Mathematical Methods

Test 2

Time 1 hour 5 min

Pg-1

Q1. (a) The BVP $y''+y=x$, $y(0)=0$ and $y'(1)=0$ is reduced to the Fredholm integral equation

$$y(x)=\int_0^1 G(x,t)\,y(t)\,dt-\frac{1}{6}\left(k_1 x-\frac{5}{k_2}x^3\right)$$

then $k_1 = \underline{\ 3\ }$ $\quad k_2 = \underline{\ 5\ }$.

Here $G(x,t)$ is the Greens function of the associated homogeneous BVP and

$$G(x,t)=\begin{cases}\frac{2}{k_3}x & 0\le x<t\\ k_4 t & t<x\le 1\end{cases}$$

$\boxed{2M}$

Then $k_3 = \underline{\ 2\ }$ and $k_4 = \underline{\ 1\ }$

Sol^n:

$$G(x,t)=\begin{cases}a_1x+a_2 & 0\le x<t\\ b_1x+b_2 & t<x\le 1\end{cases}$$

$b_1t+b_2=a_1t+a_2 \Rightarrow t(b_1-a_1)+b_2-a_2=0$

$\left(\frac{\partial G}{\partial x}\right)_{x=t+}-\left(\frac{\partial G}{\partial x}\right)_{x=t-}=-1 \Rightarrow b_1-a_1=-1$

$G(0,t)=0 \therefore a_2=0$; $G(1,t)=0 \therefore b_1=0$

Solving $b_2=t$, $a_1=1$

$$\therefore G(x,t)=\begin{cases}x & 0\le x<t\\ t & t<x\le 1\end{cases}$$

Comparing $y''+y-x=0$ with $y''+\phi(x)=0$, $\phi(x)=y(x)-x$

$\therefore \phi(t)=y(t)-t$

$$\therefore y(x)=\int_0^1 G(x,t)\,\phi(t)\,dt$$

$$=\int_0^1 G(x,t)\,[y(t)-t]\,dt$$

$$\int_0^1 t\,G(x,t)\,dt = \int_0^x t\,G(x,t)\,dt + \int_x^1 t\,G(x,t)\,dt$$

$$= \int_0^x t^2\,dt + \int_x^1 xt\,dt$$

$$= \left[\frac{t^3}{3}\right]_0^x + x\left[\frac{t^2}{2}\right]_x^1$$

$$= \frac{x^3}{3} + \frac{x}{2}(1-x^2) = \frac{1}{6}(3x-x^3)$$

$$\therefore y(x,t) = \int_0^1 G(x,t)\,y(t)\,dt - \frac{1}{6}(3x-x^3)$$

Q1.(b) In problem Q.1.(a), if the first boundary condition is changed to $y(0)=1$, then the Fredholm integral equation takes the form $\boxed{1M}$

$$y(x) = \int_0^1 G(x,t)\,y(t)\,dt + \frac{1}{k_6}(x^3 + k_7 x + k_8)$$

Then $k_6 =$ 6, $k_7 =$ −3 and $k_8 =$ 6

Sol^n:
$$y(x) = P(x) + \int_0^1 G(x,t)\,\phi(t)\,dt$$

$$\Rightarrow y(x) = P(x) + \int_0^1 G(x,t)\,y(t)\,dt - \int_0^1 t\,G(x,t)\,dt$$

$P''(x)=0 \quad P(0)=1 \quad P'(1)=0$

$P(x) = A'x+B' \quad P(0)=1 \Rightarrow B'=1$

$P'(1)=0 \Rightarrow A'=0$

$\therefore P(x)=1$

$$\therefore y(x) = 1 + \int_0^1 G(x,t)\,y(t)\,dt - \frac{1}{6}(3x-x^3)$$

$$= \frac{1}{6}(x^3-3x+6) + \int_0^1 G(x,t)\,y(t)\,dt$$

Q2.(a) To solve $y'' - 2y' + y = xe^x \log x$, $x > 0$, method of variation of parameter is adopted. The P.I. is of the form $\frac{1}{k_1} x^3 e^x \log x - \frac{k_2}{k_3} x x^3 e^x$.

Then $k_1 = $ <u>6</u>, $k_2 = $ <u>5</u> and $k_3 = $ <u>36</u>

$\boxed{2M}$

Solⁿ:

$m^2 - 2m + 1 = 0 \qquad m = 1, 1$

C.F. $\quad (c_1 + c_2 x) e^x = c_1 e^x + c_2 x e^x$

Let $u = e^x$, $v = xe^x \qquad R = xe^x \log x$

$$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$$

P.I. $= u f(x) + v g(x)$

$$f(x) = -\int \frac{vR}{W} dx = -\int \frac{xe^x \, xe^x \log x}{e^{2x}} dx$$

$$= -\int x^2 \log x \, dx$$

$$= -\left[\log x \cdot \frac{x^3}{3} - \int \frac{1}{x} \frac{x^3}{3} dx\right] = -\left[\frac{1}{3} x^3 \log x - \frac{1}{9} x^3\right]$$

$$g(x) = \int \frac{uR}{W} dx = \int \frac{e^x \, xe^x \log x}{e^{2x}} dx$$

$$= \int x \log x \, dx$$

$$= \frac{x^2}{2} \log x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \frac{x^2}{2} \log x - \frac{x^2}{4}$$

$$\text{P.I.} = -e^x \left\{ \frac{x^3}{3} \log x - \frac{x^3}{9} \right\} + xe^x \left\{ \frac{x^2}{2} \log x - \frac{x^2}{4} \right\}$$

$$= x^3 e^x \log x \left(\frac{1}{2} - \frac{1}{3}\right) - x^3 e^x \left(\frac{1}{4} - \frac{1}{9}\right)$$

$$= \frac{1}{6} x^3 e^x \log x - \frac{5}{36} x^3 e^x$$

Q2-(6) Using the method of variation of parameter, the solution of $y'' + 9y = \phi(x)$ satisfying the initial conditions $y(0)=0$, $y'(0)=0$ is obtained as

$$y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{c_3}\int_0^x \phi(t) \sin c_4(x-t)\, dt$$

Then $c_1 = \underline{0}$, $c_2 = \underline{0}$, $c_3 \underline{3}$ and $c_4 = \underline{3}$

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Sol^n: A.E. $m^2+9=0$ $m=\pm 3i$ $y = c_1 \cos 3x + c_2 \sin 3x$

$u = \cos 3x$ $v = \sin 3x$

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$$

$$P.I. = u f(x) + v g(x)$$

$$f(x) = -\int \frac{vR}{W}\, dx = -\int_0^x \frac{\sin 3x\, \phi(x)}{3}\, dx = -\frac{1}{3}\int_0^x \phi(t) \sin 3t\, dt$$

$$g(x) = \int \frac{uR}{W}\, dx = \int_0^x \frac{\cos 3x\, \phi(x)}{3}\, dx = \frac{1}{3}\int_0^x \phi(t) \cos 3t\, dt$$

$$\therefore P.I. = -\frac{1}{3}\cos 3x \int_0^x \phi(t) \sin 3t\, dt + \frac{1}{3}\sin 3x \int_0^x \phi(t) \cos 3t\, dt$$

$$= \frac{1}{3}\int_0^x \phi(t) \sin 3(x-t)\, dt$$

$$y = c_1 \cos kx + c_2 \sin kx + \frac{1}{3}\int_0^x \phi(t) \sin 3(x-t)\, dt$$

$y(0)=0$ $\therefore c_1 = 0$ $\therefore y = c_2 \sin 3x + \frac{1}{3}\int_0^x \phi(t) \sin 3(x-t)\, dt$

$$y'(x) = c_2 \cdot 3\cos 3x + \frac{1}{3}\left[\int_0^x \frac{\partial}{\partial x}\{\phi(t) \sin 3(x-t)\}\, dt + \phi(x) \sin 3(x-x)\frac{dx}{dx}\right]$$

$$= c_2\, 3\cos 3x + \int_0^x \phi(t) \cos 3(x-t)\, dt$$

$y'(0)=0$ $\therefore$ $0 = c_2 \cdot 3$ $\therefore c_2 = 0$

Q3. (a) With the help of $1, x, x^2$ three functions φ_0, φ_1 and φ_2 are constructed which are orthogonal with respect to e^{-x} over $0 \le x < \infty$. If $\varphi_2(x)$ has the form $k_1 x^2 - k_2 x + k_3$, then

$k_1 = $ 1 $\quad k_2 = $ 4 $\quad$ and $\quad k_3 = $ 2

$\boxed{2M}$

Sol^n: $\varphi_0(x) = 1$

$\varphi_1(x) = x + c\,\varphi_0(x) = x + c$

$$\int_0^\infty e^{-x}(x+c)\,dx = 0 \Rightarrow \int_0^\infty e^{-x}x\,dx + c\int_0^\infty e^{-x}\,dx = 0$$

$$\int_0^\infty e^{-x}x^n\,dx = n! \qquad n = 0, 1, 2, 3, \dots$$

$$\therefore\ 1! + c\,0! = 0 \Rightarrow 1 + c = 0 \quad \therefore c = -1$$

$$\therefore \varphi_1 = x - 1$$

$$\varphi_2 = x^2 + c_1\varphi_0 + c_2\varphi_1 = x^2 + c_1 + c_2(x-1)$$

$$\int_0^\infty e^{-x}\varphi_2\varphi_0\,dx = 0 \Rightarrow \int_0^\infty e^{-x}(x^2 + c_2 x + c_1 - c_2)\,dx = 0 \quad -(1)$$

and $$\int_0^\infty e^{-x}\varphi_2\varphi_1\,dx = 0 \Rightarrow \int_0^\infty e^{-x}(x^2 + c_2 x + c_1 - c_2)(x-1)\,dx = 0 \quad -(2)$$

From (1),
$$2! + c_2 \times 1! + (c_1 - c_2)\,0! = 0$$
$$\Rightarrow 2 + \cancel{c_2} + c_1 - \cancel{c_2} = 0 \quad \therefore c_1 = -2$$

From (2),
$$\int_0^\infty e^{-x}\{x^3 + (c_2 - 1)x^2 + (c_1 - 2c_2)x + (c_2 - c_1)\}\,dx = 0$$
$$\Rightarrow 3! + (c_2 - 1)\times 2! + (c_1 - 2c_2)\times 1! + (c_2 - c_1)\times 0! = 0$$
$$\Rightarrow 6 + 2(c_2 - 1) + c_1 - 2c_2 + c_2 - c_1 = 0 \Rightarrow c_2 = -4$$

$$\therefore \varphi_2(x) = x^2 - 4x + 2$$

Q3.(b) Consider three functions $f_1(x) = a_0$, $f_2(x) = b_0 + b_1 x$ and $f_3(x) = c_0 + c_1 x + c_2 x^2$, where $a_0, b_0, b_1, c_0, c_1, c_2$ are real constants. If the given functions form an orthonormal set on the interval $-1 \le x \le 1$, then value of b_0 is <u>0</u>, value of c_1 is <u>0</u>; If c_2 and c_0 is connected by $c_2 = -k c_0$, then k is <u>3</u>.

The value of a_0 can be (i) zero ✓(ii) non-zero 3M

" " " b_1 " " (i) zero ✓(ii) non-zero

" " " c_0 " " (i) zero ✓(ii) non-zero

Sol^n: Since f_1, f_2, f_3 form an orthonormal set on $-1 \le x \le 1$,

(i) $\int_{-1}^{1} f_1 f_2 \, dx = 0$ (ii) $\int_{-1}^{1} f_2 f_3 \, dx = 0$ (iii) $\int_{-1}^{1} f_3 f_1 \, dx = 0$

(iv) $\|f_1\| \neq 0$ (v) $\|f_2\| \neq 0$ (vi) $\|f_3\| \neq 0$

(i) $\Rightarrow \int_{-1}^{1} a_0 (b_0 + b_1 x)\, dx = 0 \Rightarrow \left[a_0 b_0 x + \frac{1}{2} a_0 b_1 x^2\right]_{-1}^{1} = 0$

$\Rightarrow a_0 b_0 = 0$ — (1)

(ii) $\Rightarrow \int_{-1}^{1} (b_0 + b_1 x)(c_0 + c_1 x + c_2 x^2)\, dx = 0$

$\Rightarrow \left[b_0 c_0 x + \frac{1}{2} b_0 c_1 x^2 + \frac{1}{3} b_0 c_2 x^3 + \frac{1}{2} b_1 c_0 x^2 + \frac{1}{3} b_1 c_1 x^3 + \frac{1}{4} b_1 c_2 x^4\right]_{-1}^{1} = 0$

$\Rightarrow 2 b_0 c_0 + \frac{2}{3} b_0 c_2 + \frac{2}{3} b_1 c_1 = 0 \Rightarrow b_0 (3c_0 + c_2) + b_1 c_1 = 0$ — (2)

(iii) $\Rightarrow \int_{-1}^{1} a_0 (c_0 + c_1 x + c_2 x^2)\, dx = 0 \Rightarrow \left[a_0 c_0 x + \frac{1}{2} a_0 c_1 x^2 + \frac{1}{3} a_0 c_2 x^3\right]_{-1}^{1} = 0$

$\Rightarrow 2 a_0 c_0 + \frac{2}{3} a_0 c_2 = 0 \Rightarrow a_0 (3c_0 + c_2) = 0$ — (3)

(iv) $\Rightarrow \left[\int_{-1}^{1} a_0^2 \, dx\right]^{1/2} \neq 0 \Rightarrow (2a_0^2)^{1/2} \neq 0 \Rightarrow a_0 \neq 0$ — (4)

From (1) and (4) $b_0 = 0$ — (5)

$\therefore$ By (2), $b_1 c_1 = 0$ — (6)

(v) $\Rightarrow \int_{-1}^{1} \left[(b_0 + b_1 x)^2 \, dx\right]^{1/2} \neq 0 \Rightarrow \left[\int_{-1}^{1} b_1^2 x^2 \, dx\right]^{1/2} \neq 0$

$\Rightarrow \left[\frac{2}{3} b_1^2\right]^{1/2} \neq 0 \Rightarrow b_1 \neq 0$

$\therefore$ From (6), $c_1 = 0$ — (8)

From (3) and (4), $3c_0 + c_2 = 0$ $\therefore c_2 = -3c_0$ — (9)

(vi) $\Rightarrow \int_{-1}^{1} \left[(c_0 + c_1 x + c_2 x^2)^2 \, dx\right]^{1/2} \neq 0$ Using (9), $c_0 \neq 0$

Q 4-(a) Consider a 2nd order eigenvalue problem

$$-P(x)y'' - q(x)y' + R(x)y = \lambda y \quad \text{— (1)}$$

which is converted into Strum-Liouville eigenvalue problem

$$Ly = \lambda r y \quad \text{where } Ly = -(py')' + qy \quad \text{— (2)}$$

with boundary conditions

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0, \quad \beta_1 y(l) + \beta_2 y'(l) = 0, \quad p(x) > 0, r(x) > 0$$

Multiplying by a suitable factor $\mu(x)$, Eq (1) is converted in Eq. (2). Then the form of $\mu(x)$ is

(i) $\dfrac{e^{\int \frac{p}{q} dx}}{q}$ (ii) $\dfrac{e^{-\int \frac{p}{q} dx}}{q}$ ✓(iii) $\dfrac{e^{\int \frac{q}{p} dx}}{P}$ (iv) $\dfrac{e^{-\int \frac{q}{p} dx}}{P}$

$\boxed{2M}$

4. (b) Hence if $y'' + xy' + \lambda y = 0$, $y(0) = 0 = y(1)$ is reduced in the form of Eq (2) of Q 4 (a), then

$p(x) = e^{\frac{x^2}{k_1}}$ $r(x) = e^{\frac{2x^2}{k_2}}$.

Here $k_1 = \underline{2}$ $k_2 = \underline{4}$ $\boxed{1M}$

Solⁿ: (a) $-P(x)y'' - q(x)y' + R(x)y = \lambda y$

$$-\mu(x)P(x)y'' - \mu(x)q(x)y' + \mu(x)R(x)y = \lambda\mu(x)y \quad \text{—(1)}$$

$$-py'' - p'y' + \quad qy \quad = \lambda r y \quad \text{—(2)}$$

Comparing (1) and (2), $p = \mu P$ $p' = \mu q$

$\therefore p' = \mu' P + \mu P' = \mu q \Rightarrow \mu' P + \mu P' - \mu q = 0$

$\Rightarrow \mu' + \frac{P'}{P}\mu - \mu\frac{q}{P} = 0$

$\Rightarrow \mu' + \left(\frac{P'}{P} - \frac{q}{P}\right)\mu = 0$

I.F. $e^{\int\left(\frac{P'}{P} - \frac{q}{P}\right)dx} = P e^{-\int \frac{q}{P} dx}$

$\therefore \mu P e^{-\int \frac{q}{P} dx} = c$ $\mu = \dfrac{e^{\int \frac{q}{P} dx}}{P}$

(b) $y'' + xy' + \lambda y = 0$ $P = 1$ $q = x$

$\mu = \frac{1}{P} e^{\int \frac{q}{P} dx} = e^{\int x\, dx} = e^{\frac{x^2}{2}}$

$e^{\frac{x^2}{2}}y'' + e^{\frac{x^2}{2}}xy' + \lambda e^{\frac{x^2}{2}}y = 0 \Rightarrow -\left(e^{\frac{x^2}{2}}y'\right)' = \lambda e^{\frac{x^2}{2}}y$

$\therefore p(x) = e^{\frac{x^2}{2}}$ $r(x) = e^{\frac{x^2}{2}}$

Q 4-(c) Define the Strum-Liouville problem as given in Q 4-(a). If u and v satisfy the S-L ODE, then $\int_0^l (vLu - uLv)\,dx =$

(i) $pvu'|_0^l - puv'|_0^l$

(ii) $-qvu'|_0^l + quv'|_0^l$

✓(iii) $-pvu'|_0^l + puv'|_0^l$

(iv) $+qvu'|_0^l - quv'|_0^l$

$\boxed{2M}$

Solⁿ: $\int_0^l vLu\,dx$

$= \int_0^l v\{-(pu')' + qu\}\,dx = -\int_0^l v(pu')'\,dx + \int_0^l quv\,dx$

$= -vpu'|_0^l + \int_0^l v'pu'\,dx + \int_0^l quv\,dx$

$= -vpu'|_0^l + upv'|_0^l + \int_0^l u\{-(pv')'\}\,dx + \int_0^l quv\,dx$

$= -vpu'|_0^l + upv'|_0^l + \int_0^l u\{-(pv')' + qv\}\,dx$

$= -pvu'|_0^l + puv'|_0^l + \int_0^l uLv\,dx$

$\therefore \int_0^l (vLu - uLv)\,dx = -pvu'|_0^l + puv'|_0^l$

4-(d) In continuation to Q 4-(c), if u and v both satisfy SL B.C given in Q 4 (a), then the value of $\int_0^l (vLu - uLv)\,dx$ is $\underline{0}$

$\boxed{1M}$

Solⁿ: u & v satisfy SL B.C $\left.\begin{array}{l}\alpha_1 u(0) + \alpha_2 u'(0) = 0\\ \alpha_1 v(0) + \alpha_2 v'(0) = 0\end{array}\right\}$ $\begin{array}{l}\beta_1 u(l) + \beta_2 u'(l) = 0\\ \beta_1 v(l) + \beta_2 v'(l) = 0\end{array}$

$\int_0^l (vLu - uLv)\,dx = -p(l)v(l)u'(l) + p(l)u(l)v'(l)$
$\qquad + p(0)u'(0)v(0) - p(0)u(0)v'(0)$

$= p(l)\left\{\frac{\beta_1}{\beta_2}u(l)v(l) + u(l)\left(-\frac{\beta_1}{\beta_2}v(l)\right)\right\}$
$+ p(0)\left\{-\frac{\alpha_1}{\alpha_2}u(0)v(0) - u(0)\left(-\frac{\alpha_1}{\alpha_2}\right)v(0)\right\}$
$= 0$

Q5.(a) Consider the ODE

$$(5-x^2)y'' + \frac{1+x}{x}y' - \left(\frac{3}{x^2}+x\right)y = 0$$

The number of singular points are $\underline{3}$. $(0, \pm 5)$ $\boxed{1M}$

$y(x) = \sum_{n=0}^{\infty} c_n x^n$

Q5.(b) If a power series solution for $(x^2+1)y'' - 4xy' + 6y = 0$ about $x=0$ is obtained, then the number of terms in the power series will be $\underline{4}$.

Q.5(c) If a recurrence relation between c_{n+2} and c_n is obtained as

$$c_{n+2} = -\frac{(n-k_1)(n-k_2)}{(n+k_3)(n+k_4)} c_n \quad n \geq 2$$

then $k_1 = \underline{2}$ $k_2 = \underline{3}$ $k_3 = \underline{2}$ $k_4 = \underline{1}$

$(k_1 < k_2,\ k_4 < k_3)$. $\boxed{2M}$

Sol^n:

$$(x^2+1)y'' - 4xy' + 6y = 0$$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 4x\sum_{n=1}^{\infty} n c_n x^{n-1} + 6\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 4\sum_{n=1}^{\infty} n c_n x^n + 6\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - 4\sum_{n=1}^{\infty} n c_n x^n + 6\sum_{n=0}^{\infty} c_n x^n = 0$$

$$2c_2 + 6c_0 + (6c_3 - 4c_1 + 6c_1)x + \sum_{n=2}^{\infty}\Big[n(n-1)c_n + (n+2)(n+1)c_{n+2} - 4nc_n + 6c_n\Big]x^n = 0$$

$2c_2 + 6c_0 = 0$. $c_2 = -3c_0$; $6c_3 + 2c_1 = 0 \Rightarrow 3c_3 + c_1 = 0$

$\Rightarrow c_3 = -\frac{1}{3}c_1$

$$n(n-1)c_n + (n+2)(n+1)c_{n+2} - 4nc_n + 6c_n = 0$$

$$\Rightarrow (n^2 - n - 4n + 6)c_n + (n+2)(n+1)c_{n+2} = 0$$

$$\Rightarrow (n+2)(n+1)c_{n+2} = -(n^2 - 5n + 6)c_n$$

$$\Rightarrow c_{n+2} = -\frac{(n-2)(n-3)}{(n+2)(n+1)}c_n \quad n \geq 2$$

$c_4 = 0,\ c_5 = 0,\ c_6 = 0,\ c_7 = 0$ $\therefore y = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

Q 6 (a) If a power series solution $y = \sum_{n=0}^{\infty} c_n x^{r+n}$

is found for $2x^2y'' - xy' + (1+x)y = 0$, then roots of the indicial equation are k_1 and $\frac{1}{k_2}$. Then

$k_1 = \underline{1}$ $\qquad k_2 = \underline{2}$ $\qquad (k_1 < k_2)$

Solⁿ: $y = \sum_{n=0}^{\infty} c_n x^{r+n}$ $\qquad y' = \sum_{n=0}^{\infty} c_n (r+n) x^{r+n-1}$

$y'' = \sum_{n=0}^{\infty} c_n (r+n)(r+n-1) x^{r+n-2}$

$$\sum_{n=0}^{\infty} 2c_n (r+n)(r+n-1) x^{r+n} - \sum_{n=0}^{\infty} c_n (r+n) x^{r+n} + \sum_{n=0}^{\infty} c_n x^{r+n} + \sum_{n=0}^{\infty} c_n x^{r+n+1} = 0$$

$$\Rightarrow c_0 \left[2r(r-1) - r + 1\right] x^r + \sum \left[\{2(r+n)(r+n-1) - (r+n) + 1\} c_n + c_{n-1} \right] x^{r+n} = 0$$

$2r(r-1) - r + 1 = 2r^2 - 3r + 1 = (r-1)(2r-1) = 0$

$r = 1, \frac{1}{2}$

$\boxed{2M}$

Q6.(b) If an ascending power series solution $y = \sum_{m=0}^{\infty} c_m x^{k+m}$, $c_0 \neq 0$

is obtained ~~from~~ for Legendre's eqn.

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$

and the roots of the indicial eqns. are k_1 and k_2, then

$k_1 = \underline{0}$ $\qquad k_2 = \underline{1}$ $\qquad (k_1 < k_2)$

Q6(c) If a recurrence relation between c_m and c_{m-2} is obtained as

$$c_m = \frac{(k+m-a-n)(k+m-b+n)}{(k+m)(k+m-1)} c_{m-2}$$

$\boxed{2M}$

then $a = \underline{2}$ $\qquad b = \underline{1}$

Sol^n: $y = \sum_{m=0}^{\infty} c_m x^{k+m}$ $\quad c_0 \neq 0$

$y' = \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1}$ $\quad y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2}$

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$

$\Rightarrow \sum c_m (k+m)(k+m-1) x^{k+m-2} - x^2 \sum c_m (k+m)(k+m-1) x^{k+m-2}$

$- 2x \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1} + n(n+1) \sum_{m=0}^{\infty} c_m x^{k+m} = 0$

$\Rightarrow \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} c_m \{(k+m)(k+m-1) + 2(k+m)$
$- n(n+1)\} x^{k+m} = 0$

$\Rightarrow \sum c_m (k+m)(k+m-1) x^{k+m-2} - \sum c_m \{(k+m)^2 + (k+m) - n^2 - n\} x^{k+m} = 0$

$\Rightarrow \sum c_m (k+m)(k+m-1) x^{k+m-2} - \sum c_m \{(k+m+n)(k+m-n)$
$+ (k+m-n)\} x^{k+m} = 0$

$\Rightarrow \sum_0^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} - \sum_0^{\infty} c_m (k+m-n)(k+m+n+1) x^{k+m} = 0$

Equating to 0, the coeff. of smallest power of x, i.e x^{k-2}

$c_0 k(k-1) = 0$ $\quad k(k-1) = 0$ $\quad \because c_0 \neq 0$ $\therefore k = 0, 1$

Equating to 0, the coeff. of x^{k+m-2}

$c_m (k+m)(k+m-1) - c_{m-2} (k+m-2-n)(k+m-2+n+1) = 0$

$$\Rightarrow c_m = \frac{(k+m-2-n)(k+m-1+n)}{(k+m)(k+m-1)} c_{m-2}$$