lecture-2 (09-01-2024).

Note:- Union of any two topologies need not be a topology on a nonempty set X.

Ex: $X = \{a, b, c\}$

$T_1 = \{X, \varphi, \{a\}\}$, $T_2 = \{X, \varphi, \{b\}\}$

then T_1 and T_2 are topologies on X.

Now

$T_1 \cup T_2 = \{X, \varphi, \{a\}, \{b\}\}$.

$\because \{a\} \cup \{b\} = \{a, b\} \notin T_1 \cup T_2$

$\therefore T_1 \cup T_2$ is not a topology on X.

Problem: Let $X = \mathbb{R}$, the set of real numbers

and

$T = \{E_a = \{(a, \infty) / a \in R\}, \mathbb{R}, \varphi\}$.

Then show that T is a topology on $\mathbb{R}$.

Sol: Clearly $\mathbb{R}, \phi \in T$.

Let $A = \{E_i \in T \mid i \in I\}$, I is the index set

be any arbitrary class of sets of T.

Claim: $\bigcup_i E_i \in T$.

If the index set I is not bounded below,

Then $\inf(I) = -\infty$

$E_i = (a_i, \infty)$, $a_i \in \mathbb{R}$

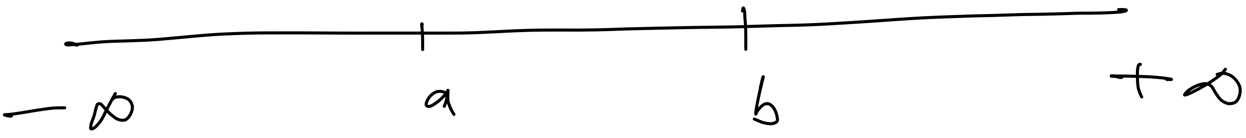
Then $\bigcup_i E_i = (-\infty, \infty) = \mathbb{R} \in T$

If the index set I is bounded below, then $\inf\{I\} = i_0$ (say)

Then $\bigcup_i E_i = E_{i_0} = (i_0, \infty) \in T$

Illy we can show that if

$E_a, E_b \in T \Rightarrow E_a \cap E_b \in T$.



Then

$$E_a \cap E_b = (a, \infty) \cap (b, \infty)$$

$$= \begin{cases} E_b = (b, \infty) & \text{if } a < b \\ E_a = (a, \infty) & b < a \end{cases}$$

$\therefore$ $T = \{\{E_a = (a, \infty) / a \in R\}, R, \phi\}$

is a topology on $\mathbb{R}$.

Problem: let $X = \mathbb{R}$, and

$T = \{\{E_a = (-\infty, a) / a \in R\}, R, \phi\}$.

Show that T is a topology on R.

Coarser and finer topologies :

Let T_1 and T_2 be any two topologies on a nonempty set X. Suppose each T_1-open set is also a T_2-open set, i.e., $T_1 \subseteq T_2$, we say T_1 is coarser or smaller or weaker topology than T_2 or we say T_2 is finer/larger/stronger than T_1.

Ex: Let $(X, \mathcal{D})$ be a discrete topological space and $(X, \mathcal{I})$ be indiscrete topological space, and (X, T) be any topological space.

$$\mathcal{I} \subset T \subset \mathcal{D}$$

i.e., T is coarser than $\mathcal{D}$ and finer than $\mathcal{I}$.

Ex: $X = \mathbb{R}^2$ and U be the class of open sets in $\mathbb{R}^2$.

i.e., let $(\mathbb{R}^2, U)$ be usual topological space.

Also let $(\mathbb{R}^2, T)$ be the cofinite topological space.

★ Since every finite subset of $\mathbb{R}^2$ is a U-closed set, hence the complement of any finite subset of $\mathbb{R}^2$ is a U-open set.

$$T = \{ A \subseteq \mathbb{R}^2 \mid A^c \text{ is finite} \}$$

$$U = \{ G \subseteq \mathbb{R}^2 \mid G \text{ is open} \}$$

$T \subset U$

$A \in T \Rightarrow A^c$ is finite

$\Rightarrow A^c$ is a closed set

$\Rightarrow (A^c)^c$ is open set

$\Rightarrow A \in U$

$\therefore T$ is coarser than U or

U is finer than T.

Subspace topology:

let (X, T) be a topological space, and A be any subset of X.

let $T_A = \{A \cap G \mid G \in T\}$.

then T_A is a topology on A.

Since $X, \varphi \in T$

$\Rightarrow A = A \cap X \in T_A$

$\varphi = A \cap \varphi \in T_A$

Now let $\{H_i \mid i \in I\}$ be any arbitary subclass of T_A.

$\because H_i \in T_A \Rightarrow \exists G_i \in T$ such that $H_i = A \cap G_i \ \forall i$

Now, $G_1, G_2, G_3 \ldots \in T$ and T is topology on X, implies

$$\bigcup_i G_i \in T.$$

Now

$$\bigcup_i H_i = \bigcup_i (A \cap G_i)$$

$$= A \cap (\bigcup_i G_i) \in T_A.$$

Now let $H_1, H_2 \in T_A$

$\Rightarrow \exists\ G_1, G_2 \in T \ni H_1 = A \cap G_1$

$H_2 = A \cap G_2$

Now $\because G_1, G_2 \in T \Rightarrow G_1 \cap G_2 \in T$

$$H_1 \cap H_2 = (A \cap G_1) \cap (A \cap G_2)$$

$$= A \cap (G_1 \cap G_2) \in T_A.$$

Thus T_A is a topology on A and (A, T_A) is a topological space. This topology is called subspace topology on A.

Def :- let A be a non-empty subset of a topological space (X, T). Then the class T_A of all intersections of A with T- open subsets of X is a topology on A. This topology T_A is called relative topology and the topological space (A, T_A)

is called sub-space of (X, T).

Ex: $X = \{a, b, c, d, e\}$.

$T = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

let $A = \{a, d, e\} \subset X$

Then

$T_A = \{A \cap G \mid G \in T\}$

$= \{A \cap X, A \cap \phi, A \cap \{a\}, A \cap \{c, d\}, A \cap \{a, c, d\}, A \cap \{b, c, d, e\}\}$

$= \{A, \phi, \{a\}, \{d\}, \{a, d\}, \{d, e\}\}$

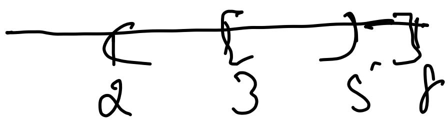
Ex: (R, U) be usual topological space.

$A = [3, 8]$

$U_A = \{A \cap G \mid G \in U\}$

let $G = (2, 5) \in U$

Then



$$A \cap G = [3, 8] \cap (2, 5)$$

$$= [3, 5) \text{ is } U_A\text{-open set}$$

But $[3, 5) \notin U$.

Note: (1) A set may be open relative to a subspace topology, but neither open nor closed in the entire space.

(2) Let (X, T) be a topological space and $A \in T$. Let Y be any subset of X such that $A \subset Y \subset X$. Then A is T_Y-open set.

$\because T_Y = \{Y \cap G \mid G \in T\}$

$\because A \subset Y$ and $A \in T$, then $A = Y \cap A \in T_Y$

Ex: $X = \{a, b, c, d, e\}$

$T = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

$Y = \{a, d, e\}$.

let $A = \{a\} \in T$

Then $A \subset Y \subset X$.

Then $A \cap Y \in T_Y$

$\Rightarrow \{a\} \cap \{a, d, e\} = \{a\} \in T_Y$.

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Problem: Let $f : X \longrightarrow Y$ be a function from a nonempty set X into a topological space (Y, T^*). Let $T = \{f^{-1}(G) \subset X \mid G \in T^*\}$. Then show that T is a topology on X.

$\therefore \quad f^{-1}(Y) = X$

$\quad f^{-1}(\varphi) = \varphi$

$\because Y, \varphi \in T^{*}$

$\Rightarrow \quad X = f^{-1}(Y) \in T$

$\quad \varphi = f^{-1}(\varphi) \in T$

let $f^{-1}(G_1), f^{-1}(G_2), \ldots \ldots \in T$.

Claim: $\bigcup_i f^{-1}(G_i) \in T$.

[$\bigcup_i f^{-1}(G_i) = f^{-1}(\bigcup_i G_i)$

let $x \in \bigcup_i f^{-1}(G_i) \Leftrightarrow x \in f^{-1}(G_{i_0})$

atleast for some i_0.

$\Leftrightarrow f(x) \in G_{i_0}$

$\Leftrightarrow f(x) \in \bigcup_i G_i$

$\Leftrightarrow x \in f^{-1}(\bigcup_i G_i)$]

$\because G_1, G_2, G_3, \ldots \in T^*$

$\Rightarrow \bigcup_i G_i \in T^*$

$\Rightarrow f^{-1}(\bigcup_i G_i) \in T$

$\Rightarrow \bigcup_i f^{-1}(G_i) = f^{-1}(\bigcup_i G_i) \in T$

Illy we can prove that

if $f^{-1}(G_1), f^{-1}(G_2) \in T$

$\Rightarrow f^{-1}(G_1) \cap f^{-1}(G_2) = f^{-1}(G_1 \cap G_2) \in T$ $\left\{ \because G_1, G_2 \in T^* \Rightarrow G_1 \cap G_2 \in T^* \right.$

$\therefore (X, T)$ is a topological space

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Attendance

[32, 27, 57, 06, 11, 23, 41, 26, 43, 60, 46]