KEERTI P CHARANTIMATH (19MA20059)

Numerical solutions of Ordinary and partial differential equations

Assignment - 4

1. Given

$x^2 y'' - 2y + x = 0 \rightarrow (1)$ $\quad y(2) = 0$

$h = 1/4 = 0.25$ $\quad y(3) = 0$

We know that

General solution of the equation

$\boxed{y(x) = \lambda y_1(x) + (1-\lambda) y_2(x)} \rightarrow 2$

Boundary value problem of first kind

$y_1(2) = \gamma_1 = 0 \quad , y_1'(2) = 0 \quad ; x^2 y_1'' - 2y_1 + x = 0$

$y_2(2) = \gamma_1 = 0 \quad y_2'(2) = 1 \quad ; x^2 y_2'' - 2y_2 + x = 0$

We know that

Second order Taylor's series method

$$y_{j+1} = y_j + h y_j' + \frac{h^2}{2!} y_j'' \rightarrow (3)$$

differentiating

$$y_{j+1}' = y_j' + h y_j'' + \frac{h^2}{2!} y_j''' \rightarrow (4)$$

from eq^n (1) $x^2 y'' - 2y + x = 0 \quad y'' = \frac{2y - x}{x^2} \rightarrow (5)$

differentiating

$x^2 y''' + 2xy'' - 2y' + 1 = 0 \Rightarrow y''' = \frac{2y' - 1 - 2xy''}{x^2} \rightarrow (6)$

substituting (5) in (6)

$$y''' = 2y' - 1 - 2x\left(\frac{2y - x}{x^2}\right) = \frac{2xy' + x - 4y}{x^3}$$

$$\therefore y_{j+1} = y_j + h y_j' + \frac{h^2}{2!}\left(\frac{2y_j - x_j}{x_j^2}\right)$$

$$y_{j+1}' = y_j' + h\left(\frac{2y_j - x_j}{x_j^2}\right) + \frac{h^2}{2!}\left(\frac{2x_j y_j' + x_j - 4y_j}{x^3}\right)$$

for y_1

$j = 0$, $x_0 = 2$, $y_1(2) = 0$, $y_1'(2) = 0$

$$y_1 = \cancelto{0}{y_0} + h\cancelto{0}{y_0'} + \frac{h^2}{2!}\left(\frac{\cancelto{0}{2y_0} - x_0}{x_0^2}\right)$$

$$y_1' = \cancelto{0}{y_0'} + h\left(\frac{\cancelto{0}{2y_0} - x_0}{x_0^2}\right) + \frac{h^2}{2!}\left(\frac{2x_0\cancelto{0}{y_0'} + x_0 - 4\cancelto{0}{y_0}}{x_0^3}\right)$$

$$y_1 = y_1(2.25) = -0.015625$$

$$y_1' = y_1'(2.25) = -0.1171875$$

$j = 1$

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}\left(\frac{2y_1 - x_1}{x_1^2}\right)$$

$$y_2' = y_1' + h\left(\frac{2y_1 - x_1}{x_1^2}\right) + \frac{h^2}{2!}\left(\frac{2x_1y_1' + x_1 - 4y_1}{x_1^3}\right)$$

$$y_2 = y_1(2.5) = -0.0590036651$$

$$y_2' = y_1'(2.5) = -0.2249442730$$

x	$y_1(x)$	$y_1'(x)$
2	0	0
2.25	-0.015625	-0.1171875
2.5	-0.0590036651	-0.2249442730
2.75	-0.12832977	-0.3264419796
3	-0.222364477	-0.423629912

Similarly for y_2

x	$y_2(x)$	$y_2'(x)$
2	0	1
2.25	0.234375	0.8984375
2.5	0.4479890046	0.8251671811
2.75	0.6462606899	0.7706740612
3	0.8329065663	0.7291100265

We know that

$$\boxed{\gamma_2 = u(b) = \lambda u_1(b) + (1-\lambda) u_2(b)},$$

$$\therefore \lambda = \frac{\gamma_2 - u_2(b)}{u_1(b) - u_2(b)} \qquad y(3) = 0$$

$$\lambda = \frac{y(3) - y_2(3)}{y_1(3) - y_2(3)} = 0.7892821201$$

$\therefore$ General solution would be

$$\boxed{y(x) = 0.7892821201 \quad y_1(x) + (0.2107178799) y_2(x)}$$

x	$y_1(x)$	$y_2(x)$	(y)
2	0	0	0
2.25	-0.15625	0.234375	0.03705447
2.5	-0.0590036651	0.447989	0.04782875
2.75	-0.12832977	0.6462606899	0.03489028
3	-0.222364477	0.8329065663	0

6) $u'' = 2uu'$, $0 < x < 1$, $u(0) = 0.5$, $u(1) = 0.5$

Let us define

$u_s(0) = 0.5$, $u_s(0) = s$

$u_s'' = 2(0.5)s = s$

Now

$u_s'' = f(x, u_s, u_s')$

Differentiating partially wrt s,

$$\frac{\partial}{\partial s}(u_s'') = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial u_s} \cdot \frac{\partial u_s}{\partial s} + \frac{\partial f}{\partial u_s'} \cdot \frac{\partial u_s'}{\partial s}$$

$$= \frac{\partial f}{\partial u_s} \cdot \frac{\partial u_s}{\partial s} + \frac{\partial f}{\partial u_s'} \cdot \frac{\partial u_s'}{\partial s}$$

Again

$$\frac{\partial}{\partial s}\left(U_s(0)\right) = 0$$

$$\frac{\partial}{\partial s}\left(U_s'(0)\right) = 1$$

$$v = \frac{\partial U_s}{\partial s} \quad (\text{let})$$

Differentiating partially wrt x,

$$v' = \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial U_s}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial U_s}{\partial x}\right) = \frac{\partial}{\partial s}\left(U_s'\right)$$

$$v'' = \frac{\partial v'}{\partial x} = \frac{\partial}{\partial s}\left(U_s''\right)$$

$$= \frac{\partial f}{\partial U_s} \cdot v + \frac{\partial f}{\partial U_s'} \cdot v'$$

$$= 2\left(U_s v' + v U_s'\right)$$

first variational Equation is

$$v'' = 2\left(u v' + v u'\right)$$

x	v	v'
0	0	1
0.25	0.28125	1.2425
0.5	0.64882156	1.71536687
0.75	1.14342292	2.35302340
1	1.83362015	3.36393594

Now

$$\phi(s) = 0.65496682 - 0.5$$

$$= 0.15496682$$

$$\phi'(s) = 1.83362015$$

Applying Newton - Ralphson method

$$s^{(k+1)} = s^{(k)} - \frac{\phi(s^{(k)})}{\phi'(s^{(k)})}$$

$k=0$

$$s^{(1)} = s^{(0)} - \frac{\phi(s^{(0)})}{\phi'(s^{(0)})}$$

$$= 0.005486$$

Applying Taylor series Method of 2n order

$$U_{j+1} = U_j + hU'_j + \frac{h^2}{2} U''_j$$

$v(0) = 0, \quad v'(0) = 1$

$v''(0) = 0.5$

Applying Taylor series Method of 2^{nd} order

$U_{s,j+1} = U_{s,j} + hU'_{s,j} + \frac{h^2}{2} u''_{s,j}$

$\Rightarrow U'_{j+1} = U'_{s,j} + hU''_{j} + \frac{h^2}{2} u'''_{s,j}$

$= U'_{s,j} + h(2U_{sj}U'_{sj}) + \frac{h^2}{2}(2U_{sj}U'_{sj} + 2(U'_{sj})^2)$

$= U'_{s,j} + 2h\,U_{sj}U'_{sj} + h^2\{U_{sj}(U_j U'_j + (U'_j)^2\}$.

$= U'_{sj} + 2h\,U_{sj}U'_{sj} + h^2(U_j^2 U'_j + (U'_j)^2)$

Let us assume $U'_s(0) = s = 0.09$

x	U_s	U'_s
0	0.5	0.09
0.25	0.5253150	0.11441280
0.5	0.55767202	0.14725508
0.75	0.59961829	0.19283260
1	0.65496682	0.25689891

Applying Again Taylor series Method of 2^{nd} order

$V_{j+1} = V_j + hV'_j + \frac{h^2}{2} V''_j$

$\Rightarrow V'_{j+1} = V'_j + hV''_j + h^2\{U_jV''_j + V'_jU'_j + U_jU''_j + U'_jU'_j\}$

$= V'_j + hV''_j + h^2(U''_jV_j + 2U'_jV'_j + U_jV_j$

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x	u
0	0.5
0.25	0.50154294
0.5	0.50349692
0.75	0.50597354
1	0.50911598

7) $y'' = \frac{3}{2} y^2 \qquad 0 < x < 1 \quad , y(0) = 1 \quad , y(1) = 4$

$h = 1/4$

$s^{(0)} = 0.9$

We need to solve the following IVP

$$\left.\begin{array}{l} u'' = \frac{3}{2} u^2 \\ u(0) = 1, \quad u'(0) = 0.9 \, (s^{(0)}) \end{array}\right\} 1$$

and

$$\left.\begin{array}{l} v'' = 3uv \\ v(0) = 0 \quad , \quad v'(0) = 1 \end{array}\right\} 2$$

The two IVPs in the system of equation form

$$\begin{bmatrix} u \\ \bar{u} \end{bmatrix}' = \begin{bmatrix} \bar{u} \\ 3/2 \, u^2 \end{bmatrix} = \begin{bmatrix} f_1 (x, u, \bar{u}) \\ f_2 (x, u, \bar{u}) \end{bmatrix} \quad \begin{array}{l} u(0) = 1 \\ \bar{u}(0) = 0.9 \end{array}$$

and

$$\begin{bmatrix} v \\ \bar{v} \end{bmatrix}' = \begin{bmatrix} \bar{v} \\ 3uv \end{bmatrix} = \begin{bmatrix} g_1 (x, v, \bar{v}) \\ g_2 (x, v, \bar{v}) \end{bmatrix} \quad \begin{array}{l} v(0) = 0 \\ \bar{v}(0) = 1 \end{array}$$

Runge-kutta method for Second order problem

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$$\bar{k_1} = \begin{bmatrix} k_1^{(1)} \\ k_1^{(2)} \end{bmatrix} = \begin{bmatrix} f_1(x_j, v_j, \bar{v}_j) \\ f_2(x_j, v_j, \bar{v}_j) \end{bmatrix} = \begin{bmatrix} \bar{v}_j \\ 3/2\, v_j^2 \end{bmatrix}$$

$$\bar{k_2} = \begin{bmatrix} k_2^{(1)} \\ k_2^{(2)} \end{bmatrix} = \begin{bmatrix} f_1(x_j + h, v_j + hk_1^{(1)}, \bar{v}_j + hk_1^{(2)}) \\ f_2(x_j + h, v_j + hk_1^{(1)}, \bar{v}_j + hk_1^{(2)}) \end{bmatrix}$$

$$\bar{k_2} = \begin{bmatrix} \bar{v}_j + hk_1^{(2)} \\ \frac{3}{2}(v_j + hk_1^{(1)})^2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} v_{j+1} \\ \bar{v}_{j+1} \end{bmatrix} = \begin{bmatrix} v_j \\ \bar{v}_j \end{bmatrix} + \frac{h}{2}[\bar{k_1} + \bar{k_2}]$$

fourth Order Runge Kutta Method for problem 2

$$\bar{k_1} = \begin{bmatrix} k_1^{(1)} \\ k_1^{(2)} \end{bmatrix} = \begin{bmatrix} g_1(x_j, v_j, \bar{v}_j \\ g_2(x_j, v_j, \bar{v}_j \end{bmatrix} = \begin{bmatrix} \bar{v} \\ 3v_j v_j \end{bmatrix}$$

$$\bar{k_2} = \begin{bmatrix} k_2^{(1)} \\ k_2^{(2)} \end{bmatrix} = \begin{bmatrix} g_1(x_j + h, v_j + hk_1^{(1)}, \bar{v}_j + hk_1^{(2)}) \\ g_2(x_j + h, v_j + hk_1^{(1)}, \bar{v}_j + hk_1^{(2)}) \end{bmatrix}$$

$$\bar{k_2} = \begin{bmatrix} \bar{v} + hk_1^{(2)} \\ 3v_j(v_j + hk_1^{(1)}) \end{bmatrix}$$

$$\begin{bmatrix} \underline{v}_{j+1} \\ v_{j+1} \end{bmatrix} = \begin{bmatrix} v_j \\ \bar{v}_j \end{bmatrix} + \frac{h}{2} [k_1 + k_2]$$

$$g'(s^{(0)}) = V_y \qquad g(s^{(0)}) = U_y - y$$

$$s^{(1)} = s^{(0)} - \frac{g(s^{(0)})}{g'(s^{(0)})}$$

$u_1 = 1.27187500$

$\bar{u}_1 = 1.36886719$

$u_2 = 1.68991989$

$\bar{u}_2 = 2.16067188$

$u_3 = 2.36395486$

$\bar{u}_3 = 3.62863208$

$u_4 = 3.53306362$

$\bar{u}_y = 6.68271871$

and

$v_1 = 0.2500000$

$\bar{v}_1 = 1.09375000$

$v_2 = 0.55324707$

$\bar{v}_2 = 1.46264343$

$v_3 = 1.00655886$

$\bar{v}_3 = 2.39557744$

$v_4 = 1.82852756$

$\bar{v}_4 = 4.71108192$

$$\therefore \; s^{(1)} = 0.9 - \frac{(3.53306362 - y)}{1.82852756}$$

$$\therefore \; s^{(1)} = 1.15536195$$

2) Given BVP $y'' = 2y - y'$

$y(1) = 2e + e^{-2}$

$y(2) = 2\,8^{2} + 8^{-4}$

Use shooting Method

<u>Soln</u> We have to solve the two initial value Problems

i) $y_1'' + y_1' - 2y_1 = 0 \quad Y_1(1) = 2e + e^{-2} \quad Y_1'(1) = 0$

ii) $Y_2'' + Y_2' - 2Y_2 = 0 \quad Y_2(1) = 2e + e^{-2} \quad Y_2'(1) = 1$

Using Taylor Series Method Order 3

$$Y_{j+1} = Y_j + h y_j' + \frac{h^2}{2!} Y_j'' + \frac{h^3}{3!} Y_j'''$$

$$Y_{j+1}' = Y_j' + h y_j'' + \frac{h^2}{2!} y_j''' + \frac{h^3}{3!} y_j^{iv}$$

<u>Solving for $Y_1(8)$ & $Y_1(2)$</u>

$j=1 \quad Y(4/3) = Y(1) + 1/3\, Y'(1) + \frac{(1/3)^2}{2} y''(1) + \frac{(1/3)^3}{3!} Y'''(1)$

$$\boxed{Y_1(4/3) = 6.122210}$$

$Y'(4/3) = Y'(1) + 1/3\, Y''(1) + 1/18\, Y'''(1)$

$Y'(4/3) = 0 + 1/3 \left(2(2e + 0^{c})\right) + \frac{1}{18}(2Y' - Y) + \frac{1}{162}(2Y'' - Y''')$

$Y'(4/3) = 1/3\left[2(2e + 8^{-2})\right] + \frac{1}{18}\left(0 - 2\left[2e + e^{-2}\right]\right) + \frac{1}{162}\left(6(2e + e^{-2})\right)$

$$\boxed{Y_1'(4/3) \cong 3.0954}$$

$j=2 \quad Y(5/3) = Y(4/3) + 1/3\, Y'(4/3) + \frac{(1/3)^2}{2} Y''(4/3) + \frac{(1/3)^3}{3!} Y'''(4/3)$

$$\boxed{Y_1(5/3) \cong 7.644058}$$

$Y'(5/3) = Y'(4/3) + 1/3\, Y''(4/3) + \frac{(1/3)^2}{2} Y''(4/3)$

$$\boxed{Y_1'(5/3) \cong 5.9808}$$

$j=3 \quad Y(2) = Y(5/3) + 1/3\, Y'(5/3) + \frac{(1/3)^2}{2} Y''(5/3) + \frac{(1/3)^3}{3!} Y'''(5/3)$

$$\boxed{Y_1(2) \cong 10.17111}$$

$Y'(2) = Y'(5/3) + 1/3\, Y''(5/3) + \frac{(1/3)^2}{2} Y'''(5/3)$

$$\boxed{Y_1'(2) \cong 9.2307}$$

Solving for $Y_2(2)$ and $Y_2'(2)$

On solving similarly as for Y_1 we get

$$\boxed{\begin{aligned} Y_2(4/3) &\cong 6.418506 \\ Y_2'(4/3) &\cong 3.928833 \end{aligned}}$$

$$\boxed{\begin{aligned} Y_2(5/3) &\cong 8.216531 \\ Y_2'(5/3) &\cong 6.839864 \end{aligned}}$$

$$\boxed{\begin{aligned} Y_2(2) &\cong 11.054667 \\ Y_2'(2) &\cong 10.264626 \end{aligned}}$$

$$Y(b) = \gamma_2 = \lambda Y_1(b) + (1-\lambda) Y_2(b)$$

$$Y(2) = 2e + e^4 = \lambda(10.17111) + (1-\lambda)(11.054667)$$

$$\boxed{\lambda = -4.23927}$$

Using $Y(x) = \lambda Y_1(x) + (1-\lambda) Y_2(x)$

$$Y(x) = -4.239270 Y_1(x) + 5.23927 Y_2(x)$$

$$\left.\begin{aligned} Y(4/3) &\cong 7.673298 \\ Y(5/3) &\cong 10.640912 \\ Y(2) &\cong 14.79642 \end{aligned}\right] \leftarrow \text{Ans.}$$

3) Solve the B.V.P

$$y'' = y \qquad y'(0) = 3$$
$$y'(1) = e + \frac{2}{e}$$

Using Taylor series Method of Order 3 $h = 1/4$

Soln We solve the two initial value problem

i> $y_1'' = y_1 \qquad y_1(0) = 0 \qquad y_1'(0) = 3$

ii> $y_2'' = y_2 \qquad y_2(0) = 1 \qquad y_2'(0) = 3$

Using taylor series of Order 3 we get

$$y_{j+1} = y_j + h y_j' + \frac{h^2}{2!} y_j'' + \frac{h^3}{3!} y_j'''$$

$$y_{j+1}' = y_j' + h y_j'' + \frac{h^2}{2} y_j'''$$

Calculating for y_1

$j = 0$

$y_1 = y(0.25) \qquad y_0 = y(0) = 0$

$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0'''$

$y_1 = 0 + (1/4)(3) + \frac{(1/4)^2}{2}(0) + \frac{(1/4)^3}{3}(3)$

$\boxed{y_1 \cong 0.757813}$

$y_1' = y_0' + h y_0'' + \frac{h^2}{2} y_0'''$

$y_1' = 3 + 1/4 (0) + \frac{(1/4)^2}{2}(3)$

$\boxed{y_1' \cong 3.0937}$

$j = 1$

$y_1(0.5) = y_{12} \cong 1.562989$

$y_1'(0.5) = y_{12}' \cong 3.379883$

$j = 2$

$y_1(0.75) = y_{12} \cong 2.465605$

$y_1'(0.75) = y_{13} \cong 3.87625$

$j = 3$

$$\boxed{\begin{array}{l} y_1(1.0) = y_{14} \cong 3.5021813 \\ y_1'(1.0) = y_{14}' \cong 4.613796 \end{array}}$$

Calculating for Y_2 Similarly calculating like Y_1 we get

$Y_2(0.25) \cong 1.789063$

$Y_2'(0.25) \cong 3.34375$

$Y_2(0.50) \cong 2.689616$

$Y_2'(0.50) \cong 3.895508$

$Y_2(0.75) \cong 3.757688$

$Y_2'(0.75) \cong 4.689647$

$$\boxed{\begin{aligned} Y_2(1.0) &\cong 5.059740 \\ Y_2'(1.0) &\cong 5.775620 \end{aligned}}$$

$$Y'(x) = r_2 = \lambda Y_1' + (1-\lambda) Y_2'$$

$$Y'(1) = e + \frac{2}{e} = \lambda(4.613796) + (1-\lambda)(5.775620)$$

$$\boxed{\lambda = 1.998219}$$

$$Y(x) = 1.998219\, Y_1(x) + (-0.99821)\, Y_2(x)$$

$$\boxed{\begin{aligned} Y(0.25) &\cong -0.271600 \\ Y(0.50) &\cong 0.438369 \\ Y(0.75) &\cong 1.175823 \\ Y(1.0) &\cong 1.986625 \end{aligned}} \leftarrow \text{Ans.}$$

4) Shooting Method to solve B.V.P.

$$y'' = xy + 1$$

<u>B.C</u>

$$\left.\begin{array}{r} y(0) + y'(0) = 1 \\ y(1) = 1 \end{array}\right]$$

Using Taylor Series Method of Order 3

<u>Soln</u> We solve the two initial value Problem

i) $y_1'' = xy_1 + 1$ $\quad y_1(0) = 0 \quad y_1'(0) = 1$

ii) $y_2'' = xy_2 + 1$ $\quad y_2(0) = 1 \quad y_2'(0) = 0$

Using taylor series expansion of order 3

$$\left[\begin{array}{l} y_{j+1} = y_j + h y_j' + \frac{h^2}{2!} y_j'' + \frac{h^3}{3!} y_j''' \\ y_{j+1}' = y_j' + h y_j'' + \frac{h^2}{2!} y_j''' + \frac{h^3}{3!} y_j^{iv} \end{array}\right]$$

<u>On solving for y_1</u>

$j = 0$ $\quad y_1(0.25) \cong 0.28125$

$\quad y_1'(0.25) \cong 1.25$

$j = 1$ $\quad y_1(0.50) \cong 0.628745$

$\quad y_1'(0.50) \cong 1.536148$

$j = 2$ $\quad y_1(0.75) \cong 1.057494$

$\quad y_1'(0.75) \cong 1.908392$

$j = 3$ $\quad \boxed{\begin{array}{l} y_1(1.00) \cong 1.597108 \\ y_1'(1.00) \cong 2.434447 \end{array}}$

5) Use shooting Method to solve BVP

$y'' = 6y^2 \quad y(0) = 1$

$y(\frac{3}{10}) = \frac{100}{169}$

Use Taylor Series method order 3 $h = \frac{1}{10}$

Secant method for iteration

Soln following is Non-Linear BVP

We solve the two Initial Value Problems

i) $y_1'' = 6y_1^2 \quad Y_1(0) = 1 \quad Y_1'(0) = 3^{(0)} = -1.8$

ii) $y_2'' = 6Y_2^2 \quad Y_2(0) = 1 \quad Y_2'(0) = 3^{(2)} = -1.9$

Using Taylor Series method of Order 3, We get

$y_{j+1} = y_j + hy_j' + \frac{h^2}{2!} y_j'' + \frac{h^3}{3!} y_j''' \quad h = 0.1$

$y_{j+1}' = y_j' + hy_j'' + \frac{h^2}{2!} y_j''' + \frac{h^3}{3!} y_j^{iv}$

On solving for Y_1 :-

On solving for values we get

$j = 0 \quad Y_1(0.1) \cong 0.8464$

$Y_1'(0.1) \cong -1.308$

$j = 1 \quad Y_1(0.2) \cong 0.734878$

$Y_1'(0.2) \cong -0.94459$

$j = 2$ $\boxed{\begin{array}{l} Y_1(0.3) \cong 0.655232 \\ Y_1'(0.3) \cong -0.578913 \end{array}}$ or $Y(3, 0.3) = 0.655232$

we define $\phi(s) = y_1(s, x) - y(x)$

$\phi(s^{(0)}) = y_1(s^{0}, 0.3) - y(0.3)$

$\phi(s^{(1)}) = 0.655232 - \frac{100}{169}$

$\boxed{\phi(s^{(1)}) = 0.063516}$

On solving for Y_2 similarly as of Y_1

$j=0$ $Y_2(0.25) \cong 1.033854$

$Y_2'(0.25) \cong 0.28125$

$j=1$ $Y_2(0.50) \cong 1.146369$

$Y_2'(0.50) \cong 0.630371$

$j=2$ $Y_2(0.75) \cong 1.356930$

$Y_2'(0.75) \cong 1.069341$

$j=3$

$$\boxed{\begin{aligned} Y_2(1.0) &\cong 1.692941 \\ Y_2'(1.0) &\cong 1.641232 \end{aligned}}$$

$$Y(x) = \lambda Y_1(x) + (1-\lambda) Y_2(x)$$

$$Y(1) = \gamma_2 = \lambda y_1(1) + (1-\lambda) Y_2(1)$$

$$1 = \lambda(1.597108) + (1-\lambda)(1.692941)$$

$$\boxed{\lambda \cong 7.230714}$$

$$Y(x) = 7.230714\, Y(x) - 6.230714\, Y_2(x)$$

$$\boxed{\begin{aligned} Y(0.25) &\cong -4.408010 \\ Y(0.50) &\cong -2.596422 \\ Y(0.75) &\cong -0.808206 \\ Y(1.0) &\cong 1.0 \end{aligned}}$$

← Ans

On solving for Y_2

$j=0$ $\quad Y_2(0.1) \cong 0.8362$

$\quad\quad Y_2'(0.1) \cong -1.414$

$j=1$ $\quad Y_2(0.2) \cong 0.713412$

$\quad\quad Y_2'(0.2) \cong -1.065405$

$j=2$ $\quad \boxed{\begin{array}{l} Y_2(0.3) \cong 0.620620 \\ Y_2'(0.3) \cong -0.805635 \end{array}} \Rightarrow Y_2(s,0.3) = 0.620620$

$$\phi(s^{(1)}) = Y_2(s,0.3) - Y(0.3)$$

$$\phi(s^{(1)}) = 0.620620 - \frac{100}{169}$$

$$\boxed{\phi(s^{(1)}) = 0.028904}$$

Using the secant method to find s such that $\phi(s) \cong 0$

given $s^{(0)} = -9/5$ $\quad s^{(1)} = -19/10$

$$s^{(k+1)} = s^{(k)} - \left[\frac{s^{(k)} - s^{(k-1)}}{\phi(s^{(k)}) - \phi(s^{(k-1)})}\right]\phi(s^{(k)})$$

$$s^{(2)} = s^{(1)} - \left[\frac{s^{(1)} - s^{(0)}}{\phi(s^{(1)}) - \phi(s^{(0)})}\right]\phi(s^{(1)})$$

On substituting values

$$s^{(2)} = -\frac{19}{10} - \left[\frac{-19/10 + 9/5}{0.028904 - 0.063516}\right](0.028904)$$

$$\boxed{s^{(2)} \cong -1.983504}$$

Now On solving for Y_3

$$Y_3'' = 6Y_3^2 \quad Y_3(0) = 1$$

$$Y_3'(0) = s^{(2)} = -1.983504$$

Again using Taylor series method we get

$Y_3(0.1) \cong 0.835616$

$Y_3'(0.1) \cong -1.502520$

$Y_3(0.2) \cong 0.703801$

$Y_3'(0.2) \cong -1.158899$

$$\boxed{\begin{array}{l} Y_3(0.3) \cong 0.601140 \\ Y_3'(0.3) \cong -0.910636 \end{array}} \Rightarrow Y_3(3, 0.3)$$

$$\phi(s^{(2)}) = Y_3(3, 0.3) - Y(0.3)$$

$$\phi(s^{(2)}) = 0.009424$$

$$s^{(3)} = s^{(2)} - \left[\frac{s^{(2)} - s^{(1)}}{\phi(s^{(2)}) - \phi(s^{(1)})}\right] \phi(s^{(2)})$$

$$s^{(3)} = -1.983504 - \left[\frac{-1.983504 + 19/10}{0.009424 - 0.028904}\right] \times 0.009424$$

$$\boxed{s^{(3)} = -2.023909}$$

Exact value of $Y'(0) = -2$

Finally solving IVP

$$Y'' - 6Y^2 \quad Y(0) = 1$$

$$Y'(0) = -2.023909$$

On solving using 3rd order Taylor series Method We get

$$\boxed{\begin{array}{l} Y(0.1) \cong 0.823561 \\ Y(0.2) \cong 0.686829 \\ Y(0.3) \cong 0.577837 \end{array}} \leftarrow \text{Ans}$$

8) (i) $y'' = y + x \quad y(0) = 0 \quad y(1) = 0$

Substitute $Y_i'' = \left(\dfrac{Y_{i+1} - 2Y_i + Y_{i-1}}{h^2}\right)$ in $(y_i'' = y_i + x_i)$

$$\Rightarrow \frac{Y_{i+1} - 2Y_i + Y_{i-1}}{h^2} = Y_i + x_i$$

$$\Rightarrow \boxed{Y_{i+1} - (2+h^2)Y_i + Y_{i-1} = h^2 x_i} \quad i = 1, 2, 3, \ldots$$

a) $h = 1/2$ $\quad x_0 = 0 \quad x_1 = 1/2 \quad x_2 = 1$

$Y_0 = 0 \quad Y_2 = 0$

$i=1 \quad Y_2 - (2 + 1/4)Y_1 + Y_0 = 1/4(1/2)$

$$Y_1 = \left[\frac{0 + 0 - 1/8}{(2 + 1/4)}\right] = \left(\frac{1}{-18}\right) = -0.05556$$

$$\boxed{Y(1/2) = -0.05556} \leftarrow \text{Ans}$$

b) $h = 1/3$ $\quad x_0 = 0 \quad x_1 = 1/3 \quad x_2 = \frac{2}{3} \quad x_3 = 1$

$y_0 = 0 \quad y_3 = 0$

$\underline{i=1} \quad Y_2 = (2 + 1/9)Y_1 + Y_0 = \frac{1}{9}(1/3)$

$$\boxed{Y_2 - \frac{19}{9}Y_1 = \left(\frac{1}{27}\right)} \text{ — (1)}$$

$i=2 \quad Y_3 = (2 + 1/9)Y_2 + Y_1 = 1/9(2/3)$

$$\boxed{-\frac{19}{9}Y_2 + Y_1 = \left(\frac{2}{27}\right)} \text{ — (11)}$$

Solving (1) and (11) simultaneously we get

$Y_1 \cong -0.044048$

$Y_2 \cong -0.055952$

$$\boxed{\begin{aligned} Y(1/3) &= -0.044048 \\ Y(2/3) &= -0.055952 \end{aligned}} \leftarrow \text{Ans}$$

c) $h = 1/4$ $\quad x_0 = 0 \quad x_1 = 1/4 \quad x_2 = 2/4 \quad x_3 = \frac{3}{4} \quad x_4 = 1$

$y_0 = 0 \qquad y_4 = 0$

$i = 1 \quad Y_2 - \left(2 + \frac{1}{16}\right) Y_1 + \cancel{Y_0}^{\,0} = \left(\frac{1}{4}\right)^2 \left(\frac{1}{4}\right)$

$i = 2 \quad Y_3 - \left(2 + \frac{1}{16}\right) Y_2 + Y_1 = \left(\frac{1}{4}\right)^2 \left(\frac{2}{4}\right)$

$i = 3 \quad \cancel{Y_4}^{\,0} - \left(2 + \frac{1}{16}\right) Y_3 + Y_2 = \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)$

$$\begin{pmatrix} -\left(\frac{33}{16}\right) & 1 & 0 \\ 1 & -\left(\frac{33}{16}\right) & 1 \\ 0 & +1 & \left(-\frac{33}{16}\right) \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{64} \\ \frac{2}{64} \\ \frac{3}{64} \end{pmatrix}$$

On solving we get

$Y_1 \cong -0.034885$

$Y_2 \cong -0.056326$

$Y_3 \cong -0.050037$

$$\boxed{\begin{aligned} Y(1/4) &\cong -0.034885 \\ Y(1/2) &\cong -0.056326 \\ Y\left(\tfrac{3}{4}\right) &\cong -0.050037 \end{aligned}}$$ ← Ans