Sylow's Thm Application

Lecture 13

10/02/2022

$$A = \{ (x_1, \dots, x_p) \in G \times \cdots \times G \mid x_1 \cdots x_p = 1 \}.$$

$$H = \langle \sigma \rangle \quad \text{where} \quad \sigma = (1\ 2\ 3 \cdots p) \in S_p,$$

$$H \times A \longrightarrow A$$

$$(\sigma^i, (x_1, \dots, x_p)) \longmapsto (x_{\sigma^i(1)}, \dots, x_{\sigma^i(p)})$$

$$p = 5. \qquad \sigma = (1\ 2\ 3\ 4\ 5).$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}.$$

$$\sigma^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}, \quad \dots$$

$$x_1 x_2 \cdots x_p = 1.$$

$$(x_1 x_2 \cdots x_r)(x_{r+1} \cdots x_p) = 1.$$

$$1 = x_{r+1} \cdots x_p x_1 x_2 \cdots x_r \Leftarrow x_{r+1} \cdots x_p = (x_1 x_2 \cdots x_r)^{-1}$$

Sylow's Thm.

$|G| = p^\alpha . m.$

Let P be a Sylow p-subgp of G.

$$gPg^{-1} \subseteq G \quad \text{where } g \in G.$$

Note that gPg^{-1} is a subgp of G.

$$|gPg^{-1}| = |P| = p^\alpha$$

Thus gPg^{-1} is a Sylow p-subgp.

Suppose G has only 1 Sylow p-subgp. say P.

$$gPg^{-1} = P \quad \forall\, g \in G.$$

Therefore P is a normal subgp of G.

<u>Remark</u>: If there is only 1 Sylow p-subgp then it must be a normal subgp of G.

Group of order 15:

Let G be a gp of order 15.

$$15 = 3.5.$$

n_3 : # of Sylow 3-subgp

n_5 : # of Sylow 5-subgp.

$n_3 \mid 5$ and $n_3 \equiv 1 \pmod 3 \Rightarrow n_3 = 1.$

$n_5 \mid 3$ and $n_5 \equiv 1 \pmod 5 \Rightarrow n_5 = 1.$

Let H & K be subgps of order 3 and 5 respectively.

$H = \langle x \rangle$ with $|x| = 3$

and $K = \langle y \rangle$ with $|y| = 5$.

and $H \trianglelefteq G$, $K \trianglelefteq G$ & $H \cap K = \{1\}$.

$|HK| = 15$. $\therefore$ $HK = G \cong H \times K$
$\cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$

$$G \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}/15\mathbb{Z}.$$

i.e G is a cyclic grp of order 15.

<u>Example</u>. Let G be a grp of order 21.
Then classify G.

$$21 = 3 \cdot 7.$$

$n_3 \to$ # of Sylow 3-subgrp

$n_7 \to$ # of Sylow 7-subgrp.

$n_3 \mid 7$ and $n_3 \equiv 1 \pmod 3 \Rightarrow n_3 = 1, 7$

$n_7 \mid 3$ and $n_7 \equiv 1 \pmod 3 \Rightarrow n_7 = 1.$

There exists only one Sylow 7 subgrp.
Say H. Let $H = \langle x \rangle$ s.t $|x| = 7$.
and $H \trianglelefteq G$.

<u>Case 1</u>: Let $n_3 = 1$

$\Rightarrow \exists$ only one Sylow 3-subgp of G.

Let K be the Sylow 3-subgp

$K = \langle y \rangle$, $|y| = 3$. $K \trianglelefteq G$.

$|HK| = 21$, $H \cap K = \{1\}$.

and both H & K are normal

$$\boxed{G \cong H \times K \cong \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/21\mathbb{Z}}.$$

<u>Case 2</u>. Let $n_3 = 7$. i.e $\exists$ 7 Sylow 3-subgp.

Note that H is a normal subgp.

Say y is an elt of order 3.

$yxy^{-1} \in H = \{1, x, x^2, \ldots, x^6\}$.

Let $yxy^{-1} = x^r$ $\quad 1 \leq r \leq 6$.

$$y^2 x y^{-2} = y\,(y x y^{-1})\,y^{-1}$$

$$= y\, x^{r} y^{-1}$$

$\overset{y^{-1}y}{x \; x \; x} \cdots$

$$= \underbrace{y x y^{-1}}\,\underbrace{y x y^{-1}} \cdots \underbrace{y x y^{-1}}$$

$$= x^{r^2}$$

$$x = y^3 x y^{-3} = x^{r^3} \qquad \text{i.e. } x^{r^3} = x$$

$$\Rightarrow x^{r^3 - 1} = 1.$$

$$\Rightarrow r^3 \equiv 1 \pmod 7.$$

$$\Rightarrow r = 1, 2, 4$$

If $r = 1$ then $y x y^{-1} = x$. i.e. $yx = xy$.

Verify G is abelian

If $r = 2$ then $y x y^{-1} = x^2$

$$G \cong \langle x, y \mid \underline{|x| = 7}, \ |y| = 3, \ yx = x^2 y \rangle.$$

If $r=4$, $\quad yxy^{-1} \neq x^4$.

But if $\quad yxy^{-1} = x^2$

$$\Rightarrow y^2xy^{-2} = x^4.$$

As $|y^2| = 3$, so it the same

case as $\quad yxy^{-1} = x^4$.

If G is a gp of order 21 then

(1) $G \cong \mathbb{Z}/21\mathbb{Z}$

(2) $G \cong \langle x, y \mid |x| = 7, |y| = 3, yx = x^2y \rangle$.

__Propn.__ Let p & q be distinct primes and $q < p$ and let G be a gp order pq. Then

(1) If $q \nmid p-1$ then $G \cong \mathbb{Z}/pq\mathbb{Z}$.

(2) If $q \mid p-1$ and G is not cyclic

$$G = \langle x, y \mid |x| = p, \ |y| = q, \ yxy^{-1} = x^s \text{ where } s \not\equiv 1 \pmod p \ \& \ s^q \equiv 1 \pmod p \rangle$$

__Cor.__ Let p be an odd prime and G be a gp of order $2p$. Then either $G \cong \mathbb{Z}/2p\mathbb{Z}$ or $G \cong D_p$.

Group of order 12:

(1) $\mathbb{Z}/12\mathbb{Z}$

(2) $A_4 \rightsquigarrow$ gp of even permutation of S_4.

(3) $D_6 = \langle x, y \mid |x| = 6, |y| = 2, yxy^{-1} = x^{-1} \rangle$.

(4) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

(5) $G \cong \langle x, y \mid |x| = 4, |y| = 3, xy = y^2 x \rangle$.

Pf: $12 = 2^2 \cdot 3$.