Integral Equations:

Def^n An integral equ^n is an equ^n in which an unknown fn appears under one or more integral signs.

eg. $t, s \in [a,b]$

$$f(s) = \int_a^b k(s,t)\, g(t)\, dt$$

$$g(s) = f(s) + \int_a^b k(s,t)\, g(t)\, dt$$

$$g(s) = \int_a^b k(s,t)\, [g(t)]^2\, dt.$$

$g(s) \rightarrow$ unknown, rest known

* Linear Integral equn:

$L[g(s)] = f(s)$ } linear integral equn

$\hookrightarrow$ integral operator

for any c_1, c_2 we have, if L is linear,

$$L[c_1 g_1(s) + c_2 g_2(s)] = c_1 L g_1(s) + c_2 L g_2(s)$$

$$h(s)\, g(s) = f(s) + \lambda \int_a \underbrace{k(s,t)}_{\text{kernel}}\, g(t)\, dt \quad \text{---} \; (*)$$

f, h, k are known fns; g unknown fn

λ non-zero real/complex param.

Upper limit may be fixed/variable

$\rightarrow$ Special cases in $(*)$

(1) Fredholm integral equns. — Upper limit is fixed (say, b)

(a) Fredholm integral equn. of 1st kind :— $h(s) = 0$

$$f(s) + \lambda \int_a^{b \rightarrow \text{fixed}} k(s,t)\, g(t) = 0$$

(b) Fredholm integral equn. of 2^{nd} kind :- $h(s) = 1$

$$g(s) = f(s) + \lambda \int_a^b k(s,t)\, g(t)\, dt$$

~~Volt~~ (c) Homogeneous Fredholm Integral equn :- Spcl case of (b); $f(s) = 0$

$$g(s) = \lambda \int_a^b k(s,t)\, g(t)\, dt$$

(2) Volterra Integral Equations : Defined as above the " " " of 1^{st}, 2^{nd} kind & homogeneous equns, except $b = s$ is the variable upper limit of integration.

(3) Singular Integral Equations :- When one or both limits of integration becomes infinite, or when the kernel becomes infinite at one or more points within the range of integration, the integral is called singular.

eg. $g(s) = f(s) + \lambda \int_{-\infty}^{\infty} \exp(-|s-t|)\, g(t)\, dt$

& $f(s) = \int_0^s \frac{1}{(s-t)^{\alpha}}\, g(t)\, dt$, $\alpha \in (0,1)$

are S.I.E.s.

Separable: $f(x,y) = f_1(x) f_2(y)$

LI: if $\sum \alpha_i a_i = 0 \Rightarrow \alpha_i = 0 \ \forall i$

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* **Regularity condⁿ:** We assume that the fns are either continuous or integrable or square-integrable

→ **Square-integrable:** $g(t)$ is ~ if
$$\int_a^b |g(t)|^2 dt < \infty$$

This is called $\mathcal{L}_2$-functions (L^2-functions)

→ **Square integrable Kernel:** A kernel $K(s,t)$ as a fn of two variables is said to be ~ / L^2-fn if

(a) $\forall s,t$ in the square s.t. $s,t \in [a,b]$
$$\int_a^b \int_a^b |K(s,t)|^2 ds\, dt < \infty$$

(b) $\forall s \in [a,b]$
$$\int_a^b |K(s,t)|^2 ds < \infty$$

(c) $\forall t \in [a,b]$
$$\int_a^b |K(s,t)|^2 dt < \infty$$

* **Special kinds of kernels**

1. **Separable or Degenerate Kernels:**
$K(s,t)$ can be expressed as a sum of finite no. of terms, each of which is the product of ~~the~~ a fn of s only & a fn of t only, ie,
$$K(s,t) = \sum_{i=1}^{n} a_i(s) b_i(t) \quad \text{--- (5)}$$

The fns. $a_i(s)$ (or $b_i(t)$) can be assumed to be LI, otherwise the no. of terms in (5) can be reduced.

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Note: LI : $\{a_i(s)\}_{i=1}^{n}$ or $\{a_1(s) - a_n(s)\}$ is LI if
$\sum \alpha_i a_i(s) = 0 \Rightarrow \alpha_i = 0 \ \forall i = 1$ to n

② Symmetric Kernel :
A complex valued fn $K(s,t)$ is called symmetric/ Hermitian if $K(s,t) = K^*(t,s)$, where $K^*(t,s)$ is complex conjugate of $K(t,s)$. For real kernel, $K(s,t) = K(t,s)$

→ Eigenvalue & Eigenvector $Ax = \lambda x$
When the eqn $Ax = \lambda x$ has a non-trivial soln, then the λ & x are known as ~.

$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$
By Cramer's rule, $\det(A - \lambda I) = 0$ to avoid only trivial soln.

* Eigenvalues & Eigenfns :
If the linear homogeneous integral eqn.

$$\lambda \int_a^b K(s,t)\, g(t)\, dt = g(s) \quad — ⑥$$

has non-trivial solution $g(s)$, the the param. λ is called eigenvalue & the non-trivial soln $g(s)$ is called the corresp. eigenfn.

→ Note:- In operator form :
$Lg(s) = \mu g(s)$, ⊛ where L is integral operator
$L = \int_a^b k(s,t)(\cdot)\, dt$

if ⊛ has non-trivial soln., then we say that μ is eigenvalue

& corresp. non-trivial soln. is eigenfn.

The homogeneous eqn. ⑥ can be written as

$$\int_a^b k(s,t)\, g(t)\, dt = \mu g(s), \quad \mu = 1/\lambda$$

If we follow operator defn then $\mu (= 1/\lambda)$ is the eigenvalue, but linear integral eqn. is studied in the form ⑥ so instead of $\frac{1}{\lambda}$ we use λ as eigenvalue.

* Convolution Integral:
If $K(s,t) = k(s-t)$, where k is a fn of one variable, then the kernel is called convoln kernel & corresp. integral eqns.

$$g(s) = f(s) + \lambda \int_a^b k(s-t)\, g(t)\, dt$$

or,

$$g(s) = f(s) + \lambda \int_a^s k(s-t)\, g(t)\, dt$$

$\int_0^s k(s-t) g(t) dt$

$s - t = u$

$t = s - u$ | $u = s$ to 0

$dt = -du$

$-\int_s^0 k(u) g(s-u) du$

$= \int_0^s k(u) g(s-u) du$

$u \to t$

$$\boxed{\int_0^s k(s-t) g(t) dt = \int_0^s k(t) g(s-t) dt}$$

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Inner Product of two fns:

* Inner product: of two fns ϕ & ψ is

$$\langle \phi, \psi \rangle \triangleq \int \phi(t)\, \psi^*(t)\, dt$$

Denoted by $\langle \phi, \psi \rangle^a$

Here $\phi(s)$ & $\psi(s)$ are fns of real variable $s \in [a, b]$

* Orthogonal fns: The two L^2 fns ϕ & ψ are ~ if $\langle \phi, \psi \rangle = 0$.

* Norm of an L^2 fn ($\|\phi\|$)

$$\|\phi\| = \left[\int_a^b \phi(t)\phi^*(t)\,dt\right]^{1/2} = \left(\int |\phi(t)|^2 dt\right)^{1/2}$$

L^2-norm

* Schwarz inequ.

$$|\langle \phi, \psi \rangle| \leq \|\phi\| \, \|\psi\|$$

Minkowski inequ.

$$\|\phi + \psi\| \leq \|\phi\| + \|\psi\|$$

→ $Ax = B$ → consistent when rank A = rank B

→ unique soln. when rows/cols are all indep.

→ Integral equns w/ separable Kernel:

$$g(s) = f(s) + \lambda \int_a^b K(s,t)\,g(t)\,dt \quad ⓵ \; ; \quad K(s,t) = \sum_{i=1}^{n} a_i(s) b_i(t) \quad ⓶$$

$$\cancel{g(s) = f(s) + \lambda \sum_{i=1}^{n} a_i(s) \int_a^b b_i(t) g(t)\,dt}$$

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} \int_a^b a_i(s)\, b_i(t)\, g(t)\,dt$$

or, $$g(s) = f(s) + \lambda \int_a^b \sum_{i=1}^{n} a_i(s)\, b_i(t)\, g(t)\,dt$$

(exchange only when not infinite sum)

$$\Rightarrow g(s) = f(s) + \lambda \int_a \left(\sum_{i=1}^{n} a_i(s)\, b_i(t)\right) g(t)\,dt$$

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} \left\{ a_i(s) \left(\int_a^b b_i(t)\, g(t)\,dt \right) \right\} \quad — ⓷$$

Let $c_i = \int_a^b b_i(t) g(t)\,dt$ (unknown loc. of g) — ⓸

$$\Rightarrow g(s) = f(s) + \lambda \sum_{i=1}^{n} a_i(s)\, c_i \quad — ⓹$$

$$\Rightarrow g(s) - f(s) = \lambda \sum_{i=1}^{n} a_i(s)\, c_i \qquad (a_i\text{'s are LI})$$

$$g(s)-f(s) = \lambda \sum_{i=1}^{n} c_i\, a_i(s) \quad \text{--- } 6$$

Substituting 5 in 1,

$$g(s) = f(s) + \lambda \int_a^b K(s,t)\left(f(t) + \lambda \sum_{i=1}^{n} c_i b_i(t)\right) dt$$

$$\text{"} = f(s) + \lambda \int_a^b K(s,t) f(t)\, dt + \lambda^2 \int_a^b \left(K(s,t) \sum_{i=1}^{n} c_i b_i(t) dt\right)$$

or, $$g(s) = f(s) + \lambda \sum_{i=1}^{n} a_i(s) \int_a^b b_i(t) f(t)\, dt + \lambda^2 \sum_{i=1}^{n} a_i(s) c_i \int_a^b b_i^2(t)\, dt$$

from 2,

$$g(s) - f(s) = \lambda \sum_{i=1}^{n} a_i(s) \int_a^b b_i(t) g(t)\, dt \quad \text{--- } 2$$

<u>From 3 & 6,</u>

$$\lambda \sum_{i=1}^{n} c_i a_i(s) = \lambda \sum_{i=1}^{n} a_i(s) \int_a^b b_i(t) g(t)\, dt$$

$$\Rightarrow \lambda \sum_{i=1}^{n} a_i(s) \left(c_i - \int_a^b b_i(t) g(t)\, dt\right) = 0$$

$\because$ All $a_i(s)$'s are LI

$$\Rightarrow \quad c_i = \int_a^b b_i(t) g(t)\, dt \quad \forall\ i = 1 \text{ to } n$$

or, $$c_i = \int_a^b b_i(t)\left[f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t)\right] dt \quad \forall\ i = 1 \text{ to } n \quad \text{--- } 7$$

(from 5)

Let $f_i = \int_a^b b_i(t) f(t)\, dt$, $a_{ik} = \int_a^b b_i(t) a_k(t)\, dt$

$$\Rightarrow \quad c_i = f_i + \lambda \sum_{k=1}^{n} c_k a_{ik}$$

or, $$f_i = c_i - \lambda \sum_{k=1}^{n} a_{ik} c_k \quad \forall\ i = 1 \text{ to } n \quad \text{--- } 8$$

$i=1 \quad f_1 = c_1 - \lambda \sum_{k=1}^{n} a_{1k} c_k = (1-\lambda a_{11})c_1 - \lambda a_{12} c_2 - \lambda a_{13} c_3 - \cdots \lambda a_{1n} c_n$

$i=2 \quad f_2 = c_2 - \lambda \sum_{k=1}^{n} a_{2k} c_k = -\lambda a_{21} c_1 + (1-\lambda a_{22}) c_2 - \lambda a_{23} c_3 - \cdots$

$\vdots \quad \vdots$

$i=n \quad f_n = c_n - \lambda \sum_{k=1}^{n} a_{nk} c_k = -\lambda a_{n1} c_1 - \lambda a_{n2} c_2 - \cdots + (1-\lambda a_{nn}) c_n$

→ c_i's are unknown here ($c_i = \int_a^b b_i(t) g(t) dt$)

$$\boxed{D(\lambda) C = F}$$

where,

$F =$ $\quad D(\lambda)$ $\quad$ Learn

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} (1-\lambda a_{11}) & -\lambda a_{12} & \cdots & -\lambda a_{1n} \\ -\lambda a_{21} & (1-\lambda a_{22}) & \cdots & -\lambda a_{2n} \\ \vdots & & & \vdots \\ \vdots & & & (1-\lambda a_{nn}) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

here, $a_{ik} = \int_a^b b_i(t) a_k(t) dt$ | $c_i = \int_a^b b_i(t) g(t) dt$

$f_i = \int_a^b b_i(t) f(t) dt$ $\quad$ unknown

$$D(\lambda) = (d_{ij}) = \begin{cases} -\lambda a_{ij}, & i \neq j \\ 1-\lambda a_{ij}, & i = j \end{cases}$$

$C = (c_1 \; c_2 \cdots c_n)^T \qquad F = (f_1 \; f_2 \cdots f_n)^T$

$$\boxed{D(\lambda) C = F}$$

→ Homogeneous ⇒ $DC = 0$ $\quad \left(c_i - \lambda \sum_{k=1}^{n} a_{ik} c_k = 0, \; i = 1(1)n \right)$

~~$\lambda = 0 \Rightarrow D = I$~~

→ $\lambda = \frac{1}{\mu}$ $(\lambda \neq 0)$, then $(\mu I - A) C = 0$ (homog.)

Homogeneous, if $\lambda \neq 0$

↳ $(\mu I - A) C = 0$ is an eigen-value problem, corresp. to $AC = 0$.
The eigenvalues are given by $|\mu I - A| = 0$ or $D(\lambda) = 0$.
They are eigenvalues of corresp. integral eqn.

→ $\lambda = 0 \Rightarrow D = I$

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Examples:

Q1. Fredholm ~~eqn.~~ IE of 2^{nd} kind

$$g(s) = s + \lambda \int_0^1 (st^2 + s^2 t)\, g(t)\, dt \quad \text{--- (1)}$$

Soln. $K(s,t) = st^2 + s^2 t = \cancel{st(t+s)} \sum_{i=1}^{2} a_i(s) b_i(t)$

$a_1(s) = s \qquad b_1(t) = t^2$

$a_2(s) = s^2 \qquad b_2(t) = t$

$$(1) \rightarrow g(s) = s + \lambda \left[s \int_0^1 t^2 g(t)\, dt + s^2 \int_0^1 t\, g(t)\, dt \right]$$

Take $c_1 = \int_0^1 t^2 g(t)\, dt$, $c_2 = \int_0^1 t\, g(t)\, dt$

$$g(s) = s + \lambda [s c_1 + s^2 c_2] \quad \text{--- (2)}$$

From (1) & (2),

$$\lambda \int_0^1 (st^2 + s^2 t)\, g(t)\, dt = \lambda [s c_1 + s^2 c_2]$$

$$\Rightarrow s\left[c_1 - \int_0^1 t^2 g(t)\, dt\right] + s^2 \left[c_2 - \int_0^1 t\, g(t)\, dt\right] = 0$$

from (2),

$$s\left[c_1 - \int_0^1 t^2 \left(t + \lambda (t c_1 + t^2 c_2)\right) dt\right] +$$

$$s^2 \left[c_2 - \int_0^1 t \left(t + \lambda (t c_1 + t^2 c_2)\right) dt\right] = 0 \quad \text{--- (3)}$$

Comparing coeffs we get,

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$$c_1 = \int_0^1 t^3 dt + \lambda c_1 \int_0^1 t^3 dt + \lambda c_2 \int_0^1 t^4 dt$$

or, $c_1 = \frac{1}{4} + c_1 \frac{\lambda}{4} + \frac{\lambda}{5} c_2$

or, $\left(1 - \frac{\lambda}{4}\right) c_1 - \frac{\lambda}{5} c_2 = \frac{1}{4}$ — (4)

Similarly, $c_2 = \int_0^1 t^2 dt + \lambda c_1 \int_0^1 t^2 dt + \lambda c_2 \int_0^1 t^3 dt$

or, $c_2 = \frac{1}{3} + \frac{\lambda c_1}{3} + \frac{\lambda c_2}{4}$

or, $-\frac{\lambda}{3} c_1 + \left(1 - \frac{\lambda}{4}\right) c_2 = \frac{1}{3}$ — (5)

$$\begin{bmatrix} 1-\frac{\lambda}{4} & -\frac{\lambda}{5} \\ -\frac{\lambda}{3} & 1-\frac{\lambda}{4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/3 \end{bmatrix}$$

$$c_1 = \begin{vmatrix} 1/4 & -\lambda/5 \\ 1/3 & 1-\lambda/4 \end{vmatrix} \Bigg/ \begin{vmatrix} 1-\lambda/4 & -\lambda/5 \\ -\lambda/3 & 1-\lambda/4 \end{vmatrix}$$

Cramer's Rule

$$c_2 = \begin{vmatrix} 1-\lambda/4 & 1/4 \\ -\lambda/3 & 1/3 \end{vmatrix} \Bigg/ \begin{vmatrix} 1-\lambda/4 & -\lambda/5 \\ -\lambda/3 & 1-\lambda/4 \end{vmatrix}$$

→ Alternate method: directly form the matrix by formulating a_i, b_i, f_i, c_i, a_{ik} ...

$g(s) = s + \lambda s c_1 + \lambda s^2 c_2$ (from (2))

Q2. $g(s) = f(s) + \lambda \int_0^1 (s+t) g(t) dt$ — ①

Solve the above IE & find the eigen values.

$K(s,t) = s + t$

$a_1(s) = s \qquad b_1(t) = 1$

$a_2(s) = 1 \qquad b_2(t) = t$

$g(s) = f(s) + \lambda s \int_0^1 g(t) dt + \lambda \int_0^1 t\, g(t) dt$

Let $c_1 = \int_0^1 g(t) dt$, $c_2 = \int_0^1 t\, g(t) dt$

$g(s) = f(s) + \lambda [s c_1 + c_2]$ — ②

From ① & ②,

$\lambda \int_0^1 (s+t) g(t) dt = \lambda (s c_1 + c_2)$

or, $\int_0^1 (s+t)(f(t) + \lambda t c_1 + \lambda c_2) dt = s c_1 + c_2$

a_i's are LI

$\Rightarrow \cancel{s \left[c_1 \left(\int_0^1 f(t) dt + \lambda c_1 \int_0^1 \right.\right.}$

$\Rightarrow s \int_0^1 f(t) dt + \lambda c_1 s \int_0^1 dt + \lambda c_2 s \int_0^1 dt +$

$\int_0^1 t f(t) dt + \lambda c_1 \int_0^1 t\, dt + \lambda c_2 \int_0^1 t\, dt = s c_1 + c_2$

$-6 \pm 4\sqrt{3}$

$\Rightarrow s \int_0^1 f(t) dt + \lambda c_1 s + \lambda c_2 s + \int_0^1 t f(t) dt + \frac{\lambda}{2} c_1 + \frac{\lambda}{2} c_2 = s c_1 + c_2$

$\lambda \neq$ above, then can apply Cramer's rule.

$\Rightarrow s \left[\int_0^1 f(t) dt + \lambda c_1 + \lambda c_2 - c_1 \right] + \left[\int_0^1 t f(t) dt + \frac{\lambda}{2} c_1 + \frac{\lambda}{2} c_2 - c_2 \right] = 0$

Let $f_1 = \int_0^1 f(t) dt$, $f_2 = \int_0^1 t f(t) dt$

$\Rightarrow s \left[f_1 + (\lambda - 1) c_1 + \lambda c_2 \right] + \left[f_2 + \frac{\lambda}{2} c_1 + \left(\frac{\lambda}{2} - 1 \right) c_2 \right] = 0$

$$\Rightarrow (\lambda-1)c_1 + \lambda c_2 = -f_1 \quad \text{— (3)}$$

$$\& \quad \frac{\lambda}{2}c_1 + \left(\frac{\lambda}{2}-1\right)c_2 = -f_2 \quad \text{— (4)}$$

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25/01/23 $g(s) = f(s) +$

$$\frac{\lambda}{\det(D(\lambda))}\int_a^b \left[\sum_{i=1}^{n} \{D_{1i}\, b_1(t) + D_{2i}\, b_2(t) + \dots + D_{ni}\, b_n(t)\}\, a_i(s)\right] f(t)\,dt \quad \text{--- (4)}$$

Now, consider the determinant of $(n+1)$ order

$$D(s,t;\lambda) = - \begin{vmatrix} 0 & a_1(s) & a_2(s) & \cdots & a_n(s) \\ b_1(t) & 1-\lambda a_{11} & -\lambda a_{12} & \cdots & -\lambda a_{1n} \\ b_2(t) & -\lambda a_{21} & 1-\lambda a_{22} & \cdots & -\lambda a_{2n} \\ \vdots & & & & \\ b_n(t) & -\lambda a_{n1} & \cdots & \cdots & 1-\lambda a_{nn} \end{vmatrix}$$

$$= - \left| \begin{array}{c|c} 0 & \text{---}\; a_n(s) \\ \hline b_1(t) & \\ \vdots & D(\lambda) \\ b_n(t) & \end{array} \right| \quad \text{--- (5)}$$

Take $\Gamma(s,t;\lambda) = D(s,t;\lambda) / \det D(\lambda)$ — ⑥

Then equn. ④ becomes
$$g(s) = f(s) + \lambda \int_a^b \underbrace{\Gamma(s,t;\lambda)}_{\text{Resolvent kernel}} f(t)\,dt \quad —⑦$$

* Rewrite $K(s,t)$ for $K(t,s)$
$D(\lambda) = d_{ij}$ for $K(s,t)$
$P(\lambda) = p_{ij}$ for $K(t,s)$ then, $p_{ij} = d_{ji}$

* $$g(s) = f(s) + \lambda \int_a^b K(s,t) g(t)\,dt \quad —①$$

Then, the transpose of above IE is defined as:

$$\psi(s) = f(s) + \lambda \int_a^b K(t,s) \psi(t)\,dt \quad —②$$

$I - \lambda A \to I - \lambda A^T$

* Fredholm Theorem: If $\det D(\lambda) \neq 0$, the ~~non~~inhomogeneous Fredholm IE
$$g(s) = f(s) + \lambda \int_a^b K(s,t) g(t)\,dt$$

w/ separable kernel
$$K(s,t) = \sum_{i=1}^{n} a_i(s) b_i(t)$$

has one & only one soln.
$$g(s) = f(s) + \lambda \int_a^b \Gamma(s,t;\lambda) f(t)\,dt,$$

Rank $p \in [1, n]$ then # LI solns $= n-p$.

where $\Gamma(s, t; \lambda)$ is defined by (5) - (6)

* **Fredholm Alternative Th.** : Either the IE

$$g(s) = f(s) + \lambda \int_a^b K(s,t)\, g(t)\, dt \quad \text{— (1)}$$

w/ fixed λ possesses one & only one function soln $g(s)$ for arbitrary L^2-fns $f(s)$ & $K(s,t)$, in particular the soln. $g = 0$ for $f = 0$; or the homogeneous eqn.

$$g(s) = \lambda \int_a^b K(s,t)\, g(t)\, dt \quad \text{— (2)}$$

possesses a finite no. of r linearly indep. solns g_{0i}, $i = 1(1)r$.

In the 1st case, the ~~transformed~~ transposed inhomogeneous eqn.
(eqn. (1))

$$\psi(s) = f(s) + \lambda \int_a^b K(t,s)\, \psi(t)\, dt \quad \text{— } 1^*$$

also possesses a unique soln. In the 2nd case, the transposed homogeneous eqn.

$$\psi(s) = \lambda \int_a^b K(t,s)\, \psi(t)\, dt \quad \text{— } 2^*$$

has r - LI solns ψ_{0i}, $i = 1(1)r$; the inhomogeneous eqn (1) has a soln. iff the given fn $f(s)$ satisfies the r conditions

$$\langle f, \psi_{0i} \rangle = \int_a^b f(s)\, \psi_{0i}(s)\, ds = 0, \quad i = 1(1)r$$

orthogonal

In this case the soln of (1) is determined only up to an additive linear combination: $\sum_{i=1}^{r} c_i g_{0i}$

Read: range of r for multiplicity 'm'