Ring Theory

Lecture 17

03/03/2022

Example $\phi : \mathbb{Q}[x] \longrightarrow \mathbb{R}$ by

$$\phi(f(x)) = f(\sqrt{2})$$

Then ϕ is a ring homo.

$$\ker \phi = \{ f(x) \in \mathbb{Q}[x] \mid \phi(f(x)) = 0 \}$$

$$= \{ f(x) \in \mathbb{Q}[x] \mid f(\sqrt{2}) = 0 \}.$$

note that f can't be a linear poly.

Note that $(x^2 - 2) \in \ker \phi$

WTS. $\ker \phi = \langle (x^2-2) \rangle$.

WTS every elt in the kernel will be of the form $(x^2-2)\, g(x)$ where $g(x) \in \mathbb{Q}[x]$.

Note that $(x^2-2)g(x) \subseteq \ker \phi$

for all $g(x) \in \mathbb{Q}[x]$.

WTS $\ker\phi \subseteq \underline{\langle (x^2-2) \rangle}$.

$$\langle (x^2-2) \rangle = \{ (x^2-2)g(x) \mid g(x) \in \mathbb{Q}[x] \}$$

Let $h(x) \in \ker\phi$. $\Rightarrow h(\sqrt{2}) = 0$.

By division algorithm we have

$$h(x) = (x^2-2)g(x) + r(x).$$

where either $r(x) = 0$ or

$\deg r(x) < 2$.

Note that $h(x) - (x^2-2)g(x) \in \ker\phi$.

$\Rightarrow r(x) \in \ker\phi$.

$\Rightarrow r(x) = 0$ as $\deg r(x) < 2$.

$\therefore \quad \ker \phi = \{ (x^2-2)\, g(x) \mid g(x) \in \mathbb{Q}[x] \}$

R is commutative ring from now onwards.

Let R be a ring and I is an ideal of R if $\exists$ an elt $a \in R$ s.t

$$I = \{ ra \mid r \in R \} = (a).$$

then I is called a principal ideal.

How to construct ideal in a ring?

The set consisting of zero alone (0) is an ideal called the zero ideal and it is principal ideal.

The whole ring is also an ideal which is called the unit ideal and it is gen by (1).

$\mathbb{Z}$. $\qquad \langle 5 \rangle = \{ 5n \mid n \in \mathbb{Z} \}$.

$\mathbb{R}$ $\qquad \underline{\underline{\langle 5 \rangle}} = \mathbb{R}$.

$$\begin{array}{c} \shortparallel \\ \{ 5x \mid x \in \mathbb{R} \} \end{array}$$

If $x = 1/5$ then $5 \cdot \frac{1}{5} = 1 \in \underline{\underline{\langle 5 \rangle}}$.

$\therefore \langle 5 \rangle = \langle 1 \rangle = \mathbb{R}$.

If R is any ring then the unit ideal is gen by an unit elt.

An ideal I is said to be proper ideal if it is not (0) nor (1).

If R is any non-zero ring and $0 \neq a$ is a non unit elt then $(a) = \{ ra \mid r \in R \}$. is an ideal of R.

We may consider the ideal I gen by a set of elts $a_1, a_2 \ldots a_n$ of R which is defined to be the smallest ideal containg the elts and denoted by

$$(a_1, a_2, \ldots, a_n) = \{ r_1 a_1 + \ldots + r_n a_n \mid r_i \in R \}.$$

Q. Let F be a field. Identify the ideals of F.

The only ideals of F are zero and unit ideal.

Q. If R is a ring that has exactly two ideals then is R is field?

Ans is Yes.

Let R has exactly two ideals.

Then $1 \neq 0$ (because if $1 = 0$ then it is the zero ring and it has only one ideal.)

Then (1) & (0) are two diff ideals.

Let $0 \neq a \in R$ then $(a) = (1)$

$\therefore \exists\, r \in R$ s.t $ra = 1$. $\Rightarrow a$ is an unit in R. Therefore R is a field.

Q. Let F be a field and R be a non-zero ring and $\phi : F \longrightarrow R$. is a ring homo. $\ker \phi$?

Propn. Let F be a field and R be a non-zero ring. Then every ring hom $\phi : F \longrightarrow R$ is injective.

Pf: Since $\ker\phi$ is an ideal and F is a field. so either $\ker\phi = (0)$ or $\ker\phi = (1)$.

$$\left\{ \begin{array}{l} \phi(1) = 1 \\ \phi(1) = 0 \end{array} \right\}$$

If $\ker\phi = (1)$ then R is the zero ring which is a contradiction.

Hence $\ker\phi = (0)$ thus ϕ is inj.

Propn. Let F be a field. Every ideal in $F[x]$ is a principal ideal.

Pf: Let $(0) \subsetneq I \subsetneq F[x]$ be a proper ideal. WTS $I = (f(x))$

Since $I \neq F[x]$, $\exists$ a poly of (+)ve deg in I. Let $f(x) \in I$ having smallest (+)ve deg.

Let $g(x) \in I$. Then by division algorithm, $g(x) = f(x)\, q(x) + r(x)$ where $q(x), r(x) \in F[x]$ and either $\deg r(x) < \deg f(x)$ or $r(x) = 0$.

$r(x) = g(x) - f(x)\, q(x) \in I$.

$\Rightarrow r(x) = 0$ by minimality of $\deg f(x)$

$\therefore g(x) = f(x)\, q(x) \in (f(x))$.

Remark: Note that in $\mathbb{Z}[x]$ not every ideal is gen by a single poly. $I = (2, x) \subseteq \mathbb{Z}[x]$.

Suppose $I = (2, x) = (f)$.

for some $f \in \mathbb{Z}[x]$,