GRAPH THEORY

* Books

Mostly used & exercise

easy to understand

1 Intro to Graph theory by D. B. West

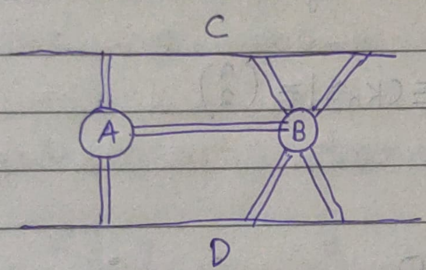
2 " " by G. Chartrand & P. Zhang

3 Graph Theory with appl. to Engg & CS by N. Deo

4 Graph theory by Bondy & USR Murty

* Origin of Graph theory

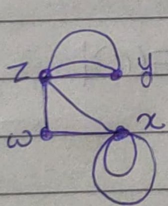
Konigsberg 7 bridges problem



Euler in 1736

* Representation

1

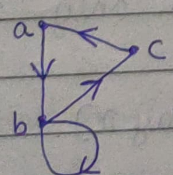


$V = \{x, y, z, w\}$ → vertex set

$E = \{\{x\}, \{x\}, \{x, w\}, \{x, z\}, \{w, z\}$
$\{z, y\}, \{z, y\}\}$ → edge multiset
(repetitions allowed)

E is set of subsets of V with cardinality 1 or 2

2 Directed Graphs



$V = \{a, b, c\}$

$E = \{(a, b), (b), (b, c), (c, a)\}$

↑ ordered pair

* Undirected Graph

G is pair $G = (V, E)$ where G is non empty set and E is a multiset of 1-subsets and 2-subsets of V.

Elements of V are called vertices and E are edges of G

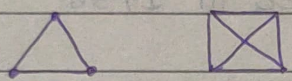
* Simple Graph

Graph free from self loops & multiple edges

Simple graph having n vertices have at max $^nC_2$

* $l = \{x, y\} \in E(G)$ then we say x & y are adjacent or neighbours. We also say l is incident on x & y.

* Complete graph means simple graph of n vertices and $^nC_2$ edges. They are denoted by K_n

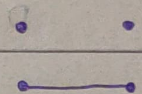


K_3 K_4

$|E(K_n)| = \binom{n}{2}$

Isolated vertex have degree 0.

Pendant vertex have degree 1



* 1^{st} theorem of GT

For every graph G, $\sum_{x \in V(G)} \deg x = 2|E(G)|$

* Two ways counting

All ordered pairs (x, l) where $x \in V(G)$ & $l \in E(G)$ incident on x.

Way 1 : fix x and count $l_1, l_2, \ldots$ incident on x. It is basically counting degrees

Way 2 : fix l and count x, y as only 2 vertex for given edge.

Then we obtain #(Way 1) = #(Way 2)

* Corollary In every graph there are even no. of vertices with odd degree

Proof

$V(G) \to V_1$: all odd degree vertices

$V(G) \to V_2$: all even degree vertices

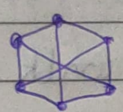
To prove $|V_1| = 2k$, $k \in \mathbb{Z}$

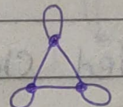
$$\sum_{x \in V_1} \deg x + \underbrace{\sum_{x \in V_2} \deg x}_{\text{each degree is even for vertex of } V_2} = \underbrace{2e}_{\text{even no. on RHS}}, \quad e = |E(G)|$$

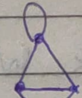
So $\sum_{x \in V_1} \deg x$ is even integer But $\deg x_i$ is odd as

$x_i \in V_1$ so in total there would be even no. of odd terms
Hence even no. of v vertices of odd degree.

* Regular graph degree of all vertices is same


3-regular


4-regular


non regular

Every complete graph is regular with degree $= n-1$
So Every K_n is $(n-1)$ regular graph

* Compliment of Graph
Let G be simple graph. Then $\overline{G}$ or G^c is complement of G if $V(\overline{G}) = V(G)$ and $\{x, y\} \in E(\overline{G})$ iff $\{x, y\} \notin E(G)$
So basically just flipping — if edge then remove & if not edge then add edge

- If G is regular then $\overline{G}$ is also regular. More, if G has n vertices and is d-regular then $\overline{G}$ is $(n-d-1)$ regular.
- Complement of Complete graph is fully isolated vertices

$$\overline{K_n} = \begin{matrix} & \cdot & \cdot & \\ \cdot & \cdot & & \cdot \end{matrix} = \text{0-regular graph}$$

* Subgraphs
H is subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

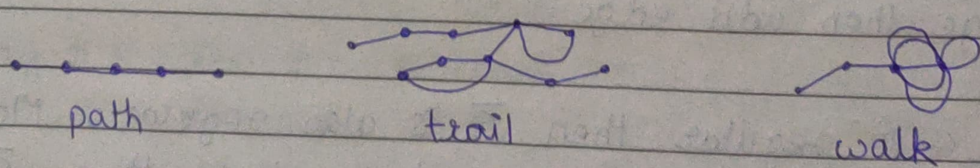
* Induced subgraph
H is induced subgraph of G if H contains all edges of G whose whose end vertices (both) are in $V(H)$

* Walk (W)
Let G be graph and $u, v \in V(G)$. Then $u-v$ walk in G is alternating sequence $u - e_1 - u_1 - e_2 - u_2 - \cdots u_{k-1} - e_k - v$ where $u_i \in V(G)$ and $e_i \in E(G)$ and $e_i = \{u_{i-1}, u_i\}$
Neither vertices nor edges need to be distinct

If $u = v$ then walk W is called Closed walk else it is called Open walk

* Trail
In a walk W if all the edges are distinct then it is called trail. However vertices may repeat

* Path
In a walk if all vertices are distinct then it is path
Since we have distinct vertices, edges will also be distinct



Path on n vertices is denoted by P_n

* Cycle
Closed walk in which all vertices are distinct except the end vertices.
C_n = cycle with n vertices

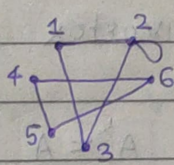
* **Length of Walk**

Number of edges present in W with counting multiplicity ie if an edge l appears k times in W then it is counted for k times

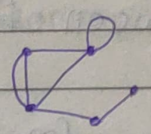
- Every u-v walk contains u-v path and u-v trail also (as every path is a trail)

* **Connected graph**

For every pair of distinct vertices x & y, there is x-y path in G then its called connected else disconnected



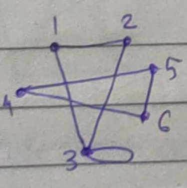
Disconnected



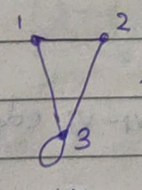
Connected

* **Maximal connected subgraph**

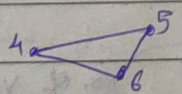
It is connected component (or simply component) of G



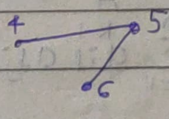
G



H_1



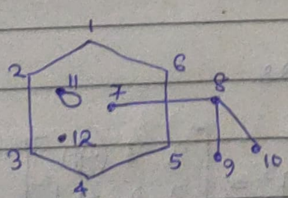
H_2



H_3

H_1, H_2 are connected component of G but not H_3

Eg



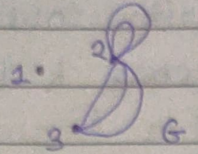
G

G has 4 connected components

* **Adjacency matrix of G**

A(G) is $n \times n$ matrix where row & columns indexed by vertices of G. A_{ij} = no. of edges between i^{th} & j^{th} vertex

- For undirected graph, $A(G)$ is symmetric matrix

- 

$$A(G) = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 3 \\ 3 & 0 & 3 & 0 \end{array}$$

- If we do different ordering of vertices we get other matrix B ~~A~~(G) which is similar to $A(G)$
  Similar Matrix : A, B if $A = P^{-1}BP$ for invertible P

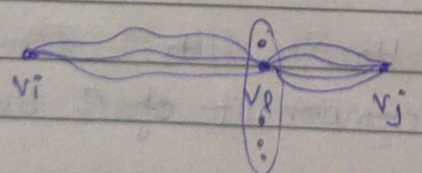
- For simple graph A_{ij} is either 0 or 1

- A^k_{ij} = no. of walks between i^{th} & j^{th} vertex of length k

Proof : Induction on k, for $k=1$, $A^k = A$
Assume result is true for A^{k-1}
Let $A = (a_{ij})$ and $A^k = (a^k_{ij})$
To prove a^k_{ij} is no. of k length $v_i - v_j$ walks

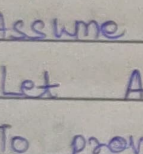
$$A^k = A^{k-1} A$$
$$a^k_{ij} = \sum_{l=1}^{n} a^{k-1}_{il}\, a_{lj}$$

no. of $v_l - v_j$ edges



where $l = 1, \dots, n$

Eg find 3 length $v_1 - v_3$ walks



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

ans = 4

$v_1 - v_2 - v_1 - v_3$ $\qquad$ $v_1 - v_3 - v_4 - v_3$

$v_1 - v_3 - v_2 - v_3$ $\qquad$ $v_1 - v_3 - v_1 - v_3$

* Incidence matrix

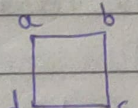
G be loop free graph with $|V(G)| = n$ and $|E(G)| = m$
$I(G)$ is $n \times m$ matrix which is row indexed by vertices and column indexed by edges and

$I_{ij} = 1$ if i^{th} vertex is end vertex of j^{th} edge
$= 0$ otherwise

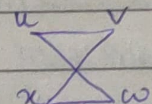
* Isomorphic Graphs

G_1 and G_2 are isomorphic if there exists bijection mapping $f: V(G_1) \to V(G_2)$ such that $\{x, y\} \in E(G_1)$ iff $\{f(x), f(y)\} \in E(G_2)$ ie it preserves adjacency
This f is called isomorphism between G_1 & G_2.

Eg

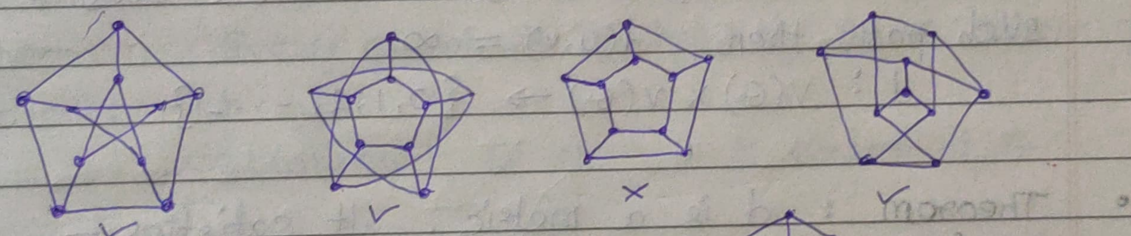


G_1

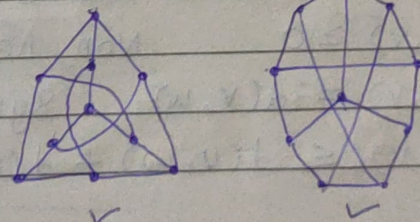


G_2

$f: \{a, b, c, d\} \to \{u, v, x, w\}$
$f(a) = u$, $f(b) = v$, $f(c) = x$, $f(d) = w$



Peterson graphs



* Girth

length of smallest cycle present in graph.
Girth of Peterson graph is 5

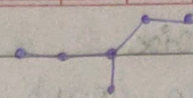
* Degree sequence

Degree sequence of graph G is non increasing seq. of degree of vertices in G

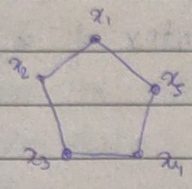
~~Pf~~ ~~$|V(G)| = n$~~ ~~then~~ ~~we~~ ~~write~~

Eg

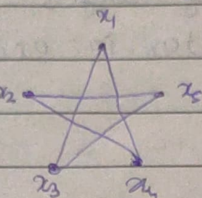
(3, 2, 2, 1, 1, 1)



(3, 2, 2, 1, 1, 1)



C_5



$\overline{C_5}$

$C_5 \simeq \overline{C_5}$
↑
isomorphic

* Self Complementary Graph

$G \simeq \overline{G}$. ie graph is isomorphic to its compliment

* Distance in graph

distance between u & v denoted by $d(u,v)$ is length of shortest u-v path in G. If there doesn't exist such path then $d(u,v) = \infty$.

$$d : V(G) \times V(G) \to \{0, 1, 2, \ldots\}$$

- Theorem : d is a metric. It satisfies –
  1) $d(u,v) \geq 0$ Non negativity
  2) $d(u,v) = d(v,u)$ Symmetry
  3) $d(u,v) \leq d(u,w) + d(w,v)$ Triangle Inequality

* Eccentricity of vertex

$e(v) = \max \{ d(v,x) : x \in V(G) \}$

* Radius of graph

$rad(G) = \min \{ e(v) \mid v \in V(G) \}$

* Diameter of graph

$diam(G) = \max \{ e(v) \mid v \in V(G) \}$

- **Theorem** $rad(G) \le diam(G) \le 2\,rad(G)$

first inequality is obvious we will prove second inequality

let $diam(G) = d(u,v)$ and $e(w) = rad(G)$

By triangle inequality,
$$d(u,v) \le d(u,w) + d(w,v)$$
$$diam(G) \le e(w) + e(w)$$
$$diam(G) \le 2rad(G)$$

* **Central vertices & Center of graph**

Vertices with minimum eccentricity are central vertices

Subgraph induced by all central vertices is center of G

* **Peripheral vertices & Periphery**

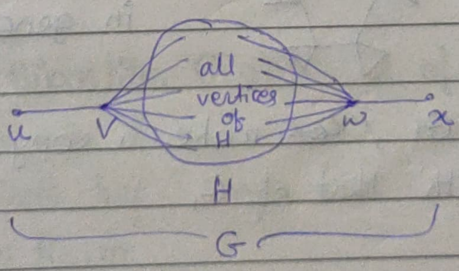
vertices with maximum eccentrity are peripheral vertices

Subgraph induced by them is periphery of G

- **Theorem** Every graph is center of some other graph

Proof: let given H

we construct G st. H is center of G



$V(G) = V(H) \cup \{u, v, w, x\}$

$E(G) = E(H) \cup \{\{u,v\}, \{w,x\}\}$
$\cup \{\{v,z\}, \{w,z\} \mid z \in V(H)\}$

$e(u) = d(u,x) = 4$ $(u-v-z-w-x) = e(x)$

$e(v) = 3$ $(v-z-w-x) = e(w)$

$e(z) = 2 \quad \forall\, z \in H$

Hence center of G is H

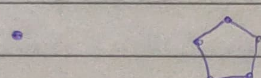
* Operations on Graph

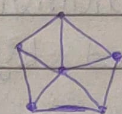
Let $G_1(V_1, E_1)$ & $G_2(V_2, E_2)$

1) Union : $G_1 \cup G_2 = G(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$
Note G_1, G_2 need not be disjoint. If $V_1 \cap V_2 = \phi$ and $E_1 \cap E_2 = \phi$ then its called disjoint union

2) Join : $G_1 \vee G_2 = G(V_1 \cup V_2, E_1 \cup E_2 \cup \{\{x,y\} \mid x \in V_1, y \in V_2\})$
Its like disjoint union & then join each vertex from G_1 to G_2
Note that $V_1 \cap V_2 = \phi$ is needed

$K_1 \vee C_5 =$





& W_6 wheel graph

In general $K_1 \vee C_n = W_{n+1}$

3) Corona : $G_1 \circ G_2 = G$
$V(G) = V(G_1) \cup |V(G_1)| * V(G_2)$

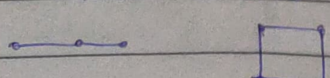
$C_5 \circ K_2 =$

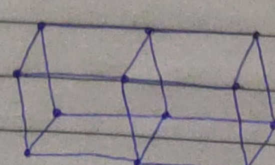
Note $G_1 \circ G_2 \neq G_2 \circ G_1$ in general

For every vertex of G_1, make a copy of G_2 and join all vertex of G_2 with that of G_1

4) Cartesian Product : $G_1 \times G_2 = G$
$V(G) = V(G_1) \times V(G_2)$ note $V(G_1) \cap V(G_2) = \phi$
every vertex of G is pair of vertex $\{\{x,y\} \mid x \in G_1, y \in G_2\}$
Two vertex (u_1, v_1) and (u_2, v_2) have edge in G if either $u_1 = u_2$ & $v_1 - v_2$ or $v_1 = v_2$ and $u_1 - u_2$

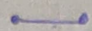
$P_3 \times C_4 =$

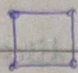


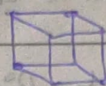
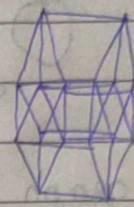


$G_1 \times G_2 = G_2 \times G_1$

* N- dimensional hypercube (Q_n)

$Q_1 = K_2$ 

$Q_2 = Q_1 \times K_2$ 

⋮  

$Q_n = Q_{n-1} \times K_2$

* Construction of graph G_n

$V(G_n) = \{ (a_1\, a_2 \cdots a_n) \mid \text{where } a_i = 0 \text{ or } 1 \}$

two vertices have edge if differ in exactly one place.

Fact: G_n is isomorphic to Q_n

* Bipartite graphs

G can be partitioned as $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$

and ~~$E(G) = \{(x,y) \mid x \in V_1 \;\&\; y \in V_2\}$~~ every edge has end points in two different parts

Every even cycle is bipartite & odd cycle is not bipartite

Complete Bipartite Graph is of form $E(G) = \{(x,y) \mid x \in V_1, y \in V_2\}$

edge between every vertex possible.

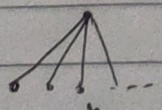
Its denoted by K_{mn} as $|V(G_1)| = m$ & $|V(G_2)| = n$

$K_{m,n} \simeq K_{n,m}$

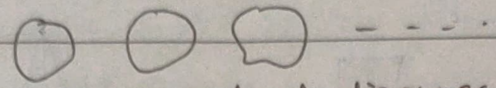
K- Partite graph : $V(G) = V_1 \cup V_2 \cdots V_k$ and $\bigcap_i V_i = \phi$

and no edge between vertices of same group

Complete K-Partite graph : all possible edges

$K_{1,n}$  $K_{1,n}$ is called star graph

Theorem : Disconnected graph is bipartite if all components are bipartite

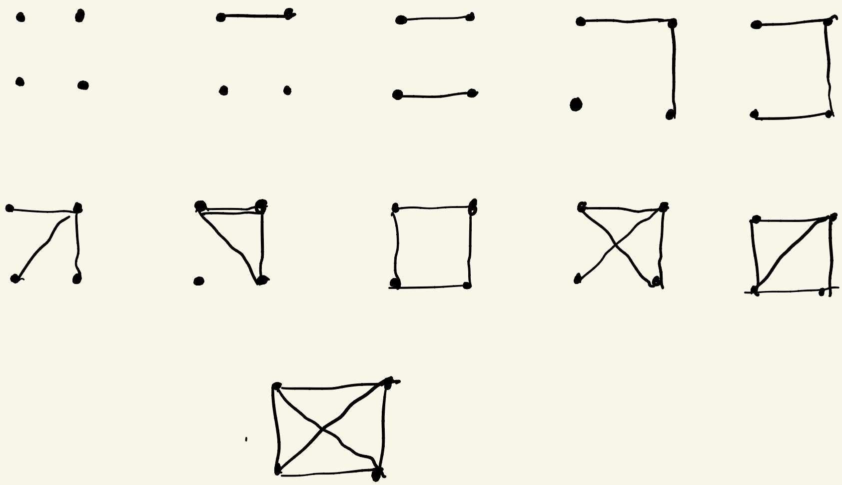


components of disconnected graph

Peterson graph is not Bipartite. All Q_n are bipartite

Tutorial-2

(1) $n = 4$, simple graph



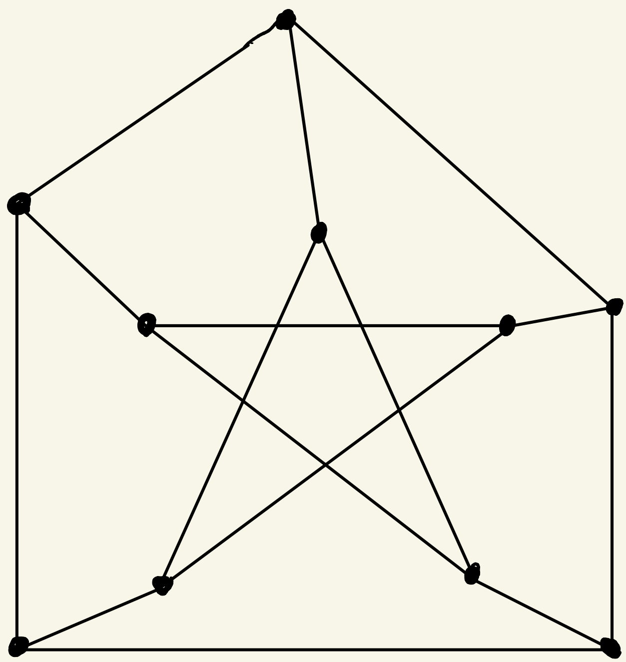
(2) $d_1 \geq d_2 \geq d_3 \geq d_4$ (degree sequences)

0	0	0	0					
1	1	0	0		2	2	2	2
1	1	1	1		3	2	2	1
2	1	1	0		3	3	2	2
2	2	1	1		3	3	3	3
3	1	1	1					
2	2	2	0					

3. $S = \{1, 2, 3, 4, 5\}$

$$V(G) = \{(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$$

petersen graph:



(4)

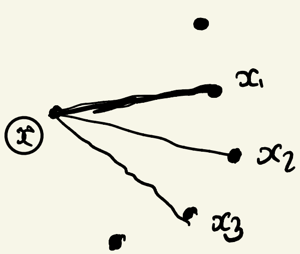
H is a subgraph of G with $E(H) = \phi$, then H is called independent set.

"every simple graph on 6 vertices contain either K_3 or $\overline{K_3}$ as an induced subgraph".

$\Updownarrow$

every simple graph (G) on 6 vertices, either G or $\overline{G}$ will contain K_3 as induced graph.

Let $x \in V(G)$, x has atleast 3 neighbours in G or $\overline{G}$



case 1: x has atleast 3 neig. in G

then x_1, x_2, x_3 might be $\overline{K_3}$

$\overline{G}$ will have K_3

if x_1, x_2, x_3 is not $\overline{K_3}$

edge exists between x_i and x_j

then x, x_i, x_j will form K_3

---

case-2: if $\overline{G}$ has 3 neighbours for x

then if x_1, x_2, x_3 is $\overline{K_3}$

then G will have K_3

else $\overline{G}$ will have K_3

So G or $\overline{G}$ will have K_3 for all simple graphs with 6 vertices

5) $\delta(G) \geq k$

(i) conside a path P of maximum length in G

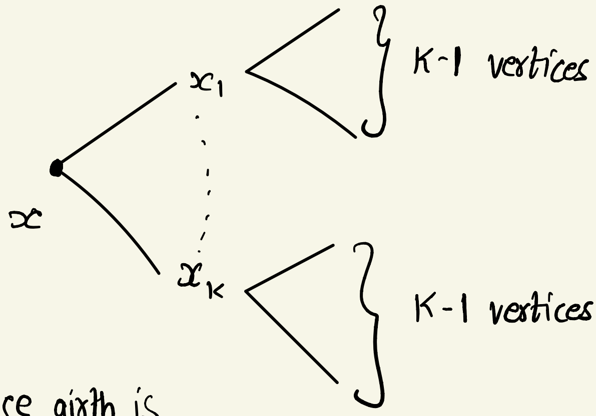
(6) $E(G) + E(\bar{G}) = \frac{n(n-1)}{2}$

$E(G) = E(\bar{G}) \rightarrow$ self complementary

$E(G) = \frac{n(n-1)}{4}$

4 must divide $n / n-1$

(7)



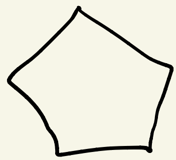
so min. vertices

$= K(K-1) + K + 1$

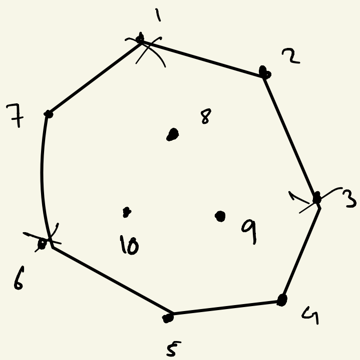
$= K^2 + 1$

Since girth is 5 x_1, x_2 cannot be neigh.

$K = 2$



(8).



vertex from C_7 will have neigh. only in $\{8, 9, 10\} = S$

$\exists\, x \in S$ x has 3 neighbours on C_7

(which is not possible)

10. radius $= 2$

diametre $= 2$

for petersen graph.

$C(G) = G$ then G is self-centred graph.

centre: { set of all vertices with min. eccentricity } ,,

Operations on graph

$$G_1 = (V_1, E_1) \quad , \quad G_2 = (V_2, E_2)$$

(1) Union of G_1, G_2

$$G = G_1 \cup G_2$$

$$V(G) = V(G_1) \cup V(G_2)$$

$$E(G) = E(G_1) \cup E(G_2)$$

Union is disjoint union if

$$V_1 \cap V_2 = \emptyset,$$

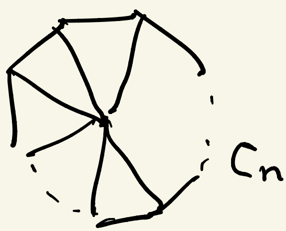
$$E_1 \cap E_2 = \emptyset$$

(2) $G_1 \vee G_2$, when

$$V(G_1 \vee G_2) = V_1 \cup V_2$$

$$E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{(x,y) : x \in V_1, y \in V_2\}$$

$K_1 \vee C_n =$

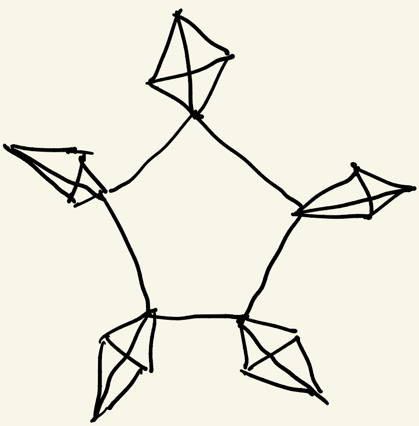


W_{n+1} - wheel of $n+1$

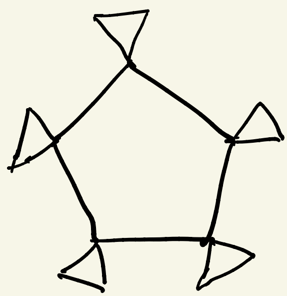
3. Corona of $G_1 \circ G_2$

$G_1 \circ G_2 \neq G_2 \circ G_1$

$C_5 \circ K_3$



$C_5 \circ K_2$



4. $G_1 \times G_2$ / cartesian product:

$$V(G_1 \times G_2) = V(G_1) \times V(G_2)$$

$\{V(G_1) \cap V(G_2) = \phi\}$

two vertices (u_1, v_1), (u_2, v_2)

are adjacent in $G_1 \times G_2$

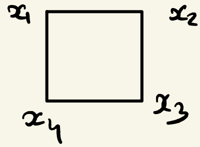
if

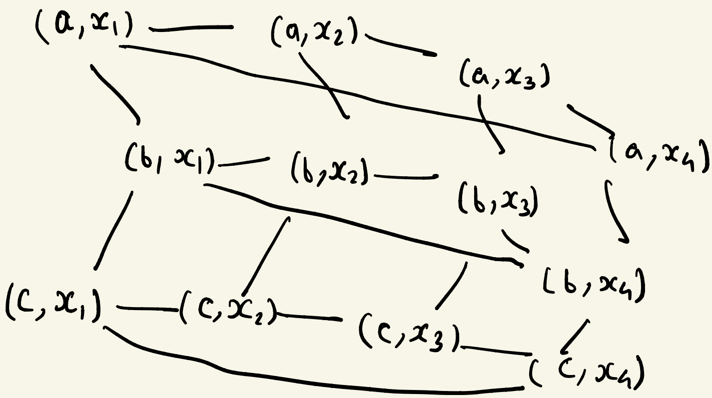
u_1 is adjacent of u_2 & $v_1 = v_2$

v_1 is adjacent of v_2 $u_1 = u_2$

$P_3 \times C_4$:

$V(P_3) = \{a, b, c\}$

$C_4 =$ 



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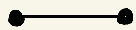
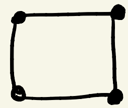
n-dimensional hyper cubes (Q_n)

K - complete graph

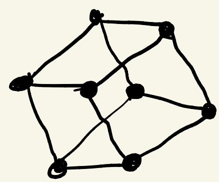
$Q_1 = K_2$ 

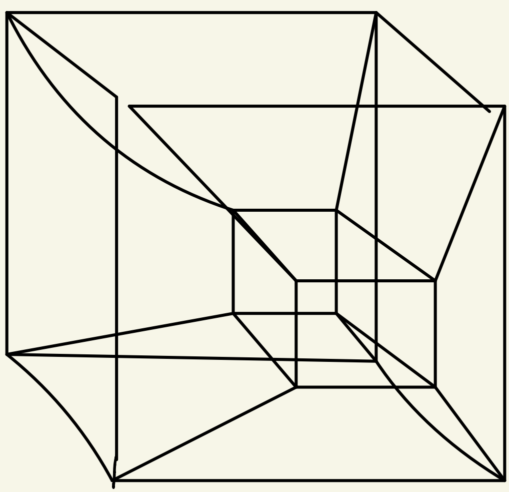
$Q_2 = Q_1 \times K_2$

 $= Q_2$

$Q_3 = Q_2 \times K_2$

$Q_n = Q_{n-1} \times K_2$

 $= Q_3$

 $= Q_4$

construction of a graph G_n

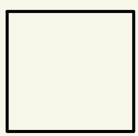
$$V(G_n) = \{(a_1 \dots a_n) : a_i = 0/1\}$$ binary strings of length $= n$

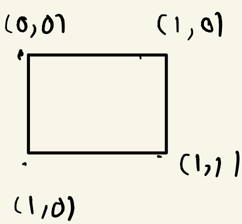
2 nodes are adjacent, if they differ exactly in 1 place.

$1111 \sim 0111$

$1111 \not\sim 0011$

Ex: prove that G_n is isomorphic to Q_n

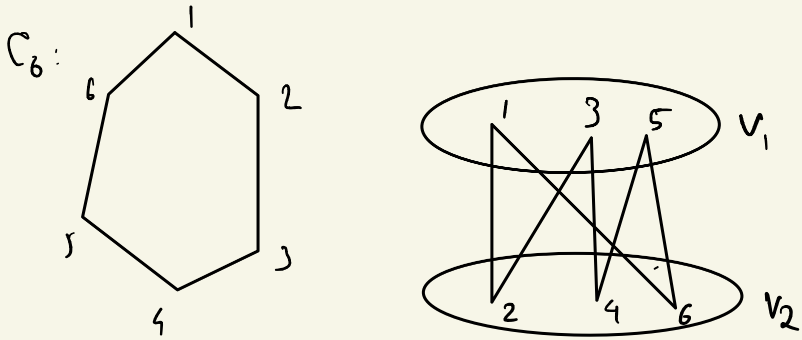
$Q_2 =$ 

$G_2 =$ 

$V(G_2) = \{(0,0), (0,1), (1,0), (1,1)\}$

Bi-partite graphs

A graph G is bi-partite if $V(G)$ can be partitoned as V_1 and V_2 and no two edges are adjacent in V_1/V_2.

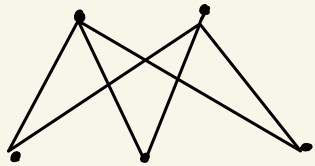


C_n : n is even $\Rightarrow$ bi-partite graphs

V_1, V_2 are also partite sets

A bi-partite graph is complete bi-partite if every vertex of V_1 is adjacent to every vertex of V_2.

$K_{2,3}$



$K_{2,3} \approx K_{3,2}$

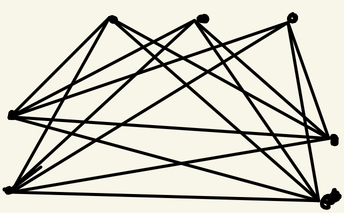
$K_{n,m} \Rightarrow |V_1| = n,$
$|V_2| = m$

K-partite graph:

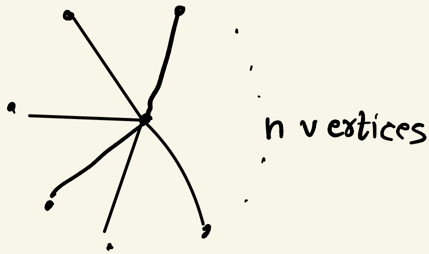
$$V(G) = V_1 \cup V_2 \ \cdots \cup V_K$$

no two vertices in V_i are adjacent.

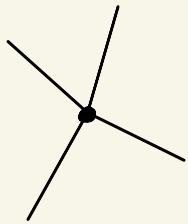
$K_{2,3,2}$



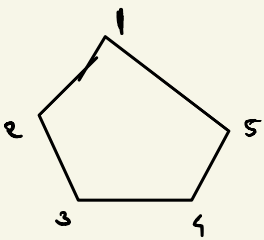
$K_{1,n}$

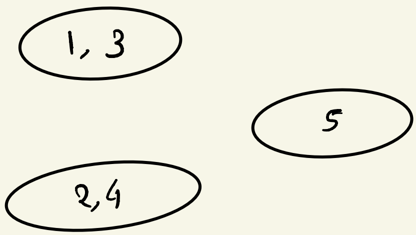


$K_{1,4}$:



C_5 :





C_5 is a tri-partite graph.

observation: A disconn. graph is bipart. iff all its components are bipartite.

Theorem : A graph is bipart. iff G contains no odd cycle.

Proof:

consider bipart. (V_1, V_2)

$$V(G) = V_1 \cup V_2$$

Let C be a cycle in graph G

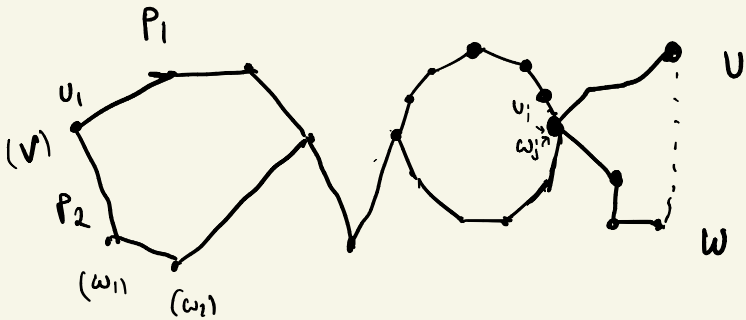
$$C = v_1 \, v_2 \, \ldots \, v_r \, v_1$$

r should be an even-integer

$v_1 \in V_1 \quad v_2 \in V_2 \quad v_3 \in V_1 \; \ldots \ldots \quad v_r \in V_2$

v_r must belong to V_2

so $r = 2k$ $\quad r$ is even.



Let u_i be last common vertex of

P_1 and P_2 So $u_i = w_j$

if $u_i = w_j$

then $i = j$

because if $i > j$ or $j > i$

P_1 / P_2 will not be

shortest paths.

So $u_i = w_i$

So length of Final - cycle $\{v_i, v_{i+1}, \ldots\ldots, v_{2r-1}, v, w, w_{2s-1}, \ldots\ldots, w_i)$

$$2r - i + 1 + 2s - j$$

$$= 2(r + s - i) + 1$$

which is odd-cycle

(contradiction) to assumption.

So no two vertices in V_1 are adjacent.

Tutorial-3 28/08

1\.

(iii) cartesian product of 2 bipartite graphs will be a bipartite graph.

$$G_1 = U_1 \cup V_1$$

$$G_2 = U_2 \cup V_2$$

$$V(G_1 \times G_2) = (U_1 \times U_2) \cup (U_1 \times V_2) \cup$$

ω_2

$$(V_1 \times U_2) \cup (V_1 \times V_2)$$

ω_1

$$\omega_1 = (U_1 \times U_2) \cup (V_1 \times V_2)$$

$$\omega_2 = (U_1 \times V_2) \times (U_2 \times V_1)$$

(ω_1, ω_2) is a bipartition for $(G_1 \times G_2)$

(5) $Q_n = Q_{n-1} \times K_2$

prove $Q_n \simeq G_n$

$G_n = \{(a_1, a_2, \dots, a_n) : a_i = 0/1\}$

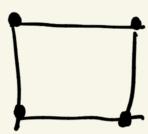
induction on n

$n=1$

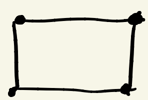
Q_1

G_1

$n=2$



Q_2

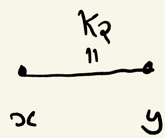


G_2

assume that result is true till $n-1$

$Q_{n-1} \simeq G_{n-1}$

$Q_n = Q_{n-1} \times K_2$



$V(Q_n) = \{(u,x), (u,y) : u \in Q_{n-1}\}$

$$V(G_n) = \{(V,0), (V,1) : V \in V(G_{n-1})\}$$

we can define

a mapping $f : V(Q_n) \to V(G_n)$

$$f(U,x) = (g(V),0)$$

$$f(u,y) = (g(v),1)$$

adjacency in

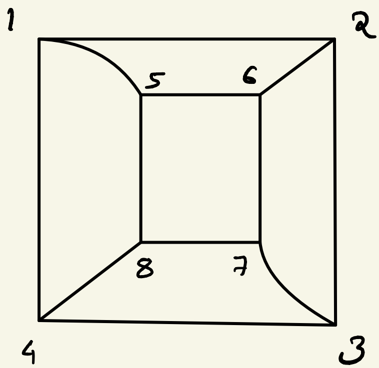
Q_n $(U_1,x), (U_2,x)$

if U_1, U_2 are adjacent

(U_1,x) (U_3,y) For adjacency

$U_1 = U_3$

(7)



$1 \not\sim 6$ (2,5) are 2 common neighbours

$1 \not\sim 7$ no - common neighbours

$u, v \in V(Q_n)$, $u \sim v$

the they differ

atleast in 2 positions.

Q_2 - ✓

Q_3 - ✓

So if they differ at 2 positions

we will have 2 common

neighbour case

if they differ at more than 2 positions

they will have no neighbours.

consider $\omega_1 = (a_1, a_2, \ldots, d_1, \ldots \ldots d_2, \ldots a_n)$

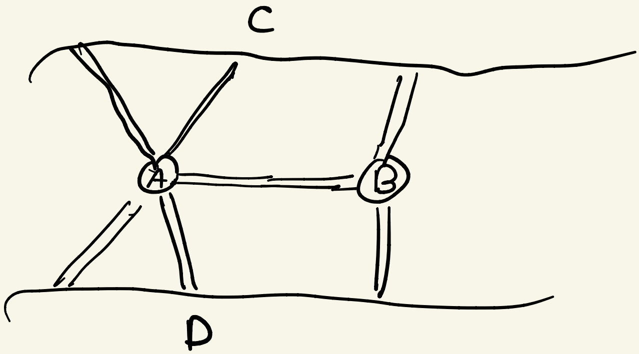
$\omega_2 = (a_1, a_2, \ldots, d_1', \ldots \ldots, d_2', \ldots a_n)$

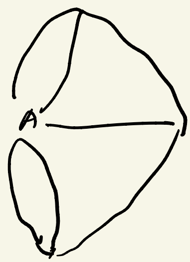
ω_1, ω_2 differ at (d_1, d_1'), (d_2, d_2')

ω_1, ω_2 will have 2 common neighbours

Eulerian graphs

Konigberg's 7- bridge problem





Is it possible to get a closed trail containing all edges of graph

↓

Such trails are eulerian trail / eulerian circuit

graph with eulerian trial is called eulerian graphs, all the vertices have even degree in eulerian graph.

(every time you enter a vertex, you will have an outward edge for that node)

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Theorem:

A connected graph G is eulerian iff the degree of all vertices in G is an even integer.

Proof: G is eulerian $\Rightarrow$ G contains eulerian trail W

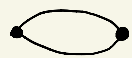
$$E(G) = E(W)$$

$$V(G) = V(W)$$

every time we enter a vertex on this trail we have a different edge to come out of the vertex so every vertex has degree a multiple of 2.

conversely let $\deg x$ is an even integer for all $x \in V(G)$, G contains an eulerian trial induction on no. of edges

if $e=1$, then G is 

$e=2$,  

assuming that result is true till $e-1$ edges.

consider G with e edges

$\deg x$ is even $\forall x \in V(G)$

G contains a cycle, say C

$G_1 = G - E(C)$

degree $(x)\ \forall x \in G_1$ is also even.

G is simple graph

if $\deg x \geq K$

ie. $\delta(x) \geq K$

G contains a cycle of length $K+1$

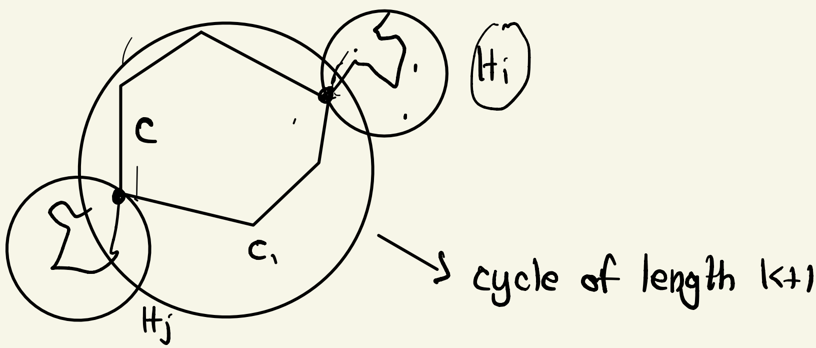
Let $H_1, H_2, \dots\dots, H_k$ be non-trivial components of G.

each H_i is connected & $\deg v$ is even $\forall\, v \in V(H_i)$

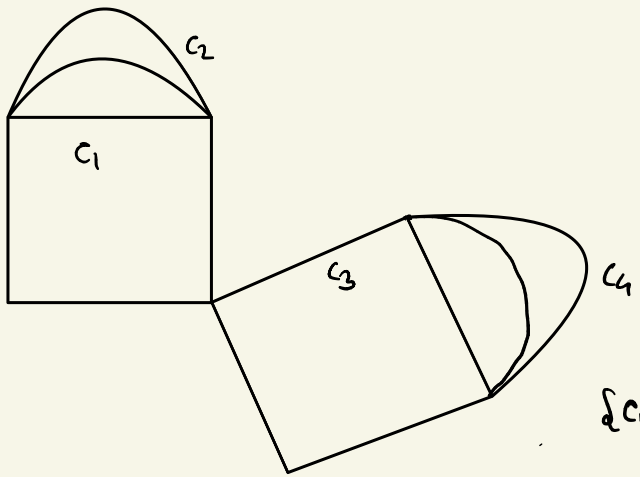
so from the assumption each H_i has an eulerian trial. (C_i)

$$W = C \cup \left(\bigcup_{i=1}^{k} C_i\right)$$

$E(\omega) = E(G)$ $\qquad$ ω is a trial



start from vertex c_1 and go along cycle and then finish trail of H_i whenever possible and end at c_1.



$G = C_1 \cup C_2 \cup C_3 \cup C_4$

$E(C_i) \cup E(C_j) = \phi$

$\{C_1, C_2, C_3, C_4\}$ is a decomposition of G

- Let G be a graph K, $G_1, G_2, \ldots, G_K$ be a subgraphs of G, if $E(G) = \bigcup_{i=1}^{K} E(G_i)$ and

$$E(G_i) \cap E(G_j) = \phi$$

then $\{G_1, G_2, \ldots, G_K\}$ be a decomposition of G //

Theorem:

A connected graph is eulerian iff it decomposes into cycles.

Proof:

every vertex $v \in V(G)$ must be a part of k cycles $k > 1$

and each cycle contributes 2 edges

so

every vertex has $2k$ edges

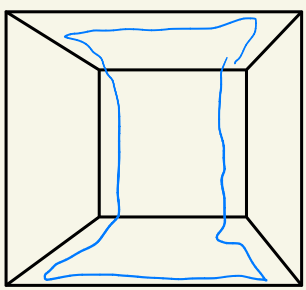
which is even so every graph which decomposes into cycles is eulerian.

Hamiltonian cycles

A cycle C in a graph G is called a hamiltonian cycle if C contains all vertices of G if $V(C) = V(G)$.

A path containing all the vertices of G is hamiltonian path

A graph G is called hamiltonian if it contains a hamiltonian cycle.



Theorem:

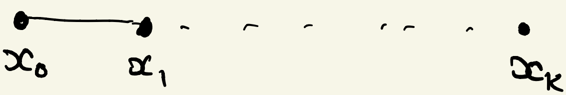
Let G be a simple graph on n vertices, if $\delta(G) \geq n/2$

then G is a hamiltonian graph.

Proof:

Let P be a path of maximum length in G

$$P = (x_0, \dots\dots, x_k)$$



(all neighbours of x_0, x_k are on path P)

$$S = \{0, 1, \dots\dots, k-1\}$$

x_k is having atleast $n/2$ neighbours in S. $(x_k \sim x_j)$

Same goes with x_0. $x_0 \sim x_{j+1}$

$|S| = k \le n-1 < n$

$S_1 \to$ set of $n/2$ indices i $\quad x_0 \sim x_i$

$S_2 \to$ set of $n/2$ indice j $\quad x_k \sim x_j$

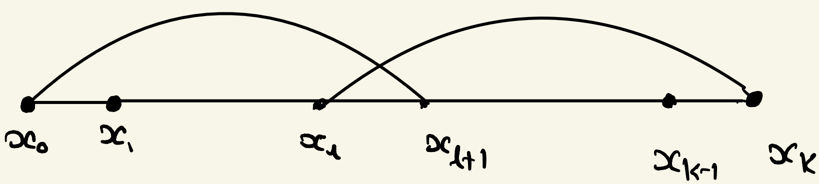
$|S_1| \ge n/2$, $|S_2| \ge n/2$

$S_1, S_2 \in S$

So $\quad S_1 \cap S_2 \ne \emptyset$

$l \in S_1 \cap S_2$

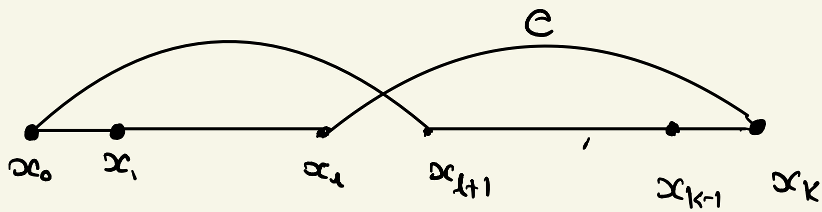
$x_k \sim x_l$, $x_0 \sim x_{l+1}$



$$C = P \cup \{x_0, x_{l+1}\} \cup \{x_k, x_l\} - \{x_l, x_{l+1}\}$$

does C contain all the vertices?

claim : $V(C) = V(G)$



if $V(C) \neq V(G)$

then $\exists\ x_t \in V(G)$, $x_t \notin V(C)$

Since G is connected, $\exists$ a path

P_1 between x_t and a vertex

x_i in C, then

max length will be not be

Let

$k+1 \leq n$,

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$S = \{0, 1, 2, \ldots, k-1\}$

All the neighbours of x_0 and x_k lie on path P.

Let $S_1 = \{i \in S : x_k \sim x_i\}$, $S_2 = \{j \in S : x_0 \sim x_{j+1}\}$

$|S| = k \leq n-1$, $|S_1| \geq \frac{n}{2}$, $|S_2| \geq \frac{n}{2}$

$S_1 \cap S_2 \neq \phi$ ($\because$ $|S| = n-1$, $|S_1| + |S_2| = n$; $|S| = |S_1 \cup S_2|$

$|S_1 \cap S_2| = |S_1| + |S_2| - |S_1 \cup S_2| \geq \frac{n}{2} + \frac{n}{2} - (n-1) = 1$

$\Rightarrow \exists\, l \in S$ s.t $l \in S_1 \cap S_2$.

$\Rightarrow x_k \sim x_l$ and $x_0 \sim x_{l+1}$

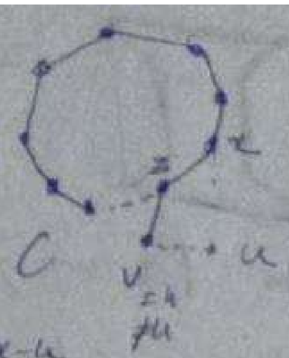
Now $C = P - \{x_l, x_{l+1}\} \cup \{\{x_0, x_{l+1}\}, \{x_k, x_l\}\}$ is a cycle on $k+1$ vertices.

Claim: C contains all the vertices of G.

Suppose $\exists\, u \in V(G)$ s.t. $u \notin C$.

v may or may not be equal to u.

Then we get a path longer than length k.



Since G is connected, for $x \in C$, $\exists$ a x-u path in G.

Then $\exists$ a vertex $v \notin V(C)$ s.t $v \sim z$, for some $z \in V(C)$.

(z may be equal to x and v may be equal to u).

Let $z = x_i$. Then $P' = C - \{x_i, x_{i+1}\} \cup \{v, x_i\}$ is a path with $L(P') = k+1$, which is longer than path P, and is a contradiction.

Hence the claim is true. i.e. C is a Hamiltonian cycle.

<u>Theorem (Ore, 1960)</u>

Let G be a simple connected graph on n vertices, $n \geq 3$. If $\deg x + \deg y \geq n$, for every pair of non adjacent vertices x and y in G, then G is a Hamiltonian graph.

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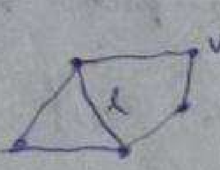
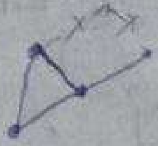
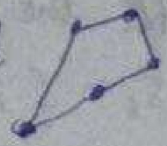


G

G-v 

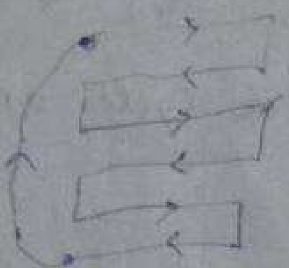
G-e 

① petersen graph.

② $u_1, u_2, u_3, \dots, u_m, u_1$ Hamiltonian cycle in G_1

$v_1, v_2, v_3, \dots, v_n, v_1$ Hamiltonian cycle in G_2

$V(G_1 \times G_2)$

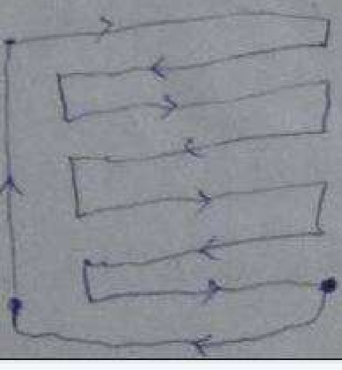


$(u_1, v_1) \rightarrow (u_1, v_2) \rightarrow \dots (u_1, v_n)$

$(u_2, v_1) \leftarrow (u_2, v_2) \dots \leftarrow (u_2, v_n)$

if m is even

$(u_m, v_1) \leftarrow (u_m, v_2) \leftarrow \dots (u_m, v_n)$



$(u_1, v_1) - (u_1, v_2) \dots - (u_1, v_n)$

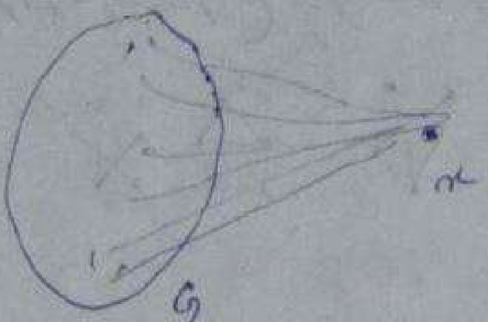
$(u_2, v_1) \quad (u_2, v_2) - - - (u_2, v_n)$ if m is odd

$(u_{m-1}, v_1) \quad (u_{m-1}, v_2) \quad (u_{m-1}, v_n)$

$(u_m, v_1) \quad (u_m, v_2) \dots (u_m, v_n)$

(3) $\delta(G) \geq \frac{n-1}{2} \rightarrow G$ contains a Hamiltonial path.

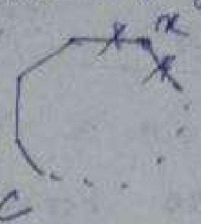
$\delta(G) \geq \frac{n}{2} \rightarrow G$ is Hamiltonian.



$\hat{G} = G \vee K_1$

$\delta_{\hat{G}}(v) = \frac{n-1}{2} + 1 = \frac{n+1}{2} \geq \frac{n}{2}$

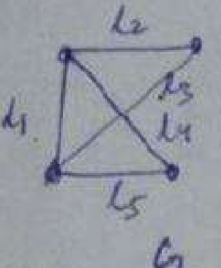
Hamiltonian cycle



$\Rightarrow \hat{G}$ is Hamiltonian

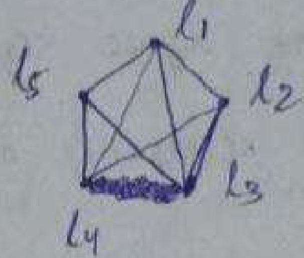
$C - x$ is a Hamiltonian path in G.

(4)



Line graph of G

$L(G)$



If G is Eulerian then prove that $L(G)$ is Hamiltonian.

??

(5) $Q_n = Q_{n-1} \times K_2$

Induction on n.

Assume that Q_{n-1} is Hamiltonian.

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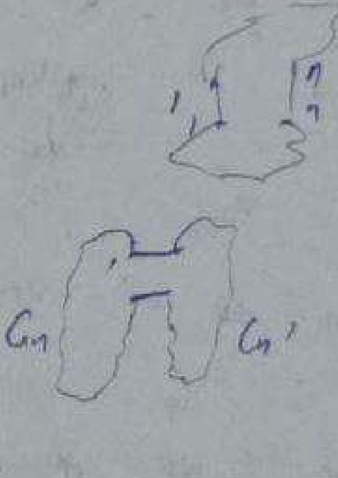
Induction on n.

Assume that Q_{n-1} is Hamiltonian.

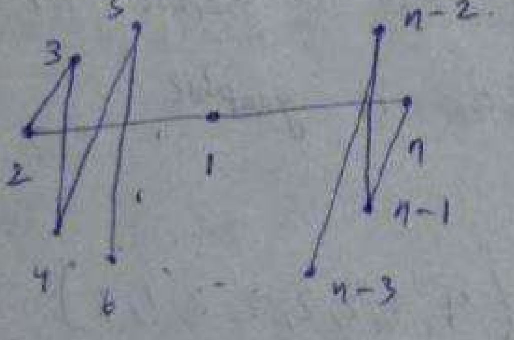


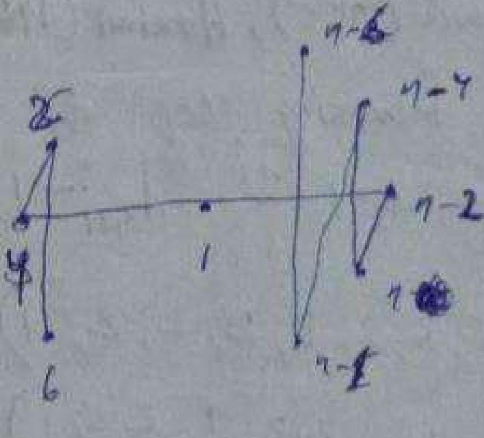
(6) K_n, n odd. ans: $\frac{n-1}{2}$

$|E(K_n)| = \frac{n(n-1)}{2}$

Every Hamiltonian cycle in K_n contains n edges.

$\Rightarrow$ K_n contains at most $\frac{n-1}{2}$ no of edge disjoint Hamiltonian cycles





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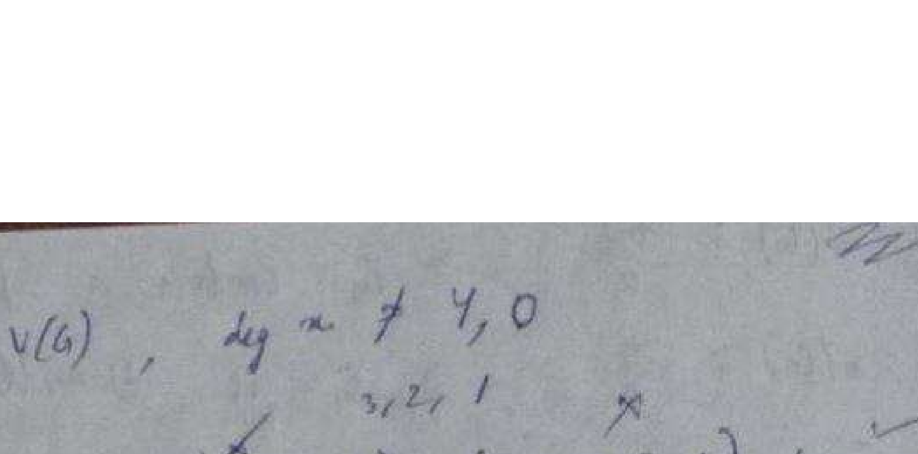
③ $G \simeq \bar{G}$ $x \in V(G)$, $\deg x \neq 4, 0$

$|E(G)| = 5$

deg sequences

3, 2, 1

$(3, 3, 2, 1, 1)$, $(3, 2, 2, 2, 1)$, $(2,2,2,2,2)$



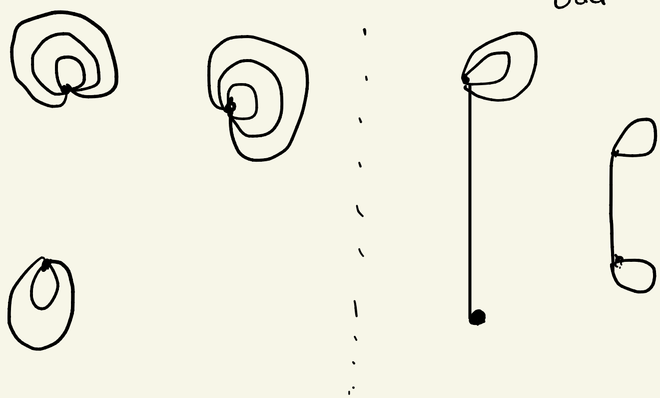
Graphic sequences

for every $d: d_1 \geq d_2 \geq d_3 \ldots d_n$ (non-increasing)

$\exists$: a graph

ex: $d: 6, 6, 5, 4, 3, 3, 1$

even odd



Let $d: d_1 \geq d_2 \cdots \geq d_n$ be a non-increasing sequences of non-negative integers with $\sum d_i$ even. If $\exists$ a simple graph G with deg. sequence d, then d is a graphic sequence.

---

example:

whether sequence below is graphic?

a) $(5,5,5,4,2,1,1,1)$

$d: \cancel{5}, 5, 5, 4, 2, 1, 1, 1$

$d': 4, 4, 3, 1, 0, 1, 1 \quad (4,4,3,1,1,1,0)$

$d'': (3,2,0,0,1,0) \quad (3,2,1,0,0,0)$

$d''': (1,0,-1,0,0) \quad (1,0,0,0,-1)$

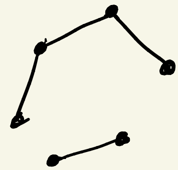
(ii) d: 5, 5, 4, 4, 2, 2, 1, 1

d' (4, 3, 3, 1, 1, 1, 1)

d'' (2, 2, 1, 1, 0, 0)

d''' (1, 1, 0, 0, 0)

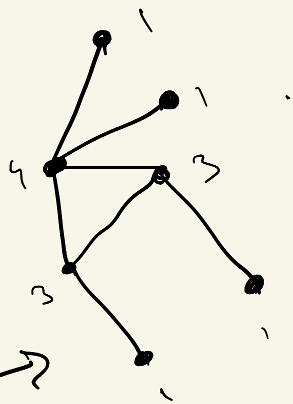
(0, 0, 0, 0)



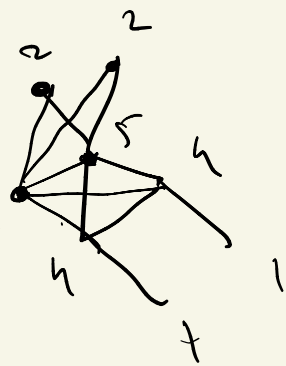
: d''

(given seq. is graphic)

d':



d:

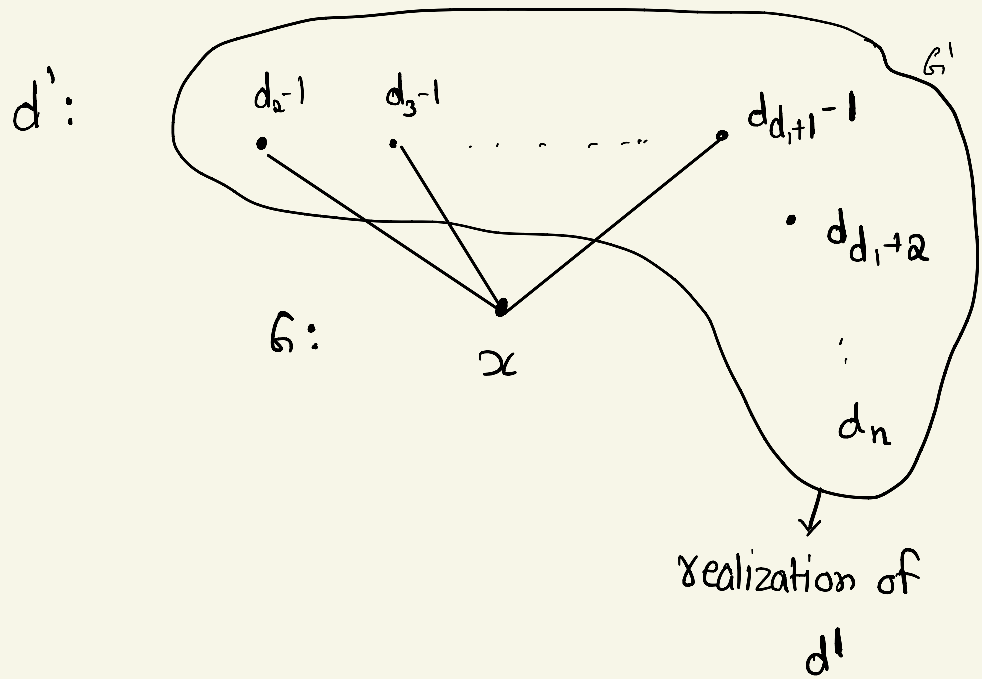


Theorem: A non-increasing sequence.

$d: d_1 \geq d_2 \geq \ldots \ldots \geq d_n$ of non-neg integers is graphic, iff the seq.

$$d' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1}, \ldots, d_n)$$

Pf: if d' is graphic $\Rightarrow$ d is graphic



$$V(G) = V(G') \cup \{v_1\}$$

$$E(G) = E(G') \cup \{(v_1, v_2), \dots, (v_1, v_{d_1+1})\}$$

convorsely let d be graphic, then d' is graphic

$$V(G) = \{v_1, v_2, \dots, v_n\}$$

$\deg \cdot v_i = d_i$ $\quad (\deg v_1 \geq \deg v_2 \geq \dots > \deg v_n)$

Let $S = \{v_2, \dots, v_{d_1+1}\}$

if v_1 is adjacent to all the vertices in S, then $G - v_1$ is a realization of d'.

if V_1 is not adjacent to all vertices in S.

Let $V_r \in S$, $V_r \nsim V_1$

construct a graph G_1 (simillar to G)

in which $(V_1 \sim V_r)$. then we can construct

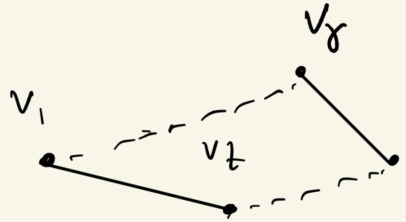
realization for D.

in G $\exists V_t$ s.t $V_1 \sim V_t$ $t > d_1+1$

$V_1 \nsim V_r$

$2 \le r \le d_1+1$

$\deg V_r \ge \deg V_t$ | $\exists$ a vertex V_s st. $V_r \sim V_s$ & $V_t \nsim V_s$



V_s in G

$V(G_1) = V(G)$

$E(G_1) = E(G) - \{(V_r, V_s), (V_1, V_t)\}$
$\cup \{(V_r, V_1), (V_s, V_t)\}$

check V_1 adjacency to set S, if V_1 is not adjacent to any vertex in S repeat the process.

after finite number of steps we get G_k st. $V_1 \sim S$

then $G_k - V_1$ is realization of d'.