Lecture 28

Stage three: Non-negative functions:

To study the Lebesgue integral of functions that are measurable & non-negative but not necessarily bounded.

Def:- For a non-negative measurable function f (f can be extended real valued) its Lebesgue integral is defined as

$$\int f \doteq \sup_{g} \left(\int g \right)$$

where the supremum is taken over all measurable functions g such that $0 \le g \le f$ & where g is bounded & supported on a set of finite measure.

Remark:-

$$\int f := \sup\left(\left\{ \int g \;\middle|\; \begin{array}{l} 0 \le g \le f,\ g \text{ is measurable, bounded \& supported on a set} \\ \text{of finite measure} \end{array} \right\}\right).$$

Def:- We say that a measurable non-negative function f is Lebesgue-integrable, if

$$\int f < \infty .$$

Def:- For any measurable set $E \subseteq \mathbb{R}^d$, & $f \geq 0$ measurable, we define

$$\int_E f := \int f \chi_E$$

Proposition:- Let $f \geq 0$, $g \geq 0$ be measurable functions

Then

(i) (Linearity) For any $a, b \in \mathbb{R}_+$,

$$\int (af + bg) = a \int f + b \int g.$$

(ii) (additivity) If E, F are disjoint sets, then

$$\int_{E \cup F} f = \int_E f + \int_F f.$$

(iii) (Monotonicity) If $0 \le f \le g$, then

$$\int f \le \int g.$$

(iv) If g is Lebesgue integrable & $0 \le f \le g$, then f is Lebesgue integrable.

(v) If f is Lebesgue integrable, the $f(x) < \infty$ a.e

(vi) If $\int f = 0$, then $f = 0$ a.e.

proof:- (i) Take $a=b=1$.

To show: $\int(f+g) = \int f + \int g$

Let $0 \le \varphi \le f$, $0 \le \psi \le g$, where φ, ψ are bounded measurable funtions supported on a set of finite measure.

Then $\underline{\varphi+\psi \le f+g}$ & $\varphi+\psi$ is bounded, measurable & supported on a set of finite measure.

$$\Rightarrow \int(\varphi+\psi) \le \int(f+g)$$

$$\int(\varphi+\psi) = \int\varphi + \int\psi$$

$$\Rightarrow \int\varphi + \int\psi \le \int(f+g)$$

True $\forall \varphi \ldots$ &

$\forall \psi \ldots$

$$\Rightarrow \int f + \int g \le \int(f+g)$$

For reverse inequality, let η be a <u>bounded</u> measurable function supported on a set of finite measure with $0 \le \eta \le f+g$.

Define $\eta_1(x) = \min\left(f(x), \eta(x)\right) \geq 0 \;\; \forall x$.

& $\eta_2 = \eta - \eta_1$. Such that $\eta_2 \geq 0$.

$$\eta_2 = \eta - \eta_1 = \begin{cases} \eta - f \\ \eta - \eta \end{cases} \text{ or } \le \begin{cases} f+g-f = g \\ 0 \end{cases}$$

Then by the definition of η_1, $\eta_1 \le f$

& $0 \le \eta_2 \le g$

Then both η_1, η_2 are bounded & supported on a set of finite measure.

$$\therefore \int \eta = \int (\eta_1 + \eta_2)$$

$$= \int \eta_1 + \int \eta_2$$

$$\le \int f + \int g$$

True for any such η.

$$\therefore \int (f+g) \le \int f + \int g.$$

$\therefore \quad \int (f+g) = \int f + \int g.$

(ii) to (iv) : EXERCISE.

Let f be L-integrable. That is $\int f < \infty$

(v). To show : $m\left(\{x \mid f(x) = \infty\}\right) = 0$

Suppose $E_k = \{x \mid f(x) \geq k\}, \quad \forall k \geq 1,$

$E_\infty = \{x \mid f(x) = \infty\}$

Then

$$\int f \chi_{E_k} = \int_{E_k} f \chi_{E_k} = \int_{E_k} f \geq \int_{E_k} k = k\, m(E_k)$$

$$\Rightarrow \int f \geq \int_{E_k} f \chi_{E_k} \geq \int_{E_k} k$$

$$\Rightarrow k\, m(E_k) \leq \int f \chi_{E_k} \leq \int f < \infty \quad \forall k \geq 1$$

$$\Rightarrow k\, m(E_k) < \infty \quad \forall k \geq 1.$$

$$\Rightarrow \lim_{k\to\infty}\left(k\, m(E_k)\right) < \infty$$

$$\Rightarrow \quad m(E_k) = 0 \quad \underline{\forall k \gg 0}.$$

$$\left(E_{k+1} \subseteq E_k \quad \forall k \geq 1.\right)$$

$$\Rightarrow \quad m(E_\infty) = 0.$$

(if $\underset{k \gg 0.}{m(E_k) \neq 0}$, then $\left(\lim_{k\to\infty} k\right)\left(\underbrace{\lim_{k\to\infty} m(E_k)}_{\parallel}\right) < \infty$

which implies that $\qquad m\left(\underbrace{\lim_{k\to\infty} E_k}_{<\infty.}\right).$

$$\infty \cdot \underset{\substack{\neq \\ 0.}}{m(E_\infty)} < \infty.$$

$$\infty. < \infty \quad \text{which is absured.}$$

$\therefore \; m(E_\infty) = 0.$

(iv) We proved this for bounded measurable functions supported on a set of finite measure.

We have $\int f = \sup_g \left(\int g\right).$

(EXERCISE)

Qn:- Suppose $f_n \geq 0$ measurable function. &

$f_n \to f$ p.w a.e. Does $\int f_n \longrightarrow \int f$

as $n \to \infty$?

Ans: NO.
