## Continuum Hypothesis:

It is well-known that the matter is made up of molecules/atoms which are always in motion. We consider the microscopic/continuum behaviour of the fluid by assuming the fluid particles to be continuously distributed in the given region/space/domain. This is called continuum hypothesis.

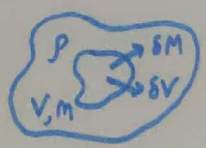
## Isotropic Fluids:

A fluid is said to be isotopic with respect to some properties such as pressure, density, etc. if that property is same in all directions at a point. A fluid is said to be anisotropic if that property is not same in all directions.

## Density:

Mass per unit volume, $\rho = \frac{m}{V}$ where $\rho$ be the density, $m$ be the mass, and $V$ be the volume of fluid region

$$\rho = \lim_{\delta V \to 0} \frac{\delta m}{\delta V} = \frac{M}{V}$$




## Pressure:

When a fluid contained in a closed region/fixed region/vessel, it exerts a force at each point of the inner side of the region/vessel. Such force is called pressure. Mathematically,

$$P = \lim_{\delta s \to 0} \frac{\delta F}{\delta s} = \text{force per unit area}$$

## Compressible and Incompressible Fluids:

If the density of a fluid changes with respect to time 't', temperature, pressure, etc., then it is called compressible fluid.
For e.g. - gases.
If the density does not change with respect to time, pressure, temperature, etc., then it is called incompressible fluid.
For e.g. - liquid.

## Viscous and Inviscous / Non-Viscous Fluid:

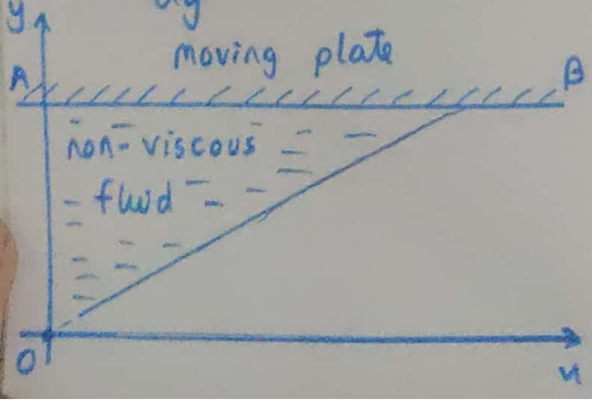
A fluid is said to be viscous when both normal and shearing stresses exist.
A fluid is inviscous when shearing stress is absent.

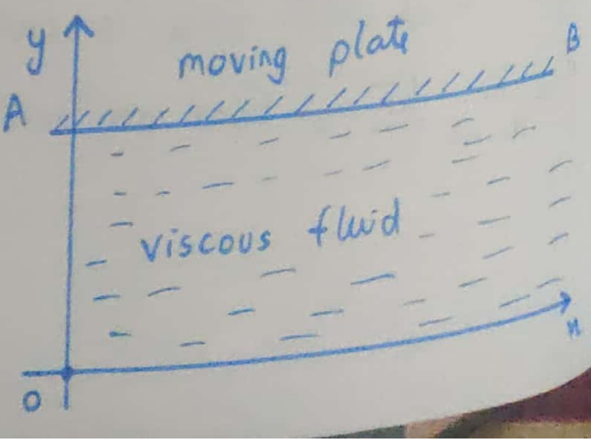
## Newtonian and Non-Newtonian Fluid:

Inviscous fluids ~~which obey Newtons~~ are often considered as Newtonian fluids, i.e., they obey Newton's law of viscosity. Viscous fluids such as polymer, tar, etc. are called non-Newtonian fluids, i.e., they do not obey Newton's law of viscosity.

## Newton's Law of Viscosity:

Let us consider a small fluid element in touch with a moving plate. Let $\tau$ be the shearing of the top, then $\tau \propto \frac{du}{dy}$, where $u$ is the velocity of the plate in x-direction



$\Rightarrow \tau = \mu \frac{du}{dy}$ , Newton's Law of Viscosity

$\mu$ — viscosity constant / coefficient of viscosity

Case I: $\tau = 0 \Rightarrow \mu = 0$

This represents ideal / perfect fluid (incompressible, homogeneous, frictionless)

Case II: $\frac{du}{dy} = 0 \Rightarrow \mu = \infty$

This represents a solid

Case III:

A fluid for which $\mu$ does not change with respect to shearing stress, then it is called Newtonian fluid.

Case IV:

If $\mu$ varies with respect to shearing stress, then it is called non-Newtonian fluid.

# Kinematics of Fluids:

They can be described by:

1. Eulerian Description  2. Lagrangian Description

## Eulerian Description:

In this method, we select any point fixed in space occupied by a fluid and study the changes which takes place in velocity, pressure, density, etc. of the fluid as the fluid passes through the point. Let $\vec{q}$ be the velocity of a fluid particle at point $P(x, y, z)$, then

$$u = f(x, y, z, t)$$
$$v = g(x, y, z, t) \qquad \text{where } \vec{q} = (u, v, w)$$
$$w = h(x, y, z, t)$$

## Lagrangian Description:

Let $P(x_0, y_0, z_0)$ be the initial point at $t = t_0$, then the fluid properties are expressed in terms of initial configuration, that is:

$$u = f(x_0, y_0, z_0, t), \quad v = g(x_0, y_0, z_0, t), \quad w = h(x_0, y_0, z_0, t)$$

## Lagrangian → Eulerian:

Suppose $\phi = \phi(x_0, y_0, z_0, t)$ be some fluid property associated with the flow, then by Lagrangian description the point $P(x, y, z)$ can be expressed as:

$$x = f_1(x_0, y_0, z_0, t), \quad y = f_2(x_0, y_0, z_0, t), \quad z = f_3(x_0, y_0, z_0, t)$$

$$\Rightarrow x_0 = f_4(x, y, z, t), \quad y_0 = f_5(x, y, z, t), \quad z_0 = f_6(x, y, z, t)$$

Then, $\phi(x,y,z) = \phi(f_4(x,y,z,t), f_5(x,y,z,t), f_6(x,y,z,t))$

↳ Eulerian description

Ex. The velocity components for a 2D-flow in Eulerian description is given by:

$u = 2x + 2y + 3t$, $v = x + y + t/2$

Then find the displacement of the fluid particle in Lagrangian description.

Ans. $\frac{dx}{dt} = 2x + 2y + 3t$, $\frac{dy}{dt} = x + y + \frac{t}{2}$

$\Rightarrow (D^2 - 3D)y = \frac{1}{2} + 2t$ where $D \equiv \frac{d}{dt}$

General solution, CF + PI

$CF = y(t) = (c_1 + c_2 e^{3t})$

$PI = \frac{1}{D^2 - 3D}\left(\frac{1}{2} + 2t\right) = \frac{-1}{3D}\frac{1}{\left(1 - \frac{D}{3}\right)}\left(\frac{1}{2} + 2t\right)$

$= \frac{-1}{3D}\left(1 - \frac{D}{3}\right)^{-1}\left(\frac{1}{2} + 2t\right) = \frac{-1}{3D}\left(1 + \frac{D}{3} + \frac{D^2}{9} + \dots\right)\left(\frac{1}{2} + 2t\right)$

$= \frac{-1}{3D}\left(\frac{7}{6} + 2t\right) = \frac{-7}{18}t - \frac{t^2}{3}$

$\Rightarrow y(t) = (c_1 + c_2 e^{3t}) - \frac{7}{18}t - \frac{t^2}{3}$

Similarly, we get, $x(t) = -c_1 + 2c_2 e^{3t} + \frac{t^2}{3} - \frac{7t}{9} - \frac{7}{18}$

Let $x(t_0) = x_0$, $y(t_0) = y_0$, then

$x_0 = -c_1 + 2c_2 e^{3t_0} + \frac{t_0^2}{3} - \frac{7t_0}{9} - \frac{7}{18}$

$y_0 = c_1 + c_2 e^{3t_0} - \frac{7}{18}t_0 - \frac{t_0^2}{3}$

Solving for $c_1, c_2$, we get:

$$c_1 = \frac{2y_0 - x_0}{3} - \frac{7}{54} \quad , \quad c_2 = \frac{x_0 + y_0}{3} + \frac{7}{54}$$

The required displacement in Lagrangian description is:

$$x(t) = -\left(\frac{2y_0 - x_0}{3} - \frac{7}{54}\right) + 2\left(\frac{x_0 + y_0}{3} + \frac{7}{54}\right)e^{3t} + \frac{t^2}{3} - \frac{7t}{9} - \frac{7}{18}$$

$$y(t) = \left(\frac{2y_0 - x_0}{3} - \frac{7}{54}\right) + \left(\frac{x_0 + y_0}{3} + \frac{7}{54}\right)e^{3t} - \frac{7}{18}t - \frac{t^2}{3}$$

**Lagrangian and Eulerian Approach to describe a flow:**

Let a fluid particle be initially located at $P_0(x_0)$ at time $t_0$ and at any instant of time $t$, the particle is at $P(x)$. By Lagrangian description, $\vec{x} = \vec{x}(\vec{x_0}, t)$, i.e., $\vec{r} = \vec{r}(\vec{r_0}, t)$

In Eulerian description of a field, it is represented by the position vector $\vec{x}$ and time $t$. The flow/velocity is represented by $\vec{q}(\vec{x}, t) = \frac{\partial}{\partial t}(\vec{x}) = \frac{\partial}{\partial t}(\vec{x}(\vec{x_0}, t))$

**Steady and Unsteady Flow:**

Steady flow occurs when at various points of the flow field, the properties associated with the flow, remain constant or unaltered with respect to time. Mathematically, if $A$ be any fluid property, then in steady state, $\frac{\partial A}{\partial t} = 0$

On the other hand, if the fluid property changes with respect to time '$t$', i.e., $\frac{\partial A}{\partial t} \neq 0$, then it is an unsteady flow.

## Uniform Flow:

If at every point the velocity is identical in magnitude and direction at any given instant of time 't', then it is termed as uniform flow.

If the velocity changes then it is a non-uniform flow.

## 1D, 2D and 3D Flows:

The motion of a fluid at any point in space can be specified with respect to a coordinate system. Let $\vec{q} = (u, v, w)$ be the fluid velocity. Then,

1D: 1D flow neglects the variation in other two direction, i.e.,
$v = 0,\ w = 0 \Rightarrow \vec{q} = (u, 0, 0)$, i.e., $\vec{q} : \mathbb{R}^3 \to \mathbb{R}$

2D: $u \neq 0,\ v \neq 0$ and $w = 0$, $\vec{q} = (u, v, 0)$, i.e., $\vec{q} : \mathbb{R}^3 \to \mathbb{R}^2$

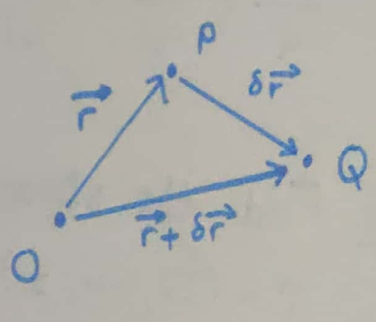
3D: $u, v, w \neq 0$, $\vec{q} = (u, v, w)$, i.e., $\vec{q} : \mathbb{R}^3 \to \mathbb{R}^3$

## Velocity of a fluid particle at a point:

Consider $P(\vec{r})$ and $Q(\vec{r} + \delta\vec{r})$ be the positions of a fluid particle at a time 't' and '$t + \delta t$'. Then,

$$\vec{q} = \lim_{\delta t \to 0} \frac{\vec{OQ} - \vec{OP}}{\delta t} = \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$$

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt}$$




## Path Lines:

The curve described in space by the moving fluid element is known as path lines.

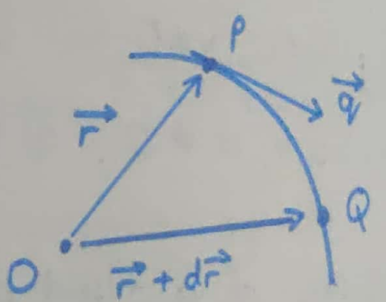
## Stream lines of a flow:

A streamline is a continuous line of flow drawn in the fluid so that the tangent at every point of it at any instant of time 't', coincides with the direction of motion of the fluid.

## Lines of flow:

A line of flow is a line whose direction coincides with the direction of the resultant velocity of the fluid.

Consider a fluid element ds with volume dv is located at the point P in the region/domain/Fluid. Let $\vec{q}$ be the fluid velocity. Since the direction of velocity and tangent coincides.




$$\vec{q} \times d\vec{r} = 0$$

$$\Rightarrow (u, v, w) \times (dx, dy, dz) = 0$$

$$\Rightarrow (w\,dy - v\,dz,\ u\,dz - w\,dx,\ v\,dx - u\,dy) = 0$$

$$\Rightarrow \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

**Ex.** Find the streamlines of 3D flow given by

$$\vec{q} = (2x, -y, z)$$

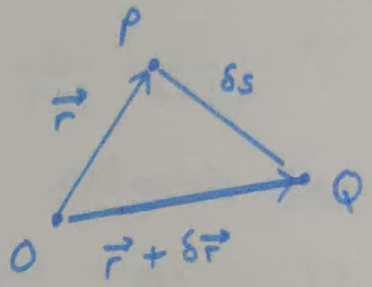
**Ans.** $$\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{z}$$

$$\Rightarrow xy^2 = c_1$$

$$yz = c_2$$

## Local, material, Convective Derivatives:

Consider $\vec{F}(\vec{r}, t)$ be some fluid property associated with the flow. Let us consider a fluid element at point $P$ whose position vector is $\vec{r}$ and at any time 't', let $Q$ be its position with position vector $\vec{r} + \delta\vec{r}$ relative to an origin $\vec{O}$. Let the element move through a distance $\delta s = \vec{q}.\delta t$ in any time interval $\delta t$. Then, the rate of change of fluid property is




$$\frac{d\vec{F}}{dt} = \lim_{\delta t \to 0} \frac{\delta \vec{F}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{F}(\vec{r}+\delta\vec{r}, t+\delta t) - \vec{F}(\vec{r}, t)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\vec{F}(\vec{r}+\delta t.\vec{q}, t+\delta t) - F(\vec{r}, t)}{\delta t}$$

$$= \lim_{\delta t \to 0} \left[ \frac{\cancel{\vec{F}(\vec{r}, t)} + \vec{q}\,\delta t \frac{\partial F}{\partial t} + (\vec{q}\,\delta t)^2 \frac{\partial^2 F}{\partial t^2} + \dots + \frac{\partial F}{\partial r}\delta t + \frac{\partial^2 F}{\partial r^2}(\delta t)^2 + \dots - \cancel{\vec{F}(\vec{r}, t)}}{\delta t} \right]$$

$$= \lim_{\delta t \to 0} \left[ \vec{q}.\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r} \right] + \delta t \left[ \vec{q}^{\,2} \frac{\partial^2 F}{\partial t^2} + \dots + \dots \frac{\partial^2 F}{\partial r^2} + \dots \right]$$

$$= \frac{\partial \vec{F}}{\partial t} + \vec{q}.\frac{\partial \vec{F}}{\partial \vec{r}} = \frac{D\vec{F}}{Dt} \longrightarrow \text{Material derivative}$$

$\frac{\partial \vec{F}}{\partial t} \downarrow$ Local derivative

$\vec{q}.\frac{\partial \vec{F}}{\partial \vec{r}} \downarrow$ Convective derivative

$\vec{F} = \vec{F}(\vec{r}, t)$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= \vec{F}(x, y, z, t)$

$$\frac{\partial \vec{F}}{\partial \vec{r}} = \frac{\partial \vec{F}}{\partial x}.\frac{\partial x}{\partial \vec{r}} + \frac{\partial \vec{F}}{\partial y}.\frac{\partial y}{\partial \vec{r}} + \frac{\partial \vec{F}}{\partial z}.\frac{\partial z}{\partial \vec{r}}$$

$$= \hat{i}.\frac{\partial \vec{F}}{\partial x} + \hat{j}\frac{\partial \vec{F}}{\partial y} + \hat{k}\frac{\partial \vec{F}}{\partial z} = \nabla(\vec{F})$$

$$\Rightarrow \frac{d\vec{F}}{dt} = \frac{\partial \vec{F}}{\partial t} + \underset{1\times3}{\vec{q}} \cdot \underbrace{(\underset{3\times1}{\vec{\nabla}} \underset{1\times3}{\vec{F}})}_{3\times3}$$

$$\therefore \frac{d}{dt} = \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{q}\cdot\vec{\nabla}$$

**Ex.** Let the fluid flow velocity be given by:

$\vec{q} = \left(\frac{x}{1+t}, \frac{y}{1+t}, \frac{z}{1+t}\right)$, then determine pathlines and streamlines.

**Ans.** i) Path lines

$$\frac{dx}{dt} = \frac{x}{1+t}, \quad \frac{dy}{dt} = \frac{y}{1+t}, \quad \frac{dz}{dt} = \frac{z}{1+t}$$

ii) Streamlines

$$\frac{dx\,(1+t)}{x} = \frac{dy\,(1+t)}{y} = \frac{dz\,(1+t)}{z}$$

**Note:**

- Lagrangian Description: $\vec{q}(t) = 3t^2\hat{i} + (6t-2)\hat{j} + t^3\hat{k}$
- Eulerian Description: $\vec{q}(t, x, y, z) = 6xyz\,\hat{i} + 2t^2xy\,\hat{j} + (3t^2-2)xy$

**Ex.** Let $T = (xyz + 4y + 9xy)$ denote the temperature, and $\vec{q} = xy\hat{i} + y\hat{j} + 5t^2\hat{k}$ denote the velocity of the fluid particle. Then calculate the rate of change of $T$ at $(1, 1, -2)$ at $t = 5$ sec.

**Ans.** $$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \vec{q}\cdot\vec{\nabla}T$$

$$\frac{dT}{dt} = 0 + (xy\,\hat{i} + y\,\hat{j} + 5t^2\,\hat{k}).\left[\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right]$$

$$= xy.(yz + 9y) + y.(xz + 4 + 9x) + 5t^2.(xy)$$

$(x, y, z) = (1, 1, -2)$ ; $t = 5$

$$= 1.(-2+9) + 1.(-2+4+9) + 5\times 5^2(1.1)$$ $\begin{array}{r} 125 \\ +1 \\ \hline 136 \\ -1 \end{array}$

$$= 7 + 11 + 125 = 143$$

Ex. Find the acceleration of a fluid flow given by
$\vec{q} = (xyz + t^2,\ t^2 - xy,\ 6xyt)$ at $(1, 1, -1)$ at $t = 2$ sec.

Ans. $$\frac{D\vec{q}}{Dt} = \frac{\partial \vec{q}}{\partial t} + \vec{q}.\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}[\vec{q}]$$

$$\frac{d\vec{q}}{dt} = (2t, 2t, 6xy) + \vec{q}.\begin{bmatrix} yz - y + 6yt \\ xz - x + 6xt \\ xy \end{bmatrix}$$

$$\frac{d\vec{q}}{dt} = (2t, 2t, 6xy)$$

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$a_1 = \frac{\partial u}{\partial t} + u.\frac{\partial u}{\partial x} + v.\frac{\partial u}{\partial y} + w.\frac{\partial u}{\partial z}$$

$$a_2 = \frac{\partial v}{dt} + u.\frac{\partial v}{dx} + v.\frac{\partial v}{\partial y} + w.\frac{\partial v}{\partial z}$$

$$a_3 = \frac{\partial w}{\partial t} + u.\frac{\partial w}{\partial x} + v.\frac{\partial w}{\partial y} + w.\frac{\partial w}{\partial z}$$

$$a_1 = 2t + (xyz + t^2)yz + (t^2 - xy)xz + (6xyt)xy$$

$$a_1\Big|_{\substack{(1,1,-1)\\ t=2}} = 4 + (-1+4)(-1) + (4-1)(-1) + (6\times 2).1$$

$$= 4 - 3 - 3 + 12 = 10$$

Similarly, we calculate $a_2$ and $a_3$

## Rotational and Irrotational Flow:

A fluid flow is said to be irrotational if $\vec{\nabla}\times\vec{q} = \vec{0}$

It is said to be rotational if $\vec{\nabla}\times\vec{q} \neq \vec{0}$

**Vorticity:** $\Omega = \vec{\nabla}\times\vec{q}$

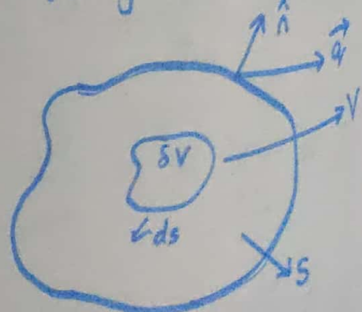
If $\vec{\nabla}\times\vec{q} = \vec{0}$, then there exists a function $\phi$ called velocity potential such that $\vec{q} = \nabla.\phi$, where $\nabla^2\phi = 0$ if the flow is incompressible.

$\vec{\nabla}.\vec{q} = -\nabla^2\phi = 0$ is called incompressibility condition

## Conservation of Mass:

The mathematical expression of conservation of mass is known as equation of continuity. In general, the conservation of mass relates to

Fluid in − Fluid out + Sources − Sinks = Fluid accumulated in the given region




Consider a fluid region V in the flow with surface area S. Let $\vec{q}$ be the fluid velocity and $\hat{n}$ be the unit outward drawn normal.

Let $\delta v$ be an infinitesimal volume element whose surface area is $\delta s$. The normal component of $\vec{q}$ measured outward from $V$ is $\vec{q}.\hat{n}$

Rate of mass flow across the surface $\delta s = \rho.(\vec{q}.\hat{n})\,\delta s$

Total rate of mass flow across the surface $S = -\iint\limits_{S} \rho(\vec{q}.\hat{n})\,ds$

$$= -\iiint\limits_{V} \vec{\nabla}.\vec{q}\,dV$$

This is called Gauss Divergence Theorem

## Equation of Continuity:

Mass of fluid in $V = \iiint\limits_{V} \rho\,dV$

Rate of change of mass through $V = \frac{d}{dt}\left(\int\limits_{V} \rho\,dV\right)$

$$= -\int\limits_{V} \vec{\nabla}.(\rho\vec{q})\,dV$$

$$\Rightarrow \frac{d}{dt}\int\limits_{V} \rho\,dV + \int\limits_{V} \vec{\nabla}.(\rho\vec{q})\,dV = 0$$

$$\Rightarrow \int\limits_{V}\left(\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho\vec{q})\right)dV = 0$$

Since $V$ is arbitrary, $\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho\vec{q}) = 0$

Equation of Continuity of a fluid flow

<u>Case I</u>: Incompressible fluid

$\rho(t, x) = \text{constant} \quad \Rightarrow \vec{\nabla}\cdot(\rho\vec{q}) = 0$

$\vec{q}\cdot\vec{\nabla}\rho + \rho\,\vec{\nabla}\cdot\vec{q} = 0$

$\Rightarrow \rho\,\vec{\nabla}\cdot\vec{q} = 0 \quad , \quad \vec{\nabla}\cdot\vec{q} = 0$

<u>Case II</u>: Irrotational and incompressible fluid

$\vec{q} = -\vec{\nabla}\phi \qquad \Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\phi) = 0$

$\qquad\qquad\qquad \Rightarrow \nabla^2\phi = 0$

<u>Ex.</u> Verify whether the flow $\vec{q} = \dfrac{K^2(x\hat{j} - y\hat{i})}{x^2+y^2}$ is incompressib[le]

<u>Ans.</u> $\vec{\nabla}\cdot\vec{q} = \cancel{\dfrac{\partial \vec{q}}{\partial x}} \quad a_1\hat{i} + a_2\hat{j}$

$u\cancel{a_x} = \dfrac{-K^2}{x^2+y^2}\,y$

$\dfrac{\partial u}{\partial x} = K^2 \cdot \dfrac{2xy}{(x^2+y^2)^2} \quad , \quad \dfrac{\partial u}{\partial y} = -K^2\left[\dfrac{(x^2+y^2)\cdot 1 - y\times 2y}{(x^2+y^2)^2}\right]$

$\vec{\nabla}\cdot\vec{q} = 0$

$\vec{\nabla}\times\vec{q} = \vec{0}$

$\exists\ \phi$ such that $\vec{q} = -\nabla\phi$

$\phi(x, y) = K^2\tan^{-1}\left(\dfrac{x}{y}\right) + c$

## Alternative Derivation:

Let $m$, $V$ and $\rho$ be the mass, volume and density of a fluid, respectively.

$$m = \rho V$$

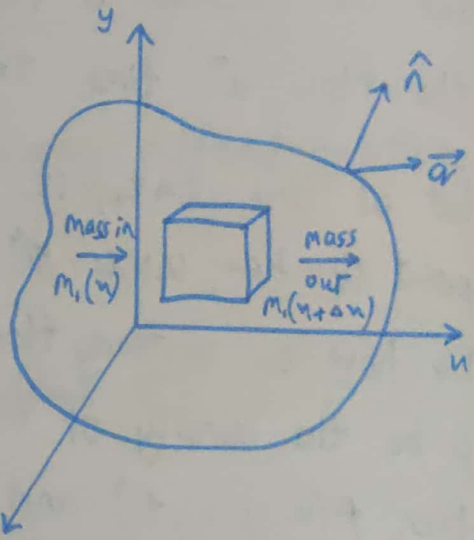
$$\log_e m = \log_e \rho + \log_e V$$

$$\frac{1}{m}\underbrace{\frac{dm}{dt}}_{\to 0} = \frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{V}\underbrace{\frac{dV}{dt}}_{\to 0} \Rightarrow \frac{d\rho}{dt} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{q}) = 0$$

## 2nd Alternative Method:

Let us consider a fluid region $V$ of surface area $S$. Let us take a fluid element $\Delta x\,\Delta y\,\Delta z$ at any point $P$ within the flow




rate of mass in $-$ mass out $= $ mass within fluid element

$$\Rightarrow m'(x) - m'(x+\Delta x) = \frac{\partial}{\partial t}(\rho\,\Delta x\,\Delta y\,\Delta z)$$

$$\Rightarrow m'(x) - m'(x) - \frac{\partial m'}{\partial x}\Delta x - \frac{\partial^2 m'}{\partial x^2}\Delta x^2 - \dots = \frac{\partial}{\partial t}(\rho\,\Delta x\,\Delta y\,\Delta z)$$

$$\Rightarrow -\frac{\partial m'}{\partial x}\Delta x - \frac{\partial^2 m'}{\partial x^2}\Delta x^2 - \dots = \frac{\partial}{\partial t}(\rho\,\Delta x\,\Delta y\,\Delta z)$$

Neglecting higher order terms:

$$-\frac{\partial}{\partial x}(\rho u\,\Delta y\,\Delta z)\,\Delta x = \frac{\partial}{\partial t}(\rho\,\Delta x\,\Delta y\,\Delta z)$$

We write similar expressions for $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$

Adding all, we get

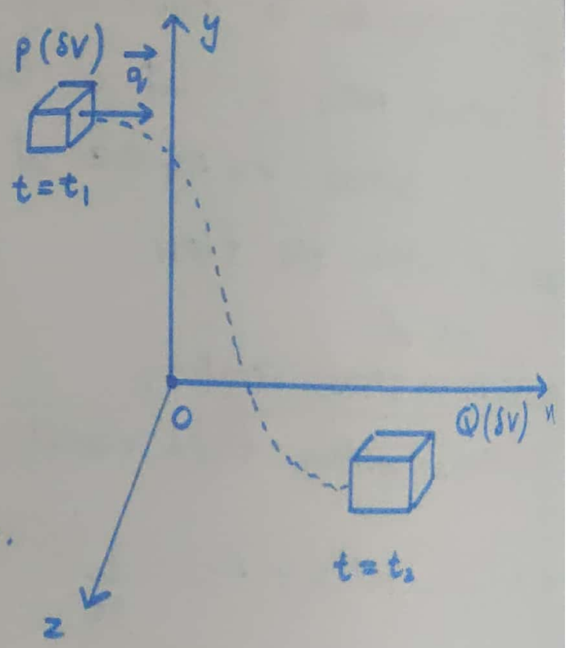
$$\Delta x \Delta y \Delta z \left[ -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w) \right] = \frac{\partial \rho}{\partial t}(\Delta x \Delta y \Delta z)$$

$$\Rightarrow -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w) = \frac{\partial \rho}{\partial t}$$

## Difference between Spatial/Material/Total derivative & local derivate:

Let us consider a fluid element $\delta v$ at a point $P$ within the flow at time $t = t_1$ and the velocity $\vec{q} = (u, v, w)$

Let at time $t = t_2$, it is at point $Q$. Here, $u, v, w : \mathbb{R}^4 \to \mathbb{R}$. We have an unsteady flow. Let $\rho$ be the density of the fluid, then $\rho(x_1, y_1, z_1, t_1)$ and $\rho(x_2, y_2, z_2, t_2)$ be the densities at $P$ and $Q$ respectively, $\rho_1$ & $\rho_2$.




$$\rho_2 = \rho(x_2, y_2, z_2, t_2) = \rho_1 + \left(\frac{\partial \rho}{\partial x}\right)_P (x_2 - x_1) + \left(\frac{\partial \rho}{\partial y}\right)_P (y_2 - y_1) + \left(\frac{\partial \rho}{\partial z}\right)_P (z_2 - z_1) + \left(\frac{\partial \rho}{\partial t}\right)_P (t_2 - t_1) + \text{higher powers}$$

Expanding density at $Q$ as a Taylor series expansion about $P$

$$\Rightarrow \frac{\rho_2-\rho_1}{t_2-t_1} = \left(\frac{\partial \rho}{\partial t}\right)_P + \left(\frac{\partial \rho}{\partial x}\right)_P \cdot \frac{x_2-x_1}{t_2-t_1} + \left(\frac{\partial \rho}{\partial y}\right)_P \cdot \frac{y_2-y_1}{t_2-t_1} + \left(\frac{\partial \rho}{\partial z}\right)_P \frac{z_2-z_1}{t_2-t_1}$$

↳ average time-rate of change of density of the fluid element as it moves from P to Q

If $t_2 \to t_1$, i.e., $Q \to P$

$$\frac{D\rho}{Dt} = \frac{d\rho}{dt} = \frac{\Delta \rho}{\Delta t} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} u + \frac{\partial \rho}{\partial y} v + \frac{\partial \rho}{\partial z} w$$

$$= \frac{\partial \rho}{\partial t} + \nabla \rho \cdot \vec{q}$$

$\frac{D\rho}{Dt}$ → Spatial / Material / Total derivative

$\frac{\partial \rho}{\partial t}$ → Local derivative

$\nabla \rho \cdot \vec{q}$ → Convective derivative

$\frac{D\rho}{Dt}$ and $\frac{\partial \rho}{\partial t}$ are numerically and physically different quantities

__Ex.__ Let $\vec{q} = \left(\frac{ax-by}{x^2+y^2}, \frac{ay+bx}{x^2+y^2}, 0\right)$. Check $\vec{\nabla}\cdot\vec{q}$ & $\vec{\nabla}\times\vec{q}$

__Ans.__ $$\vec{\nabla}\cdot\vec{q} = \frac{\partial}{\partial x}\left(\frac{ax-by}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{ay+bx}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)a - (ax-by)2x}{(x^2+y^2)^2} + \frac{(x^2+y^2)a - (ay+bx)2y}{(x^2+y^2)^2}$$

$$= \frac{2x^2a + 2y^2a - 2ax^2 + 2xyb - 2ay^2 - 2xyb}{(x^2+y^2)^2}$$

$$= 0$$

$$\vec{\nabla}\times\vec{q} = 0$$

Ex. For an irrotational ~~fluid~~ flow, velocity potential is $\phi(x,y,z) = \frac{a}{2}(x^2+y^2-2z^2)$. Then determine $\vec{q}$

Ans. $\vec{q} = -\nabla\phi$

$$= -(ax, ay, -2az) = -a(x, y, -2z)$$

Ex. Let the velocity potential of an irrotational & incompressible flow satisfy:

$\phi(x,0) = \sin x \quad , \; 0 \le x \le a$

$\phi(x,b) = 0 \quad , \; 0 \le x \le a$

$\phi(0,y) = 0 \quad , \; 0 \le y \le b$

$\phi(a,y) = 0 \quad , \; 0 \le y \le b$

Then determine $\vec{q} = (u,v)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$

Ans. $\exists\, \phi \ni \vec{q} = -\nabla\phi \Rightarrow \nabla^2\phi = 0$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad \text{Dirichlet BVP} \qquad -①$$

$$\phi(x,0) = \sin x, \; \phi(x,b) = 0 \quad , \; 0 \le x \le a \qquad -②$$

$$\phi(0,y) = \phi(0,y) = 0 \quad , \; 0 \le y \le b \qquad -③$$

Let us consider $\phi(x,y) = X(x)\,Y(y)$ be the solution of ①

$$X''(x)\,Y(y) + Y''(y)\,X(x) = 0 \Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} = k$$

Solution for $k<0$

By superposition principle,

$$\phi(x,y) = \sum_{n=1}^{\infty} \hat{a}_n \frac{\sinh \frac{n\pi(y-b)}{a}}{\sinh \frac{n\pi}{b}} \sin\left(\frac{n\pi x}{a}\right)$$

where $\hat{a}_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$

$$\Rightarrow \vec{q} = -\nabla\phi$$

$$= -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\left[\sum_{n=1}^{\infty} \hat{a}_n \frac{\sinh\left(\frac{n\pi(y-b)}{a}\right)}{\sinh\left(\frac{n\pi}{b}\right)} \sin\left(\frac{n\pi x}{a}\right)\right]$$

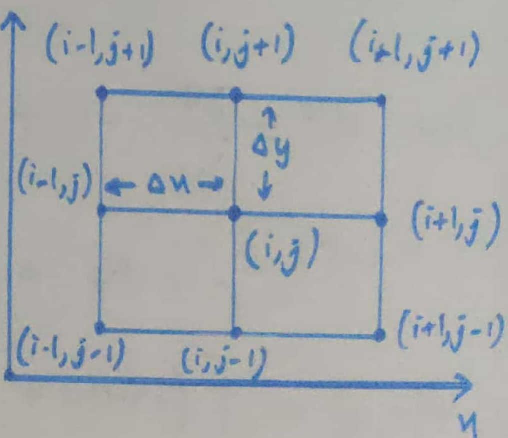
$$= -\left[\sum_{n=1}^{\infty} \hat{a}_n \frac{\sinh\left(\frac{n\pi(y-b)}{a}\right)}{\sinh\left(\frac{n\pi}{b}\right)} \cos\left(\frac{n\pi x}{a}\right) \cdot \frac{n\pi}{a}, \quad \sum_{n=1}^{\infty} \hat{a}_n \frac{\sinh\left(\frac{n\pi x}{a}\right)}{\sinh\left(\frac{n\pi}{b}\right)} \cosh\left(\frac{n\pi(y-b)}{a}\right) \cdot \frac{n\pi}{a}\right]$$

—— X — X ——

## Alternative Method to Solve Dirichlet BVP

$$u_{i+1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2!} + \dots$$

Forward Diff




$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x)$$

$$u_{i-1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j}(-\Delta x) + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(-\Delta x)^2}{2!} + \dots$$

Backward Diff

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} + O(\Delta x)$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x^2)$$ Central Difference

Similarly, $$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta x)^2}$$

—— x —— x ——

## Stream Function:

Let us consider a 2D flow which is irrotational and incompressible

$\vec{\nabla}\cdot\vec{q} = 0$ , $\vec{\nabla}\times\vec{q} = \vec{0}$ , $\vec{q} = -\nabla\phi$

where $\vec{q} = (u, v)$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Stream Function

If $\psi$ is a function such that: $u = \frac{\partial \psi}{\partial y}$ , $v = -\frac{\partial \psi}{\partial x}$

then $$d\psi = \frac{\partial \psi}{\partial x}.dx + \frac{\partial \psi}{\partial y}.dy$$

$$= -v\,dx + u\,dy$$

$$\psi(x,y) = \int(-v\,dx + u\,dy) + \text{constant}$$

$$\vec{q} = -\nabla\phi \Rightarrow \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right) = -\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)$$

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$$\Rightarrow \frac{\partial \phi}{\partial x} = \frac{-\partial \psi}{\partial y} \quad , \quad \frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x} \qquad \text{CR - equations}$$

$\nabla^2 \phi = 0$ & $\nabla^2 \psi = 0$

$W = \phi + i\psi$ = Complex Potential

Complex Conjugates

## Continuity Equation in Polar Coordinates:

In 3D: $$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho \sin\theta\, q_\theta)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho q_\phi) = 0 \quad \text{when}$$

$\rho$: density of the fluid,

Position, $P(r, \theta, \phi)$

$\vec{q} = (q_r, q_\theta, q_\phi)$

In 2D: $$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho \sin\theta\, q_\theta) = 0$$

## Continuity in Cylindrical Coordinates:

In 3D: $$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0$$

Position = $P(r, \theta, z)$

$\vec{q} = (q_r, q_\theta, q_z)$

Ex. Let us consider a two-dimensional flow (incompressible)

$$q_r = a\left(1 - \frac{3}{2}\frac{b}{r} + \frac{1}{2}\frac{b^3}{r^3}\right)\cos\theta$$

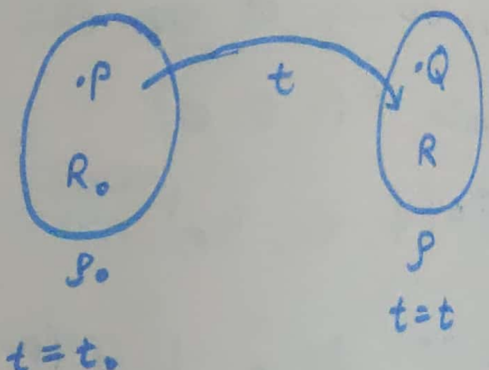
$$q_\theta = -a\left(1 - \frac{3}{4}\frac{b}{r} - \frac{1}{4}\frac{b^3}{r^3}\right)\sin\theta$$

$$q_\phi = 0, \quad r \geq b > 0$$

Then, verify the continuity equation

## Continuity Equation in Lagrangian Form:

Let $R_0$ be the region occupied by a fluid at an initial time $t_0$ and $R$ be the region occupied by the same fluid at an instant of time $t$.




Let $P(x_0, y_0, z_0)$ be the position of a fluid element $\delta x_0 \, \delta y_0 \, \delta z_0$ and $Q(x, y, z)$ be the position of the same fluid element $\delta x \, \delta y \, \delta z$ at time $t$

The mass of the fluid at $P = \rho_0 \, \delta x_0 \, \delta y_0 \, \delta z_0$

Similarly, mass at $Q = \rho \, \delta x \, \delta y \, \delta z$

By conservation of mass,

$$\rho_0 \, \delta x_0 \, \delta y_0 \, \delta z_0 = \rho \, \delta x \, \delta y \, \delta z$$

$$\int_{R_0} \rho_0 \, \delta x_0 \, \delta y_0 \, \delta z_0 = \int_{R} \rho \, \delta x \, \delta y \, \delta z$$

Using change of variables, $\delta x\, \delta y\, \delta z = J\, \delta x_0\, \delta y_0\, \delta z_0$

where $J = \dfrac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}$

$$\Rightarrow \int_{R_0} \rho_0\, \delta x_0\, \delta y_0\, \delta z_0 = \int_{R_0} \rho\, J\, \delta x_0\, \delta y_0\, \delta z_0$$

$$\Rightarrow \int_{R_0} (\rho_0 - \rho J)\, \delta x_0\, \delta y_0\, \delta z_0 = 0$$

Since $R_0$ is closed area, $\rho_0 - \rho J = 0$

$\Rightarrow \rho_0 = \rho J$ Continuity Equation in Lagrangian Form

Ex. $\dfrac{dJ}{dt} = J\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right)$

Prove the above

Corollary 1: Lagrangian $\rightarrow$ Eulerian

$$\rho_0 = \rho J$$

$$\Rightarrow \frac{d}{dt}(\rho J) = \frac{d}{dt}(\rho_0) = 0$$

$$\Rightarrow J \frac{d\rho}{dt} + \rho \frac{dJ}{dt} = 0$$

$$\Rightarrow J \frac{d\rho}{dt} + \rho J\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\Rightarrow \frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho\vec{q}) = 0$$

**Corollary 2:** Eulerian $\rightarrow$ Lagrangian

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho\vec{q}) = 0$$

$$\Rightarrow \frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\Rightarrow \frac{d\rho}{dt} + \frac{\rho}{J}\frac{dJ}{dt} = 0 \quad \Rightarrow \frac{d}{dt}(\rho J) = 0, \quad \rho_0 = \rho J$$

**Ex.** Show that $\phi(x, y, t) = (x-t)(y-t)$ represents the velocity potential of an incompressible flow. Determine the streamlines.

**Ans.** $\vec{q} = -\vec{\nabla}\phi$

$$= [-(y-t), -(x-t)]$$

Incompressibility condition, $\vec{\nabla}\vec{q} = 0$

The flow is incompressible

For streamlines, $\frac{dx}{u} = \frac{dy}{v} \Rightarrow \frac{dx}{-(y-t)} = \frac{dy}{-(x-t)}$

$(x-t)\,dx = (y-t)dy \quad \Rightarrow (x-t)^2 - (y-t)^2 = \text{constant}$

For path lines, $\frac{dx}{dt} = u = -(y-t)$

$$\frac{dy}{dt} = v = -(x-t)$$

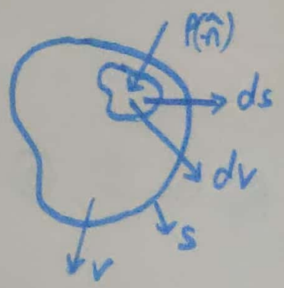
## Momentum Equation:

[Pressure: When a fluid is contained in a vessel, it exerts a force at each points of the inner side of the vessel. Such a force is called pressure. It is given by:

$$P = \lim_{\delta s \to 0} \frac{\delta F}{\delta s} \begin{matrix} \to \text{Force} \\ \to \text{Surface Area} \end{matrix}$$
]

- Motion of an inviscous / inviscial fluid

Consider an arbitrary closed surface S drawn in the region occupied by an incompressible invisid fluid, of volume V.




S moves with V so as to contain same amount of fluid at any instant of time 't'.

Let P be the pressure, $\vec{q}$ be the fluid velocity and $\hat{n}$ be the outward drawn normal.

By Newton's 2nd Law, force acting on S = rate of change of inner momentum of S — ①

The forces acting on S can be resolved into two parts:

1. The normal pressure / thrust acting on the boundary / surface area
2. External / body force acting on the entire volume such as gravity

Let $\rho$ be the fluid density of the particles within region S and ds be the arbitrary / elementary surface with volume dv in S. The mass of the fluid with $dv = \rho dv$

This gives the linear momentum of the fluid element as $\vec{q}\rho dv$

The linear momentum of the region V is $p = \int_V \vec{q}\rho dv$ - ②

Let $\vec{F}$ be the external force acting on dv, then the total external force on the mass of the fluid within V

$= \int_V \vec{F}\rho dv$ - ③

Let P be the pressure on ds, the force on $ds = P(-\hat{n})ds$

Total force on $S = \iint_S P(-\hat{n})\,ds$

$$= -\int_V \nabla \vec{P}\, dv \quad -④$$

By ①, ②, ③ and ④, we obtain

$$\frac{d\lambda}{dt} = \frac{d}{dt}\int_V \vec{q}\rho dv = \int_V \left(\rho\frac{d\vec{q}}{dt} + \vec{q}\underbrace{\frac{d\rho}{dt}}_{0}\right)dv$$

$$= \int_V \vec{F}\rho dv - \int_V \nabla\vec{P}\, dv$$

$$\Rightarrow \int_V \left[\frac{d\vec{q}}{dt}\rho - \vec{F}\rho dv - \nabla\vec{P}\, dv\right] = \vec{0}$$

As V is arbitrary, $\rho\left[\frac{d\vec{q}}{dt} - F + \frac{1}{\rho}\nabla P\right] = 0$

$$\Rightarrow \frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\nabla P$$

1D : $\rho\left(\frac{du}{dt} - f\right) = \frac{dP}{dx}$

**Corollary 1:** Bernoulli's equation of a one-dimensional inviscid irrotational incompressible irrotational flow in a conservative force field:

Let $\vec{q}$ be the velocity such that $\exists$ a potential function $\phi$

$\vec{q} = -\vec{\nabla}\phi$

Since $F$ is conservative, $\exists$ a $V \ni \vec{F} = -\nabla V = (X, Y, Z)$

We have, $u = -\frac{\partial \phi}{\partial x}$, $v = -\frac{\partial \phi}{\partial y}$, $w = -\frac{\partial \phi}{\partial z}$

$$X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y}, \quad Z = -\frac{\partial V}{\partial z}$$

Now, $\frac{\partial u}{\partial y} = -\frac{\partial^2 \phi}{\partial y \partial x} = -\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial v}{\partial x}$

Similarly, $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$, $\frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}$

From Euler's equation,

$$\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\nabla P$$

$$\Rightarrow \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial P}{\partial t}$$

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When resolving along x-axis

$$\frac{\partial}{\partial t}\left(-\frac{\partial \phi}{\partial x}\right) + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} = X - \frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$\Rightarrow -\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2+w^2) = X - \frac{1}{\rho}\frac{\partial P}{\partial x} \quad -①$$

Along y-axis: $-\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial y}(u^2+v^2+w^2) = Y - \frac{1}{\rho}\frac{\partial P}{\partial y}$ ②

Along z-axis: $-\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial z}(u^2+v^2+w^2) = Z - \frac{1}{\rho}\frac{\partial P}{\partial z}$ -③

Now, ① $dx$ + ② $dy$ + ③ $dz$

$$-d\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}d(u^2+v^2+w^2) = X\,dx + Y\,dy + Z\,dz - \frac{1}{\rho}dP$$

$$\Rightarrow -d\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}d(q^2) = -dV - \frac{1}{\rho}dP$$

$$\Rightarrow \int -d\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}\int d(q^2) = -\int dV - \int \frac{1}{\rho}dP$$

$$\Rightarrow -\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 = -V - \int \frac{dP}{\rho}$$

Bernoulli's equation / Pressure equation

Case I: $\rho$ is constant

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2} q^2 = -V - \frac{P}{\rho} + c$$

Case II: $P$ is constant steady flow

$$\frac{1}{2} q^2 = -V - \frac{P}{\rho} + c$$

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## Motion in 2-dimension:

Let $\vec{q} = (u, v)$ be the velocity of a fluid, where

$u, v : \mathbb{R}^2 \to \mathbb{R}$

The equation of streamlines: $\dfrac{dx}{u} = \dfrac{dy}{v} \Rightarrow v\,dx - u\,dy$ — ①

If the fluid is incompressible, then

$\vec{\nabla} \cdot \vec{q} = 0$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

In equation ①, we find that it is exact. Therefore, $\exists$ a function $\psi$ such that

$$d\psi = v\,dx - u\,dy$$

$$\Rightarrow \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = v\,dx - u\,dy$$

$$\Rightarrow u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \quad - ②$$

The function $\psi$ is called stream function. The streamlines are given by $d\psi = 0$

$\Rightarrow \psi(t, x, y) = \text{constant}$

The stream function is constant along a streamline

• For irrotational flow:

$$\vec{\nabla} \times \vec{q} = 0$$

$\exists$ a $\phi$ such that $u = -\frac{\partial \phi}{\partial x}$, $v = -\frac{\partial \phi}{\partial y}$ — ③

From ② & ③

$$\Rightarrow \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}$$

$\Rightarrow$ $\psi$ and $\phi$ satisfy the CR-equation, that is, they are harmonic conjugate:

$$\nabla^2 \phi = 0 \quad \text{and} \quad \nabla^2 \psi = 0$$

elliptic PDE

Then, we introduce complex potential:

$$w(z) = w(x, y)$$
$$= \phi(x, y) + i\psi(x, y), \quad i = \sqrt{-1}$$

Q. We are given complex potential and we need to obtain CR equation, so the given flow is irrotational and incompressible.

$$w(x+iy) = \phi(x+iy) + i\psi(x+iy)$$

$$\Rightarrow \frac{\partial w}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$i\frac{\partial w}{\partial y} = i\frac{\partial \phi}{\partial y} + i^2\frac{\partial \psi}{\partial y}$$

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CR equation in Polar form:

$$\phi + i\psi = f(z) = f(re^{i\theta})$$

$$\frac{\partial \phi}{\partial r} + i\frac{\partial \psi}{\partial r} = f'(re^{i\theta})\, e^{i\theta} \quad -①$$

$$\frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta} = f'(re^{i\theta})\, re^{i\theta}\, i \quad -②$$

① × ri = ②

$$\Rightarrow ri\frac{\partial \phi}{\partial r} - r\frac{d\psi}{\partial r} = \frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta}$$

$$\Rightarrow \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad \frac{\partial \phi}{\partial \theta} = -r\frac{\partial \psi}{\partial r}$$

<u>Magnitude of velocity from Complex Potential:</u>

Let $w = f(z)$ be the complex potential, then

$$w = \phi + i\psi, \quad z = x + iy \quad -①$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial x} = \frac{-\partial \psi}{\partial x} \quad -②$$

From ①

$$\frac{dw}{dz} \cdot \frac{\partial z}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$\Rightarrow \frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$\Rightarrow \frac{dw}{dz} = -u - i\frac{\partial \phi}{\partial y} = -u + iv$$

$$\Rightarrow \left|\frac{dw}{dz}\right| = \sqrt{u^2+v^2} = |\vec{q}|$$

Ex.-1 $\quad w = e^z = e^x(\cos y + i\sin y)$

$\phi = e^x \cos y, \quad \psi = e^x \sin y$

$\Rightarrow \vec{q} = -\nabla\phi = -(e^x \cos y, -e^x \sin y)$

Ex.-2 Motion of an uniform flow

$$w = ikz, \quad z = x+iy$$

$\dfrac{dw}{dz} = ik = -u + iv \quad , \quad v = k, \; u = 0$

$$|\vec{q}| = |k|$$

Ex.-3 $\quad \cancel{w = ike^{iz}} \quad w = -ke^{-i\alpha} z$

$\dfrac{dw}{dz} = -ke^{-i\alpha} = -k(\cos\alpha - i\sin\alpha)$

$u = k\cos\alpha, \quad v = k\sin\alpha$

$|\vec{q}| = |k|$

**Ex.-4** $\phi(x,y) = c(x^2 - y^2)$, then ...
function

**Ans.** $\frac{\partial \psi}{\partial y} = 2cx$, $\frac{\partial \psi}{\partial x} = 2cy$

$\downarrow$

$\Rightarrow \psi(x,y) = 2cxy + f(x)$

## Flow Equations:

1. Continuity equation: $\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{q}) = 0$

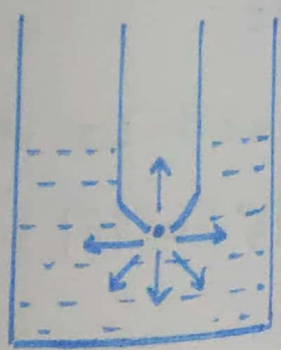
2. Euler's equation: $\frac{d\vec{q}}{dt} - \vec{F} = \frac{1}{\rho} \nabla \vec{P}$

3. Bernoulli's equation: $-\frac{\partial \phi}{\partial t} + \frac{1}{2} q^2 + V + \int \frac{dP}{\rho} = \text{constant}$

## Sources and Sinks:

**Sources:** If the motion of a fluid consists of symmetrical radial flow in all directions preceding from a point, then the point is called source.

Similarly, if the flow is such that the fluid is directed inwards at that point, then the point is called sink.

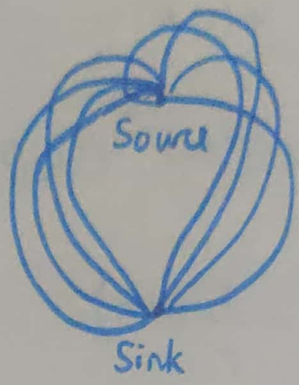


Furthermore, if we consider a source at origin, then the mass 'm' of the fluid coming out from the origin in a unit time is known as strength of the source.

Similarly, the amount of fluid going into the sink is called strength of the sink.

Without source / sink

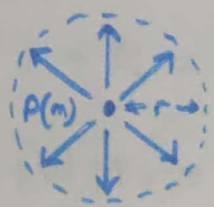
With source / sink




## Sources and Sinks in two-dimensions:

In two dimensions, a source of strength $m$ is ~~said~~ such that the flow across any small curve surrounding is $2\pi m$. Sinks are regarded as sources of strength $-m$.

Consider a circle of radius $r$ with source at its centre. Then, the radial velocity $q_r$ is given by:




C

$$q_r = \frac{-1}{r}\frac{\partial \psi}{\partial \theta} \quad - ①$$

Also, $q_r = \dfrac{-\partial \phi}{\partial r} = \dfrac{-1}{r}\dfrac{\partial \psi}{\partial \theta} \Rightarrow \dfrac{\partial \phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta} \quad - ②$

But the flow across the circle is $2\pi r q_r$, then

$$2\pi r q_r = 2\pi m$$

$$\Rightarrow q_r = \frac{m}{r} \quad - ③$$

From ① and ③,

$$-\frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{m}{r}$$

$$\Rightarrow \frac{\partial \psi}{\partial \theta} = -m$$

$$\Rightarrow \psi(r,\theta) = -m\theta + \text{constant}$$

Omitting the constant, $\psi(r,\theta) = -m\theta$

From ②, $\frac{\partial \phi}{\partial r} = \frac{1}{r}(-m)$

$$\Rightarrow \phi(r,\theta) = -m\log r + \text{constant}$$

Omitting the constant, $\phi(r,\theta) = -m\log r$

The required complex potential for a source of strength m is

$$w = \phi + i\psi = -m\log r - mi\theta = -m(\log r + i\theta)$$

$$= -m\log(re^{i\theta}) = -m\log z \quad , \text{ where } z = re^{i\theta}$$

If the source is at $z = z'$,

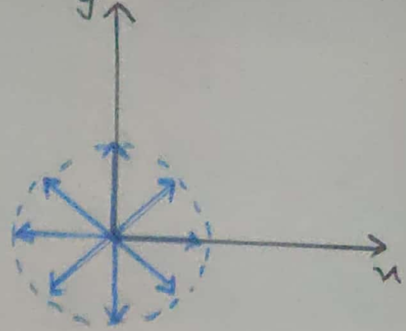
$$w(z) = -m\log(z - z')$$

If there is a sink of strength $(-m)$,

$$w(z) = m\log z$$

The radial and tangential components of the velocities:

$$q_r = \frac{m}{2\pi r}, \quad q_\theta = 0 \qquad - (1)$$




Now the volume flow rate per unit area across the circle of radius r is computed as:

$$V = \int_0^{2\pi} \vec{q}.n\,ds = \int_0^{2\pi} (q_r, q_\theta).\vec{n}\,ds = \int_0^{2\pi} q_r . r\,d\theta = \int_0^{2\pi} \frac{m}{2\pi r}\,r\,d\theta$$

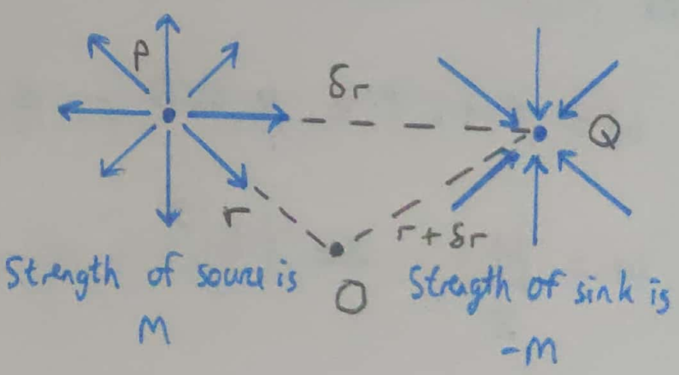
$$= \int_0^{2\pi} \frac{m}{2\pi}\,d\theta = m$$

Hence, we see that the source of strength m specifies the rate of volume flow issuing outward from the source

$$2\pi r q_r = 2\pi m$$

<u>Doublet:</u>

Let us assume that the source is at point $P(r,\theta)$ and sink is at Q $(r+\delta r, \theta + \delta\theta)$




Let $\phi$ be the velocity potential, then

$$\phi(r,\theta) = -m \log r + m \log(r+\delta r)$$

$$= m \log\left(1 + \frac{\delta r}{r}\right) = m\frac{\delta r}{r} + \text{higher order terms}$$

$$\approx m\frac{\delta r}{r} \qquad - (1)$$

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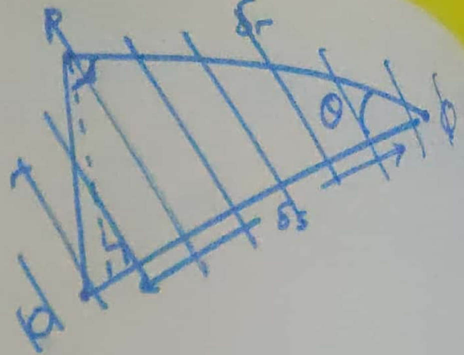
$$\cos\theta = \frac{\delta r}{\delta s}$$




$$\delta r = \cos\theta \, \delta s$$

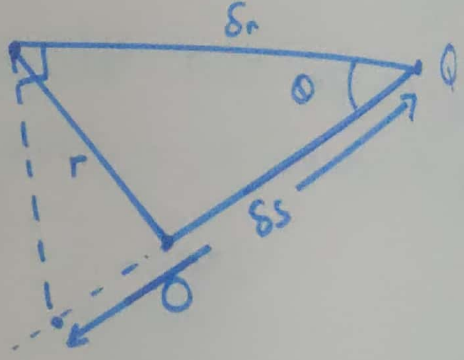
$$\Rightarrow \phi(r,\theta) = m\frac{\delta r}{r} = \frac{\delta s \cos\theta}{r} m$$

$$= \frac{\mu}{r}\cos\theta, \quad \mu = m\,\delta s$$

= strength of the doublet

From CR-equation, $\frac{1}{r}\frac{\partial \psi}{\partial \theta} = +\frac{\partial \phi}{\partial r} = -\frac{\mu \cos\theta}{r^2}$

$$\Rightarrow \frac{\partial \psi}{\partial \theta} = -\frac{\mu\cos\theta}{r}$$




$$\psi(r,\theta) = -\frac{\mu \sin\theta}{r}$$

The required complex ~~potential~~ potential for doublet

$$w = \phi + i\psi = \frac{\mu}{r}\cos\theta - i\frac{\mu}{r}\sin\theta = \frac{\mu}{r}e^{-i\theta}$$

$$w = \frac{\mu}{z}, \quad z = re^{i\theta}$$

Ex. Determine the acceleration at the point $P(2,1,3)$ at $t = 0.5$ sec. if the velocity is given by:

$$\vec{q}(t,x,y,z) = (yz+t,\ xz-t,\ xy)$$

Ans. $$\frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + \vec{q}\,\nabla\vec{q}$$

$$\vec{q}\cdot\nabla\vec{q} = u\frac{\partial \vec{q}}{\partial x} + v\frac{\partial \vec{q}}{\partial y} + w\frac{\partial \vec{q}}{\partial z}$$

$$= (yz+t)(0,z,y) + (xz-t)(z,0,x) + xy(y,x,0)$$

$$\vec{q}\cdot\nabla\vec{q}\Big|_{t=0.5}^{P(2,1,3)} = [6.5(0,3,1) + 5.5(3,0,2) + 2(1,2,0)]$$

$$= (17.5,\ 23.5,\ 9.5)$$

$$\frac{\partial \vec{q}}{\partial t} = (1,-1,0)$$

$$\vec{a} = \frac{d\vec{q}}{dt}\Big|_{P(2,1,3)}^{t=0.5} = (18.5,\ 22.5,\ 9.5)$$

Ex. In a 3-dimensional flow, the velocity components $V(x,y) = ax^3 - by^2 + cz^2$, $w(x,y) = bx^3 - cy^2 + az^2x$

Determine the $x$-component of the velocity vector $\vec{q}$

Ans. Let $\vec{q} = (u,v,w)$, then $\vec{\nabla}\cdot\vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

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$$\Rightarrow \frac{\partial u}{\partial x} - 2by + 2azx = 0$$

$$\Rightarrow u(x,y) = 2bxy + azx^2 + f(z,y)$$

<u>Ex.</u> Verify the conservation of mass for an incompressible flow $\vec{q} = (-ay, -bx^2, xy)$

<u>Ans.</u> $\vec{\nabla}.\vec{q} = 0$

$\Rightarrow$ Conservation of mass = Continuity equation

<u>Streamlines:</u>

<u>Ex.</u> If a flow has constant velocity, then determine the curve ~~which~~ such that the tangent to ~~it~~ it is $\parallel$ to $\vec{q}$

<u>Ans.</u> Let $\vec{q} = (u, v, w) = (\alpha, \beta, \gamma) \in \mathbb{R}^3$

By definition; $\dfrac{dx}{\alpha} = \dfrac{dy}{\beta} = \dfrac{dz}{\gamma} = c$

$$\Rightarrow \beta x - \alpha y = c_1$$
$$\gamma y - \beta z = c_2$$

<u>Ex.</u> Determine the pathlines and streamlines of a flow

$$\vec{q} = \left(\frac{x}{2+t}, \frac{y}{4+t}, \frac{z}{2+t}\right)$$

<u>Ans.</u> Path lines: $\dfrac{dx}{dt} = u$, $\dfrac{dy}{dt} = v$, $\dfrac{dz}{dt} = w$

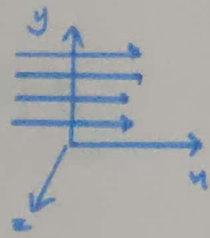
Stream lines : $\dfrac{dx}{u} = \dfrac{dy}{v} = \dfrac{dz}{w}$

(i) $\vec{q} = \left( \dfrac{2xt}{1+t^2}, \dfrac{2yt}{1+t^2}, \dfrac{2zt}{1+t^2} \right)$

Ex. Give an example of an irrotational and a rotational flow

Ans. $\Omega = \dfrac{1}{2} \vec{\nabla} \times \vec{q}_1 \neq 0$ | $\vec{q}_1 = (0, kx, 0)$ , $k>0$

$\Omega = \dfrac{1}{2} \vec{\nabla} \times \vec{q}_2 = 0$ | $\vec{q}_2 = (kx, 0, 0)$ , $k>0$




Ex. Determine the velocity vector of the flow

$\vec{q} = (ax^2yt, by^2zt, cz^2t)$

Ans.

Ex. Let $\vec{q} = \left( \dfrac{-k^2 y}{x^2+y^2}, \dfrac{k^2 x}{x^2+y^2}, 0 \right)$, then

(i) verify continuity equation

Ans. True since $\vec{\nabla} \cdot \vec{q} = 0$

(ii) Determine streamlines

Ans. $\dfrac{dx}{\frac{-k^2 y}{x^2+y^2}} = \dfrac{dy}{\frac{k^2 x}{x^2+y^2}} \Rightarrow \dfrac{k^2(x\,dx + y\,dy)}{x^2+y^2} = 0$

$\Rightarrow x^2+y^2 = \text{constant}$

(iii) Verify irrotationality

Ans. True

(iv.) Determine velocity potential

Ans. Let $\phi$ be the velocity potential, then

$$\vec{q} = -\nabla \phi$$

$$\Rightarrow \frac{-\partial \phi}{\partial x} = \frac{-k^2 y}{x^2+y^2} \quad -(1), \qquad \frac{-\partial \phi}{\partial y} = \frac{k^2 x}{x^2+y^2} \quad -(2)$$

Integrating (1), will give

$$\phi(x,y) = k^2 \tan^{-1}\left(\frac{x}{y}\right) + f(y)$$

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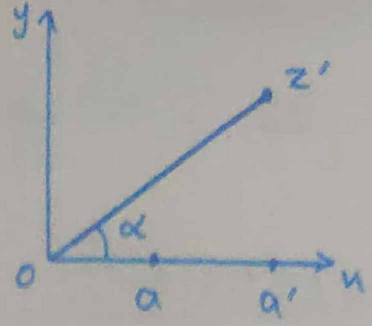
Source: $w = -m \log(z-a)$

Sink: $w = m \log(z-a')$

Doublet: $w = \frac{\mu}{z}$




Case I:

If the doublet is located at $z = z'$, then

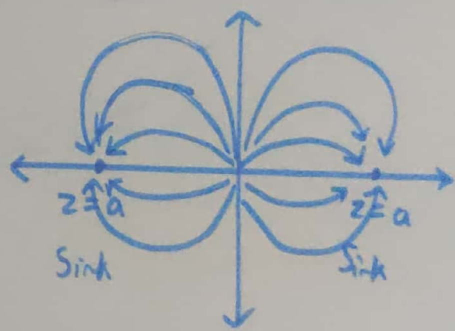
$$w = \frac{\mu e^{i\alpha}}{z-a}$$

The doublet makes an angle $\alpha$ with x-axis, then, we write $\theta - \alpha$ instead of $\theta$

$$w = \frac{\mu}{re^{i(\theta-\alpha)}} = \frac{\mu e^{i\alpha}}{re^{i\theta}} = \frac{\mu e^{i\alpha}}{z}$$

Since the doublet is located at $z = z'$, then

$$w = \frac{\mu e^{i\alpha}}{z - z'}$$




Ex.-1. What arrangements of sources and sinks give rise to the complex potential $w = \log_e\left(z - \frac{a^2}{z}\right)$. Draw the stream lines.

Ans. $w = \log\left(z - \frac{a^2}{z}\right) = \log\left(\frac{z^2 - a^2}{z}\right)$

$= \log(z-a) + \log(z+a) - \log(z)$

$\log z = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$

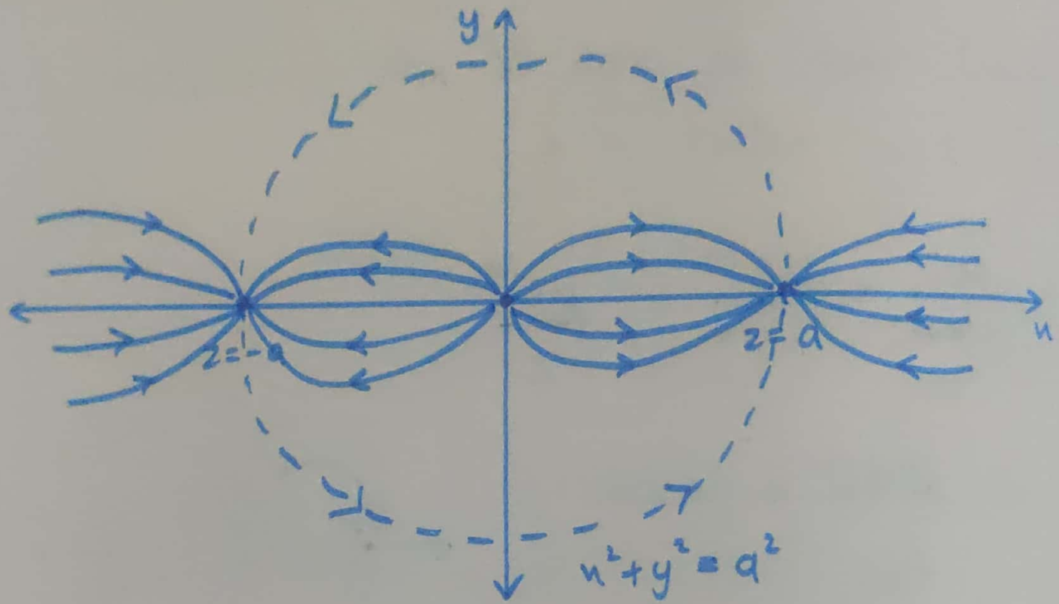
For streamlines,

$$\tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x} = C$$

$$\Rightarrow \frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)} = C'$$

When $C' = \infty$, on y-axis, $x^2+y^2=a^2 \rightarrow$ circle

When $C' = 0$, $y=0$, so x-axis is the streamline




## Milne-Thompson Theorem:

Let $f(z)$ be the complex potential for a flow having no rigid boundaries and such that there are no singularities of flow within the circle $|z|=a$. Then, on introducing a solid cylinder $|z|=a$, into the flow, the new complex potential is given by:

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) \quad \text{for } |z| \geq a$$

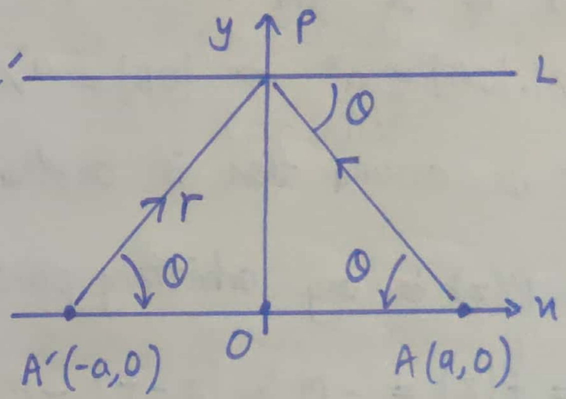
## Images:

If in a fluid a surface S can be drawn across which there is no flow, then any system of sources, sinks, or doublets on the opposite side of the surface is known as image of the system w.r.t. the surface.

Moreover, if the surface S is treated as a rigid boundary and the liquid removed from one side of it, the motion on the other side will remain unchanged.

## Image of a source w.r.t. a line:

Suppose that the image of a source of strength m at (a,0) on the x-axis is required w.r.t. the line OP.




Take an equal source at A'(-a, 0) of strength m. Let P be any point on OY such that $AP = A'P = r$

Then, the velocity at P due to the source at A is $m/r$. Similarly, it is also $m/r$ due to the source at A'.

Resultant velocity at P due to source at A, A' $= \frac{m}{r}\cos\theta + \frac{m}{r}\cos(\pi-\theta)$

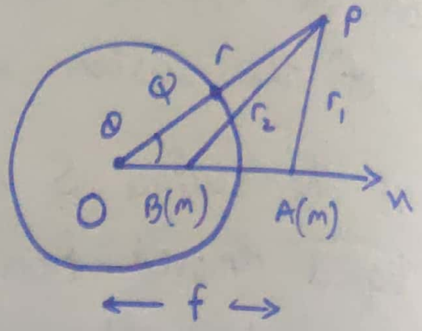
$= 0$

This shows that there is no flow across OY. Hence, by definition, the image of a source w.r.t. a line in 2D flow is an equal source equidistant from the line and opposite to the source

Similarly, image of a sink w.r.t. a line in 2D flow is an equal sink equidistant from the line & opp. to the sink.

## Image of a ~~circle~~ source w.r.t. a circle:

Consider a circle with centre O and radius r. Given any poin P in the plane, the inverse of P w.r.t. the circle is the unique point P' on the line OP such that $|OP|.|OP'| = r^2$




Let us determine the image of a source at $A(f,0)$ of strength m w.r.t. a circle centered at O. Let $|OA| = f$ and B be the inverse point of A. Let $|OQ| = a$, then

$|OA|.|OB| = a^2 \Rightarrow |OB| = a^2/f$

Let us assume there is another source at $B\left(\frac{a^2}{f}, 0\right)$

Let $P(z)$ be any arbitrary point outside the circle in the flow,

$$w = f(z) = -m\log(z-f) - m\log\left(z - \frac{a^2}{f}\right)$$

$$= -m\left[\log[(r\cos\theta - f) + ir\sin\theta]\right.$$

$$\left. + \log\left[\left(r\cos\theta - \frac{a^2}{f}\right) + ir\sin\theta\right]\right]$$

$$= (\phi_1 + \phi_2) + i(\psi_1 + \psi_2)$$

$$\phi = -\frac{m}{2}\left[\log(r\cos\theta - f)^2\right] - \frac{m}{2}\left[\log\left(r\cos\theta - \frac{a^2}{f}\right)^2\right]$$

$$\frac{\partial\phi}{\partial r} = -\frac{m}{2}\left[\frac{2(r - f\cos\theta)}{r^2 + f^2 - 2rf\cos\theta} + \frac{2\left(r - \frac{a^2}{f}\cos\theta\right)}{r^2 + \frac{a^4}{f^2} - 2r\frac{a^2}{f}\cos\theta}\right]$$

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$$-\frac{\partial \phi}{\partial r}\bigg|_{r=a} = m\left[\frac{a - f\cos\theta + f^2/a - f\cos\theta}{a^2 + f^2 - 2fa\cos\theta}\right] = \frac{m}{a}$$

Now if we place a source of strength $-m$ at the origin, the normal velocity at Q will be $-m/a$, hence normal velocity of the system $= 0$

"The image system for a source outside a circle consists of an equal source at the inverse point and an equal ~~square~~ ~~ad~~ sink at the center of the circle."

## Alternative Derivation of Image of a source via Milne-Thompson:

Let $OA = f$. Suppose there is a source of strength $m$ at $A$, where $z = f$. Let the radius of the circle is 'a' centered at O. Let $P(z)$ be any point in the flow outside of circle.

In the absence of the circle, the complex potential

$$w = -m\log(z-f)$$

Then, $f(z) = -m\log(z-f)$, $\bar{f}(z) = -m\log(z-f)$

Now let us introduce a circle/cylinder, $|z| = a$

$$w = f(z) + \bar{f}(a^2/z)$$

$$= -m\log(z-f) - m\log\left(\frac{a^2}{z} - f\right)$$

$$= -m\log(z-f) - m\log\left(z - \frac{a^2}{f}\right) + m\log z + \text{constant}$$

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## Blasius Theorem:

In a steady 2D irrotational motion of an incompressible flow under no external force given by the complex potential $w = f(z)$. If the pressure thrusts on the fixed cylinder of any shape are represented by $(X, Y)$ and a couple of moment $M$ about the origin. Then,

$$X - Yi = \frac{1}{2} i\rho \int_c \left(\frac{dw}{dz}\right)^2 dz$$

$$M = Re\left[-\frac{1}{2} i\rho \int_c z \left(\frac{dw}{dz}\right)^2 dz\right]$$

where $\rho$ is the density, $c$ is the centre of cylinder

——— x ———

**Ex.** $\psi(x, y, t) = xy$. Find:

a) $\phi$

b) Verify whether $\phi$ & $\psi$ satisfy Laplace's equation

c) Find streamlines and potential lines

**Ans.** $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

$u = x$, $v = -y$

$$\vec{q} = (x, -y) = \left(-\frac{\partial \phi}{\partial x}, -\frac{\partial \phi}{\partial y}\right)$$

$$\vec{\nabla} \times \vec{q} = 0$$

a) $\phi(x, y) = \frac{y^2}{2} + f(x)$ | $\frac{\partial \phi}{\partial x} = f'(x) = -x$

$f(x) = -\frac{x^2}{2} + c$

$$\Rightarrow \phi(x, y) = \frac{y^2 - x^2}{2} + c$$

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b.) $\nabla^2 \phi = 0$

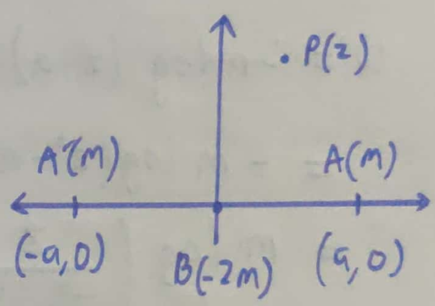
$\nabla^2 \psi = 0$

c.) $\psi \Rightarrow xy = k$ Streamlines

$\phi \Rightarrow x^2 - y^2 = c$ Potential lines

Ex. Let the sources of equal strength 'm' be at $(\pm a, 0)$, $a > 0$ and a sink of strength '2m' is at origin. Determine the streamlines, potential lines and magnitude of velocity

Ans. $W = -m \log(z-a)$
$\quad -m \log(z+a)$
$\quad +2m \log z$




$= \phi + i\psi$

$$= -m \left[ \frac{1}{2} \log\left[(x-a)^2 + y^2\right] + i \tan^{-1} \frac{y}{x-a} \right]$$
$$- m \left[ \frac{1}{2} \log\left[(x+a)^2 + y^2\right] + i \tan^{-1} \frac{y}{x+a} \right]$$
$$+ 2m \left[ \frac{1}{2} \log(x^2 + y^2) + i \tan^{-1} \frac{y}{x} \right]$$

(a.) $\phi(x,y) = c_1 \rightarrow$ potential lines

$\psi(x,y) = c_2 \rightarrow$ streamlines

b.) $$\frac{dw}{dz} = \frac{-m}{z-a} - \frac{m}{z+a} + \frac{2m}{z}$$

$$|\vec{q}| = \left| \frac{dw}{dz} \right|$$

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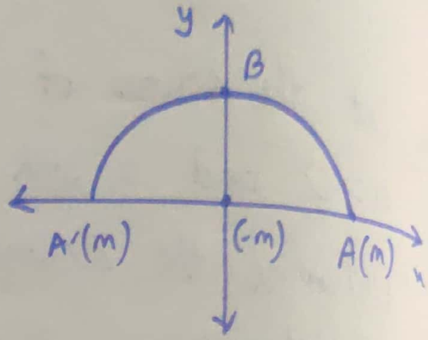
Ex. In a region bounded by a fixed arc and its radii, deduce the motion due to source of an equal sink situated at the ends of one of its bounding radii. If the streamlines at either end are leaving at angle $\alpha$, determine the streamlines.

Ans. Let $AOA'B$ be the circular arc in a 2D plane. Consider a source of strength 'm' at $A(a,0)$, a sink at $(0,0)$ and another source at $A'(-a,0)$




$$w = -m\log(z-a) - m\log(z+a) + m\log(z)$$

$$= -m\log(z^2-a^2) + m\log(z)$$

$$= m\log\left(\frac{z}{z^2-a^2}\right) = -m\log\left(z-\frac{a^2}{z}\right)$$

$$= -m\log\left[\left(r-\frac{a^2}{r}\right)\cos\theta + i\left(r+\frac{a^2}{r}\right)\sin\theta\right]$$

$$= -m\left[\frac{1}{2}\log\left[\left\{\left(r-\frac{a^2}{r}\right)\cos\theta\right\}^2 + \left\{\left(r+\frac{a^2}{r}\right)\sin\theta\right\}^2\right]\right]$$

$$- m\,i\tan^{-1}\frac{\left(r+\frac{a^2}{r}\right)\sin\theta}{\left(r-\frac{a^2}{r}\right)\cos\theta}$$

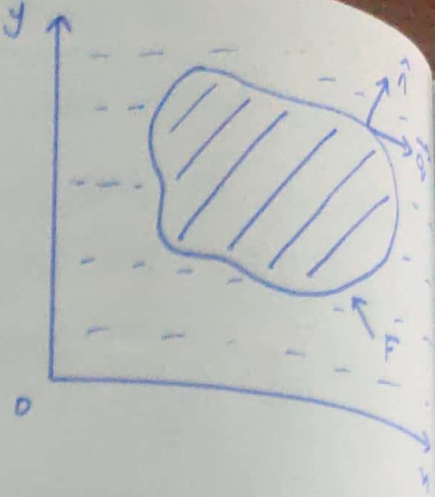
$$\psi(r,\theta) = -m\tan^{-1}\left(\frac{r^2+a^2}{r^2-a^2}\cdot\tan\theta\right) = -m(\pi-\alpha)$$

$$\left(\frac{r^2+a^2}{r^2-a^2}\right)\cdot\tan\theta = \tan(\pi-\alpha) = -\tan\alpha$$

$$(r^2+a^2)\tan\theta + (r^2-a^2)\tan\alpha = 0$$

## Blasius Theorem:

$$X - iY = \frac{i\rho}{2}\int_c \left(\frac{dw}{dz}\right)^2 dz$$




$M$ = couple of moment

$$= \text{Real}\left\{\frac{i\rho}{2}\int_c z\left(\frac{dw}{dz}\right)^2 dz\right\}$$

Let $P$ be the pressure acting on the surface of the cylinder. Let $P(x,y)$ be any arbitrary point on the surface, $S$ and let $c$ be the contour of the cylinder measured in the clockwise direction. Let $\delta s$ be any arbitrary surface element on the cylinder, then the force acting on $\delta s$ per unit area $= p\delta s$

Then the force on the entire cylinder $= -\int_c p\vec{n}\, ds$

where $n_i = \varepsilon_{ik3}\dfrac{dx_i}{ds}$

$$= -\int_c p n_i\, ds, \quad i = 1,2,3$$

The $x$-component of force, $F_x = F_{x_1} = F_1 = -\int_c p\frac{dx_2}{ds}ds$

$$= -\int_c p\, dx_2 = -\int p\, dy$$

$$F_y = F_{x_2} = F_2 = +\int_c p\, dx_1 = \int_c p\, dx$$

$$\hat{e}_i \times \hat{e}_j = \sum_{k=1}^{3} \varepsilon_{ijk}\hat{e}_k$$

$\downarrow$

Levi-Civita symbol

$\varepsilon_{ijk} = 1$ when $i, j, k$ are cyclic

$= -1$ when $i, j, k$ are acyclic

$= 0$ $\quad i = j,\ j = k$

$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$

The required force is given by:

$$X - iY = F_x - iF_y = -\int_c p\,dy - i\int_c p\,dx$$

$$= -i\int_c p(dx - i\,dy) = -i\int_c p\,d\bar{z} \quad -①$$

The moment of force on cylinder has component only along z-axis.

Therefore, $M = \vec{M}_F \cdot \hat{e}_k = -\int_c \varepsilon_{ij3}\, x_i\, n_j\, p\, ds$

$$= -\int_c \varepsilon_{ij3}\, x_i\, p\, \varepsilon_{jk3}\, dx_k$$

$$= -\int_c -\delta_{ik}\, x_i\, p\, dx_k$$

$$= \int_c \delta_{ik}\, x_i\, p\, dx_k$$

$$= \int_c (x_1 p\, dx_1 + x_2 p\, dx_2)$$

$$= \int_c p\,(x\,dx + y\,dy)$$

$$= \int_c p\, \mathrm{Re}(z\, d\bar{z}) \quad - ②$$

From Bernoulli's equation, we have

$$p + \frac{\rho}{2}\underbrace{(u^2 + v^2)}_{\left(\frac{dw}{dz}\right)^2} = \text{constant} = p_0$$

$$\because \frac{dw}{dz} = -u + iv$$

$$\Rightarrow p = p_0 - \frac{\rho}{2}\left(\frac{dw}{dz}\right)\cdot\overline{\left(\frac{dw}{dz}\right)} \quad - ③$$

From ① and ②, we have

$$X - iY = F_x - iF_y = -i\int_c \cancel{p_0}\, d\bar{z} + \frac{i\rho}{2}\int_c \frac{dw}{dz}\cdot\overline{\frac{dw}{dz}}\cdot d\bar{z}$$

neglected

$$= \frac{i\rho}{2}\int_c \frac{dw}{dz}\cdot\frac{d\bar{w}}{d\bar{z}}\, d\bar{z}$$

$$= \frac{i\rho}{2}\int_c \frac{dw}{dz}\, d\bar{w}$$

$w = \phi + i\psi$

$\bar{w} = \phi - i\psi$

$$= \frac{i\rho}{2}\int_c \frac{dw}{dz}\cdot\frac{dw}{dz}\cdot dz$$

$$= \frac{i\rho}{2}\int_c \left(\frac{dw}{dz}\right)^2 dz$$

To prove: $M = Re\left[\frac{\rho}{2}\int_c z\left(\frac{dw}{dz}\right)^2 dz\right]$

$$M = \int_c p\, Re(z\, d\bar{z}) \quad - ①$$

$$= Re\left(\int_c \left(p_0 - \frac{\rho}{2}(u^2+v^2)\right) z\, d\bar{z}\right)$$

$$= Re\left[\int_c \left(p_0 - \frac{\rho}{2}\frac{dw}{dz}\times\overline{\frac{dw}{dz}}\right) z\, d\bar{z}\right]$$

$$= Re\left[\int_c (p_0\, z\, d\bar{z}) - \frac{\rho}{2}\int_c \frac{dw}{dz}\times\frac{d\bar{w}}{d\bar{z}}\times d\bar{z}\times z\right]$$

$$= Re\left[\int_c p_0(x\,dx + y\,dy) - \frac{\rho}{2}\int_c z\,\frac{dw}{dz}\times\frac{dw}{dz}\times dz\right]$$

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$$= Re\left[\underbrace{\oint_c p_0(x\,dx + y\,dy)}_{=0} - \frac{\rho}{2}\oint_c z\left(\frac{dw}{dz}\right)^2 dz\right]$$

since $c$ is closed container

$$= Re\left[-\frac{\rho}{2}\oint_c z\left(\frac{dw}{dz}\right)^2 dz\right]$$

Ex. Verify that the $w = ik\log\frac{z-ia}{z+ia}$, $a, k > 0$ is the complex potential of a steady flow of a liquid about a circular cylinder and the plane $y = 0$. Find the forces acting on the cylinder by the fluid.

Ans. $\phi(x,y) + i\psi(x,y) = ik\log\left[\frac{x+i(y-a)}{x+i(y+a)}\right]$

$$= ik\log\left[\frac{(x+i(y-a))(x-i(y+a))}{x^2+(y+a)^2}\right]$$

$$= ik\log\left[\frac{x^2+(y-a)(y+a)+i[(y-a)x - x(y+a)]}{x^2+(y+a)^2}\right]$$

$$= ik\log\left[\frac{(x^2+(y-a)(y+a)) - 2iax}{x^2+(y+a)^2}\right]$$

$$= ik\left\{\frac{1}{2}\log\left[\left(\frac{x^2+y^2-a^2}{x^2+(y+a)^2}\right)^2 + \left(\frac{2ax}{x^2+(y+a)^2}\right)^2\right] + i\tan^{-1}\left(\frac{\cdots}{\cdots}\right)\right\}$$

$$\psi(x,y) = \frac{k}{2}\log\left[\left(\frac{x^2+y^2-a^2}{x^2+(y+a)^2}\right)^2 + \left(\frac{2ax}{x^2+(y+a)^2}\right)^2\right] = C$$

$$\left[\left(\frac{x^2+y^2-a^2}{x^2+(y+a)^2}\right)^2+\left(\frac{2ax}{x^2+(y+a)^2}\right)^2\right]=e^{\frac{2c}{k}}=\lambda$$

$$\Rightarrow x^2+(y-a)^2=\lambda\left[x^2+(y+a)^2\right]$$

If $\lambda=1$, $y=0$

$$W=ik\log\left(\frac{z-ia}{z+ia}\right), \quad \frac{dw}{dz}=ik\left(\frac{1}{z-ia}+\frac{1}{z+ia}\right)$$

$$X-iY=\frac{+1}{2}i\rho\int_c\left(\frac{dw}{dt}\right)^2 dz \quad -①$$

$$\Rightarrow \frac{dw}{dz}=\frac{2ka}{z^2+a^2} \quad -②$$

From ① and ②,

$$X-iY=\frac{+i\rho}{2}\int_c\frac{(2ka)^2}{(z^2+a^2)^2}dz$$

$$=\frac{2i\rho k^2a^2}{a^4}\int_c\left(1+\frac{z^2}{a^2}\right)^{-2}dz$$

$$=\frac{2i\rho k^2}{a^2}\int_c\left(1+\frac{z^2}{a^2}\right)^{-2}dz$$

$$=\frac{2i\rho k^2}{a^2}\times 2\pi i\times(\text{sum of residuals})$$

$$=-\frac{4\pi\rho k^2}{a^2}\times \text{Residual of } \left(1+\frac{z^2}{a^2}\right)^{-2} \text{ at } z=ai$$

$$=-4\pi a^2\rho k^2\times\frac{1}{8ia^3}=\frac{-\pi}{2i}\frac{\rho k^2}{a}$$

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$$X - iY = \frac{-\pi \rho k^2}{2ia} = \frac{\pi \rho k^2}{2a} i$$

$$X = 0, \quad Y = -\frac{\pi \rho k^2}{2a}$$

$$F = \left| \sqrt{X^2 + Y^2} \right| = \frac{\pi \rho k^2}{2a}$$

## Circulation:

The flow around a closed curve is known as the circulation around the curve C.

Let C be any closed curve and $\Gamma$ be the circulation then,

$$\Gamma = \int_C \vec{q}\cdot d\vec{r} = \int_C (u\,dx + v\,dy + w\,dz)$$

## Kelvin Circulation Theorem:

When the external force are conservative and derivable from a potential function and density is the function of pressure only. Then, the circulation in any closed curve moving with fluid is constant all the time.

Ans. We know $\Gamma = \int_C \vec{q}\cdot d\vec{r}$

We aim to show $\frac{d\Gamma}{dt} = 0$

$$\frac{d\Gamma}{dt} = \int_C \frac{d}{dt}(\vec{q}\cdot d\vec{r}) = \int_C \frac{d\vec{q}}{dt}\cdot d\vec{r} + \int_C \vec{q}\cdot d\vec{q}$$

$$= \underbrace{\int_C \left(\vec{F} - \frac{1}{\rho}\nabla P\right) d\vec{r} + \int_C \vec{q}\cdot d\vec{q}}_{\text{Single valued function}}$$

$$\Rightarrow \frac{d\Gamma}{dt} = 0 \quad \Rightarrow \Gamma = \text{constant}$$

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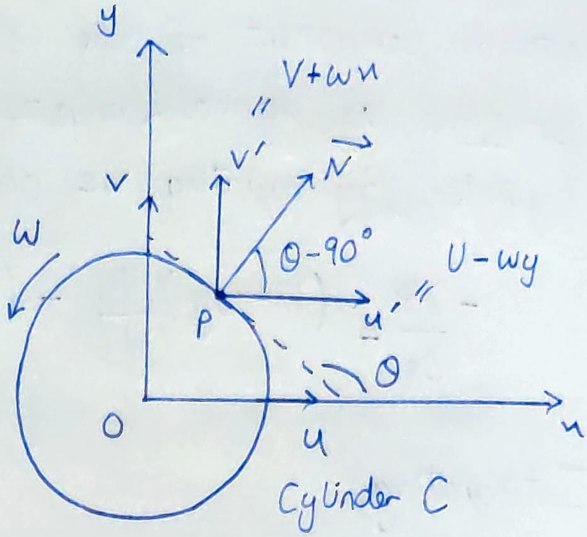
## Flow around a cylinder:




Let the cylinder 'C' be placed at the origin. Let u and v be the velocity components parallel to the coordinate axes.

Due to flow, let the angular velocity of the cylinder is w. Here, the cylinder is placed in an infinite mass of fluid which is at rest in infinity.

Since the fluid is at rest at infinity,

$$\vec{q} = 0 \Rightarrow u, v = 0 \Rightarrow \frac{\partial \psi}{\partial x} = 0, \frac{\partial \psi}{\partial y} = 0 \Rightarrow \psi = \text{constant} \quad - (1)$$

Let us consider P to be the position of a particle. On the contour of the cylinder and at any time "t"

Let the particle is at Q such that $\vec{PQ} = ds$, then

$$\cos\theta = \frac{dx}{ds}, \quad \sin\theta = \frac{dy}{ds}$$

Moverover, the velocity components at P are $(U-wy, V+wx)$ respectively.

Therefore, outward normal velocity at P

$$= (U-wy)\cos(\theta-90) + (V+wx)\sin(\theta-90)$$

$$= (U-wy)\sin\theta - (V+wx)\cos\theta$$

$$= (U-wy)\frac{dy}{ds} - (V+wx)\frac{dx}{ds} \quad - (2)$$

Now at the boundary of the moving cylinder, the normal component of the velocity of the liquid must be same as normal component of the velocity of the cylinder. By equating, we obtain,

$$-\frac{\partial \psi}{\partial s} = (U - \omega y)\frac{dy}{ds} - (V + \omega x)\frac{dx}{ds}$$

Integrating,

$$\psi(x, y) = Vx - Uy + \frac{1}{2}\omega(x^2 + y^2) + C$$

$$= V\left(\frac{z + \bar{z}}{2}\right) + iU\left(\frac{z - \bar{z}}{2}\right) + \frac{1}{2}\omega z\bar{z} + C$$

## Motion of a circular cylinder:

To determine the motion of a circular cylinder moving in an infinite mass of liquid which is at rest at infinity, with velocity $U$ in the direction of $x$-axis and for a given irrotational flow.

Ans. Since the flow is irrotational, $\exists\ \phi$ such that

$$\vec{q} = -\vec{\nabla}\phi$$

From equation of irrotationality,

$$\nabla^2 \phi = 0$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \quad -①$$

The solution of above can be of following forms

$$\phi_1(r,\theta) = A r^n \cos n\theta$$

$$\phi_2(r,\theta) = B r^n \sin n\theta$$

Now, from previous result, the normal velocity of the cylinder at any point must be equal to the velocity of the fluid along direction of normal

$$\Rightarrow -\frac{\partial \phi}{\partial r} = U\cos\theta \quad \text{at } r = a \quad -②$$

Since the fluid is at rest at infinity

$$q_r = 0 \;,\quad q_\theta = 0 \quad \text{when } r \to \infty$$

$$\Rightarrow -\frac{\partial \phi}{\partial r} = 0 \;,\; \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \quad \text{when } r \to \infty$$

$$\Rightarrow \phi(r,\theta) = Ar\cos\theta + \frac{A_1}{r}\cos\theta$$

$$-\frac{\partial\phi}{\partial r}\Big|_{r\to\infty} = 0 \Rightarrow A = 0$$

This implies, $\phi(r,\theta) = \frac{A}{r}\cos\theta$

$$\frac{\partial\phi}{\partial r} = \frac{-A}{r^2}\cos\theta$$

$$\Rightarrow -\frac{\partial\phi}{\partial r}\Big|_{r=a} = \frac{A}{r^2}\cos\theta\Big|_{r=a} = U\cos\theta$$

$$\Rightarrow \frac{A}{a^2} = U \quad (1) \rightarrow A = Ua^2$$

$$\therefore \phi(r,\theta) = \frac{Ua^2}{r}\cos\theta$$

By CR equation, $\frac{\partial\psi}{\partial r} = \frac{-1}{r}\frac{\partial\phi}{\partial\theta}$

$$\frac{\partial\psi}{\partial r} = \frac{-1}{r}\left[-\frac{Ua^2}{r}\sin\theta\right] = \frac{Ua^2}{r^2}\sin\theta$$

$$\Rightarrow \psi(r,\theta) = -\frac{Ua^2}{r}\sin\theta + \underbrace{f(\theta)}_{0} = \frac{-Ua^2}{r}\sin\theta$$

The required complex potential, $w = \phi + i\psi$

$$w = \frac{Ua^2}{r}\cos\theta - i\frac{Ua^2}{r}\sin\theta$$

$$= \frac{Ua^2}{r}(\cos\theta - i\sin\theta) = \frac{Ua^2}{r}e^{-i\theta} = \frac{Ua^2}{re^{i\theta}}$$

$$w = \frac{Ua^2}{z} \quad \text{where } z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

# Flow past a circular cylinder fixed in a stream:

$$\psi(x,y) = Vx - Uy + \frac{1}{2}\omega(x^2+y^2) + c \quad \text{[for flow around a cylinder]}$$

$y =$ constant (taking flow along $x$-axis) and $\omega = 0$ (fixed)

$$\Rightarrow \psi(x,y) = Vx + c_1$$

$$\therefore \phi(x,y) = -Vy + c_2$$

Let us consider a circular cylinder and let the fluid with velocity $U$ past through the cylinder from the negative direction of $x$-axis. Then from previous derivation, we obtain

$$W = \frac{-Ua^2}{z} \quad -①$$

which is the complex potential for a moving cylinder. Now if we want to reduce it to rest then the velocity must include $+Ux$ ($= Ur\cos\theta$) or the stream function must include $Uy$ ($= Ur\sin\theta$), i.e.,

$$\phi(r,\theta) = Ux + \frac{Ua^2}{r}\cos\theta, \quad \psi(r,\theta) = Uy - \frac{Ua^2}{r}\sin\theta$$

$$\phi(r,\theta) = Ur\cos\theta + \frac{Ua^2}{r}\cos\theta, \quad \psi(r,\theta) = Ur\sin\theta - \frac{Ua^2}{r}\sin\theta$$

$$W = \phi + i\psi$$

$$= Ur\cos\theta + \frac{Ua^2}{r}\cos\theta + i\left[Ur\sin\theta - \frac{Ua^2}{r}\sin\theta\right]$$

$$= Ure^{i\theta} + \frac{Ua^2}{re^{i\theta}} = U\left[z + \frac{a^2}{z}\right]$$

$$\frac{dw}{dz} = U - \frac{Ua^2}{z^2} = -u + iv \quad \text{when } \vec{q} = (u, v)$$

$$\left|\frac{dw}{dz}\right| = |\vec{q}| = \sqrt{u^2+v^2} = |U| . |1 - \cos 2\theta + i \sin 2\theta|$$

$$= |U| \sqrt{(1-\cos 2\theta)^2 + (\sin 2\theta)^2} \quad \text{at} \quad z = ae^{i\theta}$$

$$= |U| \sqrt{2 - 2\cos 2\theta}$$

$$= |U| . 2 \sin\theta = 2 |U| \sin\theta$$

$\theta = 0$, $q_{min} = 0$

$\theta = \frac{\pi}{2}$, $q_{max} = 2 . |U|$

## Flow past a Cylinder with Circulation:

Let $\alpha$ be the constant circulation about the cylinder, then the circulation is,

$$\alpha = \int_c \vec{q} \cdot d\vec{r} = q_0 (2\pi r)$$

$$\Rightarrow \alpha = \frac{-1}{r}\frac{\partial \phi}{\partial \theta} \cdot 2\pi r$$

$$\Rightarrow \frac{\partial \phi}{\partial \theta} = \frac{-\alpha}{2\pi}, \quad \phi(r,\theta) = \frac{-\alpha\theta}{2\pi}$$

Using CR equations,

$$\psi(r,\theta) = \frac{\alpha}{2\pi}\log r$$

Complex potential due to circulation, $W_{cir} = \phi + i\psi$

$$W_{cir} = \frac{-\alpha\theta}{2\pi} + i\frac{\alpha}{2\pi}\log r = \frac{\alpha i}{2\pi}(\log r + i\theta)$$

$$= \frac{\alpha i}{2\pi}(\log r + \log e^{i\theta}) = \frac{\alpha i}{2\pi} \cdot \log(re^{i\theta})$$

$$W_{cir} = \frac{\alpha i}{2\pi}\log z \qquad \text{where } z = re^{i\theta}$$

Required complex potential, $W = W_{flow} + W_{cir}$

$$W = U\left(z + \frac{a^2}{z}\right) + \frac{\alpha i}{2\pi}\log z$$

$$\frac{dw}{dz} = U\left(1 - \frac{a^2}{z^2}\right) + \frac{i\alpha}{2\pi} \cdot \frac{1}{z}$$

On $r = a$ or $z = ae^{i\theta}$,

$$\frac{dw}{dz} = U\left(1 - \frac{\cancel{a^2}}{\cancel{a^2} e^{2i\theta}}\right) + \frac{i\alpha}{2\pi} \cdot \frac{1}{ae^{i\theta}}$$

$$= U\left(1 - e^{-2i\theta}\right) + \frac{i\alpha}{2\pi a} e^{-i\theta}$$

$$\left|\frac{dw}{dz}\right| = |\vec{q}| = \left|2U \sin\theta + \frac{\alpha}{2\pi a}\right|$$

• Note: Stagnation points, where $|\vec{q}| = 0$

Here, stagnation point is given by:

$$\left|2U \sin\theta + \frac{\alpha}{2\pi a}\right| = 0$$

$$\Rightarrow \sin\theta = -\frac{\alpha}{4\pi a U}$$

$$|\sin\theta| \leq 1 \quad \Rightarrow \quad \left|\frac{\alpha}{4\pi a U}\right| \leq 1$$

$$\Rightarrow |\alpha| \leq 4\pi a U$$

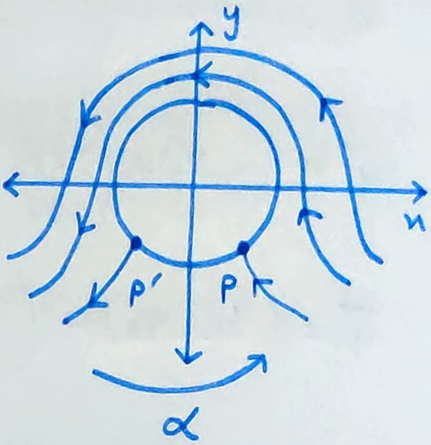
↳ $|\alpha| = 4\pi a U$

↳ $|\alpha| < 4\pi a U$

- When $|\alpha| = 4\pi a U$,

$\alpha = \pm 4\pi a U$ , $\sin \theta_{stag} = \mp 1$ , $\theta_{stag} = \frac{3\pi}{2}$.

We have two stagnation points, P and P', coinciding at the bottom of the cylinder, which is M $[\alpha = 4\pi a U]$




- When $-4\pi a U < \alpha < 4\pi a U$,

then the lines will be given by:

Ex. A circular cylinder is fixed across a stream of velocity $U$ and a circulation $\alpha$ around the cylinder. Determine the max velocity

Ans. The velocity potential is given by:

$$\phi(r,\theta) = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{\alpha\theta}{2\pi}$$

$$\vec{q} = (u, v) = \left(-\frac{\partial \phi}{\partial x}, -\frac{\partial \phi}{\partial y}\right)$$

$$\vec{q} = (q_r, q_\theta) = \left(-\frac{\partial \phi}{\partial r}, -\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$\frac{\partial \phi}{\partial r} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$\frac{\partial \phi}{\partial \theta} = -U\left(r + \frac{a^2}{r}\right)\sin\theta - \frac{\alpha}{2\pi}, \quad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\alpha}{2\pi r}$$

$\cancel{q_r \; \theta = \pi/2 =}$

By definition, $|q|^2 = \left(\frac{\partial \phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)^2$

$$= \left[1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right]U^2 + \frac{U\alpha}{\pi r}\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\alpha^2}{4\pi^2 r^2}$$

As cylinder is fixed, $r = a$ for maximum velocity

$$= \left[2 - 2\cos 2\theta\right]U^2 + \frac{U\alpha}{\pi a} \cdot 2\sin\theta + \frac{\alpha^2}{4\pi^2 a^2}$$

$$= 4\sin^2\theta\, U^2 + 2\frac{U\alpha}{\pi a}\sin\theta + \frac{\alpha^2}{4\pi^2 a^2}$$

$$= \left(2\sin\theta\, U + \frac{\alpha}{2\pi a}\right)^2$$

$$\Rightarrow |\vec{q}| = \left| 2U \sin\theta + \frac{\alpha}{2\pi a} \right|$$

$$\theta = \pi/2$$

$$|\vec{q}|_{max} = \left| 2U + \frac{\alpha}{2\pi a} \right|$$

Ex. The circular cylinder $(x+a)^2 + y^2 = a^2$ is placed in an oncoming flow with velocity $U$ and there is a circulation of $2\pi k$. Find the force and moment of force about the origin.

Ans. $W = W_1 + W_2$

$$= U\left((z+a) + \frac{a^2}{(z+a)}\right) + i\frac{2\pi k}{2\pi}\log(z+a)$$

$$= U\left((z+a) + \frac{a^2}{(z+a)}\right) + ik\log(z+a) \quad - 1$$

$$X - iY = \frac{i\rho}{2}\int_c \left(\frac{dw}{dz}\right)^2 dz \quad - 2$$

$$M = \mathrm{Re}\left\{-\frac{\rho}{2}\int_c z\left(\frac{dw}{dz}\right)^2 dz\right\} \quad - 3$$

$$\frac{dw}{dz} = U\left(1 - \frac{a^2}{(z+a)^2}\right) + \frac{ik}{z+a}$$

$$\left(\frac{dw}{dz}\right)^2 = U^2\left(1 - \frac{a^2}{(z+a)^2}\right)^2 - \frac{k^2}{(z+a)^2} + \frac{2Uik}{z+a}\left(1 - \frac{a^2}{(z+a)^2}\right)$$

**Ex.** A circular cylinder of radius $a$ is moving with velocity $U$ along x-axis in a flow. Determine the complex potential, magnitude of velocity and the direction of the velocity of the fluid.

**Ans.** $\frac{dx}{dt} = U$ , $x(t) = Ut$ , $x=0$ at $t=0$

$$W = \frac{Ua^2}{z-Ut} = \phi + i\psi$$

$$\frac{dw}{dz} = \frac{-Ua^2}{(z-Ut)^2} = -u+iv \quad , \quad \text{let } z-Ut = re^{i\theta}$$

$$u-iv = \frac{Ua^2}{r^2}(\cos 2\theta - i\sin 2\theta)$$

$$|\vec{q}| = \sqrt{u^2+v^2} = \sqrt{\left(\frac{Ua^2}{r^2}\cos 2\theta\right)^2 + \left(\frac{Ua^2}{r^2}\sin 2\theta\right)^2} = \frac{Ua^2}{r^2}$$

Direction of velocity, $\tan^{-1}\left(\frac{v}{u}\right) = 2\theta$

## Flow past Aerofoils:

**Def.** An aerofoil usually has a profile structure of the ~~type~~ fish type. Such structures are employed in the construction of aeroplanes. It has a blunt leading edge and a sharp trailing edge. We make the following assumptions:

i.) air behaves as an incompressible fluid

ii.) the aerofoil is a cylinder whose cross-section is a curve of fish type

iii.) the flow is 2D irrotational motion

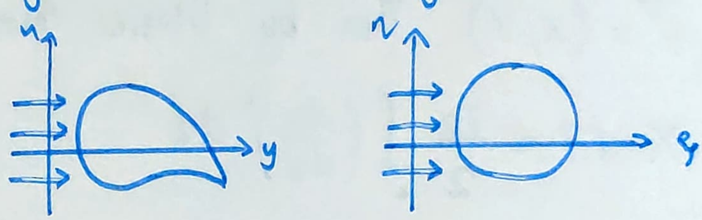
## Kutta - Joukowski Mapping:

$\zeta = z + \dfrac{a^2}{z}$ , $a$ = radius of cylinder

Helps map the aerofoil to a cylinder

↓

Conformal mapping




## Kutta - Joukowski Theorem:

When a cylinder of any shape is placed in an uniform stream of speed $U$, then the resultant thrust on the cylinder is a lift of magnitude $k\rho U$ per unit length and at right angles to the stream, $k$ is the circulation around the cylinder.

**Proof:**

Let there be a fixed cylinder in the $xy$-plane making an angle $\alpha$ with the axis with its cross-section containing the origin. Then, the disturbance caused by the cylinder is given by:

$$w_1 = Az + \frac{B}{z^2}$$

The stream makes an angle $\alpha$ where touching the surface of the cylinder

$$w_2 = Ue^{-i\alpha}z$$

Also, k is the circulation,

$$w_3 = \frac{ik}{2\pi}\log z$$

Complex potential, $w = f(z) = w_1 + w_2 + w_3$

$$= Az + \frac{B}{z^2} + Ue^{-i\alpha}z + \frac{ik}{2\pi}\log z$$

The thrust on the cylinder is denoted by:

$\vec{F} = (X, Y)$. Then by Blasius Theorem,

$$X - iY = \frac{i\rho}{2}\int_c \left(\frac{dw}{dz}\right)^2 dz$$

$$= \frac{i\rho}{2}\int_c \left[A - \frac{2B}{z^3} + Ue^{-i\alpha} + \frac{ik}{2\pi z}\right]^2 dz$$

$$= \frac{i\rho}{2} \times 2\pi i \times (\text{Sum of residuals at } z=0)$$

$$= \frac{i\rho}{\cancel{2}} \times \cancel{2}\cancel{\pi}i \times \frac{ikUe^{-i\alpha}}{\cancel{\pi}} = -ik\rho Ue^{-i\alpha}$$

$$= k\rho U[-\sin\alpha - i\cos\alpha] = -k\rho U[\sin\alpha + i\cos\alpha]$$

$$X = -k\rho U\sin\alpha$$

$$Y = k\rho U\cos\alpha$$

$$\text{Thrust} = \sqrt{X^2 + Y^2} = k\rho U$$

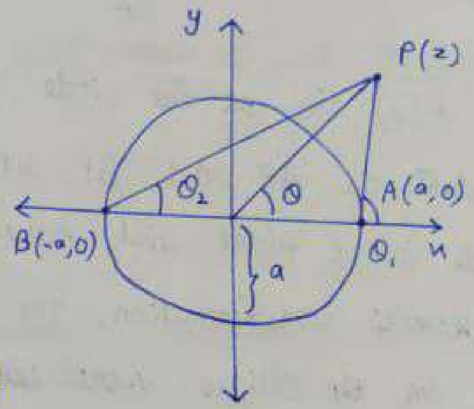
Hence, maximum lift is $k\rho U$ which always acts at right angles to the st

Thrust $= \sqrt{X^2 + Y^2} = k\rho U$

Hence, maximum lift is $k\rho U$ which always acts at right angles to the st

## Joukowski Transformation:

Let $z$ be any arbitrary point in original plane and variable $\xi$ is defined as:




$$\xi = z + \frac{a^2}{z} \quad - (1)$$

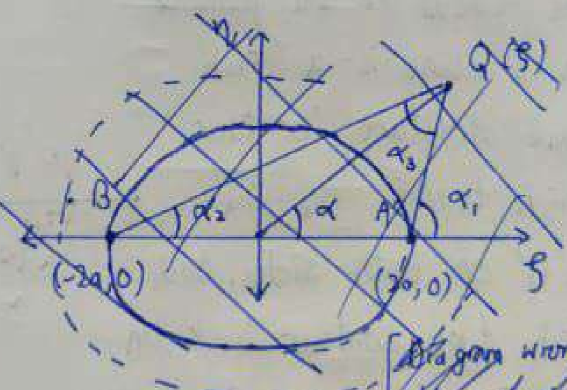
(Here $z$ is in $XY$-plane and $\xi$ is in $\xi\eta$-plane)

Let $A(a,0)$ and $B(-a,0)$ be any two points in $XY$-plane

$$\xi = a + \frac{a^2}{a} = 2a \quad , \quad \xi = -a + \frac{a^2}{-a} = -2a \quad - (2)$$

$$\xi - 2a = z + \frac{a^2}{z} - 2a$$

$$= \frac{(z-a)^2}{z}$$




[Diagram wrong, ellipse don't pass through foci]

$$A'Q\, e^{i\alpha_1} = \frac{(AP\, e^{i\theta_1})^2}{OP\, e^{i\theta}}$$

Comparing the real and imaginary parts

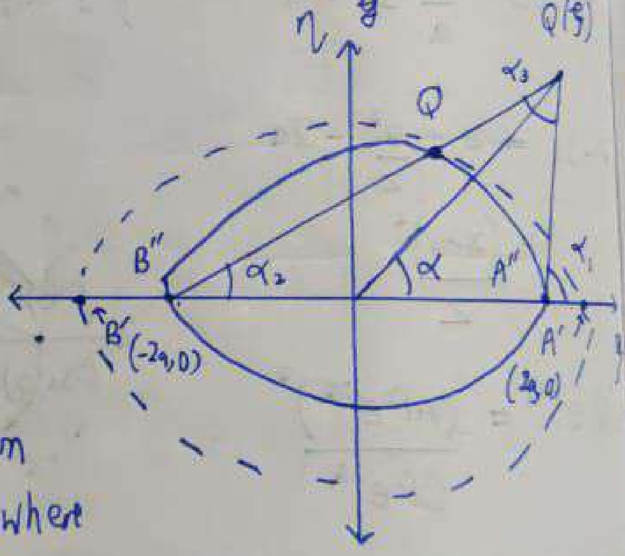
$$A'Q = \frac{AP^2}{OP} \quad \text{and} \quad e^{i\alpha_1} = e^{2i\theta_1 - i\theta}, \quad \alpha_1 = 2\theta_1 - \theta$$

Similarly, $B'Q = \dfrac{BP^2}{OP}$ , $\alpha_2 = 2\theta_2 - \theta$

Then, $\angle A'QB' = \alpha_3 = \alpha_1 - \alpha_2 = (2\theta_1 - \theta) - (2\theta_2 - \theta)$
$= 2(\theta_1 - \theta_2)$
$= 2\angle APB$

Also $A'Q + B'Q = \dfrac{AP^2 + BP^2}{OP} =$ constant quantity

If $P(z)$ lies on the circle then $A'Q + B'Q$ will be constant as $AP$ and $BP$ are constants and then the circle in $\zeta$ plane will reduce to an ellipse. Hence, via Joukowski transformation, the mapping ① maps a circle into an ~~er~~ ellipse with centre at origin and foci $A'$ and $B'$. Let $P$ be any point on the circle such that $\angle APB = \pi/2$, then $\angle A'QB' = \pi$, so $Q$ lies on the line $A'B'$




The locus of the point $Q$ in $\zeta$-plane is a fish type structure / contour which touches the line $A'B'$ on both sides. Such fish type contour is known as Joukowski aerofoil, where the point $B'$ is the trailing edge and $A'$ is the leading / blunt edge. By ①,

$$\zeta = z + \frac{a^2}{z} \quad , \quad \frac{d\zeta}{dz} = 1 - \frac{a^2}{z^2} \quad - ③$$

Now the velocity is zero at,

$$\frac{d\xi}{dz} = 0 \Rightarrow z = \pm a$$

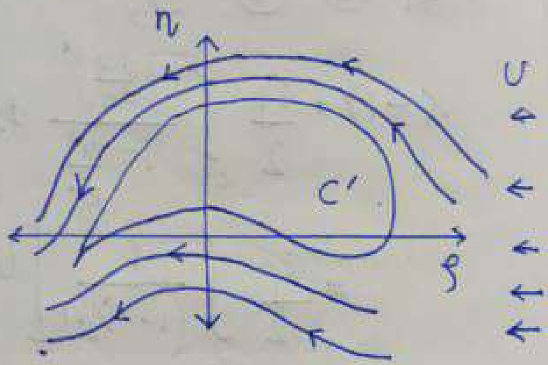
The points $(a,0)$ and $(-a,0)$ correspond to $(2a,0)$ and $(-2a,0)$ where the velocity vanishes. Now, if $\vec{q}$ velocity at the point A of the circle and $\vec{q}'$ at the point A' of the ~~circle~~ aerofoil.

$$\text{Then, } |\vec{q}'| = \left|\frac{dw}{d\xi}\right| = \left|\frac{dw}{dz}\cdot\frac{dz}{d\xi}\right| = \left|\frac{dw}{dz}\right|\cdot\left|\frac{dz}{d\xi}\right|$$

$$= |\vec{q}|\cdot\left|\frac{dz}{d\xi}\right|$$

Flow around an aerofoil:

We obtain the shape of an aerofoil via Joukowski Transformation.




$$\xi = z + \frac{a^2}{z} \quad -①$$

The stagnation points $z = \pm a$, transforms to $\xi = \pm 2a$, Let U be the fluid velocity which makes an angle $\alpha$ with $\xi$-axis, then the lift on aerofoil given by Blasius theorem,

$$X - iY = \frac{i\rho}{2}\int_{C'}\left(\frac{dw}{d\xi}\right)^2 d\xi \quad -②$$

Now, $\cancel{d\zeta}$ $\frac{dw}{d\zeta} = \frac{dw}{dz} \times \frac{dz}{d\zeta} = \frac{dw/dz}{\left(1-\frac{a^2}{z^2}\right)}$

$$\left(\frac{dw}{d\zeta}\right)^2 = \frac{dw}{d\zeta} \times \frac{dw}{d\zeta} = \frac{\left(dw/dz\right)^2}{\left(1-\frac{a^2}{z^2}\right)} \times \frac{dz}{d\zeta}$$

$$\left(\frac{dw}{d\zeta}\right)^2 d\zeta = \frac{\left(dw/dz\right)^2}{\left(1-\frac{a^2}{z^2}\right)} dz \quad -(3)$$

For circle in z-plane,

$$w = Ue^{i\alpha}(z-a) + \frac{Ua^2e^{-i\alpha}}{z-a} + \frac{ik}{2\pi}\log(z-a)$$

$$\frac{dw}{dz} = Ue^{i\alpha} - \frac{Ua^2e^{-i\alpha}}{(z-a)^2} + \frac{ik}{2\pi(z-a)} \quad -(4)$$

From (2), (3), (4),

$$X - iY = \frac{i\rho}{2}\int_{C'} \frac{\left(\frac{dw}{dz}\right)^2}{1-\frac{a^2}{z^2}}\, dz$$

$$= \frac{i\rho}{2}\int_{C'} \frac{1}{(1-a^2/z^2)} \overbrace{\left[Ue^{i\alpha} - \frac{Ua^2e^{-i\alpha}}{(z-a)^2} + \frac{ik}{2\pi(z-a)}\right]}^{(I)} dz$$

$$= \frac{i\rho}{2} \cdot 2\pi i \cdot [\text{sum of residuals of (I)}]$$

$$= \frac{i\rho}{\cancel{2}} \cdot \cancel{2}\pi i \cdot \left(-\cancel{2}Ue^{i\alpha} \cdot \frac{ik}{\cancel{2\pi}}\right) = -i\rho kUe^{i\alpha}$$

$$X - iY = \rho kU(\sin\alpha - i\cos\alpha)$$

$X = \rho kU \sin\alpha$, $Y = \rho kU\cos\alpha$, Lift $= \sqrt{X^2+Y^2} = \rho kU$

## Motion of Viscous Fluid:

### Tensors

Let $\vec{a}$ be a vector in $x_i$-plane with components $a_i$ along $x_i$-axes, then

Summation Convention

$$\vec{a} = \sum_{i=1}^{3} a_i e_i = a_1 e_1 + a_2 e_2 + a_3 e_3 = a_i e_i \quad -①$$

Let $a_i'$ be the components of $a$ in $x_i'$-system, then

$$\vec{a} = a_i' e_i' \quad -②$$

Taking the product with $e_i'$ on both sides of ①, then

$$\vec{a} . e_i' = a_i' = [a_1 e_1 + a_2 e_2 + a_3 e_3] e_i'$$

$$\left\{ \begin{aligned} e_i e_i' &= \cos(e_i \text{ and } e_i') |e_i| . |e_i| \\ \alpha_{ij} &= e_i . e_j' = \cos(e_i \text{ and } e_j') \end{aligned} \right\}$$

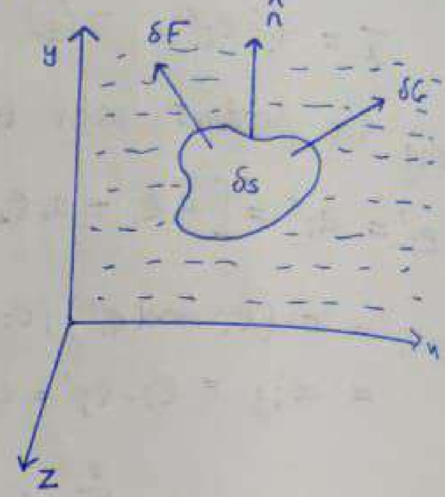
$$\Rightarrow a_i' = \alpha_{ij} a_j = \sum_{j=1}^{3} \alpha_{ij} a_j \quad \forall\, i = 1, 2, 3 \quad -③$$

Similarly, $a_i = \alpha_{ji} a_j' \quad -④$

The components $a_i$ and $a_i'$ obey the transformation in ③ and ④, respectively. The rule represents a property of the vectors and this property is used to define

## Stress at a point in a fluid:

Consider an infinitesimal area $\delta s$ surrounding a point $P(x, y, z)$ in a viscous fluid which may be at rest/motion let $\hat{n}$ be the unit outward drawn normal. Consider the forces exerted across $\delta s$ by the portion of the fluid which lies on the sides of $\hat{n}$. These internal forces are ~~not distorted~~ denoted by $\delta F$ and a couple of moment $\delta G$, about the same axis through $P$.




If the area of the plane surface ($\delta s$) shrinks into a point $P$, then both $\delta F$ and $\delta G$ will tend to zero. Hence for an infinitesimally small area, the surface force $\delta F$ may be assumed to be proportional to surface area ~~sur~~ $\delta s$. The fluid on each side of the surface exerts a force $\delta F$ on it. Thus, a stress $\tau$ is defined as the limit of the ratio of force to the area on which it acts, and when area shrinks to a point, it follows that the ratio vanishes. The creation of the internal forces in the immediate neighbourhood of a point $P$ across the surface, normal to $\hat{n}$, can be represented by the limit of the ratio of force to area:

$$\tau = \lim_{\delta s \to 0} \frac{\delta F}{\delta s} \left(\frac{\text{force}}{\text{area}}\right)$$

which is finite and non-zero

It follows that the two directions are necessary to define the stress vector:

i) direction of the normal to the surface area

ii) direction of the stress vector itself

Therefore, the stress components can be represented by $\tau_{\alpha\beta}$ where $\alpha$ denotes the direction of the normal to the area $\delta s$ and $\beta$ is the direction in which the components of the stress considered.

## Components of Stress Vector:

Consider a rectangular parallelopiped $\delta x\, \delta y\, \delta z$ at a point P within the flow having P at its center. Let the mean stresses on the surfaces to be the stresses at the middle point of the ~~selected~~ related surfaces. The co-ordinates of the middle points of the surfaces perpendicular to $x, y, z$ axes are:

$$\left(x \pm \frac{\delta x}{2}, y, z\right), \left(x, y \pm \frac{\delta y}{2}, z\right), \left(x, y, z \pm \frac{\delta z}{2}\right)$$