16/8/2022

Let Ω be any nonempty set. Then $F(\Omega)$ is the space of all K-valued functions defined Ω with addition and scalar multiplication defined by

[F has domain space sigma and range space K]

$$(f+g)(t) = f(t) + g(t), \quad t \in \Omega$$

$$(\alpha f)(t) = \alpha f(t), \quad \forall \alpha \in F, t \in \Omega$$

is a vector space.

clearly $B(\Omega)$, the space of all K-valued bounded functions on Ω is a subspace of $F(\Omega)$.

Ex: For $1 \leq p < \infty$, let

$$l^p = \left\{ x = (x_1, x_2, \ldots) \in F(\mathbb{N}) \,/\, \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty \right\}$$

x=f(1),f(2).....where f is a function on natural numbers
xi=f(i) in the above case as the notation used are wrong

elements of lp are sequences built on functions of natural numbers such that the summation f(i) till infinity is bounded

For $p \in [1, \infty]$, define

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{1/p}, & 1 \le p < \infty \\ \max_i |x(i)|, & p = \infty, \end{cases}$$

where $x = (x(1), x(2), x(3), \ldots)$.

Then clearly $\|\cdot\|_1$ is a norm on ℓ^1 and $\|\cdot\|_\infty$ is a norm on

$$\ell^\infty = \{x = (x(1), x(2) \ldots) \in F(\mathbb{N}) \mid \max_i |x(i)| < \infty\}.$$

(H.W)

Now for $1 < p < \infty$, we show that ℓ^p is a normed linear space (n.l.s)

For this, let $x = (x(1), x(2) \ldots) \in \ell^p$

$y = (y(1), y(2) \ldots) \in \ell^p$

Now for any $n \in \mathbb{N}$, we have that

$$\left(\sum_{i=1}^{n} |x(i)+y(i)|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y(i)|^p\right)^{1/p}$$

$$\leq \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y(i)|^p\right)^{1/p}$$

Let $n \to \infty$, then we get

$$\left(\sum_{i=1}^{\infty} |x(i)+y(i)|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y(i)|^p\right)^{1/p}.$$

$\Rightarrow$

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

$\forall\, x, y \in \ell^p.$

$\Rightarrow$ $\forall\, x, y \in \ell^p$, $x+y \in \ell^p$ and triangle inequality is satisfied in ℓ^p.

Please check

$\|x\|_p \geq 0$ and $\|x\|_p = 0 \iff x = 0$

$$\|\alpha x\|_p = \left(\sum_{i=1}^{\infty} |\alpha x(i)|^p\right)^{1/p}$$

$$= \left(\sum_{i=1}^{\infty} |\alpha|^p |x(i)|^p\right)^{1/p}$$

$$= |\alpha| \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{1/p}$$

$$= |\alpha| \, \|x\|_p .$$

$\therefore$ For $1 \leq p \leq \infty$, ℓ^p is a normed linear space.

Note :— For $0 < p < 1$, ℓ^p is not a n.l.s.

$\because$ $e_1 = (1, 0, \ldots \ldots)$

$e_2 = (0, 1, 0 \ldots \ldots)$

$$\|e_1\|_p = \|e_2\|_p = 1$$

$$\|e_1+e_2\|_p = 2^{1/p}.$$

$$\therefore \|e_1+e_2\|_p = 2^{1/p} > 2 = \|e_1\|_p+\|e_2\|_p$$

Let

$$C_{00} = \{x = (x(1), x(2), \ldots) \in F(\mathbb{N}) \,/\, \exists\, k \in \mathbb{N} \ni x(n) = 0,\ \forall\, n \geq k\},$$

i.e., $x = (x(1), x(2), \ldots\ldots) \in C_{00}$

$$\Rightarrow x = (x(1), x(2), \ldots x(k), 0, 0, 0, \ldots\ldots)$$

$$C_0 = \{x = (x(1), x(2), \ldots\ldots) \in F(\mathbb{N}) \,/\, x(n) \to 0 \text{ as } n \to \infty\},$$

$$C = \{x = (x(1), x(2), \ldots\ldots) \in F(\mathbb{N}) \,/\, x(n) \text{ cgs as } n \to \infty\}$$

convergent sequence

Now clearly we have

$$c_{00} \subsetneq \ell^p \subsetneq c_0 \subsetneq c \subsetneq \ell^\infty$$

$x = (1, -1, 1, -1, \ldots\ldots) \in \ell^\infty$ f(i)=-1^(i+1)

$\notin c$

$x = (1, 1, 1, \ldots\ldots) \in c$, $x \notin c_0$ f(i)=1

$x = (1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots) \in c_0$, f(i)=1/i

but $\sum_{j=1}^{\infty} \frac{1}{j}$ dgt $\Rightarrow x \notin \ell^1$

$x = (1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots) \in \ell^1$

$\because \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$

but $x \notin c_{00}$

Now let $1 \leq p < r < \infty$.

Let $x = (x(1), x(2), \ldots) \in \ell^p$

with $\|x\|_p \le 1$.

$$\Rightarrow \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{1/p} \le 1.$$

$$\Rightarrow \quad |x(i)| \le 1$$

$$\because \quad p < r \Rightarrow |x(i)|^r \le |x(i)|^p$$

$$\Rightarrow \left(\sum_{j=1}^{\infty} |x(j)|^r \right)^{\cancel{1/r}} \le \left(\sum_{j=1}^{\infty} |x(j)|^p \right)^{\cancel{1/p}},$$

$$\Rightarrow \|x\|_r^r \le \|x\|_p^p$$

$$\Rightarrow \|x\|_r^r \le \|x\|_p^p \le 1.$$

$$\Rightarrow \|x\|_r \le 1 \quad \text{——} \quad \alpha$$

Now for any $0 \neq x \in \ell^p$,

let $y = \dfrac{x}{\|x\|_p}$

Then $\|y\|_p = \left\| \dfrac{x}{\|x\|_p} \right\|_p$

$$= \frac{1}{\|x\|_p} \cdot \|x\|_p = 1$$

$\Rightarrow \|y\|_p \leq 1$.

Then by $*$ $\|y\|_r \leq 1$.

$$\Rightarrow \left\| \frac{x}{\|x\|_p} \right\|_r \leq 1$$

$$\Rightarrow \frac{1}{\|x\|_p} \cdot \|x\|_r \leq 1$$

$$\Rightarrow \|x\|_r \leq \|x\|_p \quad , p < r.$$

Thus for $1 \leq p < r \leq \infty$, $\|x\|_r \leq \|x\|_p$, this is known as Jensen's inequality.

If $x \in \ell^p \Rightarrow \|x\|_p < \infty$

$$\Rightarrow \|x\|_r \le \|x\|_p < \infty, \quad p < r,$$

$$\Rightarrow x \in \ell^r.$$

$\therefore \ \ell^p \subset \ell^r, \quad 1 \le p < r < \infty.$

Also for $p \ge 1$, $x \in \ell^p$

$$\Rightarrow \left(\sum_{j=1}^{\infty} |x(j)|^p\right)^{\frac{1}{p}} < \infty$$

$$\Rightarrow \sum_{j=1}^{\infty} |x(j)|^p < \infty$$

$$\Rightarrow |x(j)|^p \le \sum_{j=1}^{\infty} |x(j)|^p < \infty.$$

$$\Rightarrow \max_j |x(j)| \le \left(\sum_{j=1}^{\infty} |x(j)|^p\right)^{\frac{1}{p}} < \infty \quad \forall j$$

$\Rightarrow \|x\|_\infty \leq \|x\|_p < \infty$.

$\Rightarrow x \in \ell^\infty, \ \forall \ p \geq 1$.

$\therefore \forall \ p \geq 1, \ \ell^p \subset \ell^\infty$.

Convergent Sequence in a n.l.s

Let $\{x_n\}$ be a sequence in a n.l.s $(X, \|\cdot\|)$. We say $\{x_n\}$ is convergent to an element $x \in X$, if given any $\epsilon > 0$ $\exists \ n_0 \in \mathbb{N}$ such that

$$\|x_n - x\| < \epsilon, \ \forall \ n \geq n_0.$$

$\Rightarrow \|x_n - x\| \longrightarrow 0$ as $n \to \infty$.

Cauchy Sequence :— A sequence $\{x_n\}$ is a n.l.s $(X, \|\cdot\|)$ is said to be Cauchy sequence, if given any $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that $\|x_n - x_m\| < \epsilon$, $\forall\, n, m \geq n_0$.

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Suppose a sequence $\{x_n\}$ converges to x in l^p. Then

$$\|x_n - x\|_p \longrightarrow 0 \quad \text{as } n \to \infty.$$

$$\Rightarrow \quad \|x_n - x\|_\infty \leq \|x_n - x\|_p \longrightarrow 0$$

as $n \to \infty$

$\forall\, p \geq 1$.

Also for $1 \leq p < r < \infty$,

$$\|x_n - x\|_r \leq \|x_n - x\|_p \longrightarrow 0$$

Then $x_n \to x$ in ℓ^p implies

$x_n \to x$ in ℓ^r, and $x_n \to x$ in ℓ^∞, $\forall\, p \geq 1$.

Banach Space :

A normed linear space $(X, \|\cdot\|)$ is said to be a Banach space if every Cauchy sequence is a convergent sequence in X.

* $(X, \|\cdot\|)$ is a n.l.s.

define $d(x,y) = \|x-y\|$, $\forall x,y \in X$.

Show that d is a metric on X.

metric-distance function

* If every Cauchy sequence is convergent in a metric space (X, d), we say X is a complete metric space.

* A n.l.s $(X, \|\cdot\|)$ is said to be a Banach space, if it is complete in the metric d induced by the norm $\|\cdot\|$.

Ex: For each $n \in \mathbb{N}$, $(K^n, \|\cdot\|_p)$, $1 \le p \le \infty$, is a Banach space.

Sol: let $p = \infty$,

$$\|x\|_\infty = \max_{i=1..n} |x(i)|.$$

We know that $(K^n, \|\cdot\|_\infty)$ is a n.l.s.

Let $\{x_n\}$ be a Cauchy sequence in K^n w.r.t $\|\cdot\|_\infty$.

$\Rightarrow$ Given any $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ $\ni$

$$\|x_l - x_m\|_\infty < \epsilon, \quad \forall\, l, m \geq n_0.$$

$$\Rightarrow \max_{i=1..n} |x_l(i) - x_m(i)| < \epsilon$$

$$\Rightarrow |x_l(i) - x_m(i)| < \epsilon, \quad \forall\, i = 1, 2, \ldots n$$

$\Rightarrow$ $\{x_m(i)\}$ is a Cauchy sequence in the field K, $i = 1, 2, \ldots n$.

$\because$ K is a complete metric space,

$$x_m(i) \longrightarrow \alpha_i \in K, \quad i = 1, \ldots n$$

cauchy sequence in with distance function |x| i.e in 1d is always convergent let $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$= (x(1), x(2), \ldots x(n)) \in K^n$

Then $|x_m(i) - x(i)| \longrightarrow 0$

as $m \to \infty$
$i = 1, \ldots n.$

$\therefore \max_{i=1,\ldots n} |x_m(i) - x(i)| \longrightarrow 0$

as $m \to \infty$

$\Rightarrow \|x_n - x\|_\infty \longrightarrow 0.$

and $x \in K^n.$

$\therefore (K^n, \|\cdot\|_\infty)$ is a complete n.l.s.

started with given norm
converted to 1d |x| norm index wise
proved it cauchy and hence is convergent
converted the convergent sequence index wise to the required norm

i.e., $(K^n, \|\cdot\|_\infty)$ is a Banach space.

Now for $1 \leq p < \infty$, $x = (x(1), x(2), \ldots x(n)) \in K^n$,

we have

$$|x(i)| \leq \|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p}$$

$\forall i = 1, \ldots n$

$$\Rightarrow \quad \|x\|_\infty \leq \|x\|_p \quad \text{---} (1)$$

Also

$$\|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p \right)^{1/p}$$

$$\leq \max_{i=1..n} |x(i)| \cdot \left(\sum_{i=1}^{n} 1 \right)^{1/p}$$

$$= \|x\|_\infty \cdot n^{1/p} \quad \text{---} (2)$$

So combining (1) & (2) we get

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \|x\|_\infty \, , \quad 1 \leq p < \infty .$$

Then for any sequence $\{x_n\}$ in K^n, we have

$$\|x_m - x_\ell\|_\infty \leq \|x_m - x_\ell\|_p \leq n^{1/p} \|x_m - x_\ell\|_\infty$$

$$1 \leq p < \infty$$

cauchy->convergent as proved in infinity norm

Also

$$\|x_m - x\|_\infty \leq \|x_m - x\|_p \leq n^{1/p} \leq \|x_m - x\|_\infty$$

$1 \leq p < \infty$

$\therefore$ $\{x_n\}$ is Cauchy in K^n w.r.t $\|\cdot\|_p$, $1 \leq p < \infty$

iff $\{x_n\}$ is Cauchy in K^n, w.r.t $\|\cdot\|_\infty$

and

$x_n \to x \in K^n$ w.r.t $\|\cdot\|_\infty$

iff $x_n \to x \in K^n$, $\|\cdot\|_p$,

$1 \leq p < \infty$.

$\therefore$ $(K^n, \|\cdot\|_p)$ is a Banach space $1 \leq p \leq \infty$.