Lecture 12

Proposition: Let $f : E \to \mathbb{R}$ be a function, where E is measurable. Then f is measurable iff $f^{-1}([a,b))$ is measurable, $\forall\, a, b \in \mathbb{R}$

proof:

$(\Rightarrow)$: Suppose f is measurable.

Then $f^{-1}([a,b)) = f^{-1}((-\infty, b)) \cap f^{-1}([a, \infty)) \in \mathcal{M}$

$(\Leftarrow)$: Suppose $f^{-1}([a,b)) \in \mathcal{M}$ $\forall\, a, b \in \mathbb{R}$.

To show: f is measurable.

Let $a \in \mathbb{R}$,

$$(-\infty, a) = \bigcup_{n=1}^{\infty} [-n, a)$$

$$f^{-1}((-\infty, a)) = f^{-1}\left(\bigcup_{n=1}^{\infty} [-n, a) \right)$$

$$= \bigcup_{n=1}^{\infty} \underbrace{f^{-1}([-n, a))}_{\in\, \mathcal{M} \text{ by our assumption}} \in \mathcal{M}$$

Thus $f^{-1}((-\infty, a)) \in \mathcal{M}$, $\forall\, a \in \mathbb{R}$

$\therefore$ f is measurable.

Proposition:- Let $f: E \to \mathbb{R}$ be a measurable function.

Then $f \leq \text{esssup}(f)$ a.e.

proof:-

If $\text{esssup}(f) = +\infty$, then there is nothing to prove.

Suppose $\text{esssup}(f) = -\infty = \inf\left(\{\alpha \in \mathbb{R} \mid f \leq \alpha \text{ a.e on } E\}\right)$

$\Rightarrow \quad \forall n \in \mathbb{Z}, \quad f \leq n \text{ a.e}$

$\Rightarrow \quad f = -\infty \quad \text{a.e}$ $\qquad (\underbrace{-\infty \leq f \leq -\infty}_{\text{a.e}})$

$\therefore f \leq -\infty$ a.e.

Suppose $\text{esssup}(f)$ is a finite number.

Let $E_n = \left\{x \in E \,\middle|\, f(x) > \frac{1}{n} + \text{esssup}(f)\right\} \in \mathcal{M}$ $\forall n \in \mathbb{N}$.

& $F = \{x \in E \mid f(x) > \text{esssup}(f)\} \in \mathcal{M}$

To show: $m(F) = 0$. ✓

We have $F = \bigcup_{n=1}^{\infty} E_n$ ✓

$$\therefore \quad m(F) = m\left(\bigcup_{n=1}^{\infty} E_n\right)$$

$$\leq \sum_{n=1}^{\infty} m(E_n) \longrightarrow (*)$$

But by the definition of esssup(f), we have

$$\text{esssup}(f) = \inf\left\{ \alpha \in \mathbb{R} \,\middle|\, f \leq \alpha \text{ a.e on } E \right\}$$

To <u>show</u>: $m(E_n) = 0 \quad \forall n.$

Take $\alpha = \frac{1}{n} + \text{esssup}(f) > \text{esssup}(f).$

$\therefore \quad f \leq \alpha$ a.e on E. (use inf. property)??

$\Rightarrow \quad \{x \in E \mid f(x) \nleq \alpha\}$ has measure 0.

$\quad \parallel$

$\{x \in E \mid f(x) > \frac{1}{n} + \text{esssup}(f)\}$ has measure 0.

$\therefore \quad m(E_n) = 0 \quad \forall n$

$\therefore$ From $(*)$, $m(F) \leq 0$

$\therefore \quad m(F) = 0.$

Proposition:- Let $f, g : E \to \mathbb{R}$ be measurable functions.

Then $\operatorname{ess\,sup}(f+g) \leq \operatorname{ess\,sup}(f) + \operatorname{ess\,sup}(g)$.

& this inequality can be strict.

proof:- By above Proposition,

$$f \leq \operatorname{ess\,sup}(f) \text{ a.e } X$$
$$g \leq \operatorname{ess\,sup}(g) \text{ a.e}$$

$$\Rightarrow \quad f+g \leq \operatorname{ess\,sup}(f) + \operatorname{ess\,sup}(g) \quad \text{a.e}$$

$$\Rightarrow \quad \operatorname{ess\,sup}(f+g) \leq \operatorname{ess\,sup}(f) + \operatorname{ess\,sup}(g) \quad \text{a.e}$$

Example:- ① Let $f = \chi_{[-1,0]} - \chi_{[0,1]}$

& $g = -f = \chi_{[0,1]} - \chi_{[-1,0]}$

$f+g \equiv 0$

$\therefore \operatorname{ess\,sup}(f+g) = 0.$

&

$\operatorname{ess\,sup}(f) = 1$

$\operatorname{ess\,sup}(g) = 1$

$\therefore \quad \operatorname{ess\,sup}(f) + \operatorname{ess\,sup}(g) = 2 \gneq \operatorname{ess\,sup}(f+g).$

Proposition:- Let $f : E \to \mathbb{R}$ be a measurable function.

Then $\text{ess.sup}(f) = -\,\text{ess.inf}(-f)$.

Proof:-
$$\begin{aligned}
\text{ess.sup}(f) &= \inf\left(\{\alpha \in \mathbb{R} \mid f \leq \alpha \text{ a.e on } E\}\right) \\
&= \inf\left(\{\alpha \in \mathbb{R} \mid -\cancel{f} \geq -\alpha \text{ a.e on } E\}\right) \\
&= -\sup\left(\{\underbrace{-\alpha}_{\beta} \mid -f \geq \underbrace{-\alpha}_{\beta} \text{ a.e on } E\}\right) \\
&= -\,\text{ess.inf}(-f)
\end{aligned}$$

Definition:- A measurable function $f : E \to \mathbb{R}$ is said to be essentially bounded if

$\text{ess.sup}(|f|) < \infty$.
