Joint and Conditional distributions.

Vector valued random variable.

Let $(\Omega, \mathcal{A}, P)$ be a probability space.
and $\omega \in \Omega$, which is a sample space define.

$$\underset{\sim}{X}(\omega) = (x_1(\omega), x_2(\omega), x_3(\omega), \cdots, x_k(\omega))^T$$

$$\underset{\sim}{X} : \Omega \longrightarrow \mathbb{R}^k.$$

"such that we can compute the porbalities of our interest".

Ω = students in your class.

ω_i = ith student.

$$\begin{cases} x_1(\omega_i) = \text{height of ith student.} \\ x_2(\omega_i) = \text{weight of ith student.} \end{cases}$$

$x_3(\omega_i)$ = family income of ith student.

$x_4(\omega_i)$ = Male / Female.

Let (X, Y) be a pair of random variables with joint c.d.f F on some probability space $(\Omega, \mathcal{A}, P)$.

Then F is defined as

$$F(x, y) = P(X \leq x, Y \leq y) \qquad x, y \in \mathbb{R} \text{ or } (x, y) \in \mathbb{R}^2$$

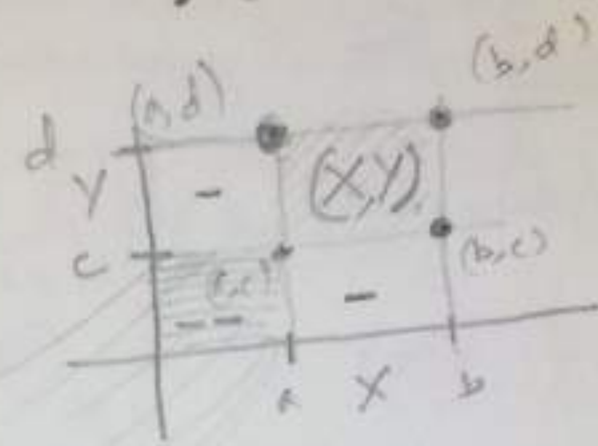
$$= P(\{\omega \mid X(\omega) \leq x, Y(\omega) \leq y\}).$$

Joint cdf of X and Y

$$= P(\{\omega \mid X(\omega) \in (-\infty, x], Y(\omega) \in (-\infty, y]\}).$$

$$= P(\{X^{-1}(-\infty, x]\} \cap \{Y^{-1}(-\infty, y]\}).$$

$$F(x,y) = P(X \le x, Y \le y) \qquad x, y \in \mathbb{R}.$$



(1) $\lim\limits_{x \downarrow -\infty} \lim\limits_{y \downarrow -\infty} F(x,y) = 0$

(2) $\lim\limits_{x \uparrow \infty} \lim\limits_{y \uparrow \infty} F(x,y) = 1.$

(3) $\lim\limits_{x \uparrow \infty} F(x,y) = P(Y \le y) = F_Y(y)$

Marginal distribution of Y.

(4) $\lim\limits_{y \uparrow \infty} F(x,y) = P(X \le x) = F_X(x)$

Marginal distribution of X.

5 $P(a < X \le b, c < Y \le d) \ge 0$ Nondecreasing property in 2-dimension.

$$= F(b,d) - F(a,d) - F(b,c) + F(a,c).$$

If (X, Y) are independent

$$= F_X(b) \cdot F_Y(d) - F_X(a) \cdot F_Y(d) - F_X(b) F_Y(c) + F_X(a) F_Y(c)$$

$$= (F_X(b) - F_X(a))(F_Y(d) - F_Y(c)).$$

If (X,Y) are discrete.

(1) $\sum_x \sum_y f(x,y) = 1 \Rightarrow \sum_x \sum_y P(X=x, Y=y)$

(2) $\sum_y f(x,y) = f_X(x)$ marginal pmf of X.

(3) $\sum_x f(x,y) = f_Y(y)$ marginal pmf of Y.

Joint pmf.

$$f(x,y) = P(X=x, Y=y).$$

If X and Y are independent

$$f(x,y) = P(X=x)\,P(Y=y)$$
$$= f_X(x)\,f_Y(y).$$

If (X,Y) are continuous.

Joint pdf

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$
$$= \frac{\partial^2}{\partial y \partial x} F(x,y).$$

(1) $\int_x \int_y f(x,y) = 1$

(2) $\int_y f(x,y)\,dy = f_X(x)$ marginal pdf

(3) $\int_x f(x,y)\,dx = f_Y(y).$

If X and Y are independent.

$$f(x,y) = f_X(x)\,f_Y(y).$$

	TrAS	TrAF	
TrBS	$p_A p_B + \epsilon$	$q_A p_B - \epsilon$	p_B
TrBF	$p_A q_B - \epsilon$	$q_A q_B + \epsilon$	q_B
	p_A	q_A	

X

$P(\text{Sneem in trA}) = p_A$

$P(\text{Sneem in trB}) = p_B$.

Y Y.

Assume TrA and TrB are independent.

$X = \begin{cases} 1 & \text{w.p. } p_A \\ 0 & \text{w.p. } q_A \end{cases}$

'ϵ' very small number.

$Y = \begin{cases} 1 & \text{w.p. } p_B \\ 0 & \text{w.p. } q_B \end{cases}$

$\Rightarrow \quad P(X=1, Y=1) = p_A p_B + \epsilon \neq P(X=1)\,P(Y=1)$

no longer independent.

but the marginal property unchanged.

If we know the joint distribution then we can extract the marginals easilly

But if we know only marginals then we may not know the exact joint distribution

Conditional pdf or pmf is defined as follows:

Conditional density of $Y \mid X = x$ is defined as.

$$f_{Y|x}(y \mid X = x) = \frac{f(x, y)}{f_x(x)} \quad \text{when } f_x(x) > 0.$$

Conditional expectation of $Y \mid X = x$ is defined as.

$$E(Y \mid X = x) = \int_y y \, f_{Y|x}(y \mid x) \, dx \quad \text{(It is a function of } x\text{.)}$$

$$= \int_y y \, \frac{f(x, y)}{f_x(x)} \, dx. \quad \text{or} \quad \sum_y y \, \frac{f(x, y)}{f_x(x)}.$$

$E(Y \mid X = x)$ is known as the regression function of Y on X.

Properties of Expectation.

Assume (X, Y) has joint density $f(x,y)$ with $E(X^2) < \infty$, $E(Y^2) < \infty$

$\Rightarrow E(|X|) < \infty$ $E(|Y|) < \infty$

(1) $E(X+Y) = E(X) + E(Y)$.

First we need to show

$E(|X+Y|) < \infty$.

$$
\begin{aligned}
& E(|X+Y|) \\
&= \iint_{x\,y} |x+y| f(x,y)\,dx\,dy. \\
&\leq \iint_{x\,y} (|x| + |y|) f(x,y)\,dx\,dy. \\
&\leq \iint_{x\,y} |x| f(x,y)\,dx\,dy + \iint_{x\,y} |y| f(x,y)\,dx\,dy. \\
&= \int_x |x| f_X(x)\,dx + \int_y |y| f_Y(y)\,dy. \\
&= \underline{E(|X|) + E(|Y|)} < \infty
\end{aligned}
$$

As a consequence

$$
\begin{aligned}
& E(X+Y) \\
&= \iint_{x\,y} (x+y) f(x,y)\,dx\,dy \\
&= \iint x f(x,y)\,dx\,dy \\
&\quad + \iint y f(x,y)\,dx\,dy \\
&= E(X) + E(Y).
\end{aligned}
$$

(Independence NOT needed)

Sum law of expectation.

② If (X, Y) are independently distributed then $f(x, y) = f_X(x) f_Y(y)$.
Which implies $E(XY) = E(X) E(Y)$.

$E\,|XY|$

$= \iint |xy|\, f(x,y)\, dx\, dy$

$= \iint |x|\,|y|\, f_X(x) f_Y(y)\, dx\, dy$

$= \int |x|\, f_X(x)\, dx \int |y|\, f_Y(y)\, dy$

$= E(|X|)\; E(|Y|) < \infty$

$E(XY)$

$= \iint xy\, f(x,y)\, dx\, dy$

$= \iint x\, y\, f_X(x) f_Y(y)\, dx\, dy$

$= \underline{E(X)\, E(Y)}$.

Under independence.

(3) $E(\alpha X) = \alpha E(X)$. when $E(|X|) < \infty$.

Dfn. (4) Covariance between X and Y are defined as.

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y).$$

$$= E\left[(X - E(X))(Y - E(Y))\right]$$

$$= \operatorname{cov}(Y,X).$$

$$\Rightarrow \operatorname{cov}(X,X) = E\left[(X-E(X))^2\right] = \underline{Var(X)}.$$

If X and Y are independent then $\operatorname{cov}(X,Y) = 0$

But $\operatorname{cov}(X,Y) = 0 \not\Rightarrow$ X,Y independent. in general.

(5) $V(a+bX) = E\left[(a+bX) - E(a+bX)\right]^2$

$= E\left[a+bX - (a+bE(X))\right]^2$

$= E\, b^2\left[X - E(X)\right]^2$

$= b^2\, Var(X).$

$\begin{cases} \text{Location invariant.} \\ \text{scale equivariant.} \end{cases}$

(6). $V(a\cancel{X} + bY)$

$= cov(aX+bY,\ aX+bY)$

$= cov(aX, aX) + cov(aX, bY) + cov(bY, aX) + cov(bY, bY).$

$= a^2 Var(X) + ab\ cov(X,Y) + ab\ cov(X,Y) + b^2 Var(Y).$

$= a^2 Var(X) + b^2 Var(Y) + \cancel{2abcov}\ \underline{2ab\ cov(X,Y)}.$

7 $E(Y) = E_X \underbrace{E_{Y|X}(Y|X=x)}_{\text{function of } X}.$

$$\text{RHS} = E_X\left(E_{Y|X}(Y|X=x)\right)$$

$$= E_X \int_y y\, f_{Y|X}(y|x)\, dy$$

$$= E_X \int_y y\, \frac{f(x,y)}{f_X(x)}\, dy.$$

$$= \int_x\int_y y\, \frac{f(x,y)}{f_X(x)} \cdot f_X(x)\, dy\, dx.$$

$$= \int_x\int_y y\, f(x,y)\, dy\, dx.$$

$$= \int_y y\, f_Y(y)\, dy = E(Y).$$

8

$$\boxed{V(Y) = E_X V_{Y|X}(Y|X) + V_X E_{Y|X}(Y|X).}$$

$EV + VE$ formula

H.W.

start from RHS.

$$V(Y) \geq V_X E_{Y|X}(Y|X).$$

V_X (Regression)

9 Correlation coefficient.

Dfn: $$\operatorname{corr}(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)\ V(Y)}}.$$

(a) If X and Y are independent then $\operatorname{corr}(X,Y) = 0$

(b) $\operatorname{corr}(X,Y) = 0 \not\Rightarrow$ X and Y are independent.

(c) $|\operatorname{corr}(X,Y)| \le 1$.

(d) Correlation measures linear dependency between X & Y.

Example when correlation zero implies independence.

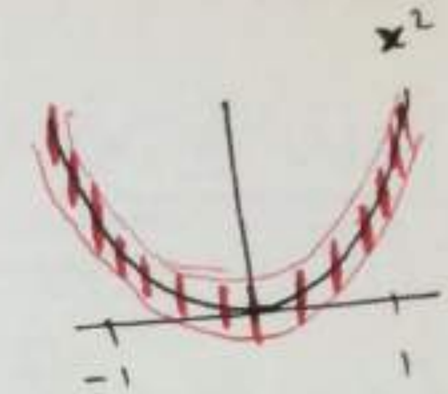
$\operatorname{cov}(XY) = (p_A p_B + \epsilon) - p_A p_B = \epsilon.$

(i) From bivariate Bernoulli
Hence $\operatorname{cov} = 0 \Rightarrow$ independent.

(ii) Bivariate normal $(X,Y) \sim N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$
correlation $\rho = 0 \Rightarrow$ independence.

2.b. $\operatorname{corr}(X,Y)=0 \not\Rightarrow X$ and Y independent.

$X \sim U(-1,1)$ [Symmetric around 0]



$\underline{Y \mid X=x \sim U\left(x^2-0.05,\ x^2+0.05\right)}$

$\operatorname{Cov}(X,Y)$

$= E(XY) - \cancel{E(X)}^{\,0}\, E(Y)$

$E(X) = 0$

$= E(XY)$

$= E_X\left(E_{Y|X}(XY)\right)$

$= E_X\left(X\, E_{Y|X}(Y)\right)$

$= E_X\left(X \cdot X^2\right)$

$= E_X\left(X^3\right) = 0$ as $X \sim \underline{U(-1,1)}$.

Cov or Correlation zero does not imply independence.

2.6 $|\mathrm{Corr}(X,Y)| \le 1.$

Let
$$\begin{cases} Z = X - E(X) \\ W = Y - E(Y). \\ E(Z) = 0 = E(W). \\ V(Z) = V(X), \quad V(W) = V(Y) \end{cases}$$

$\mathrm{cov}(Z,W) = \mathrm{cov}(X,Y).$

$$\Leftrightarrow \frac{|\mathrm{cov}(X,Y)|}{\sqrt{V(X)\,V(Y)}} \le 1$$

$$\Leftrightarrow (\mathrm{cov}(X,Y))^2 \le V(X)\,V(Y)$$

$$\Leftrightarrow \left(E\,(X-E(X))(Y-E(Y))\right)^2 \le E\,(X-E(X))^2\,E\,(Y-E(Y))^2.$$

$$\Leftrightarrow (E(ZW))^2 \le V(Z)\,V(W).$$

$$\Leftrightarrow (E(ZW))^2 \le E(Z^2)\,E(W^2).$$ <u>we need to show</u>.

Define. $T = Z - \lambda W$ $\Rightarrow E(T) = 0$ <u>λ is real</u>.

$V(T) \geqslant 0$ or $E(T^2) \geqslant 0$

$E\,(Z - \lambda W)^2 \geqslant 0$

$\Rightarrow E\,(Z^2 + \lambda^2 W^2 - 2\lambda Z W) \geqslant 0$

$\Rightarrow E(Z^2) + \lambda^2 E(W^2) - 2\lambda E(ZW) \geqslant 0.$

$$g(\lambda) = E(z^2) + \lambda^2 E(w^2) - 2\lambda E(zw) \geq 0.$$

$$= \lambda^2 E(w^2) - 2\lambda E(zw) + E(z^2) \geq 0.$$

$g(\lambda)$ will attain the minimum when $\lambda = \dfrac{E(zw)}{E(w^2)} = \hat{\lambda}$ (say).

$$g(\hat{\lambda}) \geq 0 \quad \Rightarrow \quad E(z^2) - \frac{(E(zw))^2}{(E(w^2))} \geq 0.$$

$$\Rightarrow \quad \cancel{\frac{E(z^2) - (E(zw))^2}{E\,w}}$$

$$\Rightarrow \quad (E(zw))^2 \leq E(z^2)E(w^2).$$

$$\Rightarrow \boxed{(\mathrm{cov}(x,y))^2 \leq V(x)\,V(y).}$$

formula.

"=" will hold.

when. $Z = \lambda W$

$\Rightarrow X - E(X) = \lambda(Y - E(Y))$

$\Rightarrow X = (E(X) - \lambda E(Y)) + \lambda Y.$

$$\boxed{X = a + bY.}$$

when X and Y are linearly realated

then only correlation between X and Y

are 1.

$\mathrm{cov}(X, a+bX)$

$= b\,\mathrm{cov}(X,X) = b\,\mathrm{var}(X)$

$\mathrm{corr}(X, a+bX)$

$$= \frac{b\,\mathrm{var}(X)}{\sqrt{V(X)\; b^2 V(X)}} = \frac{b}{|b|}$$

$=$ sign of b

$\underline{\underline{+1 \text{ or } -1}}$.

Ex A coin is tossed n times independently.

X = number of 'H'

Y = number of 'T'

$X + Y = n$ or $X = n - Y$.

$Y = n - X$

$\mathrm{Corr}(X, Y) = \underline{\underline{-1}}.$

Let X be a random variable with mgf. $M_X(t)$.

$Y = a + bX$.

then
$$M_Y(t) = E\left(e^{tY}\right) = E\left(e^{t(a+bX)}\right) = E\left(e^{ta+btX}\right) = e^{ta}E\left(e^{btX}\right) = e^{ta}M_X(bt).$$

Let $X_1 X_2 \cdots X_n$ iid random. variables. with mgf. $M_X(t)$.
define $Z = \sum_{i=1}^{n} X_i$

$$M_Z(t) = \left(M_X(t)\right)^n$$

We will use to prove CLT.

Let X and Y be independent random variables with mgf $M_X(t)$ & $M_Y(t)$ respectively.

then mgf of $Z = X+Y$ is

$$E\left(e^{tZ}\right) = E\left(e^{t(X+Y)}\right) = E\left(e^{tX}\cdot e^{tY}\right) = E\left(e^{tX}\right)E\left(e^{tY}\right)$$
$$= M_X(t)\,M_Y(t).$$

Ex $X \sim \text{bin}(n_1, p)$ > independent.
$Y \sim \text{bin}(n_2, p)$

Find the conditional distribution of $X \mid X+Y=k$.

$X+Y \sim \text{bin}(n_1+n_2, p)$.

$P(X=x \mid X+Y=k)$ $\qquad x = 0, 1, 2 \dots \min\{n_1, k\}$.

$$= \frac{P(X=x, X+Y=k)}{P(X+Y=k)} = \frac{P(X=x, Y=k-x)}{P(X+Y=k)}$$

$$= \frac{P(X=x)\,P(Y=k-x)}{P(X+Y=k)} = \frac{\binom{n_1}{x} p^x (1-p)^{n_1-x} \binom{n_2}{k-x} p^{k-x} (1-p)^{n_2-k+x}}{\binom{n_1+n_2}{k} p^k (1-p)^{n_1+n_2-k}}.$$

$$= \frac{\binom{n_1}{x}\binom{n_2}{k-x}}{\binom{n_1+n_2}{k}}$$ Hypergeometric distribution.

Ex $X \sim \text{poisson}(\lambda_1)$ > indeptd.
$Y \sim \text{poisson}(\lambda_2)$

Show that $X \mid X+Y = n \sim \text{bin}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$.

Ex Let $P \sim U[0,1]$ (prior) and $Y \mid P = p \sim \text{bin}(n, p)$.
Find the marginal distribution of Y. [prior predictive]

$Y \mid p \sim \text{bin}(n,p). \Rightarrow f(y \mid p) = \binom{n}{y} p^y (1-p)^{n-y}$.

$P \sim U(0,1) \quad g(p) = 1$ on $[0,1]$

Joint pdf of (y, p) is $f(y \mid p)\, g(p)$.

Marginal of $Y = h(y) = \int_0^1 f(y \mid p)\, g(p)\, dp$.

$$= \int_0^1 \binom{n}{y} p^y (1-p)^{n-y} \cdot 1 \cdot dp = \binom{n}{y} B(y+1, n-y+1)$$

$$= \frac{1}{n+1}$$

$Y \sim$ discrete uniform on $\{0, 1, 2, \dots n\}$.

$f(x, y) = f(y \mid x) f_x(x)$

Bivariate Normal distribution:

(X,Y) is said to follow bivariate normal distribution with
$E(X) = \mu_x$, $E(Y) = \mu_y$, $V(X) = \sigma_x^2$, $V(Y) = \sigma_y^2$, $\mathrm{Corr}(X,Y) = \rho$
if it has the following p.d.f.

$(X,Y) \sim N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$

$$f(x,y) = \frac{e^{-\frac{1}{2}\frac{1}{(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]}}{2\pi\,\sigma_x\,\sigma_y\sqrt{1-\rho^2}}$$

$$\begin{cases} \mu_x \in \mathbb{R} \\ \mu_y \in \mathbb{R} \\ \sigma_x > 0 \\ \sigma_y > 0 \\ \rho \in (-1,1) \end{cases}$$

when $\rho = 0$

$$f(x,y) = \frac{e^{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}}{\left(\sqrt{2\pi}\right)^2\sigma_x\,\sigma_y} = \underline{f_X(x)\cdot f_Y(y)}.$$

where. $X \sim N(\mu_x, \sigma_x^2)$ true in general
$Y \sim N(\mu_y, \sigma_y^2)$

$\rho = 0 \Rightarrow$ independence for Bivariate normal.

Eg: (Height, weight) distribution, (Palm length, Height) distribution.

when $(X,Y) \sim N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$ then the conditional distribution of $Y|X=x$ or (regression of Y on X) is given by

$$Y|X \sim N\left(\mu_y + \rho \frac{\sigma_y}{\sigma_x}(x-\mu_x),\ (1-\rho^2)\sigma_y^2\right).$$

(*) This regression is linear i.e.

$$E(Y|X=x) = \mu_y + \rho \frac{\sigma_y}{\sigma_x}(x-\mu_x)$$

$$= \left(\mu_y - \rho \frac{\sigma_y}{\sigma_x}\mu_x\right) + \rho \frac{\sigma_y}{\sigma_x} x.$$

$$\equiv a + bx.$$ linear

(*) $V(Y|X=x) = (1-\rho^2)\sigma_y^2 < \sigma_y^2 \equiv$ unconditional variance of $\underline{Y}$.

(*) $X|Y \sim N\left(\mu_x + \rho \frac{\sigma_x}{\sigma_y}(y-\mu_y),\ (1-\rho^2)\sigma_x^2\right)$

(*) Even though X, and Y individually follows Normal distribution (X,Y) jointly may NOT follow Bivariate $\underline{\text{normal}}$.

$$g = \left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)$$

$$= \left[\left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{y-\mu_y}{\sigma_y}\right)\left(\frac{x-\mu_x}{\sigma_x}\right) + \rho^2\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right] + (1-\rho^2)\left(\frac{x-\mu_x}{\sigma_x}\right)^2$$

$$f_{XY}(x,y) = \frac{e^{-\frac{1}{2}\frac{1}{(1-\rho^2)}(g)}}{2\pi\,\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

$$= \frac{e^{-\frac{1}{2(1-\rho^2)}\left(\frac{y-\mu_y}{\sigma_y} - \rho\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sqrt{2\pi}\,\sigma_y\sqrt{1-\rho^2}} \cdot \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sqrt{2\pi}\,\sigma_x}$$

$$= \left[\frac{e^{-\frac{1}{2}\frac{1}{\sigma_y^2(1-\rho^2)}\left(y-\mu_y-\rho\frac{\sigma_y}{\sigma_x}(x-\mu_x)\right)^2}}{\sqrt{2\pi}\,\sigma_y\sqrt{1-\rho^2}}\right] \cdot \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sqrt{2\pi}\,\sigma_x}$$

$$f_X(x) = \int_Y f(x,y)\,dy = \frac{e^{-1/2\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sqrt{2\pi}\,\sigma_x}$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-\frac{1}{2}\frac{1}{(1-\rho^2)\sigma_y^2}\left(y-\mu_y-\rho\frac{\sigma_y}{\sigma_x}(x-\mu_x)\right)^2}}{\sqrt{2\pi}\,\sigma_y\sqrt{1-\rho^2}}$$

$C_k = \{(x,y) \mid f(x,y) = k\}$ contour for $f(x,y) = k$.

Transformation of Random Vectors.

Let (X,Y) be continuous valued random vector with pdf $f(x,y)$ then the pdf of $(U,V) = (U(X,Y), V(X,Y))$ can be given by

$$g(u,v) = f\left(x(u,v),\ y(u,v)\right) \left\| \frac{\partial(x,y)}{\partial(u,v)} \right\|$$

where $\frac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian matrix.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$|\ |$ stands for determinant.

$|\ |$ stands for absolute value.

$x(u,v)$ is the value of x in terms of (u,v)

$y(u,v)$ is the value of y in terms of (u,v)

$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$A_1 = \pi . 1^2$

$x^2 + y^2 = 1$

$\longrightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \overset{\rightarrow T}{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$

$u = ax$
$v = by$

$A_2 = \pi a . b$
$= \pi |T|$

Ex 1 $X, Y \overset{iid}{\sim} N(0,1)$ show that $U = X+Y \sim N(0,2)$, $V = X-Y \sim N(0,2)$ > independent.

$$X = \frac{u+v}{2} \qquad J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \Rightarrow |J| = -1/2$$

$$Y = \frac{u-v}{2}$$

$$\|J\| = \text{abs}(|J|) = 1/2.$$

Joint pdf of (X,Y) is

$$f(x,y) = \frac{e^{-1/2(x^2+y^2)}}{2\pi}$$

Joint pdf of (u,v)

$$g(u,v) = f\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \cdot \|J\|$$

$$= \frac{e^{-\frac{1}{2}\left\{\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2\right\}}}{2\pi}(1/2) = \frac{e^{-\frac{1}{2}\left(\frac{u^2}{2}\right)} e^{-\frac{1}{2}\left(\frac{v^2}{2}\right)}}{2(2\pi)}$$

$$= \frac{e^{-1/2\left(\frac{u}{\sqrt{2}}\right)^2}}{\sqrt{2\pi}\sqrt{2}} \cdot \frac{e^{-1/2\left(\frac{v}{\sqrt{2}}\right)^2}}{\sqrt{2\pi}\sqrt{2}}$$

$$\boxed{U, V \overset{iid}{\sim} N(0,2)}$$

Ex 2 $X, Y \overset{iid}{\sim} N(0,1)$ Find the distribution of R and Θ where $X = R\cos\Theta$, $Y = R\sin\Theta$.

$R > 0$

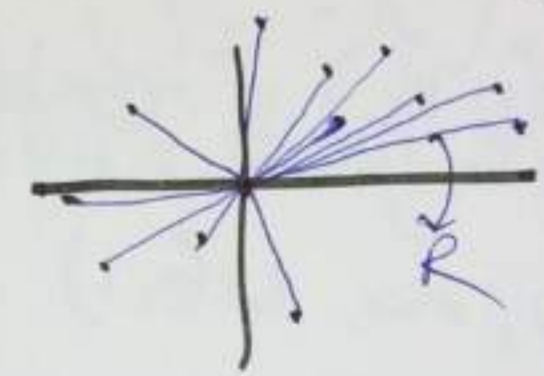
~~$0 < \Theta < 2\pi$.~~

$-\pi < \Theta < \pi$.

$$|J| = R > 0$$

Joint density of (X,Y) is

$$f(x,y) = \frac{e^{-\frac{1}{2}(x^2+y^2)}}{2\pi} =$$



Joint density of (R, Θ)

$$g(r,\theta) = \frac{e^{-\frac{1}{2}\left((r\cos\theta)^2 + (r\sin\theta)^2\right)}}{2\pi} \cdot r \qquad \begin{array}{l} r > 0 \\ -\pi < \theta < \pi \end{array}$$

$$= \left(r e^{-\frac{1}{2}r^2}\right) \cdot \left(\frac{1}{2\pi}\right)$$

$$g_R(r) = \begin{cases} r e^{-\frac{1}{2}r^2} & r > 0 \\ 0 & \text{o.w.} \end{cases}$$

Raylight distribution.

$$g_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & \end{cases}$$

uniform $(-\pi, \pi)$

26

$X \sim N(0,1)$
$Y \sim N(0,1)$ $\Rightarrow$ indepnt.

$X = R\cos\theta$
$Y = R\sin\theta$.

$R > 0$
$\theta \in (-\pi, \pi)$ or $(0, 2\pi)$

$$R^2 = X^2 + Y^2 \sim \chi^2_2 \equiv G(1, 1/2)$$

$$\boxed{\text{If } x_i \overset{iid}{\sim} N(0,1) \quad \sum_{i=1}^{k} x_i^2 \sim \chi^2_k}$$

$$G(1, 1/2) \equiv \exp(1/2)$$

$$\chi^2_k \equiv G\left(\tfrac{k}{2}, 1/2\right)$$
$\downarrow \alpha \quad \downarrow \lambda$

Let $U_1 \sim U(0,1)$ $\quad \boxed{U_1 \overset{d}{=} 1 - U_1}$ same cdf.

$$R^2 = -2\log_e U_1 \sim \exp(1/2)$$

Let $U_2 \sim U(0,1)$

$$\theta = 2\pi U_2 \sim U(0, 2\pi).$$

$(U_1, U_2) \overset{iid}{\sim} U(0,1)$.

$$X = \left(\sqrt{-2\log_e U_1}\right)\cos(2\pi U_2).$$

$$Y = \left(\sqrt{-2\log_e U_1}\right)\sin(2\pi U_2).$$

$(X, Y) \overset{iid}{\sim} N(0,1)$.

Box-Muller method.

HW Let $(X, Y) \overset{iid}{\sim} N(0,1)$.

(1) Find the distribution of $Z = \rho X + \sqrt{1-\rho^2}\, Y$

(2) Find the joint distribution of (X, Z)

$$\boxed{\rho \in (-1, 1)} \qquad \begin{pmatrix} X \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

HW

$$f(x,y) = \frac{\sqrt{3}}{4\pi}\, e^{-(x^2 - xy + y^2)/2}$$

(1) Find the marginal pdf of X and Y.

(2) Find the conditional pdf of $Y|X$ and $X|Y$.

Ex Let $X, Y \overset{iid}{\sim} N(0,1)$ Find the pdf of $U = \frac{X}{Y}$.

$$\begin{pmatrix} X \\ Y \end{pmatrix} \longrightarrow \begin{pmatrix} U \\ V \end{pmatrix} \qquad \begin{matrix} U = \frac{X}{Y} \\ V = Y \end{matrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} uv \\ v \end{pmatrix} \quad \begin{matrix} u \in \mathbb{R} \\ v \in \mathbb{R} \end{matrix}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = J = \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix} \Rightarrow |J| = v \Rightarrow \|J\| = |v|$$

$$\boxed{U = \frac{X}{Y} \sim \text{Cauchy}(0,1)}.$$

Joint density of (X,Y) is

$$f(x,y) = \frac{e^{-\frac{1}{2}(x^2+y^2)}}{2\pi}$$

Joint density of (U,V)

$$g(u,v) = \frac{e^{-\frac{1}{2}(u^2v^2+v^2)}}{2\pi}|v| = \frac{e^{-\frac{1}{2}v^2(1+u^2)}|v|}{2\pi}$$

$$g_U(u) = \int_v g(u,v)\,dv = \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}v^2(1+u^2)}|v|}{2\pi}dv = \int_0^{\infty} \frac{e^{-\frac{1}{2}v^2(1+u^2)}}{\pi}v\,dv$$

$$= \frac{1}{\pi(1+u^2)}$$

1. $X, Y \overset{iid}{\sim} N(0,1)$

(a) $\frac{X}{Y} \sim \text{Cauchy}(0,1)$.

(b) $\frac{X}{|Y|} \sim \text{Cauchy}(0,1)$.

2. $X, Y \overset{iid}{\sim} N(0,\sigma^2)$

(a) $\frac{X}{Y} \sim \text{cauchy}(0,1)$

(b) $\frac{X}{|Y|} \sim \text{Cauchy}(0,1)$.

Moment does not exists for cauchy distribut.

3. $$\frac{X}{|Y|} = \frac{X}{\sqrt{Y^2}} = \frac{X}{\sqrt{Y^2/1}} \sim t_1$$

$X, Y \overset{iid}{\sim} N(0,1)$

t_n distribution will converge to $N(0,1)$ when $n \uparrow \infty$.

$X_1 X_2 \cdots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ then show that.

① $\bar{X} \sim N(\mu, \sigma^2/n)$

(2) $S^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 \sim \sigma^2 \chi^2_{n-1}$

(3) $\bar{X}$ and S^2 are independently distributed.

$\underset{\sim}{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\underset{\sim}{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ $\underset{\sim}{Y} = A\underset{\sim}{X}$

where A is an orthogonal matrix.

$\Rightarrow A^T A = I$

$\Rightarrow |A| = |A^T| = \pm 1$

The first row of A is

$\left(\frac{1}{\sqrt{n}} \; \frac{1}{\sqrt{n}} \cdots \frac{1}{\sqrt{n}}\right)$.

① Note $Y_1 = \left(\frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} \cdots \frac{1}{\sqrt{n}}\right) \underset{\sim}{X} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i = \frac{1}{\sqrt{n}} n\bar{X} = \sqrt{n}\,\bar{X}$

② Note. $\sum_{i=1}^{n} Y_i^2 = \underset{\sim}{Y}^T \underset{\sim}{Y} = (A\underline{x})^T (A\underline{x}) = \underline{x}^T (A^T A)\underline{x} = \underline{x}^T \underline{x} = \sum_{i=1}^{n} x_i^2$

We will find the joint pdf of $\underset{\sim}{Y}$ from the joint pdf of $\underset{\sim}{X}$.

As $x_1 x_2 \cdots x_n$ iid.

$$f(\underset{\sim}{x}) = \prod_{i=1}^{n} f_{X_i}(x_i) = \prod_{i=1}^{n} \frac{e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma} = \frac{e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}}{(\sqrt{2\pi}\,\sigma)^n}$$

$$= \frac{1}{(\sqrt{2\pi}\,\sigma)^n} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n} x_i^2 - 2\mu\sum_{i=1}^{n} x_i + n\mu^2\right]\right\}$$

$$= \frac{1}{(\sqrt{2\pi}\,\sigma)^n} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n} x_i^2 - 2\mu n\bar{x} + n\mu^2\right]\right\}$$

$\text{abs}(|A|)=1$

The joint pdf of $\underset{\sim}{Y}$ is

$$g(\underset{\sim}{y}) = \frac{1}{(\sqrt{2\pi}\,\sigma)^n} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n} y_i^2 - 2\mu\sqrt{n}\,y_1 + n\mu^2\right]\right\} \cdot 1$$

$$= \frac{1}{(\sqrt{2\pi}\,\sigma)^n} \exp\left\{-\frac{1}{2\sigma^2}\left[(y_1-\sqrt{n}\mu)^2 + \sum_{i=2}^{n} y_i^2\right]\right\}$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{y_1-\sqrt{n}\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma} \cdot \prod_{i=2}^{n} \frac{e^{-\frac{1}{2}\left(\frac{y_i}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma} = \prod_{i=1}^{n} f_{Y_i}(y_i)$$

So we have

① $Y_1 \sim N(\sqrt{n}\mu, \sigma^2)$

② $Y_i \overset{iid}{\sim} N(0, \sigma^2)$ for $i = 2, 3, \ldots n$. they are independent to Y_1

(a) Now $Y_1 \sim N(\sqrt{n}\mu, \sigma^2)$

$$\Rightarrow \sqrt{n}\,\bar{x} \sim N(\sqrt{n}\mu, \sigma^2)$$

$$\Rightarrow \bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

proof of (a)

(b)

$$\left(\frac{y_i}{\sigma}\right) \overset{iid}{\sim} N(0,1) \text{ for } i = 2, 3, \ldots n.$$

$$\sum_{i=2}^{n} \frac{y_i^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\Rightarrow \sum_{i=2}^{n} y_i^2 \sim \sigma^2 \chi^2_{n-1}$$

$$\sum_{i=2}^{n} y_i^2 \sim \sigma^2 \chi^2_{n-1}$$

$$\Rightarrow \left(\sum_{i=1}^{n} y_i^2 - y_1^2\right) \sim \sigma^2 \chi^2_{n-1}$$

$$\Rightarrow \left(\sum_{i=1}^{n} x_i^2 - (\sqrt{n}\,\bar{x})^2\right) \sim \sigma^2 \chi^2_{n-1}$$

$$\Rightarrow \sum_{i=1}^{n} (x_i - \bar{x})^2 \sim \sigma^2 \chi^2_{n-1}$$

$$\Rightarrow S^2 \sim \sigma^2 \chi^2_{n-1}$$

proof of (b).

(c) As Y_1 is independent of $Y_2 \cdots Y_n \Rightarrow \bar{X}$ is independent of S^2.

$\bar{X} \sim N(\mu, \sigma^2/n)$

$\Rightarrow \dfrac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$

$S^2 \sim \sigma^2 \chi^2_{n-1}$

$\Rightarrow \dfrac{S^2}{\sigma^2} \sim \underline{\underline{\chi^2_{n-1}}}$

$\bar{X}$, S^2 are indept.

$$T = \frac{\sqrt{n}(\bar{X}-\mu)}{\sqrt{S^2/(n-1)}} \sim t_{n-1}$$

$$\Rightarrow \frac{\sqrt{n}(\bar{X}-\mu)/\sigma}{\sqrt{\left(\frac{S^2}{\sigma^2}\right)/(n-1)}} \sim t_{n-1}.$$

$$T^2 = \frac{(N(0,1))^2}{(\chi^2_{n-1})/(n-1)}$$

$$= \frac{\chi^2_1/1}{\chi^2_{n-1}/(n-1)}$$

$$\sim \underline{\underline{F_{1,n-1}}}.$$

Law of Large numbers. (LLN)

Convergence in probability.

A sequence of random variables $\{X_n\}$ for $n \in \mathbb{N}$ on a probability space $(\Omega, \mathcal{A}, P)$ is said to converge to a random variable X in $(\Omega, \mathcal{A}, P)$ if.

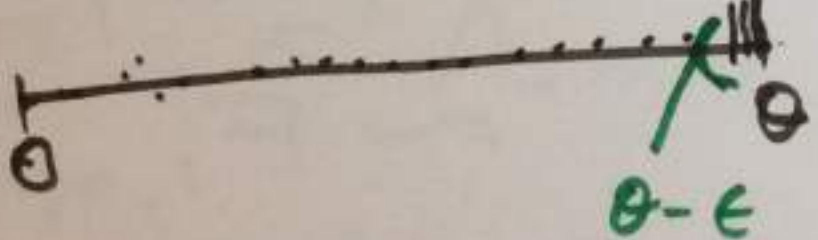
$$\lim_{n\to\infty} P\left(\{\omega \mid |X_n(\omega) - X(\omega)| < \epsilon\}\right) = 1 \qquad \forall \epsilon > 0$$

Ex $X_1, X_2, \dots, X_n$ iid $\sim U(0, \theta)$ $\theta > 0$.

$X_{(n)}$ is the maximum value of $\{X_1, X_2 \dots X_n\}$

$X_{(n)}$ converges in probability to θ which is constant.

$$\lim_{n\to\infty} P\left(|X_{(n)} - \theta| < \epsilon\right) = 1$$



In estimation this idea is known as consistency.

Weak law of large numbers. (WLLN).

$X_1, X_2 \cdots\cdots X_n \overset{iid}{\sim} E(X_i) = \mu, \quad V(X_i) = \sigma^2 < \infty.$

define sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

Then we have. $\lim_{n\uparrow\infty} P\left(|\bar{X} - \mu| < \epsilon\right) = 1 \quad \forall \epsilon > 0.$

" Sample mean converges to population for iid random variable with finite variance in large sample "

$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{n\mu}{n} = \mu.$

$V(\bar{X}) = \frac{\sigma^2}{n} = \frac{1}{n^2}\sum_{i=1}^{n} V(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n} \to 0 \text{ as } n\uparrow\infty$

By Chebyshev's inequality.

$P(|\bar{X} - \mu| > \epsilon) \leq \frac{E(\bar{X}-\mu)^2}{\epsilon^2} = \frac{V(\bar{X})}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \downarrow 0$ as

for large n.

$\Rightarrow \lim_{n\uparrow\infty} P\left(|\bar{X} - \mu| \leq \epsilon\right) = 1 \quad \forall \epsilon > 0$

Convergence in distribution

A sequence of random variables $\{X_n\}$ on prob. space $(\Omega_n, \mathcal{F}_n, P_n)$ is said to converge in distribution to another random variable Y on $(\Omega, \mathcal{F}, P)$ if.

$$\lim_{n \uparrow \infty} F_{X_n}(a) = F_Y(a)$$

(cdf overlaps in limiting sense).

for all 'a' such that $F_Y(a)$ is continuous

If X_n has MGF $M_{X_n}(t)$ which converges to $M_Y(t)$.
then we say '$\{X_n\}$ converges in distribution to Y'

NOTATION : $X_n \xrightarrow{d} Y$ or $X_n \xrightarrow{\mathcal{L}} Y$

(Continuity theorem)

Ex1 $X_n \sim \text{bin}(n, p_n)$ $n \to \infty$ $p_n \to 0$ $np_n \to \lambda$

$X_n \xrightarrow{d} \text{pois}(\lambda)$

Central limit theorem: (CLT).

Let $x_1, x_2 \dots x_n$ be iid random variables with $E(x_i) = \mu$ and $V(x_i) = \sigma^2 < \infty$ then define.

$$S_n = \sum_{i=1}^{n} x_i = n\bar{x} \text{ and}$$

$$T_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\sqrt{n}(\bar{x} - \mu)}{\sigma} = \frac{(S_n) - E(S_n)}{\sqrt{V(S_n)}}$$

$$= \frac{\bar{x} - E(\bar{x})}{\sqrt{V(\bar{x})}}$$

$$\lim_{n \uparrow \infty} P(T_n \le t) = \Phi(t) = \int_{-\infty}^{t} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx$$

$$\Leftrightarrow \quad T_n \xrightarrow{d} N(0,1). \quad \text{as } n \uparrow \infty.$$

$x_1, x_2 \cdots x_n \overset{iid}{\sim} E(x_i) = \mu, \; V(x_i) = \sigma^2$

$$T_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma} = \frac{\sum_{i=1}^{n} x_i - n\mu}{\sqrt{n}\,\sigma} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i$$

Let

$Y_i = \frac{x_i - \mu}{\sigma}$

$E(Y_i) = 0$

$V(Y_i) = 1$

$Y_i \sim$ iid.

$E\left(e^{tY_i}\right) = 1$

$M_{Y_i}(\;)$ is the MGF of Y_i

$M^{(0)}_{Y_i}(0) = 1$

$M^{(1)}_{Y_i}(0) = 0$

$M^{(2)}_{Y_i}(0) =$

Note: $E(T_n) = 0$

$V(T_n) = 1$

MGF of T_n is

$$E\left(e^{tT_n}\right) = E\left(e^{\left(t/\sqrt{n} \sum_{i=1}^{n} Y_i\right)}\right) = \prod_{i=1}^{n} M_{Y_i}\left(t/\sqrt{n}\right)$$

$$= \left(M_{Y_1}\left(t/\sqrt{n}\right)\right)^n = \left(\sum_{k=0}^{\infty} M^{(k)}_{Y_1}(0) \frac{\left(t/\sqrt{n}\right)^k}{k!}\right)^n$$

$$E\left(e^{tT_n}\right) = \left(\sum_{k=0}^{\infty} M_{Y_1}^{(k)}(0)\frac{(t/\sqrt{n})^k}{k!}\right)^n$$

$M_{Y_1}^{(0)}(0) = 1$
$M_{Y_1}^{(1)}(0) = 0$
$M_{Y_1}^{(2)}(0) = 1$

$$= \left(1 + 0 + \frac{t^2}{2n} + R\left(t/\sqrt{n}\right)\right)^n.$$

where $R\left(t/\sqrt{n}\right) = \sum_{k=3}^{\infty} M_{Y_1}^{(k)}(0)\frac{(t/\sqrt{n})^k}{k!}$

$$\lim_{n\to\infty} E\left(e^{tT_n}\right)$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\left\{\frac{t^2}{2}+n\left[R\left(t/\sqrt{n}\right)\right]\right\}\right)^n$$

$$= e^{t^2/2} \equiv \text{MGF of } N(0,1)$$

$\lim_{n\to\infty} \frac{R(t/\sqrt{n})}{(t/\sqrt{n})} \to 0$

$\lim_{n\to\infty} \frac{R(t/\sqrt{n})}{(t/\sqrt{n})^2} \to 0.$

$R\left(0/\sqrt{n}\right) = 0.$

$T_n \xrightarrow{d} N(0,1)$

$\lim_{n\to\infty}\left(1+\frac{a_n}{n}\right)^n \to e^a$ when $a_n \to a$.

Ex $\displaystyle\lim_{n\uparrow\infty} e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!}$

poisson $(\lambda = n)$.

→ Mean, Var, λ

→ lim. of bin( ).

→ additive propert.

$X_i \overset{iid}{\sim}$ pois(1)

$S_n = \sum_{i=1}^{n} X_i \sim$ pois(n).

$E(S_n) = n \quad Var(S_n) = n$

$$\lim_{n\uparrow\infty} P\left(\frac{S_n - n}{\sqrt{n}} \le 0\right) = \int_{-\infty}^{0} \frac{e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}}\,dt. \text{ by CLT.}$$

$$\Rightarrow \lim_{n\uparrow\infty} P(S_n - n \le 0) = 1/2$$

$$\lim P(S_n \le n) = 1/2$$

$$\Rightarrow \lim_{n\uparrow\infty} e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!} = 1/2$$

$\Rightarrow \lim_{n\uparrow\infty}$

Ex Let $X_1 \sim G(\alpha_1, \lambda)$ and $X_2 \sim G(\alpha_2, \lambda)$ be independently distributed. Find the distribution of $\frac{X_1}{X_1+X_2}$ and X_1+X_2.

$$Y_1 = \frac{X_1}{X_1+X_2} \in (0,1) \qquad X_1 = Y_1 Y_2$$

$$Y_2 = X_1 + X_2 \in (0,\infty) \qquad X_2 = Y_2(1-Y_1)$$

$$\frac{\partial(X_1,X_2)}{\partial(Y_1,Y_2)} = \begin{bmatrix} Y_2 & Y_1 \\ -Y_2 & (1-Y_1) \end{bmatrix} = Y_2 - Y_1Y_2 + Y_1Y_2 = Y_2 \underline{>0}$$

Joint density of $X_1 X_2$ is

$$f(x_1,x_2) = \frac{\lambda^{\alpha_1} e^{-\lambda x_1} x_1^{\alpha_1-1}}{\Gamma\alpha_1} \cdot \frac{\lambda^{\alpha_2} e^{-\lambda x_2} x_2^{\alpha_2-1}}{\Gamma\alpha_2}$$

$$= \frac{\lambda^{\alpha_1+\alpha_2} e^{-\lambda(x_1+x_2)} x_1^{\alpha_1-1} x_2^{\alpha_2-1}}{\Gamma\alpha_1 \Gamma\alpha_2}$$

Joint density of Y_1, Y_2 is

$$g(y_1, y_2) = \frac{\lambda^{\alpha_1+\alpha_2} e^{-\lambda y_2} (y_1 y_2)^{\alpha_1 - 1} (y_2(1-y_1))^{\alpha_2 - 1} y_2}{\cancel{\Gamma\alpha_1}\, \Gamma\alpha_1\, \Gamma\alpha_2}$$

$$= \frac{\lambda^{\alpha_1+\alpha_2} e^{-\lambda y_2} y_2^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \cdot \frac{y_1^{\alpha_1-1} (1-y_1)^{\alpha_2-1}}{\dfrac{\Gamma\alpha_1\, \Gamma\alpha_2}{\Gamma(\alpha_1+\alpha_2)}}$$

$$= \frac{\lambda^{\alpha_1+\alpha_2} e^{-\lambda y_2} y_2^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \cdot \frac{y_1^{\alpha_1-1} (1-y_1)^{\alpha_2-1}}{B(\alpha_1, \alpha_2)}.$$

$$\left.\begin{array}{l} Y_2 \sim G(\alpha_1+\alpha_2, \lambda) \\ Y_1 \sim B(\alpha_1, \alpha_2) \end{array}\right\} \ \underline{\text{independent}}.$$