Ring Theory

Lecture 20

14/03/2022

$\mathbb{Z} \rightsquigarrow \mathbb{Q}$

int domain

quotient field.

<u>Thm</u>: Let R be an int domain. There exists an inj homo $R \to F$ where F is a field.

<u>Pf</u>: Consider the set

$$\mathcal{F} = \{(a,b) \mid a, b \in R \text{ and } b \neq 0\}.$$

Define a relation on $\mathcal{F}$ as follows

$$(a,b) \sim (c,d) \text{ iff } ad - bc = 0.$$

Check $\sim$ is an equivalance relation.

F = Set of equivalence class.

Equivalence class of $(a,b) \in \mathcal{F}$ is denoted by a/b.

Define '+' by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

and define '$\cdot$' by

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Ex. check that these are well defi

Additive identity is '$\frac{0}{1}$'

multiplicative identity is '$\frac{1}{1}$'.

WTS. Every non-zero elt has a multiplicative inverse.

Let $\frac{a}{b} \neq \frac{0}{1}$ ie $a \neq 0$.

then $\frac{b}{a} \in F$ and $\frac{a}{b} \cdot \frac{b}{a} = \frac{1}{1}$.

Thus F is a field.

$$\phi : R \longrightarrow F$$
$$\phi(r) = \frac{r}{1}.$$

$$\ker\phi = \left\{ r \in R \;\middle|\; \phi(r) = \frac{0}{1} \right\}.$$
$$= \left\{ r \in R \;\middle|\; \frac{r}{1} = \frac{0}{1} \right\}$$
$$= \{0\}.$$

$\therefore$ ϕ is an inj ring homo.

We identify R with $\phi(R) = \left\{ \frac{r}{1} \middle| r \in R \right\}$
$\subset F$.

<u>Defn</u>. F is called the quotient field of R denoted by $\underline{\underline{Q(R)}}$.

Example (1) $\mathbb{Q}$ is the quotient field of $\mathbb{Z}$.

(2) The quotient field of $k[x]$ is the field of rational fn.

$$k(x) := \left\{ \frac{f(x)}{g(x)} \;\middle|\; f(x), g(x) \in k[x] \text{ and } g(x) \neq 0 \right\}.$$

Thm. Let R be an int domain with quotient field F and let $\varphi : R \to K$ be any inj homo of R to a field K. Then $\exists$ a unique $\varphi^* : F \to K$ homo which is extension of φ.

Pf: Define $\varphi^*: F \to K$

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "\varphi"'] & F \arrow[dl, "\varphi^*"] \arrow[dl, phantom, shift right=3, "\circlearrowleft"] \\ K & \end{tikzcd}$$

$$\varphi^*\left(\frac{a}{b}\right) = \varphi(a)\varphi(b)^{-1}$$

[$\because$ φ is inj for $b \neq 0$, $\varphi(b) \neq 0$
hence $\varphi(b)^{-1}$ exists in K].

WTS φ^* is well defined.

Let $\frac{a}{b} = \frac{c}{d}$ WTS $\varphi^*\left(\frac{a}{b}\right) = \varphi^*\left(\frac{c}{d}\right)$

$$ad - bc = 0,$$

$$\Rightarrow \varphi(ad - bc) = 0$$

$$\Rightarrow \varphi(a)\varphi(d) - \varphi(b)\varphi(c) = 0$$

$$\Rightarrow \varphi(a)\varphi(d) = \varphi(b)\varphi(c)$$

$$\Rightarrow \varphi(a)\varphi(b)^{-1} = \varphi(c)\varphi(d)^{-1}$$

$$\Rightarrow \varphi^*(a/b) = \varphi^*(c/d).$$

(uniqueness): Let $g: F \longrightarrow K$ be a homo extending φ.

i.e $\quad g\big|_R = \varphi.$

WTS $\quad g = \varphi^*.$

$$g\left(\frac{a}{b}\right) = g\left(a \cdot b^{-1}\right)$$

$$= g(a)\, g(b)^{-1}$$

$$= \varphi(a)\, \varphi(b)^{-1}$$

$$= \varphi^*\left(\frac{a}{b}\right).$$

Hence $\quad g = \varphi^*.$

Q Let R be a ring and I be an ideal of R. When R/I is an int domain?

R/I will be an int domain if $a, b \in R$ and $\bar{a} \neq 0$, $\bar{b} \neq 0$. then $\bar{a}\bar{b} = \overline{ab} \neq 0$. i.e iff $a \notin I$, $b \notin I$ then $ab \notin I$

Equivalently, if $ab \in I$ then either $a \in I$ or $b \in I$.

Defn. An ideal $I \subsetneq R$ is called a prime ideal of R if $ab \in I$ then either $a \in I$ or $b \in I$.

$\underline{\underline{6\mathbb{Z}}}$

$\downarrow$

is not a prime ideal.

$2 \cdot 3 = 6 \in 6\mathbb{Z}$.

$2 \notin 6\mathbb{Z}$ nor $3 \notin 6\mathbb{Z}$.

<u>Propn</u>: Let R be a ring. Then an ideal P is a prime ideal of R iff R/P is an int domain.

<u>propn</u>. R is an int domain iff (0) is a prime ideal.

$ab \in (0)$ either $a \in (0)$ or $b \in (0)$.

i.e $ab = 0 \Rightarrow a = 0$ or $b = 0$.

$R = R/(0)$.

(2) $n\mathbb{Z}$ is a prime ideal in $\mathbb{Z}$ iff n is a prime number.

(4) Let $k[x]$ be a ring and

$$I = (f(x)).$$

A poly is said to be irreducible over k if is non-constant and can not be factored into the product of two or more non-constant polys with coeff in k.

(x^2-1) is not a prime.

$\mathbb{R}[x] \leftarrow (x^2-1) = (x+1)(x-1)$,

$(x^2+1) \rightsquigarrow$ irreducible.

Ex. $\mathbb{Z}[i]$.

(2). is it a prime ideal?.

$\left\{ \mathbb{C}[x] \rightsquigarrow \text{linear polys. are irreducible} \right\}$.