Lecture 8

<u>Definition:-</u> A map $m : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$, defined as $m(A) := m^*(A)$, is called the Lebesgue measure. & $m(A)$ is called the <u>Lebesgue measure of</u> A or simply measure of A;

where $\mathcal{M}$ = the collection of all measurable subsets of $\mathbb{R}$.

<u>Theorem:-</u> Let $\mathcal{A}$ be a class of subsets of a metric space (X, d). Then there exists a smallest σ-algebra $\mathcal{S}$ containing $\mathcal{A}$.

We say that $\mathcal{S}$ is the <u>σ-algebra generated by</u> $\mathcal{A}$.

<u>Proof:-</u> Let $\{\mathcal{S}_\alpha\}$ be any collection of σ-algebras of subsets of X.

Then by the definition of σ-algebra,

$$\bigcap_{\alpha} S_\alpha \text{ is also a } \sigma\text{-algebra.}$$

Now take the σ-algebra S as the intersection of all σ-algebras of subsets of X which containing $\mathcal{A}$.

i.e., $$S = \bigcap_{\substack{S_\alpha \ \sigma\text{-algebra} \\ \& \ S_\alpha \supseteq \mathcal{A}}} S_\alpha \quad \sigma\text{-algebra}$$

& S is the smallest σ-algebra containing $\mathcal{A}$.

Definition:- The σ-algebra generated by the class of all intervals of the form $[a, b)$, $a, b \in \mathbb{R}$ is called the Borel σ-algebra & denote by $\mathcal{B}$.

The members of $\mathcal{B}$ are called the Borel sets of $\mathbb{R}$.

Theorem:-

① Every Borel set is measurable.

That is, $\mathcal{B} \subseteq \mathcal{M}$.

② $\mathcal{B}$ is the σ-algebra generated by each of the following class:

- The open intervals
- the open sets
- the G_δ-sets
- the F_σ-sets.

proof:-

①. We know $[a,b) \in \mathcal{M}$ $\forall\, a, b \in \mathbb{R}$

& $\mathcal{M}$ is a σ-algebra.

Then $\mathcal{A} = \{ [a,b) \mid a, b \in \mathbb{R} \} \subseteq \mathcal{M}$

$\Rightarrow$ $\mathcal{B}$ = the smallest σ-algebra containg $\mathcal{A}$

$\subseteq \mathcal{M}$

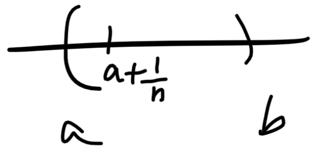
$\therefore \mathcal{B} \subseteq \mathcal{M}$.

(2) Let $\mathcal{B}_1 =$ the σ-algebra generated by the open intervals.

To show: $\mathcal{B} = \mathcal{B}_1$.

We have any open interval is a Countable union of open intervals of the form $[a, b)$

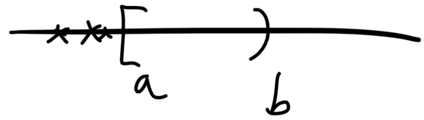
$$(a, b) = \bigcup_{n=1}^{\infty} \left[a+\tfrac{1}{n}, b\right) \in \mathcal{B}$$



$\therefore \quad \{ (a,b) \mid a, b \in \mathbb{R} \} \subseteq \mathcal{B}$

$\Rightarrow \quad \mathcal{B}_1 \subseteq \mathcal{B} \quad (\because \mathcal{B}$ is a σ-algebra$)$

Now $$[a, b) = \bigcap_{n=1}^{\infty} \left(a-\tfrac{1}{n}, b\right) \in \mathcal{B}_1$$



$\therefore \quad \{ [a,b) \mid a, b \in \mathbb{R} \} \subseteq \mathcal{B}_1$

$\Rightarrow \quad \mathcal{B} \subseteq \mathcal{B}_1$

Hence $\mathcal{B}_1 = \mathcal{B}$.

Remaining proof: EXERCISE.

Hints: • Every open set is a countable union of open intervals.

• G_δ-sets, F_σ-sets, are formed from open sets / closed sets using only countable intersections or unions.

<u>Qn</u>:- Does $\mathcal{B} = \mathcal{M}$?

Ans: No.

<u>Proposition</u>:- For any $A \subseteq \mathbb{R}$, there exists a measurable set $E \subseteq \mathbb{R}$ such that $A \subseteq E$ & $m^*(A) = m^*(E)$.

<u>proof</u>:- Let $\varepsilon > 0$.

There exists an open set $U \subseteq \mathbb{R}$ such that $A \subseteq U$ & $m^*(U) \leq m^*(A) + \varepsilon$.

Take $\varepsilon = \frac{1}{n} > 0$, then there exists an open set $U_n \subseteq \mathbb{R}$ such that $\underline{A \subseteq U_n}$ &

$$m^*(U_n) \leq m^*(A) + \frac{1}{n} \quad \forall\, n \geq 1.$$

Let $E = \bigcap_{n=1}^{\infty} U_n$. E is a G_δ-set & hence

$E \in \mathcal{M}$.

& $m^*(E) = m^*\left(\bigcap_{n=1}^{\infty} U_n\right) \leq m^*(U_n) \leq m^*(A) + \frac{1}{n}$. $\forall n$

$m^*(E) \leq m^*(A) + \frac{1}{n}, \quad \forall n \geq 1$

$\Rightarrow \quad m^*(E) \leq m^*(A)$.

Also $A \subseteq E = \bigcap_{n=1}^{\infty} U_n$, $\quad m^*(A) \leq m^*(E)$.

$\therefore \quad m^*(E) = m^*(A)$.

<u>Definition</u>:- For any sequence of sets $\{E_i\}_{i=1}^{\infty}$,

$$\limsup(E_i) \doteq \bigcap_{n=1}^{\infty}\left(\bigcup_{i\geq n} E_i\right)$$

$$= \bigcap_{n=1}^{\infty} (E_n \cup E_{n+1} \cup \cdots\cdots)$$

$$= \underline{(E_1 \cup E_2 \cup E_3 \cup \cdots\cdots)} \cap \underline{(E_2 \cup E_3 \cup \cdots)}$$

$$\cap \underline{(E_3 \cap \cdots \cdots)} \cap \cdots\cdots.$$

$$\liminf(E_i) := \bigcup_{n=1}^{\infty}\left(\bigcap_{i\geq n} E_i\right)$$

$$= \bigcup_{n=1}^{\infty}\left(E_n \cap E_{n+1} \cap \cdots\right)$$

$$= \left(E_1 \cap E_2 \cap E_3 \cap \cdots\right) \cup \left(E_2 \cap E_3 \cap \cdots\right)$$

$$\cup \left(E_3 \cap \cdots\right) \cup \cdots$$

Remark:-

① $\liminf(E_i) \subseteq \limsup(E_i)$.

proof:-

Let $a \in \liminf(E_i) = \bigcup_{n=1}^{\infty}\left(\bigcap_{i\geq n} E_i\right)$

$\Rightarrow$ $a \in \bigcap_{i\geq n_0} E_i$ for some $n_0 \in \mathbb{N}$

$\Rightarrow$ $a \in E_i \quad \forall\, i \geq n_0$.

$\Rightarrow$ $a \in E_1 \cup E_2 \cup \cdots \cup E_{n_0} \cup \cdots$

& $a \in E_2 \cup E_3 \cup \cdots$.

& $a \in E_3 \cup E_4 \cup \cdots$

$\vdots$

Thus $a \in \left(E_1 \cup E_2 \cup \cdots\right) \cap \left(E_2 \cup E_3 \cup \cdots\right)$

$\cap (E_3 \cup E_4 \cup \cdots) \cap \cdots$

$= \limsup (E_i)$

$\therefore \liminf (E_i) \subseteq \limsup (E_i)$.

If they are equal, then we denote this set as $\lim (E_i)$.

That is $\lim(E_i) = \limsup(E_i) = \liminf (E_i)$.

2. $\limsup(E_i) =$ the set of points belonging to infinitely many of the sets E_i.

& $\liminf (E_i) =$ the set of points belonging to all but finitely many of the sets E_i.

Examples:-

1. Suppose $E_1 \subseteq E_2 \subseteq \cdots$. Then

$$\limsup (E_i) = \bigcup_{i=1}^{\infty} E_i = \liminf (E_i).$$

2. Suppose $E_1 \supseteq E_2 \supseteq \cdots$. Then $\lim (E_i) = \bigcap_{i=1}^{\infty} E_i$

Theorem:- Let $\{E_i\}$ be a sequence of measurable sets in $\mathbb{R}$. Then

(i) if $E_1 \subseteq E_2 \subseteq \cdots$, then $m(\lim(E_i)) = \lim(m(E_i))$

(ii) if $E_1 \supseteq E_2 \supseteq \cdots$, & $m(E_i) < \infty \ \forall i$, then

$$m(\lim(E_i)) = \lim(m(E_i)).$$

proof:-

(i). Note that $\lim(E_i) = \bigcup_{i=1}^{\infty} E_i$

Let $F_1 = E_1$

$F_i = E_i \setminus E_{i-1} \quad \forall i \geq 2$

Then $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$

& $F_i = E_i \setminus E_{i-1} = E_i \cap E_{i-1}^c \in \mathcal{M}, \ \forall i \geq 2$

$$\therefore \ m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = m^*\left(\bigcup_{i=1}^{\infty} F_i\right), \quad F_i \in \mathcal{M}$$

$$= \sum_{i=1}^{\infty} m(F_i)$$

$$= \lim\left(\sum_{i=1}^{n} m(F_i)\right)$$

$$= \lim m\left(\dot{\bigcup_{i=1}^{n}} F_i\right)$$

$$= \lim m\left(\bigcup_{i=1}^{n} E_i\right) \qquad E_1 \subseteq E_2 \subseteq \cdots$$

$$= \lim m(E_n).$$

$$\therefore m\left(\lim(E_i)\right) = \lim\left(m(E_i)\right).$$

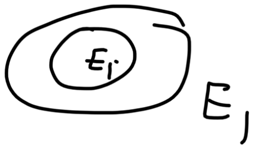
(ii). Given $E_1 \supseteq E_2 \supseteq \cdots$

$$\Rightarrow \underset{\phi}{\underbrace{E_1 \setminus E_1}} \subseteq E_1 \setminus E_2 \subseteq E_1 \setminus E_3 \subseteq \cdots$$

$\therefore$ By (i), we have

$$m\left(\lim(E_1 \setminus E_i)\right) = \lim\left(m(E_1 \setminus E_i)\right)$$

Now $m(E_1) = m(E_1 \cup E_i)$



$$= m\left((E_1 \setminus E_i) \dot{\cup} E_i\right)$$

$$= m(E_1 \setminus E_i) + m(E_i)$$

$$\Rightarrow m(E_1 \setminus E_i) = m(E_1) - m(E_i).$$

$$\therefore m\left(\lim(E_1 \setminus E_i)\right) = \lim\left(m(E_1) - m(E_i)\right)$$