Graph Theory and Algorithms

Tutorial Sheet-1

(1) Let $M = (a_{ij})$ be an $n \times m$ matrix. Consider a graph G whose vertices are the entries of M and two vertices are adjacent if they lie in the same row or same column. Then

(a) Find the total number of vertices of G?
(b) Is G a regular graph? If yes, then find the degree of regularity.
(c) Find the total number of edges of G.

(2) In any group of n persons ($n \geq 3$), show that there are at least two with the same number of friends.

(3) Prove that if a graph has exactly two vertices of odd degree then there must be a path in the graph joining them.

(4) Prove or disprove: (i) If every vertex of a simple graph G has degree 2 then G is a cycle.
(ii) A closed trail with all its vertices of degree 2 is a cycle.

(5) If G is a simple graph with n vertices and the minimum degree $\delta(G) \geq \frac{n-1}{2}$ then prove that G is connected.

(6) Show by means of an example that the condition $\delta(G) \geq \frac{n-2}{2}$ for a simple graph G, need not imply that G is connected.

(7) Let G be a graph in which there is no pair of adjacent edges. What can you say about the degrees of the vertices in G.

(8) Let G be a graph with n vertices and e edges. Let m be the smallest positive integer such that $m \geq \frac{2e}{n}$. Prove that G has a vertex of degree at least m.

(9) Let G be a graph with n vertices and exactly $n-1$ edges. Prove that G has either a pendant vertex or an isolated vertex.

(10) Prove that in a group of six people, there must be three people who are mutually acquainted or three people who are mutually non-acquainted.

(11) Prove that if a simple graph G is not connected then its complement is connected.

(12) Let G be a simple graph with $\delta(G) \geq k$. Show that: (i) G contains a path of length at least k.
(ii) If $k \geq 2$ then G contains a cycle of length at least $k+1$.

(13) Draw all possible simple non-isomorphic graphs on four vertices.

(14) Let $S = \{1, 2, 3, 4, 5\}$. Construct a graph G whose vertex set is the collection of all 2-subsets of S and two vertices are adjacent in G if and only if the corresponding 2-subsets are disjoint. Prove that G is isomorphic to the Petersen graph.