- What actually we're doing?

LPP

lecture 4 pg 44

Max $Z = CX$

s.t. $AX = b$, $X \geq 0$

$m < n$

Rank $A = m$

$C = (c_1 \quad c_2 \quad \dots \quad c_n)$

$X = (x_1 \quad x_2 \quad \dots \quad x_n)$

$A = (a_1 \quad a_2 \quad \dots \quad a_n)$

$a_i \Rightarrow$ ith element of A.

$m \times n$

$b = [b_1 \quad b_2 \quad \dots \quad b_m]$

$B = (\beta_1 \quad \beta_2 \quad \dots \quad \beta_m)$

$A = [B \mid R]$

$x = [x_B \mid x_R]$

$x_B \to$ Basic varible

$x_R \to$ non basic variables

$Ax = b \Rightarrow Bx_B + Rx_R = b$

set $x_R = 0$

$\to Bx_B = b$

$x_B = B^{-1} b$

			c_j	c_1	c_2	...	c_j	...	c_n
C_B	B	X_B	b						
C_{B1}	β_1	X_{B1}	b_1	Y_1	Y_2	...	Y_j	...	Y_n
C_{B2}	β_2	X_{B2}	b_2						
$\vdots$	$\vdots$	$\vdots$	$\vdots$						
C_{Bm}	β_m	X_{Bm}	b_m						

$$Z = CX$$
$$= C_B X_B + C_R X_R$$

$$Y_1 = \begin{pmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{m1} \end{pmatrix} \qquad Y_2 = \begin{pmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{m2} \end{pmatrix}$$

$X_B = B^{-1} b$ [Initially $B = I_m$

$z_j - c_j = C_B Y_j - c_j$, $z = C_B X_B$

$Y_j = B^{-1} a_j = a_j$

One more problem where we can have infinitely many soln :

• Example [infinitely many ~~vari~~ Soln]

Max $z = 4x_1 + 14x_2$

s.t. $2x_1 + 7x_2 \leq 21$

$7x_1 + 2x_2 \leq 21$

$x_1, x_2 \geq 0$

Std form: after introducing slack variables

Max $z = 4x_1 + 14x_2 + 0 \cdot x_3 + 0 \cdot x_4$

s.t. $2x_1 + 7x_2 + x_3 = 21$

$7x_1 + 2x_2 + x_4 = 21$

$x_1, x_2, x_3, x_4 \geq 0$

			c_j	4	14	0	0
C_B	B	X_B	b				

14	a_2	x_2	3	2/7	1	1/7	0
0	a_4	x_4	15	45/7	0	-2/7	1
		$z_j - c_j$		0	0	2	0

Optimality reached ✓ $\Rightarrow$ $z_{max} = \cdot \cdot \cdot$

another round

14	a_2	x_2	7/3	0
4	a_1	x_1	7/3	1
				0 0 2 0

find

$$\overline{x_1} = \lambda \cdot 0 + (1-\lambda)\frac{7}{3}$$

$$\overline{x_2} = \lambda \cdot 3 + (1-\lambda)\frac{7}{3}$$

$$0 < \lambda < 1$$

all values are ~~non~~ positive $\Rightarrow$ optimality reached ✓

But since the $z_j - c_j = 0$ for a basic vari[able] $\Rightarrow$ Existence of infinitely many soln

slack and artificial variables: basic
surplus: not basic

artificial added in = and >= cases

22.08.23

Penalty or big M Method

Example: Max $z = -2x_1 + x_2 + 3x_3$

s.t. $x_1 - 2x_2 + 3x_3 = 2$

$3x_1 + 2x_2 + 4x_3 = 1$

$x_1, x_2, x_3 \geq 0$

} Equality ✓ No identity matrix.

Max $Z = -2x_1 + x_2 + 3x_3 - M.x_4 - Mx_5$

s.t. $x_1 - 2x_2 + 3x_3 + x_4 = 2$

$3x_1 + 2x_2 + 4x_3 + x_5 = 1$

$x_1, x_2 \ldots x_5 \geq 0$

M → very large no.

		c_j		-2	1	3	-M	-M	Min ratio
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	
-M	a_1	x_4	2	1	-2	3	1	0	$2/3 = \frac{8}{12}$
-M	a_5	x_5	1	3	2	4	0	1	$1/4 = \frac{3}{12}$ →
		$z_j - c_j$		-4M+2	-1	-7M-3 ↑	0	0	
-M	a_4	x_4	5/4	-5/4	-7/2	0	1	-3/4	
3	a_3	x_3	1/4	3/4	2/4	1	0		
		$z_j - c_j$		5M/4 + 17/4	7M/2 + 2/2	0	0		

artificial variable occurs in the basis ⇒ No soln

Optimality is reached $z_j - c_j \geq 0 \ \forall j$

but the artificial variable x_4 remain in the basis and at the positive level, Here $x_4 = 5/4$.

Hence, no feasible soln.

Unbounded soln - case Example below :-

Example: Max $Z = 2x_1 + 3x_2 + x_3$

s.t. $-3x_1 + 2x_2 + 3x_3 = 8$

$-3x_1 + 4x_2 + 2x_3 = 7$

$x_1, x_2, x_3 \geq 0$

Std:

Max $Z = 2x_1 + 3x_2 + x_3 - Mx_4 - Mx_5$

s.t. $-3x_1 + 2x_2 + 3x_3 + x_4 = 8$

$-3x_1 + 4x_2 + 2x_3 + x_5 = 7$

$x_1, x_2, x_3, x_4, x_5 \geq 0$ artificial variables.

$M \rightarrow$ a large positive no.

			c_j	2	3	1	$-M$	$-M$	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	
$-M$	a_4	x_4	8	-3	2	3	1	0	
$-M$	a_5	x_5	7	-3	$\boxed{4}$	2	0	1	
			$z_j - c_j$	$6M-2$	$-6M-3$ ↑				
$-M$	a_4	x_4	$9/2$	$-3/2$	0	2	1		$\frac{9}{4}$ →
3	a_2	x_2	$7/4$	$-3/4$	1	$1/2$	0		$\frac{14}{4}$
			$z_j - c_j$	$3M/2 - \frac{17}{4}$	0	$-2M + \frac{1}{2}$ ↑	0		

1	a_3	x_3	$9/4$	$-3/4$	0	1
3	a_2	x_2	$5/8$	$-3/8$	1	0
		$z_j - c_j$		$-3/8$ ↑	0	0

$z_j - c_j < 0$ for a non-basic variable,

a_1 is the entering ~~vector~~ ~~variable~~ vector.

but cannot find the departing vector

as y_{11}, y_{21} both < 0

<u>Hence the soln is unbounded.</u>

~~Std max prob~~ ~~(Penalty or Big M method~~

If the optimality criterion is satisfied at any iteration and

(a) if the basis does not contain any artificial variables, then the constraint eqn are all consistent and the soln obtained is an optimal basic feasible soln of the problem

(b) if the basis contains one or more artificial variables, at zero level, then the solⁿ obtained is an optimal solⁿ and the constraints eqⁿ are consistent although there may be redundancy for one or more of them.

(c) If the basis contains one or more artificial variables at positive level, then the original problem will have no feasible soln.

(d) When min ratio cannot found, i.e. the departing variable cannot be determined, the solⁿ will be unbounded.

(e) Optimality condition is reached but $z_j - c_j$ corresponding to a non-basic variable is zero and at least one $y_{ij} > 0$, we get alternative basic optimal soln (infinitely many soln)

(f) opt. condition reached i.e. $z_j - c_j \geq 0 \ \forall j$ but $z_j - c_j = 0$ f.s. non-basic variable

and $y_{ij} < 0 \quad \forall\ i = 1, 2, \ldots m$
then we get an alternative non-basic optimal soln.

- <u>Degeneracy occurs</u> when there is a <u>tie</u> in determining min ratio, may yield <u>cycling problem</u>.

If $A_{m \times n}$

But Rank $A < m$

Degeneracy occurs

- <u>Basis</u>

$B_{m \times n} = (\beta_1, \beta_2 \ldots \beta_m) = I_m$
invertible

$a = y_1 \beta_1 + y_2 \beta_2 + \ldots y_m \beta_m$

$= B\ Y$

where $Y = (y_1\ y_2 \ldots)$

$Y = B^{-1} a$

Basis keep on changing using Replacement

$y_j = B^{-1} a_j = I^{-1} a_j = a_j$

$$B_a = (\beta_1 \quad \beta_2 \dots \beta_{r-1}, a, \beta_{r+1}, \beta_m)$$

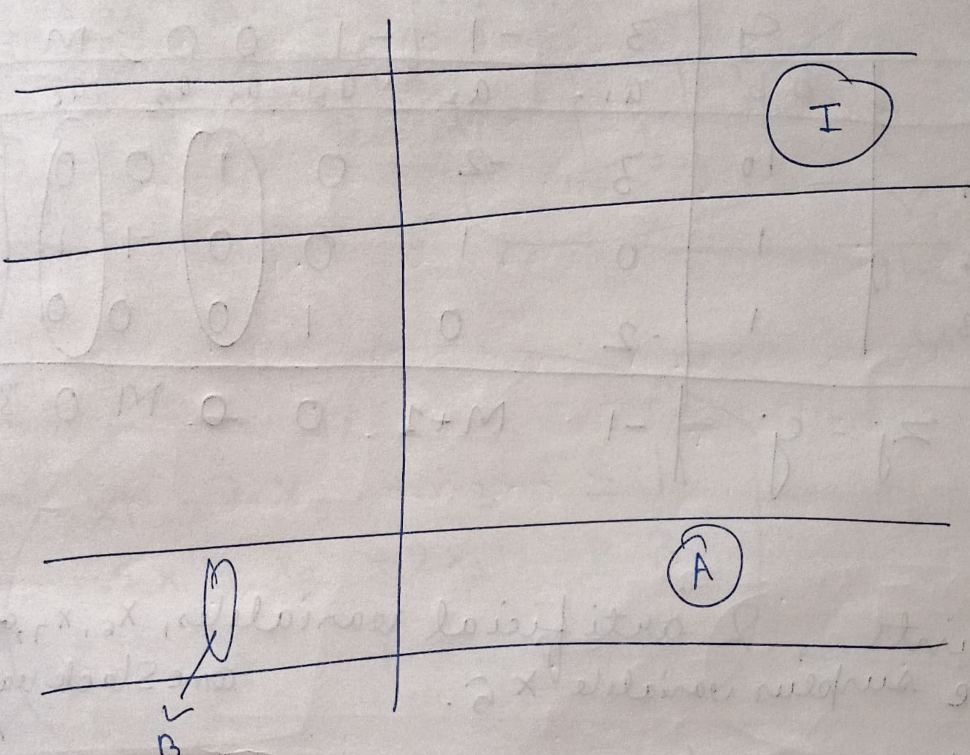
$$A = (B \mid R)$$

$$Ax = b \Rightarrow (B \mid R)\left(\frac{x_B}{x_R}\right) = b$$

$$\Rightarrow Bx_B = b$$

$$x_B = B^{-1} b$$

Terminology to understand the comming problem



← first round

Last round (final Table)

$$\boxed{B = A^{-1}}$$

Example: The given table represents a certain stage in the soln of a LPP by the simplex method.

Here M is a large +ve no. and the problem is a maximization problem.

State the original problem.

		c_j	3	-1	-1	0	0	-M	-M
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7
0	a_4	10	3	-2	0	1	0	0	-1
-M	a_6	1	0	1	0	0	-1	1	-2
-1	a_3	1	-2	0	1	0	0	0	1
		$z_j - c_j$	-1	M+1	0	0	M	0	3M-1

slack (over a_4), artificial (over a_6, a_7)

3 constraints, 2 artificial variables, x_6, x_7, one surplus variable x_5. One slack variable x_4,

(i) Eqn $+x_4 \le$

(ii) Eqn $+x_5 \ge 0$ $+x_6$

(iii) Eqn $+x_7 =$

$B = (a_4, a_6, a_3)$

So, the columns
$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$
← inverse of this $= B$.

$x_B = B^{-1} b \Rightarrow b = B x_B = \begin{pmatrix} 11 \\ 3 \\ 1 \end{pmatrix}$

$y_j = B^{-1} a_j \quad j = 1 \text{ to } 7$.

$b = B x_B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \\ 1 \end{bmatrix}$

$y_j = B^{-1} a_j, \quad j = 1 \text{ to } 7$

$a_j = B y_j \Rightarrow \underbrace{a_1 = \begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix}, \; a_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \; a_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{\text{The initial constraints.}}$

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &\le 11 \\
-4x_1 + x_2 + 2x_3 &\ge 3 \\
-2x_1 \quad \cancel{0} \quad + x_3 &= 1
\end{aligned}
$$

$\max Z = 3x_1 - x_2 - x_3$.

· Product form of the inverse of a matrix

Basis : $B = (\beta_1, \dots, \beta_{r-1}, \beta_r . \beta_{r+1} \dots \beta_m)$

$$\beta_i \in E^m$$

$a \in E^m$ s.t.

$$a = y_1\beta_1 + \dots y_m\beta_m \quad \text{——— (1)}$$

Let $B_a = (\beta_1 \dots \beta_{r-1}, a, \beta_{r+1} \dots \beta_m)$

$\hookrightarrow$ invertible, provided $y_r \neq 0$

$$\beta_r \cancel{y_r}\,\cancel{\beta_r} = -\frac{y_1}{\beta_r}\beta_1 - \dots \frac{y_{r-1}}{\beta_r}.\beta_{r-1} + \frac{a}{\beta_r} - \frac{y_{r+1}}{\beta_r}\beta_r \dots$$

$$\beta_r = B_a . \eta$$

$$\boxed{\eta = \left(\frac{-y_1}{y_r}, \dots -\frac{y_{r-1}}{y_r}, \dots \frac{1}{y_r}, \dots -\frac{y_{r-1}}{y_r}, \dots -\frac{y_m}{y_r}\right)} \quad \text{(II)}$$

$$B_a = (\beta_1 \dots \beta_{r-1}, a, \beta_{r+1}, \dots \beta_m)$$

$$\beta_1 = B_a . e_1$$

$$\beta_j = B_a . e_j \qquad j = 1, \dots r-1, r+1, \dots m$$

Hence $\beta = B_a E$ when $\boxed{E = (e_1 \ldots e_{r-1}, \eta\ e_{r+1} \ldots e_m)}$ (III)

$\Rightarrow$ $B_a^{-1} = E B^{-1}$

(1) $\Rightarrow$ $a = \sum_1^m B_i y_i = By$

$y = [y_1 \ldots y_r \ldots y_m]$

$y = B^{-1} a$.

- Start with I_m (usually the case) & $B_1^{-1}, B_2^{-1} \ldots B_i^{-1}$ represent the successive inverse upto i-th iteration and $E_1, E_2 \ldots E_i$ are the associated matrices as defined in (3). then

$B_1^{-1} = E_1 I^{-1}$

$B_2^{-1} = E_2 . B_1^{-1}$

$\vdots$

$B_i^{-1} = E_i B_{i-1}^{-1} = E_i E_{i-1} \ldots E_1$

Illustration

$\beta = (\beta_1\ \beta_2 \cdots \beta_3)$

with $B^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 5 & 1 & 3 \\ 0 & 2 & 4 \end{pmatrix}$ replace by

$B_a = (\beta_1, a, \beta_3)$ $\quad a = (1, 5, 6)$

$x = ?$ $\quad y = ?$ $\quad E = ?$ $\quad \cancel{B_a = ?}$

$B_a^{-1} = E B^{-1}$

28.08.23

Degeneracy

Max $Z = CX$

$AX = b$

$X \geq 0$

Rank of $A = m$

$A_{m \times n}$

Basic variable are zero

But if non basic variables are zero

$\Rightarrow$ degeneracy

Ex: 1 Solve the following LPP.

Max $Z = 2x_1 + 3x_2 + 10x_3$

s.t. $x_1 + 2x_3 = 0$

$x_2 + x_3 = 1$

$x_1, x_2, x_3 \geq 0$

Soln:

$x_1 = 0,\ x_2 = 1,\ x_3 = 0$

$Z_{max} = 3.$

			C_j	2	3	10	
C_B	B	X_B	b	a_1	a_2	a_3	Min Ratio
2	a_1	x_1	0	1	0	$\boxed{2}$	$2/2 = 1 \rightarrow$
3	a_2	x_2	1	0	1	1	$3/1 = 3$
	$Z_j - C_j$			0	0	-3 $\uparrow$	

10	a_3	x_3	0	$1/2$	0	1
3	a_2	x_2	1	$-1/2$	1	0
	$z_j - c_j$			$3/2$	0	0

optimality reached.

$x_1 = 0, \quad x_2 = 1, \quad x_3 = 0$

$Z_{max} = 3$. — no change in optimal value.

Theorem:
Whenever there is a tie in selecting the departing variable, the next soln is bound to be degenerate.

Example 2: Solve the following LPP.

Std points

Max $Z = 3x_1 + 9x_2 + 0 \cdot x_3 = 0 \cdot x_4$

$x_1 + 4x_2 \leq 8$
$x_1 + 2x_2 \leq 4$
$x_1, x_2 \geq 0$

s.t. $x_1 + 4x_2 + x_3 = 8$
$x_1 + 2x_2 \quad + x_4 = 4$ ← introducing artificial variable.
$x_1, x_2, x_3, x_4 \geq 0$

		c_j		3	9	0	0	Min
C_B	B	x_B	b	a_1	a_2	a_3	a_4	Ratio
0	a_3	x_3	8	1	$\boxed{4}$	1	0	$8/4 = 2$ → tie
0	a_4	x_4	4	1	2	0	1	$4/2 = 2$
	$z_j - c_j$			-3	-9 ↑	0	0	

			c_j				
9	a_2	x_2	2	1/4	1	1/4	0
0	a_4	x_4	0		0		1

Sometimes cycling problem occurs :

			c_j							
CB	B	x_B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7
0	a_5	x_5	0	1/4	-8	-1	9	1	0	0
0	a_6	x_6	0	1/2	-12	1/2	3	0	1	0
0	a_7	x_7	1	0	0	1	0	0	0	1
	$z_j - c_j$									

Charnes Perturbed technique (To avoid cycli

(applied on previous example) to previous

			c_j	3	9	0	0	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	
0	a_3	x_3	8	1	$\boxed{4}$	1	0	2 → tie
0	a_4	x_4	4	1	2	0	1	2
	$z_j - c_j$			-3	-9	0	0	

x_1' $a_1' \to a_3$

x_2' $a_2' \to a_4$

x_3' $a_3' \to a_1$

x_4' $a_4' \to a_2$

C_B	B	x_B	b \ c_j	0 a_1'	0 a_2'	3 a_3'	9 a_4'	
0	a_1'	x_1'	8	1	0	1	4	$8/4$
0	a_2'	x_2'	4	0	1	1	2	$4/2$ →
	$z_j - c_j$			0	0	-3	-9 ↑	
0	a_1'	x_1'	0	1	-2	-1	0	
9	a_4'	x_4'	2	0	$1/2$	$1/2$	1	
	$z_j - c_j$			0	$9/2$	$3/2$	0	

optimality reached

- if tie found, find min ratio by $\frac{a_2'}{a_k'}$

  if tie found again,

  find minimum $\frac{a_l'}{a_k'}$, x_k is entering variable.

  $l = 3, 4, \dots$

  $\frac{a_l'}{a_k'}$ $l = 1, 2 \dots$

Use the Charnes Perturbed Method to find the soln of Ex2.

<u>Answer</u>:

<u>opt soln</u>

$x_5 = 3/4$, $x_1 = 1$

$x_2 = 0 = x_4$, $x_3 = 1$.

Optimality Condition

Theorem:

$$\text{Max } Z = Cx$$
$$\text{s.t. } Ax = b, \quad x \geq 0$$

If for a BFS, x_B of LPP ①, we have $z_j - c_j \geq 0$ for every column a_j of A, then x_B is an optimal soln.

(Rank $A = n$)

Proof: Let $B = (\beta_1, \beta_2 \ldots \beta_m)$ be the basis corresponding to BFS x_B.

$$Ax = b \Rightarrow (B|R)\frac{(x_B)}{(x_R)} = b \quad \underline{>}_0$$

$$\Rightarrow B x_B = b$$

Let Z_B be the value of the objective func

$$\boxed{Z_B = C_B x_B}$$

$$x = [x_1 \; x_2 \ldots x_n]$$

$$x_B = (B^{-1}A)x$$
$$= Yx$$

when $B^{-1}A = Y$

$$= [y_1 \; y_2 \ldots y_n]$$

$$= \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & \cdots & \cdots & y_{2n} \\ \vdots & & & \vdots \\ y_{m1} & \cdots & \cdots & y_{mn} \end{bmatrix}$$

$x_B = [x_{B1}, x_{B2} \dots x_{Bm}]$

$$x_{Bi} = \sum_{j=1}^{n} y_{ij} x_j$$

$$z_j - c_j \geq 0$$

i.e. $z_j \geq c_j$

$$\sum_1^n z_j x_j \geq \sum_1^n c_j x_j = Z \qquad (z_j = c_B y_j)$$

$$\sum (c_B y_j) x_j \geq Z$$

$$\sum_{j=1}^{n} x_j \sum_1^m c_{Bi} y_{ij} \geq Z$$

$$\cancel{\sum_{i=1}^{n} z_j x_j \geq \sum_1^n c_j x_j = Z}$$

$$\sum c_{Bi} \sum_{j=1}^{n} x_j y_{ij} \geq Z$$

$$\Rightarrow \sum c_{Bi} x_{Bi} \geq Z$$

$$Z_B \geq Z$$