Ring Theory

Lecture 19

09/08/2022

1st isomorphism Thm: Let $f: R \to S$ be a surjective ring homo then

$$R/\ker f \cong S.$$

Example (1) $\phi: \mathbb{R}[x] \longrightarrow \mathbb{C}$

$$\phi(f(x)) = f(i).$$

ϕ is a ring homo.

Note that ϕ is surjective as $\phi(a+bx) = a+bi$

$$\ker\phi = \{ f(x) \in \mathbb{R}[x] \mid f(i) = 0 \}.$$

Note that $(x^2+1) \in \ker\phi$.

WTS $\ker\phi = \langle (x^2+1) \rangle$.

Let $f(x) \in \ker \phi$.

By division algorithm

$$f(x) = q(x)(x^2+1) + r(x)$$

where either $\deg r(x) < 2$ or $r(x) = 0$.

$$r(x) = f(x) - q(x)(x^2+1) \in \ker \phi.$$

But no poly of deg < 2 is not in the kernel. Hence $r(x) = 0$.

$$\therefore \quad f(x) = q(x)(x^2+1).$$

By 1st isomorphism Thm,

$$\frac{\mathbb{R}[x]}{(x^2+1)} \cong \mathbb{C}.$$

The remainder is of the form $a+bx$. $a, b \in \mathbb{R}$

$f(x) = \underline{x^3 + 3x + 1} \in \mathbb{R}[x] \rightsquigarrow \dfrac{\mathbb{R}[x]}{(x^2+1)}$.

You have to divide $f(x)$ by (x^2+1)

the remainder will be the

image in $\mathbb{R}[x]/(x^2+1)$.

$$\frac{\mathbb{R}[x]}{(x^2+1)}.$$

$$f(x) \in \mathbb{R}[x]$$

$$f(x) + (x^2+1).$$

$$x^3 + 3x + 1$$

$$= q(x)(x^2+1) + r(x).$$

$$x^3 + 3x + 1 + (x^2+1)$$

$$= q(x)(x^2+1) + r(x) + (x^2+1)$$

$$= r(x) + (x^2+1).$$

$\mathbb{Z}/6\mathbb{Z}$. $\quad n \in \mathbb{Z}$.

$n = 6q + r$.

$$\boxed{n + 6\mathbb{Z} = r + 6\mathbb{Z}}$$

Defn. $\mathbb{Z}[i] := \{ a+bi \mid a, b \in \mathbb{Z} \} \subseteq \mathbb{C}$.

This is a ring. This ring is known as ring of Gaussian integers.

Example. $\mathbb{Z}/10\mathbb{Z} \cong \mathbb{Z}[i]/(1+3i)$

$\phi : \mathbb{Z} \longrightarrow \mathbb{Z}[i]/(1+3i) = \overline{R}$.

$\phi(r) = r + (1+3i)$

check ϕ is a ring homo. (Ex).

$$a+bi+(1+3i)$$

$$= a+3b+(1+3i).$$

$$\therefore \phi(a+3b) = a+3b+(1+3i)$$

$$= a+bi+(1+3i).$$

$$\bar{R} = \frac{\mathbb{Z}[i]}{(1+3i)}.$$

$$1+3i=0.$$

$$\Rightarrow 3i=-1.$$

$$\Rightarrow i=3. \quad 10=0$$

$$i^2=-1 \Rightarrow 9=-1$$

$\therefore$ ϕ is a surjective ring homo.

<u>WTS</u>. $\ker\phi = 10\mathbb{Z}$.

$$\phi(10) = 10+(1+3i).$$

$$= (1+3i)(1-3i)+(1+3i)$$

$$= (1+3i).$$

$$\therefore 10\mathbb{Z} \subseteq \ker\phi.$$

Let $n \in \ker\phi$ then $n \in (1+3i)$.

$$\Rightarrow n = (1+3i)(a+bi)$$

$$\Rightarrow 3a+b=0 \Rightarrow b=-3a$$

$\therefore\ n = (1+3i)(a-3ai)$

$= a(1+3i)(1-3i) = 10a \in 10\mathbb{Z}.$

$\therefore\ \ker\phi = 10\mathbb{Z}.$

By 1st isomorphism thm,

$$\mathbb{Z}/10\mathbb{Z} \cong \frac{\mathbb{Z}[i]}{(1+3i)}.$$

Ex. Show that $\mathbb{Z}[i] \cong \dfrac{\mathbb{Z}[x]}{(x^2+1)}$.

<u>Homomorphism Thm</u>: Let $f: R \to S$ be a surjective ring homo. Then $\exists$ a bijection between the set

$$\left\{\begin{array}{l}\text{all ideals of } R\\ \text{containing } \ker f\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{all ideals}\\ \text{of } S\end{array}\right\}.$$

$J \rightsquigarrow f(J).$

$f^{-1}(I) \leftarrow I$

1st step WTS $f(J)$ is an ideal of S.

2nd Step WTS $f^{-1}(I) := \{a \in R \mid f(a) \in I\}$
is an ideal of $R \supseteq \ker f$.

$\to$ $a, b \in f(J)$. then $\exists\ a', b' \in J$

s.t $a = f(a')$ & $b = f(b')$.

$$a+b = f(a') + f(b') = f(a'+b') \in f(J).$$

Let $s \in S$. and $a \in f(J)$.

WTS $sa \in f(J)$

Since f is surjective $\exists x \in R$ s.t

$f(x) = s$. and say $a = f(a')$ where $a' \in J$.

$$sa = f(x) \cdot f(a') = f(xa') \in f(J).$$

$\therefore$ $f(J)$ is an ideal of S.

<u>2nd Step</u>, <u>WTS</u> $f^{-1}(I)$ is an ideal $\supseteq \ker f$.

Let $x \in \ker f$ then $f(x) = 0 \in I$.

$$\Rightarrow \quad x \in f^{-1}(I).$$

$\therefore$ $\ker f \subseteq f^{-1}(I)$.

<u>3rd Step</u>, <u>WTS</u> $J = f^{-1}(f(J))$.

and $I = f(f^{-1}(I))$.

It is clear that $J \subseteq f^{-1}(f(J))$.

<u>WTS</u>, $f^{-1}(f(J)) \subseteq J$.

Let $a \in f^{-1}(f(J))$ for some $y \in J$

$\Rightarrow f(a) \in f(J) \Rightarrow f(a) = f(y)$

$\Rightarrow \quad f(a-y) = 0$

$\Rightarrow \quad a-y \in \ker f \subseteq J.$

$\Rightarrow \quad a \in J$ [as $y \in J$.].

Note that it is clear $f(f^{-1}(I)) \subseteq I$

<u>WTS</u> $\quad I \subseteq f(f^{-1}(I))$.

Let $a \in I$. then $\exists\, x \in R$ s.t

$$f(x) = a \in I.$$

$$\Rightarrow \quad x \in f^{-1}(I).$$

$$\therefore \quad a \in f(f^{-1}(I)).$$

Hence $\quad f(f^{-1}(I)) = I.$

Cor. Consider the natural surjective ring homo $\pi : R \to R/I$ where I is an ideal of R. By the previous Thm, $\exists$ a bijection between

$$\left\{ \text{ideals of } R \supseteq \ker \pi = I \right\} \longleftrightarrow \left\{ \begin{matrix} \text{ideals} \\ \text{of } R/I \end{matrix} \right\}.$$

Thm. Let $f : R \to S$ be a surj ring homo and $J \subseteq S$ be an ideal. Then $f^{-1}(J)$ is an ideal of R. and

$$R/f^{-1}(J) \cong S/J.$$

Pf: $R \xrightarrow{f} S \xrightarrow{\pi} S/J$

Then $\pi \circ f : R \to S/J$ is surj ring homo.

$$\ker(\pi \circ f) = \{a \in R \mid (\pi \circ f)(a) \in J\}.$$

$$= \{a \in R \mid f(a) + J \in J\}.$$

$$= \{a \in R \mid f(a) \in J\}.$$

$$= f^{-1}(J).$$

$\therefore$ By 1st isomorphism Thm,

$$R/f^{-1}(J) \cong S/J.$$

Cor. Let $\pi : R \longrightarrow R/I$ be the natural surj ring homo. Let $J \supseteq I$ be an ideal of R. Then $\pi^{-1}(J/I) = J$.

By previous Thm,

$$R/J \cong \frac{R/I}{J/I}.$$

Remark Ideals of R/I have the form $J/I = \{ b + I \mid b \in J \}$ where J is an ideal of $R \supseteq I$.

$\mathbb{Z}$ - int domain but not a field.

$\mathbb{Z} \subseteq \mathbb{Q}$. $\rightsquigarrow$ field containg $\mathbb{Z}$.

<u>Thm</u>. Let R be an int domain. There exists an inj homo $R \longrightarrow F$ where F is a field.

<u>Pf</u>: Consider the set

$$\mathcal{F} = \{ (a,b) \mid a, b \in R \text{ and } b \neq 0 \}.$$

Define a relation on $\mathcal{F}$ as follows.

$$(a,b) \sim (c,d) \text{ if } ad - bc = 0.$$

check $\sim$ is an equivalance relation.

It is clear that $\sim$ is relflexive.

If $(a,b) \sim (c,d) \Rightarrow \underline{ad - bc = 0.}$

$\downarrow$

WTS $(c,d) \sim (a,b) \Rightarrow \underline{cb - da = 0}$

$\therefore \sim$ is symmetric.

$(a_1, b_1) \sim (a_2, b_2) \Rightarrow a_1 b_2 - a_2 b_1 = 0$ — (1)

and $(a_2, b_2) \sim (a_3, b_3) \Rightarrow a_2 b_3 - a_3 b_2 = 0$ — (2)

<u>WTS</u> $(a_1, b_1) \sim (a_3, b_3)$

$b_3 \times (1) + b_1 \times (2)$

$$a_1 b_2 b_3 - a_3 b_2 b_1 = 0$$

$$\Rightarrow b_2(a_1 b_3 - a_3 b_1) = 0.$$

Since $b_2 \neq 0 \Rightarrow a_1 b_3 - a_3 b_1 = 0.$