Lecture 33

$$f : \mathbb{R}^d \to \mathbb{C}$$

$$f(x) = u(x) + i\, v(x)$$

- f is Lebesgue integrable if $|f|$ is Lebesgue integrable

That $\int |f| < \infty$, $\quad |f(x)| = \sqrt{u(x)^2 + v(x)^2}$

- "Lebesgue integrable" shortly as "L-integrable".

We have $|u| \le |f|$, $|v| \le |f|$

& $|f| \le |u| + |v|$

Thus a complex valued function f is L-integrable iff both its real & imaginary parts are L-integrable

& the <u>Lebesgue integral</u> of f is defined as

$$\int f(x)\, dx = \int u(x)\, dx + i \int v(x)\, dx.$$

& for any $E \subseteq \mathbb{R}^d$ measurable set,

$$\int_E f := \int_E u + i \int_E v .$$

$$= \int_{\mathbb{R}^d} f \chi_E .$$

Let $V = \{ f : E \to \mathbb{C} \mid f \text{ is L-integrable} \}$.

- for any $f, g \in V$, $f+g \in V$

$$\left(\because \int_E |f+g| \leq \int_E |f| + \int_E |g| < \infty \right)$$

- for any $c \in \mathbb{C}$, $f \in V$, we have

$cf \in V$ $\left(\because \int_E |cf| = \int_E |c||f| \right.$

$$\left. = |c| \int_E |f| < \infty \right)$$

V is a vector space over $\mathbb{C}$

For any L-integrable function f on $\mathbb{R}^d$, we define a <u>norm</u> of f,

$$\|f\|_{L^1(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |f(x)|\, dx .$$

Simply we write as $\|f\|_{L^1}$.

Now define a relation on V as follows:

for any $f, g \in V$, $f \sim g$ if $f = g$ a.e

Check that $\sim$ is an equivalence relation on V.

<u>**Def:-**</u> $L^1(\mathbb{R}^d) := \{$ all equivalence classes of L-integrable functions on $\mathbb{R}^d \}$.

$= \{ [f] \,/\, f$ is L-integrable on $\mathbb{R}^d \}$.

$= \{ f \,/\, f$ is L-integrable on $\mathbb{R}^d \}$, where we denote the equivalence class by elements only.

That is, $f \in L^1(\mathbb{R}^d)$ means f is an equivalence class.

<u>Proposition:</u>- Suppose $f, g : \mathbb{R}^d \to \mathbb{C}$ are L-integrable functions. Then

(i) $\|f\|_{L^1} \geq 0$ & $\|f\|_{L^1} = 0$ iff $f = 0$ a.e

(ii) $\|af\|_{L^1} = |a| \, \|f\|_{L^1} \quad \forall a \in \mathbb{C}$.

(iii) $\|f+g\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L}$

(iv) Define $d : L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d) \to \mathbb{R}$,
$d(f,g) = \|f-g\|_{L^1} \cdot \forall f, g \in L^1(\mathbb{R}^d)$.
Then d defines a metric on $L^1(\mathbb{R}^d)$.

proof:- Suppose $f \sim g$ $\Rightarrow$ $f = g$ a.e

$\Rightarrow$ $|f| = |g|$ a.e

$\Rightarrow$ $\displaystyle\int_{\mathbb{R}^d} |f| = \int_{\mathbb{R}^d} |g|$

$\Rightarrow$ $\|f\|_{L^1} = \|g\|_{L^1}$

$\therefore$ $\| - \|_{L^1} : L^1(\mathbb{R}^d) \to \mathbb{R}$ is well defined

$$[f] \mapsto \int_{\mathbb{R}^d} |f| .$$

(i) clearly $\|f\|_{L^1} \geq 0$. $\|0\|_{L^1} = \int_{\mathbb{R}^d} |0| = 0 < \infty$

Suppose $\|f\|_{L^1} = 0$ $\Rightarrow$ $\displaystyle\int_{\mathbb{R}^d} |f(x)|\, dx = 0$.

$\Rightarrow$ $f = 0$ a.e

(proved already).

$\therefore$ $\|f\|_{L^1} = 0 \iff f = 0$ a.e.

$\iff [f] = [0]$.

(ii) & (iii): EXERCISE. ✓

(iv) $d(f,g) = \int_{\mathbb{R}^d} |f-g| \quad \forall\, f,g \in L^1(\mathbb{R}^d)$.

<u>To show</u>: (i) $d(f,g) \geq 0$ & $d(f,g) = 0$ iff $f = g$ a.e

(ii) $d(f,g) \leq d(f,h) + d(h,g)$
$\forall\, f,g,h \in L^1(\mathbb{R}^d)$.

$d(f,g) = \|f-g\|_{L^1} = 0$

$\Leftrightarrow f = g$ a.e

Thus $L^1(\mathbb{R}^d) = \{ f \mid f$ is L-integrable on $\mathbb{R}^d \}$

is a normed linear space over $\mathbb{C}$.

V. space + it has a norm $\|-\|_{L^1}$

$\therefore$ $L^1(\mathbb{R}^d)$ is a metric space with the distance function (or metric) is $d(f,g) = \|f-g\|_{L^1} \; \forall\, f,g \in L^1(\mathbb{R}^d)$

Recall :- A metric space (X, d) is said to be a complete metric space if every Cauchy sequence in X is convergent.

(Note that every convergent seq is Cauchy).

Defn.- A sequence $\{x_n\}$ in a metric space (X, d) is said a Cauchy sequence, if given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $m, n \geq N$, we have

$$d(x_m, x_n) < \varepsilon.$$

Theorem:- (Riesz-Fischer) $\left(L^1(\mathbb{R}^d), \|\cdot\|_{L^1}\right)$ is a Complete normed linear space in its metric.

(Complete normed linear space is also called as Banach Space).

proof:- Let $\{f_n\}$ be a Cauchy sequence in $L^1(\mathbb{R}^d)$ w.r.t L^1-norm.

To show: $\{f_n\}$ is Convergent in L^1-norm.

Step 1: To show: There exists a subsequence of $\{f_n\}$ that Converges to f both p.w & a.e in L^1-norm.

Since $\{f_n\}$ is a Cauchy sequence in L^1-norm, then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}$ with the following property:

$$\int_{\mathbb{R}^d} |f_{n_{k+1}} - f_{n_k}| = \| f_{n_{k+1}} - f_{n_k} \|_{L^1} \leq \frac{1}{2^k} \quad \forall k \geq 1.$$

Consider the series

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} \left(f_{n_{k+1}}(x) - f_{n_k}(x) \right)$$

& $$g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} | f_{n_{k+1}}(x) - f_{n_k}(x) |.$$

Both f, g are convergent in L^1-norm:

The sequence of partial sums of f is

$$f_{n_1}(x) + \sum_{k=1}^{m} \left(f_{n_{k+1}}(x) - f_{n_k}(x) \right)$$

$$= \cancel{f_{n_1}(x)} + \left(\cancel{f_{n_2}(x)} - \cancel{f_{n_1}(x)} \right) + \left(\cancel{f_{n_3}(x)} - \cancel{f_{n_2}(x)} \right) + \cdots$$

$$\cdots + \left(f_{n_{m+1}}(x) - \cancel{f_{n_m}(x)} \right)$$

$$= f_{n_{m+1}}(x)$$

$$\|f_{n_{m+1}} - f\|_{L^1} = \int_{\mathbb{R}^d} |f_{n_{m+1}} - f|$$

$$= \underbrace{\int |f_{n_{k+1}}|} + \int |f|$$

$$\leq \int |f_{n_{k+1}}| + \int |f_{n_1}| + \sum_{k=1}^{\infty} \int |f_{n_{k+1}} - f_{n_k}|$$

$$\leq \int |f_{n_{k+1}}| + \int | f_{n_1} | + \underline{\underline{\sum_{k=1}^{\infty} \frac{1}{2^k}}}$$

$$< \infty$$

Thus $f \in L^1(\mathbb{R}^d)$.

Also $g \in L^1(\mathbb{R}^d)$.

And $|f| \leq g$.

Thus we have $|f|_{n_k} \longrightarrow f$ a.e as $k \to \infty$

($\because \{f_{n_k}\}$ is the sequence of partial sums of f).

<u>To show</u>: $f_{n_k} \to f$ in L^1.