05/09/2022

Continuity of a linear map

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be n.l.s.

A linear map $A : X \longrightarrow Y$ is said to be continuous at $x \in X$ if

$x_n \longrightarrow x$ in $X \implies Ax_n \longrightarrow Ax$ in Y

$\|x_n - x\|_X \longrightarrow 0 \implies \|Ax_n - Ax\|_Y \longrightarrow 0$ as $n \longrightarrow \infty$.

OR

Given any $\epsilon > 0$ $\exists$ $\delta > 0$ such that $u \in X$,

$\|x - u\| < \delta \implies \|Ax - Au\| < \epsilon$

Theorem: Let X and Y be n.l.s. If X is finite dimensional n.l.s, then every linear map $A : X \longrightarrow Y$ is continuous.

Proof: $X = \{0\}$, then clearly
A is continuous [$\because A0 = 0$].

Assume $X \neq \{0\}$. $\dim X = m$.

Let $\{u_1, u_2, \ldots u_m\}$ be a basis for X.

Let $\{x_n\}$ be a sequence in X
such that $x_n \longrightarrow x \in X$

Let $x_n = \sum_{j=1}^{m} k_{nj} u_j$, $x = \sum_{j=1}^{m} k_j u_j$

$x_n \longrightarrow x \implies k_{nj} \longrightarrow k_j$
$\forall j = 1, \ldots m$.

Now

$$A x_n = A\left(\sum_{j=1}^{m} k_{nj} u_j\right)$$
$$= \sum_{j=1}^{m} k_{nj} A u_j$$
$$\longrightarrow \sum_{j=1}^{m} k_j A u_j$$
$$= A\left(\sum_{j=1}^{m} k_j u_j\right) = Ax.$$

Thus $x_n \to x \Rightarrow Ax_n \to Ax$

$\Rightarrow$ A is continuous at $x \in X$.

Since this is true for any $x \in X$,

it follows that A is continuous on X.

Problem: If $x_n \to x$, $y_n \to y$,

Show that (i) $x_n + y_n \to x + y$

(ii) $kx_n \to kx$, $\forall k \in K$

(iii) $k_n \to k$, $x_n \to x$

$\Rightarrow k_n x_n \to kx$.

Bounded linear map:

A linear map A is bounded on

$\overline{U(0,r)}$, $r > 0$ of a n.l.s X if there exists $\beta > 0$ such that

$$\|A(x)\| \leq \beta, \quad \forall\, x \in \overline{U(0,r)},$$

Theorem: Let X and Y be n.l.s and $A : X \longrightarrow Y$ be a linear map. If A is bounded on $\overline{U(0,r)}$, $r > 0$, then there exists $\alpha > 0$ such that

$$\|Ax\| \leq \alpha \|x\|, \quad \forall\, x \in X.$$

Proof: Given that $A : X \longrightarrow Y$ is bounded on $\overline{U(0,r)} = \{x \in X \mid \|x\| \leq r\}$.

$\therefore\ \exists\ \beta > 0 \ni \|Ax\| \leq \beta, \forall x \in \overline{U(0,r)}$.

If $x = 0$, then

$$0 = \|Ax\| = 0 \leq \beta \|x\|$$

So let $0 \neq x \in X$.

For any $\gamma > 0$, $y = \frac{\gamma x}{\|x\|}$

Then $\|y\| = \left\| \frac{\gamma x}{\|x\|} \right\| = \gamma$

$\Rightarrow y \in \overline{U(0, \gamma)}$

$\therefore \|Ay\| \leq \beta$

$\Rightarrow \left\| A\left(\frac{\gamma x}{\|x\|}\right) \right\| \leq \beta$

$= \frac{\gamma}{\|x\|} \|Ax\| \leq \beta$

$\Rightarrow \|Ax\| \leq \frac{\beta}{\gamma} \|x\|$

$\Rightarrow \|Ax\| \leq \alpha \|x\|, \ \forall x \in X$

$\alpha = \beta/\gamma$.

Theorem: let $A : X \longrightarrow Y$ be a linear map from a n.l.s X into a n.l.s Y. Then A is continuous on X iff there exists $\alpha > 0$ such that $\|Ax\| \leq \alpha \|x\|, \forall x \in X$.

Proof: Suppose $A : X \longrightarrow Y$ is continuous.

Claim: $\exists \alpha > 0 \ni \|Ax\| \leq \alpha \|x\|, \forall x \in X$.

Suppose there exists no $\alpha > 0 \ni$

$$\|Ax\| \leq \alpha \|x\|, \forall x \in X.$$

Then for each $n \in \mathbb{N}$, we can find an element $x_n \in X$ such that

$$\|Ax_n\| > n \|x_n\|.$$

$$\Rightarrow \left\| A\left(\frac{x_n}{n\|x_n\|}\right)\right\| > 1$$

let $y_n = \dfrac{x_n}{n\|x_n\|}$, then

$$\|y_n - 0\| = \left\|\frac{x_n}{n\|x_n\|} - 0\right\| = \frac{1}{n} \longrightarrow 0$$

as $n \to \infty$

$\therefore$ $y_n \longrightarrow 0$, but $\|Ay_n\| > 1$,

i.e., $Ay_n \not\longrightarrow 0$

which is contradiction to A is continuous on X.

$\therefore$ $\exists\ \alpha > 0 \ni \|Ax\| \leq \alpha\|x\|$,

$\forall x \in X$.

Conversely, suppose there exists $\alpha > 0$ such that $\|Ax\| \leq \alpha\|x\|$, $\forall x \in X$.

$\longrightarrow (1)$

Claim: A is Continuous on X.

Let $\{x_n\}$ be a sequence in X such that $x_n \longrightarrow x$.

$\Longrightarrow x_n - x \longrightarrow 0$ as $n \to \infty$.

$\Longrightarrow \|x_n - x\| \longrightarrow 0$

Now by (1),

$$\|Ax_n - Ax\| = \|A(x_n - x)\| \leq \alpha \|x_n - x\| \longrightarrow 0$$

$\Longrightarrow Ax_n \longrightarrow Ax$ as $n \to \infty$.

Since this is true for any $x \in X$, it follows that A is Continuous on X.

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* If A is Continuous at the origin, then A is Continuous everywhere on a n.l.s X. and vice versa.

Assume $x_n \to 0 \Rightarrow Ax_n \to 0$.

Now let $x_n \to x \in X$

$$\Rightarrow \underbrace{x_n - x}_{y_n} \to 0 \text{ as } n \to \infty$$

$y_n \to 0$ as $n \to \infty$.

$\Rightarrow Ay_n \to 0$ as $n \to \infty$

$\Rightarrow A(x_n - x) \to 0$ as $n \to \infty$

$\Rightarrow Ax_n - Ax \to 0$ as $n \to \infty$

$\Rightarrow Ax_n \to Ax$,

A is continuous at $x \in X$

$\because$ x is arbitrary element of X,

A is continuous on X.

Def: A linear map $A : X \to Y$ from a n.l.s $(X, \|\cdot\|_X)$ into a n.l.s $(Y, \|\cdot\|_Y)$ is said to be bounded on X iff there exists some $\alpha \geq 0$ such that

$$\|Ax\| \leq \alpha \|x\|, \quad \forall x \in X$$

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Theorem: Let X and Y be n.l.s, and $A : X \to Y$ be a linear map. Then A is bounded iff A maps bounded sets in X to a bounded sets in Y.

Proof:

Assume that $A : X \to Y$ is a bounded linear map.

Then there exists $m > 0$ such that

$$\|Ax\| \leq m\|x\|, \quad \forall x \in X. \quad \text{---(1)}$$

Suppose E is a bounded set in X

Then for all $x \in E$ $\exists$ $\alpha > 0$ $\ni$

$$\|x\| \leq \alpha, \ \forall x \in E$$

Let

$$B = \overline{U(0,r]}, \text{ where}$$

$r >> \alpha$.

[$x \in E \Rightarrow \|x\| \leq \alpha \leq r$

So if we prove $\|Ax\| \leq \beta, \forall x \in \overline{U(0,r]}$,

then $\|Ax\| \leq \beta, \ \forall x \in E \subseteq \overline{U(0,r]}$.

Now $x \in B = \overline{U(0,r]} \Rightarrow \|x\| \leq r$. ---(2)

But $\|Ax\| \leq m\|x\|, \ \forall x \in X$ by (1)

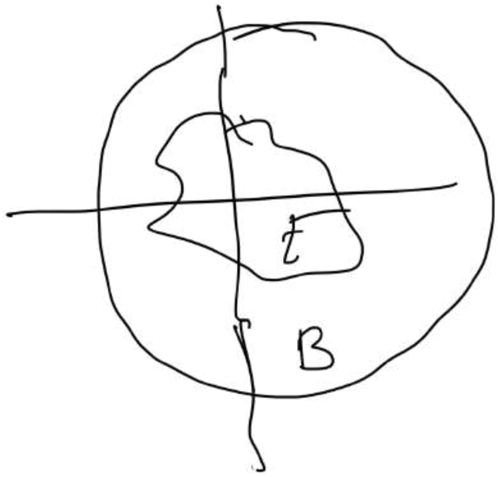
$\leq mr, \ \forall x \in B = \overline{U(0,r]}$ ---(2)

$\Rightarrow \|Ax\| \leq mr, \ \forall x \in \overline{U(0,r]}$.

$\therefore \{Ax / x \in B = \overline{U(0,r]}\}$ is a bounded subset of Y.

$\Rightarrow A(B)$ is bounded in Y

$\because E \subseteq B \Rightarrow A(E)$ is also bounded in Y.



Conversely, let $A : X \to Y$ maps bounded sets in X to a bounded sets in Y.

Claim: A is bounded linear map.

let $\overline{U(0,1)}$ be a closed unit ball in X, which is bounded in X.

Then by the assumption $A(\overline{U(0,1)})$ is bounded set in Y.

$\therefore \exists\, k > 0 \ni \|Ax\| \leq k, \ \forall x \in \overline{U(0,1)}$.

Let $0 \neq x \in X$, then $y = \frac{x}{\|x\|} \in \overline{U(0,1)}$

$\because \|y\| = 1$.

$\therefore \|Ay\| \leq k$

$\Rightarrow \left\|A\left(\frac{x}{\|x\|}\right)\right\| \leq k, \quad \forall\, 0 \neq x \in X$

$= \frac{1}{\|x\|} \|A(x)\| \leq k$

$\Rightarrow \|A(x)\| \leq k\|x\|, \forall x \in X$.

$\therefore$ $A : X \to Y$ is a bounded linear map.

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Note :- Let X and Y be n.l.s

If $A : X \to Y$ is a continuous linear map, then it is uniformly continuous.

Proof :- Given that $A : X \longrightarrow Y$ is continuous on X. $\therefore$ A is continuous at the origin also.

$\therefore$ given $\epsilon > 0$ $\exists$ $\delta > 0$ $\ni$

$$\|x\| < \delta \implies \|Ax\| < \epsilon \quad \text{--- (1)}$$

Now for any $u \in X$, replace x by $x-u$ in (1), we get

$$\|x-u\| < \delta \implies \|A(x-u)\| < \epsilon$$

$$\implies \|Ax - Au\| < \epsilon.$$

Since δ is independent of $u \in X$, it follows that A is uniformly continuous on X.

Combining all the above results, we have the following theorem.

Theorem: Let X and Y be n.l.s and $A : X \to Y$ be linear map. Then the following are equivalent.

(i) A is continuous at the origin

(ii) A is continuous at every $x \in X$.

(iii) A is uniformly continuous on X.

(iv) There exists $\alpha > 0$ such that

$$\|Ax\| \leq \alpha \|x\|, \quad \forall x \in X$$

(v) $\{Ax \mid \|x\| \leq 1, x \in X\}$ is a bounded set in Y.

(vi) For every bounded set $E \subseteq X$, the set $A(E) = \{Ax \mid x \in E\}$ is bounded in Y.

Theorem: Let X and Y be n.l.s and $A : X \to Y$ be a linear map. Let $Z(A)$ be a null space of X.

If A is continuous then, $Z(A)$ is closed in X. And $\tilde{A} : \frac{X}{Z(A)} \to Y$ be the map defined by

$$\tilde{A}(x + Z(A)) = Ax, \quad \forall x \in X$$

is also continuous linear map.

Proof: Let $A : X \to Y$ be continuous linear map. Then

$Z(A) = A^{-1}\{0\}$ is closed in X, since $\{0\}$ is closed in Y.

Let x be a limit point of $Z(A)$.

Then $\exists$ a sequence $\{x_n\}$ in $Z(A)$ such that $x_n \rightarrow x$.

$\because x_n \in Z(A) \Rightarrow Ax_n = 0 \;\; \forall n.$

Also $A : X \rightarrow Y$ is continuous

$\therefore x_n \rightarrow x \Rightarrow \underset{\overset{\shortparallel}{0}}{Ax_n} \rightarrow Ax$

$\Rightarrow Ax = 0 \Rightarrow x \in Z(A)$

$\therefore Z(A)$ is closed in X.

$\therefore \left(\frac{X}{Z(A)}, |||\cdot|||\right)$ is n.l.s.

Define $\tilde{A} : \frac{X}{Z(A)} \rightarrow Y$ by

$\tilde{A}(x + Z(A)) = Ax, \;\; \forall x \in X.$

Claim: $\tilde{A}$ is a continuous linear map.

Given that $A: X \to Y$ is continuous.

$\therefore \exists\ \alpha > 0 \ni \|Ax\|_Y \leq \alpha \|x\|_X, \forall x \in X.$

Now for any $x \in X$, $z \in Z(A)$, we have

$$\|\tilde{A}(x + Z(A))\|_Y = \|Ax\|_Y$$

$$\leq \alpha \|x\|_X, \quad \forall x \in X$$

$$\leq \alpha \|x+z\|_X$$

This is true for all $z \in Z(A)$.

Since above inequality is true for any $z \in Z(A)$, it follows that

$$\|\tilde{A}(x + Z(A))\| \leq \alpha \inf\{\|x+z\| \mid z \in Z(A)\}$$

$$= \alpha \, |||x + Z(A)|||, \ \alpha > 0$$

$\Rightarrow \tilde{A} : \dfrac{X}{Z(A)} \to Y$ is Continuous.

Conversely if $\tilde{A} : \dfrac{X}{Z(A)} \to Y$ is Continuous, then $A : X \to Y$ is also Continuous

$\because$ For any $x \in X$,

$$\|Ax\| = \|\tilde{A}(x + Z(A))\|$$
$$\leq \alpha |||x + Z(A)|||$$
$$\leq \alpha \|x\|, \quad [\because 0 \in Z(A)]$$

$\Rightarrow A$ is Continuous.

$$\tilde{A}[\alpha(x_1 + Z(A)) + \beta(x_2 + Z(A))]$$
$$= \tilde{A}(\alpha x_1 + \beta x_2 + Z(A))$$
$$= A(\alpha x_1 + \beta x_2)$$

$$= \alpha A x_1 + \beta A x_2$$

$$= \alpha \tilde{A}(x_1 + Z(A)) + \beta \tilde{A}(x_2 + Z(A))$$

$\therefore \tilde{A} : \dfrac{X}{Z(A)} \longrightarrow Y$ is linear.

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Note :— A linear map $A : X \to Y$ is shown to be discontinuous, by showing that there exists a bounded set $E \subseteq X$, such that set $\{Ax \mid x \in E\}$ is not bounded in Y.

OR

Produce a bounded sequence $\{x_n\}$ in X, such that $\{Ax_n\}$ is unbounded in Y.

Ex.: Let $X = C^1[0,1]$ with $\|\cdot\|_\infty$ be a n.l.s.

Define $f : X \longrightarrow K$ by

$$f(x) = x^{(1)}(1), \quad \forall\, x \in X$$

Clearly f is linear map.

Let $\{ x_n(t) = t^n,\ t \in [0,1] \}$ be a sequence in X.

$$\|x_n\|_\infty = \max_{t \in [0,1]} |x_n(t)| = 1.$$

$$f(x_n) = x_n^{(1)}(1) = n$$

$$|f(x_n)| = |x_n'(1)| = n$$

$\therefore$ $\{ f(x_n) \}$ is unbounded set in K.

$\therefore$ $f : X \longrightarrow K$ is not continuous.

(2)

$$X = C^1[0,1], \quad \|\cdot\|_\infty.$$

$$Y = C[0,1], \quad \|\cdot\|_\infty.$$

Define $A : X \longrightarrow Y$ by

$$Ax = x^{(1)}(t), \quad t \in [0,1]$$

$$\forall x \in C^1[0,1].$$

$\therefore x_n(t) = t^n \Rightarrow \|x_n\|_\infty = 1.$

and

$$\|Ax_n\|_\infty = \max_{t\in[0,1]} |Ax_n(t)|$$

$$= \max_{t\in[0,1]} |x_n'(t)|$$

$$= \max_{t\in[0,1]} |n t^{n-1}|$$

$$= n.$$

$\therefore \{Ax_n\}$ is unbounded in Y.

$\therefore A : X \longrightarrow Y$ is dis continuous.

* A linear map on n.l.s X may be continuous w.r.t some norm on X, but discontinuous w.r.t some other norm on X.

Ex: $X = c_{00}$, $f : X \longrightarrow K$

by $f(x) = \sum_{j=1}^{\infty} x(j)$, $x = (x(1), x(2), \ldots) \in X$.

Then

$$|f(x)| = \left| \sum_{j=1}^{\infty} x(j) \right|$$

$$\leq \sum_{j=1}^{\infty} |x(j)|$$

$$= 1 \cdot \|x\|_1$$

$\therefore |f(x)| \leq 1 \|x\|_1$

$\therefore$ f is continuous w.r.t $\|\cdot\|_1$.

But f is discontinuous w.r.t $\|\cdot\|_2$

$\because x_n = (1, \frac{1}{2}, \frac{1}{3} \cdots \frac{1}{n}, 0, 0, \ldots) \in c_{00}$

Then
$$\|x_n\|_2 = \left(\sum_{j=1}^{\infty} |x_n(j)|^2\right)^{1/2}$$

$$= \left(\sum_{j=1}^{n} \frac{1}{j^2}\right)^{1/2}$$

$$\leq \left(\sum_{j=1}^{\infty} \frac{1}{j^2}\right)^{1/2}$$

$$\leq \left(\frac{\pi^2}{6}\right)^{1/2} < \infty.$$

$\therefore \{x_n\}$ is bounded w.r.t $\|\cdot\|_2$.

But
$$f(x_n) = \sum_{j=1}^{n} \frac{1}{j} \longrightarrow \infty \text{ as } n \to \infty$$

$\Rightarrow f(x_n) \longrightarrow \infty$ as $n \to \infty$

$\therefore$ f is discontinuous.

Ex : Consider the infinite matrix of scalars (a_{ij}), $a_{ij} \in K$ $\forall\, i, j$

Now for any

$$x = (x(1), x(2), x(3) \cdots) \in F(N, K)$$

be the set of all functions from N to K.

Define $A : X \longrightarrow Y$

$$Ax(i) = \sum_{j=1}^{\infty} a_{ij}\, x(j), \quad j = 1, 2, 3 \cdots$$

where X and Y are n.l.s.

$[Ax = (Ax(1), Ax(2), Ax(3), \cdots \cdots)]$

Assume $\sum_{j=1}^{\infty} |a_{ij}|\, |x(j)| < \infty$

and $\alpha = \sup_{j} \sum_{i=1}^{\infty} |a_{ij}| < \infty.$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & - & - & - \\ a_{21} & a_{22} & a_{23} & \cdots & & - & - \\ a_{i1} & a_{i2} & a_{i3} & \cdots & - & - & \\ | & | & | & & | & & \end{bmatrix}$$

Now for $X = Y = \ell^1$,

Then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| |x(j)|$$

$$\leq \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |a_{ij}| \right) |x(j)|$$

Now

$$\|Ax\|_{\ell^1} = \sum_{i=1}^{\infty} |Ax(i)|$$

$$= \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} a_{ij} x(j) \right|$$

$$\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| \, |x(j)|$$

$$= \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |a_{ij}| \right) |x(j)|$$

$$\leq \left(\sup_{j} \sum_{i=1}^{\infty} |a_{ij}| \right) \sum_{j=1}^{\infty} |x(j)|$$

$$= \alpha \, \|x\|_{\ell^1}$$

$\therefore \|Ax\| \leq \alpha \, \|x\|_{\ell^1}, \ \forall \, x \in \ell^1$

where $\alpha = \sup_{j} \sum_{i=1}^{\alpha} |a_{ij}| < \infty$.

$\therefore A : \ell^1 \to \ell^1$ is continuous.

Now let us take $X = Y = \ell^{\infty}$.

and $\beta = \sup_{i} \sum_{j=1}^{\infty} |a_{ij}| < \infty$.

Then for any $x \in \ell^{\infty}$,

$$\|Ax\|_\infty = \sup_i |Ax(i)|$$

$$= \sup_i \left| \sum_{j=1}^{\infty} a_{ij}\, x(j) \right|$$

$$\le \sup_i \sum_{j=1}^{\infty} |a_{ij}|\, |x(j)|$$

$$\le \left(\sup_i \sum_{j=1}^{\infty} |a_{ij}| \right) \cdot \sup_j |x(j)|$$

$$= \beta\, \|x\|_\infty$$

$\therefore$ $A : \ell^\infty \longrightarrow \ell^\infty$ is a bounded linear map.

Now let

$$\alpha_{p,q} = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}|^q \right)^{p/q} < \infty.$$

$$\beta = \sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty$$

$$\gamma = \sup_j \sum_{i=1}^{\infty} |a_{ij}| < \infty$$

where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

Assume $\min\{\alpha_{p,q}^{1/p}, \beta^{1/q}\gamma^{1/p}\} < \infty$.

$$Ax(i) = \sum_{j=1}^{\infty} a_{ij}x(j), \quad \forall i \in \mathbb{N}.$$

Then

$$\|Ax\|_{\ell^p} \leq \min\{\alpha_{p,q}^{1/p}, \beta^{1/q}\gamma^{1/p}\} \|x\|_{\ell^p}.$$

For any $x \in \ell^p$, consider

$$\sum_{j=1}^{\infty} |a_{ij}||x(j)| \leq \left(\sum_{j=1}^{\infty} |a_{ij}|^q\right)^{1/q} \cdot \|x\|_p$$

(By Holder's inequ)

Now

$$|Ax(i)|^p = \left|\sum_{j=1}^{\infty} a_{ij}x(j)\right|^p$$

$$\leq \left[\sum_{j=1}^{\infty} |a_{ij}||x(j)|\right]^p$$

$$\leq \left(\sum_{j=1}^{\infty} |a_{ij}|^q\right)^{p/q} \cdot \|x\|_p^p$$

$\Rightarrow$

$$\sum_{i=1}^{\infty} |Ax(i)|^p \leq \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}|^q\right)^{p/q} \cdot \|x\|_p^p$$

$$= \alpha_{p,q} \|x\|_p^p$$

$$\Rightarrow \|Ax\|_{\ell^p}^p \leq \alpha_{p,q} \|x\|_p^p$$

$$\Rightarrow \|Ax\|_{\ell^p} \leq \alpha_{p,q}^{1/p} \|x\|_p, \ \forall x \in \ell^p \quad \text{--- (1)}$$

$A : \ell^p \longrightarrow \ell^p$ is a bounded linear map.

Again by using Holder's inequality, for any $x \in \ell^p$,

$$\sum_{j=1}^{\infty} |a_{ij}| |x(j)| = \sum_{j=1}^{\infty} |a_{ij}|^{\frac{1}{q}} \cdot \left[|a_{ij}|^{\frac{1}{p}} x(j) \right]$$

$$\leq \left(\sum_{j=1}^{\infty} |a_{ij}| \right)^{\frac{1}{q}} \left(\sum_{j=1}^{\infty} |a_{ij}| |x(j)|^{p} \right)^{\frac{1}{p}}$$

By Holder
$i \sim 1$

$\Rightarrow$

$$\left(\sum_{j=1}^{\infty} |a_{ij}| |x(j)| \right)^{p}$$

$$\leq \sup_{i} \left(\sum_{j=1}^{\infty} |a_{ij}| \right)^{p/q} \cdot \sum_{j=1}^{\infty} |a_{ij}| |x(j)|^{p}$$

$$= \beta^{p/q} \cdot \sum_{j=1}^{\infty} |a_{ij}| |x(j)|^{p} \longrightarrow (*)$$

$$\therefore \sum_{i=1}^{\infty} |Ax(i)|^{p} = \sum_{i=1}^{\infty} \left(\left| \sum_{j=1}^{\infty} a_{ij} x(j) \right| \right)^{p}$$

$$\leq \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}| |x(j)| \right)^{p}$$

$$\leq \sum_{i=1}^{\infty} \beta^{p/q} \sum_{j=1}^{\infty} |a_{ij}| |x(j)|^p$$

$$\leq \beta^{p/q} \sup_{j} \sum_{i=1}^{\infty} |a_{ij}| \sum_{j=1}^{\infty} |x(j)|^p$$

$$= \beta^{p/q} \gamma \, \|x\|_{\ell^p}^p$$

$$\|Ax\|_{\ell^p}^p \leq \beta^{p/q} \gamma \, \|x\|_{\ell^p}^p$$

$$\Rightarrow \|Ax\|_{\ell^p} \leq \beta^{1/q} \gamma^{1/p} \, \|x\|_{\ell^p} \quad \text{(2)}$$

$\therefore$ From (1) & (2)

$$\|Ax\|_{\ell^p} \leq \min\{\alpha_{p,q}, \beta^{1/q}\gamma^{1/p}\} \|x\|_{\ell^p}.$$

$\therefore A : \ell^p \to \ell^p$, $1 \leq p \leq \infty$

is a bounded linear map.

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