Lecture 22

<u>Def:-</u> Let f be a bounded funtion that is supported on a measurable set of finite measure. Then the <u>Lebesgue integral</u> of f is defined as

$$\int f(x)\,dx := \lim_{n\to\infty} \int \varphi_n(x)\,dx,$$

where $\{\varphi_n\}$ is any sequence of simple functions satisfying: $|\varphi_n(x)| \leq M \ \forall x$ & each φ_n is supported on the $\text{supp}(f)$ & $\varphi_n(x) \to f(x)$ a.e as $n \to \infty$.

By above Lemma, the limit exists.

<u>Claim:-</u> $\int f$ is independent of the limiting sequence $\{\varphi_n\}$ used, in order for the integral to be well defined.

proof the claim:-

Let $\{\psi_n\}$ be another sequence of simple functions that is bounded by $M'>0$ supported on supp(f) such that

$\psi_n(x) \to f(x)$ a.e as $n\to\infty$.

Let $\eta_n := \varphi_n - \psi_n \quad \forall n\geq 1$.

Then $\{\eta_n\}$ is a sequence of simple functions bounded by $M+M'$ $\left(\because |\eta_n| \leq |\varphi_n| + |\psi_n| \leq M+M'\right)$

supported on a set of finite measure &

such that $\eta_n \longrightarrow 0$ a.e as $n\to\infty$

$\therefore$ By above Lemma (ii), we have

$$\int \eta_n \longrightarrow 0 \quad \text{as } n\to\infty$$

$$\Rightarrow \lim_{n\to\infty} \int \eta_n = 0$$

$$\Rightarrow \lim_{n\to\infty} \int (\varphi_n - \psi_n) = 0$$

$$\Rightarrow \lim_{n\to\infty}\left(\int \varphi_n - \int \psi_n\right) = 0.$$

$$\Rightarrow \lim_{n\to\infty} \int \varphi_n = \lim_{n\to\infty} \int \psi_n$$

as required.

<u>Def</u>:- Let $E \subseteq \mathbb{R}^d$ be a measurable set of finite measure. Let f be a bounded measurable function with $m(\text{supp}(f)) < \infty$. Then

$$\int_E f(x)\,dx := \int f(x)\chi_E(x)\,dx.$$

<u>Proposition</u>:- Suppose f, g are measurable bounded functions supported on sets of finite measure. Then the following properties hold:

(i) For $a, b \in \mathbb{R}$, $\int (af+bg) = a\int f + b\int g$ (linearity).

(ii) (Additive) Let $E, F \subseteq \mathbb{R}^d$ be disjoint measurable sets. Then

$$\int_{E \cup F} f = \int_E f + \int_F f.$$

(iii) (monotonicity) If $f \le g$, then $\int f \le \int g$.

(iv) (triangular inequality) If $|f|$ is bounded supported on a set of finite measure, then

$$\left| \int f \right| \le \int |f|.$$

<u>proof</u>:- We have $\int f = \lim_{n\to\infty} \int \varphi_n$,

$$\int g = \lim_{n\to\infty} \int \psi_n$$

where φ_n, ψ_n are simple functions etc...

EXERCISE.

<u>Def</u>:- We say that a bounded measurable function f supported on a set E of finite measure, is <u>Lebesgue integrable</u> (L-integrable)

if $\int_E f$ is finite.

Recall :- $\int_a^b f(x)\,dx$ Riemann integral of f where f is bounded on $[a,b]$.

Now we have the concept of Lebesgue integral of a bounded function on $[a,b]$, $\int_a^b f$.

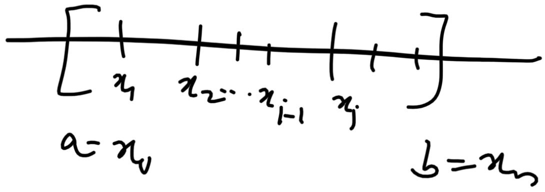
Qn:- What is the relation between $\int_a^b f(x)\,dx$ & $\int_a^b f$?

Def:- Let $f : [a,b] \to \mathbb{R}$ be a bounded function. That is, there exists $M > 0$ such that

$$|f(x)| \leq M \quad \forall x \in [a,b].$$

For a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$, define the following:

The upper Riemann sum of f with respect to the partition P



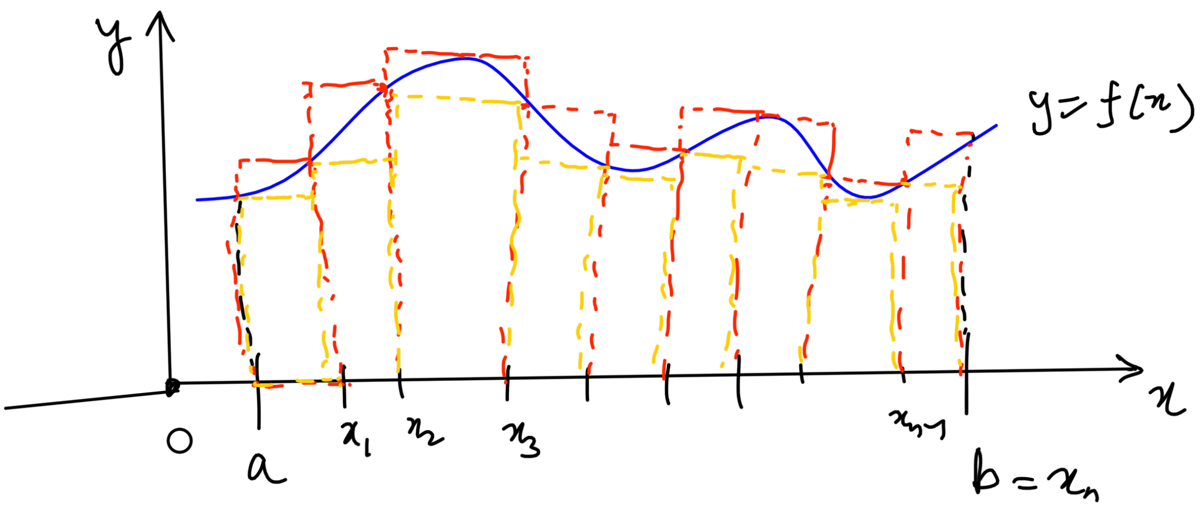
as $U(P, f) := \sum_{i=1}^{n} M_i (x_i - x_{i-1})$, where

$$M_i = \sup_{x \in [x_{i-1}, x_i]} (f(x))$$

& the Lower Riemann sum of f w.r.t P

as $$L(P,f) := \sum_{i=1}^{n} m_i (x_i - x_{i-1}),$$ where

$$m_i = \inf_{x \in [x_{i-1}, x_i]} (f(x))$$



$U(P,f)$ = the sum of areas of the red rectangles

$L(P,f)$ = the sum of the areas of the yellow rectangles.

The upper Riemann integral of f is defined as

$$\overline{\int_a^b} f := \inf_{P \in \mathcal{P}} (U(P,f)),$$ where

$\mathcal{P}$ = the collection of all possible partitions of $[a,b]$

The <u>Lower Riemann integral</u> of f is defined

as $$\underline{\int_a^b} f := \sup_{P \in \mathcal{P}} \left(L(P, f) \right).$$

<u>Def:-</u> We say that f is <u>Riemann integrable</u> (R-integrable) if $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$.