Application of Sylow's Thm.

Lecture 14

14/02/2022

Classification of gps of order 12

$$12 = 3 \cdot 4$$

n_3 # of Sylow 3-subgp

$n_3 \mid 4$ and $n_3 \equiv 1 \pmod 3$

$\Rightarrow n_3 = 1, 4$

n_2 # of Sylow 2-subgp.

$n_2 \mid 3$ and $n_2 \equiv 1 \pmod 2$

$\Rightarrow n_2 = 1, 3.$

Let H be a Sylow 2-subgp

$|H| = 4$ then

$$H \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \end{cases}$$

V_4 - Klien 4 gp.

$\underline{\underline{p^2}}$

$$\begin{cases} \mathbb{Z}/p^2\mathbb{Z} \\ \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \end{cases}$$

and K = Sylow 3-subgp

$|K| = 3$, $K \cong \mathbb{Z}/3\mathbb{Z}$.

$H \cap K = \{1\}$ and $|HK| = 12$.

__Formula__.
$$\boxed{|HK| = \frac{|H| \cdot |K|}{|H \cap K|}} \quad \text{(Ex)}.$$

__Lemma__. Either H or K is a normal subgp of G.

__Pf__: Let K be a Sylow 3-subgp which is not normal then $n_3 = 4$.
Any two Sylow 3-subgps intersect trivially.
$\therefore$ # of elts in all Sylow 3-subgp is 8. apart from identity.

Hence there is only one Sylow 2-subgp in G, $\therefore$ $H \trianglelefteq G$.

<u>case 1</u>: <u>H and K both are normal</u>

HK is a subgp and $G = HK$.

and $H \cap K = \{1\}$.

$\therefore$ $G \cong H \times K$.

Hence $G \cong$

- $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/12\mathbb{Z}$
- $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

<u>case 2</u>. let $H \trianglelefteq G$ and $K \ntrianglelefteq G$.

Hence $\exists$ 4 Sylow 3-subgp of G

say $\{K_1, K_2, K_3, K_4\} = A$.

Consider the gp action on the set A

$|A| = 4$.

$$G \times A \longrightarrow A$$

$$(g, k_i) \longmapsto g k_i g^{-1}$$

which can be represented as

$$\phi : G \longrightarrow S_4.$$

$$g \longmapsto \sigma_g$$

$\sigma_g : A \to A$

$\sigma_g(k_i) = g k_i g^{-1}$.

ϕ is a gp homo.

$$\ker \phi = \{ g \in G \mid g k_i g^{-1} = k_i \ \forall i \}.$$

$$= \bigcap_{i=1}^{4} C(k_i)$$

$$C(k_i) = \{ g \in G \mid g k_i g^{-1} = k_i \}.$$

Since all Sylow 3-subgps are conjugate so A has only one orbit

$$|A| = |O(K_1)|$$

$$= [G : C(K_1)]$$

$$= \frac{|G|}{|C(K_i)|}$$

$$\Rightarrow \quad |C(K_i)| = 3.$$

Since $K_i \subseteq C(K_i)$ and $|K_i| = 3$.

$$\therefore \quad C(K_i) = K_i$$

$$\ker \phi = \bigcap_{i=1}^{4} C(K_i) = \bigcap_{i=1}^{4} K_i = \{1\}.$$

$$\therefore \quad G \cong \phi(G) \subseteq S_4.$$

$$|G| = |\phi(G)| = 12.$$

If $x \in G$ is an elt of order 3

$\phi(x)$ is an elt of order 3 in S_4

Since 3-cycles are order 3 elts

in S_4. Therefore $\phi(x)$ will be

a 3-cycle. and a 3-cycle is

an even permutation. There are

8 elts of order 3 in G.

$$|\phi(G) \cap A_4| \geq 8.$$

Since $|A_4| = 12$ and $|\phi(G)| = 12$.

$\therefore$ $\phi(G) \subseteq A_4$ infact $\phi(G) = A_4$.

Thus $G \cong A_4$.

Case 3: $K \triangleleft G$ and $H \ntriangleleft G$.

Let $K = \langle x \rangle$ s.t $|x| = 3$

$$H \begin{cases} \langle y \rangle \text{ s.t } |y| = 4 \\ \{1, u, v, uv \mid u^2 = 1, v^2 = 1, uv = vu\} \end{cases}$$

Consider the case when $H = \langle y \rangle$.

then $yxy^{-1} \in \langle x \rangle = \{1, x, x^2\}$.

$\Rightarrow yxy^{-1} = x$ or x^2.

If $yxy^{-1} = x \Rightarrow G$ is abelian (Ex 1).

$\Rightarrow H \triangleleft G$, which is a contradiction.

$\therefore yxy^{-1} = x^2 \Rightarrow yx = x^2 y$.

$\therefore G \cong \langle x, y \mid |x| = 3, |y| = 4, yx = x^2 y \rangle$.

Consider the case when

$$H \cong \left\{ 1, u, v, uv \;\middle|\; \begin{array}{l} |u| = 2, \\ |v| = 2, \\ uv = vu \end{array} \right\}.$$

$\underline{uxu^{-1} = x^a}$, $\underline{vxv^{-1} = x^b}$

where a, b is either 1 or -1.

and check that $\boxed{uv\,x\,(uv)^{-1} = x^{ab}}$.

If $a = b = 1$. then G is abelian which is a contradiction.

Assume $a = 1$, $b = -1$. then

$ux = xu \Rightarrow |xu| = 6$.

let $z = xu$ then $|z| = 6$

and $v \notin \langle z \rangle$.

Let $v \in \langle z \rangle. \Rightarrow v = z^c$

But $v^2 = 1. \Rightarrow (z^c)^2 = 1.$

$$z^{2c} = 1.$$

$$x^{2c} u^{2c} = 1.$$

$$\Rightarrow x^{2c} = 1.$$

$\underline{|x| = 3}$.

$$\underline{\underline{3 \nmid 2c}}.$$

$$v z v^{-1} = v u x v^{-1} = u v x v^{-1}$$
$$= u x^{-1}$$
$$= u^{-1} x^{-1}$$
$$= (xu)^{-1}$$
$$= z^{-1}.$$

$$G \cong \langle v, z \mid |z| = 6, |v| = 2, vzv^{-1} = z^{-1} \rangle.$$

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