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Q5) a) $_2F_1(2,3;1;x)$

$\alpha = 2, \beta = 3, \gamma = 1$

We know that

$$\frac{d^n}{dx^n}\, {}_2F_1(\alpha,\beta;\gamma;x) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\, {}_2F_1(\alpha+n, \beta+n; \gamma+n; x)$$

∴ third derivative of given function:-

$$\frac{d^3}{dx^3}\, {}_2F_1(2,3;1;x) = \frac{(2)_3\,(3)_3}{(1)_3}\, {}_2F_1(2+3, 3+3; 1+3; x)$$

$$= \frac{\cancel{(2)}\cancel{(3)}(4)\,.\,(3)(4)(5)}{(1)\cancel{(2)}\cancel{(3)}}\, {}_2F_1(5,6;4;x)$$

$$\frac{d^3}{dx^3}\, {}_2F_1(2,3;1;x) = 240\, {}_2F_1(5,6;4;x)$$

Q5) b) $\int_0^1 \frac{u\, J_0(xu)\, du}{\sqrt{1-u^2}} = \frac{\sin x}{x}$

Now,

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \dots$$

$$\Rightarrow J_0(xu) = 1 - \frac{x^2u^2}{2^2} + \frac{x^4u^4}{2^2.4^2} - \frac{x^6u^6}{2^2.4^2.6^2} + \dots$$

$$LHS = \int_0^1 \frac{u\, J_0(xu)\, du}{\sqrt{1-u^2}}$$

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$$= \int_0 \frac{u}{\sqrt{1-u^2}}\left(1 - \frac{x^2u^2}{2^2} + \frac{x^4u^4}{2^2.4^2} - \frac{x^6u^6}{2^2.4^2.6^2} + \dots\right) du$$

let $u = \sin\theta$

$\Rightarrow \frac{du}{d\theta} \cos\theta$

Also, when ~~to~~ $u=0$, $\theta=0$ and $u=1$, $\theta = \pi/2$

$$\therefore \text{LHS} = \int_0^{\pi/2} \frac{\sin\theta}{\cancel{\sqrt{1-\sin^2\theta}}}\left(1 - \frac{x^2\sin^2\theta}{2^2} + \frac{x^4\sin^4\theta}{2^2.4^2} - \frac{x^6\sin^6\theta}{2^2.4^2.6^2} + \dots\right)\cos\theta$$

(as $\sqrt{1-\sin^2\theta} = \cos\theta$)

$$\text{LHS} = \int_0^{\pi/2} \sin\theta\, d\theta - \frac{x^2}{2^2}\int_0^{\pi/2} \sin^3\theta\, d\theta + \frac{x^4}{2^2.4^2}\int_0 \sin^5\theta\, d\theta - \dots$$

Using wallis formula $\Rightarrow \int_0^{\pi/2} \sin^n x\, dx = \frac{(n-1)(n-3)\dots 4.2}{n(n-2)(n-4)\dots 5.3}$

for odd n, $n \geq 3$

$$\text{LHS} = -\cos\theta\Big|_0^{\pi/2} - \frac{x^2}{4}.\frac{2}{3} + \frac{x^4}{2^2.4^2}.\frac{4.2}{5.3} - \frac{x^6}{2^2.4^2.6^2}\frac{6.4.2}{7.5.3}$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

$$= \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)$$

$$= \frac{1}{x}\sin x \qquad \left(\text{as } \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right.$$

$$= \frac{\sin x}{x}$$

LHS = RHS

hence proved

Q 3)a) Quotient law :- A set of N^p functions of the coordinates x^i $(i - 1 - N)$ forms the components of a tensor of order $= p$, with the condition that inner multiplication of these functions with arbitrary tensor is a tensor itself.

Proof :- Let us take N^3 functions. A^{ijk} form the components of a tensor of the type by its indices.

Let B^{p}_{ij} be any arbitrary tensor.
and inner product of A^{ijk} & B^{p}_{ij} be C^{pk}

Let us transform the coordinates from x^i to $\bar{x}^i$

$A^{ijk}\, B^{p}_{ij} = C^{pk}$ ~~tant~~ transformed to
$\bar{A}^{ijk}\, \bar{B}^{p}_{ij} = \bar{C}^{pk}$

~~xx~~ now,

$$\bar{B}^{p}_{ij} = \frac{\partial \bar{x}^p}{\partial x^l} \cdot \frac{\partial x^m}{\partial \bar{x}^i} \cdot \frac{\partial x^n}{\partial \bar{x}^j}\, B^{l}_{mn}$$

$$\bar{C}^{pk} = \frac{\partial \bar{x}^p}{\partial x^q} \cdot \frac{\partial \bar{x}^k}{\partial x^r} \cdot C^{qr}$$

$$\bar{A}^{ijk}\, \bar{B}^{p}_{ij} = \bar{C}^{pk}$$

$$\Rightarrow \bar{A}^{ijk} \frac{\partial \bar{x}^p}{\partial x^l} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} B^{l}_{mn} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} C^{qr}$$

Also, $C^{qr} = A^{ijpk} B^{q}_{ij}$

$$\Rightarrow \bar{A}^{ijk} \frac{\partial \bar{x}^p}{\partial x^l} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} B^{l}_{mn} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} A^{ijk} B^{q}_{ij}$$

Changing the dummy indices $q \to l$, $i \to m$, $j \to n$ we get

$$\bar{A}^{ijk} \frac{\partial \bar{x}^p}{\partial x^l} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} B^l_{mn} = \frac{\partial \bar{x}^p}{\partial x^l} \frac{\partial \bar{x}^k}{\partial x^r} \cancel{B} A^{mnr} B^l_{mn}$$

$$\Rightarrow \left(\frac{\partial \bar{x}^p}{\partial x^l}\right)\left(\bar{A}^{ijk} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} \cancel{B_{mn}} - \frac{\partial \bar{x}^k}{\partial x^r} A^{mnr} \cancel{B}\right) B^l_{mn} = 0$$

Multiplying $\frac{\partial x^s}{\partial \bar{x}^p}$

$$\Rightarrow \left(\frac{\partial x^s}{\partial x^l}\right) B^l_{mn} \left(\bar{A}^{ijk} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} - \frac{\partial \bar{x}^k}{\partial x^r} A^{mnr}\right) = 0$$

~~As~~ & $\frac{\partial x^s}{\partial x^l} = \delta^s_l$ & $\delta^s_l B^l_{mn}$ is 1 when $l = s$ only else it is B^s_{mn}

$$\Rightarrow B^s_{mn} \left(\bar{A}^{ijk} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} - \frac{\partial \bar{x}^k}{\partial x^r} A^{mnr}\right) = 0$$

As B^s_{mn} is arbitrary, we can arrange B^s_{mn} such that it does not vanish

$$\Rightarrow \left(\bar{A}^{ijk} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} - \frac{\partial \bar{x}^k}{\partial x^r} A^{mnr}\right) = 0$$

$$\Rightarrow \bar{A}^{ijk} \frac{\partial x^m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j} = A^{mnr} \frac{\partial \bar{x}^k}{\partial x^r}$$

Multiplying $\frac{\partial \bar{x}^s}{\partial x^m} \cdot \frac{\partial \bar{x}^t}{\partial x^n}$ on both sides.

$$\Rightarrow \bar{A}^{ijk} \frac{\partial \bar{x}^s}{\partial \bar{x}^i} \frac{\partial \bar{x}^t}{\partial \bar{x}^j} = \frac{\partial \bar{x}^s}{\partial x^m} \cdot \frac{\partial \bar{x}^t}{\partial x^n} \cdot \frac{\partial \bar{x}^k}{\partial x^r} A^{mnr}$$

$$\Rightarrow \delta^s_i \, \delta^t_j \, \bar{A}^{ijk} = \frac{\partial \bar{x}^s}{\partial x^m} \cdot \frac{\partial \bar{x}^t}{\partial x^n} \cdot \frac{\partial \bar{x}^k}{\partial x^r} A^{mnr}$$

$$\Rightarrow \boxed{\bar{A}^{stk} = \frac{\partial \bar{x}^s}{\partial x^m} \cdot \frac{\partial \bar{x}^t}{\partial x^n} \cdot \frac{\partial \bar{x}^k}{\partial x^r} A^{mnr}}$$

Thus, A^{mnr} is a tensor of third order & is contravarient on all indices.

Thus proved Quotient ~~tat~~ law for 3 indices.

This proof can be extendent to any number of indices.

Hence proved.

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Q3) b) $\lim_{a\to\infty} {}_2F_1(1, a; 1; \frac{x}{a})$

Let us first consider

${}_2F_1(1, a; 1; \frac{x}{a})$

We know that $\alpha = 1, \beta = a, \gamma = 1$ and

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{(\gamma)_r}\frac{x^r}{r!} = 1 + \frac{\alpha\beta x}{1.\gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)x^2}{1.2.\gamma(\gamma+1)} + \dots$$

$$\therefore {}_2F_1(1, a; 1; \frac{x}{a}) = \sum_{r=0}^{\infty} \frac{(1)_r(a)_r}{(1)_r}\left(\frac{x}{a}\right)^r \cdot \frac{1}{r!}$$

$$= 1 + \frac{1\cancel{a}}{1.1}\left(\frac{x}{\cancel{a}}\right) + \frac{1(\cancel{2})(\cancel{a})(a+1)}{1.\cancel{2}.1.2}\frac{x^2}{a^{\cancel{2}}} + \frac{(1)(\cancel{2})(\cancel{3})(\cancel{a})(a+1)(a+2)}{1.\cancel{2}.\cancel{3}.1.2.3}\frac{x^3}{a^{\cancel{3}\,2}} + \dots$$

$$= 1 + x + \frac{(a+1)}{2a}x^2 + \frac{(a+1)(a+2)}{3\times 2\;a^2}x^3 + \dots$$

$$= 1 + x + \frac{(a+1)}{a}\frac{x^2}{2!} + \frac{(a+1)(a+2)}{a^2}\frac{x^3}{3!} + \dots$$

applying $\lim_{a\to\infty}$ on both sides

$$\lim_{a\to\infty} {}_2F_1(1, a; 1; \frac{x}{a}) = \lim_{a\to\infty}\left(1 + x + \frac{\cancel{a}(1+\frac{1}{a})}{\cancel{a}}\frac{x^2}{2!} + \frac{\cancel{a^2}(1+\frac{1}{a})(1+\frac{2}{a})}{\cancel{a^2}}\frac{x^3}{3!} + \dots\right)$$

$$\lim_{a\to\infty} {}_2F_1(1, a; 1; \frac{x}{a}) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x$$

$$\boxed{\lim_{a\to\infty} {}_2F_1(1, a; 1; \frac{x}{a}) = e^x}$$

Q4) a) We know that product of ~~let~~ two tensors of types (r, s) & (r', s') is a tensor of type $(r+r', s+s')$. Now, a vector is a tensor of rank 1 i.e it is either of type $(1, 0)$ or $(0, 1)$.

If we multiply two vectors of type $(1, 0)$, we get a ~~vector~~ tensor of type $(1+1, 0)$ i.e. $(2, 0)$.

If we multiply two vectors of type $(0, 1)$, we ~~again~~ get a ~~vector~~ tensor of type $(0, 1+1)$ i.e. $(0, 2)$

If we multiply one vector of type $(0, 1)$ & one vector of type $(1, 0)$ we get a ~~vector~~ ~~of type~~ tensor of type $(1, 1)$

In all of the cases, the rank of the product ~~vector~~ tensor is 2.

Hence ~~pro~~ proved that the outer product of two vectors is a tensor of order/rank 2.

But the converse is not true.

Let us take a counter-example to prove that the converse is not true.

Consider $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and build a tensor

$$e_1 \otimes e_1 + e_2 \otimes e_2$$

suppose vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ & $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ are such that

$$(a_1 e_1 + a_2 e_2) \otimes (b_1 e_1 + b_2 e_2) = e_1 \otimes e_1 + e_2 \otimes e_2$$

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LHS $= a_1 b_1 (e_1 \otimes e_1) + a_2 b_2 (e_2 \otimes e_2) + a_1 b_2 (e_1 \otimes e_2) + a_2 b_1 (e_2 \otimes e_1)$

To satisfy RHS, $a_1 b_2 = a_2 b_1 = 0$ and $a_1 b_1 = a_2 b_2 = 1$ which cannot be simultaneously possible.

Thus there exist no vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Hence, through a ~~conve~~ counter-example we prove that the converse is not true

Q4) b) Suppose A^{ij} is a skew-symmetric tensor of type $(2,0)$ wrt i and j in x^i coordinate system.

We know that $A^{ij} = -A^{ji}$

We also know that, on transformation of coordinates from x^i to $\bar{x}^i$ we get

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k} \times \frac{\partial \bar{x}^j}{\partial x^l} \times A^{kl} = \frac{\partial \bar{x}^i}{\partial x^k} \times \frac{\partial \bar{x}^j}{\partial x^l} \times (-A^{lk}) = -\bar{A}^{ji}$$

Thus, $\bar{A}^{ij} = -\bar{A}^{ji}$

$\bar{A}^{ij}$ is skew-symmetric with respect to i, j in $\bar{x}^i$ coordinate system.

Hence proved that a tensor which is skew-symmetric wrt a pair of indices in one system of coordinates is skew symmetric wrt the pair of indices in every other system of coordinates.

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Q1)a) Symmetric tensor:- ~~A~~ A tensor is said to be symmetric with respect to two contravarient indices if its components ~~are~~ ~~[illegible]~~ do not change when we interchange the two indices.

Similarly a tensor is said to be symmetric wrt two covarient indices if its components remain unchanged even on the ~~[illegible]~~ interchange of the two indices

Eg:- • Any ~~vector~~ tensor A^{ij} that satisfies $A^{ij} = A^{ji}$ is a symmetric ~~vector~~ tensor

• $\delta^{ij} = \delta^{ji}$, hence δ^{ij} is a symmetric tensor

• $\delta_{ij} = \delta_{ji}$ is also true, hence δ_{ij} is a symmetric tensor

Skew-symmetric tensor:- A tensor is said to be skew-symmetric wrt two contravarient indices if its components only change to opposite signs on interchanging the indices. The magnitude of components doesn't change on interchanging the indices in skew-symmetric tensors.

Similarly, a tensor is skew-symmetric wrt two covarient indices if only the signs of the components change to the opposite sign and magnitude remains unchanged on ~~[illegible]~~ interchanging the indices.

Eg:- • Any tensor A^{ij} that satisfies $A^{ij} = -A^{ji}$ is a skew-symmetric tensor

• Any tensor A_{ij} that satisfies ~~[illegible]~~ $A_{ij} = -A_{ji}$ is a skew-symmetric tensor

A covarient tensor cannot be symmetric or skew-symmetric to a contravarient tensor.

let us consider a skew-symmetric tensor of order = 2.

$$A^{ij} = \begin{bmatrix} A^{11} & A^{12} & \dots & A^{1n} \\ \vdots & \vdots & & \vdots \\ A^{n1} & \dots & \dots & A^{nn} \end{bmatrix}$$

We know $A^{ij} = -A^{ji} \Rightarrow A^{ii} = -A^{ii} \Rightarrow A^{ii} = 0$, $i = 1, 2 \dots n$

Out of $n^2 - n$ remaining components, at most half of them are independent as

$A^{13} = -A^{31}$, $A^{12} = -A^{21}$ ....

Hence, a skew symmetric tensor of order two chan ~~co~~ have $\frac{n^2-n}{2} = n\left(\frac{n-1}{2}\right)$ independent components

Q1)b) A^{ij} is contravarient tensor, B_i is covarient tensor.

now, $A^{ij} B_k = C^{ij}_k$ i.e C^{ij}_k has contravarient order 2 and covarient order 1.

Thus, its rank $= 2+1 = 3$, which means C^{ij}_k is a tensor.

Replacing k with j, we get $A^{ij} B_j = C^i$

C^i has contravarient order 1 and covarient order 0 i.e it has rank $= 1+0 = 1$.

Thus, C^i is also a tensor.

Hence proved that $A^{ij} B_k$ & $A^{ij} B_j$ are tensors.

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$\therefore A^{ij} B_k = C^{ij}_k$ with contravarient order = 2
covarient order = 1
rank = 2+1 = 3

$\therefore A^{ij} B_j = C^i$ with contravarient order = 1
covarient order = 0
rank = 1.

Q2)a) Christoffel's symbols of 1^{st} kind :-

$$[i,j,k] = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial_{ij}}{\partial x^k}\right]$$

Christoffel's symbols of 2^{nd} kind :-

$$\left\{ {k \atop ij} \right\} = g^{kl}[i,j,l]$$, where g^{ij} are components of fundamental tensor

For each pair (i,j), there are n distinct Christoffel symbols of each kind for each g_{ij} which is independent

Now, g_{ij} is a symmetric tensor of rank = 2. Thus, it has at most $\frac{n(n+1)}{2}$ independent components.

Thus, the number of independent components of Christoffel's symbols are $n.\frac{n(n+1)}{2} = \frac{n^2(n+1)}{2}$

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Q2) b) $[i,j,m] = g_{lm}\left\{ {l \atop ij} \right\}$

$RHS = g_{lm}\left\{ {l \atop ij} \right\}$

we know that $\left\{ {l \atop ij} \right\} = g^{lk}\,[i,j,k]$

$RHS = g_{lm}\left\{ {l \atop ij} \right\} = g_{lm}\,g^{lk}\,[i,j,k]$

$= \delta^k_m\,[i,j,k]$ (as $g_{lm}\,g^{lk} = \delta^k_m$)

$RHS = [i,j,m]$

$RHS = LHS$

hence proved