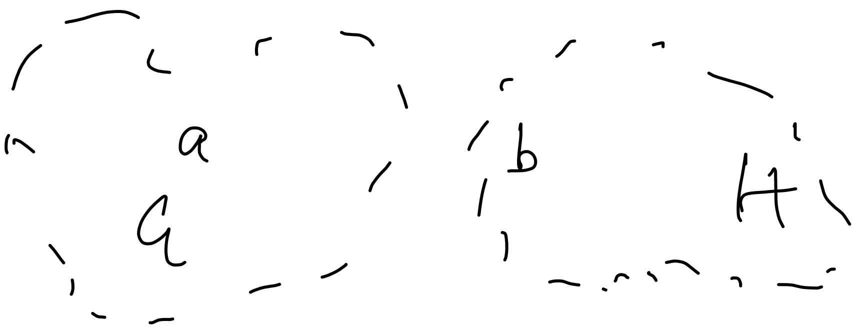
Lecture-16 (11-03-2024)

Hausdorff Space (T_2-Space) :

A topological space (X, T) is said to Hausdorff Space or T_2-Space if for each pair of distinct points $a, b \in X$, there exist open sets G and H such that

$a \in G$, $b \in H$ and $G \cap H = \phi$



Theorem: Every metric space (X, d) is a Hausdorff space.

Proof: Let (X, d) be a metric space and $a, b \in X$ with $a \neq b$.

Then $d(a, b) = \epsilon > 0$

Then let $G := S(a, \epsilon/3)$

and $H := S(b, \epsilon/3)$

Then G and H are open sets with $a \in G$ and $b \in H$.

Claim: $G \cap H = \phi$.

Suppose $p \in G \cap H$.

$\Rightarrow p \in G$ and $p \in H$.

$\Rightarrow d(p, a) < \epsilon/3$ and $d(p, b) < \epsilon/3$

Now by Δle inequality, we have

$$\epsilon = d(a, b) \leq d(a, p) + d(p, b)$$
$$< \epsilon/3 + \epsilon/3$$
$$= \frac{2\epsilon}{3}$$, which is not possible.

$\therefore G \cap H = \phi$.

$\Rightarrow (X, d)$ is a T_2- Space.

Ex: Let (R, T) be a Cofinite topological space. Then (R, T) is a T_1- Space.

But it is not a T_2- Space.

Let G and H be any two open sets in (R, T).

Then G and H are infinite sets.

If $G \cap H = \phi$, then $G \subset H^c$.

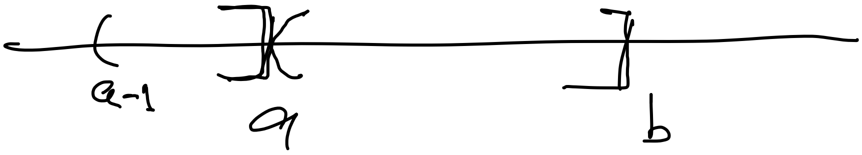
$\Rightarrow$ an infinite set G is contained in a finite set H^c, which is not true.

$\therefore G \cap H \neq \phi$

$\therefore (R, T)$ is not a Hausdorff Space.

Problem: Let (R, T) be the upper limit topological space. Then show that, it is a T_2- Space.

Let $a, b \in \mathbb{R}$, $a \neq b$.



Let $G = (a-1, a]$, $H = (a, b]$

Clearly $a \in G$ and $b \in H$,

$G \cap H = \phi$, and G, H are open sets.

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Theorem: Let $(X, \mathcal{T})$ be a Hausdorff Space. Then every convergent sequence in X has a unique limit point.

Proof: Let $\{a_n\}$ be a sequence in $(X, \mathcal{T})$ such that $a_n \to a$ and $a_n \to b$ with $a \neq b$.

$\because$ X is a Hausdorff Space, there exist two open sets G and H such that $a \in G$, $b \in H$ and $G \cap H = \phi$.

$\because$ $a_n \longrightarrow a \in G \implies \exists\ n_0 \in \mathbb{N}$ such that $a_n \in G,\ \forall\ n \geq n_0$.

$\because$ $G \cap H = \varphi$, then H contains those terms of the sequence $\{a_n\}$ which do not belongs to G.

$\implies$ H contains only finite number of terms of the sequence $\{a_n\}$.

$\implies$ $a_n \not\longrightarrow b$

This is a contradiction to $a_n \longrightarrow b$.

$\therefore$ $a = b$.

Theorem: Let (X, T) be a first countable topological space. Then the followings are equivalent.

(i) X is a Hausdorff space

(ii) Every convergent sequence has unique limit point.

Proof: (i) $\Longrightarrow$ (ii) follows from the previous theorem.

Now we prove (ii) $\Longrightarrow$ (i).

i.e, we prove (X, T) is a Hausdorff space.

Suppose (X, T) is not a Hausdorff space. Then there exist $a, b \in X$, $a \neq b$, with the property that every open set containing 'a' has non-empty intersection with every open set containing 'b'.

Since (X, T) is also a first countable topological space with $a, b \in X$, there exist nested local bases $\{G_n\}$ and $\{H_n\}$ at 'a' and 'b' respectively.

Then $G_n \cap H_n \neq \varphi$, $\forall\, n = 1, 2, 3 \ldots$

$\left[\because\ a \in G_n \atop b \in H_n\ \forall n\right]$

So there exists a sequence $\{a_n\}$ in (X, T) such that

$a_1 \in G_1 \cap H_1$, $a_2 \in G_2 \cap H_2$, $\ldots a_n \in G_n \cap H_n, \ldots$

$\Rightarrow$ $\{a_n\}$ is a sequence in a nested local bases $\{G_n\}$ and $\{H_n\}$.

$\Rightarrow$ $a_n \longrightarrow a$ and $a_n \longrightarrow b$

$\Rightarrow$ The sequence $\{a_n\}$ converging to two distinct points a and b, which is contradiction to our assumption (ii).

$\therefore$ (X, T) is a Hausdorff space

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Attendance morning $[65, 11, 42, 62, 43, 07, 27, 32, 60, 06, 57, 38, 10, 39, 26, 17, 16, 34]$

Regular Space:—

A topological space (X, T) is said to be a regular space if it satisfy the following.

" If F is a closed subset of X and $p \in X$ does not belong to F, then there exists disjoint open sets G and H such that $F \subset G$ and $p \in H$ ".

That is

F is closed in X and $p \in F^c$

$\Rightarrow \exists\ G, H \in T \ni F \subset G,\ p \in H,$

$G \cap H = \varphi.$

* A regular space need not be a T_1-space.

Ex: $X = \{a, b, c\}$

$T = \{X, \varphi, \{a\}, \{b, c\}\}$.

closed sets: $\varphi, X, \{b, c\}, \{a\}$.

let $F = \{b, c\}$ and let $a \notin F$

clearly $a \in \{a\} = H$ and $F = \{b, c\} \subset G = \{b, c\}$.

$\therefore$ (X, T) is a regular space.

But (X, T) is not a T_1-space,

since all finite sets are not closed

in X, for example $\{b\}$ is not a

closed set.

Def: A regular T_1-space is called

T_3-space.

* A T_3 - Space is also a T_2 - Space.

Let (X, T) be a T_3 - Space.

Claim: (X, T) is a T_2 - Space

So let $a, b \in X$, with $a \neq b$.

$\because$ X is also T_1 - Space. $\{a\}$ is a closed set.

Also $a \neq b \Rightarrow b \notin \{a\}$.

Since (X, T) is a regular Space, $\{a\}$ is a closed set and $b \notin \{a\}$ implies, there exist open sets G and H such that

$b \in H$ and $\{a\} \subset G$, $H \cap G = \phi$

$\Rightarrow$ For $a \neq b$, $a, b \in X$ $\exists$ $G, H \in T$ such that $a \in G$, $b \in H$, $G \cap H = \phi$

$\Rightarrow$ (X, T) is a T_2 - Space

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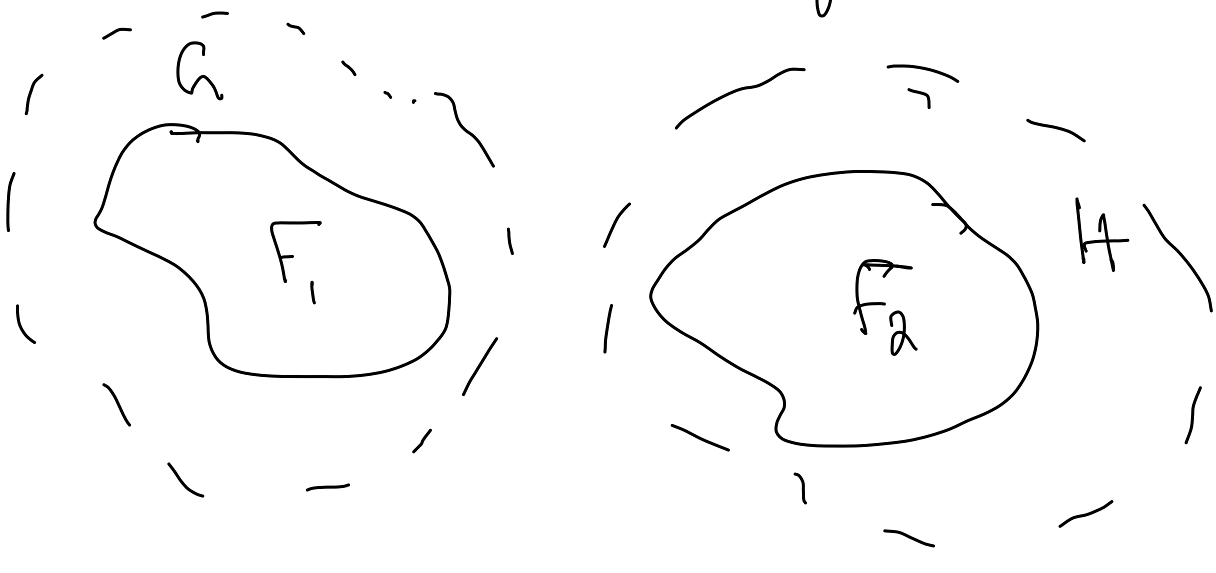
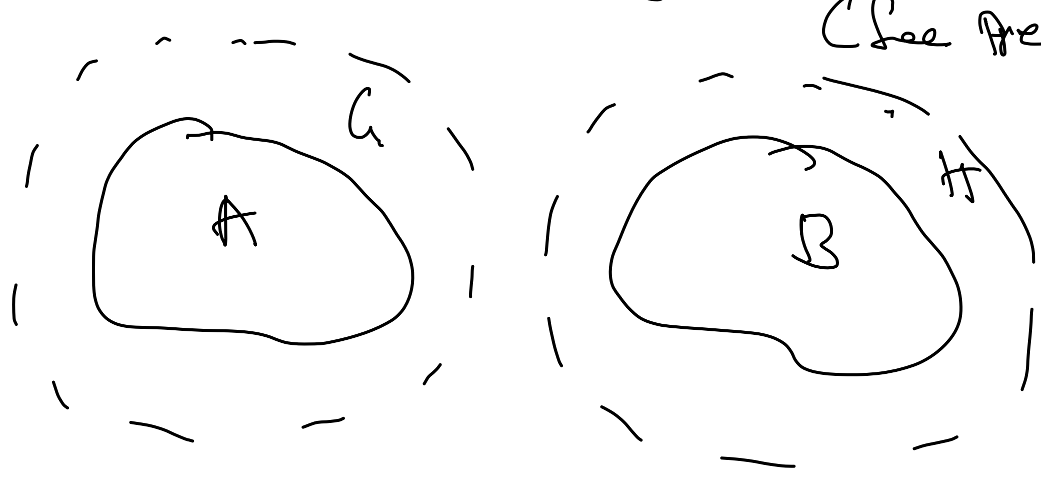
Normal Space :

A topological space (X, T) is said to be a normal space if for every disjoint closed subsets F_1 and F_2 of X, there exist two disjoint open sets G and H such that

$$F_1 \subset G \text{ and } F_2 \subset H.$$



Ex: Every metric space (X, d) is a normal space by Separation Axiom.
(See previous class notes)



Ex: let $X = \{a, b, c\}$

$T = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$.

closed sets: $\varphi, X, \{b, c\}, \{a, c\}, \{c\}$.

If F_1 and F_2 are disjoint closed subsets of X then one of them must be an open set say $F_1 = \varphi$.

Then $G = \varphi$, $H = X$ are disjoint open sets such that $F_1 \subset G$ and $F_2 \subset X$.

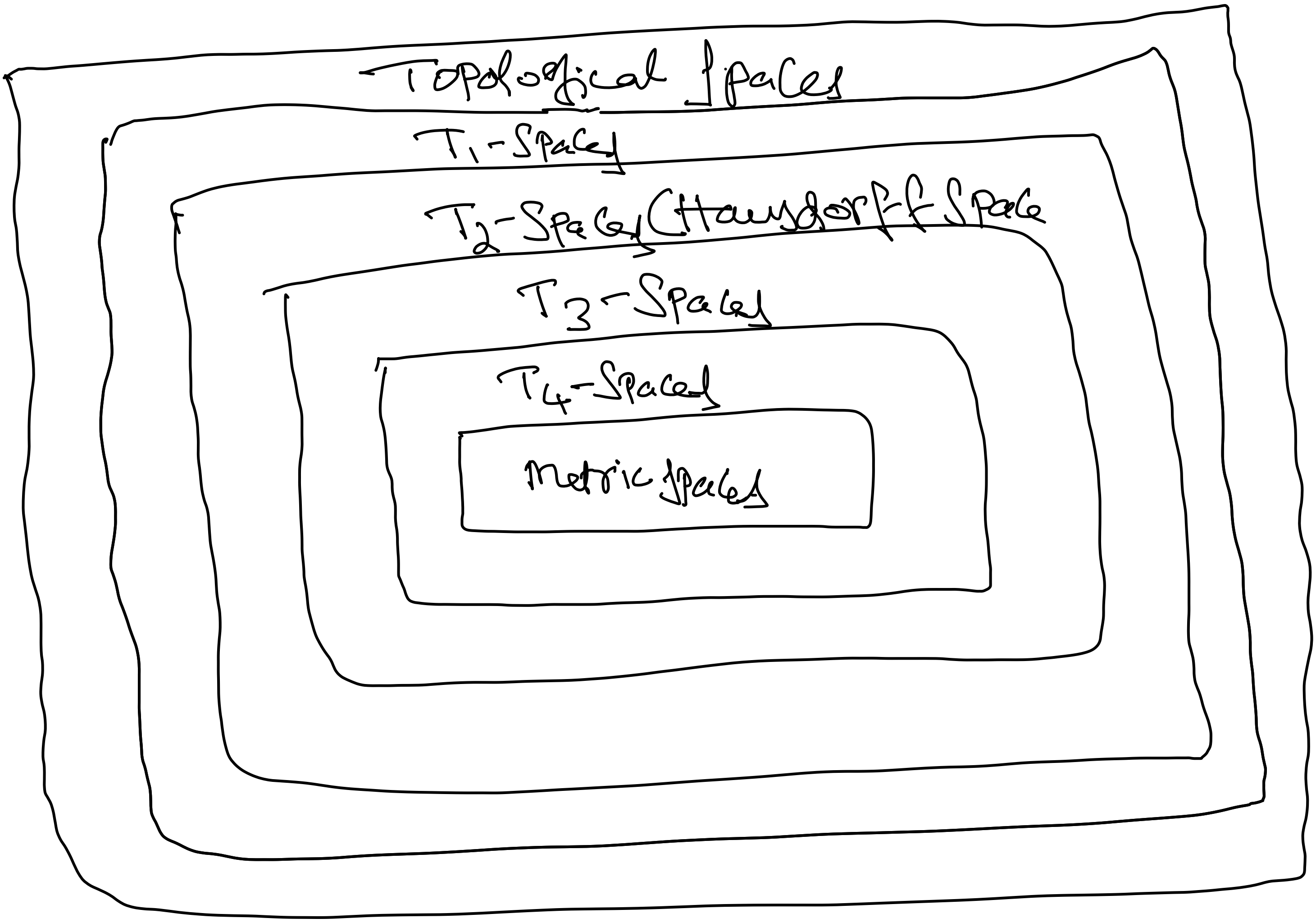
$\therefore$ (X, T) is a normal space.

But (X, T) is not a T_1-space, since $\{a\}$ is not a closed set.

Also (X, T) is not a regular space, since $a \notin \{c\}$ and the only superset of a closed set $\{c\}$ is X which also containing the point 'a'.

Def: A normal T_1-Space is called T_4-Space.

a normal space need not be a T1 space or regular space



Attendance 5PM [11, 42, 55, 27, 06, 60]

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