Assignment Problem:

n- Machines

n- Jobs

	M_1	M_2	....		M_n
J_1	c_{11}	c_{12}	..	..	c_{1n}
J_2	c_{21}	c_{22}	..	...	c_{2n}
$\vdots$	...		..	..	..
$\vdots$	..	.	..	..	....
J_n	c_{n1}	c_{n2}	...	...	c_{nn}

c_{ij} = Processing cost of ith job in jth Machine. $c_{ij} \geq 0$

$i = 1, 2, .., n$; $j = 1, 2, \ldots, n$

We like to assign a job to a machine so that total processing cost is minimum. ($n!$ ways)

$n = 3$, 6 ways

$n = 5$, 120 ways

$n = 10$, 3628800 ways

Model: $\min: Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

s.to $\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n$

$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n$

where $x_{ij} = 0/1$
Binary

Example 1: (Hungarian Method)

	M_1	M_2	M_3
J_1	5	7	9
J_2	14	10	12
J_3	15	13	16

$\sim$

	M_1	M_2	M_3
J_1	0	2	4
J_2	4	0	2
J_3	2	0	3

	M_1	M_2	M_3
J_1	$\boxed{0}$	2	2
J_2	4	$\cancel{0}$	$\boxed{0}$
J_3	2	$\boxed{0}$	1

$x_{11} = 1, \quad x_{23} = 1, \quad x_{32} = 1$

$Z = 5 + 12 + 13 = 30$

Minimum cost $= 30$

Optimal Assignment for Minimum cost.

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	M_1	M_2	M_3
J_1	-5	-7	-9
J_2	-14	-10	-12
J_3	-15	-13	-16

	M_1	M_2	M_3
J_1	4	2	0
J_2	0	4	2
J_3	1	3	0

	M_1	M_2	M_3
J_1	4	$\boxed{0}$	$\cancel{0}$
J_2	$\boxed{0}$	2	2
J_3	1	1	$\boxed{0}$

$x_{12} = 1$

$x_{21} = 1$

$x_{33} = 1$

$z = 7 + 14 + 16 = 37$

Optimal Assignment for Maximum Cost.

Assignment Matrix:

Jobs		Machines			
		M_1	M_2	M_3	M_4
	J_1	1	4	6	3
	J_2	9	7	10	9
	J_3	4	5	11	7
	J_4	8	7	8	5

Find the minimum element in each row/column. Then subtract it from the row/column to get at least one zero in every row/column.

I

	M_1	M_2	M_3	M_4
J_1	0	3	5	2
J_2	2	0	3	2
J_3	0	1	7	3
J_4	3	2	3	0

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II

	M_1	M_2	M_3	M_4
J_1	$\boxed{\cancel{0}}$	3	2	2
J_2	2	$\cancel{0}$	$\boxed{0}$	2
J_3	$\cancel{0}$	1	4	3
J_4	3	2	$\cancel{0}$	$\boxed{0}$

III

	M_1	M_2	M_3	M_4
J_1	0	3	2	2
J_2	2	0	0	2
J_3	0	1	4	3
J_4	3	2	0	0

IV

	M_1	M_2	M_3	M_4
J_1	$\boxed{0}$	2	1	1
J_2	3	$\cancel{0}$	$\boxed{0}$	2
J_3	$\cancel{0}$	$\boxed{0}$	3	2
J_4	4	2	$\cancel{0}$	$\boxed{0}$

$Z = 1 + 10 + 5 + 5 = 21$ units

$x_{11} = x_{23} = x_{32} = x_{41} = 1$

Rest zero.

If we subtract a constant from any row or column of an Assignment Matrix, then the optimal sol^n will not change.

Problem:

$$\min: Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} x_{ij}$$

s.to

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n$$

$$x_{ij} = 0/1 \quad \forall\, i, j$$

Binary variable

(b)

min: $Z' = \sum_{i=1}^{n} \sum_{j=1}^{n} c'_{ij} x_{ij} + k$

We consider only 1st row.

$$Z' = \sum_{j=1}^{n} c'_{1j} x_{1j} + \sum_{i=2}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} + k$$

$$= \sum_{j=1}^{n} (c'_{1j} + k - k) x_{1j} + \sum_{i=2}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} + k$$

But $c'_{1j} + k = c_{1j} \quad j = 1, 2, \dots, n$

$$= \sum_{j=1}^{n} c_{1j} x_{1j} + \sum_{i=2}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - k \sum_{j=1}^{n} x_{ij} + k$$

$$= \sum_{j=1}^{n} c_{1j} x_{1j} + \sum_{i=2}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - K + K$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} = Z$$

$$\Rightarrow \min: Z' + K = \min: Z$$

Hence it is clear that optimal solution remain same. However, the minimum value of Z and Z' are different.

Assignment Problem:

$$\min: \ z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, .., n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, .., n$$

$$x_{ij} = 0/1, \quad \forall \ i, j$$

We solve an Assignment problem by Hungarian Method.

The basic principle of the Hungarian method is that the optimal assignment is not affected if a constant k is added or subtracted from any row or column of an assignment cost matrix $(c_{ij})_{n \times n}$.

$$\text{Min: } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - k$$

$$= \sum_{j=1}^{n} (c_{1j} - k + k) x_{1j} + \sum_{i=2}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - k$$

$$= \sum_{j=1}^{n} (c_{1j} - k) x_{1j} + \sum_{i=2}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$= \sum_{j=1}^{n} c'_{1j} x_{1j} + \sum_{i=2}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Hungarian Method:

Step 1: Reduce the initial matrix by subtracting the smallest element in each row from every element in that row. Then, using the row reduced matrix, subtract the smallest element in each column from every element in that column. We will find at least one zero in any row and at least one zero in any column.

Step 2: Find the minimum no. of lines that must be drawn through the rows and the columns of the current matrix so that all the zeros in the matrix will be covered. If the minimum no. of lines is the same as order of the matrix, an optimal assignment can be made. If the minimum no. of lines is less than the order of the matrix, go to step 3.

Step 3: Subtract the value of the smallest uncover element from every uncover element. Add this value to every element at the intersection of two lines. All other elements remain unchanged. Return to Step 2 and continue until the minimum number of lines necessary to cover all the zeros in the matrix is equal to the order of the matrix.

Example

	M_1	M_2	M_3	M_4
J_1	10	9	8	7
J_2	3	4	5	6
J_3	2	1	1	2
J_4	4	3	5	6

	M_1	M_2	M_3	M_4
J_1	3	2	1	$\boxed{0}$
J_2	$\boxed{0}$	1	2	3
J_3	1	$\cancel{0}$	$\boxed{0}$	1
J_4	1	$\boxed{0}$	2	3

$x_{14} = 1$, 7
$x_{21} = 1$, 3
$x_{33} = 1$, 1
$x_{42} = 1$, 3

14 units.

$J_1 \rightarrow M_4$, $J_2 \rightarrow M_1$, $J_3 \rightarrow M_3$, $J_4 \rightarrow M_2$

Example 2 :

	M_1	M_2	M_3	M_4
J_1	10	9	7	8
J_2	5	8	7	7
J_3	5	4	6	5
J_4	2	3	4	5

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	M_1	M_2	M_3	M_4
J_1	3	2	0	1
J_2	0	3	2	2
J_3	1	0	2	1
J_4	0	1	2	3

	M_1	M_2	M_3	M_4
J_1	3	2	0	0
J_2	0	3	2	1
J_3	1	0	2	0
J_4	0	1	2	2

3 lines to cover all zeros

	M_1	M_2	M_3	M_4
J_1	4	2	$\boxed{0}$	$\cancel{0}$
J_2	$\boxed{0}$	2	1	0
J_3	2	$\cancel{0}$	2	$\boxed{0}$
J_4	$\cancel{0}$	$\boxed{0}$	1	1

$J_1 \to M_3$

$J_2 \to M_1$

$J_3 \to M_4$

$J_4 \to M_2$

$Z = 20$ units

4 lines to cover all zeros

Travelling Salesperson Problem:

$$\min: Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

s.to

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, .., n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, .., n$$

$$x_{ij} = 0/1 \quad \forall i, j$$

$$\underset{\sim}{X} = \{ x_{ij} \} \text{ forms a tour.}$$

Example 1:

City

City	1	2	3	4	5
1	∞	c_{12}	c_{13}	c_{14}	c_{15}
2	c_{21}	∞	c_{23}	c_{24}	c_{25}
3	c_{31}	c_{32}	∞	c_{34}	c_{35}
4	c_{41}	c_{42}	c_{43}	∞	c_{45}
5	c_{51}	c_{52}	c_{53}	c_{54}	∞

Let $c_{ij} = d_{ij}$ distance between city i & j

Tour: $c_1 \longrightarrow c_2 \longrightarrow c_3 — c_4 — c_5 — c_1$

Tour $c_1 \longrightarrow c_3 \longrightarrow c_5 \longrightarrow c_4 — c_2 \rightarrow c_1$

He has to visit all the cities exactly once and return to the home city.