Lecture - 18 (18-03-2024)

Urysohn's Metrization Theorem :

Every second countable normal $T_1$-space is metrizable.

Proof: Let $(X, T)$ be a second countable normal $T_1$-space.

Claim: $(X, T)$ is metrizable.

Since $(X, T)$ is second countable, it has a countable base say

$\mathcal{B} = \{G_1, G_2, G_3, \ldots\}$ where none of the members of $\mathcal{B}$ are equal to $X$.

[* For any $p \in G_i \in \mathcal{B}$, $\exists$ some $G_j \in \mathcal{B}$ $\ni$ $p \in \overline{G_j} \subset G_i$

$\because$ $X$ is a $T_1$-space, $p \in G_i \subset X$
$\Rightarrow \{p\}$ is a closed set.

$\because$ $\{p\} \subset G_i$
$\Rightarrow G_i$ is an open superset of a closed set $\{p\}$.

Since $X$ is also normal space, there exists an open set $G$ such that

$$\{p\} \subset G \subset \overline{G} \subset G_i \quad \text{---} (1)$$

Now, since $p \in G$, $\mathcal{B}$ is a base for $T$ on $X$, there exists a member $G_j \in \mathcal{B}$ such that

$p \in G_j \subset G$ (by def of base)

$\Rightarrow \{p\} \subset G_j \subset \overline{G_j} \subset \overline{G}$ —(2)

So from (1) and (2) we have

$$\{p\} \subset \overline{G_j} \subset \overline{G} \subset G_i$$

$$\Rightarrow p \in \overline{G_j} \subset G_i \; ]$$

We know that in a normal $T_1$-Space for any $p \in G_i \in \mathcal{B} = \{G_1, G_2, \dots\}$, then exists $G_j \in \mathcal{B}$ such that

$$p \in \overline{G_j} \subset G_i, \quad \forall i, j. \quad \text{—} \; (*)$$

Then the class pairs $(G_j, G_i)$ with $\overline{G_j} \subset G_i$, $G_i, G_j \in \mathcal{B}$ is countable and we denote these pairs by $P_1, P_2, P_3, \dots$,

where $P_n = (G_{j_n}, G_{i_n})$ with $\overline{G_{j_n}} \subset G_{i_n}$

Now since $\overline{G_{jn}} \subset G_{in}$

$$\Rightarrow \overline{G_{jn}} \cap G_{in}^{c} = \varphi.$$

Thus $\overline{G_{jn}}$ and $G_{in}^{c}$ are disjoint closed subsets of a normal space $X$. So by Urysohn's lemma, there exists a continuous function $f_n : X \longrightarrow [0\ 1]$ such that

$$f_n(\overline{G_{jn}}) = \{0\} \text{ and } f_n(G_{in}^{c}) = \{1\}, \quad \forall\, n = 1, 2, 3 \ldots \quad (**)$$

Now let $I = \{ (a_1, a_2, a_3, \ldots) \mid a_n \in \mathbb{R},\ \forall n \in \mathbb{N},\ 0 \leq a_n \leq \frac{1}{n} \}$

be a Hilbert Cube.

Define $f : X \longrightarrow I$

by $f(x) = \left( \frac{f_1(x)}{2}, \frac{f_2(x)}{2^2}, \ldots \frac{f_n(x)}{2^n}, \ldots \right)$,

$\forall x \in X$.

$\because \quad 0 \le f_n(x) \le 1, \quad \forall\, n = 1, 2, 3 \ldots$

we have

$$\left| \frac{f_n(x)}{2^n} \right| \le \frac{1}{2^n} \le \frac{1}{n}$$

$$\Rightarrow f(x) = \left( \frac{f_1(x)}{2}, \frac{f_2(x)}{2^2}, \ldots, \frac{f_n(x)}{2^n}, \ldots \right) \in I$$

Claim: $f$ is $1-1$.

Let $x, y \in X$ with $x \neq y$.

$\because$ $X$ is also $T_1$-space,

there exists a member $G_i \in B$

$\ni$ $x \in G_i$, but $y \notin G_i$

[$y, x \in X \Rightarrow \exists\, G \in T \ni x \in G,\ y \notin G$

$\because$ $B$ is a base, so $\exists\, G_i \in B$

$\ni x \in G_i \subset G$]

Now by $\circledast$, there exists a pair $P_m = (G_j, G_i)$ with $x \in \overline{G_j} \subset G_i$

Then by $\circledast\circledast$, there exists a continuous function $f_m : X \to [0,1]$ such that

$$f_m(\overline{G_j}) = \{0\}, \quad f_m(G_i^c) = \{1\}.$$

$$\therefore f_m(x) = 0, \quad \because x \in \overline{G_j} \subset G_i$$

$$\because y \notin G_i \Rightarrow y \in G_i^c$$

$$\Rightarrow f_m(y) = 1.$$

$$\Rightarrow f_m(x) \neq f_m(y).$$

Now

$$f(x) = \left(\frac{f_1(x)}{2}, \frac{f_2(x)}{2^2}, \dots \frac{f_m(x)}{2^m}, \dots\right)$$

$$f(y) = \left(\frac{f_1(y)}{2}, \frac{f_2(y)}{2^2}, \dots \frac{f_m(y)}{2^m}, \dots\right)$$

$\therefore f_M(x) \neq f_M(y)$

$\Rightarrow f(x) \neq f(y)$.

$\therefore f$ is $1-1$.

[Attendance 11AM : 11, 17, 19, 61, 44, 58, 35, 32, 27, 06, 57, 38, 07, 43, 23, 39, 10, 45, 60]

Next we prove $f : X \longrightarrow I$ is continuous.

Let $\epsilon > 0$ be given.

Note that $f$ is continuous at $p \in X$ iff there exists an open nbd $G$ of $p$ such that $x \in G$ and

$$\|f(x) - f(p)\| < \epsilon$$

$$\Rightarrow \|f(x) - f(p)\| < \epsilon^2$$

{$\|\cdot\| = \|\cdot\|_{\ell^2}$

[ $y = (y_1, y_2, \ldots)$

$\|y\|_2 = \left(\sum_{i=1}^{\infty} |y_i|^2\right)^{1/2}$ ]

Now

$$\|f(x)-f(p)\|^2 = \sum_{n=1}^{\infty} \frac{|f_n(x)-f_n(p)|^2}{2^{2n}}.$$

$\because \quad 0 \le f_n(x) \le 1, \ \forall\, n, \forall\, x \in X,$

implies

$$\frac{|f_n(x)-f_n(p)|}{2^{2n}} \le \frac{1}{2^{2n}}$$

$\because \sum \frac{1}{2^{2n}}$ is a Convergent Series,

implies

$$\sum_{n=1}^{\infty} \frac{|f_n(x)-f_n(p)|^2}{2^{2n}}$$ is also

Convergent.

Hence there exists an integer $n_0 = n_0(\epsilon)$ which is independent of $x$ and $p$ such that

$$\sum_{n>n_0}^{\infty} \frac{|f_n(x)-f_n(p)|^2}{2^{2n}} < \epsilon^2/2.$$

$\therefore$

$$\|f(x)-f(p)\|^2 \leq \sum_{n=1}^{n_0} \frac{|f_n(x)-f_n(p)|^2}{2^{2n}} + \epsilon^2/2$$

Now since each $f_n : X \longrightarrow [0,1]$ is continuous, there exists a nbd $G_n$ of $p$ such that for all $x \in G_n$,

$$|f_n(x)-f_n(p)|^2 < \frac{\epsilon^2}{2n_0}, \quad n=1,2,\dots n_0.$$

Let $G = \bigcap_{n=1}^{n_0} G_n$, each $G_n$ is a nbd of $p$.

$\Rightarrow$ $G$ is also nbd of $p$.

~~Hence~~ if $x \in G$, we have

$$\|f(x)-f(p)\|^2 \leq \sum_{n=1}^{n_0} \frac{|f_n(x)-f_n(p)|^2}{2^{2n}} + \epsilon^2/2$$

$$\leq \sum_{n=1}^{n_0} \frac{\epsilon^2}{2n_0} + \epsilon^2/2$$

$$= n_0 \cdot \frac{\epsilon^2}{2n_0} + \epsilon^2/2$$

$$= \epsilon^2$$

Thus $\|f(x) - f(p)\| < \epsilon$

Thus for every $x$ in a nbd $G$ of $p$, we have $\|f(x) - f(p)\| < \epsilon$

$\Rightarrow f$ is continuous at $p \in X$.

Since $p$ is an arbitrary point of $X$, it follows that $f$ is continuous on $X$.

Next let $Y = f(X) \subset I$

Claim: $f^{-1} : f(X) \longrightarrow X$ is continuous.

Since continuity in $Y = f(X)$ is equivalent to sequentially continuity, hence $f^{-1} : f(X) \longrightarrow X$ is continuous at $f(p) \in Y$ if for every sequence $\{f(y_n)\}$ converges to $f(p)$ implies $y_n \longrightarrow p$ in $X$.

$[\because \{f(y_n)\}$ in $f(X) \ni f(y_n) \longrightarrow f(p) \in Y$

$\overset{?}{\Longrightarrow} f^{-1}(f(y_n)) \to f^{-1}(f(p))$

$\Longrightarrow y_n \longrightarrow p$ in $X]$

Suppose $f^{-1}$ is not sequentially continuous

i.e., $f(y_n) \longrightarrow f(p)$ in $Y$,

but $y_n \not\longrightarrow p$ in $X$.

Then there exists an open nbd $G$ of $p$ such that $G$ does not contain an infinite number of terms of the sequence $\{y_n\}$.

Hence we can choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that all the terms of $\{y_{n_j}\}$ lie outside of $G$.

Now $p \in G \Rightarrow \exists\, G_i \in \mathcal{B}$ such that $p \in G_i \subset G$.

Then by $(*)$, $\exists$ a pair $P_m = (G_j, G_i)$ with $p \in \overline{G_j} \subset G_i$

Now since $y_{n_k} \notin G$, $\forall\, n_k$,

$$\Rightarrow y_{n_k} \notin G_i \, , \forall\, n_k$$

$$\Rightarrow y_{n_k} \in G_i^c$$

$$\because \overline{G_j} \subset G_i \Rightarrow \overline{G_j} \cap G_i^c = \varphi$$

Hence by $(**)$, $\exists\, f_m : X \longrightarrow [0,1]$

$$f_m(\overline{G_j}) = \{0\}, \quad f_m(G_i^c) = \{1\}.$$

$$\Rightarrow f_m(p) = 0, \quad f_m(y_{n_k}) = 1.$$

$$\Rightarrow |f_m(y_{n_k}) - f_m(p)|^2 = 1$$

and

$$\|f(y_{n_k}) - f(p)\|^2 = \sum_{k \geq 1}^{\infty} \frac{|f_k(y_{n_k}) - f_k(p)|^2}{2^{2k}}$$

$$\geq \frac{|f_m(y_{n_k}) - f_m(p)|^2}{2^{2m}}$$

$$= \frac{1}{2^{2m}}$$

$$\Rightarrow \|f(y_{n_k}) - f(p)\| \geq \frac{1}{2^m}$$

$$\Rightarrow f(y_{n_k}) \not\rightarrow f(p)$$

as $n_k \rightarrow \infty$.

which is contradiction to the sequence $\{f(y_n)\}$ converges to $f(p)$ and

$\{f(y_{n_k})\}$ is a subsequence of $\{f(y_n)\}$.

$\therefore$ our assumption that $y_n \not\to p$ is wrong.

Hence $f(y_n) \longrightarrow f(p)$

$\Rightarrow y_n \longrightarrow p$.

$\Rightarrow f^{-1} : f(X) \longrightarrow X$ is continuous.

$\therefore f : X \longrightarrow f(X) \subset I$ is a Homeomorphism and $X$ is homeomorphic to a subset $f(X)$ of a Hilbert Cube $I$.

$\therefore X \cong f(X)$

$\Rightarrow (X, T)$ is metrizable

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Attendance [ 11, 61, 27, 03, 06, 45, 60 ]

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