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Two phase Method

Example: Use two phase method to solve the following:

$$\text{Max } Z = 3x_1 - x_2$$

$$\text{s.t. } 2x_1 + x_2 \geq 2$$

$$x_1 + 3x_2 \leq 2$$

$$x_1 \leq 4$$

$$x_1, x_2 \geq 0$$

Std. prb. introducing slack, surplus & artificial variables:

$$\text{Max } z = 3x_1 - x_2$$

$$\text{s.t. } 2x_1 + x_2 - x_3 + x_6 = 2$$

$$x_1 + 3x_2 + x_4 = 2$$

$$x_1 + x_5 = 4$$

$$x_i \geq 0 \quad i = 1 \text{ to } 6.$$

Soln: The auxiliary objective func. to be maximize is:

Phase I: $\text{Max } Z^* = 0.x_1 + 0.x_2 + 0.x_3 + 0.x_4 + 0.x_5 - 1x_6$

		c_j		0	0	0	0	0	-1	Min Ratio
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	a_6	
-1	a_6	x_6	2	$\boxed{2}$	1	-1	0	0	1	$\boxed{2/2}$ →
0	a_4	x_4	2	1	3	0	1	0	0	2/1
0	a_5	x_5	4	1	0	0	0	1	0	4/1
		$z_j - c_j$		-2 ↑	-1	1	0	0	0 ↓	

0	a_1	x_1	1	1	1/2	-1/2	0	0	1/2
0	a_4	x_4	1	0	5/2	1/2	1	0	-1/2
0	a_5	x_5	3	0	-1/2	1/2	0	1	-1/2
	$z_j - c_j$			0	0	0	0	0	0

we will consider this table in phase two whith Z (basis, slack, surplu) (No artifici)

Cases :-

- $Z^*_{max} = 0$
  - → no artificial vetors in the final basic yeilding a b.f.s.
  - → one or more artificial vectors appear in final basis at 0 level, → b.f.s.
- $Z^*_{max} < 0$, one or more artificial variables ~~occur~~ appear in the final basis of positive level, yielding no f.s. of the original problem.

Phase II :

C_B	B	x_B	c_j / b	3 / a_1	-1 / a_2	0 / a_3	0 / a_4	0 / a_5	Min Ratio
3	a_1	x_1	1	1	1/2	-1/2	0	0	
0	a_4	x_4	1	0	5/2	$\boxed{1/2}$	1	0	$1/\frac{1}{2} = 2$
0	a_5	x_5	3	0	-1/2	1/2	0	1	$3/\frac{1}{2} = 6$
	$z_j - c_j$			0	5/2	-3/2 ↑	0 ↓	0	
3	a_1	x_1	2	1	3	0	1	0	
0	a_3	x_3	2	0	5	1	2	0	
0	a_5	x_5	2	0	-3	0	-1	1	
	$z_j - c_j$			0	10	0	3	0	

optimization reached.

Optimal soln

$x_1 = 2, \; x_2 = 0$

$Z_{max} = 6 - 0 = 6$

Example : Use two phase method to solve the following LPP:

$$\text{Max } Z = 2x_1 + x_2 + x_3$$

$$\text{s.t. } 4x_1 + 6x_2 + x_3 \le 8$$

$$3x_1 - 6x_2 - 4x_3 \le 1$$

$$2x_1 + 3x_2 - 5x_3 \ge 4$$

$$x_1, x_2, x_3 \ge 0$$

Std form

$$\text{Max } Z = 2x_1 + x_2 + x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6$$

$$\text{s.t. } 4x_1 + 6x_2 + x_3 + x_4 = 8$$

$$3x_1 - 6x_2 - 4x_3 + x_5 = 1$$

$$2x_1 + 3x_2 - 5x_3 - x_6 + x_7 = 4$$

$$x_1, x_2 \ldots x_7 \ge 0$$

Soln : The auxiliary objective func. to be maximiz

Phase I : $\text{Max } Z^* = 0 \cdot x_1 + 0 \cdot x_2 + \ldots 0 \cdot x_6 - 1 \cdot x_7$

		c_j		0	0	0	0	0	0	-1	Min Ratio
C_B	B	X_B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	
0	a_4	x_4	8	4	$\boxed{6}$	1	1	0	0	0	$\frac{8}{6} \rightarrow$
0	a_5	x_5	1	3	-6	-4	0	1	0	0	$\frac{4}{3}$
-1	a_7	x_7	4	2	3	-5	0	0	-1	1	
		$z_j - c_j$		-2	-3 $\uparrow$	5	0	0	1	0	

0	a_2	x_2	4/3	2/3	1	1/6	1/6	0	0	0
0	a_5	x_5	9	7	0	-3	1	1	0	0
-1	a_7	x_7	0	0	0	-11/2	-1/2	0	-1	1
	$z_j - c_j$			0	0	11/2	1/2	0	0	0

Phase II :

			c_j	2	1	1	0	0	0	0	Min Ratio
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	
0	a_2	x_2	4/3	2/3	1	1/6	1/6	0	0	0	$4/2 = 2$
0	a_5	x_5	9	7	0	-3	1	1	0	0	$9/7 = 1.1$
-1	a_7	x_7	0	0	0	-11/2	-1/2	0	-1	1	
	$z_j - c_j$			-4/3 ↑	0	1/6	1/6	0 ↓	0	0	
0	a_2	x_2	10/21	0	1			0		0	
2	a_1	x_1	9/7	1	0			1/7		0	
-1	a_7	x_7	0	0	0			0		1	
	$z_j - c_j$			0	0			4/21		0	

optimality reached:

optimal soln : $x_1 = 9/7$ $\quad x_2 = \quad x_3 = 0$

$$Z_{max} = \frac{64}{21}$$

H.W. Max $Z = 5x_1 + 3x_2$

s.t. $3x_1 + x_2 \leq 1$

$3x_1 + 4x_2 \geq 12$

x_1, x_2

Show that there is no feasible soln.

Artificial variable appear in phase I final table (optimality condition reached) at positive level.

(iii) Max $Z = 2x_1 + 3x_2 + x_3$

s.t. $-3x_1 + 2x_2 + 3x_3 = 8$

$-3x_1 + 4x_2 + 2x_3 = 7$

$x_1, x_2, x_3 \geq 0$

Phase I: No artificial variable. after Phase I

Pass to Phase II.

(Unbounded soln) ⟶ min ratio cannot be determined.

departing vector cannot be determined.

Mam may ask problems like solving a system of linear eqn's, finding the inverse. (rather than optimization)

Example: Solve the following system of linear Simultaneous eqn using Simplex Method

$x_1 + x_2 = 1$

$2x_1 + x_2 = 4$

x_1, x_2 unrestricted in sign

let $x_1 = x_1' - x_1''$

$x_2 = x_2' - x_2''$

Soln Reduced problem.

$x_1', x_2', x_1'', x_2'' \geq 0$

dummy objective fun

Max $Z = 0.x_1' + 0.x_1'' + 0.x_2' + 0.x_2'' - 1x_3 - 1x_4$

s.t. $x_1' - x_1'' + x_2' - x_2'' + x_3 = 1$

$2x_1' - 2x_1'' + x_2' - x_2'' + x_4 = 4$

$Z = 0.x - 1x_a$

Apply Phase I.

$\boxed{Ax = b}$ $A_{m \times n}$

$X_{n \times 1}$

$b_{m \times 1}$

Apply Phase I using dummy obj. func:

$Z = 0 \cdot x - 1 \cdot x_a$, $x_a \rightarrow$ vector giving the artificial variables.

$x \rightarrow$ original variable

s.t. $A(x_r' - x_r'') + x_a = b$

where $x_r = x_r' - x_r''$, $r = 1, 2 .. n$

$x_r', x_r'' \geq 0$

<u>Example</u>: Use simplex method to obtain the inverse of the matrix $\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}$

<u>Sol^n</u>: <u>Reduced Problem</u>

Dummy objective funct^n:

Max $Z = 0 \cdot x_1 + 0 \cdot x_2 - 1 x_3 - 1 x_4$

$\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ how to choose this

$\hookrightarrow$ dummy column

$x_1, x_2 \geq 0$

s.t. $4x_1 + 2x_2 + x_3 = 6$

$x_1 + 5x_2 + x_4 = 4$

:(

Phase I will give the sol^n, Phase II not needed.

Reduced problem for phase I (A) To find A^{-1}

Max $z = 0 \cdot x - 1 \cdot x_a$ (dummy objective fun^c)

s.t. $Ax + I x_a = b, \quad x, x_a \geq 0$

$b \rightarrow$ dummy column vector.

			c_j	0	0	-1	-1	Min Ratio
C_B	B	X_B	b	a_1	a_2	a_3	a_4	
-1	a_3	x_3	6	4	2	1	0	$6/2 = 3$
1	a_4	x_4	4	1	5	0	1	$4/5 \rightarrow$
		$z_j - c_j$		-5	-7 ↑	0	0	
-1	a_3	x_3	$22/5$	$18/5$	0	1	$-2/5$	$22/18 \rightarrow$
0	a_2	x_2	$4/5$	$1/5$	1	0	$1/5$	$4/1$
		$z_j - c_j$		$-18/5$ ↑	0	0	Removed	
0	a_1	x_1	$11/9$	1	0	$5/18$	$-1/9$	
0	a_2	x_2	$5/9$	0	1	$-1/18$	$2/9$	
		$z_j - c_j$		0	0	0	0	

Inverse

To show: ??

$$\mathcal{B} = (\beta_1 . \beta_2 \ldots \beta_m)$$

If $a_j \in B$ say $a_j = \beta_i$ for $i = 1 \ldots m$

$$a_j = \beta_i = 0\beta_1 + 0 \cdot \beta_2 + \ldots 0 \beta_{i-1} + 1 \beta_i + 0 \cdot \beta_{i+1} \ldots 0 \cdot \beta_m$$

$$= B \cdot Y_j \text{ when } Y_j = (0, 0 \ldots \overset{i^{th}}{1}, 0, \ldots 0)$$

$$= e_i$$

Also $q_j = \beta_j \Rightarrow c_j = c_B$

a_1 is L.C. of y_{11} to y_{m1}

$$\begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & & & \vdots \\ y_{31} & & & \vdots \\ \vdots & & & \vdots \\ y_{m1} & \cdots & \cdots & y_{mn} \end{array}$$

$$\begin{aligned} z_j - c_j &= c_B y_j - c_j \\ &= c_B(e_i) - c_j \\ &= c_{B_i} - c_j \\ &= c_j - c_j = 0 \end{aligned}$$

Proved!

These are always 0, 0.. is proved above.

Optimality condition:

for maximization, $z_j - c_j \geq 0$

Rank of A $= m$ (non-degeneracy Case)

$$\begin{cases} z_j - c_j > 0 & \forall \text{ non-basic variables} \\ z_j - c_j = 0 & \forall \text{ basic variable} \end{cases}$$

Theorem: (Unboundedness)

If at any iteration of the simplex algo, we get (a_j entering vector) $\boxed{z_j - c_j < 0 \text{ for at least one } j}$ and for this j, $y_{ij} < 0 \ \forall \ i = 1, 2 \ldots m$, then the LPP admits of a unbounded solⁿ in maximization problem.

(having trouble in determining the min ratio i.e. the departing vector)

$\longrightarrow \quad B = (\beta_1, \beta_2 \ldots, \beta_m)$

$x_B, \quad Z_B = C_B x_B$ — ①

$B x_B = b \Rightarrow \boxed{\sum_{1}^{m} x_{Bi} \cdot \beta_i = b}$ — ②

(x_{B1}, x_{B2}, x_{Bm})

$Ax = b$

$(B|R)\left(\frac{x_B}{x_R}\right) = b$, $\quad x_R = 0$

$$\sum_{1}^{m} x_{Bi}\beta_i + \theta a_j - \theta a_j = b \quad \text{— ③}$$

$\theta > 0$

$$a_j = \sum_{i=1}^{m} y_{ij}\beta_i$$

l.c. of basis vector

$$\sum_{i=1}^{m} x_{Bi}\beta_i + \theta a_j - \theta \sum_{i=1}^{m} y_{ij}\beta_i = b$$

$$\sum_{i=1}^{m} (x_{Bi} - \theta y_{ij})\beta_i + \theta a_j = b$$

Another soln : $\begin{cases} x_{Bi} - \theta y_{ij} > 0 & i = 1, 2 \ldots m \\ \theta > 0 \end{cases}$

m+1 non-zero variables

$$z' = \sum_{i=1}^{m} C_{Bi}(x_{Bi} - \theta y_{ij}) + c_j\theta$$

$$= \sum C_{Bi} \cdot x_{Bi} - \theta \sum_{i=1}^{m} C_{Bi} y_{ij} + c_j\theta$$

$$= Z_B - \theta(z_j - c_j)$$

You can always construct (under these conditions) ~~two~~ such case . . . ∴ the soln goes unbounded.