Lecture - 8 (30-01-2024)

Note :— The previous theorem gives how to obtain a unique topology on a nonempty set X from the given a collection of subsets of X whose union is X.

Ex: $X = \{1, 2, 3, 4, 5\}$

$S = \{\{1\}, \{1,2,3\}, \{2,3,4\}, \{3,5\}\}$,

clearly $X = \bigcup_{s \in S} s = \{1\} \cup \{2,3,4\} \cup \{3,5\}$

Let $\mathcal{B}$ be the collection of all intersection of finite number of members of S.

$\mathcal{B} = \{\{1\}, \{1,2,3\}, \{2,3,4\}, \{3,5\}, \{2,3\}, \{3\}\}$

Let $T = \{\phi$ and all possible subsets of X which can be expressed as union of members of $\mathcal{B}\}$.

$= \{ X, \phi, \{1\}, \{1,2,3\}, \{2,3,4\},$
$\{3,5\}, \{2,3\}, \{3\}, \{2,3,5\},$
$\{1,2,3,4\}, \{2,3,4,5\}, \{1,3,5\},$
$\{1,3\}, \{1,2,3,5\} \}.$

H.W Let $X = \{a, b, c, d\}$

and $S = \{ \{a,b\}, \{b,c\}, \{d\} \}$.

Find the unique topology generated by S.

Subbase for the subspace topology:

Theorem: Let S be a subbase for a topology T on X and A be a subset of X. Then $S_A = \{ s \cap A \mid s \in S \}$ is a subbase for the relative topology T_A on A.

Proof: Let $H \in T_A$.

Then there exists an open set G in T such that $H = A \cap G$.

$\because$ $G \in T$ and S is a sub-base for T on X, implies

$$G = \bigcup_i (S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k})$$

where $S_{i_1}, S_{i_2}, \ldots S_{i_k} \in S$ $\forall\, i, k$.

$\therefore$

$$\begin{aligned} H &= A \cap G \\ &= A \cap \left(\bigcup_i (S_{i_1} \cap S_{i_2} \cdots \cap S_{i_k}) \right) \\ &= \bigcup_i \left[(A \cap S_{i_1}) \cap (A \cap S_{i_2}) \cap \cdots \cap (A \cap S_{i_k}) \right] \end{aligned}$$

$\therefore$ $H \in T_A$ is the union of intersection of finite number of members of S_A.

$\therefore$ S_A is a sub-base for T_A.

Base for the subspace topology :—

Theorem: If $\mathcal{B}$ is a base for a topology τ on a nonempty set X and $Y \subset X$, then $\mathcal{B}_Y = \{ Y \cap B \mid B \in \mathcal{B} \}$ is a base for a subspace topology τ_Y on Y.

Proof: Let $H \in \tau_Y$.

Then $H = Y \cap G$, for some $G \in \tau$.

Now if $x \in H = Y \cap G$

$\Rightarrow x \in Y$ and $x \in G$

$\because x \in G$ and $\mathcal{B}$ is a base for τ,

$\exists\ B \in \mathcal{B} \ni x \in B \subset G$

or $G = \bigcup_i B_i, \quad B_i \in \mathcal{B}$.

$\because$ $x \in Y$ and $x \in B \Rightarrow x \in Y \cap B$

And

$x \in Y \cap B \subset Y \cap G = H$.

$\Rightarrow$ $\mathcal{B}_Y$ is a base for T_Y.

OR

$$H = Y \cap G = Y \cap \left(\bigcup_i B_i \right)$$

$$= \bigcup_i (Y \cap B_i)$$

= union of members of $\mathcal{B}_Y$

$\therefore$ $\mathcal{B}_Y$ is a base for T_Y on Y.

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Ex:- let $(\mathbb{R}, \mathcal{U})$ be a usual topological space. Then

$\mathcal{B} = \{ (a, b) \,/\, a < b,\ a, b \in \mathbb{R} \}$ is a base for $\mathcal{U}$ on $\mathbb{R}$.

let $A = [0, 1) \subset \mathbb{R}$.

Consider the subspace topology (A, T_A), where

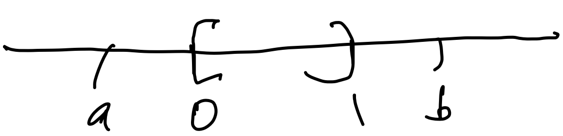
$$T_A = \{ A \cap G \mid G \in U \}.$$

Now

$$B_A = \{ A \cap (a,b) \,/\, a, b \in \mathbb{R},\ a < b \}$$

$$= \{ [0,b),\ \{(a,b),\ 0 < a < b \le 1\} \}$$

is a base for T_A on A.



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Theorem: Let A be a class of subsets of a nonempty set X. Then the topology T on X generated by A is the intersection of topologies on X which contain A.

Proof: Let $\{T_i\}$ be a collection of topologies on X such that $A \subset T_i,\ \forall i$.

Let $\mathcal{T}^* := \bigcap_i \mathcal{T}_i$.

$\because \mathcal{A} \subset \mathcal{T}_i, \forall i \Rightarrow \mathcal{A} \subset \bigcap_i \mathcal{T}_i = \mathcal{T}^*$

$\therefore \mathcal{A} \subset \mathcal{T}^*$.

We Prove $\mathcal{T} = \mathcal{T}^*$.

$\because \mathcal{T}$ is a topology Containing $\mathcal{A}$ and $\mathcal{T}^*$ is the intersection of all those topologies on X containing $\mathcal{A}$, it follows that

$$\mathcal{T}^* \subseteq \mathcal{T} \quad \text{———} (1).$$

On the other hand, let $G \in \mathcal{T}$, then

$$G = \bigcup_i (S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}),$$

$S_{i_k} \in \mathcal{A}$.

$\because \mathcal{A} \subset \mathcal{T}^*$ and $S_{i_1}, S_{i_2}, \ldots S_{i_k} \in \mathcal{A}$

$\Rightarrow S_{i_1}, S_{i_2} \ldots S_{i_k} \in \mathcal{T}^* \quad \forall i, k$

$(\forall i, k$.

$\because \mathcal{T}^*$ is a topology on X, we have $g_{i_1} \cap g_{i_2} \cap \cdots \cap g_{i_k} \in T^*$

$$\Rightarrow \bigcup_i (g_{i_1} \cap g_{i_2} \cdots \cap g_{i_k}) \in T^*$$

$$\Rightarrow G \in T^*$$

$$\therefore T \subseteq T^* \text{ --- } (2)$$

$\therefore$ from (1) & (2) we have

$$T = T^*$$

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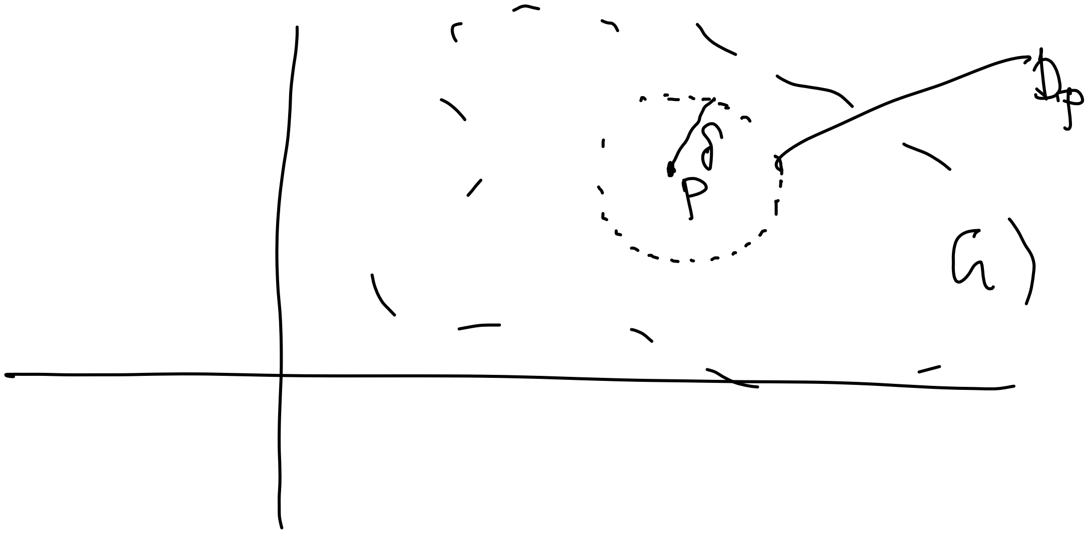
* The topology generated by a given class of sets in X is the coarsest topology on X containing that class.

Local Base :—

Let (X, T) be a topological space, and 'p' be any arbitrary point in X. A class B_p of open sets containing 'p' is called local base at 'p'

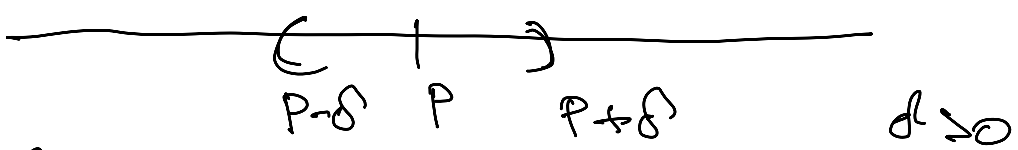
if for each open set G containing 'p', there exists $G_p \in \mathcal{B}_p$ such that
$$p \in G_p \subset G.$$

Ex: $(\mathbb{R}^2, U)$ be usual topological space and $p \in \mathbb{R}^2$.



let $\mathcal{B}_p$ be the class of all open discs centered at p with radius $\delta \in \mathbb{R}$.
Then $\mathcal{B}_p$ is a local base at $p \in \mathbb{R}^2$.

Illy in $(\mathbb{R}, U)$, for any $p \in \mathbb{R}$,



$\mathcal{B}_p = \{ (p-\delta, p+\delta) \mid \delta > 0, \delta \in \mathbb{R} \}$ is a local base at $p \in \mathbb{R}$.

Theorem: A point 'p' in a topological space (X, T) is an accumulation point of a subset A of X iff each member of some local base B_p at 'p' contains a point of A different from 'p'.

Proof: Let $p \in A'$. Then for every open G with $p \in G$, we have

$$(G - \{p\}) \cap A \neq \phi.$$

But $B_p \subset T$, so in particular, we have

$$(B - \{p\}) \cap A \neq \phi, \forall B \in B_p.$$

Conversely assume that

$$(B - \{p\}) \cap A \neq \phi, \forall B \in B_p. \quad \text{———} \ (*)$$

Claim: $p \in A'$.

let G be any open set containing p.

$\because$ $\mathcal{B}_p$ is a local base at p, and G is an open set containing p, by definition, there exists some $B \in \mathcal{B}_p$ such that

$$p \in B \subset G.$$

Now

$$(G - \{p\}) \cap A \supset \big((B - \{p\}) \cap A\big) \neq \varphi$$

(by $*$)

$$\Rightarrow \quad p \in A'$$

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Theorem: A sequence $\{a_1, a_2, a_3, \ldots\}$ of points in a topological space $(X, \mathcal{T})$ converges at $p \in X$ iff each member of some local base at $\mathcal{B}_p$ at p contains almost all terms of the sequence $\{a_n\}$.

Proof: Let $a_n \longrightarrow p$.

Then every open set G containing p contains almost all terms of the sequence $\{a_n\}$.

i.e., $\exists\ n_0 \in N \ni a_n \in G,\ \forall\ n \geq n_0$.

$\because \mathcal{B}_p \subset T$, so in particular, for every member $B \in \mathcal{B}_p$ contains almost all terms of the sequence $\{a_n\}$, i.e. $\exists\ n_1 \ni$

$$a_n \in B,\ \forall\ n \geq n_1.$$

$$\forall\ B \in \mathcal{B}_p.$$

Conversely assume that every member $B \in \mathcal{B}_p$ contains almost all terms of the sequence $\{a_n\}$.

Let G be an open set containing p. Then by definition of local base, there exists some $B \in \mathcal{B}_p$ such that $p \in B \subset G$.

$\Rightarrow$ G also contains almost all terms of the sequence $\{a_n\}$ containing p.

$$\therefore a_n \longrightarrow p$$

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Def :— Let (X, T_1) and (Y, T_2) be any two topological spaces.

Let $\mathcal{B} = \{ U \times V \;/\; U \text{ is open in } X, V \text{ is open in } Y \}$

$= \{ U \times V \;/\; U \in T_1, V \in T_2 \}$.

The Product topology on $X \times Y$ is the topology generated by $\mathcal{B}$.

$\because X \in T_1$ and $Y \in T_2$

$\Rightarrow X \times Y \in \mathcal{B}$.

$\therefore X \times Y$ itself is a basis element.

Now $B_1 = u_1 \times v_1$, $B_2 = u_2 \times v_2$ be any two elements of $\mathcal{B}$, for any $u_1, u_2 \in T_1$, $v_1, v_2 \in T_2$.

Consider

$$B_1 \cap B_2 = (u_1 \times v_1) \cap (u_2 \times v_2)$$

$$= (u_1 \cap u_2) \times (v_1 \cap v_2)$$

$$\in \underline{B}$$

[$\because$ $u_1 \cap u_2 \in T_1$ and $v_1 \cap v_2 \in T_2$

$\Rightarrow (u_1 \cap u_2) \times (v_1 \cap v_2) \in \underline{B}$]

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[Attendence:

65, 11, 17, 19, 39, 27, 06, 09, 38].