ASSIGNMENT - 01

1. Normed Linear spaces and Banach spaces

(1) (a) Show that a bounded metric on a linear space $X \neq \{0\}$ can never be induced by a norm.

(b) Show that a metric d induced by a norm on vector space X satisfies :

(i) $d(x+a, y+a) = d(x, y)$,

(ii) $d(\beta x, \beta y) = |\beta| \, d(x, y)$ $\quad \forall \; x, y, a \in X$, $\beta \in \mathbb{K}$, the field of scalars.

(c) Show that the metric defined by

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{3^i} \frac{|\xi_i - \zeta_i|}{1 + |\xi_i - \zeta_i|}$$

can not be induced by a norm, where $x = (\xi_i)$ and $y = (\zeta_i)$ belong to the space of all sequences of complex numbers.

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(2) (a) Let X be a normed linear space with norm $\|\cdot\|$ and for some $0 \neq a \in \mathbb{K}$, let

$$\|x\|_a = \|ax\| \quad \forall \; x \in X.$$

Show that $\|\cdot\|_a$ is a norm on X.

(b) Let $t_1, t_2, \ldots, t_j$ be distinct points in $[a, b]$ and for $f \in \mathcal{P}_n[a, b]$, the space of all $\mathbb{K}$-valued polynomials of degree at most n on $[a, b]$, we define:

$$\eta(f) = \sum_{i=1}^{j} |f(t_i)|.$$

Show that η is a norm on $\mathcal{P}_n[a, b]$ iff $j \geq n+1$.

(3) Show that the map

$$x \mapsto \|x'\|_2 , \quad x \in C'[0,1] \text{ is not a}$$

norm on $C'[0,1]$.

Here x' denotes the derivative of x

and $\|y\|_2 = \left(\int_0^1 |y(t)|^2 dt\right)^{\frac{1}{2}}$, $y \in C[0,1]$.

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(4) Prove that the norm $x \mapsto \|x\|$ is a continuous map of $(X, \|\cdot\|)$ into $\mathbb{R}$.

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(5) Let $C[a,b]$ denote the space of all real valued functions of an independent real variable t, that are defined and continuous on a given closed interval $[a,b]$.

Let us define

(i) $\|x\|_\infty = \max\limits_{t \in [a,b]} |x(t)|$, $x \in C[a,b]$

(ii) $\|x\|_1 = \int_0^1 |x(t)|\, dt$, $\cancel{x \in C[0,1]}$ $x \in C[0,1]$.

Show that $C[a,b]$ is a Banach space with respect to $\|\cdot\|_\infty$ norm but $C[0,1]$ is not with respect to $\|\cdot\|_1$ norm defined above.

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(6) Prove that the closure of c_{00} in

(i) $(l^1, \|\cdot\|_1)$ is l^1,

(ii) $(l^2, \|\cdot\|_2)$ is l^2,

(iii) $(l^\infty, \|\cdot\|_\infty)$ is c_0.

Let $x := (x(1), x(2), \dots)$

Here, $l^1 := \{x : x(j) \in \mathbb{K} \ \forall \ j \in \mathbb{N}$ and $\sum_{j=1}^{\infty} |x(j)| < \infty\}$,

$l^2 := \{x : x(j) \in \mathbb{K} \ \forall \ j \in \mathbb{N}$ and $\sum_{j=1}^{\infty} |x(j)|^2 < \infty\}$,

$l^\infty := \{x : x(j) \in \mathbb{K} \ \forall \ j \in \mathbb{N}$ and $\sup_{j \in \mathbb{N}} |x(j)| < \infty\}$,

$c_0 := \{x : x(j) \in \mathbb{K} \ \forall \ j \in \mathbb{N}$ and $x(j) \to 0$ as $j \to \infty\}$,

$c_{00} := \{x : (x(j)) \in \mathbb{K} \ \forall \ j \in \mathbb{N}$ and $\exists \ N \in \mathbb{N}$ s.t. $x(n) = 0 \ \forall \ n \geq N\}$.

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(7) Prove that a normed space X is finite dimensional iff the closed unit ball $M = \{x : \|x\| \leq 1\}$ in X is compact.

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(8) Show that the interior of a proper subspace of a normed linear space is empty.

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(9) Using the above result (8) and Baire category theorem or otherwise, prove that a Banach space cannot have a countably infinite basis.

Hence show that c_{00} is not a Banach space with respect to any norm.

(10.) Give an example of an absolutely convergent series in c_{00} with $\|\cdot\|_2$ that is not convergent. Is this possible if c_{00} is replaced by l^2?

(11) For $1 \le p \le r \le \infty$, show that

$l^p(\mathbb{N}) \subseteq l^r(\mathbb{N})$, $L^r[a,b] \subseteq L^p[a,b]$.

Also show that the above inclusions are strict if $p < r$.

(12) Prove that $C'[a,b]$, the space of all continuously differentiable functions $f : [a,b] \to \mathbb{K}$ is a Banach space with respect to the norm

$$\|f\|_* = \|f\|_\infty + \|f'\|_\infty, \quad \forall f \in C'[a,b].$$

(13) Show that for $x \in \mathbb{K}^n$, $\|x\|_p \to \|x\|_\infty$ as $p \to \infty$.

(14) Show that there does not exist any $c > 0$ s.t. $\|x\|_\infty \le c\|x\|$, $\forall x$ in $C[a,b]$.

(15) Let $t_1, t_2, \ldots, t_n$ be distinct points in $[a,b]$. For $x \in C[a,b]$, let

$$\eta_p = \begin{cases} \left(\sum_{j=1}^{n} |x(t_j)|^p\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \max\{|x(t_j)| : j = 1, 2, \ldots, n\} & \text{if } p = \infty. \end{cases}$$

Show that η_p is a seminorm on $C[a,b]$ but not a norm on $C[a,b]$.