Lecture 6

Proposition:– Suppose that in the definition of outermeasure, $m^*(E) := \inf\left(\left\{\sum_n \ell(I_n) \,\middle|\, E \subseteq \bigcup_n I_n, \{I_n\} \text{ collection of left closed right open intervals}\right\}\right)$

for $E \subseteq \mathbb{R}$, we stipulate

(i) I_n's open, i.e. $I_n = (a_n, b_n)$

(ii) $I_n = [a_n, b_n)$ $\forall n$

(iii) $I_n = (a_n, b_n]$, $\forall n$

(iv) $I_n = [a_n, b_n]$ $\forall n$

(v) mixture of all above four, for different n, of the various types of intervals,

then the same m^* is obtained.

proof:– In case (ii), we obtain m^* as definition.

In case (i), denote

$$m_0^*(E) = \inf\left(\left\{\sum_n \ell(I_n) \,\middle|\, E \subseteq \bigcup I_n, \text{ where } I_n\text{'s are open intervals}\right\}\right)$$

Illy $m^*_{oc}(E)$ in Case (iii),

$m^*_c(E)$ in Case (iv)

& $m^*_m(E)$ in (v).

We show that $m^*(E) = m^*_o(E) = m^*_{oc}(E) = m^*_c(E)$

$= m^*_m(E)$.

To show: $m^*_o(E) = m^*_m(E)$.

From definition, $m^*_m(E) \leq m^*_o(E)$.

Let $\varepsilon > 0$. For each I_n any interval, let

I_n' be an open interval such that

$I_n \subseteq I_n'$ & $l(I_n') = (1+\varepsilon)\, l(I_n)$

(If $I_n = [a_n, b_n]$ or $[a_n, b_n)$ or $(a_n, b_n]$ or (a_n, b_n),

choose $I_n' = (a_n', b_n')$, where

$$a_n' = a_n - \frac{(b_n - a_n)}{2}\varepsilon$$

$$b_n' = b_n + \frac{(b_n - a_n)}{2}\varepsilon. \;)$$

Suppose for $\{I_n\}$ such that $E \subseteq \bigcup_n I_n$

& $m^*_m(E) + \varepsilon \geq \sum_n l(I_n)$. $\longrightarrow$ (*)

We know $I_n \subseteq I_n' \ \forall n$

$$\Rightarrow E \subseteq \bigcup_n I_n \subseteq \bigcup_n I_n'$$

$$\therefore E \subseteq \bigcup_n I_n'$$

$$\therefore m_0^*(E) \leq \sum_n l(I_n') \quad \left(\text{by def. of } m_0^*\right)$$

$$= \sum_n (1+\varepsilon)\, l(I_n)$$

$$= (1+\varepsilon) \sum_n l(I_n)$$

$$\leq (1+\varepsilon)\left(m_m^*(E)+\varepsilon\right)$$

$$= m_m^*(E) + \varepsilon\left(m_m^*(E)+\varepsilon\right)$$

true for any $\varepsilon > 0$.

$$\therefore m_0^*(E) \leq m_m^*(E).$$

$$\therefore m_0^*(E) = m_m^*(E).$$

Remaining part: EXERCISE.

Definition:- Let $E \subseteq \mathbb{R}$. We say that E is <u>Lebesgue measurable</u> or simply <u>measurable</u>, if for each $A \subseteq \mathbb{R}$, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Remark: Since m^* is subadditive & $A = (A \cap E) \cup (A \cap E^c)$

$$\therefore \quad m^*(A) \le m^*(A \cap E) + m^*(A \cap E^c).$$

Thus to show a subset $E \subseteq \mathbb{R}$ is measurable it is enough to show

$$m^*(A) \ge m^*(A \cap E) + m^*(A \cap E^c).$$

Proposition:- Let $E \subseteq \mathbb{R}$. Suppose $m^*(E) = 0$. Then E is measurable.

proof:- Given $m^*(E) = 0$.

<u>To show</u>: E is measurable.

i.e., to show: for any $A \subseteq \mathbb{R}$,

$$m^*(A) \ge m^*(A \cap E) + m^*(A \cap E^c).$$

Let $A \subseteq \mathbb{R}$. We know

$$m^*(A \cap E) \leq m^*(E) = 0 \quad (\because A \cap E \subseteq E)$$

$$\Rightarrow m^*(A \cap E) = 0.$$

& we have $A \cap E^c \subseteq A$

$$\Rightarrow m^*(A \cap E^c) \leq m^*(A)$$

$$\Rightarrow \underbrace{m^*(A \cap E)}_{\text{is } 0} + m^*(A \cap E^c) \leq m^*(A)$$

$\therefore$ E is measurable.

<u>Example</u>:- ① $E = \{x\} \subseteq \mathbb{R}$ is measurable.

$(\because m^*(E) = 0)$

② Any finite subset of $\mathbb{R}$ is measurable.

Let $E = \{x_1, \dots, x_n\} \subseteq \mathbb{R}$

$$E = \bigcup_{i=1}^{n} \{x_i\}$$

$$m^*(E) = m^*\left(\bigcup_{i=1}^{n} \{x_i\}\right) \leq \sum_{i=1}^{n} m^*(\{x_i\})$$

$$= \sum_{i=1}^{n} 0$$

$$= 0.$$

$\therefore m^*(E) = 0.$

$\therefore$ E is measurable.

③ $\mathbb{Q} \subseteq \mathbb{R}$, $m^*(\mathbb{Q}) = ?$

Let $\mathbb{Q} = \{q_1, q_2, \cdots\cdots\}$.

$$= \bigcup_{n=1}^{\infty} \{q_n\}$$

$$m^*(\mathbb{Q}) = m^*\left(\bigcup_{n=1}^{\infty} \{q_n\}\right)$$

$$\leq \sum_{n=1}^{\infty} m^*(\{q_n\})$$

$$= \sum_{n=1}^{\infty} 0 = 0$$

$\therefore m^*(\mathbb{Q}) = 0$.

$\therefore$ $\mathbb{Q}$ is measurable.

In fact, any countable subset of $\mathbb{R}$ is measurable.

Qn:- What are all the measurable subsets of $\mathbb{R}$?

Definition:- A class of subsets $\mathcal{F}$ of an arbitrary space X is said to be a σ-algebra or a σ-field if it satisfies the following conditions :

(i) $X \in \mathcal{Y}$

(ii) If $A \in \mathcal{Y}$, then $A^c \in \mathcal{Y}$

& (iii) If $E_i \in \mathcal{Y}$, $\forall i \geq 1$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{Y}$.

Let $\mathcal{M}$:= the class of all Lebesgue measurable subsets of $\mathbb{R}$.

<u>Theorem</u>:- $\mathcal{M}$ is a σ-algebra.

<u>proof</u>:- From the definition of Lebesgue measure, if $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$.

$$\Big[\because m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

$$= m^*(A \cap E^c) + m^*(A \cap (E^c)^c)$$

$$\Rightarrow E^c \text{ is measurable} \Big)$$

$\therefore$ (ii) is true.

Since $m^*(\phi) = 0$, $\phi \in \mathcal{M}$

$$\Rightarrow \phi^c = \mathbb{R} \in \mathcal{M}.$$

$\therefore$ (i) is true.

Remains to show: If $\{E_i\}_{i=1}^{\infty}$ is a sequence of measurable sets, then $\bigcup_{i=1}^{\infty} E_i$ is also measurable.

Let $A \subseteq \mathbb{R}$.

To show: $m^*(A) \geq m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) + m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)^c\right)$

Given E_1 is measurable,

$\Rightarrow m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c)$.

Now E_2 is measurable, & replace E_1 by E_2 & A by $A \cap E_1^c$ in the above equality, we get

$$m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$$

$$\therefore m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$$

Continuing in this way we get

$$m^*(A) = m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\left(A \cap E_i \cap \left(\bigcap_{j<i} E_j^c\right)\right)$$

$$+ \; m^*\Big(A \cap \underbrace{\left(\bigcap_{j=1}^{n} E_j^c\right)}_{= \left(\bigcup E_j\right)^c}\Big) \longrightarrow \S \quad \forall n \geq 1$$

$$\Rightarrow m^*(A) \geq m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\left(A \cap E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right)$$

$$+ m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right)$$

$$\left(\because \left(\bigcup_{j=1}^{\infty} E_j\right)^c \subseteq \left(\bigcup_{j=1}^{n} E_j\right)^c, \forall n\right).$$

$\forall\, n \geq 1.$

$$\Rightarrow m^*(A) \geq m^*(A \cap E_1) + \sum_{i=2}^{\infty} m^*\left(A \cap E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right)$$

$$+ m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right).$$

$$\Rightarrow m^*(A) \geq m^*(A \cap E_1) + m^*\left(\bigcup_{i=2}^{\infty}\left(A \cap E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right)\right)$$

$$+ m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right).$$

(by subadditive property of m^*)

Now $\bigcup_{i=2}^{n}\left(E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) = \bigcup_{i=2}^{n} E_i$ (check it!) ??

$\forall\, n \geq 2$

$\therefore m^*(A) \geq m^*(A \cap E_1) + m^*\left(A \cap \left(\bigcup_{i=2}^{\infty} E_i\right)\right)$

$$+ m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right).$$

$$\Rightarrow m^*(A) \geq m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) + m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_j\right)^c\right)$$

$\therefore \bigcup_{i=1}^{\infty} E_j$ is measurable

$\Rightarrow \bigcup_{i=1}^{\infty} E_j \in \mathcal{M}$.

$\therefore \mathcal{M}$ is a σ-algebra.
