MTI Assignment 2   Keerti P. Charantimath   19MA20059

Solution 1:- (i)

Monotone Convergence Theorem - If $\{f_n\}$ is a sequence in L^+ such that $f_j \leq f_{j+1}$ $\forall j$. & $f = \lim_{n \to \infty} f_n \ (= \sup_n f_n)$ then $\int f = \lim_{n \to \infty} \int f_n$

Fatou's lema - If $\{f_n\}$ is any sequence in L^+ then

$$\int f = \int (\lim \text{if } f_n) \leq \lim \inf \int f_n$$

Let $\{f^n\}_{n \in \mathbb{N}} \subset L^+$, then by Fatou's lemma

$$\int f = \int \left(\lim_{n \to \infty} f_n \right) \leq \lim_{n \to \infty} \inf_{k \geq n} \int f_k$$

We know that $f = \lim_{n \to \infty} f_n \ (= \sup_n f_n)$ and that $f_n \leq f$ hence $\int f_n \leq \int f$ $\forall n \in \mathbb{N}$.

So, $\sup_{k \geq n} \int f_k \leq \int f$ $\forall n \in \mathbb{N}$

Then it is clear that $\lim_{n \to \infty} \sup_{k \geq n} \int f_k \leq f$

$$\therefore \lim_n \sup \int f_n = \lim_{n \to \infty} \sup_{k \geq n} \int f_k \leq \int f = \int \lim_n \inf f_n \leq \lim_{n \to \infty} \inf_{k \geq n} \int f_k$$

$$= \lim_n \inf \int f_n \quad \hookrightarrow ①$$

Since we also know that

$$\lim_n \inf \int f_n \leq \lim_n \sup \int f_n$$

Then, from ①, we get $\int f = \lim_n \inf \int f_n = \lim_n \sup \int f_n$

which means $\int f = \lim_{n \to \infty} \int f_n$

(ii) For a sequence of $fn^{\underline{ns}}$ $(f_n)_{n\geq 1}$, let
$A := \int \lim_{n\to\infty} \inf f_n \, d\lambda$ & $B := \lim_{n\to\infty} \inf \int f_n \, d\lambda$

Fatal's lemmas theorem states that we always have $A \leq B$

Another eg:- $A=0$, $B=+\infty$, if $f_n = 1_{[2^n, 2^{n+1}]}$

or $f_n(x) = \begin{cases} 1_{[0, 1/2]}(x) & , n \text{ is even} \\ 1_{[1/2, 1]}(x) & , n \text{ is odd} \end{cases}$

(iii) Fatou's Lemma is not true for any sequence of measurable functions

let $f_n(x) = \begin{cases} -n & , 1/n \leq x \leq 2/n \\ 0 & \text{otherwise} \end{cases}$

$\lim_{n\to\infty} f_n(n) = 0 \quad \forall x \in [0, 1$
$= f$ (say)

$$\int_{[0,1]} f_n = \int_0^1 f_n = \int_{1/n}^{2/n} (-n)\, dx$$

$$= -nx \Big|_{x=1/n}^{2/n}$$

$$= -1 \text{ for any } n \in \mathbb{N}$$

$\therefore \lim_{n\to\infty} \inf \int_{[0,1]} f_n = \lim_{n\to\infty} \int_{[0,1]} f_n = -1$

$\int_{[0,1]} f = \int_{[0,1]} 0 = 0 \nleq -1 = \lim_{n\to\infty} \inf \int_{[0,1]} f_n$

(iv) Fatou's lemma

Dominated Convergence Theorem

Bounded Convergence Theorem.

Fatou's lemma $\Rightarrow$ DCT

Assume $\{fn\}$ is a sequence of measurable fn$^{\underline{s}}$ such that $fn \to f$ i.e. as $n \to \infty$ & $(fn) \leq g$ where $g \in L^1(\mathbb{R}^d)$

(g is L - integrable)

To show: $\int fn \to \int f$ as $n \to \infty$

we have $|f| \leq g$ & $g - fn \to g - f$ as $n \to \infty$ a-e

$g + fn \to g + f$ as $n \to \infty$ a-e

& $g - fn$, $g + fn$ are non -ve fn$^{\underline{s}}$, $n \geq 1$

$\therefore$ By Fatou's lemma

$$\left.\begin{aligned} \liminf_{n \to \infty} \left(\int (g - fn) \right) &\geq \int (g - f) \\ \liminf_{n \to \infty} \left(\int (g + fn) \right) &\geq \int (g + f) \end{aligned}\right\} \textcircled{*}$$

But $\liminf_{n \to \infty} \left(\int (g - fn) \right) = \liminf_{n \to \infty} \int g + \liminf_{n \to \infty} \left(-\left(\int fn \right) \right)$

$= \int g - \limsup_{n \to \infty} \left(\int fn \right)$

& $\liminf_{n \to \infty} \left(\int (g + fn) \right) = \liminf_{n \to \infty} \int g + \liminf_{n \to \infty} \left(\int fn \right)$

$= \int g + \liminf_{n \to \infty} \left(\int fn \right)$

$\therefore$ from $\circledast$ we have

$$\cancel{\int g} - \limsup_{n\to\infty} \left(\int f_n\right) \geq \cancel{\int g} - \int f$$

& $\cancel{\int g} + \liminf_{n\to\infty} \left(\int f_n\right) \geq \cancel{\int g} + \int f$

$$\Rightarrow \limsup_{n\to\infty} \left(\int f_n\right) \leq \int f \leq \liminf_{n\to\infty} \left(\int f_n\right)$$

$$\Rightarrow \limsup_{n\to\infty} \left(\int f_n\right) = \int f = \liminf_{n\to\infty} \int f_n$$

$$\Rightarrow \operatorname{limit}_{n\to\infty} \left(\int f_n\right) = \int f$$

DCT $\Rightarrow$ BCT

let $|f_n| \leq M$ $\forall n \geq 1$, supported on a set E of infinte measure

$f_n \to f$ a.e as $n \to \infty$

$\int |f_n - f| \to 0$ as $n \to \infty$

To. show : $\int |f_n - f| \to 0$ as $n \to \infty$

let $g \equiv M$ constant $f_m \triangleq M$

& $\int_E g = \int_E M = M$

$m(E) < \infty$ (i.e g is L integrable)

$\therefore$ by DG, $\int |f_n - f| \to 0$ as $n \to \infty$

solution 4:

$f\ [0,1] \to R$

$f(x) \begin{cases} 0 & \text{if } x \text{ is a cantor point} \\ p & \text{if } x \text{ belongs to open interval } \overset{\text{each at}}{\text{~~tat~~}} \text{ length } 3^{-p} \end{cases}$

f is the sum of the non-negative step fn^{DS} & so it is non -ve

Hence f is measurable

$$\int_0^1 f\,dx = \sum_{p=1}^{\infty} p\, g^{p-1} 3^{-p}$$

$$= \sum_{p=1}^{\infty} \left(p/2\right)\left(2/3\right)^{p} = \sum_{p=1}^{\infty} p \frac{2^{p-1}}{3p}$$

$$= \sum_{p=1}^{\infty} \left(p/3\right)\left(\frac{2}{3}\right)^{p-1}$$

$$= \sum_{p=1}^{\infty} p\left(1-\frac{2}{3}\right)\left(\frac{2}{3}\right)^{p-1} = \sum_{p=1}^{\infty} p\left(\left(2/3\right)^{p-1} - \left(2/3\right)^{p}\right)$$

$$= 1\left(1-\frac{2}{3}\right) + 2\left(\frac{2}{3} - \left(\frac{2}{3}\right)^2\right) + 3\left(\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3\right) + \dots$$

$$= 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3$$

$$= \frac{1}{1-\frac{2}{3}}$$

$$\int_0 f\,dx = 3$$

Solution 8 :-

DCT ⇒ BCT

let $|fn| \leq M \quad \forall n \geq 1$. supported on a set E of finite measure

$$fn \to f \quad \text{a.e as } n \to \infty$$

let $g = M$ constant $fn^{\underline{n}}$ M

& $\int_E g = \int_E M = M$

$m(E) < \infty$ (ie g is L-integrable)

DCT $|fn| \leq g \Rightarrow |fn - f) \to 0$ as $n \to \infty$

∴ by DCT $\int |fn - f| \to 0$ as $n \to \infty$

Solution 9 :- (i)

$$f_n(x) = \frac{n^{3/2}}{1+n^2x^2}$$

$$\lim_{n\to\infty} \frac{n^{3/2}x}{1+n^2x^2} = \lim_{n\to\infty} \frac{1}{n^{1/3}} \frac{x}{1/n^2 + x^2} \to 0 \text{ as } n \to \infty$$

(ii) $(nx-1)^2 \geq 0$

$$n^2x^2 - 2nx + 1 \geq 0$$

$$1 + n^2x^2 \geq 2nx$$

$$f_n(x) = \frac{n^{3/2}x}{1+n^2x^2}$$

$$\frac{n^{3/2}x}{1+n^2x^2} \leq \frac{n^{3/2}x}{2nx}$$

$$\frac{n^{3/2}x}{1+n^2x^2} \leq \frac{n^{1/2}}{2}$$

∴ sequence $\{f_n\}$ is not uniformly bounded

(iii) $(1+n^2x^2)^2 \geq n^3x^3 \qquad x \in (0,1)$

$$1+n^2x^2 \geq n^{3/2}\, x^{3/2}$$

$$\Rightarrow \frac{n^{3/2} \cdot x^{3/2}}{1+n^2x^2} \leq 1$$

$$\Rightarrow \frac{n^{3/2}x}{1+n^2x^2} \leq \frac{1}{\sqrt{x}}$$

Solution 5 :-

Let g be a step fn $\left[\frac{1}{x}\right]^{-1}$ on $(0,1)$

Then $\int_0^1 g\,dx > \int_{1/n+1}^{1} g\,dx = \sum_{n+1}^{N} \frac{1}{n+1}$

So $\int_0^1 g\,dx = \infty$

Also $g-f = 0$ a.e

If f is a non-ve measurable fn

Then $f=0$ a.e if & only if $\int f\,dx = 0$

So $\int (g-f)\,dx = 0$

But $\int g\,dx = \int (g-f)\,dx + \int f\,dx$.

$\int f\,dx = \infty$

Solution 2:-

x^{-1} is a continuous fn$^{\underline{n}}$ for $x>0$
so it is measurable

$$\int_{-1}^{\infty} x^{-1}\,dx > \int_{1}^{n} x^{-1}\,dx$$

But $x^{-1} > k^{-1}$ on $[k-1, k)$ so

$$\int_{1}^{n} x^{-1}\,dx > \sum_{k=2}^{n} \int_{1}^{n} k^{-1}\,\chi(k-1, k)\,dx$$

$$\int_{1}^{n} x^{-1}\,dx > \sum_{k=2}^{n} \int_{1}^{n} k^{-1}\,\chi(k-1, k) > \sum_{k=2}^{n} k^{-1} \longrightarrow \infty \text{ as } n \to \infty$$

$$\therefore \int_{1}^{\infty} x^{-1}\,dx = \infty$$

Solution 3:-

For $n \in [0, 1]$
Let $g(x) = 0$ if $10^{-(n+1)} \leq x < 10^{-n}$, $n = 0, 1 \cdots$
& $g(1) = 0$
Then $f \leq g$, $f = g$ a.e
so f is measurable

$$\int_{0}^{1} f\,dx = \int_{0}^{1} g\,dx$$

$$\text{But } \int_{0}^{1} g\,dx = \sum_{n=0}^{\infty} n\left(\frac{1}{10^{n}} - \frac{1}{10^{n+1}}\right)$$

$$= \frac{1}{10} + \frac{1}{10^{2}} + \frac{1}{10^{3}} \cdots$$

$$= \frac{\frac{1}{10}}{1 - \frac{1}{10}} = \frac{1}{9} = \int_{0} f\,dx = 1/9$$