NSOPDE ASSIGNMENT - 3

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1) $u_{n+2} + 28 u_{n+1} - 28 u_{n-1} - u_{n-2} =$

$$h(12 f_{n+1} + 36 f_n + 12 f_{n-1})$$

test eqn $y' = \lambda y$ $y(t_0) = y_0$ $t \in [t_0, b]$

LMM $\Rightarrow$ $\rho(E)\, u_{n-k+1} = h\, \sigma(E)\, f_{n-k+1}$

(i) for convergence $\rho(1) = 0$ $\rho'(1) = \sigma(1)$
& root condition must be satisfied.

$\rho(\xi) = \xi^4 + 28\xi^3 - 28\xi - 1$ $\sigma(\xi) = 12\xi^3 + 36\xi^2 + 12\xi$

$\rho'(\xi) = 4\xi^3 + 84\xi^2 - 28$

$\rho(1) = 1 + 28 - 28 - 1 = 0$ (satisfied)

$$\left.\begin{aligned} \rho'(1) &= 4 + 84 - 28 = 60 \\ \sigma(1) &= 12 + 36 + 12 = 60 \end{aligned}\right\} \begin{array}{c} \rho'(1) = \sigma(1) \\ \text{(satisfied)} \end{array}$$

$\rho(\xi) = \xi^4 + 28\xi^3 - 28\xi - 1$

$\Rightarrow \rho(\xi) = (\xi - 1)(\xi + 1)(\xi^2 + 28\xi + 1)$

$\Rightarrow \xi = 1, -1, -0.036, -27.96$

root condition $|\xi| \leq 1$

root condition is not satisfied $\Rightarrow$ not convergent

Date

(ii) growth factor $K_i = \dfrac{\sigma(\xi_i)}{\xi_i \, p'(\xi_i)}$

$\Rightarrow K_1 = 1 \qquad K_3 = -0.3854$

$K_2 = -\dfrac{3}{13} \qquad K_4 = -0.3848$

2) $p(\xi) = \xi^2 - 1$

$\Rightarrow p(\xi) = (\xi-1)^2 + 2(\xi-1)$

$\sigma(\xi) = \dfrac{p(\xi)}{\ln \xi} = \dfrac{(\xi-1)^2 + 2(\xi-1)}{\ln(\xi-1+1)}$

$\because \ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dots$

$\Rightarrow \sigma(\xi) = \dfrac{(\xi-1)^2 + 2(\xi-1)}{(\xi-1) - \dfrac{(\xi-1)^2}{2} + \dfrac{(\xi-1)^3}{3} - \dfrac{(\xi-1)^4}{4} \dots}$

$\Rightarrow \sigma(\xi) = (\xi-1)^2 + 2(\xi-1)\left[(\xi-1) - \dfrac{(\xi-1)^2}{2} + \dfrac{(\xi-1)^3}{3} \dots\right]^{-1}$

$\Rightarrow \sigma(\xi) = (\xi+1)\left[1 - \dfrac{(\xi-1)}{2} + \dfrac{(\xi-1)^2}{3} - \dots\right]^{-1}$

$= (\xi+1)\left(1 + \dfrac{\xi-1}{2} - \dfrac{(\xi-1)^2}{12} + \dots\right)$

$= ((\xi-1)+2)\left(1 + \dfrac{(\xi-1)}{2} - \dfrac{(\xi-1)^2}{12} + \dots\right)$

$= 2 + 2(\xi-1) + \dfrac{(\xi-1)^2}{3}$

$= \dfrac{6\xi + \xi^2 - 2\xi + 1}{3}$

Date

$$\Rightarrow \sigma(\xi) = \frac{\xi^2 + 4\xi + 1}{3}$$

$$P(E)\, u_{n-k+1} = h\sigma(E)\, f_{n-k+1}$$

$$(E^2 - 1)\, u_{n-1} = \frac{h}{3}(E^2 + 4E + 3)\, f_{n-1}$$

$$u_{n+1} - u_{n-1} = \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1})$$

$P(1) = 0 \quad \xi = \pm 1 \Rightarrow |\xi| \leq 1$ The method is stable

The corresponding implicit LMM is "Milne Simpsons method"

3) $u' = t + u \quad u(0) = 1 \quad u(0.1) = \alpha \quad u(0.2) = \beta \quad u(0.3) = \gamma$

$$u' = t + u$$
$$u'' = 1 + u' = 1 + t + u$$
$$u''' = 1 + u' = 1 + t + u$$

$$u_{n+1} = u_n + hu'_n + \frac{h^2}{2} u''_n + \frac{h^3}{6} u'''_n$$ 3rd order taylor series method

$$u(0.1) = u(0) + (0.1)\, u'(0) + \frac{(0.1)^2}{2} u''(0) + \frac{(0.1)^3}{6} u'''(0)$$

$$= 1 + (0.1) + \frac{(0.1)^2 \times 2}{2} + \frac{(0.1)^3}{6} \times 2$$

$$= 1 + 0.1 + 0.01 + \frac{0.001}{3}$$

$$\alpha = u(0.1) = 1.11033$$

using

$$u_{n+1} = u_{n-1} + \frac{h}{3}\left(f_{n+1} + 4f_n + f_{n-1}\right)$$

$$u(0.2) = 1.2428 = \beta$$

$$u(0.3) = 1.3997 = \gamma$$

4) $$u_{n+3} = u_n + \frac{3h}{8}\left[u'_n + 3u'_{n+1} + 3u'_{n+2} + u'_{n+3}\right]$$

$$\Rightarrow u_{n+1} - u_{n-2} = \frac{3h}{8}\left[u'_{n+1} + 3u'_n + 3u'_{n-1} + u'_{n-2}\right]$$

$$(E^3 - 1)\, u_{n-2} = h\left[\frac{3}{8}\left(E^3 + 3E^2 + 3E + 1\right)\right] u'_{n-2}$$

$$p(\xi) = \xi^3 - 1 \qquad p'(\xi) = 3\xi^2$$

$$\sigma(\xi) = \frac{3}{8}(\xi+1)^3$$

$$p(\xi) = (\xi - 1)(\xi^2 + \xi + 1)$$

$$\xi = 1, \ \frac{-1+\sqrt{3}i}{2}, \ \frac{-1-\sqrt{3}i}{2}$$

growth parameters $\Rightarrow$ $K_i = \dfrac{\sigma(\xi_i)}{\xi_i \, p'(\xi_i)}$

$$\Rightarrow K_i = \frac{\frac{3}{8}(\xi_i+1)^3}{\xi_i \; 3\xi_i^2} = \left(\frac{\xi_i + 1}{2\xi_i}\right)^3$$

$$\Rightarrow K_1 = 1 \qquad K_2 = -0.125$$

$$K_3 = -0.125$$

5) $u_{n+1} = u_n + \frac{h}{12}\left[5u'_{n+1} + 8u'_n - u'_{n-1}\right]$

$y' = -y \qquad y(x_0) = y_0$

using the test equation

$\Rightarrow u_{n+1} = u_n + \frac{h}{12}\left[-5u_{n+1} - 8u_n + u_{n-1}\right]$

$\Rightarrow \left(1 + \frac{5h}{12}\right)u_{n+1} + \left(\frac{8h}{12} - 1\right)u_n - \frac{h}{12}u_{n-1} = 0$

characteristic eqn $\Rightarrow \left(1+\frac{5h}{12}\right)\xi^2 + \left(\frac{2h}{3} - 1\right)\xi - \frac{h}{12} = 0$

$\xi = \frac{1+z}{1-z}$

$(1+z)^2\left(1+\frac{5h}{12}\right) + (1-z^2)\left(\frac{2h}{3} - 1\right) - \frac{h}{12}(1-z)^2 = 0$

$z^2\left(1 + \frac{5h}{12} - \frac{2h}{3} + 1 - \frac{h}{12}\right) + z\left(2 + \frac{5h}{6} + \frac{h}{6}\right) + 1 + \frac{5h}{12} + \frac{2h}{3} - 1$

$- \frac{h}{12} = 0$

$\Rightarrow z^2\left(2 - \frac{h}{3}\right) + z(2+h) + h = 0$

<u>Routh-Hurwitz criterion</u>

$b_0 = 2 - \frac{h}{3} \qquad b_1 = 2+h \qquad b_2 = h \qquad D = \begin{bmatrix} 2+h & 0 \\ 2-\frac{h}{3} & h \end{bmatrix}$

$2+h > 0 \qquad 2 - \frac{h}{3} > 0 \qquad h > 0$

$h > -2 \qquad h < 6 \qquad h > 0$

$\Rightarrow \boxed{h \in (0, 6)}$ Interval of absolute stability.

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6) (i) $\Delta^2 u_n - 3\Delta u_n + 2u_n = 0$

$\xi^2 - 3\xi + 2 = 0 \Rightarrow \xi = 1, 2$

$\Rightarrow \boxed{u_n = c_1 + c_2 2^n}$

(ii) $\Delta^2 u_n + \Delta u_n + \frac{1}{4} u_n = 0$

$\xi^2 + \xi + \frac{1}{4} = 0 \Rightarrow (2\xi)^2 + 2\cdot 2\xi + 1 = 0$

$\Rightarrow (2\xi + 1)^2 = 0 \Rightarrow \xi = -\frac{1}{2}$

$u_n = (c_1 + nc_2)\left(-\frac{1}{2}\right)^n$

(iii) $\Delta^2 u_n - 2\Delta u_n + 2u_n = 0$

$\xi^2 - 2\xi + 2 = 0$

$\Rightarrow (\xi - 1)^2 + 1 = 0 \Rightarrow \xi = 1 \pm i$

$\xi_1 = 1 + i = \sqrt{2}\, e^{\frac{i\pi}{4}}$ $\quad \xi_2 = 1 - i = \sqrt{2}\, e^{-\frac{i\pi}{4}}$

$u_n = \left[c_1 \cos\left(\frac{n\pi}{4}\right) + c_2 \sin\left(\frac{n\pi}{4}\right)\right] 2^{n/2}$

(iv) $\Delta^2 u_{n+1} - \frac{1}{3}\Delta^2 u_n = 0$

$\xi^3 - \frac{\xi^2}{3} = \frac{\xi^2}{3}(3\xi - 1) \Rightarrow \xi = 0, 0, \frac{1}{3}$

$u_n = \frac{c_1}{3^n}$

7) (i) $(1-5\alpha)\, y_{n+2} - (1+8\alpha)\, y_{n+1} + \alpha y_n = 0$

Characteristic equation $\Rightarrow$

$$\xi^2(1-5\alpha) - \xi(1+8\alpha) + \alpha = 0 \qquad \xi = \frac{1+z}{1-z}$$

$$\Rightarrow (1+z)^2(1-5\alpha) - (1-z^2)(1+8\alpha) + \alpha(1-z)^2 = 0$$

$$\Rightarrow (z^2+2z+1)(1-5\alpha) - (1-z^2)(1+8\alpha) + \alpha(z^2-2z+1) = 0$$

$$\Rightarrow z^2(1-5\alpha+1+8\alpha+\alpha) + z(2-10\alpha-2\alpha) + 1-5\alpha-1-8\alpha + \alpha = 0$$

$$\Rightarrow z^2(2+4\alpha) + z(2-12\alpha) - 12\alpha = 0$$

$$z^2(1+2\alpha) + z(1-6\alpha) - 6\alpha = 0$$

$$\left.\begin{array}{lll} 1+2\alpha > 0 & 1-6\alpha > 0 & -6\alpha > 0 \\ \alpha > -\frac{1}{2} & \alpha < \frac{1}{6} & \alpha < 0 \end{array}\right\} \text{Routh Hurwitz criteria}$$

$$\alpha \in \left(-\frac{1}{2}, 0\right)$$

$\Rightarrow$ for $|\xi| \leq 1$ $\quad \alpha \in \left(-\frac{1}{2}, 0\right)$

(ii) $(1-9\alpha)\, y_{n+3} - (1+19\alpha)\, y_{n+2} + 5\alpha y_{n-1} - \alpha y_n = 0$

Characteristic equation $\Rightarrow$

$$\xi^4(1-9\alpha) - \xi^3(1+19\alpha) - \alpha\xi + 5\alpha = 0 \qquad \xi = \frac{1+z}{1-z}$$

$$(1+z)^4(1-9\alpha) - (1+z)^3(1-z)(1+19\alpha) - (1+z)(1-z)^3\alpha + 5\alpha(1-z)^4 = 0$$

$$(1-9\alpha)(z^4+4z^3+6z^2+4z+1) + (1+19\alpha)(z^4+2z^3-2z-1) + \alpha(z^4-2z^3+2z-1) + 5\alpha(z^4-4z^3+6z^2-4z+1) = 0$$

$\Rightarrow z^4(2+16\alpha) + z^3(6-20\alpha) + z^2(6-24\alpha)$
$+ z(2-92\alpha) - 24\alpha = 0$

$\Rightarrow z^4(1+8\alpha) + z^3(3-10\alpha) + z(3-12\alpha) + z(1-46\alpha)$
$-12\alpha = 0$

$b_0 = 1+8\alpha, \; b_1 = 3-10\alpha \quad b_2 = 3-12\alpha$
$b_3 = 1-46\alpha \quad b_4 = -12\alpha$

$$D = \begin{bmatrix} 3-10\alpha & 1-46\alpha & 0 & 0 \\ 1+8\alpha & 3-12\alpha & -12\alpha & 0 \\ 0 & 3-10\alpha & 1-46\alpha & 0 \\ 0 & 1+8\alpha & 3-12\alpha & -12\alpha \end{bmatrix}$$

$1+8\alpha > 0 \quad 3-10\alpha > 0 \quad 3-12\alpha > 0 \quad 1-46\alpha > 0$
$\Rightarrow \alpha > -\frac{1}{8} \quad \Rightarrow \alpha < \frac{3}{10} \quad \Rightarrow \alpha < \frac{1}{4} \quad \Rightarrow \alpha < \frac{1}{46}$

$-12\alpha > 0 \quad\quad \Rightarrow \alpha > -\frac{1}{8} \quad \alpha < 0 \Rightarrow \alpha \in \left(-\frac{1}{8}, 0\right)$
$\Rightarrow \alpha < 0$

$\Rightarrow$ for $|\xi| \le 1 \quad \alpha \in \left(-\frac{1}{8}, 0\right)$

8) $y_{j+1} - 2\lambda y_j + y_{j-1} = 0$, ST $\begin{cases} \text{is bounded when } j \to \infty \\ -1 < \lambda < 1 \end{cases}$

$\Rightarrow \xi^2 - 2\lambda\xi + 1 = 0$
$(\xi - \lambda)^2 = \lambda^2 - 1$

$\Rightarrow \xi = \lambda \pm \sqrt{\lambda^2 - 1}$

for $\lambda \in (-1, 1)$ $\quad \lambda^2 - 1 < 0$

$\Rightarrow \xi = \lambda \pm i\sqrt{1-\lambda^2}$

$\Rightarrow \xi = \lambda + i\sqrt{1-\lambda^2}, \quad \lambda - i\sqrt{1-\lambda^2}$

$|\xi| = 1 \Rightarrow \xi = e^{i\phi}, e^{-i\phi} \qquad \phi = \tan^{-1}\left(\frac{\sqrt{1-\lambda^2}}{\lambda}\right)$

$y_j = \left[c_1 \cos(j\phi) + c_2 \sin(j\phi)\right](1)^j \leq |c_1| + |c_2|$

$\Rightarrow y_j$ is bounded for any value of j by $|c_1| + |c_2|$

$\underline{\lambda = 1}$ $\quad y_j = (c_1 + jc_2)(1)^j = c_1 + jc_2$

as $j \to \infty$ $\quad y_j$ is unbounded $(c_2 \neq 0)$

$\underline{\lambda = -1}$ $\quad y_j = (c_1 + jc_2)(-1)^j$

as $j \to \infty$ $\quad y_j$ is unbounded $\quad (c_2 \neq 0)$

$\underline{|\lambda| > 1}$ $\quad \xi_1 = \lambda + \sqrt{\lambda^2 - 1}, \quad \xi_2 = \lambda - \sqrt{\lambda^2 - 1}$

$y_j = c_1\left(\lambda + \sqrt{\lambda^2-1}\right)^j + c_2\left(\lambda - \sqrt{\lambda^2-1}\right)^j$

for $\lambda > 1$

$\lambda + \sqrt{\lambda^2 - 1} > 1$ $\qquad (\lambda - \sqrt{\lambda^2-1})^j$ is bounded

as $j \to \infty$ $(\lambda + \sqrt{\lambda^2-1})^j \to \infty$ $\qquad$ for $\lambda > 1$

y_j is unbounded

for $\lambda < -1$

$\lambda - \sqrt{\lambda^2 - 1} < -1$ $\quad (\lambda + \sqrt{\lambda^2-1})^j$ is bounded

as $j \to \infty$ $|(\lambda - \sqrt{\lambda^2-1})^j| \to \infty$ $\quad$ as $j \to \infty$

unbounded

but y_j is unbounded

9) $u_{j+1} = \frac{4}{3} u_j - \frac{1}{3} u_{j-1} + \frac{2}{3} h u'_{j+1}$

$u' = \lambda u \quad \lambda < 0$

using characteristic equation

$u_{j+1} = \frac{4}{3} u_j - \frac{1}{3} u_{j-1} + \frac{2}{3} h \lambda u_{j+1}$

$\Rightarrow u_{j+1}\left(1 - \frac{2h\lambda}{3}\right) - \frac{4}{3} u_j + \frac{u_{j-1}}{3} = 0$

let $h\lambda = \bar{h}$

$\Rightarrow \xi^2\left(1 - \frac{2\bar{h}}{3}\right) - \frac{4}{3}\xi + \frac{1}{3} = 0 \qquad \xi = \frac{1+z}{1-z}$

$(1 + z^2 + 2z)(3 - 2\bar{h}) - 4(1 - z^2) + 1 + z^2 - 2z = 0$

$z^2(8 - 2\bar{h}) + z(4 - 4\bar{h}) - 2\bar{h} = 0$

$z^2(4 - \bar{h}) + 2z(1 - \bar{h}) - \bar{h} = 0$

$4 - \bar{h} > 0 \qquad 2(1-\bar{h}) > 0 \qquad -\bar{h} > 0$

$\bar{h} < 4 \qquad \bar{h} < 1 \qquad \bar{h} < 0$

$\Rightarrow \bar{h} < 0 \quad \Rightarrow \bar{h} \in (-\infty, 0) \rightarrow$ absolute stability

$\therefore$ The given method is A-stable.

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10) $u_{j+1} - (1+a)u_j + a u_{j-1} = h\left[\left\{\frac{1}{2}(1+a)+b\right\} u'_{j+1}\right.$

$\left. + \left\{\frac{1}{2}(1-3a)-2b\right\} u'_j + b u'_{j-1}\right]$

test eqn $\Rightarrow$ $u' = \lambda u$ $\quad \lambda < 0$

$u_{j+1} - (1+a)u_j + a u_{j-1} = h\left[\left\{\frac{1}{2}(a+1)+b\right\} \lambda u_{j+1}\right.$

$\left. + \left\{\frac{1}{2}(1-3a)-2b\right\} \lambda u_j + b\lambda u_{j-1}\right]$

$u_{j+1}\left(1 - h\lambda\left(\frac{1+a}{2}+b\right)\right) - u_j\left(1+a-h\lambda\left(\frac{1-3a}{2}-2b\right)\right)$

$+ u_{j-1}(a - h\lambda b) = 0$

let $h\lambda = \bar{h}$

$\xi^2\left(1-\bar{h}\left(\frac{1+a}{2}+b\right)\right) - \xi\left(1+a-\bar{h}\left(\frac{1-3a}{2}-2b\right)\right)$

$+ a - \bar{h}b = 0$ $\qquad \xi = \frac{1+z}{1-z}$

$(z^2+2z+1)\left(1-\bar{h}\left(\frac{1+a}{2}+b\right)\right) - (1-z^2)\left(1+a-\bar{h}\left(\frac{1-3a}{2}-2b\right)\right)$

$+ (a-\bar{h}b)(z^2-2z+1) = 0$

$z^2\left(1-\bar{h}\left(\frac{1+a}{2}+b\right) + 1 + a - \bar{h}\left(\frac{1-3a}{2}-2b\right) + a - \bar{h}b\right)$

$+ z\left(2-\bar{h}(1+a+2b) - 2a + 2\bar{h}b\right) + \left(1-\bar{h}\left(\frac{1+a}{2}+b\right)\right.$

$\left. - 1 - a + \bar{h}\left(\frac{1-3a}{2}-2b\right) + a - \bar{h}b\right) = 0$

$z^2(2+2a+\bar{h}(a-1)) + z(2-2a-\bar{h}(1+a)) - \bar{h}2(a+2b) = 0$

- $2+2a-\bar{h}(1-a)>0$

  $a<1$ & $2(1+a)>0$

  $a>-1$ $(\bar{h}<0)$

  $a\in(-1,1)$

- $2(1+a)>\bar{h}(1+a)$

  $a>-1$ & $2(1-a)>0$

  $a<1$ $(\bar{h}<0)$

  $\Rightarrow a\in(-1,1)$

- $-2\bar{h}(a+2b)>0$

  $\Rightarrow \bar{h}(a+2b)<0$ $(\bar{h}<0)$

  $\Rightarrow a+2b>0$

$\therefore$ For the method to be A-stable

$a\in(-1,1)$ & $a+2b>0$