Lecture 25

$\Leftarrow$: Assume $m(D)=0$.

that is, f is continuous a.e on $[a,b]$.

To show: f is R-integrable.

Let $D_r = \left\{ x \in [a,b] \;\middle|\; \omega_f(x) \geq \frac{1}{r} \right\}$, $\forall r \geq 1$

Then $D = \bigcup_{r=1}^{\infty} D_r$

Since each $D_r \subseteq D$ & $m(D)=0$, implies that

$m(D_r) = 0 \quad \forall r \geq 1$.

$\Rightarrow$ there exists $\{I_{n,r}\}_{n=1}^{\infty}$ open intervals such that $D_r \subseteq \bigcup_{n=1}^{\infty} I_{n,r}$ &

$\sum_{n=1}^{\infty} l(I_{n,r}) < \frac{1}{r}$.

Let $A_r = \bigcup_{n=1}^{\infty} I_{n,r} \quad \forall r \geq 1$.

Then A_r is an open set.

Let $B_r = [a,b] \setminus A_r$

$$= [a,b] \setminus \left(\bigcup_{n=1}^{\infty} I_{n,r} \right)$$

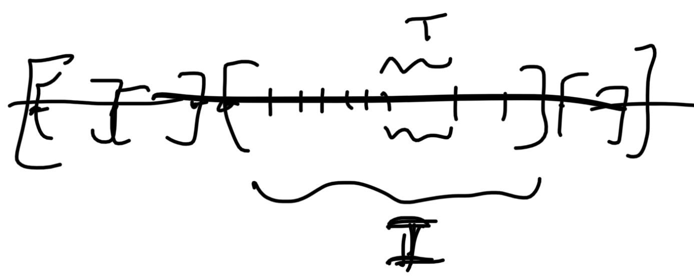
$$= \bigcap_{n=1}^{\infty} \underbrace{\left([a,b] \setminus I_{n,r} \right)}_{\text{closed set \& bounded.}}$$

= a union of finite number of closed subintervals of $[a,b]$
(we can do always this).

Let I be a typical closed subinterval of B_r.

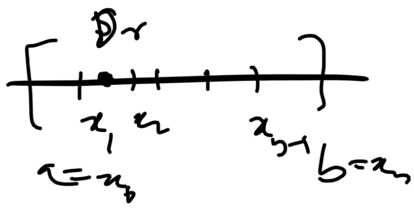
For $x \in I \subset B_r$, then $x \notin A_r$ which implies $x \notin D_r$ & hence $\omega_f(x) < \frac{1}{r}$.

Then by the Lemma 2, there exists a $\delta > 0$ (depends only on r) such that I can be further subdivided into a finite no. of subintervals T of length $< \delta$ in which $\Omega_f(T) < \frac{1}{r}$.



The end points of all these subintervals T_i determine a partition P_γ of $[a,b]$.

If $P \supseteq P_\gamma$ is any partition of $[a,b]$, finer than P_γ, we can write



$$U(P,f) - L(P,f) = \sum_{k=1}^{n} \left(M_k(f) - m_k(f)\right)\left(x_k - x_{k-1}\right).$$

$$= S_1 + S_2$$

where S_1 = the sum of those terms coming from subintervals containing points of D_γ in its interior.

& S_2 = the sum of the remaining terms not in S_1.

In the k^{th} term of S_2 we have

$$M_k(f) - m_k(f) < \frac{1}{\gamma}$$

& $S_2 = \sum_{k}' \left(M_k[f] - m_k[f]\right)(x_k - x_{k-1})$

$[x_{k-1}, x_k]$ contains pts of D_r not in its interior

$$< \frac{1}{r} \sum_{''}' (x_k - x_{k-1})$$

$$< \frac{b-a}{r}.$$

Since A_r covers all the intervals contributing to S_1, we have $S_1 \le \frac{M-m}{r}$,

where $M = \sup_{x\in[a,b]} (f(x))$,

$m = \inf_{x\in[a,b]} (f(x))$.

$\therefore$ $U(P,f) - L(P,f) = S_1 + S_2$

$$\le \frac{M-m}{r} + \frac{b-a}{r} = \frac{M-m+b-a}{r}.$$

True for all $r \geq 1$.

Thus $U(P,f) - L(P,f)$ sufficiently small enough.

$\therefore$ f satisfies Riemann's condition.

$\Rightarrow$ f is R-integrable.

Coroll:- Suppose $f : [a, b] \to \mathbb{R}$ is bounded &

R-integrable on $[a,b]$. Then f is measurable.

proof:- By Lebesgue's criterion, if f is R-integrable,

then f is continuous a.e on $[a,b]$

$\Rightarrow$ $f = g$ a.e, on $[a, b]$ where g is a continuous function.

($\because$ g is measurable)

$\Rightarrow$ f is measurable.

Definitions:- (Convergence in measure)

We say that a sequence of measurable functions $\{f_n\}$ defined on a measurable set E, converges to f in measure, denote by

$f_n \xrightarrow{m} f$ as $n \to \infty$, if for any $\varepsilon > 0$,

$$m\Big(\underbrace{\{x \in E \mid |f_n(x) - f(x)| \geq \varepsilon\}}_{\mathcal{M}}\Big) \to 0 \text{ as } n \to \infty.$$

$= g(x)$ $\quad g^{-1}([\varepsilon, \infty))$

That is, $\displaystyle \lim_{n \to \infty} m\big(\{x \in E \mid |f_n(x) - f(x)| \geq \varepsilon\}\big) = 0.$

Remark: $f_n \not\xrightarrow{m} f$ on E if and only if there exists $\varepsilon > 0$ such that

$$m\Big(\underbrace{\{x \in E \mid |f_n(x) - f(x)| \geq \varepsilon\}}_{\mathcal{M}}\Big) \not\to 0 \text{ as } n \to \infty.$$

Defn (Almost uniform Convergence)

We say that a sequence of measurable functions $\{f_n\}$ converges almost uniform (a.u) to f on E, if for given $\varepsilon > 0$,

there exists $A_\varepsilon \subseteq E$ measurable subset such that $m(A_\varepsilon) < \varepsilon$ & $f_n \to f$ uniformly on $E \setminus A_\varepsilon$.

<u>Remark</u> $f_n \not\to f$ a.u on E, if there exists $\varepsilon > 0$ & for any $A \subseteq E$ measurable subset with $m(A) < \infty$ we have $f_n \not\to f$ uniformly on $E \setminus A$.