$ for step size $h = \frac{1}{2}$.
Compare the solution with that obtained using finite difference method, and also the analytic solution.

<u>Soln:</u> Using Spline interpolation, we have:
$y_k'' - y_k = 0 \quad \Rightarrow M_k - y_k = 0 \quad ; k = 0, 1, 2$

Given $y_0 = 1$, $y_2 = 1$. So, $M_0 = y_0 = 1$ and $M_2 = y_2 = 1$

And, we also have :

$$M_{k-1} + 4M_k + M_{k+1} = \frac{6}{h^2} \times (y_{k-1} - 2y_k + y_{k+1}) \quad ; \; k=1$$

$$\Rightarrow M_0 + 4M_1 + M_2 = 24(1 - 2y_1 + 1) \qquad [\because y_0 = y_2 = 1]$$

$$\Rightarrow 2 + 4M_1 = 24 - 48y_1 + 24$$

$$\Rightarrow 2 + 4y_1 = 24 - 48y_1 + 24 \qquad [\because M_1 = y_1]$$

$$\Rightarrow 52y_1 = 46 \quad \Rightarrow \quad y_1 = \frac{46}{52} = \frac{23}{26} = 0.8846154$$

[Ans.] Using Spline

Now, using finite difference method :

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} = y_k \qquad ; \; k=1$$

$$\Rightarrow \frac{y_2 - 2y_1 + y_0}{(\frac{1}{2})^2} = y_1 \quad \Rightarrow 4(y_2 + y_0) - 8y_1 = y_1$$

$$\Rightarrow 9y_1 = 8 \quad \Rightarrow y_1 = \frac{8}{9} = 0.8888\ldots$$

[Ans.] Using

Now we find the analytic solution :

$y'' - y = 0 \longrightarrow$ Characteristic polynomial : $m^2 - 1 = 0 \Rightarrow m = \pm 1$.

Soln. is of the form : $y = A e^{m_1 x} + B e^{m_2 x}$

$$\Rightarrow y = Ae^x + Be^{-x}$$

Using the BCs : $\left.\begin{array}{l} y(0) = 1 \Rightarrow A + B = 1 \\ y(1) = 1 \Rightarrow Ae + \frac{B}{e} = 1 \end{array}\right\}$ On solving, $A = \frac{1}{e+1}$, $B = \frac{e}{e+1}$

$$\therefore y = \left(\frac{1}{e+1}\right)e^x + \left(\frac{e}{e+1}\right)e^{-x}$$

Then, $y(\frac{1}{2}) = 0.8868189 \longrightarrow$ Actual value.

7) Solve the BVP : $y'' + 2y' + y = 30x \quad ; \quad y(0) = y(1) = 0$

for step size $h = \frac{1}{2}$.

Soln: We know, for the BVP :

$$y'' + A(x)\cdot y' + B(x)\cdot y = C(x) \quad ; \quad y(0) = y_0 \ , \ y(a) = y_a$$

with $A(x) \neq 0$ , then , $y_k'' + A_k y_k' + B_k y_k = C_k$

$$\Rightarrow A_k y_k' = C_k - y_k'' - B_k y_k = C_k - M_k - B_k y_k$$

Now, equating $P_k'(x_k) = y_k'$ and $P_{k-1}'(x_k) = y_k'$

$k = 0, 1, \ldots, n-1$ $\qquad$ $k = 1, 2, \ldots, n$

$$\Rightarrow \left(1 - \frac{h}{3}A_k\right) M_k - \frac{h}{6} A_k M_{k+1} = C_k - B_k y_k - \frac{A_k}{h}(y_{k+1} - y_k) \quad \ldots \text{(i)}$$

$; \ k = 0, 1, \ldots, n-1$

And, $$\frac{h}{6} A_k M_{k-1} + \left(1 + \frac{h}{3}A_k\right) M_k = C_k - B_k y_k - \frac{A_k}{h}(y_k - y_{k-1}) \quad \ldots \text{(ii)}$$

$; \ k = 1, 2, \ldots, n$

In this problem, using (i) and (ii), we get :

$$\frac{2}{3} M_k - \frac{1}{6} M_{k+1} = 30x_k - y_k - 4(y_{k+1} - y_k)$$

$k=0$ : $\frac{2}{3} M_0 - \frac{1}{6} M_1 = -4y_1$ $\quad [\because y_0 = 0]$ $\ldots$ (iii)

$k=1$ : $\frac{2}{3} M_1 - \frac{1}{6} M_2 = 15 + 3y_1$ $\ldots\ldots$ (iv)

And, $$\frac{1}{6} M_{k-1} + \frac{4}{3} M_k = 30x_k - y_k - 4(y_k - y_{k-1})$$

$k=1$ : $\frac{1}{6} M_0 + \frac{4}{3} M_1 = 15 - 5y_1$ $\ldots\ldots$ (v) $\qquad [\because y_2 = 0]$

$k=2$ : $\frac{1}{6} M_1 + \frac{4}{3} M_2 = 30 + 4y_1$ $\ldots\ldots$ (vi)

On solving the system of equations, $M_0 = 16.4876033$ ,

$M_1 = 16.85950413$ , $M_2 = 14.25619835$ , and

$y_1 = -2.0454545 \simeq y(0.5)$ [Ans.]

8) Solve: $3yy'' + (y')^2 = 0$ ; $y(0) = 0$, $y(1) = 1$ using Newton's Linearization technique for step size $h = \frac{1}{3}$.

Soln: We have, $3y_i y_i'' + (y_i')^2 = 0$

$$\Rightarrow 3y_i \times \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + \left(\frac{y_{i+1} - y_{i-1}}{2h}\right)^2 = 0$$

$$\Rightarrow 12y_i(y_{i+1} - 2y_i + y_{i-1}) + y_{i+1}^2 + y_{i-1}^2 - 2y_{i+1}y_{i-1} = 0$$

Then, $f_i = y_{i-1}^2 - 24y_i^2 + y_{i+1}^2 + 12y_i y_{i-1} + 12y_i y_{i+1} - 2y_{i+1}y_{i-1} = 0$

Then, $\frac{\partial f_i}{\partial y_{i-1}} = 2y_{i-1} + 12y_i - 2y_{i+1}$, $\frac{\partial f_i}{\partial y_i} = 12y_{i-1} - 48y_i + 12y_{i+1}$,

$$\frac{\partial f_i}{\partial y_{i+1}} = -2y_{i-1} + 12y_i + 2y_{i+1}$$

So, by Newton Linearization technique,

$$\left.\frac{\partial f_i}{\partial y_{i-1}}\right|^{(k)} \times \Delta y_{i-1} + \left.\frac{\partial f_i}{\partial y_i}\right|^{(k)} \times \Delta y_i + \left.\frac{\partial f_i}{\partial y_{i+1}}\right|^{(k)} \times \Delta y_{i+1} = -f_i^{(k)} \quad \left[\text{At } (k+1)\text{th iteration}\right]$$

$$\Rightarrow \left[2y_{i-1}^{(k)} + 12y_i^{(k)} - 2y_{i+1}^{(k)}\right] \times \Delta y_{i-1} + \left[12y_{i-1}^{(k)} - 48y_i^{(k)} + 12y_{i+1}^{(k)}\right] \times \Delta y_i$$

$$+ \left[-2y_{i-1}^{(k)} + 12y_i^{(k)} + 2y_{i+1}^{(k)}\right] \times \Delta y_{i+1} = -y_{i-1}^{2(k)} + 24y_i^{2(k)} - y_{i+1}^{2(k)} - 12y_i^{(k)}y_{i-1}^{(k)}$$
$$-12y_i^{(k)}y_{i+1}^{(k)} + 2y_{i+1}^{(k)}y_{i-1}^{(k)}$$

Initial guess: $y^{(0)}(x) = x$ [Satisfying BCs] $\Rightarrow y_i^{(0)} = x_i$

And, $\Delta y_0 = 0$, $\Delta y_3 = 0$. $\Rightarrow y_0^{(0)} = 0$, $y_1^{(0)} = \frac{1}{3}$, $y_2^{(0)} = \frac{2}{3}$, $y_3^{(0)} = 1$

1st iteration: (i.e. $k = 0$)

$i = 1$: $\left[2y_0^{(0)} + 12y_1^{(0)} - 2y_2^{(0)}\right] \times \Delta y_0 + \left[12y_0^{(0)} - 48y_1^{(0)} + 12y_2^{(0)}\right] \times \Delta y_1$

$$+ \left[-2y_0^{(0)} + 12y_1^{(0)} + 2y_2^{(0)}\right] \times \Delta y_2 = -y_0^{2(0)} + 24y_1^{2(0)} - y_2^{2(0)} - 12y_1^{(0)}y_0^{(0)} - 12y_1^{(0)}y_2^{(0)}$$
$$+ 2y_2^{(0)}y_0^{(0)}$$

$$\Rightarrow (-16+8)\times\Delta y_1 + \left(4+\frac{4}{3}\right)\times \Delta y_2 = \frac{-4}{9} \Rightarrow -8\,\Delta y_1 + \frac{16}{3}\Delta y_2 = \frac{-4}{9} \quad \ldots\text{(i)}$$

$\underline{\underline{i=2:}}$ $\left[2y_1^{(0)} + 12y_2^{(0)} - 2y_3^{(0)}\right]\times\Delta y_1 + \left[12y_1^{(0)} - 48y_2^{(0)} + 12y_3^{(0)}\right]\times\Delta y_2 + 0$

$$= -y_1^{2(0)} + 24y_2^{2(0)} - y_3^{2(0)} - 12y_1^{(0)}y_2^{(0)} - 12y_2^{(0)}y_3^{(0)} + 2y_3^{(0)}y_1^{(0)} \quad [\because \Delta y_3 = 0]$$

$$\Rightarrow \left(\frac{2}{3}+6\right)\Delta y_1 + (4-32+12)\Delta y_2 = \frac{95}{9} - 1 - \frac{8}{3} - 8 + \frac{2}{3}$$

$$\Rightarrow \frac{20}{3}\Delta y_1 - 16\Delta y_2 = \frac{-4}{9} \quad \ldots\text{(ii)}$$

Solving (i) and (ii), we get: $\Delta y_1 = \frac{4}{39}$ and $\Delta y_2 = \frac{11}{156}$

So, $y_1^{(1)} = y_1^{(0)} + \Delta y_1 = \frac{1}{3} + \frac{4}{39} = \frac{17}{39} = 0.4359$

$y_2^{(1)} = y_2^{(0)} + \Delta y_2 = \frac{2}{3} + \frac{11}{156} = \frac{115}{156} = 0.7372$

So, $y_1 \simeq y\left(\frac{1}{3}\right) \simeq 0.4359$ and $y_2 \simeq y\left(\frac{2}{3}\right) \simeq 0.7372$ [Ans.]

9) $f''' + ff'' + 1 - (f')^2 = 0$ ; $f(0) = 0 = f'(0)$ and $f'(10) = 1$

Get the reduced block tridiagonal system which needs to be solved at every iteration.

$\underline{\underline{\text{Soln:}}}$ Let $p = f'$ $\Rightarrow \int f'\cdot df = \int p\cdot dx$ [Using Trapezoidal rule]

$$\Rightarrow f_i - f_{i-1} - \frac{h}{2}\times(p_i + p_{i-1}) = 0 \quad \ldots\ldots\text{(i)}$$

And, now the ODE becomes: $p'' + fp' + 1 - p^2 = 0$ ; $\Delta f_0 = \Delta p_0 = \Delta p_n = 0$

$$\Rightarrow \frac{p_{i+1} - 2p_i + p_{i-1}}{h^2} + f_i \times \frac{p_{i+1} - p_{i-1}}{2h} - p_i^2 = -1 \quad \ldots\text{(ii)}$$

Then, at any iteration $(k+1)$, $f_i^{(k+1)} = f_i^{(k)} + \Delta f_i$ and $p_i^{(k+1)} = p_i^{(k)} + \Delta p_i$

So, (i) becomes:

$$f_i^{(k)} + \Delta f_i - f_{i-1}^{(k)} - \Delta f_{i-1} - \frac{h}{2}\times\left[p_i^{(k)} + p_{i-1}^{(k)} + \Delta p_i + \Delta p_{i-1}\right] = 0$$

As coeff. of $f_n$ will be zero, we can have condition $\Delta f_n = 0$ also.

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$$\Rightarrow -\Delta f_{i-1} + \Delta f_i - \frac{h}{2}\times \Delta p_{i-1} - \frac{h}{2}\times \Delta p_i = \frac{h}{2}\times\left(p_i^{(k)} + p_{i-1}^{(k)}\right) - f_i^{(k)} + f_{i-1}^{(k)} \quad \ldots\ (iii)$$

And, we have:

$$\left(\frac{1}{h^2} - \frac{f_i}{2h}\right)p_{i-1} - \frac{2}{h^2}p_i - p_i^2 + \left(\frac{1}{h^2} + \frac{f_i}{2h}\right)p_{i+1} = -1 \qquad \text{[From (ii)]}$$

Then at any $(k+1)^{th}$ iteration, this becomes:

$$\left(\frac{1}{h^2} - \frac{f_i^{(k)}}{2h} - \frac{\Delta f_i}{2h}\right)\left(p_{i-1}^{(k)} + \Delta p_{i-1}\right) - \frac{2}{h^2}\left(p_i^{(k)} + \Delta p_i\right) - \left(p_i^{(k)} + \Delta p_i\right)^2$$
$$+ \left(\frac{1}{h^2} + \frac{f_i^{(k)}}{2h} + \frac{\Delta f_i}{2h}\right)\left(p_{i+1}^{(k)} + \Delta p_{i+1}\right) = -1$$

Simplifying this, and ignoring the terms involving $\Delta f_i \times \Delta p_i$ and similar products as they are negligibly small, we get:

$$\left(\frac{-p_{i-1}^{(k)}}{2h} + \frac{p_{i+1}^{(k)}}{2h}\right)\Delta f_i + \left(\frac{1}{h^2} - \frac{f_i^{(k)}}{2h}\right)\Delta p_{i-1} + \left(\frac{-2}{h^2} - 2p_i^{(k)}\right)\Delta p_i$$
$$+ \left(\frac{1}{h^2} + \frac{f_i^{(k)}}{2h}\right)\Delta p_{i+1} = -1 - \left(\frac{1}{h^2} - \frac{f_i^{(k)}}{2h}\right)p_{i-1}^{(k)} + \frac{2}{h^2}p_i^{(k)} + \left(p_i^{(k)}\right)^2$$
$$- \left(\frac{1}{h^2} + \frac{f_i^{(k)}}{2h}\right)p_{i+1}^{(k)} \quad \ldots\ (iv)$$

Take $X_i = \begin{bmatrix} \Delta f_i \\ \Delta p_i \end{bmatrix}$. Then the system (iii) and (iv) can be written as:

$$\underbrace{\begin{bmatrix} -1 & \frac{-h}{2} \\ 0 & \left(\frac{1}{h^2} - \frac{f_i^{(k)}}{2h}\right) \end{bmatrix}}_{A_i} X_{i-1} + \underbrace{\begin{bmatrix} 1 & \frac{-h}{2} \\ \left(\frac{p_{i+1}^{(k)} - p_{i-1}^{(k)}}{2h}\right) & \left(\frac{-2}{h^2} - 2p_i^{(k)}\right) \end{bmatrix}}_{B_i} X_i + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{1}{h^2} + \frac{f_i^{(k)}}{2h}\right) \end{bmatrix}}_{C_i} X_{i+1}$$

$= D_i$, where $D_i$ is given by RHS of (iii) and (iv).

Hence we get the reduced block tridiagonal form: $A_i X_{i-1} + B_i X_i + C_i X_{i+1} = D_i$. [Ans]

10) $F''' + (2F+4)F' = 0$ ; $F(0)=0$, $F''(0) = -k$, $F'(\omega) = 0$

where $k = 0.1$ and $\omega = 0.087$.

Get the reduced block tridiagonal form.

Soln: Let $F' = g$. Then, $g'' + (2F+4)g = 0$ with

$F(0)=0$, $g'(0) = -k$, $g(\omega) = 0$

Using forward difference approximation,

$$g_0' = \frac{-3g_0 + 4g_1 - g_2}{2h} = -k$$

$$\Rightarrow 3g_0 = 4g_1 - g_2 + 2hk \Rightarrow g_0 = \frac{2hk + 4g_1 - g_2}{3} \quad \cdots (i)$$

Now, $F' = g$ → Integrating using Trapezoidal rule gives:

$$F_i - F_{i-1} - \frac{h}{2} \times (g_i + g_{i-1}) = 0 \quad ; i = 1, 2, \ldots, n-1 \quad \cdots (ii)$$

And, discretizing $g'' + (2F+4)g = 0$ using central difference scheme:

$$\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} + (2F_i + 4)g_i = 0 \quad ; i = 1, 2, 3, \ldots, n-1 \quad \cdots (iii)$$

And in (iii), for $i=1$, we'll substitute value of $g_0$ from (i).

Now, at any $(k+1)^{th}$ iteration, (ii) becomes:

$$-\Delta F_{i-1} + \Delta F_i - \frac{h}{2}\Delta g_{i-1} - \frac{h}{2}\Delta g_i = \frac{h}{2}\left(g_i^{(k)} + g_{i-1}^{(k)}\right) - F_i^{(k)} + F_{i-1}^{(k)} \quad \cdots (iv)$$

And, eqn. (iii) becomes, at any $(k+1)^{th}$ iteration:

For $i=1$: $\dfrac{g_2 - 2g_1 + g_0}{h^2} + (2F_1 + 4)g_1 = 0$

$$\Rightarrow \frac{2}{3h^2}(g_2 - g_1) + (2F_1 + 4)g_1 = \frac{-2k}{3h}$$

Then, $\dfrac{2}{3h^2}\left(g_2^{(k)} + \Delta g_2 - g_1^{(k)} - \Delta g_1\right) + \left(2F_1^{(k)} + 2\Delta F_1 + 4\right)\left(g_1^{(k)} + \Delta g_1\right) = \dfrac{-2k}{3h}$

$$\Rightarrow \left(\frac{-2}{3h^2} + 2F_1^{(k)} + 4\right)\Delta g_1 + \frac{2}{3h^2}\Delta g_2 + 2g_1^{(k)}\Delta F_1 = \frac{-2k}{3h} - \frac{2}{3h^2}\left(g_2^{(k)} - g_1^{(k)}\right) - \left(2F_1^{(k)} + 4\right)g_1^{(k)} \quad \cdots\cdots (v)$$

[On ignoring terms involving products of $\Delta g_i$'s and $\Delta F_i$'s]

Now, for $i = 2, 3, \ldots, n-1$, we get:

$$\frac{1}{h^2}\left(g_{i-1} + g_{i+1} - 2g_i\right) + (2F_i + 4)g_i = 0$$

$$\Rightarrow \frac{1}{h^2} \times \left(g_{i-1}^{(k)} + \Delta g_{i-1} + g_{i+1}^{(k)} + \Delta g_{i+1} - 2g_i^{(k)} - 2\Delta g_i\right) + \left(2F_i^{(k)} + 2\Delta F_i + 4\right)\left(g_i^{(k)} + \Delta g_i\right) = 0$$

$$\Rightarrow \frac{1}{h^2}\Delta g_{i-1} + \left(\frac{-2}{h^2} + 2F_i^{(k)} + 4\right)\Delta g_i + \left(\frac{1}{h^2}\right)\Delta g_{i+1} + 2g_i^{(k)}\Delta F_i$$

$$= \frac{-1}{h^2}\left(g_{i+1}^{(k)} - 2g_i^{(k)} + g_{i-1}^{(k)}\right) - \left(2F_i^{(k)} + 4\right)g_i^{(k)} \quad \cdots\cdots (vi)$$

System of equations (iv), (v), (vi) can be written as: Taking $X_i = \begin{bmatrix} \Delta F_i \\ \Delta g_i \end{bmatrix}$

$$\underbrace{\begin{bmatrix} -1 & -\frac{h}{2} \\ 0 & \frac{1}{h^2} \end{bmatrix}}_{A_i} X_{i-1} + \underbrace{\begin{bmatrix} 1 & -\frac{h}{2} \\ 2g_i^{(k)} & \left(\frac{-2}{h^2} + 2F_i^{(k)} + 4\right) \end{bmatrix}}_{B_i} X_i + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} \end{bmatrix}}_{C_i} X_{i+1} = \underbrace{\begin{bmatrix} \text{RHS of (iv)} \\ \text{RHS of (vi)} \end{bmatrix}}_{D_i}$$

: for $i = 2, 3, \ldots, n-1$

And, using (v), we get: (For $i = 1$)

$$A_1 = \begin{bmatrix} -1 & -\frac{h}{2} \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & -h/2 \\ 2g_1^{(k)} & \left(\frac{-2}{3h^2} + 2F_1^{(k)} + 4\right) \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{2}{3h^2} \end{bmatrix},$$

$$D_1 = \begin{bmatrix} \text{RHS of (iv)} \\ \text{RHS of (v)} \end{bmatrix}$$

Thus we get the block tridiagonal system to be solved as:

$$A_i X_{i-1} + B_i X_i + C_i X_{i+1} = D_i \quad ; i = 1, 2, \ldots, n-1$$

[Ans.]

11) $y'' - (y')^2 - y^2 + y + 1 = 0$ ; $y(0) = 0.5$ , $y(\pi) = -0.5$

Convert the non-linear BVP to linear BVP to be solved at every iteration, using Quasi Linearization technique.

Soln: Let $F(x, y, y', y'') = y'' - (y')^2 - y^2 + y + 1 = 0$

Then, $\frac{\partial F}{\partial y} = -2y + 1$ , $\frac{\partial F}{\partial y'} = -2y'$ , $\frac{\partial F}{\partial y''} = 1$

Then by Quasi linearization technique, we get: (At any $(k+1)^{th}$ iteration)

$$F(x, y^{(k)}, y'^{(k)}, y''^{(k)}) + (y^{(k+1)} - y^{(k)}) \times (-2y^{(k)} + 1) + (y'^{(k+1)} - y'^{(k)}) \times (-2y'^{(k)})$$

$$+ (y''^{(k+1)} - y''^{(k)}) \times 1 = 0$$

$$\Rightarrow y''^{(k+1)} - 2y'^{(k)} \times y'^{(k+1)} + (1 - 2y^{(k)})\, y^{(k+1)} = y''^{(k)} - 2(y'^{(k)})^2 + (1 - 2y^{(k)})\, y^{(k)}$$
$$- y''^{(k)} + (y'^{(k)})^2 + (y^{(k)})^2 - y^{(k)}$$
$$-1$$

$$\Rightarrow y''^{(k+1)} - 2y'^{(k)} \times y'^{(k+1)} + (1 - 2y^{(k)})\, y^{(k+1)} = -(y'^{(k)})^2 - (y^{(k)})^2 - 1$$

$$\Rightarrow y''^{(k+1)} - 2y'^{(k)} \times y'^{(k+1)} + (1 - 2y^{(k)})\, y^{(k+1)} + \left[(y'^{(k)})^2 + (y^{(k)})^2 + 1\right] = 0$$

and, BCs: $y^{(k+1)}(0) = 0.5$ and $y^{(k+1)}(\pi) = -0.5$ , which is a linear BVP. [Ans.]

12) $f''' + ff'' + 1 - (f')^2 = 0$ ; $f(0) = 0$ , $f'(0) = 0$ , $f'(10) = 1$

Convert the non-linear BVP to linear BVP using Quasi linearization. Derive the block tri-diagonal system which needs to be solved at every iteration.

Soln: Let $F(x, f, f', f'', f''') = f''' + ff'' + 1 - (f')^2 = 0$

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Then, $\frac{\partial F}{\partial f} = f''$ , $\frac{\partial F}{\partial f'} = -2f'$ , $\frac{\partial F}{\partial f''} = f$ , $\frac{\partial F}{\partial f'''} = 1$

Then, by Quasi linearization, at any $(k+1)^{th}$ iteration,

$$f'''^{(k)} + f^{(k)} \cdot f''^{(k)} + 1 - (f'^{(k)})^2 + (f^{(k+1)} - f^{(k)}) \times f''^{(k)} + (f'^{(k+1)} - f'^{(k)}) \times (-2f'^{(k)})$$

$$+ (f''^{(k+1)} - f''^{(k)}) \times f^{(k)} + (f'''^{(k+1)} - f'''^{(k)}) \times 1 = 0$$

$$\Rightarrow f'''^{(k+1)} + f^{(k)} \times f''^{(k+1)} - 2f'^{(k)} \times f'^{(k+1)} + f''^{(k)} \times f^{(k+1)}$$

$$= f^{(k)} \cdot f''^{(k)} - (f'^{(k)})^2 - 1 \quad \cdots (i)$$

with BCs : $f^{(k+1)}(0) = 0$ , $f'^{(k+1)}(0) = 0$ , $f'^{(k+1)}(10) = 1$

which is a linear BVP.

Now, let $g = f'$ ..(ii). So, this linear BVP becomes :

$$g''^{(k+1)} + f^{(k)} \times g'^{(k+1)} - 2g^{(k)} \times g^{(k+1)} + g'^{(k)} \times f^{(k+1)} = f^{(k)} \times g'^{(k)} - (g^{(k)})^2 - 1 \quad \cdots (iii)$$

Integrating (ii) using Trapezoidal rule, $f_i - f_{i-1} - \frac{h}{2}(g_i + g_{i-1}) = 0$ ... (iv)

And using central difference scheme to discretize (iii),

$$\frac{g_{i+1}^{(k+1)} - 2g_i^{(k+1)} + g_{i-1}^{(k+1)}}{h^2} + f_i^{(k)} \times \frac{g_{i+1}^{(k+1)} - g_{i-1}^{(k+1)}}{2h} - 2g_i^{(k)} \times g_i^{(k+1)} + g_i'^{(k)} \times f_i^{(k+1)}$$

$$= f_i^{(k)} \times g_i'^{(k)} - (g_i^{(k)})^2 - 1 \quad \cdots (v)$$

Let $X_i = \begin{bmatrix} f_i \\ g_i \end{bmatrix}^{(k+1)}$ , then we get the system :

$A_i X_{i-1} + B_i X_i + C_i X_{i+1} = D_i$ ; which is a block tri diagonal system, where

$$A_i = \begin{bmatrix} -1 & -h/2 \\ 0 & \left(\frac{1}{h^2} - \frac{f_i^{(k)}}{2h}\right) \end{bmatrix}, \quad B_i = \begin{bmatrix} 1 & -h/2 \\ g_i'^{(k)} & \left(\frac{-2}{h^2} - 2g_i^{(k)}\right) \end{bmatrix}, \quad C_i = \begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{1}{h^2} + \frac{f_i^{(k)}}{2h}\right) \end{bmatrix}$$

and $D_i = \begin{bmatrix} 0 \\ f_i^{(k)} g_i'^{(k)} - (g_i^{(k)})^2 - 1 \end{bmatrix}$ . This is the required block tri diagonal system. [Ans.]