Lecture-14 (04-03-2024)

Countability :

A topological space (X, T) is said to be first countable topological space if it satisfies the following:

" For each point $p \in X$, there exists a countable class $\mathcal{B}_p$ of open sets containing p such that every open set G containing p also contains a member of $\mathcal{B}_p$ ".

That is (X,T) is first countable if there exists a countable local base at every point $p \in X$.

Ex: Let (X, d) be a metric space and $p \in X$. Then $Z = \{S(p, \frac{1}{n}) \mid n \in \mathbb{N}\}$ is a countable local base at $p \in X$.

$\therefore$ (X, d) is a first countable topological space.

Ex: Let (X, D) be a discrete topological space. Now the singleton set $\{p\}$ is open in X for every $p \in X$. So if G is any open set containing p, then $p \in \{p\} \subset G$.

So (X, D) is a first countable topological space.

Theorem: A function f defined on a first countable topological space (X, T) is continuous iff it is sequentially continuous at $p \in X$.

Proof: We know that if f is continuous at $p \in X$, then f is sequentially continuous at $p \in X$.

Conversely suppose f is sequentially continuous at P in a first countable topological space (X, T).

[$\because (X, T)$ is first countable topological space, let $B_p = \{G_1, G_2, G_3, \ldots\}$ be a countable local base at $P \in X$.

let $B_1 := G_1$, $B_2 := G_1 \cap G_2$, $B_3 := G_1 \cap G_2 \cap G_3$

$\ldots\ldots B_n = \bigcap_{i=1}^{n} G_i, \ldots\ldots$

Then $B_1 \supseteq B_2 \supseteq B_3 \supseteq B_4 \ldots \supseteq B_n \supseteq \ldots$

and each B_K containing $P \in X$. [$\because P \in G_i \; \forall i$]

Further if G is any open set containing P, $\exists \; n_0 \in \mathbb{N} \ni$

$$P \subset G_{n_0} \subset G.$$

$\because P \in B_{n_0} \subset G_{n_0} \subset G$

$\Rightarrow P \in B_{n_0} \subset G$.

Then $\{B_1, B_2, B_3 \dots \dots\}$ is also a countable local base at $P \in X$.

$\because B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots \supseteq B_n \supseteq \dots$, implies $\{B_1, B_2, B_3 \dots\}$ is a nested countable local base at $P \in X$.

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Now suppose $f:(X,\tau) \to (Y,\tau^*)$ is not continuous at $P \in X$, where (Y,τ^*) is any topological space. Then there exists an open subset H of Y such that

not open

$f(P) \in H$, but $B_n \not\subset f^{-1}(H)$, $\forall n \in \mathbb{N}$.

So for every $n \in \mathbb{N}$, $\exists\ a_n \in B_n$ such that $a_n \notin f^{-1}(H)$.

$\therefore$ $a_n \in B_n$, $n = 1, 2, 3, \ldots$

and $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots \supseteq B_n \supseteq \cdots$

implies that the sequence $a_n \to p$.

But $f(a_n) \not\to f(p)$,

because $a_n \notin f^{-1}(H)$, $\forall n$

$\Rightarrow$ $f(a_n) \notin H$, $\forall n$ and $f(p) \in H$.

$\Rightarrow$ f is not sequentially continuous at $p \in X$, which is contradiction.

$\therefore$ Our assumption is wrong.

Hence f is continuous at $p \in X$.

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Problem:(3) Let $B_p = \{G_1, G_2, G_3, \ldots\}$ be a countable local base at $p \in X$. Show that there exists a nested local base at $p \in X$.

(2) Let $B_p = \{B_1, B_2, \ldots\}$ be a nested local base at $p \in X$ and $\{a_1, a_2, a_3, \ldots\}$ be a sequence such that $a_i \in B_i$, $\forall i = 1, 2, \ldots$; Then show that $a_n \longrightarrow p$.

Second Countable topological space :—

A topological space (X, T) is said to be a second countable topological space if there exists a countable base B for the topology T on X.

Ex: let (R, U) be usual topological space. Then the class

$$B = \{ (a, b) \mid a, b \in Q \}$$ is

a countable base for U on R.

$\therefore$ (R, U) is a second countable topological space.

Attendance [11, 55, 32, 35, 06, 60, 58, 57, 10, 43, 07, 17, 16, 19, 63, 23].

Ex (2) Let (R, D) be a discrete topological space.

If B is a base for D, then B containing all the singleton sets $\{\{p\} \mid p \in R\}$, which is uncountable.

$\therefore (R, D)$ is not a second countable topological space.

Note :— A second countable topological space is also a first countable topological space.

Because if B is a countable base for the topology T on X and if B_p consists of members of B which contain $p \in X$, then B_p is a countable local base at $p \in X$.

i.e., X is also a first countable topological space.

Converse of above need not be true since $(\mathbb{R}, \mathcal{D})$ is a first countable space, but not a second countable space.

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Let A be a subset of a topological space $(X, \mathcal{T})$ and $\mathcal{A}$ be a class of subsets of X such that

$$A \subseteq \cup \{ E \mid E \in \mathcal{A} \}.$$

$$\left(A \subseteq \bigcup_{E \in \mathcal{A}} E \right)$$

Then $\mathcal{A}$ is called cover of the set A. If each member of $\mathcal{A}$ is an open subset of X, then $\mathcal{A}$ is called an open cover of A.

Furthermore, if $\mathcal{A}$ contains a countable (finite) subclass which also covers A, then $\mathcal{A}$ is said to be reducible to a countable (finite) cover or $\mathcal{A}$ is said to contain a countable (finite) subcover.

[$(\mathbb{R}, U)$

$\mathcal{A} = \{(-\frac{1}{n}, \frac{1}{n}) / n \in \mathbb{N}\}$

$A = \{0\}$

$$A \subset \bigcup_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$$

$$A \subset \left(-\frac{1}{n_0}, \frac{1}{n_0}\right)$$

$A = (2, 3)$ $\qquad \mathcal{A} = \{(2-\frac{1}{n}, 3+\frac{1}{n}) / n \in \mathbb{N}\}$

$$(0, 1) \subseteq \bigcup_{n \in \mathbb{N}} \left(0, \frac{1}{n}\right).$$]

Theorem: (Lindelöf): Let A be any subset of a second countable topological space (X, T). If $\mathcal{S}$ is an open cover of the set A, then $\mathcal{S}$ is reducible to a countable cover.

Proof: let (X, T) be a second countable topological space and $\mathcal{B}$ be a countable base for T on X.

Given that

$$A \subseteq \cup\{G \mid G \in \Sigma\}$$

and each $G \in \Sigma$ is an open set.

So for any $p \in A$, there exists $G_p \in \Sigma$ such that $p \in G_p$.

Since G_p is an open set containing p and $\mathcal{B}$ is a base for (X, T), so there exists a member $B_p \in \mathcal{B}$ such that

$$p \in B_p \subset G_p.$$

Then

$$A \subseteq \cup\{B_p \mid p \in A, B_p \in \mathcal{B}\}$$
$$\subseteq \{G_p \mid G_p \in \Sigma\}.$$

Since $\mathcal{B}$ is countable and $\{B_p \mid p \in A\}$ is subset of $\mathcal{B}$ implies $\{B_p \mid p \in A\}$ is also countable. So we can write

$$\{B_p \mid p \in A\} = \{B_n \mid n \in I\},$$

where I is the index set, which is countable.

Now for each $n \in \mathbb{N}$, choose an open set $G_n \in \mathcal{S}$ such that $B_n \subseteq G_n$.

$\therefore$

$$A \subseteq \bigcup\{B_n \mid n \in I\} \subseteq \bigcup\{G_n \mid n \in I\}$$

$$\Rightarrow A \subseteq \bigcup\{G_n \mid n \in I\}$$

$\Rightarrow \{G_n \mid n \in I\}$ is a countable open cover of A.

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Theorem (Lindelöf) Let S be a base for a second countable topological space (X, T). Then S is reducible to a countable base for T on X.

Proof: Given that (X, T) is a second countable topological space. So (X, T) has a countable base say $\mathcal{B} = \{B_n \mid n \in I\}$, where I is the index set.

Also given that S is a base for (X, T).

Since each $B_n \in \mathcal{B}$ is an open set, by the definition of a base, we have $B_n = \cup \{G / G \in S_n\}$, where $S_n \subseteq S$.

This implies that S_n is an open cover for B_n. So by the previous theorem, S_n is reducible to a countable cover S_n^* for B_n.

That is

$$B_n \subseteq \bigcup \{G \mid G \in S_n^*\} \text{ and } S_n^*$$

is a countable set.

Now let

$$S^* = \{S_1^*, S_2^*, S_3^*, \dots, S_n^*, \dots\}$$

$$= \{G \mid G \in S_n^*, n \in I\}$$

Then S^* is a base for T on X, since B is a base and S^* is countable —//—

Attendance

[11, 62, 42, 40, 32, 06, 57, 60].