TOPICS COVERED

1. Solving 3rd order BVP by BLOCK TRIDIAGONAL METHOD.
2. Solving special cases of 4th order BVPs
3. Introduction of non-linear BVP.

   Recap of Newton Raphson method.

HIGHER ORDER BVP

$$y''' + A(x)y'' + B(x)y' + C(x)y = D(x)$$

$$y(0) = \alpha \,, \quad y'(0) = \varepsilon$$

$$y'(L) = \lambda$$

$$0 < x < L$$

Using central diff method, 2 issues:

(1) Do not get tri-diagonal system.

(2) # unknowns > # equations

We reduce 3rd order eq^n to 2nd order:

COUPLED EQ^{n}s,

note: z depends on y

$$\begin{cases} z = y' & \text{(i)} \\ z'' + A(x)z' + B(x)z + C(x)y = D(x) & \text{(ii)} \end{cases}$$

$$y(0) = \alpha$$

$$z(0) = \varepsilon$$

$$z(L) = \lambda$$

- Integrate (i) between x_{i-1}, x_i using TRAPEZOIDAL RULE ($\because$ z is an unknown function of y)

$$\frac{dy}{dx} = z$$

$$\Rightarrow \int_{x_{i-1}}^{x_i} dy = \int_{x_{i-1}}^{x_i} z\,dx$$

$$\Rightarrow y_i - y_{i-1} - \frac{\delta x_{i-1}}{2}(z_i + z_{i-1}) = 0$$

$$\Rightarrow y_i - y_{i-1} - \frac{h}{2}(z_i + z_{i-1}) = 0$$

- Use central diff. to discretise (ii)

You will have $2(n-1)$ eq^n involving $y_1, y_2, \ldots, y_{n-1}$

& $z_1, z_2, \ldots, z_{n-1}$

$\rightarrow 2(n-1)$ var,

$$\frac{z_{i+1} - 2z_i + z_{i-1}}{h^2} + A_n \frac{z_{i+1} - z_{i-1}}{2h} + B_i z_{i-1} + C_i y_i = D_i$$

$i = 1, \ldots, n-1$

Combine I and II into a matrix equation.

$$Z_i = \begin{pmatrix} y_i \\ z_i \end{pmatrix}$$

and express the paired equations in matrix form as follows:

$$\bar{A}_i \bar{Z}_{i-1} + \bar{B}_i \bar{Z}_i + \bar{C}_i \bar{Z}_{i+1} = \bar{D}_i$$

$$\Rightarrow \begin{pmatrix} -1 & -h/2 \\ 0 & \frac{1}{h^2} - \frac{A_i}{2h} \end{pmatrix} \begin{pmatrix} y_{i-1} \\ z_{i-1} \end{pmatrix} + \begin{pmatrix} 1 & -h/2 \\ C_i & -\frac{2}{h^2} + B_i \end{pmatrix} \begin{pmatrix} y_i \\ z_i \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{A_i}{2h} \end{pmatrix} \begin{pmatrix} y_{i+1} \\ z_{i+1} \end{pmatrix} = \begin{pmatrix} 0 \\ D_i \end{pmatrix}$$

(D_i)

Here

$$\bar{A} = \begin{pmatrix} -1 & -h/2 \\ 0 & \frac{1}{h^2} - \frac{A_i}{2h} \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 1 & -h/2 \\ C_i & -\frac{2}{h^2} + B_i \end{pmatrix}$$

$$\bar{C} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{A_i}{2h} \end{pmatrix}, \quad \bar{D} = \begin{pmatrix} 0 \\ D_i \end{pmatrix}$$

$$\Rightarrow \quad \bar{A}_i \bar{Z}_{i-1} + \bar{B}_i \bar{Z}_i + \bar{C}_i \bar{Z}_{i+1} = \bar{D}_i$$

$\because$ y_0, z_0 are known (given as B.C.s)

and y_n does not enter this system,

while z_n is known (B.C.),

we take $i = 1$ to $n-1$

At $i=1$

$$\bar{A}_1 \bar{Z}_0 + \bar{B}_1 \bar{Z}_1 + \bar{C}_1 \bar{Z}_2 = \bar{D}_1$$

$\bar{A}_1$ and $\bar{Z}_0$ are known

$$\Rightarrow \bar{B}_1 \bar{Z}_1 + \bar{C}_1 \bar{Z}_2 = \bar{D}_1 - \bar{A}_1 \bar{Z}_0$$

For $i = 2$ to $n-2$

$$\bar{A}_i \bar{Z}_{i-1} + \bar{B}_i \bar{Z}_i + \bar{C}_i \bar{Z}_{i+1} = \bar{D}_i$$

At $i = n-1$

$$\bar{A}_{n-1} \bar{Z}_{n-2} + \bar{B}_{n-1} \bar{Z}_{n-1} + \bar{C}_{n-1} \bar{Z}_n = \bar{D}_{n-1}$$

Now here, in the product $\bar{C}_{n-1}\bar{Z}_n$, all values are known.

$$\therefore \bar{A}_{n-1} \bar{Z}_{n-2} + \bar{B}_{n-1} \bar{Z}_{n-1} = \bar{D}_{n-1} - \bar{C}_{n-1} \bar{Z}_n$$

This can be expressed as a tri-diagonal system.

$$\begin{bmatrix} \bar{B}_1 & \bar{C}_1 & 0 & \cdots & 0 & 0 & 0 \\ \bar{A}_2 & \bar{B}_2 & \bar{C}_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \bar{A}_{n-1} & \bar{B}_{n-1} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_{n-1} \end{bmatrix} = \begin{bmatrix} \bar{D}_1 - \bar{A}_1 \bar{Z}_0 \\ \bar{D}_2 \\ \vdots \\ \bar{D}_{n-1} - \bar{C}_{n-1} \bar{Z}_n \end{bmatrix}$$

$$\mathbf{A} \qquad \mathbf{X} = \mathbf{B}$$

This kind of matrix is called a BLOCK-TRIDIAGONAL MATRIX.

which can be reduced to

$$\begin{bmatrix} I & M_1 & O & \cdots & O & O & O \\ O & I & M_2 & \cdots & O & O & O \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & O & O & I \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_{n-1} \end{bmatrix} = \begin{bmatrix} \bar{D}'_1 \\ \bar{D}'_2 \\ \vdots \\ \bar{D}'_{n-1} \end{bmatrix}$$

ALGORITHM for BLOCK ELIMINATION METHOD:

Here

$$M_1 = (\bar{B}_1)^{-1} \bar{C}_1 \quad \text{and } \bar{D}'_1 = (\bar{B}_1^{-1})(\bar{D}_1 - \bar{A}_1 \bar{Z}_0)$$

and for $i = 2, \ldots, n-1$,

$$\bar{B}'_i = \bar{B}_i - \bar{A}_i M_{i-1}$$

$$M_i = (\bar{B}'_i)^{-1} \bar{C}_i \quad , \bar{D}'_i = (\bar{B}'_i)^{-1} (\bar{D}_i - \bar{A}_i \bar{D}'_{i-1})$$

where

$$Z_{n-1} = \bar{D}'_{n-1}$$

$$Z_i = \bar{D}'_i - M_i Z_{i+1}$$

$$i = n-2, n-3, \cdots, 2, 1$$

This method does not give us $\begin{pmatrix} y_n \\ z_n \end{pmatrix}$ directly.

Will have to do a substitution to get the value.

Q. $y''' + 4y'' + y' - 6y = 1$

$y(0) = y'(0) = 0, \quad y(1) = 1$

$h = 0.25$

NEXT CLASS: 4th order coupled system.

Q. $y''' + A(x)y' + B(x)y' + C(x)\,y = D(x)$

$y'(0) = \alpha$

$y(L) = \beta$

$y'(L) = \gamma$

Block condition can also be acquired in other cases,

eg.

$$y^{(iv)} + 81y = 81x^2$$

$$y(0) = y(1) = y''(0) = y''(1) = 0$$

Let $\quad p = y'' \quad \Rightarrow \quad y'' - p = 0$

$$p'' + 81y = 81x^2$$

$$y_i'' - P_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - P_i = 0$$

$$\frac{P_{i+1} - 2P_i + P_{i-1}}{h^2} + 81y_i = 81x_i^2$$

$$\underbrace{\begin{pmatrix} \frac{1}{h^2} & 0 \\ 0 & \frac{1}{h^2} \end{pmatrix}}_{\bar{A}} \begin{pmatrix} y_{i-1} \\ P_{i-1} \end{pmatrix} + \underbrace{\begin{pmatrix} -\frac{2}{h^2} & -1 \\ 81 & -\frac{2}{h^2} \end{pmatrix}}_{\bar{B}} \begin{pmatrix} y_i \\ P_i \end{pmatrix} + \overbrace{\begin{pmatrix} \frac{1}{h^2} & 0 \\ 0 & \frac{1}{h^2} \end{pmatrix}}^{\bar{C}} \begin{pmatrix} y_{i+1} \\ P_{i+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 81x_i^2 \end{pmatrix}$$

$h = 1/3 \quad , \; n = 3 \; , \quad n-1 = 2$

$i = 1$

$$\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} y_0 \\ P_0 \end{pmatrix} + \begin{pmatrix} -18 & -1 \\ 81 & -18 \end{pmatrix} \begin{pmatrix} y_1 \\ P_1 \end{pmatrix} + \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} y_2 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 9\,\cancel{81}\left(\frac{1}{\cancel{9}}\right) \end{pmatrix}$$

$$\cancel{9y_0} - 18y_1 - P_1 + 9y_2 = 0$$

$$\cancel{9P_0} + 81y_1 - 18P_1 + 9P_2 = 9$$

$$\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} y_1 \\ P_1 \end{pmatrix} + \begin{pmatrix} -18 & -1 \\ 81 & -18 \end{pmatrix}\begin{pmatrix} y_2 \\ P_2 \end{pmatrix} + \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} y_3 \\ P_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \overset{9}{\cancel{81}}\left(\frac{4}{\cancel{9}}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 36 \end{pmatrix}$$

$$9y_1 - 18y_2 - P_2 + \cancel{9y_3} = 0$$

$$9P_1 + 81y_2 - 18P_2 + \cancel{9P_3} = 36$$

- <u>STRAIGHT-FORWARD METHOD</u>

Q. $$y_i^{(iv)} = y_i'' = \frac{y_{i+1}'' - 2y_i'' + y_{i-1}''}{h^2}$$

$$= \frac{1}{h^2}\left[\frac{1}{h^2}\left\{ y_{i+2} - 2y_{i+1} + y_i - 2y_{i+1} + 4y_i - 2y_{i-1} + y_i - 2y_{i-1} + y_{i-2} \right\}\right]$$

$$= \frac{1}{h^4}\left[y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2} \right]$$

Substituting this in

$$y^{(iv)} + 81y = 81x^2$$

$$y(0) = y(1) = y''(0) = y''(1) = 0$$

and multiplying by h^4 on both sides,

we get

$$y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2} + 81h^4 y_i = 81h^4 x_i^2$$

$$y_0'' = y_n'' = 0 = y_0 = y_n$$

By introducing y_{-1} and y_{n+1},

$$y_0'' = \frac{y_1 - 2y_0 + y_{-1}}{h^2} \Rightarrow y_{-1} = 2y_0 - y_1$$
$$= -y_1$$

$$y_n'' = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \Rightarrow y_{n+1} = -y_{n-1}$$

Now, taking $h = 1/3$, write the equation for $i = 1, 2$

$\underline{i=1}$

$$\cancel{y_3} - 4y_2 + 6y_1 - \cancel{4y_0} + \overset{-y_1}{\cancel{y_{-1}}} + \cancel{81}\left(\frac{1}{\cancel{81}}\right)\cancel{y_1} = \cancel{81}\left(\frac{1}{\cancel{81}}\right)\left(\frac{1}{3}\right)^2$$

$$-4y_2 + 6y_1 = \frac{1}{9} \qquad \ldots (a)$$

$\underline{i=2}$

$$\overset{-y_2}{\cancel{y_4}} - \cancel{4y_3} + 6y_2 - 4y_1 + \cancel{y_0} + \cancel{81h^4 y_2} = \cancel{81}\left(\frac{1}{\cancel{81}}\right)\left(\frac{4}{9}\right)$$

$$6y_2 - 4y_1 = \frac{4}{9} \qquad \ldots (b)$$

$$6(a) + 4(b) \Rightarrow (36 - 16)y_1 = \frac{6 + 16}{9}$$

$$\Rightarrow \quad \overset{10}{\cancel{20}}\, y_1 = \frac{\overset{11}{\cancel{22}}}{9}$$

$$\Rightarrow \quad y_1 = 11/90$$

$$\Rightarrow \quad y_2 = \frac{1}{4}\left(\frac{66}{90} - \frac{10}{90}\right) = \frac{1}{\cancel{4}} \times \frac{\overset{\overset{7}{\cancel{14}}}{\cancel{56}}}{\cancel{90}\,45}$$

Ans. $y_1 = \frac{11}{90}$, $y_2 = \frac{7}{45}$

Q. $y^{(iv)} + 81y = \phi(x)$

$\phi(x)$ is given by the table :

Our goal is to get a compact system.

x	$1/3$	$2/3$	1
$\phi(x)$	81	162	243

$$y(0) = y'(0) = y''(1) = y'''(1) = 0$$

Take $h = 1/3$

Ans. Introduce fictitious points y_{-1}, y_{n+1}

i.e. y_{-1} and y_4

$$y_0' = 0 \Rightarrow y_{-1} = y_1$$

$$y_3'' = 0 \Rightarrow \frac{y_4 - 2y_3 + y_2}{h^2} = 0$$

$$\Rightarrow y_4 = 2y_3 - y_2$$

$$y_3''' = 0 \Rightarrow (y_3'')' = \frac{y_4'' - y_2''}{2h}$$

$$= \frac{y_5 - 2y_4 + \cancel{y_3} - \cancel{y_3} + 2y_2 - y_1}{2h^3}$$

$$\Rightarrow y_3''' = \frac{y_5 - 2y_4 + 2y_2 - y_1}{2h^3} = 0$$

$$\Rightarrow y_5 = 2y_4 - 2y_2 + y_1$$

Discretise for $i = 1, 2, 3$

$$y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2} + 81h^4 y_i = h^4 \phi(x_i)$$

$\underline{i=1}$

$$y_3 - 4y_2 + \underset{\sim}{6y_1} \cancel{-4y_0}^{=0} + \cancel{y_{-1}}^{=y_1} + 81\left(\frac{1}{81}\right) \underset{\sim}{y_1} = \frac{1}{81} \cdot 81$$

$$\Rightarrow y_3 - 4y_2 + 8y_1 = 1$$

$\underline{i=2}$

$$y_4 - 4y_3 + 6y_2 - 4y_1 \cancel{+y_0}^{=0} + \cancel{81h^4} y_2 = \frac{1}{\cancel{81}} \cdot \cancel{162}^{2}$$

$$\Rightarrow 2y_3 - y_2 - 4y_3 + 7y_2 - 4y_1 = 2$$

$$\Rightarrow -2y_3 + 6y_2 - 4y_1 = 2$$

$\underline{i=3}$

$$y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 + \cancel{81h^4} y_3 = \frac{1}{\cancel{81}} \cancel{243}^{3}$$

$$\Rightarrow 2y_4 - 2y_2 + y_1 - 4y_4 + 6y_3 - 4y_2 + y_1 + y_3 = 3$$

$$\Rightarrow -2y_4 + 7y_3 - 6y_2 + 2y_1 = 3$$

$$\Rightarrow -4y_3 + 2y_2 + 7y_3 - 6y_2 + 2y_1 = 3$$

$$\Rightarrow 3y_3 - 4y_2 + 2y_1 = 3$$

which upon solving gives us

$$y_1 = \frac{8}{13}, \quad y_2 = \frac{22}{13}, \quad y_3 = \frac{37}{13}$$

Q. $y''' + A(x)y'' + B(x)y' + C(x)\,y = D(x)$

$y'(0) = \alpha$

$y(L) = \beta$

$y'(L) = \gamma$

How to reduce the system ?

$y' - z = 0$

TRICK:

Integrate between x_i and x_{i+1} instead.

$$\int_{x_i}^{x_{i+1}} dy - \int_{x_i}^{x_{i+1}} z\,dx = 0$$

$$y_{i+1} - y_i - \frac{h}{2}(z_i + z_{i+1}) = 0$$

$i = 1, 2, \ldots, (n-1)$

Q. $y''' + 4y'' + y' - 6y = 1$

$y(0) = y'(0) = 0, \quad y'(1) = 1$

$h = 0.1, 0.05, 0.01$

"optimal step size".

Q. $y^{iv} + 2y = \frac{x^2}{9} + \frac{2}{3}x + 4$

$y(0) = y'(0) = y(3) = y'(3) = 0$

$h = 1$.

Q. $y^{IV} - y''' + y = x^2$

$y(0) = y'(0) = 0$, $y(1) = 2$, $y'(1) = 0$

$h = 1/3$

NON-LINEAR BVP

$$G(y, y', y'') = 0$$

eg. (arbitrary)

$$y'' + (y')^2 + y^2 = x \quad , \quad a < x < b$$

$$y(a) = y_a \, , \quad y(b) = y_b$$

$$x_i = a + i\delta x = a + ih \quad , \quad i = 1, 2, \ldots, N-1$$

$$y_0 = y_a \quad ; \quad y_N = y_b$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{(y_{i+1} - y_{i-1})^2}{4h^2} + y_i^2 = x_i$$

$$i = 1, 2, \ldots, N-1$$

Forms a system of $(N-1)$ equations with $(N-1)$ variables.

PRO → a compact system

CON → cannot be expressed in matrix form due to non-linear nature.

NEWTON RAPHSON METHOD for solving nonlinear equations iteratively.

$$\phi(x) = 0 \quad , \quad \alpha \text{ is the root.}$$

Let α_k be an approximation of α.

$$\alpha = \alpha_k + \text{Error} = \alpha_k + \Delta\alpha$$

$$\Delta\alpha \rightarrow \text{unknown.}$$

$$\phi(\alpha_k + \Delta\alpha) = 0 \quad \ldots \text{(i)}$$

Expand (i) by Taylor Series -

$$\phi(\alpha_k) + \Delta\alpha\, \phi(\alpha_k) + \frac{(\Delta\alpha)^2}{2!}\phi''(\alpha_k) + \cdots = 0$$

The unknown $\Delta\alpha$ satisfies the infinite degree polynomial given above.

If we retain only the terms upto linear order of $\Delta\alpha$, we get

$$\phi(\alpha_k) = \Delta\alpha + \phi'(\alpha_k) \simeq 0$$

$$\Delta\alpha \quad - \frac{\phi(\alpha_k)}{\phi'(\alpha_k)}$$

$$\alpha_{k+1} = \alpha_k + \Delta\alpha$$

$$\Rightarrow \alpha_{k+1} = \alpha_k - \frac{\phi(\alpha_k)}{\phi'(\alpha_k)} \quad , k = 0, 1, 2, \ldots$$

$\alpha_{k+1} \rightarrow$ next approximation for α.

We repeat the above procedure till $|\phi(\alpha_k)| < \varepsilon$.

To start the iteration we consider an initial approximation α_0.

NR method has a quadratic order of convergence.

$\hookrightarrow$ Convergence depends on initial guess (α_0).

CAUCHY SEQUENCE (recall - Real Analysis)

$$\{\alpha_k \mid k \geq 0\} \quad , \quad \alpha_k \to \alpha \text{ as } k \to \infty$$

A sequence $X = (x_n) \in \mathbb{R}$ is said to be a Cauchy sequence if $\forall \varepsilon > 0 \; \exists \; H(\varepsilon) \in \mathbb{N}$ s.t. $\forall \; \mathbb{N} \ni n, m \geq H(\varepsilon)$,

$$|x_n - x_m| < \varepsilon .$$

Similarly here, for convergence we can check

$$|\alpha_{k+1} - \alpha_k| < \varepsilon$$

For a pre-assigned $\varepsilon > 0$.