Group Action

Lecture 11

07/02/2022

Example. $G_1 = \mathbb{Z}$, $H = 6\mathbb{Z} \subseteq G_1$

$$G_2 = \mathbb{Z}/6\mathbb{Z}.$$

$$\phi : G_1 \longrightarrow G_2.$$

$$\phi(g) = \bar{g}$$

$$\ker \phi = H.$$

> Say G is any gp and H is normal subgp of G, $\phi : G \to G/H$. Then subgps of G/H are subgp of G containing H

Subgps of G_2 are in one to one correspondence with the subgps of G_1 containing $H = 6\mathbb{Z}$.

Want find n s.t $6\mathbb{Z} \subseteq n\mathbb{Z}$.

$6\mathbb{Z} \subseteq 2\mathbb{Z}$. and $6\mathbb{Z} \subseteq 3\mathbb{Z}$.

There are 2 non-trivial subgps of G_2 which corresponds to $2\mathbb{Z}$ & $3\mathbb{Z}$.

Group Action :

A gp action of a gp G on a set A is a map from $G \times A \longrightarrow A$.

$$(g, s) \longmapsto gs.$$

satisfying the following properties:

(1) $g_1(g_2 s) = (g_1 g_2) s \quad \forall g_1, g_2 \in G, s \in A.$

(2) $1_G \cdot s = s \quad \forall s \in A.$

Example 1 (i) Let $G = \{1, r\}$ where r is the reflection wrt x-axis.

and $A = \mathbb{C}$. Then G acts on $\mathbb{C}$

as
$$G \times \mathbb{C} \longrightarrow \mathbb{C}$$
$$(g, z) \longmapsto gz = \begin{cases} z & \text{if } g = 1. \\ \bar{z} & \text{if } g = r. \end{cases}$$

Thus this is a gp action.

Example 2. $G = GL_n(\mathbb{R})$, $A = \mathbb{R}^n$.

$$G \times A \longrightarrow A$$

$$\left(B, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) = B \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

$$B \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$$

Then this is a gp action.

Example 3. $G \times G \longrightarrow G$,

$$(g, a) \longmapsto ga$$ gp action by left multiplication

Example 4. $G \times G \longrightarrow G$.

$$(g, x) \longmapsto gxg^{-1}$$

This is a gp action by conjugation.

Example 5. Let G be a gp and H is a subgp of G.

$G/H \rightsquigarrow$ set of all left cosets of H.

$$G \times G/H \longrightarrow G/H.$$

$$(g, aH) \longmapsto gaH.$$

Note that G/H is a set may not be a gp as we have not assumed H is a normal subgp.

Defn. The kernel of a gp action is

$$= \{ g \in G \mid gs = s \ \ \forall s \in A \}.$$

Defn. For each $s \in A$ the stabilizer of s in G $= \{ g \in G \mid gs = s \}$ which is denoted by $G_s \subseteq G$.

Q Is G_s a subgp of G?

$g_1, g_2 \in G_s$ WTS $g_1^{-1} g_2 \in G_s$.

$$g_1^{-1} g_2 s = g_1^{-1} s = s.$$

Therefore G_s is a subgp of G.

$g_1 s = s$,

$g_1^{-1}(g_1 s) = g_1^{-1} s$.

$\Rightarrow (g_1^{-1} g_1) s = g_1^{-1} s$

$\Rightarrow 1_G \cdot s = g_1^{-1} s$

$\Rightarrow s = g_1^{-1} s$.

If G acts on A we define an equivalance relation on A as follows:
$s \sim t$ if $t = gs$ for some $g \in G$.

The equivalance class of $s \in A$ is defined as orbit of s and denoted by $O(s)$.

$$O(s) = \{ gs \mid \forall g \in G \}.$$

Thus $$A = \bigsqcup_{s \in A} O(s).$$

If A is a finite set then and $O(s_1), \ldots, O(s_k)$ are distinct orbits then

$$A = O(s_1) \sqcup \cdots \sqcup O(s_k).$$

Q what is the relation between $O(s)$ and G_s?

<u>Propn</u>. Let G be a gp acting on a set A. For each $s \in A$, the no. of elts in $O(s)$ is the index of the stabilizer i.e $[G : G_s]$.

<u>Pf</u>: Define $\phi : G/G_s \longrightarrow O(s) = \{gs \mid g \in G\}$.

$$\phi(gG_s) = gs.$$

<u>check</u> ϕ is well defined.

<u>WTS</u> ϕ is inj

$$\phi(g_1 G_s) = \phi(g_2 G_s).$$

$$\Rightarrow \quad g_1 s = g_2 s$$

$$\Rightarrow \quad g_2^{-1}(g_1 s) = g_2^{-1}(g_2 s)$$

$$\Rightarrow \quad g_2^{-1} g_1 s = s. \Rightarrow g_2^{-1} g_1 \in G_s.$$

Note ϕ is surjective

$$g_1 G_s = g_2 G_s.$$

$$\Uparrow$$