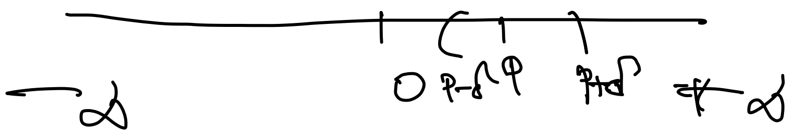
Lecture-13 (27-02-2024)

Ex: $X = \mathbb{R}$,

for $a, b \in X$, let $d(a,b) = |a-b|$.

Then $d$ is a metric on $\mathbb{R}$.




The open spheres in $\mathbb{R}$ are finite open intervals.

Hence metric $d$ induces a usual topology on $\mathbb{R}$.

Illy usual metric $d(p,q) = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}$, $\forall\ p = (a_1, a_2),\ q = (b_1, b_2) \in \mathbb{R}^2$ induces a usual topology on $\mathbb{R}^2$.

Theorem: Let $(X, d)$ be a metric space and the $\mathcal{D}_p$ denotes the class of open spheres with center at $p \in X$. Then $\mathcal{D}_p$ is a local base at $p$.

Proof: Let $G$ be any open set containing $p \in X$.
Since the open spheres form a base for a metric topology on $X$, so there exists an open sphere $S$ such that

$$p \in S \subset G$$

Since $p \in S$ and $S$ is an open sphere, so there exists an open sphere $S_p$ with center $p$ such that

$$p \in S_p \subset S.$$

$$\Rightarrow \quad p \in S_p \subset S \subset G$$

Thus for any point $p$ in the open set $G$, there exists an open sphere $S_p$ with center $p$ such that

$$p \in S_p \subset G.$$

$\therefore$ $D_p = \{ S(p, \delta) \mid \delta \in \mathbb{R}, \delta > 0 \}$ form a local base at $p \in X$.

—c—

Theorem: Let $(X, d)$ be a metric space. Then the countable class of open spheres $Z = \{ S(p, \frac{1}{n}) / n \in \mathbb{N} \}$ with center at $p \in X$ is a countable local base at $p \in X$.

Proof: Let $G$ be any open set in $X$ with $p \in X$. Then by previous theorem, the class

$$D_p = \{ S(p, \delta) / \delta \in \mathbb{R}, \delta > 0 \}$$

is a local base at $p \in X$.

Hence for $p \in G$, $\exists$ open sphere $S_p = S(p, \delta)$ in $D_p$ such that

$$p \in S(p, \delta) \subset G.$$

$\because \delta > 0$, $\exists$ $n_0 \in \mathbb{N} \ni \frac{1}{n_0} < \delta$.

$\Rightarrow p \in S(p, \frac{1}{n_0}) \subset S(p, \delta) \subset G$

$\Rightarrow Z = \{ S(p, \frac{1}{n}) / n \in \mathbb{N} \}$ is a countable local base at $p \in X$.

interior plus limit point of A wrt metric

Theorem: The closure $\bar{A}$ of a subset $A$ of a metric space $(X,d)$ is the set of points whose distance from $A$ is zero, i.e.,

$$\bar{A} = \{x \mid d(x,A) = 0\},$$

where $d(x,A) = \inf\{d(x,a) \mid a \in A\}$.

Proof: Suppose $d(P,A) = 0$, for some $P \in X$.

$\Rightarrow d(P,A) = \inf\{d(P,x) \mid x \in A\} = 0.$

Then every open sphere with center at $P$, contains at least one point of $A$.

Hence every open set $G$ containing $P$ also contains at least one point of $A$.

This implies either $P \in A$ or $P$ is the limit point of $A$.

Hence $P \in \bar{A}$

$\therefore \{x \mid d(x,A) = 0\} \subseteq \bar{A}$. — (1)

On the other hand suppose $d(P,A) = \epsilon > 0$, for some $P \in X$.


Then the open sphere $S(P, \frac{\epsilon}{2})$ with center at $P$ containing no point of $A$.

$\Rightarrow$ $P$ is the exterior point of $A$.

$\Rightarrow$ $P \notin \overline{A}$.

Then by (1), we have

$$\overline{A} = \{x \in X \mid d(x, A) = 0\}.$$

— // —

We know that for any $a, b$ in a metric space $(X, d)$ we have

$$d(a, b) > 0, \text{ for } a \neq b.$$

Now let $A = \{p\}$. Then

$$\overline{\{p\}} = \{x \in X \mid d(x, \{p\}) = 0\}$$
$$= \{x \in X \mid d(x, p) = 0\}$$
$$= \{x \in X \mid x = p\} = \{p\}.$$

Thus for any $p \in X$, $\overline{\{p\}} = \{p\}$.

Hence all the singleton sets $\{p\}$, $p \in X$ are closed sets.

If $A = \{a_1, a_2, \ldots a_n\}$, then

$$A = \bigcup_{i=1}^{n} \{a_i\}.$$

Since finite union of closed sets is also a closed set, it follows that all finite sets in a metric space $(X, d)$ are closed sets.

Thus we notice that a metric topological space $(X, d)$ has certain topological properties which do not hold for other topological space in general.

—— // ——

Theorem: Let $A$ and $B$ be disjoint closed sub-sets of a metric space $(X, d)$. Then there exist disjoint open sets $G$ and $H$ such that $A \subset G$ and $B \subset H$


Proof: If either $A$ or $B$ is an empty set say $A = \phi$, then $\phi$ and $X$ are the disjoint open sets such that

$$A \subseteq \phi \quad \text{and} \quad B \subseteq X.$$

So in this case theorem is proved.

Now assume that $A \neq \phi$ and $B \neq \phi$

Let $a \in A$.

Since $A \cap B = \phi \Rightarrow a \notin B$

$$\Rightarrow d(a, B) = \delta_a > 0.$$

Illy for $b \in B$, $d(b, A) = \delta_b > 0$.

Set $S_a := S(a, \frac{\delta_a}{3})$, $a \in A$

$S_b := S(b, \frac{\delta_b}{3})$, $b \in B$.

Clearly $a \in S_a$ and $b \in S_b$

and $S_a \cap S_b = \phi$.

Thus $\{S_a \mid a \in A\}$ and

$\{S_b \mid b \in B\}$ are the class of open spheres with $S_a \cap S_b = \phi$

$\forall a \in A$
$b \in B$.

Let

$$G := \bigcup_{a \in A} S_a, \quad H := \bigcup_{b \in B} S_b.$$

Then $G$ and $H$ are open sets with $A \subset G$ and $B \subset H$.

Claim: $G \cap H = \phi$.

Suppose $G \cap H \neq \phi$.

Let $p \in G \cap H \Rightarrow p \in G$ and $p \in H$.

Then there exist $a_0 \in A$ and $b_0 \in B$ such that $P \in S_{a_0} = S(a_0, \frac{\delta_{a_0}}{3})$

and $P \in S_{b_0} = S(b_0, \frac{\delta_{b_0}}{3})$.

$\Rightarrow P \in S_{a_0} \cap S_{b_0}$ which is Contradiction to $S_{a_0} \cap S_{b_0} = \varphi$.

$\therefore G \cap H = \varphi$

or

$P \in S_{a_0} \Rightarrow d(P, a_0) < \frac{\delta_{a_0}}{3}$

$P \in S_{b_0} \Rightarrow d(P, b_0) < \frac{\delta_{b_0}}{3}$.

$\because A \cap B = \varphi \Rightarrow d(a_0, b_0) = \epsilon > 0$.

Now

$d(a_0, B) = \delta_{a_0} \leq \epsilon$

$d(b_0, A) = \delta_{b_0} \leq \epsilon$.

$\left[ \begin{array}{l} \because d(a_0, B) \\ = \inf\{d(a_0, b) \mid b \in B\} \\ \leq d(a_0, b_0) \end{array} \right]$

$\Delta$le inequality we have

$$\epsilon = d(a_0, b_0) \leq d(a_0, P) + d(P, b_0)$$
$$< \frac{\delta_{a_0}}{3} + \frac{\delta_{b_0}}{3}$$
$$\leq \epsilon/3 + \epsilon/3 = 2\epsilon/3$$

which is not true

$\therefore G \cap H = \varphi$.

—————//—————

Equivalent Metrics :—

Two metrics $d$ and $d^*$ on a nonempty set $X$ are said to be equivalent if they induce the same topology on $X$. That is $d$- open spheres and $d^*$- open spheres in $X$ are bases for the same topology on $X$.

Ex: $X = \mathbb{R}^2$, $\left.\begin{array}{l} p = (a_1, a_2) \\ q = (b_1, b_2) \end{array}\right\} \in \mathbb{R}^2$,

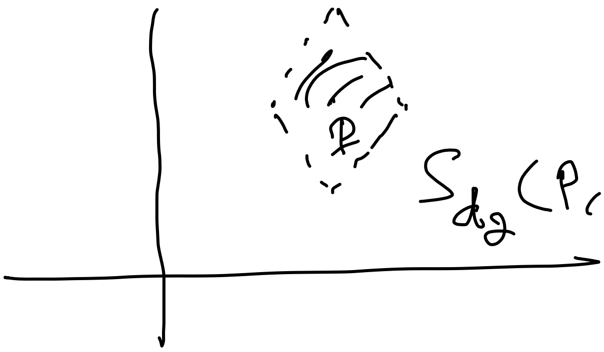
$$d(p,q) = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}$$

$$d_1(p,q) = \max\{|a_1-b_1|, |a_2-b_2|\}$$

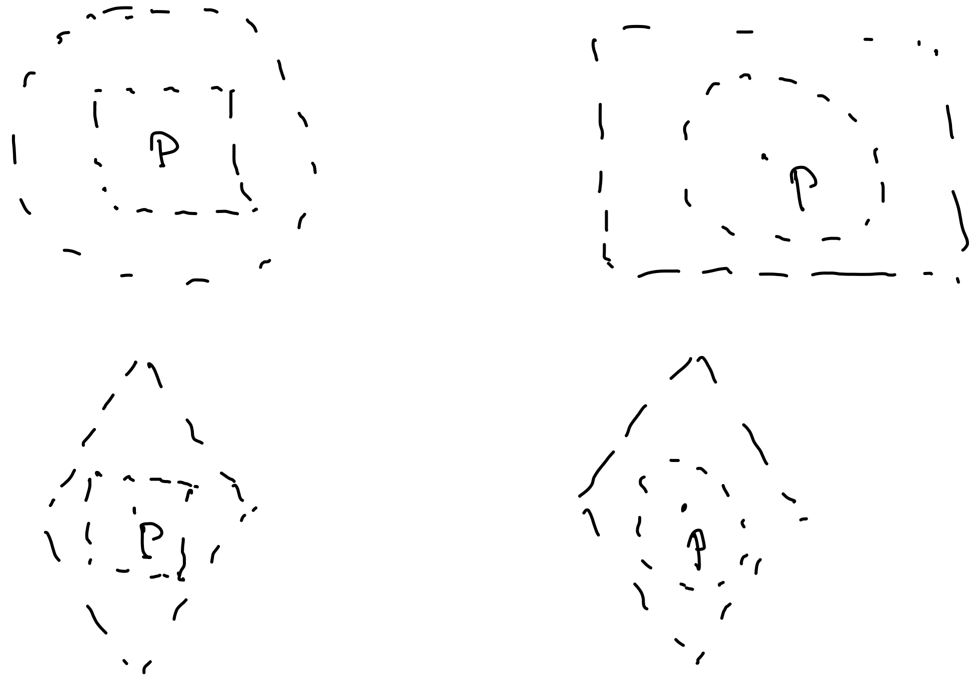
$$d_2(p,q) = |a_1-b_1| + |a_2-b_2|.$$

$S_d(p, \delta]$

$S_{d_1}(p, \delta]$



$S_{d_2}(p, \delta]$.



Hence the metric $d$, $d_1$ and $d_2$ induce the same topology on $\mathbb{R}^2$.
Hence $d$, $d_1$ and $d_2$ are equivalent.

## Metrization :

Given any topological space $(X, T)$, if there exists a metric $d$ on $X$ which induce the topology $T$ on $X$, then the topological space $(X, T)$ is called metrizable.

Ex : let $(\mathbb{R}, U)$ be usual topological space.

Since the usual metric $d(a,b) = |a-b|$, $\forall\, a, b \in \mathbb{R}$ induce the usual topology on $\mathbb{R}$, we see that $(\mathbb{R}, U)$ is metrizable.

Hly $(\mathbb{R}^2, U)$ is metrizable.

---

[Attendance: 65, 62, 41, 27, 06, 60]