Lecture 11

<u>Theorem</u>:- Let $f_n : E \to \mathbb{R}$ be a sequence of Borel measurable functions. Then

(i) $\sup\limits_{1\le i\le n}(f_i) = \max\{f_1, \dots, f_n\}$ is Borel measurable $\forall n \geq 1$

(ii) $\inf\limits_{1\le i\le n}(f_i) = \min\{f_1, \dots, f_n\}$ " "

(iii) $\sup\limits_n (f_n)$ is Borel measurable.

(iv) $\inf\limits_n (f_n)$ " " .

(v) $\limsup (f_n)$ " "

(vi) $\liminf (f_n)$ " " .

<u>Definition</u>:- We say that a property P holds <u>almost everywhere</u> (a.e), if P holds except on a set of measure zero.

Example:— ① $f : E \to \mathbb{R}$ "continuous a.e"

means that $m\left(\{a \in E \mid f \text{ is not continuous at } x = a\}\right) = 0$.

that is the set of all points where f is not continuous, has measure zero.

② $f : E \to \mathbb{R}$ differentiable a.e.

That $\{a \in E \mid f \text{ is not differentiable at } x = a\}$ has measure 0.

③ $\{f_n\}$ seq. of functions, $f_n : E \to \mathbb{R}$, $\forall n$

"$f_n \to f$ pointwise a.e" as $n \to \infty$ means that

$\{a \in E \mid f_n(a) \not\to f(a) \text{ as } n \to \infty\}$ has measure zero.

Theorem:- Let $f: E \to \mathbb{R}$ be a measurable function & let $g: E \to \mathbb{R}$ be any function.

Suppose $f = g$ a.e. Then g is measurable.

proof:- Given f is measurable. Then for any $\alpha \in \mathbb{R}$,

$$\{x \in E \mid f(x) > \alpha\} \in \mathcal{M}$$

Given $f = g$ a.e

$$\Rightarrow m(\{x \in E \mid f(x) \neq g(x)\}) = 0.$$

Look at $\underbrace{\{x \in E \mid f(x) > \alpha\}}_{=A} \,\Delta\, \underbrace{\{x \in E \mid g(x) > \alpha\}}_{=B}$,

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

This set is a subset of $\{x \in E \mid f(x) \neq g(x)\}$.

$\{a \in E \mid f(a) > \alpha\} \,\Delta\, \{a \in E \mid g(a) > \alpha\} \subseteq \{a \in E \mid f(a) \neq g(a)\}$.

has measure 0.

$$\Rightarrow \quad m\left(\{a \in E \mid f(a) > \alpha\} \,\Delta\, \{a \in E \mid g(a) > \alpha\}\right) = 0.$$

Thus $m(A \Delta B) = 0$ & $A \in \mathcal{M}$

$\Rightarrow \quad B \in \mathcal{M}$

$\Rightarrow \quad \{a \in E \mid g(a) > \alpha\} \in \mathcal{M}, \quad \forall \alpha \in \mathbb{R}$

$\therefore$ g is measurable.

<u>Proposition</u>:- Let $f_n : E \to \mathbb{R}$, $\forall n$, $\{f_n\}$ be a sequence of measurable functions. Suppose $f_n \to f$ pointwise a.e. Then f is measurable.

<u>proof</u>:- Given $f_n \to f$ a.e (pointwise)

Then $f = \limsup(f_n)$ a.e

($\because$ $f_n(x) \to f(x)$, then $f(x) = \limsup(f_n(x))$
$= \liminf(f_n(x))$)

But $\limsup(f_n)$ is measurable.

$\therefore$ By above theorem, f is measurable.

(FACT: $r_n \to r$. Then $\lim(r_n) = \limsup(r_n)$
$= \liminf(r_n)$
$= r$.)

Substitute $r_n = f_n(x) \in \mathbb{R}$
& $r = f(x) \in \mathbb{R}$

Proposition:- Let $f_n : E \to \mathbb{R}$ be a sequence of measurable functions. Then

$\{ x \in E \mid f_n(x) \text{ converges} \}$ is a measurable set.

proof:-

$$\{ x \in E \mid f_n(x) \text{ converges} \} = \{ x \in E \mid \limsup(f_n(x)) = \liminf(f_n(x)) \}$$

$$= \{ x \in E \mid (\limsup(f_n) - \liminf(f_n))(x) = 0 \}.$$

$\in \mathcal{M}$ ($\because$ $\limsup(f_n)$, $\liminf(f_n)$ are measurable).

<u>Definition</u>:- Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable function. Then the <u>essential supremum</u> of f is defined as

$$\operatorname{esssup}(f) := \inf \{ \alpha \in \mathbb{R} \mid f \leq \alpha \text{ a.e on } E \}.$$

$$= \inf(U_f),$$

where $U_f = \{ \alpha \in \mathbb{R} \mid m(f^{-1}((\alpha, \infty))) = 0 \}.$

($\because$ $f \leq \alpha$ a.e on E $\Rightarrow$ $m(\{ a \in E \mid f(a) \nleq \alpha \}) = 0$

$\qquad \| $

$m(\{ a \in E \mid f(a) > \alpha \}) = 0.$

$\qquad \| $

$m(f^{-1}((\alpha, \infty))) = 0.$),

if $U_f \neq \phi$, otherwise the essup(f) is defined as $+\infty$.

Examples:- ① $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} -1 & \text{if } x = 1 \\ 30 & \text{if } x = 0 \\ 10 & \text{otherwise.} \end{cases}$$

$\sup(f) = 30$, $\inf(f) = -1$

$\text{esssup}(f) = 10$?

f takes the values -1 & 30 on the sets $\{1\}, \{0\}$ respectively which have measure 0. & everywhere else f takes the value 10.

$f \leq 10$ a.e & $f \geq 10$ a.e

$\therefore$ $\text{esssup}(f) = 10 = \text{essinf}(f)$

② $$f(x) = \begin{cases} x^5 & \text{if } x \in \mathbb{Q} \\ \tan^{-1}(x) & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

$$\operatorname{esssup}(f) = \inf\left(\{\alpha \in \mathbb{R} \mid f \le \alpha \text{ a.e}\}\right)$$
$$= \inf\left(\{\alpha \in \mathbb{R} \mid \tan^{-1} x \le \alpha \text{ a.e}\}\right)$$
$$= \pi/2$$

$\operatorname{essinf}(f) = -\pi/2$.

<u>Definition</u>:- Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable function. Then the <u>essential infimum</u> of f is defined as

$$\operatorname{essinf}(f) := \sup\left(\{\alpha \in \mathbb{R} \mid f \ge \alpha \text{ a.e on } E\}\right)$$

Remark:-

f is measble $\Longrightarrow \{x \in E \mid f(x) > \alpha\} \in \mathcal{M}$ $\forall \alpha \in \mathbb{R}$

$\Downarrow$ $\nexists$??

$$\{x \in E \mid a < f(x) < b\} = \{x \in E \mid a < f(x)\} \cap \{x \in E \mid f(x) < b\} \in \mathcal{M} \quad \forall a, b \in \mathbb{R}$$

$\boxed{V = \text{Non-measurable set}}$ $V \subseteq [0,1]$

$$f(x) = \begin{cases} 1 & \text{if } x \in V \\ 0 & \text{if } x \in [0,1] \setminus V \\ 3 & \text{if } x \in \mathbb{R} \setminus [0,1] \end{cases}$$

f is not measurable.. ✓✓.

But $f^{-1}\big((2,4]\big) = \{ x \in \mathbb{R} \,/\, 2 < f(x) \leq 4 \}$
$= \mathbb{R} \setminus [0,1] \in \mathcal{M}$

$f^{-1}(a,\infty) = f^{-1}(a, b_1) \cup f^{-1}([b_1, b_2)) \cup \ldots$

$\mathcal{M}$ $\qquad$ $\mathcal{M}$?! ...

$A_n \in \mathcal{M} \implies \boxed{\bigcup A_n} \in \mathcal{M}.$

$\Leftarrow$??

$V \cup V^c = \mathbb{R} \in \mathcal{M}$

But $V \notin \mathcal{M}$.