Lecture 21

proof of Littlewood 2nd principle:-

Given $f : E \to \mathbb{R}$ is a measurable function &
$m(E) < \infty$.

We have that there exists a sequence of step functions $\{f_n\}$ such that
$f_n \to f$ a.e on E (p.w).

Each f_n is a step function, therefore by Proposition (2), there exists a set $E_n \subseteq E$ such that
$m(E_n) < \frac{1}{2^n}$ & f_n is continuous on $E \setminus E_n$.

By Egorov's theorem, there exists a closed set $A_{\varepsilon/3} \subseteq E$ on which $f_n \to f$ uniformly
& $m(E \setminus A_{\varepsilon/3}) \leq \varepsilon/3$.

Consider $F' = A_{\varepsilon/3} \setminus \left(\bigcup_{n \geq N} E_n \right) \subseteq E \setminus \bigcup_{n \geq N} E_n \subseteq E \setminus E_n \quad \forall n \geq N$, where
$N \in \mathbb{N}$ so that $\sum_{n \geq N} \frac{1}{2^n} < \frac{\varepsilon}{3}$

Now for any $\underline{n \geq N}$, the function $\underline{f_n}$ is Continuous on F'.

$\left(\because F' \subseteq \underline{E \setminus E_n} \right)$

$\therefore$ f_n is continuous on F' $\forall\ n \geq N$

& $f_n \to f$ uniformly on F' $\quad \left(F' \subseteq A_{\varepsilon/3} \right)$

$\Rightarrow$ f is Continuous on F'

(Fact: The uniform limit of seq. of Continuous functions is Continuous)

Note that F' is measurable.

Then there is a closed subset $\underline{F_\varepsilon \subseteq F'}$

such that $m(F' \setminus F_\varepsilon) < \frac{\varepsilon}{3}$.

Now $\quad E \setminus F_\varepsilon \subseteq \left(E \setminus A_{\varepsilon/3}\right) \cup \left(A_{\varepsilon/3} \setminus F'\right) \cup \left(F' \setminus F_\varepsilon\right)$

$\left(F_\varepsilon \subseteq F' \subseteq A_{\varepsilon/3} \subseteq E \right)$

$$\Rightarrow \quad m(E \setminus F_\varepsilon) \leq m\left(\left(E \setminus A_{\varepsilon/3}\right) \cup \left(A_{\varepsilon/3} \setminus F'\right) \cup \left(F' \setminus F_\varepsilon\right)\right)$$

$$\leq m\left(E \setminus A_{\varepsilon/3}\right) + m\Big(\underbrace{A_{\varepsilon/3} \setminus F'}_{= \bigcup_{n \geq N} E_n}\Big) + m\left(F' \setminus F_\varepsilon\right)$$

$$< \frac{\varepsilon}{3} + m\left(\bigcup_{n \geq N} E_n\right) + \frac{\varepsilon}{3}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{n \geq N} m(E_n)$$

$$< \frac{2\varepsilon}{3} + \sum_{n \geq N} \frac{1}{2^n}$$

$$< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus f is continuous on F_ε &

$$m(E \setminus F_\varepsilon) < \varepsilon.$$

Recall:- $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ simple function. Then

$$\mathcal{L}\int f \quad \text{or} \quad \int^{\mathcal{L}} f = \sum_{i=1}^{n} a_i \, m(A_i)$$

Stage two: Bounded functions supported on a set of finite measure:

Def:- The support of a function $f: E \to \mathbb{R}$ is defined as

$$\operatorname{supp}(f) := \{ x \in E \mid f(x) \neq 0 \}.$$

We say that f is supported on set A, if $f(x) = 0$ whenever $x \notin A$.

Example:- Let $f = \chi_{[0,1]} + 2\chi_{[2,3]}$. Then f is supported on $[0,1] \cup [2,3]$

Remark:- If f is measurable, then $\operatorname{supp}(f)$ is measurable.

Lemma:- Let $f: E \to \mathbb{R}$ be a bounded function, where $E \subseteq \mathbb{R}^d$ is a measurable set of finite measure. If $\{\varphi_k\}_{k=1}^{\infty}$ is any sequence of simple functions bounded by M, that is, $\underline{|\varphi_k(x)| \leq M}$ $\forall x, \forall k \geq 1$;

& supported on E & with $\varphi_k(x) \to f(x)$ for a.e x.

Then (i) $\lim\limits_{k\to\infty} \int \varphi_k$ exists

(ii) If $f = 0$ a.e, then $\lim\limits_{k\to\infty} \int \varphi_k = 0$.

proof:-

(i) By Littlewood 3rd principle, given $\varepsilon > 0$,

there exists a closed set $A_\varepsilon \subseteq E$ such that

$m(E \setminus A_\varepsilon) \leq \varepsilon$ & $\varphi_n \to f$ uniformly on A_ε.

Set $\quad I_n = \int_E \varphi_n \quad \forall n \geq 1$

Now
$$\left| \int_E \varphi_n - \int_E \varphi_m \right| = | I_n - I_m |$$
$$\leq \int_E |\varphi_n - \varphi_m| \quad \text{(by triangular inequality)}$$
$$= \int_{A_\varepsilon \cup (E \setminus A_\varepsilon)} |\varphi_n - \varphi_m|$$
$$= \int_{A_\varepsilon} |\varphi_n - \varphi_m| + \int_{E \setminus A_\varepsilon} |\varphi_n - \varphi_m|$$

$$\leq \int_{A_\varepsilon} |\varphi_n - \varphi_m| + \int_{E \setminus A_\varepsilon} \left(|\varphi_n| + |\varphi_m|\right)$$

$$\leq \int_{A_\varepsilon} |\varphi_n - \varphi_m| + \int_{E \setminus A_\varepsilon} 2M$$

$$\leq \int_{A_\varepsilon} |\varphi_n - \varphi_m| + 2M\, m(E \setminus A_\varepsilon)$$

By uniform convergence, $\forall\, x \in A_\varepsilon$ & for m, n sufficiently large, we have

$$|\varphi_n(x) - \varphi_m(x)| < \varepsilon$$

$$\therefore\ |I_n - I_m| \leq \int_{A_\varepsilon} \varepsilon + 2M\varepsilon$$

$$= \varepsilon\, m(A_\varepsilon) + 2M\varepsilon$$

$$\therefore\ |I_n - I_m| \leq \varepsilon \cdot \left(\underbrace{m(A_\varepsilon)}_{\checkmark} + 2M\right)$$

for m, n sufficiently large.

$\therefore$ $\{I_n\}$ is a Cauchy sequence in $\mathbb{R}$.

Since $\mathbb{R}$ is complete, $\{I_n\}$ is convergent.

$$\Rightarrow \lim_{n\to\infty} I_n \text{ exists}$$

$$\| $$

$$\lim_{n\to\infty} \int \varphi_n$$

(ii) Suppose $f = 0$ a.e.

& $\varphi_k \to 0$ uniformly on A_ε

& $m(E \setminus A_\varepsilon) \le \varepsilon$.

$$|I_n| = \left| \int_E \varphi_n \right|$$

$$\le \int_E |\varphi_n| = \int_{A_\varepsilon} \underline{\underline{|\varphi_n|}} + \int_{E\setminus A_\varepsilon} |\varphi_n|$$

$$\le \varepsilon\, m(A_\varepsilon) + M\, m(E \setminus A_\varepsilon)$$

$$< m(A_\varepsilon)\,\varepsilon + M \cdot \varepsilon$$

$$= \varepsilon \cdot (m(A_\varepsilon) + M)$$

for n sufficiently large.

$\therefore$ $I_n \to 0$ as $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} \int \varphi_n = 0.$$