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Ex: l^p, $1 \le p \le \infty$ is a Banach space.

Sol

Let $1 \le p < \infty$

we know that l^p is a n.l.s.

Let $\{x_n\}$ be a Cauchy sequence in l^p

{xn}=sequence of elements of lp
elements of lp are already bounded sequences on N
hence {xn} is sequence of sequences

Then given any $\epsilon > 0$ $\exists$ $n_0 \in N$ such that

$$\|x_n - x_m\|_p < \epsilon \quad , \forall\, n, m \ge n_0.$$

$$\Rightarrow \sum_{j=1}^{\infty} |x_n(j) - x_m(j)|^p < \epsilon^p$$

For each j,

$$\Rightarrow |x_n(j) - x_m(j)|^p \le \sum_{j=1}^{k} |x_n(j) - x_m(j)|^p$$

$$\le \sum_{j=1}^{\infty} |x_n(j) - x_m(j)|^p < \epsilon^p$$

$\rightarrow$ (*)

$\Rightarrow \{x_n(j)\}_{j=1}^{\infty}$ is a Cauchy Sequence in the field K.

$[x_n = (x_n(1), x_n(2), \ldots x_n(j), \ldots)]$

$\because$ The field K is Complete,

$x_n(j) \longrightarrow \alpha_j \quad , \quad \forall n \geq n_0$

$[\alpha_j = \lim_{m\to\infty} x_m(j)]$

Define $x(j) = \alpha_j \quad \forall j = 1, 2 \ldots,$

Then and let $x = (x(1), x(2), x(3), \ldots)$.

$$\sum_{j=1}^{K} |x_n(j) - x(j)|^p = \lim_{m\to\infty} \sum_{j=1}^{K} |x_n(j) - x_m(j)|^p$$

$< \epsilon^p$ by $(*)$

$\forall n \geq n_0$

Letting $K \longrightarrow \infty$, we get

$$\sum_{j=1}^{\infty} |x_n(j) - x(j)|^p < \epsilon^p$$

$\Rightarrow \quad \|x_n - x\|_p < \epsilon \quad \forall\, n \geq n_0$

$\Rightarrow \quad x_n \longrightarrow x$ in ℓ^p

and $x_n - x \in \ell^p$.

Now since

$x = x - x_{n_0} + x_{n_0}$, $\quad x - x_{n_0}, x_{n_0} \in \ell^p$

$\Rightarrow \quad x - x_{n_0} + x_{n_0} \in \ell^p \Rightarrow x \in \ell^p$, $1 \leq p < \infty$.

l^∞: $\{x_n\}$ be a Cauchy sequence in l^∞. Then given $\epsilon > 0$ $\exists\, n_0 \ni$

$$\|x_n - x_m\|_\infty < \epsilon \quad \forall\, n, m \geq n_0$$

$$\Rightarrow \max_j |x_n(j) - x_m(j)| < \epsilon$$

$$\Rightarrow |x_n(j) - x_m(j)| \leq \|x_n - x_m\|_\infty < \epsilon \quad (*)$$

$\Rightarrow \{x_n(j)\}$ is a Cauchy sequence in the field K.

Since K is complete, let $x_n(j) \to \alpha_j \in K$.

$[\because x_n - x_m = (x_n(1) - x_m(1), x_n(2) - x_m(2), \ldots, x_n(j) - x_m(j), \ldots)]$

Denote $x(j) = \alpha_j$ and $x = (x(1), x(2), \ldots x(j), \ldots)$

Now for $j \in \mathbb{N}$, keep $n \geq n_0$ and letting $m \to \infty$ in the inequality $(*)$ we have

$$|x_n(j) - x(j)| = \lim_{m \to \infty} |x_n(j) - x_m(j)| < \epsilon$$

$\Rightarrow |x_n(j) - x(j)| < \epsilon \quad \forall\, n \geq n_0$ and $j \in \mathbb{N}$.

In particular $|x_{n_0}(j) - x(j)| < \epsilon, \forall j \in \mathbb{N}$

$\Rightarrow \max_j |x_{n_0}(j) - x(j)| < \epsilon, \forall j \in \mathbb{N}$

$\Rightarrow \|x_{n_0} - x\|_\infty < \epsilon$

$\Rightarrow x_{n_0} - x \in \ell^\infty$

Since $x = x - x_{n_0} + x_{n_0}$, $x - x_{n_0}, x_{n_0} \in \ell^\infty \Rightarrow x \in \ell^\infty$.

$\therefore \ell^\infty$ is a Banach space.

Problem: Show that

Proof?

$c = \{x = (x(1), x(2), \ldots) \mid \{x(n)\}$ is cgt in $\mathbb{K}\}$

&

$c_0 = \{x = (x(1), x(2) \ldots) \mid x(n) \to 0$ as $n \to \infty\}$

are closed subspaces of ℓ^∞.

Since ℓ^∞ is a Banach space, it follows that c & c_0 are also Banach spaces.

as C, Co are spaces convergent sequences, cauchy sequences in these spaces are also convergent and hence they are banach spaces.

Ex: c_{00} is not a closed subspace of ℓ^∞.

Are all banach spaces subspaces of l-infinity when infinity norm is concerned?

$\because \quad x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, 0, 0, \dots\right) \in C_{00}$

and

$x_n \to x = \left(1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \frac{1}{n+1}, \dots\right)$

$\notin C_{00}$

$\Rightarrow$ C_{00} is not a closed subspace of ℓ^∞, hence C_{00} is not a Banach space.

Ex: $\quad C^1[a, b] = \{x \in C[a, b] \,/\, x$ is differentiable and x' is continuous on $[a, b]\}$

Also $C^1[a, b]$ contains all the polynomials defined on $[a, b]$.

Note that for every $x \in C[a, b]$, there exists a sequence $\{P_n\}$ of polynomials defined on $[a, b]$ such that

$$\|p_n - x\|_\infty \longrightarrow 0$$

Also $C^1[a,b] \subsetneq C[a,b]$

Also $x \in C[a,b] \Rightarrow \exists \{p_n\}$ ~~sequence~~ of polynomials in $C^1[a,b]$ $\ni$

$$\|p_n - x\|_\infty \longrightarrow 0$$

But x need not be in $C^1[a,b]$

$\therefore$ $C^1[a,b]$ is not a closed subspace of $(C[a,b], \|\cdot\|_\infty)$.

$\Rightarrow$ $C^1[a,b]$ is not a Banach space w.r.t $\|\cdot\|_\infty$.

Now for any $x \in C^1[a,b]$, let

$$\|x\|_{1,\infty} = \max\{\|x\|_\infty, \|x'\|_\infty\}$$

the above norm is infinity norm with xi and xi' .... basically more elements

Clearly $(C^1[a,b], \|\cdot\|_{1,\infty})$ is a n.l.s.

Claim: $(C^1[a,b], \|\cdot\|_{1,\infty})$ is a Banach space.

[Well known result, Rudin, Real analysis, theorem 7.17.

" If $\{f_n\}$ is a sequence of continuous differentiable functions such that $f_n \to f$ pointwise on $[a,b]$,

$f_n^{(1)} \longrightarrow g$ uniformly, then

$f_n \longrightarrow f$ uniformly and f is differentiable $f^{(1)} = g$,]

Let $\{x_n\}$ be a Cauchy sequence in the n.l.s $(C^1[a,b], \|\cdot\|_{1,\infty})$.

Then given any $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ $\ni$

$\|x_n - x_m\|_{1,\infty} < \epsilon, \quad \forall \; n, m \geq n_0.$

$\Rightarrow \max\{\|x_n - x_m\|_\infty, \; \|x_n^{(1)} - x_m^{(1)}\|_\infty\} < \epsilon$

$\Rightarrow \quad \|x_n - x_m\|_\infty < \epsilon$ and $\|x_n^{(1)} - x_m^{(1)}\|_\infty < \epsilon$

$\Rightarrow \quad \{x_n\}$ and $\{x_n^{(1)}\}$ are Cauchy sequences in $(C[a,b], \|\cdot\|_\infty)$, which is a Banach space.

$\therefore \; \exists \; x, y \in C[a,b]$ such that

$\|x_n - x\|_\infty \to 0, \quad \|x_n^{(1)} - y\|_\infty \to 0$

as $n \to \infty$.

$\Rightarrow x_n \to x$ and $x_n^{(1)} \to y$ uniformly

$\therefore$ By above result, x is differentiable and $x^{(1)} = y$.

$\therefore \; \|x_n - x\|_\infty \to 0 \;$ & $\; \|x_n^{(1)} - x^{(1)}\|_\infty \to 0$

$\Rightarrow \max\{\|x_n - x\|_\infty, \|x_n^{(1)} - x^{(1)}\|_\infty\} \to 0$

$\Rightarrow \|x_n - x\|_{1,\infty} \to 0$ and

$x \in C^1[a,b]$.

$\therefore (C^1[a,b], \|\cdot\|_{1,\infty})$ is a

Banach space.

Ex: $B(\Omega)$ is a Banach space,

where Ω is a nonempty set

Let $\{x_n\}$ be a Cauchy sequence in

$B(\Omega)$ and $\epsilon > 0$ be given, then

there exists $n_0 \in N \ni$

$\|x_n - x\|_\infty < \epsilon, \ \forall n \geq n_0$.

$\Rightarrow \max\{|x_n(t) - x(t)| \ / \ t \in \Omega\} < \epsilon$

$\forall n \geq n_0$

{xn}=sequence of f(i) defined for all i belonging to omega
i.e sequence of sequences of f(i)

In particular for each $t \in \Omega$, we have

$$|x_n(t) - x(t)| < \epsilon \quad , \quad \forall\, n \geq n_0$$

$\Rightarrow$ $\{x_n(t)\}$ is a Cauchy sequence in the field K, for each $t \in \Omega$.

Since K is complete, $\exists\, \alpha_t \in K$ such that $x_n(t) \longrightarrow \alpha_t$, $t \in \Omega$.

Then let $x(t) = \alpha_t$, $\forall\, t \in \Omega$.

$$|x_n(t) - x(t)| = \lim_{m \to \infty} |x_n(t) - x_m(t)| \longrightarrow 0$$

$$\Rightarrow \max_t |x_n(t) - x(t)| \longrightarrow 0$$

$$\Rightarrow \|x_n - x\|_\infty \longrightarrow 0 \text{ as } n \to \infty.$$

i.e given $\epsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ $\ni$

$$\|x_n - x\|_\infty < \epsilon \quad \forall\, n \geq n_0.$$

in particular $\|x_{n_0} - x\|_\infty < \epsilon$.

$\because$ x_{n_0}, $x_{n_0} - x \in B(\Omega)$,

$\Rightarrow$ $x = x - x_{n_0} + x_{n_0} \in B(\Omega)$

[$\because$ $B(\Omega)$ is n.l.s]

$\therefore$ $B(\Omega)$ is a Banach space.

In particular $\Omega = \mathbb{N}$,

$B(\Omega) = l^\infty(\mathbb{N})$ is a Banach space.

Ex: Let Ω be a metric space. Then $C(\Omega)$, the space of all K-valued continuous functions on Ω is a sub-space of $F(\Omega)$. ?????

Denote

$C_b(\Omega) = C(\Omega) \cap B(\Omega)$, the space of all K-valued bounded and

Continuous functions on Ω.

Claim: $C_b(\Omega)$ is a closed subspace of $B(\Omega)$.

Let $\{x_n\}$ be a sequence in $C_b(\Omega)$ such that $\|x_n - x\|_\infty \longrightarrow 0$, for some $x \in B(\Omega)$.

Claim: $x \in C_b(\Omega)$.

Let $\epsilon > 0$ be given, then $\exists$ $n_0 \in \mathbb{N}$ such that $\|x_n - x\|_\infty < \epsilon$ $\forall$ $n \geq n_0$.

$\Rightarrow \max_t \{ |x_n(t) - x(t)| \} < \epsilon$

Hence for each $t \in \Omega$, $n \geq n_0$

$$|x(t) - x(\tau)| \leq |x(t) - x_n(t)| + |x_n(t) - x_n(\tau)| + |x_n(\tau) - x(\tau)|.$$

$$< \epsilon + |x_n(t) - x_n(\tau)| + \epsilon$$

$\longrightarrow$ (*)

$\because \{x_n\}$ is continuous on Ω,

in particular x_{n_0} is continuous and bounded.

$\therefore \exists$ an open set $G \subseteq \Omega$ containing τ such that

$$|x_{n_0}(t) - x_{n_0}(\tau)| < \epsilon \quad \forall\, t \in G$$

$\therefore$ From $(*)$ we have

for given $\epsilon > 0$ $\exists$ an open set $G \subseteq \Omega$ such that

$$|x(t) - x(\tau)| < \epsilon + |x_{n_0}(t) - x_{n_0}(\tau)| + \epsilon$$

$$< \epsilon + \epsilon + \epsilon = 3\epsilon$$

$\Rightarrow$ x is continuous

$\Rightarrow$ $x \in C_b(\Omega)$.

$\Rightarrow$ $C_b(\Omega)$ is a closed subspace of a Banach space $B(\Omega)$.

$\therefore C_b(\Omega)$ is a Banach space

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* In particular $C_b(\Omega) = C[a,b]$ is a Banach space w.r.t $\|\cdot\|_\infty$.

H.W:

(1) $(C[a,b], \|\cdot\|_1)$ is n.l.s.

when $\|x\|_1 = \int_a^b |x(t)|\,dt$.

S.T $(C[a,b], \|\cdot\|_1)$ is not a Complete n.l.s.

(2) $(C[a,b], \|\cdot\|_p)$ is not a Banach space $1 \le p < \infty$.

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Ex: $X = P[a,b]$ of all Polynomials on $[a,b]$ is not a Banach space w.r.t $\|\cdot\|_\infty$.

We know that by Weistrass approximation theorem, for every $x \in C[a,b]$, there exists a sequence $\{x_n\}$ of polynomials such that $\|x_n - x\|_\infty \longrightarrow 0$.

$\therefore \overline{P[a,b]} = C[a,b]$, w.r.t $\|\cdot\|_\infty$

$\therefore P[a,b]$ is not a closed subspace of $C[a,b]$. $\left[\because P[a,b] \subsetneq C[a,b]\right]$

$\therefore P[a,b]$ is not a Banach space.

Also $P[a,b]$ is not a Banach space w.r.t any $\|\cdot\|_p$, $1 \le p < \infty$.

Let $x \in C[a,b] - P[a,b]$, let $\{x_n\}$ be a sequence in $P[a,b]$ such that

$$\|x_n - x\|_\infty \longrightarrow 0 \text{ as } n \to \infty.$$

Then

$$\|x_n - x\|_p^p = \int_a^b |x_n(t) - x(t)|^p \, dt$$

$$\leq \max_t |x_n(t) - x(t)| \int_a^b dt$$

$$= \|x_n - x\|_\infty (b-a)$$

$$\longrightarrow 0 \text{ as } n \to \infty$$

$$\Rightarrow \|x_n - x\|_p \longrightarrow 0 \text{ as } n \to \infty.$$

$\therefore$ $P[a,b]$ is dense in $C[a,b]$ w.r.t $\|\cdot\|_p$, $1 \leq p < \infty$.

But $P[a,b]$ is not closed w.r.t $\|\cdot\|_p$, $1 \leq p < \infty$, since $x \notin P[a,b]$.

A dense in B :
A is a subset of B such that for some epsilon & for every b there exists an a such that ||a-b||< epsilon

$\therefore$ $P[a,b]$ is not a Banach space

w.r.t $\|\cdot\|_p$, $1 \leq p \leq \infty$.

Ex: $X = C_{00}$ with $\|\cdot\|_p$, $1 \leq p \leq \infty$,

is not a Banach space.

Let $x \in C_0$ and $n \in \mathbb{N}$,

$x = (x(1), x(2), \ldots \ldots)$

Let $x_n = (x(1), x(2), \ldots x(n), 0, 0, \ldots \ldots) \in C_{00}$

Then

$$\|x_n - x\|_\infty = \sup\{|x(j)| \;/\; j > n\}$$

as |xn-x|=(0,0,.. n-times, x(n+1),x(n+2)+...)
thus, max of the above is max(x(j) | j>n) i.e sup{ x(j) | j>n}
whic is equal to ||xn-x||

$\longrightarrow 0$ $\quad [\because x \in C_0]$

$\therefore$ C_{00} is dense in C_0 w.r.t $\|\cdot\|_\infty$.

Also for $1 \leq p < \infty$, consider $x \in \ell^p$,

and $x_n = (x(1), x(2), \ldots x(n), 0, 0, \ldots)$

$\in C_{00}$

Then

$$\|x_n - x\|_p^p = \sum_{j=n+1}^{\infty} |x(j)|^p \longrightarrow 0$$

as $n \to \infty$

$\therefore$ $\overline{C_{00}} = \ell^p$, $1 \le p < \infty$.

the above stands for closure of Coo

[Dense :- we say a set A is dense in X, if every $x \in X$ $\exists$ $\{a_n\}$ in A $\ni$ $\|a_n - x\| \longrightarrow 0$]

But C_{00} is not closed, otherwise

$$C_{00} = \overline{C_{00}} = \ell^p$$

$$C_{00} = \overline{C_{00}} = C_0 \quad \text{which is not true.}$$

How do we say closure of Coo is lp/Co wrt p-norm/inf-norm????

$\therefore$ C_{00} is not a Banach space for $1 \le p \le \infty$.

$\left[x_n - x = (0, 0 \cdots -x(n+1), -x(n+2), \cdots)\right]$

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Quotient Space:

Let X be a n.l.s, Y be a closed subspace of X. The coset of an element $x \in X$ w.r.t Y is defined as

x1+Y=x2+Y iff (x1-x2) belong to Y
else x1+Y is disjoint from x2+Y

$$x + Y = \{x + y \mid y \in Y\}.$$

affine space

* Any two cosets are either identical or disjoint. These cosets form the partition of the n.l.s X.

$$\frac{X}{Y} = \{x + Y \mid x \in X\}$$

Quotient space = set of all affine spaces of Y

Define linear operations on $\frac{X}{Y}$ as

$$(x_1+Y)+(x_2+Y) = x_1+x_2+Y$$

$$k(x+Y) = kx+Y, \quad \forall\, x_1, x_2 \in X$$
$$k \in K.$$

Then $\frac{X}{Y}$ is a vector space.

Y is the additive identity.

and

$-(x+Y) = -x+Y$ is the additive inverse of $x+Y$ in $\frac{X}{Y}$

Now define a function $|||\cdot||| : \frac{X}{Y} \to \mathbb{R}$

by

$$|||x+Y||| = \inf\{\|x+y\| \mid y \in Y\}$$

Claim: $\frac{X}{Y}$ is a n.l.s.

$\because \|x+y\| \geq 0 \quad \forall\, y \in Y,\ x \in X$

$\Rightarrow \inf\{\|x+y\| \,/\, y \in Y\} \geq 0.$

$\Rightarrow |||x+Y||| \geq 0.$

Now let $|||x+Y||| = 0$

$\Rightarrow \inf\{\|x+y\| \,/\, y \in Y\} \to 0$

$\Rightarrow \exists$ a sequence $y_n \in Y$ such that

$\|x+y_n\| \to 0$ as $n \to \infty$.

$\Rightarrow x+y_n \to 0$ as $n \to \infty$

$\Rightarrow y_n \to -x \in Y$ [$\because Y$ is closed]

$\therefore \quad -x \in Y \Rightarrow x \in Y$

$\Rightarrow \quad x + Y = Y$

$\therefore \; |||x+Y||| = 0 \Rightarrow x + Y = Y$

the above happens iff x belongs to Y i..e -x also belongs to Y

identity in Y.

Now for $k \in K$

$$|||k(x+Y)||| = |||kx + Y|||$$

$$= \inf\{ \|kx + y\| \;/\; y \in Y \}$$

$$= \inf\{ \underline{|k| \; \|x + (y/k)\|} \;/\; y \in Y \}$$

$$= |k| \inf\{ \|x + y_1\| \;/\; y_1 \in Y \}$$

$$= |k| \; |||x+Y|||.$$

Now for any $x_1, x_2 \in X$, Consider

$$|||(x_1+Y)+(x_2+Y)|||$$

$$= |||x_1+x_2+Y|||$$

$$= \inf\{\|x_1+x_2+y\| \mid y \in Y\}$$

$$= \inf\{\|x_1+x_2+y_1+y_2\| \mid y = y_1+y_2 \in Y\}$$

$$= \inf\{\|x_1+y_1+x_2+y_2\| \mid y_1, y_2 \in Y\},$$

$$\leq \inf\{\|x_1+y_1\| \mid y_1 \in Y\}$$

$$+ \inf\{\|x_2+y_2\| \mid y_2 \in Y\}.$$

$$= |||x_1+Y||| + |||x_2+Y|||$$

$\therefore \frac{X}{Y}$ is a n.l.s.

Dim(X/Y)=dim(X)-dim(Y)

Def :- Let X be a n.l.s.

A series $\sum_{n=1}^{\infty} x_n$ in X is said to be absolutely summable if $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Def :- A series $\sum_{n=1}^{\infty} x_n$ in a n.l.s X is said to be summable if $s_n = \sum_{j=1}^{n} x_j \longrightarrow x \in X$.