Lecture-11 (12-02-2024)

Homeomorphic topological spaces :-

continuous (lecture 9, page 10)
f:X->Y is continuous iff f^-1 on any open set of Y is open in X

Two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}^*)$ are said to be homeomorphic or topologically equivalent if there exists a bijective function $f: X \longrightarrow Y$ such that f and $f^{-1}: Y \longrightarrow X$ are continuous.

The function f is called homeomorphism.

In this case we write $(X, \mathcal{T}) \cong (Y, \mathcal{T}^*)$.

H.W.
Problem: Show that homeomorphism is an equivalence relation. ?

https://proofwiki.org/wiki/Homeomorphism_Relation_is_Equivalence#:~:text=Let%20T1%20and%20T2%20be%20topological%20spaces%20such,been%20shown%20to%20b

Def: A function $f: (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}^*)$ is called topological or bicontinuous if f is an open map and continuous.

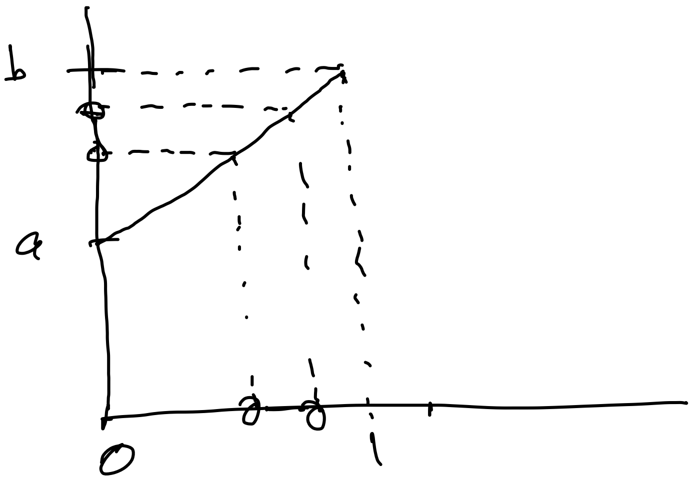
open map
f:X->Y is open if it maps all open sets of X to open sets of Y

Thus $f: (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}^*)$ is homeomorphism iff f is bicontinuous and f is bijective.

Ex: $(0,1) \cong (a,b)$, $a<b$.

Define $f : (0,1) \longrightarrow (a,b)$

by $f(x) = a(1-x) + bx$.



Clearly image of any open interval in $(0,1)$ is an open set in (a,b). Also, any open set in $(0,1)$ is union of open intervals.

$\therefore$ $G = \bigcup_i (a_i, b_i)$, $0 \leq a_i < b_i \leq 1$

Then $f(G) = f\left(\bigcup_i (a_i, b_i)\right)$

$= \bigcup_i f(a_i, b_i)$

$\Rightarrow f$ is an open map.

$\Rightarrow f^{-1} : (a,b) \longrightarrow (0,1)$ is continuous.

Clearly f is $1-1$ and onto, f is continuous.

$\therefore$ $f: (0,1) \to (a,b)$ is homeomorphism.

$$(0,1) \cong (a,b).$$

Problem: (1) Show that $\mathbb{R} \cong (a,b)$, $a<b$, $a,b \in \mathbb{R}$.

(2) Define $f: (-1,1) \to \mathbb{R}$ by $f(x) = \dfrac{x}{1-|x|}$, $x \in (-1,1)$.

Show that $(-1,1) \cong \mathbb{R}$.

(3) $(c,d) \cong (a,b)$, $a,b,c,d \in \mathbb{R}$, $a<b$, $c<d$.

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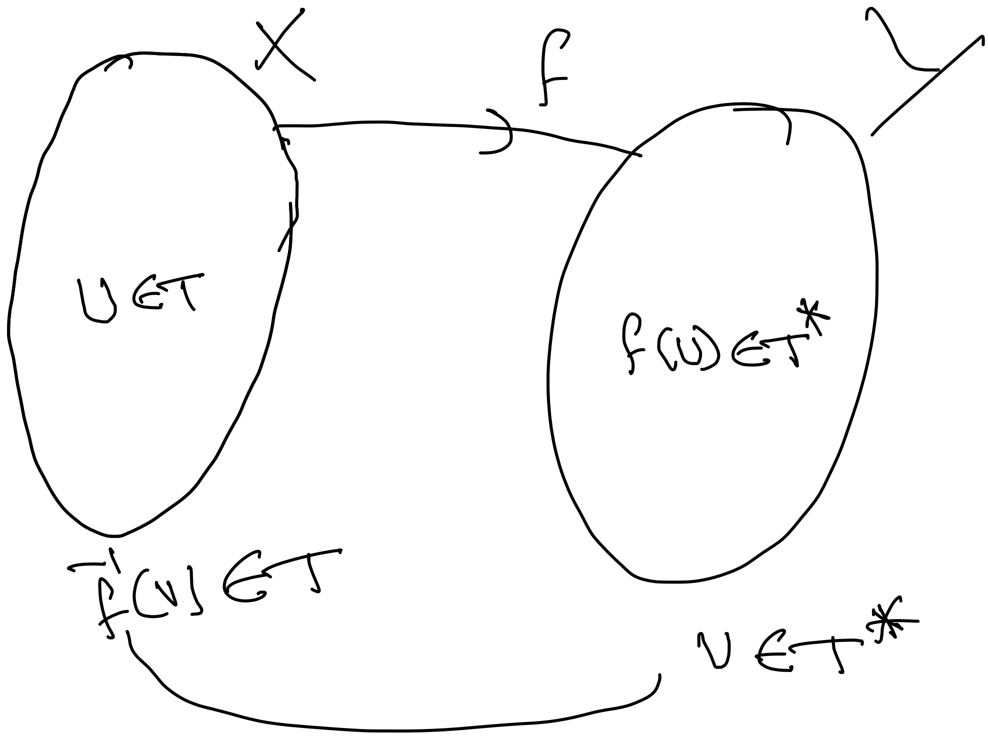
Now let $f:(X,T)\to(Y,T^*)$ be a bijective map.



$\therefore$ A bijective function $f:(X,T)\to(Y,T^*)$ is homeomorphism equivalent to say that

(a) A set V is open in Y.

$\Leftrightarrow$ $f^{-1}(V)$ is open in X.

[f is homeomorphism $\Leftrightarrow$ f is open map and f is continuous bijective]

$f : X \longrightarrow Y$ Continuous $\Leftrightarrow$ $f^{-1}(V)$ open in X $\forall$ open sets V in Y.

Also

$f^{-1} : Y \longrightarrow X$ is continuous for any open set U in X,

$(f^{-1})^{-1}(U) = f(U)$ is open in Y.

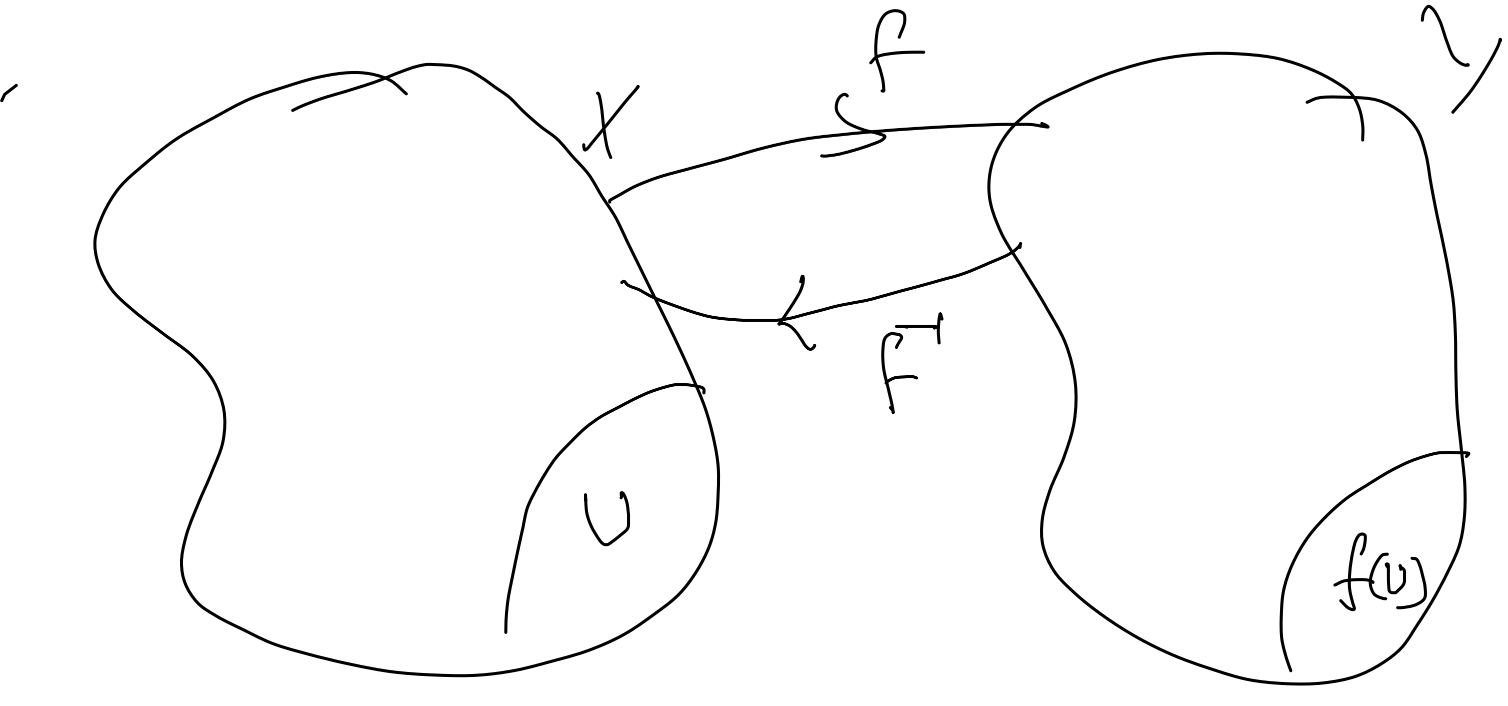
Now each set $U \subset X$ has the form $U = f^{-1}(V)$, for some set V in Y.

Thus $f^{-1} : Y \longrightarrow X$ continuous is equivalent to $U = f^{-1}(V)$ is open in X implies V is open in Y.

$\therefore$ A bijection map $f : X \longrightarrow Y$ is homeomorphism equivalent to say that " a set V is open in Y $\Leftrightarrow$ $f^{-1}(V)$ is open in X.

Illy replacing f by f^{-1} in the above, we see that a bijective function $f : X \longrightarrow Y$ is homeomorphism equivalent to, say that

" A set U is open in X $\Longleftrightarrow$ $f(U)$ is open in Y".



This shows that a homeomorphism $f : X \longrightarrow Y$ gives bijective correspondence not only between

X and Y, but also between the collection of open sets of X and Y.

As a result any property of X that is entirely expressed in terms of topology of X gives the corresponding property of the space Y via function f.

[Attendance: 11, 41, 63, 62, 19, 42, 55, 32, 24, 27, 06, 60, 57, 09, 64, 17, 61, 10, 43, 07, 58, 01]

" A property 'P' of sets is called topological property or topologically invarient if whenever a topological space (X, T) has the property 'P', then every space homeomorphic to (X, T) also has the property 'P'.

Ex: $X = (0, \infty) \subset \mathbb{R}$

Define $f : X \longrightarrow X$ by

$f(x) = \frac{1}{x}$.

Let $\{a_n = \frac{1}{n}\}$ be a sequence in X.

Then $\{f(a_n)\} = \{1, 2, 3, 4, 5, \ldots\}$.

The sequence $\{a_n\}$ is a Cauchy sequence, but $\{f(a_n)\}$ is not a Cauchy sequence.

$\therefore$ The property of being a Cauchy sequence is not a topological property.

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Topologies Induced by functions:

Let (Y_i, T_i), $i = 1, 2, 3, \ldots$, be a collection of topological spaces and for each $i = 1, 2, \cdots$, let $f_i : X \longrightarrow Y_i$ be a function defined on some arbitrary non-empty set X.

We want find a topology on X w.r.t which all the functions $f_i : X \longrightarrow Y_i$ are continuous.

So

Let $S = \bigcup_i \{ f_i^{-1}(H) \,/\, H \in T_i \}$.

The topology T on X generated by S is called the topology induced by the functions f_i, $i = 1, 2, 3, \cdots$.

We call S, the defining subbase for the topology T induced by the function f_i, $i = 1, 2, 3, \cdots$.

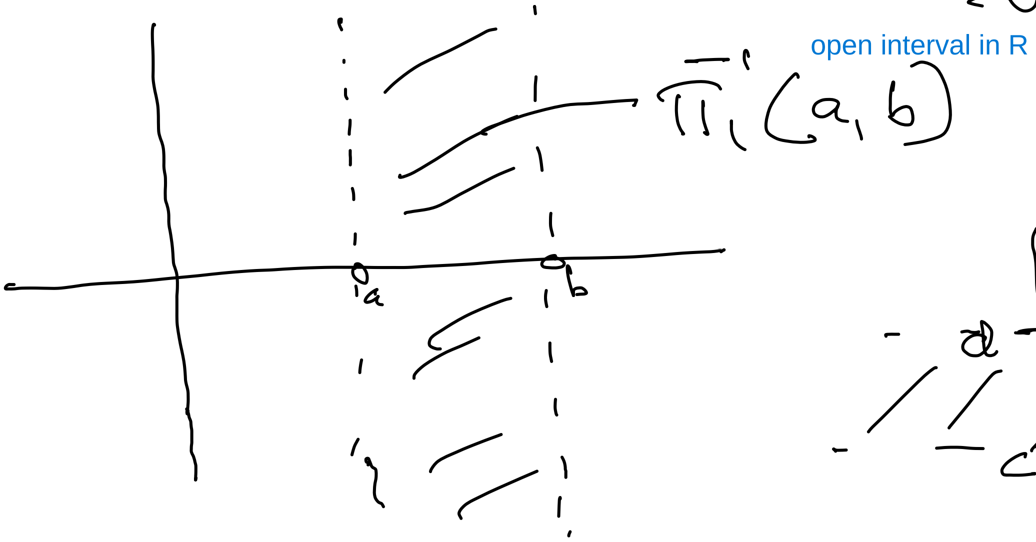
The topology T is the coarsest topology on X containing S w.r.t which f_i, $i = 1, 2, \cdots$ are continuous, $[S \subset T]$.

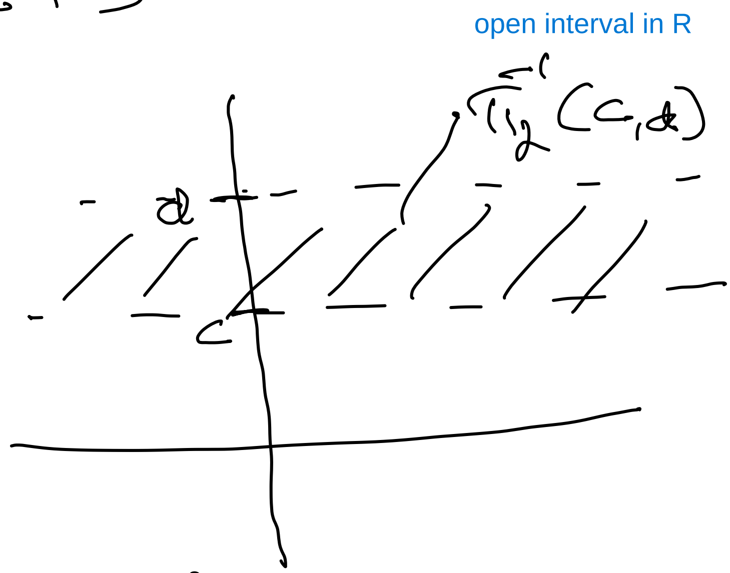
Ex: $\pi_i : (\mathbb{R}^2, U) \longrightarrow (\mathbb{R}, U)$, $i=1,2$

point $\quad\quad\quad\quad$ point

by $\pi_1(x,y) = x$, $\pi_2(x,y) = y$

$\forall\, (x,y) \in \mathbb{R}^2$.





$$S = \{\pi_1^{-1}(G) \mid G \in U\}$$
$$\cup \{\pi_2^{-1}(G) \mid G \in U\}.$$

Since the infinite strips form a subbase for the usual topology on $\mathbb{R}^2$, so the usual topology on $\mathbb{R}^2$ is the coarsest topology on $\mathbb{R}$ w.r.t which the projections $\pi_i : (\mathbb{R}^2, U) \longrightarrow (\mathbb{R}, U)$, $i=1,2$ are continuous.

Theorem: let $f_i : X \longrightarrow (Y_i, T_i)$, $i = 1, 2, \dots$, be a collection of functions defined on an arbitrary non empty set X, to topological space (Y_i, T_i), $i = 1, 2, \dots$

let $S = \bigcup_i \{ f_i^{-1}(H) \mid H \in T_i \}$

and let T be the topology generated by S. Then

(i) All the functions f_i are continuous w.r.t. T.

[$\because S \subset T$]

(ii) If T^* is the intersection of all the topologies on X, w.r.t which all the functions $f_i : X \longrightarrow (Y_i, T_i)$ are continuous, then $T^* = T$.

Proof: Clearly $\mathcal{S} \subset \mathcal{T}^*$.

Also $\mathcal{T}$ is the topology generated by $\mathcal{S}$. $\therefore \mathcal{T} \subset \mathcal{T}^*$

On the other hand $\mathcal{T}$ is one of the topologies containing $\mathcal{S}$ and $\mathcal{T}^*$ is the intersection of all those topologies containing $\mathcal{S}$.

$$\therefore \mathcal{T}^* \subset \mathcal{T}$$

$$\therefore \mathcal{T} = \mathcal{T}^*$$

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Problem: Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}^*)$ be 1-1 and open map. Let $A \subseteq X$ and $f(A) = B$. Let $f_A = f|_A : A \to Y$ be the restriction of f to the set A.

Let (A, T_A) and (B, T_B^*) be relative topological spaces. Then show that $f_A : (A, T_A) \longrightarrow (B, T_B^*)$ also 1–1 and open map.

Sol: If f is 1–1, then the restriction of f any subset of X is also 1–1. So we only prove f_A is open map.

So let $H \in T_A \Rightarrow \exists\, G \in T$ such that $H = A \cap G$.

$\because f$ is 1–1, we have

$$f(A \cap G) = f(A) \cap f(G).$$

$\therefore$

$$f_A(H) = f(H) = f(A \cap G)$$
$$= f(A) \cap f(G)$$
$$= B \cap f(G)$$
$$\in T_B^* \quad [\because G \in T \Rightarrow f(G) \in T^*]$$

Thus $H \in T_A \Rightarrow f_A(H) \in T_B^*$

$\therefore f_A : (A, T_A) \longrightarrow (B, T_B^*)$

is an open map.

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* Problem
In the above Problem if $f : (X, T) \to (Y, T^*)$ is homeomorphism, Prove that $f_A : (A, T_A) \to (B, T_B^*)$ is also homeomorphism.

[Hint: $f_A^{-1}(G) = \{x \in A \mid f_A(x) \in G\}$

$= \{x \in A \mid f(x) \in G\}$

$= A \cap \{x \in X \mid f(x) \in G\}$

$= A \cap f^{-1}(G)$, for any $G \subseteq Y$]

If G is any open set in Y, $f^{-1}(G) \in T$, then $A \cap f^{-1}(G) \in T_A$

$\Rightarrow f_A^{-1}(G) \in T_A$]

Problem: Let $I : (X, T) \longrightarrow (X, T^*)$ be the identity map.

Then I is continuous iff $T^* \subset T$.

[∵ For $G \in T^*$, $I^{-1}(G) = G \in T$

$\Leftrightarrow T^* \subset T$].

Problem: Let $f : (X, T_1) \longrightarrow (Y, T_2)$ be continuous.

Let T_1^* be a topology on X coarser than T_1

and T_2^* " on Y finer than T_2.

Then discuss the continuity of

$f : (X, T_1^*) \longrightarrow (Y, T_2^*)$.

[Attendance 5 PM ——//——

11, 62, 27, 06, 60, 01, 02, 23, 41]

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