Mathematical Methods

Test - I

7.9.2021

Time 55 min

12 to 12:55 PM

Pg-1

Q1. For the IVPs given below, find the largest interval in which a unique solution is guaranteed to exist

(a) $(x-3)y' + \ln(x)y = 2x$, $y(1) = 2$ Ans: $(0,3)$

(b) $(x^2-81)y' + 5e^{3x}y = \sin x$, $y(1) = 10\pi$ Ans: $(-9,9)$

(c) $\sqrt{(16-x^2)}\,y'' + \ln(x+1)y' + \cos(x)y = 0$

$y(0) = 2$, $y'(0) = 0$ Ans: $(-1,4)$

Q2. Consider the set of functions

(1) $f(x) = 9\cos 2x$ $g(x) = 2\cos^2 x - 2\sin^2 x$ $\forall x$

(2) $f(t) = 2t^2$ $g(t) = t^4$ $\forall t$

Which one of the following options is correct?

(i) Both are L.I.

(ii) " " L.D.

(iii) The 1st set is L.I. and the 2nd set is L.D.

(iv) " " " " L.D. and " " " " L.I.

Solⁿ:

$9c_1\cos 2x + 2c_2(\cos^2 x - \sin^2 x) = 0$

$\Rightarrow 9c_1\cos 2x + 2c_2\cos 2x = 0$

$\Rightarrow (9c_1 + 2c_2)\cos 2x = 0$

Take $c_1 = 2$, $c_2 = -9$. Hence f and g are L.D.

For the 2nd case,

$$W = \begin{vmatrix} 2t^2 & t^4 \\ 4t & 4t^3 \end{vmatrix} = 4t^5 \neq 0 \text{ if } t \text{ is not } 0.$$

$\therefore$ L.I.

Q3. Consider the non-homogeneous BVP posed on $[0,\pi]$ as $y''+y=0$ with $y(0)=0$ and $y(\pi)=1$. Then which of the following is true?

(i) Both the non-h BVP and the h BVP have no solution

(ii) The non-h BVP has unique solⁿ., h BVP has trivial solⁿ.

(iii) Both have infinite number of solutions

(iv) The non-h BVP has no solⁿ., h BVP has inf. no. of solⁿs.

Q4. (a) Let y_1 and y_2 be solutions of the diff. eqn. $y''+p(t)y'+q(t)y=0$ where p and q are continuous on $[a,b]$. Then the Wronskian $W(y_1,y_2)(t)$ is given by (i) $ce^{\int q(t)dt}$ (ii) $ce^{-\int \frac{p(t)}{q(t)}dt}$ (iii) $ce^{\int \frac{q(t)}{p(t)}dt}$

(iv) $ce^{-\int p(t)dt}$

Solⁿ:

$$W = y_1y_2' - y_1'y_2$$

$$W' = y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2'$$

$$= y_1y_2'' - y_1''y_2$$

$y_1'' + p(t)y_1' + q(t)y_1 = 0$ —(1) as y_1 is a solⁿ. of the ODE

$y_2'' + p(t)y_2' + q(t)y_2 = 0$ —(2) as y_2 " " " " ODE

Eq (1) × $(-y_2)$ + Eq. (2) × y_1

$$(y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0$$

$$\Rightarrow W' + p(t)W = 0$$

$\therefore \frac{dW}{W} = -p(t)\,dt$ The result follows.

Q4 (b) If the Wronskian of two solutions of $t^4y'' - 2t^3y' - t^8y = 0$ on $[1,5]$ is $ct^{\frac{m}{n}}$, c being a constant, then $m+n$ is $\underline{\quad 3 \quad}$.

Sol^n:

$$t^4y'' - 2t^3y' - t^8y = 0$$

$$y'' - \frac{2}{t}y' - t^4y = 0$$

$$W = ce^{-\int -\frac{2}{t}dt} = ce^{2\ln t} = ct^2$$

Q5. The adjoint equation of $x^2y'' + (2x^3+1)y' + y = 0$ is

(i) $x^2y'' + (2x + 4x^3 - 2)y' - 2y(1-3x^2) = 0$

(ii) $x^2y'' - (3x + 2x^3 - 1)y' + 3y(1+2x^2) = 0$

~~(iii)~~ $x^2y'' + (4x - 2x^3 - 1)y' + 3y(1-2x^2) = 0$

(iv) $x^2y'' - (4x + 2x^3 + 1)y' - 2y(1+3x^2) = 0$

Sol^n: $x^2y'' + (2x^3+1)y' + y = 0$ —(1)

Comparing (1) with $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$

we have $a_0(x) = x^2$, $a_1(x) = 2x^3+1$ and $a_2(x) = 1$

The adjoint eqn. is of the form

$$\frac{d^2}{dx^2}[a_0(x)y] - \frac{d}{dx}[a_1(x)y] + a_2(x)y = 0$$

$$\Rightarrow \frac{d^2}{dx^2}(x^2y) - \frac{d}{dx}\{(2x^3+1)y\} + y = 0$$

$$\Rightarrow \frac{d}{dx}\left[2xy + x^2\frac{dy}{dx}\right] - 6x^2y - (2x^3+1)y' + y = 0$$

$$\Rightarrow 2y + 2xy' + 2xy' + x^2y'' - 6x^2y - (2x^3+1)y' + y = 0$$

$$\Rightarrow x^2y'' + (4x - 2x^3 - 1)y' + 3y(1-2x^2) = 0$$

Q6. Use method of variation of parameter to find a P.I. $y_p(t)$ of the ODE $y''-2y'+y=\frac{e^t}{1+t^2}+3e^t$. The answer will be

$\boxed{\text{Option (ii)}}$ $y_p(t)=-\frac{1}{2}e^t\ln(1+t^2)+te^t\tan^{-1}t+\frac{3}{2}t^2e^t$

Q7. Consider the BVP $y''-y=0$ in $[0,l]$ with $y(0)=y'(0)$, $y(l)+\lambda y'(l)=0$, λ is a constant. Then which one of the following is true for the Green's function of the BVP?

$\boxed{\text{Option (iv)}}$ $G(x,t)=-\frac{1}{2}\left(\frac{1-\lambda}{1+\lambda}\right)e^{x+t-2l}+\frac{1}{2}e^{x-t},\ 0\le x<t$

Sol^n) $G(x,t)=\begin{cases} a_1e^x+a_2e^{-x} & 0\le x<t \\ b_1e^x+b_2e^{-x} & t<x\le l\end{cases}$

(i) $G(x,t)$ is continuous at $x=t$ i.e. $(b_1-a_1)e^t+(b_2-a_2)e^{-t}=0$

(ii) $\left(\frac{\partial G}{\partial x}\right)_{x=t+}-\left(\frac{\partial G}{\partial x}\right)_{x=t-}=-1$

$\Rightarrow (b_1-a_1)e^t-(b_2-a_2)e^{-t}=-1$

(iii) $G(x,t)_{x=0}=\left(\frac{\partial G}{\partial x}\right)_{x=0} \Rightarrow \left[a_1e^x+a_2e^{-x}\right]_{x=0}=\left[a_1e^x-a_2e^{-x}\right]_{x=0}$

$\Rightarrow a_1+a_2=a_1-a_2 \Rightarrow a_2=0$

$\left[G(x,t)\right]_{x=l}+\lambda\left(\frac{\partial G}{\partial x}\right)_{x=l}=0$

$\Rightarrow \left[b_1e^x+b_2e^{-x}\right]_{x=l}+\lambda\left[b_1e^{x}-b_2e^{-x}\right]_{x=l}=0$

$\Rightarrow b_1e^l+b_2e^{-l}+\lambda(b_1e^l-b_2e^{-l})=0 \Rightarrow (1+\lambda)b_1e^l+(1-\lambda)b_2e^{-l}=0$

Setting $b_1-a_1=c_1,\ b_2-a_2=c_2$; $\left.\begin{aligned} c_1e^t+c_2e^{-t}&=0\\ c_1e^t-c_2e^{-t}&=-1\end{aligned}\right\}$ Solving $c_1=-\frac{1}{2}e^{-t}$, $c_2=\frac{1}{2}e^t$

$\therefore b_1-a_1=-\frac{1}{2}e^{-t}$; $a_2=0$

$b_2-a_2=\frac{1}{2}e^t$; $\therefore b_2=\frac{1}{2}e^t$

$(1+\lambda)b_1e^l+(1-\lambda)\frac{1}{2}e^te^{-l}=0 \Rightarrow b_1=-\frac{1}{2}\left(\frac{1-\lambda}{1+\lambda}\right)e^{t-2l}$

$a_1=-\frac{1}{2}\left(\frac{1-\lambda}{1+\lambda}\right)e^{t-2l}+\frac{1}{2}e^{-t}$