Lecture-17 (12-03-2024)

Theorem: Let (X, T) be a topological space. Then the followings are equivalent.

(i) (X, T) is a normal space

(ii) If H is an open superset of a closed set F, then there exists an open set G such that $F \subset G \subset \bar{G} \subset H$.

Proof: (i) $\Rightarrow$ (ii)

Suppose (X, T) is a normal space. Let F be a closed subset of X, H be an open superset of F.

That is we have

$$F \subset H \Rightarrow F \cap H^c = \varphi,$$

Also F and H^c are closed subsets of a normal space X.

by definition of Normal space.

~~Hence~~ There exist two open sets G and G^* such that

$F \subset G$, $H^c \subset G^*$, $G \cap G^* = \varphi$,

$\because G \cap G^* = \varphi \implies G \subset G^{*c}$

Also $H^c \subset G^* \implies G^{*c} \subset H$ and G^{*c} is a closed set.

Combining all these we obtain

$F \subset G \subset G^{*c} \subset H$

But $\overline{G}$ is the smallest closed superset of G and contained in any closed superset of G

$\therefore F \subset G \subset \overline{G} \subset G^{*c} \subset H$

$\implies F \subset G \subset \overline{G} \subset H.$

Now assume (ii), we prove (X, T) is a normal space.

Let F_1 and F_2 be two disjoint closed subsets of X.

$\because F_1 \cap F_2 = \phi \Rightarrow F_1 \subset F_2^c$.

Thus F_2^c is an open superset of a closed set F_1.

Hence by (ii), there exists an open set G such that

$$F_1 \subset G \subset \overline{G} \subset F_2^c$$

But $\overline{G} \subset F_2^c \Rightarrow F_2 \subset \overline{G}^c$

and $G \cap \overline{G}^c = \phi$, $(\because G \subset \overline{G})$

$\overline{G}^c$ is an open set.

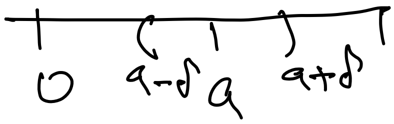
Thus $F_1 \subset G$, $F_2 \subset \overline{G}^c$, $G \cap \overline{G}^c = \phi$, and $G, \overline{G}^c$ are open sets

$\therefore (X, T)$ is a normal space

* The set

$D = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \ldots \right\}$ is the set of dyadic fractions, i.e., the fractions whose denominators are powers of 2 in $[0,1]$. Then $\overline{D} = [0, 1]$.

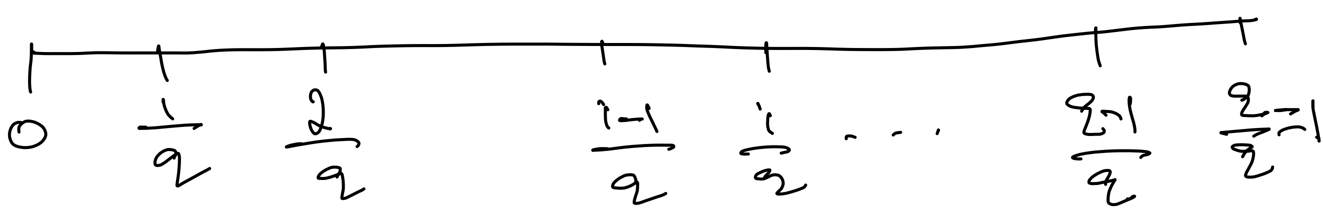
Let $a \in [0,1]$.



Then we prove that open interval $(a-\delta, a+\delta)$ contains a point of D.

$\because \lim\limits_{n \to \infty} \frac{1}{2^n} = 0 \Rightarrow \exists\ q = 2^{n_0},\ n_0 \in \mathbb{N}$

such that $0 < \frac{1}{q} < \delta$. —(1)



$\left[0, \frac{1}{q}\right], \left[\frac{1}{q}, \frac{2}{q}\right], \ldots \left[\frac{i-1}{q}, \frac{i}{q}\right], \ldots \left[\frac{q-2}{q}, \frac{q-1}{q}\right],$

$\left[\frac{q-1}{q}, \frac{q}{q}\right].$

$$\therefore [0, 1] = \bigcup_{m=0}^{q-1} \left[\frac{m}{q}, \frac{m+1}{q}\right]$$

$$\because a \in [0, 1] = \bigcup_{m=0}^{q-1} \left[\frac{m}{q}, \frac{m+1}{q}\right]$$

$$\Rightarrow a \in \left[\frac{m}{q}, \frac{m+1}{q}\right], \text{ for some } m \text{ in } \{0, \ldots, q-1\},$$

$$\Rightarrow \frac{m}{q} \le a \le \frac{m+1}{q} \quad \text{---} \quad (*)$$

Now $a \le \frac{m+1}{q} \Rightarrow a - \frac{1}{q} \le \frac{m}{q}$

But by $(*)$ $\frac{m}{q} \le a$

$$\therefore a - \frac{1}{q} \le \frac{m}{q} \le a < a + \frac{1}{q}$$

$\therefore$ by (1), we have

$$a - \delta < a - \frac{1}{q} \le \frac{m}{q} \le a < a + \frac{1}{q} < a + \delta$$

$$\Rightarrow \frac{m}{q} \in (a - \delta, a + \delta)$$

$\therefore (a-\delta, a+\delta)$ contains a point of D.

$\left[\because a - \delta < \frac{m}{q} < a + \delta \right.$

$\left. \Rightarrow \left|a - \frac{m}{q}\right| < \delta \right]$

* Note that the set $\{ [0,a), (b,1] \mid a,b \in \mathbb{R},\ 0<a,b<1 \}$ is a subbase for the relative topology on the interval $I = [0,1]$ in $(\mathbb{R}, \mathcal{U})$.

Urysohn's Lemma:

Let F_1 and F_2 be disjoint closed subsets of a normal space $(X, \mathcal{T})$. Then there exists a continuous function $f : X \longrightarrow [0,1]$ such that

$$f(F_1) = \{0\} \text{ and } f(F_2) = \{1\}.$$

Proof: Since F_1 and F_2 are disjoint closed subsets of a normal space $(X, \mathcal{T})$, $F_1 \cap F_2 = \emptyset \Rightarrow F_1 \subset F_2^c$. We have by one of the previous theorems that, there exists an open set $G_{\frac{1}{2}}$ such that

$$F_1 \subset G_{\frac{1}{2}} \subset \overline{G_{\frac{1}{2}}} \subset F_2^c.$$

From this we see that

$G_{\frac{1}{2}}$ is an open superset of a closed set F_1

F_2^c " " " $\overline{G_{\frac{1}{2}}}$

So again applying (same theorem), we get open sets $G_{\frac{1}{4}}$ and $G_{\frac{3}{4}}$ such that

$$F_1 \subset G_{\frac{1}{4}} \subset \overline{G_{\frac{1}{4}}} \subset G_{\frac{1}{2}} \subset \overline{G_{\frac{1}{2}}} \subset G_{\frac{3}{4}} \subset \overline{G_{\frac{3}{4}}} \subset F_2^c$$

We continue in this manner and obtain for each $t \in D$, the set of dyadic fractions in $[0,1]$, an open set G_t with the property that

if $t_1 < t_2 \Rightarrow \overline{G_{t_1}} \subset G_{t_2}$, $t_1, t_2 \in D$.

Now we define a function on X as follows:

$$f(x) = \begin{cases} \inf\{t \in D \mid x \in G_t\}, & \text{if } x \notin F_2 \text{ i.e. } x \in F_2^c \\ 1 & \text{if } x \in F_2 \end{cases}$$

Then we observe that

$$0 \leq f(x) \leq 1, \quad \forall\, x \in X.$$

$\therefore$ $f : X \longrightarrow [0,1]$ is a mapping,

and $f(F_1) = \{0\}$ and $f(F_2) = \{1\}$.

Hence it is enough to prove that
$f : X \longrightarrow [0,1]$ is continuous.

To prove this, we prove that inverse image of sub-base elements $\{[0,a), (b,1] \mid 0 < a, b < 1\}$ for the relative topology on $[0,1]$ w.r.t the usual topology U on $\mathbb{R}$ are open in X.

That is we show that

$f^{-1}([0,a))$, $f^{-1}((b,1])$, for $0<a,b<1$

are open in X.

To prove this we show that

$$f^{-1}([0,a)) = \cup\{G_t \mid t<a\} \quad -(1)$$

$$f^{-1}((b,1]) = \cup\{\bar{G}_t^{\,c} \mid t>b\} \quad -(2)$$

let $x \in f^{-1}([0,a)) \Rightarrow f(x) \in [0,a)$

$$\Rightarrow 0 \leq f(x) < a$$

$\because \bar{D} = [0,1] \Rightarrow \exists\, t_x \in D$ such that

$$0 \leq f(x) < t_x < a.$$

$$\Rightarrow f(x) = \inf\{t \mid x \in G_t\} < t_x < a$$

$$\Rightarrow x \in G_{t_x}, \quad t_x < a$$

$\Rightarrow x \in \cup \{ G_t \mid t < a \}$

$\Rightarrow f^{-1}([0,a)) \subseteq \cup \{ G_t \mid t < a \}$ —— $(*)$

On the other hand, suppose

$y \in \cup \{ G_t \mid t < a \}$

$\Rightarrow \exists\, t_y \in D \ni y \in G_{t_y},\ t_y < a.$

$\therefore f(y) = \inf \{ t \mid y \in G_t \} \leq t_y < a.$

$\in [0, a)$

$\Rightarrow y \in f^{-1}([0,a))$

$\Rightarrow \cup \{ G_t \mid t < a \} \subseteq f^{-1}([0,a))$ —— $(**)$

Combing $(*)$ and $(**)$, we prove (1).

$\therefore f^{-1}([0,a)) = \cup \{ G_t \mid t < a \}$ and

and each G_t is open and union of

open sets is an open set, it follows that $f^{-1}([0,a))$ is an open subset of X.

Now to prove (2),

let $x \in f^{-1}((b,1])$

$$\Rightarrow f(x) \in (b,1]$$

$$\Rightarrow b < f(x) \le 1.$$

~~Again~~ Since $\bar{D} = [0,1]$, $\exists\ t_1, t_2 \in D$ such that

$$b < t_1 < t_2 < f(x) \le 1$$

$$\therefore f(x) = \inf \{t \mid x \in G_t\} > t_2$$

$$\Rightarrow x \notin G_{t_2}.$$

Now for $t_1 < t_2 \Rightarrow \bar{G}_{t_1} \subset G_{t_2}$

$$\Rightarrow x \notin \bar{G}_{t_1} \quad [\because x \notin G_{t_2}]$$

$$\Rightarrow x \in \bar{G}_{t_1}^{c}, \quad t_1 > b$$

$$\Rightarrow x \in \cup \{ \overline{G_t}^c \mid t > b \}$$

$$\Rightarrow f^{-1}((b,1]) \subseteq \cup \{ \overline{G_t}^c \mid t > b \}$$

$\ast\ast\ast$

On the other hand, let

$$y \in \cup \{ \overline{G_t}^c \mid t > b \}$$

$\Rightarrow \exists\ t_y \in D$ such that

$$t_y > b, \quad y \in \overline{G_{t_y}}^c$$

$$\Rightarrow y \notin \overline{G_{t_y}}.$$

But $t < t_y \Rightarrow G_t \subset G_{t_y} \subset \overline{G_{t_y}}$

$$\Rightarrow y \notin G_t, \ \forall\ t < t_y$$

$$\Rightarrow f(y) = \inf \{ t \mid y \in G_t \} \geq t_y > b$$

$$= (b, 1].$$

$\Rightarrow \quad y \in f^{-1}((b,1])$

$\therefore \bigcup\{\overline{G}_t^{\,c} \mid t > b\} \subset f^{-1}((b,1])$ ——— ✱✱✱✱

Combining ✱✱✱ and ✱✱✱✱,

we get

$f^{-1}((b,1]) = \bigcup\{\overline{G}_t^{\,c} \mid t > b\}$,

Since R.H.S is union of open sets, and union of sets is open, it follows that R.H.S is an open set and hence $f^{-1}((b,1])$ is an open subset of X.

Thus $f : X \longrightarrow [0,1]$ is continuous, and $f(F_1) = \{0\}$ and $f(F_2) = \{1\}$

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Attendance

[65, 11, 27, 32, 06, 60, 58]

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