Lecture 20

Theorem (Egorov) (Littlewood 3rd principle) :-

Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with finite measure ($\underline{m(E) < \infty}$). Assume $f_k \to f$ a.e on E (p.w).

Given $\varepsilon > 0$. There exists a closed set $A_\varepsilon \subseteq E$ such that $m(E \setminus A_\varepsilon) \leq \varepsilon$ & $f_k \to f$ uniformly on A_ε.

proof:- Given that $f_k(x) \to f(x)$ a.e on E.

We may assume without loss of generality that $f_k(x) \to f(x)$ $\forall x \in E$, as $k \to \infty$

$$\Big(\because m\big(\underbrace{\{x \in E \,/\, f_k(x) \not\to f(x)\}}_{=F}\big) = 0.$$

Now replace E by $E \setminus F$ & we prove the Theorem on $E \setminus F$ $\Big)$

For each pair $n, k \geq 1$, let us define

$$E_k^{(n)} := \left\{ x \in E \,\middle|\, |f_j(x) - f(x)| < \frac{1}{n}, \ \forall j > k \right\}$$

For a fixed n, $\quad E_k^{(n)} \subseteq E_{k+1}^{(n)} \quad \forall k \geq 1$

$$\therefore \lim_{k \to \infty} E_k^{(n)} = \bigcup_{k=1}^{\infty} E_k^{(n)} \underset{\supseteq}{\overset{\subseteq}{=}} E \ni x_0$$

$\left(\because f_k(x) \to f(x) \ \forall x \in E \right)$.

$f_k(x_0) \to f(x_0)$

$\Rightarrow$ For $\varepsilon = \frac{1}{n} > 0$, $\exists\, k_0 \in \mathbb{N}$ s.t. $|f_j(x_0) - f(x_0)| < \frac{1}{n}$ $\forall j > k_0$

$\Rightarrow x_0 \in E_{k_0}^{(n)}$

$\Rightarrow x_0 \in$ LHS.

$$\Rightarrow \lim_{k \to \infty} m(E_k^{(n)}) = m(E)$$

$$\Rightarrow \lim_{k \to \infty} \left(m(E_k^{(n)}) - m(E) \right) = 0$$

$$\Rightarrow \lim_{k \to \infty} \left(\underbrace{m(E) - m(E_k^{(n)})}_{= \, m(E \setminus E_k^{(n)})} \right) = 0.$$

$\left(\because m(E) < \infty \right)$

$$\Rightarrow \lim_{k \to \infty} m(E \setminus E_k^{(n)}) = 0$$

$\Rightarrow$ There exists $k_n \in \mathbb{N}$ such that

$$\boxed{m(E \setminus E_k^{(n)}) < \frac{1}{2^n} \quad \forall k \geq k_n.} \longrightarrow (*)$$

For $x \in E_{k_n}^{(n)}$, we have

$$|f_j(x) - f(x)| < \frac{1}{n}, \quad \forall j > k_n.$$

Choose $N \in \mathbb{N}$ so that $\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}$.

Let $\tilde{A}_\varepsilon := \bigcap_{n \geq N} E_{k_n}^{(n)}$ ✓

Now
$$m(E \setminus \tilde{A}_\varepsilon) = m\left(E \setminus \bigcap_{n \geq N} E_{k_n}^{(n)}\right)$$
$$= m\left(\bigcup_{n \geq N} \left(E \setminus E_{k_n}^{(n)}\right)\right)$$
$$\leq \sum_{n=N}^{\infty} m\left(E \setminus E_{k_n}^{(n)}\right)$$
$$\leq \sum_{n=N}^{\infty} \frac{1}{2^n} \quad \left(\text{by using } (*)\right)$$
$$< \frac{\varepsilon}{2}.$$

$\therefore$ $\underline{m(E \setminus \tilde{A}_\varepsilon) < \frac{\varepsilon}{2}}$.

If $\varepsilon' > 0$, we can choose $\underset{\in \mathbb{N}}{n_0} \geq N$ such that $\frac{1}{n_0} < \varepsilon'$.

if $x \in \tilde{A}_\varepsilon$, then $x \in E_{k_n}^{(n)}$ $\forall n \geq N$.

In particular $x \in E_{k_{n_0}}^{(n_0)}$ ✓

$\therefore$ $|f_j(x) - f(x)| < \frac{1}{n_0} < \varepsilon'$, $\forall j > k_{n_0}$.
$\forall x \in \tilde{A}_\varepsilon$

Thus $|f_j(x) - f(x)| < \varepsilon'$, $\forall j \geq k_{n_0}$, $\forall x \in \tilde{A}_\varepsilon$.

$\Rightarrow$ $f_k \to f$ uniformly on $\tilde{A}_\varepsilon$.

Note that $\tilde{A}_\varepsilon$ is measurable

$\Rightarrow$ there exists a closed set $A_\varepsilon \subseteq \tilde{A}_\varepsilon$

& $m(\tilde{A}_\varepsilon \setminus A_\varepsilon) < \frac{\varepsilon}{2}$.

Now $m(E \setminus A_\varepsilon) \leq m(E \setminus \tilde{A}_\varepsilon) + m(\tilde{A}_\varepsilon \setminus A_\varepsilon)$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\quad (\because A_\varepsilon \subseteq \tilde{A}_\varepsilon \subseteq E)$

$\therefore$ $m(E \setminus A_\varepsilon) < \varepsilon$ & $f_k \to f$ uniformly on A_ε.

<u>Proposition 1</u>: Let $f : \bigcup_{i=1}^{s} B_i \longrightarrow \mathbb{R}$, where B_i are closed.

be a function such that $f|_{B_i}$ is constant/continuous.

Then f is continuous.

<u>proof</u>:- Let $U \subseteq \mathbb{R}$ be an open set.

<u>To show</u>: $f^{-1}(U)$ is an open set.

Now
$$f^{-1}(U) = \left\{ x \in \bigcup_{i=1}^{s} B_i \,\middle|\, f(x) \in U \right\}.$$
$$= \bigcup_{i=1}^{s} \left(f^{-1}(U) \cap B_i \right)$$
$$= \bigcup_{i=1}^{s} \left\{ x \in B_i \,\middle|\, f(x) \in U \right\}$$
$$= \bigcup_{i=1}^{s} \left\{ x \in B_i \,\middle|\, f|_{B_i}(x) \in U \right\}$$
$$= \bigcup_{i=1}^{s} \underbrace{f|_{B_i}^{-1}(U)}_{\text{open } (\because f|_{B_i} \text{ is continuous})} \qquad \therefore \text{open}$$

$\therefore f^{-1}(U)$ is an open set.

$\therefore f$ is continuous.

Propositoon 2: Suppose f is a step funtion (simple funtion). Then f is continuous a.e.

proof:- Let $f = \sum_{i=1}^{s} a_i \chi_{A_i}$, where $a_i \in \mathbb{R}$ & A_i are measurable $m(A_i) < \infty$ & disjoint. $1 \le i \le s$

Since each A_i is measurable, there exists a closed set $B_i \subseteq A_i$ such that $m(A_i \setminus B_i) < \frac{1}{s 2^n}$.

Let $E \doteq \bigcup_{i=1}^{s} A_i$ &

$$E_1 \doteq E \setminus \bigcup_{i=1}^{s} B_i = \bigcap_{i=1}^{s} (E \setminus B_i).$$

Then $E \setminus E_1 = \bigcup_{i=1}^{s} B_i$

Now f is continuous on each B_i $\left(\because f|_{B_i} = a_i \right)$ $1 \le i \le s$.

$\therefore$ By the proposition (1) above, f is continuous on $\bigcup_{i=1}^{s} B_i = E \setminus E_1$

& $\quad m(E_1) = m\Big(\underbrace{\bigcup_{i=1}^{n} A_i \setminus \big(\bigcup_{i=1}^{n} B_i\big)}\Big)$

$$\leq m\Big(\bigcup_{i=1}^{n} (A_i \setminus B_i)\Big)$$

$$\leq \sum_{i=1}^{n} m(A_i \setminus B_i)$$

$$< \sum_{i=1}^{n} \frac{1}{2^n s} = \frac{1}{2^n}.$$

$\therefore \quad m(E_1) < \frac{1}{2^n} \qquad \forall n \geq 1$

$\Rightarrow \quad m(E_1) = 0$

& we have f is continuous on $E \setminus E_1$.

Thus f is continuous a.e.

Theorem (Lusin) (Littlewood 2nd principle) :—

Let E be a measurable set of finite measure. Let $f : E \to \mathbb{R}$ be a measurable function. Then for any $\varepsilon > 0$, there exists a closed set F_ε with $F_\varepsilon \subseteq E$ & $m(E \setminus F_\varepsilon) \leq \varepsilon$ & such that $f\big|_{F_\varepsilon}$ is continuous.