Lecture 18

Proposition:- Let $E = E_1 \cup E_2$, where $E_1, E_2 \subseteq \mathbb{R}^d$.

Suppose $\text{dist}(E_1, E_2) > 0$. Then

$$m^*(E) = m^*(E_1) + m^*(E_2).$$

proof:-

$$\text{dist}(E_1, E_2) := \inf\left(\left\{ \underset{\substack{\| \\ |\underline{x} - \underline{y}|}}{d(\underline{x}, \underline{y})} \,\middle/\, \underline{x} \in E_1, \underline{y} \in E_2 \right\}\right)$$

By subadditivity, $m^*(E) \le m^*(E_1) + m^*(E_2)$.

To prove the reverse inequality, let $\varepsilon > 0$.

Select a $\delta > 0$ such that $\text{dist}(E_1, E_2) > \delta > 0$.

Choose a covering $E \subseteq \bigcup_{j=1}^{\infty} Q_j$ by closed cubes

with $\displaystyle\sum_{j=1}^{\infty} |Q_j| \le m^*(E) + \varepsilon. \longrightarrow (*)$

We may subdivide the cubes Q_j such that

diameter $(Q_j) < \delta$. $\left(\text{diameter}(A) := \sup\left\{ |\underline{x} - \underline{y}| \,/\, \underline{x}, \underline{y} \in A \right\}\right)$

In this case, each Q_j can intersects

<u>at most one of</u> the two sets E_1 or E_2.

If we denote by J_1 & J_2, the sets of those indices j for which Q_j intersects E_1 & E_2 respectively

Then $J_1 \cap J_2 = \phi$ & we have

$$E_1 \subseteq \bigcup_{j \in J_1} Q_j \quad \& \quad E_2 \subseteq \bigcup_{j \in J_2} Q_j .$$

$$\Rightarrow \quad m^*(E_1) \leq m^*\Big(\bigcup_{j \in J_1} Q_j\Big) \quad \& \quad m^*(E_2) \leq m^*\Big(\bigcup_{j \in J_2} Q_j\Big)$$

$$\Rightarrow m^*(E_1) + m^*(E_2) \leq \sum_{j \in J_1} m^*(Q_j) + \sum_{j \in J_2} m^*(Q_j)$$

$$\leq \sum_{j=1}^{\infty} m^*(Q_j) .$$

$$= \sum_{j=1}^{\infty} |Q_j|$$

$$\Rightarrow m^*(E_1) + m^*(E_2) \leq m^*(E) \quad \Big| \quad \leq m^*(E) + \varepsilon \ (\text{by } \circledast)$$

$$\therefore \quad m^*(E) = m^*(E_1) + m^*(E_2).$$

<u>Proposition</u>:- Let K be a compact set & F be a closed set in $\mathbb{R}^d$. Then $\operatorname{dist}(K, F) > 0$. & $K \cap F = \phi$

proof:- Suppose $\text{dist}(K, F) = 0$

$$= \inf \left\{ |\underline{x} - \underline{y}| \;\middle|\; \begin{array}{l} \underline{x} \in K \\ \underline{y} \in F \end{array} \right\}.$$

Let $\underline{x}_n \in K$ & $\underline{y}_n \in F$ be such that

$$|\underline{x}_n - \underline{y}_n| \to 0 \quad \text{as} \quad n \to \infty.$$

Since K is compact, there exists a convergent subsequence of $\{x_n\}$ say $\{x_{n_k}\}_{k=1}^{\infty}$

such that $x_{n_k} \to \underline{x}$ as $k \to \infty$ & $\underline{x} \in K$.

($\because$ Thm: K is compact in $\mathbb{R}^d$ $\Leftrightarrow$ every sequence in K has a convergent subsequence)

Now
$$|\underline{x} - \underline{y}_{n_k}| = |\underline{x} - \underline{x}_{n_k} + \underline{x}_{n_k} - \underline{y}_{n_k}|$$

$$\leq \underbrace{|\underline{x} - \underline{x}_{n_k}|}_{\downarrow \, 0} + \underbrace{|\underline{x}_{n_k} - \underline{y}_{n_k}|}_{\downarrow \, 0}$$

as $k \to \infty$.

$$\Rightarrow \quad \underline{y}_{n_k} \to \underline{x} \quad \text{as} \quad k \to \infty$$

$\Rightarrow$ $\underline{x}$ is a limit point of F

Since F is closed, $\underline{x} \in F$.

Thus $\underline{x} \in K \cap F = \phi$
which is a contradiction.

$\therefore \operatorname{dist}(K, F) > 0$.

Theorem:— Suppose f is a measurable function on $\mathbb{R}^d$. Then there exists a sequence of step functions $\{\psi_k\}_{k=1}^{\infty}$ that converges pointwise to $f(x)$, for almost every x.

Step fn $= \sum_{k=1}^{N} a_k \chi_{R_k}$

R_k = rectangles.

Littlewood's three principles.

(1) Every measurable set is nearly a finite union of intervals/cubes.

(2) Every measurable function is nearly a continuous function.

(3) Every convergent sequence of measurable

functions is nearly uniformly convergent.

<u>Theorem (3rd principle)</u> (Egorov)

Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$. Assume that $f_k \to f$ a.e on E pointwise.

Given $\varepsilon > 0$, we can find a closed set $A_\varepsilon \subseteq E$ such that $m(E \setminus A_\varepsilon) \leq \varepsilon$ & $f_k \to f$ uniformly on A_ε.

<u>Theorem (2nd principle)</u> (Lusin)

Suppose f is measurable & finite valued on E with E of finite measure. Then for every $\varepsilon > 0$ there exists a closed $F_\varepsilon \subseteq E$ such that $m(E \setminus F_\varepsilon) \leq \varepsilon$ & $f|_{F_\varepsilon}$ is continuous.

Examples:

① $f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \notin [0,1]. \end{cases}$ measurable.

f is not continuous at 0 & 1.

Try to find a closed set F_ε such that $\underline{m(\mathbb{R} \setminus F_\varepsilon) \leq \varepsilon}$ & $f|_{F_\varepsilon}$ is continuous.

Choose any closed set $F_\varepsilon \subseteq \mathbb{R} \setminus \{0,1\}$
(clearly $f|_{F_\varepsilon}$ is continuous).

$F_\varepsilon = ?$

The Lebesgue integral

The general notion of the Lebesgue integral on $\mathbb{R}^d$ will be defined in a step-by-step fashion:

(1) Simple functions
(2) Bounded functions supported on a set of finite measure.
(3) Non-negative functions
(4) Integrable functions (the general case).

Let $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ be a simple function, where E_k are measurable sets of finite measure & a_k are constants.

<u>Def</u>:- The <u>canonical form</u> of φ is the unique decomposition $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$, where a_k are distinct & the sets E_k are disjoint.

<u>Remark</u>:- Let $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$

range of $\varphi = \{c_1, \dots, c_M\}$ distinct.

Let $F_k = \{x \mid \varphi(x) = c_k\} = \varphi^{-1}(\{c_k\})$. $\forall k = 1, \dots, M$.

Then F_k are disjoint &

$$\varphi = \sum_{k=1}^{M} c_k \chi_{F_k} \quad \text{canonical form.}$$

Def:- Let φ be a simple function with canonical form $\varphi(x) = \sum_{k=1}^{M} c_k \chi_{F_k}(x)$. Then the Lebesgue integral of φ is defined as

$$\int_{\mathbb{R}^d} \varphi(x)\, dx := \sum_{k=1}^{M} c_k\, m(F_k)$$

Other notation: $\int_{\mathbb{R}^d} \varphi$ or $\int \varphi$.