it is case of unbounded solution. (And $z_j - c_j$ is neg. for some

Alt. optimal Soln.

Q. Max. $z = x_1 + 2x_2 + 3x_3$

s.t. $x_1 + 2x_2 + 3x_3 \le 10$

$x_1 + x_2 \le 5$

$x_1 \le 1$

$x_1, x_2, x_3 \ge 0.$

$x_1 + 2x_2 + 3x_3 + x_4 + 0.x_5 = 10.$

$x_1 + x_2 + x_5 = 5$

$x_1 + x_6 = 1$

After phase I

			1	2	3	0	0	0
C_b	X_b	b	x_1	x_2	x_3	x_4	x_5	x_6
3	x_3	10/3	1/3	2/3	1	1/3	0	0
0	x_5	5	1	1	0	0	1	0
0	x_6	1	1	0	0	0	0	1
			0	0	0	1	0	0

So, $z_j - c_j \ge 0 \ \forall j$ & $z_j - c_j = 0$ for non basic x_1

We can change of them to basis to get alt soln.

Currently $X = \begin{pmatrix} 0 \\ 0 \\ 10/3 \end{pmatrix}$ if $x_5 \to x_1$ then $X = \begin{pmatrix} 0 \\ 5/3 \end{pmatrix}$

$z = 10$ $Z = 10$

Linear combination of these is also soln.

Post Mids

Q. Use simplex to find soln of dual form of this LPP by final table

$$\text{Max } z = 30x_1 + 23x_2 + 29x_3$$

s.t. $6x_1 + 5x_2 + 3x_3 \le 26$

$4x_1 + 2x_2 + 5x_3 \le 7$ $\forall_i\ x_i \ge 0$

St. form $z = 30x_1 + 23x_2 + 29x_3 + 0x_4 + 0x_5$

$6x_1 + 5x_2 + 3x_3 + x_4 = 26$

$4x_1 + 2x_2 + 5x_3 + x_5 = 7$

			30	23	29	0	0	
C_B	X_B	b	a_1	a_2	a_3	a_4	a_5	
0	x_4	26	6	5	3	1	0	26/6
0	x_5	7	$\boxed{4}$	2	5	0	1	7/4
			−30 ↑	−23	−29	0	0	

			30	23	29	0	0
C_B	X_B	b	a_1	a_2	a_3	a_4	a_5
0	x_4	31/2	0	2	−9/2	1	−3/2
30	x_1	7/4	1	$\boxed{1/2}$	5/4	0	1/4
			0	−8	17/2	0	15/2

C_B	X_B	b	a_1	a_2	—	a_5
0	x_4	17/2				
23	x_2	7/2				

all +ve

Currently $x = \begin{pmatrix} 0 \\ 0 \\ 10/3 \end{pmatrix}$ if $x_5 \to x_1$ then $x = \begin{pmatrix} 5 \\ 0 \\ 5/3 \end{pmatrix}$

$z = 10$ $z = 10$

Linear combination of these is also soln.

Post Mids

Q. Use simplex to find soln of dual form of this LPP by final table

$$\text{Max } z = 30x_1 + 23x_2 + 29x_3$$

s.t.
$$6x_1 + 5x_2 + 3x_3 \le 26$$
$$4x_1 + 2x_2 + 5x_3 \le 7 \qquad \forall_i\ x_i \ge 0$$

St. form
$$z = 30x_1 + 23x_2 + 29x_3 + 0x_4 + 0x_5$$
$$6x_1 + 5x_2 + 3x_3 + x_4 = 26$$
$$4x_1 + 2x_2 + 5x_3 + x_5 = 7$$

			30	23	29	0	0	
C_B	X_B	b	a_1	a_2	a_3	a_4	a_5	
0	x_4	26	6	5	3	1	0	26/6
0	x_5	7	$\boxed{4}$	2	5	0	1	7/4
			-30 ↑	-23	-29	0	0	

			30	23	29	0	0
C_B	X_B	b	a_1	a_2	a_3	a_4	a_5
0	x_4	31/2	0	2	-9/2	1	-3/2
30	x_1	7/4	1	$\boxed{1/2}$	5/4	0	1/4
			0	-8	17/2	0	15/2

C_B	X_B	b	a_1	a_2	—	a_5
0	x_4	17/2				
23	x_2	7/2				

all +ve.

Optimal Solution: $x_1 = 0$, $x_2 = 1/2$ $x_3 = 0$

$$Z_{max} = \frac{161}{2}$$

Optimal solution for dual: $V = \left[z_j - c_j \text{ of slack} \right]$ $V_1 = 0$, $V_2 = \frac{23}{2}$

Q. By solving the dual of the following problem, show that the given problem has NO feasible soln.

min $z = x_1 - x_2$ s.t. $2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$ $x_1, x_2 \geq 0$

Dual form

Max $W = 2V_1 + 2V_2$

s.t. $2V_1 - V_2 \leq 1$ $V_1, V_2 \geq 0$

$-V_1 + V_2 \geq 1$

Max $W = 2V_1 + 2V_2 + 0V_3 + 0V_4 - MV_5$

$2V_1 - V_2 + V_3 = 1$

$-V_1 + V_2 - V_4 + V_5 = 1$ $V_i \geq 0 \ \forall i$

Primal	Dual	Conclusion
FS	FS	Both have optimal FS
No FS	FS	Dual has unbdd.
FS	NO FS	Primal " "
NO FS	NO FS	No Soln

	C_j		2	2	0	0	$-M$
C_B	X_B	b	a_1	a_2	a_3	a_4	a_5
0	x_3	1	2	-1	1	0	0
0	x_5	1	-1	1	0	-1	1
			-2	-2	0	0	0

↳ After iterations, $z_j - c_j < 0$ for some j but there wouldn't be any leaving var. Hence, unbdd soln.

Unbdd soln. ⇒ No feasible soln of original problem.

Duality Theorem

Theorem: If any of constraint in the primal problem is a perfect equality, then the corresponding dual variable is unrestricted in sign.

Theorem: If any variable of the primal problem is unrestricted in sign then the corresponding contraint of the dual is an equality.

Theorem: Dual of the dual is a primal.

Theorem: If x is any f.s to the primal problem and v is any fs to the dual problem, then $cx \leq b^T v$
(If primal is maximization problem)

Theorem: If x^* is a f.s of the primal problem and v^* is the f.s to the dual problem s.t

$$\boxed{cx^* = b^T v^*}$$

then both x^* and v^* are optimal soln.

Theorem: (Fundamental Theorem of duality)
A f.s x^* to the primal is optimal iff $\exists$ a f.s v^* to the dual problem s.t $cx^* = b^T v^*$

Revised Simplex Method (Solving LPP with less labour)

Example: Solve the following LPP by Revised simplex.

$$\text{Max } z = x_1 + x_2$$

$$\text{s.t} \quad 3x_1 + 2x_2 \leq 6$$

$$x_1 + 4x_2 \leq 4 \qquad x_1, x_2 \geq 0$$

Transformations in Simplex : $y_j = B^{-1} a_j$, $x_B = B^{-1} b$, $z_j - c_j$ ✓

If a_k is entering vector then calculate only y_k

Std. form Max $z = x_1 + x_2 + 0x_3 + 0x_4$

s.t $3x_1 + 2x_2 + x_3 = 6$

$x_1 + 4x_2 + x_4 = 4$ $x_i \geq 0$

$z - x_1 - x_2 = 0$

$a_0^* = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $a_1^* = \begin{pmatrix}-1\\3\\1\end{pmatrix}$ $a_2^* = \begin{pmatrix}-1\\2\\4\end{pmatrix}$ ...

Initial Basis: $B^* = (a_0^*, a_3^*, a_4^*)$

$(B^*)^{-1} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = (\beta_0^*, \beta_1^*, \beta_2^*)$

$x_B^* = (B^*)^{-1} b^* = \begin{pmatrix}0\\6\\4\end{pmatrix}$

first row of $(B^*)^{-1}$ by a_i^*

Finding Δ, $z_1 - c_1 = (1\ 0\ 0)\begin{pmatrix}-1\\3\\1\end{pmatrix} = -1$

$z_2 - c_2 = (1\ 0\ 0)\begin{pmatrix}-1\\2\\4\end{pmatrix} = -1$

$\min\{-1, -1\} \xRightarrow{\text{tie}}$ Take $k=1$ $\Rightarrow$

$\left[y_j^* = (B^*)^{-1} a_j^* = \begin{pmatrix} z_j - c_j \\ y_j \end{pmatrix} \right.$

Compute

$y_1^* = (B^*)^{-1} a_1^* = \begin{pmatrix}-1\\3\\1\end{pmatrix}$

$= \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}-1\\3\\1\end{pmatrix}$

Revised Simplex Method

Computation of inverse by Partitioning

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ $a_{l\times l}$, $b_{l\times m}$

$c_{m\times l}$, $d_{m\times m}$

$n = m + l$

$$M^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$MM^{-1} = I_n$

⇒ if d has an inverse then we get,

$A = (a - bd^{-1}c)^{-1}$ $B = -Abd^{-1}$

$C = -d^{-1}cA$ $D = d^{-1} - d^{-1}cB$

$$M = \begin{bmatrix} I & Q \\ O & R \end{bmatrix}$$ & R^{-1} exists

then $$M^{-1} = \begin{bmatrix} I & -QR^{-1} \\ O & R^{-1} \end{bmatrix} \longrightarrow (11)$$

Revised Simplex

$$\left.\begin{array}{l} \text{Max } Z = Cx \\ Ax = b \quad A_{m\times n} \\ x \geq 0 \end{array}\right\} \longrightarrow \text{Initial Basis } B$$

New set of equations : $Z - Cx = 0$

$0.Z + Ax = b$

↳ $A^* x^* = b^*$

$$A^* = \begin{bmatrix} 1 & -c \\ 0 & A \end{bmatrix} \quad b^* = \begin{bmatrix} 0 \\ b \end{bmatrix} \quad x^* = \begin{bmatrix} Z \\ x \end{bmatrix}$$

↳ a_j^*

Basis, $B^* = \begin{bmatrix} 1 & -C_B \\ 0 & B \end{bmatrix}$

$(B^*)^{-1}$ from (11)

$$x_B^* = (B^*)^{-1} b^* = \begin{bmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{pmatrix} C_B(B^{-1}b) \\ B^{-1}b \end{pmatrix}$$

$$\Rightarrow x_B^* = \begin{bmatrix} z \\ x_B \end{bmatrix}$$

$a_j = $ "A" matrix

$$y_j^* = (B^*)^{-1} (a_j)^*$$

$$\Rightarrow y_j^* = \begin{bmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} -c_j \\ a_j \end{bmatrix}$$

$$\Rightarrow y_j^* = \begin{bmatrix} z_j - c_j \\ y_j \end{bmatrix}$$

Q. Solve the following LPP by revised simplex

$$\text{Min } z = x_1 + 2x_2$$
$$2x_1 + 5x_2 \geq 6$$
$$x_1 + x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

Std. form $\quad \text{Max. } z' = -x_1 - 2x_2$

$$\text{Max } z_a = -x_{a_1} - x_{a_2}$$

$$2x_1 + 5x_2 - x_3 + x_{a_1} = 6$$
$$x_1 + x_2 - x_4 + x_{a_2} = 2 \qquad x_i \geq 0 \; i \in \Sigma$$

Set of eq^n is :

$$\begin{array}{lllllllll}
\downarrow \alpha_0 & \downarrow \alpha_1 & \downarrow \alpha_2 & \downarrow \alpha_3 & \downarrow \alpha_4 & \downarrow \alpha_5 & \downarrow \alpha_6 & \downarrow \alpha_7 & \\
z' + & x_1 + & 2x_2 & & & & & & = 0 \\
 & & & & & z_a + & x_{a_1} + & x_{a_2} & = 0 \\
 & 2x_1 + & 5x_2 & -x_3 & & & + x_{a_1} & & = 6 \\
 & x_1 + & x_2 & & -x_4 & & & + x_{a_2} & = 2
\end{array}$$

$$\alpha_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 5 \\ 1 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \alpha_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\alpha_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \alpha_6 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \alpha_7 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 2 \end{pmatrix}$$

Phase 1

Initial Basis, $S = \begin{array}{cccc} \alpha_1 & \alpha_5 & \alpha_6 & \alpha_7 \end{array}$

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S^{-1} = \begin{array}{cccc} g_0 & g_1 & g_2 & g_3 \end{array}$$

$$S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

g_0 would always be (1 0 0 ...) we don't care abt it.

$$x_s = S^{-1}d = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -8 \\ 6 \\ 2 \end{pmatrix}$$

Initial table

Min ratio $\frac{x_s}{n_2}$

Basis	g_1	g_2	g_3	x_s	n_2	
α_0	0	0	0	0	2	
α_5	1	-1	-1	-8	-6	
α_6	0	1	0	6	$\boxed{5}$	6/8 → Min.
α_7	0	0	1	2	1	

Determining entering vari.

Calculating $z_j - c_j$; $z_j - c_j = (2^{nd}$ row of S^{-1} $\times$ $\alpha_j)$

$$z_j - c_j = \begin{pmatrix} -3 \\ -6 \\ 1 \\ 1 \end{pmatrix} \rightarrow \text{Most } -ve$$

α_2 is entering

Compute $\eta_2 = S^{-1}\alpha_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 5 \\ 1 \end{pmatrix}$

~~Considering~~ α_6 is departing cuz of its min ratio ~~is~~ ~~min~~.

Basis	$\bar{g}_1$	$\bar{g}_2$	$\bar{g}_3$	x_S	η_1	Min Ratio
α_0	0	$-2/5$	0	$-12/5$	$1/5$	
α_5	1	$1/5$	-1	$-4/5$	$-3/5$	
α_2	0	$1/5$	0	$6/5$	$2/5$	3
α_7	0	$-1/5$	1	$4/5$	$\boxed{3/5}$	$4/5$

α_7 → Artificial

$z_j - c_j = 2^{nd}$ row of $S^{-1} \times \alpha_1 = \begin{pmatrix} 0 & 1 & \frac{1}{5} & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$

$= (-3/5)$

α_1 is entering

$\eta_1 = S^{-1}\alpha_1 = \begin{pmatrix} 1 & 0 & 2/5 & 0 \\ 0 & 1 & 1/5 & -1 \\ 0 & 0 & 1/5 & 0 \\ 0 & 0 & -1/5 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} = \begin{matrix} 1/5 \\ -3/5 \\ 2/5 \\ 3/5 \end{matrix}$

α_7 is leaving $\because$ (Min Ratio)
$= 4/5$

Basis	$\bar{g}_1$	$\bar{g}_2$	$\bar{g}_3$	x_S
α_0	0	$-1/3$	$-1/3$	$-8/3$
α_5	1	0	0	0
α_2	0	$1/3$	$-2/3$	$2/3$
α_1	0	$-1/3$	$5/3$	$4/3$

$z_3 - c_3 = 0$

$z_4 - c_4 = 0$ $\qquad z_j - c_j \geq 0 \ \forall j$

$z_6 - c_6 = 1$ $\qquad$ Optimality Reached.

$z_7 - c_7 = 1$

Optimal Value : $\left(\frac{-8}{3}\right)$

Max $z = cx$

$Ax = b$

$x \geq 0$

$z_a = -\sum_{i=1}^{m} x_{ai}$ $\qquad x_{ai} \to$ artificial variables

Rewrite as,

$$\begin{array}{cccccccc}
\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_n & \alpha_{n+1} & \cdots\ \alpha_{n+m+1} & d \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\
z & -c_1x_1 & -c_2x_2 & \cdots & -c_nx_n & & & = 0 \\
 & & & & & z_a + x_{a_1} + x_{a_2} \cdots x_{am} & & = 0 \\
 & a_{11}x_1 & +a_{12}x_2 & \cdots & a_{1n}x_n & x_{a_1} & & \\
 & \vdots & \vdots & & & \ddots & & b \\
 & a_{m1}x_1 & a_{m2}x_2 & \cdots & a_{mn}x_n & & x_{am} & =
\end{array}$$

$\alpha_j = [-c_j \ \ 0 \ \ a_j]$, $j = 1, 2, \ldots n$

$\alpha_j = [0 \ . \ 1 \ \ e_i]$ $\quad j = n+1+i$, $i = 1, 2 \ldots m$

$$S_{m+2} = (\underset{z}{e_1}, \underset{z_a}{S_1}, \underset{x_{a_1}}{S_2}, \ldots \underset{x_{am}}{S_{m+1}}) = \cancel{\begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}}$$

$d = [0 \ \ 0 \ \ b]$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & & \\ 0 & 1 & 1 & 1 & \cdots & \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$ Identity

$$S_{m+2} = \left(\begin{array}{cc|c} 1 & 0 & -C_B \\ 0 & 1 & -C_l \\ \hline 0 & 0 & B \end{array}\right)$$

$C_l = [C_{l1} \; C_{l2} \ldots C_{l}]$ where $C_{li} = \begin{cases} 0 & \text{corresp to } \ldots \; j \le n \\ -1 & \text{" " } \; j \ge n+1 \end{cases}$

$$S^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\begin{array}{cc|c} 1 & 0 & C_B B^{-1} \\ 0 & 1 & C_l B^{-1} \\ \hline 0 & 0 & B^{-1} \end{array}\right)$$

$$x_r = S^{-1} d = \begin{pmatrix} 1 & 0 & C_B B^{-1} \\ 0 & 1 & C_l B^{-1} \\ 0 & 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} C_B B^{-1} b \\ C_l B^{-1} b \\ B^{-1} b \end{pmatrix}$$

$$\Rightarrow x_r = \begin{pmatrix} Z \\ Z_a \\ x_B \end{pmatrix}$$

$$\eta_j = S^{-1} \alpha_j = \begin{pmatrix} 1 & 0 & C_B B^{-1} \\ 0 & 1 & C_l B^{-1} \\ 0 & 0 & B^{-1} \end{pmatrix} \begin{pmatrix} -c_j \\ a \\ a_j \end{pmatrix} = \begin{pmatrix} -c_j + C_B \underbrace{B^{-1} a_j}_{y_j} \\ C_l B^{-1} a_j \\ B^{-1} a_j \end{pmatrix}$$

$$\Rightarrow \eta_j = \begin{pmatrix} z_j - c_j \\ (z_j - c_j)_a \\ y_j \end{pmatrix}$$

→ If all artificials are removed from the final table when optimality is reached then we stop

→ If artificial variable is present at zero level then we go to Phase II and

i) Remove Z_a column

ii) 1st column corresponding to Z never leaves

iii) $z_j - c_j$ → Multiply 1st row of S^{-1} with α_j Non-basic vector.

Dual Simplex Method

Example: Use dual simplex method to solve the LPP:

$$\text{Max } z = -2x_1 - 3x_2 - x_3$$

s.t. $2x_1 + x_2 + 2x_3 \geq 3$

$3x_1 + 2x_2 + x_5 \geq 4$ $\qquad x_1, x_2, x_3 \geq 0$

Std. form $\quad \text{Max } z = -2x_1 - 3x_2 - x_3 + 0 \cdot x_4 + 0x_5$

s.t. $2x_1 + x_2 + x_3 - x_4 = 3$

$3x_1 + 2x_2 + x_3 - x_5 = 4$ $\qquad \underbrace{\begin{pmatrix} 3 \\ 4 \end{pmatrix}}_{b}$ $\qquad x_i \geq 0 \quad i \in \{1,2,3,4,5\}$

Select the basis $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2$

$$B^{-1} = -I_2$$

$x_B = B^{-1} b = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$ $\qquad C_B = 0$

$z_j - c_j = C_B y_j - c_j = -c_j \geq 0 \quad \forall j$

- Conditions for using dual simplex:
  - → Maximization
  - → Initial Basis yields $z_j - c_j \geq 0 \ \forall j$ and atleast one element of x_B is -ve.

New set of eqn,

$-2x_1 - x_2 - x_3 + x_4 = -3$

$-3x_1 - 2x_2 - x_3 \qquad x_5 = -4$

			c_j	-2	-3	-1	0	0	Max Ratio
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	
0	a_4	x_4	-3	-2	-1	-1	1	0	
0	a_5	x_5	-4	$\boxed{-3}$	-2	-1	0	1	←
	$z_j - c_j$			2	3	1	0	0	
				$2/-3$	$3/-2$	$1/-1$			

↳ Max ratio

→ Leaving variable is determined first by choosing the most −ve component of b. → a_r

Ratio is calculated as $\left\{ \dfrac{z_j - c_j}{y_{rj}},\ y_{rj} \le 0 \right\} \rightarrow \text{max}$

			c_j	−2	−3	−1	0	0
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5
0	a_4	x_4	$-1/3$	0	$1/3$	$\boxed{-4/3}$	1	$-2/3$
−2	a_1	x_1	$4/3$	1	$2/3$	$1/3$	0	$-1/3$
			$z_j - c_j$	0	$5/3$	$1/3$	0	$2/3$
	Max Ratio					$\frac{1}{-4}$		$\frac{2}{-2}$

				−2	−3	−1	0	0
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5
−1	a_3	x_3	$1/4$	0	$-1/4$	1	$-3/4$	$1/2$
−2	a_1	x_1	$3/4$	1	$3/4$	0	$1/4$	$-1/2$

All b are positive. Optimality Reached.
and $\Delta_j \geq 0$

Optimal solution is : $\left\{ \begin{array}{l} x_1 = 5/4 \\ x_2 = 0 \\ x_3 = 1/4 \end{array} \quad Z_{max} = \dfrac{-11}{4} \right\}$

form

$$\max z^* = -7x_1 - 3x_2 + 0x_3 + 0x_4 + 0x_5$$

S.t. $x_1 - 3x_2 - x_3 = 1$

$x_1 + x_2 - x_4 = 2$

$-2x_1 + 2x_2 - x_5 = 1$ $\quad x_i \geq 0$

Initial Basis $B = [a_3 \; a_4 \; a_5]$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -I_3$$

$B^{-1} = -I_3$

$x_B = B^{-1}b = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \rightarrow -ve$

$C_B = (c_3 \; c_4 \; c_5) = 0$

$z_j - c_j = c_B y_j - c_j = -c_j \geq 0$

		c_j	-7	-3	0	0	0
C_B	x_B	b	a_1	a_2	a_3	a_4	a_5
0	x_3	-1	-1	3	1	0	0
0	x_4	-2	-1	$\boxed{-1}$	0	1	0
0	x_5	-1	2	-2	0	0	1
	$z_j - c_j$		7	3	0	0	0

Max. ratio $\frac{7}{-1}$ $\frac{3}{-1}$

		c_j	-7	-3	0	0	0
C_B	x_B	b	a_1	a_2	a_3	a_4	a_5
0	x_3	-7	$\boxed{-4}$	0	1	3	0
-3	x_2	2	1	1	0	-1	0
0	x_5	3	4	0	0	-2	1
	$z_j - c_j$		4	0	0	3	0

$\frac{4}{-4}$

C_B	X_B	c_j / b	-7 a_1	-3 a_2	0 a_3	0 a_4	0 a_5
-7	x_1	7/4	1	0	-1/4	-3/4	0
-3	x_2	1/4	0	1	1/4	-1/4	0
0	x_5	-4	0	0	1	1	1
	$z_j - c_j$		0	0	1	6	0

b is still -ve but max ratio is invalid since $y_{rj} \geq 0$

This is case of <u>NO SOLUTION</u>

Example: (Artificial Constraint Method)

Add a constraint, $\Sigma x_i \leq M$

Summation is over all the j's

for which $z_j - c_j < 0$

& M is sufficiently large +ve no.

$\Sigma x_j + x_M = M$ $\quad$ $j = p \rightarrow z_p - c_p$ -ve most

$$x_p = M - \left(\sum_{j \neq p} x_j + x_M \right)$$

Q. Use the artificial constraint method to find the initial basis soln. of the following problem and then apply dual simplex algm to solve it.

Max $z = 2x_1 - 3x_2 - 2x_3$

s.t. $x_1 - 2x_1 - 3x_3 = 8$

$2x_2 + x_3 \leq 10$ $\quad$ $x_i \geq 0$

$x_2 + 2x_3 \geq ~~8~~ 4$

Std. form, Max. $z = 2x_1 - 3x_2 - 2x_3$

s.t. $x_1 - 2x_2 - 3x_3 = 8$

$2x_2 + x_3 + x_4 = 10$ $\quad$ $x_i \geq 0$

$-x_2 - 2x_3 + x_5 = -4$

Basis, $B = [a_1 \ a_4 \ a_5] = I_3$

$\Rightarrow B^{-1} = I_3$

$x_B = B^{-1}b = \begin{pmatrix} 8 \\ 10 \\ -4 \end{pmatrix}$

$z_j - c_j = C_B y_j - c_j$ $\quad C_B = [2 \ 0 \ 0]$

		c_j	2	-3	-2	0	0
C_B	x_B	b	a_1	a_2	a_3	a_4	a_5
2	x_1	8	1	-2	-3	0	0
0	x_4	10	0	2	1	1	0
0	x_5	-4	0	-1	-2	0	1
			0	-1	-4	0	0

~~Option~~ Clearly, optimality is not reached.

x_3 has the most -ve Δj

Add new constraint.

$x_2 + x_3 \leq M$

Add artificial variable x_M

$x_2 + x_3 + x_M = M$

$\Rightarrow x_3 = M - x_2 - x_M$

The augmented problem then becomes

Max $z = 2x_1 - 3x_2 - 2(M - x_2 - x_M)$

$= 2x_M + 2x_1 - x_2 - 2M$

s.t

$x_1 - 2x_2 - 3(M - x_2 - x_M) = 8$

$2x_2 + (M - x_2 - x_M) + x_4 = 10$

$-x_2 - 2(M - x_2 - x_M) + x_5 = 4$

OR, Max $z = 2x_M + 2x_1 - x_2 - 2M$

$3x_M + x_1 + x_2 \ldots = 8 + 3M$

$-x_M + x_2 + x_4 = 10 - M$

$2x_M + x_2 + x_5 = -4 + 2M$

$x_M + x_2 + x_3 = M$

		C_j	2	2	-1	0	0	0
C_B	X_B	b	a_M	a_1	a_2	a_3	a_4	a_5
2	x_1	$8+3M$	3	1	1	0	0	0
0	x_4	$10-M$	-1	0	1	0	1	0
0	x_5	$-4+2M$	-2	0	-3	0	0	1
0	x_3	M	1	0	1	1	0	0
	$Z_j - C_j$		4	0	3	0	0	0

?

X_B	b					
x_1	$14/5$					
x_M	$\frac{5M-26}{5}$					
x_2	$24/5$					
x_3	$2/5$					
$Z_j - C_j$		0	0	....	$6/5$	$7/5$

Eliminate the extra column and row corresponding to a_M and put the original cost.

		C_j	2	-3	-2	0	0
C_B	B	b	a_1	a_2	a_3	a_4	a_5
-2	a_1	$14/5$	1	0	0	$7/5$	$4/5$
-3	a_2	$24/5$	0	1	0	$2/5$	$-1/5$
-2	a_3	$2/5$	0	0	1	$1/5$	$7/5$
		$Z_j - C_j$	0	0	0	$6/5$	$7/5$

Complementary Slackness Theorem

For any pair of optimal soln. to a LPP and its associated dual.

i) The product of the j^{th} variable of the primal and the j^{th} surplus variable of the dual is zero. for each $j = 1, 2, \ldots n$

ii) The product of the i^{th} variable of the dual and i^{th} slack variable of the primal is zero for each $i = 1, 2, \ldots n$.

Proof:

Primal

$$\left.\begin{aligned} \text{Max } & z = cx \\ \text{s.t } & Ax \le b \\ & x \ge 0 \end{aligned}\right\} - (1)$$

Std. form

$$(3) - \left\{\begin{aligned} \text{Max } & Z = cx \\ \text{s.t } & \underset{m\times n}{A}\,\underset{n\times 1}{x} + \underset{n\times 1}{x_s} = \underset{m\times 1}{b} \\ & x_i \ge 0 \quad x_s \to \text{Slack} \end{aligned}\right.$$

Dual

$$\left.\begin{aligned} \text{Min } & w = b^T v \\ \text{s.t } & A^T v \ge c^T \\ & v \ge 0 \end{aligned}\right\} - (2)$$

$$(4) - \left\{\begin{aligned} \text{Max } & w = b^T v \\ \text{s.t } & A^T v - v_s = c^T \\ & v_s \to \text{Surplus} \end{aligned}\right.$$

From (3) $\quad v^T(Ax + x_s) = v^T b$

$\Rightarrow \quad v^T A x + v^T x_s = v^T b$

$\Rightarrow \quad x^T A^T v + x_s v = b^T v \quad - (5)$

From (4), $\quad x^T(A^T v - v_s) = x^T c^T$

$\Rightarrow \quad x^T A^T v - x^T v_s = x^T c^T$

$\Rightarrow \quad v^T A x - v_s^T x = cx \quad - (6)$

Since v and x are optimal soln. So $b^T v = cx$ (by theorem)

So ~~sub~~ (5) $-$ (6) $\Rightarrow \quad x_s v + v_s^T x = 0$

But $x_s, v_s, x, v \ge 0$ so ~~all~~ both of them must be zero

$x_s v = 0$ and $v_s^T x = 0$

Theorem

If (x, x_s), (v, v_s) are feasible soln to the primal ① and associated dual ② under conditions where complementary slackness holds then (x, x_s) and (v, v_s) are also their respective optimal solutions.

Proof: Complementary slackness holds

$$\Rightarrow v^T x_s + x^T v_s = 0$$

$$\Rightarrow v^T x_s = -x^T v_s = -v_s^T x$$

Also $v^T A x \Rightarrow$

$$v^T A x + v^T x_s = v^T A x - v_s^T x$$

OR, $\underbrace{v^T (Ax + x_s)}_{b} = x^T A^T v - x^T v_s$

$$= x^T \underbrace{(A^T v - v_s)}_{\to c^T}$$

$$\to v^T b = x^T c^T = L(x)^T$$

$$cx = b^T v$$

Now according to fundamental theorem of duality this solution is also optimal.

Assignment Problem

$$\begin{vmatrix} 8 & 4 & 2 & 6 & 1 \\ 0 & 9 & 5 & 5 & 4 \\ 3 & 8 & 9 & 2 & 6 \\ 4 & 3 & 1 & 0 & 3 \\ 9 & 5 & 8 & 9 & 5 \end{vmatrix} \Rightarrow \begin{vmatrix} 7 & 3 & 0 & 5 & 0 \\ 0 & 9 & 1 & 5 & 4 \\ 1 & 6 & 6 & 0 & 4 \\ 1 & 3 & 0 & 0 & 3 \\ 4 & 0 & 2 & 4 & 0 \end{vmatrix}$$

a_{ij} -

→ Row operation → $a_{rj} = \min(a_{ij+1}, a_{ij+2} --)$

→ Colum op.

Example: Five jobs and five man cost is as follows

Cost matrix

	V	W	X	Y	Z
A	3	5	10	15	8
B	4	7	15	18	8
C	8	12	20	20	12
D	5	5	8	10	6
E	10	10	15	25	10

$\rightarrow$

$$\begin{matrix} 0 & 2 & 7 & 12 & 5 \\ 0 & 3 & 11 & 14 & 4 \\ 0 & 4 & 12 & 12 & 4 \\ 0 & 0 & 3 & 5 & 1 \\ 0 & 0 & 5 & 15 & 0 \end{matrix}$$

$\downarrow$

$$\begin{matrix} 0 & 2 & 4 & 7 & 5 \\ 0 & 3 & 8 & 9 & 4 \\ 0 & 4 & 9 & 7 & 4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 10 & 0 \end{matrix}$$

```
While (#lines < n){
    temp = min(ai) | ai ∈ Uncovered element
    ai -= temp;
    Intersection += temp;
}
```

Transportation Problem

Example: For the following problem, obtain the different starting solution by adapting

i) the North-west Corner method

ii) the vogel's approximation method

iii) The matrix minima method (least cost entry method)

	D_1	D_2	D_3	
O_1	5	1	8	12
O_2	2	4	0	14
O_3	3	6	7	4
	9	10	11	

By VAM:

$\rightarrow$ Taking the (i, j) with min cost use all min(bj, ai)

This will give BFS not Optimal Soln.

Theorem: The no. of basic variables in Transportation Problem is at most $(m+n-1)$