Thus we have $|f|_{n_k} \longrightarrow f$ a.e as $k \to \infty$

($\because \{f_{n_k}\}$ is the sequence of partial sums of f).

Step 2:

To show: $f_{n_k} \to f$ in L^1-metric.

Lecture 34

Now $|f - f_{n_k}| \leq |f| \leq g \quad \forall k \geq 1.$

$\therefore$ By dominated convergence thm,

$$\lim_{k\to\infty} \int |f - f_{n_k}| = 0 \quad \text{as } k \to \infty.$$

$$\Rightarrow \lim_{k\to\infty} \|f - f_{n_k}\|_{L^1} = 0$$

Since $\{f_n\}$ is Cauchy, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$,

$$\|f_n - f_m\|_{L^1} < \varepsilon/2$$

Choose $n_k > N$ & $\|f_{n_k} - f\|_{L^1} < \varepsilon/2$

Then $\|f_n - f\|_{L^1} = \|f_n - f_{n_k} + f_{n_k} - f\|_{L^1}$

$$\leq \|f_n - f_{n_k}\|_{L^1} + \|f_{n_k} - f\|_{L^1}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\, n > N.$$

$\therefore$ $f_n \to f$ in L^1-norm.

Thus every Cauchy seq. in $L^1(\mathbb{R}^d)$ is Convergent.

$\therefore$ $L^1(\mathbb{R}^d)$ is a complete normed space & hence it is a Complete metric space w.r.t L^1-metric.

$d(f,g) := \|f-g\|_{L^1} \quad \forall\, f,g \in L^1(\mathbb{R}^d)$.

distance function.

Let

$C[a,b]$ = the space of all continuous functions on $[a,b]$.

$\mathcal{R}[a,b]$ = the space of all Riemann integrable functions on $[a,b]$.

Then $$C[a,b] \subseteq \mathcal{R}[a,b] \subseteq \left(L^1([a,b]),\ L^1\text{-metric}\right)$$

<u>Qn</u>:- does $\left(C[a,b],\ L^1\text{-metric}\right)$ Complete ?

or

does $\left(\mathcal{R}[a,b],\ L^1\text{-metric}\right)$ Complete ?

<u>Ans</u>:- No!

<u>Example</u>:- Let $$f_n(x) = \begin{cases} 0 & \text{if } -1 \le x < \frac{-1}{n} \\ \frac{n}{2}\left(x+\frac{1}{n}\right), & \text{if } \frac{-1}{n} \le x < \frac{1}{n} \\ 1, & \text{if } \frac{1}{n} \le x \le 1, \end{cases}$$

$\forall\, n \geq 1.$

& $$f(x) = \begin{cases} 0 & \text{if } -1 \le x < 0 \\ 1 & \text{if } 0 \le x \le 1, \end{cases}$$

<u>claim</u>:- $f_n \to f$ in L^1-metric as $n \to \infty$.

That is, $\|f_n - f\|_{L^1} \longrightarrow 0$ as $n \to \infty$.

proof of claim:

$$\|f_n - f\|_{L^1} = \int_{[-1,1]} |f_n - f|$$

$$= \int_{-1}^{-1/n} |0 - 0| + \int_{-1/n}^{0} \left|\frac{n}{2}\left(x+\frac{1}{n}\right) - 0\right| dx$$

$$+ \int_{0}^{1/n} \left|\frac{n}{2}\left(x+\frac{1}{n}\right) - 1\right| dx + \int_{1/n}^{1} |1-1|$$

$$= 0 + \int_{-1/n}^{0} \frac{n}{2}\left(x+\frac{1}{n}\right) dx + \int_{0}^{1/n} \left(1 - \frac{n}{2}\left(x+\frac{1}{n}\right)\right) dx + 0$$

$$= \frac{n}{2}\left(\frac{x^2}{2} + \frac{x}{n}\right)\Big|_{-1/n}^{0}$$

$$+ \left(x - \frac{n}{2}\left(\frac{x^2}{2} + \frac{x}{n}\right)\right)\Big|_{0}^{1/n}$$

$$\boxed{\begin{aligned} &\frac{n}{2}\left(x+\frac{1}{n}\right) \le 1 \\ &x + \frac{1}{n} \le 2/n \\ &\Rightarrow x \le 2/n - \frac{1}{n} = 1/n \\ &\underline{1/n \ge x} \end{aligned}}$$

$$= \frac{n}{2}\left(0 - \left(\frac{1}{2n^2} - \frac{1}{n^2}\right)\right) + \left[\frac{1}{n} - \frac{n}{2}\left(\frac{1}{2n^2} + \frac{1}{n^2} - 0\right)\right]$$

$$= \frac{n}{2}\left(\frac{+1}{2n^2}\right) + \left[\frac{1}{n} - \frac{1}{4n} - \frac{1}{2n}\right]$$

$$= \frac{1}{4n} + \frac{4-3}{4n}$$

$$= \frac{1}{2n}$$

$\therefore \quad \|f_n - f\|_{L^1} = \frac{1}{2n} \longrightarrow 0 \quad$ as $n \to \infty$.

$\therefore \quad f_n \to f$ in L^1-metric.

Note that each f_n is continuous but f is not continuous.

$\therefore \quad C[a,b]$ is not complete w.r.t L^1-norm.

Qn:- By what metric/norm on $C[a,b]$ so that it is complete w.r.t this norm?

- $C[a,b]$ is complete with sup norm. $\| - \|_\infty$.

where $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$

& $d(f,g) = \|f-g\|_\infty$

$= \sup_{x \in [a,b]} |f(x) - g(x)|$.

In sup-norm, $f_n \to f$ in sup-norm means that $f_n \to f$ uniformly as $n \to \infty$.

Know result:- If $\{f_n\}$ Continuous funtions, & $f_n \to f$ uniformly, then f is Continuous.

Riemann-Lebesgue Lemma:- (weaker version).

Let f be a bounded measurable funtion defined on $[a,b]$. Show that

$$\lim_{\beta \to \infty} \int_a^b f(x) \sin \beta x \, dx = 0.$$

proof:- Given f is bounded & measurable on $[a,b]$

$$\Rightarrow \quad f \in L^1([a,b]).$$

$\Rightarrow$ given $\varepsilon > 0$ there exists a simple function $h = \sum_{i=1}^{n} c_i \chi_{(a_i, b_i)}$ such that

$$\int_a^b |f-h|\, dx < \varepsilon.$$

$$\boxed{\begin{array}{l} h = \varphi_k \to f \\ \& \ \boxed{\int \varphi_k \to \int f} \end{array}}$$

Now $\left| \int_a^b f(x) \sin \beta x \, dx \right|$

$$= \left| \int_a^b (f - h + h) \sin \beta x \, dx \right|$$

$$\leq \int_a^b |f-h| \, |\sin \beta x| \, dx + \left| \int_a^b h \sin \beta x \right| dx$$

$$\leq \int_a^b |f-h| \, dx + \left| \int_a^b h \sin \beta x \, dx \right|$$

$$< \varepsilon + \left| \int_a^b h(x) \sin\beta x \, dx \right|$$

Now $\left| \int_a^b h(x) \sin\beta x \, dx \right| = \left| \int_a^b \left(\sum_{i=1}^n c_i \chi_{(a_i, b_i)} \right) \sin\beta x \, dx \right|$

$$= \left| \sum_{i=1}^n c_i \left(\int_a^b \chi_{(a_i, b_i)} \sin\beta x \, dx \right) \right|$$

$$= \left| \sum_{i=1}^n c_i \int_{a_i}^{b_i} \sin\beta x \, dx \right|$$

Let $\beta x = t$

$\beta dx = dt$.

$$= \left| \sum_{i=1}^n c_i \int_{\beta a_i}^{\beta b_i} \frac{\sin t}{\beta} dt . \right|$$

$$\leq \sum_{i=1}^n \frac{|c_i|}{|\beta|} \left| \int_{\beta a_i}^{\beta b_i} \sin t \, dt \right|$$

$$= \sum_{i=1}^n \left| \frac{c_i}{\beta} \right| \left| \left(-\cos\beta b_i + \cos\beta a_i \right) \right|$$

$$\leq \sum_{i=1}^n \frac{|c_i|}{|\beta|} 2 = \frac{2}{|\beta|} \sum_{i=1}^n |c_i|$$

$\leq \dfrac{2}{|\beta|} nM$, where

$M = \max\{|c_1|, \dots, |c_n|\}$.

$\therefore \left| \int_a^b f(x) \sin(\beta x)\, dx \right| < \varepsilon + \dfrac{2}{|\beta|} nM$

as $\beta \to \infty$, we have $\left| \int_a^b f(x) \sin \beta x \, dx \right| < \varepsilon$

for $\beta \gg 0$ sufficiently large.

$\therefore \lim_{\beta \to \infty} \int_a^b f(x) \sin(\beta x)\, dx = 0.$

Announcement

Class Test - 3 | End exam., Marks = 30.

date: 8th April 2022 (Friday)

@ 9 AM.

- Online exam in moodle.

Syllabus! whole syllabus. Stress on end portions.