Lecture-21 (01-04-2024)

Theorem: Let (X, T) be a compact topological space. Then X is also a countably compact space.

Proof: Let (X, T) be a compact space.

Suppose A be an infinite subset of X with no accumulation point in X.

Then for each point $p \in X$ belongs to an open set G_p which contains atmost one point of A

i.e., $$G_p \cap A = \begin{cases} \phi, & p \notin A \\ \{p\}, & p \in A \end{cases}$$

Then $\{G_p \mid p \in X\}$ is an open cover for the compact space X.

Hence there exist a finite sub-cover say $G_{p_1}, G_{p_2}, G_{p_3}, \ldots G_{p_m}$ such that

$$X = \bigcup_{j=1}^{m} G_{p_j}$$

$$\Rightarrow A \subseteq X = \bigcup_{j=1}^{m} G_{p_j} \quad \text{———} \quad (*)$$

and $G_{p_j} \cap A = \begin{cases} \varphi, & p_j \notin A \\ \{p_j\}, & p_j \in A \end{cases}$

Since each open set G_{p_j} contains atmost one point of the set A, $\bigcup_{j=1}^{m} G_{p_j}$ contains atmost m points of the set A

But by $(*)$ $A \subseteq \bigcup_{j=1}^{m} G_{p_j}$,

which implies A is a finite set which is contradiction to A is an infinite subset of X.

$\therefore$ Our assumption is wrong. Hence Every infinite subset of X has an accumulation point in X. $\therefore X$ is countably compact.

$$\boxed{Cpt \implies C.C}$$

Next we prove

$$S.C \implies C.C.$$

Theorem: let (X, T) be a sequentially compact topological space. Then X is countably compact space.

Proof: Let (X, T) be a sequatially topological space.

Claim: X is countably compact.

let A be an infinite subset of X.

Then there exists a sequence $\{a_1, a_2, a_3, \ldots\}$ in A with distinct terms.

Since X is a sequentially compact, the sequence $\{a_n\}$ contains a

Convergent subsequence say
$\{a_{i_1}, a_{i_2}, a_{i_3}, \ldots\ldots\}$ with
distinct terms and $a_{n_j} \longrightarrow p \in X$.

$\Rightarrow$ Every nbd of p contains infinitely many terms of the subsequence $\{a_{n_j}\}$.

Hence Every nbd of p contains an infinite number of points of the set A.

This implies $p \in X$ is an accumulation point of the set A.

$\Rightarrow$ X is countably compact

$\therefore$ $\boxed{S.C \Longrightarrow C.C}$

* Converse of above two theorems need not be true.

Ex: Let (N, T) be a topological space generated by the class of sets of the form

$$\mathcal{A} = \{ \{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \ldots \}$$

Let A be any nonempty infinite subset of N.

and let $n_0 \in A$.

If n_0 is odd, then n_0+1 is a limit point of A

For, if G be any open set containing n_0+1.

Then $(G - \{n_0+1\}) \cap A \neq \phi$

Ily if n_0 is even, then $n_0 - 1$ is limit point of A.

$\therefore$ (N, T) is countably compact set.

Since $\mathcal{A} = \{ \{1,2\}, \{3,4\}, \{5,6\}, \ldots \}$

is an open cover for N.

$$N = \bigcup_{j=1}^{\infty} G_j, \quad G_j \in \mathcal{A}.$$

But there is no finite sub-cover of N.

$\therefore$ N is not compact.

Also since $\{1, 2, 3, \ldots\} = \{n\}$ is a sequence in (N, T), and it has no convergent sub-sequence in N.

$\therefore$ (N, T) is not sequentially compact

[Attendance (morning)] ——— II ———

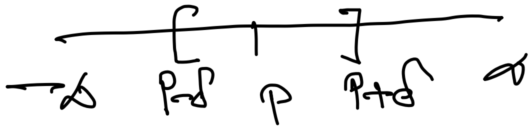
[65, 11, 03, 27, 06, 10, 60, 17, 58, 19]

Locally Compact Sets :—

A topological space (X, T) is said to be locally compact topological space if every point in X has a compact neighborhood.

Ex: $(\mathbb{R}, U)$ be usual topological space.

Let $p \in \mathbb{R}$



Then since

$$p \in (p-\delta, p+\delta) \subset [p-\delta, p+\delta], \quad \delta > 0$$

$\Rightarrow$ $[p-\delta, p+\delta]$ is a nbd of $p \in \mathbb{R}$.

Also $[p-\delta, p+\delta]$ is a compact set.

$\therefore$ $[p-\delta, p+\delta]$ is a compact nbd of $p \in \mathbb{R}$. This is true for all $p \in \mathbb{R}$.

$\therefore$ $(\mathbb{R}, U)$ is locally compact.

But $(\mathbb{R}, U)$ is not a Compact Space.

Since $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-n, n)$

But $\{(-n,n) / n \in \mathbb{N}\}$ has no finite Sub-cover for $\mathbb{R}$.

$\therefore$ Locally Compact Space need not be a Compact Space, but every Compact Space is a locally Compact Space,

Since a topological Space X is nbd of each of its points.

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Compactness in the metric Spaces

ϵ-net :—

Let (X, d) be a metric Space and A be a Subset of X, and let $\epsilon > 0$.

A finite subset $S = \{e_1, e_2, \cdots e_m\}$ of X is called an ϵ-net for the set A if every point $p \in A$, there exists an $e_{i_0} \in S$ with $d(p, e_{i_0}) < \epsilon$

i.e.,

$$A \subseteq \bigcup_{i=1}^{m} S(e_i, \epsilon)$$

$$= S(e_1, \epsilon) \cup S(e_2, \epsilon) \cup \cdots \cup S(e_m, \epsilon)$$

$$= \bigcup_{i=1}^{m} \{q \,/\, d(e_i, q) < \epsilon\}.$$

Ex: $A = \{(x, y) \,/\, x^2 + y^2 < 4\}$ be the open disc centered at the origin and radius 2 in $(\mathbb{R}^2, U)$.

If $\epsilon = \frac{3}{2}$, then the set

$S = \{(1,-1), (1,0), (1,1), (0,-1), (0,0), (0,1), (-1,-1), (-1,0), (-1,1)\}$

is an $\frac{3}{2}$-net for the set A.

If $\epsilon = \frac{1}{2}$, then S is not an $\frac{1}{2}$-net for the set A.

$\because$ $p = (\frac{1}{2}, \frac{1}{2}) \in A$,

but distance of $p = (\frac{1}{2}, \frac{1}{2})$ and any point of S is greater than $\frac{1}{2}$.

Diameter of a set :—

Let A be any subset of a metric space (X, d). Then the diameter of the set A is defined as

$$d(A) = \sup \{ d(a, a') \mid a, a' \in A \}.$$

If $d(A) < \infty$, then we say A is a bounded set.

Totallary bounded set :—

A subset Y of a metric space (X, d) is said to be totally bounded if Y has an ϵ-net for every $\epsilon > 0$.

Note :— A set Y in a metric space (X, d) is totally bounded set if for every $\epsilon > 0$, there exist a decomposition of the set Y into a finite number of sets each with diameter less than ϵ.

* Bounded sets need not be totally bounded sets.

Ex: let Y be a subset of a metric space l^2, consisting of the

Points $e_1 = (1, 0 \ldots \ldots), e_2 = (0, 1, 0 \ldots \ldots)$

$\ldots \ldots e_i (0 \ldots \underset{i\text{th place}}{1}, 0 \ldots), \ldots \ldots$

Then

$$d(e_i, e_j) = \begin{cases} \sqrt{2}, & i \neq j \\ 0, & i = j \end{cases}$$

$\therefore$ Y is is bounded.

But Y is not totally bounded.

For if $\epsilon = \frac{1}{2}$, then the only nonempty subset of Y with diameter less than $\frac{1}{2}$ are singleton sets.

Accordingly the infinite set Y cannot be decomposed into a finite number of disjoint subsets each with diameter less than $\frac{1}{2}$.

$\therefore$ Bounded sets need not be totally bounded.

[Attendance 65, 11, 27, 06, 60, 07, 45].