PDE:

$$\frac{\partial^2 z}{\partial y^2} = k\left(\frac{\partial^3 z}{\partial x^3}\right)^2$$ order = 3, degree = 2

DERIVATION OF PDE:

ⓐ By elimination of arb. constants:

$F(x, y, z, a, b) = 0$

diff. partially w.r.t x & y and equate to 0.

$\left(\frac{\partial z}{\partial x}\right) = p \quad \left(\frac{\partial z}{\partial y}\right) = q$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0.$$

Eliminate a & b.

we get, $F(x, y, z, p, q)$.

★ if more constants, double diff. with x, y finding out r, s, t.

ⓑ By elimination of arb. functions:

$f(u, v) = 0$ u & v are functions of x and y and z.

$u = \phi(x, y, z)$; $v (= \psi(x, y, z)$

diff. f w.r.t. x, y. and equate to 0.

w.r.t x:

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

w.r.t y.

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = 0$$

$$\boxed{Pp + Qq = R}$$

where $P = \frac{\partial(u, v)}{\partial(y, z)} = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial z}$

$Q = \frac{\partial(u, v)}{\partial(z, x)} = \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}$

$R = \frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$

TYPES OF PDE:

① Quasi linear: $P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$

② Semi linear: $P(x, y)p + Q(x, y)q = R(x, y, z)$

③ Linear: $P(x, y)p + Q(x, y)q = R(x, y)z^{\star} + S(x, y)$

④ Non-linear: any other combination.

↓

if coeff. depends on p, q, higher degree term, or product like pq.

CLASSIFICATION OF INTEGRALS:

Integral surface of pde: A soln. $z = z(x,y)$ when interpreted as in 3-D.

① Complete integral: $F(x, y, z, a, b) = 0$. contains arbitrary const

② General integral / General soln: Involves arbitrary fn.

③ Singular integral: obtained from complete integral by eliminating F= b by $\frac{\partial F}{\partial a} = 0$ and $\frac{\partial F}{\partial b} = 0$ & then checking

LAGRANGE'S EQN: (for quasi linear)

Step 1: Put in standard form: $Pp + Qq = R$

Step 2: Write Lagrange Aux eqn:

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$$

Step 3: Use any of the 4 methods to find:

$$\boxed{u(x, y, z) = c_1 \quad ; \quad v(x, y, z) = c_2}$$

Step 4: The general soln. is written as:

$$\boxed{\phi(u,v) = 0 \; ; \; u = \phi(v) \; ; \; v = \phi(u)}$$

4 METHODS:

TYPE 1: If any variable is absent or cancels out from 2 fractions. Integral can be obtained by normal means of integration.

Note: linear diff: $\frac{dy}{dx} + yP(x) = Q(x)$ $\quad y \cdot I.F = \int Q(x)dx \cdot IF$

$IF = e^{\int P(x) \cdot dx}$.

TYPE 2: One integral found normally.
Second integral found using 1st integral.

TYPE 3: $P_1, Q_1, R_1 \rightarrow$ fns. of x, y, z or const. *can also be const.

$$\boxed{\text{each fraction} = \frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}}$$

$$\boxed{P_1 P + Q_1 Q + R_1 R = 0}$$

eg. $\frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)zx} = \frac{c\,dz}{(a-b)xy}$ use multipliers x, y, z
ax, by, cz.

TYPE 4: each fraction $= \frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R} = \frac{P_2 dx + Q_2 dy + R_2 dz}{P_2 P + Q_2 Q + R_2 R}$.

⊛ numerator should be exact differential of denominator.

To determine the integral surface passing through a given curve:

Method 1: Find $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$

Suppose we wish to obtain the integral surface whose eqn. in parametric form is given by:

$$x = x(t)\ ;\ y = y(t)\ ;\ z = z(t).$$

Express u & v in terms of t. and then eliminate t.

(we get a reln. in c_1 & c_2).

substitute the value of c_1 & c_2.

eg. Find the integral surface of pde:

$$x(y^2+z)p - y(z^2+z)q = (x^2-y^2)z$$

which contains: $x+y=0;\ z=1$.

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(z^2+z)} = \frac{dz}{(x^2-y^2)z}$$

$$\frac{xdx+ydy}{z(x^2-y^2)} = \frac{dz}{(x^2-y^2)z}$$

$$x^2+y^2-2z=c_1$$

multipliers $yz,\ xz,\ xy$.

$$\frac{yzdx + xzdy + xydz}{0}$$

$$xyz = c_2.$$

$x=t,\ y=-t,\ z=1$

$$-t^2 = c_2$$

$$2t^2-2=c_1 \quad -2c_2-c_1-2=0$$

$$\therefore\ 2xyz + x^2+y^2-2z+2=0$$

Method 2:

Suppose we wish to obtain the integral surface passing through the (and determined by) the two eqn: $\phi(x,y,z)=0$ and $\psi(x,y,z)=0$.

We eliminate x, y, z from 4 eqns. (u, v, ϕ, ψ)

COMPATIBILITY CONDITION:

2 PDEs are compatible if they have a common soln.

$f(x,y,z,p,q)=0 \qquad g(x,y,z,p,q)=0$

A necessary & sufficient condition for integrability of

$dz = \phi(x,y,z)dx + \psi(x,y,z)dy$

$$\boxed{[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0}$$

For first order PDE: $p = P(x,y)$ compatible iff. $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$

$q = Q(x,y)$

If compatible, (on checking $[f,g]$)

we find p, and q w.r.t. x and y.

using p & q,

$$\boxed{dz = p\,dx + q\,dy}$$

integrate & find z.

CHARPIT'S METHOD → complete integral.

$f(x, y, z, p, q) = 0$

to find: $g(x, y, z, p, q, a) = 0$ st. they can be solved to give $p = p(x, y, z, a)$ and $q = q(x, y, z, a)$

using compatibility conditions:

Charpit's eqn:

$$\boxed{\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}} \quad -①$$

p & q must occur is the soln. found.

(from aux., find the simplest reln. involving atleast one of p & q.

STEPS:

① Take everything to LHS & write as $f = 0$.

② Substitute $f_p, f_q \ldots$ in ① (Charpit's auxilary eqn.)

③ Select two of the fractions so that resulting integral has atleast one of p and q.

④ From f & new relation, find p & q.

⑤ $\boxed{dz = p\,dx + q\,dy}$ On integration, gives complete integral of given eqn.
↓
has arb. constant.

To check whether singular integral:

diff. z partially w.r.t a & b and equate to 0.

put the attained values of a & b into z.

if $z = 0 \Rightarrow$ singular integral.

SPECIAL TYPES OF 1st ORDER EQNS:

① Eqn. with only p & q. $\left(\frac{dp}{0} = \frac{dq}{0}\right) \Rightarrow \boxed{p \text{ or } q = a}$

② Eqn. with p, q and z. $\left(\frac{dp}{-pf_z} = \frac{dq}{-qf_z}\right) \Rightarrow \boxed{p = aq}$

③ Separable eqn. $\Rightarrow f(x, p) = g(y, q)$ $\frac{dx}{f_p} = \frac{dp}{-f_x} \Rightarrow \boxed{f(x, p) = a}$

④ Clairout's Eqn: $z = px + qy + f(p,q)$

Substitute $p = a$; $q = b$

substituting; $\boxed{z = ax + by + f(a,b)}$

general (for singular integral):

if $\frac{\partial z}{\partial a} = x - \sqrt{\frac{b}{a}}$ $\quad \frac{\partial z}{\partial b} = y - \sqrt{\frac{a}{b}}$

$\therefore xy = 1 \quad \underbrace{xy-1=0}$

No z in this.

∴ not an integral.

HOMOGENOUS LINEAR PDE (with constant coeff.)

$$\frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots \dots A_n \frac{\partial^n z}{\partial y^n} = f(x,y)$$

$\frac{\partial}{\partial x} : D \qquad \frac{\partial}{\partial y} : D'$

$$(D + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots A_n D'^n) z = f(x,y)$$

2 parts:

① Complimentary fn. (C.F)

② Particular integral (P.I)

Auxiliary eqn:

$D = m$; $D' = 1$

$m^n + A_1 m^{n-1} + A_2 m^{n-2} \dots + A_n = 0$

Let $m_1, m_2 \dots m_n$ be soln. of aux. eqn.

C.F: $z = \psi_1(y + m_1 x) + \psi_2(y + m_2 x) + \dots \psi_n(y + m_n x)$

if 'r' equal roots, then:

$z = f_1(y+mx) + x f_2(y+mx) + x^2 f_3(y+mx) + \dots x^{r-1} f_r(y+mx)$

② PARTICULAR INTEGRAL:

$f(D, D')z = \phi(x,y)$

$$\boxed{P.I = \frac{1}{f(D,D')} \phi(x,y)}$$

$\boxed{G.S = C.F + P.I}$

PARTICULAR INTEGRAL (other method)

when $f(x,y)$ is of form $f(ax+by)$

THEOREM: if $f(D, D')$ is a homogenous fn. of degree 'n':

$$\boxed{\frac{1}{f(D,D')}\phi^n(ax+by) = \frac{1}{f(a,b)}\phi(ax+by)}$$

when $F(a, b)=0$

$$\boxed{\frac{1}{(bD-aD')^n}\phi(ax+by) = \frac{x^n\phi(ax+by)}{b^n n!}}$$

eg. $(D^2+3DD'+2D'^2)z = x+y$

$$\frac{1}{(D^2+3DD'+2D'^2)}x+y$$

$D=a=1 \quad D'=b=1$

$$\frac{1}{6}\iint(x+y) = \frac{1}{6}\frac{(x+y)^3}{6}$$ we integrate twice ∵ degree of homogenity is 2.

eg. 2: $r-t = x-y$ $\quad r=\frac{\partial^2 z}{\partial x^2}$; $s=\frac{\partial^2 z}{\partial x \partial y}$; $t=\frac{\partial^2 z}{\partial y^2}$

$D^2 - D'^2 = x-y$

$$= \frac{1}{D^2-D'^2}(x-y) = \frac{1}{(D-D')(D+D')}(x-y)$$

we need to create $bD-aD'$ $\quad b=-1$, $a=1$

$\Rightarrow -D-D'$

$$= \frac{1}{(D+D')}\frac{x-y}{D-D'}$$ (solve normally)

$$= \frac{1}{(D+D')}\frac{\int x-y}{2} = \frac{1}{(D+D')}\frac{(x-y)^2}{2\times 2}$$

$$\frac{1}{4(-1)}\frac{1}{(-D-D')}(x-y)^2$$

$$= \frac{-1}{4}\cdot\frac{x^1}{(-1)^1 \cdot 1!}(x-y)^2 = \frac{x(x-y)^2}{4}$$

Other method: (general)

Take the P.I corresponding to:

$\frac{1}{(D-mD')}\phi$ as $\int\phi(x, a-mx)dx$ & then replace a by $y+mx$ after integration.

$x,\ a-mx$

eg. $(D^2 - 2DD' - 15D'^2) = 12xy$

$$PI = \frac{1}{(D-5D')(D+3D')} 12xy \qquad m = 5, -3$$

$$= \frac{1}{(D-5D')} \times 12 \times \int 12x(a+3x)\,dx$$

$$= \frac{12}{D-5D'} (6x^2 a + 12x^3)$$

$$= \frac{12}{D-5D'} [6x^2(y-3x) + 12x^3]$$

$y \to a - 5x$

$$= \int [6x^2(a-8x) + 12x^3]\,dx$$

$$= 2x^3(a) - 12x^4 + \frac{x^4}{2\times 2} \times 12$$

$$= 2ax^3 - 9x^4 = 2(y+5x)x^3 - 9x^4$$

$$= 2x^3 y + x^4$$

PDE (of order 2)

Reduction to canonical form:

$$\boxed{R\frac{\partial^2 z}{\partial x^2} + S\frac{\partial^2 z}{\partial x \partial y} + T\frac{\partial^2 z}{\partial y^2} + F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0}$$

$$\boxed{Rr + Ss + Tt + F(x, y, z, p, q) = 0}$$

1. Hyperbolic : $S^2 - 4RT > 0$ $u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ OR $u_{\xi\xi} - u_{\eta\eta}$
2. Parabolic : $S^2 - 4RT = 0$ $u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ OR $u_{\xi\xi}$
3. Elliptical : $S^2 - 4RT < 0$ $u_{\alpha\alpha} + u_{\beta\beta} = \psi(\alpha, \beta, u_\alpha, u_\beta, u)$

Hyperbolic $\longrightarrow$ canonical:

1. Write λ quadratic eqn. $R\lambda^2 + S\lambda + T = 0$: λ_1, λ_2 are distinct real roots.
2. $\frac{\partial y}{\partial x} + \lambda_1 = 0$; $\frac{\partial y}{\partial x} + \lambda_2 = 0$
3. Solving $f_1(x,y) = c_1$ and $f_2(x,y) = c_2$

$$\left.\begin{array}{l} \xi = f_1(x,y) \\ \eta = f_2(x,y) \end{array}\right\} \rightarrow \text{we write } z(x,y) \text{ as } u(\xi,\eta)$$

note: $J = \frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$

note: $\frac{\partial z}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = u_\xi \xi_x + u_\eta \cdot \eta_x$

$$\therefore \frac{\partial^2 z}{\partial x^2} = \xi_x \frac{\partial u_\xi}{\partial x} + u_\xi \frac{\partial \xi_x}{\partial x} + \frac{\partial u_\eta}{\partial x}\cdot \eta_x + u_\eta \frac{\partial \eta_x}{\partial x}$$

$$= \xi_x\left[\frac{\partial u_\xi}{\partial \xi}\times\frac{\partial \xi}{\partial x} + \frac{\partial u_\xi}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right] + u_\xi\left[\frac{\partial \xi_x}{\partial x}\right] + \eta_x\left[\frac{\partial u_\eta}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} + \frac{\partial u_\eta}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}\right] + ()$$

$$= \xi_x u_{\xi\xi}\xi_x + \xi_x \cdot u_{\xi\eta}\cdot\eta_x + u_\xi \xi_{xx} + u_{\eta\eta}\cdot\eta_x^2 + u_{\eta\xi}\cdot\xi_x\eta_x \pm u_\eta \eta_{xx}.$$

$$= \xi_x^2 u_{\xi\xi} + 2\eta_x\xi_x u_{\xi\eta} + u_\xi \xi_{xx} + u_\eta \eta_{xx} + u_{\eta\eta}\eta_x^2$$

$\therefore$ find all values like this & substitute in og. eqn.

④ Substitute p, q, r, s, t in terms of ξ, η.

on simplifying, we get

canonical as: $\frac{\partial^2 u}{\partial \xi \partial \eta} = \phi\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right)$

(II) if $S^2-4RT=0$,

$\lambda_1 = \lambda_2$.

$\therefore$ we get only one fn. of (x, y) (from $\frac{dy}{dx} + \lambda = 0$) $= \xi$

we assume another fn. of (x, y) INDEPENDANT of ξ

$\xi = x - y$

we assume $\eta = x + y$.

(on substitution)

note: eg: if we get $u_{\eta\eta} = 0$

$\Rightarrow \frac{\partial u}{\partial \eta} = \phi_1(\xi)$

$\Rightarrow u = \eta\phi_1(\xi) + \phi_2(\xi)$

$z = (x+y)\phi_1(x-y) + \phi_2(x-y)$

(III) if $S^2-4RT<0$,

$\xi = \alpha + i\beta$

$\eta = \alpha - i\beta$.

we find α, β in terms of x, y.

& then solve considering $u(\alpha, \beta)$.

eg. type (II) $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ $R=1$; $S=2$; $T=\cancel{x}\ 1$

$S^2 - 4RT = 0$

$\lambda^2 + 2\lambda + 1 = 0$ $\lambda = -1$

$\therefore \frac{dy}{dx} - 1 = 0$ $\therefore y - x = c_1 = \xi$

$\eta = y + x$ (we assume).

$\frac{\partial^2 u}{\partial \eta^2} = 0$

One dimensional wave eqn:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

initial conditions: $u(x,0) = f(x)$ $\quad$ $u(x,y) \rightarrow u(\xi, \eta)$ (to solve)

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\cdot ds}$$

D'Alembert's soln. of wave eqn.

How to solve:

SEPARATION OF VARIABLES: (via example) ☆ remember method for paper.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad -\text{Ⓐ} \qquad 0 < x < l \;;\; t > 0$$

BC: $u(0,t) = 0$, $u(l,t) = 0$ — Ⓑ $\quad\Big|\quad$ $u(x,0) = f(x)$, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$ — Ⓒ

$$\boxed{u(x,t) = F(x)T(t) = FT}$$

where F is a fn. of x & T is a fn. of t only.

substitute in Ⓐ

$$F \cdot \frac{\partial^2 T}{\partial t^2} = c^2 \frac{\partial^2 F}{\partial x^2} \cdot T \implies \underbrace{\frac{1}{F}\frac{\partial^2 F}{\partial x^2}}_{\text{fn. of } x.} = \underbrace{\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2}}_{\text{fn. of } t.}$$

cannot be equal unless both reduce to const. value.

case 1: $= 0$

$$\frac{\partial^2 F}{\partial x^2} = 0 \quad \text{and} \quad \frac{\partial^2 T}{\partial t^2} = 0$$

$\implies F = Ax + B$ $\qquad$ $\implies T = Ct + D$

case 2: $= k^2$

$$\frac{\partial^2 F}{\partial x^2} - k^2 F = 0 \qquad\qquad \frac{\partial^2 T}{\partial t^2} - c^2k^2 t = 0$$

$F = Ae^{kx} + Be^{-x\cdot k}$ $\qquad$ $T = \cancel{Ae}\; Ce^{t\cdot kc} + De^{-t\cdot kc}$

case 3: $= -k^2$

$$\frac{\partial^2 F}{\partial x^2} + k^2 F = 0 \qquad\qquad \frac{\partial^2 T}{\partial t^2} + k^2c^2 T = 0$$

$F = A\cos kx + B\sin kx$ $\qquad$ $T = C\cos(ckt) + D\sin(ckt)$

$u(x,t) = F(x)\,T(t)$

B.C. $u(0,t) = 0$
$u(l,t) = 0$

Case 1: $F = Ax + B$ $\quad T = cT + D$

$u(0,t) = 0 \Rightarrow T(t) = 0$ or $F(0) = 0$

$u(l,t) = 0 \Rightarrow T(t) = 0$ or $F(l) = 0$

$T(t) \neq 0 \Rightarrow u(x,t) = 0 \cdot F(x) = 0$ (always a soln. $(u(x,t) = 0)$)
$\therefore$ trivial, not useful soln.

$F(0) = 0$ and $F(l) = 0$

$\Rightarrow B = 0 \qquad \Rightarrow Al + B = 0$

$\therefore A = 0 \quad \because (B = 0)$

$\Rightarrow F = 0 \Rightarrow u(x,t) = 0$ X

Case 2: $F = Ae^{kx} + Be^{-x\cdot k}$ $\quad T = Ce^{t\cdot ck} + De^{-t\cdot ck}$

$u(0,t) = 0$

$\Rightarrow (A+B)(Ce^{tck} + De^{-tck}) = 0 \Rightarrow A + B = 0$

$u(l,t) = 0$

$Ae^{kl} + Be^{-kl} = 0 \Rightarrow A = 0 = B.$

$\therefore u(x,t) = 0 \quad (\because F = 0)$

Case 3: $F = A\cos kx + B\sin kx$ $\quad T = \cancel{A}\, C\cos kct + D\sin kct.$

$u = (A\cos kx + B\sin kx)(C\cos kct + D\sin kct)$

$u(0,t) = A(\ )$

$\Rightarrow A = 0$

$u(l,t) = B\sin kx\,(C\cos kct + D\sin kct)$

$B\sin kl = 0 \qquad B \neq 0$

$\Rightarrow \sin kl = 0 \Rightarrow kl = n\pi$

$$\therefore u_n(x,t) = \cancel{B}\sin\left(\frac{n\pi x}{l}\right)\left[C_n\cos\left(\frac{n\pi ct}{l}\right) + D_n\sin\left(\frac{n\pi ct}{l}\right)\right]$$

(By principle of superposition)

$$\sum u_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left[C_n\cos\left(\frac{n\pi ct}{l}\right) + D_n\sin\left(\frac{n\pi ct}{l}\right)\right]$$

substitute in this: gives required $u(x,t)$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = \sum_{n=1} \frac{n\pi c}{l} D_n \sin\left(\frac{n\pi x}{l}\right)$$

comparing with Fourier Series: (next page)

$$C_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad \frac{n\pi c}{l} D_n = \int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

FOURIER SERIES:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right] \qquad \alpha < x < \alpha + 2\pi$$

$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin(nx)\,dx.$$

EVEN FUNCTION: F.S. of even fn. only consists of cosine terms.

ODD FUNCTION: F.S. of odd fn. only consists of sine terms.

FUNCTIONS WITH ARBITRARY PERIOD $(-l, l)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx. \qquad a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx.$$

HALF RANGE FOURIER SERIES: $(0, l)$: depending on odd/even. used on prev. page.

Half range sine: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ $\boxed{b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx}$

Half range cosine: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$ $\qquad a_0 = \frac{2}{l}\int_0^l f(x)\,dx.$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

eg. if $f(x) = \sin^3 \frac{\pi x}{2}$.

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right)\cos\left(\frac{n\pi c t}{2}\right)$$

$$u(x,0) = \sum A_n \sin\left(\frac{n\pi x}{2}\right) = f(x) = \sin^3\frac{\pi x}{2}$$

$$= \frac{3}{4}\sin\frac{\pi x}{2} - \frac{1}{4}\sin\frac{3\pi x}{2}$$

$$\therefore A_1 = \frac{3}{4};\quad A_3 = \frac{-1}{4}.$$

HEAT EQUATION: (heat flows in x- direction only)

$u(x,t)$: temp. at point x at time t.

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}} \quad 0<x<l \quad t>0.$$

B.C $\{u(0,t)=u(l,t)=0$

I.C $\{u(x,0)=f(x)$. Also u remains finite as $t \longrightarrow \infty$.

To solve: consider $\boxed{u(x,t)=F(x).T(t)}$

(solve same way as heat, equating the three cases to $0, k^2, -k^2$)

for case ②:

$F=Ae^{kx}+Be^{-kx}$ $T=ce^{k^2c^2t}$.

reject as: u should be finite when $t \longrightarrow \infty$.

for case ③, we get : $\boxed{\text{general}: \theta(x,t)=e^{-c^2k^2t}(A\cos kx + B\sin kx)}$

$$u(x,t)=\sum_{n=1} u_n(x,t) = \sum_{n=1} B_n \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-n^2\pi^2c^2}{l^2}.t}$$

$$u(x,0)=f(x)=\sum_{n=1} B_n \sin\left(\frac{\pi n x}{l}\right) \quad \therefore B_n = \frac{1\times 2}{l}\int_0^l f(x).\sin\left(\frac{n\pi x}{l}\right)dx.$$

LAPLACE EQUATION: (we consider steady state T in a rect. sheet of metal). T everywhere independent of time.

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \quad \text{Laplace Equation} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

DIRICHLET PROB. FOR RECTANGLE: $0 \le x \le a$
$0 \le y \le b$.

BC. $u(0,y)=0$
$u(a,y)=0$
$u(x,b)=0$
$u(x,0)=f(x)$.

again, assume soln as $u(x,y)=F(x).G(y)$

case ③ : $\boxed{u(x,y)=(A\cos kx + B\sin kx)(Ce^{ky}+De^{-ky})}$

remember: $\dfrac{e^{\frac{n\pi(y-b)}{a}} - e^{-\frac{n\pi(y-b)}{a}}}{2} = \sinh\left(\frac{n\pi(y-b)}{a}\right)$ $(\sinh x)$

Laplace in polar:

$$\boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0} \quad 0 \le r \le a \quad 0 \le \theta \le 2\pi$$

assume $u(r,\theta)=R(r)H(\theta)$

COMPLEMENTARY FN & PARTICULAR INTEGRAL:

Non homogenous linear with const. coeff:

$$\boxed{\begin{aligned}(D - mD' - \alpha)z &= 0 \\ z &= e^{\alpha x} f(y+mx)\end{aligned}}$$

$$\therefore (D - m_1D' - \alpha_1)(D - m_2D' - \alpha_2)\dots(D - m_nD' - \alpha_n)z = 0$$

$$Z = e^{\alpha_1 x} f(y+m_1x) + e^{\alpha_2 x} f_2(y+m_2x) + \dots e^{\alpha_n x} f_n(y+m_nx)$$

$(D - mD' - \alpha)^r z = 0$ (repeated factors):

$$Z = e^{\alpha x} f_1(y+mx) + xe^{\alpha x} f_2(y+mx) + \dots x^{r-1}e^{\alpha x} f_r(y+mx)$$

PARTICULAR INTEGRAL:

case ① RHS: e^{ax+by}.

$$\frac{1}{f(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}.$$

if $f(D,D')$ is O at (a,b)

$D \to D+a$; $D' = D'+b$

$\Rightarrow e^{ax+by} \cdot \left(\frac{1}{f(D+a, D'+b)}\right)$

case ② RHS: $\sin(ax+by)$ or $\cos(ax+by)$

put $D^2 = -a^2$; $D'D = -ab$; $D'^2 = -b^2$

$$\frac{1}{f(D,D')} \sin(ax+by).$$

case ③ RHS: $x^m y^n$ m, n are +ve integers.

$$\frac{1}{f(D,D')} x^m y^n = [f(D,D')]^{-1} x^m y^n.$$

$$\frac{1}{\phi(D, D')} F(x,y) = \frac{1}{\phi(D,D')} e^{ax+by} \cdot V(x,y) = e^{ax+by} \frac{1}{\phi(D_x+a, D_y+b)} V(x,y)$$