Keerti P. Charantimath

19MA20059

DM - Test 3

Page No

Keerti P. Charantimath

19MA20059

DM - Test 3

Q4) Given:- a is an integer, n is any non-negative integer

To prove:- a & a^{4n+1} have the same last digit

Proof:-

To show that a & a^{4n+1} have same last digit i.e same digit in one's place, we need to show that

$$a^{4n+1} \equiv a \pmod{10}$$

which is same as showing

$$a^{4n+1} \equiv a \pmod 2 \text{ and}$$

$$a^{4n+1} \equiv a \pmod 5$$ according to Chinese remainder theorem.

Part 1:- $a^{4n+1} \equiv a \pmod 2$

$a \equiv 0 \pmod 2$ and $a \equiv 1 \pmod 2$ are both true

There is no other possibility hence

$a^{4n+1} \equiv a \pmod 2$ is proved to be true

Part 2:- $a^{4n+1} \equiv a \pmod 5$

We know by Fermat's little theorem that

$a^4 \equiv 1 \pmod 5$ when $\gcd(a,5) = 1$.

Then

$$a^{4n+1} = (a^4)^n \cdot a \equiv a \pmod 5$$

But we are now left with case $\gcd(a,5)>1$
That is true,
if and only if $a \equiv 0 \pmod 5$.
In this case, we have $a^{(4n+1)} \equiv a \pmod 5$
as both are congruent to 0.
So, $a^{4n+1} \equiv a \pmod 5$ for all integers a.

Hence, by the Chinese remainder theorem,
we show that $a^{4n+1} \equiv a \pmod{10}$ for all integers a.

Hence proved that a and a^{4n+1} have same digit in the one's position

Q3) $17x \equiv 1 \bmod (101)$

$\gcd(17, 101) = 1 = 17y + 101x$

$u = [a, 1, 0] = [101, 1, 0]$
$V = [b, 0, 1] = [17, 0, 1]$

Q	u_1	u_2	u_3	V_1	V_2	V_3
-	101	1	0	17	0	1
5	17	0	1	16	1	-5
1	16	1	-5	1	-1	6
16	1	-1	6	0		

gcd = 1, $x = -1$, $y = 6$

19MA20059

Now,

$$1 = 17(6) + 101(-1)$$

$$17(6) \equiv 1 \mod (101)$$

$$\boxed{6 = 17^{-1} \mod (101)}$$

Ans = 6

Q2) Given:- n

To prove :- $n!+1$ & $(n+1)!+1$ are relatively prime

Proof :-

$$(n+1)!+1 = (n+1)n!+1 = n.n! + (n!+1)$$

If p is a prime factor of $n!+1$, then we know that it is not a prime factor of any integer $q \leq n$.

Due to this fact, p is not a factor of $n.n!$ and so p is not a factor of $n.n!+(n!+1)$

So, $n!+1$ & $(n+1)!+1$ have no common prime factors.

Thus $n!+1$ & $(n+1)!+1$ are relatively prime.
Hence, proved.

19MA20059

Q5) Given:- n is any natural number

To prove:- $(n)(n+1)(n+2)(n+3) = (n^2+3n+1)^2 - 1$

Proof:-

for $n = 1$

$LHS = 1.(1+1).(1+2).(1+3) = 1\cdot 2\cdot 3.4 = 24$

$RHS = (1+3\times1+1)^2 - 1 = 5^2 - 1 = 24$

$LHS = RHS$

$\therefore$ the given statement is true for $n=1$

Let us assume that the statement is true for some $n = k \in \mathbb{N}$

$$k(k+1)(k+2)(k+3) = (k^2+3k+1)^2 - 1 \longrightarrow (1)$$

We have to prove that ~~our assumpt~~ the given statement is true for $n = k+1$

$$LHS = (k+1)(k+1+1)(k+1+2)(k+1+3)$$
$$= (k+1)(k+2)(k+3)(k+4)$$
$$= \frac{[(k^2+3k+1)^2 - 1]}{k}(k+4) \quad (\text{from } (1)\ (k)(k+1)(k+2)(k+3) = (k^2+3k+1)^2 - 1)$$
$$= (k^4+9k^2+2(3k^3+k^2+3k))\frac{(k+4)}{k}$$
$$= (k^3+9k+6k^2+2k+6)(k+4)$$
$$= (k+1)(k^3+6k^2+11k+6)$$

$$RHS = ((k+1)(k+4)+1)^2 - 1$$

19MA20059

$$=\left(\left((k+1)(k+4)\right)^2+1+2(k+1)(k+4)-1\right.$$
$$=(k+4)\left((k+1)^2(k+4)+2(k+1)\right)$$
$$=(k+4)\left((k^2+1+2k)(k+4)+2k+2\right)$$
$$=(k+4)\left(k^3+4k^2+k+4+2k^2+8k+2k+2\right)$$
$$=(k+4)(k^3+6k^2+11k+6)$$

LHS = RHS

$\therefore$ Our given statement is true for $n=k+1$ our assumption is right.

Hence, by the method of induction, it is true that $(n)(n+1)(n+2)(n+3)=(n^2+3n+1)^2-1$ $\forall\ n \in \mathbb{N}$

Q1) Inductive case:- we use a^k & a^{k-1} equal to 1

Basis case:- we show that the basis case holds only for a^0.

To prove this result, we have to show that the base case is TRUE for both a^0 & a^1 and then ~~shows that~~ use inductive step to prove for a^k and a^{k-1}

19MA20059

Q6)

a) Finding primitive roots mod 98

$\phi(98) = 2(7^2) = 98(1-1/2)(1-1/7) = 42$

Since $(a,98)=1$ by EGCF $a^{\phi(n)} = a^{42} = 1$

$\phi(\phi(96)) = \phi(42) = \phi(2\times3\times4)$
$= 42(1-1/2)(1-1/3)(1-1/7)$
$= 42(2/7) = 12$

There are 12 primitive roots

b) $a^{42} \longrightarrow$ primitive root

c) Using a^{42}, we can get a^{84} as the second primitive root as
$42\times2 = 84 < 98$

$\therefore$ second primitive root $\longrightarrow a^{84}$