

## Queuing Theory:

$N(t)$  : # of customers in queuing system at time  $t$

$N_q(t)$  : " + queuing in - - - - -  $t$

$N_s(t)$  : " - setting service - - - - -  $t$ .

$$N(t) = N_q(t) + N_s(t)$$

$N$  : steady state # of customers in queuing system

$N_q$  : " " " " , queuing " - - - - -

$N_s$  : " " " " - setting service - - - - -

$$\begin{aligned} N &= N_q + N_s \\ \Rightarrow E(N) &= E(N_q) + E(N_s) \\ L &= L_q + L_s \end{aligned}$$

$$\left| \begin{array}{l} L = E(N) \\ L_q = E(N_q) \\ L_s = E(N_s) \end{array} \right.$$

Relationship between time

$w$  : total time a customer spends in queuing system

$$\begin{aligned} w &= q + s \\ &\quad \downarrow \quad \rightarrow \text{service time} \\ &\quad \text{time spent} \\ &\quad \text{in queue} \end{aligned}$$

$$\Rightarrow E(w) = E(q) + E(s)$$

$$w = w_q + w_s$$

$$\left| \begin{array}{l} w = E(w) \\ w_q = E(q) \\ w_s = E(s) \end{array} \right.$$

Cdf

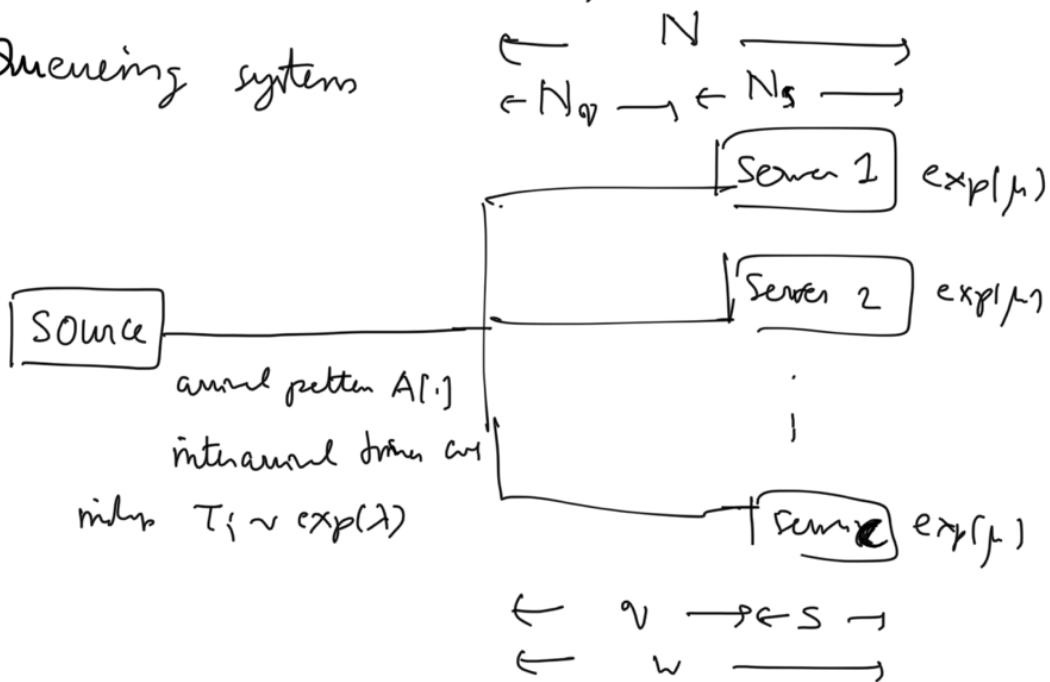
$$W(t) = P(w \leq t)$$

$$W_{\dots}(t) = P(w < t)$$

Queueing - I

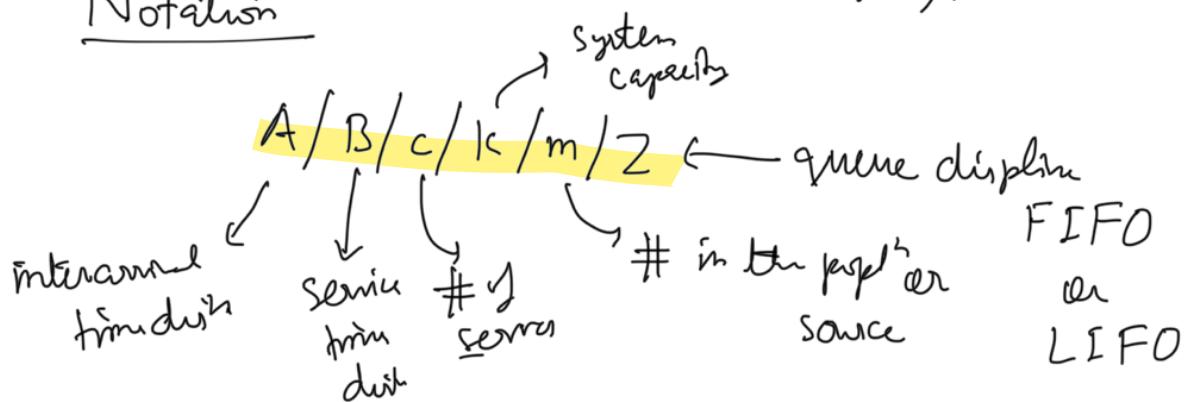
$$W_s(t) = P(s \leq t)$$

Queuing systems

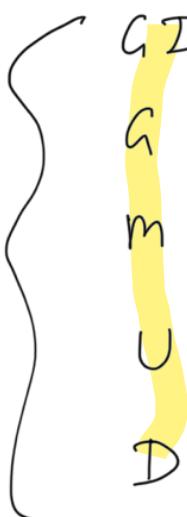


If  $N$  servers are busy, the  
T time for next server complete  
 $\sim \exp(n\mu)$

### Notation



$A, B$



- GI** general interarrival times
- G** general service times
- M** expo. interarrival or service times
- U** uniform
- D** deterministic

$N(t)$  # of customers arrived by time  $t$

av. arrival rate of the customer

$$\lambda_a = \lim_{t \rightarrow \infty} \frac{N(t)}{t}$$

Basic cost identity:

$$\left[ \begin{array}{l} \text{av. rate at which} \\ \text{the system earns} \end{array} \right] = \lambda_a \times \left[ \begin{array}{l} \text{av. amount an entrant} \\ \text{customer pays} \end{array} \right]$$

$$L = \lambda_a W$$

$$L_q = \lambda_a W_q$$

Little's law

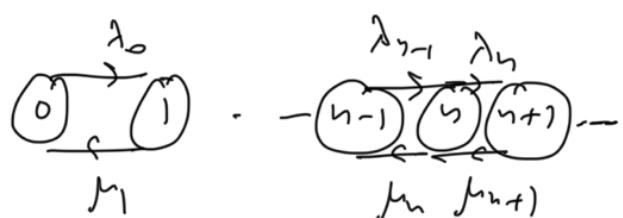
$$L_s = \lambda_a W_s$$

Birth and death process:

In steady state

$$N = n$$

$$P_n = P(N=n)$$

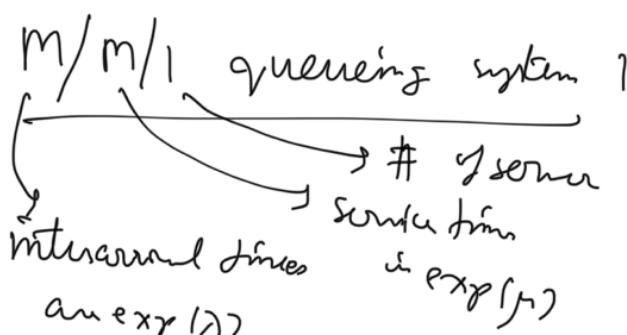


$$S = 1 + c_1 + c_2 + \dots$$

$$P_0 = \frac{1}{S}$$

$$c_n = \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n}$$

$$P_n = c_n P_0, \quad n = 1, 2, \dots$$



If  $N=n$ , then

prob. of an arrival in time interval of length  $h$  is

$$e^{-\lambda h} \lambda h = \lambda h \left( 1 - \lambda h + \frac{(\lambda h)^2}{2!} \dots \right)$$

$$= \lambda h + o(h)$$

$$\therefore \lambda_n \leq \lambda, n = 0, 1, \dots$$

$N=n$ , service times  $\mu_i$ 's

$$W_S(t) = P(S \leq t) \leq 1 - e^{-\mu t}$$

$\square N=n$ , prob. Service completion in a small interval of length  $h$

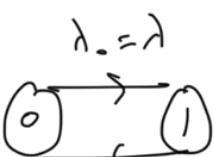
$$= 1 - e^{-\mu h} \leq 1 - (1 - \mu h + \frac{(\mu h)^2}{2!} \dots)$$

$$= \mu h + o(h)$$

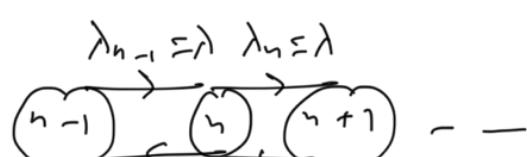
B & D process

$$\therefore \mu_n \leq \mu, n = 1, 2, \dots$$

$m/m/1$



$$\mu_1 = \mu$$



$$\mu_n = \mu, \mu_{n+1} = \mu$$

$$\text{let } \frac{\lambda}{\mu} = \beta$$

$$P_0 = \frac{1}{S}; S = 1 + c_1 + c_2 + \dots$$

$$P_0 = 1 - \beta \quad = 1 + \beta + \beta^2 + \dots = \frac{1}{1 - \beta}$$

$$P_n = P(N=n) = C_n P_0 = \beta^n (1 - \beta), n = 0, 1, 2, \dots$$

$$L \leq E(N) = \frac{\beta}{1 - \beta}$$

$$\sigma_N^2 = \frac{\beta}{(1 - \beta)^2}$$

$$\lambda_a = \lambda \times 1 = \lambda$$

$$S \sim \exp(\mu) \quad W_S = E(S) = \frac{1}{\mu} = \frac{\beta}{\lambda}$$

$$\Rightarrow \beta = \lambda W_S$$

$$\frac{1}{M(1 - \beta)}$$

$$L = \lambda W \Rightarrow W = \frac{L}{\lambda} = \frac{\beta}{\lambda} = \frac{W_S}{\lambda}$$

$$\lambda \quad \lambda(1-\rho) \quad 1-\rho$$

$$W_q = W - W_s = \frac{W_s}{1-\rho} - W_s = \frac{\rho}{1-\rho} W_s$$

$$L_q = \lambda W_q = \frac{\lambda \rho}{1-\rho} W_s = \frac{\rho^2}{1-\rho}$$

$$L_s = L - L_q$$

$$\begin{aligned} P(\text{server is busy}) &= 1 - P(\text{server is empty}) \\ &= 1 - P(N=0) = 1 - (1-\rho) \\ &= \rho \rightarrow \text{server utilization} \\ &\quad (\text{fraction of time server is busy}) \end{aligned}$$

arrival/service ratio  $\rho = \frac{\lambda}{\mu}$

Example Suppose that customers arrive at a Poisson rate of one per every 12 min, and that the service time is expo. at a rate of one service per 8 min. Find the parameters of M/M/1 system.

Given

$$\lambda = \frac{1}{12} \quad ; \quad \mu = \frac{1}{8}$$

$$\rho = \frac{\lambda}{\mu} = \frac{1}{12} = \frac{2}{3} < 1 \quad \lambda_q = \lambda = \frac{1}{12}$$

$$L = \frac{\rho}{1-\rho} = \frac{2}{3} \times 3 = 2$$

$$W = \frac{L}{\lambda} = 2 \times 12 = 24 \text{ min}$$

$$L_q = \frac{\rho^2}{1-\rho} = \frac{4}{9} \times 3 = \frac{4}{3}$$

$$W_q = W - W_s = \frac{24-8}{1} = 16 \text{ min}$$

$$W_S = \frac{1}{\lambda} = 8 \text{ min}$$

$$L_S = L - L_V = 2 - \frac{4}{3} = \frac{2}{3}$$

$$P_0 = 1 - \beta = \frac{1}{3}$$

$$P_n = P(N=n) = C_n P_0 = \beta^n (1-\beta) = \left(\frac{2}{3}\right)^n \frac{1}{3}; n=0,1,2,\dots$$

When there is  $N=0$  no customer  $x$  is queuing system then no queuing time  
so  $W_V(0) = P(V=0) = P(N=0) = 1 - \beta$

$$N=n, V = S_1 + S_2 + \dots + S_n, S_n \sim \text{IID Exp}(\mu)$$

but

$$f_{V|N=n}(t) = \frac{\mu^n}{\Gamma(n)} e^{-\mu t} t^{n-1}, t > 0$$

For  $n > 0$

$$P(V \leq t | N=n) = \int_0^t f_{V|N=n}(x) dx$$

$$P(0 < V \leq t) = E(P(0 < V \leq t | N))$$

$$= \sum_{n=1}^{\infty} P(0 < V \leq t | N=n) P(N=n)$$

$$= \sum_{n=1}^{\infty} \int_0^t \frac{\mu^n}{(n-1)!} e^{-\mu n} x^{n-1} \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n dx$$

$$= \int_0^t \lambda e^{-\mu n} \left(1 - \frac{\lambda}{\mu}\right) \underbrace{\left( \sum_{n=1}^{\infty} \frac{(\lambda x)^{n-1}}{(n-1)!} \right)}_{e^{\lambda x}} dx$$

$$= \frac{\lambda}{\mu} \int_0^t (\mu - \lambda) e^{-(\mu - \lambda)x} dx$$

$$= \frac{\lambda}{\mu} (1 - e^{-(\mu-\lambda)t}) = \beta [1 - e^{-\mu(1-\beta)t}]$$

$$= \beta (1 - e^{-t/\lambda})$$

For  $t > 0$

$$W_{\mathcal{N}}(t) = P(\eta = 0) + P(0 < \eta \leq t)$$

$$= 1 - \beta + \beta (1 - e^{-t/\lambda}) = 1 - \beta e^{-t/\lambda}$$

Given  $N = n$

bdf

$$w = s_1 + s_2 + \dots + s_{n+1} \sim \text{Gamma}(n+1, \mu) \quad s_i \sim \text{iid exp}(\mu)$$

$$f_{w|N=n}(t) = \frac{\mu^{n+1}}{\sqrt{n+1}} e^{-\mu t} t^n, \quad t \geq 0 \quad \frac{d^n}{dx^n} e^{-\lambda x} x^n$$

$$W(t) = P(w \leq t) = E(P(w \leq t | N))$$

$$= \sum_{n=0}^{\infty} P(w \leq t | N=n) P(N=n)$$

$$= \sum_{n=0}^{\infty} \int_0^t \frac{\mu^{n+1}}{n!} e^{-\mu x} x^n \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n dx$$

$$= \int_0^t \mu e^{-\mu x} \left(1 - \frac{\lambda}{\mu}\right) \left(\sum_{n=0}^{\infty} \frac{(\lambda x)^n}{n!}\right) dx$$

$$\rightarrow e^{\lambda x}$$

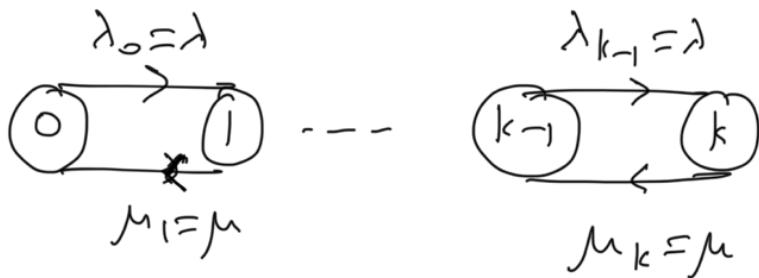
$$= \int_0^t (\mu - \lambda) e^{-(\mu - \lambda)x} dx$$

$$= 1 - e^{-(\mu - \lambda)t} = 1 - e^{-\mu(1-\beta)t} = 1 - e^{-t/\lambda}$$

$$w \sim \exp\left(\frac{1}{w}\right) \quad E(w) = w \quad ; \quad \sigma_w^2 = w^2$$

$$s \sim \exp(\mu) \quad E(s) = \frac{1}{\mu}$$

M/M/1/k queuing system:



B & D process

$$\lambda_n = \begin{cases} \lambda, & n = 0, 1, \dots, k-1 \\ 0, & n = k, k+1, \dots \end{cases}$$

$$\mu_n = \begin{cases} \mu, & n = 1, 2, \dots, k, \\ 0, & n = k+1, k+2, \dots \end{cases}$$

$$a = \frac{\lambda}{\mu}$$

$$\checkmark P_n = C_n P_0 = a^n P_0, \quad n = 0, 1, 2, \dots$$

$$\sum_{n=0}^k P_n = 1 \Rightarrow P_0 + a P_0 + \dots + a^k P_0 = 1$$

$$\frac{1 - a^{k+1}}{1 - a} P_0 = 1 \Rightarrow P_0 = \frac{1 - a}{1 - a^{k+1}}$$

if  $a \neq 1$

$$\text{if } a = 1 \Rightarrow P_0 = \frac{1}{k+1}$$

$$P(N=n) = P_n = \begin{cases} \frac{(1-a)a^n}{1 - a^{k+1}} & \text{if } a \neq 1 \\ \frac{1}{k+1} & \text{if } a = 1 \end{cases}$$

$$\left\{ \begin{array}{l} \frac{1}{k+1} \\ \text{if } a=1 \end{array} \right.$$

$a \neq 1$

$$L = \sum_{n=0}^k n p_n = \frac{(1-a)a}{1-a^{k+1}} \sum_{n=0}^k n a^{n-1}$$

$$= \frac{(1-a)a}{1-a^{k+1}} \cdot \frac{d}{da} \left( \sum_{n=0}^k a^n \right) \rightarrow \frac{1-a^{k+1}}{1-a}$$

$$= \frac{(1-a)a}{1-a^{k+1}} \times \left[ \frac{-(1-a)(k+1)a^k + (1-a^{k+1})}{(1-a)^2} \right]$$

$$= \frac{a}{1-a} - \frac{(k+1)a^{k+1}}{1-a^{k+1}}$$

if  $a=1$

$$L = \sum_{n=0}^k n \times \frac{1}{k+1} = \frac{k}{2}$$

$$L \leq \begin{cases} \frac{a}{1-a} - \frac{(k+1)a^{k+1}}{1-a^{k+1}} & \text{if } a \neq 1 \\ \frac{k}{2} & \text{if } a=1 \end{cases}$$

$$a = \lambda / \mu$$

$$L_s = E(N_s) = E(E(N_s | N))$$

$$= E(N_s | N=0) P(N=0) + E(N_s | N>0) P(N>0)$$

$$= 0 \times P_0 + 1 \times (1-P_0) = 1 - P_0$$

$$L_V = L - L_s = L - (1 - P_0)$$

$$P_{\text{blocking}} = P_k$$

$$\lambda_a = \lambda \times (1 - P_{\text{blocking}}) = \lambda \times (1 - P_k)$$

$$W = \frac{L}{\lambda_a} , \quad W_V = \frac{W}{\lambda_a} \quad W_S = \frac{1}{\mu} = \frac{\alpha}{\lambda} \quad \alpha = \frac{\lambda}{\mu}$$

$$\text{true server utilization} = \rho = \lambda_a W_S = \lambda (1 - P_k) W_S \\ = (1 - P_k) \alpha$$

For  $n = 0, 1, \dots, k-1$

$q_n$  prob. that an arriving customer who enters the system finds  $n$  customers already in the system

event  $A_n$ : there are  $n$  customers in the system

$A$ : an arrival is about to occur

$$P(A|A_n) = \lambda h + o(h)$$

$$P(A) = \sum_{n=0}^{k-1} (P(A|A_n)) P(A_n) \\ = (\lambda h + o(h)) \left( \sum_{n=0}^{k-1} P(A_n) \right)$$

$$= \left( \lambda h + o(h) \right) (1 - P_k)$$

$$\begin{aligned} P(N_a = n) &= q_n = P(A_n | A) = \frac{P(A \cap A_n)}{P(A)} \\ &= \frac{P(A | A_n) P(A_n)}{P(A)} \\ &= \frac{(\lambda h + o(h)) P_n}{(\lambda h + o(h)) (1 - P_k)} = \frac{P_n}{1 - P_k}, \quad n = 0, 1, \dots, k-1 \end{aligned}$$

$N_a$  r.v. that counts # of customers in an  $M/m/1/k$  system just before a customer arrives to enter the system

$$\begin{aligned} W(t) &= P(w \leq t) = E(P(w \leq t | N_a)) \\ &= \sum_{n=0}^{k-1} P(w \leq t | N_a = n) \underbrace{P(N_a = n)}_{q_n} \\ &= \sum_{n=0}^{k-1} q_n \int_0^t \frac{\mu^{n+1}}{\Gamma(n+1)} e^{-\mu u} u^n du \\ &= 1 - \sum_{n=0}^{k-1} q_n \int_t^\infty \frac{\mu^{n+1}}{\Gamma(n+1)} e^{-\mu u} u^{n+1-1} du \end{aligned}$$

$$\begin{cases} S_n > t \equiv N(t) \leq n-1 \\ \hline t & N(t) \sim PP(\mu) \\ & S_i \sim \text{Geom.}(1/n, 1) \end{cases}$$

$$= 1 - \sum_{n=0}^{k-1} q_n$$

$\sum_{k=0}^n \frac{e^{-\mu t} (\mu t)^k}{k!}$

$\otimes(n; \mu t)$

$$= 1 - \sum_{n=0}^{k-1} q_n \otimes(n; \mu t)$$

$$W_q(t) = P(q \leq t)$$

$$= W_q(0) + \sum_{n=1}^{k-1} P(q \leq t | N_q = n) q_n$$

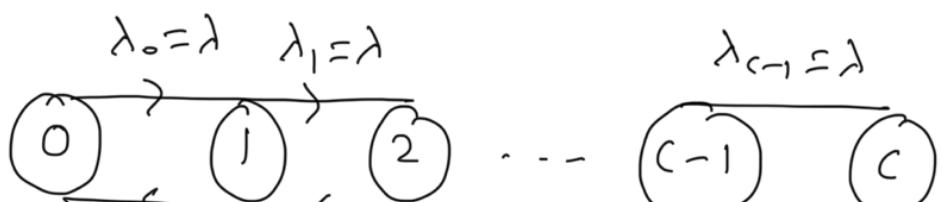
$$= q_0 + \sum_{n=1}^{k-1} q_n \int_0^t \frac{\mu^n}{\Gamma_n} e^{-\mu u} u^{n-1} du$$

$$= q_0 + \sum_{n=1}^{k-1} q_n \left[ 1 - \int_t^\infty \frac{\mu^n}{\Gamma_n} e^{-\mu u} u^{n-1} du \right]$$

$$= q_0 + (1 - q_0) - \sum_{n=1}^{k-1} q_n \otimes(n-1; \mu t)$$

$$= 1 - \sum_{n=0}^{k-2} q_{n+1} \otimes(n; \mu t)$$

M/M/c/c queuing system?



$$\mu_1 = \mu \quad \mu_2 = 2\mu \quad \mu_c = c\mu$$

B & D power

$$C_n = \frac{a^n}{n!}, n = 1, 2, \dots, c \quad ; \quad a = \frac{\lambda}{\mu}$$

$$w_s = \frac{1}{c}$$

$$S = \frac{1}{P_0} = 1 + C_1 + \dots + C_c$$

$$= 1 + a + \frac{a^2}{2!} + \dots + \frac{a^c}{c!}$$

$$P_n = C_n P_0 = \frac{a^n / n!}{1 + a + \frac{a^2}{2!} + \dots + \frac{a^c}{c!}}, n = 0, 1, \dots, c$$

$$P_{\text{blocking}} = P_c$$

$$\lambda_a = \lambda \times (1 - P_c)$$

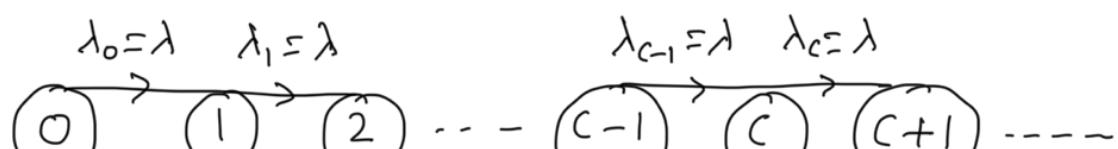
$$W_q = 0 \quad ; \quad L_q = 0$$

$$L = E(N) = \sum_{n=0}^c n P_n = P_0 \sum_{n=1}^c n \frac{a^n}{n!} = a P_0 \sum_{n=1}^c \frac{a^{n-1}}{(n-1)!}$$

$$= a P_0 \sum_{n=0}^{c-1} \frac{a^n}{n!}$$

$$W(t) = W_S(t) = 1 - e^{-\mu t} = 1 - e^{-b/w_s}$$

M/M/c queuing system:





$$\mu_1 = \mu, \mu_2 \leq 2\mu$$



$$\mu_c = c\mu, \mu_{c+1} = c\mu$$

$$\lambda_n = \lambda, n = 0, 1, 2, \dots$$

B&D process

$$\mu_n = \begin{cases} n\mu, & n = 1, 2, \dots, c \\ c\mu, & n = c+1, c+2, \dots \end{cases}$$

$$C_n = \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n}$$

$$a = \frac{\lambda}{\mu}$$

$$\beta = \frac{a}{c}$$

$$C_n = \begin{cases} \frac{a^n}{n!}, & n = 1, 2, \dots, c \\ \frac{\beta^{n-c} a^c}{c!}, & n = c+1, c+2, \dots \end{cases}$$



$$S = \frac{1}{P_0} = 1 + C_1 + C_2 + \dots$$

$$= 1 + a + \frac{a^2}{2!} + \dots + \frac{a^{c-1}}{(c-1)!} + \frac{a^c}{c!} [1 + \beta + \beta^2 + \dots]$$

$$= \sum_{n=0}^{c-1} \frac{a^n}{n!} + \frac{a^c}{c! (1-\beta)}$$

$$P_0 = \frac{1}{S}$$

$$P_n = C_n P_0 = \frac{a^n}{n!} P_0, n = 0, 1, \dots, c$$

$$= \frac{\beta^{n-c} a^c}{c!} P_0, n = c+1, c+2, \dots \checkmark$$

$$C[c, a] = P(N \geq c) = \sum_{n=c}^{\infty} P_n = 1 - \sum_{n=0}^{c-1} P_n$$

Entropy formula

$$= 1 - P_0 \sum_{n=0}^{c-1} \frac{a^n}{n!} = \frac{a^c}{c! (1-\beta)} P_0$$

$$\begin{aligned}
 L_V &= \sum_{n=c}^{\infty} (n-c) P_n = P_0 \frac{a^c}{c!} \sum_{n=c}^{\infty} (n-c) \int^{n-c} \\
 &= P_0 \frac{a^c}{c!} \beta \sum_{k=0}^{\infty} \underbrace{k \beta^{k-1}}_{\frac{d}{d\beta} \beta^k} \\
 &= P_0 \frac{a^c}{c!} \beta \underbrace{\frac{d}{d\beta} \left( \sum_{k=0}^{\infty} \beta^k \right)}_{\frac{1}{1-\beta}} \\
 &= P_0 \frac{a^c}{c!} \beta \frac{1}{(1-\beta)^2} = \frac{P_0 a^c \beta}{c! (1-\beta)^2} = C[c, q] \frac{\beta}{1-\beta}
 \end{aligned}$$

$$\lambda_a = \lambda \times 1 = \lambda$$

$$W_V = \frac{L_V}{\lambda} \quad , \quad W_S = \frac{1}{\mu}$$

$$W = W_V + W_S \quad , \quad L = \lambda W$$

$$\begin{aligned}
 W_V(0) &\leq P(V=0) = P(N < c) = 1 - P(N \geq c) \\
 &= 1 - C[c, q] \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 W_V(t) &= W_V(0) + P(0 < V \leq t) \\
 &= W_V(0) + \sum_{n=c}^{\infty} P(V \leq t | N=n) P_n \\
 &= W_V(0) + \sum_{n=c}^{\infty} \left( \int_0^t \frac{(c\mu)^{n-c+1}}{\Gamma(n-c+1)} e^{-c\mu u} u^{n-c+1-1} du \right) \frac{P_0 a^n}{c! c^{n-c}}
 \end{aligned}$$

$$= W_V(0) + \frac{P_0 a^c}{(c-1)!} \int_0^t \mu e^{-\mu n} \left( \sum_{n=c}^{\infty} \frac{(a \mu n)^{n-c}}{(n-c)!} \right) dn$$

$\downarrow e^{a \mu n}$

$$= W_V(0) + \frac{P_0 a^c}{(c-1)!} \left( \int_0^t \mu(c-a) e^{-\mu n(c-a)} dn \right) \times \frac{1}{c-a}$$

$\downarrow$

$$1 - e^{-\mu t(c-a)}$$

$$= 1 - c[c,a] + \frac{P_0 a^c}{(c-a)(c-1)!} (1 - e^{-\mu t(c-a)})$$

$\downarrow c[c,a]$

$$= 1 - c[c,a] e^{-\mu t(c-a)}$$

$$W(t) = P(W \leq t)$$

$$\begin{cases} 1 - \frac{a-c + W_V(0)}{a+1-c} e^{-\mu t} - \frac{c[c,a]}{a+1-c} e^{-c \mu t (1-\gamma)} & \text{if } a \neq c \\ 1 - (1 + c[c,a] \mu t) e^{-\mu t} & \text{if } a = c-1 \end{cases}$$

M/M/ $\infty$  queueing system



$\curvearrowleft \curvearrowleft \curvearrowleft$

$$\mu_1 = \mu \quad \mu_2 = 2\mu$$

$\curvearrowleft \curvearrowleft \curvearrowleft$

$$\mu_n = n\mu \quad \mu_{n+1} = (n+1)\mu$$

$$C_n = \frac{a^n}{n!}, \quad n=1, 2, \dots$$

$$a = \frac{\lambda}{\mu}$$

$$S = \frac{1}{P_0} = 1 + C_1 + C_2 + \dots = 1 + a + \frac{a^2}{2!} + \dots \\ = e^a$$

$$P_n = C_n P_0 = e^{-a} \frac{a^n}{n!}, \quad n=0, 1, 2, \dots$$

$$L = E(N) = a, \quad T_N^2 = a$$

$$L_q = 0, \quad W_q = 0$$

$$L = L_s = a \quad W = W_s = \frac{1}{\mu}$$

$$W(t) = W_s(t) = 1 - e^{-\mu t}$$

Example: Calls in a telephone system arrive

randomly at an exchange at the rate of 140 per hr. If there is very large number of lines available to handle the calls, that last on average 3 min, what is the av. number of lines in use?

Sol  $m/m/\infty$

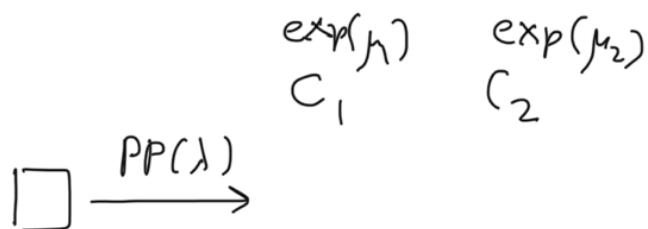
$$\lambda = 140 \text{ per hr} = \frac{140}{60} \text{ per min}$$

$$M = \frac{1}{3} \text{ pm m/s}$$

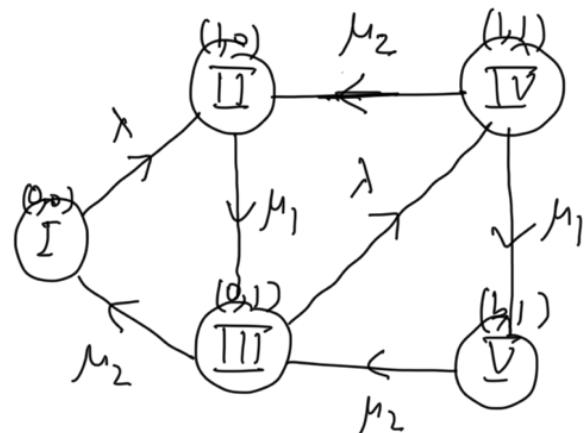
$$L_s = a = \frac{\lambda}{\mu} \leq \frac{14}{6} \times 3^1 = 7$$

Av. # of lines in use is 7.

Example Shoeshine shop (Extreme)



		State	
		$X$	$X$
I	(0,0)	$C_1$	$C_2$
II	(1,0)	✓	$C_2$
III	(0,1)	$C_1$	✓
IV	(1,1)	✓	✓
V	(b,1)	$C_1$	✓



$$\nu_I = \lambda, \nu_{II} = \mu_1, \nu_{III} = \lambda + \mu_2$$

$$\nu_{IV} = \mu_1 + \mu_2, \nu_V = \mu_2 \quad \text{midp} < \frac{Y_2 - \exp(\mu_1)}{X - \exp(-\lambda)}$$

$$P_{I,II} = 1, P_{II,III} = 1, P_{III,I} = P(Y_2 \leq X)$$

$$P_{III,IV} = \frac{\lambda}{\lambda + \mu_2}$$

$$\begin{aligned} &= \int P(X \geq y) \mu_2 e^{-\mu_2 x} dx \\ &= \int_0^\infty e^{-\lambda y} \mu_2 e^{-\mu_2 x} dx \end{aligned}$$

$$P_{\text{IV}, \text{II}} = \frac{\mu_2}{\lambda + \mu_2} \quad \boxed{= \frac{\mu_2}{\lambda + \mu_2}}$$

$$P_{\text{IV}, \text{V}} = \frac{\mu_1}{\lambda + \mu_2}, \quad P_{\text{V}, \text{II}} = 1$$

$$\nu_{\text{I}, \text{II}} = \nu_{\text{I}} P_{\text{I}, \text{II}} = \lambda \times 1$$

$$\nu_{\text{II}, \text{III}} = \mu_1, \quad \nu_{\text{III}, \text{V}} = (\cancel{\lambda + \mu_2}) \times \frac{\mu_2}{\cancel{\lambda + \mu_2}} = \mu_2$$

$$\nu_{\text{III}, \text{IV}} = \lambda, \quad \nu_{\text{IV}, \text{V}} = \mu_1, \quad \nu_{\text{V}, \text{III}} = \mu_2$$

$$\mathcal{D} = \begin{matrix} & \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \\ \text{I} & -\lambda & \lambda & 0 & 0 & 0 \\ \text{II} & 0 & -\mu_1 & \mu_1 & 0 & 0 \\ \text{III} & \mu_2 & 0 & -(\lambda + \mu_2) & \lambda & 0 \\ \text{IV} & 0 & \mu_2 & 0 & -(\mu_1 + \mu_2) & \mu_1 \\ \text{V} & 0 & 0 & \mu_2 & 0 & -\mu_2 \end{matrix}$$

balance equation

~~★~~

$$\pi_I + \pi_{\text{II}} + \pi_{\text{III}} + \pi_{\text{IV}} + \pi_{\text{V}} = 1$$

proportion of customers entering the system =  $\pi_I + \pi_{\text{III}}$

$L_s$  = av # of customers in the system

$$= \pi_{\text{II}} + \pi_{\text{III}} + 2(\pi_{\text{IV}} + \pi_{\text{V}})$$

$$\lambda_a = \lambda \times (\pi_I + \pi_{II})$$

$$W = \frac{L}{\lambda_a}$$

—x—