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Theorem: A normed linear space X is a Banach space iff every absolutely summable series is summable.

Proof: Let X be a Banach space.

Suppose $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Claim $\sum_{n=1}^{\infty} x_n$ is summable.

Let $S_n = \sum_{j=1}^{n} x_j$, then for $m >> n$

$$\|S_m - S_n\| = \left\| \sum_{j=n+1}^{m} x_j \right\|$$

$$\leq \sum_{j=n+1}^{m} \|x_j\| \quad \text{———} \quad (*)$$

[Define $\alpha_n = \sum_{j=1}^{n} \|x_j\|$, $m >> n$,

if $\{\alpha_n\}$ is convergent, then $\{\alpha_n\}$ is a

we can say a(n) is convergent as a(infinity)<infinity and a(n) tends to a(infinity) as n tends to infinity, this is in vector space K, hence it is cauchy sequence as well

Cauchy sequence.

$\therefore \exists\ n_0 \in \mathbb{N} \ \ni$

$|\alpha_m - \alpha_n| < \epsilon$]

Now from (*)

$$\|s_n - s_m\| \leq \sum_{j=n+1}^{m} \|x_j\|$$

$$= |\alpha_m - \alpha_n|$$

$$< \epsilon \quad \forall\ n, m \geq n_0$$

$\Rightarrow \{s_n\}$ is a Cauchy sequence in a Banach space X.

$\therefore\ s_n \longrightarrow x \in X$.

$$s_n = \sum_{j=1}^{n} x_j \longrightarrow x \in X.$$

$\Rightarrow$ the series $\sum_{j=1}^{\infty} x_j < \infty$.

Conversely assume that every absolutely summable series is summable in X.

Claim: X is a Banach space.

Let $\{s_n\}$ be a Cauchy sequence in X

Then there exists $m_1 \in \mathbb{N}$ $\ni$

$\forall\ m \geq m_1$

$\| s_m - s_{m_1} \| < 1$

Choose $m_2 > m_1$ $\ni$

$\| s_m - s_{m_2} \| < \frac{1}{2^2}$, $\forall\ m \geq m_2$

$\vdots \qquad \vdots \qquad \vdots$

Choose $m_n > m_{n-1}$, such that

$\| s_m - s_{m_n} \| < \frac{1}{n^2}$, $\forall\ m \geq m_n$.

Now for all n, $m_{n+1} > m_n$,

let $x_n = s_{m_{n+1}} - s_{m_n}$, $n = 1, 2, 3, \ldots$

$$\Rightarrow \|x_n\| = \|s_{m_{n+1}} - s_{m_n}\| < \frac{1}{2^n}$$

$$\Rightarrow \sum_{n=1}^{\infty} \|x_n\| = \sum_{n=1}^{\infty} \|s_{m_{n+1}} - s_{m_n}\| < \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \|x_n\| < \infty .$$

Thus the series $\sum_{n=1}^{\infty} x_n$ is absolutely summable.

$\therefore \sum_{n=1}^{\infty} x_n$ is summable by the assumption.

Hence let

$$\sum_{n=1}^{\infty} x_n = x \in X .$$

Since $s_{m_n} = s_{m_1} + \sum_{j=1}^{n-1} x_j$

$\therefore \quad x_n = f_{m_{n+1}} - f_{m_n}, \quad n = 1, 2, 3, \ldots$

$\therefore \quad x_1 = f_{m_2} - f_{m_1}, \quad x_2 = f_{m_3} - f_{m_2} \ldots$

$\ldots \quad x_{n-1} = f_{m_n} - f_{m_{n-1}}$.

$$\therefore \sum_{i=1}^{n-1} x_i = f_{m_n} - f_{m_1}$$

$\therefore$ It follows that subsequence $\{f_{m_n}\}$ of a Cauchy sequence $\{f_n\}$ is convergent. Hence $\{f_n\}$ itself is a convergent sequence.

$\therefore X$ is a Banach space.

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Theorem: Let X be a normed linear space and Y be a closed subspace of X.

Then X is a Banach space iff Y and $\frac{X}{Y}$ are Banach spaces in the induced norm, respectively.

Proof: let X be a Banach space and Y be a closed subspace of X.
Since every closed subspace of a Banach space is a Banach space $\Rightarrow$ Y is a Banach space.

Claim: $\frac{X}{Y}$ is a Banach space.

let $\{x_n + Y\}$ be a sequence in a n.l.s $\frac{X}{Y}$ such that,

$$\sum_{n=1}^{\infty} |||x_n + Y||| < \infty.$$

Then by definition of $|||\cdot|||$, $\exists$ some $y_n \in Y$ such that

$$\|x_n + y_n\| < |||x_n + Y||| + \frac{1}{n^2}$$

$$[\because |||x_n + Y||| = \inf\{\|x_n + y\| / y \in Y\}]$$

$$\Rightarrow \sum_{n=1}^{\infty} \|x_n + y_n\| < \sum_{n=1}^{\infty} |||x_n + Y||| + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$< \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \|x_n + y_n\| < \infty .$$

Hence the series $\sum_{n=1}^{\infty} (x_n + y_n)$ is a absolutely summable series in a Banach space X.

By Previous theorem, it follows that

$$\sum_{n=1}^{\infty} (x_n + y_n) = s \in X. \quad (*)$$

Now for $m = 1, 2, 3, \ldots$,

$$\left|\left|\left| \sum_{n=1}^{m} (x_n + Y) - (s + Y) \right|\right|\right|$$

$$= \left|\left|\left| \sum_{n=1}^{m} (x_n + y_n + Y) - (s + Y) \right|\right|\right|$$

[$\because y_n \in Y \Leftrightarrow y_n + Y = Y$]

$$= \left|\left|\left| \sum_{n=1}^{m} (x_n + y_{n}) - s + Y \right|\right|\right|$$

$$\leq \left\| \sum_{n=1}^{m} (x_n + y_{n}) - s) \right\| \longrightarrow 0$$

by $\textcircled{*}$

$m \to \infty$.

$[\because 0 \in Y, \quad |||x+Y||| = \inf\{\|x+y\| \mid y \in Y\}$

$\leq \|x - 0\|$

$= \|x\|]$

$$\Rightarrow \left|\left|\left| \sum_{n=1}^{\infty} (x_n + Y) - (s + Y) \right|\right|\right| = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} (x_n + Y) = s + Y \in \frac{X}{Y}.$$

$\therefore$ Every absolutely summable series is summable in $\frac{X}{Y}$.

$\therefore \frac{X}{Y}$ is a Banach space.

Conversely assume Y and $\frac{X}{Y}$ are Banach spaces.

Claim: X is a Banach space.

Let $\{x_n\}$ be a Cauchy sequence in X.

Then $\|x_n - x_m\| \to 0$ as $n, m \to \infty$.

$$\therefore \quad |||(x_n + Y) - (x_m + Y)|||$$

$$= |||(x_n - x_m) + Y|||$$

$$\leq \|x_n - x_m\| \to 0$$

as $m, n \to \infty$

$\Rightarrow \{x_n + Y\}$ is a Cauchy sequence in a Banach space $\frac{X}{Y}$.

$\therefore \{x_n + Y\}$ is Convergent in $\frac{X}{Y}$.

$\Rightarrow \exists\ x + Y \in \frac{X}{Y} \ni$

$$|||(x_n + Y) - (x + Y)||| \longrightarrow 0$$

as $n \to \infty$.

$$\Rightarrow |||(x_n - x) + Y||| \longrightarrow 0$$

as $n \to \infty$.

$$\Rightarrow \inf\{\|(x_n - x) + y\| \,/\, y \in Y\} \longrightarrow 0$$

as $n \to \infty$.

$\Rightarrow \exists$ a sequence $\{y_n\}$ in Y

such that

$$x_n - x + y_n \longrightarrow 0$$

$$\Rightarrow x_n + y_n - x \longrightarrow 0$$

$$\Rightarrow x_n + y_n \longrightarrow x \in X.$$

Now

$$y_n - y_m = y_n + x_n - x - x_n + x_m - x_m - y_m + x$$

$$\|y_n - y_m\| \leq \|y_n + x_n - x\| + \|x_m - x_n\| + \|x_m + y_m - x\| \longrightarrow 0$$

$\Rightarrow \{y_n\}$ is Cauchy sequence in Y.

Since Y is a Banach space,

$y_n \longrightarrow y \in Y$.

Now

$$x_n = (x_n + y_n) - y_n \longrightarrow x - y \in X.$$

$\therefore \{x_n\}$ is a Cgt sequence in X.

$\therefore X$ is a Banach space

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$$\left[\left\| \sum_{i=1}^{n} x_i \right\| \le \sum_{i=1}^{n} \|x_i\| \right].$$

Denote

$$U(0,1) = \{ x \in X \mid \|x\| < 1 \}$$

is a open unit ball in X.

$$\overline{U(0,1)} = \{ x \in X \mid \|x\| \le 1 \}$$ is a closed unit ball in X.

IIIly

$$B(x_0, r) = \{ x \in X \mid \|x - x_0\| < r \}$$

is an open ball with center x_0 and radius r.

$$\overline{B(x_0, r)} = \{ x \in X \mid \|x - x_0\| \le r \}$$

is a closed ball in X.

Riesz lemma :—

Let $(X, \|\cdot\|)$ be a normed linear space, Y be a closed sub-space of X and $Y \neq X$. Let r be a real number such that $0 < r < 1$. Then there exists some $x_r \in X$ such that $\|x_r\| = 1$ and $r \leq \text{dist}(x_r, Y) \leq 1$.

Proof: Since $Y \neq X$, consider

$x \in X$ and $x \notin Y$

$\because$ Y is a closed sub-space, implies

$\text{dist}(x, Y) > 0$.

Also as $r < 1$, there exists some $y_0 \in Y$ such that

$$\|x - y_0\| \le \frac{\text{dist}(x, Y)}{r} \quad \text{——— (1)}$$

$[\because r < 1 \Rightarrow \frac{1}{r} \geq 1$

$d = \text{dist}(x, Y)$, then $\frac{d}{r} \geq d]$

$$\underline{\underline{\text{dist}(x, Y)}} \le \frac{\text{dist}(x, Y)}{r}$$

$$\|x - y_0\| \le \frac{d}{r} \quad]$$

Let $x_r = \frac{x - y_0}{\|x - y_0\|}$

$$\|x_r\| = \left\| \frac{x - y_0}{\|x - y_0\|} \right\|$$

$$= \frac{\|x - y_0\|}{\|x - y_0\|} = 1$$

Also $0 \in Y$, we see that

$$\text{dist}(x_r, Y) \le \|x_r - 0\| = \|x_r\| = 1$$

Now consider

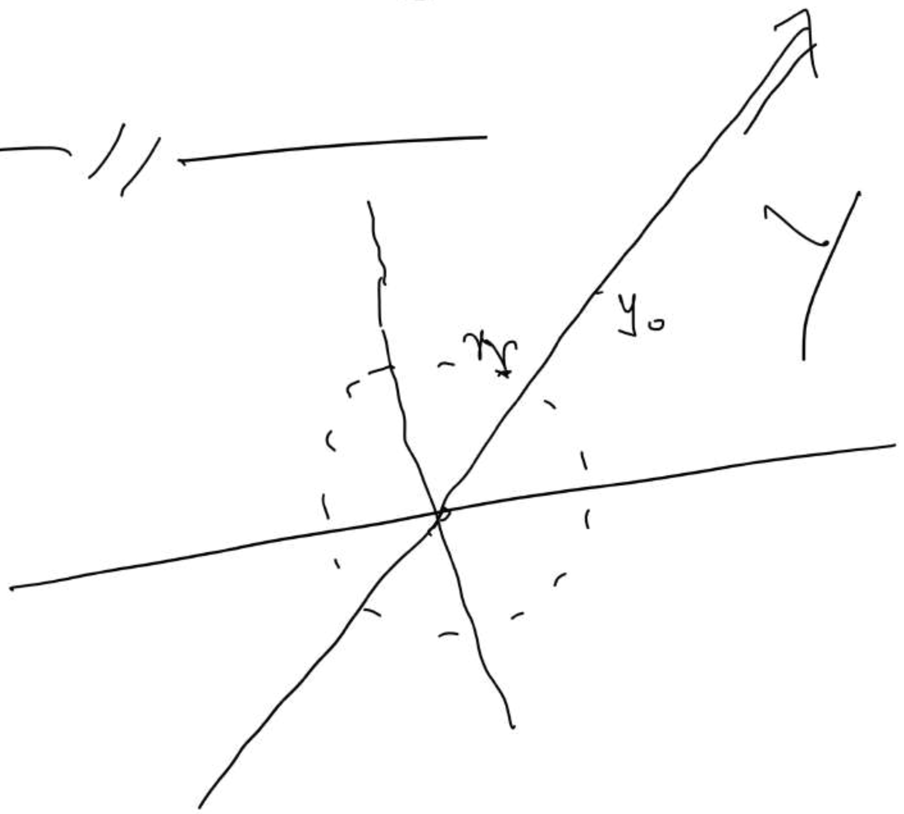
$$\operatorname{dist}(x_r, Y) = \inf\{\|x_r - y\| \mid y \in Y\}$$

$$= \inf\left\{\left\|\frac{x - y_0}{\|x - y_0\|} - y\right\| \mid y \in Y\right\}$$

$$= \frac{1}{\|x - y_0\|} \inf\{\|x - (\underbrace{y_0 + y\|x - y_0\|}_{\in Y})\| \mid y \in Y\}$$

$$= \frac{1}{\|x - y_0\|} \operatorname{dist}(x, Y)$$

$$\geq r \quad \text{by (1)}$$



Lemma : Let $(X, \|\cdot\|)$ be n.l.s and Y be a subspace of X.

(a) For $x \in X$, $y \in Y$ and $k \in K$,

$$\|kx+y\| \geq |k| \operatorname{dist}(x, Y).$$

Proof: If $k=0$, result is true.

If $k \neq 0$,

$$\|kx+y\| = \|k(x+y/k)\|$$

$$= |k| \, \|x-(-y/k)\|$$

$$\geq |k| \operatorname{dist}(x, Y) \qquad [\because -y/k \in Y]$$

$[\because \operatorname{dist}(x, Y) = \inf\{\|x-y\| \,/\, y \in Y\}$
$\leq \|x-y\|]$

(b) Let Y be a finite dimensional subspace of a n.l.s $(X, \|\cdot\|)$. Then Y is complete. In particular Y is closed in X.

Let $\{y_1, y_2 \ldots y_m\}$ be a basis for Y and $\{x_n\}$ is a sequence in Y.

If $x_n = \sum_{j=1}^{m} k_{nj} y_j$, $\forall$ $n = 1, 2, 3 \ldots$

then $x_n \to x = \sum_{j=1}^{m} k_j y_j$

iff $k_{nj} \to k_j$ $\forall$ $j = 1 \ldots n$.

Also $\{x_n\}$ is bounded iff $\{k_{nj}\}$ is bounded for $j = 1, 2 \ldots m$.

closed: has all limit points (point which has at least one point in the set from its epsilon boundary)eg: (-inf,2], [a,b], (-inf,inf) (all converging sequence converge to an element inside the set)
open: has all its interior points (point whose epsilon boundary is completely inside the set) eg: (a,b), (-inf,inf)
bounded: has a closed boundary eg: [a,b], (a,b)
unbounded: (-inf,0] (is open at left), (-inf,inf)
complete: banach space i.e all converging sequence converge to an element inside the set wrt ||.||

Proof: let $\dim Y = 1$ (Proof by induction)

let $Y = [y] = \text{span}\{y\}$

$= \{ky \mid k \in K\}$.

let $\{x_n\}$ be a sequence in Y, then

$$x_n = k_n y, \quad k_n \in K.$$

Then

$$\|x_n - x_m\| = \|(k_n - k_m)y\|$$

$$= |k_n - k_m| \, \|y\|$$

$$\Rightarrow \frac{\|x_n - x_m\|}{\|y\|} = |k_n - k_m| \longrightarrow (*)$$

$\because$ $\{x_n\}$ is a Cauchy, given $\epsilon > 0$

$\exists \; n_0 \in \mathbb{N} \ni \forall \; n, m \geq n_0$

$$\|x_n - x_m\| \leq \epsilon \|y\|.$$

$\therefore$ From $(*)$, we see that

$$|k_n - k_m| < \epsilon, \ \forall \ n, m \geq n_0$$

$\Rightarrow \{k_n\}$ is a Cauchy sequence in K. But K is complete

$\therefore \ k_n \longrightarrow k \in K$.

let $x = ky \in Y$

Then

$$\|x_n - x\| = |k_n - k| \, \|y\| \longrightarrow 0$$

$\Rightarrow \ x_n \longrightarrow x \in Y$.

$\therefore$ Y is complete if $\dim Y = 1$.

Now assume that every $m-1$ dimensional space is complete.

let Y be a m-dimensional space with basis $\{y_1, y_2, \ldots, y_m\}$

Let $\{x_n\}$ be a Cauchy sequence in Y.

Let $Z = [y_2, y_3 \ldots y_m]$ be an $m-1$ dimensional space which is complete by induction.

$\because \{x_n\}$ is a Cauchy sequence in Y, we have

$$x_n = k_n y_1 + z_n \in \text{span}\{y_1, Z\}$$

$k_n \in K$, $z_n \in Z$.

Now for any $n, p \in \mathbb{N}$ by (a)

$$\|x_n - x_p\| = \|\underbrace{(k_n - k_p)}\, y_1 + z_n - z_p\|$$

$$\geq |k_n - k_p| \underbrace{\text{dist}(y_1, Z)}_{>0} \quad \text{— (2)}$$

$\because y_1 \notin Z$

$\therefore$ we have

$$\|x_n - x_p\| \longrightarrow 0$$

$$\Rightarrow |k_n - k_p| \longrightarrow 0$$

as $n, p \to \infty$.

$\Rightarrow \{k_n\}$ is a Cauchy in K

$\Rightarrow k_n \longrightarrow k \in K$.

Now $z_n = x_n - k_n y_1$

$\Rightarrow \{z_n\}$ is also Cauchy sequence in Z, which is complete.

$$\left[\|z_n - z_p\| = \|x_n - k_n y_1 - x_p + k_p y_1\| \right.$$

$$\le \|x_n - x_p\| + |k_n - k_p| \|y_1\|$$

$$\left. \longrightarrow 0 \right]$$

$\therefore z_n \longrightarrow z \in Z$.

$\therefore x_n = k_n y_1 + z_n \longrightarrow k y_1 + z \in Y$.

$\therefore$ Y is Complete.

In Particular Y is a closed.

Next let $x_n \in Y$ and $x_n = \sum_{j=1}^{m} k_{nj} y_j$

$\forall\ n = 1, 2, 3 - - -$

If $k_{nj} \to k_j$, $j = 1, 2, 3 - - M$

as $n \to \infty$.

let $x = \sum_{j=1}^{M} k_j y_j$

Then

$$\|x_n - x\| = \left\| \sum_{j=1}^{M} (k_{nj} - k_j) y_j \right\|$$

$$\leq \sum_{j=1}^{M} |k_{nj} - k_j| \, \|y_j\|$$

$$\longrightarrow 0$$

$\Rightarrow x_n \to x \in X.$

shouldnt this be Y???

Conversely assume that

$$x_n \to x = \sum_{j=1}^{m} k_j y_j$$

Then

$$\|x_n - x\| = \left\|\sum_{j=1}^{m} (k_{n_j} - k_j)\, y_j\right\| \qquad (**)$$

Let $Y_j = \text{Span}\{y_1, y_2, \dots, y_{j-1}, y_{j+1}, \dots, y_m\}$

$\forall\, j = 1, 2, \dots, m$

$\because\ y_j \notin Y_j \Rightarrow \text{dist}(y_j, Y) > 0.$

Now from (**)

$$\|x_n - x\| = \left\|\sum_{j=1}^{m} (k_{n_j} - k_j)\, y_j\right\|$$

$$= \left\|(k_{n_j} - k_j)\, y_j + \underbrace{\sum_{\substack{i=1 \\ i \neq j}}^{m} (k_{n_i} - k_i)\, y_i}_{\in Y_j}\right\|$$

$$\geq |k_{n_j} - k_j|\ \text{dist}(y_j, Y_j)$$

$\therefore$ From above if $x_n \longrightarrow x$

implies $k_{nj} \longrightarrow k_j \; \forall j = 1 - m$.

Now assume

$$x_n = \sum_{j=1}^{m} k_{nj} y_j \quad \text{is}$$

bounded. Then for each $j = 1, 2, \ldots m$,
let Y_j be defined as above.

Then

$$\|x_n\| = \left\| \sum_{j=1}^{m} k_{nj} y_j \right\|$$

$$= \left\| k_{nj} y_j + \sum_{\substack{i=1 \\ i \neq j}}^{m} k_{ni} y_i \right\|$$

$$\geq |k_{nj}| \, \operatorname{dist}(y_j, Y_j)$$

$\Rightarrow$ if $\|x_n\| \leq \alpha < \infty$

$\Rightarrow$ $|k_{nj}| < \infty$

$\Rightarrow \{k_{nj}\}$ bounded $j = 1 - m$.

Conversely if $\{k_{nj}\}$ bounded

Then

$$\|x_n\| = \left\| \sum_{j=1}^{M} k_{nj} y_j \right\|$$

$$\leq \sum_{j=1}^{M} |k_{nj}| \, \|y_j\|$$

$$\leq \alpha \underbrace{\sum_{j=1}^{M} \|y_j\|}_{< \infty},$$

where $\alpha = \max\{ |k_{nj}| \, / \, j = 1, \dots, m \ \ \forall n \}$.

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Remark: An infinite dimensional subspace of a n.l.s X need not be closed in X.

Ex: $X = \ell^{\infty}$ with $\|\cdot\|_{\infty}$

$Y = c_{00}$ is a subspace of ℓ^{∞} which is not closed in X.

$\therefore$ $x_n = (1, \frac{1}{2}, \cdots \frac{1}{n}, 0, 0, \cdots) \in c_{00}$

$x_n \to x = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots \frac{1}{n}, \frac{1}{n+1} \cdots)$

$\notin c_{00}$.

Theorem:

Let $(X, \|\cdot\|)$ be a n.l.s. Then the following are equivalent.

(i) Every closed and bounded subset of X is compact

(ii) The subset $\{x \in X / \|x\| \leq 1\}$ of X is compact.

(iii) X is finite dimensional.

Proof: clearly (i) $\Longrightarrow$ (ii)

$\because$ $\{x \in X / \|x\| \leq 1\}$ is a closed and bounded subset of X.

statement 1 is only true in Rn and Cn

compact set: in a nls (X,||.||), a subset A of X is said to be (sequentially) compact if for every sequence {xn} in A, we can find a subsequence {xnk} that converges to x belonging to A

(ii) $\Longrightarrow$ (iii)

Assume $\{x \in X \,/\, \|x\| \leq 1\}$ is compact set.

Claim: X is a finite dimensional space.

Suppose X is not finite dimensional space.

Let $\{y_1, y_2, y_3 \ldots\}$ be an infinite linear indepent subset of X.

Now for each $n \in \mathbb{N}$, let

$Z_n = \operatorname{Span}\{y_1, y_2 \ldots y_n\}$.

Then Z_n is a finite dimensional, hence it is closed subspace.

Also for all n, $Z_n \neq Z_{n+1}$.

Now by Riesz lemma, there exists

$x_n \in Z_{n+1}$ $\ni$ $\|x_n\| = 1$

and $\operatorname{dist}(x_n, Z_n) \geq \frac{1}{2}$.

If we apply this for all $n = 1, 2, 3 - - -$,

we obtain a sequence $\{x_n\}$

on the closed unit ball

$\{x \in X \,/\, \|x\| \leq 1\}$ such that

$\|x_n - x_m\| \geq \frac{1}{2}$, $\forall\, n, m$, $n \neq m$.

$\Rightarrow$ $\{x_n\}$ is not Cauchy

Thus we obtained a sequence $\{x_n\}$ in $\{x \in X \,/\, \|x\| \leq 1\}$ with no convergent subsequence, which is contradiction to $\{x \in X \,/\, \|x\| \leq 1\}$ is a compact set.

$\therefore$ X has to be finite dimensional.

(iii) $\implies$ (i)

Suppose X is a finite dimensional space.

Claim: Every closed and bounded subset of X is a compact set.

Let E be a closed and bounded subset of X.

Let $\{x_n\}$ be a sequence in E

Let $\{y_1, y_2 \ldots y_m\}$ be a basis for X.

$\because E \subseteq X \implies \{x_n\} \subseteq X.$

$\therefore x_n = \sum_{j=1}^{m} K_{nj} y_j$, $\forall n = 1, 2, 3 \ldots$

$\because E$ is bound $\Rightarrow \|x\| \leq \alpha \ \forall x \in E$
in particular $\{x_n\}$ is bounded
$\therefore \|x_n\| \leq \alpha \ \forall n$

$\Rightarrow \{K_{nj}\}_{n=1}^{\infty}$ is bounded for $j = 1, \ldots m$

By Bolzano-weistrass theorem for K, and passing to subsequency of subsequency several times, we find $n_1 < n_2 < \ldots$ such that

Bolzano Weierstrass theorem is a theorem that states that a convergent subsequence, or subsequential limit, exists for every bounded sequence of real/complex numbers

$\{K_{n_p j}\}$ converges in K, $j = 1 \ldots m$

Then by previous lemma, the subsequence $\{x_{n_p}\}$ converges to some $x \in X$.

Since E is closed and $\{x_{n_p}\}$ is a convergent subsequence, it follows that $x \in E$.

Thus every sequence in E has a convergent subsequence in E

$\therefore$ E is compact

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