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Integer linear Programming

Gomory cutting Plane Method
↓
Pure ILPP

Example-

Solve the LPP

$$\text{Max } Z = 2x_1 + 2x_2$$

s.t.
$$5x_1 + 3x_2 \le 8$$
$$x_1 + 2x_2 \le 4$$
$x_1, x_2 \ge 0$ & are integer

Std form-

$$\text{Max } Z = 2x_1 + 2x_2 + 0 \cdot x_3 + 0 \cdot x_4$$

s.t.
$$5x_1 + 3x_2 + x_3 \le 8$$
$$x_1 + 2x_2 + x_4 \le 4$$
$$x_1, x_2, x_3, x_4 \ge 0$$

		C_j	2	2	0	0	
C_B	x_B	b	a_1	a_2	a_3	a_4	Min Ratio.
0	x_3	8	$\boxed{5}$	3	1	0	8/5 →
0	x_4	4	1	2	0	1	4/1
		$z_j - c_j$	−2	−2	0	0	
2	x_1	8/5	1	3/5	1/5	0	8/3
0	x_4	12/5	0	$\boxed{7/5}$	−1/5	1	12/7 →
		$z_j - c_j$	0	−4/5 ↑	2/5	0	
2	x_1	4/7	1	0	2/7	−3/7	
2	x_2	12/7	0	1	−1/7	5/7	
		$z_j - c_j$	0	0	2/7	4/7	

$$-1 + \frac{6}{7}$$

$x_{B_1} = x_1 = \frac{4}{7}$

$x_{B_2} = x_2 = \frac{12}{7} = 1 + \frac{5}{7}$

has most fractional part $f_{20} = 5/7$

$f_{23} = \frac{6}{7} \quad f_{24} = \frac{5}{7}$

corresponding to non-basic variables x_3, x_4 in x_2 row.

Then the cutting plane is.

$$f_{20} - (f_{23} x_3 + f_{24} x_4) \leq 0$$

$$\frac{5}{7} - \left(\frac{6}{7} x_3 + \frac{5}{7} x_4\right) \leq 0 \quad -(1)$$

$$\frac{5}{7} - \left(\frac{6}{7}(8 - 5x_1 - 3x_2) + \frac{5}{7}(4 - x_1 - 2x_2)\right) \leq 0$$

$$5x_1 + 4x_2 \leq 9 \quad -(2)$$

Now $x_1^* = 1 \quad x_2^* = 1 \quad z^*_{max} = 4$

$$\frac{5}{7} - \left(\frac{6}{7} x_3 + \frac{5}{7} x_4\right) + s_1 = 0$$

Include this in simplex table & solve again

$$\frac{5}{6} - \left(\frac{5}{6} x_1 + \frac{5}{6} s_1\right) \leq 0.$$

Cutting Plane 2 $\rightarrow$ $\frac{5}{6} x_1 - \frac{5}{2} s_1 + s_2 = -\frac{5}{6}$

Now u will get optimal solution

$x_1 = 1 \quad x_2 = 1$

$z_{max} = 4.$

Proof →

Max $z = Cx$

$Ax = b$

$x \geq 0$

x_j are integer $j = 1, 2, \ldots, n$

$A = [B \mid R]$ $\quad x^* [x_B^* \mid x_R^*]$ $\quad$ 1 – m basic variables

m+1 – n NBV

$$B x_B^* + R x_R^* = b$$

$$x_B^* = B^{-1} b - B^{-1} R x_R^*$$

we take $x_R^* = 0$ so $x_B^* = B^{-1} b$

But in case of $B^{-1} b$ not an integer then

$$x_B^* = y_0 - \sum_{j=m+1}^{n} y_j x_j \qquad y_0 = B^{-1} b$$

the value of x_B^* is non-integral

So, $$x_{Bi} = y_{i0} - \sum_{j=m+1}^{n} y_{ij} x_j$$

$$y_{i0} = \underbrace{d_{i0}}_{\text{integer part}} + \underbrace{f_{i0}}_{\text{fractional part}}$$

$$y_{ij} = d_{ij} - f_{ij}$$

$$0 < f_{i0} \leq 1$$

$$\underbrace{x_{Bi}}_{\text{for all integer soln}} = \underbrace{\left(d_{i0} - \sum_{j=m+1}^{n} d_{ij} x_j\right)}_{\text{integer}} + \underbrace{\left(f_{i0} - \sum_{j=m+1}^{n} f_{ij} x_j\right)}_{\text{integer}}$$

$f_{ij} \geq 0 \qquad x_j \geq 0$

$0 < f_{i0} \leq 1$

$$\boxed{f_{i0} - \sum_{j=m+1}^{n} f_{ij} x_j \leq 0}$$

Cutting plane constraint of Gomory

$\Rightarrow$ If you have constraint

$$\frac{3}{11}x_1 + \frac{2}{9}x_2 \leq 1$$

$$\hookrightarrow 27x_1 + 22x_2 \leq 99$$

Sensitivity Analysis

A → coefficient matrix
b → Requirement vector
c → cost vector

Change in the objective func.
Variation of requirement vector
~~Chaar~~ Change in the coefficient matrix
Addition of a variable
Addition of a constraint

Example →
find the limits of variations of the costs $c_1, c_2, c_3, \ldots$ $\ldots c_6$ respectively for the LPP whose optimal table is given below so that the optimal solution remains optimal.

c_B	B	x_B	c_j / b	-1 / a_1	-1 / a_2	3 / a_3	0 / a_4	-3 / a_5	0 / a_6
-1	a_2	x_2	5	$2/5$	1	0	$1/10$	$4/5$	0
3	a_3	x_3	6	$1/5$	0	1	$3/10$	$2/5$	0
0	a_6	x_6	8	1	0	0	$-1/2$	10	1
			$z_j - c_j$	$6/5$	0	0	$4/5$	$17/5$	0

$$\bar{c}_k = c_k + \delta_k$$

$$\text{Max}\left\{\frac{-(z_j-c_j)}{y_{kj}} \,\middle|\, y_{kj}<0\right\} \ge \delta_k \ge \text{Max}\left\{\frac{-(z_j-c_j)}{y_{kj}} \,\middle|\, y_{kj}>0\right\}$$

$$\left.\begin{array}{l} \delta_1 \le z_2 - c_2 \\ \delta_4 \le z_3 - c_3 \\ \delta_5 \le z_6 - c_6 \end{array}\right\} \rightarrow \text{for Non Basic Variable}$$

Date

For non-basic variables

$j = 1, 4, 5$

$c_j \to c_j + \delta_j$

$\delta_1 \leq z_1 - c_1 = 6/5$

$\delta_4 \leq z_4 - c_4 = 4/5$

$\delta_5 \leq z_5 - c_5 = 17/5$

$x_1 = 0 \quad x_2 = 5 \quad x_3 = 6$

$z_{max} = -1 \times 0 - 1 \times 5 + 3 \times 6 = 18 - 5 = 13$

for Basic Variable -

$y_{ij} = \left(\frac{2}{5}, 1, 0, 1/10, 4/5, 0\right)$

$\max\left\{\frac{-6/5}{2/5}, \frac{-4/5}{1/10}, \frac{-17/5}{4/5}\right\}$

$= \max\{-3, -8, -17/5\}$

$\infty \geq \delta_2 \geq -3$

$\infty \geq \delta_3 \geq -8/3$

$\frac{8}{5} \geq \delta_5 \geq -17/10$

$b \to b + \alpha$

$x_B = B^{-1} b \to \bar{x}_B = B^{-1}(b+\alpha)$

$\downarrow$

i^{th} component is changed

$\bar{b}_i = b_i + d_i$

$d = [d_1, d_2, \dots, d_m]$

$B^{-1} = [y_1, y_2, \dots, y_m]$

$\bar{x}_B = x_B^* + \sum_{j=1}^{m} d_j y_j$

$\bar{x}_{Bi} = x_{Bi}^* + \sum_{j=1}^{m} d_j y_{ij}$

$\bar{x}_{Bi} = x_{Bi} + d_k y_{ik} \geq 0$

$d_k \geq \frac{x_{Bi}}{y_{ik}}$ if $y_{ik} > 0$

$\leq \frac{-x_{Bi}}{y_{ik}}$ if $y_{ik} < 0$

$$\max_{y_{ik}>0} \left\{ -\frac{x_{Bi}}{y_{ik}} \right\} \leq d_k \leq \min_{y_{ik}<0} \left\{ \frac{-x_{Bi}}{y_{ik}} \right\}$$

Q Find the optimal solution of the LPP

$$\text{Max } z = 4x_1 + 3x_2$$

s.t.

$$x_1 + x_2 \leq 5$$

$$3x_1 + x_2 \leq 7$$

$$x_1 + 2x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

Show how to find the optimal solution of the

(i) b_1 increase by one unit & b_3 decrease by one unit

(ii) b_2 is decreased by two unit.

(i) $\bar{x}_B = x_B + B^{-1} d \qquad d = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

3	x_2	4	0	1	3/2	-1/2	0
4	x_1	1	1	0	-1/2	1/2	0
0	x_5	1	0	0	-5/2	1/2	1
		$z_j - c_j$	0	0	3/2	1/2	0

$B = (a_2 \ a_1 \ a_5)$

$$B^{-1} = \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ -5/2 & 1/2 & 1 \end{bmatrix} \qquad d = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\bar{x}_B = x_B + B^{-1} d$$

$$\bar{x}_B = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + B^{-1} d = \begin{pmatrix} 11/2 \\ 1/2 \\ -5/2 \end{pmatrix} \begin{matrix} x_2 \\ x_1 \\ x_5 \end{matrix}$$

$\downarrow$

Solve it using dual simplex