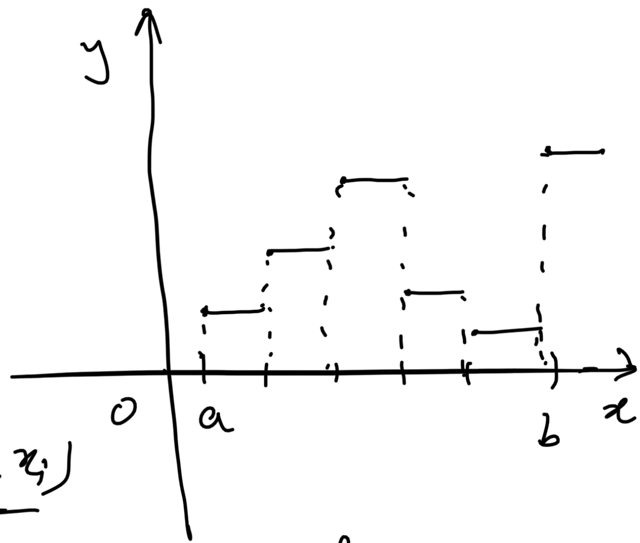
Lecture 23

Recall:– A function $\varphi : [a, b] \longrightarrow \mathbb{R}$ is called a step function or a piecewise constant function if there exists number

$a = x_0 < x_1 < \cdots < x_n = b$ such that

φ is constant on each of the intervals $[x_{i-1}, x_i)$, $1 \le i \le n$.

That is, $\varphi = \sum_{i=1}^{n} a_i \chi_{[x_{i-1}, x_i)}$



Def:– The Riemann integral of a step function $\varphi = \sum_{i=1}^{n} a_i \chi_{[x_{i-1}, x_i)}$

is

$$\int_a^b \varphi(x)\,dx = \sum_{i=1}^{n} a_i (x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} \varphi(x_{i-1}) \cdot (x_i - x_{i-1})$$

$$\int \varphi(x)\,dx = \int \sum a_i \chi_{[x_{i-1}, x_i)} = \sum a_i \int \chi_{[x_{i-1}, x_i)} \overset{?}{=} \sum a_i \int_{x_{i-1}}^{x_i} 1\,dx = \sum a_i (x_i - x_{i-1}).$$

Then the Riemann upper integral of f on $[a,b]$ is

$$\overline{\int_a^b} f(x)\,dx = \inf_{P \in \mathcal{P}} \left(U(P,f) \right)$$

$$= \inf_{\substack{P \in \mathcal{P} \\ \| \\ \{x_0, \dots, x_n\}}} \left(\underline{\sum_{i=1}^{n} M_i \, (x_i - x_{i-1})} \right)$$

$$= \inf \left(\left\{ \underline{\underline{\int_a^b}} \varphi(x)\,dx \;\middle|\; \begin{array}{l} \varphi \text{ is a step function} \\ \text{with } \varphi \geq f \end{array} \right\} \right)$$

& the Riemann lower integral of f on $[a,b]$ is

$$\underline{\int_a^b} f(x)\,dx = \sup_{P \in \mathcal{P}} \left(\sum_{i=1}^{n} m_i \, (x_i - x_{i-1}) \right)$$

$$= \sup \left(\left\{ \int_a^b \psi(x)\,dx \;\middle|\; \begin{array}{l} \psi \text{ is a step function } \& \\ \psi \leq f \end{array} \right\} \right)$$

Thus f is Riemann integrable if

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

$$\Rightarrow \boxed{ \inf_{\substack{\varphi \geq f \\ \varphi \text{ step function}}} \left(\int_a^b \varphi \right) = \sup_{\substack{\psi \leq f \\ \psi \text{ step function}}} \left(\int_a^b \psi \right) }$$

Def:- We say that a function $f: [a,b] \to \mathbb{R}$ is said to satisfy the Riemann's Condition if for any $\varepsilon > 0$, there exists a partition P_ε of $[a,b]$ such that for any P finer than P_ε (i.e., $P \supseteq P_\varepsilon$), we have

$$0 \leq U(P,f) - L(P,f) < \varepsilon$$

Theorem:- The following statements are equivalent for any $f: [a,b] \to \mathbb{R}$ bounded function.

(i) f is Riemann integrable on $[a,b]$

(ii) f satisfies the Riemann's Condition on $[a,b]$.

Examples:- ① Any Continuous function is R-integrable on $[a,b]$

② $f = \chi_{\mathbb{Q} \cap [0,1]}$

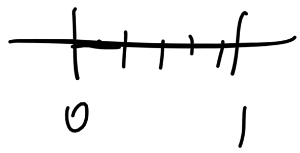
$$f(x) = \begin{cases} 1 & \text{if } x \in Q \cap [0,1] \\ 0 & \text{if } x \notin Q \cap [0,1]. \end{cases}$$

Infact, f is no where Continuous on $[0, 1]$.

For any partition P of $[a,b]$,

$$U(P,f) = 1$$

$$\& \quad L(P,f) = 0$$



$$U(P,f) - L(P,f) = 1 \quad \text{or}$$

$$\overline{\int_0^1} f = 1 \quad \& \quad \underline{\int_0^1} f = 0$$

$\therefore$ f is not Riemann integrable.

$$\mathcal{L}\int f = \mathcal{L}\int \chi_{\underline{Q \cap [0,1]}} \qquad \left(\because \mathcal{L}\int \left(\sum_i a_i \chi_{E_i}\right) := \sum_i a_i \, m(E_i) \right)$$

$$= m(Q \cap [0,1])$$

$$= 0 < \infty$$

$\therefore$ f is not R-integrable but f is $\mathcal{L}$-integrable

Lebesgue Criterion for Riemann integration

Def:- Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Let $A \subseteq [a,b]$. Then the number

$$\Omega_f(A) := \sup\left(\left\{ |f(x) - f(y)| \;\middle|\; x, y \in A \right\}\right)$$

is called the <u>oscillation</u> of f on A

Def:- The <u>oscillation</u> of f at <u>$x \in [a,b]$</u> is defined as the number

$$\omega_f(x) := \lim_{r \to 0^+} \Omega_f(B(x; r)),$$

where $B(x; r) = \{ y \in [a,b] \mid |x - y| < r \}$

$$= (x - r, x + r) \quad \text{open interval.}$$

The above limit exists always.

In fact, if $A \subseteq B$, then $\Omega_f(A) \leq \Omega_f(B)$.

if $0<r_1 \leq r_2$, then $B(x; r_1) \subseteq B(x; r_2)$

$$\Rightarrow \quad \Omega_f(B(x, r_1)) \leq \Omega_f(B(x, r_2))$$

Lemma 1:- Let $f : [a, b] \to \mathbb{R}$ be a bounded funtion.

Then

f is Continuous at $x \in [a, b] \iff \omega_f(x) = 0$.

proof:-

$\Rightarrow$: Assume f is Continuous at x.

$\Rightarrow$ given $\varepsilon > 0$, there exists $\delta_0 > 0$

such that $|f(y) - f(x)| < \frac{\varepsilon}{2}$,

whenever $|y - x| < \delta_0$

$$\Rightarrow \quad |f(y) - f(x)| < \frac{\varepsilon}{2}, \quad \forall y \in B(x, \delta_0) = (x - \delta_0, x + \delta_0)$$

Let $0 < \delta < \delta_0$

Then $\Omega_f(B(x, \delta)) = \sup\left(\left\{ |f(y) - f(z)| \,\middle|\, y, z \in B(x, \delta) \right\}\right)$

$$\leq \sup\left(\left\{ |f(y)-f(x)| + |f(x)-f(z)| \,\middle/\, y,z \in B(x,\delta) \right\}\right)$$

(by triangular ineq)

$$\leq \sup\left(\left\{ \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \,\middle/\, y,z \in B(x,\delta) \right\}\right)$$

$$\left(\because \begin{array}{l} y \in B(x,\delta) \subseteq B(x,\delta_0) \\ z \in B(x,\delta) \subseteq B(x,\delta_0) \end{array}\right)$$

$$= \varepsilon$$

$$\therefore \quad 0 \leq \Omega_f(B(x,\delta)) \leq \varepsilon$$

$$\Rightarrow \quad \Omega_f(B(x,\delta)) \to 0 \quad \text{as} \quad \delta \to 0^+.$$

$$\Rightarrow \quad \lim_{\delta \to 0^+} \Omega_f(B(x,\delta)) = 0$$

$$\Rightarrow \quad \Omega_f(x) = 0.$$

$\underline{\Leftarrow]:}$ Suppose $\Omega_f(x) = 0$.

$$\Rightarrow \lim_{r \to 0^+} \Omega_f(B(x,r)) = 0.$$

$\Rightarrow$ given $\varepsilon > 0$, there $\delta_0 > 0$ such that

$$|\Omega_f(B(x,r)) - 0| < \varepsilon \quad \text{whenever } 0 < r < \delta_0.$$

$$\Rightarrow \quad \Omega_f(B(x,r)) < \varepsilon, \quad \text{for } 0 < r < \delta_0.$$

$$\Rightarrow \sup\left(\left\{ |f(y) - f(z)| \;/\; y, z \in B(x,r) \right\}\right) < \varepsilon \quad \text{for } 0 < r < \delta_0$$

$$\Rightarrow |f(y) - f(x)| \le \sup\left(\left\{ |f(y) - f(z)| \;/\; y, z \in B(x,r) \right\}\right) < \varepsilon, \quad \text{for } y \in B(x,r),\ 0 < r < \delta_0.$$

$$\Rightarrow \quad |f(y) - f(x)| < \varepsilon \quad \text{whenever } |y - x| < r < \delta_0$$

$$|\mathcal{U}| = ??$$

$$\Rightarrow \quad \lim_{y \to x} f(y) = f(x)$$

$\Rightarrow$ f is continuous at x.