

# Simulation 1

## Monte Carlo Simulation approach

$$\underline{X} = (X_1, \dots, X_n) \stackrel{\text{density}}{\sim} f(x_1, \dots, x_n)$$

compute  $E[g(\underline{X})] = \int \dots \int g(x_1, \dots, x_n) f(x_1, \dots, x_n) dx_1 \dots dx_n$

$$\underline{X}^{(1)} = (\underline{X}_1^{(1)}, \dots, \underline{X}_n^{(1)}) \stackrel{\text{density}}{\sim} f(x_1, \dots, x_n)$$

compute  $\underline{Y}^{(1)} = g(\underline{X}^{(1)})$

$$\underline{X}^{(2)} \stackrel{\text{density}}{\sim} f(x_1, \dots, x_n)$$

$$\underline{Y}^{(2)} = g(\underline{X}^{(2)})$$

$$\underline{Y}^{(i)} = g(\underline{X}^{(i)}) \quad , i = 1, 2, \dots, n$$

IID

$$\lim_{n \rightarrow \infty} \frac{\underline{Y}^{(1)} + \dots + \underline{Y}^{(n)}}{n} = E(Y) = E[g(\underline{X})]$$

—X—

init seed  $X_0$

$$X_{n+1} = (a X_n + c) \text{ modulo } m \quad , n \geq 0$$

PRN

$$U_n = \frac{X_n}{m}$$

Eg 1  $X_i = (5 X_{i-1} + 1) \text{ mod } 8$  seed  $X_0 = 0$

$$X_1 = 1 \text{ mod } 8 = 1 \quad ; U_1 = 1/8$$

$$X_2 = 6 \text{ mod } 8 = 6 \quad ; U_2 = 6/8$$

$$\begin{array}{r} 8 \overline{) 170} \\ \underline{16} \phantom{0} \\ 10 \phantom{0} \end{array}$$

$$\underline{1} = X_1$$

$$\begin{array}{r} 8 \overline{) 17170} \\ \underline{16} \phantom{0} \\ 117 \phantom{0} \end{array}$$

$$X_3 = 31 \bmod 8 = 7 \quad ; \quad U_3 = 7/8$$

$$\frac{24}{7} = X_3$$

$$X_4 = 4 \quad , \quad U_4 = 4/8$$

$$X_5 = 5 \quad U_5 = 5/8$$

$$X_6 = 2 \quad U_6 = 2/8$$

$$X_7 = 3 \quad U_7 = 3/8$$

$$X_8 = 0 \quad U_8 = 0$$

Full period generator

$$(2) \quad X_i = (3X_{i-1} + 1) \bmod 7, \quad X_0 = 3$$

$$X_1 = 3 = X_2 = \dots \quad \text{Not full period}$$

$$(3) \quad \text{generator} \quad X_i = 16807 X_{i-1} \bmod (2^{31} - 1)$$

seed  $X_0 \neq 0$

is full period and used in many real world applications and passes most statistical tests for uniformity and randomness.

Example (1) Monte Carlo integration

$$I = \int_a^b f(x) dx = (b-a) \int_0^1 f(a + (b-a)u) du$$

$$x = a + (b-a)u$$

$$dx = (b-a)du$$

$$\hat{I}_n = \frac{b-a}{n} \sum_{i=1}^n f(a + (b-a)U_i)$$

$$E(\hat{I}_n) = \frac{b-a}{n} \sum_{i=1}^n E(f(a + (b-a)U_i))$$

$U_i \sim U(0,1)$   
 $\text{density } f_{U_i}(u) = \begin{cases} 1 & 0 \leq u < 1 \\ 0 & \text{or} \end{cases}$

$$= \frac{b-a}{n} \sum_{i=1}^n \int_0^1 f(a+(b-a)u) x_i du$$

$$= (b-a) \int_0^1 f(a+(b-a)u) du = I$$

$\hat{I}_n$  UE for  $I$ .

$$I = \int_0^1 (1 + \cos(\pi u)) du (= 1)$$

$n=4$      $U(0,1)$      $u_1: 0.419, u_2: 0.109, u_3: 0.732, u_4: 0.893$

$$\hat{I}_4 = \frac{1-0}{4} \sum_{i=1}^4 [1 + \cos(\pi(1-0)u_i)]$$

$$= 0.896$$

② it flip H if  $U < 0.5$

T if  $U \geq 0.5$

0.12    0.52    0.73    0.05 —  
H    T    T    H —

③ Single-server queue

6 customers arrive at a bank at the following times which have been generated from some appropriate prob. dist 3, 4, 6, 10, 15, 20 ( $A_i$ )

1 server FCFS service times corresponding to the arriving customers are 7, 6, 7, 6, 1, 2 ( $S_i$ )

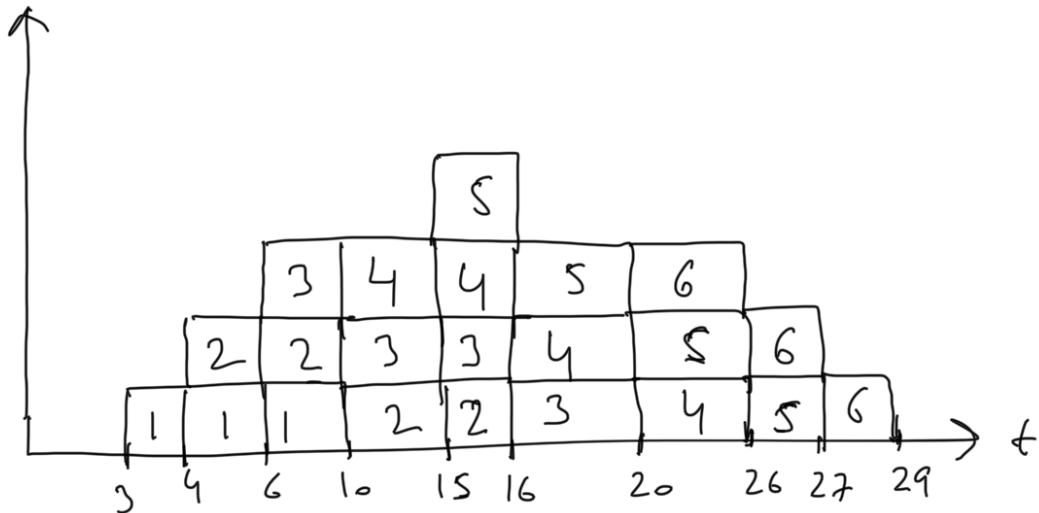
Customer $i$	arrival time $A_i$	begin service ( $B_i$ )	service time ( $S_i$ )	departure time $D_i = B_i + S_i$	wait $W_i = B_i - A_i$
1	3	3	7	10	0

2	4	10	6	16	6
3	6	16	4	20	10
4	10	20	6	26	10
5	15	26	1	27	11
6	20	27	2	29	7

$$\sum_i w_i = 44$$

av. waiting time for six customers =  $\frac{\sum w_i}{6} = \frac{44}{6}$

# of customers in system at time  $t$   
 $L(t)$



av. # of customers in the system =  $\frac{1}{29} \int_0^{29} L(t) dt$

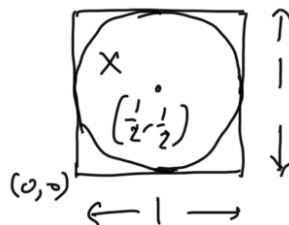
$$= \frac{1}{29} \left( \int_0^3 0 dt + \int_3^4 1 dt + \int_4^6 2 dt + \int_6^{10} 3 dt + \int_{10}^{15} 4 dt + \int_{15}^{16} 5 dt + \int_{16}^{20} 6 dt + \int_{20}^{26} 5 dt + \int_{26}^{27} 4 dt + \int_{27}^{29} 3 dt \right)$$

$$= \frac{70}{29}$$

(L) Estimate  $\pi$

⑦

$$\frac{\text{Circle area}}{\text{Square area}} = \frac{\pi (\frac{1}{2})^2}{1} = \frac{\pi}{4}$$



→ prob. that a dart will land in the circle

$$(U_{i1}, U_{i2}), (U_{21}, U_{22}), \dots$$

$$(U_{i1} - \frac{1}{2})^2 + (U_{i2} - \frac{1}{2})^2 \leq \frac{1}{4} \quad (U_{i1}, U_{i2}) \text{ will fall in the circle of } \textcircled{1}$$

$$X_i = \begin{cases} 1 & \text{if pair } (U_{i1}, U_{i2}) \text{ satisfy } \textcircled{1} \\ 0 & \text{o.w.} \end{cases}$$

$$X = \sum_{i=1}^n X_i \sim \text{Bin}(n, p) \quad , p = \frac{\pi}{4}$$

$$\hat{p} = \frac{X}{n} \text{ is mle for } p = \frac{\pi}{4}$$

$$\text{mle of } \pi \text{ is } 4\hat{p}$$

eg  $n = 1000$  ,  $X = 753$  darts are landing in the circle

$$\hat{\pi} = 4 \times \frac{753}{1000} = 3.12$$

—X—  
Probability Integral Transform

cont.  $X \stackrel{\text{cdf}}{\sim} F(x)$  , then  $U = F(X) \sim U(0,1)$

Sol, 
$$\underline{F(x) = P(X \leq x)}$$

cdf of  $U$  is  $0 \leq u \leq 1$

$$G(u) = P(U \leq u) = P(F(X) \leq u)$$

$$= P(X \leq F^{-1}(u))$$

$$= F(F^{-1}(u))$$

$$= u$$

cdf of  $U$

$$U \sim U(0,1) \quad g(u) = \begin{cases} 1 & \text{if } 0 \leq u \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

Inverse transform method

$$X \stackrel{\text{cdf}}{\sim} F(\cdot)$$

$$U = F(X) \sim U(0,1)$$

$$\Rightarrow X = F^{-1}(U)$$

① Simulate expo. r.v. with mean 1.

$$\text{cdf } X \sim F(x) = \begin{cases} 1 - e^{-x}, & x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

$$u = F(x) = 1 - e^{-x} \Rightarrow x = -\ln(1-u)$$

$$X = -\ln(1-U), \quad U \sim U(0,1)$$

$$\sim \exp(1)$$

$$u_1 = 0.1, \quad x = -\ln(1-0.1) \quad \underline{X = -\ln U \sim \exp(1)}$$

$$1-U \stackrel{d}{=} U$$

$$P(1-U \leq x) = P(U \geq 1-x) = 1 - P(U \leq 1-x)$$

$$= 1 - (1-x) = x = P(U \leq x)$$

②  $X_1, \dots, X_n \stackrel{i.i.d}{\sim} \exp(\lambda)$

$$Y = \sum_{i=1}^n X_i \sim \text{Gamma}(n, \lambda)$$

$$Y = -\sum_{i=1}^n \frac{1}{\lambda} \ln U_i = -\frac{1}{\lambda} \ln \left( \prod_{i=1}^n U_i \right) \sim \text{Gamma}(n, \lambda)$$

③ generate a Standard normal r.v.  $Z$

$$\Phi(Z) = U \quad Z \stackrel{\text{cdf}}{\sim} \Phi$$

$$\Rightarrow Z = \Phi^{-1}(U)$$

$$u = 0.72$$

$$P(Z \leq z) = 0.72$$

$$\Phi^{-1}(u) = z = 0.59$$

④  $Z \sim N(0, 1) \quad ; \quad X = \mu + \sigma Z \sim N(\mu, \sigma^2)$

$$\mu = 1, \sigma^2 = 4 \quad \rightarrow \quad X = 1 + 2 \times 0.59 \sim N(1, 4)$$

⑤  $U_1, \dots, U_n \sim \text{i.i.d. PRN's}$

largen CLT

$$\frac{\sum_{i=1}^n U_i - \sum_{i=1}^n E(U_i)}{\sqrt{\sum_{i=1}^n \text{Var}(U_i)}} = \frac{\sum_{i=1}^n U_i - \frac{n}{2}}{\sqrt{\frac{n}{12}}} \approx N(0, 1)$$

$$n=12 \quad \sum_{i=1}^{12} U_i - 6 \approx N(0, 1)$$

Simulate from discrete dists:

simulate  $X$  pmf  $P(X = x_j) = P_j \quad j=1, 2, \dots, \sum_j P_j = 1$

$$U \sim U(0, 1)$$

$$\text{Set } X = \begin{cases} x_1, & U < P_1 \\ x_2, & P_1 < U < P_1 + P_2 \\ \vdots & \\ x_j, & \sum_{i=1}^{j-1} P_i < U < \sum_{i=1}^j P_i \\ \vdots & \end{cases}$$

$$P(X = x_j) = P\left(\sum_{i=1}^{j-1} P_i < U < \sum_{i=1}^j P_i\right) = P_j$$

$\therefore X$  is the desired dist.

eg  $X$  pmf  $p(x) = \begin{cases} 0.3, & x = -1 \\ 0.6, & x = 2.3 \\ 0.1, & x = 7 \\ 0, & \text{o.w.} \end{cases}$

$x$	$p(x)$	$U$	
-1	0.3	$[0, 0.3)$	If $U \leq 0.4$ , then
2.3	0.6	$[0.3, 0.9)$	$x = 2.3$
7	0.1	$[0.9, 1]$	

determining the number of runs

generate  $n$  i.i.d. obs  $y^{(1)}, \dots, y^{(n)}$  having  
mean  $\mu$ , var  $\sigma^2$ .

$$\bar{y}_n = \frac{y^{(1)} + \dots + y^{(n)}}{n} \quad \text{as an estimate to } \mu$$



$\sigma^2 \rightarrow \text{unknown}$   
 $\text{Var}(\bar{Y}_n) = \frac{\sigma^2}{n}$  Choose  $n$  s.t.  $\frac{\sigma^2}{n}$  is small (acceptable)  
 Initially simulate  $k$  runs  $y^{(1)}, \dots, y^{(k)}$

$$\hat{\sigma}^2 = \frac{1}{k-1} \sum_{i=1}^k (y^{(i)} - \bar{y}_k)^2$$

Choose  $n$  s.t.  $\frac{\hat{\sigma}^2}{n}$  is small (acceptable)

remaining  $n-k$  simulate runs

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