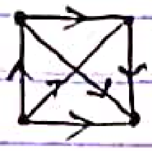
~~Class Test~~

(3) **Tournament:** An orientation of a complete graph.



digraphs $D_1 = (V_1, E_1)$ & $D_2 = (V_2, E_2)$ are isomorphic iff $\exists$ a bijection
$f : V_1 \to V_2$ s.t.
$(u,v) \in E_1$ iff $(f(u), f(v)) \in E_2$.

Q. Upto isomorphism find all tournaments on 4 vertices.

$u \longrightarrow v$ Team having max. outdegree
u defeats v   will be winner. Team → Vertex

Defn: A vertex with maximum outdegree in a tournament is called a king.

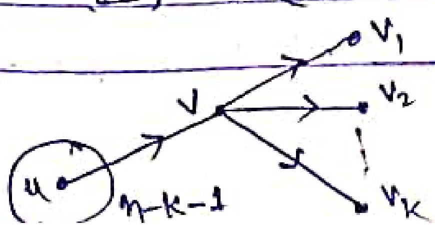
$D \to$ tournament
v is a king in D
$u \in V(D)$; $u \neq v$

$v \longrightarrow u$ or $v \longrightarrow w \longrightarrow u$

Theorem: Let D be a tournament and $v \in V(D)$ be a king. For any vertex $u \in V(D)$, there is a $v-u$ directed path of length at most 2.

Proof: If $(v,u) \in E(D)$ then we get a $v-u$ directed path of length 1.
Next let $(v,u) \notin E(D)$.
Since D is a tournament, we get
$(u,v) \in E(D)$ $u \longrightarrow v$
let $d^+(v) = k$, which is the max. out degree in D.

~~$|V(D)| = n$; $u \in V(D)$~~



~~$(v_i, u) \in E(D)$~~

If $(v_i, u) \in E(D)$ for some i, $1 \le i \le k$
then a directed path of $v \to v_i \to u$
length 2.
If $(u, v_i) \in E(D)$, $i = 1, 2, \ldots k$
then $d^+(u) \ge k+1$, which is a contradiction.
Hence, $\exists$ i, $1 \le i \le k$ s.t.
$(v_i, u) \in E(D)$

$v \to v_i \to u$

Defn: The out-degree seq. of a tournament is called its score sequence.

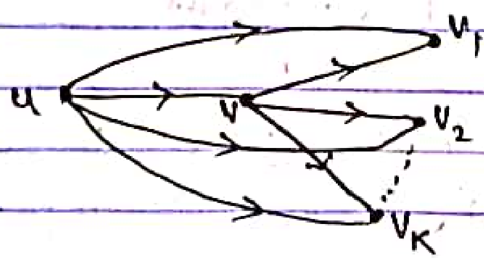
Transitive Relation
$(u,v), (v,w) \in R$
$\Rightarrow (u,w) \in R$

A digraph D is said to be transitive if $(u,v), (v,w) \in E$
$\Rightarrow (u,w) \in E(D)$

Theorem: A tournament D is transitive iff the score sequence of D is $n-1, n-2, \ldots, 1, 0$.
where $n = |V(D)|$.

Proof: Suppose D is transitive. Possible outdegrees are $0, 1, 2, \ldots, n-2, n-1$.

Claim: No two vertices in D have the same outdegree.
Let $u, v \in V(D)$, $u \ne v$.
We show that $d^+(u) \ne d^+(v)$ w.l.g. let $(u,v) \in E$
Let $d^+(v) = k$.

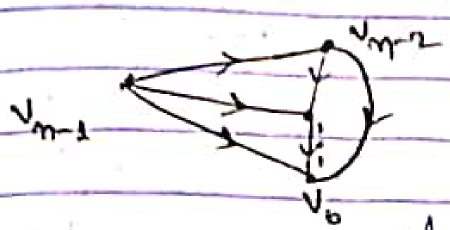


$\Rightarrow d^+(u) \ge k+1$
$\Rightarrow d^+(u) \ne d^+(v)$
Hence, the score seq. of D is $n-1, n-2, \ldots, 1, 0$.

Conversely, let the score seq. of D be $n-1, \ldots, 1, 0$.
To show that D is transitive.

Let $V(D) = \{v_0, v_1, \ldots, v_{n-1}\}$ with $d^+(v_i) = i$, $i = 0, 1, \ldots, n-1$.

$d^+(v_{n-1}) = n-1$



$d^+(v_{n-2}) = n-2$

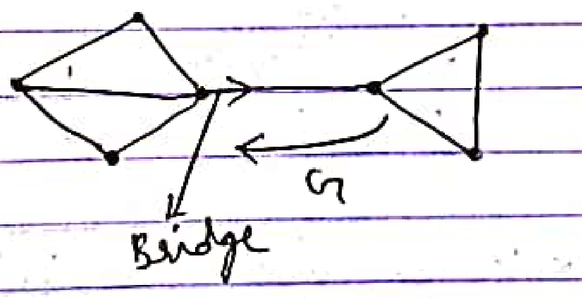
$\Rightarrow (v_i, v_j) \in E(D)$ iff $i > j$

which is a transitive relation. Hence D is transitive.

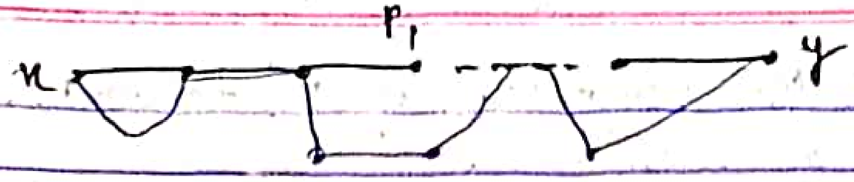
Defn: A digraph D is said to be weakly connected if the underlying graph of D is connected.

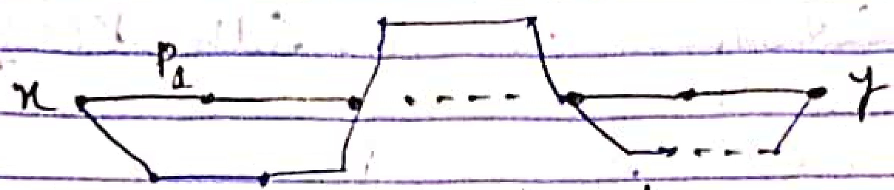
D is called strongly connected if for every pair $x, y \in V(D)$, $x \neq y$, $\exists$ a $x-y$ directed path in D.

Defn: A graph G is said to be orientable if G has a strongly connected orientation.



Lemma: A graph G has no bridge iff for every pair of vertices x & y, there is at least two edge disjoint $x-y$ paths.





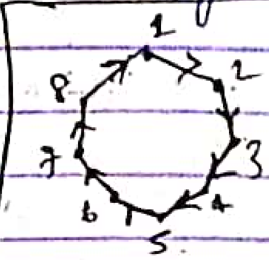
Edge disjoint

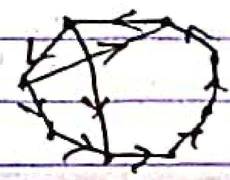
Theorem: A connected graph G is orientable iff G has no bridge.

Proof: If G is orientable then G has no bridge.

Next let G is bridgeless.

$$\deg x \geq 2 \quad \forall \; x \in V(G).$$





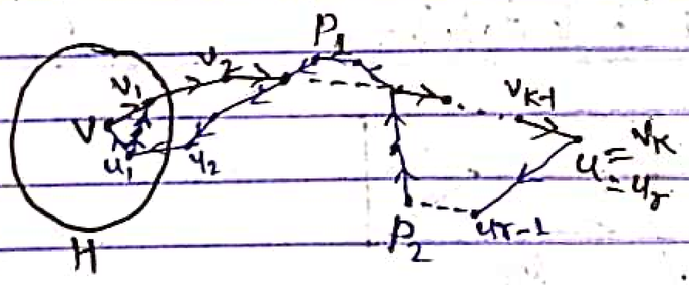
G contains a cycle C. Let H be a maximal induced orientable subgraph of G.

$$H \neq \phi, \quad C \subseteq H$$

Claim: $\quad V(H) = V(G)$

Suppose $\quad V(H) \neq V(G)$

Then $\exists \; u \in V(G)$ s.t. $u \notin V(H)$.



Let $v \in V(H)$

Since G is connected & bridgeless there are at least two edge disjoint v–u paths, P_1 & P_2, in G.

$$H_1 = H \cup P_1 \cup P_2$$

Let $P_1 : \; v (= v_0), v_1, \ldots\ldots, v_k (= u)$

$P_2 : \; v (= u_0), u_1, \ldots\ldots, u_r (= u)$

Let v_i & u_j be the last vertices of P_1 & P_2 respectively s.t. $v_i, v_j \in V(H)$.
(may be $v_i = v_j = v$)
$v_i = v_j \neq v$.

<u>Give</u>: orientation of $P_1 \cup P_2$

$(v_i, v_{i+1}), \ldots\ldots, (\vec{v_{k-1}}, v_k = v)$

$(v, u_{r-1}), (u_{r-1}, u_{r-2}) \ldots\ldots\ldots (u_{j+1}, u_j)$

We have a directed $u_j - v_i$ path in H.

$\Rightarrow$ $H_1 = H \cup P_1 \cup P_2$ is orientable.

$\underset{\text{properly}}{H_1 \supset H}$, which is a contradiction $\boxed{\supset \Rightarrow \text{superset}}$

to the assumption that H is maximal orientable subgraph.

Hence, $V(H) = V(G) \Rightarrow H = G$.