Group Theory

Lecture 5

13/01/2022

Group Homomorphism :

$f : G_1 \to G_2$ is a group homo if

$$f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$$

Properties of gp homo :

(1) If $f : G_1 \to G_2$ is a gp homo then $f(1_{G_1}) = 1_{G_2}$ and $f(a^{-1}) = f(a)^{-1}$

[Hint : use $1 \cdot 1 = 1$. ; use $a \cdot a^{-1} = 1$.]

(2) Let $G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3$ where f and g are group homo. Then $g \circ f : G_1 \to G_3$ is also a gp homo.

Symmetric Group :

Ex 1. S_n is gen by trasposition (2 cycles)

$$(a_1\ a_2 \cdots a_k) = (a_1\ a_k)(a_1\ a_{k-1}) \cdots (a_1\ a_3)(a_1\ a_2)$$

(*).

Defn. A permutation in S_n is called an even permutation if it is a product of even number of traspositions. Otherwise it is called an odd permutation.

$n=3$

$$P(x_1, x_2, x_3) = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$$

Justification :

Consider the poly

$$P(x_1, \cdots, x_n) = \prod_{n \geq i > j \geq 1} (x_i - x_j)$$

Let $\sigma \in S_n$ we define

$$\sigma P(x_1, \dots, x_n) = \prod_{n \geq i > j \geq 1} \left(x_{\sigma(i)} - x_{\sigma(j)}\right)$$

Let $n = 3$, $\sigma = \left(1\ 2\ 3\right)$.

$$\sigma P(x_1, x_2, x_3) = \left(x_{\sigma(3)} - x_{\sigma(2)}\right)\left(x_{\sigma(3)} - x_{\sigma(1)}\right)\left(x_{\sigma(2)} - x_{\sigma(1)}\right).$$

$$= \left(x_1 - x_3\right)\left(x_1 - x_2\right)\left(x_3 - x_2\right).$$

$$= P(x_1, x_2, x_3).$$

If σ is a transposition then $\sigma P = -P$

Defn $\sigma \in S_n$ is called an even permutation if $\sigma P = P$ and σ is called an odd permutation if $\sigma P = -P$.

Remark. Let $\sigma \in S_n$ be a k-cycle. If k is even the σ is an odd permutation and if k is odd then σ is an even permutation. This follows from (*).

$(1) \in A_n \subseteq S_n$.

A_n = The set of all even permutations.

$\sigma_1, \sigma_2 \in A_n$.

$\sigma_1 \cdot \sigma_2 \in A_n$.

$\sigma_1 \in A_n$. $\sigma_1^{-1} \in A_n$.

Propn The set of all even permutations denoted by A_n is a subgp of S_n.

$|S_n| = n!$ $\qquad |A_n| = n!/2$.

Let X = Set of all even permutations

Y = set of all odd permutations.

$$f : X \longrightarrow Y$$

$$f(\sigma) = (12)\sigma$$

Check that f is inj

Let $\tau \in Y$ then consider $(12)\tau \in X$,

$$f((12)\tau) = \tau .$$

f is surjective. f is a bijection.

$$\therefore |X| = |Y| = n!/2 .$$

Hence The no. of even permutations is $\frac{n!}{2}$,

$$\text{sign} : S_n \longrightarrow \{1, -1\}$$ is a gp wrt multipl.

$$\sigma \longmapsto \text{sig}\,\sigma .$$

$$\text{sign}\,\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

is a gp homo.

Ex. If σ is a k-cycle then $|\sigma| = k$ and if π is a product of disjoint cycles of length $k_1, \ldots, k_r$

$$|\pi| = \mathrm{l.c.m}(k_1, \ldots, k_r).$$

Every group homo $\phi : G_1 \to G_2$ determines two important subgps namely $\ker \phi$ and $\operatorname{im} \phi$.

The image of a gp homo is defined as

$$\operatorname{im} \phi = \{x \in G_2 \mid x = \phi(a) \text{ for some } a \in G_1\}.$$

Ex check $\operatorname{im}\phi$ is a subgp of G_2.

The kernel of a gp homo is defined as

$$\ker \phi = \{x \in G_1 \mid f(x) = 1_{G_2}\}.$$

Q. Is $\ker\phi$ a subgp of G_1?

$1_{G_1} \in \ker\phi$ as $\phi(1_{G_1}) = 1_{G_2}$.

Let $f, g \in \ker\phi$

$$\phi(fg) = \phi(f)\phi(g) = 1_{G_2} \cdot 1_{G_2} = 1_{G_2}$$

If $f \in \ker\phi$. wT $f^{-1} \in \ker\phi$.

$$\phi(f^{-1}) = \phi(f)^{-1} = 1_{G_2}^{-1} = 1_{G_2}.$$

$\therefore$ $f^{-1} \in \ker\phi$.

Hence $\ker\phi$ is a subgp of G_1.

Note: Let $g \in \ker\phi$ and $h \in G_1$.

$$\phi(hgh^{-1}) = \phi(h)\phi(g)\phi(h)^{-1}$$
$$= \phi(h) \cdot 1_{G_2}\phi(h)^{-1} = 1_{G_2}$$

$\Rightarrow$ $hgh^{-1} \in \ker\phi$ $\forall h \in G_1$.

Defn. A subgp N of G is called a normal subgp of G ($N \trianglelefteq G$) if $ghg^{-1} \in N$ $\forall\, h \in N$ and $g \in G$.

Note: The kernel of a gp homo is always a normal subgp.