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1) $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$, $P_4(x) = (35x^4 - 30x^2 + 3)/8$

$\Rightarrow$ From the generating funⁿ of $P_n(x)$, we have:

$$\sum_{n=0}^{\infty} P_n(x) t^n = (1 - 2xt + t^2)^{-1/2} = (1 - t(2x - t))^{-1/2}$$

Expanding RHS :-

$$\sum_{n=0}^{\infty} P_n(x) t^n = 1 + \frac{1}{2} t(2x - t) + \frac{\frac{1}{2} \times \frac{3}{2}}{2!} \cdot t^2 (2x - t)^2 + \frac{\frac{1}{2} \times \frac{3}{2} \times 5/2}{3!} t^3 (2x-t)^3 + \frac{\frac{1}{2} \times \frac{3}{2} \times \frac{5}{2} \times \frac{7}{2}}{4!} t^4 (2x - t)^4 + \dots$$

$$= 1 + \frac{1}{2} t(2x - t) + \frac{3}{8} t^2 (2x - t)^2 + \frac{5}{16} t^3 (2x - t)^3 + \frac{35}{128} t^4 (2x - t^4) + \dots$$

Comparing powers of t from LHS & RHS

$P_0(x) = 1$, $P_1(x) = \frac{1}{2} \times 2x = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$,
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$; $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$
where $P_n(x)$ is the coefficient of t^n.

Hence Proved

2) $P = x^4 + 2x^3 + 2x^2 - x - 3$

$\Rightarrow$ ~~$P_0 = 1$, P_1~~, $P_0(x) = 1$, $P_1(x) = x$

$P_2(x) = \frac{1}{3}(3x^2 - 1) \Rightarrow \frac{2}{3} P_2(x) = x^2 - 1/3 \Rightarrow x^2 = \frac{2}{3} P_2(x) + 1/3 P_0(x)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow \frac{2P_3(x) + 3P_1(x)}{5} = x^3$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \Rightarrow \frac{1}{35}\left[8 P_4(x) - 3P_0(x) + 30\left(\frac{2}{3} P_2(x) + \frac{1}{3} P_0(x)\right)\right] = x^4$

$$\Rightarrow x^4 = \frac{8}{35} P_4(x) + \frac{\cancel{20}\,4}{\cancel{35}\,7} P_2(x) + \frac{\cancel{7}\,3}{\cancel{35}\,35} P_0(x)$$

Now, $P = x^4 + 2x^3 + 2x^2 - x - 3$

$$= \left(\frac{8}{35}P_4 + \frac{4}{7}P_2 + \frac{7}{35}P_0\right) + 2\left(\frac{2}{5}P_3 + \frac{3}{5}P_1\right)$$

$$+ 2\left(\frac{P_0}{3}\right) - 3P_0 + \frac{4}{3}P_2(x) - P_1$$

$$= \frac{8}{35}P_4 + \left(\frac{4}{7} + \frac{4}{3}\right)P_2 + \frac{4}{5}P_3 + \left(\frac{6}{5} - 1\right)P_1 + \left(\frac{7}{35} + \frac{2}{3} - 3\right)P_0$$

$$P = \frac{8}{35}P_4 + \frac{4}{5}P_3 + \frac{40}{21}P_2 + \frac{1}{5}P_1 - \frac{224}{105}P_0$$

hence proved.

3) $P'_{n+1} + P'_n = P_0 + 3P_1 + 5P_2 + \dots + (2n+1)P_n$

$\Rightarrow$ $P_0 = P_1'$ as $P_0 = 1$ & $P_1 = x$.

Also, from recurrence relation $(2n+1)P_n = P'_{n+1} - P'_{n-1}$.

$\therefore$ $3P_1 = P_2' - P_0'$, $5P_2 = P_3' - P_1'$, $7P_3 = P_4' - P_2'$ .....

... $(2n-3)P_{n-2} = P'_{n-1} - P'_{n-3}$, $(2n-1)P_{n-1} = P_n' - P'_{n-2}$, $(2n+1)P_n = P'_{n+1} - P'_{n-1}$

Adding all of the above, we get

$P_0 + 3P_1 + 5P_2 + \dots + (2n+1)P_n = P'_{n+1} + P_n' - P_n' = P'_{n+1} + P_n'$

hence proved

4) $\int_{-1}^{+1} (1-x^2)\, P_m'\, P_n'\, dx = 0$

$\Rightarrow$ Integrating by parts,

$LHS = \left[(1-x^2)P_m' P_n\right]_{-1}^{1} - \int_{-1}^{1} \frac{d}{dx}\left\{(1-x^2). P_m'\right\} P_n . dx$

$= 0 - \int_{-1}^{1} \left\{(1-x^2)P_m'' - 2x . P_m'\right\}. P_n . dx$ ..... (i)

Now, as $P_m(x)$ is a solution of Legendre's equation, it satisfies: $(1-x^2). P_m'' - 2x P_m' + m(m+1)P_m = 0$

$\Rightarrow (1-x^2)P_m'' - 2xP'(x) = -m(m+1)P_m$ ..... (ii)

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Putting (ii) in (i), we get:

$$\int_{-1}^{1}(1-x^2)\,P_m'\,P_n'\,dx = \int_{-1}^{1} m(m+1)\,P_m.P_n\,dx$$
$$= m(m+1)\int_{-1}^{1} P_m(x).P_n(x)\,dx$$
$$= 0$$ as $m \neq n$ is given, so P_m & P_n are orthagonal on $-1\leq x\leq 1$

hence proved

5) $\int_{-1}^{1} x^2 P_{n+1} P_{n-1}\,dx = \dfrac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$

$\Rightarrow$ From recurrance relation: $(2n+1)x P_n = (n+1)P_{n+1} + n P_{n-1}$

we get: $x P_{n-1} = \dfrac{1}{2n-1} \times (n P_n + (n-1)P_{n-2}) \longrightarrow (1)$

$$x(P_{n+1}) = \frac{1}{(2n+3)} \times \left((n+2)P_{n+2} + (n+1)(P_n)\right) \rightarrow (2)$$

(1) $\times$ (2) $\Rightarrow$

$$x^2 P_{n+1}P_{n-1} = \frac{1}{(2n-1)(2n+3)} \times \left(n(n+2)P_nP_{n+2} + n(n+1)P_n^2 + (n-1)(n+2)P_{n-2}P_{n+2} + (n^2-1)P_nP_{n-2}\right)$$

Integrating both sides, we get

$$\int_{-1}^{1} x^2 P_{n+1}P_{n-1} = \frac{n(n+1)}{(2n-1)(2n+3)}\int_{-1}^{1} P_n^2\,dx \quad \text{(as } P_m(x) \text{ \& } P_n(x) \text{ are orthagonal)}$$
$$= \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

hence proved

6) $\dfrac{1-h^2}{(1-2xh+h^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)P_n(x)h^n$

$\Rightarrow$ we know, $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$

Partially d.w.r.t t, $-\frac{1}{2}(1-2xt+t^2)^{3/2}(2t-2x) = \sum_{n=1}^{\infty} n P_n(x) t^{n-1}$

Multiplying by t on both sides.

$t(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=1}^{\infty} n P_n(x) t^n$

$\Rightarrow (xt-t^2)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} n(P_n(x)) t^n \longrightarrow (1)$

we know,

$t(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^{n+1}$

partially d.w.r.t t,

$(1-2xt+t^2)^{-1/2} - \frac{1}{2}t(1-2xt+t^2)^{-3/2} \times (2t-2x) = \sum_{n=0}^{\infty} P_n(x)(n+1)t^n \longrightarrow (2)$

$(1)+(2) \Rightarrow$

$$\sum_{n=0}^{\infty}(2n+1)(P_n(x)t^n) = (1-2xt+t^2)^{-1/2} + 2x(tx-t^2)(1-2xt+t^2)^{-3/2}$$
$$= (1-2xt+t^2)^{-3/2} \cdot (1-2xt+t^2+2xt-2t^2)$$
$$= (1-2xt+t^2)^{-3/2} \cdot (1-t^2)$$

hence proved.

7. $P_n(1) = 1$

$\Rightarrow$ we know, $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$

put $x=1$

$$(1-2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1) t^n$$

$$\Rightarrow ((1-t)^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1) t^n$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(1) t^n = 1 + t + t^2 + t^3 + \dots$$

Equating the coefficient of t^n from both sides, we have

$P_n(1) = 1 \quad \forall \; n = 0, 1, 2 \dots$

hence proved