Sensitivity Analysis

Example: Find the limits of variations of the costs c_1, c_2, c_3, c_4, c_5 and c_6 respectively for the LPP whose optimal table is given below, so that the optimal soln remains optimal.

Soln.

C_B	B	x_B	b \ c_j	-1	-1	3	0	-3	0
				a_1	a_2	a_3	a_4	a_5	a_6
-1	a_2	x_2	5	$2/5$	1	0	$1/10$	$4/5$	0
3	a_3	x_3	6	$1/5$	0	1	$3/10$	$2/5$	0
0	a_6	x_6	8	1	0	0	$-1/2$	10	1
			$z_j - c_j$	$6/5$	0	0	$4/5$	$17/5$	0

Soln. optimal soln $\rightarrow$ $x_1 = 0,\ x_2 = 5,\ x_3 = 6,\ x_4 = x_5 = 0, x_6 = 8$.

$Z_{max} = 13$.

$x_1, x_4, x_5 \rightarrow$ non-basic variables in the optimal table.

Hence, permissible limits of the variations of c_1, c_4, c_5 which are $\delta_1, \delta_4, \delta_5$ resp^{ly}, so that the above soln remains optimal, are given by

$$\delta_1 \le z_1 - c_1 = 6/5$$

$$\delta_4 \le z_4 - c_4 = 4/5$$

$$\delta_5 \le z_5 - c_5 = 17/5$$

To find the permissible limits for the variations of c_2, c_3, c_6 which are the costs corresponding to the basic variables x_2, x_3, x_6.

c_2

$$\text{Min}\left\{ -\frac{(z_j - c_j)}{y_{2j}},\ y_{2j} < 0 \right\} \ge \delta_2 \ge \text{Max}\left\{ -\frac{(z_j - c_j)}{y_{2j}},\ y_{2j} > 0 \right\}$$

for j corresponding to non-basic vectors.

$y_{2j} = \left(\frac{2}{5}, 1, 0, \frac{1}{10}, \frac{4}{5}, 0\right)$

basic.

Non-basic $\rightarrow$ $j = 1, 4, 5$

$y_{2j} \rightarrow \frac{2}{5}, \frac{1}{10}, \frac{4}{5}$

As no $y_{2j} < 0$, we get $\text{Min}\left\{-\frac{(z_j - c_j)}{y_{2j}} \middle| y_{2j} < 0\right\} = \infty$.

Also $\text{Max}\left\{-\frac{(z_j - c_j)}{y_{2j}} \middle| y_{2j} > 0\right\}$

$= \text{Max}\left\{-\frac{6/5}{2/5}, -\frac{4/5}{1/10}, -\frac{17/5}{4/5}\right\}$

$= \text{Max}\left\{-3, -8, -\frac{17}{4}\right\}$

$-\frac{17}{4} = -4.25$

$= -3$.

$\therefore\ -\infty \geq \delta_2 \geq -3$.

C_3

$\min\left\{-\frac{(z_j - c_j)}{y_{3j}} \middle| y_{3j} < 0\right\} \geq \delta_3 \geq \max\left\{-\frac{(z_j - c_j)}{y_{3j}} \middle| y_{3j} > 0\right\}$

$y_{4j} = \left(\frac{1}{5}, 0, 1, 3/10, 2/5, 0\right)$

non-basic

$-\infty \geq \delta_3 \geq \max\left\{-\frac{6/5}{1/5}, -\frac{4/5}{3/10}, -\frac{17/5}{2/5}\right\}$

$= \text{Max}\left\{-6, -8/3, -\frac{17}{2}\right\}$

$-8/3 = -3.9$, $-\frac{17}{2} = -8.5$

$= -8/3$.

c_5 $\text{Min}\left\{-\frac{(z_j-c_j)}{y_{5j}} \middle| y_{5j}<0\right\} \geq \delta_5 \geq \text{Max}\left\{-\frac{(z_j-c_j)}{y_{5j}} \middle| y_{5j}>0\right\}$

$$y_{5j} = (1, 0, 0, -1/2, 10, 1)$$

non-basic

$$\text{Min}\left\{-\frac{4/5}{-1/2}\right\} \geq \delta_5 \geq \text{Max}\left\{-\frac{6/5}{1}, -\frac{17/5}{10}\right\}$$

$$-1.2 \qquad -.34 \qquad 3.4$$

$$\frac{8}{5} \geq \delta_5 \geq -\frac{17}{50}$$

variation in c_2, c_3, c_4 by amounts specified by $\delta_2, \delta_3, \delta_4$ as given above will not disturb the optimality of the present soln.

Example: Find the optimal s.ln. of the LPP

$$\text{Max} \quad z = 4x_1 + 3x_2$$

$$\text{s.t.:} \quad x_1 + x_2 \leq 5$$
$$3x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \leq 10$$
$$x_1, x_2 \geq 0.$$

Show how to find the optimal soln. of the problem if

i) the first component of the original requirement vector is increased by one unit and the third component is decreased by one unit;

ii) the 2nd component of the original requirement vector is decreased by two units.

Soln i)

$$\text{Max } z = 4x_1 + 3x_2 + 0.x_3 + 0.x_4 + 0.x_5.$$

s.t.

$$x_1 + x_2 + x_3 = 5$$

$$3x_1 + x_2 + x_4 = 7$$

$$x_1 + 2x_2 + x_5 = 10$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

			c_j	4	3	0	0	0	
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	min ratio
0	a_3	x_3	5	1	1	1	0	0	5/1
0	a_4	x_4	7	$\boxed{3}$	1	0	1	0	7/3 →
0	a_5	x_5	10	1	2	0	0	1	10/1
			$z_j - c_j$	−4 ↑	−3	0	0 ↓	0	
0	a_3	x_3	8/3	0	$\boxed{2/3}$	1	−1/3	0	8/2 = 4 →
4	a_1	x_1	7/3	1	1/3	0	1/3	0	7
0	a_5	x_5	23/3	0	5/3	0	−1/3	1	23/5 = 4.6
			$z_j - c_j$	0	−5/3 ↑	0 ↓	4/3	0	
3	a_2	x_2	4	0	1	3/2	−1/2	0	
4	a_1	x_1	1	1	0	−1/2	1/2	0	
0	a_5	x_5	1	0	0	−5/2	1/2	1	
			$z_j - c_j$	0	0	5/2	1/2	0	

$5 - \frac{7}{3}$ $\quad 10 - \frac{7}{3}$ $\quad 1 - \frac{1}{3}$ $\quad 2 - \frac{1}{3}$ $\quad \frac{4}{3} - 3$ $\quad \frac{7}{3} - \frac{8}{3}\cdot\frac{1}{2}$ $\quad \frac{23}{3} - \frac{8}{3}\cdot\frac{5}{2}$

basis $B = (a_2, a_1, a_5)$

$$\Rightarrow B^{-1} = \begin{pmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ -5/2 & 1/2 & 1 \end{pmatrix}$$

$-\frac{3}{2} + \frac{4}{2}$ $\quad \frac{9}{2} - \frac{4}{2}$ $\quad -\frac{1}{3} + \frac{5}{3}\cdot\frac{3}{2}$ $\quad \frac{1}{3} + \frac{1}{3}\cdot\frac{3}{2}$

Now the requirement vector b becomes b+d when $d = (1, 0, -1)$

$$\therefore \bar{x}_B = x_B^* + B^{-1}d = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{3}{2} - 0 \\ -\frac{1}{2} - 0 \\ -\frac{5}{2} - 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3/2 \\ -1/2 \\ -7/2 \end{pmatrix} = \begin{pmatrix} 11/2 \\ 1/2 \\ -5/2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ x_5 \end{pmatrix}$$

⇒ The new optimal soln. is not feasible as $x_1 = 11/2, x_2 = 1/2, x_3 = -5/2$ (slack variable)

..., we ~~need~~ take the help of the dual simplex method to find the new feasible soln.

So a_5 leaves the basis as $x_5 = -5/2 < 0$.

To determine the entering vector, find Max ratio

$$\text{Max}\left\{ \frac{z_j - c_j}{y_{5j}} \,\middle|\, y_{5j} < 0 \right\}$$

C_B	B	x_B	c_j / b	4 / a_1	3 / a_2	0 / a_3	0 / a_4	0 / a_5	Max ratio
3	a_2	x_2	11/2	0	1	3/2	-1/2	0	
4	a_1	x_1	1/2	1	0	-1/2	1/2	0	$\text{Max}\left\{\frac{5/2}{-5/2}\right\} = -1$
0	a_5	x_5	-5/2	0	0	$\boxed{-5/2}$	1/2	1	
			$z_j - c_j$	0	0	5/2	1/2	0	a_3 enters
3	a_2	x_2	4	0	1	0	-1/5	3/5	
4	a_1	x_1	1	1	0	0	2/5	-1/5	
0	a_3	x_3	1	0	0	1	-1/5	-2/5	
			$z_j - c_j$	0	0	0	1	1	

$\frac{11}{2} - \frac{8}{2}\cdot\frac{3}{2}\cdot\frac{2}{5}$

$\frac{8}{2} = 4$

$\frac{1}{2} + \frac{5}{2}\cdot\frac{1}{2}\cdot\frac{2}{5}$

optimality is reached.

The new optimal Soln. is

$x_1 = 1, x_2 = 4, x_3 = 1; Z_{max} = 16$

~~4+12~~

$\frac{9}{5} - \frac{4}{5}$ $\frac{8}{2}$

$-\frac{3}{5} + \frac{8}{5}$ $-\frac{1}{2} + \frac{1}{2}\cdot\frac{3}{2}$

$0 + \frac{3}{2}\cdot\frac{2}{5}$ $\frac{-2}{2\cdot 5}$

$0 + \frac{1}{2}\cdot\frac{2}{5}$ $\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2}$

(ii) Here $d = [0, -2, 0]$

$$\therefore x_B = x_B^* + B^{-1}d = \begin{pmatrix}4\\1\\1\end{pmatrix} + \begin{pmatrix}2/2\\-2/2\\-2/2\end{pmatrix} = \begin{pmatrix}4\\1\\1\end{pmatrix} + \begin{pmatrix}1\\-1\\-1\end{pmatrix} = \begin{pmatrix}5\\0\\0\end{pmatrix} = \begin{pmatrix}x_2\\x_1\\x_5\end{pmatrix}$$

Hence, the optimal Soln. is

$x_1 = 0, x_2 = 5, x_5 = 0; Z_{max} = 15$

Note. A change in only one component of the requirement vector may change all the components of the optimal Soln.

... to be [3, 2].

Example: The optimal simplex table for a given LPP is given below.

		c_j		4	3	4	6	0	0	0
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7
6	a_4	x_4	$18/13$	0	$14/13$	0	1	$4/13$	$-5/13$	$2/13$
4	a_3	x_3	$20/13$	0	$-23/13$	1	0	$-1/13$	$11/13$	$-7/13$
4	a_1	x_1	$28/13$	1	$16/13$	0	0	$-1/13$	$-2/13$	$6/13$
		$z_j - c_j$		0	$17/13$	0	0	$16/13$	$6/13$	$8/13$

The given problem is a maximization problem with all the constraints '$\leq$' type.

Determine the separate ranges of discrete changes in a_{12}, a_{22} and a_{32} consistent with the optimal soln. of the given problem.

Soln.

$B \to$ optimal basis, $B = [a_4, a_3, a_1]$

$$\therefore B^{-1} = \begin{pmatrix} 4/13 & -5/13 & 2/13 \\ -1/13 & 11/13 & -7/13 \\ -1/13 & -2/13 & 6/13 \end{pmatrix}$$

[$\because$ x_5, x_6, x_7 are slack variables, so initial basis was $[a_5, a_6, a_7]$]

$$= [\delta_1, \delta_2, \delta_3] \text{ (say)}.$$

$$C_B\delta_1 = (6,4,4)\begin{pmatrix} 4/13 \\ -1/13 \\ -1/13 \end{pmatrix} = \frac{24}{13} - \frac{4}{13} - \frac{4}{13} = \frac{16}{13}$$

$$C_B\delta_2 = -\frac{30}{13} + \frac{44}{13} - \frac{8}{13} = 6/13$$

$$C_B\delta_3 = \frac{12}{13} - \frac{28}{13} + \frac{24}{13} = 8/13.$$

a_{12}, a_{22}, a_{32} are the elements of the vector a_2 which is non-basic as evident from the optimal table.

- Thus change in a_2 will only violate the conditions of optimality.

$$a_k = (a_{1k}, a_{2k}, \cdots)$$
$$\delta_{1k} \quad \delta_{2k} \cdots$$

$$z_2 - c_2 = 17/13.$$

$$\underset{l}{\text{Min}} \frac{-(z_2-c_2)}{c_B\delta_l < 0} \geq \delta_{l2} \geq \text{Max}_{l} \frac{-(z_2-c_2)}{(c_B\delta_l) > 0}$$

bound does not exist for non-existent denominator.

$$\text{Change in } a_{12} = \delta_{12} \geq \frac{-17/13}{16/13} = -17/16$$

$$\text{Change in } a_{22} = \delta_{22} \geq -\frac{17/13}{6/13} = -17/6$$

$$\text{Change in } a_{32} = \delta_{32} \geq \frac{-17/13}{8/13} = -17/8.$$

Example: The optimum simplex table for a maximization problem (with all constraints '≤' type) is

C_B	B	b	y_1	y_2	y_3	y_4	y_5
12	a_2	8/5	0	1	1/5	2/5	-1/5
5	a_1	9/5	1	0	7/5	1/5	2/5
			0	0	17/5	29/5	M-2/5

$12+5-4=0$

$5-4=0$
$4=5$

where x_4 is the slack variable and x_5 is the artificial variable.

Let a new variable $x_6 \geq 0$ be introduced in the problem with a cost 18 assigned to it in the objective fn. Suppose that the new vector corresponding to the variable x_6 be $[3,2]$.

Discuss the effect of this addition of a variable on the optimality of the optimal soln. of the given problem.

Soln. x_4 and x_5 being the slack and artificial variables, $[a_4, a_5]$ constituted the initial basis.

If B is the optimal basis $[a_2, a_1]$ then $B^{-1} = \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix}$

Let a_6 be the vector corresponding to the variable x_6.

$$\therefore y_6 = B^{-1} a_6 = \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{6}{5} - \frac{2}{5} \\ \frac{3}{5} + \frac{4}{5} \end{pmatrix} = \begin{pmatrix} 4/5 \\ 7/5 \end{pmatrix}$$

The cost vector $C_B = [12, 5]$ and $C_6 = 18$

$$\therefore z_6 - c_6 = C_B y_6 - C_6 = (12, 5) \begin{pmatrix} 4/5 \\ 7/5 \end{pmatrix} - 18$$

$$= \frac{48}{5} + \frac{35}{5} - 18 = \frac{83}{5} - 18$$

$$= -\frac{7}{5} < 0$$

Thus the optimality condition is violated and the new simplex table is

C_B	B	x_B	b \ c_j	a_1 (5)	a_2 (12)	a_3 (6)	a_4 (0)	a_5 (−M)	a_6 (18)	Min ratio
12	a_2	x_2	8/5	0	1	1/5	4/5	−1/5	4/5	8/4 = 2
5	a_1	x_1	9/5	1	0	7/5	1/5	2/5	$\boxed{7/5}$	9/7 = 1.1
			$z_j - c_j$	0	0	17/5	29/5	M − 2/5	−7/5	
12	a_2	x_2	4/7	−4/7	1	−3/5	2/7	−3/7	0	
18	a_6	x_6	9/7	5/7	0	1	1/7	2/7	1	
			$z_j - c_j$	1	0	24/5	6	M	0	

$-\frac{48}{7} + \frac{90}{7} = \frac{42}{7}$

$\frac{12}{5} + \frac{35}{5} = \frac{47}{5}$

Since all $z_j - c_j \geq 0$, the optimal ~~condition~~ $= \frac{306}{7}$

Sol^n is obtained.

Hence, the new optimal b.f.s. is

$x_1 = 0,\ x_2 = 4/7,\ x_3 = 0.$

~~$Z_{max} = 12 \times \frac{4}{7} + \frac{18 \times 9}{7} = \frac{48}{7}$~~

$= \frac{210}{7} = 30.$

$Z_{max} = 5x_1 + 12x_2 + 6x_3$

$= 12 \cdot \frac{4}{7} = \frac{48}{7}.$

Previous Sol^n was

$x_1 = 9/5,\ x_2 = 8/5,\ x_3 = 0$

$Z_{max} = 5 \cdot \frac{9}{5} + 12 \cdot \frac{8}{5}$

$= \frac{141}{5}.$

$-\frac{36}{5} + 18 - 6$

$\frac{16}{25} + \frac{9}{25} - \frac{29}{5}$ $\frac{60-36}{5}$

$1 - \frac{29}{5}$

$-\frac{12}{5} + \frac{10}{5} - M + \frac{2}{5} = \frac{-5}{5}$

$\frac{29}{5}$

$\frac{24}{7} + \frac{18}{35} + \frac{35}{5} - c_j$

$= \frac{17}{5}$

$\frac{426}{7}$

$\frac{47}{5} - \frac{17}{5} = 9$

$\frac{306}{7} = c_3$

48
162
210

$-\frac{36}{7} + \frac{36}{7}$

$\frac{24}{5} + \frac{5}{5} - \frac{29}{5}$

max. $z = 5x_1 + 12x_2 + 6x_3$.

$\frac{24}{5} + \frac{5}{5} - \frac{29}{5}$

45
96
141

Theorem The set of all feasible sol^ns of a LPP is a convex set.

Proof Consider the LPP in which constraints are

$$Ax = b, \; x \geq 0.$$

Let X = the set of all feasible sol^n. of this problem.

Let $x_1, x_2 \in X$.

Then $Ax_1 = b$, $Ax_2 = b$, $x_1, x_2 \geq 0$.

Let $x_3 = \lambda x_1 + (1-\lambda) x_2, \; 0 \leq \lambda \leq 1.$

Then
$$\begin{aligned} Ax_3 &= A\left[\lambda x_1 + (1-\lambda) x_2\right] \\ &= \lambda A x_1 + (1-\lambda) A x_2 \\ &= \lambda b + (1-\lambda) b \\ &= b. \end{aligned}$$

Also $x_3 = \lambda x_1 + (1-\lambda) x_2 \geq 0$ as $0 \leq \lambda \leq 1$ & $x_1 \geq 0, x_2 \geq 0$.

$\Rightarrow x_3 \in X$. as $\lambda x_1 \geq 0$, & $(1-\lambda) x_2 \geq$

as $\lambda, (1-\lambda) \geq$

Hence, X is a convex set.

Corollary: If a LPP has two feasible soln., it has an infinite no. of feasible soln. as any convex combination of the two feasible soln. is a feasible soln.

Note: This convex set is bdd. below as $x \geq 0$.
Again, all the pts. of the hyperplane $Ax = b$ form a closed set and the set of all pts. x for which $x \geq 0$ is also closed.
Thus the convex set X being the intersection of these two closed set is also closed.

Theorem: The objective fn. of a LPP assumes its ~~optimal soln.~~ optimum value at an extreme pt. of the convex set of feasible solns.

$$\boxed{\begin{aligned} \text{Max } Z &= cx \\ Ax &= b \\ x &\geq 0 \end{aligned}}$$

proof: We consider a maximisation problem.
A minimization problem will have similar proof.
Let the # of extreme pts. be finite and they are

$$x_1, x_2, \ldots, x_k.$$

Suppose the set of all feasible solns. is strictly bdd.
Let x_m be the optimal soln. of the LPP.

- if x_m is an extreme pt. of the convex set of feasible solns., then the theorem is proved.
- Suppose x_m is not an extreme pt. and it gives the optimal value Z_m of the objective fn. Z.

$$\therefore Z_m = c x_m.$$

Since x_m is not an extreme pt., it can be expressed as a convex combination of the extreme pts. $x_1, x_2, \ldots, x_k$

i.e. $x_m = \sum_{i=1}^{k} \lambda_i x_i$, $\lambda_i \geq 0$, $i = 1, 2, \ldots, k$, $\sum_{i=1}^{k} \lambda_i = 1$

$$\therefore z_m = c x_m = \sum_{i=1}^{k} c \lambda_i x_i = \sum_{i=1}^{k} \lambda_i (c x_i)$$

let $c x_p = \max_i \{c x_i\}$ where x_p is an extreme pt.

$$\therefore z_m = \sum_{i=1}^{k} \lambda_i (c x_i) \leq \sum_{i=1}^{k} \lambda_i (c x_p) = c x_p \sum_{i=1}^{k} \lambda_i = c x_p$$
$$= z_p.$$

$\therefore z_p \geq z_m$

But z_m is the maximum value of the objective fun. z.

$\therefore z_p = z_m = c x_p$

Hence ~~option~~ maximum value of the objective fun. z occurs at some extreme pt. x_p

Note: Since the set of all convex combinations of a finite no. of pts. is the convex polyhedron generated by these pts., we can restate the theorem as:

Theorem: The objective fun. has its optimal value at an extreme pt. of the convex polyhedron generated by the set of feasible soln. of the LPP.

Theorem A basic feasible soln. to a LPP corresponds to an extreme pt. of the convex set of feasible soln.

Proof

Let there be m linearly independent columns of $A = (a_{ij})_{m\times n}$, namely, $a_1, a_2, \ldots a_m$ s.t.

$$x_1 a_1 + x_2 a_2 + \cdots + x_m a_m = b$$

and suppose x, an n-component vector, given by

$$x = (x_1, x_2, \ldots, x_m, \underbrace{0, 0, \ldots 0}_{n-m})$$

be a basic feasible soln. of

$$Ax = b, \quad x \geq 0 \quad \text{--- (1)}$$

Claim x is an extreme pt.

proof of the claim

- Suppose x is not an extreme pt. of the convex set X of the feasible solns. of (1).
- As x ~~being~~ is a feasible soln, it can be expressed as a convex combination of two other pts., $u, v \in X$ s.t. $x = \lambda u + (1-\lambda) v$, $0 < \lambda < 1$ --- (2)
- All the components of x are non-negative.

Since $0 < \lambda < 1$, and $(n-m)$ components of x are zero, it can be stated from (2) that $(n-m)$ components of u and v must also be zero.

Hence, $u = (u_1, u_2, \ldots, u_m, 0, \ldots, 0)$
$v = (v_1, v_2, \ldots, v_m, 0, \ldots, 0)$

Since u, v are feasible sol^n of $Ax = b, x \geq 0$, we have

$$a_1 u_1 + a_2 u_2 + \cdots + a_m u_m = b$$
$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = b.$$

Also $a_1 x_1 + a_2 x_2 + \cdots + a_m x_m = b$.

$$\Rightarrow a_1(x_1 - u_1) + a_2(x_2 - u_2) + \cdots + a_m(x_m - u_m) = 0$$
$$a_1(x_1 - v_1) + a_2(x_2 - v_2) + \cdots + a_m(x_m - v_m) = 0.$$

As $a_1, a_2, \ldots, a_m$ are linearly independent, we have

$$\left.\begin{array}{l} x_j - u_j = 0 \\ u_j - v_j = 0 \end{array}\right\} \quad j = 1, 2, \ldots, m$$

$$\Rightarrow x_j = u_j = v_j \quad \forall j = 1, 2, \ldots, m.$$

Thus x cannot be expressed as a convex combination of two other pts. u and $v \in X$.

Hence, x is an extreme pt. of X, the ~~set~~ convex set of feasible sol^n. of $Ax = b, x \geq 0$

Note. We have proved the theorem for non-degenerate basic feasible sol^n. In case of degenerate basic feasible sol^n, the # of non-zero variable will be $< m$ and that will create ~~not much~~ not much difficulty for the proof.

Theorem Every extreme pt. of the convex set of all feasible sol^{ns}. of the system

$$Ax = b, x \geq 0$$

corresponds to a basic feasible sol^n.

Proof.

Let $x = (x_1, x_2, \dots, x_n)$ be an extreme pt. of X, the convex set of feasible sol^{ns}. of the given system of equations.

Claim x is a basic feasible sol^n.

i.e. to prove that the columns of A associated with the non-zero x_j are linearly independent with at most m of x_j's being positive.

proof of claim

Without any loss of generality, we assume that the 1st. m of the x_j's are non-zero and positive, so we can write

$$x_1 a_1 + x_2 a_2 + \dots + x_m a_m = b \quad \text{--- (1)}$$

where a_j is the j-th column of A associated with x_j.

If the vectors a_j are not linearly independent, then they will be linearly dependent & we have

$$\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_m a_m = 0, \quad \text{--- (2)}$$

with at least one $\lambda_j \neq 0$, $j = 1, 2, \dots, m$.

Let $\beta > 0$ be a scalar.

Then (1), (4) $\Rightarrow$

$$\sum_{j=1}^{m} (x_j + \lambda_j \beta) a_j = b$$

$$\sum_{j=1}^{m} (x_j - \lambda_j \beta) a_j = b$$

$\Rightarrow \quad u = (x_1 + \beta\lambda_1, x_2 + \beta\lambda_2, \dots, x_m + \beta\lambda_m, 0, \dots, 0)$

and $v = (x_1 - \beta\lambda_1, x_2 - \beta\lambda_2, \dots, x_m - \beta\lambda_m, 0, \dots, 0)$

may both feasible solns. of

$$Ax = b, x \geq 0.$$

provided the first m components of u and v are positive and $u, v \in X$.

This happends if β is so chosen that

$$0 < \beta < \min_{j \in S} \frac{x_j}{|\lambda_j|}, S = \{j \mid \lambda_j \neq 0\}$$

With this choice of β, u, v are feasible solns. & we can write

$$x = \frac{1}{2}u + \frac{1}{2}v.$$

$u + v = (2x_1, 2x_2, \dots, 2x_m, 0, 0, \dots, 0)$

$\frac{1}{2}u + \frac{1}{2}v = x$

But this is a contradiction to the assumption that x is an extreme pt.

This contradiction is due to the assumption that the vectors $a_1, a_2, \dots, a_m$ are linearly dependent.

Hence, $a_1, a_2, \dots, a_m$ must be linearly independent.

Now every set of $(p+1)$ vectors in a p-dimensional space is linearly dependent set.

Hence, by the above result and by the fact that a_j vectors are each m-component, we cannot have more than m positive x_j.

Thus every extreme pt. corresponds to a basic feasible soln.

Theorem If the objective fun. assumes its optimal value at more than one extreme pt, then every convex combination of these extreme pts. also gives the optimal value of the objective fun.

proof

Let the objective fun. Cx assume its optimal value z^* at the extreme pts. $x_1, x_2, \ldots, x_k$ of the convex set of feasible solns.

Then $$z^* = Cx_1 = Cx_2 = \cdots = Cx_k$$

If x_0 is the convex combination of these extreme pts., we have

$$x_0 = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k$$

where $\sum_{i=1}^{k} \lambda_i = 1, \quad \lambda_i \geq 0, \quad i = 1, 2, \ldots, k.$

$$\therefore Cx_0 = C(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k)$$

$$= \lambda_1 Cx_1 + \lambda_2 Cx_2 + \cdots + \lambda_k Cx_k$$

$$= \lambda_1 z^* + \lambda_2 z^* + \cdots + \lambda_k z^*$$

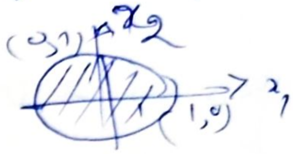
$$= z^* \sum_{i=1}^{k} \lambda_i = z^* \quad \text{as} \quad \sum_{i=1}^{k} \lambda_i = 1.$$

This proves the theorem.

Note

- This theorem shows that either the optimal soln. is unique or infinite in number.

Point sets → sets whose ~~are~~ elements are pts. or vectors in E^n.



e.g. $X = \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\}$

represents a pt. set in E^2 lying inside a circle of unit radius with centre at the origin

2D

for fixed c_1, c_2, z

linear equ^n. in x_1, x_2

$$c_1 x_1 + c_2 x_2 = z$$

represents a straight line.

A line is a set of pts. in E^2 satisfying $c_1 x_1 + c_2 x_2 = z$

3D

for fixed c_1, c_2, c_3, z

linear equ^n. in x_1, x_2, x_3

$$c_1 x_1 + c_2 x_2 + c_3 x_3 = z$$

represents a plane.

A plane is a set of pts. in E^3 satisfying $c_1 x_1 + c_2 x_2 + c_3 x_3 = z$.

Compact form

$cx = z$, $c = (c_1, c_2)$ or (c_1, c_2, c_3)

$x = (x_1, x_2)$ or (x_1, x_2, x_3)

in 2D or 3D respectively.

Generalization

A set of pts. in n-dimentional space whose co-ordinates satisfy the linear equ^n. of the form

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = z$$

is called a hyperplane for fixed values of z and c_i, $i = 1, 2, \ldots, n$.

In short

$$\boxed{cx = z}$$ when $c = (c_1, c_2, \ldots, c_n)$, $x = (x_1, x_2, \ldots, x_n)$, not all $c_i = 0$

$\swarrow$

hyperplane.

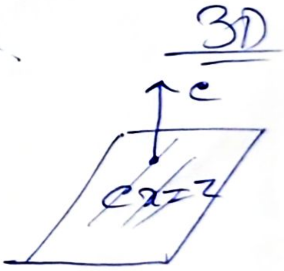
- For different values of z, we get different hyperplanes.

- $H = \{x \mid cx = z\}$, $c \neq 0$, z real scalar, $x \in E^n$
is a hyperplane whose equ^n is $cx = z$.

- $z = 0 \Rightarrow cx = 0 \Rightarrow$ the hyperplane passes through the origin.

$\Rightarrow c \perp x \ \forall x \in H$.

- c is called normal to the hyperplane.



- $x_1, x_2 \in H \Rightarrow cx_1 = z = cx_2$
$\Rightarrow c(x_1 - x_2) = 0$
$\Rightarrow c \perp (x_1 - x_2)$.

- $\dfrac{\pm c}{|c|} \rightarrow$ two unit vectors normal to H.

Parallel hyperplanes

- ~~Two~~ Hyperplanes having the same unit normals
- are said to be parallel. By increasing or decreasing the value of z, we can get hyperplanes parallel to $cx = z$.

The hyperplane $cx = z$ in E^n divides E^n into 3 mutually disjoint sets:

$X_1 = \{x \mid cx < z\}$ open half space

$X_2 = \{x \mid cx = z\}$

$X_3 = \{x \mid cx > z\}$ open half space

$X_4 = \{x \mid cx \le z\}$, $X_5 = \{x \mid cx \ge z\}$

are closed half spaces.

e.g. $x = (1, 2, 3, 4)$ lies in the open half-space $cx > z$ generated by the hyperplane

$2x_1 + 3x_2 + 4x_3 + 5x_4 = 7$

as $2 + 6 + 12 + 20 = 40 > 7$.

But $x = (1, 2, 3, -4)$ lies in the open half-space $cx < z$ as

$2 + 6 + 12 - 20 = 0 < 7$.

In LPP

optimize $z = cx$

s.t. $Ax (\le = \ge) b$, $x \ge 0$

the objective fun. as well as the constraints with equality sign represents hyperplanes ($z = cx$, $Ax = b$).

The constraints with $\le$ or $\ge$ are the half-spaces produced by the hyperplane with the sign of equality only.

A line in the n-dimentional Euclidean space passing through pts. x_1, x_2 $(x_1 \neq x_2)$ is defined to be the set of pts.

$$X = \{x \mid x = \lambda x_1 + (1-\lambda)x_2, \ \}$$

λ real.

If the restriction $0 \le \lambda \le 1$ is imposed on λ, then the pt. x on the above line is constrained to lie within the line segment joining the pts. x_1, x_2.

- $X = \{x \mid x = \lambda x_1 + (1-\lambda)x_2, \ 0 \le \lambda \le 1\}$ represents line segment joining the pts. x_1, x_2.

Hyper sphere $\to$ the set of pts.

$$X = \{x \mid |x-a| = \varepsilon > 0\}$$

is a hypersphere in E^n with centre at a and radius $= \varepsilon$.

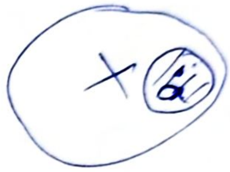
- $n=2$, it is a circle in E^2
- $n=3$, it is a sphere in E^3.

ε-nbd about a pt a, $\varepsilon > 0$. pts. inside

$X = \{x \mid |x-a| < \varepsilon\} \to$ hypersphere with centre at a and radius $\varepsilon > 0$.

Interior pt. of a set X

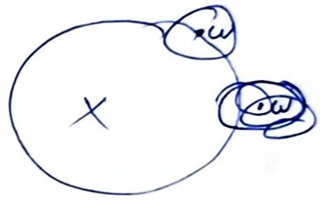
- A pt. a is said to be an interior pt. of a set X if an ε-nbd. about the pt. a contains only pts. of the set X.



- An interior pt. of X must be an element of X.

Boundary pt. of a set X

A pt. ω is a boundary pt. of a set X if every ε-nbd. about ω contains pts. of the set and also pts. not of the set.



- A boundary pt. may or may not belong to the set, but an interior pt. must belong to the set.

closed set

→ Contains all the boundary pts.

$$\text{e.g. } X = \{(x_1, x_2) \mid x_1^2 + x_2^2 \le 4\}$$

open set

→ contains only the interior pts.

$$\text{e.g. } X = \{(x_1, x_2) \mid x_1^2 + x_2^2 < 4\}$$

Strictly bdd. set

A set X is said to be strictly bdd. if $\exists$ a +ve number r s.t. for every $x \in X$, $|x| < r$.

if $\forall$ $x = (x_1, x_2, \dots, x_n) \in X$, each component has a lower bound, the set X is bounded below.

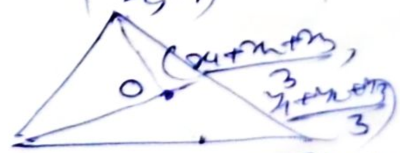
Convex Combination and convex set

- If a pt. x is expressed as
$$x = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_p x_p,\ \lambda_i \geq 0$$
when x_i's are a finite no. of pts. in E^n for all $i = 1, 2, \dots, p$ and
$$\sum_{i=1}^{p} \lambda_i = 1$$
then x is said to be a convex combination of the pts. $x_1, x_2, \dots, x_p$.

- If we consider 2 pts. x_1 and x_2, their convex combination is a pt. 'x' given by
$$x = \lambda x_1 + (1-\lambda) x_2, \quad 0 \leq \lambda \leq 1.$$
or
$$x = \lambda_1 x_1 + \lambda_2 x_2 \text{ with } \lambda_1 + \lambda_2 = 1 \ \& \ \lambda_1, \lambda_2 \geq 0.$$

- Thus the line segment joining 2 pts. x_1, x_2 is the set of all possible convex combinations of these 2 pts.

- In a triangle, the centroid is the convex combination of the vertices.



- A set X is said to be convex set if for any 2 pts. x_1, x_2 in the set, the line segment joining them 2 pts. also in the set

$\Rightarrow$ if X convex then every pt. $x = \lambda x_1 + (1-\lambda) x_2$, $0 \leq \lambda \leq 1$ when $x_1, x_2 \in X$, must also be in the set X

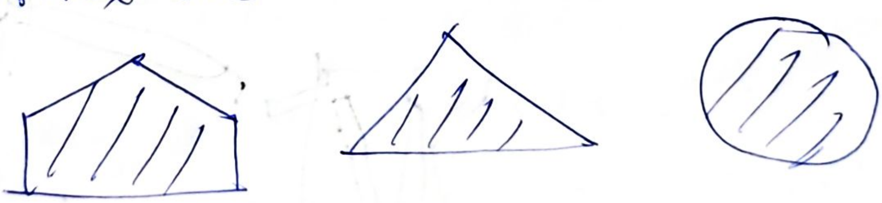
- A circle and a triangle in E^2 are convex sets
- A sphere & a cube in E^3 are convex sets.
- (Convention) A set containing only one pt. is a convex set.
- The set given by

$$X = \{ (x_1, x_2) \mid x_1^2 + x_2^2 \le 4 \}$$

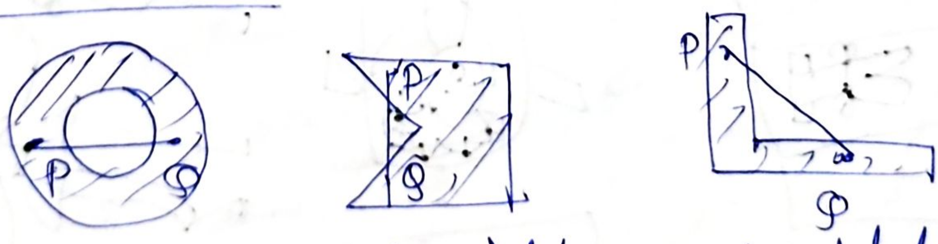
is convex

↓ a circle and its interior constitutes a convex set.

- But the set of pts. that form the boundary of a circle is not a convex set.
- A convex set according to the defⁿ cannot have holes or ~~re-entrance~~ re-entrants in it.
- 2D convex sets.



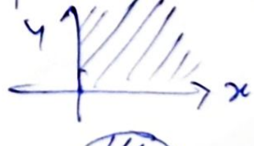
Not convex sets



- A convex set may be bdd. or unbdd.

e.g. The 1st. quadrant of the 2D plane is unbdd.

The circle is bdd.



A few important results

a) A hyperplane is a convex set.

proof Let $X = \{x \mid cx = z\}$ be a hyperplane.

& $x_1, x_2 \in X$.

$\therefore cx_1 = z = cx_2$.

Let $x_3 = \lambda x_1 + (1-\lambda)x_2$, $0 \leq \lambda \leq 1$

$\Rightarrow cx_3 = \lambda cx_1 + (1-\lambda) cx_2 = \lambda z + (1-\lambda) z = z$

$\Rightarrow x_3 \in X$

Thus $x_3 \in X$ being a convex combination of $x_1, x_2 \in X$

Hence, X is a convex set.

Note- In the same way, we can show that a half-space (open or closed) is a convex set.

$X = \{x \mid cx \leq z\}$ or $cx \geq z$ or $cx < z$ or $cx > z$

b) Intersection of 2 convex sets is also a convex set.

proof: Let X_1, X_2 be 2 convex sets & $X = X_1 \cap X_2$

Let $x_1, x_2 \in X \Rightarrow x_1, x_2 \in X_1$ and $x_1, x_2 \in X_2$.

$$\left.\begin{array}{l} \Rightarrow \lambda x_1 + (1-\lambda) x_2 \in X_1 \\ \text{and} \\ \lambda x_1 + (1-\lambda) x_2 \in X_2 \end{array}\right\} \text{for } 0 \leq \lambda \leq 1.$$

$\Rightarrow \lambda x_1 + (1-\lambda) x_2 \in X_1 \cap X_2 = X$.

Thus if $x_1, x_2 \in X$, then their convex combination also $\in X = X_1 \cap X_2$

Hence, $X = X_1 \cap X_2$ is a convex set.

Ex. Show that $X = \{x \mid |x| \le 2\}$ is a convex set.

Soln. Let $x_1, x_2 \in X$

Then $|x_1| \le 2$, $|x_2| \le 2$.

Let us consider the convex combination of x_1 and x_2 —

$$\lambda x_1 + (1-\lambda) x_2, \quad 0 \le \lambda \le 1$$

Then $|\lambda x_1 + (1-\lambda) x_2| \le \lambda |x_1| + (1-\lambda)|x_2| \le \lambda \cdot 2 + (1-\lambda) \cdot 2$
$= 2$.

Thus the pt. determined by the convex combination of $x_1, x_2 \in X$ is also in X.

$\Rightarrow$ The given set X is convex

Ex. Show that the set $X = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 16\}$ is not a convex set.

Soln. Here X represents the perimeter of the circle $x_1^2 + x_2^2 = 16$ and not the pts. inside.

Hence it is not a convex set.

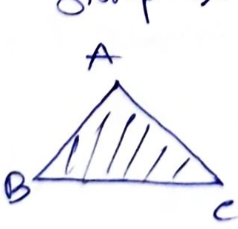
Ex. Give examples of

i) Convex hulls in E^2 and E^3.

ii) Convex polyhedron in $E2$

iii) Simplex in zero and one dimension.

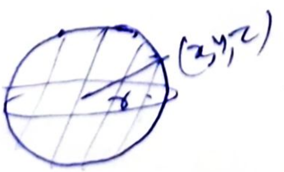
Soln. i)



$X = \{A, B, C\}$.

$\hookrightarrow$ set of 3 vertices of a triangle in E^2

Then the whole triangle is its convex hull $C(X)$.

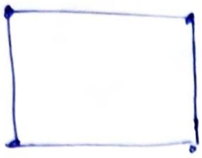


$X = \{(x, y, z) \mid x^2 + y^2 + z^2 = r^2\}$

$\hookrightarrow$ set of all pts. on the surface of a sphere in $E3$

Then the whole sphere is its convex hull $C(X)$

ii) A square is the convex ~~polygon~~ polyhedron generated by the vertices in E^2.

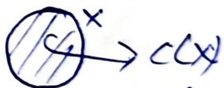


iii) A ~~pt~~ point is a simplex in zero dimension
A line segment is a simplex in ~~zero~~ one dimension.

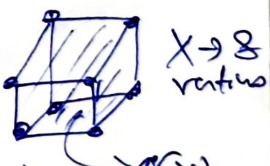
Convex hull and convex polyhedron

- Let X be a given set of pts.
The set of all convex combinations of the set of pts. from X constitutes a convex hull of the given set X and is denoted by $C(X)$.

- Thus if X is the set of pts. lying on the boundary of a circle, then the circle with its interior is its convex hull in E^2 (E^2)



- The whole cube is a convex hull of the set of all pts. consisting of only the eight vertices of the cube in E^3.



- $C(X)$ is the smallest convex set containing X.
- The set of all convex combinations of a finite no. of pts. is called the convex polyhedron generated by these pts.

$X \to$ finite.
$C(X) \to$ convex polyhedron.

- The convex hull of the eight vertices of a cube is a convex polyhedron, but the circle is not convex polyhedron as it has an infinite no. of extreme pts.

In the defn. of a convex set, it has been shown that the set of all convex combinations of a finite no. of pts. is a convex set.

Hence, the convex polyhedron is a convex set.

- An n dimensional convex polyhedron having exactly $(n+1)$ vertices is called a simplex

e.g A tetrahedron is a simplex in 3D.

— Generalising this result, we can show that the intersection of a finite no. of convex sets is also a convex set.

<u>Note</u>- Union $(X_1 \cup X_2)$ or the difference $(X_1 \sim X_2)$ ~~may not~~ of 2 convex sets may not be convex.

<u>Note</u>- X_1, X_2 closed set $\Rightarrow X_1 \cap X_2$ also closed.
Can be generalized to a finite no. of closed sets.

<u>Note</u>
Hyperplanes and half-spaces are convex sets (already proved).
In a LPP, the set of feasible sol^ns is given by

$$Ax\ (< = >)\ b,\ x \geq 0$$

which will be nothing but the intersection of a finite no. of hyperplanes or half-spaces or both as given by the constraints.

Now the intersection of a finite no. of hyperplanes or closed half-spaces or both is a closed convex set.
Hence, the set of feasible sol^ns to a linear programming problem (if exists) is also a closed convex set.
(Will be proved later in the form of Theorem).

⇒ The set of all convex combinations of a finite no. of pts. is a convex set.

<u>Proof</u> Let X be the set of the convex combinations of the finite no. of pts. $x_1, x_2, \dots, x_n$

s.t. $X = \left\{ x \mid x = \sum_{i=1}^{n} \mu_i x_i \ \forall \mu_i \geq 0 \text{ and } \sum_{i=1}^{n} \mu_i = 1 \right\}$

<u>Claim</u> X is a convex set.

<u>proof of the claim</u> Let $u, v \in X$ with

$u = \sum_{i=1}^{n} \mu_i' x_i$ and $v = \sum_{i=1}^{n} \mu_i'' x_i$, $\mu_i', \mu_i'' \geq 0$, $\sum \mu_i' = \sum \mu_i'' = 1$

Let $0 \le \lambda \le 1$.

Then $\lambda u + (1-\lambda) v = \sum_{i=1}^{n} \left[\lambda \mu_i' + (1-\lambda)\mu_i''\right] x_i$

where $\lambda\mu_i' + (1-\lambda)\mu_i'' \geq 0$ as $\mu_i', \mu_i'' \geq 0$ and $0 \le \lambda \le 1$.

~~and~~

~~$\lambda u + (1-\lambda) v = \lambda \sum_{i=1}^{n} \mu_i' + (1-\lambda) \sum_{i=1}^{n} \mu_i''$~~

~~$= \lambda + (1-\lambda) = 1.$~~

and $\sum_{i=1}^{n} \left[\lambda \mu_i' + (1-\lambda)\mu_i''\right] = \lambda \sum_{i=1}^{n} \mu_i' + (1-\lambda) \sum_{i=1}^{n} \mu_i''$

$= \lambda + (1-\lambda) = 1$.

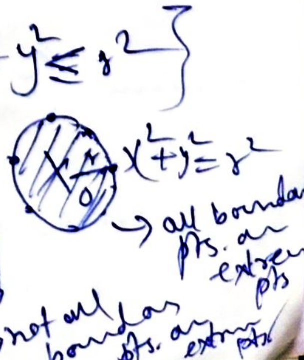
$\Rightarrow \lambda u + (1-\lambda) v \in X$.

Extreme pt

- A pt. x of a convex set X is an extreme pt. if it cannot be expressed as a convex combination of two other pts. in X.
- Geometrically, a pt. x in a convex set is said to be an extreme pt. if it does not lie on the line segment joining any two pts. other than x in the set.
- e.g. Any pt. on the circumference of a circle is an extreme pt. of the convex set of pts. within & on the circle.

$X = \{(x,y) \mid x^2 + y^2 \le r^2\}$

- e.g. The vertices of a polygon are the extreme pts. of the convex set of pts. within and on the boundary of the polygon.



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The vertices are the only extreme pts. ~~If it cannot be expressed as a convex combination~~ of the convex set formed by a triangle and its interior.

$$X : \{(x,y) \mid (x,y) \in A\}$$

Def^n. Analytically, a pt. x is an extreme pt. of a convex set iff there do not exist pts. x_1 and x_2 ($x_1 \neq x_2$) in the set s.t.

$$x = \lambda x_1 + (1-\lambda)x_2, \quad \cancel{0 \le \lambda \le 1} \quad 0 < \lambda < 1$$

λ is neither equal to 0, nor equal to 1, so x_1, x_2 both are different from x.

e.g. every pt. on the circumference of a circle

$$x_1^2 + x_2^2 \le 1$$

is an extreme pt.

- ~~An~~ Extreme pt. is a boundary pt. of the set although all boundary pts. are not necessarily extreme pts.
- Some of the boundary pts. lie between two other boundary pts.
- A convex set containing a single pt. will have that pt. as its extreme pt. by convention.
- A straight line or a hyperplane has no extreme pt. although they are each a convex set.

$ax + by = c$ $\qquad$ $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

- 2 Adjacent extreme pts. $\qquad$ $ax = b$ the convex set
  - 2 extreme pts. connected by an edge.