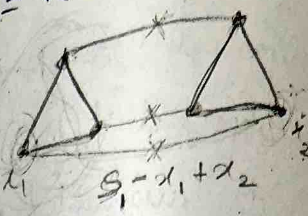
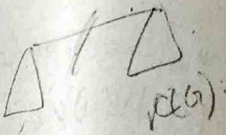
$l = \{x_1, x$

Next we have $S_1 = V(G_1)$ & $S_2 = V(G_2)$.



$|S| \geq 2$ W.L.O.G. Let $l \in S$ with $l = \{x_1, x_2\}$. $S_1' = S_1 - x_1 + x_2$ is a ~~set~~ vertex-cut.

and $|S_1'| = |S_1| \leq m$

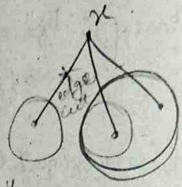


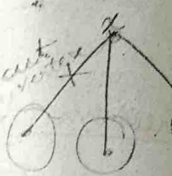
Tutorial 7 (30th Oct 2023):

7\. $\kappa(G) \leq \kappa'(G) \leq 3$

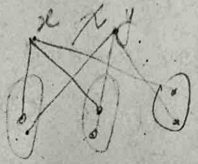
If $\kappa(G) = 3 \Rightarrow \kappa'(G) = 3$

Case I - If $\kappa(G) = 1$,





Case II - $\kappa(G) = 2$



- Let G be a graph on n vertices of 'e' edges, then
  $\delta(G) \leq \frac{2e}{n}$, $\sum_{x \in V(G)} \deg(x) = 2e$

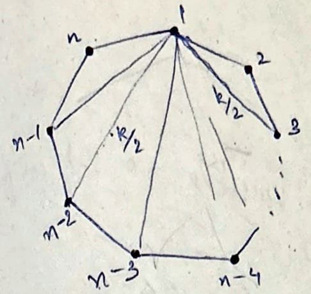
  Avg. vertex degree is $\frac{1}{n} \sum \deg(x) = \frac{2e}{n}$

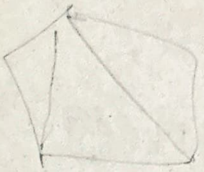
  $\Rightarrow \delta(G) \leq \frac{2e}{n}$

Q) For a k- dimensional hypercube Q_k, prove that
$\kappa(Q_k) = \kappa'(Q_k) = \delta(Q_k) = k$? (Harary graphs)

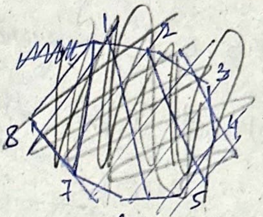
- Harary Graphs - Let k & n be integers with $2 \leq k \leq n-1$.
  The Harary graphs $H_{k,n}$ are constructed as below -

Case ... even & any n. $H_{k,n}$

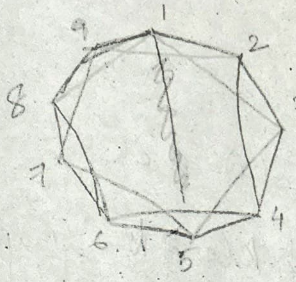




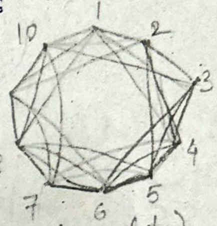
Consider the cycle C_n. Then for every vertex x on C_n, x is adjacent with $\frac{k}{2}$ no. of vertices on left side of x & $\frac{k}{2}$ no. of vertices on right side of x.



e.g $H_{4,9}$, $\frac{k}{2} = 2$

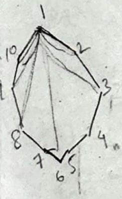


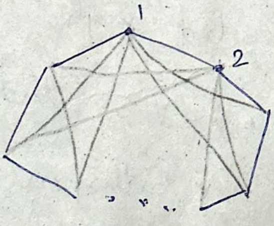
$H_{6,10}$ - $\frac{k}{2} = \frac{6}{2} = 3$



(complete)

Case II - k is odd, n is even.



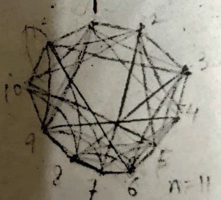


C_n

Consider C_n & for every vertex x in C_n, x is adjacent with $\frac{k-1}{2}$ consecutive vertices on both LHS & RHS, and

x is also adjacent with its opposite vertex.

e.g. $H_{5,10}$ (draw kar le)



Case III - ~~k odd~~ k is odd, n is odd.

$H_{k,n}$ - First we construct $H_{k-1,n}$

e.g. $H_{7,11}$ Now we construct $H_{6,11}$.

$n = 11$

$\frac{n-1}{2} = 5$

$H_{k-1,n}$. vertex i ~~adj~~ $(i + \frac{n-1}{2})$ vertex for $i = 1, 2, \dots, \frac{n+1}{2}$

Minimum possible no. of edges in a graph on n vertices and with $\delta(G) = k$,

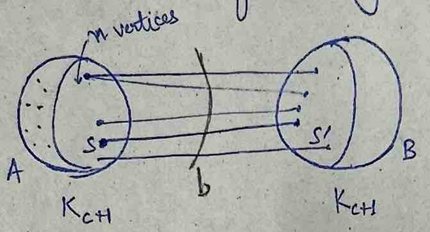
$$nk \le \sum \deg(x) = 2e$$

$$\Rightarrow e \ge \frac{nk}{2} \text{ (minimum possible no. of edges)}$$

- **Theorem:** $\kappa(H_{k,n}) = \kappa'(H_{k,n}) = \delta(H_{k,n}) = k$
- **Theorem:** For any ~~possible~~ positive integers $a \le b \le c$, $\exists$ a graph G with $\kappa(G) = a$, $\kappa'(G) = b$ and $\delta(G) = c$.

Proof: If $a=b=c$, then take $G = K_{c+1}$.

All three integers are not the same. Then $a<c$. Then construct G in the following way -

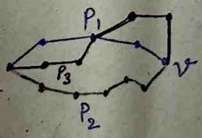


$S \subset A$, $|S| = a = |S'|$, $S' \subset B$, $\delta(G) = c$

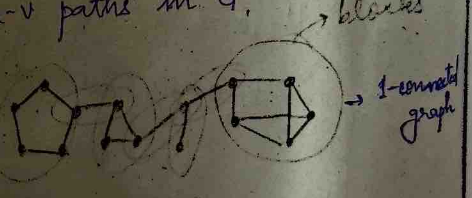
Def^n: A graph G is called k-connected if $\kappa(G) \ge k$. Graph G is called k-edge connected if $\kappa'(G) \ge k$.

- **Menger's Theorem:** Let G be a connected graph.

i) G is k-connected iff for every $u, v \in V(G)$, there are atleast k no. of vertex disjoint u-v paths in G.



ii) G is k-edge connected iff for every $u, v \in V(G)$, there are atleast k no. of ~~vertices~~ edge disjoint u-v paths in G.

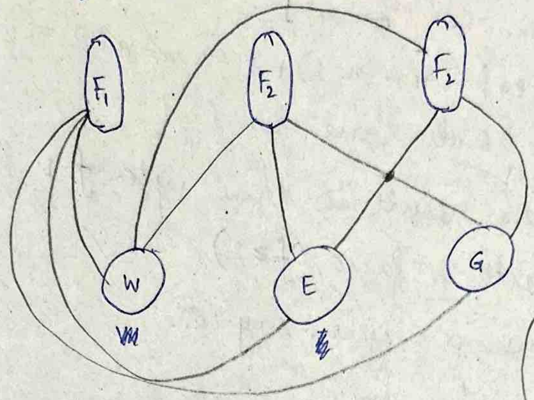


- **Def^n:** Block - Either a bridge or a maximal connected subgraph H with $\kappa(H) \ge 2$.

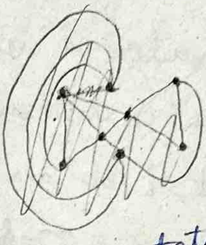
* Planar Graphs :

A graph G is called planar if G can be drawn on a plane with no edge crossing.

The crossing no. of G, denoted by $\nu(G)$, is the minimum no. of edge ~~cros~~ crossing in a drawing of G on a plane. If G is planar, then $\nu(G) = 0$.

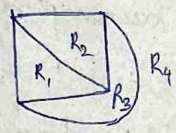


3 - families
3 - necessities
Find minimum no. of crossings.

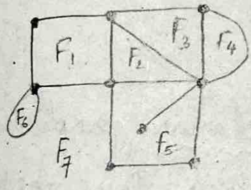


Is $K_{3,3}$ planar ? NO , $\nu(K_{3,3}) = ?1$

K_4 -



Planar representation of a planar graph partitions the plane into some regions or faces.



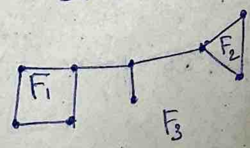
length of face 1, $l(F_1) = 4$, $\sum_{i=1}^{7} l(F_i) = 28 = 2e$

$l(F_5) = 6$, $l(F_7) = 9$ where e=14.

Def^n: Length of a face F is the length of a smallest closed walk containing all the edges in the boundary of F.

Theorem: $\sum l(F) = 2e$ for a planar graph G.

e.g.



$l(F_1) = 4$, $l(F_2) = 3$, e= 10

$l(F_3) = 13$

$\sum l(F) = 4+3+13 = 20 = 2e$

Also , $n - \underbrace{e+f}_{\text{no. of faces}} = 9 - 10 + 3 = 2$

Theorem (Euler) (1752): Let G be a connected planar graph on n vertices & e edges. If f is the no. of faces in a planar representation of G then

$$n - e + f = 2 \quad \text{(Euler's formula)}$$

Proof: (using induction on f)

For $f = 1$, then G contains no cycle. Hence G is a tree in this case. So $e = n-1$

Then $n - e + f = n - (n-1) + 1 = \cancel{n} - \cancel{n} + 2 = 2$

The result holds true for $f = 1$.

Assume that the result is true upto $f-1$ faces.

Consider G with f faces $(f \geq 2)$

Then G contains a cycle, say C.

Let l be an edge in C.

Let $G_1 = G - l$. Then G_1 is connected, planar with $f-1$ faces. Then from the assumption G_1 satisfies the formula. That is, $n - (e-1) + (f-1) = 2 \Rightarrow n - e + f = 2$

$(|N(G_1)| = n, \quad |E(G_1)| = e-1, \quad \text{No of faces in } G_1 = f-1)$

Hence, proved.

Corollary 1: Let G be a disconnected planar graph with k components, n vertices, e edges & f faces. Then

$$n - e + f = k + 1.$$

Proof: Let the components of G be $G_1, G_2, \ldots, G_k$.

Let n_i, e_i, f_i be no. of vertices, edges & faces of G_i, $i = 1, 2, \ldots, k$.

Every G_i is connected & planar. So from Euler's theorem, we get –

$$n_i - e_i + f_i = 2, \quad \forall\, i = 1, 2, \ldots, k$$

Then $\sum_{i=1}^{k}(n_i - e_i + f_i) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} e_i + \sum_{i=1}^{k} f_i = 2k$

$\Rightarrow n - e + (f + k - 1) = 2k$

$\Rightarrow n - e + f = k + 1$

Hence, proved.

Observation - Every planar representation of a planar graph has the same no. of faces.

Corollary 2: Let G be a simple connected planar graph on n vertices $(n \geq 3)$ and e edges and f faces.

Then

i) $e \geq \frac{3f}{2}$, $e \leq 3n - 6$

ii) Further if G is bipartite then $e \geq 2f$, $e \leq 2n - 4$.

Proof: i) Since G is simple & connected, length of every face is atleast 3.

~~$\sum l(f_i)$~~ $\sum_i l(F) = 2e$

Now, $2e = \sum_i l(F) \geq 3f$

$\Rightarrow e \geq \frac{3f}{2}$

From Euler's theorem, $n - e + f = 2$

$\Rightarrow f = 2 - n + e$

Now $e \geq \frac{3f}{2} = \frac{3}{2}(2 - n + e) = 3 - \frac{3}{2}n + \frac{3}{2}e$

$\Rightarrow 2e \geq 6 - 3n + 3e$

$\Rightarrow e \leq 3n - 6$

ii) $l(F) \geq 4 \Rightarrow 2e = \sum l(F) \geq 4f$

$\Rightarrow e \geq 2f$

Similarly, $e \geq 2(2 - n + e) = 4 - 2n + 2e$

$\Rightarrow e \leq 2n - 4$

~~Corollary 3: Corollary 3: K_5 & $K_{3,3}$ are non-planar~~

Corollary 3: K_5 & $K_{3,3}$ are non-planar.

Proof: Suppose K_5 is planar. Then for K_5,

$$e \leq 3n-6$$

$$10 \leq 3\times 5-6=9 \rightarrow ※$$

If $K_{3,3}$ is planar, we get

$$e \leq 2n-4$$

$$\Rightarrow 9 \leq 2(6)-4=8 \longrightarrow ※$$

Hence K_5 and $K_{3,3}$ are non-planar.

Corollary 4: For every simple, connected planar graph, $\delta(G) \leq 5$.

Proof: Suppose $\delta(G)=6$, i.e. $deg(x) \geq 6 \ \forall \ x \in V(G)$.

Then ~~$\sum deg$~~ $2e = \sum_i deg(x) \geq 6n$

$$\Rightarrow e \geq 3n$$

But we have $e \leq 3n-6$ for G.

So we get a contradiction.

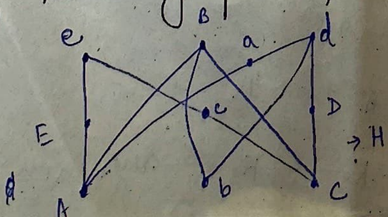
Hence $\delta(G) \leq 5$.

Defn: Subdivision of a graph G is obtained from G by replacing edges of G by paths.

Observation – G is planar iff its subdivision is planar.

• Theorem (Kuratowski, 1930): A graph G is planar iff it does not contain K_5, $K_{3,3}$ or any subdivision of them as a subgraph.

Q) Check if Peterson graph is planar? No.



H is a subdivision of $K_{3,3}$.

H is a subgraph of the Peterson graph.

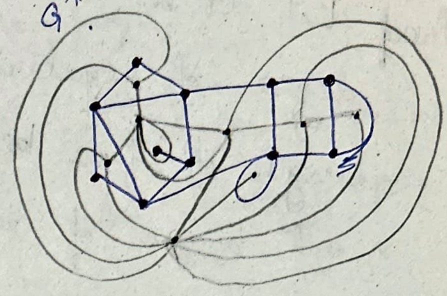
• **Dual of a planar graph** -

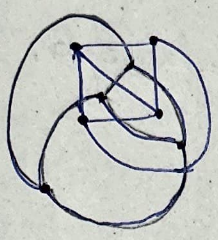
The dual graph G^* of G is constructed as below -

i) Consider a planar ~~ett~~ representation of G.

ii) ~~Corr Corr~~ Corresponding to every face of G consider a vertex in G^*, i.e. $V(G^*)$ = the set of all faces of G.

iii) Let X & Y be any two vertices in G^*, then for every edge e ~~considered~~ common in the boundary of X & Y draw an edge e^* of G^*.





$G^* \cong K_4 = G$

K_4 is ~~dtt~~ self-dual.

• **Properties of dual graph G^*** -

Let n, e, f be the no. of vertices, edges & faces of G.

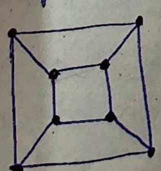
i) $|V(G^*)| = n^* = f$

$|E(G^*)| = e^* = e$, $f^* = n$.

ii) G^* is planar & is connected.

iii) For $X \in V(G^*)$, $\deg_{G^*} X$ is the length of the face X in G.

iv) Every pendant edge of ~~G~~ G corresponds to a self-loop in G^*. ~~and vice versa.~~ Every self-loop in G corresponds to a pendant edge in G^*.

$Q_3 \rightarrow$  $\rightarrow$ Completely regular graph

Def^n: A connected, simple and planar graph G is called completely regular if -

i) G is a regular graph.

ii) all the faces of G are of the same length.

- **Theorem (Plato)**: There are exactly five completely regular graph with degree of regularity atleast 3.

Proof: Let G be a completely regular graph with degree of regularity $k \geq 3$.

Length of faces be $l \geq 3$.

n, e, f be no. of vertices, edges & faces of G.

$$kn = \sum_{x \in V(G)} deg(x) = 2e = \sum_{\substack{F \text{ is a} \\ \text{face in } G}} l(F) = lf$$

($\because$ length of all faces are same)

$n - e + f = 2$, since G is planar.

$$\frac{2e}{k} - e + \frac{2e}{l} = 2$$

$$\Rightarrow e\left(\frac{2}{k} + \frac{2}{l} - 1\right) = 2$$

$$\Rightarrow \frac{2}{k} + \frac{2}{l} - 1 > 0$$

$$\Rightarrow \frac{2}{k} + \frac{2}{l} > 1$$

$$\Rightarrow 2l + 2k > kl \quad \text{— (I)}$$

$(k-2)(l-2)$
$= kl - 2l - 2k + 4$

$kl - 2l - 2k + 4 < 4$ from (I)

$k, l \geq 3$, G^* is l-regular graph, so $k, l \leq 5$. because $\delta(G^*) \leq 5$

and $(k-2)(l-2) < 4$

$\therefore 3 \leq k, l \leq 5$

k	l	e	n	f	
3	3	6	4	4	Tetrahedron (K_4)
3	4	12	8	6	Cube Q_3
4	3	12	6	8	Octahedron
3	5	30	20	12	Dodecahedron

k	l	e	n	f	
5	3	30	12	20	Icosahedron

G is a planar graph. Is $(G^*)^* = G$?

True if G is a simple connected & planar graph. ~~Then~~

Then $G^{**} = G$.

- Coloring of Graphs -

1. Vertex coloring
2. Edge coloring
3. Face coloring for a planar graph.

- Vertex coloring - Assignment of colours to vertices s.t adjacent vertices get different ~~colo~~ colors (proper coloring)

G is said to be a k-colorable graph if there is a coloring of G with k colors.

We consider G is a simple graph.

Chromatic no. of a graph G, ~~(X(G)) is the~~, denoted by $\chi(G)$, is the minimum number of colors needed for a vertex coloring of G.

$$\chi(G) = \min \{ k : G \text{ is } k\text{-colorable} \}$$

Observations -

i) $\chi(K_n) = n$

ii) $\chi(P_n) = 2$

iii) $\chi(C_n) = \begin{cases} 2, & n \text{ is even} \\ 3, & n \text{ is odd} \end{cases}$

iv) If $G \cong \overline{K_n}$, $E(G) = \phi$, $\chi(G) = 1$

v) If H is a subgraph G then $\chi(H) \leq \chi(G)$.

vi) $\chi(G) = 2$ iff G is a bipartite graph.

The set of vertices which get the same color in a coloring of a graph, is called color class.

$H \subseteq G$, $H \cong K_m$.

Clique no. of a graph G, denoted by $\omega(G)$, is the maximum no. of vertices in G which induces a complete graph. (subgraph of G that is a complete graph, maximum integer $k = \omega(G)$)

• **Theorem:** For any n-vertex graph G

i) $\chi(G) \geq \omega(G)$

ii) $\chi(G) \geq \dfrac{n}{\alpha(G)}$

Proof: (of (ii)) G has $\chi(G)$ no. of color classes, say $C_1, C_2, \ldots, C_{\chi(G)}$. Each C_i is an independent set. $\forall\, i = 1, 2, \ldots, \chi(G)$.

So $|C_i| \leq \alpha(G)$

$\{C_1, C_2, \ldots, C_{\chi(G)}\}$ is a partition of $V(G)$.

$$n = \sum_{i=1}^{\chi(G)} |C_i| \leq \alpha(G)\chi(G)$$

$$\Rightarrow \chi(G) \geq \frac{n}{\alpha(G)}$$

• Greedy Algorithm -

L: ordering of vertices, say $v_1, v_2, \ldots, v_n$.

$S = \{1, 2, \ldots, n\}$

Input: An n vertex graph G.

Output: A coloring of G.

Step 1 - Let $S = \{1, 2, \ldots, n\}$

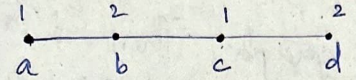
Step 2 - Consider an ordering of vertices, say,

$L: v_1, v_2, \ldots, v_m$.

Step 3 - Assign color 1 to v_1.

Step 4 - For v_i, $i \geq 2$, assign smallest possible color in S, that is not used by any of the lower indexed neighbours of v_i in L.

e.g.



$L: a, b, c, d$
$v_1 - v_2 - v_3 - v_4$
1 2 1 2

$L: a, b, d, c$
$v_1 - v_2 - v_3 - v_4$
1 2 1 2

• **Theorem:** For any graph G, $\chi(G) \leq \Delta(G) + 1$.

Proof: $L: v_1, v_2, \ldots, v_i, \ldots, v_n$.

v_i can have at the most $\Delta(G)$ lower ~~st~~ indexed neighbours in L. In worst case, all those $\Delta(G)$ neighbours of G use different colors. So smallest possible order available for v_i is $\Delta(G)+1$. This is true for all v_i, $i = 1, 2, \ldots, n$.

So the maximum ~~sta~~ color no. we need in the greedy algorithm is $\Delta(G) + 1$. Hence,

$\chi(G) \leq \Delta(G) + 1$.

For K_n & an odd cycle C_n,

$$\chi(K_n) = \underbrace{\Delta(G) + 1}_{= n-1+1} = n.$$

$$\chi(C_n) = \underbrace{\Delta(G) + 1}_{= 2+1} = 3 \quad , \; n \text{ is odd}$$

• **Theorem** (Brook's, 1941): Let G be a graph different from plete graph & an odd cycle. Then $\chi(G) \le \Delta(G)$.

Proof: W.L.O.G. let G be connected.

(If G has k components $G_1, \ldots, G_k$ then

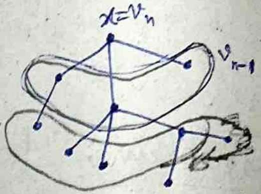
$$\chi(G) = \max_{1 \le i \le k} \chi(G_i) \text{)}$$

$\Delta(G) \neq 1$, ~~other~~ as $G \not\cong K_2$

If $\Delta(G) = 2$, then G is a path P_n, $n \geq 3$ or an even cycle. In this case, $\chi(G) = \Delta(G) = 2$.

Next we take $\Delta(G) \geq 3$.

Case I – G is non-regular graph. Then $\exists$ a vertex x in G with ~~deg(x)~~ $\deg(x) < \Delta(G)$. Consider a spanning tree T of G and then take x as the root vertex of T.



Let h be the height of this rooted tree.

We order the vertices in the following way –

Take $v_n = x$. Then name the vertices at level 1 by $v_{n-1}, v_{n-2}, \ldots, v_{n-k}$. Next name the vertices in level 2 as $v_{n-k-1}, v_{n-k-2}, \ldots$

In this way, name the vertices up to level $h-1$. Finally the vertices in level h (which are pendant vertices) get names as $v_{n-l}, \ldots, v_2, v_1$.

Consider $L: v_1, v_2, \ldots, v_i, \ldots, v_n$

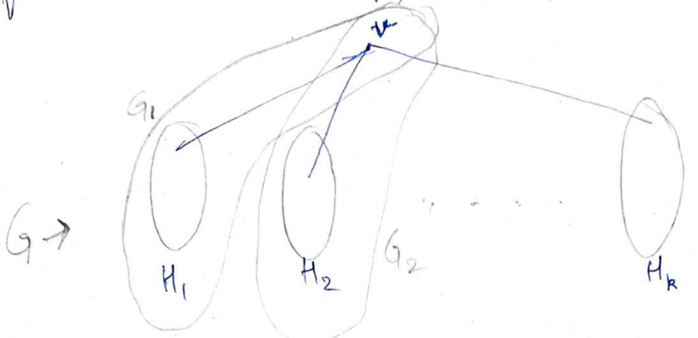
Every v_i, $1 \le i \le n-1$ has a higher indexed neighbour (i.e. parent in T) in L. So every v_i, $1 \le i \le n-1$, has at the most $\Delta(G) - 1$ lower indexed neighbour in L.

Greedy algorithm assigns color no. at the most $\Delta(G)$ to v_i, $1 \le i \le n-1$.

Since $deg(v_n) = deg(x) < \Delta(G)$, color no. $\Delta(G)$ is available for v_n also in the greedy algorithm.
Hence $\chi(G) \leq \Delta(G)$.

Case II - G is a regular graph, i.e. G is $\Delta(G)$-regular graph, $\Delta(G) \geq 3$.

~~Subgraph~~ Subcase I - G has a cut vertex, i.e. $\kappa(G) = 1$.
Let v be a cut vertex of G. Let the components of $G - v$ be $H_1, H_2, \ldots, H_k$.



Let $G_i = H_i \cup \{v\}$, $i = 1, 2, \ldots, k$

Each G_i is non-regular.
From case I, G_i needs at the most $\Delta(G)$ colors, for its coloring, i.e., color each G_i by using at the most $\Delta(G)$ colors. If needed, permute colors in G_i so that v will get the same color in each G_i.
Now we have a coloring of G with at the most $\Delta(G)$ colors. Hence $\chi(G) \leq \Delta(G)$.

Subcase II - $\kappa(G) \geq 2$.
Claim - G has a vertex v, v has two neighbours v_1, v_2, $v_1 \not\sim v_2$, and $G - \{v_1, v_2\}$ is connected.
If the claim is true then we consider an ordering of vertices as $L: v_1, v_2, \ldots, v_n = v$.
~~$G - \{v_1, v_2\}$~~ $G - \{v_1, v_2\}$ is connected.

We ~~consi~~ consider an ordering of vertices in $G - \{v_1, v_2\}$ in the same order as in Case I, taking v as the root vertex. Say this ordering is L_1,

$$L: v_1, v_2, \underbrace{\ldots\ldots}_{L_1}, v_n = v$$

→ Complete proof?

$$v_1 \nsim v_2, \quad v_1, v_2 \sim v$$

- Edge coloring - It is an assignment of colors to edges in such a way that adjacent edges get different colors.

Minimum no. of colors needed for an edge coloring of G is called edge-chromatic number of G, denoted by $\chi_1(G)$. The set of edges which get the same color is a matching of G.

- **Lemma:** $\chi_1(G) \geq \Delta(G)$ and $\chi_1(G) \geq \dfrac{|E(G)|}{\alpha'(G)}$

where $\alpha'(G)$ is the size of a maximum matching.

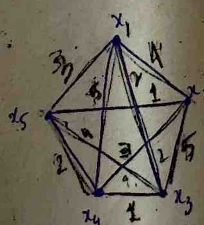
Let G be a graph, then line graph of G, i.e. $L(G)$, is $V(L(G)) = E(G)$, $l_1, l_2 \in$ ~~V(G₂)~~ $V(L(G))$, $l_1 \sim l_2$, iff l_1, l_2 have a common end vertex in G.

Edge coloring of G is the same as vertex coloring of L(G).

example - i) $\chi_1(C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$

ii) $\chi_1(P_n) = 2$

iii) $\chi_1(K_n) = \begin{cases} n-1, & \text{if } n \text{ is even} \\ n, & \text{if } n \text{ is odd} \end{cases}$



Case I - If n is odd,

Draw a cycle C_n of K_n as a regular n-gon.

Let the vertices on C_n be $x_1, x_2, \ldots, x_n$. For any vertex x_i ~~and~~ let e_i be the edge on C_n which is opposite to x_i. Assign color no. i to all the edges parallel to e_i (including). This gives an edge coloring of K_n with n colors. So $\chi_1(\underset{\overset{\|}{K_n}}{G}) \le n$.

Note: In this edge coloring of K_n, none of the edges incident on x_i recieve color 'i'.

$\chi_1(K_n) \le n$, for n odd.

~~$|E(G)|$~~ $|E(K_n)| = \frac{n(n-1)}{2}$, $\alpha'(K_n) = \frac{n-1}{2}$

$\frac{5-1}{2} = \frac{4}{2} = 2$

$\Rightarrow \chi_1(K_n) \ge n$

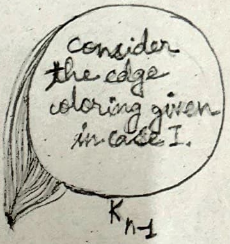
Hence, $\chi_1(K_n) = n$, when n is odd.

Case II – n is even

$E(K_n) = \frac{n(n-1)}{2}$, $\alpha'(K_n) = \frac{n}{2}$

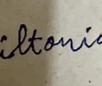
$\chi_1(K_n) \ge n-1$

From case I, $\chi_1(K_{n-1}) = n-1$



We get an edge coloring of K_n with $n-1$ colors..

~~$\chi_1(K_n) \ge n-1$~~ $\chi(K_n) \ge n-1$



$2e = 3n$

• **Theorem** (Viging, 1964): For any graph G,

$\chi_1(G) = \Delta(G)$ or $\Delta(G) + 1$.

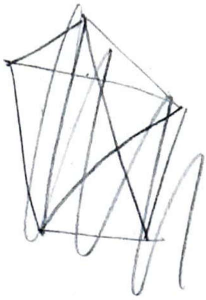
example- For every simple 3-regular Hamiltonian graph G, $\chi_1(G) = 3$.

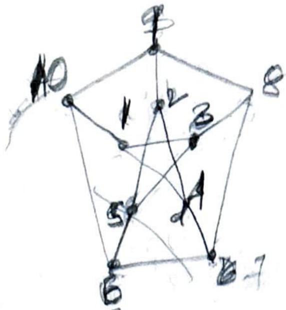
Since, G is 3-regular, n is an even integer.

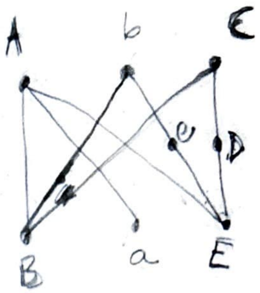
Let C be a Hamiltonian cycle in G. C is even cycle.

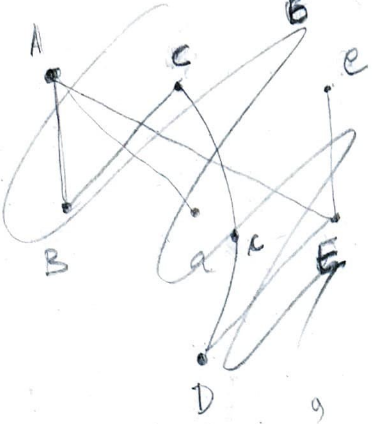
So $\chi_1(G) \leq 3$

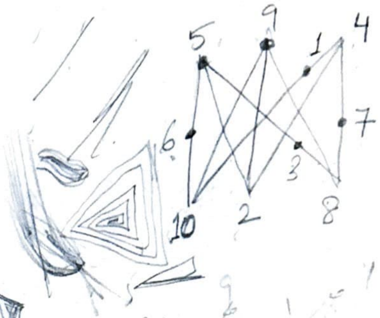
Also, $\chi_1(G) \geq \Delta(G) = 3$

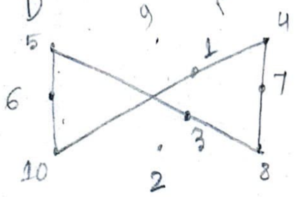


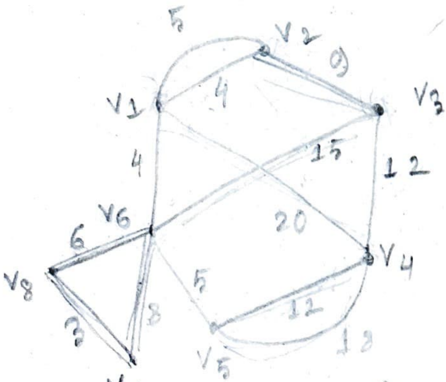


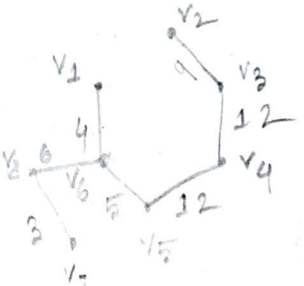


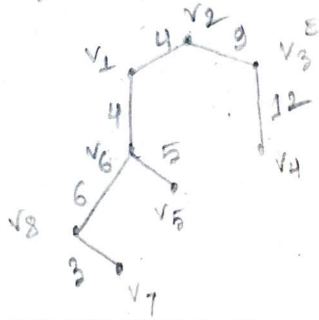












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