$L_q = 0, L_s = L,$

$W_q = 0, W_s = W = \frac{1}{\mu}$. ; $W(t) = W_s(t) = 1 - e$

→ Simulation - Monte Carlo simulation: estimating theoretical mean by using multiple observations and averaging them

$$I = \int_0^b f(x)\,dx = (b-a)\int_0^1 f(a + (b-a)u)\,du$$ replace x by a+(b-a)u

$$\boxed{\hat{I}_n = \frac{b-a}{n}\sum_{i=1}^{n} f(a + (b-a)u_i)}$$ Monte Carlo Integration (ui are observations)

E(I cap)=I

X0 is the seed, n>=0

Pseudo Random generator - $x_{n+1} = (ax_n + c)\,\%\,m$

$$\boxed{U_i = \frac{x_i}{m}}$$

full period generator - generates all numbers in range before repeating.

(0 to m-1)

Prob - Integral transfer (PIT)

If $X \overset{iid}{\sim} F$, then $U = F(x) \sim U(0,1)$

~~If F is th cdf of X, then~~

$U = F(x) \Rightarrow \boxed{x = F^{-1}(U)}$

$\hookrightarrow$ CDF

cdf of exp(lambda)=
1-exp(-lambda * x)

- $\ln U \sim$ exponentl mean 1 exp(lambda)=-(1/lambda)ln(U)

- if $Z \sim N(0,1)$

$$\boxed{\Phi(Z) = U \Rightarrow Z = \Phi^{-1}(U)}$$

- if $X \sim N(\mu, \sigma^2)$

$$\boxed{X = \mu + \sigma Z}$$

- if $X \sim \exp(\lambda)$

$$\boxed{X = -\frac{1}{\lambda} \ln(U)}$$

$Y = \text{Gamma}(n, \lambda)$

$\therefore Y = \sum_{i=1}^{n} X_i$

$$Y = -\frac{1}{\lambda} \ln\left(\prod_{i=1}^{n} U_i\right)$$

$\Rightarrow$ $Y^1 \ldots X^{(n)}$ n = no of runs. , $X^i \sim \mu \sigma^2$

$$\boxed{Var(\bar{Y}_n) = \frac{\sigma^2}{n}}$$

Discrete dist^n -

$$X = \begin{cases} x_1 & U < p_1 \\ x_2 & p_1 < U < p_1 + p_2 \\ \vdots \\ \boxed{x_j , \sum_{i=1}^{j-1} p_{\theta i} < U < \sum_{i=1}^{j} p_i} \\ \vdots \end{cases}$$

Accuracy and no of runs -

If σ^2 is not known,

$Y^{(1)} \dots Y^{(k)}$ to estimate σ^2 by sample variance.

$$\sigma^2 = \frac{1}{k-1} \sum_{i=1}^{k} \left(Y^{(i)} - \bar{Y}_k\right)^2$$

Martingales

- A SP $(X_n, n = 0, 1, 2, \dots)$ is martingale
  a) $E(X_n) < \infty$
  b) $E(X_m \mid X_0, X_1, \dots X_n) = X_n$ for $m \geq n$.

$\boxed{E(X_{n+1}) = E(X_n)}$ $\quad E(X_{n+1} \mid X_0, X_1, \dots X_n) = X_n$

i.e. martingale has const mean.

→ Brownian Motion -

$X(t) \equiv$ pos of particle at time t.

$$\boxed{X(t) = \Delta x \left(X_1 + X_2 + \dots X_{\left[\frac{t}{\Delta t}\right]} \right)}$$

If $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$

$E(X_i) = 0$, $E(X_i^2) = 1$, $V(X_i) = 1$

$$\boxed{E(X(t)) = 0}$$

$$\boxed{V(X(t)) = \Delta x^2 \left[\frac{t}{\Delta t} \right]}$$

If $\Delta x = \sigma\sqrt{\Delta t}$, $\sigma > 0$

$E(X(t)) = 0$, $V(X(t)) = \sigma^2 t$

SP is a BM if:

i) $X(t) \sim N(0, \sigma^2 t)$

ii) $X(t)$ has ind. increment

iii) " " stationary increment

iv) $X(0) = 0$

If $\sigma = 1$, BM is called standard BM / Wiener process

$X(t) \equiv W(t) \equiv W_t$ $\quad$ $W_t \sim N(0, t)$

W(t)=B(t)/sigma

$X(t) \sim N(0, \sigma^2 t)$

$W(t) = \frac{X(t)}{\sigma} \sim N(0, t)$ $\quad f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$

$\cancel{W(t_k) - W(t_{k-1}) \sim N(0, t_k - t_{k-1})}$

$$\boxed{[W(s) \mid W(t) = B] \sim N\left(\frac{s}{t} B, \frac{s(t-s)}{t}\right)}$$

, $s < t$

→ GBM

drift coeff μ

$$Y(t) \sim N(\mu t, \sigma^2 t)$$

$$X(t) = e^{Y(t)} \text{ ; } X(t) \text{ is called GBM}$$

$$E(X(t) \mid X(s)) = X(s)\, e^{(t-s)\left(\mu + \frac{\sigma^2}{2}\right)} \quad (s < t)$$

$$E(X(t)) = E(X(s))\, e^{(t-s)\left(\mu + \frac{\sigma^2}{2}\right)}$$

Renewal theory -

- if interarrival time are IID $\to$ renewal process
- X_n: interarrival time b/w $(n-1)^{th}$ and $(n)^{th}$ event (renewal)
- $N(t) = \#$ of events at time $t = \max(n : S_n \leq t)$
  S_n = time for n^{th} event/renewal.
  $= \sum_{i=1}^{n} X_i$
- $0 < E(X_n) = \mu < \infty$
- $X_n \overset{CDF}{\sim} F()$, $F(0) = P(X_n = 0) < 1$
- Only finite no. of renewals can occur in finite time ($\frac{S_n}{n} \to \mu$, $S_n \to \infty$ if $n \to \infty$. $\therefore S_n < \infty$ if n finite).
- $m(t) = E(N(t)) \to$ renewal function, mean value fn
- $\boxed{m(t) = \sum_{n=1}^{\infty} F_n(t)}$, $m(t) < \infty$ for $t < \infty$ P(Sn<=t)=Fn(t)
- $\boxed{m(t) = F(t) + \int_0^t m(t-x) f(x) dx}$ $\to$ Fundamental renewal eqn
  $m(t) = E(N(t)) = E(E(N(t) | X_1 = x))$
- $m(0) = E(N_0) = 0$
- $P(N(\infty) = \infty) = 1$, sandwich theorem $\frac{1}{S_n}$ $\frac{1}{t}$ $\frac{1}{S_{n+1}}$
- $\boxed{\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu}}$ = rate of renewal process, $\mu = E(X_n)$
- $\boxed{E(S_{N(t)+1}) = \mu(m(t)+1)}$
- $Y(t)$ - excess or residual life at t.
  $S_{N(t)+1} = t + Y(t) \Rightarrow \lim_{t \to \infty} \frac{Y(t)}{t} \to 0$ (divide by t)
- Renewal Reward Process - $R(t) = \sum_{n=1}^{N(t)} R_n(t)$

$$\boxed{\lim_{t \to \infty} \frac{R(t)}{t} = \frac{E(R(t))}{E(X(t))}} \quad \text{and} \quad \lim_{t \to \infty} \frac{E(R(t))}{t} = \frac{E(R(t))}{E(X(t))}$$