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$u_i = 0u_1 + 0u_2 + \cdots + 1u_i + 0u_{i+1} + \cdots + 0u_n$

$T(u_i) = 1v_i = v_i \quad \forall i$

Suppose there exists another linear map $S: U \to V$ such that $S(u_i) = v_i \quad i = 1, 2 \dots n$

Then for any $u \in U$, we have

$$S(u) = S\left(\sum_{i=1}^{n} \alpha_i u_i\right) = \sum_{i=1}^{n} \alpha_i S(u_i)$$

$$S(u) = \sum_{i=1}^{n} \alpha_i v_i \qquad \text{[Since } S \text{ is a linear map]}$$

$$= T\left(\sum_{i=1}^{n} \alpha_i u_i\right)$$

$$\therefore \boxed{S(u) = T(u)} \quad \forall u \in U.$$

Range and Null Space :-

let $T: U(F) \to V(F)$ be a linear map from a vector space $U(F)$ to $V(F)$.

* Then Range of T is defined as :-

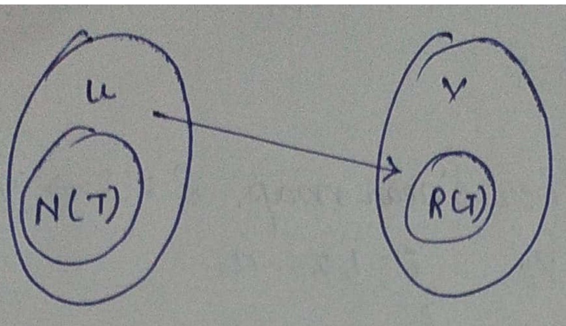
$$R(T) = \text{Range } T = \{T(u) \mid u \in U\}.$$

* Null Space of T is defined as :- (Kernel)

$$N(T) = \text{Null Space of } T = \{u \in U \mid T(u) = 0_V\}.$$

Theorem: Let $T: U(F) \to V(F)$ be a linear map. Then Range of T is a Subspace of V. and Null Space of T is Subspace of U.

Proof



let $v_1, v_2 \in R(T)$.

Then there exists $u_1, u_2 \in U$

such that $T(u_1) = v_1$

$T(u_2) = v_2$

Now for any $\alpha, \beta \in F$

$$\alpha v_1 + \beta v_2 = \alpha T(u_1) + \beta T(u_2)$$

$$= T(\alpha u_1 + \beta u_2).$$

$\because$ $u_1, u_2 \in U$ $\alpha, \beta \in F$

$\Rightarrow \alpha u_1 + \beta u_2 \in U$.

$\therefore$ $\alpha v_1 + \beta v_2 = T(\alpha u_1 + \beta u_2) \in R(T)$

$\therefore$ $R(T)$ is a subspace of V.

Now let $u_1, u_2 \in N(T)$

Then $T(u_1) = 0_V$

$T(u_2) = 0_V$.

Now for any $\alpha, \beta \in F$ $T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2)$

$$T(\alpha u_1 + \beta u_2) = \alpha 0_V + \beta 0_V$$

$$= 0_V.$$

$\alpha u_1 + \beta u_2 \in N_T$

$\therefore$ $N(T)$ is a subspace of U.

Theorem. let $T: U \to V$ be a linear map.

Then (i) T is 1-1 iff $N(T) = \{0_u\}$.

(ii) If $U = [u_1\, u_2 \ldots u_n]$, then.

$$R(T) = [T(u_1), T(u_2) \ldots T(u_n)]$$

(iii) If U is finite dimensional. then $\dim R(T) \leq \dim U$.

Proof:- (i) Suppose T is $1-1$

Then $T(u) = T(v) \Rightarrow (u = v)$.

Now for any $u \in N(T)$

$$T(u) = 0_V = T(0_U)$$

Conversely suppose $N(T) = \{0_u\}$.

we prove T is $1-1$

So let $T(u) = T(v)$

$\Rightarrow T(u) - T(v) = 0_V$

$\Rightarrow T(u-v) = 0_V$

$\Rightarrow u - v \in N_T = \{0_u\}$

$\Rightarrow u - v = 0_u$

$\Rightarrow u = v$.

$\Rightarrow T$ is one-one.

(ii) let $\mathcal{U} = [u_1\ u_2\ \ldots\ u_n]$.

then every $u \in \mathcal{U}$ can be written as

$$u = \sum_{i=1}^{n} \alpha_i u_i \quad \alpha_i \in F$$

Also $u_1\ u_2\ \ldots\ u_n \in \mathcal{U}$.

$\Rightarrow T(u_1), T(u_2), \ldots T(u_n) \in R(T)$.

let $v \in R(T)$ then there exists $u \in \mathcal{U} \ni$
$v = T(u)$.

$$v = T(u) = T(\sum \alpha_i u_i)$$
$$= \sum \alpha_i T(u_i)$$
$$\in [T(u_1)\ T(u_2)\ \ldots\ T(u_n)]$$

$$\Rightarrow R(T) \subseteq [T(u_1)\ T(u_2)\ \ldots\ T(u_n)]$$

Also NOTE $[T(u_1)\ T(u_2) \ldots T(u_n)] \subseteq R(T)$

Since $R(T)$ is a subspace $\therefore R(T) = [T(u_1)\ T(u_2)\ \ldots\ T(u_n)]$

(iii) In the above we proved that if $\mathcal{U} = [u_1 u_2 \ldots u_n]$ $\dim(\mathcal{U}) = n$ Basis for $\mathcal{U}$ $B = \{u_1 u_2 \ldots u_n\}$

then

$$R(T) = [T(u_1)\ T(u_2)\ \ldots\ T(u_n)]$$

$$\dim R(T) \leq n = \dim(\mathcal{U})$$
$$\dim(R(T)) \leq \dim(\mathcal{U})$$

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Def^n let $T: U \to V$ be a linear map.

Then (a) If $R(T)$ is finite dimensional. then dimension of $R(T)$ is called rank of T. and it is denoted by. $r(T)$

(b) If $N(T)$ is finite dimensional. then dimension of $N(T)$ is called nullity of T and is denoted by $n(T)$.

Ex. $T: V_3 \longrightarrow V_3$ be defined by $T(x_1, x_2, x_3) = (x_1, x_2, 0)$

$$R(T) = \{ T(x_1, x_2, x_3) \in V_3 \,/\, (x_1, x_2, x_3) \in V_3 \}.$$
$$= \{ (x_1, x_2, 0) \,/\, x_1, x_2 \in K \}$$

$\because$ $\left\{ \begin{array}{l} e_1 = (1,0,0) \\ e_2 = (0,1,0) \\ e_3 = (0,0,1). \end{array} \right\}$ is basis for V_3.

$$R(T) = [T(e_1), T(e_2), T(e_3)].$$
$$= [(1,0,0), (0,1,0), (0,0,0)].$$
$$= [(1,0,0), (0,1,0)].$$

$\therefore$ $r(T) = 2$.

$$N(T) = \{ (x_1, x_2, x_3) \in V_3 \,/\, T(x_1, x_2, x_3) = (0,0,0) \}$$
$$= \{ (x_1, x_2, x_3) \,/\, (x_1, x_2, 0) = (0,0,0) \}$$
$$= \{ (x_1, x_2, x_3) \,/\, x_1 = 0 \;\; x_2 = 0 \}$$
$$= \{ (0, 0, x_3) \} \quad x_3 \in \mathbb{R}$$
$$= X_3 \text{ axis.}$$
$$= [(0,0,1)].$$

$\therefore$ $n(T) = 1 = \dim(N(T))$.

T is not onto since. $R(T) = [e_1, e_2] \neq V_3$.

Ex: $T: V_3 \to V_2$ be defined by $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3)$

What is $R(T)$ & $N(T)$.

$$R(T) = \{T(x_1, x_2, x_3) / (x_1, x_2, x_3) \in V_3\}.$$

$$= \{(x_1 - x_2, x_1 + x_3) / \cancel{(x_1, x_2, x_3) \in} \; x_1, x_2, x_3 \in R\}.$$

Let $(a, b) \in V_2$ ~~then consider~~.

such that $T(x_1, x_2, x_3) = (a, b)$

$$(x_1 - x_2, x_1 + x_3) = (a, b)$$

$$x_1 - x_2 = a$$
$$x_1 + x_3 = b.$$

$$\Rightarrow x_2 = (x_1 - a)$$
$$x_3 = (b - x_1).$$

$$\therefore T(x_1, x_1 - a, b - x_1) = (a, b).$$

$$R(T) = [T(1,0,0), T(0,1,0), T(0,0,1)]$$
$$= [(1,1), (-1,0), (0,1)].$$
$$= [(-1,0) \; (0,1)]$$

$\therefore r(T) = 2$.

$\because R(T) = V_2 \Rightarrow T$ is ONTO.

$$N(T) = \{(x_1, x_2, x_3) / T(x_1, x_2, x_3) = \{0_v\}\}$$
$$\{(x_1, x_2, x_3) / (x_1 - x_2) = 0 \; (x_1 + x_3) = 0\}$$

$N(T) = \{ (x_1, x_2, x_3) \;/\; x_1 = x_2 \;\; x_1 = -x_3 \}$

$N(T) = \{ (x_1, x_1, -x_1) \} \;\; x_1 \in \mathbb{R}$

$N(T) = [1, 1, -1]$

$\therefore \eta(T) = 1$

$\therefore$ T is not 1-1. sint $N(T) \neq \{0_V\}$.

<u>Theorem</u> :- let $T: U \to V$ be a linear map. Then

(i) If T is one-one and $u_1 \; u_2 \ldots u_n$ are L. I. vectors of U. Then $T(u_1), T(u_2), \ldots T(u_n)$ are L. I.

(ii) If $v_1, v_2, \ldots v_n$ are L.I. vectors of $R(T)$ and $u_1, u_2, \ldots u_n$ are vectors of U. s.t

$T(u_1) = v_1$
$T(u_2) = v_2$
$\vdots$
$T(u_n) = v_n$.

Then. $u_1, u_2, \ldots u_n$ are linearly Independent in U.

<u>proof</u> Let T is (1-1) and $u_1 \; u_2 \; u_3 \ldots u_n$ are

(i) L.I. vectors in U.

we prove $T(u_1) \; T(u_2) \ldots T(u_n)$ are L. I.

so consider $\alpha_1 T(u_1) + \alpha_2 T(u_2) + \ldots + \alpha_n T(u_n) = 0_V$

since T is linear map.

$\Rightarrow T(\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n) = 0_V$.
$= T(0_u)$

$\because$ T is 1-1, we have.

$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0_u$ since $u_1, u_2 \ldots u_n$ are L. I.

$\Rightarrow \alpha_1 = 0 \quad \alpha_2 = 0 \; \ldots \; \alpha_n = 0$.

$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$

$T(u_1)\ T(u_2) \ldots T(u_n)$ are L.I.

(iv) let $v_1\ v_2 \ldots v_n$ be L.I in $R(T)$
and $u_1\ u_2 \ldots u_n$ in U $\exists$ $T(u_i) = v_i$
$i = 1, 2, 3 \ldots n$.

We prove $u_1\ u_2 \ldots u_n$ are L.I.

consider $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = O_u$

$T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n) = O_v = T(O_u)$

Since T is linear

$\alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_n T(u_n) = O_v$

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = O_v$.

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

Since $v_1, v_2, v_3, \ldots v_n$ are L.I.

Ex: let $T: V_3 \to V_3$ be defined by $T(e_1) = e_1 - e_2$, $T(e_2) = 2e_2 + e_3$, $T(e_3) = e_1 + e_2 + e_3$.
Show that: T is neither onto or 1-1.

HINT:- [dim $R(T) = 2$. $R(T) \neq V_3 \Rightarrow T$ is not onto.]
[$N(T) = [(1,1,-1)] \Rightarrow T$ is not 1-1]

In the above problem. $T: V_3 \to V_3$.

~~so under the above map. dim $R(T) = 2$. dim V_3 reduces to 2 of domain space~~

The effect of linear map T on V_3 shrinks to a 2-dimensional subspace $R(T)$ of $V = V_3$.

$\because N(T) = [(1,1,-1)]$ $n(T) = 1 = \dim(N(T))$

$\dim R(T) + \dim N(T) = 2 + 1 = 3 = \dim$ of domain space V_3

So we have the following theorem

RANK NULLITY THEOREM

Let $T: U \to V$ be a linear map.
and U is finite dimensional vector space.
Then $\boxed{\dim R(T) + \dim N(T) = \dim U}$

ie $r(T) + n(T) = \dim U$
rank T + nullity T = dim of domain space

proof Since $N(T)$ is a subspace of a finite dimensional vector space. U. so $N(T)$ is itself a finite dimensional.

let $\dim N(T) = n(T) = n$.
and $\dim U = p$. $(p \geq n)$
let $B = \{u_1 u_2 \dots u_n\}$ be a Basis for $N(T)$

~~Then~~ Since $u_i \in N(T) \Rightarrow T(u_i) = 0 \quad i = 1$ to n
$\therefore$ B is linearly Independent in $N(T)$
$\Rightarrow$ B is also L.I. in U. Extend B to a basis of U.
let $B_1 = \{u_1, u_2, \dots u_n, u_{n+1} \dots u_p\}$ be a extended basis for U.

consider $A = \{T(u_{n+1}), T(u_{n+2}) \dots T(u_p)\}$
claim :- A is Basis for $R(T)$.
So that $\dim R(T) = p - n$.
$\Rightarrow \dim R(T) + n = p$.
$\Rightarrow \dim R(T) + \dim N(T) = \dim U$.

First we prove $[A] = R(T)$
$\because [B_1] = U$
Then $R(T) = [T(u_1), T(u_2) \dots T(u_n), T(u_{n+1}) \dots T(u_p)]$

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$$R(T) = [T(u_{n+1}) \ T(u_{n+2}) \ \cdots \ T(u_p)]$$

$$R(T) = [A]$$ ($\because T(u_i) = 0$ for $i = 1$ to n as u_i belons to Nullspace.)

Now we prove Linearly Indp.

Consider. $\alpha_{n+1} T(u_{n+1}) + \alpha_{n+2} T(u_{n+2}) + \cdots + \alpha_p T(u_p) = 0_V$ — ⊛

$\Rightarrow T(\alpha_{n+1} u_{n+1} + \alpha_{n+2} u_{n+2} + \cdots + \alpha_p u_p) = 0_V$ [$\because$ T is linear].

$\Rightarrow \alpha_{n+1} u_{n+1} + \alpha_{n+2} u_{n+2} + \cdots + \alpha_p u_p \in N(T) = [u_1, u_2, \cdots u_n]$

$\Rightarrow$ There exists scalars $\beta_1 \ \beta_2 \cdots \beta_n \in K$.

s.t. $\alpha_{n+1} u_{n+1} + \alpha_{n+2} u_{n+2} + \cdots + \alpha_p u_p$

$$= \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n = \sum_{i=1}^{n} \beta_i u_i$$

unique expression.

$$(\alpha_{n+1} u_{n+1} + \alpha_{n+2} u_{n+2} + \cdots + \alpha_p u_p) - (\beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n) = 0$$

$\Rightarrow \alpha_{n+1} = 0, \alpha_{n+2} = 0, \cdots \alpha_p = 0$

$\beta_1 = 0, \ \beta_2 = 0, \cdots \beta_p = 0.$

$\because \{u_1 \ u_2 \cdots u_n, u_{n+1} \cdots u_p\}$ are L.I. set.

using ⊛ we see that.

$T(u_{n+1}), T(u_{n+2}) \cdots T(u_p)$ are linearly independent.

$\Rightarrow$ A is L.I.

Ex $T: V_4 \to V_3$. be linear Map defined by

$T(e_1) = (1,1,1)$

$T(e_2) = (1,-1,1)$

$T(e_3) = (1,0,0)$

$T(e_4) = (1,0,1)$

verify $r(T) + n(T) = \dim V_4 = 4$

Inverse of a Linear Map :-

Let U and V be vector spaces over same field.

Let $T: U \to V$ be a linear map.

such that T is 1-1 and onto.

Then T is called invertible.

It is also called non-singular linear map.

We say T is 1-1 if $u_1, u_2 \in U$

$u_1 \neq u_2 \Rightarrow T(u_1) \neq T(u_2)$

as $T(u_1) = T(u_2) \Rightarrow u_1 = u_2$

T is onto for every $v \in V$

$\exists\ u \in U \ni T(u) = v$.

we define T^{-1} as follows. :-

let $v \in V$ Since T is onto there exists

$u \in U \ni T(u) = v$

Also u is determined in this way is unique element of U.

because T is 1-1.

$u_1, u_2 \in U$ and $u_1 \neq u_2 \Rightarrow v = T(u) \neq T(u_1)$

So we define $T^{-1}(v)$ to be u.

$\therefore T^{-1}: V \to U$. such that.

$T^{-1}(v) = u \Leftrightarrow T(u) = v$.

$T^{-1}: V \to U$. $\quad v_1, v_2 \in V$.

$\because v_1, v_2 \in V$ with $v_1 \neq v_2$.

$\because T: U \to V$ is onto $\exists$ unique $u_1\ u_2 \in U$.

$T(u_1) = v_1$

$T(u_2) = v_2$. & $u_1 \neq u_2$.

$T^{-1}(v_1) \neq T^{-1}(v_2)$

Thus $v_1 \neq v_2 \Rightarrow T^{-1}(v_1) \neq T^{-1}(v_2)$

$\therefore$ <u>$T^{-1}: V \to U$ is 1-1.</u>

$T^{-1}: V \to U$. let $u \in U$, claim $\exists\ v \in V$

$\ni T^{-1}(v) = u$.

$T(u) \in V$

$\exists\ v \in V \ni T(u) = v. \Leftrightarrow T^{-1}(v) = u$.

$\therefore$ <u>$T^{-1}: V \to U$ is onto</u>

<u>prove that</u> :- $T: U \to V$ be a linear map. which is $1-1$ and onto.

then prove that $T^{-1}: V \to U$ is a linear map.

<u>problem</u> :- $T: V_2 \to V_2$

$T(x_1\ x_2) = (x_1, -x_2)$.

find T^{-1}

$T^{-1}: V_2 \to V_2$

$T^{-1}(y_1 - y_2) = (y_1, +y_2)$.

$T(u) = v \Leftrightarrow T^{-1}(v) = u$

let $(y_1 y_2) \in \cancel{V_2}\ V_2 \ni T^{-1}(y_1, y_2) = (x_1\ x_2)$.

$\Leftrightarrow (y_1\ y_2) = T(x_1\ x_2)$

$T^{-1}(y_1, y_2) = (y_1, -y_2) \quad \leftarrow y_1 = x_1\ \ y_2 = -x_2$

Problem $T: P_2 \to V_3$

$T(\alpha_0 + \alpha_1 x + \alpha_2 x^2) = (\alpha_0, \alpha_1, \alpha_2)$

find T^{-1}?

$I : U \to U.$

$I(u) = u, \ \forall u \in U$

1-1 & onto.

$I(u) = u. \iff I^{-1}(u) = u$

$\therefore I^{-1} = I.$

Ex let $U = \{(x, x_2 \ldots) / x_i \in \mathbb{R}\}$.
be infinite sequence of scalars.

$x = (x_1, x_2, x_3 \ldots\ldots)$
$y = (y_1, y_2, y_3 \ldots\ldots)$

$x+y = (x_1+y_1, x_2+x_2, \ldots\ldots)$
$\alpha x = (\alpha x_1, \alpha x_2, \ldots\ldots)$

Then U is a vector space.
Define $T: U \to U$ by

$T(x) = T(x_1, x_2, x_3 \ldots\ldots)$
$= (x_2, x_3, x_4 \ldots\ldots).$

clearly $R(T) = U.$
let $(z, y_1, y_2, y_3 \ldots\ldots) \in U$
Then $T(x, y_1, y_2, \ldots\ldots) = (y_1, y_2, y_3 \ldots\ldots).$

$N(T) = \{(1, 0, 0, \ldots\ldots)\} \neq \{0_U\}$

$\therefore T$ is 1-1.
T^{-1} does not exists.

If $T: U \to U$.

$T(x_1, x_2, x_3 \dots) = (0, x_1, x_2, \dots)$.

It is not onto, but 1-1.

$\therefore T^{-1}$ does not exists.