Keerti P. Charantimath 19MA20059

ANT Assignment 1

Q. Find Error in the differenciation of lagrange polynomial

⇒ We know

$$E(x) = f(x) - p_n(x) = \frac{\prod_{0}^{n}(x-x_i)}{(n+1)!} f^{n+1}(\xi)$$

$$\Rightarrow E'(x) = f'(x) - p_n'(x) = \frac{1}{(n+1)!}\frac{d}{dx}\prod_{i=0}^{n}(x-x_i)$$

Error in differenciation

$$= f'(x) - p_n'(x) = \frac{1}{(n+1)!}\frac{d}{dx}\prod_{i=0}^{n}(x-x_i)$$

Q Find $y = y(x)$ in the BVP

$$y'' + A(x)\, y' + B(x)\, y = C(x), \quad y(0) = y_0, \; y(a) = y_a$$

→ $$y'_i = \frac{y_{i+1} - y_{i-1}}{2h}, \quad y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$y'' + Ay' + By = C$$

Discretizing

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + A_i \frac{y_{i+1} - y_{i-1}}{2h} + B_i y_i = C_i$$

$$\Rightarrow \left(\frac{1}{h^2} + \frac{A_i}{2h}\right) y_{i+1} + \left(\frac{-2}{h^2} + B_i\right) y_i + \left(\frac{1}{h^2} - \frac{A_i}{2h}\right) y_{i-1} = C_i$$

$$\Rightarrow \alpha_i y_{i+1} + \beta_i y_i + \gamma_i y_{i-1} = C_i$$

for $i = 1$, $\alpha_1 y_2 + \beta_1 y_1 = C_1 - \gamma_1 y_0$, as y_0 is known

$i = n-1$, $\beta_{n-1} y_{n-1} + \gamma_{n-1} y_{n-2} = C_{n-1} - \alpha_{n-1} y_n$ as, y_n is known

$$\begin{bmatrix} \beta_1 & \gamma_1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & 0 \cdots & 0 \\ 0 & \alpha_3 & \beta_3 & \gamma_3 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \alpha_{n-2} & \beta_{n-2} & \gamma_{n-2} \\ 0 & 0 & 0 & & \alpha_{n-1} & \beta_{n-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} C_1' \\ \vdots \\ \vdots \\ C_{n-1}' \end{bmatrix}$$

for $i = 2$ to $n-2$, $C_i' = C_i$

$i = 1$, $C_1' = C_1 - \gamma_1 y_0$,

$i = n-1$, $C_{n-1}' = C_{n-1} - \alpha_{n-1} y_n$

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Using ~~Eo~~ Thomas algo we get

$$\begin{bmatrix} 1 & \gamma_1' & & & \\ 0 & 1 & \gamma_2' & & \vdots \\ \vdots & 0 & \ddots & & 0 \\ \vdots & \vdots & & 1 & \gamma_{n-2}' \\ \vdots & \vdots & \cdots & & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} c_1'' \\ \vdots \\ \vdots \\ c_{n-1}'' \end{bmatrix}$$

here

$$\gamma_1' = \frac{\gamma_1}{\beta_1}, \quad c_1'' = \frac{c_1'}{\beta_1}$$

& $$\gamma_i' = \frac{\gamma_i}{\beta_i - \alpha_i \gamma_{i-1}'}, \quad c_i'' = \frac{c_i' - \alpha_{i-1} c_{i-1}'}{\beta_i - \alpha_i \gamma_{i-1}'} \quad \text{for } i = 2 \text{ to } n-1$$

Solving above we get

$$y_{n-1} = c_{n-1}''$$

$$y_i = c_i'' - \gamma_i' y_{i+1} \quad \text{for } i = 1 \text{ to } n-2$$

8) Discretize $y'' - 2xy' - 2y = -4x$, $h = 0.1$, $y(0) - y'(0) = 0$
$2y(1) - y'(1) = 1$

→ $n = 10$

2^{nd} Order forward difference

$$y_0 + \frac{3y_0 - 4y_1 + y_2}{2h} = 0$$

$$\Rightarrow (2h + 3) y_0 - 4y_1 + y_2 = 0 \rightarrow (1)$$

Backward diff. (2^{nd} Order)

$$2y_{10} - y'_{10} = 1 \Rightarrow 2y_{10} - \frac{3y_{10} - 4y_9 + y_8}{2h} = 1$$

$$\Rightarrow \frac{-1}{2h} y_8 + \frac{4}{2h} y_9 + \left(2 - \frac{3}{2h}\right) y_{10} = 1 \rightarrow (2)$$

$$y''_i - 2x_i y'_i - 2y_i = -4x_i$$

$$\Rightarrow \frac{y_{i+1} - 2y_i + y_{i-1}}{2h} - 2x_i \frac{y_{i+1} - y_{i-1}}{2h} - 2y_i = -4x_i$$

$$\Rightarrow \left(\frac{1}{h^2} + \frac{x_i}{h}\right) y_{i-1} + \left(\frac{-2}{h^2} - 2\right) y_i + \left(\frac{1}{h^2} + \frac{x_i}{h}\right) y_{i+1} = -4x_i$$

for $i - 1$ to $n-1$ → (3)

Using (1), (2), & (3), we obtain a tridiagonal system of $(n+1) \times (n+1)$ dimensions, which can be solved using Thomas algorithm

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Q $y'' = x+y$, $y(0) = y(1) = 0$, $h = 0.25$

→

$A_i = 0$, $B_i = -1$, $C_i = x_i = x_0 + ih$

$$\left(\frac{1}{h^2} + \frac{A_i}{2h}\right) y_{i+1} + \left(\frac{-2}{h^2} + B_i\right) y_i + \left(\frac{1}{h^2} - \frac{A_i}{2h}\right) y_{i-1} = C_i$$

for $i = 1$

$$\frac{1}{h^2} y_2 - \left(\frac{2}{h^2} + 1\right) y_1 = x_0 + h$$

$i = 2$

$$\frac{1}{h^2} y_3 - \left(\frac{2}{h^2} + 1\right) y_2 + \frac{1}{h^2} y_1 = x_0 + 2h$$

$i = 3$

$$-\left(\frac{2}{h^2} + 1\right) y_3 + \frac{1}{h^2} y_2 = x_0 + 3h$$

$$\begin{bmatrix} -(2/h^2+1) & 1/h^2 & 0 \\ 1/h^2 & -(2/h^2+1) & 1/h^2 \\ 0 & 1/h^2 & -(2/h^2+1) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_0 + h \\ x_0 + 2h \\ x_0 + 3h \end{bmatrix}$$

for $h = 0.25$ we get

$$\begin{bmatrix} -33 & 16 & 0 \\ 16 & -33 & 16 \\ 0 & 16 & -33 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.75 \end{bmatrix}$$

solving above we get

$y(0.25) = (y_1) = -0.03488$

$y(0.5) = y_2 = -0.0563258$

$y(0.75) = y_3 = -0.05003676$

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Q Derive Newton-cote's formula for integration & corresponding error for $n=1$ (Trapezoidal) & $n=2$ (Simpson's $1/3^{rd}$)

⇒ Trapezoidal rule formula derivation:-

Using linear lagrange polynomial

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} P_1(x)\,dx = \int_{x_0}^{x_1}\left[\frac{x-x_1}{x_0-x_1}f(x_0) + \frac{x-x_0}{x_1-x_0}f(x_1)\right]dx$$

$$= \frac{1}{2}\left[\frac{(x-x_1)^2}{x_0-x_1}f(x_0) + \frac{(x-x_0)^2}{x_1-x_0}f(x_1)\right]_{x_0}^{x_1}$$

$$= \frac{1}{2}\left[(x_1-x_0)f(x_0) + (x_1-x_0)f(x_1)\right]$$

Let $x_1 - x_0 = h$

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}\left(f(x_0) + f(x_1)\right) \longrightarrow \text{Trapezoidal rule.}$$

For error :-

Using lagrange polynomial interpolation error :-

$$E(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(n+1)!} f^{n+1}(\xi)$$

$E(t)$ for $n = 1$

$$\frac{1}{2}\int_{x_0}^{x_1}(x-x_0)(x-x_1)f''(\xi)\,dx = \frac{1}{2}f''(\xi)\int_{x_0}^{x_1}(x-x_0)(x-x_1)\,dx$$

$$= \frac{1}{2}f''(\xi)\left[\frac{x^3}{3} + \frac{1}{2}(x_0-x_1)x^2 + (x_0x_1)x\right]_{x_0}^{x_1}$$

$$= \frac{1}{2}f''(\xi)\left(\frac{-1}{6}\right)(x_1-x_0)^3 = \frac{-h^3}{12}f''(\xi)$$

Error of Trapezoidal formula $= \frac{-h^3}{12} f''(\xi)$

Simpsons $1/3$ rule derivation:-

Taylor expansion:-

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2} \left[f(x_1) + (x-x_1) f'(x_1) + \frac{(x-x_1)^2}{2!} f''(x_1) + \frac{(x-x_1)^3}{3!} f'''(x_1) + \frac{(x-x_1)^4}{4!} f^{iv}(\xi) \right] dx$$

here $x_1 = (x_0 + x_2)/2 = (x_0 + x_0 + 2h)/2 = x_0 + h$

$$\int_{x_0}^{x_2} f(x)\,dx = \left[f(x_1)x + \frac{(x-x_1)^2}{2!} f'(x_1) + \frac{(x-x_1)^3}{3!} f''(x_1) + \frac{(x-x_1)^4}{4!} f'''(x_1) + \frac{(x-x_1)^5}{5!} f^4(\xi) \right]_{x_0}^{x_2}$$

$x_0 + 2h = x_2$ & $x_0 + h = x_1$

$$\int_{x_0}^{x_2} f(x)\,dx = 2h\, f(x_1) + \frac{h^3}{3} f''(x_1) + \frac{h^5}{60} f^4(\xi)$$

$$= 2h\, f(x_1) + \frac{h^3}{3}\left(\frac{f(x_0) - 2f(x_1) + f(x_2)}{h^2} \right) + \frac{h^5}{60} f^4(\xi)$$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2) \right] - \frac{h^5}{90} f^4(\xi)$$

here $\frac{h^5}{90} f^4(\xi)$ is the error

Q. Truncation error for CDS & its consistancy

⇒ Using Taylor expansion

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!} y_i'' + \dots \rightarrow (1)$$

$$y_{i-1} = y_i - hy_i' + \frac{h^2}{2!} y_i'' - \dots \rightarrow (2)$$

(1) - (2), we get

$$y_{i+1} - y_{i-1} = 2hy_i' + 2h^3 y_i''' + 2h^5 y_i^{v} + \dots$$

$$\Rightarrow y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2)$$

$$\Rightarrow TE = 2h^2 [y_i''' + h^2 y_i^{(5)} + h^7 y_i^{(7)} \dots]$$

$$\lim_{h \to 0} TE = 0$$

∴ TE is consistant.

Q. $x^2 y'' + xy' = 1$, $h = 0.1$, $y(1) = 0$, $y(1.4) = 0.0566$

→ Using central Difference scheme

$$y_{i+1}\left(\frac{1}{h^2} + \frac{1}{x_i \cdot 2h}\right) + y_i\left(\frac{-2}{h^2}\right) + y_{i-1}\left(\frac{1}{h^2} - \frac{1}{2hx_i}\right) = \frac{1}{x_i^2}$$

$$\Rightarrow y_{i-1}\left(\frac{x_i^2}{h^2} - \frac{x_i}{2h}\right) + y_i\left(-\frac{2x_i^2}{h^2}\right) + y_{i+1}\left(\frac{x_i^2}{h^2} + \frac{x_i}{2h}\right) = 1$$

for $i = 1$ to $n-1$ & $h = 0.1$ we have

$$\begin{bmatrix} -220 & 115.5 & 0 \\ 60 & -240 & 180 \\ 0 & 123.5 & -260 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -6.7259 \end{bmatrix}$$

On solving the above, we get

$y_1 = y(1.1) = 0.00990$
$y_2 = y(1.2) = 0.02751$
$y_3 = y(1.3) = 0.03894$

Q. $y^{iv} + 2y = \frac{x^2}{9} + \frac{2}{3}x + 4$, $h = 1$, $y(0) = y'(0) = y(3) = y'(3) = 0$

$\Rightarrow$ $y'(0) = 0 \Rightarrow y_{-1} = y_1$ | $y(0) = y_0 = 0$

$y'(3) = y_3' = 0 \Rightarrow y_4 = y_2$ | $y(3) = y_3 = 0$

discretizing with $h = 1$

$$y_{i+2} - 4y_{i+1} + 8y_i - 4y_{i-1} + y_{i-2} = \frac{x^2}{9} + \frac{2}{3}x + 4$$

$i = 1$

$$y_3 - 4y_2 + 8y_1 - 4y_0 + y_{-1} = \frac{43}{9}$$

$$\Rightarrow 9y_1 - 4y_2 = \frac{43}{9} \longrightarrow (1)$$

$i = 2$

$$y_4 - 4y_3 + 8y_2 - 4y_1 + y_0 = \frac{52}{9}$$

$$\rightarrow 9y_2 - 4y_1 = \frac{52}{9} \longrightarrow (2)$$

Solving (1) & (2)

$y(2) = y_2 = 128/117 = 1.09401$

$y(1) = y_1 = 119/117 = 1.017094$

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Q. $y^{iv} - y''' + y = x^2$, $h = 1/3$, $y(0) = y'(0) = 0$, $y(1) = 2$, $y'(1) = 0$

$\rightarrow$ $y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2} - \frac{h}{2}[y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}] + y_i h^4 = x_i^2 h^4$

$i = 1, 2$

$i = 1$

$y_3 - 4y_2 + 6y_1 - 4y_0 + y_{-1} - \frac{1}{6}[y_3 - 2y_2 + 2y_0 - y_{-1}] + \frac{y_1}{81} = \frac{1}{729}$

~~now~~ now, $y'(0) = 0 \Rightarrow y_{-1} = y_1$ similarly, $y_4 = y_2$

& $y_0 = 0$, $y_3 = 2$

$\frac{1163}{2} y_1 - 297 y_2 = \frac{1}{9} - 135 \rightarrow (1)$

$i = 2$

$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 - \frac{1}{6}[y_4 - 2y_3 + 2y_1 - y_0] + \frac{y_2}{81} = \frac{4}{729}$

$\frac{1109}{2} y_2 - 351 y_1 = \frac{4}{9} + 594 \rightarrow (2)$

Solving (1) & (2)

$y_1 = y(\frac{1}{3}) = 0.466347$

$y_2 = y(\frac{2}{3}) = 1.367235$

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Q. Discretize $y'' + Ay' + By = C$ wrt $\alpha_0 y_0 + \alpha_1 y_0' = \gamma_1$

$\beta_0 y_n + \beta_1 y_n' = \gamma_2$

Find the reduced Tridiagonal system for it

$\Rightarrow y_i' = \frac{y_{i+1} - y_{i-1}}{2h} \Rightarrow y_0' = \frac{y_1 - y_{-1}}{2h} \Rightarrow y_{-1} = -2h y_0' + y_1$

$y_{-1} = \frac{\gamma_1 - \alpha_0 y_0}{\alpha_1}(-2h) + y_1$

Similarly $y_{n+1} = y_{n-1} + (2h)\frac{(\gamma_2 - \beta_0 y_n)}{\beta_1}$

now, $y''_i + A_i y'_i + B_i y_i = C_i$

$\Rightarrow \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + A_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + B_i y_i = C_i$

$\Rightarrow y_{i-1}\left(\frac{1}{h^2} - \frac{A_i}{2h}\right) + y_i\left(B_i + \frac{2}{h^2}\right) + y_{i+1}\left(\frac{1}{h^2} + \frac{A_i}{2h}\right) = C_i$

The above is valid for $i = 1$ to $n-1$

for $i = 0$

$y_{-1}\left(\frac{1}{h^2} - \frac{A_0}{2h}\right) + y_0\left(B_0 + \frac{2}{h^2}\right) + y_1\left(\frac{1}{h^2} + \frac{A_0}{2h}\right) = C_0$

$(y_1 - 2h y_0')\left(\frac{1}{h^2} - \frac{A_0}{2h}\right) + y_0\left(B_0 + \frac{2}{h^2}\right) + y_1\left(\frac{1}{h^2} + \frac{A_0}{2h}\right) = C_0$

$y_0\left(B_0 + \frac{2}{h^2} - \frac{2}{h} + A_0\right) + y_1\left(\frac{2}{h^2}\right) = C_0$

for $i = n$

$y_{n-1}\left(\frac{1}{h^2} - \frac{A_n}{2h}\right) + y_n\left(B_n + \frac{2}{h^2}\right) + y_{n+1}$

$\left(y_1 - \frac{2h\gamma_1}{\alpha_1} + \frac{2h\alpha_0}{\alpha_1} y_0\right)\left(\frac{1}{h^2} - \frac{A_0}{2h}\right) + y_0\left(B_0 + \frac{2}{h^2}\right) + y_1\left(\frac{1}{h^2} + \frac{A_0}{2h}\right) = C_0$

$y_0\left(B_0 + \frac{2}{h^2} + \frac{2\alpha_0}{\alpha_1 h} - \frac{\alpha_0}{\alpha_1} A_0\right.$

Let

$$a_i = \frac{1}{h^2} - \frac{A_i}{2h}, \quad b_i = B_i - \frac{2}{h^2}, \quad c_i = \frac{1}{h^2} + \frac{A_i}{2h}, \quad d_i = C_i$$

for $i = 1$ to $n-1$

$$y_{i-1} a_i + y_i b_i + y_{i+1} c_i = d_i$$

for $i = 0$

$$y_{-1} a_0 + y_0 b_0 + y_1 c_0 = d_0$$

$$\left(y_1 - 2h\frac{\gamma_1}{\alpha_1} + 2h\frac{\alpha_0}{\alpha_1} y_0\right) a_0 + y_0 b_0 + y_1 c_0 = d_0$$

$$y_1(a_0 + c_0) + y_0\left(2h\frac{\alpha_0}{\alpha_1} a_0 + b_0\right) = d_0 + 2h\frac{\gamma_1}{\alpha_1} a_0$$

for $i = n$

$$y_{n-1} a_n + y_n b_n + y_{n+1} c_n = d_n$$

$$y_{n-1} a_n + y_n b_n + \left(y_{n-1} + 2h\frac{\gamma_2}{\beta_1} - 2h\frac{\beta_0}{\beta_1} y_n\right) c_n = d_n$$

$$y_{n-1}(a_n + c_n) + y_n\left(b_n - \frac{\beta_0}{\beta_1}(2h) c_n\right) = d_n - 2h\frac{\gamma_2}{\beta_1} c_n$$

The above equations form the following tridiagonal system

$$\begin{bmatrix} 2h\frac{\alpha_0}{\alpha_1}a_0 + b_0 & a_0 + c_0 & \cdots & & & \\ a_1 & b_1 & c_1 & \cdots & & \\ 0 & a_2 & b_2 & \cdots & & \\ \vdots & \vdots & \vdots & & & \\ \vdots & \vdots & \vdots & & & 0 \\ \vdots & \vdots & \cdots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 \cdots 0 & a_n + c_n & & b_n - \frac{\beta_0}{\beta_1}(2h c_n) \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} d_0 + \frac{2h\gamma_1 a_0}{\alpha_1} \\ d_1 \\ \vdots \\ \vdots \\ \vdots \\ d_{n-1} \\ d_n - \frac{2h\gamma_2 c_n}{\beta_1} \end{bmatrix}$$

$$[A]_{n\times n}[y]_{n\times 1} = [B]_{n\times 1}$$

Prove the following can be solved using Block Diagonal Method

Q $y''' + 4y'' + y' - 6y = 1$, $y(0) = y'(0) = 0$, $y(1) = 1$, $h = 0.25$

⇒ let $z = y'$ → $\int_{x_{i-1}}^{x_i} z\,dx = \int_{x_{i-1}}^{x_i} y'\,dx \Rightarrow y_i - y_{i-1} = \frac{h}{2}(z_i - z_{i-1})$

$z_0 = 0$, $y_0 = 0$, $z_n = 1$ $(n = 4)$

$y_i - y_{i-1} = \frac{h}{2}(z_i - z_{i-1})$ for $i = 1$ to n gives n equations

Discretizing $z'' + 4z' + z - 6y = 1$

$$\Rightarrow \frac{z_{i+1} - 2z_i + z_{i+1}}{h^2} + \frac{\cancel{4}^{2}(z_{i+1} - z_{i-1})}{\cancel{2}h} + z_i - 6y_i = 1$$

$$\Rightarrow z_{i+1} - 2z_i + z_{i+1} + 2z_{i+1} - z_{i-1} + z_i - 6y_i = 1$$

for $i = 1$ to $n-1$ gives $n-1$ equations

In total we have $n + n - 1 = 2n - 1$ equations in variables $z_1, z_2 \dots z_{n-1}, y_1, y_2 \dots y_n$.

In total we have $2n-1$ variables & $2n-1$ equations.

Hence, we can solve this system to ~~fing~~ find the solution to the given problem

Q. $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = r(x)$; $0<x<a$

Show that if equi-spaced grid pts are used & p_i, q_i, r_i are continuous, control vol. method reduces to central diff. method

→ $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = r(x)$; $0<x<a$ ⟶ ①

$\Rightarrow p(x)y'' + p'(x)y' + q(x)y = r(x)$

$\Rightarrow y'' + \frac{p'(x)}{p(x)}y' + \frac{q(x)}{p(x)}y = \frac{r(x)}{p(x)}$ ⟶ ②

Applying central difference scheme

$$\left(\frac{1}{h^2} - \frac{p_i'}{2hp_i}\right)y_{i-1} + \left(\frac{q_i}{p_i} - \frac{2}{h^2}\right)y_i + \left(\frac{1}{h^2} + \frac{p_i'}{2hp_i}\right)y_{i+1} = \frac{r_i}{p_i} \longrightarrow ③$$

Discretizing ① by control volume.

$$p_{i+\frac{1}{2}}\left(\frac{y_{i+1}-y_i}{\delta x_i}\right) - p_{i-\frac{1}{2}}\left(\frac{y_i - y_{i-1}}{\delta x_{i-1}}\right) + y_i\left(\frac{q_{i-}\delta x_{i-1} + q_{i+}\delta x_i}{2}\right)$$
$$= \frac{r_{i-}\delta x_{i-1} + r_{i+}\delta x_i}{2}$$

~~As / p,q,r~~

As p, q, r are cont., $q_{i-} = q_{i+} = q_i$. Similarly for r_i

Also, step size is uniform. Hence $\delta x_i = \delta x_{i-1} = h$

We get

$$\left(\frac{p_{i-\frac{1}{2}}}{h^2}\right)y_{i-1} + \left(q_i - \frac{p_{i+\frac{1}{2}}}{h^2} - \frac{p_{i-\frac{1}{2}}}{h^2}\right)y_i + \left(\frac{p_{i+\frac{1}{2}}}{h^2}\right)y_{i+1} = r_i$$

As $p_{i-\frac{1}{2}} \simeq \frac{p_{i-1}+p_i}{2}$ & $p_{i+\frac{1}{2}} \simeq \frac{p_i + p_{i+1}}{2}$

we get

$$\left(\frac{p_{i-1}+p_i}{2h^2}\right) y_{i-1} + \left(q_i - \frac{p_i + p_{i+1}}{2h^2} - \frac{p_{i-1}+p_i}{2h^2}\right) y_i$$
$$+ \left(\frac{p_i + p_{i+1}}{2h^2}\right) y_{i+1} = r_i$$

Dividing throughout by p_i

$$\left(\frac{1}{h^2} - \frac{p_i - p_{i-1}}{2h^2 p_i}\right) y_{i-1} + \left(\frac{q_i}{p_i} - \frac{1}{h^2} - \frac{\frac{p_{i+1}+p_{i-1}}{2}}{h^2 p_i}\right) y_i + \left(\frac{1}{h^2} + \frac{p_{i+1}-p_i}{2h^2 p_i}\right) y_{i+1}$$
$$= \frac{r_i}{p_i}$$

from forward & backward approximation we know:- $\frac{p_i - p_{i-1}}{h} \simeq p_i'$ & $\frac{p_{i+1} - p_i}{h} \simeq p_i'$

Also, $\frac{p_{i+1} + p_{i-1}}{2h} \simeq p_i$

Using above approximations we get

$$\left(\frac{1}{h^2} - \frac{p_i'}{2hp_i}\right) y_{i-1} + \left(\frac{q_i}{p_i} - \frac{2}{h^2}\right) y_i + \left(\frac{1}{h^2} + \frac{p_i'}{2hp_i}\right) y_{i+1} = \frac{r_i}{p_i} \rightarrow 4$$

Clearly $3 = 4$.

Hence proved

Q) $y'' + 2xy' + 2y = 4x$; $y(0) = 1$, $y(0.5) = 1.279$, $h = 0.1$

→ Using Central diff. scheme

$$\left(\frac{1}{h^2} - \frac{\cancel{2}x_i}{\cancel{2}h}\right) y_{i-1} + \left(2 - \frac{2}{h^2}\right) y_i + \left(\frac{1}{h^2} + \frac{\cancel{2}x_i}{\cancel{2}h}\right) y_{i+1} = 4x_i \longrightarrow (1)$$

For $h = 0.1$, we have 4 unknowns - we get the following System

$$\begin{bmatrix} -198 & 101 & 0 & 0 \\ 98 & -198 & 102 & 0 \\ 0 & 97 & -198 & 103 \\ 0 & 0 & 96 & -198 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -493/5 \\ 4/5 \\ 6/5 \\ -131.416 \end{bmatrix}$$

Solving the above, we get

$y_1 = y(0.1) = 1.0902945$

$y_2 = y(0.2) = 1.1611743$

$y_3 = y(0.3) = 1.2[illegible]4344$

$y_4 = y(0.4) = 1.25249$

9) Solve $y'' - 2y = 0$; $y(0) = 1$, $y'(1) = 0$, $h = 0.1$

→ Central difference scheme

$$\left(\frac{1}{h^2}\right) y_{i-1} - \left(2 + \frac{2}{h^2}\right) y_i + \left(\frac{1}{h^2}\right) y_{i+1} = 0 \longrightarrow (1)$$

y_1 to $y_n \rightarrow n$ unknowns.

$$y_n' = y'(1) = 0 = \frac{3y_n - 4y_{n-1} + y_{n-2}}{2h}$$

$\Rightarrow 3y_n - 4y_{n-1} + y_{n-2} = 0$ (from backward diff.) → (2)

for $i = 1$ to $n-1$ in (1) & (2) we get the following system. (taking $h = 0.2$)

$$\begin{bmatrix} -52 & 25 & 0 & 0 & 0 \\ 25 & -52 & 25 & 0 & 0 \\ 0 & 25 & -52 & 25 & 0 \\ 0 & 0 & 25 & -52 & 25 \\ 0 & 0 & 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} -25 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above, we get

$y_1 = y(0.2) = 0.7861564$

$y_2 = y(0.4) = 0.6352053$

$y_3 = y(0.6) = 0.5350707$

$y_4 = y(0.8) = 0.4777417$

$y_5 = y(1) = 0.4586320$

[Solution using 2^{nd} order backward difference]

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→ Using central diffrence scheme

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - 2y_i = 0 \Rightarrow \left(\frac{1}{h^2}\right) y_{i-1} - \left(2 + \frac{2}{h^2}\right) y_i + \left(\frac{1}{h^2}\right) y_{i+1} = 0 \rightarrow (1)$$

y_1 to $y_n \rightarrow n$ unknowns.

$$y'(1) = y_n' = 0 \Rightarrow \frac{y_{n+1} - y_{n-1}}{2h} = 0 \Rightarrow \underset{\text{fictitious pt.}}{\underset{\downarrow}{y_{n+1}}} = y_{n-1} \rightarrow (2)$$

for $i = 1$ to n in (1) & using (2) for $i = n$ we get
$(h = 0.2)$

$$\begin{bmatrix} -52 & 25 & 0 & 0 & 0 \\ 25 & -52 & 25 & 0 & 0 \\ 0 & 25 & -52 & 25 & 0 \\ 0 & 0 & 25 & -52 & 25 \\ 0 & 0 & 0 & 50 & -52 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} -25 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving above we get,

$y_1 = y(0.2) = 0.7865088$

$y_2 = y(0.4) = 0.6359383$

$y_3 = y(0.6) = 0.536243$

$y_4 = y(0.8) = 0.4794468$

$y_5 = y(1) = 0.46100658$

[Solution using ficticious pts]