Page No.
Date

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Linear Algebra
Test 3

Q1. $\|v\| = \sqrt{17}$

Q2. Orthonormal basis wrt $\{1, x, x^2\}$ is
$$\left\{ \frac{1}{\sqrt{2}}, \left(\sqrt{\frac{3}{2}}\right)x, \left(\sqrt{\frac{45}{8}}\right)\left(x^2 - \frac{1}{3}\right) \right\}$$

Q3. $T^*(x, y) = ((1-i)x + 2y, -ix - iy)$

T is normal = <u>FALSE</u>

Q4. T is self-adjoint operator = <u>NO</u>

Q5. Projection of v onto $W = \frac{19}{41}(2, -1, 6)$

Projection of v onto $W^{\perp} = \frac{1}{41}(126, 60, -32)$

Page No.
Date

Q1 : $\|u\| = 3$

$\|u+v\| = 4$

$\|u-v\| = 6$

$\|v\| = ?$

→ According to parallelogram equality

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

$$\Rightarrow (4)^2 + (6)^2 = 2((3)^2 + \|v\|^2)$$

$$\Rightarrow \cancel{16}^{8} + \cancel{36}^{18} = \cancel{2}(9 + \|v\|^2)$$

$$\Rightarrow 8 + 18 - 9 = \|v\|^2$$

$$\Rightarrow 8 + 9 = \|v\|^2$$

$$\Rightarrow 17 = \|v\|^2$$

$$\Rightarrow \sqrt{17} = \|v\|$$

$\therefore$ Ans $\Rightarrow$ $\|v\| = \sqrt{17}$

Page No.
Date

Q2. $P_2(\mathbb{R})$

$\langle p, q\rangle = \int_{-1}^{1} p(t)\, q(t)\, dt$

$\{1, x, x^2\}$

Now, $\{1, x, x^2\}$ are ~~are~~ linearly independent,

Thus, we can apply the Gram-Schmidt Procedure.

$e_1 = \frac{v_1}{\|v_1\|} \Rightarrow e_1 = \frac{1}{\left[\int_{-1}^{1} 1.1\, dx\right]^{1/2}} \Rightarrow \boxed{e_1 = \frac{1}{\sqrt{2}}}$

$e_2 = \frac{v_2 - \langle v_2, e_1\rangle e_1}{\|v_2 - \langle v_2, e_1\rangle e_1\|}$ & , $v_2 = x$, $e_1 = \frac{1}{\sqrt{2}}$

$v_2 - \langle v_2, e_1\rangle e_1 = x - \left[\int_{-1}^{1} \frac{x}{\sqrt{2}}\, dx\right] \frac{1}{\sqrt{2}} = x - \frac{1}{2}\left[\frac{x^2}{2}\right]_{-1}^{1} = x - 0 = x$

$\|x\|^2 = \int_{-1}^{1} x^2\, dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1-(-1)}{3} = \frac{2}{3}$

$\therefore e_2 = \frac{x}{\sqrt{\frac{2}{3}}} \Rightarrow \boxed{e_2 = \left(\sqrt{\frac{3}{2}}\right) x}$

Now, $e_3 = \frac{v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2}{\|v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2\|}$, $v_3 = x^2$, $e_1 = \frac{1}{\sqrt{2}}$, $e_2 = \sqrt{\frac{3}{2}}\, x$

Numerator of $e^3 = x^2 - \left(\int_{-1}^{1} x^2 \sqrt{\frac{1}{2}}\, dx\right) \frac{1}{\sqrt{2}} - \left(\int_{-1}^{1} x^3 \sqrt{\frac{3}{2}}\, dx\right) \sqrt{\frac{3}{2}}\, x$

$= x^2 - \frac{1}{3}$

Denominator of $e^3 = \|x^2 - \frac{1}{3}\| = \left[\int_{-1}^{1} \left(x^4 + \frac{1}{9} - \frac{2}{3}x^2\right) dx\right]^{\frac{1}{2}}$

$= \sqrt{\frac{8}{45}}$

Thus,

$$e_3 = \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}} \Rightarrow e_3 = \left(\sqrt{\frac{45}{8}}\right)\left(x^2 - \frac{1}{3}\right)$$

Hence, orthonormal bases wrt $\{1, x, x^2\}$ is $\left\{ \frac{1}{\sqrt{2}}, \left(\sqrt{\frac{3}{2}}\right)x, \left(\sqrt{\frac{45}{8}}\right)\left(x^2 - \frac{1}{3}\right) \right\}$

Q3. $T : V_2(\mathbb{C}) \to V_2(\mathbb{C})$, $T(1,0) = (1+i, 2)$
$T(0,1) = (i, i)$

$T(1,0) = (1+i)(1,0) + (2)(0,1)$
$T(0,1) = (i)(1,0) + (i)(0,1)$

$$M(T) = \begin{bmatrix} (1+i) & (i) \\ (2) & (i) \end{bmatrix}$$

Now, for T^*, we know that $M(T^*)$ would be conjugate transpose of $M(T)$.

Thus,

$$M(T^*) = \begin{bmatrix} (1-i) & (2) \\ (-i) & (-i) \end{bmatrix}$$

$$T^*(x,y) = \begin{bmatrix} 1-i & 2 \\ -i & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \left((1-i)x + 2y,\ (-i)x + (-i)y\right)$$

The condition for T to be normal is that $\|Tv\| = \|T^*v\|$, Thus the following has to be true

$$\|((1+i)x + iy,\ 2x + iy)\| = \|((1-i)x + 2y,\ -ix - iy)\|$$

This calculation is not easy to be executed, hence we use another condition, i.e $TT^* = T^*T$

Let us check TT^*

$$M(TT^*) = M(T)\,M(T^*)$$
$$= \begin{bmatrix} 1+i & i \\ 2 & i \end{bmatrix} \begin{bmatrix} 1-i & 2 \\ -i & -i \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & 2(1+i)+1 \\ 2(1-i)+1 & 4+1 \end{bmatrix} = \begin{bmatrix} 3 & 3+2i \\ 3-2i & 5 \end{bmatrix}$$

$$M(T^*T) = M(T^*)\,M(T)$$
$$= \begin{bmatrix} 1-i & 2 \\ -i & -i \end{bmatrix} \begin{bmatrix} 1+i & i \\ 2 & i \end{bmatrix}$$

$$= \begin{bmatrix} 2+\cancel{2}4 & i+1+2i \\ -i+1-2i & 1+1 \end{bmatrix}$$

Clearly $M(T^*T) \neq M(TT^*)$
Thus, $T^*T \neq T^*T$

Thus, T is not Normal.

Ans;- $T^*(x,y) = ((1-i)x + 2y,\ -ix - iy)$

T is NOT Normal

Page No.
Date

Q4. $T: R^3(R) \longrightarrow R^3(R)$
$T(a,b,c) = (b+c, -a+2b+c, a-3b-2c), \quad (a,b,c) \in R^3$

$T(1,0,0) = (0,-1,1) = 0e_1 + (-1)e_2 + (1)e_3$
$T(0,1,0) = (1,2,-3) = 1e_1 + 2e_2 + (-3)e_3$
$T(0,0,1) = (1,1,-2) = 1e_1 + 1e_2 + (-2)e_3$

Thus,

$$M(T) = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix}$$

We know that $M(T^*)$ is conjugate transpose of $M(T)$

$$M(T^*) = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{bmatrix}$$

Now, $M(T^*) \neq M(T)$, Thus $T^* \neq T$

This proved that T is not an self adjoint operator.

Page No.

Date

Q5. $W = \{\text{span}\{u = (2,-1,6)\}\}$,
$v = (4,1,2)$
Projection of on W, $W^\perp$ =?

$\rightarrow$ $$v = \left(\frac{\langle v,u \rangle}{\|u\|^2} u\right) + \left(v - \frac{\langle v,u \rangle}{\|u\|^2} u\right)$$

$$\frac{\langle v,u \rangle}{\|u\|^2} u = \frac{\langle (4,1,2),(2,-1,6) \rangle}{\langle 2,-1,6,(2,-1,6) \rangle} (2,-1,6)$$

$$= \frac{19}{41}(2,-1,6)$$

Thus, projection of v on $W = \frac{19}{41}(2,-1,6)$

Now,
$$v - \frac{\langle v,u \rangle}{\|u\|^2} u = (4,1,2) - \frac{19}{41}(2,-1,6)$$
$$= \frac{1}{41}(126, 60, -32)$$

Thus projection of v on $W^\perp = \frac{1}{41}(126, 60, -32)$