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Modern Algebra : Assignment

Keerti P. Charantimath 19MA20059

1.i. let $a = 82 + 67i$ and $I = (a)$

Then $a \in \mathbb{Z}[i]$

also $N(a) = 82^2 + 67^2 = 11213$ which is a prime number

This means that a is an irreducible element of $\mathbb{Z}[i]$

since $\mathbb{Z}[i]$ is a principal ideal domain (PID), I is a maximal ided in ~~Z~~ $\mathbb{Z}[i]$

Thus, I is a ~~f~~ maximal ideal in $\mathbb{Z}[i]$

Also, $11213 \in I$ as $11213 = (82 + 67i)(82 - 67i) = ab$,

where $a, b \in \mathbb{Z}[i]$

1.ii. let α be an irreducible element of $\mathbb{Z}[i]$ and it divides 11213. This means $11213 \in (\alpha)$. Suppose $\varepsilon \in U(\mathbb{Z}[i]) = \{\pm 1, \pm i\}$

This means $(\varepsilon\alpha) = (\alpha)$ & thus $11213 \in (\varepsilon\alpha)$. Also $\varepsilon\alpha$ divides 11213. We already know that $\varepsilon\alpha$ is an irreducible element of $\mathbb{Z}[i]$

Let β & γ be a & b from part (i) respectively.

Thus, $\beta = 82 + 67i$, $\gamma = 82 - 67i$

Now, $N(\beta) = N(\gamma) = 11213 \Rightarrow \beta$ & γ are irreducible elements of $\mathbb{Z}[i]$

As $\beta\gamma = 11213$, it follows that the following elements are irreducible elements which divide 11213:

$$S = \{\pm\beta, \pm i\beta, \pm\gamma, \pm i\gamma\}$$

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suppose α is an irreducible element in $\mathbb{Z}[i]$ which divides 11213. $\Rightarrow$ α divides $\beta\gamma$.
As $\mathbb{Z}[i]$ is a PID, we know α is prime element in $\mathbb{Z}[i]$.
$\Rightarrow$ α divides β or α divides γ.
Let us suppose α divides β, then $\beta = \alpha\delta$ & $\delta \in \mathbb{Z}[i]$.
As β is an irreducible element of $\mathbb{Z}[i]$ & α is not a unit in that ring, δ must be a unit in $\mathbb{Z}[i]$.
Let $\varepsilon = \delta^{-1}$ which is also a unit in the ring.
Then $\alpha = \varepsilon\beta$ an element in the set S.
Similarly, if α divides γ, $\alpha = \varepsilon\gamma$ where ε is unit in $\mathbb{Z}[i]$ and thus α is ~~th~~ an element in set S.

Thus, S is the set of irreducible elements in $\mathbb{Z}[i]$ which divide 11213.

1.iii) As $I = (\beta)$ & $N(\beta) = p$, where $p = 11213$ (p is prime), the quotient ring $S = \mathbb{Z}[i]/I$ is a ring with p elements. As I is maximal ideal, S is also a field.

Thus, S is an integral domain.
We know that every integral domain either contains a subring isomorphic to $\mathbb{Z}$ or a subring isomorphic to $\mathbb{Z}/q\mathbb{Z}$ where q is some prime number.

As S is finite, every subring is finite. Thus, S will contain a subring isomorphic to $\mathbb{Z}/q\mathbb{Z}$ (q is prime).

The additive group of S has p elements, ~~to~~ hence q must divide p. As q & p are both prime, $q = p$.
Thus the ~~sub~~ subring coincides with S.
Hence, S is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ where $p = 11213$

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2) Any element of $\mathbb{Z}[\sqrt{-5}]$ is $a+b\sqrt{-5}$, $a, b \in \mathbb{Z}$
Also, $N(a+b\sqrt{-5}) = (a+b\sqrt{-5})(a-b\sqrt{-5}) = a^2 - b^2(-5) = a^2+5b^2$

Consider, $a=2$, $b=1$
norm $= (2+\sqrt{-5})(2-\sqrt{-5}) = 9 = 3 \cdot 3$

The claim is that $2+\sqrt{-5}$, $2-\sqrt{-5}$ & 3 are irreducible elements in the ring $\mathbb{Z}[\sqrt{-5}]$

We need to prove that any element of norm 9 is irreducible.

let $\alpha, \beta, \gamma \in \mathbb{Z}[\sqrt{-5}]$ st $N(\alpha) = 9$ & $\alpha = \beta\gamma$.
We need to show either β or γ is a unit in order to prove α is irreducible.

Now, $N(\alpha) = 9 = N(\beta) N(\gamma)$
$N(\beta) = 1, 3, 9$ (as norms are non-ve)

Case 1 :- $N(\beta) = 1 \Rightarrow \beta$ is a unit
Case 2 :- $N(\beta) = 3 \Rightarrow \beta = u + v\sqrt{-5}$, $u, v \in \mathbb{Z}$ st $N(\beta) = u^2+5v^2 = 3$

No integers satisfy $u^2 + 5v^2 = 3$
Impossible case.

Case 3 :- $N(\beta) = 9 \Rightarrow N(\gamma) = 1 \Rightarrow \gamma$ is a unit.

Thus, $\alpha = \beta\gamma$ where β or γ is a unit.
Hence, $2 \pm \sqrt{-5}$, 3 are irreducible
The factorization of $9 = (2+\sqrt{-5})(2-\sqrt{-5}) = 3 \cdot 3$ is not unique
Thus, $\mathbb{Z}[\sqrt{-5}]$ is not a UFD

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$3 \in R = \mathbb{Z}[\sqrt{-5}]$

we know that $(2+\sqrt{-5})(2-\sqrt{-5}) = 3 \cdot 3$, But 3 divides neither $(2+\sqrt{-5})$ nor $(2-\sqrt{-5})$

$\Rightarrow$ 3 is not prime in $\mathbb{Z}[\sqrt{-5}]$

Thus 3 is an example of an element in R, which is irreducible but not prime

3) Intersection of two ideals in a ring R is also an ideal in R

$\Rightarrow (a) \cap (b)$ is an ideal in R

As R is a PID $\rightarrow (a) \cap (b) = (k) \quad k \in R$

Now, $k \in (k)$ & $(k) \subseteq (b) \Rightarrow k \in (b)$ & $b | k \in R \Rightarrow k = bc, c \in R$

and $k \in (k)$ & $(k) \subseteq (a) \Rightarrow k \in (a)$ & $a | k \in R \Rightarrow a | bc \in R$

a, b are co-primes & $a | bc \in R \Rightarrow a | c \in R \Rightarrow c = ad, d \in R$

Thus $k = bc = bad = dab \in (ab) \Rightarrow k \in (ab) \rightarrow (k) \subseteq (ab) \rightarrow (1)$

Also, $ab \in (a)$ & $ab \in (b) \Rightarrow (ab) \subseteq (a)$ & $(ab) \subseteq (b)$

$\Rightarrow (ab) \subseteq (a) \cap (b) \Rightarrow (ab) \subseteq (k) \longrightarrow (2)$

using (1) & (2), $(ab) = (k) \rightarrow (ab) = (a) \cap (b)$

Hence proved.

Let us consider

$\psi : R \longrightarrow R/(a) \times R/(b)$

$\psi(r) = (r + (a), r + (b)) \quad r \in R$

We need to prove: ψ is a surjective ring homomorphism

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let $r, s \in R$, then $\psi(r+s) = (r+s+(a),\ r+s+(b))$
$= (r+(a)+s+(a),\ r+(b)+s+(b))$
$= (r+(a), r+(b)) + (s+(a), s+(b))$

$\Rightarrow \psi(r+s) = \psi(r) + \psi(s) \longrightarrow (1)$

Similarly $\psi(rs) = (rs+(a),\ rs+(b)) = ((r+(a))(s+(a)),$
$(r+(b))(s+(b)))$
$= (r+(a), r+(b))(s+(a), s+(b))$

$\Rightarrow \psi(rs) = \psi(r)\psi(s) \longrightarrow (2)$

Thus, $\psi(r)$ is a ring homomorphism

To prove surjectivity,
as a & b are coprimes $\Rightarrow (a)+(b) = R$
Thus, $\exists\ u, v \in R$ st $ua + vb = 1_R$
$\Rightarrow \psi(ua) = (ua+(a), ua+(b)) = (0_R+(a),\ 1_R+(b)) \longrightarrow (3)$
$(ua \in (a)$ & $ua - 1_R = -vb \in (b))$

Similarly,
$\psi(vb) = (vb+(a), vb+(b)) = (1_R+(a),\ 0_R+(b)) \rightarrow (4)$
$(vb \in (b)$ & $vb - 1_R = -ua \in (a))$

Every element in $R/(a) \times R/(b)$ is of the form
$(s+(a), t+(b))$, $s, t \in R$

let $R \ni r = sua + tvb$
$\Rightarrow \psi(r) = \psi(s)\psi(ua) + \psi(t)\psi(vb)$ (as ψ is homomorphism)
$= (s+(a), s+(b))(0_R+(a), 1_R+(b)) + (t+(a), t+(b))$
$(1_R+(a), 0_R+(b))$
$= (s+(a), 0_R+(b)) + (0_R+(a), t+(b)) = (s+(a), t+(b))$
Thus, $\psi(r)$ is surjective

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For kernel of ψ.
let $r \in R$ is in $\ker(\psi)$

$\Rightarrow \psi(r) = (r+(a), r+(b)) = (0_R+(a), 0_R+(b))$
$\Rightarrow r+(a) = 0_R+(a)$ & $r+(b) = 0_R+(b)$ if $r \in \ker(\psi)$
or $\ker(\psi) = \{r \mid r \in (a)$ & $r \in (b)\} = (a) \cap (b) = (ab)$

As ψ is a surjective ring homomorphism
$\phi = R/\ker(\psi) \to R/(a) \times R/(b)$
OR
$\phi : R/(ab) \to R/(a) \times R(b)$ is also a ring homomorphism

hence proved.

4) $f(x) = 3x^5 + 15x^4 - 20x^3 + 10x + 20 \in \mathbb{Z}[x]$
$a_5 = 3$, $a_4 = 15$, $a_3 = -20$, $a_2 = 0$, $a_1 = 10$, $a_0 = 20$

Acc. to Einstein's irreducibility criteria,
if $\exists$ p st $p \mid a_i$, $i = n-1$ to 0
$p \nmid a_n$, $p^2 \nmid a_0$

Then $f(x)$ is irreducible over $\mathbb{Q}[x]$

Let $p = 5$, $\Rightarrow$ p divides a_{n-1} to a_0
But $p \nmid 3$, $p^2 \nmid 20$

$\therefore$ $f(x)$ is 5-Einstein and hence is irreducible over $\mathbb{Q}[x]$

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Using the corollary of Gauss lemma, let R be an UFD with quotient field K.

Let $f(x) \in R[x]$ & $c(f(x)) = 1$.
Then $f(x)$ is irreducible in $R[x]$ iff it is irreducible in $K[x]$

Here $\mathbb{Z}$ is the UFD & $\mathbb{Q}$ is its quotient field.
We proved that $f(x) = 3x^5 + 15x^4 - 20x^3 + 10x + 20$ is irreducible in $\mathbb{Q}[x]$

Thus it is irreducible in $\mathbb{Z}[x]$

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