Lecture-19 (19-03-2024)

Compactness

Let $\mathcal{A} = \{G_i\}$ be a class of subsets of a topological space (X, T) and let A be any subset of X such that

$$A \subseteq \bigcup_i G_i .$$

Then $\mathcal{A} = \{G_i\}$ is called cover for the set A. If each G_i is open, we say $\mathcal{A} = \{G_i\}$ is an open cover for the set A.

If a finite number of members of class $\mathcal{A} = \{G_i\}$ also covers the set A, i.e., if there exists $G_{i_1}, G_{i_2}, G_{i_3}, \ldots G_{i_n}$ in $\mathcal{A} = \{G_i\}$

such that $A \subseteq \bigcup_{j=1}^{m} G_{i_j}$, then the class of sets $\mathcal{A} = \{G_i\}$ is said to be reducible to a finite cover for the set A or $\mathcal{A} = \{G_i\}$ contains a finite subcover for the set A.

Compactness :— A subset A of a topological space (X, T) is said to be a compact set if every open cover of the set A is reduced to a finite subcover.

That is the set A is a compact set if

$$A \subseteq \bigcup_i G_i, \quad G_i \text{ are open}$$

implies $\exists\, G_{i_1}, G_{i_2} \ldots G_{i_m} \ni$

$$A \subseteq G_{i_1} \cup G_{i_2} \cup \cdots \cup G_{i_m}.$$

Ex: (1) Every closed and bounded interval $[a,b]$, $a,b \in \mathbb{R}$ is a compact set.

(2) Let $A = \{a_1, a_2, \dots a_m\}$ be any finite subset of a topological space (X,T). Let $\mathcal{A} = \{G_i\}$ be any open cover for the set A.

Then

$$A = \{a_1, a_2, \dots a_m\} \subseteq \bigcup_i G_i$$

$$\Rightarrow \exists\ G_{i_1}, G_{i_2} \dots G_{i_m} \ni$$

$$a_j \in G_{i_j},\ j = 1, 2, \dots m.$$

$$\Rightarrow A \subseteq G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_m}.$$

$\Rightarrow$ A is a compact set.

Thus every finite set is a compact set.

Note :— In order to show that a set A is not a compact set, it is enough to show that there exists one open cover of the set A with no finite subcover.

Ex : Let $A = (0,1)$ in the usual topological space $(\mathbb{R}, U)$.

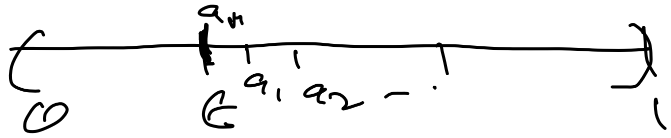
Let $\mathcal{A} = \{ (\frac{1}{n+2}, \frac{1}{n}) \,/\, n \in \mathbb{N} \}$ be an open cover for $A = (0,1)$

i.e., $A = (0,1) \subseteq \bigcup_{n=1}^{\infty} (\frac{1}{n+2}, \frac{1}{n})$.

But $\mathcal{A}$ has no finite subcover for $A = (0,1)$, since if $\mathcal{A}$ contains a finite subcover say $\{ (a_1, b_1), (a_2, b_2), \ldots (a_m, b_m) \}$.

let $\epsilon := \min\{a_1, a_2, \dots a_M\}$

Then $\epsilon > 0$. ($\because$ each $a_i > 0$)



Then

$$(a_1, b_1) \cup (a_2, b_2) \cdots \cup (a_M, b_M) \subset (\epsilon, 1)$$

and $(0, \epsilon] \cap (\epsilon, 1) = \varphi$

$$\therefore (0,1) \not\subset (a_1, b_1) \cup (a_2, b_2) \cdots \cup (a_M, b_M).$$

$\Rightarrow \{(a_1, b_1), (a_2, b_2) \dots (a_M, b_M)\}$ is not a cover for the set $A = (0,1)$.

$\therefore$ $(0,1)$ is not a compact set.

Theorem: Continuous image of a compact set is compact set.

Proof: Let (X, T) and (Y, T^*) be any two topological spaces.

Let A be any compact subset of (X, T).

and let $f : (X, T) \longrightarrow (Y, T^*)$ be a continuous map.

Claim: $f(A)$ is a compact subset of (Y, T^*),

Let $\{G_i\}$ be any open cover for the set $f(A)$. That is

$$f(A) \subseteq \bigcup_i G_i$$

Then

$$A \subseteq f^{-1}(f(A)) \subset f^{-1}\left(\bigcup_i G_i\right)$$

$$= \bigcup_i f^{-1}(G_i) \quad \text{---(1)}$$

Since each G_i is an open subset of (Y, T^*) and $f : (X, T) \longrightarrow (Y, T^*)$

is continuous map, it follows that each $f^{-1}(G_i)$ is an open subset of X.

$\therefore$ By (1) we have that $\{f^{-1}(G_i)\}$ is an open cover for the set A.

That is $A \subseteq \bigcup_i f^{-1}(G_i)$.

But A is a compact subset of (X, τ). Hence $\{f^{-1}(G_i)\}$ must have a finite subcover say $f^{-1}(G_{i_1}), f^{-1}(G_{i_2}), \ldots, f^{-1}(G_{i_m})$.

$$\therefore \quad A \subseteq \bigcup_{j=1}^{m} f^{-1}(G_{i_j})$$

$$= f^{-1}\left(\bigcup_{j=1}^{m} G_{i_j}\right)$$

$$\Rightarrow \quad f(A) \subset \bigcup_{j=1}^{m} G_{i_j}$$

$\therefore$ $f(A)$ is a compact subset of (Y, τ^*).

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Theorem: Let A be any subset of a topological space (X, T). Then the following are equivalent:

(i) A is compact w.r.t T

(ii) A is compact w.r.t the relative topology T_A on A.

Proof: $(i) \Longrightarrow (ii)$

Assume (i), i.e., let A be a compact subset of (X, T).

Claim: A is compact w.r.t T_A.

Let $\{G_i\}$ be any T_A open cover for the set A.

Then $A \subseteq \bigcup_i G_i, \quad G_i \in T_A$.

$\because$ each $G_i \in T_A \Longrightarrow \exists\, H_i \in T$ such that $G_i = A \cap H_i, \quad i = 1, 2 \ldots$

This implies $G_i = A \cap H_i \subseteq H_i$
$\forall\, i = 1, 2, \ldots$

$\therefore\; A \subseteq \bigcup_i G_i$

$= \bigcup_i (A \cap H_i)$

$\subseteq \bigcup_i H_i$

$\Rightarrow \{H_i\}$ is T - open cover for the set A. But A is compact w.r.t T.

$\therefore$ there exists $H_{i_1}, H_{i_2}, \ldots H_{i_m}$ in $\{H_i\}$ such that

$$A \subseteq \bigcup_{j=1}^{m} H_{i_j}.$$

$$\Rightarrow A \subseteq A \cap \left(\bigcup_{j=1}^{m} H_{i_j}\right)$$

$$= \bigcup_{j=1}^{m} (A \cap H_{i_j})$$

$$= \bigcup_{j=1}^{m} G_{i_j}.$$

$\Rightarrow \{G_{i_j}\}_{j=1}^{n}$ is a T_A-open finite sub cover of A.

$\therefore A$ is compact w.r.t T_A.

Now we prove (ii) $\Longrightarrow$ (i).

Assume (A, T_A) is a compact space.

Claim: A is compact set w.r.t T.

Let $\{H_i\}$ be any T-open cover for the set A.

$$\therefore \quad A \subseteq \bigcup_i H_i .$$

Set $G_i := A \cap H_i, \quad i = 1, 2, 3, \ldots$

Then $G_i \in T_A, \quad i = 1, 2, 3, \ldots$

Now $A \subseteq \bigcup_i H_i$

$$\Rightarrow A \subseteq \left(\bigcup_i H_i\right) \cap A$$

$$= \bigcup_i (H_i \cap A)$$

$$= \bigcup_i G_i$$

$\Rightarrow \{G_i\}$ is T_A-open cover for the compact space (A, T_A).

$\therefore$ there exist $G_{i_1}, G_{i_2}, \ldots G_{i_M}$ in $\{G_i\}$ such that

$$A \subseteq G_{i_1} \cup G_{i_2} \cup \ldots \cup G_{i_M}$$

$$= (A \cap H_{i_1}) \cup (A \cap H_{i_2}) \cup \ldots \cup (A \cap H_{i_M})$$

$$= A \cap (H_{i_1} \cup H_{i_2} \cup \ldots \cup H_{i_M})$$

$$\subseteq H_{i_1} \cup H_{i_2} \cup \ldots \cup H_{i_M}$$

$\Rightarrow \{H_i\}$ is reducible to a finite subcover $H_{i_1}, H_{i_2}, \ldots H_{i_M}$ such that

$$A \subseteq H_{i_1} \cup H_{i_2} \ldots \cup H_{i_M}$$

$\Rightarrow A$ is compact set w.r.t T.

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Note :- A subset of a Compact set need not be a Compact set.

Ex: $[0,1]$ is Compact set in $(\mathbb{R}, U)$, but $(0,1) \subset [0,1]$ is not compact.

But we can prove that every closed subset of a Compact set is Compact.

Theorem: Let F be a closed subset of a compact topological space (X, T). Then F is also Compact set.

Proof: Let $\mathcal{A} = \{G_i\}$ be any open cover for the closed set F.

Then $F \subseteq \bigcup_i G_i$, $G_i \in T$.

$\because$ F is closed, F^c is an open set, Now

$$X = F \cup F^c$$
$$\subseteq \bigcup_i G_i \cup F^c$$

$\Rightarrow \{G_i, F^c\}$ is an open cover for the compact space X.

Then there exists a finite open subcover say $G_{i_1}, G_{i_2}, \ldots G_{i_m}$ and F^c such that

$$X = G_{i_1} \cup G_{i_2} \cup \cdots \cup G_{i_m} \cup F^c$$

$$\Rightarrow F \cup F^c = X = G_{i_1} \cup G_{i_2} \cdots \cup G_{i_m} \cup F^c$$

$$\Rightarrow F \subseteq G_{i_1} \cup G_{i_2} \cup \cdots \cup G_{i_m}$$

$\Rightarrow F$ is a compact set.

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Finite Intersection Property (FIP)

A class of set $\{A_i\}$ in a topological space (X, T) is said to have finite intersection property (FIP) if every finite sub class $\{A_{i_1}, A_{i_2}, \ldots A_{i_m}\}$ has a non-empty intersection. That is

$$A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_m} \neq \varphi$$

for any m.

Ex: let $\mathcal{A} = \{(0, \frac{1}{n}) \mid n \in \mathbb{N}\}$

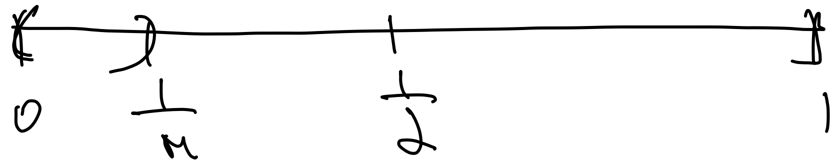
$$= \{(0,1), (0, \tfrac{1}{2}), (0, \tfrac{1}{3}), \ldots \ldots (0, \tfrac{1}{n}), \ldots\}.$$

Then $\mathcal{A}$ has FIP.

Since $(0, a_1) \cap (0, a_2) \cap \dots (0, a_M)$

$= (0, b)$, $b = \min\{a_1, a_2, \dots a_M\}$

$\neq \varphi$ $\quad a_i \in \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$.



But $\bigcap\limits_{n=1}^{\infty} (0, \frac{1}{n}) = \varphi$.

So $\mathcal{A} = \{(0, \frac{1}{n}) \mid n \in \mathbb{N}\}$ has empty intersection.

Theorem: Let (X, T) be a topological space. Then the following are equivalent.

(i) X is compact space

(ii) For every class $\{F_i\}$ of closed subsets of X with $\bigcap_i F_i = \varphi$, implies

$\{F_i\}$ contains a finite subclass
$\{F_{i_1}, F_{i_2}, \ldots F_{i_M}\}$ with

$$F_{i_1} \cap F_{i_2} \cap F_{i_3} \ldots \cap F_{i_M} = \varphi.$$

Proof:

(i) $\Rightarrow$ (ii)
Assume X is a compact space.

Given that

$$\varphi = \bigcap_i F_i, \quad \text{each } F_i \text{ is closed}$$

$$\Rightarrow \varphi^c = \left(\bigcap_i F_i\right)^c$$

$$\Rightarrow X = \bigcup_i F_i^c$$

$\Rightarrow$ $\{F_i^c\}$ is an open cover for compact space X. So there exists $F_{i_1}^c, F_{i_2}^c, \ldots F_{i_M}^c$ such that

$$X = F_{i_1}^c \cup F_{i_2}^c \ldots \cup F_{i_M}^c$$

$\Rightarrow \quad X^c = (F_{i_1}^c \cup F_{i_2}^c \cdots \cup F_{i_M}^c)^c$

$\Rightarrow \quad \varphi = (F_{i_1}^c)^c \cap (F_{i_2}^c)^c \cdots \cap (F_{i_M}^c)^c$

$\Rightarrow \quad F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_M} = \varphi$.

$\Rightarrow$ (i) $\Rightarrow$ (ii)

Now we prove (ii) $\Rightarrow$ (i).

Assume (ii)

Claim: X is compact space.

Let $\{G_i\}$ be any open cover for X,

$$\Rightarrow \quad X = \bigcup_i G_i, \quad G_i \in T$$

$\Rightarrow \varphi = X^c = (\bigcup_i G_i)^c$

$\Rightarrow \varphi = \bigcap_i G_i^c$.

$\Rightarrow \{G_i^c\}$ is a class of closed sets with empty intersection.

$\therefore$ By (ii), $\exists$ $G_{i_1}^c, G_{i_2}^c, \dots G_{i_M}^c$ in $\{G_i^c\}$ with $G_{i_1}^c \cap G_{i_2}^c \dots \cap G_{i_M}^c = \varphi$

Now

$$\varphi = G_{i_1}^c \cap G_{i_2}^c \dots \cap G_{i_M}^c$$

$$\Rightarrow \varphi^c = \left(G_{i_1}^c \cap G_{i_2}^c \dots \cap G_{i_M}^c\right)^c$$

$$\Rightarrow X = (G_{i_1}^c)^c \cup (G_{i_2}^c)^c \dots \cup (G_{i_M}^c)^c$$

$$\Rightarrow X = G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_M}$$

$\Rightarrow X$ is compact space

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Attendance [11, 61, 27, 32, 06, 38, 60]

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