Fundamental Thm of f-g abelian gp.

Lecture 15

28/02/2022

Fundamental Thm of finitely generated abelian groups:

Defn. A gp G is said to be finitely generated if ~~there~~ there is a finite subset A of G s.t $G = \langle A \rangle$.

Defn. For each $r \in \mathbb{N}$, $\mathbb{Z}^r = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ be the direct product of r copies of the gp $\mathbb{Z}$. $\mathbb{Z}^r$ is called the free abelian gp of rank r.

Thm. Let G be a finitely generated abelian gp. Then

(1) $G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s}$

for some integers $r, n_1, n_2, \ldots, n_s$ satisfying the following condn.s

(a) $r \geq 0$ and $n_j \geq 2$ $\forall j$ and

(b) $n_{i+1} \mid n_i$ for $1 \leq i \leq s-1$.

(2) The expression in (1) is unique i.e if $G \cong \mathbb{Z}^t \times \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_u}$ where $t, m_1, \ldots, m_u$ satisfy condn. (a) & (b) then

$r = t$, $u = s$ and $m_i = n_i$ $\forall i$.

The integer r is called the free rank of G and the integers $n_1, n_2, \ldots, n_s$ are called invariant factors of G and it is called invariant factor decomposition of G. Finitely generated abelian is finite iff its free rank is 0. and the order is $n_1 n_2 \cdots n_s$.

If G is a finite abelian gp with invariant factors $n_1, n_2, \ldots, n_s$ where $n_{i+1} \mid n_i$, then G is said to be of type $(n_1, n_2, \ldots, n_s)$

$$|G| = n_1 n_2 \cdots n_s.$$

In order to find out different abelian gps of a particular order we must find all finite seqn's $n_1, n_2, \ldots, n_s$ s.t

(1) $n_j \geq 2 \quad \forall j$

(2) $n_{i+1} \mid n_i$ for $1 \leq i \leq s-1$.

and (3) $n = n_1 n_2 \ldots n_s$.

where n is the order of the gp.

If p is a prime divisor of n then $p \mid n_i$ for some i thus by (2) $p \mid n_j \quad \forall j \leq i$.

$\therefore$ It follows that every prime divisor of n must divide the first invariant factor n_1.

<u>Cor</u>. If n is the product of distinct primes then upto isomorphism the only abelian gp of order n is the cyclic gp of order n i.e. $\mathbb{Z}_n$.

<u>Example</u>. $n = 180 = 2^2 \cdot 3^2 \cdot 5$.

By above observation $2 \cdot 3 \cdot 5 \mid n_1$

$\therefore$ The possible values of n_1 are

$n_1 = 2^2 \cdot 3^2 \cdot 5$; $n_1 = 2^2 \cdot 3 \cdot 5$,

$n_1 = 2 \cdot 3^2 \cdot 5$, $n_1 = 2 \cdot 3 \cdot 5$.

Now if $n_1 = 2^2 \cdot 3 \cdot 5$ then only possibility for $n_2 = 3$.

and if $n_1 = 2 \cdot 3^2 \cdot 5$ then only possibility for $n_2 = 2$.

If $n_1 = 2 \cdot 3 \cdot 5$, then possibility for n_2 are $2, 3$ or 6

But n_2 cannot be 2 or 3

Hence $n_2 = 6$.

Invariant factors	Abelian gps
$2^2 \cdot 3^2 \cdot 5$	$\mathbb{Z}_{180}$
$2^2 \cdot 3 \cdot 5, \ 3$	$\mathbb{Z}_{60} \times \mathbb{Z}_3$
$2 \cdot 3^2 \cdot 5, \ 2$	$\mathbb{Z}_{90} \times \mathbb{Z}_2$
$2 \cdot 3 \cdot 5, \ 6$	$\mathbb{Z}_{30} \times \mathbb{Z}_6$.

Thm Let G be an abelian grp of order $n > 1$ and let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

Then

(1) $G \cong A_1 \times \cdots \times A_k$ where $|A_i| = p_i^{\alpha_i}$

(2) For each $A \in \{A_1, \ldots, A_k\}$ with $|A| = p^{\alpha}$

$$A \cong \mathbb{Z}_{p^{\beta_1}} \times \mathbb{Z}_{p^{\beta_2}} \cdots \times \mathbb{Z}_{p^{\beta_t}}$$

with $\beta_1 \geq \beta_2 \cdots \geq \beta_t \geq 1$ and

$$\beta_1 + \cdots + \beta_t = \alpha.$$

(3) The decomposition in (1) & (2) is unique.

Defn. The integers p^{β_j} are called elementry divisors of G. This is called elementry divisor decomposition

Remark: The subgps A_i are actually the Sylow p_i-subgps of G (Note that they are normal since G is abelian and hence unique).

Example. $n = 1800 = 2^3 \cdot 3^2 \cdot 5^2$.

$G \cong A_1 \times A_2 \times A_3$. s.t. $|A_1| = 2^3$

$|A_2| = 3^2$

$|A_3| = 5^2$.

order of p^β	partition of β	Abelian gp
2^3	3 ; 2+1 ; 1+1+1	$\mathbb{Z}_8$; $\mathbb{Z}_4 \times \mathbb{Z}_2$; $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.
3^2	2 ; 1+1	$\mathbb{Z}_9$; $\mathbb{Z}_3 \times \mathbb{Z}_3$
5^2	2 ; 1+1	$\mathbb{Z}_{25}$; $\mathbb{Z}_5 \times \mathbb{Z}_5$

Total 12 distinct abelian gps of order 1800 is possible.

$\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_{25}$; $\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \times \mathbb{Z}_5$

$\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$; - - - - -

<u>Propn</u>. $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$

iff $\gcd(m, n) = 1$.

If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ then

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \mathbb{Z}_{p_k^{\alpha_k}}.$$

<u>Obtain elementary divisors from invariant factors</u>:

Let G be an abelian gp of type $(n_1, n_2, \ldots, n_s)$. i.e $n = n_1 \cdots n_s = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

i.e $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s}$.

For each $n_i = p_1^{\beta_{i1}} \cdots p_k^{\beta_{ik}}$

where $\beta_{ik} \geq 0$.

Then by above propn.

$$\mathbb{Z}_{n_i} \cong \mathbb{Z}_{p_1^{\beta_{i1}}} \times \cdots \times \mathbb{Z}_{p_k^{\beta_{ik}}}$$

Example. $G \cong \mathbb{Z}_{30} \times \mathbb{Z}_{30} \times \mathbb{Z}_2$

$\mathbb{Z}_{30} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$. $\qquad 30 = 2.3.5$

$$G \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_2.$$

This is the elementary factor decomposition of G.