$Ax = b$

Left inverse :- Any matrix C is called a left inverse of A if $CA = I$

$Ax = b \qquad A \in \mathbb{R}^{m \times n}, \; b \in \mathbb{R}^m$

Let C be a left inverse of A.

$Cb = C(Ax) = (CA)x = x$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$C = A^T, \quad Cb = A^T b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x$$

check $Ax = b$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

{Hence no solution exist for $x$ such that $Ax = b$}

If any "$x$" exist by using left inverse then that $x$ is a unique solution.

We can use this condition to check for solution existence.

(#) Right Inverse

→ Tall matrix can not have a right inverse as rows are linearly dependent $\begin{bmatrix} & \\ & \end{bmatrix}$ ✗

→ Square matrix and wide matrix can have right inverse if the rows are linearly independent.

$\begin{bmatrix} & \end{bmatrix}$ ✓ $\begin{bmatrix} & & \end{bmatrix}$ ✓

something wrong (see saurabh's notes)

$A \in \mathbb{R}^{m \times n}, \; b \in \mathbb{R}^m$

A is right invertible. Let D be a right inverse

$AD = I_{m \times m}$

$Ax = b$

let $x = Db$

$Ax = A(Db) = (AD)b = b$

- Since, rows are linearly independent Hence rank of A is "m".
- Hence columns are spanning the entire $\mathbb{R}^m$.
- Hence, if a right inverse exist then there always exist a solution.

(*) {Row dimension = column dimension}

If $D$ is the right inverse of $A$
$\Rightarrow$ $D^T$ is the left inverse of $A^T$.

Right invertibility :- rows are linearly independent.

Left invertibility :- columns are linearly independent

(#) Invertibility of a Matrix

↳ If a matrix is both left and right invertible $\Rightarrow$ the matrix is invertible

Hence, both rows and columns are linearly independent.

- If left inverse and right inverse exist then the matrix is invertible (Prove that)
- Show that if the matrix is invertible then the left and right inverse is same.
- Prove that inverse is unique.