Lecture - 22 (02-04-2024)

* Totally bounded sets are bounded.

Proof: Let (X, d) be a metric space which is totally bounded.

Let $F = \{a_1, a_2, \ldots, a_m\}$ be 1-net for X $(\epsilon = 1)$.

Let $M = \max\{d(p, q) \mid p, q \in F\}$.

Now for any $x, y \in X$, we have $d(x, a_i) < 1$ and $d(y, a_j) < 1$, for some $a_i, a_j \in F$.

[$\because X = \bigcup_{i=1}^{m} S(a_i, 1)$, $F = \{a_1, \ldots, a_m\}$]

Then by Δle inequality we have

$$d(x, y) \leq d(x, a_i) + d(a_i, a_j) + d(a_j, y)$$

$$< 1 + M + 1$$

$$= M + 2 < \infty$$

$\Rightarrow X$ is bounded.

Lebesgue number for Covers :

Let $\mathcal{A} = \{G_i\}$ be a cover for a subset S of a metric space (X, d).

A real number $\delta > 0$ is called the Lebesgue number for the cover $\mathcal{A}$ if for each subset S_1 of S with diameter less than δ, there is a member in the cover $\mathcal{A}$ which contains S_1.

[if $S_1 \subset S$ with $d(S_1) < \delta$ and $S \subseteq \bigcup_i G_i \Rightarrow \exists\, G_j \in \{G_i\}$ with $S_1 \subset G_{i_j}$]

Theorem: Let Y be a sequentially compact subset of a metric space (X, d). Then Y is totally bounded.

Proof: We prove the contrapositive of the above statement, i.e., Y is not totally bounded, implies it is not sequentially compact.

If Y is not totally bounded, then there exists some $\epsilon > 0$ such that Y has no finite ϵ-net.

Let $a_1 \in Y \Longrightarrow \exists\ a_2 \in Y$ such that

$d(a_1, a_2) \geq \epsilon$ otherwise $\{a_1\}$ would be an ϵ-net for Y.

Similarly for $a_1, a_2 \in Y$, $\exists\ a_3 \in Y$ such that $d(a_1, a_3) \geq \epsilon$, $d(a_2, a_3) \geq \epsilon$ otherwise $\{a_1, a_2\}$ would be an ϵ-net for Y.

Continuing in this manner, we obtain a sequence $\{a_1, a_2, a_3, \ldots\}$ in Y with $d(a_i, a_j) \geq \epsilon$, $\forall\ i \neq j$.

Thus the sequence $\{a_n\}$ in Y cannot contain a convergent subsequence.

$\therefore$ Y is not sequentially compact

Lebesgue (Lemma) :

Let $\mathcal{A} = \{G_i\}$ be an open cover of a sequentially compact set Y in a metric space (X, d). Then $\mathcal{A}$ has a Lebesgue number.

Proof: Suppose $\mathcal{A} = \{G_i\}$ does not have a Lebesgue number.

Then for each $n \in \mathbb{N}$, $\exists$ a subset B_n of Y with the property that

$0 < d(B_n) < \frac{1}{n}$ and $B_n \not\subset G_i$, $\forall G_i \in \mathcal{A}$.

Now for each $n \in \mathbb{N}$, choose a point $b_n \in B_n$ and form a sequence

$\{b_1, b_2, b_3, \ldots\}$ in Y.

Since Y is a sequentially compact set, the sequence $\{b_1, b_2, b_3 \ldots\}$ contains a convergent subsequence say $\{b_{i_1}, b_{i_2}, b_{i_3}, \ldots\}$ converging to a point $p \in Y$.

Now since

$$p \in Y \subseteq \bigcup_i G_i$$

$\Rightarrow \exists$ some G_p such that $p \in G_p$.

Since X is a metric space, there exists an open sphere $S(p, \epsilon)$ with center p and radius ϵ such that

$p \in S(p, \epsilon) \subset G_p$. [$\because G_p$ is an open set].

$\because$ the subsequence $\{b_{i_n}\}$ converging to p, there exists an integer i_{n_0} such that $d(p, b_{i_{n_0}}) < \epsilon/2$, $b_{i_{n_0}} \in B_{i_{n_0}}$ and $d(B_{i_{n_0}}) < \epsilon/2$. [Choose i_{n_0} large enough so that

$$\frac{1}{i_{n_0}} < \epsilon/2$$

$$\therefore d(B_{i_{n_0}}) < \frac{1}{i_{n_0}} < \epsilon/2 \,]$$

Now by Δle inequality, for any $x \in B_{i_{n_0}}$, we have

$$d(x, p) \leq d(x, b_{i_{n_0}}) + d(b_{i_{n_0}}, p)$$

$$< \epsilon/2 + \epsilon/2 \qquad \left[\because d(B_{i_{n_0}}) < \epsilon/2, \; x, b_{i_{n_0}} \in B_{i_{n_0}}\right]$$

$$= \epsilon$$

Thus for any $x \in B_{i_{n_0}}$, we have

$$d(x, p) < \epsilon$$

$$\Rightarrow x \in S(p, \epsilon) \subset G_p$$

$$\Rightarrow B_{i_{n_0}} \subset S(p, \epsilon) \subset G_p.$$

$\Rightarrow B_{i n_0} \subset G_p$ which is Contradiction to $B_{i n_0} \not\subset G_i \ \forall \ G_i \in \mathcal{A}$.

$\therefore$ Our assumption is wrong

$\therefore$ The open Cover $\mathcal{A} = \{G_i\}$ of a sequentially compact set in a metric space (X, d) has a Lebesgue number.

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Now we Prove C.C $\Rightarrow$ S.C in a metric Space.

Theorem: Let Y be a Countably Compact subset of a metric space (X, d). Then Y is sequentially compact.

Proof: Let $B = \{a_1, a_2, a_3 \dots\}$ be a sequence in Y.

If $B = \{a_n\}$ is finite sequence, then one of the point say a_{i_0} in $\{a_n\}$ is repeated infinite number of times.

$\therefore \{a_{i_0}, a_{i_0}, a_{i_0}, \ldots\}$ is a convergent subsequence of $\{a_n\}$ converging to a_{i_0}.

If $B = \{a_n\}$ is an infinite sequence in Y, then $B = \{a_1, a_2, a_3 \ldots\}$ is an infinite subset of a countably compact set Y.

— Then $B = \{a_1, a_2 \ldots\}$ has a limit point in Y.

Since (X, d) is a metric space, we can choose a subsequence say $\{a_{i_1}, a_{i_2}, \ldots\}$ of $\{a_n\}$ converging to a point $P \in Y$.

$\therefore$ A is sequentially compact.

$\therefore$ C.C $\Rightarrow$ S.C in a Metric Space.

Theorem: In a Metric Space (X, d), sequentially compact implies compact.

Proof: Let Y be a sequentially compact set in a metric space (X, d).

Claim: Y is compact set.

Let $\{G_i\}$ be an open cover for Y.

$$\therefore Y \subseteq \bigcup_i G_i.$$

Since Y is sequentially compact, Y is totally bounded and the open cover $\{G_i\}$ has a Lebesgue number say $\delta > 0$.

Now since Y is totally bounded, there is decomposition of Y into a finite number of sets $B_1, B_2, \ldots B_m$ with $d(B_i) < \delta$, $\forall\, i = 1, 2 - m$.

i.e., $Y \subseteq \bigcup_{i=1}^{M} B_i$, $d(B_i) < \delta$, $i = 1, 2, \ldots M$

But δ is a Lebesgue number for the open cover $\{G_i\}$.

Hence there exist open sets $G_{i_1}, G_{i_2}, \ldots G_{i_m}$ in $\{G_i\}$ such that $B_j \subset G_{i_j}$, $j = 1, 2, \ldots m$

$$\therefore\; Y \subseteq \bigcup_{j=1}^{m} B_j \subset \bigcup_{j=1}^{m} G_{i_j}$$

$$\Rightarrow\; Y \subseteq \bigcup_{j=1}^{m} G_{i_j}.$$

$\Rightarrow$ Y is Cpt.

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Cpt $\Rightarrow$ C.C $\Leftrightarrow$ S.C $\Leftrightarrow$ Cpt

$$\begin{tikzcd}
\text{Cpt} \arrow[r] \arrow[r, bend right] \arrow[rr, bend left] & \text{C.C} \arrow[r, leftrightarrow] \arrow[r, bend right] & \text{S.C} \arrow[r, leftrightarrow] & \text{Cpt}
\end{tikzcd}$$

$\therefore$ We have the following theorem:

Theorem: In a metric space (X, d), the following are equivalent

(i) Compact (ii) Sequentially compact

(iii) Countably compact.

H.W problem: Let Y be a sequentially compact subset of a metric space (X, d). Then prove that every countable open cover of Y is reducible to a finite cover.

H.W Problem: Let $f: (X, d) \longrightarrow (Y, d^*)$ be a continuous map from a compact metric space (X, d) into a metric space (Y, d^*). Then prove that f is uniformly continuous.

Sol: Since $f : (X, d) \longrightarrow (Y, d^*)$ is Continuous, for each point $P \in X$, there exists an open sphere $S(P, \delta_p)$ such that

$x \in S(P, \delta_p)$, $f(x) \in S(f(P), \epsilon/2)$. $\longleftarrow$ $*$

Then

$$\mathcal{A} = \{ S(P, \delta_p) / P \in X \}$$

is an open Cover for the compact metric space (X, d).

Since in a metric space compact implies sequentially compact, it follows that X is sequentially compact.

$\therefore$ The open Cover $\mathcal{A} = \{ S(P, \delta_p) | P \in X \}$ has a Lebesgue number say $\delta > 0$

Now for any $x, y \in X$, $d(x,y) < \delta$,
we have

$$d(\{x, y\}) = d(x, y) < \delta$$

$\Rightarrow$ $\{x, y\}$ is Contained in some member $S(p_0, \delta_0)$ of the Open Cover $\mathcal{A} = \{S(p, \delta_p) \mid p \in X\}$.

Now $\{x, y\} \subset S(p_0, \delta_0)$

$\Rightarrow$ $x, y \in S(p_0, \delta_0)$

$\Rightarrow$ $f(x), f(y) \in S(f(p_0), \epsilon/2)$

But the diameter of $S(f(p), \epsilon/2)$ is ϵ

$\therefore$ $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$.

$\Rightarrow$ f is uniformly continuous

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Attendance [65, 27, 06]