Syllabus

Linear Programming Problem (LPP)

- Motivation & Mathematical formulation
- Graphical Method
- Simplex Method
  - Big M method
  - Two phase method

Standardization of LPP by introducing slack and surplus variables

- Degeneracy Resolution
- Duality
- Dual simplex method
- Revised simplex

Basic Theory

- Basic soln
- Basic feasible sol
- optimal soln
- convex set
- Related Theorem

Mid Sem

- Trasportation Problem
- Assignment Problem
- Integer Programming (Linear)
- Parametric Programming
- cutting plane method
- Branch & bound algorithm
- 0-1 implicit enumeration
- Interior pt. method for LPP
- Game Theory
- Queuing Models (If time permits) (deterministic and Probabilistic)

Books

1) H. A. Taha – operation Research – an Intro
2) F. Hiller & G. J. Liberman – Intro to OR.
3) G. Hadley – Linear Programming.

Mid sem – 30%.
End sem – 50%.
Class test – 10%.
TA – 10%.

1 August

Graphical Method for solving LPP

Example: Solve the following LPP graphically

For graphical method we've 2 variable, may be extend

$$\text{Max } Z = 150x + 100y$$

$$\text{s.t. } 8x + 5y \le 60$$

$$4x + 5y \le 40$$

$$x, y \ge 0$$

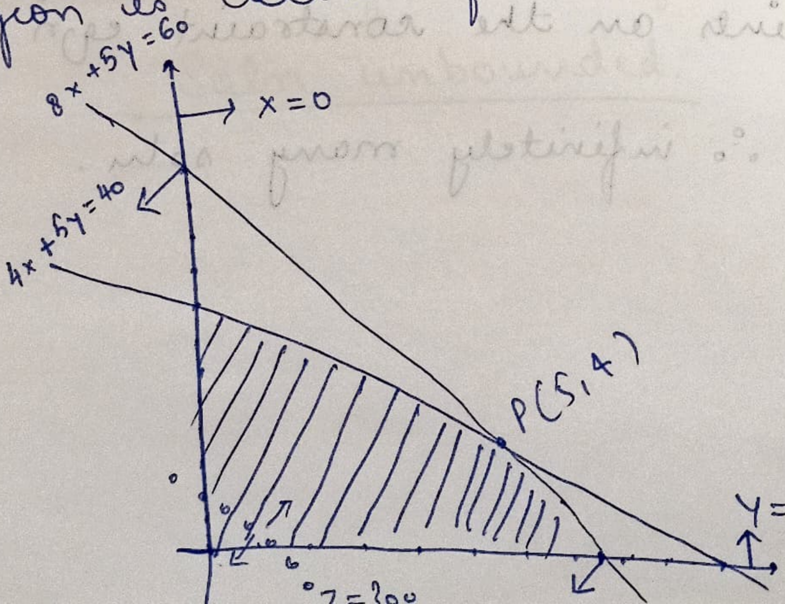
Here the ~~set~~ eqn of of constraints can be written as:

$$Ax \begin{matrix}\le \\ \ge \\ =\end{matrix} b$$

$\therefore$ Max Z s.t. $Ax \begin{matrix}\ge \\ \le\end{matrix} b$ $\longrightarrow$ LPP looks like this.

Area bounded by the constraints is called feasible region. all soln in the feasible region is called feasible points.

Soln:



$$\frac{8x}{60} + \frac{5y}{60} = 1$$

$$\frac{x}{15/2} + \frac{y}{12} = 1$$

$$\frac{4x}{40} + \frac{5y}{40} = 1$$

$$\frac{4x}{10} + \frac{y}{8} = 1$$

On shifting the cost line we get the solution (max or min)

Extremes occur on the edge points.

Z_{max} occurs at (5, 4)

$x = 5$, $y = 4$

$$Z_{max} = 150 \times 5 + 100 \times 4$$
$$= 1150$$

In case of bounded region we took solution on upper edge, How about when the ~~solution~~ region would be unbounded?

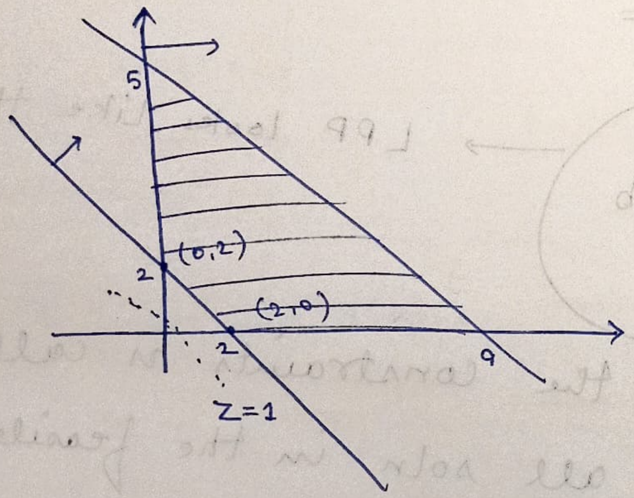
Example : Min $z = x_1 + x_2$

s.t. $5x_1 + 9x_2 \leq 45$

$x_1 + x_2 \geq 2$

$x_2 \leq 4$

$x_1, x_2 \geq 0$



The cost line $z = 1$ intersects with infinitely many points on the constraint eqn $x_1 + x_2 \geq 2$

$\therefore$ infinitely many soln.

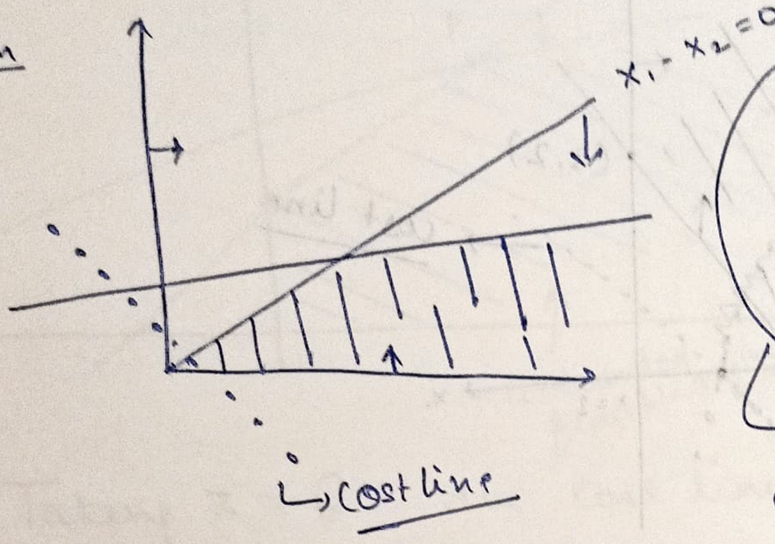
Example Max $z = 3x_1 + 4x_2$

s.t. $x_1 - x_2 \geq 0$

$-x_1 + 3x_2 \leq 3$

$x_1, x_2 \geq 0$

Soln



To choose what direction, Here its tricky since the line passes throug zero.

↳ To continue, you may check a point ~~and~~

$(1, 0)$

$1 - 0 \geq 0$

∴ Lies downside the line

∴ ↓

∵ The region is unbounded on the right,

∴ Max z will be unbounded.

Min z might have a ~~bounded~~ unique soln.

Unbdd feasible region.

Soln unbounded.

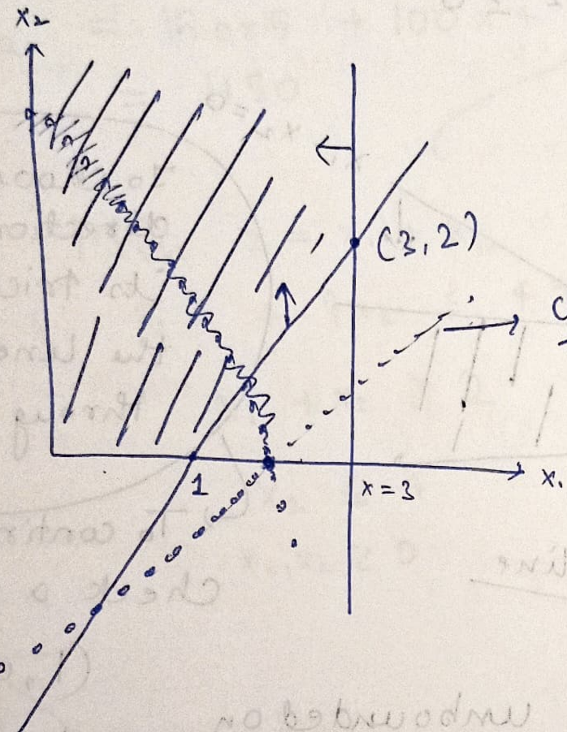
what is cost line?

Example : Max $z = 2x_1 - x_2$

s.t. $x_1 - x_2 \le 1$

$x_1 \le 3$

$x_1, x_2 \geq 0$



Let $z = 4$

$$\frac{x_1}{2} - \frac{x_2}{4} = 1$$

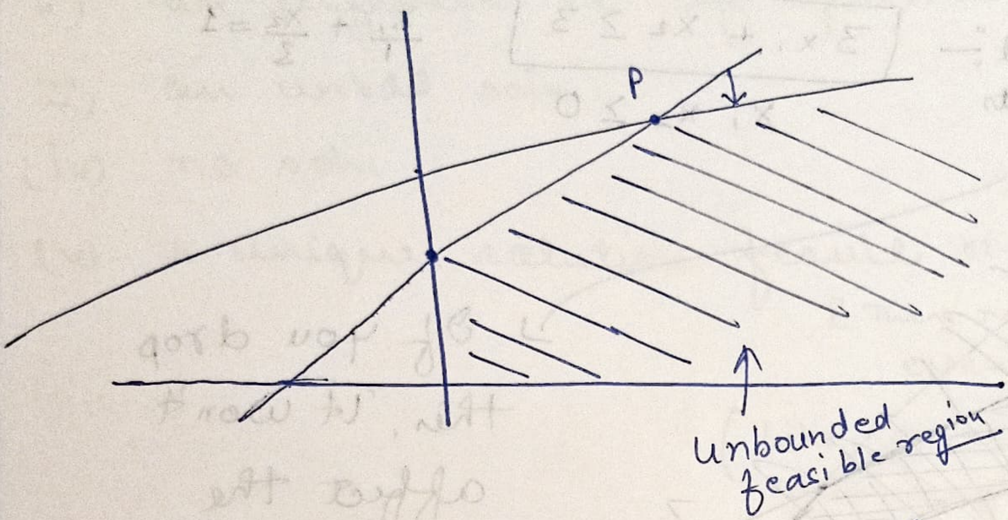
at (3, 2) the optimum soln lies. (maxima)

Example: Max $Z = -x_1 + 3x_2$

s.t. $x_1 - x_2 \geq -1$ $\qquad x_2 - x_1 \leq 1$

$-0.5x_1 + 1.5x_2 \leq 3$

$x_1, x_2 \geq 0$



$x_2 - x_1 = 1$

$\frac{x_1}{-6} + \frac{x_2}{2} = 1$

$x_2 - x_1 = 1$

$x_1 - 3x_2 = -6$

Taking $Z = 6$ the cost line is same as constraint eqn $-0.5x_1 + 1.5x_2 \leq 3$.

$\therefore$ Every point on the line will give a solut

$\Longrightarrow$ infinitely many soln.

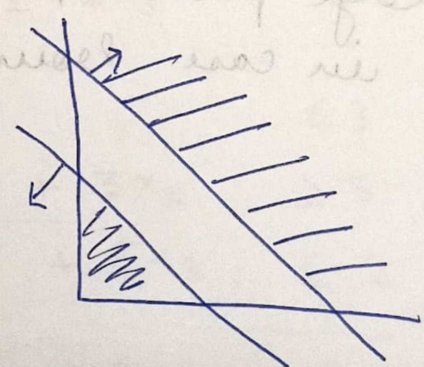
Example (No soln Case)

Max $Z = 2x_1 - 3x_2$

s.t. $x_1 + x_2 \leq 2$

$2x_1 + 2x_2 \geq 8$

$x_1, x_2 \geq 0$



No feasible region.

<u>Example</u> Max $z = 6x_1 + 4x_2$

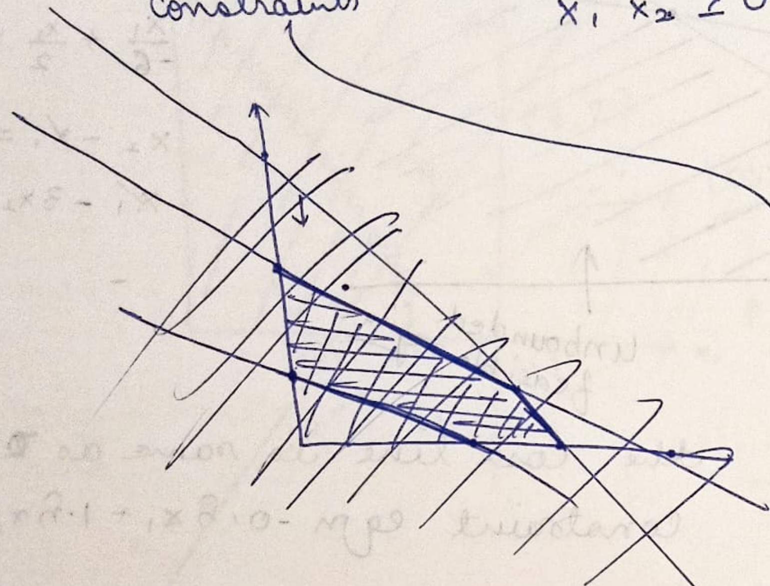
s.t. $7x_1 + 5x_2 \le 35$

$5x_1 + 7x_2 \le 35$

$4x_1 + 3x_2 \ge 12$

Reductant :- constraints $\boxed{3x_1 + x_2 \ge 3}$ $\frac{x_1}{1} + \frac{x_2}{3} = 1$

$x_1\ x_2 \ge 0$



If you drop this, it won't affect the ~~so~~ feasible region.

Another method (than moving cost line)
find edge points, comput z value on the edge points, find max min
(only in case bounded regions)

Nature of solution of a LPP

A LPP may have.

i) a unique soln and finite optimum soln

ii) an infinite no. of optimal soln.

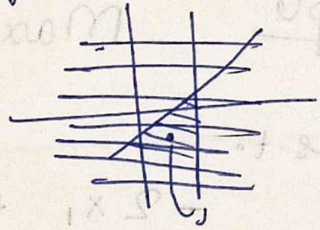
iii) an unbdd soln.

(iv) no soln

(v) a unique solution feasible or not.

[There may be some problem even if constraints satisfied]

(Like constraints satisfied, but non-negativity not satisfied



• Standardization operations

Example: $\text{Min } z = 3x_1 + x_2 + 2x_3$

s.t. $-2x_1 + 4x_2 \leq 3$

$x_1 + 2x_2 + 3x_3 \geq 5$

$2x_1 + 5x_3 \leq 2$

$x_1, x_2, x_3 \geq 0$

Standard for of LP

$$\text{Min } Z = Cx$$
s.t. $Ax = b$
$x \geq 0$

or $\text{Max } Z = Cx$
s.t. $Ax = b$
$x \geq 0$

There are no inequalities

Changing into the standard form (the previous example)

Example $\text{Max } Z' = -Z = -3x_1 - x_2 - 2x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6$

s.t.

$$-2x_1 + 4x_2 + x_4 = 3$$
$$x_1 + 2x_2 + 3x_3 - x_5 = 5$$
$$2x_1 + 5x_3 + x_6 = 2$$

$x_1, x_2, x_4, x_5, x_6 \geq 0$

surplus

slack variable

Recasting a LPP

A general LPP in std form has the following characteristics:

- all constraints are equiv except for non-negativity constraints which remains inequality.

- RHS elements of each constraint eqn ≥ 0

- all variables are non-negative
- obj. function either max or min

Example: Min $Z = 3x_1 - 4x_2 - x_3$

s.t. $x_1 + 3x_2 - 4x_3 \leq 12$

$2x_1 - x_2 + x_3 \leq 20$

$x_1 - 4x_2 - 5x_3 \geq 5$

$x_1 \geq 0$, x_2, x_3 are unrestricted in sign

Let $x_2 = x_2' - x_2''$ $\qquad x_2', x_2'', x_3', x_3'' \geq 0$

$x_2 = x_3' - x_3''$

7 August

- Standard form of a LPP

Example: Consider the LPP

Max $Z = 2x_1 + 3x_2 + x_3$

s.t. $\begin{cases} x_1 + x_2 - 2x_3 \geq -5 \\ -6x_1 + 5x_2 - 3x_3 = 12 \\ 12x_1 - 5x_2 + 5x_3 \leq 12 \\ x_1, x_2, x_3 \geq 0 \end{cases}$

Standard form $\begin{cases} Ax = b \\ x_i \geq 0 \end{cases}$

transforms to Max Z ..

$-x_1 - x_2 + 2x_3 + x_4 = 5$

$-6x_1 + 5x_2 - 3x_3 = 12$

$12x_1 - 5x_2 + 5x_3 + x_5 = 12$

$\underbrace{x_1, x_2, x_3}_{\text{original}}, \underbrace{x_4, x_5}_{\text{slack variable}} \geq 0$

<u>Example</u> : Max $z = 2x_1 + 3x_2 + 7x_3$

s.t. $x_1 + x_2 + 2x_3 \leq 40$

$2x_1 + x_2 - 3x_3 \geq 50$

$|5x_2 + 8x_3| \leq 60$ $\begin{cases} 5x_2 + 8x_3 \leq 60 \\ 5x_2 + 8x_3 \geq -60 \end{cases}$

<u>Standard form</u>: Max $Z = 2x_1 + 3x_2 + 7x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7$

s.t. $x_1 + x_2 + 2x_3 + x_4 = 40$

$2x_1 + x_2 - 3x_3 - x_5 = 50$

$5x_2 + 8x_2 + x_6 = 60$

$-5x_2 - 8x_3 + x_7 = 60$

$\underbrace{x_1, x_2, x_3}_{\text{original}} \geq 0$ $\quad \underbrace{x_4, \overbrace{x_5, x_6, x_7}^{\text{surplus}}}_{\text{slack variable}} \geq 0$

Now we will show that the solution of the converted standard form is same as that of the question given.

- <u>Theorem</u>: There is a one to one correspondence between the opt. solution of the original LPP and that of the new problem in the std form, when slack and surplus variables are introduced

$$X = (X_{original}, X_{slack}, X_{surplus})$$

$$C = (C_{original}, C_{slack}, C_{surplus})$$

$$X' = (X'_{original}, X'_{slack}, X'_{surplus})$$

Claim : ($X'_{original}$ is an ~~original~~ optimal soln of the original problem)

New problem : x' z'

$\downarrow$

$x'_{original}$

$\downarrow$

old problem: feasible soln of the Original problem

100% in exam

Suppose it is not the optimal soln.

Let $x''_{original}$ be the opt. soln. of the original $\neq x'_{original}$

$x'' = (x''_{original}, x''_{slack}, x''_{surplus})$ with

$z'' > z'$

contradiction.

Example :

$$\text{Max } z = 3x_1 + 4x_2 - 4x_3 + 2x_4 + 9x_5$$

s.t.

$$3x_1 - 7x_2 - 9x_3 + x_4 - 2x_5 \leq 6$$

$$2x_1 + 5x_2 + 4x_3 - 3x_4 + x_5 \leq 8$$

$$x_1 + 2x_2 - 5x_3 - 2x_4 + 11x_5 \leq 10$$

(1) $x_1, x_2, x_3 \geq 0$, x_3, x_5 unrestricted in sign.

Conversion :-

~~Standard form~~: Let $x_3 = x_3' - x_3''$ $\quad x_3', x_3'' \geq 0$

$x_5 = x_5' - x_5''$ $\quad x_5', x_5'' \geq 0$

$$\text{max } z = 3x_1 + 4x_2 - 4x_3' + 4x_3'' + 2x_4 + 9x_5' - 9x_5''$$

$$\text{s.t. } 3x_1 - 7x_2 - 9x_3' + 9x_3'' + x_4 - 2x_5' + 2x_5'' \leq 6$$

⋮

converting into standard form

Standard form:

$$
\text{Max} \begin{cases} Z' = 3y_1 + 4y_2 - 4y_3 + 4y_4 + 2y_5 + 9y_6 - 9y_7 \\ \text{s.t. } 3y_1 - 7y_2 - 9y_3 + 9y_4 + y_5 - 2y_6 + 2y_7 + y_8 = 6 \\ \vdots \\ y_i \geq 0 \end{cases} \quad (2)
$$

Now all $y_1, y_2 \ldots$ determine uniquely $x_1, x_2 \ldots$

Claim: if $y_1, y_2, \ldots$ opt. solⁿ for (2)
then $x_1, x_2, \ldots$ opt. " " (1)

Proof of the Claim.

?

• $x_K \rightarrow$ unrestricted variable.

$$x_K^+ = \begin{cases} x_K & \text{if } x_K \geq 0 \\ 0 & \text{if } x_K < 0 \end{cases}$$

$$x_K^- = \begin{cases} 0 & \text{if } x_K > 0 \\ -x_K & \text{if } x_K \leq 0 \end{cases}$$

Then. $x_K = x_K^+ - x_K^-$

$|x_K| = x_K^+ + x_K^-$

<u>Example</u>: Max $Z = 6x_1 + 4x_2 + 10x_3$

s.t. $2x_1 + 3x_2 + 5x_3 \leq 50$

$4x_1 + 2x_2 + 7x_3 \leq 6$

$x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_2 \leq 20$

$x_1 \geq 5, \quad x_2 \geq 4, \quad x_2 \geq 3.75$

<u>Let</u> $y_1 = x_1 - 5$ sly y_2, y_3

$x_1 = y_1 + 5$

$\therefore x_2 = y_2 + 4$

$x_3 = y_3 + 3.75$

Convert entire question to y_1, y_2, y_3

only the cost function will be shifted by a fixed amount.

Example: Max $Z = x_1 + 2x_2 - 3x_3$

s.t. $-3 \leq 3x_1 - 5x_2 \leq 15$ — break into 2 diff. ineq.

$x_2 \geq 0$, $x_3 \leq 0$ → $y_3 = -x_3$

std. form

Max Z

s.t $Ax = b$

Example: Min $Z = |x_1 - 3| + |x_2 + 4|$

$y_1 = x_1 - 3 = y_1^+ - y_1^-$

$y_2 = x_2 + 4 = y_2^+ - y_2^-$

$y_1^+ \; y_1^- \; y_2^+ \; y_2^- \geq 0$

Duality

8 August

Example: Obtain the dual problem of the following LPP.

$$\text{Max } Z = 2x_1 + 5x_2 + 6x_3$$

$$\text{s.t. } 5x_1 + 6x_2 - x_3 \le 3$$
$$-2x_1 + x_2 + 4x_3 \le 4$$
$$x_1 - 5x_2 + 3x_3 \le 1$$
$$-3x_1 - 3x_2 + 7x_3 \le 6$$

Primal problem

$$\text{Max } Z = Cx$$
$$Ax \le b$$
$$x \ge 0$$

Dual: $\text{Min } w = 3v_1 + 4v_2 + v_3 + 6v_4$

$$5v_1 - 2v_2 + v_3 - 3v_4 \ge 2$$
$$6v_1 + v_2 - 5v_3 - 3v_4 \ge 5$$
$$-v_1 + 4v_2 + 3v_3 + 7v_4 \ge 6$$

Dual Problem

$$\left.\begin{aligned} &\text{Min } w = b^T v \\ &\text{s.t. } A^T v \ge C^T \\ &v \ge 0 \end{aligned}\right\}$$ No. of variables and no. of constraints have been reversed.

Dual of Dual Problem gives Primal again

Example $\quad \text{Max } Z = 2x_1 - 6x_2$

$$\text{s.t.} \quad x_1 - 3x_2 \leq 6$$
$$x_1 + 4x_2 \geq 8$$
$$x_1 - 3x_2 \geq -6$$
$$x_1, x_2 \geq 0$$

Formulate the dual of the above primal problem

Primal: $\quad \text{Max } Z = 2x_1 - 6x_2$

$$\text{s.t.} \quad x_1 - 3x_2 \leq 6$$
$$-x_1 - 4x_2 \leq -8$$
$$-x_1 + 3x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

Dual: $\quad \text{Min } w = 6v_1 - 8v_2 + 6v_3$

$$\text{s.t.} \quad v_1 - v_2 - v_3 \geq 2$$
$$-3v_1 - 4v_2 + 3v_3 \geq -6$$
$$v_1, v_2, v_3 \geq 0$$

Duality   Ex: $\text{Max } z = 2x_1 + 3x_2 + x_3$

s.t. $4x_1 + 3x_2 + x_3 = 6 \;<\; \begin{matrix} \geq 6 \\ \leq 6 \end{matrix}$

$x_1 + 2x_2 + 5x_3 = 4 \;<\; \begin{matrix} \leq 4 \\ \geq 4 \end{matrix}$

$x_1, x_2, x_3, x_4 \geq 0$

Std primal problem:

$\text{Max } z = 2x_1 + 3x_2 + x_3$

s.t. $4x_1 + 3x_2 + x_3 \leq 6$

$-4x_1 - 3x_2 - x_3 \leq -6$

$x_1 + 2x_2 + 5x_3 \leq 4$

$-x_1 - 2x_2 - 5x_3 \leq -4$

$x_1, x_2, x_3 \geq 0$

Dual $\text{Min}: w = 6v_1 - 6v_2 + 4v_3 - 4x_4$

s.t. $4v_1 - 4v_2 + v_2 - v_1 \geq 2$

$3v_1 - 3v_2 + 2v_2 - 2v_1 \geq 3$

$v_1 - v_2 + 5v_3 - 5v_4 \geq 1$

$v_1, v_2 \ldots v_4 \geq 0$

Let $v_1' = v_1 - v_2$, $v_2' = v_3 - v_4$

($\because$ we said variables and constraints are interchanged, but in original eqn. constraints are 2, so should be (2) no. of ~~varia~~ total no. of variables here)

$(v_1 - v_2) = v_1'$ $(v_3 - v_4) = v_2'$

v_1', v_2' are unrestricted variables.

Example Max $z = 2x_1 + 3x_2 + 4x_3$

s.t. $x_1 - 5x_2 + 3x_3 = 7 \rightarrow v_1$

$2x_1 - 5x_2 \leq 3 \rightarrow v_2$

$3x_1 - x_2 \geq 5$

$x_1, x_2 \geq 0$ x_3 is unrestricted.

Std primal Let $x_3 = x_3' - x_3''$

~~Let = x_3' − x_3''~~

$$\text{Max } z = 2x_1 + 3x_2 + 4x_3' - 4x_3''$$

s.t. $x_1 - 5x_2 + 3x_3' - 3x_3'' \leq 7$

$-x_1 + 5x_2 - 3(x_3' - x_3'') \leq -7$

$2x_1 - 5x_2 \leq 3$

$-3x_1 + x_2 \leq -5$

$x_1 \ x_2 \ x_3' \ x_3'' \geq 0$

Dual

$$\text{Min } w = \underline{7v_1 - 7v_2} + 3v_3 - 5v_4$$

s.t. $\underline{v_1 - v_2} + 2v_3 - 3v_4 \geq 2$

$-5v_1 + 5v_2 - 5v_3 + v_4 \geq 3$

$3v_1 \cancel{+} - 3v_2 \geq 4$

$-3v_1 + 3v_2 \geq -4$

$v_1 \ v_2 \ v_3 \ v_4 \geq 0$

(Let $v_1' = v_1 - v_2$)

$[A|I] \longrightarrow \longrightarrow \longrightarrow \cdots [I|B]$

$Ax = b$

$A_{m\times n}$, $m < n$

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix}_{m\times n} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m\times 1}$$

If rank $(A) = m$ = rank (A/b)

$a_1 \; a_2 \; \cdots \; a_n$

$\downarrow$

m subset

$(a_{i1} \; a_{i2} \cdots a_{im})_{m\times n}$

Basic sol^n

Partition A :

$$A = [B|R], \quad x = [x_B \,|\, x_R]$$

$m\times n \qquad m\times(n-m)$

$Ax = b \Rightarrow Bx_B + Rx_R = b$

Setting $x_R = 0$

$Bx_B = b$

$x_B = B^{-1} b$

· Basic feasible soln

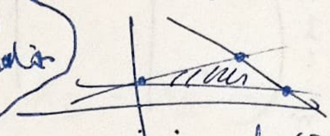
$$Ax = b \quad , \quad x \geq 0$$

one one correspond

LPP

Optimal soln $\longrightarrow$ Extreme pt $\rightleftarrows$ Basic feasible soln

of the set of all feasible condition



Extreme point of a convex set X.

$x = (x_1 \; x_2 \; \ldots \; x_n)$

$y = (y_1 \; y_2 \; \ldots \; y_n)$

?

P

X is an extreme pt of a convex set X if x cannot be written as.

for any,

$x_1 \; x_2 \in X$

$x = \lambda x_1 + (1-\lambda) x_2$

* All the feasible soln always form a convex sets.

Example [Simplex Algorithm]

Solve the following LPP

$$\text{Max } Z = 60x_1 + 50x_2$$

$$\text{s.t. } x_1 + 2x_2 \le 40$$
$$3x_1 + 2x_2 \le 60$$
$$x_1, x_2 \ge 0$$

Std form by introducing slack and surplus variables.

$$\text{Max } Z = 60x_1 + 50$$

$$\text{s.t. } x_1 + 2x_2 + x_3 = 40$$
$$3x_1 + 2x_2 + x_4 = 60$$
$$x_1, x_2, \underbrace{x_3, x_4}_{\text{slack variable}} \ge 0$$

$$A = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array} \right) \quad 2\times 2 \ (\text{l.I.})$$

		C_j		60	50	0	0	Min Ratio
C_B	B	X_B	b	a_1	a_2	a_3	a_4	
0	a_3	x_3	40	1	2	1	0	$40/1 = 40$
0	a_4	x_4	60	3	2	0	1	$60/3 = 20$
		$Z_j - C_j$		-60	-50	0	0	

0	a_3	x_3						
60	a_1	x_1						

$$Z_{max} = 60 \times 10 + 50 \times 15 = 1350$$

Simplex algo can be asked in exam to find the inverse of the given matrix. By heart ü.

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21 August 23

Simplex Algo : Solve the LPP

$$\text{Max. } Z = x_1 + x_2 + 3x_3$$

$$\text{s.t. } 3x_1 + 2x_2 + x_3 \le 3$$

$$2x_1 + x_2 + 2x_3 \le 2$$

$$x_1, x_2, x_3 \ge 0$$

$$\text{Max } Z = x_1 + x_2 + 3x_3 + 0\cdot x_4 + 0\cdot x_5$$

$$\text{s.t. } 3x_1 + 2x_2 + x_3 + x_4 = 3$$

$$2x_1 + x_2 + 2x_3 + x_5 = 2$$

$$x_i \ge 0 \quad i = 1, 2, 3, 4, 5$$

			c_j						
C_B	B	X_B	b	a_1	a_2	a_3	a_4	a_5	Min Ratio
0	a_4	x_4	3	3	2	1	1	0	
0	a_5	x_5	2	2	1	$\boxed{2}$	0	1	$\rightarrow$
(index) $\rightarrow$	$z_j - c_j$	=		-1	-1	-3 ↑	0	0	

a_3 - entering vector.

If 2 constrain standard basis is I_2, 3 then I_3 . . .

$$\text{Min}\left\{ \frac{X_{Bi}}{Y_{1K}}, \frac{X_{Bi}}{Y_{2K}} \right\} = \frac{X_{Bi}}{Y_{rK}} \quad r = 5.$$

$$\text{Min}\left\{ \frac{3}{1}, \frac{2}{2} \right\} = 1$$

$\therefore$ $\boxed{2}$ is the pivot.

Divide every element of the row by 2.