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1) $\frac{dy}{dx} = \frac{1}{x^2+y}$, $y(4)=4$, $y(4.2)=?$ $h=0.1$

$\Rightarrow$ Second Order Taylor Series

$$y_{n+1} = y_n + h y'_n + \frac{h^2}{2!} y''_n$$

$$y' = \frac{1}{x^2+y}$$

$$y'' = \frac{d}{dx}\left(\frac{1}{x^2+y}\right) = \frac{-1}{(x^2+y)^2}\frac{d}{dx}(x^2+y) = -\frac{(2x+y')}{(x^2+y)^2}$$

$$y'' = -\frac{2x}{(x^2+y)^2} - \frac{1}{(x^2+y)^3} = -\frac{(2x^3+2xy+1)}{(x^2+y)^3}$$

Now

$$y_0 = y(4) = 4 \quad \text{at } x=4$$

$$y_1 = y_0 + h y'_0 + \frac{h^2}{2!} y''_0$$

$$y_1 = 4 + 0.1\left(\frac{1}{x^2+y}\right)\Bigg|_{x=4,y=4} + \frac{(0.1)^2}{2!}\frac{(+2x^3+2xy+1)(-1)}{(x^2+y)^3}\Bigg|_{x=4,y=4}$$

$$y_1 = 4 + 0.005 + (-1)(0.000100625)$$

$$y_1 = 4.004899$$

$$y(4.1) \approx y_1 = 4.004899 \quad \text{at } x = 4.1$$

$$y_2 = y_1 + h y'_1 + \frac{h^2}{2!} y''_1$$

$$y_2 = 4.004899 + 0.1\times\left(\frac{1}{x^2+y}\right)\Bigg|_{x=4.1,\ y=4.004899} - \frac{(0.1)^2}{2!}\frac{(2x^3+2xy+1)}{(x^2+y)^3}\Bigg|_{x=4.1,\ y=4.004899}$$

$$y_2 = 4.004899 + \cancel{0.005613}\ 0.00480425 - 0.00009518$$

$\cancel{y_2 = 4.0104168}$ $y_2 = 4.009608071$

$\cancel{y(4.2) \approx y_2 = 4.0104168}$

$$\boxed{y(4.2) \approx y_2 = 4.009608071}$$

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2) $\frac{dy}{dx} = 3x + y^2$, $y(0) = 1$ in interval $[0, 0.4]$, $h = 0.2$

$\Rightarrow$ $3^{rd}$ Order Taylor series

$$y_{n+1} = y_n + h y'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''_n$$

$$y' = 3x + y^2$$

$$y'' = 3 + 2y(y') = 3 + 2y(3x+y^2) = 3 + 6xy + 2y^3$$

$$y''' = 6y^2(y') + 6y + 6xy' = 6y^2(3x+y^2) + 6y + 6x(3x+y^2)$$

$$= 18y^2x + 6y^4 + 6y + 18x^2 + 6y^2x$$

$$y''' = 6y + 18x^2 + 6y^4 + 24y^2x$$

Third Order taylor series with $h = 0.2$

$$y_1 = y_0 + 0.2 y'_0 + \frac{(0.2)^2}{2!} y''_0 + \frac{(0.2)^3}{3!} y'''_0$$

$$= 1 + 0.2(1) + \frac{(0.2)^2}{2!}(3+2) + \frac{(0.2)^3}{3!}(12)$$

$$y_1 = 1 + 0.2 + 0.1 + 0.016 = 1.316$$

$$y(0.2) \simeq y_1 = 1.316.$$

Now

$$y_2 = y_1 + 0.2 y'_1 + \frac{(0.2)^2}{2!} y''_1 + \frac{(0.2)^3}{3!} y'''_1$$

$$= 1.316 + 0.2(2.331856) + \frac{(0.2)^2}{2!}(9.137444992) \cancel{+\frac{(0.2)^3}{3!}}$$

$$+\frac{(0.2)^3}{3!} \times \cancel{34.18246841844} \; 34.89961203$$

$$= 1.316 + 0.4663712 + 0.18274889 98 + \cancel{0.04648428144 2} \; 0.04653281604$$

$$\cancel{y(0.4) \simeq y_2 = 2.010885532}$$

$$y(0.4) \simeq y_2 = 2.011652916$$

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3) $\frac{dy}{dx} = 2y + 3e^{x}$ with $x_0 = 0,\ y_0 = 0$

$y$ at $x = 0.1$ ?

$y$ at $x = 0.2 = ?$

$\rightarrow$ $2^{nd}$ Order Taylor series

$$y(x) = y(x_j) + (x - x_j)\, y'(x_j) + \frac{(x - x_j)^2}{2!}\, y''(x_j)$$

$$y' = 2y + 3e^{x}$$

$$y'' = 2y' + 3e^{x} = 4y + 6e^{x} + 3e^{x} = 4y + 9e^{x}$$

$$y(0.1) \cong y(x=0) + (0.1 - 0)\, y'(x=0) + \frac{(0.1 - 0)^2}{2}\, y''(x=0)$$

$$= 0 + (0.1)(2y + 3e^{x})\Big|_{\substack{x=0\\y=0}} + \frac{(0.1)^2}{2}(4y + 9e^{x})\Big|_{\substack{x=0\\y=0}}$$

$$= 0 + 0.1(3) + 0.045$$

$$y(0.1) \cong 0.345$$

$$\cancel{y(0.2) \cong y(x=0) + (0.2 - 0)\, y'(x=0) + \frac{(0.2 - 0)^2}{2}\, y''(x=0)}$$

$$\cancel{= 0 + 0.2(2y + 3e^{x})\Big|_{\substack{x=0\\y=0}} + \frac{(0.2)^2}{2}(4y + 9e^{x})\Big|_{\substack{x=0\\y=0}}}$$

$$\cancel{= 0 + 0.2(3) + 0.18}$$

$$\cancel{y(0.2) \cong 0.78}$$

$$y(0.2) \cong y(x=0.1) + 0.1\, y'(x=0.1) + \frac{(0.1)^2}{2}\, y''(x=0.1)$$

$$= 0.345 + 0.1\left(2(0.345) + 3e^{0.1}\right) + \frac{(0.1)^2}{2}\left(4(0.345) + 9e^{0.1}\right)$$

$$= 0.345 + 0.40055127 54 + 0.0566322913$$

$$\cancel{y(0.2) \cong 0.8.02}$$

$$y(0.2) \cong 0.8021839$$

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4) $\frac{dy}{dx} = y - x$ , $y(0) = 2$, $y(0.1)$, $y(0.2)$ $h = 0.05$

$\Rightarrow$ Forward Euler Method

$y_{j+1} = y_j + h\, f(x_j, y_j)$

<u>Iteration 1 :-</u>

$x_0 = 0$, $y_0 = 2$

$y_1 = y_0 + h f(x_0, y_0)$
$= 2 + 0.05 (y - x)\Big|_{\substack{x=0 \\ y=2}}$

$= 2 + 0.05 (2)$
$= 2.1$

$y(0.05) \cong y_1 = 2.1$

<u>Iteration 2 :-</u>

$x_1 = 0.05$, $y_1 = 2.1$

$y_2 = y_1 + h f(x_0, y_0)$
$= 2.1 + 0.05 (y - x)\Big|_{\substack{x=0.05 \\ y=2.1}}$

$y_2 = 2.2025$

$y(0.1) \cong y_2 = 2.2025$

<u>Iteration 3 :-</u>

$x_2 = 0.1$, $y_2 = 2.2025$

$y_3 = y_2 + h f(x_2, y_2)$
$= 2.2025 + 0.05 (2.2025 - 0.1)$
$= 2.307625$

$y(0.15) \cong y_3 = 2.307625$

Iteration 4 :-

$x_3 = 0.15$ ; $y_3 = 2.307625$

$y_4 = y_3 + hf(x_3, y_3)$

$= 2.307625 + 0.05(2.307625 - 0.15)$

$y_4 = 2.41550625$

$y(0.2) \simeq 2.41550625$

After rounding up

$y(0.1) \simeq 2.20$

$y(0.2) \simeq 2.42$

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5) $y' = x - y^2 \quad y(0) = 1 \quad x \in [0, 0.6] \quad h = 0.2$

$\Rightarrow$ Forward Euler method

$y_{j+1} = y_j + h f(x_j, y_j)$

where $f(x_j, y_j) = y'\big|_{x_j, y_j}$

$y_0 = 1$ at $x = 0$, $h = 0.2$

$y_1 = y_0 + 0.2 f(x_0, y_0) = 1 + 0.2 (x - y^2)_{\substack{x=0 \\ y=1}}$

$y_1 = 0.8$

$\boxed{y(0.2) \cong y_1 = 0.8}$

$y_2 = y_1 + 0.2 f(x_1, y_1) = 0.8 + 0.2 (x - y^2)\big|_{\substack{x=0.2 \\ y=0.8}}$

$y_2 = 0.712$

$\boxed{y(0.4) \cong y_2 = 0.712}$

$y_3 = y_2 + 0.2 f(x_2, y_2) = 0.712 + 0.2 (x - y^2)\big|_{\substack{x=0.4 \\ y=0.712}}$

$y_3 = 0.6906112$

$\boxed{y(0.6) \cong y_3 = 0.6906112}$

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5) $\frac{dy}{dx} = x + y^2$, $y(0) = 1$

$\Rightarrow$ <u>Backward Euler Method:</u>

$y_{j+1} = y_j + h f(x_{j+1}, y_{j+1})$

<u>Iteration 1 :-</u>

$x_0 = 0$, $y_0 = 1$

$y_1 = y_0 + 0.1 (0.1 + y_1^2)$

$y_1 = 1 + 0.01 + 0.1 y_1^2$

$y_1 = 1.14$

<u>Iteration 2</u>

$x_1 = 0.1$, $y_1 = 1.14$

$y_2 = y_1 + 0.1 (0.2 + y_2^2)$

$y_2 = 1.14 + 0.02 + 0.1 y_2^2$

$\boxed{y_2 = 1.339}$

$\therefore$ ~~y(0.2)~~

$y(0.2) \cong y_2 = 1.339$

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7) $\frac{dy}{dx} = \frac{x-y}{1+x}$ , $y(0) = 1$ , $h = 0.05$

$\Rightarrow$ Backward Euler Method.

$y_n = y_{n-1} + h\, f(x_n, y_n)$

Iteration 1 :- $y_0 = 1$, $x_0 = 0$, $x_1 = 0.05$

$y(0.05) \simeq y_1 = y_0 + h\, f(x_1, y_1)$

$y_1 = 1 + 0.05\left(\frac{0.05 - y_1}{1+0.05}\right)$

$y_1 = 1 + 0.00238095 - 0.047619\, y_1$

$y_1 = 0.95681818$

Iteration 2 :- $y_1 = 0.95681818$, $x_1 = 0.05$, $x_2 = 0.1$

$y(0.1) \simeq y_2 = y_1 + 0.05\, f(x_2, y_2)$

$y_2 = 0.95681818 + 0.05\left(\frac{0.1 - y_2}{1+0.1}\right)$

$y_2 = 0.95681818 + 0.004545454 - 0.04545454\, y_2$

$y_2 = 0.919565$

Or. $\boxed{y(0.1) \simeq 0.919565}$

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8) $\frac{dy}{dx} = x^2+y$ , $y = 0.94$ at $x=0$

$y = ?$ at $x=0.1$

$\Rightarrow$ $h = 0.1$

Modified Euler Method

$y_{j+1}^{(s+1)} = y_j + \frac{h}{2}\left[f(x_j, y_j) + f(x_{j+1}, y_{j+1}^{(s)})\right]$

~~Let~~ / ~~by~~ $y_1^0 = y_0 = 0.94$

$y_1^1 = y_0 + \frac{0.1}{2}\left[f(x_0, y_0) + f(x_1, y_1^0)\right]$

$y_1^1 = 0.94 + \frac{0.1}{2}\left[x_0^2 + y_0 + x_1^2 + (y_1^0)\right]$

$= 0.94 + \frac{0.1}{2}\left[0 + 0.94 + (0.1)^2 + 0.94\right]$

$\boxed{y_1^1 = 1.0345}$

$y_1^2 = y_0 + \frac{0.1}{2}\left[f(x_0, y_0) + f(x_1, y_1^1)\right]$

$= 0.94 + \frac{0.1}{2}\left[0 + 0.94 + (0.1)^2 + 1.0345\right]$

$\boxed{y_1^2 = 1.039225}$

$y_1^3 = 0.94 + \frac{0.1}{2}\left[0 + 0.94 + (0.1)^2 + 1.039225\right]$

~~$y_1^2 = 1.03946125$~~ $\boxed{y_1^3 = 1.03946125}$

$y_1^4 = 0.94 + \frac{0.1}{2}\left[0 + 0.94 + (0.1)^2 + 1.03946125\right]$

$\boxed{y_1^4 = 1.039473063}$

$y_1^4$ & $y_1^3$ have 5 significant figures common

$\therefore$ $y(x=0.1) \cong y_1 = 1.0395$

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9) $\frac{dy}{dx} = x + |\sqrt{y}|$ $\quad y(x=0) = 1, \; x \in [0, 0.6]$

$h = 0.2$

→ Modified Euler method

$$y_{j+1}^{(s+1)} = y_j + \frac{h}{2}\left[f(x_j, y_j) + f(x_{j+1}, y_{j+1}^{(s)})\right]$$

Let

$$y_1^0 = y_0 = 1$$

$$y_1^1 = y_0 + \frac{0.2}{2}\left(0 + |\sqrt{1}| + 0.2 + |\sqrt{1}|\right) = 1.22$$

$$y_1^2 = 1 + \frac{0.2}{2}\left(0 + |\sqrt{1}| + 0.2 + \sqrt{1.22}\right) = 1.23045361$$

$$y_1^3 = 1 + \frac{0.2}{2}\left(0 + \sqrt{1} + 0.2 + \sqrt{1.23045361}\right) = 1.230925$$

$$\boxed{y(x=0.2) \approx y_1 = 1.230}$$

$$y_2^0 = y_1 = 1.23$$

$$y_2^1 = y_1 + \frac{0.2}{2}\left(0.2 + \sqrt{1.23} + 0.4 + \sqrt{1.23}\right) = 1.51181073$$

$$y_2^2 = 1.23 + \frac{0.2}{2}\left(0.2 + \sqrt{1.23} + 0.4 + \sqrt{1.51181073}\right) = 1.523861078$$

$$y_2^3 = 1.23 + \frac{0.2}{2}\left(0.2 + \sqrt{1.23} + 0.4 + \sqrt{1.523861078}\right) = 1.524350133$$

$$y_2^4 = 1.23 + \frac{0.2}{2}\left(0.2 + \sqrt{1.23} + 0.4 + \sqrt{1.524350133}\right) = 1.52436996$$

$$\boxed{y(x=0.4) \cong y_2 = 1.524}$$

$$y_3^0 = y_2 = 1.524$$

$$y_3^1 = y_2 + \frac{0.2}{2}\left(0.4 + \sqrt{1.524} + 0.6 + \sqrt{1.524}\right) = 1.87090079$$

$$y_3^2 = 1.524 + \frac{0.2}{2}\left(0.4 + 0.6 + \sqrt{1.524} + \sqrt{1.87090079}\right) = 1.88423127$$

$$y_3^3 = 1.524 + \frac{0.2}{2}\left(0.4 + 0.6 + \sqrt{1.524} + \sqrt{1.88423127}\right) = 1.884717699$$

$$\boxed{y(x=0.6) \cong y_3 = 1.884}$$

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10) $y' = xy \quad , \quad y(1) = 1 \quad , \quad x \in [1, 1.4] \quad , \quad h = 0.2$

→ $2^{nd}$ Order Runge Kutta Method.

$$y_{j+1} = y_j + \frac{h}{2}[k_1 + k_2] \; ; \quad k_1 = f(x_j, y_j)$$
$$k_2 = f(x_j + h, y_j + hk_1)$$

$$y(1.2) \simeq y_1 = y_0 + \frac{1}{2}\left(0.2 \times xy\Big|_{\substack{x=1\\y=1}} + 0.2\; xy\Big|_{\substack{x=1+0.2\\y=1+0.2(1)}}\right)$$

$$\boxed{y(1.2) \simeq y_1 = 1.244}$$

$$y(1.4) \simeq y_2 = y_1 + \frac{0.2}{2}\left(xy\Big|_{\substack{x=1.2\\y=1.244}} + xy\Big|_{\substack{1.2+0.2\\1.244+0.2(1.2)(1.244)}}\right)$$

$$\boxed{y(1.4) \simeq y_2 = 1.6092384}$$

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11) $\frac{dy}{dx} = \frac{1}{x+y}$, $y(0)=1$, $x \in [0,2]$, $h=0.5$

$\Rightarrow$ Fourth Order Runge Kutta Method.

$$y_{j+1} = y_j + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$k_1 = f(x_j, y_j)$
$k_2 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2} k_1)$
$k_3 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2} k_2)$
$k_4 = f(x_j + h, y_j + h k_3)$

Sol/:-

Iteration 1 :-

$y_0 = y(0) = 1$, $y_1 = ?$ $h = 0.5$, $x_1 = 0.5$

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(x_0, y_0) = \left.\frac{1}{x+y}\right|_{\substack{x=0 \\ y=1}} = 1$$

$$k_2 = \left.\frac{1}{x+y}\right|_{\substack{x=0+\frac{0.5}{2} \\ y=1+\left(\frac{0.5}{2}\right)1}} = \frac{1}{\frac{1}{4}+1+\frac{1}{4}} = \frac{1}{\frac{1}{2}+1} = \frac{2}{3}$$

$$k_3 = \left.\frac{1}{x+y}\right|_{\substack{x=0+\frac{0.5}{2} \\ y=1+\frac{0.5}{2}\times\frac{2}{3}}} = \frac{1}{\frac{1}{4}+1+\frac{1}{6}} = \frac{12}{17}$$

$$k_4 = \left.\frac{1}{x+y}\right|_{\substack{x=0+0.5 \\ y=1+0.5\times\frac{12}{17}}} = \frac{1}{\frac{1}{2}+1+\frac{6}{17}} = \frac{34}{63}$$

$$y_1 = 1 + \frac{0.5}{6}\left(1 + \frac{4}{3} + \frac{24}{17} + \frac{34}{63}\right) = 1.357065048$$

$$\boxed{y(x=0.5) \cong y_1 = 1.357065048}$$

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**Iteration 2 :-**

$y_1 = y(0.5) = 1.3570650 48$

$y_2 = ?$ $\quad h = 0.5$ $\quad x_2 = 1$

$$y_2 = y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\cancel{k_1 = f(x_1, y_1)} = \frac{1}{x+y}\Bigg|_{\substack{x=0.5\\ y=y_1}} = \frac{1}{\frac{0.5}{2} + 1.357065048} = 0.5384840995 \quad \cancel{= 0.4242564289}$$

$$k_2 = \frac{1}{x+y}\Bigg|_{\substack{x = 0.5 + 0.5/2 \\ y = y_1 + \frac{1}{4}\times 0.5384840995}} = \frac{1}{\frac{1}{2} + \frac{1}{4} + y_1 + \frac{1}{4} k_1}$$

$= 0.4460927924$

$$k_3 = \frac{1}{x+y}\Bigg|_{\substack{x = 0.5 + 0.5/2 \\ y = y_1 + \frac{1}{4}k_2}} = \frac{1}{\frac{1}{2} + \frac{1}{4} + y_1 + \frac{1}{4}k_2} = 0.4507370855$$

$$k_4 = \frac{1}{x+y}\Bigg|_{\substack{x = 0.5 + 0.5 \\ y = y_1 + \frac{1}{2}k_3}} = \frac{1}{1 + y_1 + \frac{k_3}{2}} = 0.3872316421$$

$$y_2 = 1.357065048 + \frac{0.5}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\boxed{y(x=1) \simeq y_2 = 1.583679673}$$

**Iteration 3 :-**

$y_2 = y(1) = 1.583679673$

$y_3 = ?$ $\quad h = 0.5$ $\quad x_3 = 1.5$

$$y_3 = y_2 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = \frac{1}{x+y}\Bigg|_{\substack{x=1\\ y=y_2}} = \frac{1}{1+y_2} = 0.3870448843$$

$$k_2 = \frac{1}{x+y}\Bigg|_{\substack{x = 1 + \frac{1}{4} \\ y = y_2 + \frac{1}{4}k_1}} = \frac{1}{1 + \frac{1}{4} + y_2 + \frac{1}{4}k_1} = 0.3412455791$$

$= (2.918991068)$

$$k_3 = \frac{1}{x+y}\Bigg|_{\substack{x = 1 + \frac{1}{4} \\ y = y_2 + \frac{1}{4}k_2}} = \frac{1}{1 + \frac{1}{4} + y_2 + \frac{k_2}{4}} = 0.3425841247$$

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$$k_4 = \left.\frac{1}{x+y}\right|_{\substack{x=1+\frac{1}{2}\\ y=y_2+\frac{k_3}{2}}} = \frac{1}{1+\frac{1}{2}+y_2+\frac{k_3}{2}} = 0.3072223298$$

$$y_3 = y_2 + \frac{h}{6}\left(k_1+2k_2+2k_3+k_4\right)$$

$$\boxed{y(1.5) \cong y_3 = 1.755506891}$$

Iteration # 4

$y_3 = y(x=1.5) = 1.755506891$

$y_4 = ?$ $h=0.5$ $x_4 = 2.$

$$y_4 = y_3 + \frac{h}{6}\left(k_1+2k_2+2k_3+k_4\right)$$

$$k_1 = \left.\frac{1}{x+y}\right|_{\substack{x=1.5\\ y=y_3}} = \frac{1}{1.5+y_3} = 0.3071718271$$

$$k_2 = \left.\frac{1}{x+y}\right|_{\substack{x=1.5+\frac{1}{4}\\ y=y_3+\frac{1}{4}k_1}} = \frac{1}{1.5+\frac{1}{4}+y_3+\frac{k_1}{4}} = 0.2791502784$$

$$k_3 = \left.\frac{1}{x+y}\right|_{\substack{x=1.5+\frac{1}{4}\\ y=y_3+\frac{k_2}{4}}} = \frac{1}{1.5+\frac{1}{4}+y_3+\frac{k_2}{4}} = 0.2796972447$$

$$k_4 = \left.\frac{1}{x+y}\right|_{\substack{x=1.5+\frac{1}{2}\\ y=y_3+\frac{k_3}{2}}} = \frac{1}{1.5+\frac{1}{2}+y_3+\frac{k_3}{2}} = 0.2567159778$$

$$y_4 = y_3 + \frac{h}{6}\left(k_1+2k_2+2k_3+k_4\right)$$

$$\boxed{y(2) \cong y_4 = 1.895638795}$$

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12) $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ , $h=0.1$ , $y(0)=1$ , $y(0.2)=?$
$y(0.4)=?$

→ Fourth order Runge-kutta method

$$y_{j+1} = y_j + \frac{h}{6}(k_1+2k_2+2k_3+k_4)$$

$$k_1 = f(x_j, y_j)$$
$$k_2 = f(x_j+\frac{h}{2}, y_j+\frac{h}{2}k_1)$$
$$k_3 = f(x_j+\frac{h}{2}, y_j+\frac{h}{2}k_2)$$
$$k_4 = f(x_j+h, y_j+h k_3)$$

Iteration 1

$y_0 = y(0) = 1$ , $y_1 = ?$ , $h=0.1$ , $x_1 = 0.1$

$$y_1 = y_0 + \frac{h}{6}(k_1+2k_2+2k_3+k_4)$$

$$k_1 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{y=1\\x=0}} = 1$$

$$k_2 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{y=1+\frac{0.1}{2}(1)\\x=0+\frac{0.1}{2}}} = 0.995475113$$

$$k_3 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{x=0+\frac{0.1}{2}\\y=1+\frac{0.1}{2}k_2}} = 0.99547113$$

$$k_4 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{x=0+0.1\\y=1+0.1k_3}} = 0.98359367$$

$$y_1 = 1+\frac{0.1}{6}(k_1+2k_2+2k_3+k_4)$$

$$\boxed{y(0.1) \simeq y_1 = 1.09942483}$$

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Iteration 2

$y_1 = y(0.1) = 1.09942483$ , $y_2 = ?$ , $h = 0.1$ , $x_2 = 0.2$

$$y_2 = y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = \left.\frac{y^2 - x^2}{y^2 + x^2}\right|_{\substack{x = 0.1 \\ y = y_1}} = 0.983589541$$

$$k_2 = \left.\frac{y^2 - x^2}{y^2 + x^2}\right|_{\substack{x = 0.1 + \frac{0.1}{2} \\ y = y_1 + \frac{0.1}{2}k_1}} = 0.966462757$$

$$k_3 = \left.\frac{y^2 - x^2}{y^2 + x^2}\right|_{\substack{x = 0.1 + 0.1/2 \\ y = y_1 + 0.1/2\, k_2}} = 0.966413534$$

$$k_4 = \left.\frac{y^2 - x^2}{y^2 + x^2}\right|_{\substack{x = 0.1 + 0.1 \\ y = y_1 + 0.1k_3}} = 0.945599485$$

$$y_2 = y_1 + \frac{0.1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y(0.2) \simeq y_2 = 1.19600719$$

$$\boxed{y(0.2) = 1.19600719}$$

Iteration 3

$y_2 = y(0.2) = 1.19600719$ , $y_3 = ?$ , $h = 0.1$ , $x_3 = 0.3$

$$y_3 = y_2 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = \left.\frac{y^2 - x^2}{y^2 + x^2}\right|_{\substack{x = 0.2 \\ y = y_2}} = 0.945594264$$

$$k_2 = \left.\frac{y^2 - x^2}{y^2 + x^2}\right|_{\substack{x = 0.2 + \frac{0.1}{2} \\ y = y_2 + \frac{0.1}{2}k_1}} = 0.922276361$$

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$$k_3 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{x=0.2+0.1/2 \\ y=y_2+\frac{0.1}{2}k_2}} = 0.922136068$$

$$k_4 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{x=0.2+0.1 \\ y=y_2+0.1k_3}} = 0.897114216$$

$$y_3 = y_2 + \frac{0.1}{6}(k_1+2k_2+2k_3+k_4)$$

$$\boxed{y(0.3) \simeq y_3 = 1.28819941}$$

Iteration 4

$y_3 = y(0.3) = 1.28819941$, $y_4 = ?$, $h = 0.1$, $x_4 = 0.4$

$$y_4 = y_3 + \frac{h}{6}(k_1+2k_2+2k_3+k_4)$$

$$k_1 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{x=0.3 \\ y=y_3}} = 0.897109$$

$$k_2 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{x=0.3+0.1/2 \\ y=y_3+0.1/2\,k_1}} = 0.87021102$$

$$k_3 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{x=0.3+0.1/2 \\ y=y_3+0.1/2\,k_2}} = 0.870784633$$

$$k_4 = \left.\frac{y^2-x^2}{y^2+x^2}\right|_{\substack{x=0.3+0.1 \\ y=y_3+0.1k_3}} = 0.844008119$$

$$y_4 = y_3 + \frac{0.1}{6}(k_1+2k_2+2k_3+k_4)$$

$$\boxed{y(0.4) \simeq y_4 = 1.37527825}$$

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13) $\frac{dy}{dx} = x^2+y^2$, $y(0.2) = ?$ $y(0.4) = ?$
$y(0) = 1$, $h=0.2$

⇒ Implicit Runge-kutta method of order 4.

$$y_{j+1} = y_j + \frac{h}{2}(k_1+k_2)$$

$$k_1 = f\left(x_j + \frac{3-\sqrt{3}}{6}h,\ y_j + \frac{h}{4}k_1 + \frac{3-2\sqrt{3}}{12}hk_2\right)$$

$$k_2 = f\left(x_j + \frac{3+\sqrt{3}}{6}h,\ y_j + \frac{3+2\sqrt{3}}{12}hk_1 + \frac{h}{4}k_2\right)$$

Iteration 1.

$y_0 = y(0) = 1$, $y_1 = y(0.2) = ?$ ; $x_1 = 0.2$, $x_0 = 0$, $h = 0.2$

$$y_1 = y_0 + \frac{h}{2}(k_1+k_2)$$

$$k_1 = x^2+y^2 \Big|_{\substack{x = 0 + \frac{3-\sqrt{3}}{6}(0.2) \\ y = 1 + \frac{0.2}{4}k_1 + \frac{3-2\sqrt{3}}{12}\times 0.2 k_2}}$$

$$k_1 = 0.00178632795 + (1+0.05k_1 - 0.0077350269k_2)^2$$

$$k_2 = x^2+y^2 \Big|_{\substack{x = 0 + \frac{3+\sqrt{3}}{6}(0.2) \\ y = 1 + \frac{3+\sqrt{3}}{12}(0.2)k_1 + \frac{0.2k_2}{4}}}$$

$$k_2 = 0.02488033 + (1+0.0788675k_1 + 0.05k_2)^2$$

$k_1 = 1.092$
$k_2 = 1.356$

$$y(0.2) \simeq y_1 = 1 + \frac{0.2}{2}(1.092+1.356) = 1.2448$$

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Iteration 2

$y_1 = y(0.2) = 1.2448$ , $y_2 = y(0.4) = ?$ , $x_2 = 0.4$ , $x_1 = 0.2$

$h = 0.2$

$y_2 = y_1 + \frac{h}{2}(k_1 + k_2)$

$k_1 = x^2 + y^2 \Big|_{\substack{x = 0.2 + \frac{3-\sqrt{3}}{6}(0.2) \\ y = 1.2448 + \frac{0.2}{4} k_1 + \frac{3-2\sqrt{3}}{12} k_2 \times 0.2}}$

$k_1 = 0.0586923171 8 + (1.2448 + 0.05 k_1 - 0.0077350269 k_2)^2$

$k_2 = x^2 + y^2 \Big|_{x = 0.2 + \frac{3+\sqrt{3}}{6}(0.2)}$

$k_2 = 0.1279743495 + (1.2448 + 0.0788675 k_1 + 0.05 k_2)^2$

$k_1 = 1.78994$

$k_2 = 2.39519$

$y(0.4) \cong y_2 = 1.2448 + \frac{0.2}{2}(1.78994 + 2.39519)$

$\boxed{y(0.4) \cong 1.66313}$

$\boxed{y(0.2) \cong 1.2448}$