(#) Linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$

A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if $\forall\, x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$

$$\boxed{T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)}$$

Example :-

1) "0" map

2) $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$ if $m \leq n$

3) Zero padding

4) $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_n \xrightarrow{T} \begin{pmatrix} x_1 \\ \frac{x_1 + x_2}{2} \\ x_2 \\ \frac{x_2 + x_3}{2} \\ \vdots \\ x_n \end{pmatrix}_{2n-1}$

Proof of 4

$$T(\alpha x + \beta y) = T\left(\begin{pmatrix} \alpha x_1 + \beta y_1 \\ \vdots \\ \alpha x_n + \beta y_n \end{pmatrix}\right)$$

$$= \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \frac{\alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2}{2} \\ \vdots \\ \alpha x_n + \beta y_n \end{pmatrix}$$

$\alpha T(x) + \beta (T y)$

$$= \begin{pmatrix} \alpha x_1 \\ \frac{\alpha x_1 + \alpha x_2}{2} \\ \vdots \\ \alpha x_n \end{pmatrix} + \begin{pmatrix} \beta y_1 \\ \frac{\beta y_1 + \beta y_2}{2} \\ \vdots \\ \beta y_n \end{pmatrix}$$

LHS = RHS

5) $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \xrightarrow{T} \begin{pmatrix} |x_1| \\ \vdots \\ |x_2| \end{pmatrix}$

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

This is not a linear transformation

(#) Let $e_1, \dots, e_n \in \mathbb{R}^n$ be the standard basis vectors

$T: \mathbb{R}^n \to \mathbb{R}^m$ Linear.

$T(e_1), \dots\dots T(e_n) \in \mathbb{R}^m$

Take any $x \in \mathbb{R}^n$, $x \in \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$T(x) = T(x_1 e_1 + x_2 e_2 + \dots + x_n e_n)$$

$$= x_1 T(e_1) + \dots + x_n T(e_n)$$

$$\underset{m\times 1}{=} \underbrace{\left[T(e_1) \;\; T(e_2) \dots\dots T(e_n) \right]}_{m\times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

This is linear combination of n vectors

→ linear combination of

1) columns → multiply on right
2) rows → multiply on left

(#) Matrix Vector multiplication

$A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$

$y = Ax$, $y \in \mathbb{R}^m$

$$y = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ \vdots & \vdots \\ a_{m1} \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix}$$

Computational Complexity

$2n-1$ (Floating point op. per inner product)

{ n multiplication an $2n-1$ additions }

$$\boxed{\text{Total :- } m(2n-1)}$$

$$= \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n$$

Computation

$mn + (n-1)m = m(2n-1)$

(#) Range of T

$T: \mathbb{R}^n \to \mathbb{R}^m \leftarrow$ linear

$R(T) =$ range of $T = \{T(x) \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$

Is $R(T)$ a subspace?

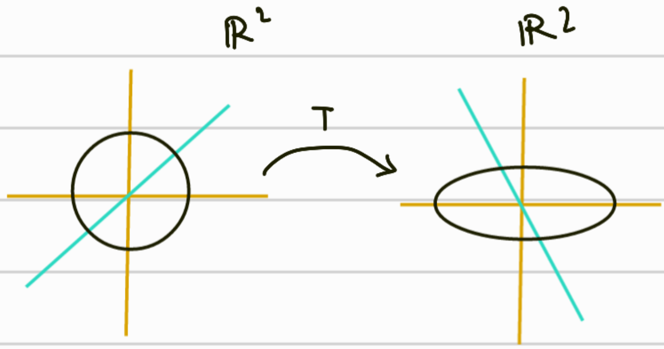
$$T(x) = \{x_1 T(e_1) + x_2 T(e_2) + \cdots x_n T(e_n) \mid x_1, \ldots x_n \in \mathbb{R}\}$$

$$= \text{span}\{T(e_1), T(e_2), \ldots T(e_n)\}$$

(#) Geometric Interpretation

Ex:- $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Q) Find for general case

$$T(x) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

for a circle

>> $T(x) = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$

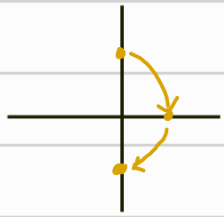
$$T(x) = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

with $x_1^2 + x_2^2 = 1$

(#) Special class of LTs

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad (n=2)$

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$$



$a, b \in \mathbb{R}^n$

$$\cos\theta = \frac{a^T b}{\|a\|_2 \|b\|_2} = \frac{\langle a, b\rangle}{\|a\|_2 \|b\|_2}$$

correlation coefficient (-1 and 1)

(Prove that $\cos\theta$ and correlation concept is same)

- a and b are orthogonal if $a^T b = 0$ / $\langle a, b\rangle = 0$

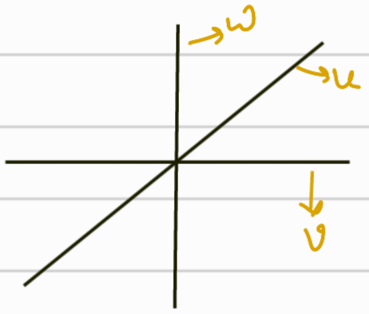
- A matrix $Q \in \mathbb{R}^{n \times n}$ is orthogonal matrix if $Q^T Q = Q Q^T = I$

  $Q = [q_1\ q_2\ \dots\ q_n]$

Q) Prove if any matrix Q which is orthogonal is either rotators or reflectors.

$T = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

Invariant subspace



$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| x_2 = 0 \right\}$

$T(v) = v$

$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| x_1 = 0 \right\}$

$T(w) = w$

$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| x_1 = x_2 \right\}$

But $T(u) \neq u$

Ex) Reflection :-

$T(u) = u$

$T(v) = -v$

For any $x \in \mathbb{R}^2$, $\exists$ unique α and β such that $x = \alpha u + \beta v$ and

$T(x) = T(\alpha u + \beta v) = \alpha v - \beta v$

if T is linear

Let $A = uu^T$

$\Rightarrow Au = (uu^T)u = u\underbrace{(u^Tu)}_{1} = u$

$Av = (uu^T)v = u\underbrace{(u^Tv)}_{0} = 0$ (scalar)

Try $A = I - 2uu^T$

(#) Eigenspace

- We may not have eigen vectors if we talk only about real values
- Geometrically eigen vectors don't change (direction) while applying Linear Transformation (They are invariance)
- Eigen values are the scaling factor of eigen vectors

For a symmetric matrix we will always have eigen values (spectral decomposition theorem)

A matrix may not have a eigen vector.

HW) Give a matrix such that it have 2 invariance subspace and they don't be a coordinate axes. (Take l_1 and l_2 to be orthogonal for easiness)