# Graph theory aug 26-31

Graph Theory And Algorithms (Indian Institute of Technology Kharagpur)

26/8/22

Induction on $n$.

Assume $Q_{n-1} \simeq G_{n-1}$

$f : V(G_{n-1}) \to V(Q_{n-1})$

$Q_n = Q_{n-1} \times K_2$ $\qquad K_2$ : $x$ — $y$

$V(Q_n) = \{ \{v, x\}, \{v, y\} : v \in Q_{n-1} \}$

$u \to n-1$ tuple

$V(G_n) = \{ \{u, 0\}, \{u, 1\} : u \in G_{n-1} \}$

$g : V(G_n) \to V(Q_n)$

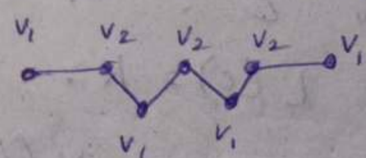
$g(\{u, 0\}) = \{ f(u), x \}$, $\quad g(\{u, 1\}) = \{ f(u), y \}$

Verify that $g$ is an isomorphism.

## Bipartite graphs

A graph $G$ is called a bipartite graph if $V(G)$ can be partitioned as $V(G) = V_1 \cup V_2$ s.t every edge in $G$ has one end vertex in $V_1$ and the other is in $V_2$.

→ All paths are bipartite graphs.

ex 

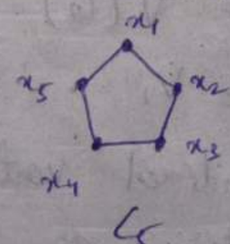

ex 


$C_6$

$V_1$ : 1, 3, 5

$V_2$ : 2, 4, 6

✓




$C_5$

✗

→ Even cycles are bipartite graphs and odd cycles are not bipartite graphs.

• $(V_1, V_2)$ is called bipartition.

• $V_1$ and $V_2$ are called partite sets.

→ All hypercubes $Q_n$ are bipartite graphs

even parity vertices connect to only odd parity vertices- proof by induction

• A complete bipartite graph is a bipartite graph with a bipartition $(V_1, V_2)$ such that all the vertices in $V_1$ are adjacent with all the vertices in $V_2$.

Further, if $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$.

$$K_{m,n} \simeq K_{n,m}.$$

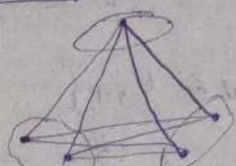
k-Partite graph, $k \geq 2$

$$V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$$

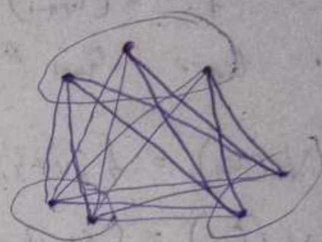
$$V_i \cap V_j = \phi, \quad \forall\, i, j$$

so that no two vertices within the same part are adjacent.

Complete K-partite graphs



$K_{1,2,2}$.



$K_{3,2,2}$

Lemma: Let $G$ be a disconnected graph. Then $G$ is bipartite iff all its components are bipartite.

Thm: Let $G$ be a loop free graph. Then $G$ is bipartite iff it contains no odd cycle.

Pf: w.l.o.g Let $G$ be a connected graph.

Suppose $G$ is a bipartite graph with a bipartition $(V_1, V_2)$.

Let $C: u_1 u_2 \cdots u_k u_1$ be an arbitrary cycle in $G$.

$L(C) = K$. we prove that $K$ is even.

$V_1 \cup V_2 = V(G)$.

Let $u_1 \in V_1$. Then $u_2 \in V_2$, $u_3 \in V_1$, $\ldots$, $u_k \in V_2$, $u_1 \in V_1$

All even indexed vertices are in $V_2$. So $K$ is even.

so $G$ contains no odd cycle.

Conversely assume that $G$ contains no odd cycle. To prove that $G$ is a bipartite graph. Let $v \in V(G)$.

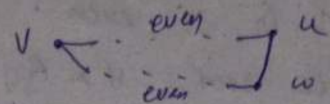
Let $V_1 = \{x \in V(G) : d(v,x) \text{ even}\}$.

$V_2 = V(G) \setminus V_1$ i.e. vertices in $V_2$ are at odd distance from $v$.

Claim: $(V_1, V_2)$ is a bipartition of $G$.

Suppose $u, w \in V_1$ s.t $u \sim w$.

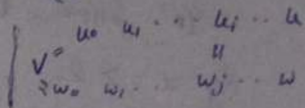
$v \neq u, w$



since $v \in V_1$, $d(v,u) = 2r$ and $d(v,w) = 2s$.

$P_1$ is shortest $v-u$ path $\ell(P_1) = 2r$

$P_2$ is shortest $v-w$ path $\ell(P_2) = 2s$



if $P_1 \cap P_2 = \{v\}$ then it forms a odd cycle $\Rightarrow\Leftarrow$.

$(P_1 \cup (u,w) \cup P_2)$

suppose $P_1 \cap P_2 \neq \{v\}$.

Let $u_i$ be the last vertex of $P_1$ that belongs to $P_2$.

we have also $u_i = w_j$ for some $j$.

Claim: $j = i$

if $i \neq j$, then $i > j$ or $i < j$.

if $i > j$, then we can construct a $v-u$ path whose length is less than the length of $P_1$. This contradicts that $P_1$ is a shortest $v-u$ path.

Similar contradiction we get if $i < j$. i.e. we get a $v-w$ path whose length is $< \ell(P_2)$.

Hence the claim is true. i.e. $i = j$.

Consider the cycle $C$ : $u_i \, u_{i+1} \cdots u \, w \cdots w_{j+1} \, w_j = u_i$

$\ell(C) = 2r - i + 1 + 2s - i$ which is an odd integer.

i.e. $G$ contains an odd cycle.

So no two vertices in $V_1$ are adjacent.

Similarly no two vertices in $V_2$ are adjacent.

Hence $G$ is a bipartite graph.

# Eulerian Graphs

An **Eulerian trail** $T$ in a graph $G$ is a closed trail s.t $T$ contains all the edges of $G$.
$E(T) = E(G)$ i.e. $T$ contains all the edges of $G$.

A graph $G$ is called Eulerian if $G$ contains an Eulerian trail

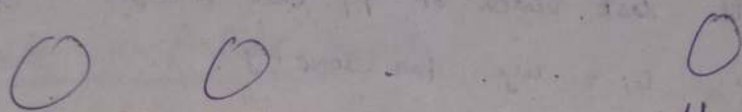
**Thm** A connected graph $G$ is Eulerian iff the degree of all the vertices in $G$ is even.

**Pf** Let $\deg x$ be even, $\forall x \in V(G)$.
To show that $G$ contains an Eulerian trail.

$\deg x \geq 2$, $\forall x \in V(G)$.

$\boxed{\text{If } \delta(G) \geq 2, \text{ then } G \text{ contains a cycle}}$.

$G$ contains a cycle, say $C$.
Consider the graph $G_1$, s.t
$E(G_1) = E(G) - E(C)$.



$H_1$ $H_2$ $H_k$

components of $G_1$.

proof???

$y \in V(H_i)$, $\deg y \geq 2$ even.

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31/8/22 (1) $Q_n = Q_{n-1} \times K_2$, $Q_1 = K_2$

$Q_n = \{(a_1, a_2, \dots, a_n) : a_i = 0 \text{ or } 1\}$

$(a_1, a_2, \dots, a_n) \sim (b_1, b_2, \dots, b_n)$ if they differ in exactly one position.

$|V(Q_n)| = 2^n$

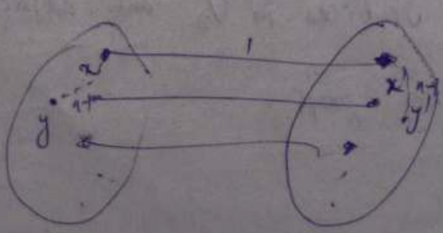
$x \in V(Q_n)$, $\deg x = n$, $\forall x$

$x = (a_1, a_2, \dots, a_n)$ $x$ is adjacent to $n$ other vertices

$|E(Q_n)| = \dfrac{\sum \deg x}{2} = \dfrac{n \cdot 2^n}{2} = n \cdot 2^{n-1}$

$r(Q_n) = n$ $\quad \operatorname{diam}(Q_n) = n$.

induction



$Q_{n-1}$ $Q_{n-1}$

$e_{Q_{n-1}}(x) = n-1 \qquad e_{Q_n}(x) = n-1+1 = n \quad \forall x \in V(Q_n)$

$C(Q_n) =$ center of $Q_n = Q_n$

- $\boxed{C(G) = G, \text{ then } G \text{ is called a self-centered graph}}$

girth of $Q_n = 4$ ; $n \geq 2$
by construction, no triangle is formed (only corresponding vertices are joined)

(2) $V(Q_n) = V_1 \cup V_2 \qquad V_1 \cap V_2 = \phi$.

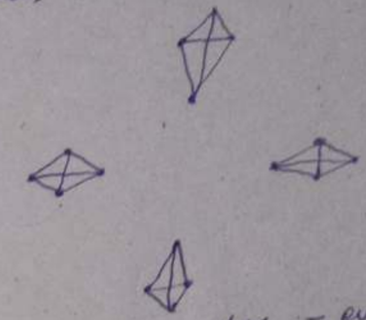
$V_1 \rightarrow$ consists of all $n$-tuples with even no of zeroes.

$V_2 \rightarrow$ consists of all $n$-tuples with odd no of zeroes.

$x, y \in V_1$, to show that $x \not\sim y$. $x \neq y$

$a_1 a_2 \cdots a_n \qquad k$ no of zeroes.

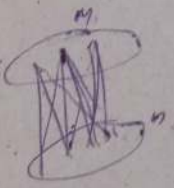
$b_1 b_2 \cdots b_n$



(3) $G_1 = \overline{K_n} \qquad G_2 = K_m$

$n = 4, m = 3 \longrightarrow$

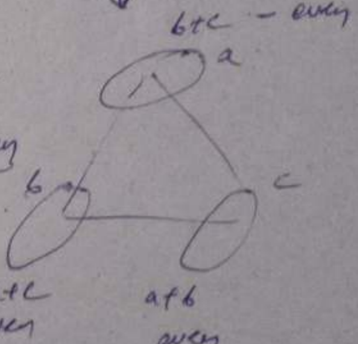
(4) $K_{m,n}$



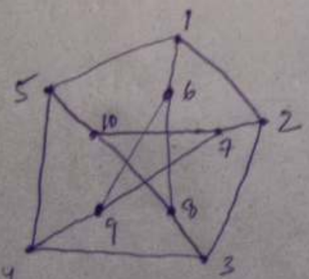

$m, n$ even.

$K_{a,b,c}$

$a, b, c$ even
or
$a, b, c$ odd




(5)




$V_1 = \{1, 9, 8\}$

$V_2 = \{5, 6, 7, 3\}$

$V_3 = \{2, 4, 10\}$

(6) $G_1 \vee G_2$

ex $P_3 \vee K_3$

rad $(G_1 \vee G_2) = 2$ or 1 if $\exists$ a vertex $v$ in $(G_1$ or $G_2)$ which is adjacent to every other vertex in ($G_1$ or $G_2$)

diam $(G_1 \vee G_2) = 2$