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For any real number a, b

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$1 \leq p \leq \infty.$$

Holder's Inequality:

Let p and q be +ve real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

For any $x = (x_1, x_2 \ldots x_n)$ and $y = (y_1, y_2 \ldots y_n)$ in K^n,

$$\sum_{i=1}^{n} |x(i) y(i)| \leq \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p} \times \left(\sum_{i=1}^{n} |y(i)|^q\right)^{1/q}.$$

Proof: For $x \in K^n$, let

$$\|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p}.$$

Clearly above inequality is true for any $x = 0$.

So assume that $x \neq 0$ & $y \neq 0$.

$$\Rightarrow \|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p} \neq 0$$

and $\|y\|_q = \left(\sum_{i=1}^{n} |y(i)|^q\right)^{1/q} \neq 0$

let $a = \dfrac{|x(i)|}{\|x\|_p}$, $b = \dfrac{|y(i)|}{\|y\|_q}$.

Then using

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

we have

$$\frac{|x(i)|}{\|x\|_p} \frac{|y(i)|}{\|y\|_q} \leq \frac{1}{p} \frac{|x(i)|^p}{\|x\|_p} + \frac{1}{q} \frac{|y(i)|^q}{\|y\|_q}$$

Take the summation from $i=1$ to n,

$$\sum_{i=1}^{n} \frac{|x(i)y(i)|}{\|x\|_p \|y\|_q} \le \frac{1}{p} \frac{\sum_{i=1}^{n} |x(i)|^p}{\|x\|_p^p} + \frac{1}{q} \frac{\sum_{i=1}^{n} |y(i)|^q}{\|y\|_q^q}$$

$$= \frac{1}{p} \frac{\|x\|_p^p}{\|x\|_p^p} + \frac{1}{q} \frac{\|y\|_q^q}{\|y\|_q^q}$$

$$= \frac{1}{p} + \frac{1}{q}$$

$$= 1$$

$$\therefore \sum_{i=1}^{n} |x(i)y(i)| \le \|x\|_p \|y\|_q,$$

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Theorem: For $1 \le p < \infty$, let $\|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p \right)^{1/p}$. Then $\|\cdot\|_p$ is a norm on K^n.

Proof: For $p=1$, the theorem follows from the last class.

Now assume $1<p<\infty$, $x, y \in K^n$

Consider

$$\|x+y\|_p^p = \sum_{i=1}^{n} |x(i)+y(i)|^p$$

$$= \sum_{i=1}^{n} |x(i)+y(i)|\, |x(i)+y(i)|^{p-1}$$

$$\leq \sum_{i=1}^{n} |x(i)|\, |x(i)+y(i)|^{p-1}$$

$$+ \sum_{i=1}^{n} |y(i)|\, |x(i)+y(i)|^{p-1}. \quad \text{---}(1)$$

Now by Holder's inequality, we have

$$\sum_{i=1}^{n} |x(i)|\, |x(i)+y(i)|^{p-1}$$

$$\leq \left(\sum_{i=1}^{n} |x(i)|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} \left(|x(i)+y(i)|^{p-1}\right)^q\right)^{\frac{1}{q}}$$

$$= \|x\|_p \left(\sum_{i=1}^{n} |x(i)+y(i)|^{q(p-1)} \right)^{1/q}$$

$$= \|x\|_p \left(\sum_{i=1}^{n} |x(i)+y(i)|^{p} \right)^{1/q} \quad \left[\because \frac{1}{p} + \frac{1}{q} = 1 \quad pq = q + p \quad (p-1)q = p \right]$$

$$= \|x\|_p \left[\|x+y\|_p^p \right]^{1/q}$$

$$= \|x\|_p \, \|x+y\|_p^{p/q}$$

$\|$ly we can show that

$$\sum_{i=1}^{n} |y(i)| \, |x(i)+y(i)|^{p-1} \leq \|y\|_p \cdot \|x+y\|_p^{p/q}$$

Substituting these in (1), we get

$$\|x+y\|_p^p \leq \left[\|x\|_p + \|y\|_p \right] \|x+y\|_p^{p/q}$$

$$\Rightarrow \|x+y\|_p^{p(1-1/q)} \leq \|x\|_p + \|y\|_p.$$

$$\Rightarrow \|x+y\|_p \leq \|x\|_p + \|y\|_p.$$

Now for any $\alpha \in K$,

$$\|\alpha x\|_p = \left(\sum_{i=1}^{n} |\alpha x(i)|^p\right)^{1/p}$$

$$= \left(\sum_{i=1}^{n} |\alpha|^p |x(i)|^p\right)^{1/p}$$

$$= |\alpha| \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p}$$

$$= |\alpha| \, \|x\|_p.$$

$\because$ $|x(i)| \geq 0 \quad \forall \, i = 1 \ldots n$

$$\left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p} \geq 0$$

$$\Rightarrow \|x\|_p \geq 0.$$

$\|\!\|ly$ $\|x\|_p = 0 \iff \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p} = 0$

$\iff |x(i)| = 0, \ \forall i = 1 \dots n$

$\implies x(i) = 0, \ i = 1 \dots n$

$\implies x = (0, 0 \dots 0)$.

$\therefore (K^n, \|\cdot\|_p)$ is a normed linear space.

For $p = 2$, $\|\cdot\|_2$ is called **Euclidean norm on K^n**.

Remarks:– For $n > 1$, $0 < p < 1$,

$\|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p\right)^{1/p}$ is **not a norm on K^n**.

$\because$ $e_1 = (1, 0 \dots 0)$

$e_2 = (0, 1, \dots 0)$

$$\|e_1\|_p = 1 = \|e_2\|_p$$

$$\|e_1+e_2\|_p = 2^{1/p}$$

$$\therefore \|e_1+e_2\|_p = 2^{1/p} > 2 = \|e_1\|_p + \|e_2\|_p.$$

Now we consider p norm on $C[a,b]$.

Theorem (Holder's inequality)

Let p and q be positive real numbers satisfying $\frac{1}{p}+\frac{1}{q}=1$.

Then for all $x, y \in C[a,b]$,

$$\int_a^b |x(t)\,y(t)|\,dt \le \left(\int_a^b |x(t)|^p\right)^{1/p} \left(\int_a^b |y(t)|^q\right)^{1/q},$$

Proof: Let $\|x\|_p = \left(\int_a^b |x(t)|^p\right)^{1/p}$, $1 \le p < \infty$.

Proof follows by replacing the summation by integration in the previous theorem.

* Using above Holder's inequality, we can show that $C[a,b]$ is normed linear space with the norm
$$\|x\|_p = \left(\int_a^b |x(t)|^p\right)^{1/p}, \quad 1 \le p < \infty.$$

* if $\frac{1}{p} + \frac{1}{q} = 1$, we say p is the conjugate of q and vice versa.

$$\left[\ \|x+y\|_p^p = \int_a^b |x(t)+y(t)|^p dt \right.$$
$$\left. = \int_a^b |x(t)+y(t)| \cdot |x(t)+y(t)|^{p-1} dt \right]$$

$\left(\|x\|_p = 0 \implies x = 0 \right)$

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Ex: Let Ω be any nonempty set.
Denote $B(\Omega)$ the set of all bounded functions on Ω.
It is a vector space w.r.t addition and scalar multiplication
$x, y \in B(\Omega)$,

$$(x+y)(t) = x(t) + y(t).$$

$$(\alpha x)(t) = \alpha x(t), \quad \alpha \in K.$$

We denote $B(\Omega) = \ell^\infty(\Omega)$.
For $x \in B(\Omega)$, Define

$$\|x\|_\infty = \sup_{t \in \Omega} |x(t)|.$$

Thus $B(\Omega)$ is a normed linear space w.r.t the norm $\|\cdot\|_\infty$.

In particular, if $\Omega = \mathbb{N}$, then $B(\mathbb{N}) = \ell^\infty(\mathbb{N})$ is normed linear space w.r.t

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x(n)| \,, \quad x \in B(\mathbb{N}).$$

* $C[a,b]$, $R[a,b]$ are subspaces of $B([a,b])$.