Lecture 29

Lemma (Fatou's Lemma):—

Suppose $\{f_n\}$ is a sequence of measurable functions & $f_n \geq 0 \quad \forall n \geq 1$. If $\lim_{n\to\infty} f_n(x) = f(x)$ a.e,

then $\int f \leq \liminf_{n\to\infty} \left(\int f_n\right)$

proof:— Note that $f \geq 0$ on some domain E.

Recall:
$\liminf_{n\to\infty}(a_n) = \lim_{n\to\infty}\left(\inf_{k\geq n}(a_k)\right)$

Suppose $0 \leq g \leq f$,
where g is a bounded measurable function supported on a set of finite measure.

Set $g_n(x) = \min\left(g(x), f_n(x)\right) \quad \forall n \geq 1$
$\forall x$.

Then g_n is measurable, $g_n \geq 0$, supported on a set of finite measure $\left(\because 0 \leq g_n(x) \leq g(x)\right)$
$\forall n \geq 1$.

Claim:- $g_n(x) \longrightarrow g(x)$ as $n \to \infty$.

$$\boxed{\int_{E \cup A} f = \int_E f}$$

If $m(A) = 0$

proof of Claim:-

We have $g(x) \leq f(x) = \lim_{n\to\infty} f_n(x)$ a.e

Let $\varepsilon > 0$.

There exists $n_0 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq n_0$$

a.e

$$\Rightarrow \quad f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \quad \forall n \geq n_0.$$

But $g(x) \leq f(x)$

$$\Rightarrow \quad g(x) - \varepsilon \leq f(x) - \varepsilon < f_n(x) \quad \forall n \geq n_0.$$

$$\Rightarrow \quad g(x) - \varepsilon < f_n(x) \quad \forall n \geq n_0$$

$$\Rightarrow \quad g(x) \leq f_n(x) + \varepsilon \quad \forall n \geq n_0$$

Thus for $n \gg 0$, $g(x) \lesssim f_n(x)$

$\Rightarrow g_n(x) = \min(g(x), f(x)) \longrightarrow g(x)$ as $n \to \infty$.
a.e

Thus we have

$\{g_n\}$ is a sequence bounded measurable functions supported on a set of finite measure. & $g_n(x) \to g(x)$ a.e as $n \to \infty$.

$\therefore$ By Bounded Convergence theorem,

$$\int g_n \longrightarrow \underline{\underline{\int g}} \quad \text{as } n \to \infty$$

Also, $g_n \leq f_n$ which implies that

$$\int g_n \leq \int f_n$$

Take the $\liminf\limits_{n \to \infty}$ both sides, then

$$\liminf_{n \to \infty} \int g_n \leq \liminf_{n \to \infty} \int f_n$$

$$\|$$

$$\limsup_{n \to \infty} \left(\int g_n \right)$$

$$\text{\rotatebox{90}{$\leq$}}$$

$$\lim_{n \to \infty} \int g_n \leq \liminf_{n \to \infty} \int f_n$$

$$\Rightarrow \quad \int g \leq \liminf_{n\to\infty} \left(\int f_n\right)$$

This is true for any g such that $0 \leq g \leq f$, g is bounded, measurable & supported on a set of finite measure.

$$\Rightarrow \sup_{\substack{0 \leq g \leq f \\ \cdots}} \left(\int g\right) \leq \liminf_{n\to\infty} \left(\int f_n\right)$$

$$\|$$

$$\Rightarrow \quad \int f \leq \liminf_{n\to\infty} \left(\int f_n\right).$$

Corollary :- Suppose $f \geq 0$ is measurable & $\{f_n\}$, a sequence of non-ve measurable functions with $f_n(x) \leq f(x)$ $\forall n \geq 1$ &

$f_n(x) \longrightarrow f(x)$ a.e as $n \to \infty$.

Then $$\lim_{n\to\infty} \int f_n = \int f.$$

proof: Given $0 \le f_n(x) \le f(x) \quad \forall n \ge 1$.

$$\Rightarrow \int f_n \le \int f \quad \forall n$$

Take $\limsup_{n \to \infty}$ both sides, we get

$$\limsup_{n \to \infty} \int f_n \le \lim_{n \to \infty} \int f = \int f$$

$$\le \liminf_{n \to \infty} \int f_n$$

(by Fatou's Lemma)

But $\liminf \left(\int f_n\right) \le \limsup \left(\int f_n\right)$,

therefore

$$\limsup_{n \to \infty} \left(\int f_n\right) = \int f = \liminf_{n \to \infty} \left(\int f_n\right)$$

$$\Rightarrow \lim_{n \to \infty} \left(\int f_n\right) = \int f.$$

Theorem:- (Monotone Convergence Theorem):-

Suppose $\{f_n\}$ is a sequence of non-negative measurable functions with $\{f_n\}$ is increasing & $f_n \to f$ a.e (That is, $f_n \uparrow f$ a.e)

Then $\lim_{n\to\infty} \int f_n = \int f$.

proof:- $\{f_n\}$ is increasing & $f_n \to f$ a.e

$$\Rightarrow 0 \le f_n(x) \le f(x) \ \forall n \ge 1, \text{ a.e}$$

$$\& \ f_n(x) \to f(x) \text{ a.e as } n \to \infty.$$

$\therefore$ By above Corollary, we have

$$\int f_n \to \int f \text{ as } n \to \infty.$$

Corollary:— Consider a series $\sum_{k=1}^{\infty} a_k(x)$, where

$a_k(x) \geq 0$ is measurable, $\forall\, k \geq 1$. Then

$$\int \left(\sum_{k=1}^{\infty} a_k(x) \right) dx = \sum_{k=1}^{\infty} \left(\int a_k(x)\, dx \right).$$

If $\sum_{k=1}^{\infty} \left(\int a_k(x)\, dx \right)$ is finite, then the

series $\sum_{k=1}^{\infty} a_k(x)$ Converges a.e.

proof:—

Let $f_n(x) = \sum_{k=1}^{n} a_k(x)$, $\forall\, n \geq 1$.

& $f(x) = \sum_{k=1}^{\infty} a_k(x)$.

Then $f_n \geq 0$, f_n is measure & $\{f_n\}$ is

an increasing sequence & $f_n(x) \to f(x)$

as $n \to \infty$.

Also $\int f_n = \int \left(\sum_{k=1}^{n} a_k \right)$

$$= \sum_{k=1}^{n} \int a_k .$$

$\therefore$ By Monotone Converges theorem,

$$\lim_{n \to \infty} \int f_n = \int f .$$

$$\Rightarrow \lim_{n \to \infty} \left(\sum_{k=1}^{n} \int a_k \right) = \int \left(\sum_{k=1}^{\infty} a_k \right)$$

$$\Rightarrow \sum_{k=1}^{\infty} \int a_k = \int \left(\sum_{k=1}^{\infty} a_k \right) .$$

If $\sum_{k=1}^{\infty} \int a_k < \infty$, then by above equality $\int \sum_{k=1}^{\infty} a_k < \infty$, this implies that

$\sum_{k=1}^{\infty} a_k$ is finite a.e.

Lemma (Borel – Cantelli) :-

Let $E_1, E_2, \cdots$ be a Collection of measurable sets with $\sum_{k=1}^{\infty} m(E_k) < \infty$.

Then the set of all points that belong to infinitely many sets E_k has measure 0.

That is, $m(\{x \mid x \in E_k \text{ for infinitely many } k's\}) = 0$.

proof:- Let $a_k(x) = \chi_{E_k}(x) \geq 0$, $\forall x$, $\forall k \geq 1$.

Then

$x \in$ infinitely many E_k's

$\Updownarrow$

$$\sum_{k=1}^{\infty} a_k(x) = \infty \;;$$

Given that $\sum_{k=1}^{\infty} m(E_k) < \infty$

Now $\underline{\sum_{k=1}^{\infty} \left(\int a_k(x)\, dx \right)} = \sum_{k=1}^{\infty} \left(\int \chi_{E_k}(x)\, dx \right)$

$$= \sum_{k=1}^{\infty} m(E_k) < \infty.$$

We have

$$\underline{\sum_{k=1}^{\infty} \left(\int a_k(x)\, dx \right)} = \int \left(\sum_{k=1}^{\infty} a_k(x) \right) dx < \infty$$

By above Corollary, $\sum_{k=1}^{\infty} a_k(x) < \infty$, a.e

$$\Rightarrow m\left(\left\{ x \,/\, \sum_{k=1}^{\infty} a_k(x) = \infty \right\} \right) = 0.$$

$$\Rightarrow m\left(\left\{ x \,/\, x \in \text{infinitely many } E_k\text{'s} \right\} \right) = 0.$$
