Lecture-3 (15-01-2024)

Accumulation Point :

Let (X, T) be a topological space.
A point $p \in X$ is an accumulation point of a subset A of X if every open set G containing 'p' contains a point of A different from 'p'.

$G \in T \ni p \in G, \ (G - \{p\}) \cap A \neq \phi.$

The set of all limit points / accumulation points of a set A is denoted by A' and it is called derived set of A.

Ex: $X = \{a, b, c, d, e\}.$
$T = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}.$

Then (X, T) is a topological space.

$A = \{a, b, c\} \subset X$

Consider $b \in X$:

The open sets containing 'b' are

$X, \{b, c, d, e\}$.

$\because X \cap A = \{a, b, c\}$

$\{b, c, d, e\} \cap A = \{b, c\}$

$\therefore b \in A'$.

Now consider $a \in X$:

The open sets containing 'a' are

$X, \{a\}, \{a, c, d\}$.

$X \cap A = \{a, b, c\}$

$\{a\} \cap A = \{a\}$

$\{a, c, d\} \cap A = \{a, c\}$.

$\because \{a\} \cap \mathcal{A} = \{a\}$, So 'a' is not a limit point of $\mathcal{A}$ as the open set $\{a\}$ does not contain any other point of $\mathcal{A}$ different from 'a'.

Ily we can check that $d, e \in \mathcal{A}'$.

$\therefore \mathcal{A}' = \{b, d, e\}$.

Problem: If $A \subset B$, then $A' \subset B'$ in a topological space (X, T).

Sol:

Let $p \in A' \Rightarrow (G - \{p\}) \cap A \neq \phi$ for all open sets G containing "p".

$\because A \subset B$, we have

$(G - \{p\}) \cap B \supset (G - \{p\}) \cap A \neq \phi$

$\forall G \in T, p \in G$.

$\Rightarrow (G - \{p\}) \cap B \neq \phi, \ \forall G \in T$
$p \in G$.

$\Rightarrow p \in B'$.

$\Rightarrow A' \subset B'$.

Problem: (X, D) be a discrete topological space. Then for any $A \subset X$, find A'.

Consider any $p \in X$, $G = \{p\} \in D$.

Then $(G - \{p\}) \cap A = \phi$

$\Rightarrow p \notin A'$.

Hence we have $A' = \phi$.

Problem: (X, I) indiscrete topological space. For any $A \subset X$, find A'.

Sol: $A' = X$.

Theorem: Let A and B be any two subsets of a topological space (X, T). Then

$$(A \cup B)' = A' \cup B'.$$

Proof:

$$\because A, B \subset A \cup B$$

$$\Rightarrow A', B' \subset (A \cup B)'.$$

$$\Rightarrow A' \cup B' \subset (A \cup B)' \quad \text{——— (1).}$$

Next we prove

$$(A \cup B)' \subset A' \cup B'.$$

Let $p \in (A' \cup B')^c$.

$$\Rightarrow p \notin A' \cup B'.$$

$$\Rightarrow p \notin A' \text{ and } p \notin B'.$$

$\because p \notin A' \Rightarrow \exists\, G \in T$ such that $p \in G$ and $G \cap A \subset \{p\}$.

Illy $p \notin B' \Rightarrow \exists\, H \in T$ such that $p \in H$ and $H \cap B \subset \{p\}$.

$\because G, H \in T \Rightarrow G \cap H \in T$

and $p \in G \cap H$ $[\because p \in G, p \in H]$.

Consider

$$(G \cap H) \cap (A \cup B) = (G \cap H \cap A) \cup (G \cap H \cap B)$$

$$\subset (G \cap A) \cup (H \cap B)$$

$$\subset \{p\} \cup \{p\}$$

$$= \{p\}.$$

Thus $G \cap H$ is the open set containing 'p' does not contain any other point of $A \cup B$.

$$\Rightarrow p \notin (A \cup B)'$$

$$\Rightarrow p \in [(A \cup B)']^c$$

$$\therefore (A' \cup B')^c \subset ((A \cup B)')^c$$

$$\Rightarrow (A \cup B)' \subset A' \cup B' \quad \text{---(2)}$$

Hence from (1) & (2) we have

$$(A \cup B)' = A' \cup B'.$$

Closed set :-

Let (X, T) be a topological space. A sub-set H of X is said to be a closed set if its complement H^c is an open set.

Ex: $X = \{a, b, c, d, e\}$

$T = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$.

Now closed sets are:

ϕ, X, $\{b, c, d, e\}$, $\{a, b, e\}$, $\{b, e\}$, $\{a\}$.

Clearly X, ϕ, $\{b, c, d, e\}$ and $\{a\}$ are both open and closed sets.

$\{a, b\}$ is neither an open set nor a closed set.

Ex: (X, D) discrete topological space.

Then all the subsets of X are both open and closed.

Closure of a set :-

[Attendance morning class.
03, 57, 06, 32, 27, 58, 35, 15, 52, 50,
60, 56, 51, 23, 26, 41, 10, 21,
34, 11, 08, 62, 63, 65].

Lecture – 4 (15-01-2024, 5PM).

Closure of a set :-

Let A be any subset of a topological space (X, T). The closure of A is denoted by $\overline{A}$ and is defined as the intersection of all closed supersets of A.

That is, if $\{F_i \mid i \in I\}$ is the class of closed supersets of A, then

$$\bar{A} = \bigcap_{i \in I} F_i \, .$$

Clearly $\bar{A}$ is the closed set as it is the arbitrary intersection of closed sets in a topological space (X, T).

Further $\bar{A}$ is the smallest closed superset of A.

Thus if F is any closed superset of A, then

$$A \subset \bar{A} \subset F \, .$$

* A set A is closed set iff
$$A = \bar{A} \, .$$

[$\because$ If A is closed, we have
$A \subset \bar{A} \subset A$]

Ex: $X = \{a, b, c, d, e\}$.

$T = \{X, \varphi, \{c, d\}, \{a, c, d\}, \{a\},$
$\{b, c, d, e\}\}$.

Closed sets: $\varphi, X, \{a, b, e\}, \{b, e\},$
$\{b, c, d, e\}, \{a\}$.

let $A = \{b\} \subset X$.

Then $\overline{A} = \overline{\{b\}} = X \cap \{a, b, e\} \cap \{b, e\}$
$\cap \{b, c, d, e\}$
$= \{b, e\}$.

IIIly $\overline{\{a, c\}} = X$.

IIIly $\overline{\{b, d\}} = X \cap \{b, c, d, e\} = \{b, c, d, e\}$.

Theorem: A subset A of a topological space (X, T) is a closed set iff A contains each of its accumulation points, i.e., $A' \subset A$.

Proof: Suppose A is a closed set.

Claim: $A' \subseteq A$.

i.e, we prove $A^c \subseteq A'^c$.

let $p \in A^c$.

$\because$ $A \cap A^c = \phi$ and A^c is an open set containing the point 'p' implies that p is not a limit point of A.

$\therefore$ $p \notin A'$

$\Rightarrow p \in (A')^c$

$\therefore$ $A^c \subseteq (A')^c \Rightarrow A' \subseteq A$.

Conversely assume that $A' \subseteq A$.

Claim: A is a closed set, that is we prove A^c is an open set.

Let $P \in A^c \implies P \notin A$

$\implies P \notin A' \quad [\because A' \subseteq A]$

$\implies \exists$ an open set G such that

$P \in G$, $(G - \{P\}) \cap A = \varphi$ $\quad \left[\because G \cap A = \varphi = \{P\}\right.$

But $P \notin A$

$\therefore G \cap A = \varphi$.

$\implies G \subset A^c$

Thus for every $P \in A^c$, $\exists$ an open set G such that $P \in G \subset A^c$.

$\implies A^c$ is an open set.

$\implies A$ is a closed set.

Problem: If F is any closed superset of A, then $A' \subset F$.

$\because A \subseteq F \Rightarrow A' \subseteq F'$

So if F is closed, then $F' \subseteq F$

$\therefore A' \subseteq F' \subseteq F$

$\Rightarrow A' \subseteq F$.

Theorem: Let A be any subset of a topological space (X, τ). Then

$$\overline{A} = A \cup A'.$$

Proof: First we prove $A \cup A'$ is a closed set.

Let $p \in (A \cup A')^c = A^c \cap (A')^c$.

$\Rightarrow p \in A^c$ and $p \in (A')^c$

$\Rightarrow p \notin A$ and $p \notin A'$.

$\Rightarrow \exists$ an open set G such that
$$G \cap A = \varphi \quad \text{---} (1)$$
Now we prove $G \cap A' = \varphi$.
Let g be any point of the open set G.
$\because G \cap A = \varphi$, implies '$g$' cannot be a limit point of A.
Thus $g \notin A'$.
$$\therefore G \cap A' = \varphi \quad \text{---} (2)$$
Now from (1) & (2) we have
$$G \cap (A \cup A') = (G \cap A) \cup (G \cap A')$$
$$= \varphi \cup \varphi$$
$$= \varphi.$$
$$\Rightarrow G \subset (A \cup A')^c.$$

Thus for every $p \in (A \cup A')^c$, $\exists$ an open set G such that

$$p \in G \subset (A \cup A')^c$$

$\Rightarrow$ p is an interior point of $(A \cup A')^c$.

$\therefore$ $(A \cup A')^c$ is an open set.

$\Rightarrow$ $A \cup A'$ is a closed set.

Claim: $\bar{A} = A \cup A'$

$\because$ $\bar{A}$ is a closed set, $A \subset \bar{A}$

$\Rightarrow$ $A' \subset (\bar{A})' \subset \bar{A}$

$\Rightarrow$ $A \cup A' \subseteq \bar{A}$ —— (3).

But $A \cup A'$ is a closed set containing A

$\therefore$ $A \subseteq \bar{A} \subseteq A \cup A'$ —— (4)

$\therefore$ By (3) and (4) we have

$$\overline{\mathcal{A}} = \mathcal{A} \cup \mathcal{A}'$$

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[Attendance: 27, 60, 57, 41, 23, 62, 11, 65].