Group Theory

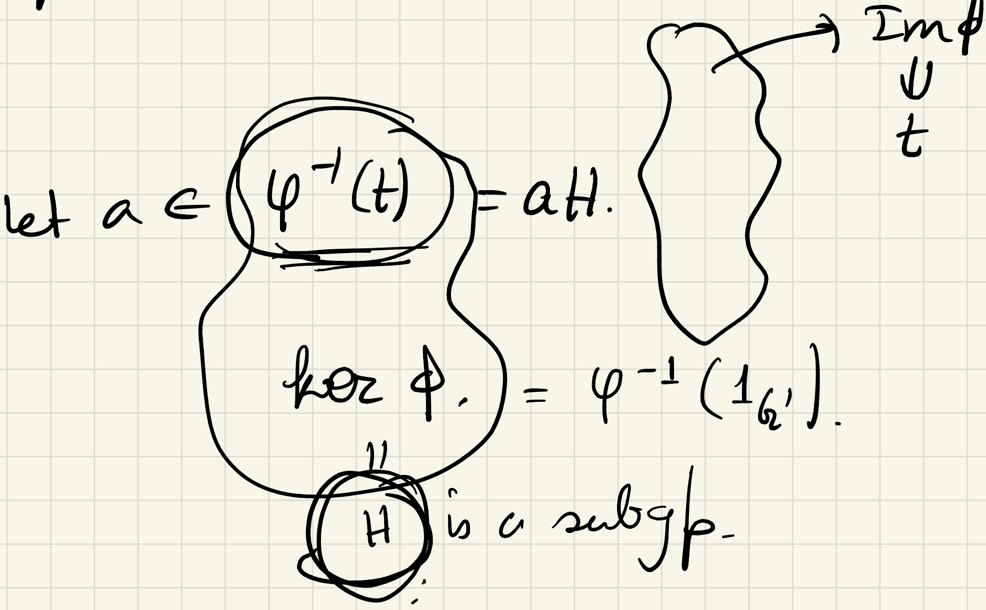
Lecture 7

20/01/2022

G is finite gp and H is a subgp of G. Then $|H| \mid |G|$.

<u>Q</u> Let $\phi : G \longrightarrow G'$ be a gp homo of finite gps. Then what is the relation among $|G|$, $|\ker\phi|$ & $|\mathrm{Im}\,\phi|$?

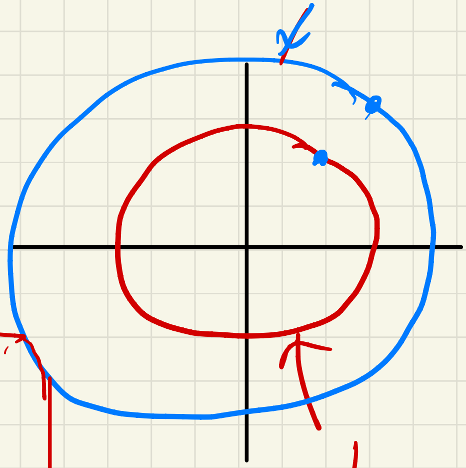
$$\phi : G \longrightarrow G'$$



$$f : \mathbb{C}^{\times} \longrightarrow \mathbb{R}^{\times}_{>}$$

$$f(z) = |z|.$$

$$f^{-1}(r) = rU.$$



$U = f^{-1}(1)$.

<u>Remark</u>: If $\phi : G \to G'$ is a gp homo then $\phi(a) = \phi(b)$ iff $b = an$ for some $n \in \ker\phi$

Note that the left cosets of $\ker\phi$ are the fibers of the map ϕ. The number of left cosets are in bijective correspondence with the elts in the image.

$$|\operatorname{Im}\phi| = [G : \ker\phi].$$

↳ no. of left cosets of the subgp $\ker\phi$.

$$\Rightarrow |\operatorname{Im}\phi| = \frac{|G|}{|\ker\phi|}$$

$$\therefore \boxed{|G| = |\ker\phi| \cdot |\operatorname{Im}\phi|}$$

Right coset:

Let G be a gp and H is a subgp of G. Then $aH = \{ah \mid h \in H\}$ is a left coset of H.

Similarly we can define right coset as $Ha = \{ha \mid h \in H\}$.

__Example__. $H = \{(1), (12)\} \subset S_3 = G$.

$$(13)\,H = \{(13), (123)\}$$

$$(23)\,H = \{(23), (132)\}.$$

$$H(13) = \{(13), (132)\}.$$

$$H(23) = \{(23), (123)\}.$$

__Q__ If H is a normal subgp then is it true that left coset of H is equal to a right coset of H?

$aHa^{-1} \in H \quad \forall a \in G$. if H is normal subgp

$$aH \ni ah = \underbrace{aha^{-1}}_{h'}a = h'a \in Ha. \checkmark$$

Propn: A subgp H of G is normal iff every left coset is also a right coset.

Pf: We have already proved that if H is normal subgp then
$$aH = Ha \quad \forall a \in G.$$
Conversely, suppose that every left coset is equal to some right coset i.e $aH = Hb$.

$a \in Ha \cap Hb \Rightarrow Hb$ and Ha are not disjoint hence $Hb = Ha$

$\therefore aH = Ha. \Rightarrow aHa^{-1} = H.$

$\therefore$ H is a normal subgp of G.

Cor. If $[G:H]=2$ then $H \trianglelefteq G$.

Pf: For any $a \notin H$.
$$G = H \sqcup aH = H \sqcup Ha.$$

$\Rightarrow \quad aH = Ha$

$\Rightarrow \quad aHa^{-1} = H.$

$\therefore \quad H \trianglelefteq G.$

Product of Groups:

G_1, G_2 are two gps then

$$G_1 \times G_2 = \{(a,b) \mid a \in G_1,\ b \in G_2\}.$$

is a gp wrt the operation

$$(a_1, b_1)\cdot(a_2, b_2) = (a_1 \cdot a_2,\ b_1 \cdot b_2)$$

and identity $(1_{G_1}, 1_{G_2})$.

$$(a,b)^{-1} = (a^{-1}, b^{-1}).$$

Suppose G is a gp and H, K are subgps of G. Then when can we write $G \cong H \times K$?

$$\phi : H \times K \longrightarrow G.$$

$$\phi(h, k) = hk$$

Let $\phi(h_1, k_1) = \phi(h_2, k_2)$.

If ϕ is inj then WTS $h_1 = h_2$
$k_1 = k_2$.

$$h_1 k_1 = h_2 k_2.$$

$$\Rightarrow h_2^{-1} h_1 = k_2 k_1^{-1} \in H \cap K = \{1\}.$$

$$\Rightarrow h_1 = h_2 \quad \& \quad k_1 = k_2.$$

<u>Propn</u> (1) If $H \cap K = \{1\}$ then the map ϕ is inj and its image is HK

(2) If either H or K is a normal subgp of G then the sets HK and KH are equal and in this case HK is a subgp of G.

Pf: Let H be a normal subgp of G.

Let $h \in H$ & $k \in K$.

Then $hk = k k^{-1} h k \in KH$

as H is normal so $k^{-1} h k \in H$.

$\Rightarrow HK \subseteq KH$.

Similarly $KH \subseteq HK$.

Hence $HK = KH$.

WTS HK is a subgp of G.

$$(h_1 k_1) \cdot (h_2 k_2) = h_1 k_1 h_2 k_2$$

$$= h_1 h_2' k_1' k_2 \in HK.$$

Since $HK = KH$ thus $k_1 h_2 = h_2' k_1'$ $\in HK$.

and $1 = 1 \cdot 1 \in HK$.

$$(hk)^{-1} = k^{-1} h^{-1} \in KH = HK.$$

$\therefore$ HK is a subgp of G.

Q Under which condn. the map $\phi : H \times K \to G$ defined by
$$\phi(h,k) = hk$$ is a gp homo?