Group Theory

Lecture 2

06/01/202

<u>Defn</u>. A group is a non-empty set G together with a binary operation which is associative and has identity elt and every elt of G has an inverse.

<u>Remark</u>: (1) If the binary operation is written additively '+' then

$$na = \underbrace{a + \cdots + a}_{n\text{-times}}$$

and its inverse is denoted by $-a$.

(2) If the binary operation is written multiplicatively '$\cdot$' then

$$a^n = \underbrace{a \cdot a \cdots a}_{n\text{-times}}$$

and inverse is denoted by a^{-1}.

Symmetric Group:

Let $T = \{1, 2, \ldots, n\}$.

$$S_n = \{ f : T \longrightarrow T \mid f \text{ is 1-1 and onto} \}.$$

S_n is a group wrt composition operation. This group S_n is known as Symmetric group and the elts of this group are known as permutations.

$$S_3 = \{ f : \{1,2,3\} \longrightarrow \{1,2,3\} \mid f \text{ is bijection} \}.$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}}_{}^{Id}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$

$f(1)$ $f(2)$ $f(3)$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \rightsquigarrow (1\ 2\ 3)$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \rightsquigarrow (1\ 2)(3) \rightsquigarrow (1\ 2).$$

$$S_3 = \{ (1), (1\ 2\ 3), (1\ 3\ 2), (1\ 2), (2\ 3), (1\ 3) \}.$$

$$S_6 \ni \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 6 & 1 & 2 \end{pmatrix}$$

We write it as $(1\ 5)(2\ 4\ 6)$ it is known as cycle decomposition of σ.

$$\underset{\substack{\downarrow \\ f_2}}{(1\ 2\ 3)} \cdot \underset{\substack{\downarrow \\ f_1}}{(2\ 3)} = (1\ 2)$$

$$f_2 \circ f_1$$

$$(1\,2\,3) \circ (1\,2)(3\,4) = (1\,3\,4) \quad (\text{in } S_4)$$

If $\sigma \in S_n$. then the inverse of σ is denoted by σ^{-1} s.t. $\sigma \circ \sigma^{-1} = \mathrm{Id}$.

$$\underset{\sigma}{\underset{\|}{(1\,2\,3)}} \circ (1\,3\,2) = (1)$$

$$\sigma^{-1} \hookrightarrow (3\,2\,1).$$

The inverse of $(1\,5)(2\,4\,6)$ is $(5\,1)(6\,4\,2)$.

<u>Defn</u>. The permutation $(a_1\, a_2 \, \ldots \, a_k)$ where $a_i \in [n]$ are distinct is called a k-cycle. which is described as $\sigma(a_i) = a_{i+1}$ for $i \in \{1, \ldots, k-1\}$. $\sigma(a_k) = a_1$ and $\sigma(a_i) = a_i$ if $i \notin \{1..k\}$.

A two cycle is called transposition.

Ex. Every permutation is a product of disjoint cycles. Disjoint cycles commutes with each other.

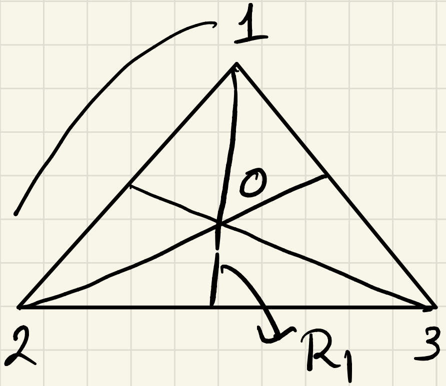
$$(1\ 2)\cdot(3\ 4) = (3\ 4)\cdot(1\ 2).$$

$$(1\ 2)\cdot(2\ 3\ 4) \neq (2\ 3\ 4)\cdot(1\ 2)$$

Ex 1 Prove that every k-cycle is a product of transpositions.

Ex 2. $S_n = \langle (1\ 2), (1\ 3), \ldots, (1, n) \rangle$.

Ex 3 $S_n = \langle (1\ 2), (2\ 3) \ldots (n-1, n) \rangle$.



The rotations about the centre O through angle $0°$, $120°$, $240°$ are symmetries.

Also three reflections along three bisectors are symmetries.

Suppose consider the rotation by $120°$ and then apply reflection by R_1 then what we will get.

$(1\ 2\ 3) \rightsquigarrow$ rotation by $120°$.

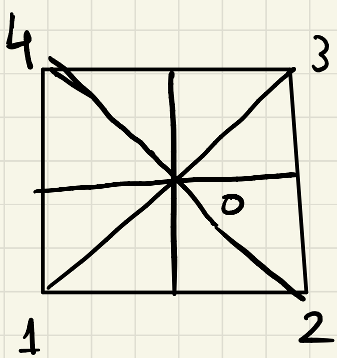
$(2\ 3) \rightsquigarrow$ reflection wrt R_1

$(23) \cdot (1\ 2\ 3) = (1\ 3)$

$$S_3 = \left\{ \underset{Id}{\underset{\downarrow}{(1)}},\ \underset{y}{\underset{\downarrow}{(1\,2\,3)}},\ \underset{y^2}{\underset{\downarrow}{(1\,3\,2)}},\ \underset{x}{\underset{\downarrow}{(1\,2)}},\ (2\,3)^{\nearrow xy^2},\ \underset{xy}{\underset{\downarrow}{(1\,3)}} \right\}.$$

$$S_3 = \{1, y, y^2, x, xy, xy^2\}.$$

$$S_3 = \langle x, y \mid y^3 = Id,\ x^2 = Id, \text{ and } xy = y^2x. \rangle.$$



(1) The four rotations through O about angle $0^\circ, 90^\circ, 180^\circ, 270^\circ$.

(2) There will four reflections.

This group is denoted by D_4.

Ex. Write down all the elts of D_4.