Lecture 26

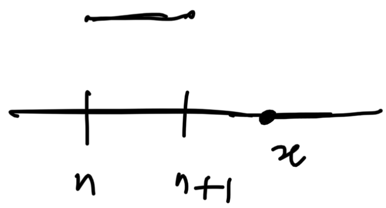
① Let $f_n(x) = \chi_{[n,n+1]}(x) \quad \forall x \in \mathbb{R}, \quad \forall n \geq 1.$

<u>pointwise Convergence:</u> $f_n \to 0$ p.w.

For $x \in \mathbb{R}$, choose $N \in \mathbb{N}$ such that $N > x$. (by Archimedian property)

Then for all $n \geq N$,

$$f_n(x) = \chi_{[n,n+1]}(x) = 0$$



Thus $f_n(x) = 0 \quad \forall n > N$

$\Rightarrow \quad f_n(x) \to 0$ as $n \to \infty$.

$\therefore \quad f_n \to 0$ p.w.

<u>$f_n \not\to 0$ uniformly:-</u>

$$\|f_n - 0\|_\infty \doteq \sup_{x \in \mathbb{R}} |f_n(x) - 0|$$

$$= \sup_{x \in \mathbb{R}} |f_n(x)|$$

$$= 1$$

$\therefore \; \|f_n - 0\| \not\to 0.$ as $n \to \infty$

$\Rightarrow \; f_n \not\to 0$ uniformly.

$\|f_n\|_\infty = 1$ $\forall n \geq 1$

$\|f_n\|_\infty \to 1$ as $n \to \infty$.

$f_n \overset{m}{\not\to} 0$ in measure:

Let $\varepsilon > 0$.

$$m\left(\{x \in \mathbb{R} \mid |f_n(x) - 0| \geq \varepsilon\}\right)$$

$$= m\left(\{x \in \mathbb{R} \mid \chi_n(x) \geq \varepsilon\}\right)$$

$$= m([n, n+1]) \qquad (\text{choose } \varepsilon < 1)$$

$$= 1 \not\to 0 \quad \text{as } n \to \infty.$$

$\therefore \; f_n \overset{m}{\not\to} 0$ as $n \to \infty$.

$f_n \not\to 0$ a.u on $\mathbb{R}$:

choose $\varepsilon = \frac{1}{2} > 0$.

Let $A \subseteq \mathbb{R}$ measurable such that $m(A) < \frac{1}{2} = \varepsilon$

Now for any $n \geq 1$,

$$m\left([n, n+1] \cap A\right) \leq m(A) < \varepsilon = \tfrac{1}{2}.$$

$$\Rightarrow \quad m\left([n, n+1] \setminus A\right) > \tfrac{1}{2}.$$

$-m(A) > -\frac{1}{2}$

$m\left([n, n+1] \setminus A\right)$

$= m([n, n+1]) - m(A)$

$= 1 - m(A)$

$> \frac{1}{2}.$

$$\Rightarrow \quad \underline{[n, n+1] \setminus A \neq \phi} \quad \forall n \geq 1$$

$\Rightarrow$ there exists $x \in [n, n+1] \setminus A$ such that

$$f_n(x) = \chi_{[n, n+1]}(x) = 1, \quad \text{for any } n \geq 1$$

For any $n \geq 1$,

$$\therefore \quad \| f_n - 0 \|_\infty = \sup_{x \in \mathbb{R}} |f_n(x)|$$

$$= 1 \quad \forall n \geq 1$$

& on $\underline{[n, n+1] \setminus A}$, $\forall \underline{n \geq 1}$.

$$\Rightarrow \quad \|f_n\|_\infty = 1 \quad \text{on } A^c = \mathbb{R} \setminus A. \quad \forall n \geq 1.$$

$$= \bigcup_{n=1}^{\infty} \left([n, n+1] \setminus A\right)$$

$$\Rightarrow \quad f_n \not\to 0 \text{ uniformly on } A^c$$

as $n \to \infty$.

② Let $f_n = \frac{1}{n}\chi_{[0,n]}$ $\forall n \geq 1$.

$f_n \to 0$ p.w. :

Let $x \in \mathbb{R}$

choose $N \in \mathbb{N}$ such that $N > x$.

Then for any $n \geq N$, $f_n(x) = \frac{1}{n}\chi_{[0,n]}(x)$

$= 0$

$\therefore$ $f_n(x) = 0$ $\forall n \geq N$.

$\Rightarrow \lim_{n\to\infty} f_n(x) = 0$.

$\Rightarrow f_n(x) \to 0$ as $n \to \infty$.

$\therefore$ $f_n \to 0$ p.w.

$f_n \to 0$ uniformly on $\mathbb{R}$:

$$\|f_n - 0\|_\infty = \sup_{x\in\mathbb{R}} |f_n(x)|$$

$$= \sup_{x\in\mathbb{R}} \left|\frac{1}{n}\chi_{[0,n]}(x)\right|$$

$$= \frac{1}{n}$$

$\therefore$ $\|f_n\|_\infty = \frac{1}{n} \to 0$ as $n \to \infty$.

$\therefore \quad f_n \to 0$ uniformly.

$f_n \xrightarrow{m} 0$: Let $\varepsilon > 0$.

$$m\left(\{x \in \mathbb{R} \mid |f_n(x) - 0| \geq \varepsilon\}\right)$$

$$= m\left(\{x \in \mathbb{R} \mid \frac{1}{n}\chi_{[0,n]}(x) \geq \varepsilon\}\right)$$

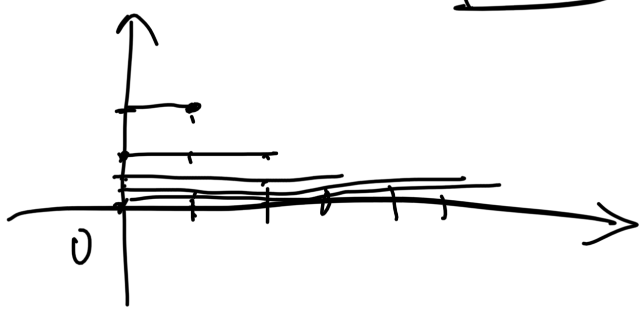
$\varepsilon > 0.$

$\boxed{\frac{1}{n} \geq \varepsilon}$

As $n \to \infty$

$$\{x \in \mathbb{R} \mid \frac{1}{n}\chi_{[0,n]}(x) \geq \varepsilon\}$$

$$= \phi \quad \cancel{\text{or } \{0\}}$$



As $n \to \infty$,

$$m\left(\{x \in \mathbb{R} \mid |f_n(x)| \geq \varepsilon\}\right) = 0.$$

$\therefore f_n \xrightarrow{m} 0$ as $n \to \infty$.

$f_n \to 0$ a.u.: choose $A_\varepsilon = \phi$.

True.

Qn:- Suppose $f, f_n : E \to \mathbb{R}$ is a bounded measurable functions & $m(E) < \infty$. Suppose $f_n \to f$ p.w on E.

When does $\displaystyle\int_E f_n \to \int_E f$ as $n \to \infty$?

That is, $$\lim_{n\to\infty} \int_E^{\mathcal{L}} f_n = \int_E^{\mathcal{L}} \left(\lim_{n\to\infty} f_n\right) \quad ?$$

Example:- (1) $f_n(x) = \chi_{[n,n+1]}(x) \quad \forall x \in \mathbb{R}, \ \forall n \geq 1.$

we have $f_n \to 0$ p.w.

$$\int^{\mathcal{L}} f_n = \int^{\mathcal{L}} \chi_{[n,n+1]}$$

$$= m([n,n+1])$$

$$= 1 \qquad \forall n \geq 1$$

$$\int^{\mathcal{L}} f = \int^{\mathcal{L}} 0 = 0$$

$$\therefore \int^{\mathcal{L}} f_n \not\to \int f \quad \text{as } n \to \infty.$$

(2) $f_n = \frac{1}{n} \chi_{[0,n]}$, $f_n \to 0$ p.w.

$$\int^{\mathcal{L}} f_n = \int^{\mathcal{L}} \frac{1}{n} \chi_{[0,n]} = \frac{1}{n} m([0,n])$$

$$= \frac{1}{n}(n) = 1$$

$$\int^{\mathcal{L}} f = 0.$$

$$\therefore \int f_n \not\to \int f = 0 \text{ as } n \to \infty$$

Theorem (Bounded Convergence Theorem) :-

Suppose that $\{f_n\}$ is a sequence of measurable functions that are all bounded by $M>0$. & are supported on a set E of finite measure.

Suppose $f_n(x) \to f(x)$ $_{pw}$ a.e on E, as $n \to \infty$.

Then f is measurable, bounded & supported on E for a.e x, & $\int_E |f_n - f| \to 0$ as $n \to \infty$.

Consequently, $\int_E f_n \to \int f$ as $n \to \infty$.