Digraphs :-

tournament : An orientation of a complete graph.

Digraphs $D_1 = (V_1, E_1)$ & $D_2 = (V_2, E_2)$ are isomorphic iff $\exists$ a bijection

$f : V_1 \to V_2$ s.t $(u, v) \in E_1$ iff ~~(f(u), f(v)) ∈ E₂~~ $(f(u), f(v)) \in E_2$.

Q: Upto ismorphism, find [total no of all tournaments on 4 vertices

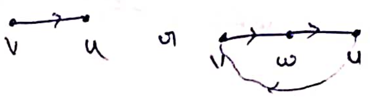
Defn :. A vertex with maximum outdegree in a tournment is called a king.

D → tournament

v is a king in D

$v \to u$ (u?) u defeats v

$u \in V(D)$ $u \neq v$



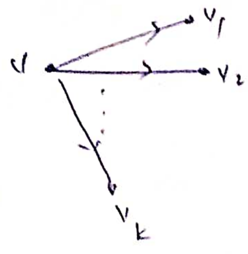
Thm :- Let D be a tournament and $v \in V(D)$ be a king. For any $u \in V(D)$, there is a $v \to u$ directed path of length at most 2.

Pf :- If $(v, u) \in E(D)$, then we get a $v - u$ directed path of length 1.

Next let $(v, u) \notin E(D)$

Since D is a tournament we get $(u, v) \in E(D)$

Let $d^+(v) = k$, which is the max. out degree in D.



$|V(D)| = n$

$u \in V(D)$

u ~~A to A~~

$n-k-1$

If $(v_i, u) \in E(D)$ for some i, $1 \le i \le k$ then a directed path of length 2 $v \to v_i \to u$.

If $(u, v_i) \in E(D)$ ~~for all~~ $i = 1, 2, \dots, k$

then $d^+(u) \geq k+1$, which is a contradiction.

Hence $\exists\, i, 1 \le i \le k$, s.t $(v_i, u) \in E(D)$.

$v \to v_i \to u$

Def^n:- The out-degree seq. of a tournament is called its score sequence.

Transitive relation

$(u,v), (v,w) \in R$

$\Rightarrow (u,w) \in R$

Def^n A digraph D is said to be transitive if $(u,v), (v,w), \in E(D) \Rightarrow (u,w) \in E(D)$

Theorem:- A tournament D is transitive iff the score sequence of D is $n-1, n-2, \dots, 1, 0$ where $n = |V(D)|$

Pf:- Suppose D is transitive.

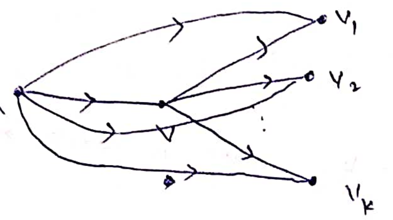
Possible out degrees are $0, 1, 2, \dots, n-2, n-1$.

Claim: No two vertices in D have the same out degree.

We show that $d^+(u) \neq d^+(v)$

w.l.g let $(u,v) \in E(D)$

let $d^+(v) = k$.



$\Rightarrow d^+(u) \geq k+1$

$\Rightarrow d^+(u) \neq d^+(v)$

Hence the score seq. of D is $n-1, n-2, \ldots, 1, 0$

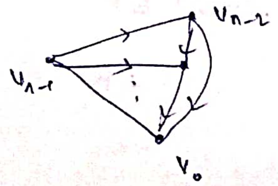
Conversely let the score seq. of D be $n-1, \ldots, 1, 0$.

To show that D is transitive.

Let $V(D) = \{v_0, v_1, \ldots, v_{n-1}\}$

with $d^+(v_i) = i$, $i = 0, 1, \ldots, n-1$

$d^+(v_{n-1}) = n-1$



$d^+(v_{n-2}) = n-2$

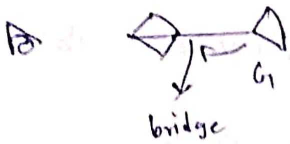
$\rightarrow (v_i, v_j) \in E(D)$ iff $i > j$

which is a transitive relation. Hence D is transitive.

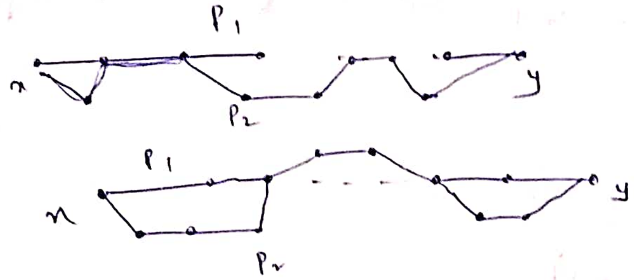
Defn :- A digraph D is said to be weakly connected if the underlying graph of D is connected.

D is called strongly connected if for every pair $x, y \in V(D), x \neq y$, $\exists$ a x-y directed path in D.

Defn :- A graph G is said to be orientable if G has a strongly connected orientation.



Lemma : A graph G has no bridge iff ~~betw~~ between every pair of vertices x & y there are atleast two edge disjoint x-y paths

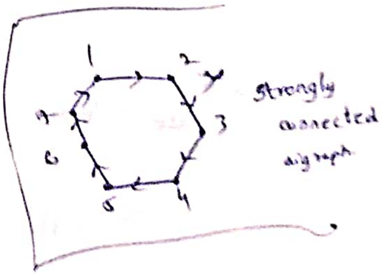


edge disjoint [No edges are same in two paths]

Theorem :- A graph G is orientable iff G has no bridge.

Pf :- If G is orientable then G has no bridge.

Next let G is bridge less.



$\deg x \geq 2 \quad \forall x \in V(G)$

G contains a cycle C.

Let H be a maximal induced orientable subgraph of G.

$H \neq \phi$, $C \subseteq H$.

Wed Tomorrow
3 - 6 pm

Claim: $V(H) = V(G)$

Suppouse $V(H) \neq V(G)$.

Then $\exists\, u \in V(G)$ s.t $u \notin V(H)$.

Let $v \in V(H)$

Since G is connected and bridgeless there are atleast two edge disjoint $v-u$ paths, P_1 & P_2 in G

$H_1 = H \cup P_1 \cup P_2$

Let $P_1 : v = v_0, v_1, \ldots, v_k = u$

$P_2 : v = u_0, u_1, \ldots, u_r = u$

Let v_i & u_j be the last vertices of P_1 & P_2 respectively

s.t $v_i, v_j \in V(H)$

(May be $v_i = v_j = v$)

$v_i = v_j \neq v$)

Give orientation of $P_1 \cup P_2$

$(v_i, v_{i+1}), \ldots, (v_{k-1}, v_k = v)$

$(u, u_{r-1}), (u_{r-1}, u_{r-2}), \ldots, (u_{j+1}, u_j)$

We have a directed $u_j - v_i$ path in H.

$\Rightarrow H_1 = H \cup P_1 \cup P_2$ is orientable.

$H_1 \supset H$ properly, which is a contradiction to the assumption that H is maximal orientable subgraph.

Hence $V(H) = V(G)$

$\Rightarrow H = G$.