Group Theory

Lecture 9

28/01/2022

$$\phi : H \times K \longrightarrow G.$$

where H & K are subgps of G.

$$\phi(h, k) = hk.$$

<u>WTS</u> ϕ is a gp homo

$$\phi((h_1, k_1) \circ (h_2, k_2))$$

$$= \phi(h_1 h_2, k_1 k_2)$$

$$= h_1 h_2 k_1 k_2.$$

$$\phi(h_1, k_1) \cdot \phi(h_2, k_2)$$

$$= h_1 k_1 \cdot h_2 k_2.$$

We want $hk = kh$

i.e $h^{-1} k^{-1} h k = 1.$

$h^{-1} k^{-1} h k = h^{-1} (k^{-1} h k) \in H$ if H is normal.

$h^{-1}k^{-1}hk = (h^{-1}k^{-1}h)k \in K$ if K is normal.

$h^{-1}k^{-1}hk \in H \cap K$

If $H \cap K = \{1\}$ then $h^{-1}k^{-1}hk = 1$.

$\Rightarrow hk = kh$

Propn. If H and K are both normal subgps of G and $H \cap K = \{1\}$ then ϕ is a gp homo.

Moreover if $G = HK$ then ϕ is an isomorphism i.e $G \cong H \times K$.

Pf: $\phi : H \times K \longrightarrow G$

$\phi(h,k) = hk$.

We have already proved ϕ is a gp homo.

Since $G = HK$ thus ϕ is surjective and since $H \cap K = \{1\}$ we have observed that ϕ is injective

Thus ϕ is a gp homo.

Ex Show that $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.

Quotient Group:

Let G be a gp and H is a normal subgp of G.

G/H = Set of all left cosets of H.

Then $aH \cdot bH = abH$ defines a binary operation on G/H. and

(1) Note that $aH \cdot H = H \cdot aH = aH$.

Thus H is the identity of G/H.

(2) $(aH)^{-1} = a^{-1}H$.

because $a^{-1}H \cdot aH = aa^{-1}H = H$.

(3) Associativity:

$$a_1H(a_2H \cdot a_3H) = a_1H \cdot (a_2a_3H)$$
$$= a_1a_2a_3H.$$

$$(a_1H \cdot a_2H) \cdot a_3H = (a_1a_2H) \cdot a_3H$$
$$= a_1a_2a_3H.$$

Thus G/H forms a gp wrt the operation defined as

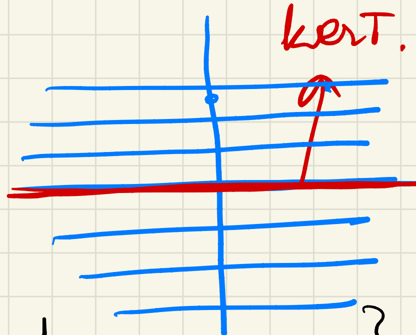
$$aH \cdot bH = abH.$$

Recall:

In Linear Algebra if $T : V \to W$ be a surjective linear transformation.

Then $$V/\ker T \cong W.$$

Example. $V = \mathbb{R}^2$, $W = \mathbb{R}$.



$$T : V \longrightarrow W$$

$$T(x, y) = y.$$

$$\ker T = \{(x, y) \in V \mid T(x, y) = 0\}$$

$$= \{(x, y) \in V \mid y = 0\}$$

$$= \{(x, 0) \mid x \in R\}$$

$$V/\ker T = \{(\alpha, \beta) + \ker T \mid (\alpha, \beta) \in R^2\}.$$

$$V/\ker T \cong \mathbb{R}.$$

First Isomorphism Thm:

Let $\phi : G \to G'$ be a surjective grp homo. Then $G/\ker\phi \cong G'$.

Pf: Let $N = \ker\phi$. Then define

$$\bar{\phi} : G/N \longrightarrow G'$$

$$\bar{\phi}(gN) = \phi(g).$$

First we show $\bar{\phi}$ is well defined.

Let $gN = g'N$ then wTS $\bar{\phi}(gN) = \bar{\phi}(g'N)$.

Since $gN = g'N \Rightarrow g'^{-1}g \in N$.

$\Rightarrow \phi(g'^{-1}g) = 1$. $(\because N = \ker\phi)$.

$\Rightarrow \phi(g')^{-1}\phi(g) = 1 \Rightarrow \phi(g) = \phi(g')$.

$\therefore$ $\bar{\Phi}$ is well defined.

Since ϕ is surjective thus $\bar{\Phi}$ is also surjective.

<u>WTS</u> $\bar{\Phi}$ is inj.

Let $\bar{\Phi}(gN) = \bar{\Phi}(g_1 N)$.

<u>WTS</u> $gN = g_1 N$.

$$\phi(g) = \phi(g_1)$$

$$\Rightarrow \phi(g_1)^{-1}\phi(g) = 1.$$

$$\Rightarrow \phi(g_1^{-1})\phi(g) = 1.$$

$$\Rightarrow \phi(g_1^{-1}g) = 1.$$

$$\Rightarrow g_1^{-1}g \in \ker\phi = N,$$

$\therefore$ $\bar{\Phi}$ is an isomorphism. $\Rightarrow$ $gN = g_1 N$. $\therefore$ $\bar{\Phi}$ is inj