Date 11/1/23

Class-1

Integral Equations:

An integral equation is an equation in which an unknown function appears under one or more integral signs.

eg: for $a \le s \le b$, $a \le t \le b$ the equations

$$f(s) = \int_a^b k(s,t)\, g(t)\, dt \quad \text{--- (1)}$$

$$g(s) = f(s) + \int_a^b k(s,t)\, g(t)\, dt \quad \text{--- (2)}$$

$$g(s) = \int_a^b K(s,t)\, [g(t)]^2\, dt \quad \text{--- (3)}$$

where the function $g(s)$ is the unknown function while all other functions are known functions are an integral equations.

Linear Integral eq^n: An integral equation can be written as

$$L[g(s)] = f(s)$$

where L is an appropriate integral operation. Then for any c_1 and c_2 if we have

$$L[c_1 g_1(s) + c_2 g_2(s)] = c_1 L[g_1(s)] + c_2 L[g_2(s)]$$

then the operation is a linear operator and hence corresponding integral equation is called Linear integral equation.

Next, we consider the general form of integral equation

$$h(s)\, g(s) = f(s) + \lambda \int_a^b K(s,t)\, g(t)\, dt \quad \text{--- (4)}$$

where the upper limit may be either variable or fixed.

- f, h and $K(s,t)$ are known functions
- g unknown function (is to be determined)

- λ is a non-zero, real or complex, parameter

- The function $k(s,t)$ is called the kernel.

Next we define some special cases of (4)

1) Fredholm integral equations: In all fredholm integral eqn, the upper limit of the integration, b, say, is fixed.

a) Fredholm integral equation of first kind:

In the fredholm integral eqn of first kind $h(s) = 0$ in (4) i.e.,

$$f(s) + \lambda \int_a^b k(s,t)\, g(t) = 0$$

b) Fredholm Integ^n of 2^{nd} kind: Here $h(s) = 1$

eg: $$g(s) = f(s) + \lambda \int_a^b k(s,t)\, g(t)\, dt$$

(c) Homogenous Fredholm Integral eq^n

special case of (b), in this case $f(s) = 0$

$$g(s) = \lambda \int_a^b k(s,t)\, g(t)\, dt$$

2) Volterra Integral Eq^n: Volterra I. E. of 1^{st}, 2^{nd} kind and homogeneous eq^n are defined same as above except that $b = s$, is the variable upper limit of integration.

3) Singular Integral Equations: when one or both limits of the integration becomes infinite or when the kernel becomes infinite at one or more points within the range of integration, the integral is called singular.

eg:

$$g(s) = f(s) + \lambda \int_{-\infty}^{\infty} \exp(-|s-t|)\, g(t)\, dt$$

and

$$f(s) = \int_0^s \frac{1}{(s-t)^\alpha}\, g(t)\, dt \quad , \quad 0 < \alpha < 1$$

are singular integral equations.

Date 12/1/23 Class-2

Regularity conditions: we assume that the functions are either continuous, or integrable or square integrable.

square integrable: A function $g(t)$ is square integrable if

$$\int_a^b |g(t)|^2\, dt < \infty$$

This is called L_2-function (or L^2-function).

Square integrable kernel: A kernel $K(s,t)$ as a function of two variables is said to be square integrable (or L^2-function) if

(a) for each set of values s,t in the square $a \le s \le b$, $a \le t \le b$,

$$\int_a^b \int_a^b |K(s,t)|^2\, ds\, dt < \infty$$

b) for each value of s in $a \le s \le b$

$$\int_a^b |K(s,t)|^2 \, dt < \infty$$

(c) for each value of t in $a \le t \le b$

$$\int_a^b |K(s,t)|^2 \, ds < \infty$$

Special Kinds of kernels

1) Separable or Degenerate kernel

A kernel $K(s,t)$ is called separable or degenerate kernel if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of s only and a function of t only i.e.

$$K(s,t) = \sum_{i=1}^{n} a_i(s)\, b_i(t) \quad \text{--- (5)}$$

The functions $a_i(s)$ can be assumed to be linearly independent, otherwise the number of terms in relation (5) can be reduced. V. Imp

Note: Linear Independence: $\{a_i(s)\}^n$ or $\{a_1(s), \ldots, a_n(s)\}$ is said to be linearly independent if

$$\sum_{i=1}^{n} \alpha_i a_i(s) = 0 \implies \alpha_i = 0 \;\forall\; i = 1, 2, \ldots n$$

then $\{a_i(s)\}_{i=1}^{n}$ is said to be linearly independent.

2) Symmetric kernel : A complex valued functions $k(s,t)$ is called symmetric (or Hermitian) if $k(s,t) = k^*(t,s)$ where $k^*(t,s)$ is complex conjugate of $k(t,s)$. For real kernel

$$k(s,t) = k(t,s).$$

For matrix A, $Ax = \lambda x$, when this has a non-trivial solution, then we say λ is eigen value & corresponding vector is eigen vector.

Eigen Values and Eigenfunctions : If the linear homogeneous integral eq^n

$$\lambda \int_a^b k(s,t)\, g(t)\, dt = g(s) \quad \text{--- (6)}$$

has a non-trivial solution $g(s)$, then the parameter λ is called eigen value and the non-trivial solution $g(s)$ is called corresponding eigen function.

Note: using operators we have

$$L\,g(s) = \int_a^b k(s,t)\, g(t)\, dt$$

$$L = \int_a^b k(s,t)\, (\;)\, dt$$

So, In operator form

$$Lg(s) = \mu g(s), \text{ where } L \text{ is integral operator}$$

μ = Constant $= \frac{1}{\lambda}$

If (*) has non-trivial soln, then we say that μ is eigen value and corresponding non-trivial soln is eigen function.

The homogeneous eqn (6) can be written as

$$\int_a^b K(s,t)\, g(t)\, dt = \mu\, g(s) \quad , \quad \mu = \frac{1}{\lambda}$$

If we follow operator defination, then μ $\left(=\frac{1}{\lambda}\right)$ is eigen value but linear integral eqn is studied in the form (6) so instead of $\frac{1}{\lambda}$ we take λ as eigen value.

Convolution Integral :

If $K(s,t) = K(s-t)$, where K is a function of one variable, then the kernel is called convolution kernel and corresponding integral eqn

Substitution $s-t = t$

$\int_a^b K(s-t)\, g(t)\, dt$

$= \int_a^b K(t)\, g(s-t)\, dt$

$$g(s) = f(s) + \lambda \int_a^b K(s-t)\, g(t)\, dt$$

or

$$g(s) = f(s) + \lambda \int_a^s K(s-t)\, g(t)\, dt$$

are called integral eqn of convolutions type.

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~~Euclid~~ L_2 norm $= \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$

$$\|\phi\| = \left(\int_a^b |\phi|^2 dt \right)^{1/2}$$

Inner product of two functions

Def: The inner product of 2 complex L^2-function ψ & ϕ is defined by

$$\int_a^b \phi(t)\, \psi^*(t)\, dt \doteq \langle \phi, \psi \rangle$$

and is denoted by $\langle \phi, \psi \rangle$. ~~Here~~. $\phi(s)$ and $\psi(s)$ are functions of real variable s, $a \le s \le b$.

Def: orthogonal functions : The two L^2 functions ϕ and ψ are called orthogonal if $\langle \phi, \psi \rangle = 0$.

Def: Norm of an L^2-function (denoted by $\|\phi\|$)

$$\|\phi\| = \left[\int_a^b \phi(t)\, \phi^*(t)\, dt \right]^{1/2} = \left[\int_a^b |\phi(t)|^2 dt \right]^{1/2}$$

Note: This is called L^2-norm

Schwarz Inequality :

$$|\underbrace{\langle \phi, \psi \rangle}_{\text{inner product}}| \le \|\phi\| \|\psi\|$$

Minkowski Inequality:

$$\|\phi + \psi\| \le \|\phi\| + \|\psi\|$$

eg ① Seperable Kernels:

$$g(s) = f(s) + \lambda \int_a^b k(s,t)\, g(t)\, dt \qquad — ①$$

and

$$k(s,t) = \sum_{i=1}^{n} a_i(s)\, b_i(t) \qquad — ②$$

$$g(s) = f(s) + \lambda \left(\sum_{i=1}^{n} a_i(s) \int_a^b b_i(t)\, g(t)\, dt \right) \qquad — ③$$

for finite ~~categ~~ summation, order of integral & summation can be interchanged but not for infinite terms summation.

let

$$c_i = \int_a^b b_i(t)\, g(t)\, dt \qquad — ④$$

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} a_i(s)\, c_i$$

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} a_i(s)\, c_i \qquad — ⑤$$

we assumed that $a_i(s)$ are linearly independent.

using ⑤ in ③

$$g(s) = f(s) + \lambda \left[\int_a^b k(s,t) \left(f(s) + \lambda \sum_{i=1}^{n} a_i(s)\, c_i \right) dt \right]$$

$$\Rightarrow g(s) = f(s) + \lambda \int_a^b k(s,t)\, f(s)\, dt + \lambda^2 \int_a^b k(s,t) \sum a_i(s)\, c_i\, dt$$

$$= f(s) + \lambda \int_a^b k(s,t)\, f(s)\, dt + \sum a_i(s) \left\{ \lambda^2 \int_a^b k(s,t)\, c_i\, dt \right\}$$

$$g(s) = f(s) + \lambda \left[\sum_{i=1}^{n} a_i(s) \int_a^b b_i(t) \left(f(t) + \lambda \sum_{k=1}^{n} a_k(t)\, c_k \right) dt \right]$$

using ⑤ in above

$$g(s) - f(s) = \lambda \sum_{i=1}^{n} a_i(s)\, c_i = \lambda \left[\sum_{i=1}^{n} a_i(s) \int_a^b b_i(t) \underbrace{\left(f(t) + \lambda \sum_{k=1}^{n} a_k(t)\, c_k \right)}_{g(t)} dt \right]$$

$$\sum_{i=1}^{n} a_i(s) \left\{ c_i - \int_a^b b_i(t) \left[f(t) + \lambda \sum_{k=1}^{n} a_k(t)\, c_k \right] dt \right\} = 0$$

Since a_i is are L.I. so,
so, $\forall\, i = 1, 2, \ldots n$

$$c_i - \int_a^b b_i(t)\left[f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t)\right] dt = 0$$

let

$$f_i = \int_a^b b_i(t)\, f(t)\, dt$$

$$a_{ik} = \int_a^b b_i(t)\, a_k(t)\, dt$$

both known quantities.

So,

$$c_i - \lambda \sum_{k=1}^{n} a_{ik} c_k = f_i \quad \forall\, i = 1, \ldots n$$

or $i = 1(1)n$
initial value
Increment
last value

$$\Rightarrow \quad c_1 - \lambda \sum_{k=1}^{n} a_{1k} c_k = f_1$$

$$c_2 - \lambda \sum_{k=1}^{n} a_{2k} c_k = f_2$$

$$c_1 - \lambda\,(a_{11} c_1 + \ldots)$$

so, combine first 2 terms

$$(1 - \lambda a_{11})\, c_1 + \ldots$$

now for $\underline{i=2}$

$$c_2 - \lambda\left(a_{21} c_1 + a_{22} c_2 + \ldots\right)$$

$$\Rightarrow (1 - \lambda a_{22})\, c_2 - \lambda\left(a_{21} c_1 + a_{23} c_3 + \ldots\right)$$

similarly combine for all i

$$(1-\lambda a_{11})c_1 - \lambda a_{12}c_2 - \lambda a_{13}c_3 + \cdots - \lambda a_{1n}c_n = f_1$$

$$-\lambda a_{21}c_1 + (1-\lambda a_{22})c_2 - \lambda a_{23}c_3 + \cdots - \lambda a_{2n}c_n = f_2$$

$$\vdots$$

$$-\lambda a_{n1}c_1 - \lambda a_{n2}c_2 + \cdots - \lambda a_{n\,n-1}c_{n-1} + (1-\lambda a_{nn})c_n = f_n$$

system of Linear equations with unknowns c_i.

$$D(\lambda)\, C = F, \quad F = [f_1, f_2, f_3 \ldots f_n]^T$$

$$\begin{bmatrix} 1-\lambda a_{11} & -\lambda a_{12} & -\lambda a_{13} & \cdots & -\lambda a_{1n} \\ -\lambda a_{21} & (1-\lambda a_{22}) & -\lambda a_{23} & \cdots & -\lambda a_{2n} \\ \vdots & & & & \\ -\lambda a_{i1} & -\lambda a_{i2} & \cdots & (1-\lambda a_{ii}) + \cdots & -\lambda a_{in} \\ \vdots & & & & \\ -\lambda a_{n1} & -\lambda a_{n2} & \cdots & \cdots & (1-\lambda a_{nn}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix}$$

$$D(\lambda)\, C = F \quad \text{--- (10)}$$

$$C = [c_1, c_2 \ldots c_n]^T$$

$$F = [f_1, f_2 \ldots f_n]^T$$

$D(\lambda) = (a_{ii})\ d_{ij}$ Find c_i & use (5) to get $g(s) = f(s) + \lambda \sum_{i=1}^{n} a_i(s)\, c_i$

we want unique solution.

If ~~the~~ it is Homogenous, then $F = 0$

$$D(\lambda)C = 0$$

$$c_i - \lambda \sum a_{ik} c_k = 0 \qquad i = 1(1)n$$

$\boxed{\text{If } \lambda = 0}$ $D(\lambda) = I$ (identity matrix) ~~$\det|D(\lambda)| \neq 0$ in general.~~

let $\lambda = \frac{1}{\mu}$ $(\lambda \neq 0)$

$$(\mu I - A)C = 0$$
$\hookrightarrow a_{ij}$ matrix

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Note: The homogeneous system corresponding to (9) is written as

$$c_i - \lambda \sum_{k=1}^{n} a_{ik} c_k = 0 \quad , \; i = 1(1)n \quad — (12)$$

Taking $\lambda = \frac{1}{\mu}$ $(\lambda \neq 0)$, we may write (12) as

$$\mu c_i - \sum_{k=1}^{n} a_{ik} c_k = 0 \quad , \; i = 1(1)n \quad — (13)$$

Take $A = (a_{ij})$, then (13) can be written as

$$(\mu I - A)C = 0 \quad — (14)$$

which is an eigen value problem (corresponding to $AC = 0$) of matrix theory. The eigenvalues are given by $|\mu I - A| = 0$ or $D(\lambda) = 0$. They are eigenvalues of corresponding integral equations.

Examples

1) Solve the Fredholm IE of 2nd kind

$$g(s) = s + \lambda \int_0^1 (st^2 + s^2 t)\, g(t)\, dt \quad — (1)$$

Sol Here, $f(s) = s$, $K(s,t) = st^2 + s^2 t = \sum_{i=1}^{2} a_i(s) b_i(t)$.

$a_1(s) = s$, $a_2(s) = s^2$

$b_1(t) = t^2$, $b_2(t) = t$

Then (1) can be written as

$$g(s) = s + \lambda\left[s\int_0^1 t^2 g(t)\,dt + s^2\int_0^1 t\, g(t)\,dt\right] \quad -(A)$$

Take $c_1 = \int_0^1 t^2 g(t)\,dt$ & $c_2 = \int_0^1 t\,g(t)\,dt$, then

$$g(s) = s + \lambda\left(sc_1 + s^2 c_2\right) \quad - ②$$

putting it back to IE (A),

$$g(s) = s + \lambda\int_0^1 (st^2 + s^2 t)\left[t + \lambda\left(tc_1 + t^2 c_2\right)\right]dt$$

$$\cancel{s} + \lambda\left[c_1 s + s^2 c_2\right] = \cancel{s} + \lambda\left[s\int_0^1 t^2\left(t + \lambda(tc_1 + t^2 c_2\right)dt + s^2\int_0^1 t\left(t + \lambda(tc_1 + t^2 c_2)\right)dt\right]$$

Now collect the coefficients of s & s^2

$$\left(\lambda c_1 - \lambda\int_0^1 t^2\left(t + \lambda(tc_1 + t^2 c_2\right)dt\right)s + \left(\lambda c_2 - \lambda\int_0^1 t\left(t + \lambda(tc_1 + t^2 c_2\right)dt\right)s^2 = 0$$

comparing LHS & RHS, coefficients need to be zero

$$\lambda c_1 = \lambda\int_0^1 t^2\left(t + \lambda(tc_1 + t^2 c_2)\right)dt \quad - ③$$

&

$$\lambda c_2 = \lambda\int_0^1 t\left(t + \lambda(tc_1 + t^2 c_2)\right)dt \quad - ④$$

$$\Rightarrow \quad c_1 = \int_0^1 t^2\left(t + \lambda(tc_1 + t^2 c_2)\right)dt$$

$$\& \quad c_2 = \int_0^1 t\left(t + \lambda(tc_1 + t^2 c_2)\right)dt$$

we get

$$\left(1-\frac{\lambda}{4}\right)c_1 - \frac{\lambda}{5}c_2 = \frac{1}{4}$$

& $-\frac{\lambda}{3}c_1 + \left(1-\frac{\lambda}{4}\right)c_2 = \frac{1}{3}$

Solve these 2 to get c_1 & c_2

$$\left(1-\frac{\lambda}{4}\right)\left(-\frac{\lambda}{3}\right)c_1 + \frac{\lambda^2}{15}c_2 = \frac{-\lambda}{12}$$

and $\left(-\frac{\lambda}{3}\right)\left(1-\frac{\lambda}{4}\right)c_1 + \left(1-\frac{\lambda}{4}\right)^2 c_2 = \left(1-\frac{\lambda}{4}\right)\frac{1}{3}$

$$\Rightarrow \frac{\lambda^2}{15}c_2 - \left(1-\frac{\lambda}{4}\right)^2 c_2 = -\frac{\lambda}{12} - \left(1-\frac{\lambda}{4}\right)\frac{1}{3}$$

$$\Rightarrow \left[\frac{\lambda^2}{15} - \left(1-\frac{\lambda}{4}\right)^2\right]c_2 = -\frac{\lambda}{12} - \left(1-\frac{\lambda}{4}\right)\frac{1}{3}$$

$$\Rightarrow c_2 = \frac{\left(-\frac{\lambda}{12} - \left(1-\frac{\lambda}{4}\right)\frac{1}{3}\right)}{\left[\frac{\lambda^2}{15} - \left(1-\frac{\lambda}{4}\right)^2\right]}$$

and $\left(1-\frac{\lambda}{4}\right)c_1 = \frac{1}{4} + \frac{\lambda}{5}\frac{\left(\frac{-\lambda}{12} - \left(1-\frac{\lambda}{4}\right)\frac{1}{3}\right)}{\left[\frac{\lambda^2}{15} - \left(1-\frac{\lambda}{4}\right)^2\right]}$

$$\Rightarrow c_1 = \frac{\frac{1}{4} + \frac{\lambda}{5}\frac{\left(-\frac{\lambda}{12} - \left(1-\frac{\lambda}{4}\right)\frac{1}{3}\right)}{\left[\frac{\lambda^2}{15} - \left(1-\frac{\lambda}{4}\right)^2\right]}}{\left(1-\frac{\lambda}{4}\right)}$$

Solving using Crammers rule would have been fast.

Solution is

$$c_1 = \frac{60+\lambda}{240-120\lambda-\lambda^2}, \quad c_2 = \frac{80}{240-120\lambda-\lambda^2}$$

Then from (2)

$$g(s) = s + \lambda c_1 s + \lambda c_2 s^2$$

$$g(s) = \left[\frac{(240-60\lambda)s + 80\lambda s^2}{(240-120\lambda-\lambda^2)}\right]$$

Alternate method (formulae)

$$c_i - \lambda \sum_{k=1}^{n} a_{ik} c_k = f_i, \quad i = 1(1)n \quad \text{--- (1)}$$

where $c_i = \int_a^b b_i(t)\, g(t)\, dt$, $a_{ik} = \int_a^b b_i(t)\, a_k(t)\, dt$

$$f_i = \int_a^b b_i(t) f(t)\, dt$$

$f(s) = s$, $a_1(s) = s$, $a_2(s) = s^2$

$b_1(t) = t^2$, $b_2(t) = t$

$$f_1 = \int_0^1 b_1(t) f(t)\, dt, \quad f_2 = \int_0^1 b_2(t) f(t)\, dt$$

$$f_1 = \int_0^1 t^2 \cdot t\, dt = 1/4, \quad f_2 = \int_0^1 t \cdot t\, dt = 1/3$$

$$a_{11} = \int_0^1 t^3 dt = 1/4, \quad a_{12} = \int_0^1 t^4 dt = \frac{1}{5}$$

$$a_{21} = \int_0^1 t^2 dt = 1/3, \quad a_{22} = \int_0^1 t^3 dt = 1/4$$

Putting back in (6) we get

$$c_1 = \lambda\left[a_{11}c_1 + a_{12}c_2\right] + f_1$$

$$c_2 = \lambda\left[a_{21}c_1 + a_{22}c_2\right] + f_2$$

or

$$c_1 = \lambda\left[\frac{1}{4}c_1 + \frac{1}{5}c_2\right] + \frac{1}{4}$$

$$c_2 = \lambda\left[\frac{1}{3}c_1 + \frac{1}{3}c_2\right] + \frac{1}{3}$$

or

$$c_1 = \frac{1}{4} + \frac{1}{4}\lambda c_1 + \frac{1}{5}\lambda c_2$$

$$c_2 = \frac{1}{3} + \frac{1}{3}\lambda c_1 + \frac{1}{4}\lambda c_2$$

Example (2) solve the IE

$$g(s) = f(s) + \lambda\int_0^1 (s+t)g(t)\,dt$$

and find the eigen values.

Soln

$$K(s,t) = s+t = \sum_{i=1}^{2} a_i(s)b_i(t)$$

$$a_1(s) = s,\quad b_1(t) = 1,\quad a_2(s) = 1,\quad b_2(t) = t$$

$$a_{ik} = \int_0^1 b_i(t)\,a_k(t)\,dt$$

$$a_{11} = \frac{1}{2},\quad a_{12} = 1,\quad a_{21} = \frac{1}{3},\quad a_{22} = \frac{1}{2}$$

$$f_1 = \int_0^1 b_1(t)f(t)\,dt = \int_0^1 f(t)\,dt$$

$$f_2 = \int_0^1 b_2(t)f(t)\,dt = \int_0^1 t f(t)\,dt$$

now

$$c_i = \lambda\sum_{k=1}^{2} a_{ik}c_k = f_i,\qquad i = 1,2$$

we got

$$c_1 - \lambda[a_{11} c_1 + a_{12} c_2] = f_1$$

$$c_2 - \lambda[a_{21} c_1 + a_{22} c_2] = f_2$$

or

$$\left.\begin{aligned} (1 - \tfrac{\lambda}{2}) c_1 - \lambda c_2 &= f_1 \\ -\tfrac{1}{3}\lambda c_1 + (1 - \tfrac{\lambda}{2}) c_2 &= f_2 \end{aligned}\right\} \text{— (7)}$$

and

$$D(\lambda) = \begin{vmatrix} 1 - \tfrac{1}{2}\lambda & -\lambda \\ -\tfrac{\lambda}{3} & 1 - \tfrac{\lambda}{2} \end{vmatrix}$$

find soln using Crammer's rule.

and $\det(D(\lambda)) = 0$ gives

$$\lambda = \frac{-6 \pm \sqrt{64 \times 3}}{2} = \frac{-6 \pm 8\sqrt{3}}{2}$$

or $\boxed{\lambda = -6 \pm 4\sqrt{3}}$

So, if λ is not equal to these values then only Crammer's rule can be applied.

for $\lambda \neq -6 \pm 4\sqrt{3}$

$$c_1 = [-12 f_1 + \lambda(6 f_1 - 12 f_2)] / (\lambda^2 + 12\lambda - 12)$$

$$c_2 = [-12 f_2 - \lambda(4 f_1 - 6 f_2)] / (\lambda^2 + 12\lambda - 12)$$

where $f_1 = \int_0^1 f(t)\,dt$, $f_2 = \int_0^1 t f(t)\,dt$

$$g(s) = f(s) + \lambda \sum_{i=1}^{2} c_i a_i(s) \qquad , a_1(s) = s, \; a_2(s) = 1$$

$$= f(s) + \lambda[c_1 a_1(s) + c_2 a_2(s)]$$

$$= f(s) + \frac{\lambda}{\lambda^2 + 12\lambda - 12} \left[s\{-12 f_1 + \lambda(6 f_1 - 12 f_2)\} + \{-12 f_2 - \lambda(4 f_1 - 6 f_2)\} \right]$$

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$$g(s) = f(s) + \frac{\lambda}{\lambda^2+12\lambda-12}\Big[\{-12s + 6\lambda s - 4\lambda\} f_1 + \{-12\lambda s - 12 + 6\lambda\} f_2\Big]$$

$$\Rightarrow g(s) = f(s) + \frac{\lambda}{\lambda^2+12\lambda-12}\Big[\int_0^1 \{-12s + 6\lambda s - 4\lambda\} f(t)\,dt + \int_0^1 \{-12\lambda s - 12 + 6\lambda\} t f(t)\,dt\Big]$$

$$\Rightarrow g(s) = f(s) + \frac{\lambda}{\lambda^2+12\lambda-12}\Big[\int_0^1 \{[-12s + 6\lambda s - 12t + 6\lambda t] - 12\lambda st - 4\lambda\} f(t)\,dt\Big]$$

$$\Rightarrow g(s) = f(s) + \frac{\lambda}{\lambda^2+12\lambda-12}\int_0^1 [6(\lambda-2)(s+t) - 12\lambda st - 4\lambda] f(t)\,dt$$

$$g(s) = f(s) + \lambda \int_0^1 \frac{6(\lambda-2)(s+t) - 12\lambda st - 4\lambda}{\lambda^2+12\lambda-12} f(t)\,dt \quad -(8)$$

Now take $\Gamma(s,t;\lambda)$ (gamma)

$$= [6(\lambda-2)(s+t) - 12\lambda st - 4\lambda] / (\lambda^2+12\lambda-12)$$

then (8) can be written as

$$g(s) = f(s) + \lambda \int_0^1 \Gamma(s,t;\lambda) f(t)\,dt \quad -(9)$$

$\Gamma(s,t;\lambda)$ is called resolvent kernel. Thus we have succeeded in inverting the integral equation as the right hand side is a known quantity.

Example : ① Solve homogeneous Fredholm IE

$$g(s) = \lambda \int_0^1 e^s e^t g(t)\,dt \quad \text{— (10)}$$

Sol:

Take $c := \int_0^1 e^t g(t)\,dt$

So,

$$g(s) = \lambda c e^s \quad \text{— (11)}$$

putting (11) in (10)

$$g(s) = \lambda c e^s = \lambda \int_0^1 e^s e^t \lambda c e^t\,dt$$

$$\Rightarrow \lambda c e^s = \lambda^2 e^s c \int_0^1 e^{2t}\,dt$$

$$\lambda c e^s = \tfrac{1}{2}(\lambda^2 e^s c)(e^2 - 1)$$

$$\Rightarrow \lambda c e^s = \tfrac{1}{2}\lambda^2 c e^s (e^2 - 1)$$

$$\Rightarrow \lambda c[2 - \lambda(e^2 - 1)] = 0 \quad \text{— (12)}$$

If either $c = 0$ or $\lambda = 0$ then from $g(s) = \lambda c e^s$
$g \equiv 0$

now if $c \neq 0$ & $\lambda \neq 0$ then we have the eigen value

$$\lambda = \frac{2}{e^2 - 1}$$

and eigen function is (or ~~corresponding~~ soln of homogeneous IE)

$$g(s) = \lambda c e^s = \frac{2c}{e^2 - 1} e^s$$

This is corresponding to eigen value $\lambda = \dfrac{2}{e^2 - 1}$

the eigen function is e^s [As $\dfrac{2c}{e^2-1}$ is constant]

Fredholm Alternative

consider the Fredholm IE of 2nd kind with separable kernel

$$K(s,t) = \sum_{i=1}^{n} a_i(s)\, b_i(t)$$

$$g(s) = f(s) + \lambda \int_a^b K(s,t)\, g(t)\, dt$$

Reducing this to system of algebric equations we get

$$c_i - \lambda \sum_{k=1}^{n} a_{ik} c_k = f_i \qquad i = 1(1)n \quad \text{— } (*)$$

and

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} c_i a_i(s) \quad \text{— } (**)$$

where

$$c_i = \int_a^b b_i(t) g(t) dt, \quad f_i = \int_a^b b_i(t) f(t)\, dt$$

$$a_{ik} = \int_a^b b_i(t)\, a_k(t)\, dt$$

This in matrix form can be written as

$$D(\lambda)\, C = F \quad \text{— } (*)'$$

where $D(\lambda)$ is $n \times n$ matrix, C and F are vectors given by

$$D(\lambda) = \begin{bmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \cdots & -\lambda a_{1n} \\ -\lambda a_{21} & (1-\lambda a_{22}) & \cdots & -\lambda a_{2n} \\ \vdots & & & \\ -\lambda a_{n1} & \cdots & \cdots & (1-\lambda a_{nn}) \end{bmatrix}$$

$$C = (c_1 \text{ — } c_n)^T, \quad F = (f_1, f_2 \text{ — } , f_n)^T$$

If $D(\lambda) \neq 0$, then from $(*)'$

$$c_i = \frac{1}{\det D(\lambda)} \begin{bmatrix} \cdots & f_1 & \cdots \\ \cdots & f_i & \cdots \\ \cdots & f_n & \cdots \end{bmatrix}$$

this determinant is formed from matrix $D(\lambda)$ by replacing ith col by vector F.

By expanding the determinant along i^{th} col then

$$\begin{array}{c} \text{ith col} \\ \begin{bmatrix} - & - & f_1 & - \\ - & - & f_2 & - \\ - & - & \vdots & \\ & & f_n & \end{bmatrix} \end{array}$$

$= f_1 D_{1i} + f_2 D_{2i} + \cdots + f_n D_{ni}$ and

$c_i = [f_1 D_{1i} + \cdots f_n D_{ni}] / \det D(\lambda)$ —— ①

where D_{ji} denotes the cofactor of $(j,i)^{th}$ element of the determinant $D(\lambda)$.

Now from (**) the soln of inhomogenous IE is given by

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} \frac{D_{1i} f_1 + \cdots D_{2i} f_2 + \cdots D_{ni} f_n}{\det D(\lambda)} a_i(s)$$

—— ②

while the corresponding homogeneous eqn

$$g(s) = \lambda \int_a^b K(s,t)\, g(t)\, dt \quad —— ③$$

has only trivial solution as $g(s) = 0$ as $\det D(\lambda) \neq 0$

substituting f_i in (2) we can write soln $g(s)$ as

$$g(s) = f(s) + \left[\frac{\lambda}{\det D(\lambda)}\right] \int_a^b \left[\sum_{i=1}^{n} \left(D_{1i} b_1(t) + D_{2i} b_2(t) + \cdots D_{ni} b_n(t)\right] a_i(s)\right] f(t)\, dt$$

—— ④

now consider the determinant of (n+1) order

$$D(s,t;\lambda) = \begin{vmatrix} 0 & a_1(s) & a_2(s) & - & - & a_n(s) \\ b_1(t) & 1-\lambda a_{11} & -\lambda a_{12} & & \text{——} & -\lambda a_{1n} \\ b_2 & -\lambda a_{21} & 1-\lambda a_{22} & & \text{——} & -\lambda a_{2n} \\ \vdots & & & & & \\ b_n(t) & -\lambda a_{n1} & - & - & \text{——} & 1-\lambda a_{nn} \end{vmatrix}$$

$$= \begin{array}{c|c} 0 & \\ \hline & D(\lambda) \end{array}$$

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$$g(s) = f(s) + \lambda \int_a^b K(s,t)\, g(t)\, dt \quad \text{—} \quad (*)$$

$$\psi(s) = f(s) + \lambda \int_a^b K(t,s)\, \psi(t)\, dt$$

Eigen functions $g(s)$ & $\psi(s)$ corresponding to different eigen values λ_1 & λ_2 respectively of the homogenous integral equations. $(\lambda_1 \neq \lambda_2)$

$$g(s) = \lambda_1 \int_a^b K(s,t)\, g(t)\, dt \quad \text{—} \quad (1)$$

$$\psi(s) = \lambda_2 \int_a^b K(t,s)\, \psi(t)\, dt \quad \text{—} \quad (2)$$

Prove they are orthogonal. ?

sol

$(1) \times \lambda_2 \psi(s)$ & $(2) \times \lambda_1 g(s)$ then integrate

$$\int_a^b \lambda_2 \psi(s)\, g(s) = \lambda_2 \lambda_1 \int_a^b \left(\int_a^b K(s,t)\, g(t)\, dt \right) \psi(s)\, ds$$

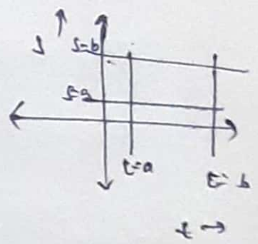
$$\int_a^b \lambda_1 g(s) \times \psi(s)\, ds = \lambda_1 \lambda_2 \int_a^b \left(\int_a^b K(t,s)\, \psi(t)\, dt \right) g(s)\, ds$$

now substract them

$$\Rightarrow (\lambda_1 - \lambda_2) \int_a^b g(s)\psi(s)\, ds$$

$$= \lambda_1 \lambda_2 \int_a^b \left(\int_a^b K(s,t)\, \psi(s)\, ds \right) g(t)\, dt$$

$$- \lambda_1 \lambda_2 \int_a^b \int_a^b \left(K(t,s)\, g(s)\, ds \right) \psi(t)\, dt$$



$$(\lambda_2-\lambda_1)\int_a^b g(s)\psi(s)\,ds = \lambda_1\lambda_2\Big[\int_a^b \psi(s)\int_a^b K(s,t)g(t)\,dt\,ds$$

$$\cancel{\int_a^b g(s)\int_a^b K(t,s)\,\psi(t)}$$

$$-\int_a^b \psi(t)\left(\int_a^b K(t,s)g(s)\,ds\right)dt\Big]$$

If $g(s)$ is a solution of inhomogeneous integral equation ②

then $\int_a^b f(s)\psi_{0i}(s)\,ds = 0$ $\forall\ i = 1(1)r$, r is rank of

$$f(s) = g(s) - \lambda\int_a^b K(s,t)g(t)\,dt$$

$$\int_a^b f(s)\psi_{0i}(s)\,ds = \int_a^b\left(g(s) - \lambda\int_a^b K(s,t)g(t)\,dt\right)\psi_{0i}(s)\,ds$$

$$= \int_a^b g(s)\psi_{0i}(s)\,ds - \lambda\int_a^b\left(\int_a^b K(s,t)g(t)\,dt\right)\psi_{0i}(s)\,ds$$

$$= \int_a^b g(s)\psi_{0i}(s)\,ds - \lambda\int_a^b g(t)\left(\int_a^b K(s,t)\psi_{0i}(s)\,ds\right)dt$$

$$= \int_a^b g(t)\left[\psi_{0i}(t) - \lambda\int_a^b K(s,t)\psi_{0i}(s)\,ds\right]dt$$

$$= 0$$

In fact, this is necessary and sufficient condition as well.
i.e. necessary & sufficient condition.

slide notes Lect-3

For $\det D(\lambda) = 0$, we consider the corresponding algebraic eqn
used to find eigenvalues?

$$(I - \lambda A) = F \quad — (1)$$

where $A = (a_{ij})$

1. If λ coincides with a certain eigen value λ_0 for which the determinant

how to find eigenvalues

$\det D(\lambda)$ has the rank p, $1 \le p < n$,

then there are $r = n - p$ L.I. solutions of the homogeneous algebraic system.

$$(I - \lambda A) C = 0 \quad — (2)$$

The number r is called index of the eigen value λ_0.

Let $g_{01}^{(s)}, g_{02}^{(s)}, \ldots, g_{0r}^{(s)}$ be r L.I. soln corresponding homogeneous integral eqn

$$g(s) = \lambda \int_a^b K(s,t)\, g(t)\, dt \quad — (3)$$

then the general solution of the homogeneous I E $g_0(s)$ is given by

$$g_0(s) = \sum_{k=1}^{r} \alpha_k\, g_{0k}(s) \quad — (4)$$

where α_k are arbitary constants.

-if lambda=eigenvalue-> |D(lamba)|=0-> contains infinite solutions for f satisfying fredholm alt thm
-in only this case if f does not satisfy fredholm alt thm, no solution exists

Note: If m be the ~~multiplicity~~ multiplicity of the eigen value λ_0 i.e. $\det D(\lambda)$ has m equal roots λ_0, then the number of linearly independent solutions of homogeneous eq (2) will be less than or equal to m, i.e.

$$1 \le r \le m$$

r and m are called geometric and algebric multiplicity of λ_0

Note: In general the inhomogeneous eq^n (1) (corresponding inhomogeneous IE) has no solution for $\det D(\lambda) = 0$. As an algebric system with vanishing determinant can be solved for particular F.

Next consider the transpose equation corresponding to fredholm IE.

$$g(s) = f(s) + \lambda \int_a^b K(s,t)\, g(t)\, dt \qquad \text{--- (5)}$$

given by

$$\psi(s) = f(s) + \lambda \int_a^b K(s,t)\, \psi(t)\, dt \qquad \text{--- (6)}$$

Note that kernel for transpose eq^n is $K(t,s)$ instead of $K(s,t)$.

If $K(s,t) = \sum_{i=1}^{n} a_i(s)\, b_i(t)$ --- (7)

then $K(t,s) = \sum_{i=1} a_i(t)\, b_i(s)$ --- (8)

Algebraic equation corresponding to transpose eq^n (6) is given by

$$(I - \lambda A^T)\, C = F \qquad \text{--- (9)}$$

where A^T is transpose of the matrix A and

$$c_i = \int_a^b a_i(t)\, \psi(t)\, dt \quad , \quad f_i = \int_a^b a_i(t)\, f(t)\, dt \quad \text{--- (10)}$$

Now if we consider algebraic equations corresponding to homogeneous IE and its transpose we get

now, as $\det(I-\lambda A) = \det(I-\lambda A^T)$ and hence eigen values of Homogeneous IE and transpose Homogeneous IE are same and the transpose eq^n (6) possesses unique solution whenever (5) does.

But the eigen functions of transpose equations are different from eigen function of homogeneous IE.

Not: eigen functions $g(s)$ and $\psi(s)$ corresponding to different eigenvalues λ_1 and λ_2 respectively, of the homogeneous IE.

$$g(s) = \lambda \int_a^b K(s,t)\, g(t)\, dt \quad \text{--- (11)}$$

and its transpose

$$\psi(s) = \lambda \int_a^b K(t,s)\, \psi(t)\, dt \quad \text{--- (12)}$$

are orthogonal.

from (11) & (12)

$$g(s) = \lambda_1 \int_a^b K(s,t)\, g(t)\, dt \quad \text{--- (13)}$$

and $$\psi(s) = \lambda_2 \int_a^b K(t,s)\, \psi(t)\, dt \quad \text{--- (14)}$$

Multiplying both sides of (13) by $\lambda_2 \psi(s)$ and those of 2nd eq^n (14) by $\lambda_1 g(s)$, integrating and then subtracting we get

$$(\lambda_1 - \lambda_2) \int_a^b g(s)\, \psi(s)\, ds = 0$$

As $\lambda_1 \neq \lambda_2$, we get $g(s)$ and $\psi(s)$ are orthogonal.

Next we establish that if $g(s)$ is solution of inhomogeneous equation

$$g(s) = f(s) + \lambda \int_a^b K(s,t)\, g(t)\, dt \qquad \text{--- (15)}$$

then

$$\int_a^b f(s)\, \psi_{0i}(s)\, ds = 0 \qquad \text{for } i = 1(1)r.$$

where $\{\psi_{0i}\}_{i=1}^{r}$ are r linearly independent eigen-functions of adjoint equation corresponding to λ_0 i.e.

$$\psi(s) = \lambda \int_a^b K(t,s)\, \psi(t)\, dt \qquad \text{--- (16)}$$

From (15)

$$f(s) = g(s) - \lambda \int_a^b K(s,t)\, g(t)\, dt$$

$$\Rightarrow \int_a^b f(s)\, \psi_{0i}(s)\, ds = \int_a^b g(s)\, \psi_{0i}(s)\, ds - \lambda \int_a^b \psi_{0i}(s) \int_a^b K(s,t)\, g(t)\, dt\, ds$$

$$= \int_a^b g(s)\, \psi_{0i}(s)\, ds - \lambda \int_a^b g(t) \left(\int_a^b K(s,t)\, \psi_{0i}(s)\, ds \right) dt$$

$$= \int_a^b g(t) \left[\psi_{0i}(t) - \lambda \int_a^b K(s,t)\, \psi_{0i}(s)\, ds \right] dt$$

$$= 0$$

In fact, this is necessary and sufficient condition as well. That is, the necessary and sufficient condition that the inhomogeneous IE

$$g(s) = f(s) + \lambda \int_a^b K(s,t)\, g(t)\, dt \quad \text{has a solution}$$

in case $\lambda = \lambda_0$, a root of $D(\lambda) = 0$, is that $f(s)$ to be orthogonal to r eigen-function ψ_{0i} of transposed equation

$$\psi(s) = \lambda \int_a^b k(t,s)\,\psi(t)\,dt$$

<u>Example</u> $\quad g(s) = f(s) + \lambda \int_0^{2\pi} \cos(s+t)\, g(t)\,dt$

and find the condition that $f(s)$ must satisfy in order that this eqn has a solution, when λ is an eigenvalue. obtain the general soln if $f(s) = \sin s$, considering all possible cases.

<u>Sol</u> :

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$

$$(1-\lambda a_{11})c_1 - \lambda a_{12} c_2 = f_1 \quad - \quad (1)$$

$$-\lambda a_{21} c_1 + (1-\lambda a_{22}) c_2 = f_2 \quad - \quad (2)$$

$$g(s) = \lambda \int_0^{2\pi} \cos(s+t)\, g(t)\, dt \qquad \text{eigen function}$$

for homogeneous

$$(1-\lambda a_{11})c_1 - \lambda a_{12} c_2 = 0$$

$$-\lambda a_{21} c_1 + (1-\lambda a_{22}) c_2 = 0$$

} find c_1, c_2 then eigen functions.

$$\Rightarrow \quad D(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} \\ -\lambda a_{21} & 1-\lambda a_{22} \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda a_{11})(1-\lambda a_{22}) + (+\lambda a_{12})(-\lambda a_{21}) = 0$$

$$\Rightarrow 1 - \lambda a_{22} - \lambda a_{11} + \lambda^2 a_{11} a_{22} - \lambda^2 a_{12} a_{21} = 0$$

(2) In the Integral equation

$$g(s) = e^{s} - s - \int_0^1 s(e^{st} - 1)\, g(t)\, dt$$

replace $s(e^{st} - 1)$ by

$$s(e^{st} - 1) = s^2 t + \frac{1}{2} s^3 t^2 + \frac{1}{6} s^4 t^3$$

obtain an approximate solution and find $g(0)$, $g(0.5)$ & $g(1)$.