Lesson 1

Introduction

Ordinary Differential Equation

An ordinary differential equation (ODE) is an equation stating a relationship between a function of a single independent variable and the derivatives of this functions with respect to the independent variable. For example:

$$\psi(t, y, y' \ldots, y^m)$$

The order of an ODE is the order of the highest order derivative in the differential equation.

If no product of the dependent variable $y(t)$ with itself or any of its derivatives occurs, then the equation is called linear, otherwise it is non-linear.
Examples are

$y'' + y = 0$	linear
$y' + y^2 = 0$	nonlinear
$y' + t^2 y = 0$	linear
$y'' + \sin(y) = 0$	nonlinear

The general first order ODE is of the form

$$\frac{dy}{dt} = f(t, y)$$

A general solution of an ODE of order m contains m arbitrary constants that can be determined by prescribing m conditions. There are two different classes of of ODE, depending on the type of auxiliary conditions specified.

Initial and Boundary Value Problem (IVP & BVP)

If all the auxiliary conditions are specified at the same value of the independent variable and the solution is to be marched forward from that initial point, the differential equation together with the specified condition is called an IVP.

If the auxiliary conditions are specified at two different values of the independent variable, the end point or at the boundaries of the domain of interest, the differential equation is called boundary value problem. For example:

$$y'' + P(t,y)y' + Q(t,y)y = F(t) \quad y(t_0) = c_1 \text{ and } y'(t_0) = c_2 \qquad (IVP)$$

$$y'' + P(t,y)y' + Q(t,y)y = F(t) \quad y(t_1) = d_1 \text{ and } y(t_2) = d_1 \qquad (BVP)$$

Reduction of higher order equations to the system of first order differential equations

Suppose that an n-th order equation can be solved for the n-th derivative, i.e., it can be written in the form:

$$y^{(n)} = f(t, y', y'', \ldots, y^{n-1})$$

This equation can now be written in a system of first order differential equations by a standard change of variables:

$$\begin{aligned} y_1 &= y \\ y_2 &= y' \\ y_3 &= y'' \\ &\vdots \\ y_n &= y^{n-1}. \end{aligned}$$

Then, the resulting first-order system is the following:

$$
\begin{aligned}
y_1' &= y' = y_2 \\
y_2' &= y'' = y_3 \\
y_3' &= y''' = y_4 \\
&\vdots \\
y_n' &= y^n = f(t, y_2, y_3, \ldots, y_n).
\end{aligned}
$$

In vector form this can simply be written as

$$\mathbf{y}' = \mathbf{f}(\mathbf{t}, \mathbf{y})$$

where $\mathbf{y} = [y_1, y_2, \ldots, y_n]^T$ and $\mathbf{f} = [y_2, y_3, \ldots y_n, f]^T$.

Let us assume that the initial values for the nth order problem are given as

$$y(t_0) = y_0, \;\; y'(t_0) = y_1, \ldots, y^{n-1}(t_0) = y_{n-1}$$

Clearly, it follows

$$\mathbf{y}(\mathbf{t_0}) = [y_0, y_1, \ldots, y_{n-1}]^T.$$

Example 1.1 *Convert the second order IVP into a system of first order IVP*

$$2y'' - 5y' + y = 0$$

$$y(0) = 6; \quad y'(0) = -1;$$

Sol: Let

$$y_1 = y \quad y_2 = y'.$$

It follows then

$$
\begin{aligned}
y_1' &= y_2 \\
y_2' &= -\frac{1}{2}y_1 + \frac{5}{2}y_2
\end{aligned}
$$

and

$$y_1(0) = 6; \qquad y_2(0) = -1$$

Remark 1.2 *The methods of solution of first order initial value problem may be used to solve the system of first order initial value problems and the* n*th order initial value problem.*

Suggested Readings

A. Quarteroni, R. Sacco, F. Saleri (2007). Numerical Mathematics. Second Edition. Springer Berlin.

M.K. Jain, S.R.K. Iyengar, R.K. Jain (2009). Numerical Mathematics. Fifth Edition. New age international publishers, New Delhi.

Lesson 2

Numerical Solutions of IVP

Let us consider the following IVP and discuss its existence and uniqueness of solution:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0 \quad t \in [t_0, b] \tag{2.1}$$

Existence and Uniqueness of the Solution of IVP

The IVP (2.1) admits a unique solution $y(t)$ if $f(t, y)$ is uniformly Lipschitz continuous, that is,

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2| \text{ for any } t \in [t_0, b] \text{ and any } y_1 \text{ and } y_2.$$

Here L is called Lipschitz constant.

Finding exact solution of a practical problem is hardly possible, therefore numerical solutions are required.

Numerical Solutions of IVP

The numerical methods produce approximate values of $y(t)$ at certain points along the t coordinate called grids or mesh points. In case of equally spaced grid points we have

$$t_{i+1} = t_i + h, \;\; i = 0, 1, \ldots, N-1; \quad t_N = b$$

where h is called the step size.

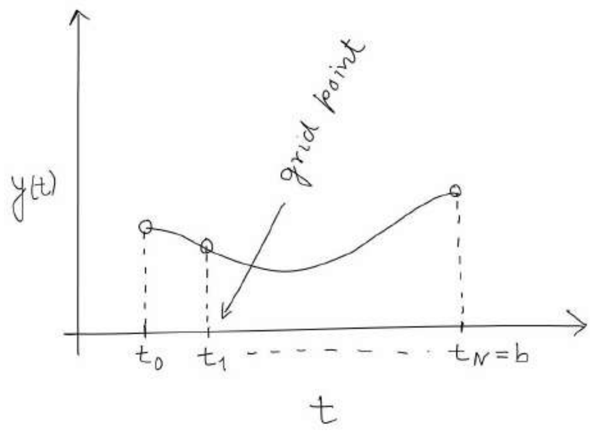


Figure 2.1: Grid points

We shall use the following notation for the the approximation of solution of IVPs

$$u_n \approx y(t_n) =: y_n.$$

Single or Multi Step Methods: If the method advances the solution from one grid point to the next grid point using only data at the single grid point, that is, u_{n+1} depends only on u_n, it is called one-step or single step method otherwise it is called multistep method.

Explicit and Implicit Methods: A method is called *explicit* method if u_{n+1} can be computed directly in terms of the previous values u_k, $k \leq n$, *implicit* if u_{n+1} depends implicitly on itself through f.

Single Step Method

A single step method can be written as

$$u_{n+1} = u_n + h\phi(t_n, u_n, f_n, h)$$

where ϕ is called an increment function.

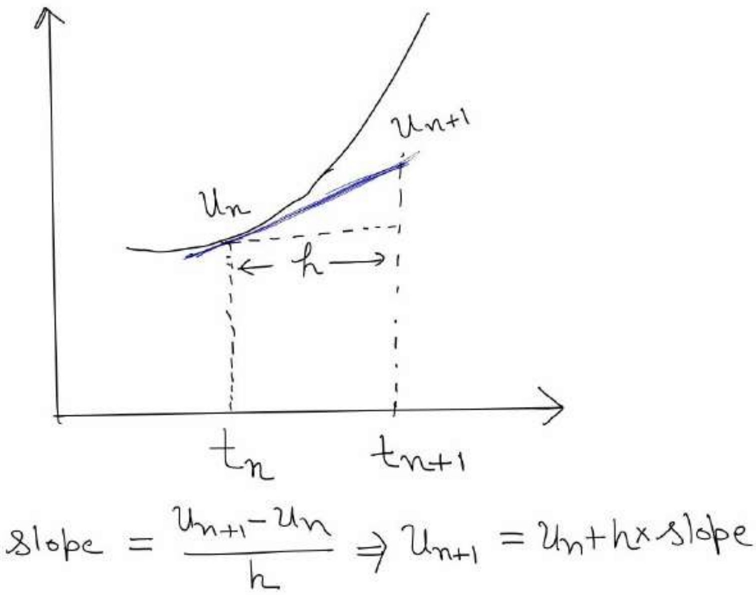


Figure 2.2: Increment function

Consistency, Stability and Convergence:

Consider a single step Method:

$$u_{n+1} = u_n + h\,\Phi(t_n, u_n, f_n, h)$$

Consider:

$$y_{n+1} = y_n + h\,\Phi(t_n, y_n, f(t_n, y_n), h) + \underbrace{\tau_{n+1}}_{\text{consistency error}}$$

If we define

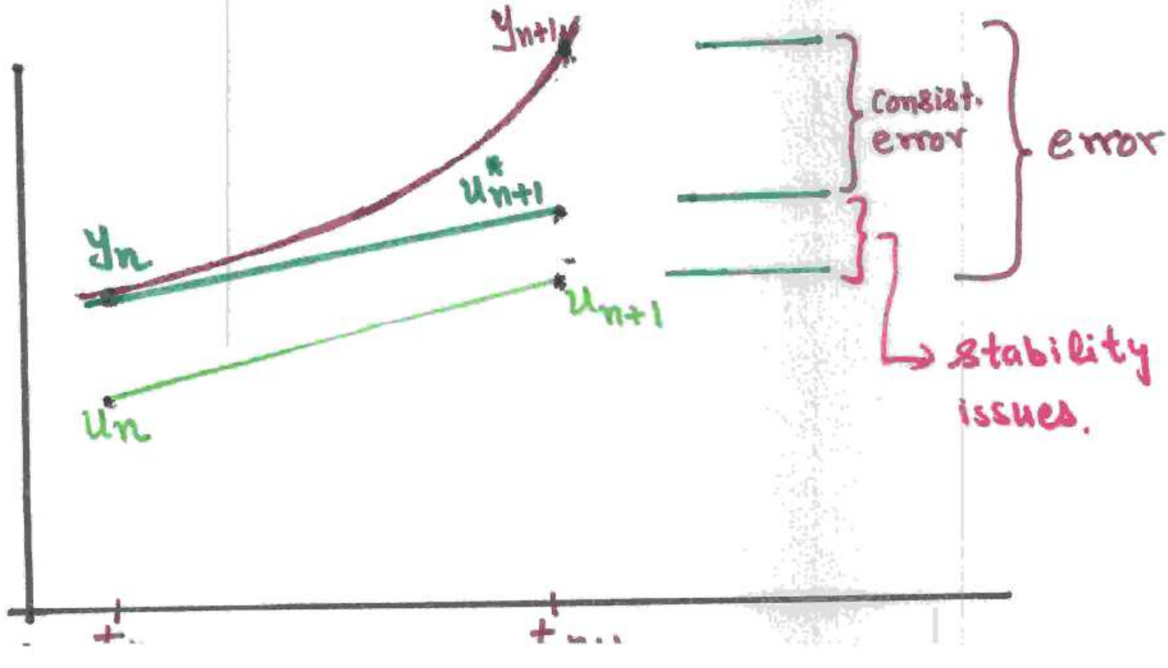
$$u^*_{n+1} = y_n + h\,\Phi(t_n, y_n, f(t_n, y_n), h)$$

Then

$$\boxed{\tau_{n+1} = y_{n+1} - u^*_{n+1}}$$

Define the error at node t_{n+1} as

$$e_{n+1} = y_{n+1} - u_{n+1}$$
$$= (y_{n+1} - u^*_{n+1}) + (u^*_{n+1} - u_{n+1})$$



Lesson 3

Numerical Solutions of IVP

Test Problem

Let us consider the following IVP:

$$\frac{du}{dt} = f(t,u), \quad u(t_0) = u_0 \quad t \in [t_0, b] \tag{2.1}$$

The behavior of solution of IVP (2.1) in the neighborhood of any point $(\bar{t}, \bar{u})$ can be predicted by the linearized form of the differential equation.

The nonlinear function $f(t,u)$ can be linearized in the neighborhood of the point $(\bar{t}, \bar{u})$ by expending it into the Taylor series as

$$f(t,u) = f(\bar{t},\bar{u}) + (t-\bar{t})\frac{\partial f}{\partial t}(\bar{t},\bar{u}) + (u-\bar{u})\frac{\partial f}{\partial u}(\bar{t},\bar{u}) + \text{higher order terms}$$

Defining

$$\lambda = \frac{\partial f}{\partial u}(\bar{t},\bar{u}), \quad \mu = \frac{\partial f}{\partial t}(\bar{t},\bar{u})$$
$$c = f(\bar{t},\bar{u}) - \bar{u}\lambda + (t-\bar{t})\mu$$

The differential equation can be written as

$$u' \approx \lambda u + c$$

Substituting $w = u + (c/\lambda) + (\mu/\lambda^2)$ in the above linearized differential equation, we get

$$w' - \mu/\lambda = \lambda\left[w - (c/\lambda) - (\mu/\lambda^2)\right] + c$$

This implies

$$w' = \lambda w$$

The exact solution of the test problem is

$$w(t) = ke^{\lambda t}$$

where the constant k can be evaluated by the given initial condition.

Order of a Method

Note that the consistency error is given as

$$\tau_{n+1} = y_{n+1} - y_n - h\phi(t_n, y_n, f(t_n, y_n), h)$$

The order of a method is the largest integer p such that

$$\left|\frac{1}{h}\tau_{n+1}\right| = \mathcal{O}(h^p)$$

The big O Notation

If a is some real number (typically 0), we write

$$f(x) = \mathcal{O}(g(x)) \text{ for } x \to a$$

if and only if there exist constants $d > 0$ and C such that

$$|f(x)| \leq C|g(x)| \text{ for all } x \text{ with } |x - a| < d.$$

For example, we write

$$e^x = 1 + x + \frac{x^2}{2} + \mathcal{O}(x^3) \text{ for } x \to 0$$

TAYLOR SERIES METHOD:

Let the solution $y(t)$ of the IVP

$$\frac{dy}{dt} = f(t,y) \,, \quad y(t_0) = y_0 \quad , \quad t \in [t_0, b]$$

exists uniquely such that $y(t) \in C^{(p+1)}[t_0, b]$.

Expand the solution $y(t)$ in a Taylor series about any point t_n

$$y(t) = y(t_n) + (t-t_n)\, y'(t_n) + \frac{(t-t_n)^2}{\underline{|2}}\, y''(t_n) + \cdots + \frac{1}{\underline{|p}}(t-t_n)^p\, y^{(p)}(t_n)$$
$$+ \frac{(t-t_n)^{p+1}}{\underline{|p+1}}\, y^{(p+1)}(\xi_n)$$

where $t \in [t_0, b]$; $t_n < \xi_n < t$.

Substituting $t = t_{n+1}$

$$y(t_{n+1}) = y(t_n) + (t_{n+1} - t_n)\, y'(t_n) + \frac{(t_{n+1}-t_n)^2}{\underline{|2}}\, y''(t_n) + \cdots\cdots$$
$$+ \frac{1}{\underline{|p}}(t_{n+1}-t_n)^p\, y^{(p)}(t_n) + \frac{(t_{n+1}-t_n)^{p+1}}{\underline{|p+1}}\, y^{(p+1)}(\xi_n)$$

Let $t_{n+1} - t_n = h$, then

$$y(t_{n+1}) = y(t_n) + \underbrace{h\, y'(t_n) + \frac{h^2}{\underline{|2}}\, y''(t_n) + \cdots + \frac{1}{\underline{|p}}\, h^p\, y^{(p)}(t_n)}_{=:\, h\,\Phi(t_n, y_n, f_n, h)} + \frac{h^{p+1}}{\underline{|p+1}}\, y^{(p+1)}(\xi_n)$$

Hence the numerical scheme to approximate $y(t_{n+1})$ is given as

$$u_{n+1} = u_n + h\, \Phi(t_n, u_n, f_n, h) \,, \quad n = 0, 1, 2, \cdots N-1 .$$

This is called Taylor's series method of order p.

For $p=1$:

$$\boxed{u_{n+1} = u_n + h\, f(t_n, u_n)} \qquad n = 0, 1, \ldots N-1.$$

is known as Euler method.

How to get $y'(t_n)$, $y''(t_n)$ ..... ?

Notice that

$$y' = f(t, y)$$

$$y'' = \frac{d}{dt}\left(f(t, y)\right) = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= f_t + f_y f.$$

& $$y''' = \frac{d}{dt}\left(f_t\right) + \frac{d}{dt}\left(f_y f\right)$$

$$= f_{tt} + f_{ty} f + f_y\left(f_t + f_y f\right) + f\left(f_{yt} + f_{yy} f\right)$$

$$= f_{tt} + 2 f f_{ty} + f^2 f_{yy} + f_y\left(f_t + f f_y\right)$$

$$\vdots$$

The consistency or truncation error is given by

$$\tau_{n+1} = \frac{h^{p+1}}{\lfloor p+1}\, y^{(p+1)}(\xi_n)$$

The number of terms to be included in the Taylor series can be obtained for a given accuracy ε as

$$\left| \frac{h^{p+1}}{\underline{|p+1}} y^{(p+1)}(\xi_n) \right| < \varepsilon$$

$$\Rightarrow \quad h^{p+1} \left| y^{(p+1)}(\xi_n) \right| < \varepsilon \, \underline{|p+1}$$

Since ξ_n is unknown, we replace $|y^{(p+1)}(\xi_n)|$ by its Maximum value in $[t_0, b]$, i.e.,

$$h^{p+1} \max_{t \in [t_0, b]} \left| y^{(p+1)}(t) \right| < \varepsilon \, \underline{|p+1}$$

With this relation, for a given h, the number of term to be included in the Taylor's series can be obtained.

OR

if the number of terms are fixed, then h can be estimated for given accuracy.

Example: The following IVP is given as

$$y' = t + y \qquad y(0) = 1.$$

a) If the error in $y(t)$ obtained from the first four terms of the Taylor series is to be less than 1×10^{-6}, find t.

b) Determine the number of terms, in the Taylor series required to obtain results with error less than 5×10^{-6} for $t\le 0.1$

c) Use Taylor's series method (second order) to get $y(0.3)$ with step size $h = 0.1$.

<u>Solution</u>: a) $\quad y' = t + y \Rightarrow y'(0) = 0 + 1 = 1$

$$y'' = 1 + y' \Rightarrow y''(0) = 1 + 1 = 2$$

$$y''' = y'' \Rightarrow y'''(0) = 2$$

$$\vdots$$

$$y^{(r)} = y^{(r-1)} \Rightarrow y^{(r)}(0) = 2, \quad r = 2, 3, \ldots$$

Writting the full Taylor series

$$y(t) = y(0) + t\,y'(0) + \frac{t^2}{\underline{|2}}y''(0) + \frac{t^3}{\underline{|3}}y'''(0) + \cdots + \frac{t^p}{\underline{|p}}y^{(p)}(0) + \cdots$$

$$= 1 + t + \frac{t^2}{\underline{|2}}2 + \frac{t^3}{\underline{|3}}\cdot 2 + \cdots + \frac{t^p}{\underline{|p}}2 + \cdots$$

We further note that

$$y^{(p+1)}(t) = 2 + 2\cdot\frac{t}{\underline{|1}} + 2\frac{t^2}{\underline{|2}} + \cdots$$

$$y^{(p+1)}(t) = 2e^t$$

The error relationship gives :

$$\left| \frac{t^4}{\lfloor 4} y^{(4)}(\xi) \right| < 10^{-6} \quad ; \quad \xi \in [0,t)$$

$$\Rightarrow \frac{t^4}{\lfloor 4} \cdot 2 \cdot e^t < 10^{-6} \qquad \text{as } y^4(\xi) < 2e^t$$

$$\Rightarrow t^4 e^t < 12 \times 10^{-6}$$

$$\Rightarrow t^4 e^t < 1.2 \times 10^{-5}$$

$$\Rightarrow \boxed{t \leq 0.058}$$

b) Again with the error formula, we have

$$\frac{t^{p+1}}{\lfloor p+1} \max_{t \in [0,0.1]} \left| y^{(p+1)}(t) \right| < 5 \times 10^{-6}$$

$$\Rightarrow \frac{(0.1)^{p+1}}{\lfloor p+1} \cdot 2e^{0.1} < 5 \times 10^{-6}$$

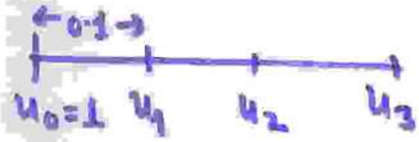
$$\Rightarrow \frac{\lfloor p+1}{(0.1)^{p+1}} > \frac{2e^{0.1}}{5} \times 10^6 = 4.42 \times 10^5$$

$$\Rightarrow p \geq 4$$

The number of terms required $= 5$.

c) The second order Taylor series method is given as

$$u_{n+1} = u_n + h\,u_n' + \frac{h^2}{2!}u_n'' \quad ; \quad n = 0, 1, 2$$



$$u_n' = t_n + u_n$$

$$u_n'' = 1 + u_n' = 1 + t_n + u_n$$

n=0:

$$u_1 = u_0 + h\,u_0' + \frac{h^2}{2!}u_0''$$

$$= 1 + 0.1 \times (0+1) + \frac{(0.1)^2}{2!}\cdot(1+0+1)$$

$$= 1 + 0.1 + \frac{0.01}{2} \times 2 = 1.11$$

n=1:

$$u_2 = u_1 + h\,u_1' + \frac{h^2}{2!}\cdot u_1''$$

$$= 1.11 + (0.1) \times (0.1 + 1.11) + \frac{(0.1)^2}{2!}(1 + 0.1 + 1.11)$$

$$= 1.24205$$

n=2:

$$u_3 = u_2 + h\,u_2' + \frac{h^2}{2!}u_2''$$

$$= 1.24205 + 0.1 \times (0.2 + 1.24205) + \frac{0.1^2}{2!}(1 + 0.2 + 1.24205)$$

$$= 1.39846525$$

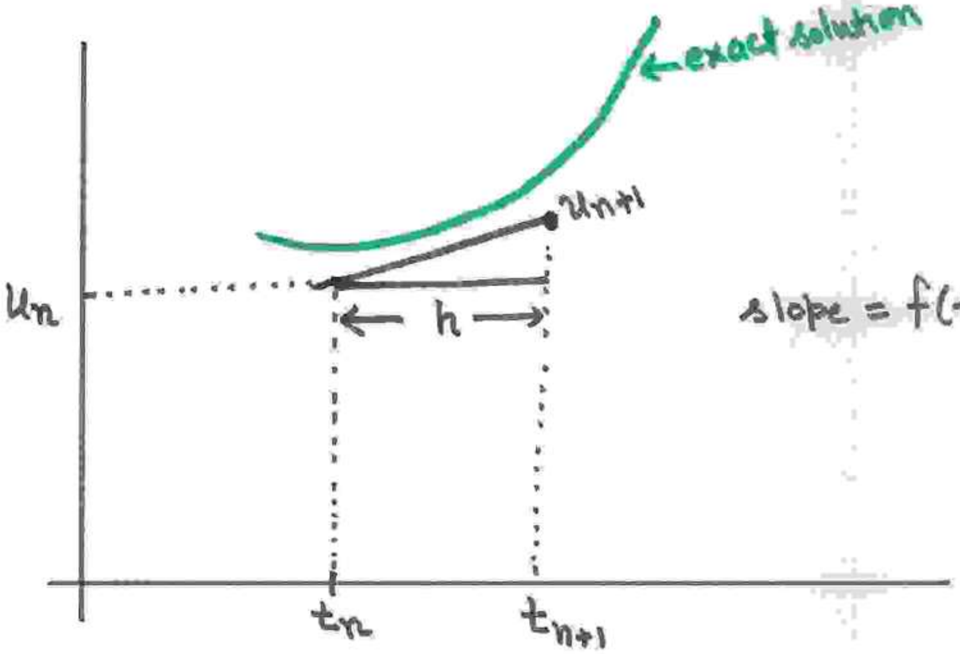
t	exact y	Numerical y
0.1	1.110341836	1.11
0.2	1.242805516	1.24205
0.3	1.399717615	1.39846525

exact solution

$$y = -1 - t + 2e^t$$

Euler Method:

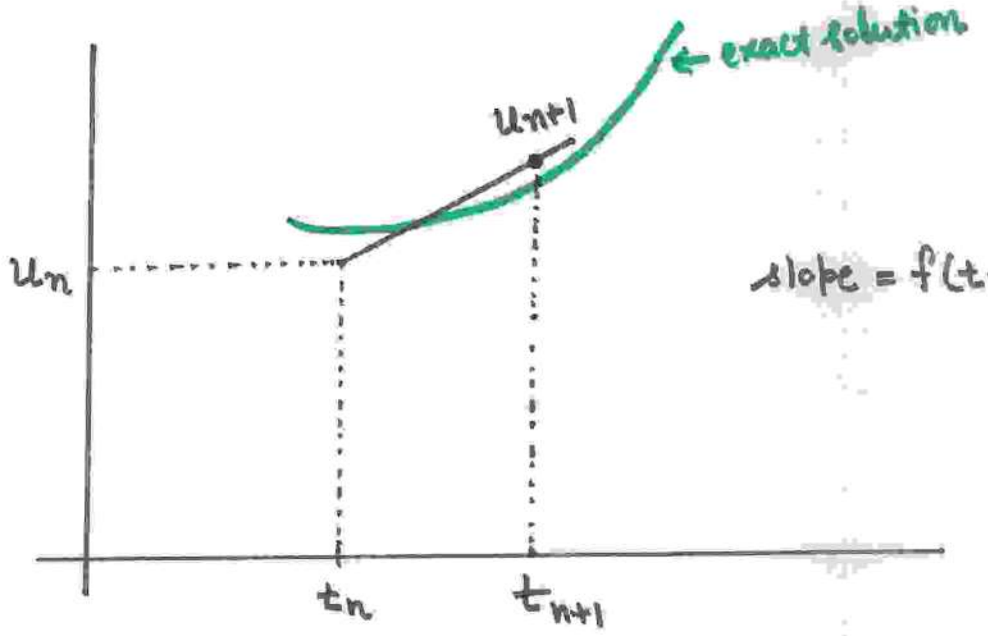
$$u_{n+1} = u_n + h\, f(t_n, u_n)$$



$$\text{slope} = f(t_n, u_n) = \frac{u_{n+1} - u_n}{h}$$

Backward Euler Method:

$$u_{n+1} = u_n + h\, f(t_{n+1}, u_{n+1})$$



$$\text{slope} = f(t_{n+1}, u_{n+1}) = \frac{u_{n+1} - u_n}{h}$$

Since evaluation of u_{n+1} requirs u_{n+1}, backward Euler method is an implicit method.

For the solution of nonlinear equation $u_{n+1} = u_n + h f(t_{n+1}, u_{n+1})$, we can use Newton-Raphson method.

Example: Solve the IVP

$$u' = -2tu^2 \qquad u(0) = 1$$

with $h = 0.2$ on the interval $[0, 0.4]$ using the backward Euler method.

Sol: Backward Euler Method $u_{n+1} = u_n + h f(t_{n+1}, u_{n+1})$

$$\Rightarrow u_{n+1} = u_n + h\left(-2t_{n+1} u_{n+1}^2\right) \quad ; \quad n = 0, 1.$$

OR $\quad u_{n+1} = u_n - 2h t_{n+1} u_{n+1}^2$

We can solve the above quadratic equation directly or by NR Method as follows: Define

$$F(u_{n+1}) = u_{n+1} - u_n + 2h t_{n+1} u_{n+1}^2$$

$$F'(u_{n+1}) = 1 + 4h t_{n+1} u_{n+1}$$

Thus, NR:

$$u_{n+1}^{(s+1)} = u_{n+1}^{(s)} - \frac{F(u_{n+1}^{(s)})}{F'(u_{n+1}^{(s)})} \quad ; \quad s = 0, 1, 2 \cdots\cdots \quad (*)$$

Take $\quad u_{n+1}^{(0)} = u_n$

For $n = 0$: using $(*)$: $\quad u_1^{(1)}, u_1^{(2)}, u_1^{(3)}, \cdots\cdots$

$$u(0.2) \simeq u_1 = u_1^{(3)} = 0.93070331$$

For $n = 1$: Using $(*)$: $\quad u_2^{(1)}, u_2^{(2)}, u_2^{(3)}, \cdots\cdots$

$$u(0.4) \simeq u_2 = u_2^{(3)} = 0.82247016 .$$

Consistency of Backward Euler Method:

$$\tau_{n+1} = y(t_{n+1}) - y(t_n) - h f(t_{n+1}, y(t_{n+1}))$$

$$= y(t_{n+1}) - y(t_{n+1} - h) - h f(t_{n+1}, y(t_{n+1}))$$

$$= \cancel{y(t_{n+1})} - \left\{ \cancel{y(t_{n+1})} - h \cancel{y'(t_{n+1})} + \frac{h^2}{\lfloor 2} y''(\xi) \right\} - h \cancel{f(t_{n+1}, y(t_{n+1}))}$$

$$= -\frac{h^2}{\lfloor 2} y''(\xi), \quad t_n < \xi < t_{n+1}$$

Thus, $\left| \frac{1}{h} \tau_{n+1} \right| = O(h)$

Similar to Euler method, Backward Euler method is a first order single step method.

Runge-Kutta Methods:

Consider the IVP:

$$\frac{dy}{dt} = f(t,y), \qquad y(t_0) = y_0$$

Integrating the above differential equation from t_j to t_{j+1}:

$$\int_{t_j}^{t_{j+1}} \frac{dy}{dt}\, dt = \int_{t_j}^{t_{j+1}} f(t,y)\, dt$$

Applying mean value theorem in the integral on the R.H.S.

$$y(t_{j+1}) - y(t_j) = h f\big(t_j + \theta h,\, y(t_j + \theta h)\big), \quad 0 < \theta < 1.$$

Different value of θ gives us a new numberical method.

Case-I: $\theta = 0$:

$$u_{j+1} = u_j + h \underbrace{f(t_j, u_j)}_{\text{slope at } t_j} \qquad \text{Euler Method}$$

Case-II: $\theta = 1$: $u_{j+1} = u_j + h \underbrace{f(t_{j+1}, u_{j+1})}_{\text{slope at } t_{j+1}}$ Backward Euler Method

Case III: $\theta = 1/2$: $y(t_{j+1}) \simeq y(t_j) + h \underbrace{f\left(t_j + \frac{h}{2},\, y\left(t_j + \frac{h}{2}\right)\right)}_{\text{slope at midpoint}}$

However, $t_{j+1/2}$ is not a nodal point.

How to evaluate (approximate) $f(t_j + h/2,\, y(t_j + h/2))$?

IDEA - 1:

$$y(t_j+\frac{h}{2}) \simeq y_j + \frac{h}{2} f(t_j, y_j) \rightarrow \text{Euler Method}$$

Then the numerical method becomes:

$$u_{j+1} = u_j + h\, f\left(t_j+\frac{h}{2},\ u_j + \frac{h}{2} f_j\right), \quad f_j = f(t_j, u_j)$$

We can rewrite the above formula as:

Set $\quad K_1 = f_j$

$$K_2 = f\left(t_j+\frac{h}{2},\ u_j + \frac{h}{2} K_1\right)$$

$$u_{j+1} = u_j + h K_2$$

This method is called Modified Euler-Cauchy method or Midpoint method.

IDEA - 2: $f\left(t_j+\frac{h}{2}, y(t_j+\frac{h}{2})\right) = y'(t_j+\frac{h}{2}) \simeq \frac{1}{2}\left[y'(t_j) + y'(t_{j+1})\right]$

$$= \frac{1}{2}\left[f(t_j, y_j) + f(t_{j+1}, y_{j+1})\right]$$

Using Euler Method:

$$f\left(t_j+\frac{h}{2}, y(t_j+\frac{h}{2})\right) \simeq \frac{1}{2}\left[f(t_j, y_j) + f\left(t_{j+1},\ y_j + h f_j\right)\right]$$

Then the numerical method becomes:

$$u_{j+1} = u_j + \frac{h}{2}\left[f(t_j, u_j) + f(t_{j+1}, u_j + h f_j)\right]$$

OR: $\quad K_1 = f_j \qquad K_2 = f(t_{j+1}, u_j + h K_1)$

$$u_{j+1} = u_j + \frac{h}{2}\left[K_1 + K_2\right]$$

This method is called Euler-Cauchy method (Heun's method)

Runge-Kutta Methods

Runge-Kutta methods use weighted average of slopes instead of a single slope.

A general Runge-Kutta method is defined as

$$u_{j+1} = u_j + h\,[\text{weighted average of slopes on the given interval}]$$

Consider n slopes in $[t_j, t_{j+1}]$:

$$k_1 = f(t_j + c_1 h,\ u_j + h a_{11} k_1 + h a_{12} k_2 + \cdots + h a_{1n} k_n)$$

$$k_2 = f(t_j + c_2 h,\ u_j + h a_{21} k_1 + h a_{22} k_2 + \cdots + h a_{2n} k_n)$$

$$\vdots$$

$$k_n = f(t_j + c_n h,\ u_j + h a_{n1} k_1 + h a_{n2} k_2 + \cdots + h a_{nn} k_n)$$

The method will be given as

$$u_{j+1} = u_j + h\,[w_1 k_1 + w_2 k_2 + \cdots + w_n k_n]$$

This is called n-stage fully implicit Runge-Kutta Method.

To formulate a Runge-Kutta Method we need:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$[c_1, c_2, \ldots, c_n]$$

$$[w_1, w_2, \ldots, w_n]$$

Semi-implicit methods:

The upper triangular part of A is zero.

$$K_1 = f(t_j + c_1 h,\ u_j + h\, a_{11} K_1)$$

$$K_2 = f(t_j + c_2 h,\ u_j + h\, a_{21} K_1 + h\, a_{22} K_2)$$

$$\vdots$$

$$K_n = f(t_j + c_n h,\ u_j + h\, a_{n1} K_1 + \cdots + h\, a_{nn} K_n)$$

$$u_{j+1} = u_j + h\left[W_1 K_1 + W_2 K_2 + \cdots + W_n K_n\right]$$

Explicit method:

The upper triangular part including diagonal is zero.

$$K_1 = f(t_j + c_1 h,\ u_j)$$

$$K_2 = f(t_j + c_2 h,\ u_j + h\, a_{21} K_1)$$

$$\vdots$$

$$K_n = f(t_j + c_n h,\ u_j + h\, a_{n1} K_1 + \cdots + h\, a_{n,n-1} K_{n-1})$$

$$u_{j+1} = u_j + h\left[W_1 K_1 + W_2 K_2 + \cdots + W_n K_n\right]$$

Explicit Runge-Kutta Methods (DERIVATION)

Consider the Runge-Kutta Method with two slopes:

$$K_1 = f(t_j, u_j)$$

$$K_2 = f(t_j + c_2 h,\ u_j + h a_{21} K_1)$$

$$u_{j+1} = u_j + h[\omega_1 K_1 + \omega_2 K_2] \quad \text{—— (1)}$$

where the parameters $c_2, a_{21}, \omega_1, \omega_2$ will be determined so that the error $u_{j+1} - y(t_{j+1})$ becomes small.

First, we write the Taylor's series of the solution

$$y(t_{j+1}) = y(t_j) + h\,y'(t_j) + \frac{h^2}{\underline{|2}}\,y''(t_j) + \frac{h^3}{\underline{|3}}\,y'''(t_j) + \cdots \quad \text{— (2)}$$

where

$$y' = f(t, y)$$

$$y'' = f_t + f f_y$$

$$y''' = f_{tt} + 2f f_{ty} + f_{yy} f^2 + f_y(f_t + f f_y)$$

Expand K_2 about (t_j, u_j):

$$K_2 = f(t_j, u_j) + c_2 h f_t + h a_{21} f_j f_y + \frac{1}{\underline{|2}} h^2 \big(c_2^2 f_{tt} + 2 c_2 a_{21} f_j f_{ty} + a_{21}^2 f_j^2 f_{yy} \big) + \cdots$$

Substituting K_1 and K_2 in (1)

$$u_{j+1} = u_j + h\Big[\omega_1 f_j + \omega_2 \Big\{ f_j + h(c_2 f_t + a_{21} f_j f_y) + \frac{h^2}{2}\big(c^2 f_{tt} + 2 c_2 a_{21} f_j f_{ty} + a_{21}^2 f_j^2 f_{yy}\big)\Big\}\Big]$$

$\Rightarrow$

$$u_{j+1} = u_j + (\omega_1+\omega_2) f_j h + (\omega_2 c_2 f_t + \omega_2 a_{21} f_j f_y) h^2 + + \frac{h^3}{2}\omega_2\left(c_2^2 f_{tt} + 2c_2 a_{21} f_j f_{ty} + a_{21}^2 f_j^2 f_{yy}\right) + \cdots \quad \text{—③}$$

Comparing ② & ③

$$\omega_1 + \omega_2 = 1$$

$$\omega_2 c_2 = \frac{1}{2}$$

$$\omega_2 a_{21} = \frac{1}{2}$$

If c_2 is chosen arbitoarily then

$$\omega_2 = \frac{1}{2c_2} \; ; \qquad a_{21} = c_2$$

$$\omega_1 = 1 - \frac{1}{2c_2} \qquad 0 \le c_2 \le 1$$

Now ③ becomes:

$$u_{j+1} = u_j + \underbrace{h f_j + \frac{h^2}{2}(f_t + f_j f_y)}_{h\Phi(t_j, y_j, h)} + \frac{h^3}{4} c_2 \left(f_{tt} + 2 f_j f_{ty} + f_j^2 f_{yy}\right) + \cdots$$

TRUNCATION ERROR: $\tau_{j+1} = y(t_{j+1}) - y(t_j) - h\Phi(t_j, y(t_j), h)$

$$\Rightarrow \tau_{j+1} = \frac{h^3}{6}\left(f_{tt} + 2 f f_{ty} + f_{yy} f^2 + f_y(f_t + f f_y)\right)_{t=t_j} - \frac{h^3}{4} c_2 \left(f_{tt} + 2 f(t_j) f_{ty} + f(t_j)^2 f_{yy}\right) + \cdots$$

$$= h^3\left[\left(\frac{1}{6} - \frac{c_2}{4}\right)\left(f_{tt} + 2 f f_{ty} + f^2 f_{yy}\right)_{t=t_j} + \frac{1}{6} f_y (f_t + f f_y)_{t=t_j}\right] + \cdots$$

ORDER OF THE METHOD $= 2$

Special cases:

1: $c_2 = \frac{1}{2}$; $w_2 = 1$ $w_1 = 0$ $a_{21} = \frac{1}{2}$

Method:

$$k_1 = f(t_j, u_j)$$

$$k_2 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} k_1\right)$$

$$u_{j+1} = u_j + h\, k_2$$

Coeff. in Table form:

$$\begin{array}{c|cc} c_2 & a_{21} & \\ \hline & w_1 & w_2 \\ \hline \end{array} \qquad \begin{array}{c|cc} \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \\ \hline \end{array}$$

This method is called Modified Euler-Cauchy method.

2: $c_2 = 1$ $a_{21} = 1$ $w_1 = \frac{1}{2}$ $w_2 = \frac{1}{2}$

Method:

$$k_1 = f(t_j, u_j)$$

$$k_2 = f(t_j + h, u_j + h k_1)$$

$$u_{j+1} = u_j + h\left(\frac{k_1 + k_2}{2}\right)$$

In Table form:

$$\begin{array}{c|cc} 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

This method is called as Euler-Cauchy method.

FOURTH ORDER METHOD: (EXPLICIT)

$$u_{j+1} = u_j + h\left[w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4\right]$$

$$k_1 = f(t_j, u_j)$$

$$k_2 = f(t_j + c_2 h,\ u_j + h a_{21} k_1)$$

$$k_3 = f(t_j + c_3 h,\ u_j + h a_{31} k_1 + h a_{32} k_2)$$

$$k_4 = f(t_j + c_4 h,\ u_j + h a_{41} k_1 + h a_{42} k_2 + h a_{43} k_3)$$

Classical Runge-Kutta Method:

$$u_{j+1} = u_j + h \cdot \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$k_1 = f(t_j, u_j)$$

$$k_2 = f\left(t_j + \frac{h}{2},\ u_j + \frac{h}{2} k_1\right)$$

$$k_3 = f\left(t_j + \frac{h}{2},\ u_j + \frac{h}{2} k_2\right)$$

$$k_4 = f(t_j + h,\ u_j + h k_3)$$

Table form:

$$\begin{array}{c|cccc} c_2 & a_{21} & & & \\ c_3 & a_{31} & a_{32} & & \\ c_4 & a_{41} & a_{42} & a_{43} & \\ \hline & w_1 & w_2 & w_3 & w_4 \end{array}$$

$$\begin{array}{c|cccc} \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \end{array}$$

← Classical

$$\begin{array}{c|cccc} \frac{1}{3} & \frac{1}{3} & & & \\ \frac{2}{3} & -\frac{1}{3} & 1 & & \\ 1 & 1 & -1 & 1 & \\ \hline & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$$

← Kutta

Minimum number of function evaluation versus order

ORDER	2	3	4	5	6	7	8	.....
MNFE	2	3	4	6	7	9	11	

Remark: The order of an s-stage explicit method (RK) can not be greater than s.

Also, there does not exist a s-stage method (explicit RK) with order s if $s \geq 5$.

Example: Apply Classical Runge-Kutta Method to get $y(0.3)$ with step size $h=0.1$ for the problem

$$\frac{dy}{dt} = t + y, \qquad y(0) = 1.$$

Sol:

Classical Runge-Kutta Method

$$K_1 = f(t_j, u_j)$$

$$K_2 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} K_1\right)$$

$$K_3 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} K_2\right)$$

$$K_4 = f(t_j + h, u_j + h K_3)$$

$$u_{j+1} = u_j + \frac{h}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right).$$

$h = 0.1$

$u_0 = 1.$

u_1, u_2, u_3 ?

$\underline{j=0}$:

$$k_1 = f(0,1) = 0+1 = 1$$

$$k_2 = f\left(\frac{0.1}{2}, 1+\frac{0.1}{2}\right) = 1.1$$

$$k_3 = f\left(\frac{0.1}{2}, 1+\frac{0.1}{2}\times 1.1\right) = 1.105$$

$$k_4 = f(0.1, 1+0.1\times 1.105) = 1.2105$$

$$u_1 = 1 + \frac{0.1}{6}\left[1 + 2\times 1.1 + 2\times 1.105 + 1.2105\right]$$

$$= 1.110341667$$

$\underline{j=1}$:

$$k_1 = 1.210341667$$

$$k_2 = 1.320858750$$

$$k_3 = 1.326384604$$

$$k_4 = 1.442980127$$

$$u_2 = 1.242805142$$

$j=2$: $k_1 = 1.442805142$

$$k_2 = 1.564945399$$

$$k_3 = 1.57105241195$$

$$k_4 = 1.69991038319$$

$$u_3 = 1.39971699 44$$

t	exact y	Numerical y
0.1	1.110341836	1.110341667
0.2	1.242805516	1.242805142
0.3	1.399717615	1.399716994

exact solution

$$y = -1-t+2e^t$$

Ex. Use the Runge-Kutta Method to approximate the particular solution at $x=1$ of the differential equation $y' = xy$ through $(0,1)$.

Sol: $h = 1$: $y(0) = 1$

$$K_1 = f(0,1) = 0.$$

$$K_2 = f(0.5, 1) = 0.5 \times 1 = 0.5.$$

$$K_3 = f(0.5, 1 + \tfrac{1}{2} \times 0.5) = (0.5)(1.25) = 0.625$$

$$K_4 = f(1, 1 + 0.625) = (1)(1.625) = 1.625$$

$$u_1 = u_0 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1 + \frac{1.0}{6}(0 + 2 \times 0.5 + 2 \times 0.625 + 1.625)$$

$$= 1.646$$

Ex: Consider a scalar problem

$$y' = -y^2 \qquad y(1) = 1$$

The exact solution is $y(t) = \frac{1}{t}$. Compute the numerical solution at $t = 1.5$ using $h = 0.5$.

Sol:

$$k_1 = f(\underset{1}{\overset{\shortparallel}{t}}, \underset{1}{\overset{\shortparallel}{y}}) = -1$$

$$k_2 = f\left(1 + \frac{0.5}{2}, 1 + \frac{0.5}{2} \times (-1)\right)$$

$$= -\left(\frac{1.5}{2}\right)^2 = -0.5625$$

$$k_3 = f\left(1 + \frac{0.5}{2}, 1 + \frac{0.5}{2} \times (-0.5625)\right)$$

$$= -0.738525390625$$

$$k_4 = f(1 + 0.5, 1 + 0.5 \times -0.738525390625$$

$$= -0.3978295475244452$$

$$u_1 = u_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{0.5}{6}(\ldots\ldots\ldots)$$

$$= 0.\underline{6666}766639268796$$

EXACT SOLUTION: $0.\underline{6666}666\ldots\ldots$

Implicit Runge-Kutta Methods:

The implicit Runge-Kutta method using n-slopes is given as

$$K_i = f\left(t_j + c_i h,\ u_j + h\sum_{m=1}^{n} a_{im} K_m\right) \qquad i = 1, 2, \dots, n.$$

$$u_{j+1} = u_j + h\sum_{m=1}^{n} \omega_m K_m$$

Case $n=1$:

$$u_{j+1} = u_j + h\,\omega_1 K_1 \quad \text{---(1)}$$

$$K_1 = f(t_j + c_1 h,\ u_j + h\,a_{11} K_1)$$

Taylor's series expansion of K_1:

$$K_1 = f(t_j, u_j) + (c_1 h f_t + h a_{11} K_1 f_y)_{t_j} + O(h^2)$$

Substituting in (1):

$$u_{j+1} = u_j + h\omega_1\left(f(t_j, u_j) + h\,(c_1 f_t + a_{11} K_1 f_y)_{t_j}\right) + O(h^3) \quad \text{---(2)}$$

Taylor's series of the solution:

$$y(t_{j+1}) = y(t_j) + h\,y'(t_j) + \frac{h^2}{\lfloor 2}\,y''(t_j) + \cdots$$

$$= y(t_j) + h f(t_j, y(t_j)) + \frac{h^2}{2}\,(f_t + f f_y)_{t_j} + \cdots \quad (3)$$

Comparing (2) & (3):

$$\omega_1 = 1. \qquad \omega_1 c_1 = \frac{1}{2} \qquad a_{11}\,\omega_1 = \frac{1}{2}$$

$$\Rightarrow \quad \omega_1 = 1 \qquad c_1 = \frac{1}{2} \qquad a_{11} = \frac{1}{2}.$$

Hence the second order Runge-Kutta method becomes:

$$u_{j+1} = u_j + K_1 h \ ; \quad K_1 = f\left(t_j + \frac{h}{2},\ u_j + \frac{h}{2} K_1\right)$$

To obtain K_1, we need to solve the nonlinear equation for K_1.

Case $n=2$:

$$u_{j+1} = u_j + \frac{h}{2}(K_1 + K_2)$$

$$K_1 = f\left(t_j + \frac{3-\sqrt{3}}{6}h,\ u_j + \frac{h}{4}K_1 + \frac{3-2\sqrt{3}}{12}hK_2\right)$$

$$K_2 = f\left(t_j + \frac{3+\sqrt{3}}{6}h,\ u_j + \frac{3+2\sqrt{3}}{12}hK_1 + \frac{h}{4}K_2\right)$$

The order of the method is 4.

Ex: Using the implicit Runge-Kutta method

$$u_{n+1} = u_n + K_1 h$$

$$K_1 = f\left(t_n + \frac{h}{2},\ u_n + \frac{h}{2}K_1\right)$$

to find the solution of the initial value problem

$$y' = t^2 + y^2,\quad y(1) = 2,\quad 1 \le t \le 1.2 \text{ with } h = 0.1.$$

Solution: Since $f(t,y) = t^2 + y^2$, we have

$$K_1 = \left(t_n + \frac{h}{2}\right)^2 + \left(u_n + \frac{h}{2}K_1\right)^2$$

$n=0$: $h = 0.1$, $t_0 = 1$, $u_0 = 2$

$$K_1 = (1 + 0.05)^2 + (2 + 0.05 \times K_1)^2$$

$$\Rightarrow K_1 = (1.05)^2 + (2 + 0.05K_1)^2$$

$$\Rightarrow K_1 = 1.1025 + 4 + .0025K_1^2 + 0.2K_1$$

$$\Rightarrow 0.0025K_1^2 - 0.8K_1 + 5.1025 = 0$$

This can be solved by Newton's Raphson method:

Let $F(K_1) = 0.0025\,K_1^2 - 0.8\,K_1 + 5.1025$

$F'(K_1) = 0.0050\,K_1 - 0.8$

NR iterations:

$$K_1^{(s+1)} = K_1^{(s)} - \frac{F(K_1^{(s)})}{F'(K_1^{(s)})} \qquad s = 0, 1, 2 \ldots$$

$K_1^{(0)} = f(t_0, u_0) = 1 + 4 = 5$

$K_1^{(1)} = 6.5032258$

$K_1^{(2)} = 6.51058650$

$K_1^{(3)} = \underline{6.510586}68$

$u_1 = u_0 + h\,K_1$

$= 2 + 0.1 \times 6.510586$

$= 2.6510586$.

$\underline{n=1}$: $K_1 = f(1.1 + 0.05,\ 2.6510586 + 0.05\,K_1)$

$\Rightarrow K_1 = (1.15)^2 + (2.6510586 + 0.05\,K_1)^2$

$\Rightarrow 0.0025\,K_1^2 + 0.26510586\,K_1 - K_1 + 8.350611701 = 0$

$\Rightarrow 0.0025\,K_1^2 - 0.73489414\,K_1 + 8.350611701 = 0$

$\underline{\text{NR Method}}$: $K_1^{(1)} = 11.793142933$

$K_1^{(2)} = 11.839886962$

$K_1^{(3)} = 11.8398950463$

$K_1^{(0)} = f(1.1, 2.6510586)$

$= 8.238111701$

$u_2 = u_1 + h\,K_1 = 2.6510586 + 0.1 \times 11.8398950$

$= \underline{3.8350481}$.

System of equations:

Consider:

$$\frac{d\bar{y}}{dt} = \bar{f}(t, y_1, y_2, \dots, y_n)$$

$$\bar{y}(t_0) = \bar{\eta}$$

where:

$$\bar{y} = [y_1, y_2, \dots, y_n]^T$$

$$\bar{f} = [f_1, f_2, \dots, f_n]^T$$

$$\bar{\eta} = [\eta_1, \eta_2, \dots, \eta_n]^T.$$

Taylor's series method:

$$\bar{u}_{j+1} = \bar{u}_j + h\,\bar{u}'_j + \frac{h^2}{\underline{|2}}\,\bar{u}''_j + \dots + \frac{h^p}{\underline{|p}}\,\bar{u}_j^{(p)}$$

$j = 0, 1, 2 \dots, N-1.$

In particular, the Euler method:

$$\bar{u}_{j+1} = \bar{u}_j + h\,\bar{f}_j \qquad j = 0, 1, 2, \dots, N-1$$

OR

$$\begin{bmatrix} u_{1,j+1} \\ \vdots \\ u_{n,j+1} \end{bmatrix} = \begin{bmatrix} u_{1,j} \\ \vdots \\ u_{n,j} \end{bmatrix} + h \begin{bmatrix} f_{1,j} \\ \vdots \\ f_{n,j} \end{bmatrix}$$

$$f_{ij} = f_i(t_j, u_{1,j}, \dots, u_{n,j})$$

Runge-Kutta Method of second order (Euler-Cauchy)

$$\bar{u}_{j+1} = \bar{u}_j + \frac{h}{2}(\bar{K}_1 + \bar{K}_2)$$

$$K_{i1} = f_i(t_j, u_{1,j}, u_{2,j}, \dots, u_{n,j}) \quad i = 1, 2, \dots, n.$$

$$K_{i2} = f_i(t_j + h, u_{1,j} + hK_{11}, u_{2,j} + hK_{21} + \dots, u_{nj} + hK_{n1})$$

Note that

$$\bar{u}_j = [u_{1,j}, u_{2,j}, \dots, u_{n,j}]^T$$

&

$$\bar{K}_1 = [K_{11}, K_{21}, \dots, K_{n1}]^T$$

$$\bar{K}_2 = [K_{12}, K_{22}, \dots, K_{n2}]^T$$

Similarly Runge-Kutta method of higher order can be formulated.

Example: Compute an approximation to $u(1)$, $u'(1)$ and $u''(1)$ with the Taylor's series method of second order and step length $h = 1$ for the IVP

$$u''' + 2u'' + u' - u = \cos t \qquad 0 \le t \le 1$$

$$u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 2.$$

Sol: We can reduce the second order or higher order equations to an equivalent system of first order equations by setting

$v_1 = u \qquad v_2 = u' \qquad v_3 = u''$

System of equations:

$$v_1' = v_2$$

$$v_2' = v_3 \qquad\qquad v_1(0) = 0$$

$$v_3' = \cos t - 2v_3 - v_2 + v_1 \qquad v_2(0) = 1$$

$$v_3(0) = 2$$

Therefore the Taylor's series method gives:

$$\bar{u}_{j+1} = \bar{u}_j + h\bar{u}_j' + \frac{h^2}{2}\bar{u}_j'' \qquad j = 0;$$

$$\bar{u}_1 = \begin{bmatrix} v_{1,1} \\ v_{2,1} \\ v_{3,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 1\cdot\begin{bmatrix} v_2(0) \\ v_3(0) \\ \cos(0) - 2v_3(0) - v_2(0) + v_1(0) \end{bmatrix}$$

$$+ \frac{1}{2}\begin{bmatrix} v_3(0) \\ \cos(0) - 2v_3(0) \dots \\ \sin(0) - 2v_3'(0) - v_3(0) + v_2(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 1 - 2\times 2 - 1 + 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ -4 \\ 0 + 8 - 2 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 7/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3/2 \end{bmatrix}$$

Hence: $u(1) \simeq 2$

$$u'(1) \simeq 1$$

$$u''(1) \simeq 3/2$$

Ex: Use the Runge-Kutta Method to approximate the particular solution at $x=2$ of the differential equation

$$y'' = x + yy'$$

if $y' = 0$ and $y = 1$ when $x = 0$.

Sol:

let $y_1 = y$

$y_2 = y'$

then:

$$y_1' = y_2 =: f_1(x, y_1, y_2)$$

$$y_2' = x + y_1 y_2 =: f(x, y_1, y_2)$$

$$\bar{u}_1 = \bar{u}_0 + \frac{h}{6}\left[\bar{k}_1 + 2\bar{k}_2 + 2\bar{k}_3 + \bar{k}_4\right] \qquad h = 2$$

$$\bar{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \bar{k}_1 = \begin{bmatrix} y_2(0) \\ 0 + y_1(0)\, y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{k}_2 = \begin{bmatrix} f_1(1, 1 + 1\times 0, 0 + 1\times 0) \\ f_2(1, 1, 0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\bar{k}_3 = \begin{bmatrix} f_1(1, 1 + 1\times 0, 0 + 1\times 1) \\ f_2(1, 1, 1) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\bar{k}_4 = \begin{bmatrix} f_1(2, 1 + 2\times 1, 0 + 2\times 2) \\ f_2(2, 3, 4) \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}$$

$$\bar{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{2}{6}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 14 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 6 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 \\ 20/3 \end{bmatrix} \qquad \boxed{y(2) \approx 3}$$

Recapitulation (Single step Methods)

$$y' = f(t, y), \quad y(t_0) = y_0$$

Explicit method:

$$u_{n+1} = u_n + h\,\phi(t_n, u_n, f_n, h)$$

i) Taylor's series method of order p:

$$u_{n+1} = u_n + h u_n' + \frac{h^2}{\lfloor 2} u_n'' + \cdots\cdots + \frac{h^p}{\lfloor p} u_n^{(p)}$$

$$u_n' = f(t_n, u_n) \quad \cdots\cdots$$

Error: $\left| \frac{h^{p+1}}{\lfloor p+1} y^{(p+1)}(\xi) \right| < \varepsilon$

ii) Euler method:

$$u_{n+1} = u_n + h f(t_n, u_n)$$

iii) Runge-Kutta method of second order

a)

$$K_1 = f(t_j, u_j)$$

$$K_2 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} K_1\right)$$

$$u_{j+1} = u_j + h K_2$$

modified Euler-Cauchy method

b)

$$K_1 = f(t_j, u_j)$$

$$K_2 = f(t_j + h, u_j + h K_1)$$

$$u_{j+1} = u_j + h\left(\frac{K_1 + K_2}{2}\right)$$

Euler-Cauchy method

IV) Fourth order Runge-Kutta Method:

$$k_1 = f(t_j, u_j)$$

$$k_2 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} k_1\right)$$

$$k_3 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} k_2\right)$$

$$k_4 = f(t_j + h, u_j + h k_3)$$

$$u_{j+1} = u_j + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Implicit methods:

i) Backward Euler method

$$u_{n+1} = u_n + h f(t_{n+1}, u_{n+1})$$

ii) Second order Runge-Kutta method:

$$u_{j+1} = u_j + k_1 h$$

$$k_1 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} k_1\right)$$

Single step methods for solving higher order differential equations / system of first order differential equation.

Consistency + Stability = Convergence

Consistency Error:

$$\tau_{n+1} = y_{n+1} - y_n - h\,\Phi(t_n, y_n, f(t_n, y_n), h)$$

Order of a method:

$$\left| \frac{1}{h} \tau_{n+1} \right| = O(h^p)$$

Stability:

A single step method when applied to $y' = \lambda y$ leads to a first order difference equation

$$u_{j+1} = E(\lambda h) u_j$$

Def: We call a single step method

- absolute stable: if $|E(\lambda h)| < 1$, $\lambda < 0$ or $Re(\lambda) < 0$
- relatively stable: if $|E(\lambda h)| < e^{\lambda h}$, $\lambda > 0$

Multistep Method

The general multistep method or K-step method can be written as:

$$u_{j+1} = a_1 u_j + a_2 u_{j-1} + \cdots\cdots + a_K u_{j-K+1}$$
$$+ h\left(b_0 u'_{j+1} + b_1 u'_j + b_2 u'_{j-1} + \cdots\cdots + b_K u'_{j-K+1}\right)$$

OR $\qquad j = K-1, K, \ldots\ldots$

$$\boxed{u_{j+1} = \sum_{i=1}^{K} a_i u_{j-i+1} + h \sum_{i=0}^{K} b_i u'_{j-i+1}}$$

Define a shift operator.

$$E f(x_i) = f(x_{i+1})$$
$$E^2 f(x_i) = f(x_{i+2})$$
$$\vdots$$
$$E^K f(x_i) = f(x_i + K)$$

OR in discrete form

$$E^K f_i = f_{i+K}$$

With the shift operator the multistep method can be rewritten as

$$E^K u_{j-K+1} = a_1 E^{K-1} u_{j-K+1} + a_2 E^{K-2} u_{j-K+1} + \cdots\cdots + a_K u_{j-K+1}$$
$$+ h\left(b_0 E^K u'_{j-K+1} + b_1 E^{K-1} u'_{j-K+1} + \cdots\cdots + b_K u'_{j-K+1}\right)$$

OR

$$\left(E^{K}-a_1 E^{K-1}-a_2 E^{K-2}+\cdots\cdots-a_K\right) u_{j-K+1}$$

$$-h\left(b_0 E^{K}+b_1 E^{K-1}+\cdots\cdots+b_K\right) u'_{j-K+1}=0$$

$$\Rightarrow \quad \rho(E)\, u_{j-K+1}-h\,\sigma(E)\, u'_{j-K+1}=0$$

wher ρ & σ are polynomials defined by

$$\rho(\xi)=\left(\xi^{K}-a_1 \xi^{K-1}-a_2 \xi^{K-2}-\cdots\cdots-a_K\right)$$

and

$$\sigma(\xi)=\left(b_0 \xi^{K}+b_1 \xi^{K-1}+b_2 \xi^{K-2}+\cdots\cdots+b_K\right)$$

Note that, if $b_0=0$, the method is called an explicit or predictor method. When $b_0 \neq 0$, it is called an implicit method.

Example: Midpoint Method:

$$u_{n+1}=u_{n-1}+2h f_n$$

where u_0, u_1 is to be known before applying the above two-step explicit method.

The local truncation error will be given as:

$$\tau_{j+1} = y(t_{j+1}) - \sum_{i=1}^{k} a_i\, y(t_{j-i+1}) - h\sum_{i=0}^{k} b_i\, y'(t_{j-i+1})$$

Further simplifications using Taylor's series expansion of $y(t_{j+1})$, $y(t_{j-i+1})$ & $y'(t_{j-i+1})$ give:

$$\tau_{j+1} = y(t_j) + h\,y'(t_j) + \frac{h^2}{\underline{|2}}\, y''(t_j) + \cdots + \frac{h^p}{\underline{|p}}\, y^{(p)}(t_j) + \frac{h^{p+1}}{\underline{|p+1}}\, y^{(p+1)}(t_j) + O(h^{p+2})$$

$$- \sum_{i=1}^{k} a_i \Big[y(t_j) + \underbrace{(t_{j-i+1} - t_j)}_{\substack{= t_0 + (j-i+1)h - t_0 - jh \\ = (1-i)h}}\, y'(t_j) + \frac{(1-i)^2 h^2}{\underline{|2}}\, y''(t_j)$$

$$+ \cdots + \frac{(1-i)^p}{\underline{|p}}\, h^p\, y^{(p)}(t_j) + \frac{(1-i)^{p+1}}{\underline{|p+1}}\, h^{p+1}\, y^{(p+1)}(t_j) + O(h^{p+2}) \Big]$$

$$- h\sum_{i=0}^{k} b_i \Big[y'(t_j) + (1-i)h\, y''(t_j) + \cdots + \frac{(1-i)^{p-1}}{\underline{|p-1}}\, h^{p-1}\, y^{(p)}(t_j)$$

$$+ \frac{(1-i)^p}{\underline{|p}}\, h^p\, y^{(p+1)}(t_j) + O(h^{p+1}) \Big]$$

This can be rewritten in the following form

$$\tau_{j+1} = c_0\, y(t_j) + c_1 h\, y'(t_j) + c_2 h^2 y''(t_j) + \cdots + c_p h^p y^{(p)}(t_j) + T_{p+1}$$

Where

$$C_0 = 1 - \sum_{i=1}^{K} a_i$$

$$C_q = \frac{1}{\lfloor q}\left[1 - \sum_{i=1}^{K} a_i (1-i)^q\right] - \frac{1}{\lfloor (q-1)} \sum_{i=0}^{K} b_i (1-i)^{q-1}$$

$$q = 1, 2, \ldots, p+1 .$$

$$T_{p+1} = C_{p+1} h^{p+1} y^{(p+1)}(t_j) + O(h^{p+2})$$

Definition: 1. The linear multistep method is said to be consistent if it has order $p \geq 1$.

2. The linear multistep method is said to be of order p if $C_0 = C_1 = \cdots = C_p = 0$ & $C_{p+1} \neq 0$

Hence for a consistent method C_0 & C_1 must be ZERO.

Ex: For a consistent method, show that

$$\rho'(1) = a_1 + 2a_2 + \cdots + ka_k$$

Sol:

$$\rho(\xi) = \xi^k - a_1 \xi^{k-1} - a_2 \xi^{k-2} - \cdots - a_k$$

$$\rho'(\xi) = k\xi^{k-1} - a_1(k-1)\xi^{k-2} - a_2(k-2)\xi^{k-3} - \cdots - a_{k-1}$$

Then

$$\rho'(1) = k - a_1(k-1) - a_2(k-2) - \cdots - a_{k-1}$$

$$= k - a_1(k-1) - a_2(k-2) - \cdots - a_{k-1}(k-(k-1))$$

$$= k(1 - a_1 - a_2 - \cdots - a_{k-1} - a_k + a_k)$$

$$+ a_1 + 2a_2 + \cdots + (k-1)a_{k-1}$$

We know that for a consistent method c_0 must be 0, that means

$$1 - \sum_{i=1}^{k} a_i = 0$$

Then.

$$\rho'(1) = k(0 + a_k) + a_1 + 2a_2 + \cdots + (k-1)a_{k-1}$$

$$= a_1 + 2a_2 + \cdots + ka_k$$

$\square$.

Ex: Prove that if a method is consistent then

$$\rho(1) = 0 \quad \&$$

$$\rho'(1) = \sigma(1)$$

Proof: For a consistent method:

$$C_0 = 0 \quad \& \quad C_1 = 0.$$

$$C_0 = 0 \Rightarrow 1 - (a_1 + a_2 + \cdots + a_K) = 0$$

$$\Rightarrow \rho(1) = 0$$

$$C_1 = 0 \Rightarrow \left[1 + a_2 + 2a_3 + \cdots + (K-1)a_K\right] - \left[b_0 + b_1 + \cdots + b_K\right] = 0$$

$$\Rightarrow 1 + \left[a_1 + 2a_2 + 3a_3 + \cdots + Ka_K\right] - \left[a_1 + a_2 + \cdots + a_K\right] - \left[b_0 + b_1 + \cdots + b_K\right] = 0$$

$$\Rightarrow 1 + \rho'(1) - 1 - \sigma(1) = 0$$

$$\Rightarrow \rho'(1) = \sigma(1)$$

Therefor a method is consistent if

$$\rho(1) = 0 \quad \text{and}$$

$$\rho'(1) = \sigma(1).$$

Definition: The multistep method

$$\rho(E)u_{j-k+1} - h\sigma(E)u'_{j-k+1} = 0$$

is said to satisfy the root condition if all roots of the equation $\rho(\xi)=0$ are contained within the unit circle centered at the origin of the complex plane, otherwise, if they fall on its boundary, they must be simple roots of ρ. Equivalently

let r_j be the roots of $\rho(\xi)$ then

$$\begin{cases} |r_j| \leq 1 \quad j = 1, 2, \cdots k. \\ \text{furthermore, for those } j \text{ such that } |r_j| = 1 \text{ then } \rho'(r_j) \neq 0 \end{cases}$$

Remark: For a consistent method, the root condition is equivalent to zero-stability. More stronger versions of stability will be considered later.

Theorem: The linear multistep method is convergent iff the method is consistent and satisfies the root condition.

Ex: Show that the method

$$u_{j+1} - 3u_j + 2u_{j-1} = \frac{h}{2}(f_j - 3f_{j-1})$$

is not convergent.

Sol: The given multistep method can be written in the following form:

$$\left(E^2 - 3E + 2\right) u_{j-1} - h\left(\frac{E}{2} - \frac{3}{2}\right) f_{j-1} = 0$$

$$\Rightarrow \quad \rho(\xi) = \xi^2 - 3\xi + 2$$

$$\sigma(\xi) = \frac{1}{2}(\xi - 3)$$

Consistency Check: $\rho(1) = 1 - 3 + 2 = 0$

$$\sigma(1) = \frac{1}{2}(1-3) = -1$$

$$\rho'(\xi) = 2\xi - 3$$

$$\rho'(1) = -1$$

hence $\rho(1) = 0$ & $\rho'(1) = \sigma(1)$.

The method is consistent.

Root condition:

$$\rho(\xi) = \xi^2 - 3\xi + 2 = 0$$

$$\Rightarrow \xi^2 - 2\xi - \xi + 2 = 0$$

$$\Rightarrow (\xi - 2)(\xi - 1) = 0$$

$$\Rightarrow \xi = 1, \boxed{2.}$$

Hence the root condition is not satisfied.

The given method is not convergent.

Consider the general multi-step method or k-step method

$$u_{j+1} = \sum_{i=1}^{K} a_i u_{j-i+1} + h \sum_{i=0}^{K} b_i u'_{j-i+1}$$

Determination of a_i's & b_i's :

For a linear multistep method of order p, we have

$$\tau_{j+1} = y(t_{j+1}) - \sum_{i=1}^{K} a_i y(t_{j-i+1}) - h \sum_{i=0}^{K} b_i y'(t_{j-i+1})$$

$$= C_{p+1} h^{p+1} y^{(p+1)}(t_j) + O(h^{p+2})$$

Not that the coefficients a_i's and b_i's are independent of $y(t)$. These coefficients can be determined by choosing $y(t) = e^t$. Substituting $y(t) = e^t$ in the above equation to get

$$\left[e^{t_{j+1}} - a_1 e^{t_j} - a_2 e^{t_{j-1}} - \cdots - a_K e^{t_{j-K+1}}\right] - h\left[b_0 e^{t_{j+1}} + b_1 e^{t_j} + \cdots + b_K e^{t_{j-K+1}}\right] = O(h^{p+1})$$

$$\Rightarrow \left[e^{t_{j-K+1}+Kh} - a_1 e^{t_{j-K+1}+(K-1)h} - \cdots - a_K e^{t_{j-K+1}}\right] - h\left[b_0 e^{t_{j-K+1}+Kh} + b_1 e^{\cdots} + \cdots + b_K e^{t_{j-K+1}}\right] = O(h^{p+1})$$

$$\Rightarrow \left\{\left[e^{Kh} - a_1 e^{(K-1)h} - a_2 e^{(K-2)h} - \cdots - a_K\right] - h\left[b_0 e^{Kh} + b_1 e^{(K-1)h} + \cdots + b_K\right]\right\} e^{t_{j-K+1}} = O(h^{p+1})$$

$$\Rightarrow \left[\rho(e^h) - h\sigma(e^h)\right] e^{t_{j-k+1}} = C_{p+1} h^{p+1} e^{t_j} + O(h^{p+2})$$

$$\Rightarrow \quad \rho(e^h) - h\sigma(e^h) = \bar{C}_{p+1} h^{p+1} + O(h^{p+2}) \quad —①$$

Setting $e^h = \xi$, we have $h = \ln(\xi)$

Note that, as $h \to 0$, $\xi \to 1$. We rewrite ① in powers of $(\xi-1)$. We have

$$\ln\xi = \ln\left[(\xi-1)+1\right] = (\xi-1) - \frac{1}{2}(\xi-1)^2 + \cdots\cdots$$

$$h^{p+1} = \left[\ln\xi\right]^{p+1} = (\xi-1)^{p+1} + O(\xi-1)^{p+2}.$$

Now equation ① can be rewritten as

$$\rho(\xi) - \ln(\xi)\,\sigma(\xi) = \bar{C}_{p+1}(\xi-1)^{p+1} + O(\xi-1)^{p+2} \quad —②$$

If $\sigma(\xi)$ is given then this equation (2) can be used to determine $\rho(\xi)$. We expand $\ln\xi$ and $\sigma(\xi)$ in powers of $(\xi-1)$, simplify and retain the terms of required order.

For implicit method, $\rho(\xi)$ & $\sigma(\xi)$ are of same order, whereas, for explicit method $\rho(\xi)$ is one degree higher than the $\sigma(\xi)$.

If $\rho(\xi)$ is given, then the equation (2) can be rewritten as.

$$\frac{\rho(\xi)}{\ln(\xi)} - \sigma(\xi) = \bar{C}_{p+1}(\xi-1)^{p} + O(\xi-1)^{p+1} \quad —③$$

We now expand $\rho(\xi)$ & $\ln(\xi)$ in powers of $(\xi-1)$ simplify and retain the terms of required order.

Adams-Bashforth Methods (Explicit)

$$\rho(\xi) = \xi^{K-1}(\xi-1) \quad \text{and} \quad \sigma(\xi) \text{ is of degree } K-1$$

For K = 2:

$$\rho(\xi) = \xi(\xi-1) = (\xi-1+1)(\xi-1)$$

$$= (\xi-1)^2 + (\xi-1)$$

Now consider

$$\frac{\rho(\xi)}{\ln \xi} = \frac{(\xi-1)^2+(\xi-1)}{\ln[1+(\xi-1)]} = \frac{(\xi-1)^2+(\xi-1)}{(\xi-1)-\frac{1}{2}(\xi-1)^2+\cdots}$$

$$= [1+(\xi-1)]\left[1-\frac{1}{2}(\xi-1)+\cdots\right]^{-1}$$

$$= [1+(\xi-1)]\left[1+\frac{1}{2}(\xi-1)+\cdots\right]$$

$$= \underbrace{\left[1+\frac{3}{2}(\xi-1)\right]}_{\sigma(\xi)} + O(\xi-1)^2$$

Thus we have

$$\sigma(\xi) = \frac{3}{2}\xi - \frac{1}{2}$$

The numerical method becomes:

$$\rho(E)u_{j-1} - h\,\sigma(E)\,u'_{j-1} = 0$$

$$\Rightarrow (E^2-E)u_{j-1} - h\cdot\frac{1}{2}[3E-1]u'_{j-1} = 0$$

$$\Rightarrow \boxed{u_{j+1} = u_j + \frac{h}{2}\left(3u'_j - u'_{j-1}\right)}$$

ORDER OF THE METHOD is 2.

Nyström Methods (Explicit)

$\rho(\xi) = \xi^{k-2}(\xi^2-1)$ and $\sigma(\xi)$ is of order $(k-1)$.

For $k=3$: $\rho(\xi) = \xi(\xi^2-1)$

$$\rho(\xi) = [(\xi-1)+1][(\xi-1+1)^2-1]$$

$$= [(\xi-1)+1][(\xi-1)^2+1+2(\xi-1)-1]$$

$$= (\xi-1)^3+3(\xi-1)^2+2(\xi-1)$$

Now: $$\frac{\rho(\xi)}{\ln(\xi)} = \frac{2(\xi-1)+3(\xi-1)^2+(\xi-1)^3}{\ln(1+(\xi-1))}$$

$$= \frac{2(\xi-1)+3(\xi-1)^2+(\xi-1)^3}{(\xi-1)-\frac{1}{2}(\xi-1)^2+\frac{1}{3}(\xi-1)^3+\cdots}$$

$$= \left[2+3(\xi-1)+(\xi-1)^2\right]\left[1-\frac{1}{2}(\xi-1)+\frac{1}{3}(\xi-1)^2-\cdots\right]^{-1}$$

$$= \left[2+3(\xi-1)+(\xi-1)^2\right]\left[1+\frac{1}{2}(\xi-1)-\frac{1}{3}(\xi-1)^2 + \frac{1}{4}(\xi-1)^2+\cdots\right]$$

$$= 2+4(\xi-1)+(\xi-1)^2\left[1+\frac{3}{2}-\frac{1}{6}\right]+O(\xi-1)^3$$

$$= 2+4(\xi-1)+(\xi-1)^2\left[\frac{7}{3}\right]$$

$$= \frac{7}{3}\xi^2-\frac{2}{3}\xi+\frac{1}{3}.$$

The numerical method becomes

$$\rho(E)\, u_{j-2} - h\,\sigma(E)\, u'_{j-2} = 0$$

$$\Rightarrow (E^3 - E)\, u_{j-2} - h\left[\frac{7}{3}E^2 - \frac{2}{3}E + \frac{1}{3}\right] u'_{j-2} = 0$$

$$\Rightarrow u_{j+1} = u_{j-1} - \frac{h}{3}\left[7u'_j - 2u'_{j-1} + u'_{j-2}\right]$$

The order of the method is 3.

Adams-Moulton Methods (IMPLICIT)

$\rho(\xi) = \xi^{k-1}(\xi - 1)$ and $\sigma(\xi)$ is of degree k.

For $k=2$: $\rho(\xi) = \xi(\xi-1) = (\xi-1) + (\xi-1)^2$

Note that

$$\frac{\rho(\xi)}{\ln \xi} = [1 + (\xi-1)]\left[1 + \frac{1}{2}(\xi-1) - \frac{1}{3}(\xi-1)^2 + \frac{1}{4}(\xi-1)^2 - \cdots\right]$$

$$= [1 + (\xi-1)]\left[1 + \frac{1}{2}(\xi-1) - \frac{1}{12}(\xi-1)^2 + \cdots\right]$$

$$= 1 + \frac{3}{2}(\xi-1) + (\xi-1)^2\left(\frac{1}{2} - \frac{1}{12}\right) + O(\xi-1)^3$$

$$= 1 + \frac{3}{2}(\xi-1) + \frac{5}{12}(\xi-1)^2 + O(\xi-1)^3$$

Therefore we have

$$\sigma(\xi) = 1 + \frac{3}{2}(\xi-1) + \frac{5}{12}(\xi-1)^2$$

$$= \frac{5}{12}\xi^2 + \frac{2}{3}\xi - \frac{1}{12}$$

The desired Adams-Moulton method is

$$\rho(E)u_{j-1} - h\,\sigma(E)\,u'_{j-1} = 0$$

$$\Rightarrow (E^2-E)u_{j-1} - \frac{h}{12}\left[5E^2+8E-1\right]u'_{j-1} = 0$$

$$\Rightarrow \quad u_{j+1} = u_j + \frac{h}{12}\left[5u'_{j+1} + 8u'_j - u'_{j-1}\right]$$

The order of the method is 3.

Milne-Simpson Methods: (IMPLICIT)

$\rho(\xi) = \xi^{k-2}(\xi^2-1)$, $\sigma(\xi)$ is of order k.

For $k=2$: $\rho(\xi) = \xi^2 - 1 = 2(\xi-1) + (\xi-1)^2$

Now: $$\frac{\rho(\xi)}{\ln(\xi)} = \left[2+(\xi-1)\right]\left[1+\frac{1}{2}(\xi-1) - \frac{1}{12}(\xi-1)^2 + \cdots\right]$$

$$= 2 + 2(\xi-1) + \frac{1}{3}(\xi-1)^2 + 0\times(\xi-1)^3 + \mathcal{O}(\xi-1)^4$$

$$\Rightarrow \sigma(\xi) = \frac{1}{3}\left(\xi^2 + 4\xi + 1\right)$$

The method is given by $\rho(E)u_{j-1} - h\,\sigma(E)\,u'_{j-1} = 0$

$$\Rightarrow \quad u_{j+1} = u_{j-1} + \frac{h}{3}\left[u'_{j+1} + 4u'_j + u'_{j-1}\right]$$

The order of the method is 4.

Another class of Numerical methods:

Given $\sigma(\xi) = \xi^k$. Find $\rho(\xi)$ so that the resulting linear multistep method is implicit.

looking for $\rho(\xi)$ of degree k.

For $k=2$: $\quad \sigma(\xi) = \xi^2 = (\xi-1)^2 + 2(\xi-1) + 1$

$$(\ln \xi)\,\sigma(\xi) = \left[(\xi-1) - \tfrac{1}{2}(\xi-1)^2 + \cdots\right]\left[(\xi-1)^2 + 2(\xi-1) + 1\right]$$

$$= (\xi-1) + \left(2-\tfrac{1}{2}\right)(\xi-1)^2 + O(\xi-1)^3$$

$$= \frac{3}{2}\xi^2 - 2\xi + \frac{1}{2} + O(\xi-1)^3$$

The numerical method is given by

$$\rho(E)\,u_{j-1} - h\,\sigma(E)\,u'_{j-1} = 0$$

$$\Rightarrow \left[\tfrac{3}{2}E^2 - 2E + \tfrac{1}{2}\right]u_{j-1} - h\,E^2\,u'_{j-1} = 0$$

$$\Rightarrow \frac{3}{2}u_{j+1} - 2u_j + \frac{1}{2}u_{j-1} = h\,u'_{j+1}$$

ORDER OF THE METHOD $= 2$.

Ex. Given $\sigma(\xi) = (23\xi^2 - 16\xi + 5)/12$.

Find out $\rho(\xi)$ and write down an explicit linear multi-step method.

→ Adams-Bashforth method of order 3.

Ex. Derive a fourth order method of the form

$$u_{n+1} = a u_{n-2} + h(b u_n' + c u_{n-1}' + d u_{n-2}' + e u_{n-3}')$$

for the solution of $y' = f(x,y)$.

Sol: The local truncated error of the method is given by

$$\tau_{n+1} = y(x_{n+1}) - a y(x_{n-2}) - h[b y'(x_n) + c y'(x_{n-1}) + d y'(x_{n-2}) + e y'(x_{n-3})]$$

$$= y(x_n) + h y'(x_n) + \frac{h^2}{\lfloor 2} y''(x_n) + \frac{h^3}{\lfloor 3} y'''(x_n) + \frac{h^4}{\lfloor 4} y^{(iv)}(x_n) + O(h^5)$$

$$-a\left[y(x_n) - 2h y'(x_n) + \frac{4h^2}{\lfloor 2} y''(x_n) + \cdots\cdots + O(h^5)\right]$$

$$-h\left[b\{y'(x_n)\} + c\{y'(x_n) - h y''(x_n) + \frac{h^2}{\lfloor 2} y'''(x_n) - \cdots\} + d\{\cdots\cdots\} + e\{\cdots\cdots\}\right]$$

To determine a, b, c, d, e, we have

$$\left.\begin{aligned} &1 - a = 0 \\ &1 + 2a - (b + c + d + e) = 0 \\ &\tfrac{1}{2}(1 - 4a) + (c + 2d + 3e) = 0 \\ &\tfrac{1}{6}(1 + 8a) - \tfrac{1}{2}(c + 4d + 9e) = 0 \\ &\tfrac{1}{24}(1 - 16a) + \tfrac{1}{6}(c + 8d + 27e) = 0 \end{aligned}\right\} \quad \begin{aligned} a &= 1 \\ b &= \tfrac{21}{8} \\ c &= -9/8 \\ d &= 15/8 \\ e &= -3/8 \end{aligned}$$

The method can be written as

$$u_{n+1} = u_{n-2} + \frac{h}{8}(21 u_n' - 9 u_{n-1}' + 15 u_{n-2}' - 3 u_{n-3}')$$

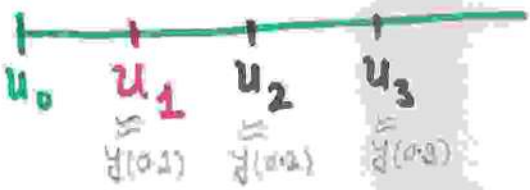
Ex. Find the solution at $x = 0.3$ for the differential equation

$$y' = x - y^2 \qquad y(0) = 1,$$

by the Adams-Bashforth Method of order two with $h = 0.1$. Determine the starting values using a second order Runge-Kutta method.

Sol: The second order Adams-Bashforth method is

$$u_{j+1} = u_j + \frac{h}{2}\left(3u'_j - u'_{j-1}\right) \qquad j = 1, 2, \ldots$$



We need to find the value of u_1 in order to start the computation. The second order Runge-Kutta method

$$u_{j+1} = u_j + \frac{h}{2}(k_1 + k_2)$$

$$k_1 = f(x_j, u_j)$$

$$k_2 = f(x_j + h, u_j + hk_1)$$

$$k_1 = 0 - 1^2 = -1$$

$$k_2 = f(0.1, 1 - 0.1) = 0.1 - (0.9)^2 = -0.71$$

$$u_1 = 1 + \frac{0.1}{2}(-1 - 0.71) = 0.9145$$

Using AB method:

$$u_2 = 0.9145 + \frac{0.1}{2}\left[3\underbrace{(0.1 - 0.9145^2)}_{=0.73631} - (0 - 1^2)\right] = 0.85405$$

$$u_3 = u_2 + \frac{h}{2}\left[3u_2' - u_1'\right]$$

$$= 0.85405 + \frac{0.1}{2}\left[3(0.2 - 0.85405^2) + 0.73631\right]$$

$$= 0.81146$$

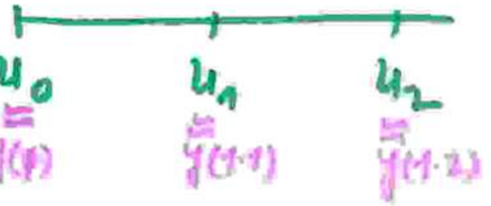
PROBLEM: For the initial value problem

$$y' = t^2 + u^2 \qquad y(1) = 2$$

find an estimate for $y(1.2)$ using the Adams-Moulton third order method with $h = 0.1$. Use Taylor's series method of order 3 in order to determine starting values.

Solution: Adams-Moulton method:

$$u_{j+1} = u_j + \frac{h}{12}\left[5u_{j+1}' + 8u_j' - u_{j-1}\right] \qquad j = 1$$



We need to get u_1 to apply Adams-Moulton Method:

Taylor series method:

$$u_1 = u_0 + h u_0' + \frac{h^2}{\lfloor 2} u_0'' + \frac{h^3}{\lfloor 3} u_0'''$$

$h = 0.1$, $u_0 = 2$

$u_0' = t_0^2 + u_0^2 = 1^2 + 4 = 5$

$$u_0'' = 2t_0 + 2u_0 u_0' = 2\times 1 + 2\times 2\times 5 = 22$$

$$u_0''' = 2 + 2u_0'^2 + 2u_0 u_0'' = 2 + 2\times 25 + 2\times 2\times 22$$
$$= 140$$

Therefore

$$u_1 = 2 + 0.1\times 5 + \frac{0.1^2}{\lfloor 2}\times 22 + \frac{0.1^3}{\lfloor 3}\times 140$$
$$= 2.633333$$

Using Adams-Moulton method:

$$u_2 = 2.633333 + \frac{0.1}{12}\left[5(1.2^2 + u_2^2) + 8(1.1^2 + 2.633333^2) - 5\right]$$

$$\Rightarrow u_2 = 0.041667\, u_2^2 + 3.194629$$

Newton Raphson method:

$$F(u_2) = 0.041667\, u_2^2 - u_2 + 3.194629$$

$$F'(u_2) = 0.083334\, u_2 - 1$$

$$u_2^{(s+1)} = u_2^{(s)} - \frac{F(u_2^{(s)})}{F'(u_2^{(s)})} \qquad s = 0, 1, 2 \ldots$$

$$u_2^{(0)} = u_1 = 2.633333$$

$$\Rightarrow u_2^{(3)} = 3.794588$$

$$y(1.2) \approx 3.794588$$

PREDICTOR-CORRECTOR METHODS:

If a predictor method (explicit method) is used to predict a value of $u_{n+1}^{(0)}$ and this value is taken as the starting approximation of the iteration for obtaining u_{n+1} using the corrector method (Implicit method), such combination of methods are called Predictor-corrector methods.

Suppose we want to use the implicit method

$$u_{j+1} = h b_0 f_{j+1} + \underbrace{\sum_{i=1}^{k} \left(a_i u_{j-i+1} + h b_i f_{j-i+1} \right)}_{=: C}$$

OR

$$u_{j+1} = h b_0 f_{j+1} + C$$

We first use the explicit (predictor) method for predicting $u_{j+1}^{(0)}$ and then use the implicit (corrector) method iteratively until the convergence is obtained.

P: Predict some value $u_{j+1}^{(0)}$

E: Evaluate $f(x_{j+1}, u_{j+1}^{(0)})$

C: Correct $u_{j+1}^{(1)} = h b_0 f(x_{j+1}, u_{j+1}^{(0)}) + C$

E: Evaluate $f(x_{j+1}, u_{j+1}^{(1)})$

C: Correct $u_{j+1}^{(2)} = h b_0 f(x_{j+1}, u_{j+1}^{(1)}) + C$

The sequence of operations PECECE....CE is denoted by $P(EC)^m E$ and is called a predictor corrector method.

Examples:

Modified Euler Method

$$P: \quad u_{j+1} = u_j + h f_j \qquad \text{(Euler method)}$$

$$C: \quad u_{j+1} = u_j + \frac{h}{2}(f_j + f_{j+1}) \qquad \text{(Euler-Cauchy)}$$

The Adams-Bashforth-Moulton method

$$P: \quad u_{j+1} = u_j + \frac{h}{2}(3u'_j - u'_{j-1})$$

$$C: \quad u_{j+1} = u_j + \frac{h}{12}(5u'_{j+1} + 8u'_j - u'_{j-1})$$

Example: Solve the IVP

$$\frac{dy}{dx} = x + y, \qquad y = 1 \text{ when } x = 0$$

with $h = 0.1$ on the interval $[0, 0.2]$ using the P-C method:

$$P: \quad u_{j+1} = u_j + h f_j$$

$$C: \quad u_{j+1} = u_j + \frac{h}{2}(f_j + f_{j+1})$$

as $P(EC)^2E$.

Sol: P: $u_1^{(0)} = u_0 + h f_0$

$$= 1 + 0.1 \times (0 + 1) = 1.1$$

E. $f(x_1, u_1^{(0)}) = (0.1 + 1.1) = 1.2$

C: $u_1^{(1)} = u_0 + \frac{h}{2}(f_0 + f_1)$

$$u_1^{(1)} = 1 + \frac{0.1}{2}(1 + 1.2) = \underline{1.11.}$$

$$E: \quad f(x_1, y_1^{(1)}) = 0.1 + 1.11 = 1.21$$

$$C: \quad u_1^{(2)} = u_0 + \frac{h}{2}\left(f(x_1, \underline{y_1^{(1)}}) + f(x_0, y_0)\right)$$

$$= 1 + \frac{0.1}{2}(1.21 + 1) = 1.1105$$

$$y_2^{(0)} = u_1 + h f_1 = 1.1105 + 0.1(0.1 + 1.1105)$$

$$= 1.2316$$

$$u_2^{(1)} = u_1 + \frac{h}{2}\left(f(x_2, u_2^{(0)}) + f(x_1, u_1)\right)$$

$$= 1.1105 + \frac{0.1}{2}\left((0.2 + 1.2316) + (0.1 + 1.1105)\right)$$

$$= 1.2426$$

$$u_2^{(2)} = 1.1105 + \frac{0.1}{2}\left((0.2 + 1.2426) + 1.2105\right)$$

$$= 1.2432.$$

Hence.

$$y(0.1) \approx 1.1105$$

$$y(0.2) \approx 1.2432.$$

Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$.

Find $y(0.1)$, $y(0.2)$, $y(0.3)$ using a sixth order Taylor's series method. Using these values, calculate $y(0.4)$ using the following P-C set

$$P: \quad u_{n+4} = u_n + \frac{4}{3}h[2f_{n+1} - f_{n+2} + 2f_{n+3}]$$

$$C: \quad u_{n+4} = u_{n+2} + \frac{h}{3}[f_{n+2} + 4f_{n+3} + f_{n+4}]$$

Sol: Note that the Taylor's formula is given by

$$y(x) \simeq y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(iv)}(0) + \frac{x^5}{5!}y^{(v)}(0) + \frac{x^6}{6!}y^{(vi)}(0)$$

Given: $y(0) = 1 \quad y'(0) = 0$

From the differential equation:

$$y'' = -xy' - y \Rightarrow y''(0) = -1$$

Diff. the given DE:

$$y''' = -xy'' - 2y' \Rightarrow y'''(0) = 0$$

$$y^{(iv)} = -xy^{(iii)} - 3y'' \Rightarrow y^{(iv)}(0) = 3$$

$$y^{(v)} = -xy^{(iv)} - 4y''' \Rightarrow y^{(v)}(0) = 0$$

$$y^{(vi)} = -xy^{(v)} - 5y^{(iv)} \Rightarrow y^{(vi)}(0) = -15$$

Hence,

$$y(x) \simeq y(0) + \frac{x^2}{2!}(-1) + \frac{x^4}{4!} \times 3 + \frac{x^6}{6!} \cdot (-15)$$

$\Rightarrow \quad y(0.1) \simeq 0.995$

$y(0.2) \simeq 0.9802$

$y(0.3) \simeq 0.956$

However, in order to compute further values using P-C method we need to transfer higher order into a system of first order DEs.

set $y' = z \Rightarrow z' = -(xz+y)$

$$\Rightarrow \begin{bmatrix} y \\ z \end{bmatrix}' = \begin{bmatrix} z \\ -(xz+y) \end{bmatrix} =: \begin{bmatrix} f \\ g \end{bmatrix} \qquad \begin{array}{l} y(0) = 1 \\ z(0) = 0 \end{array}$$

Once again using Taylor's series:

$$z(x) = y'(x) = -x + \frac{x^3}{2} - \frac{1}{4}x^5$$

$$z(0.1) = -0.0995$$

$$z(0.2) = -0.1960$$

$$z(0.3) = -0.2863$$

P: $$\begin{bmatrix} y(0.4) \\ z(0.4) \end{bmatrix} = \begin{bmatrix} y(0) \\ z(0) \end{bmatrix} + \frac{4}{3}h \begin{bmatrix} 2f_1 - f_2 + 2f_3 \\ 2g_1 - g_2 + 2g_3 \end{bmatrix}$$

where $h = 0.1$ $\quad f_i = f(x_i, y_i, z_i)$

& $g_i = g(x_i, y_i, z_i)$

$$\Rightarrow \begin{bmatrix} y(0.4) \\ z(0.4) \end{bmatrix} = \begin{bmatrix} 0.9231 \\ -0.3692 \end{bmatrix}.$$

C: $$\begin{bmatrix} y(0.4) \\ z(0.4) \end{bmatrix} = \begin{bmatrix} y(0.2) \\ z(0.2) \end{bmatrix} + \frac{h}{3} \begin{bmatrix} f_2 + 4f_3 + f_4 \\ g_2 + 4g_3 + g_4 \end{bmatrix}$$

f_4 & g_4 can be evaluated using $y(0.4), z(0.4)$ form ~~pr~~predictor formula.

$$\Rightarrow \begin{bmatrix} y(0.4) \\ z(0.4) \end{bmatrix} = \begin{bmatrix} 0.9232 \\ -0.3692 \end{bmatrix}$$

One can improve these values by repeating corrector formula.

DIFFERENCE EQUATIONS

A difference equation of order k is given as

$$F(u_n, u_{n+1}, \dots u_{n+k}) = 0 \quad -(1)$$

(a relation among $u_n, u_{n+1}, \dots u_{n+k}$)

Order = largest index − smallest index

$= n+k-n = k$

Ex: $u_{n+2} + u_{n+1} + 3u_n = 0$

order $= n+2-n = 2$

If F in (1) is linear then the difference equation is called linear, otherwise non-linear.

A general linear difference equation of order k can be written as

$$a_0 u_{n+k} + a_1 u_{n+k-1} + \dots + a_k u_n = g_n \quad a_0 \neq 0 \quad -(2)$$

if $a_0, a_1, \dots a_k$ are constants then the difference equation is called a linear difference equation with constant coefficients.

If $g_n = 0$ homogeneous

$g_n \neq 0$ non-homogeneous (inhomogeneous)

The general solution of (2) is of the form

$$u_n = u_n^{(H)} + u_n^{(P)}$$

where $u_n^{(H)}$ is the solution of the associated homogeneous difference equation

$$a_0 u_{n+k} + a_1 u_{n+k-1} + \cdots + a_k u_n = 0$$

and $u_n^{(P)}$ is any particular solution of (2).

For solving homogeneous equation we assume

$$u_n = A\xi^n \quad \text{where } A \neq 0 \text{ is constant.}$$

Substituting into the difference equation

$$A\left[a_0 \xi^k + a_1 \xi^{k-1} + \cdots + a_k\right]\xi^n = 0$$

$$\Rightarrow a_0 \xi^k + a_1 \xi^{k-1} + \cdots + a_k = 0 \qquad \text{——— (3)}$$

The equation (3) is called a characteristic equation.

Let $\xi_1, \xi_2, \cdots \xi_k$ be the roots of (3), then we have the following cases:

(I) Real and Distinct roots:

$$u_n^{(H)} = c_1 \xi_1^n + c_2 \xi_2^n + \cdots + c_k \xi_k^n.$$

(II) Real and Repeated roots:

Let $\xi_1 (= \xi_2)$ be a double root and $\xi_3, \xi_4, \cdots, \xi_k$ are distinct. Then

$$u_n^{(H)} = (c_1 + nc_2)\xi_1^n + c_3 \xi_3^n + \cdots + c_k \xi_k^n.$$

(III) Complex roots:

The complex roots occur as conjugate pair.

Let $\xi_1 = \alpha + i\beta = re^{i\theta}$ and $\xi_2 = \alpha - i\beta = re^{-i\theta}$

where $r = \sqrt{\alpha^2+\beta^2}$ $\theta = \tan^{-1}(\beta/\alpha)$.

Then $$u_n^{(H)} = \left[c_1 \cos(n\theta) + c_2 \sin(n\theta)\right]|\xi_1|^n + c_3\xi_3^n + \cdots$$
$$\cdots + c_k \xi_k^n .$$

The particular solution depends on the form of g_n.

If $g_n = g$ (a constant)

then $$u_n^{(p)} = \left(\frac{g}{a_0 + a_1 + \dots\dots + a_k}\right)$$

Some usefull observation:

- Suppose we require $u_n^h \to 0$ as $n \to \infty$,

  then the necessary and sufficient condition is:

  $$|\xi_i| < 1$$

- Suppose we require u_n^h to be BOUNDED as $n \to \infty$.

  the the necessary and sufficient condition is:

  ξ_i lie inside the unit circle in the complex plane and are simple if they lie on the unit circle.

  [Root condition].

Routh-Hurwitz Criterion :

It is not always possible to find roots of characteristic equation to check $|\xi_i| < 1$, specially when the degree of the characteristic equation is high.

This can be done without calculating roots of the characteristic equation explicitely using Routh-Hurwitz Criterion.

a) consider the following mapping :

$$\xi = \frac{1+z}{1-z} \quad \text{OR} \quad z = \frac{\xi-1}{\xi+1}$$

which maps the interior of the unit circle $|\xi| = 1$ on to the left half plane $Re(z) < 0$, and the unit circle $|\xi| = 1$ onto the imaginary axis.

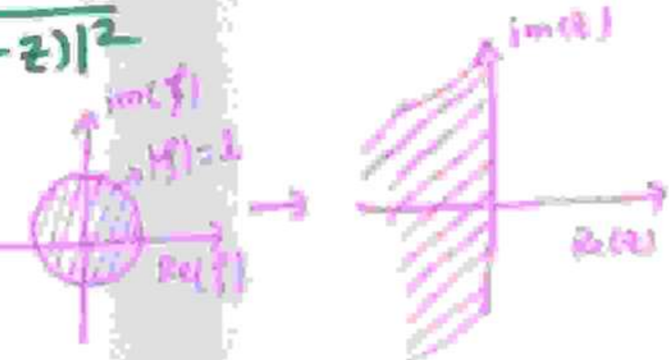
consider

$$\xi\bar{\xi} - 1 = \frac{1+z}{1-z}\,\frac{1+\bar{z}}{1-\bar{z}} - 1$$

$$= \frac{\cancel{1}+\bar{z}+z+\cancel{z\bar{z}} - (\cancel{1}-\bar{z}-z+\cancel{z\bar{z}})}{(1-z)(1-\bar{z})}$$

$$|\xi|^2 - 1 = \frac{2(z+\bar{z})}{|(1-z)|^2}$$

$$\Rightarrow \boxed{|\xi|^2 - 1 = \frac{4\,Re(z)}{|(1-z)|^2}}$$



- $|\xi| = 1 \Rightarrow Re(z) = 0$
- $|\xi| < 1 \Rightarrow Re(z) < 0$.

b) Routh-Hurwitz Criterion: substituting $\xi = \frac{1+z}{1-z}$ into

$$a_0\xi^K + a_1\xi^{K-1} + \cdots + a_K = 0, \text{ we get}$$

$$b_0 z^K + b_1 z^{K-1} + \cdots + b_K = 0 \qquad (*)$$

This is called transformed characteristic equation. Let $b_0 > 0$.

Denote:

$$D = \begin{bmatrix} b_1 & b_3 & b_5 & \cdots & b_{2K-1} \\ b_0 & b_2 & b_4 & \cdots & b_{2K-2} \\ 0 & b_1 & b_3 & \cdots & b_{2K-3} \\ 0 & b_0 & b_2 & \cdots & b_{2K-4} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & b_K \end{bmatrix}, \quad b_j = 0 \quad j > K$$

Routh-Hurwitz criterion states that the real part of the roots of (*) are negative if and only if the principal minors of D are positive, i.e.,

necessary condition for roots of (*) (real parts) to be negative is that all the coefficients b_i must be of same sign. Routh Hurwitz provides necessary and suff. condition.

K=1: $b_0 > 0, \ \underbrace{|b_1|}_{\det(b_1)} > 0 \Rightarrow b_0 > 0, \ b_1 > 0$

K=2: $b_0 > 0; \ b_1 > 0; \ \begin{vmatrix} b_1 & 0 \\ b_0 & b_2 \end{vmatrix} = b_1 b_2 > 0$

$\Rightarrow b_0 > 0; \ b_1 > 0; \ b_2 > 0.$ (assuming necessary condition)

K=3:

$$b_0 > 0; \quad b_1 > 0; \quad \underbrace{\begin{vmatrix} b_1 & b_3 \\ b_0 & b_2 \end{vmatrix} > 0}_{(b_1 b_2 - b_0 b_3) > 0}; \quad \underbrace{\begin{vmatrix} b_1 & b_3 & 0 \\ b_0 & b_2 & 0 \\ 0 & b_1 & b_3 \end{vmatrix}}_{b_3(b_1 b_2 - b_0 b_3) > 0} > 0$$

$$\Rightarrow \quad b_0 > 0; \quad b_1 > 0; \quad \underbrace{(b_1 b_2 - b_0 b_3)}_{\Rightarrow b_2 > 0} > 0; \quad b_3 > 0$$

$$\Rightarrow b_0 > 0, \ b_1 > 0, \ b_2 > 0, \ b_3 > 0, \ (b_1 b_2 - b_0 b_3) > 0.$$

K=4: Similarly:

$$b_i > 0 \ (i = 0, \ldots, 4); \ (b_1 b_2 - b_0 b_3) > 0;$$

$$(b_1 b_2 b_3 - b_1^2 b_4 - b_0 b_3^2) > 0.$$

Routh-Hurwitz Criterion

The roots of the characteristic equation

$$b_0 z^k + b_1 z^{k-1} + \dots + b_k = 0$$

have negative real part iff all the principal diagonal minors of the Hurwitz matrix are positive provided $b_0 > 0$.

If one or more of b_i's are equal to zero and other b_j's are positive, then it indicates that a root lies on the circle $|\xi| = 1$.

If one or more of b_j's are negative, then there is atleast one root for which $|\xi_i| > 1$.

Ex: Check if real part all the roots of the characteristic equation

$$z^4 + 2z^3 + 4z^2 + 7z + 3 = 0$$

are negative.

Sol:

$b_0 = 1 \quad b_1 = 2 \quad b_2 = 4 \quad b_3 = 7 \quad b_4 = 3.$

$$D = \begin{bmatrix} b_1 & b_3 & 0 & 0 \\ b_0 & b_2 & b_4 & 0 \\ 0 & b_1 & b_3 & 0 \\ 0 & b_0 & b_2 & b_4 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 0 & 0 \\ 1 & 4 & 3 & 0 \\ 0 & 2 & 7 & 0 \\ 0 & 1 & 4 & 3 \end{bmatrix}$$

$\Delta_1 = 2 > 0$; $\quad \Delta_2 = \begin{vmatrix} 2 & 7 \\ 1 & 4 \end{vmatrix} = 1 > 0$

$$\Delta_3 = \begin{vmatrix} 2 & 7 & 0 \\ 1 & 4 & 3 \\ 0 & 2 & 7 \end{vmatrix} = 2(28-6) - 7(7)$$
$$= 44 - 49 = -5 < 0.$$

Hence some root(s) has/have ~~positive~~ non-negative real part.

Ex: Find the general solution of the difference equation

$$u_{n+2} - 5u_{n+1} + 6u_n = 0$$

Sol: Substituting $u_n = A\xi^n$, we get the characteristic equation

$$\xi^2 - 5\xi + 6 = 0 \Rightarrow \xi = 2, 3.$$

general solution: $u_n = c_1 (2)^n + c_2 (3)^n$.

Ex: Find the range of α, so that the roots of the characteristic equation of the difference equations

$$(1-5\alpha) y_{n+2} - (1+8\alpha) y_{n+1} + \alpha y_n = 0$$

are less than 1 in magnitude.

Sol: The characteristic equation

$$(1-5\alpha)\xi^2 - (1+8\alpha)\xi + \alpha = 0$$

setting $\xi = \frac{1+z}{1-z}$, we get transformed characteristic equation

$$(1-5\alpha)\frac{(1+z)^2}{(1-z)^2} - (1+8\alpha)\frac{1+z}{1-z} + \alpha = 0$$

$$\Rightarrow (1-5\alpha)(1+z)^2 - (1+8\alpha)(1+z)(1-z) + \alpha(1-z)^2 = 0$$

$$\Rightarrow (2+4\alpha) z^2 + (2-12\alpha) z - 12\alpha = 0$$

The Routh-Hurwitz Criterion is satisfied if

$$2+4\alpha > 0, \qquad 2-12\alpha > 0, \qquad -12\alpha > 0$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\underbrace{\alpha > -\tfrac{1}{2} \qquad \alpha < \tfrac{1}{6} \qquad \alpha < 0}_{\alpha \in (-\frac{1}{2}, 0)}$$

Therefore $|\xi| < 1$ for all $\alpha \in (-\frac{1}{2}, 0)$

Stability Analysis:

Consider the linear multi-step method:

$$u_{j+1} = \sum_{i=1}^{k} a_i u_{j-i+1} + h \sum_{i=0}^{k} b_i u'_{j-i+1} \quad \text{—— (1)}$$

OR

$$\rho(E) u_{j-k+1} - h \sigma(E) u'_{j-k+1} = 0$$

where

$$\rho(\xi) = \xi^{k} - a_1 \xi^{k-1} - \cdots - a_{k-1}\xi - a_k$$

$$\sigma(\xi) = b_0 \xi^{k} + b_1 \xi^{k-1} + \cdots + b_{k-1}\xi + b_k$$

Applying (1) to the test equation $y' = \lambda y$, we get

$$u_{j+1} = \sum_{i=1}^{k} a_i u_{j-i+1} + h\lambda \sum_{i=0}^{k} b_i u_{j-i+1} \quad \text{—— (2)}$$

The exact solution satisfies:

$$y(t_{j+1}) = \sum_{i=1}^{k} a_i y(t_{j-i+1}) + \underbrace{\lambda h}_{=:\bar{h}} \sum_{i=0}^{k} b_i y(t_{j-i+1}) + T_{j+1} \quad \text{—— (3)}$$

Substracting (3) from (2) and setting $\varepsilon_j = u_j - y(t_j)$, we get

$$\varepsilon_{j+1} = \sum_{i=1}^{k} a_i \varepsilon_{j-i+1} + \bar{h} \sum_{i=0}^{k} b_i \varepsilon_{j-i+1} - T_{j+1}$$

OR $$\left[\rho(E) - \bar{h}\sigma(E)\right] \varepsilon_{j-k+1} + T_{j+1} = 0 \quad \text{—— (4)}$$

This is a kth order, linear, non-homogeneous difference equation with constant coefficients. For simplicity, we assume $T_{j+1} = T$ (some constant).

Solution of difference equation (4):

we first find the solution of the homogeneous equation

$$[\rho(E) - \bar{h}\sigma(E)]\, \varepsilon_{j-k+1} = 0 \quad \text{—— } (4')$$

The characteristic equation is given as

$$\rho(\xi) - \bar{h}\,\sigma(\xi) = 0 \quad \text{——— } (5)$$

Let the roots are $\xi_{1h}, \xi_{2h}, \ldots, \xi_{kh}$ and they are distinct. Then, the solution of (4') is given by

$$\varepsilon_j^H = c_1 \xi_{1h}^j + c_2 \xi_{2h}^j + \cdots + c_k \xi_{kh}^j .$$

The particular solution is given as

$$\varepsilon_j^P = -\frac{T}{[\rho(1) - \bar{h}\,\sigma(1)]}$$

For a consistent method, we have $\rho(1) = 0$ & $\sigma(1) = \rho'(1)$, then

$$\varepsilon_j^P = \frac{T}{\bar{h}\,\rho'(1)}$$

Hence, the general solution of (4) is given as

$$\varepsilon_j = c_1 \xi_{1h}^j + c_2 \xi_{2h}^j + \cdots + c_k \xi_{kh}^j + \frac{T}{\bar{h}\,\rho'(1)} \quad \text{—— } (6)$$

For $h \to 0$, the roots of the ch. equation (5) approaches to the roots of $\boxed{\rho(\xi) = 0}$ ——— (7)

The equation (7) is called reduced characteristic equation.

If $\xi_1, \xi_2, \dots, \xi_k$ are the roots of $\rho(\xi) = 0$, then for sufficiently small $\bar{h}$, we may write

$$\xi_{ih} = \xi_i \left(1 + \bar{h} K_i + O(|\bar{h}|^2)\right), \quad i = 1, 2, \dots k. \qquad -(8)$$

The coefficient K_i's are called the growth parameters.

Subst. (8) into the characteristic equation (5)

$$\rho\left(\xi_i + \bar{h} K_i \xi_i + O(|\bar{h}|^2)\right) - \bar{h}\sigma\left(\xi_i + \bar{h} K_i \xi_i + O(|\bar{h}|^2)\right) = 0$$

Expanding into the Taylor's series, we get

$$\rho(\xi_i) + \bar{h} K_i \xi_i \rho'(\xi_i) - \bar{h}\sigma(\xi_i) + O(|\bar{h}|^2) = 0$$

Since $\rho(\xi_i) = 0$ we get

$$\boxed{K_i \doteqdot \frac{\sigma(\xi_i)}{\xi_i \, \rho'(\xi_i)}}$$

Remark: since the method is consistent, $\rho(1) = 0$, $\rho'(1) = \sigma(1)$

$\Rightarrow \xi_1 = 1$ & $\rho'(1) = \sigma(1)$.

Then $K_1 \doteqdot \dfrac{\sigma(1)}{\rho'(1)} = 1$

Now consider the error equation

$$\varepsilon_j = C_1 \xi_{1h}^j + C_2 \xi_{2h}^j + \dots + C_k \xi_{kh}^j + \frac{T}{\bar{h}\,\rho'(1)}.$$

- If any of the roots $\xi_{ih}, i = 1, 2, \dots k$ satisfy $|\xi_{ih}| > 1$, then the error $|\varepsilon_j|$ grows unboundedly.
- If there is a multiple root of magnitude unity, then again $|\varepsilon_j|$ grows unboundedly.
- If the roots ξ_{ih} are simple and some of them have magnitude unity, then a fixed amount of error is retained in the numerical solution.

Stability of multi-step Method

Linear-multistep method:

$$\rho(E)u_{j-k+1} - h\,\sigma(E)\,u'_{j-k+1} = 0$$

Characteristic equation:

$$\rho(\xi) - \bar{h}\,\sigma(\xi) = 0 \quad , \quad \bar{h} = \lambda h \; , \; \text{roots } \xi_{ih} \, , \, i = 1,2 \cdots k$$

Reduced characteristic equation:

$$\rho(\xi) = 0 \qquad \text{roots } \xi_i \, , \, i = 1,2,\cdots k.$$

Growth parameter:

$$k_i \simeq \frac{\sigma(\xi_i)}{\xi_i\,\rho'(\xi)} \; ; \quad i = 1,2,\cdots k.$$

$$\xi_{ih} = \xi_i\left(1 + \bar{h}k_i + O(|\bar{h}|^2)\right) \quad i = 1,2,\cdots k.$$

Error equation

$$\varepsilon_j = c_1\xi_{1h}^j + c_2\xi_{2h}^j + \cdots + c_k\xi_{kh}^j + \frac{T}{\bar{h}\,\rho'(1)}$$

Solution of test problem using l.m.s method:

$$u_j = C_1\xi_{1h}^j + C_2\xi_{2h}^j + \cdots + C_k\xi_{kh}^j$$

Exact solution of the test problem $y' = \lambda y$:

$$y = c\,e^{\lambda t}.$$

Some more observations:

Not $$\xi_{ih}^{j} = \xi_i^{j}\left[1+\bar{h}\kappa_i+O(|\bar{h}|^2)\right]^{j}$$
$$\simeq \xi_i^{j} e^{\bar{h}\kappa_i j} \qquad i=1,\dots,k.$$

For a consistent method, $\xi_1 = 1$, $\kappa_1 = 1$, then

$$\xi_{1h}^{j} \simeq e^{j\bar{h}}$$

Here the root ξ_{1h} approximate the solution of the diff. equation $y' = \lambda y$. This root is called principal root and the remaining $(k-1)$ roots are called the extraneous roots. Therefore, for a convergent method it is essential that the principal root is dominant. This leads to a definition of relative stability of a multistep method.

Definitions: The multistep method is said to be

- Stable if $|\xi_i| < 1$, $i \neq 1$.
- Unstable if $|\xi_i| > 1$ for some i or there is a multiple root of $\rho(\xi)=0$ of magnitude unity.
- weakly stable or conditionally stable: if ξ_i's are simple and if more than one of these roots have modulus unity.
- absolutely stable: if $\exists\, h_0 > 0$ such that $|\xi_{ih}| < 1$, $i = 1,2,\dots k$ $\forall\, h \leq h_0$.
- A-stable: if the interval of absolutely stability is $(-\infty, 0)$
- relative stability if $|\xi_{ih}| < |\xi_{1h}|$, $i = 2,3,\dots k$.

The region of * stability is defined to be the set of points in the λh-plane for which the method is ~~relatively~~ * stable.

Ex: Find the interval of absolute stability for the third order Adams Moulton method

$$u_{j+1} = u_j + \frac{h}{12}\left(5u'_{j+1} + 8u'_j - u'_{j-1}\right)$$

Sol: Applying the method on the test equation

$y' = \lambda y$, $\lambda < 0$, we obtain

$$u_{j+1} = u_j + \frac{h}{12}\left[5\lambda u_{j+1} + 8\lambda u_j - \lambda u_{j-1}\right]$$

$$\Rightarrow \left[1 - \frac{5\lambda h}{12}\right]u_{j+1} - \left[1 + \frac{8\lambda h}{12}\right]u_j + \frac{\lambda h}{12}u_{j-1} = 0$$

The characteristic equation:

$$\left(1 - \frac{5\lambda h}{12}\right)\xi^2 - \left(1 + \frac{8}{12}\lambda h\right)\xi + \frac{\lambda h}{12} = 0$$

substituting $\xi = \frac{1+z}{1-z}$ and simplifying

$$\left(1 - \frac{5\lambda h}{12}\right)\left(\frac{1+z}{1-z}\right)^2 - \left(1 + \frac{8}{12}\lambda h\right)\left(\frac{1+z}{1-z}\right) + \frac{\lambda h}{12} = 0$$

$\vdots$

$$\nu_0 z^2 + 2z\nu_1 + \nu_2 = 0$$

where $\nu_0 = 1 - \frac{5\lambda h}{12} + 1 + \frac{8}{12}\lambda h + \frac{\lambda h}{12} = 2 + \frac{\lambda h}{3}$

$\nu_1 = 2 - \lambda h \qquad \nu_2 = -\lambda h$

Using Routh-Hurwitz Criterion:

$\nu_0 > 0 \Rightarrow \lambda h > -6$

$\nu_1 > 0 \Rightarrow \lambda h < 2$

$\nu_2 > 0 \Rightarrow \lambda h < 0$

Interval of absolute stability $\lambda h \in (-6, 0)$

Example: Discuss the relative and absolute stability of the second order Adams-Bashforth method

$$u_{j+1} = u_j + \frac{h}{2}(3u'_j - u'_{j-1})$$

Sol: Applying the given multistep method to the test equation $y' = \lambda y$:

$$u_{j+1} = u_j + \frac{h}{2}\left(3\lambda u_j - \lambda u_{j-1}\right)$$

$$\Rightarrow \quad u_{j+1} - \left(1 + \frac{3}{2}\lambda h\right)u_j + \frac{\lambda h}{2} u_{j-1} = 0 \qquad \text{—— (1)}$$

Characteristic equation

$$\xi^2 - \left(1 + \frac{3}{2}\bar{h}\right)\xi + \frac{\bar{h}}{2} = 0$$

Its roots:

$$\xi = \frac{\left(1 + \frac{3}{2}\bar{h}\right) \pm \sqrt{\left(1+\frac{3}{2}\bar{h}\right)^2 - 2\bar{h}}}{2}$$

$$= \frac{1}{4}\left[(2+3\bar{h}) \pm \sqrt{4 + 9\bar{h}^2 + 12\bar{h} - 8\bar{h}}\right]$$

$$= \frac{1}{4}\left[(2+3\bar{h}) \pm \sqrt{4 + 4\bar{h} + 9\bar{h}^2}\right]$$

$$= \frac{1}{4}\left[2 + 3\bar{h} \pm 2\left(1 + \bar{h} + \frac{9}{4}\bar{h}^2\right)^{1/2}\right]$$

$$= \frac{1}{4}\left[2 + 3\bar{h} \pm 2\left(1 + \frac{\bar{h}}{2} + \frac{9}{8}\bar{h}^2 + \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\bar{h}^2 + \cdots\right)\right]$$

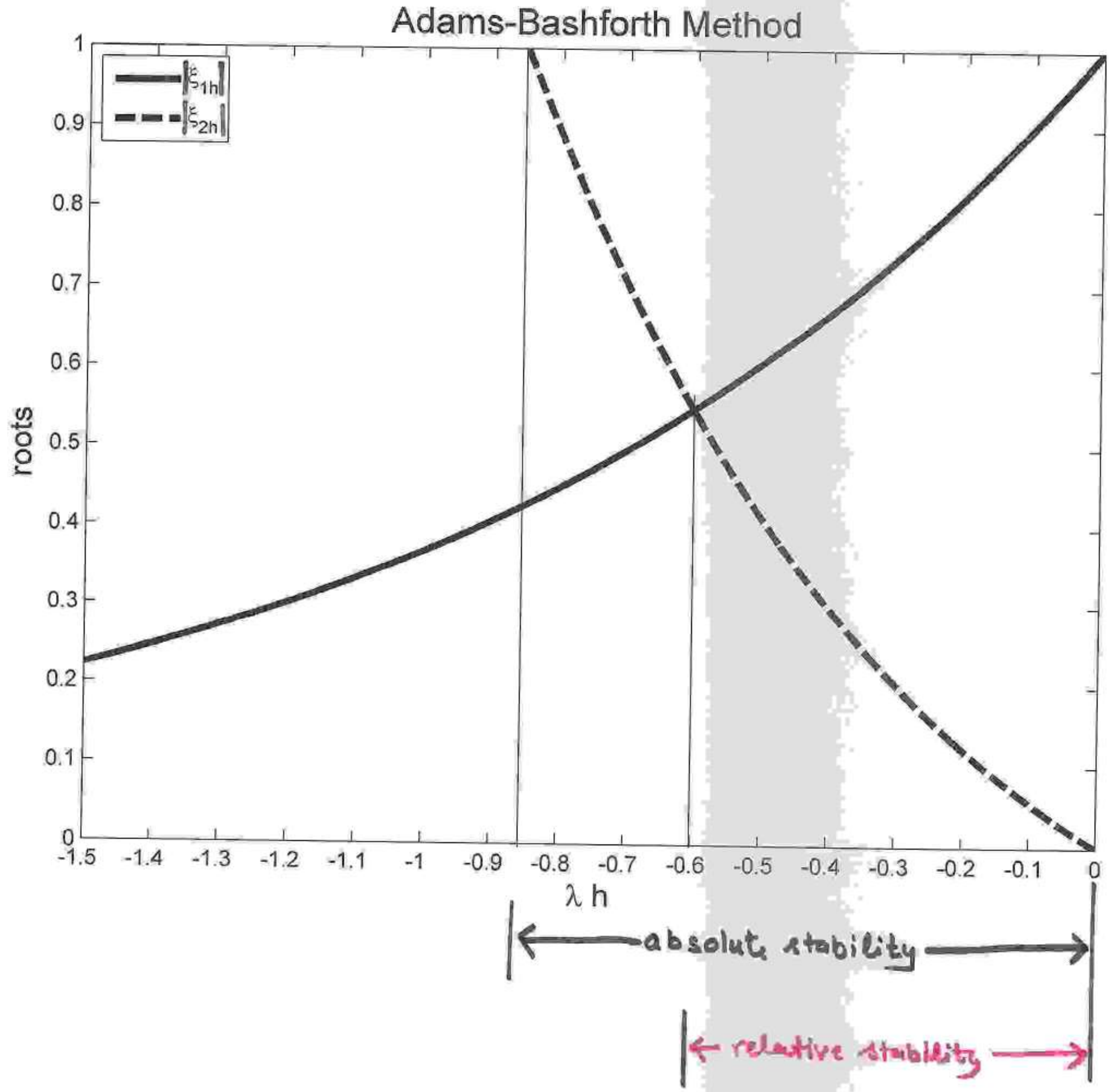
$$= \frac{1}{4}\left[2 + 3\bar{h} \pm 2\left(1 + \frac{\bar{h}}{2} + \bar{h}^2 - \cdots\right)\right]$$

$$\xi_{1h} = \frac{1}{4}\left[2 + 3\bar{h} + 2 + \bar{h} + 2\bar{h}^2 + \cdots\right] = \frac{1}{4}\left[4 + 4\bar{h} + 2\bar{h}^2\right]$$

$$= 1 + \bar{h} + \bar{h}^2/2 + \cdots \doteqdot e^{\bar{h}}$$

$$\xi_{2h} = \frac{1}{4}\left[2 + 3\bar{h} - 2 - \bar{h} - 2\bar{h}^2 \cdots\right] = \frac{\bar{h}}{2} - \frac{\bar{h}^2}{2}$$

$$= \frac{1}{2}\bar{h}e^{-\bar{h}}$$


Adams-Bashforth Method
roots
λ h
|ξ1h|
|ξ2h|
absolute stability
relative stability

Stability of Milne-Simpson method: (absolute stability)

$$u_{j+1} = u_{j-1} + \frac{h}{3}\left[u'_{j+1} + 4u'_j + u'_{j-1}\right]$$

applying the method to the test problem $y' = \lambda y$:

$$\left[1 - \frac{\lambda h}{3}\right] u_{j+1} - 4\frac{\lambda h}{3} u_j - \left(1 + \frac{\lambda h}{3}\right) u_{j-1} = 0$$

or
$$\left[1 - \frac{\bar{h}}{3}\right] u_{j+1} - \frac{4}{3}\bar{h}\, u_j - \left(1 + \frac{\bar{h}}{3}\right) u_{j-1} = 0$$

Characteristic equation

$$\left(1 - \frac{\bar{h}}{3}\right)\xi^2 - \frac{4}{3}\bar{h}\,\xi - \left(1 + \frac{\bar{h}}{3}\right) = 0$$

subst. $\xi = \dfrac{1+z}{1-z}$:

$$\left(1 - \frac{\bar{h}}{3}\right)(1 + z^2 + 2z) - \frac{4}{3}\bar{h}\,(1 - z^2) - \left(1 + \frac{\bar{h}}{3}\right)(1 - 2z + z^2) = 0$$

$$\Rightarrow \quad \frac{2}{3}\bar{h} z^2 + 4z - \underline{2\bar{h}} = 0$$

or
$$\frac{\bar{h}}{3} z^2 + 2z - \bar{h} = 0$$

Ac. to the Routh Hurwitz criterion:

$$\bar{h} > 0 \qquad 2 > 0 \qquad -\bar{h} > 0$$

never satisfied.

The method is nowhere absolute stable.

However, the reduced characteristic equation gives $\xi^2 - 1 = 0$

$\xi = \pm 1$ and hence the method is weakly stable.

Boundary Value problems

Consider a two points boundary value problem

$$y'' = f(x, y, y'), \quad x \in (a, b) \qquad \text{—— (1)}$$

with one of the three boundary conditions

a) B.C. of the first kind (DIRICHLET B.C.)

$$y(a) = r_1, \quad y(b) = r_2$$

b) B.C. of the second kind (NEUMANN B.C.)

$$y'(a) = r_1, \quad y'(b) = r_2$$

c) B.C. of the third kind (ROBIN B.C.)

$$a_0 y(a) - a_1 y'(a) = r_1$$
$$b_0 y(b) + b_1 y'(b) = r_2$$

If all terms in (1) involve only the dependent variable y & y', then the differential equation is called homogeneous, otherwise non-homogeneous.

Similarly, the BCs are homogeneous if r_1 & $r_2 = 0$ otherwise they are non homogeneous.

REMARK: A homogeneous boundary value problem, that is, a homogeneous differential equation along with homogeneous BCs, always possesses a trivial solution

$$y(x) = 0.$$

NUMERICA METHODS FOR SOLVING BVPs

i) SHOOTING METHODS — IVP Method

ii) DIFFERENCE METHODS — DIFFERENCE equation

SHOOTING METHOD

Consider the BVP:

$$y'' = f(x, y, y') \qquad y(a) = \gamma_1, \; y(b) = \gamma_2$$

In order to solve the BVP using IVP methods, we need to define the following initial values at $x = a$:

$$y(a) = \gamma_1, \qquad y'(a) = s$$

where s is unknown.

The question is: can we find the value of s for which the solution of the resulting IVP is identical to the solution of BVP?

OR

for what values of s, the IVP satisfies $y(b) = \gamma_2$.

The idea is to pick a value of s, then use the IVP method to march over to $x = b$ and see whether $y(b) = \gamma_2$. If not, then adjust the value of s and use the IVP method again and see how much close $y(b)$ is to γ_2. This is continued until $|y(b) - \gamma_2|$ is sufficiently small.

Again the question arises: how to adjust s so that $y(b)$ ends up close to γ_2? To address this, set

$$g(s) = y(s,b) - \gamma_2$$

where $y(s,b)$ is the solution of IVP corresponding to the parameter s.

The function $g(s)$ enables us to express the question of getting $y(b)$ close to γ_2 in terms of finding the value of s such that $g = 0$.

Hence we can use something such as the secant or Newton's method to improve the value of s.

For example, to use the secant method we need to specify two values for s, say s_1 & s_2. In this case, the subsequent values for s are determined using the secant method

$$s_{j+1} = s_j - \frac{g(s_j)}{g(s_j) - g(s_{j-1})} \times (s_j - s_{j-1}) \quad , \ j = 2, 3, \ldots$$

This procedure for finding s works whether the BVP is linear or non-linear. However it is possible to simplify the procedure a bit for linear problems.

If y_1 & y_2 are two solutions of a linear differential equation than their linear combination that is $(c_1 y_1 + c_2 y_2)$ will be the solution of the linear diff. equation.

Setting Initial Conditions from Boundary Conditions

i) From the first kind BCs ($y(a) = r_1$, $y(b) = r_2$)

$$\boxed{y(a) = r_1} \quad \& \quad \boxed{y'(a) = s}.$$

ii) From the second kind BCs ($y'(a) = r_1$, $y'(b) = r_2$)

$$\boxed{y'(a) = r_1} \quad \& \quad \boxed{y(a) = s}.$$

iii) BCs of the third kind: $a_0 y(a) - a_1 y'(a) = r_1$

$b_0 y(b) + b_1 y'(b) = r_2$

Here we guess $y(a)$ or $y'(a)$.

Let us guess $\boxed{y'(a) = s}$ then:

$$\boxed{y(a) = \frac{a_1 s + r_1}{a_0}}$$

Shooting Method for a linear second order problem:

$$y'' + p(x) y' + q(x) y = r(x) \qquad a < x < b \quad \text{--- (1)}$$

$$y(a) = r_1 \quad \text{--- (1a)}$$

$$y(b) = r_2 \quad \text{--- (1b)}$$

In order to use an initial value integrator for (1), we need to set $y(a)$ & $y'(a)$. By (1 a) we have $y(a) = r_1$.

Let us therefore guess a "shooting angle" s_1; i.e.,

$$y'(a) = s_1 \quad \text{--- (1c)}$$

we take $s_1 = 0$ in practice.

The IVP (1, 1a, 1c) can be solved yielding a solution, say $u(x)$.

Note that, in general, $u \not\equiv y$, because $u(b) \neq \gamma_2$.

Let us then guess another value

$$y'(a) = s_2 \qquad \text{——} \quad (1d)$$

we take $s_2 = 1$ in practice.

We call $v(x)$, the solution of the IVP (1, 1a, 1d).

Again $v(b) \neq \gamma_2$, so $y \not\equiv v$.

Linearity of the problem implies that

$$y(x) = \theta\, u(x) + (1-\theta)\, v(x), \quad 0<x<b$$

satisfies the equation (1).

Assuming that $u(b) \neq v(b)$, we can define θ by

$$\gamma_2 = y(b) = \theta\, u(b) + (1-\theta)\, v(b)$$

$$\Rightarrow \boxed{\theta = \frac{\gamma_2 - v(b)}{u(b) - v(b)}}$$

Summary: we need to solve:

$$u'' + p(x)\, u' + q(x)\, u = r(x) \qquad v'' + p(x)\, v' + q(x)\, v = r(x)$$

$$u(a) = \gamma_1 \qquad v(a) = \gamma_1$$

$$u'(a) = 0 \qquad v'(a) = 1$$

$$y(x) = \theta\, u(x) + (1-\theta)\, v(x); \quad \theta = \frac{\gamma_2 - v(b)}{u(b) - v(b)}$$

Example: Find the general solution of the boundary value problem

$$y'' = y + x \qquad x \in [0,1] \qquad \text{— (1)}$$

$$y(0) = 0 \qquad y(1) = 0 \qquad \text{— (2)}$$

with the shooting method. Use Runge-Kutta method of second order to solve the IVP with $h = 0.2$.

Sol: We set-up the two IVPs:

$$u'' = u + x; \qquad u(0) = 0 \qquad u'(0) = 0 \qquad \text{— (3)}$$

$$v'' = v + x; \qquad v(0) = 0 \qquad v'(0) = 1 \qquad \text{— (4)}$$

and write the general solution of the BVP as

$$y(x) = \theta u(x) + (1-\theta) v(x)$$

and determine θ so that

$$y(1) = 0.$$

Both the equations (3) & (4) can be converted into the following type of system of equations:

$$\begin{bmatrix} p(x) \\ q(x) \end{bmatrix}' = \begin{bmatrix} q(x) \\ p(x) + x \end{bmatrix} \qquad \text{— (5)}$$

$p(x) = y$
$q(x) = y'$

Applying the RK second order method to (5):

$$\bar{K}_1 = \begin{bmatrix} f_1(x_n, p_n, q_n) \\ f_2(x_n, p_n, q_n) \end{bmatrix} = \begin{bmatrix} q_n \\ p_n + x_n \end{bmatrix}$$

$$\bar{K}_2 = \begin{bmatrix} f_1(x_n + h, p_n + \bar{K}_1^{(1)} h, q_n + \bar{K}_1^{(2)} h) \\ f_2(x_n + h, p_n + \bar{K}_1^{(1)} h, q_n + K_1^{(2)} h) \end{bmatrix} = \begin{bmatrix} q_n + h(p_n + x_n) \\ x_n + h + p_n + q_n h \end{bmatrix}$$

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} + \frac{h}{2}\left[\bar{K}_1 + \bar{K}_2\right]$$

$$= \begin{bmatrix} p_n \\ q_n \end{bmatrix} + \frac{h}{2}\begin{bmatrix} 2q_n + h(p_n + x_n) \\ 2p_n + 2x_n + h + hq_n \end{bmatrix}$$

$$= \begin{bmatrix} p_n\left(1+\frac{h^2}{2}\right) + hq_n \\ hp_n + q_n\left(1+\frac{h^2}{2}\right) \end{bmatrix} + \begin{bmatrix} \frac{h^2}{2}x_n \\ h\,x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{h^2}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} 1.02\,p_n + 0.2\,q_n \\ 0.2\,p_n + 1.02\,q_n \end{bmatrix} + \begin{bmatrix} 0.02\,x_n \\ 0.2\,x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} 1.02 & 0.2 \\ 0.2 & 1.02 \end{bmatrix}\begin{bmatrix} p_n \\ q_n \end{bmatrix} + \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix}x_n + \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}$$

For the system (3): $p_n = u_n$, $q_n = u'_n$ $p_0 = 0$, $q_0 = 0$

At: $x_1 = 0.2$:

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.02 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}$$

At $x_2 = 0.4$:

$$\begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1.02 & 0.2 \\ 0.2 & 1.02 \end{bmatrix}\begin{bmatrix} 0 \\ 0.02 \end{bmatrix} + \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix}\begin{bmatrix} 0.2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.02 \end{bmatrix} = \begin{bmatrix} 0.008 \\ 0.0804 \end{bmatrix}$$

At $x_3 = 0.6$:

$$\begin{bmatrix} p_3 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0.03224 \\ 0.183604 \end{bmatrix}$$

At $x = 0.8$:

$$\begin{bmatrix} p_4 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0.081606 \\ 0.333728 \end{bmatrix}$$

At $x = 1$:

$$\begin{bmatrix} p_5 \\ q_5 \end{bmatrix} = \begin{bmatrix} \boxed{0.165984} = u(1) \\ 0.536724 \end{bmatrix}$$

Similarly for the system 4:

$$p_n = v_n, \quad q_n = v_n' \quad p_0 = 0 \quad q_0 = 1.$$

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 1.04 \end{bmatrix} \qquad \begin{bmatrix} p_3 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0.66448 \\ 1.367216 \end{bmatrix}$$

$$\begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0.416 \\ 1.1608 \end{bmatrix} \qquad \begin{bmatrix} p_4 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0.963213 \\ 1.667456 \end{bmatrix}$$

$$\begin{bmatrix} p_5 \\ q_5 \end{bmatrix} = \begin{bmatrix} \boxed{1.331968} - v(1) \\ 2.073448 \end{bmatrix}$$

Determination of θ:

$$y(x) = \theta u(x) + (1-\theta) v(x)$$

$$y(1) = 0 \Rightarrow \quad 0 = \theta(0.165984) + (1-\theta)(1.331968)$$

$$\Rightarrow \theta = 1.142355$$

Hence we get:

$$\boxed{y(x) = 1.142355\, u(x) - 0.142355\, v(x)}$$

So,

$y(0.2) \simeq -0.0284710$ $\qquad$ $y(0.4) \simeq -0.0500808$

$y(0.6) \simeq -0.0577625$ $\qquad$ $y(0.8) \simeq -0.0577625$

$y(1.0) = 0.$

Example: Using shooting method, solve the mixed boundary value problem

$$y'' = y - 4xe^{x} \qquad 0 < x < 1 \quad \text{---} (1)$$

$$y(0) - y'(0) = -1$$
$$y(1) + y'(1) = -e$$

Use the Taylor's series method

$$y_{j+1} = y_j + h y_j' + \frac{h^2}{2} y_j'' + \frac{h^3}{6} y_j'''$$
$$y_{j+1}' = y_j' + h y_j'' + \frac{h^2}{2} y_j'''$$

Assume $h = 0.25$. compare numerical result with the exact solution $y(x) = x(1-x)e^{x}$.

Solution:

We solve (1) with the two initial values:

a) $y(0) = 0$ & $y'(0) = 1$

b) $y(0) = 1$ & $y'(0) = 2$

If we call the solution u & v of the equation (1) associated with the ICs a) & b) respectively.

Then we can write

$$y(x) = \theta u(x) + (1-\theta) v(x)$$

The parameter θ can be calculated so that the given BC $y(1) + y'(1) = -e$ is satisfied.

Example: Using shooting method, find the solution of the boundary value problem

$$y'' = 6y^2$$

$$y(0) = 1, \quad y(0.5) = 4/9$$

Assume the initial approximation

$$y'(0) = -1.8$$

$$y'(0) = -1.9$$

and find the solution of the initial value problems using the fourth order Runge-Kutta method with $h = 0.1$. Improve the value of $y'(0)$ using the secant method once. Compare with the exact solution $y(x) = \frac{1}{(1+x)^2}$.

Sol: First we need to solve

$$y'' = 6y^2$$

$$y(0) = 1 \quad y'(0) = s \quad \text{where} \quad s = -1.8 \text{ \& } -1.9$$

set $u = y'$ then.

$$\left.\begin{array}{l} y' = u \\ \& \ u' = 6y^2 \quad \text{with} \quad y(0) = 1 \\ \qquad\qquad\qquad\qquad\quad u(0) = s \end{array}\right\} \text{system of first order equations.}$$

let $f_1 = u$ & $f_2 = 6y^2$

Runge Kutta 4th order method:

$$\begin{bmatrix} y_{j+1} \\ u_{j+1} \end{bmatrix} = \begin{bmatrix} y_j \\ u_j \end{bmatrix} + \frac{h}{6}\left(\bar{K}_1 + 2\bar{K}_2 + 2\bar{K}_3 + \bar{K}_4\right)$$

$$\bar{K}_1 = \begin{bmatrix} f_1(x_j, y_j, u_j) \\ f_2(x_j, y_j, u_j) \end{bmatrix} \qquad \bar{K}_2 = \begin{bmatrix} f_1(x_j + \frac{h}{2}, y_j + \frac{1}{2}h\bar{K}_1^{(1)}, u_j + \frac{1}{2}h\bar{K}_1^{(2)}) \\ f_2(x_j + \frac{h}{2}, y_j + \frac{1}{2}h\bar{K}_1^{(1)}, u_j + \frac{1}{2}h\bar{K}_1^{(2)}) \end{bmatrix}$$

$$\bar{K}_3 = \begin{bmatrix} f_1(x_j + \frac{h}{2}, y_j + \frac{h}{2}\bar{K}_2^{(1)}, u_j + \frac{h}{2}\bar{K}_2^{(2)}) \\ f_2(x_j + \frac{h}{2}, y_j + \frac{h}{2}\bar{K}_2^{(1)}, u_j + \frac{h}{2}\bar{K}_2^{(2)}) \end{bmatrix} \quad \bar{K}_4 = \begin{bmatrix} f_1(x_j + h, y_j + h\bar{K}_3^{1}, u_j + h\bar{K}_3^{2}) \\ f_2(x_j + h, y_j + h\bar{K}_3^{1}, u_j + h\bar{K}_3^{2}) \end{bmatrix}$$

x	$y(0)=1$, $y'(0)=-1.8$	$y(0)=1$, $y'(0)=-1.9$	$y(0)=1$, $y'(0)=-1.9990$	Y_{exact}
0.1	0.8468	0.8367	0.8266	0.8264
0.2	0.7372	0.7158	0.6947	0.6944
0.3	0.6606	0.6261	0.5922	0.5917
0.4	0.6103	0.5601	0.5108	0.5102
0.5	0.5825	0.5131	0.4453	0.4444

Secont method:

$$s^{(3)} = s^{(2)} - \frac{g(s^{(2)}) \times (s^{(2)} - s^{(1)})}{g(s^{(2)}) - g(s^{(1)})} \quad \text{—— (1)}$$

$$g(s^{(i)}) = y(s^{(i)}; 0.5) - 4/9$$

$$s^{(1)} = -1.8$$

$$s^{(2)} = -1.9$$

(1) $\Rightarrow$ $s^{(3)} = -1.9990$.

SOLVING $g(s)=0$ USING NEWTON-Raphson METHOD

$$s^{(k+1)} = s^{(k)} - \frac{g(s^{(k)})}{g'(s^{(k)})}$$

How to get $g'(s^{(k)})$?

We proceed as follows:

Suppose we want to solve:

$$y'' = f(x, y, y') \qquad a < x < b \quad \text{—— (1)}$$

$$a_0 y(a) - a_1 y'(a) = r_1 \quad \text{—— (1a)}$$

$$b_0 y(b) + b_1 y'(b) = r_2 \quad \text{—— (1b)}$$

Denote $y_s = y(x,s)$ $\quad y_s' = y'(x,s)$ $\quad y_s'' = y''(x,s)$

Then we consider

$$y_s'' = f(x, y_s, y_s') \quad \text{—— (2)}$$

$$y_s'(a) = s \; , \quad y_s(a) = \frac{a_1 s + r_1}{a_0} \quad (2')$$

Note $g(s) = b_0 y_s(b) + b_1 y_s'(b) - r_2$

$$g'(s) = b_0 \boxed{\frac{\partial y_s(b)}{\partial s}} + b_1 \boxed{\frac{\partial y_s'(b)}{\partial s}}$$

? ?

Denote $v = \dfrac{\partial y_s(x)}{\partial s}$

Then, $\boxed{g'(s) = b_0 v(b) + b_1 v'(b)}$

We need to set-up IVP for v.

Diff. (2) w.r.t. s:

$$\frac{\partial}{\partial s} y_s'' = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y_s}\frac{\partial y_s}{\partial s} + \frac{\partial f}{\partial y_s'}\frac{\partial y_s'}{\partial s}$$

as x is indep. of s.

$$= \frac{\partial f}{\partial y_s}\frac{\partial y_s}{\partial s} + \frac{\partial f}{\partial y_s'}\frac{\partial y_s'}{\partial s} \quad \text{—— (3)}$$

Diff. (2') w.r.t. s:

$$\frac{\partial}{\partial s}\left[y_s'(a)\right] = 1 ; \qquad \frac{\partial}{\partial s}\left[y_s(a)\right] = \frac{a_1}{a_0} \qquad \text{— (3')}$$

Noting $v = \frac{\partial y_s}{\partial s}$:

$$v' = \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial y_s}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial y_s}{\partial x}\right) = \frac{\partial}{\partial s} y_s'$$

$$v'' = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial s} y_s'\right) = \frac{\partial}{\partial s} y_s''$$

Then (3) $\Rightarrow$

$$v'' = \frac{\partial f}{\partial y_s}(x, y_s, y_s')\, v + \frac{\partial f}{\partial y_s'}(x, y_s, y_s')\, v' \qquad \text{— (4)}$$

(3') $\Rightarrow$

$$v'(a) = 1 ; \quad v(a) = \frac{a_1}{a_0} \qquad \text{— (4')}$$

The differential equation (4) is called the first variational equation. It can be solved step by step along with (2 & 2'). When the computation of one cycle is completed $v(b)$ & $v'(b)$ is available. Then $g'(s)$ is available form

$$g'(s) = b_0 v(b) + b_1 v'(b)$$

REMARK: If the boundary conditions of the first kind are given, then we have

$$a_0 = 1 \quad a_1 = 0$$
$$b_0 = 1 \quad b_1 = 0$$

In this case:

$$g(s) = y(b, s) - \gamma_2$$

$$\& \quad g'(s) = v(b)$$

Example: Using shooting method, solve the Boundary value problem

$$y'' = 2yy' \quad 0 < x < 1$$

$$y(0) = 0.5 \qquad y(1) = 1.$$

Use the two stage Runge-Kutta method

$$K_1 = \frac{h^2}{2} f(x_j, u_j, u_j')$$

$$K_2 = \frac{h^2}{2} f\left(x_j + \frac{2}{3}h,\ u_j + \frac{2}{3}hu_j' + \frac{2}{3}K_1,\ u_j' + \frac{4}{3h}K_1\right)$$

$$u_{j+1} = u_j + hu_j' + \frac{1}{2}(K_1 + K_2)$$

$$u_{j+1}' = u_j' + \frac{1}{2h}(K_1 + 3K_2)$$

with $h = 0.25$ to solve the corresponding IVP. Use Newton's method assuming the starting values of the slope at $x = 0$ as $s^{(0)} = u'(0) = 0.3$. Perform two iterations and compare numerical results with the exact solution $y(x) = \frac{1}{2-x}$.

Solution: We consider the following two problems for the application of NR method:

(I) $$u'' = 2uu'$$

$$u(0) = 0.5 \qquad u'(0) = 0.3 = s^{(0)}$$

(II) $$v'' = 2u'v + 2uv' = 2(u'v + uv')$$

$$v(0) = 0 \quad ; \qquad v'(0) = 1$$

Discretization:

$$u_{j+1} = u_j + h u_j' + \frac{1}{2}(k_1 + k_2)$$

$$u_{j+1}' = u_j' + \frac{1}{2h}(k_1 + 3k_2)$$

where

$$k_1 = \frac{h^2}{2} \cdot 2 u_j u_j' = h^2 u_j u_j'$$

$$k_2 = h^2 \left(u_j + \frac{2}{3} h u_j' + \frac{2}{3} k_1\right)\left(u_j' + \frac{4}{3h} k_1\right)$$

(I)

$$v_{j+1} = v_j + h v_j' + \frac{1}{2}(k_1^* + k_2^*)$$

$$v_{j+1}' = v_j' + \frac{1}{2h}(k_1^* + 3k_2^*)$$

$$k_1^* = h^2 (u_j' v_j + u_j v_j')$$

$$k_2^* = h^2 \left[u_j' \left(v_j + \frac{2}{3} h v_j' + \frac{2}{3} k_1^*\right) + u_j \left(v_j' + \frac{4}{3h} k_1^*\right)\right]$$

(II)

Newton iteration

$$g(s^{(i)}) = u_4 - 1;$$

$$g'(s^{(i)}) = v_4$$

$$s^{(i+1)} = s^{(i)} - \frac{g(s^{(i)})}{g'(s^{(i)})};$$

x = 0.00 u = 0.5000 du = 0.3000 v = 0.0000 dv = 1.0000

x = 0.25 u = 0.5858 du = 0.3918 v = 0.2856 dv = 1.3023

x = 0.50 u = 0.7005 du = 0.5364 v = 0.6743 dv = 1.8410

x = 0.75 u = 0.8631 du = 0.7833 v = 1.2561 dv = 2.8869

x = 1.00 u = 1.1122 du = 1.2543 v = 2.2417 dv = 5.1836

$g(s) = 1.1122 - 1 = 0.1123 \qquad g'(s) = 2.2417$

$$s^{(1)} = s^{(0)} - \frac{g(s^{(0)})}{g'(s^{(0)})} = 0.3 - \frac{0.1123}{2.2417} = 0.2499$$

x = 0.00 u = 0.5000 du = 0.2499 v = 0.0000 dv = 1.0000

x = 0.25 u = 0.5714 du = 0.3254 v = 0.2853 dv = 1.2988

x = 0.50 u = 0.6662 du = 0.4406 v = 0.6697 dv = 1.8066

x = 0.75 u = 0.7982 du = 0.6291 v = 1.2304 dv = 2.7407

x = 1.00 u = 0.9941 du = 0.9678 v = 2.1368 dv = 4.6538

$g(s) = 0.9941 - 1 = -0.0059$

$g'(s) = 2.1368$

$$s^{(2)} = s^{(1)} - \frac{g(s^{(1)})}{g'(s^{(1)})} = 0.2499 + \frac{0.0059}{2.1368} = +0.2527$$

x = 0.00 u = 0.5000 exact 0.5

x = 0.25 u = 0.5722 exact 0.5714

x = 0.50 u = 0.6681 exact 0.6667

x = 0.75 u = 0.8017 exact 0.8000

x = 1.00 u = 1.0004 exact 1

Ex: Solve the following nonlinear boundary value problem using shooting method:

$$y'' = \frac{3}{2} y^2 \qquad 0 < x < 1$$

$$y(0) = 1 \qquad y(1) = 4.$$

Use fourth order Runge-Kutta method to solve the initial value problems and the Newton-Raphson method (1 iter.) for iteration using the initial guess $s^{(0)} = 0.9$ and $h = 0.25$.

Sol: We need to solve the following IVPs:

$$\left.\begin{array}{l} u'' = \frac{3}{2} u^2 \\ u(0) = 1 \quad u'(0) = 0.9 \ (s^{(0)}) \end{array}\right\} - (I)$$

&

$$\left.\begin{array}{l} v'' = 3uv \\ v(0) = 0 \quad v'(0) = 1 \end{array}\right\} - (II)$$

The two IVPs in system of equation form:

$$\begin{bmatrix} u \\ \bar{u} \end{bmatrix}' = \begin{bmatrix} \bar{u} \\ \frac{3}{2}u^2 \end{bmatrix} =: \begin{bmatrix} f_1(x, u, \bar{u}) \\ f_2(x, u, \bar{u}) \end{bmatrix} \qquad \left.\begin{array}{l} u(0) = 1 \\ \bar{u}(0) = 0.9 \end{array}\right\} (I)$$

&

$$\begin{bmatrix} v \\ \bar{v} \end{bmatrix}' = \begin{bmatrix} \bar{v} \\ 3uv \end{bmatrix} =: \begin{bmatrix} g_1(x, v, \bar{v}) \\ g_2(x, v, \bar{v}) \end{bmatrix} \qquad \left.\begin{array}{l} v(0) = 0 \\ \bar{v}(0) = 1 \end{array}\right\} (II)$$

Fourth order Runge-Kutta method for the problem (I)

$$\bar{K}_1 = \begin{bmatrix} K_1^{(1)} \\ K_1^{(2)} \end{bmatrix} = \begin{bmatrix} f_1(x_j, u_j, \bar{u}_j) \\ f_2(x_j, u_j, \bar{u}_j) \end{bmatrix} = \begin{bmatrix} \bar{u}_j \\ \frac{3}{2} u_j^2 \end{bmatrix}$$

$$\bar{K}_2 = \begin{bmatrix} K_2^{(1)} \\ K_2^{(2)} \end{bmatrix} = \begin{bmatrix} f_1(x_j + \frac{h}{2}, u_j + \frac{h}{2} K_1^{(1)}, \bar{u}_j + \frac{h}{2} K_1^{(2)}) \\ f_2(x_j + h/2, u_j + h/2 K_1^{(1)}, \bar{u}_j + \frac{h}{2} K_1^{(2)}) \end{bmatrix} = \begin{bmatrix} \bar{u}_j + \frac{h}{2} K_1^{(2)} \\ \frac{3}{2}(u_j + \frac{h}{2} K_1^{(1)})^2 \end{bmatrix}$$

$$\bar{K}_3 = \begin{bmatrix} \bar{u}_j + \frac{h}{2} K_2^{(2)} \\ \frac{3}{2}(u_j + \frac{h}{2} K_2^{(1)})^2 \end{bmatrix} \qquad \bar{K}_4 = \begin{bmatrix} \bar{u}_j + h K_3^{(2)} \\ \frac{3}{2}(u_j + h K_3^{(1)})^2 \end{bmatrix}$$

$$\begin{bmatrix} u_{j+1} \\ \bar{u}_{j+1} \end{bmatrix} = \begin{bmatrix} u_j \\ \bar{u}_j \end{bmatrix} + \frac{h}{6}\left[\bar{K}_1 + 2\bar{K}_2 + 2\bar{K}_3 + K_4\right]$$

$j = 0, 1, 2, 3.$

Fourth order Runge-Kutta method for the problem (II):

$$\bar{K}_1 = \begin{bmatrix} \bar{v}_j \\ 3u_j v_j \end{bmatrix} \qquad \bar{K}_2 = \begin{bmatrix} \bar{v}_j + \frac{h}{2}K_1^{(2)} \\ 3u_j\left(v_j + \frac{h}{2}K_1^{(1)}\right) \end{bmatrix}$$

$$\bar{K}_3 = \begin{bmatrix} \bar{v}_j + \frac{h}{2}K_2^{(2)} \\ 3u_j\left(v_j + \frac{h}{2}K_2^{(1)}\right) \end{bmatrix} \qquad \bar{K}_4 = \begin{bmatrix} \bar{v}_j + h\,K_3^{(2)} \\ 3u_j\left(v_j + h\,K_3^{(1)}\right) \end{bmatrix}$$

$$\begin{bmatrix} v_{j+1} \\ \bar{v}_{j+1} \end{bmatrix} = \begin{bmatrix} v_j \\ \bar{v}_j \end{bmatrix} + \frac{h}{6}\left[\bar{K}_1 + 2\bar{K}_2 + 2\bar{K}_3 + \bar{K}_4\right]$$

$$g'(s^{(0)}) = v_4 \qquad g(s^{(0)}) = u_4 - 4;$$

$$s^{(1)} = s^{(0)} - \frac{g(s^{(0)})}{g'(s^{(0)})}$$

$u_1 = 1.2801$ $\quad u_2 = 1.7188$ $\quad u_3 = 2.4466$ $\quad u_4 = 3.7711$

$\bar{u}_1 = 1.3814$ $\quad \bar{u}_2 = 2.2114$ $\quad \bar{u}_3 = 3.8042$ $\quad \bar{u}_4 = 7.3195$

$v_1 = 0.2578$ $\quad v_2 = 0.5741$ $\quad v_3 = 1.0608$ $\quad v_4 = 1.9895$

$\bar{v}_1 = 1.0952$ $\quad \bar{v}_2 = 1.4867$ $\quad \bar{v}_3 = 2.5126$ $\quad \bar{v}_4 = 5.2062$

$$s^{(1)} = 1.01500$$

$\boxed{u_1 = 1.3099}$ $\quad \boxed{u_2 = 1.7862}$ $\quad \boxed{u_3 = 2.5754}$ $\quad \boxed{u_4 = 4.0283}$

$\bar{u}_1 = 1.5095$ $\quad \bar{u}_2 = 2.3942$ $\quad \bar{u}_3 = 4.1384$ $\quad \bar{u}_4 = 8.0981$

exact: $\boxed{1.3061}$ $\quad \boxed{1.7778}$ $\quad \boxed{2.5600}$ $\quad \boxed{4.0000}$

FINITE DIFFERENCE METHOD

(I) DISCRETIZATION OF THE DOMAIN:

We devide the interval $[a, b]$ into $(N+1)$ subintervals such that

$$x_j = a + jh \qquad j = 0, 1, 2, \cdots, N+1$$

where $\quad x_0 = a, \quad x_{N+1} = b, \quad h = \dfrac{b-a}{N+1}$

(II) FINITE DIFFERENCE APPROXIMATION OF DERIVATIVES

a) Expanding $u(x_j+h)$ in Taylor's series we get

$$u(x_j+h) = u(x_j) + h\,u'(x_j) + \frac{h^2}{2}u''(x_i) + O(h^3) \quad \text{—①}$$

$$\Rightarrow \frac{u(x_j+h) - u(x_j)}{h} = u'(x_j) + \frac{h}{2}u''(x_i) + O(h^2)$$

OR

$$\boxed{u'(x_j) \simeq \frac{u(x_j+h) - u(x_j)}{h}}$$

This is called finit forward difference formula.

This difference formula provides a first order approximation to $u'(x_j)$ with respect to h.

b) Expanding $u(x_j-h)$ in Taylor's series we get

$$u(x_j-h) = u(x_j) - h\,u'(x_j) + \frac{h^2}{2}u''(x_j) + O(h^3) \quad \text{—} (2)$$

$$\Rightarrow \quad \frac{u(x_j-h)-u(x_j)}{-h} = u'(x_j) - \frac{h}{2}u''(x_j) + O(h^3)$$

OR

$$\boxed{u'(x_j) \simeq \frac{u(x_j)-u(x_{j-1})}{h}}$$

This is called <u>backward difference formula</u>. This diff. formula provides a first order approximation to $u'(x_j)$ with respect to h.

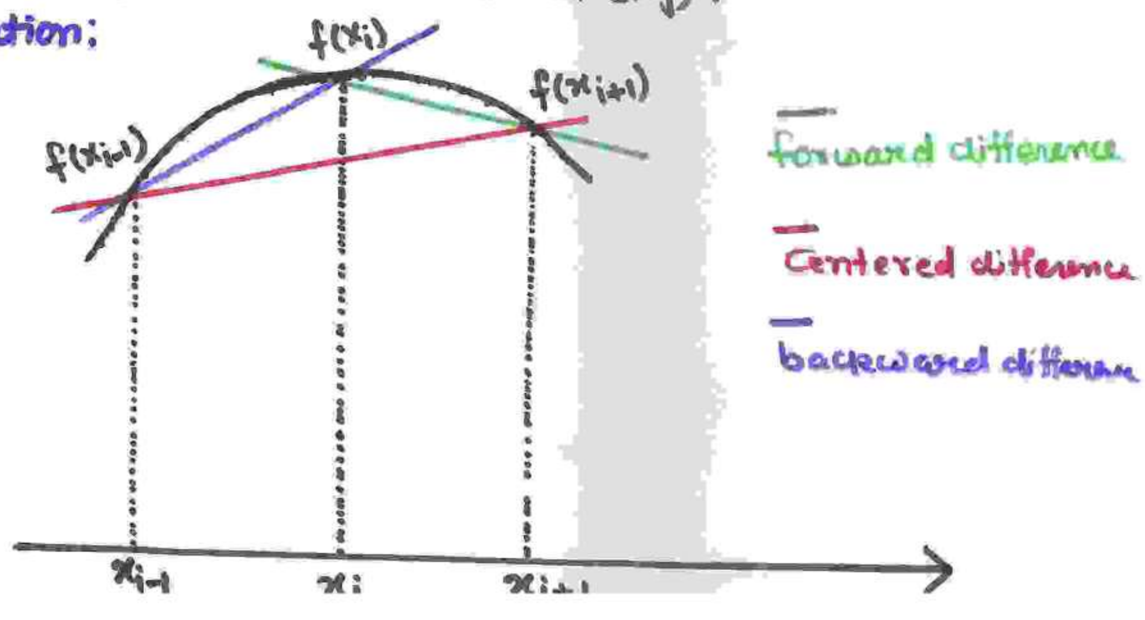
c) Subtracting (2) from (1):

$$u(x_j+h) - u(x_j-h) = 2h\,u'(x_j) + O(h^3)$$

$$\Rightarrow \quad \boxed{u'(x_j) \simeq \frac{u(x_j+h)-u(x_j-h)}{2h}}$$

This is called centered difference formula. This gives a second order approximation to $u'(x_j)$.

Physical interpretation:



d)

adding ① and ②:

$$u(x_j+h) + u(x_j-h) = 2u(x_j) + h^2 u''(x_j) + O(h^4)$$

$$\Rightarrow \quad \boxed{u''(x_j) \simeq \frac{u(x_j+h) - 2u(x_j) + u(x_j-h)}{h^2}}$$

This is called centered finite difference approximation for second order derivative.

This formula provides a second-order approximation to $u''(x_j)$ with respect to h.

$$\boxed{u'(x_j) \simeq \frac{u(x_{j+1}) - u(x_j)}{h}}$$

$$\boxed{u'(x_j) \simeq \frac{u(x_j) - u(x_{j-1})}{h}}$$

$$\boxed{u'(x_j) \simeq \frac{u(x_{j+1}) - u(x_{j-1})}{2h}}$$

$$\boxed{u''(x_j) = \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2}}$$

III) SOLVING THE EQUATIONS

Consider the second order linear differential equation:

$$-u'' + p(x)u' + q(x)u = r(x) \qquad a<x<b \quad — ①$$

$$u(a) = \gamma_1 \qquad u(b) = \gamma_2 \qquad (1a)$$

Using the second order difference approximations at $x = x_j$ we obtain the difference equation

$$-\frac{1}{h^2}\left[u_{j+1} - 2u_j + u_{j-1}\right] + p(x_j)\,\frac{u_{j+1} - u_{j-1}}{2h} + q(x_j)\,u_j = r(x_j)$$

$$j = 1, 2, \ldots N \qquad — ②$$

Note that u_j is an approximation of $u(x_j)$.

The BCs (1a) become:

$$u_0 = \gamma_1 \qquad u_{N+1} = \gamma_2$$

N-equations and N-unknowns.

Multiplying ② by $\frac{h^2}{2}$ we obtain

$$\frac{-u_{j+1} + 2u_j - u_{j-1}}{2} + \frac{h}{4}\,p(x_j)\left(u_{j+1} - u_{j-1}\right) + \frac{h^2}{2}\,q(x_j)\,u_j$$

$$= \frac{h^2}{2}\,r(x_j)$$

OR

$$-\frac{1}{2}\left(1+\frac{h}{2}p(x_j)\right)u_{j-1}+\left(1+\frac{h^2}{2}q(x_j)\right)u_j-\frac{1}{2}\left(1-\frac{h}{2}p(x_j)\right)u_{j+1}$$

$$=\frac{h^2}{2}r(x_j)$$

Definding

$$A_j=-\frac{1}{2}\left(1+\frac{h}{2}p(x_j)\right)$$

$$B_j=1+\frac{h^2}{2}q(x_j)$$

$$C_j=-\frac{1}{2}\left(1-\frac{h}{2}p(x_j)\right)$$

we get

$$A_j u_{j-1}+B_j u_j+C_j u_{j+1}=\frac{h^2}{2}r(x_j)$$

$$j=1,2,\dots,N \qquad — ③$$

The system of equations ③ can be written in matrix notation:

$$\begin{bmatrix} B_1 & C_1 & \cdots & \cdots & 0 \\ A_2 & B_2 & C_2 & \cdots & 0 \\ \vdots & & & & \\ & & A_{N-1} & B_{N-1} & C_{N-1} \\ \cdots & \cdots & \cdots & A_N & B_N \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_N \end{bmatrix}=\frac{h^2}{2}\begin{bmatrix} r(x_1)-\frac{2}{h^2}A_1\gamma_1 \\ r(x_2) \\ \vdots \\ r(x_{N-1}) \\ r(x_N)-\frac{2}{h^2}C_N\gamma_2 \end{bmatrix}$$

$$A\bar{u}=\bar{b}$$

The solution of this system gives the finite difference solution of the BVP satisfying BCs.

LOCAL TRUNCATION ERROR:

The local truncation error of the finite difference scheme discussed above is defined as

$$T_j = A_j u(x_{j-1}) + B_j u(x_j) + C_j u(x_{j+1}) - \frac{h^2}{2} r(x_j)$$

$$= -\frac{1}{2}\left[1+\frac{h}{2}p(x_j)\right]\left[\cancel{u(x_j)} - h u'(x_j) + \frac{h^2}{2}u''(x_j) - \frac{h^3}{3!}u'''(x_j) + \frac{h^4}{4!}u^{(iv)}(x_j) + \cdots\right]$$

$$+\left[\cancel{1} + \frac{h^2}{2}q(x_j)\right]u(x_j)$$

$$-\frac{1}{2}\left[1-\frac{h}{2}p(x_j)\right]\left[\cancel{u(x_j)} + h u'(x_j) + \frac{h^2}{2}u''(x_j) + \frac{h^3}{3!}u'''(x_j) + \frac{h^4}{4!}u^{(iv)}(x_j) + \cdots\right] - \frac{h^2}{2}r(x_j)$$

$$= \cancel{\frac{h}{2}u'(x_j)} + \underline{\frac{h^2}{4}p(x_j)u'(x_j)} - \underline{\frac{h^2}{4}u''(x_j)} - \cancel{\frac{h^3}{8}p(x_j)u''(x_j)}$$

$$+ \cancel{\frac{h^3}{12}u'''(x_j)} + \frac{h^4}{24}u'''(x_j)p(x_j) - \frac{h^4}{48}u^{(iv)}(x_j) + \cdots$$

$$+ \underline{\frac{h^2}{2}q(x_j)u(x_j)} - \cancel{\frac{h}{2}u'(x_j)} + \underline{\frac{h^2}{4}p(x_j)u'(x_j)} - \underline{\frac{h^2}{4}u''(x_j)}$$

$$+ \cancel{\frac{h^3}{8}u''(x_j)p(x_j)} - \cancel{\frac{h^3}{12}u'''(x_j)} + \frac{h^4}{24}p(x_j)u'''(x_j)$$

$$-\frac{h^4}{48}u^{(iv)}(x_j) - \underline{\frac{h^2}{2}r(x_j)}$$

$$= \frac{h^4}{12}p(x_j)u'''(x_j) - \frac{h^4}{24}u^{(iv)}(x_j) + O(h^5)$$

$$= O(h^4).$$

The order of a method is the largest integer p for which $\left|\frac{1}{h^2}T_j\right| = O(h^p)$.

The above method is a 2nd order method

DERIVATIVE BOUNDARY CONDITIONS:

Consider the third kind BCs:

$$a_0 u(a) - a_1 u'(a) = \gamma_1$$
$$b_0 u(b) + b_1 u'(b) = \gamma_2$$

The finite difference approximation of the linear differential equation $-u'' + p(x) u' + q(x) u = r(x)$ gives:

$$A_j u_{j-1} + B_j u_j + C_j u_{j+1} = \frac{h^2}{2} r(x_j) \quad j = 1, 2, \ldots, N \qquad (1)$$

System (1) contains $(N+2)$ unknowns. We have only N-equations. We need to have two more equations in order to solve the system uniquely.

Discretizing the BCs using a second order approximation we get at $x = x_0$:

$$a_0 u_0 - a_1 \frac{u_1 - u_{-1}}{2h} = \gamma_1$$

or $$u_{-1} = u_1 + \frac{2h}{a_1}(\gamma_1 - a_0 u_0) = -\frac{2a_0 h}{a_1} u_0 + u_1 + \frac{2h}{a_1}\gamma_1$$

At $x = x_{N+1}$:

$$b_0 u_{N+1} + b_1 \frac{u_{N+2} - u_N}{2h} = \gamma_2$$

$$\Rightarrow \quad u_{N+2} = u_N - \frac{2hb_0}{b_1} u_{N+1} + \frac{2h}{b_1}\gamma_2$$

Here u_{-1} and u_{N+2} are the approximations at x_{-1} and x_{N+2}.

The nodes x_{-1} and x_{N+2} lie outside the interval $[a, b]$ and are called fictitious nodes.

Now if we assume that the approximation (1) holds at $x = x_0$ & $x = x_{N+1}$ and use the value of fictitious nodes from above, we get

<u>$j=0$:</u>

$$A_0 u_{-1} + B_0 u_0 + C_0 u_1 = \frac{h^2}{2} r(x_0)$$

$$\Rightarrow A_0\left[-\frac{2a_0h}{a_1} u_0 + u_1 + \frac{2h}{a_1}\gamma_1\right] + B_0 u_0 + C_0 u_1 = \frac{h^2}{2} r(x_0)$$

$$\Rightarrow \left(B_0 - \frac{2ha_0}{a_1}A_0\right) u_0 + (A_0 + C_0)\, u_1 = \frac{h^2}{2} r(x_0) - \frac{2h}{a_1}\gamma_1 A_0$$

<u>$j = N+1$:</u>

$$A_{N+1} u_N + B_{N+1} u_{N+1} + C_{N+1} u_{N+2} = \frac{h^2}{2} r(x_{N+1})$$

$$\Rightarrow A_{N+1} u_N + B_{N+1} u_{N+1} + C_{N+1}\left[u_N - \frac{2hb_0}{b_1} u_{N+1} + \frac{2h}{b_1}\gamma_2\right] = \frac{h^2}{2} r(x_{N+1})$$

$$\Rightarrow [A_{N+1} + C_{N+1}]\, u_N + \left[B_{N+1} - \frac{2hb_0}{b_1} C_{N+1}\right] u_{N+1} = \frac{h^2}{2} r(x_{N+1}) - \frac{2h}{b_1}\gamma_2 C_{N+1}$$

ALTERNATIVE APPROACH: (WITHOUT USING FICTITIOUS POINTS)

We can discretize BCs using forward and backward difference formula:

For $j=0$:

$$a_0 u_0 - a_1 \left[\frac{u_1 - u_0}{h}\right] = \gamma_1$$

FORWARD DIFFERENCE

or

$$[a_0 h + a_1] u_0 - a_1 u_1 = h\gamma_1$$

For $j = N+1$:

$$b_0 u_{N+1} + b_1 \left[\frac{u_{N+1} - u_N}{h}\right] = \gamma_2$$

or

$$[b_0 h + b_1] u_{N+1} - b_1 u_N = h\gamma_2$$

Since the difference approximation used there are of first order, the method may not retain the second order.

SOLUTION OF TRIDIAGONAL SYSTEM:

a) Gauss elimination may be used.

However for a tridiagonal system much cheaper algorithm like Thomas Algorithm may be used.

THOMAS ALGORITHM:

Consider

$$\underbrace{\begin{bmatrix} b_1 & c_1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 \\ 0 & a_3 & b_3 & c_3 \\ 0 & 0 & a_4 & b_4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}}_{f}$$

The matrix A can be factorize as

$$A = LU$$

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 \\ 0 & a_3 & b_3 & c_3 \\ 0 & 0 & a_4 & b_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta_2 & 1 & 0 & 0 \\ 0 & \beta_3 & 1 & 0 \\ 0 & 0 & \beta_4 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 & c_1 & 0 & 0 \\ 0 & \alpha_2 & c_2 & 0 \\ 0 & 0 & \alpha_3 & c_3 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & c_1 & 0 & 0 \\ \beta_2\alpha_1 & \beta_2 c_1 + \alpha_2 & c_2 & 0 \\ 0 & \beta_3\alpha_2 & \beta_3 c_2 + \alpha_3 & c_3 \\ 0 & 0 & \beta_4\alpha_3 & \beta_4 c_3 + \alpha_4 \end{bmatrix}$$

Comparison gives

$$\alpha_1 = b_1 \qquad \beta_2 = \frac{a_2}{\alpha_1} \qquad \alpha_2 = b_2 - \beta_2 c_1$$

$$\beta_3 = \frac{a_3}{\alpha_2} \qquad \alpha_3 = b_3 - \beta_3 c_2$$

$$\beta_4 = \frac{a_4}{\alpha_3} \qquad \alpha_4 = b_4 - \beta_4 c_3$$

In general for $n \times n$ matrix:

$$\alpha_1 = b_1 \qquad \beta_i = \frac{a_i}{\alpha_{i-1}} \qquad \alpha_i = b_i - \beta_i c_{i-1}$$

$$i = 2, 3, \cdots, n.$$

We have

$$L \underbrace{U x}_{=: y} = f$$

Solve:

$$Ux = y \quad \& \quad Ly = f$$

$$Ly = f \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta_2 & 1 & 0 & 0 \\ 0 & \beta_3 & 1 & 0 \\ 0 & 0 & \beta_4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

$$\Rightarrow \quad y_1 = r_1$$

$$y_2 = r_2 - \beta_2 y_1$$

$$y_3 = r_3 - \beta_3 y_2$$

$$y_4 = r_4 - \beta_4 y_3$$

In general: $y_i = r_i - \beta_i y_{i-1}, \quad i = 2, \cdots, n.$

$$Ux = y \Rightarrow \begin{bmatrix} \alpha_1 & c_1 & 0 & 0 \\ 0 & \alpha_2 & c_2 & 0 \\ 0 & 0 & \alpha_3 & c_3 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$x_4 = \frac{y_4}{\alpha_4} \qquad x_i = \frac{y_i - c_i x_{i+1}}{\alpha_i} \qquad i = 3, 2, 1.$$

general form:

$$x_n = \frac{y_n}{\alpha_n}; \qquad x_i = \frac{y_i - c_i x_{i+1}}{\alpha_i}; \quad i = n-1, \cdots, 1.$$

EXAMPLE: Solve the boundary value problem

$$y'' + (1+x^2)y + 1 = 0 \qquad y(\pm 1) = 0$$

with step length $h = 0.25$. Use a second order method.

Solution: Replacing x by $-x$ the BVP remains unchanged. Thus the solution of the problem is symmetrical about y-axis. Therefore we solve the above problem in the domain $[0,1]$.

The second order method gives the difference equation:

$$\frac{1}{h^2}\left[y_{n+1} - 2y_n + y_{n-1}\right] + (1+x_n^2)\,y_n + 1 = 0$$

$$-y_{n-1} + \left[2 - (1+x_n^2)h^2\right]y_n - y_{n+1} = h^2$$

$$n = 0, 1, 2, 3$$

$\underline{n=0}$:

$$-y_{-1} + \left[2 - \frac{1}{16}\right]y_0 - y_1 = (0.25)^2$$

Since $y_{-1} = y_1$

$$\frac{31}{16}y_0 - 2y_1 = \frac{1}{16} \qquad \text{—— } (1)$$

$\underline{n=1}$: $-y_0 + \left[2 - \left(1+\frac{1}{16}\right)\frac{1}{16}\right]y_1 - y_2 = \frac{1}{16}$

$$\Rightarrow -y_0 + \left[2 - \frac{17}{256}\right]y_1 - y_2 = \frac{1}{16}$$

$$\Rightarrow -y_0 + \left[\frac{495}{256}\right]y_1 - y_2 = \frac{1}{16} \qquad \text{—— } (2)$$

$\underline{n=2}$:

$$-y_1 + \left[2 - \left(1+\frac{4}{16}\right)\frac{1}{16}\right]y_2 - y_3 = \frac{1}{16}$$

$$\Rightarrow -y_1 + \left[2 - \frac{20}{256}\right]y_2 - y_3 = \frac{1}{16}$$

$$\Rightarrow -y_1 + \frac{492}{256}y_2 - y_3 = \frac{1}{16}$$

$\underline{n=3}$:

$$-y_2 + \left[2 - \left(1+\frac{9}{16}\right)\frac{1}{16}\right]y_3 - y_4 = \frac{1}{16}$$

$$\Rightarrow -y_2 + \left[2 - \frac{25}{256}\right]y_3 - y_4 = \frac{1}{16}$$

$$\Rightarrow -y_2 + \frac{487}{256}y_3 - y_4 = \frac{1}{16}$$

$$\Rightarrow -y_2 + \frac{487}{256}y_3 = \frac{1}{16}$$

In matrix form:

$$\begin{bmatrix} \frac{31}{16} & -2 & 0 & 0 \\ -1 & \frac{495}{256} & -1 & 0 \\ 0 & -1 & \frac{492}{256} & -1 \\ 0 & 0 & -1 & \frac{487}{256} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution using Gauss elimination:

$y_0 = 0.9415$
$y_1 = 0.8808$
$y_2 = 0.6992$
$y_3 = 0.4004$

Example: Use a second order method for the solution of the boundary value problem

$$y'' = xy + 1 \qquad x \in [0, 1]$$

$$y'(0) + y(0) = 1 \qquad y(1) = 1.$$

with the step length $h = 0.25$.

Solution: Discretization at $x = x_n$ gives.

$$-\left(\frac{y_{n-1} - 2y_n + y_{n+1}}{h^2}\right) + x_n y_n + 1 = 0$$

or

$$-y_{n-1} + (2 + x_n h^2) y_n - y_{n+1} = -h^2 \qquad \text{— (1)}$$

$$n = 0, 1, 2, 3$$

BC: $$\frac{y_1 - y_{-1}}{2h} + y_0 = 1 \Rightarrow y_{-1} = y_1 + 2h y_0 - 2h$$

$$\& \; y_4 = 1.$$

At $n = 0$; (1) $\Rightarrow$

$$-y_{-1} + (2) y_0 - y_1 = -\frac{1}{16}$$

$$\Rightarrow -\left(y_1 + 2 \cdot \tfrac{1}{4} y_0 - 2 \cdot \tfrac{1}{4}\right) + 2y_0 - y_1 = -\frac{1}{16}$$

$$\frac{3}{2} y_0 - 2y_1 = -\frac{1}{16} - \frac{1}{2} = -\frac{9}{16} \qquad \text{— (1)}$$

$n = 1$:

$$-y_0 + \left(2 + \tfrac{1}{4} \cdot \tfrac{1}{16}\right) y_1 - y_2 = -\frac{1}{16}$$

$$-y_0 + \frac{129}{64} y_1 - y_2 = -\frac{1}{16} \qquad \text{— (2)}$$

<u>n = 2:</u>

$$-y_1 + \left(2 + \frac{2}{4}\cdot\frac{1}{16}\right) y_2 - y_3 = -\frac{1}{16}$$

$$\Rightarrow -y_1 + \frac{65}{32} y_2 - y_3 = -\frac{1}{16} \quad \text{— (3)}$$

<u>n = 3:</u>

$$-y_2 + \left(2 + \frac{3}{4}\cdot\frac{1}{16}\right) y_3 - y_4 = -\frac{1}{16}$$

$$\Rightarrow -y_2 + \frac{131}{16} y_3 - \underbrace{y_4}_{=1} = -\frac{1}{16}$$

$$\Rightarrow -y_2 + \frac{131}{16} y_3 = \frac{15}{16} \quad \text{— (4)}$$

In matrix form:

$$\begin{bmatrix} \frac{3}{2} & -2 & 0 & 0 \\ -1 & \frac{129}{64} & -1 & 0 \\ 0 & -1 & \frac{65}{32} & -1 \\ 0 & 0 & -1 & \frac{131}{64} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = -\frac{1}{16} \begin{bmatrix} 9 \\ 1 \\ 1 \\ -15 \end{bmatrix}$$

Using Gauss-elimination or Thomas algorithm we get:

$y_0 = -7.4616$ $\quad$ $y_1 = -5.3150$

$y_2 = -3.1889$ $\quad$ $y_3 = -1.0999$

Nonlinear second order differential equations:

$$u'' = f(x,u) \qquad a < x < b \qquad \text{—— (I)}$$

subject to the BCs:

$$u(a) = \gamma_1 \qquad u(b) = \gamma_2$$

A second order finite difference leads to:

$$u_{j-1} - 2u_j + u_{j+1} = h^2 f(x_j, u_j)\,; \quad j = 1, 2, \cdots, N \qquad \text{—(ii)}$$

with $u_0 = \gamma_1 \qquad u_{N+1} = \gamma_2$

The system of equations (ii) can be solved using Newton's method or by any other iteration method.

A simple iterative scheme:

$$u_{j-1}^{[s+1]} - 2u_j^{[s+1]} + u_{j+1}^{[s+1]} = h^2 f(x_j, u_j^{[s]})$$

$$j = 1, 2, \cdots, N.$$

This is a system of linear equations which can be solved by any known method.

Newton-Raphson-Method:

The system of equations (ii) can be written in the form

$$F(u_1, u_2, \cdots, u_N) =: F(u) = 0$$

where $F = [F_1, F_2, \cdots, F_N]^T$.

and $u = [u_1, u_2, \cdots, u_N]^T$.

Compute the Jacobian

$$J(u_1, u_2, \cdots u_N) = \frac{\partial F}{\partial u} = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \cdots & \frac{\partial F_1}{\partial u_N} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \cdots & \frac{\partial F_2}{\partial u_N} \\ \vdots & & & \\ \frac{\partial F_N}{\partial u_1} & \frac{\partial F_N}{\partial u_2} & \cdots & \frac{\partial F_N}{\partial u_N} \end{bmatrix}$$

Starting with a suitable estimate $u^{[0]}$, we define

$$u^{[s+1]} = u^{[s]} + \Delta u^{[s]} \qquad s = 0, 1, 2 \cdots$$

where $\Delta u^{[s]}$ is the solution of

$$J(u^{[s]}) \Delta u^{[s]} = -F(u^{[s]}) \qquad s = 0, 1, 2 \cdots$$

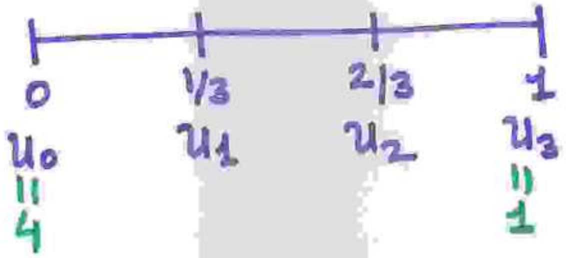
Example: Solve the boundary value problem

$$u'' = \frac{3}{2} u^2$$

$$u(0) = 4 \qquad u(1) = 1$$

with $h = \frac{1}{3}$. Use a second order finite difference method for its solution.

<u>Sol</u>:



The second order finite difference approximation

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = \frac{3}{2} u_j^2$$

$$\Rightarrow u_{j-1} - 2u_j + u_{j+1} = \frac{3}{2} \cdot \frac{1}{3} \times \frac{1}{3} \cdot u_j^2$$

$$= \frac{1}{6} u_j^2 \qquad j = 1, 2.$$

<u>For $j = 1$</u>:

$$u_0 - 2u_1 + u_2 = \frac{1}{6} u_1^2$$

Using B.C.

$$u_1^2 + 12u_1 - 6u_2 - 24 = 0$$

<u>For $j = 2$:</u>

$$u_1 - 2u_2 + u_3 = \frac{u_2^2}{6}$$

using B.C.

$$u_2^2 - 6u_1 + 12u_2 - 6 = 0.$$

So we solve:

$$F_1 = u_1^2 + 12u_1 - 6u_2 - 24$$

$$F_2 = u_2^2 - 6u_1 + 12u_2 - 6$$

$$J = \begin{bmatrix} 2u_1 + 12 & -6 \\ -6 & 2u_2 + 12 \end{bmatrix}$$

Therefore

$$J^{[s]} \Delta u^{[s]} = -F(u^{[s]})$$

$$\Rightarrow \begin{bmatrix} 2u_1^{[s]} + 12 & -6 \\ -6 & 2u_2^{[s]} + 12 \end{bmatrix} \begin{bmatrix} \Delta u_1^{[s]} \\ \Delta u_2^{[s]} \end{bmatrix} = - \begin{bmatrix} \left(u_1^{[s]}\right)^2 + 12u_1^{[s]} - 6u_2^{[s]} - 24 \\ \left(u_2^{[s]}\right)^2 - 6u_1^{[s]} + 12u_2^{[s]} - 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \Delta u_1^{[s]} \\ \Delta u_2^{[s]} \end{bmatrix} = -\frac{1}{D} \begin{bmatrix} 2u_2^{[s]} + 12 & 6 \\ 6 & 2u_1^{[s]} + 12 \end{bmatrix} \begin{bmatrix} \left(u_1^{[s]}\right)^2 + 12u_1^{[s]} - 6u_2^{[s]} - 24 \\ \left(u_2^{[s]}\right)^2 - 6u_1^{[s]} + 12u_2^{[s]} - 6 \end{bmatrix}$$

$$D = \left[2u_1^{[s]} + 12\right]\left[2u_2^{[s]} + 12\right] - 36$$

then. $$\begin{bmatrix} u_1^{[s+1]} \\ u_2^{[s+1]} \end{bmatrix} = \begin{bmatrix} u_1^{[s]} \\ u_2^{[s]} \end{bmatrix} + \begin{bmatrix} \Delta u_1^{[s]} \\ \Delta u_2^{[s]} \end{bmatrix} \qquad s = 0, 1, 2, \ldots$$

Taking $u_1^{(0)} = u_2^{(0)} = 1$

$u_1^{[1]} = 2.4500$ $\qquad$ $u_2^{[1]} = 1.5500$

$u_1^{[2]} = 2.2969$ $\qquad$ $u_2^{[2]} = 1.4691$

$u_1^{[3]} = 2.2950$ $\qquad$ $u_2^{[3]} = 1.4679$

PARTIAL DIFFERENTIAL EQUATIONS

We consider the general PDE of the form

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0 \quad -(1)$$

If A, B, C are function of $x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, then (1) is called quasilinear PDE. If A, B & C are functions of x, y and F is a linear function of $u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ then (1) is called linear.

A linear PDE is written as

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu+G=0 \quad -(2)$$

Here A, B, C, D, E, F, G are functions of x & y or they are constants.

The equation (2) is called homogeneous if $G=0$ otherwise non homogeneous.

The PDE (2) is said to be

i) hyperbolic at a point (x,y) if $B^2-4AC>0$ at (x,y)

ii) parabolic at a point (x,y) if $B^2-4AC=0$ at (x,y)

iii) elliptic at a point (x,y) if $B^2-4AC<0$ at (x,y).

Example: Classify the partial differential equation

$$y\,u_{xx} - 2u_{xy} - x\,u_{yy} - u_x + \cos(y)\,u_y - 4 = 0$$

Sol: $A = y \quad B = -2 \quad C = -x$

$$B^2 - 4AC = 4 + 4xy = 4(1+xy)$$

The equation is hyperbolic for all (x,y) such that $xy > -1$

The equation is parabolic for all (x,y) such that $xy = -1$

The equation is elliptic for all (x,y) such that $xy < -1$.

Example: Classify the region where the following PDE is hyperbolic, parabolic, elliptic.

$$(1+y)\,z_{xx} + 2x\,z_{xy} + (1-y)\,z_{yy} = z_x \quad \text{— (1)}$$

Sol: Here $A = (1+y) \quad B = 2x \quad C = (1-y)$

$$B^2 - 4AC = 4x^2 - 4(1+y)(1-y)$$

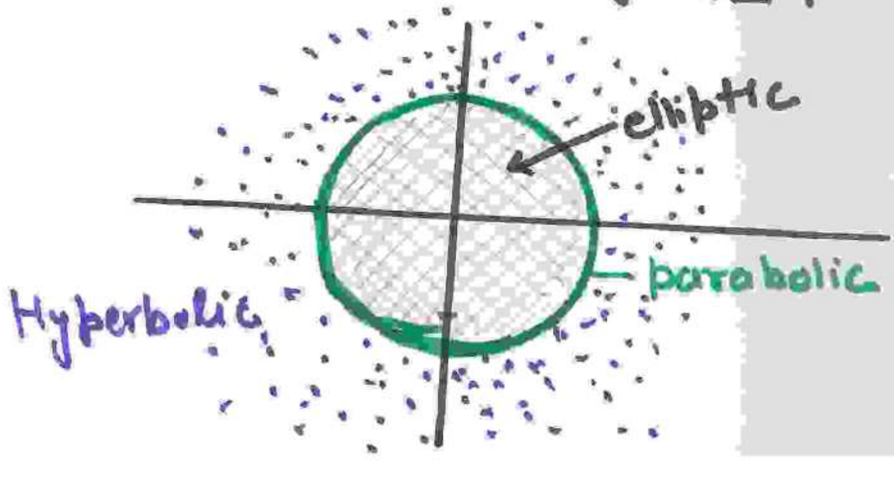
$$= 4x^2 - 4(1-y^2)$$

$$= 4(x^2+y^2-1)$$

The equation is hyperbolic in the region $x^2+y^2 > 1$,

parabolic in the region $x^2+y^2 = 1$ and

elliptic in the region $x^2+y^2 < 1$.



Example: Classify the following PDEs

a) $u_{xx} - u_{tt} = 0$ b) $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ c) $u_{xx} + u_{yy} = 0$

Sol:

a) $A = 1 \quad B = 0 \quad C = -1$

$B^2 - 4AC = 4 > 0$ hyperbolic

- Wave equation
- transverse vibration of a string

b) $A = k, \quad B = 0, \quad C = 0$

$B^2 - 4AC = 0$ parabolic

- Heat conduction in a solid
- HEAT EQUATION

c) $A = 1 \quad B = 0 \quad C = 1$

$B^2 - 4AC = 0 - 4 < 0$ (elliptic)

- Laplace equation
- Steady state heat equation

CANONICAL FORMS

We assume that the given PDE is of single type in a given domain. It does not change its nature at different point.

We shall consider

$$A u_{xx} + B u_{xy} + C u_{yy} = H(x, y, u, u_x, u_y) \quad \text{--- (1)}$$

Under a suitable transformation (non-singular)

$$\xi = \xi(x,y) \ \& \ \eta = \eta(x,y)$$

the above PDE (1) can be transformed to one of the following forms (called canonical forms):

(i) $\omega_{\xi\xi} - \omega_{\eta\eta} = \Phi(\xi, \eta, \omega, \omega_\xi, \omega_\eta)$

or $\omega_{\xi\eta} = \Phi(\xi, \eta, \omega, \omega_\xi, \omega_\eta)$ for hyperbolic case.

ii) $\omega_{\xi\xi} + \omega_{\eta\eta} = \Phi(\xi, \eta, \omega, \omega_\xi, \omega_\eta)$ for elliptic case

iii) $\omega_{\xi\xi} = \Phi(\xi, \eta, \omega, \omega_\xi, \omega_\eta)$

or

$\omega_{\eta\eta} = \Phi(\xi, \eta, \omega, \omega_\xi, \omega_\eta)$ for parabolic case.

SOLUTION OF $\underbrace{P(x,y,z)\dfrac{\partial z}{\partial x} + Q(x,y,z)\dfrac{\partial z}{\partial y} = R(x,y,z)}_{\text{OR}}$

LAGRANGE METHOD

$$Pp + Qq = R$$

I. Write Lagrange auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

II. Solve Lagrange equations to get

$u(x,y,z) = c_1$ & $v(x,y,z) = c_2$

two independent solutions

III. Write the general solution of PDE as

$$v = f(u)$$

OR

$$u = f(v)$$

or

$$f(u,v) = 0$$

where f is an aritrary function.

* For simplicity we can write a solution of the PDE as

$$v = u$$

CANONICAL FORMS (Simplify equations by coordinate transformation)

Let us consider the general second order PDE

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial y \partial x} + C \frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad -(1)$$

Let $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ be a nonsingular transformation. Note that for a nonsingular transformation we have

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$$, and then we can find $x = x(\xi,\eta)$ & $y = y(\xi,\eta)$.

So we change the independent variables (x,y) to (ξ,η). Let us write

$$\omega(\xi,\eta) = u\left(x(\xi,\eta), y(\xi,\eta)\right)$$

Using the chain rule we find

$$u_x = \omega_\xi \xi_x + \omega_\eta \eta_x$$

$$u_y = \omega_\xi \xi_y + \omega_\eta \eta_y$$

$$u_{xx} = (\omega_{\xi\xi} \xi_x + \omega_{\xi\eta} \eta_x)\xi_x + \omega_\xi \xi_{xx} + (\omega_{\eta\xi} \xi_x + \omega_{\eta\eta} \eta_x)\eta_x + \omega_\eta \eta_{xx}$$

$\Rightarrow$

$$u_{xx} = w_{\xi\xi}\,\xi_x^2 + 2w_{\xi\eta}\,\xi_x\eta_x + w_{\eta\eta}\,\eta_x^2 + w_\xi\,\xi_{xx} + w_\eta\,\eta_{xx}$$

Similarly:

$$u_{xy} = w_{\xi\xi}\,\xi_x\xi_y + w_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + w_{\eta\eta}\,\eta_x\eta_y + w_\xi\,\xi_{xy} + w_\eta\,\eta_{xy}$$

$$u_{yy} = w_{\xi\xi}\,\xi_y^2 + 2w_{\xi\eta}\,\xi_y\eta_y + w_{\eta\eta}\,\eta_y^2 + w_\xi\,\xi_{yy} + w_\eta\,\eta_{yy}$$

Substituting into (1):

$$A\left(w_{\xi\xi}\,\xi_x^2 + 2w_{\xi\eta}\,\xi_x\eta_x + w_{\eta\eta}\,\eta_x^2\right) + B\left(w_{\xi\xi}\,\xi_x\xi_y + w_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + w_{\eta\eta}\,\eta_x\eta_y\right) + C\left(w_{\xi\xi}\,\xi_y^2 + 2w_{\xi\eta}\,\xi_y\eta_y + w_{\eta\eta}\,\eta_y^2\right) + G(\xi,\eta,w,w_\xi,w_\eta) = 0$$

$$\Rightarrow w_{\xi\xi}\left(\underline{A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2}\right) + \underline{2}w_{\xi\eta}\left(\underline{A\xi_x\eta_x + \tfrac{1}{2}B\xi_x\eta_y + \tfrac{1}{2}B\xi_y\eta_x + C\xi_y\eta_y}\right) + w_{\eta\eta}\left(\underline{A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2}\right) + G(\xi,\eta,w,w_\xi,w_\eta) = 0$$

$$\Rightarrow \boxed{\bar{A}w_{\xi\xi} + \bar{B}w_{\xi\eta} + \bar{C}w_{\eta\eta} + G(\xi,\eta,w,w_\xi,w_\eta) = 0} \quad -(2)$$

Hence the problem is to choose ξ & η so that (3) takes a simple form.

Case I: If $B^2-4AC>0$ (Hyperbolic case)

We choose ξ & η so that $\bar{A}=\bar{C}=0$, i.e.,

$$A\xi_x^2+B\xi_x\xi_y+C\xi_y^2=0$$

$$\& \quad A\eta_x^2+B\eta_x\eta_y+C\eta_y^2=0$$

The equation for η is the same as for ξ; therefore we need to solve only one equation.

Solve first equation we get:

$$\frac{\xi_x}{\xi_y}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

$$\Leftrightarrow \quad 2A\xi_x-\left(-B\pm\sqrt{B^2-4AC}\right)\xi_y=0$$

In order to obtain a non-singular transformation we choose ξ to be a solution of

$$2A\xi_x-\left(-B-\sqrt{B^2-4AC}\right)\xi_y=0 \qquad \text{—— (3)}$$

and η to be a solution of

$$2A\eta_x-\left(-B+\sqrt{B^2-4AC}\right)\eta_y=0 \qquad \text{—— (4)}$$

Lagrange auxiliary equations for (3):

$$\frac{dx}{2A} = \frac{dy}{-(-B-\sqrt{B^2-4AC})} = \frac{d\xi}{0} \quad \text{——} \ (5)$$

Taking the first two fractions of (5) we get

$$\frac{dy}{dx} = -\frac{-B-\sqrt{B^2-4AC}}{2A}$$

or

$$\boxed{\frac{dy}{dx} = \frac{B+\sqrt{B^2-4AC}}{2A}} \quad \text{——} \ (6)$$

The solution of (6) may be written as

$$\varphi_1(x,y) = c_1 \quad \text{——} \ (7)$$

Taking last fraction of (5), we obtain

$$\xi = c_2$$

A solution of (3) may be written as

$$\xi = \varphi_1(x,y) \quad \text{——} \ (8)$$

Similarly for (4), we get

$$\boxed{\frac{dy}{dx} = \frac{B-\sqrt{B^2-4AC}}{2A}} \quad \text{——} \ (9)$$

& $\eta = \varphi_2(x,y)$ where $\varphi_2(x,y) =$ Constant is the solution of (9).

Hence the transformation

$$\xi = \varphi_1(x,y) \quad \text{and} \quad \eta = \varphi_2(x,y)$$

will transform equation ① to a cannonical form

$$\omega_{\xi\eta} = \Phi(\xi, \eta, \omega, \omega_\xi, \omega_\eta).$$

The equations (6) & (9) are called characteristics equations of (1).

The solution of (6) or respectively (9) is called the characteristics of the equation (1).

Case II: The parabolic case $(B^2 - 4AC = 0)$

In this case there exists only one characteristic equation

$$\frac{dy}{dx} = \frac{B}{A}$$

(assuming A or C does not vanish together otherwise $A = C = 0 \Rightarrow B = 0$ suppose $A \neq 0$.)

In this case we obtain one transformation, say

$$\xi = \varphi(x,y) \qquad (\text{or } \eta = \varphi(x,y))$$

It follows that

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0.$$

It is easy to show that

$$\bar{B}^2 - 4\bar{A}\bar{C} = J^2(B^2 - 4AC) = 0$$

$\Rightarrow$ If $\bar{A} = 0$ then $\bar{B} = 0$

So the equation (1) reduces to

$$\omega_{\eta\eta} = \Phi(\xi, \eta, \omega, \omega_\xi, \omega_\eta)$$

for arbitrary values of $\eta(x,y)$ such that $J \neq 0$.

In practice one may choose $\eta = y$ for instance to have a nonsingular transformation

$$\xi = \psi(x,y)$$
$$\eta = y$$

$$J = \begin{vmatrix} \psi_x & \psi_y \\ 0 & 1 \end{vmatrix} = \psi_x \neq 0$$

(If $\psi_x = 0$
$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow B = 0$
parabolicity $\Rightarrow A$ or $C = 0$
In that case original equation is already in canonical form)

Case III: (Similar to case I) elliptic case ($B^2 - 4AC < 0$)

Since $B^2 - 4AC < 0$, the elliptic equation has no real characteristic. Nevertheless we seek a transformation $\xi = \xi(x,y)$ & $\eta = \eta(x,y)$ which simplifies equation (1)

Proceeding in a similar fashion as in the case (I), we find ξ and η as complex conjugate.

As in case I we can arrive at

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = \Phi(\xi, \eta, w, w_\xi, w_\eta) \text{ where } \eta \,\&\, \xi \text{ are}$$

complex conjugate.

To get a real cannonical form we make further transformation

$$\alpha = \frac{1}{2}(\xi + \eta) \qquad \beta = \frac{i}{2}(-\xi + \eta)$$

$$w(\xi, \eta) = w(\xi(\alpha, \beta), \eta(\alpha, \beta)) = \bar{w}(\alpha, \beta)$$

$$\frac{\partial w}{\partial \xi} = \frac{\partial \bar{w}}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial \xi} + \frac{\partial \bar{w}}{\partial \beta} \cdot \frac{\partial \beta}{\partial \xi}$$

$$= \frac{\partial \bar{w}}{\partial \alpha} \cdot \left(\frac{1}{2}\right) + \frac{\partial \bar{w}}{\partial \beta}\left(-\frac{i}{2}\right)$$

$$\frac{\partial^2 w}{\partial \eta \partial \xi} = \frac{1}{2}\left[\frac{\partial^2 \bar{w}}{\partial \alpha^2} \cdot \frac{\partial \alpha}{\partial \eta} + \cancel{\frac{\partial^2 \bar{w}}{\partial \alpha \partial \beta}} \cdot \frac{\partial \beta}{\partial \eta}\right] - \frac{i}{2}\left[\frac{\partial^2 \bar{w}}{\partial \alpha \partial \beta} \frac{\partial \alpha}{\partial \eta} + \frac{\partial^2 \bar{w}}{\partial \beta^2} \frac{\partial \beta}{\partial \eta}\right]$$

$$= \frac{1}{2}\left[\frac{\partial^2 \bar{w}}{\partial \alpha^2} \cdot \frac{1}{2} + \cancel{\frac{\partial^2 \bar{w}}{\partial \alpha \partial \beta}}\left(\frac{i}{2}\right)\right] - \frac{i}{2}\left[\frac{1}{2} \cancel{\frac{\partial^2 \bar{w}}{\partial \alpha \partial \beta}} + \frac{i}{2} \frac{\partial^2 \bar{w}}{\partial \beta^2}\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2 \bar{w}}{\partial \alpha^2} + \frac{\partial^2 \bar{w}}{\partial \beta^2}\right]$$

So the desired canonical form is

$$\boxed{\frac{\partial^2 \bar{w}}{\partial \alpha^2} + \frac{\partial^2 \bar{w}}{\partial \beta^2} = \psi(\alpha, \beta, \bar{w}, \bar{w}_\alpha, \bar{w}_\beta)}$$

Ex: Find the canonical form of

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0$$

and solve it.

Sol:

$A = 3 \quad B = 10 \quad C = 3$

$B^2 - 4AC = 100 - 36 = 64 > 0.$

The given PDE is of hyperbolic type.

The corresponding characteristic equations are

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \frac{10 + \sqrt{100-36}}{2 \cdot 3} = \frac{10+8}{6} = 3$$

& $$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{10-8}{6} = \frac{1}{3}$$

Characteristics are $y - 3x = c_1$

& $y - \frac{x}{3} = c_2$

To find the canonical form we take the following transformation

$$\xi = y - 3x \quad \& \quad \eta = y - x/3.$$

Let $u(x,y) = w(\xi, \eta)$

$u_x = w_\xi \cdot \xi_x + w_\eta \cdot \eta_x$

$= w_\xi(-3) + w_\eta(-\frac{1}{3})$

$u_{xx} = -3\left[w_{\xi\xi} \cdot (-3) + w_{\xi\eta}(-\frac{1}{3})\right] - \frac{1}{3}\left[w_{\eta\xi}(-3) + w_{\eta\eta}(-\frac{1}{3})\right]$

$=$

$= 9w_{\xi\xi} + 2w_{\xi\eta} + \frac{1}{9}w_{\eta\eta}$

$$u_{xy} = -3\left[\omega_{\xi\xi}\,\xi_y + \omega_{\xi\eta}\cdot\eta_y\right] - \frac{1}{3}\left[\omega_{\eta\xi}\,\xi_y + \omega_{\eta\eta}\,\eta_y\right]$$

$$= -3\left[\omega_{\xi\xi}\cdot 1 + \omega_{\xi\eta}\cdot 1\right] - \frac{1}{3}\left[\omega_{\eta\xi}\cdot 1\cdot + \omega_{\eta\eta}\cdot 1\cdot\right]$$

$$= -3\,\omega_{\xi\xi} - \frac{10}{3}\,\omega_{\xi\eta} - \frac{1}{3}\,\omega_{\eta\eta}$$

$$u_y = \left(\omega_\xi\,\xi_y + \omega_\eta\,\eta_y\right)$$

$$= \omega_\xi\cdot + \omega_\eta$$

$$u_{yy} = \left(\omega_{\xi\xi} + \omega_{\xi\eta}\right) + \left(\omega_{\eta\xi} + \omega_{\eta\eta}\right)$$

$$= \omega_{\xi\xi} + 2\omega_{\xi\eta} + \omega_{\eta\eta}\,.$$

substituting in the given PDE.

$$3\left(\cancel{9\,\omega_{\xi\xi}} + 2\omega_{\xi\eta} + \cancel{\frac{1}{9}\,\omega_{\eta\eta}}\right) + 10\left(-\cancel{3\,\omega_{\xi\xi}} - \frac{10}{3}\omega_{\xi\eta} - \cancel{\frac{1}{3}\omega_{\eta\eta}}\right)$$

$$+ 3\left(\cancel{\omega_{\xi\xi}} + 2\omega_{\xi\eta} + \cancel{\omega_{\eta\eta}}\right) = 0$$

$$\Rightarrow \quad \left(6 - \frac{100}{3} + 6\right)\omega_{\xi\eta} = 0$$

$$\Rightarrow \boxed{\omega_{\xi\eta} = 0} \rightarrow \text{desired canonical form.}$$

on integration, we get

$$\omega_\xi = \Phi_1(\xi)$$

Again integrating

$$w = \int \Phi_1(\xi)\,d\xi + \psi(\eta) \Rightarrow w(\xi,\eta) = \Phi(\xi) + \psi(\eta)$$

or

$$u(x,y) = \Phi(y-3x) + \psi\left(y - \frac{x}{3}\right)$$

Ex: Reduce the equation $u_{xx} + x^2 u_{yy} = 0$ to a canonical form. (1

Sol: $A = 1. \quad B = 0 \quad C = x^2$

$$B^2 - 4AC = -4x^2 < 0$$

Hence the given PDE is elliptic.

The characteristic equations are

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \frac{0 + \sqrt{-4x^2}}{2} = +ix$$

& $$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -ix.$$

Integration gives:

$$y = \frac{ix^2}{2} - ic_1 \Rightarrow iy + \frac{x^2}{2} = c_1$$

& $$y = -\frac{ix^2}{2} + ic_1 \Rightarrow -iy + \frac{x^2}{2} = c_2$$

Hence $\xi = \frac{1}{2}x^2 + iy \quad , \quad \eta = \frac{1}{2}x^2 - iy$

Introducing the second transformation

$$\alpha = \frac{\xi + \eta}{2} \qquad \beta = \frac{i}{2}(-\xi + \eta)$$

$$\Rightarrow \alpha = \frac{x^2}{2} \qquad \beta = -\frac{i}{2} \cdot 2iy = y$$

Take. $u(x,y) = \omega(\alpha, \beta)$ and subst. in given PDE we get

$$\boxed{\omega_{\alpha\alpha} + \omega_{\beta\beta} = -\frac{\omega_\alpha}{2\alpha}}$$

Ex: Reduce the equation

$$y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} = \frac{y^2}{x} u_x + \frac{x^2}{y} u_y$$

to a canonical form and solve it.

Sol: $A = y^2$
$B = -2xy$
$C = x^2$

So $B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$ (parabolic type)

Characteristic equation

$$\frac{dy}{dx} = \frac{B}{2A} = -\frac{2xy}{2y^2} = -\frac{x}{y}$$

$$\Rightarrow \frac{dy}{dx} + \frac{x}{y} = 0$$

$$\Rightarrow y^2 = -x^2 + c_1$$

$$\Rightarrow y^2 + x^2 = c_1 \quad \text{therefore } \xi = x^2 + y^2$$

Let us choose $\eta = x^2 - y^2$

$$J = \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix} = -8xy$$

Subst. $u(x,y) = w(\xi, \eta)$ with $\xi = x^2 + y^2$ & $\eta = x^2 - y^2$

we get $\quad w_{\eta\eta} = 0$

$$\Rightarrow w_\eta = f(\xi)$$

$$\Rightarrow w = f(\xi)\eta + g(\xi)$$

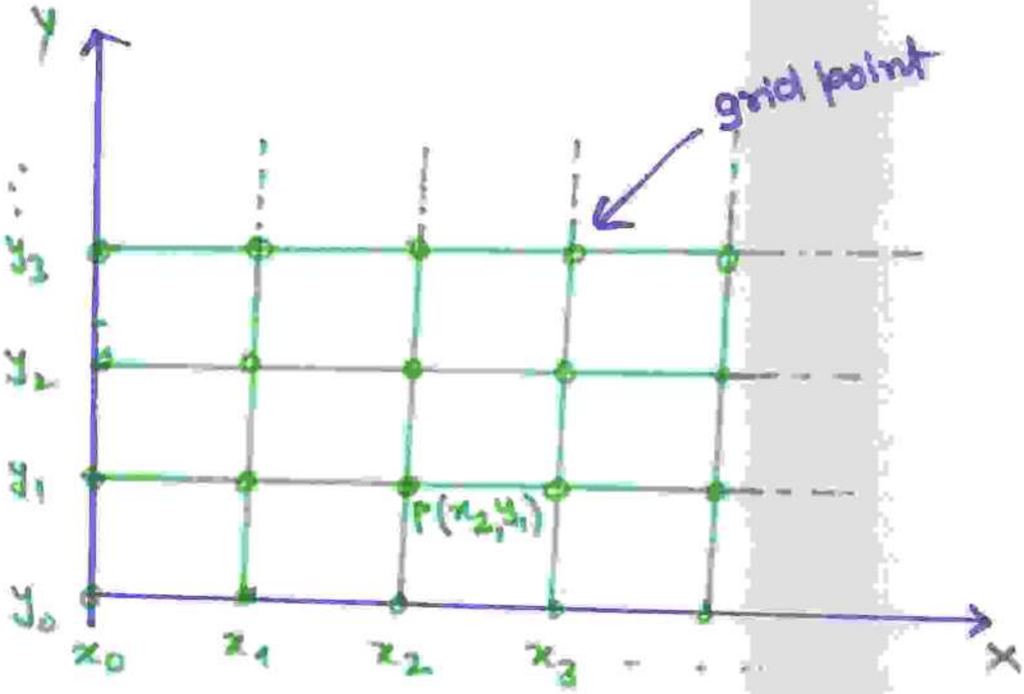
$$\Rightarrow u(x,y) = (x^2 - y^2) f(x^2+y^2) + g(x^2+y^2)$$

$\square$.

Finite difference approximations to partial derivatives

Let the xy plane be divided into a set of equal rectangles of sides $\Delta x = h$ and $\Delta y = k$ by drawing
rectangles of sides $\Delta x = h$ and $\Delta y = k$ by drawing
the equally spaced grid lines parallel to the co-ordinate axes, defined by

$$x_m = mh, \quad m = 0, 1, 2, \ldots$$
$$y_n = nk, \quad n = 0, 1, 2, \ldots$$



The approximate value of u at a grid point $P(x_m, y_n)$ is denoted by u_m^n i.e.,

$$u_m^n \fallingdotseq u(x_m, y_n) = u(mh, nk).$$

To this end, we define

$$u_x(x_m, y_n) \fallingdotseq \frac{u_{m+1}^n - u_m^n}{h} + O(h) \quad \text{(forward difference)}$$

$$u_x(x_m, y_n) \simeq \frac{u_m^n - u_{m-1}^n}{h} + O(h) \quad \begin{pmatrix}\text{backward}\\ \text{difference}\end{pmatrix}$$

$$= \frac{u_{m+1}^n - u_{m-1}^n}{2h} + O(h^2) \quad \begin{pmatrix}\text{central}\\ \text{difference}\end{pmatrix}$$

Similarly,

$$u_y(x_m, y_n) \simeq \frac{u_m^{n+1} - u_m^n}{k} + O(k) \quad \text{forward}$$

$$= \frac{u_m^n - u_m^{n-1}}{k} + O(k) \quad \text{backward}$$

$$= \frac{u_m^{n+1} - u_m^{n-1}}{2k} + O(k^2) \quad \text{central}$$

&

$$u_{xx}(x_m, y_n) \simeq \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2} + O(h^2)$$

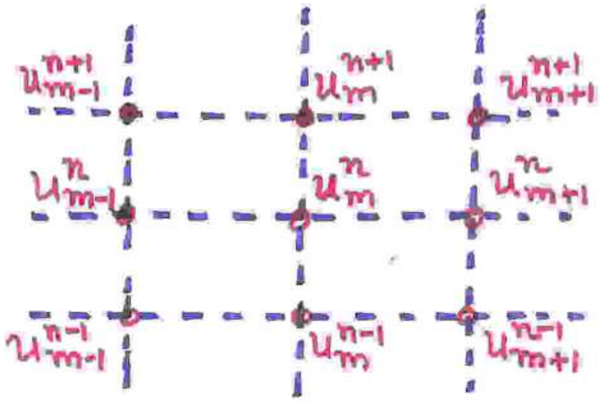
$$u_{yy}(x_m, y_n) \simeq \frac{u_m^{n-1} - 2u_m^n + u_m^{n+1}}{k^2} + O(k^2)$$

Parabolic Partial differential equation

We consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad a \le x \le b, \quad 0 \le t \le T \qquad -(1)$$

Let us denote u_m^n the approximation of u at (x_m, t_n).



The possible approximations of the equation (1) are:

2(i): $$\frac{u_m^{n+1} - u_m^n}{k} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$ Schmidt method (explicit) (two level) (cond. stable) $O(k+h^2)$

2(ii) $$\frac{u_m^{n} - u_m^{n-1}}{k} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$ Laasonen method (Implicit) (two level) uncond. stable $O(k+h^2)$

2(iii) $$\frac{u_m^{n+1} - u_m^{n-1}}{2k} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$ Richardson (Leapfrog) met (explicit) (three level) uncond. ~~stable~~ unstable $O(k^2+h^2)$.

Dufort Frankel method: modification to 2 iii):

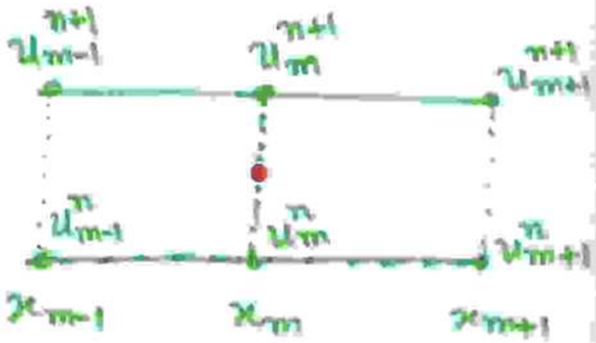
u_m^n is replaced by the average of u_m at time level $(n+1)$ and $(n-1)$; i.e.,

$$u_m^n = \frac{u_m^{n+1} + u_m^{n-1}}{2}.$$ Then the method 2(iii) becomes

2(iv) $$\frac{u_m^{n+1} - u_m^{n-1}}{2K} = \frac{u_{m-1}^{n} - (u_m^{n+1} + u_m^{n-1}) + u_{m+1}^{n}}{h^2}.$$

Now the method becomes unconditionally stable. However the method is not consistent. ORDER of the method is 0.

CRANK-NICOLSON METHOD:



Crank & Nicolson proposed approximating the partial derivatives at the point $(x_m, t_n + \frac{K}{2})$ or $(m, n+\frac{1}{2})$ as.

$$u_t\Big|_{(m,n+\frac{1}{2})} \simeq \frac{u_m^{n+1} - u_m^{n}}{K}$$

and

$$u_{xx}\Big|_{(m,n+\frac{1}{2})} \simeq \frac{1}{2}\left[u_{xx}\Big|_{(m,n)} + u_{xx}\Big|_{(m,n+1)}\right]$$

$$\simeq \frac{1}{2}\left[\frac{u_{m+1}^{n} - 2u_m^{n} + u_{m-1}^{n}}{h^2} + \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2}\right]$$

Finally, the scheme becomes:

2(v):

$$\frac{u_m^{n+1}-u_m^n}{k} = \frac{1}{2}\left[\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} + \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2}\right]$$

ORDER: $O(k^2+h^2)$

STABILITY: UNCOND. STABLE.

TRUNCATION ERROR OF THE METHOD: 2(i)

$$T.E. = \frac{u(x_m, t_n+k) - u(x_m,t_n)}{k} - \frac{1}{h^2}\left[u(x_m-h,t_n) - 2u(x_m,t_n) + u(x_m+h,t_n)\right]$$

$$= \frac{u(x_m,t_n) + k u_t(x_m,t_n) + \frac{k^2}{\lfloor 2} u_{tt}(x_m,t_n) + \cdots - u(x_m,t_n)}{k}$$

$$- \frac{1}{h^2}\Big[u(x_m,t_n) - h u_x(x_m,t_n) + \frac{h^2}{2} u_{xx}(x_m,t_n) - \frac{h^3}{2} u_{xxx} + \frac{h^4}{\lfloor 4} \cdots$$

$$-2u(x_m,t_n)$$

$$+ u(x_m,t_n) + h u_x(x_m,t_n) + \frac{h^2}{2} u_{xx}(x_m,t_n) + \frac{h^4}{4} u_{xxx} + \frac{h^4}{\lfloor 4} \cdots\Big]$$

$$= \underline{u_t} + \frac{k}{2} u_{tt} + \cdots - \underline{u_{xx}} - \frac{h^2}{12} u_{xxxx} + \cdots$$

$$= \frac{k}{2} u_{tt} + \cdots - \frac{h^2}{12} u_{xxxx} + \cdots$$

$$= \left(\frac{k}{2} - \frac{h^2}{12}\right) u_{xxxx} + O(k^2) + O(h^3) = O(k) + O(h^2)$$

$$= O(k+h^2)$$

The method is said to be second order accurate in space and 1st order in time.

The methods can be rewritten in simplified form:

2(i): $$u_m^{n+1} = (1-2\lambda)\,u_m^n + \lambda\left(u_{m-1}^n + u_{m+1}^n\right)$$

λ is called mesh ratio parameter.

$$\lambda = \frac{k}{h^2}.$$

2(ii): $$-\lambda u_{m-1}^n + (1+2\lambda)\,u_m^n - \lambda u_{m+1}^n = u_m^{n-1}$$

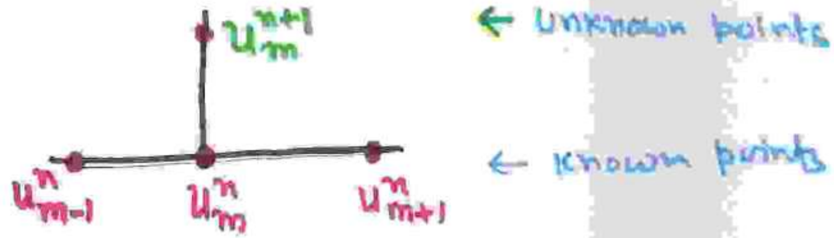
OR

$$-\lambda u_{m-1}^{n+1} + (1+2\lambda)\,u_m^{n+1} - \lambda u_{m+1}^{n+1} = u_m^n$$

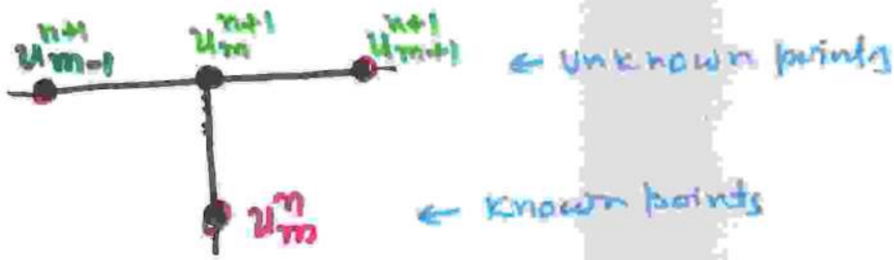
2(v): $$-\lambda u_{m-1}^{n+1} + (2+2\lambda)\,u_m^{n+1} - \lambda u_{m+1}^{n+1} = \lambda u_{m-1}^n + (2-2\lambda)\,u_m^n + \lambda u_{m+1}^n$$

SCHEMATIC DIAGRAM (STENCIL)

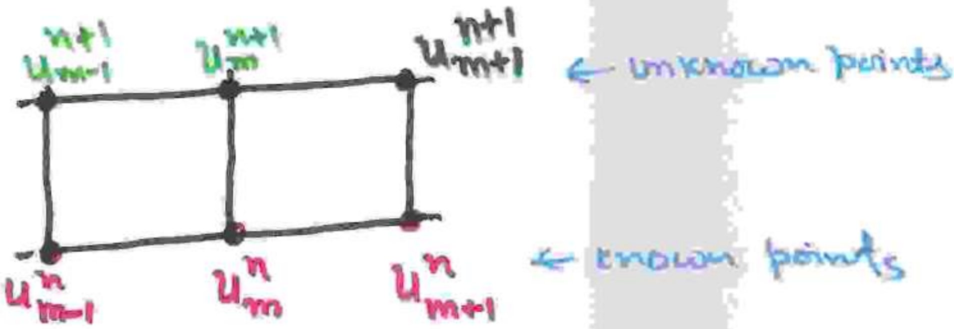
2(i):



2(ii)



2(v)



PARABOLIC PDEs $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

Possible approximations:

i) $$\frac{u_m^{n+1} - u_m^n}{k} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$ Schmidt method (explicit)

ii) $$\frac{u_m^n - u_m^{n-1}}{k} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$ Laasonen method (Implicit)

iii) $$\frac{u_m^{n+1} - u_m^{n-1}}{2k} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$ Richardson (leap frog) method

iv) $$\frac{u_m^{n+1} - u_m^{n-1}}{2k} = \frac{u_{m-1}^n - (u_m^{n+1} + u_m^{n-1}) + u_{m+1}^n}{h^2}$$ Dufort-Frankel method

v) $$\frac{u_m^{n+1} - u_m^n}{k} = \frac{1}{2}\left[\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} + \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2}\right]$$

Simplified forms by setting $\frac{k}{h^2} = \lambda$ (mesh ratio parameter)

i) $$u_m^{n+1} = (1-2\lambda)u_m^n + \lambda(u_{m-1}^n + u_{m+1}^n)$$

ii) $$-\lambda u_{m-1}^{n+1} + (1+2\lambda)u_m^{n+1} - \lambda u_{m+1}^{n+1} = u_m^n$$

iii) $$u_m^{n+1} = u_m^{n-1} + 2\lambda(u_{m-1}^n - 2u_m^n + u_{m+1}^n)$$

iv) $$u_m^{n+1} = \frac{(1-2\lambda)}{(1+2\lambda)} u_m^{n-1} + \frac{2\lambda}{1+2\lambda}(u_{m-1}^n + u_{m+1}^n)$$

v) $$-\lambda u_{m-1}^{n+1} + (2+2\lambda)u_m^{n+1} - \lambda u_{m+1}^{n+1} = \lambda u_{m-1}^n + (2-2\lambda)u_m^n + \lambda u_{m+1}^n$$

Ex Solve the heat equation by explicit method (FTCS):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1. \qquad 0 \le t \le 0.05$$

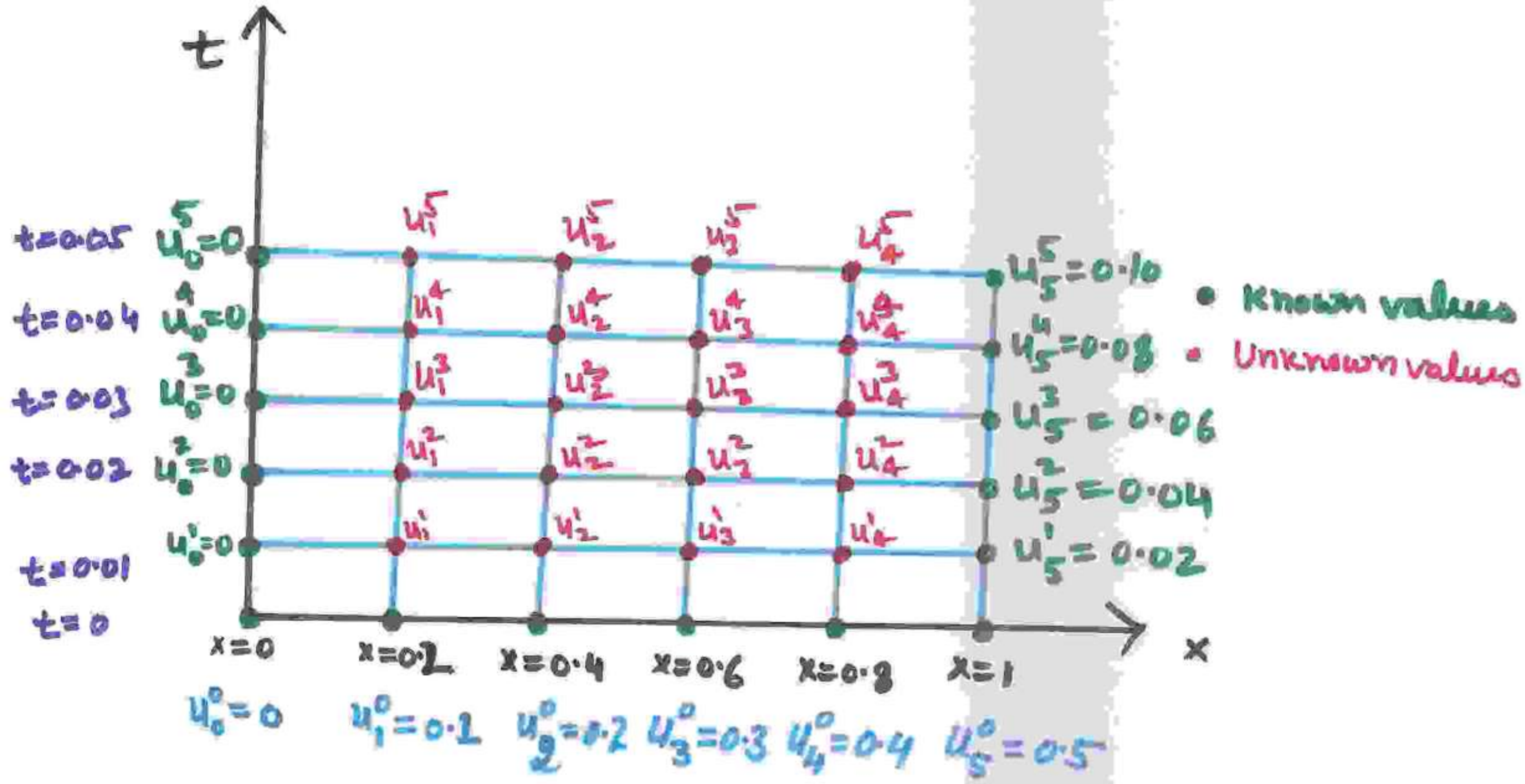
subject to the BCs: $\underbrace{u(0,t) = 0 \ \& \ u(1,t) = 2t}_{t>0}$ and

initial condition

$$u(x,0) = \frac{1}{2}x \qquad 0 \le x \le 1$$

Take $h = 0.2$, $K = 0.01$.

Solution:



$$\lambda = \frac{K}{h^2} = \frac{0.01}{0.04} = \frac{1}{4}$$

Explicit method: $u_m^{n+1} = (1-2\lambda)u_m^n + \lambda(u_{m-1}^n + u_{m+1}^n)$

$$= \frac{1}{2}u_m^n + \frac{1}{4}(u_{m-1}^n + u_{m+1}^n)$$

$$= \frac{1}{4}\left[u_{m-1}^n + 2u_m^n + u_{m+1}^n\right].$$

Then $u_1^1 = \frac{1}{4}(u_0^0 + 2u_1^0 + u_2^0) = \frac{1}{4}(0+0.2+0.2) = 0.1$

$u_2^1 = \frac{1}{4}(u_1^0 + 2u_2^0 + u_3^0) = \frac{1}{4}(0.1+0.4+0.3) = 0.2$

and so on....

$u_3^1 = 0.30 \qquad u_4^1 = 0.40$

$u_1^2 = 0.25 \qquad u_2^2 = 0.4250 \qquad u_3^2 = 0.60 \qquad u_4^2 = 0.28$

$u_1^3 = 0.9250 \qquad u_2^3 = 1.4750 \qquad u_3^3 = 1.5113 \qquad u_4^3 = 0.300$

$u_1^4 = 3.6625 \qquad u_2^4 = 5.2737 \qquad u_3^4 = 4.544 \qquad u_4^4 = 0.5428$

$u_1^5 = 14.1363 \qquad u_2^5 = 18.7825 \qquad u_3^5 = 14.7506 \qquad u_4^5 = 1.4275$

Ex: Using Crank-Nicolson method and the central differences for the boundary conditions, solve the initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = 1; \qquad 0 \le x \le 1$$

$$\frac{\partial u}{\partial x}(0,t) = u(0,t) \qquad \frac{\partial u}{\partial x}(1,t) = -u(1,t), \quad t>0$$

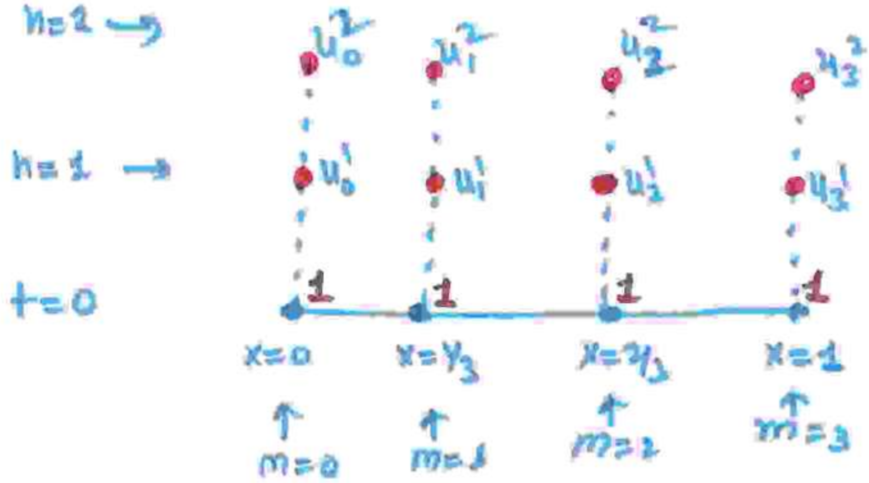
with steplength $h = 1/3$ and $\lambda = 1/3$

Integrate upto two time levels.

Sol: Crank-Nicolson Method:

$$-\lambda u_{m-1}^{n+1} + (2+2\lambda) u_m^{n+1} - \lambda u_{m+1}^{n+1} = \lambda u_{m-1}^{n} + (2-2\lambda) u_m^{n} + \lambda u_{m+1}^{n}$$

$$\Rightarrow -\frac{1}{3} u_{m-1}^{n+1} + \frac{8}{3} u_m^{n+1} - \frac{1}{3} u_{m+1}^{n+1} = \frac{1}{3} u_{m-1}^{n} + \frac{4}{3} u_m^{n} + \frac{1}{3} u_{m+1}^{n} \quad \text{— (1)}$$



equation (1) for

m=0: $$-\frac{1}{3} \underline{u_{-1}^{n+1}} + \frac{8}{3} u_0^{n+1} - \frac{1}{3} u_1^{n+1} = \frac{1}{3} \underline{u_{-1}^{n}} + \frac{4}{3} u_0^{n} + \frac{1}{3} u_1^{n} \quad \text{— (2)}$$

m=1: $$-\frac{1}{3} u_0^{n+1} + \frac{8}{3} u_1^{n+1} - \frac{1}{3} u_2^{n+1} = \frac{1}{3} u_0^{n} + \frac{4}{3} u_1^{n} + \frac{1}{3} u_2^{n} \quad \text{— (3)}$$

m=2: $$-\frac{1}{3} u_1^{n+1} + \frac{8}{3} u_2^{n+1} - \frac{1}{3} u_3^{n+1} = \frac{1}{3} u_1^{n} + \frac{4}{3} u_2^{n} + \frac{1}{3} u_3^{n} \quad \text{— (4)}$$

m=3: $$-\frac{1}{3} u_2^{n+1} + \frac{8}{3} u_3^{n+1} - \underline{\frac{1}{3} u_4^{n+1}} = \frac{1}{3} u_2^{n} + \frac{4}{3} u_3^{n} + \underline{\frac{1}{3} u_4^{n}} \quad \text{— (5)}$$

Using the central diff., the B.C. at $x=0$:

$$\frac{u_1^s - u_{-1}^s}{2h} = u_0^s \Rightarrow u_{-1}^s = u_1^s - 2h u_0^s$$

$$\Rightarrow u_{-1}^s = u_1^s - \frac{2}{3} u_0^s \quad \text{— (6)}$$

Now we can replace u_{-1}^n & u_{-1}^{n+1} in (2) using (6)

$$-\frac{1}{3}\left(u_1^{n+1}-\frac{2}{3}u_0^{n+1}\right)+\frac{8}{3}u_0^{n+1}-\frac{1}{3}u_1^{n+1}=\frac{1}{3}\left(u_1^{n}-\frac{2}{3}u_0^{n}\right)+\frac{4}{3}u_0^{n}+\frac{1}{3}u_1^{n}$$

$$\Rightarrow \frac{26}{9}u_0^{n+1}-\frac{2}{3}u_1^{n+1}=\frac{10}{9}u_0^{n}+\frac{2}{3}u_1^{n}$$

$$\Rightarrow \frac{13}{9}u_0^{n+1}-\frac{1}{3}u_1^{n+1}=\frac{5}{9}u_0^{n}+\frac{1}{3}u_1^{n} \qquad (7)$$

The second B.C. gives:

$$\frac{u_4^s-u_2^s}{2h}=-u_3^s \Rightarrow u_4^s=u_2^s-\frac{2}{3}u_3^s$$

(5) $\Rightarrow$

$$-\frac{1}{3}u_2^{n+1}+\frac{8}{3}u_3^{n+1}-\frac{1}{3}\left(u_2^{n+1}-\frac{2}{3}u_3^{n+1}\right)=\frac{1}{3}u_2^{n}+\frac{4}{3}u_3^{n}+\frac{1}{3}\left(u_2^{n}-\frac{2}{3}u_3^{n}\right)$$

$$\Rightarrow -\frac{2}{3}u_2^{n+1}+\frac{26}{9}u_3^{n+1}=\frac{2}{3}u_2^{n}+\frac{10}{9}u_3^{n}$$

$$\Rightarrow -\frac{1}{3}u_2^{n+1}+\frac{13}{9}u_3^{n+1}=\frac{1}{3}u_2^{n}+\frac{5}{9}u_3^{n} \qquad (8)$$

The equation (3) (4) (7) & (8) in matrix form

$$\begin{bmatrix} \frac{13}{9} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 8/3 & -1/3 & 0 \\ 0 & -1/3 & 8/3 & -1/3 \\ 0 & 0 & -1/3 & 13/9 \end{bmatrix} \begin{bmatrix} u_0^{n+1} \\ u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \end{bmatrix} = \begin{bmatrix} 5/9 & 1/3 & 0 & 0 \\ 1/3 & 4/3 & 1/3 & 0 \\ 0 & 1/3 & 4/3 & 1/3 \\ 0 & 0 & 1/3 & 5/9 \end{bmatrix} \begin{bmatrix} u_0^{n} \\ u_1^{n} \\ u_2^{n} \\ u_3^{n} \end{bmatrix}$$

For $n=0$:

$$\begin{bmatrix} \frac{13}{9} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 8/3 & -1/3 & 0 \\ 0 & -1/3 & 8/3 & -1/3 \\ 0 & 0 & -1/3 & 13/9 \end{bmatrix} \begin{bmatrix} u_0^1 \\ u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = \begin{bmatrix} 5/9 & 1/3 & 0 & 0 \\ 1/3 & 4/3 & 1/3 & 0 \\ 0 & 1/3 & 4/3 & 1/3 \\ 0 & 0 & 1/3 & 5/9 \end{bmatrix} \begin{bmatrix} u_0^0 = 1 \\ u_1^0 = 1 \\ u_2^0 = 1 \\ u_3^0 = 1 \end{bmatrix}$$

$$\Rightarrow \quad u_0^1 = 0.8409$$
$$u_1^1 = 0.9773$$
$$u_2^1 = 0.9773$$
$$u_3^1 = 0.8409$$

For $n=1$:

$$\begin{bmatrix} \frac{13}{9} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 8/3 & -1/3 & 0 \\ 0 & -1/3 & 8/3 & -1/3 \\ 0 & 0 & -1/3 & 13/9 \end{bmatrix} \begin{bmatrix} u_0^2 \\ u_1^2 \\ u_2^2 \\ u_3^2 \end{bmatrix} = \begin{bmatrix} 5/9 & 1/3 & 0 & 0 \\ 1/3 & 4/3 & 1/3 & 0 \\ 0 & 1/3 & 4/3 & 1/3 \\ 0 & 0 & 1/3 & 5/9 \end{bmatrix} \begin{bmatrix} 0.8409 \\ 0.9773 \\ 0.9773 \\ 0.8409 \end{bmatrix}$$

$$\Rightarrow \quad u_0^2 = 0.7629$$
$$u_1^2 = 0.9272$$
$$u_2^2 = 0.9272$$
$$u_3^2 = 0.7629$$

Elliptic Partial Differential Equation

Let us consider the two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{--- (1)}$$

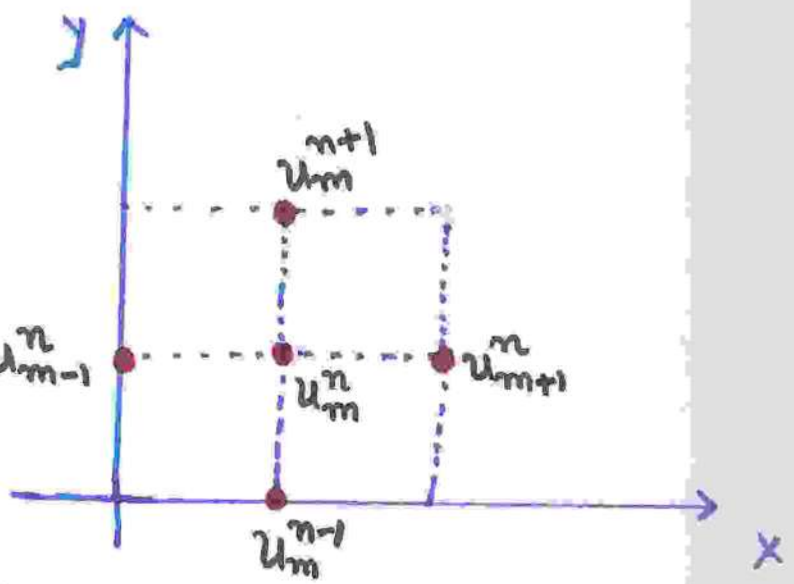
with a rectangular domain.

Using the central difference approximation to both the space and derivatives, the finite difference approximation of the above equation is given by

$$\frac{u_{m-1}^{n} - 2u_m^n + u_{m+1}^{n}}{h^2} + \frac{u_m^{n-1} - 2u_m^n + u_m^{n+1}}{k^2} = 0 \qquad \text{--- (2)}$$

If the grid points are uniform in both directions then it becomes

$$u_m^n = \frac{1}{4}\left[u_{m-1}^{n} + u_{m+1}^{n} + u_m^{n-1} + u_m^{n+1}\right] \qquad \text{--- (3)}$$



This shows that the values of u at the point (m, n) is the average of its values at the four neighbours.
This formula is know as Standard five point formula.

Remark: An equation of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

is called Poisson's equation.

Its finite difference approximation is given by

$$u_m^n = \frac{1}{4}\left[u_{m-1}^n + u_{m+1}^n + u_m^{n-1} + u_m^{n+1} - h^2 f(x_m, y_n)\right] \quad -(4)$$

Iterative Methods:

Let $u_m^{n(r)}$ denote the rth iterative value of u_m^n.

Jacobi Method:

$$u_m^{n(r+1)} = \frac{1}{4}\left[u_{m-1}^{n(r)} + u_{m+1}^{n(r)} + u_m^{(n-1)(r)} + u_m^{(n+1)(r)}\right] - (5)$$

Gauss-Seidel's Method:

In this method, the most recently computed values as soon as they are available are used and the values of u along each row are computed systematically from left to right.
The iterative formula takes the following form:

$$u_m^{n(r+1)} = \frac{1}{4}\left[u_{m-1}^{n(r+1)} + u_{m+1}^{n(r)} + u_m^{n-1(r+1)} + u_m^{n+1(r)}\right]$$

The rate of convergence of this method is twice as fast as the Jacobi's method. This method is also known as Liebmann's method.

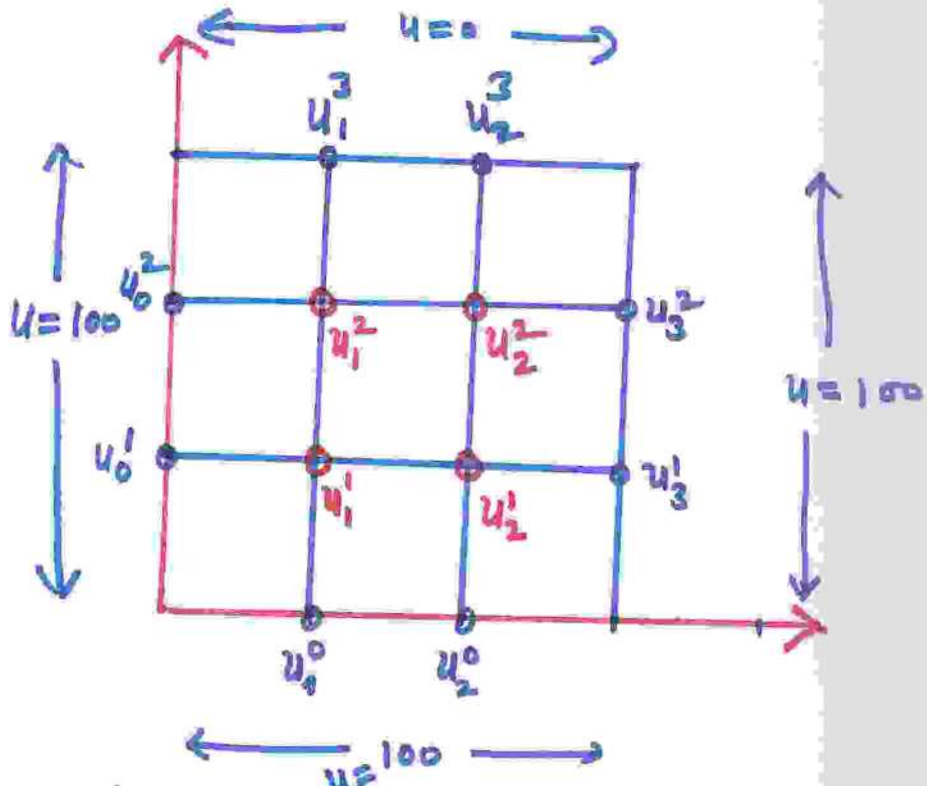
Problem: Solve the following Dirichlet problem:

$$u_{xx} + u_{yy} = 0$$

$$u(x,0) = 100$$
$$u(x,12) = 0$$
$$u(0,y) = 100$$
$$u(12,y) = 100$$

Take $h = k = 4$ and use Gauss-Seidel iteration method to solve the system of linear equations.

Solution:



Let us take initial guess as

$$u_1^{1(0)} = u_2^{1(0)} = u_1^{2(0)} = u_2^{2(0)} = 100 .$$

Then,

$$u_1^{1(1)} = \frac{1}{4}\left[u_0^1 + u_2^{1(0)} + u_1^0 + u_1^{2(0)}\right]$$

$$= \frac{1}{4}\left[100 + 100 + 100 + 100\right] = 100$$

$$u_2^{1(1)} = \frac{1}{4}\left[u_1^{1(1)} + u_3^1 + u_2^0 + u_2^{2(0)}\right]$$

$$= \frac{1}{4}\left[100 + 100 + 100 + 100\right] = 100$$

$$u_1^{2(1)} = \frac{1}{4}\left[u_0^2 + u_2^{2(0)} + u_1^{1(1)} + u_1^3\right]$$

$$= \frac{1}{4}\left[100 + 100 + 100 + 0\right] = 75$$

$$u_2^{2(1)} = \frac{1}{4}\left[u_1^{2(1)} + u_3^2 + u_2^{1(1)} + u_2^3\right]$$

$$= \frac{1}{4}\left[75 + 100 + 100 + 0\right] = 68.75$$

<u>NEXT ITERATION:</u>

$$u_1^{1(2)} = \frac{1}{4}\left[u_0^1 + u_2^{1(1)} + u_1^0 + u_1^{2(1)}\right] = 93.75$$

$$u_2^{1(2)} = \frac{1}{4}\left[u_1^{1(2)} + u_3^1 + u_2^0 + u_2^{2(1)}\right] = 90.625$$

$$u_1^{2(2)} = \frac{1}{4}\left[u_0^2 + u_2^{2(1)} + u_1^{1(2)} + u_1^3\right] = 65.625$$

$$u_2^{2(2)} = \frac{1}{4}\left[u_1^{2(2)} + u_3^2 + u_2^{1(2)} + u_2^3\right] = 64.0625$$

Remark: Use of symmetry:

Problem is symmetric about $x = 6$, i.e.,

$$u_1^1 = u_2^1 \quad \& \quad u_1^2 = u_2^2$$

$$\Rightarrow \quad u_1^1 = \frac{1}{4}\left[u_0^1 + u_1^1 + u_1^0 + u_1^2\right]$$

$$\Rightarrow \quad 3u_1^1 = 200 + u_1^2 \quad \text{———} \quad (*)$$

and

$$u_1^2 = \frac{1}{4}\left[u_0^2 + u_2^2 + u_1^1 + u_1^3\right]$$

$$= \frac{1}{4}\left[100 + u_1^2 + u_1^1 + 0\right]$$

$$\Rightarrow \quad 3u_1^2 = 100 + u_1^1 \quad \text{———} \quad (**)$$

Setup Gauss Seidel iterations:

$$u_1^{1(r)} = \frac{1}{3}\left[200 + u_1^{2(r-1)}\right]$$

$$\& \quad u_1^{2(r)} = \frac{1}{3}\left(100 + u_1^{1(r)}\right)$$

$$r = 0, 1, 2 \ldots$$

$\Rightarrow$ From $(*)$ & $(**)$, we get

$$u_1^2 = u_2^2 = 62.5$$

$$u_1^1 = u_2^1 = 87.5$$

Problem: Solve the Poisson's Equation

$$u_{xx} + u_{yy} = 12xy$$

BCs:

$$u(x,0) = 0, \quad 0<x<1.5$$

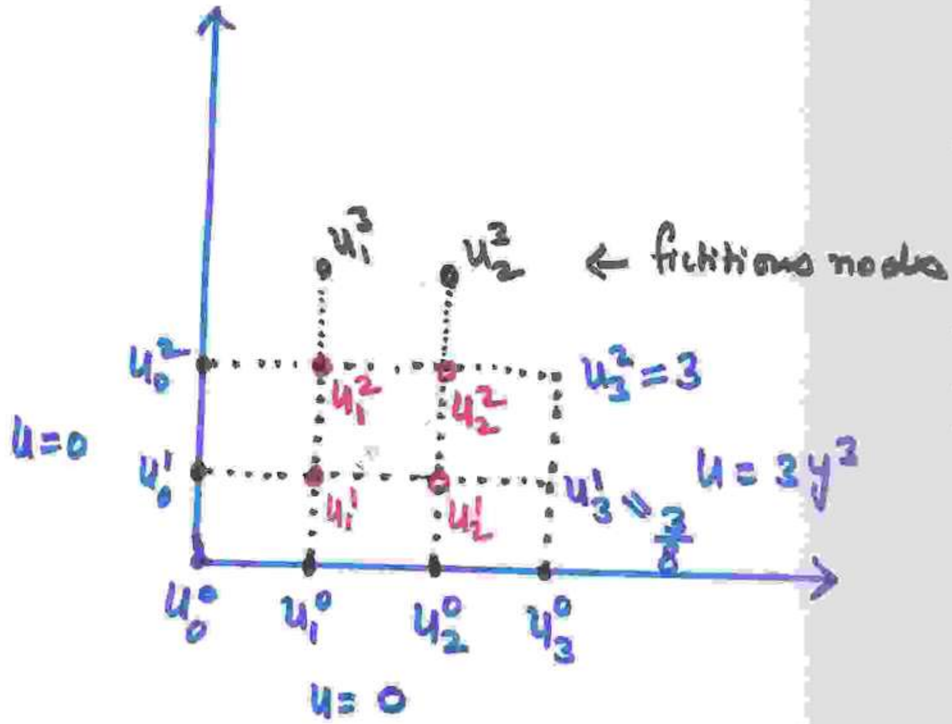
$$u(0,y) = 0, \quad 0 \le y \le 1$$

$$u(1.5,y) = 3y^3, \quad 0 \le y \le 1$$

$$\left.\frac{\partial u}{\partial y}\right|_{(x,1)} = 6x, \quad 0<x<1.5$$

Take $h = k = 0.5$.

<u>Sol</u>:



<u>At</u> u_1^1:

$$u_1^1 = \frac{1}{4}\left[0 + u_2^1 + 0 + u_1^2 - 12\left(\tfrac{1}{2}\right)^2 \tfrac{1}{2} \times \tfrac{1}{2}\right]$$

$$\Rightarrow -4u_1^1 + u_2^1 + u_1^2 = \frac{3}{4} \qquad \text{— (1)}$$

<u>At</u> u_2^1:

$$u_2^1 = \frac{1}{4}\left[u_1^1 + \frac{3}{8} + 0 + u_2^2 - 12\,\tfrac{1}{4} \times 1 \times \tfrac{1}{2}\right]$$

$$\Rightarrow 4u_2^1 = u_1^1 + u_2^2 - \frac{9}{8}$$

$$\Rightarrow -4u_2^1 + u_1^1 + u_2^2 = \frac{9}{8} \quad \text{--- (2)}$$

At: u_1^2:

$$u_1^2 = \frac{1}{4}\left[0 + u_2^2 + u_1^1 + u_1^3 - 12 \cdot \frac{1}{4} \times \frac{1}{2} \times 1\right]$$

$$\Rightarrow 4u_1^2 = u_2^2 + u_1^3 + u_1^1 - \frac{3}{2}$$

$$\Rightarrow -4u_1^2 + u_2^2 + u_1^3 + u_1^1 = 3/2 \quad \text{--- (3)}$$

At u_2^2: $u_2^2 = \frac{1}{4}\left[u_1^2 + 3 + u_2^1 + u_2^3 - 12 \cdot \frac{1}{4} \times 1 \times 1\right]$

$$\Rightarrow -4u_2^2 + u_1^2 + u_2^1 + u_2^3 = 0 \quad \text{--- (4)}$$

Use of Neumann BC.

$$\frac{\partial u_1^2}{\partial y} = 6 \times \frac{1}{2} \Rightarrow \frac{\partial u_1^2}{\partial y} = 3 \Rightarrow \frac{u_1^3 - u_1^1}{1} = 3$$

$$\Rightarrow u_1^3 = 3 + u_1^1$$

& $\frac{\partial u_2^2}{\partial y} = 6 \times 1 = 6 \Rightarrow \frac{u_2^3 - u_2^1}{1} = 6$

$$\Rightarrow u_2^3 = 6 + u_2^1$$

Substituting u_1^3 & u_2^3 in (3) and (4) we obtain

$$-4u_1^2 + u_2^2 + 2u_1^1 = -\frac{3}{2} \quad \text{— (5)}$$

& $$-4u_2^2 + u_1^2 + 2u_2^1 = -6 \quad \text{— (6)}$$

In matrix form:

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 2 & 0 & -4 & 1 \\ 0 & 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 9/8 \\ -3/2 \\ -6 \end{bmatrix}$$

Solving these equations using Gauss-elimination, we get

$$u_1^1 = 0.0769$$

$$u_2^1 = 0.1910$$

$$u_1^2 = 0.8665$$

$$u_2^2 = 1.8121.$$

HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION

Explicit method: Let us consider the following initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < 1 \qquad \text{—(1)}$$

where the initial conditions are

$$\left.\begin{aligned} u(x,0) &= f(x) \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} &= g(x) \qquad 0 < x < 1 \end{aligned}\right\} \text{—(2)}$$

and the boundary conditions

$$\left.\begin{aligned} u(0,t) &= \varphi(t) \\ u(1,t) &= \psi(t) \qquad t \geq 0 \end{aligned}\right\} \text{— (3)}$$

The central-difference approximations for u_{xx} and u_{tt} at the grid point (x_m, t_n) are

$$u_{xx} = \frac{1}{h^2}\left(u_{m-1}^{n} - 2u_m^n + u_{m+1}^{n}\right) + O(h^2)$$

$$u_{tt} = \frac{1}{k^2}\left(u_m^{n-1} - 2u_m^n + u_m^{n+1}\right) + O(k^2)$$

$$m, n = 0, 1, 2 \ldots$$

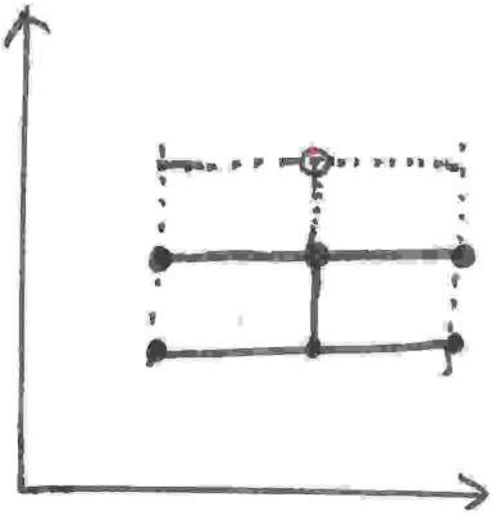
The equation (1) becomes

$$\frac{1}{k^2}\left(u_m^{n-1} - 2u_m^n + u_m^{n+1}\right) = \frac{c^2}{h^2}\left(u_{m-1}^{n} - 2u_m^n + u_{m+1}^{n}\right)$$

$$\Rightarrow u_m^{n+1} = r^2 u_{m-1}^{n} + 2(1-r^2)u_m^n + r^2 u_{m+1}^{n} - u_m^{n-1} \qquad \text{—(4)}$$

where $r = \frac{ck}{h}$

The schematic diagram:



In order to start the computations, we need the data on two previous time line.

The Information required on the line $t=K$ is obtained by using a suitable approximation to the inital condition

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

Using second order central approximation:

$$\frac{1}{2K}\left(u_m^1 - u_m^{-1}\right) = g_m \quad \text{——} \quad (4a)$$

Putting $n=0$ in (4):

$$u_m^1 = r^2 u_{m-1}^0 + 2(1-r^2)u_m^0 + r^2 u_{m+1}^0 - u_m^{-1}$$

$$= r^2 f_{m-1} + 2(1-r^2) f_m + r^2 f_{m+1} - u_m^{-1} \quad \text{—}(4b)$$

wher $f_m = u_m^0$

Eliminating u_m^{-1} between (4a) & (4b) we obtain the expression for u along $t=K$, i.e.: for $n=1$ as

$$u_m^1 = r^2 f_{m-1} + 2(1-r^2) f_m + r^2 f_{m+1} + 2K g_m - u_m^1$$

$$\Rightarrow \quad u_m' = \frac{1}{2}\left[r^2 f_{m-1} + 2(1-r^2) f_m + r^2 f_{m+1} + 2k g_m\right]$$

This gives the values of u for $n=1$. For $n=2,3,\dots$ the values are obtained from (4).

The truncation error of this method is $O(h^2+k^2)$ and the formula (4) is convergent for $0<r\le 1$.

Example: Solve the wave equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

with B.C.

$$u(0,t) = u(1,t) = 0 \qquad t>0,$$

and I.C. $u(x,0) = \frac{1}{2}\sin \pi x$

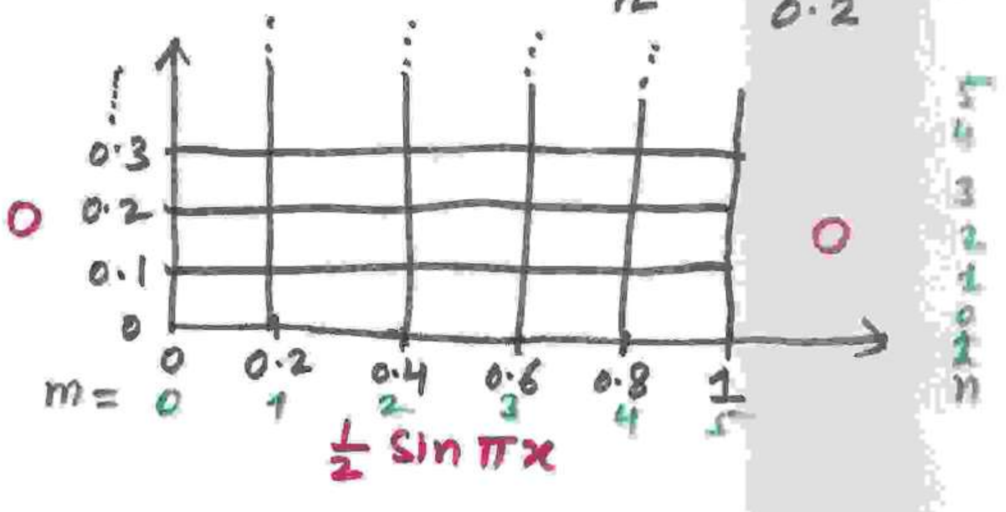
$$u_t(x,0) = 0 \qquad 0\le x\le 1$$

for $x = 0, 0.2, \dots, 1$ and $t = 0, 0.1, 0.2, \dots, 0.5$.

Sol.: The explicit formula is

$$u_m^{n+1} = r^2 u_{m-1}^n + 2(1-r^2) u_m^n + r^2 u_{m+1}^n - u_m^{n-1}$$

$h=0.2 \quad k=0.1 \qquad r = \frac{k}{h} = \frac{0.1}{0.2} = 0.5$



B.Cs: $u_0^n = u_5^n = 0, \quad n = 1, 2, \cdots 5.$

ICs: $u_m^0 = \frac{1}{2} \sin \pi x_m$

& $\frac{\partial u}{\partial t}(x_m, 0) = 0 \Rightarrow \frac{u_m^1 - u_m^{-1}}{2K} = 0$

$$\Rightarrow u_m^1 = u_m^{-1}$$

For $r = 0.5$, the finit diff. scheme becomes:

$$u_m^{n+1} = 0.25\, u_{m-1}^n + 1.5\, u_m^n + 0.25\, u_{m+1}^n - u_m^{n-1} \quad -(1)$$

For $n = 0$:

$u_m^1 = 0.25\, u_{m-1}^0 + 1.5\, u_m^0 + 0.25\, u_{m+1}^0 - u_m^{-1}$

Since $u_m^1 = u_m^{-1}$, then

$u_m^1 = 0.125\, u_{m-1}^0 + 0.75\, u_m^0 + 0.125\, u_{m+1}^0$

The above formula gives the values of u for $n = 1$

For $n = 2, 3 \cdots$ the value are obtained form (1).

See the computed values in Table 1.

t=0.5 n=5	0	0.0057	0.0093	0.0093	0.0057	0
t=0.4 n=4	0	0.0952	0.1539	0.1539	0.0952	0
t=0.3 n=3	0	0.1755	0.2840	0.2840	0.1755	0
t=0.2 n=2	0	0.2391	0.3869	0.3869	0.2391	0
t=0.1 n=1	0	0.2799	0.4528	0.4528	0.2799	0
t=0 n=0	0	0.2939	0.4755	0.4755	0.2939	0
	x=0 m=0	x=0.2 m=1	x=0.4 m=2	x=0.6 m=3	x=0.8 m=4	x=1 m=5

IMPLICIT METHODS

The difference scheme obtained in explicit method can be modified by replacing the space derivative by its central difference difference approximation at the points $(mh, (n+1)k)$, (mh, nk) and $(mh, (n-1)k)$ as a weighted sum, i.e.,

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{m,n} = c^2\left[\theta\left(\frac{\partial^2 u}{\partial x^2}\right)_{m,n+1} + (1-2\theta)\left(\frac{\partial^2 u}{\partial x^2}\right)_{m,n} + \theta\left(\frac{\partial^2 u}{\partial x^2}\right)_{m,n-1}\right]$$

$$0 \leq \theta \leq 1$$

$$\Rightarrow \frac{u_m^{n+1} - 2u_m^n + u_m^{n-1}}{k^2} = c^2\left[\theta\left[\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2}\right] + (1-2\theta)\left[\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2}\right] + \theta\left[\frac{u_{m+1}^{n-1} - 2u_m^{n-1} + u_{m-1}^{n-1}}{h^2}\right]\right]$$

$$\Rightarrow \boxed{\begin{aligned} -r^2\theta\, u_{m-1}^{n+1} + (1+2r^2\theta)\, u_m^{n+1} - r^2\theta\, u_{m+1}^{n+1} &= [2 - 2(1-2\theta)r^2]\, u_m^n + (1-2\theta)r^2 u_{m+1}^n \\ &+ (1-2\theta)r^2 u_{m-1}^n + \theta r^2 u_{m-1}^{n-1} - (1+2\theta r^2)\, u_m^{n-1} \\ &+ r^2\theta\, u_{m+1}^{n-1} \end{aligned}} \quad \text{—(1)}$$

Recall: $\Delta f(x) = f(x+h) - f(x)$ forward diff. operator

$\nabla f(x) = f(x) - f(x-h)$ backward diff. operator

$\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$ central diff. operator.

Similarly.

$$\delta^2 f(x) = \delta(\delta f(x)) = f(x+h) - 2f(x) + f(x-h) \quad ..$$

Rewriting (1) as:

$$\left(u_m^{n+1} - 2u_m^n + u_m^{n-1}\right) = r^2\theta\left(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}\right)$$

$$+ (1-2\theta)r^2\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right) + \theta r^2\left(u_{m-1}^{n-1}\right.$$

$$\left. - 2u_m^{n-1} + u_{m+1}^{n-1}\right)$$

Using notion δ, we can rewrite the above scheme as:

$$\boxed{\delta_t^2 u_m^n = r^2\delta_x^2\left[\theta u_m^{n+1} + (1-2\theta)u_m^n + \theta u_m^{n-1}\right]} \quad (2)$$

$$\delta_t^2 u_m^n = u_m^{n+1} - 2u_m^n + u_m^{n-1}$$

$$\delta_x^2 u_m^n = u_{m+1}^n - 2u_m^n + u_{m-1}^n$$

(2) may rewritten as

$$\delta_t^2 u_m^n = r^2\delta_x^2\left[u_m^n + \theta\delta_t^2 u_m^n\right]$$

or $$\boxed{\left(1-\theta r^2\delta_x^2\right)\delta_t^2 u_m^n = r^2\delta_x^2 u_m^n}$$

For $\theta = \frac{1}{4}$ this scheme is known as von-Neumann scheme:

$$\boxed{\left(1-\frac{1}{4}r^2\delta_x^2\right)\delta_t^2 u_m^n = r^2\delta_x^2 u_m^n}$$

Example: Find the solution at the FIRST time step of

$$u_{tt} = u_{xx} \qquad 0 < x < 1$$

subject to

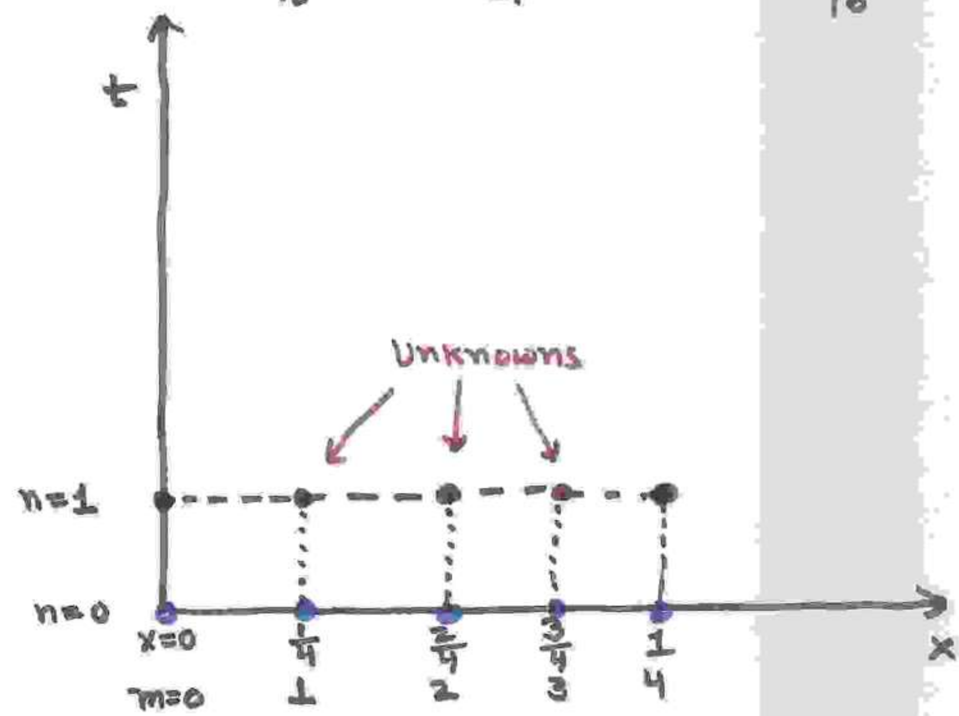
$$\left.\begin{aligned} u(x,0) &= \sin \pi x \\ u_t(x,0) &= 0 \end{aligned}\right\} \quad 0 \le x \le 1$$

$$u(0,t) = u(1,t) = 0, \; t > 0$$

by implicit scheme with $\theta = \frac{1}{2}$. Take $h = \frac{1}{4}$ & $r = \frac{3}{4}$.

Sol:

$$h = \frac{1}{4}, \quad \frac{k}{h} = r = \frac{3}{4} \Rightarrow k = \frac{3}{16}$$



IC: $\quad u_m^0 = \sin \frac{\pi m}{4}$ & $u_m^{-1} = u_m^1$, $\quad m = 0, 1, 2, 3, 4$

BCs: $\quad u_m^n = 0$ for $m = 0$ & 4.

Implicit scheme for $\theta = \frac{1}{2}$.

$$\left(1 - \frac{1}{2} r^2 \delta_x^2\right) \delta_t^2 u_m^n = r^2 \delta_x^2 u_m^n$$

$$\Rightarrow \left(1-\frac{1}{2}\frac{9}{16}\delta_x^2\right)\left(u_m^{n-1}-2u_m^n+u_m^{n+1}\right) = \frac{9}{16}\left(u_{m-1}^n-2u_m^n+u_{m+1}^n\right)$$

$$\Rightarrow \left(u_m^{n-1}-2u_m^n+u_m^{n+1}\right)-\frac{9}{32}\left(u_{m-1}^{n-1}-2u_m^{n-1}+u_{m+1}^{n-1}\right)$$

$$+\cancel{\frac{9}{16}\cdot\left(u_{m-1}^n-2u_m^n+u_{m+1}^n\right)}-\frac{9}{32}\left(u_{m-1}^{n+1}-2u_m^{n+1}+u_{m+1}^{n+1}\right)$$

$$=\cancel{\frac{9}{16}\left(u_{m-1}^n-2u_m^n+u_{m+1}^n\right)}$$

$$\Rightarrow -\frac{9}{32}u_{m-1}^{n+1}+\frac{25}{16}u_m^{n+1}-\frac{9}{32}u_{m+1}^{n+1}$$

$$=2u_m^n+\frac{9}{32}u_{m-1}^{n-1}-\frac{25}{16}u_m^{n-1}+\frac{9}{32}u_{m+1}^{n-1}$$

$m=1,2,3.$

<u>For</u> $n=0$: (using $u_m^{-1}=u_m^1$)

$$-\frac{9}{16}u_{m-1}^1+\frac{25}{8}u_m^1-\frac{9}{16}u_{m+1}^1=2u_m^0$$

For $m=1,2,3$, we have the system

$$\begin{bmatrix}\frac{25}{8} & -\frac{9}{16} & 0\\ -9/16 & 25/8 & -9/16\\ 0 & -9/16 & 25/8\end{bmatrix}\begin{bmatrix}u_1^1\\u_2^1\\u_3^1\end{bmatrix}=\begin{bmatrix}2u_1^0\\2u_2^0\\2u_3^0\end{bmatrix}$$

solving the above system we get:

$$u_1^1=u_3^1=0.60709$$

$$u_2^1=0.85855$$

Von Neumann Stability Analysis (Fourier series stability analysis)

Fourier series in complex form:

Let $f(x)$ is a periodic function over period $2l$ defined in $[-l, l]$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Using Euler formula

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

we obtain

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left\{e^{i\frac{n\pi x}{l}} + e^{-\frac{in\pi x}{l}}\right\} + \frac{b_n}{2i}\left\{e^{i\frac{n\pi x}{l}} - e^{-i\frac{n\pi x}{l}}\right\}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - ib_n)e^{i\frac{n\pi x}{l}} + \frac{1}{2}(a_n + ib_n)e^{-\frac{in\pi x}{l}}\right]$$

Denoting $\quad c_0 = \frac{a_0}{2} \quad c_n = \frac{1}{2}(a_n - ib_n)$

$$c_{-n} = \frac{1}{2}(a_n + ib_n)$$

We get

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{\frac{in\pi x}{l}} + c_{-n} e^{-\frac{in\pi x}{l}} \right)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{l}}$$

where

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Stability analysis (Boundedness of numerical solution)

Consider the explicit method for solving the heat equation

$$u_j^{n+1} = (1-2\lambda) u_j^n + \lambda (u_{j-1}^n + u_{j+1}^n) \quad \text{—— (1)}$$

The exact solution of (1) for a single step can be expressed as

$$u_j^{n+1} = G\, u_j^n$$

where G, called the amplification factor, is in general a complex constant.

The solution of the FDS at time $T = N\Delta t$ is then

$$u_j^N = G^N u_j^0$$

For u_j^N to remain bounded, we must have

$$|G| \leq 1$$

Stability analysis thus reduces to the determination of the single step exact solution of the finite difference equation (1), i.e., the amplification factor G, and an investigation of the conditions necessary to ensure that $|G| \leq 1$.

From equation (1) it is seen that u_j^{n+1} depends not only on u_j^n but also on u_{j-1}^n and u_{j+1}^n. Consequently u_{j-1}^n and u_{j+1}^n must be related to u_j^n so that equation (1) can be solved for G. It is accomplished by expressing $u(x,t^n) = F(x)$ in a complex Fourier series.

The complex Fourier series of $F(x)$ is given as

$$u(x,t^n) = F(x) = \sum_{m=-\infty}^{\infty} A_n e^{i k_m x}$$

where the wave number k_m is defined as

$$k_m = \frac{m\pi}{l}.$$

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we obtain

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{2} \left\{ e^{\frac{in\pi x}{l}} + e^{-\frac{in\pi x}{l}} \right\} + \frac{b_n}{2i} \left\{ e^{\frac{in\pi x}{l}} - e^{-\frac{in\pi x}{l}} \right\} \right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{2}(a_n - i b_n) e^{\frac{in\pi x}{l}} + \frac{1}{2}(a_n + i b_n) e^{-\frac{in\pi x}{l}} \right]$$

Denoting $\quad c_0 = \frac{a_0}{2} \quad c_n = \frac{1}{2}(a_n - i b_n)$

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We get

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{\frac{in\pi x}{l}} + c_{-n} e^{-\frac{in\pi x}{l}} \right)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{l}}$$

where

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$$u(x, t^n) = F(x) = \sum_{m=-\infty}^{\infty} A_m e^{i k_m x}$$

where the wave number k_m is defined as

$$k_m = \frac{m\pi}{l}.$$

For simplicity, let us examine the behavior of the solution by taking a single term of the series:

$$u_j^n = u(x_j, t^n) = A_m e^{i k_m x_j}$$ (one can also work with full sum)

then

$$u_{j+1}^n = A_m e^{i k_m (x_j + \Delta x)}$$

$$= A_m e^{i k_m x_j} \cdot e^{i k_m \Delta x}$$

$$= u_j^n e^{i\theta}$$ taking $k_m \Delta x = \theta$

$\theta \in [0, 2\pi]$

Note that $e^{i k_m \Delta x}$ represents sine and cosine functions, which have a period of 2π. Therefore $\theta \in [0, 2\pi]$ will covers all possible values of the $e^{i k_m \Delta x}$.

Similarly,

$$u_{j-1}^n = u_j^n e^{-i\theta}$$

&

$$u_{j\pm 1}^{n+1} = u_j^{n+1} e^{\pm i\theta}.$$

Working steps for stability

1 Substitute the complex components for $u^{n}_{j\pm1}$ & $u^{n+1}_{j\pm1}$ into the finite diff. equation, i.e.,

$$u^{n}_{j\pm1} = u^{n}_{j}\, e^{\pm i\theta}$$

$$u^{n+1}_{j\pm1} = u^{n+1}_{j}\, e^{\pm i\theta}$$

2. express $e^{\pm i\theta}$ in terms of $\sin\theta$ and $\cos\theta$, i.e.

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta,$$

and determine the amplification factor, G.

3. Analyse G (i.e., $|G|<1$) to determine the stability criteria of the finite difference equation.

Example 1. Explicit method for solving heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Consider the explicit method

$$\frac{u_m^{n+1} - u_m^n}{k} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$

Substituting $u_{m\pm1}^n = u_m^n e^{\pm i\theta}$ & $u_{m\pm1}^{n+1} = u_m^{n+1} e^{\pm i\theta}$

we obtain:

$$u_m^{n+1} = u_m^n + \lambda\left[u_m^n e^{-i\theta} - 2u_m^n + u_m^n e^{i\theta}\right]$$

$$= \left[1 + \lambda(e^{i\theta} + e^{-i\theta}) - 2\lambda\right] u_m^n$$

$$= \left[1 + 2\lambda\cos\theta - 2\lambda\right] u_m^n$$

This implies that the amplification factor is

$$G = 1 + 2\lambda\cos\theta - 2\lambda$$

$$= 1 + 2\lambda(\cos\theta - 1)$$

For stability we require

$$|G| \le 1$$

$$\Rightarrow \left|1 + 2\lambda(\cos\theta - 1)\right| \le 1$$

$$\Rightarrow -1 \le \underbrace{1 + 2\lambda(\cos\theta - 1) \le 1}_{\text{always true}}$$

The upper inequality is always true because $\lambda > 0$ and $(\cos\theta - 1)$ ranges from -2 to 0.

From the lower limit, we get

$$-1 \le 1 + 2\lambda(\cos\theta - 1)$$

$$\Rightarrow -2 \le 2\lambda(\cos\theta - 1)$$

$$\Rightarrow \lambda \le \frac{1}{1-\cos\theta} \Rightarrow \lambda \le \frac{1}{2}$$

because the largest value of $(1-\cos\theta)$ is 2.

Hence the explicit scheme is conditionally stable with the condition:

$$\frac{k}{h^2} \le \frac{1}{2}$$

$$\Rightarrow \boxed{k \le \frac{1}{2}h^2}$$

Ex: Stability of Richardson (Leapfrog) Method.

Equation $u_t = u_{xx}$

method:
$$\frac{u_m^{n+1} - u_m^{n-1}}{2K} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$

OR

$$u_m^{n+1} = u_m^{n-1} + 2\lambda\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right)$$

Substitute $u_{m\pm1}^n = u_m^n e^{\pm i\theta}$

& $\quad u_m^{n+1} = G u_m^n \Rightarrow u_m^n = G u_m^{n-1}$

$$\Rightarrow u_m^{n-1} = \frac{1}{G} u_m^n$$

we get:

$$u_m^{n+1} = \frac{1}{G} u_m^n + 2\lambda\left[u_m^n e^{-i\theta} - 2u_m^n + u_m^n e^{i\theta}\right]$$

$$= \left[\frac{1}{G} + 2\lambda\left\{e^{-i\theta} + e^{i\theta} - 2\right\}\right] u_m^n$$

$$= \left[\frac{1}{G} + 2\lambda\left(2\cos\theta - 2\right)\right] u_m^n$$

$$\Rightarrow u_m^{n+1} = \left[\frac{1}{G} + 4\lambda\left(\cos\theta - 1\right)\right] u_m^n$$

amplification factor:

$$G = \left[\frac{1}{G} + 4\lambda\left(\cos\theta - 1\right)\right]$$

$$\Rightarrow G^2 - 4\lambda(\cos\theta - 1)G - 1 = 0$$

$$\Rightarrow G_{1,2} = \frac{4\lambda(\cos\theta-1) \pm \sqrt{16\lambda^2(\cos\theta-1)^2 - 4\times -1}}{2}$$

$$= \left(2\lambda(\cos\theta-1) \pm \sqrt{4\lambda^2(\cos\theta-1)^2 + 1}\right)$$

Consider:

$$|G_2| = \left|2\lambda(\cos\theta-1) - \sqrt{1+4\lambda^2(\cos\theta-1)^2}\right|$$

$$= \left|2\lambda(1-\cos\theta) + \sqrt{1+4\lambda^2(\cos\theta-1)^2}\right| > 1$$

$\Rightarrow$ The leapfrog method is unconditionally unstable.

<u>Ex</u>: Laasonen method: (Implicit)

$$\frac{u_m^{n+1} - u_m^n}{k} = \frac{u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}}{h^2}$$

$$\Rightarrow \quad u_m^{n+1} = u_m^n + \lambda\left(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}\right)$$

$$\Rightarrow u_m^{n+1} = u_m^n + \lambda\left[u_m^{n+1}e^{-i\theta} - 2u_m^{n+1} + u_m^{n+1}e^{i\theta}\right]$$

$$\Rightarrow u_m^{n+1} = u_m^n + \lambda\left[2u_m^{n+1}\cos\theta - 2u_m^{n+1}\right]$$

$$= u_m^n + 2\lambda\left[\cos\theta - 1\right]u_m^{n+1}$$

$$\Rightarrow \quad u_m^{n+1}\left[1 + 2\lambda(1-\cos\theta)\right] = u_m^n$$

$$\Rightarrow \quad u_m^{n+1} = \frac{1}{1+2\lambda(1-\cos\theta)}\, u_m^n$$

amplification factor

$$G = \frac{1}{\underbrace{1+2\lambda(1-\cos\theta)}_{\geq 1}} \leq 1$$

Hence the method is unconditionally stable.

Ex: Stability of Cronk-Nicolson Method

method:

$$\frac{u_m^{n+1} - u_m^n}{k} = \frac{1}{2}\left[\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} + \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2}\right]$$

$$\Rightarrow u_m^{n+1} = u_m^n + \frac{\lambda}{2}\left[u_{m+1}^n - 2u_m^n + u_{m-1}^n + u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}\right]$$

$\Rightarrow$

$$u_m^{n+1} = u_m^n + \frac{\lambda}{2}\left[u_m^n e^{i\theta} - 2u_m^n + u_m^n e^{-i\theta} + u_m^{n+1} e^{i\theta} - 2u_m^{n+1} + u_m^{n+1} e^{-i\theta}\right]$$

$$\Rightarrow u_m^{n+1} = u_m^n + \frac{\lambda}{2}\left[2u_m^n \cos\theta - 2u_m^n + 2u_m^{n+1}\cos\theta - 2u_m^{n+1}\right]$$

$$\Rightarrow u_m^{n+1}\left[1 - \lambda(\cos\theta - 1)\right] = \left[1 + \lambda(\cos\theta - 1)\right] u_m^n$$

$$\Rightarrow u_m^{n+1} = \frac{1 + \lambda(\cos\theta - 1)}{1 - \lambda(\cos\theta - 1)} u_m^n$$

amplification factor:

$$G = \frac{1 - \lambda(1 - \cos\theta)}{1 + \lambda(1 - \cos\theta)}$$

$\Rightarrow |G| \leq 1$ for all λ.

$\Rightarrow$ The method is unconditionally stable.

Ex: Hyperbolic Equation (explicit method)

Equation: $u_{tt} = c^2 u_{xx}$

method: $u_m^{n+1} = r^2 u_{m-1}^n + 2(1-r^2) u_m^n + r^2 u_{m+1}^n - u_m^{n-1}$

Substituting $u_{m\pm1}^n = u_m^n e^{\pm i\theta}$ & $u_m^n = G u_m^{n-1}$

$$\Rightarrow \quad u_m^{n+1} = r^2 u_m^n e^{-i\theta} + 2(1-r^2) u_m^n + r^2 u_m^n e^{i\theta} - \frac{1}{G} u_m^n$$

$$u_m^{n+1} = 2r^2 u_m^n \cos\theta + 2(1-r^2) u_m^n - \frac{1}{G} u_m^n$$

$$\Rightarrow u_m^{n+1} = \left[2r^2(\cos\theta - 1) + 2 - \frac{1}{G}\right] u_m^n$$

amplification factor: $G = 2r^2(\cos\theta - 1) + 2 - \frac{1}{G}$

$$\Rightarrow \quad G^2 - \left(2 - 2r^2(1-\cos\theta)\right) G + 1 = 0$$

$$\Rightarrow G^2 - \left[2 - 2r^2 \, 2\sin^2\frac{\theta}{2}\right] G + 1 = 0$$

$$\Rightarrow G^2 - \left[2 - 4r^2 \sin^2 \phi\right] G + 1 = 0 \qquad \text{where } \phi = \theta/2$$

$$G_{1,2} = \frac{(2 - 4r^2\sin^2\phi) \pm \sqrt{(2-4r^2\sin^2\phi)^2 - 4}}{2}$$

$$= (1 - 2r^2\sin^2\phi) \pm \sqrt{(1-2r^2\sin^2\phi)^2 - 1}$$

Case I: If $|1-2r^2\sin^2\Phi| > 1$

In this case $|G_1| > 1$ or $|G_2| > 1$ and the scheme is unstable

Case II: If $|1-2r^2\sin^2\Phi| < 1$

then $G_{1,2}$ are complex pair whose magnitude is

$$|G_{1,2}| = \sqrt{(1-2\cancel{r^2\sin^2\Phi})^2 - (1-2\cancel{r^2\sin^2\phi})^2 + 1}$$

$$= 1$$

Hence the scheme is stable.

Case III: $|1-2r^2\sin^2\phi| = 1$

then $G_{1,2} = 1$

again the scheme is stable.

Hence the scheme is stable for

$$\underbrace{-1 \leq 1-2r^2\sin^2\phi \leq 1}_{\text{always true}}$$

The first inequality gives:

$$-1 \leq 1 - 2r^2 \sin^2 \Phi$$

$$\Rightarrow -2 \leq -2r^2 \sin^2 \Phi$$

$$\Rightarrow r^2 \sin^2 \phi \leq 1$$

$$\Rightarrow \qquad r^2 \leq \frac{1}{\sin^2 \phi}$$

$$\Rightarrow r^2 \leq 1 \Rightarrow \boxed{r \leq 1}$$