Fluid Mechanics (8.57)

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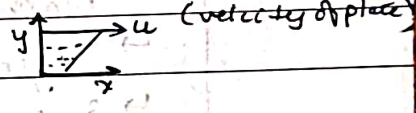
- Continuum Hypothesis
- Isotropic fluids → same prop from all dir's
- $\lim_{\delta V \to 0} \frac{\delta m}{\delta V} = \rho$ → density = $\rho = \frac{m}{V}$
- $\lim_{\delta A \to 0} \frac{\delta F}{\delta A} = P$ → pressure
- Compressible fluid :- density can change wrt temp, time, P, etc
  ↳ opp → Incompressible ($\frac{\partial \rho}{\partial t} = 0$)

- Viscous :- Normal & sheering stress exist (Stress ~ pressure)
  Non " " stress exist only

- Newtonian :- Follow Newton's law of Viscosity
  ~~all~~ else Non-Newtonian

- Newtons law of Viscosity : $\tau = \mu \frac{du}{dy}$ (μ = coeff of viscosity, τ = Sheering stress)



$\tau = 0 \Rightarrow \mu = 0 \Rightarrow$ Ideal fluid	μ const ⇒ Newtonian } Real
$du/dy = 0 \Rightarrow \mu = \infty \Rightarrow$ Solid	μ depends on τ ⇒ Non - "

- Eulerian Description :- Description of a property with respect to a fixed pt (fluid passes through chosen pt)

  velocity $\vec{q} = (u, v, w)$, Point $P = (x, y, z)$
  $u = f(x, y, z, t)$, $v = f(x, y, z, t)$, $w = f(x, y, z, t)$

- Lagrangian Description :- Fluid props. are expressed in terms of initial composition (a point moves along the flow)

  initial Pt $P(x_0, y_0, z_0)$ at t_0 ⇒ $u = f(x_0, y_0, z_0, t)$ $||^{ly}$ v & w

- Lagrangian to Eulerian :- • $\phi = \phi(x_0, y_0, z_0, t)$ prop in lag^n, $P(x, y, z)$ ...
  - $x = f_1(x_0, y_0, z_0, t)$ $||^{ly}$ y & z
    ↳ can be solved to get $x_0 = f_4(x, y, z, t)$ $||^{ly}$ y_0 & z_0
  - replace in ϕ to get eulerian $\phi = \phi'(x, y, z, t)$
    $= \phi(f_4, f_5, f_6, t)$

Binomial Thm :- $(1+x)^n = 1 + nx + \frac{(n)(n-1)}{2!}x^2 + \frac{(n)(n-1)(n-2)}{3!}x^3 + \ldots$

- **Solving :-** Eulerian to lagrangian
  - Given $u(x,y,t)$, $v(x,y,t)$
  - Using D operator, get $x(c_1, c_2, t)$, $y(c_1, c_2, t)$
  - $x_0 = x(t_0)$ → let $t_0 = 0$ and find $c_1(x_0, y_0)$, $c_2(x_0, y_0)$
  - Replace in x & y to get $x(x_0, y_0, t)$, $y(x_0, y_0, t)$
  - Substitute in $u(x,y,t)$, $v(x,y,t)$ → lagrangian

- **Steady :-** properties dont change wrt time :- $\frac{\partial A}{\partial t} = 0$

- **Uniform flow :-** velocity doesnt change wrt space $\frac{\partial q}{\partial r} = 0$
  (mag + dir)

- velocity $(\vec{q}) = \frac{d\vec{r}}{dt} \Rightarrow u = \frac{dx}{dt}, \; v = \frac{dy}{dt}, \; w = \frac{dz}{dt}$
  $\vec{q} = (u, v, w)$, $\vec{r} = (x, y, z)$

functions in terms of x, y, t

- Path line :- $\frac{dx}{dt} = \underset{u}{f}, \; \frac{dy}{dt} = \underset{v}{g}, \; \frac{dz}{dt} = \underset{w}{h}$ [motion of a particle]

- streamline :- $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$ $\boxed{t=0}$ [dir of tangent coincides with vel]
  $(\underset{\text{vel}}{\vec{q}} \times \underset{\text{tangent dir}}{d\vec{r}} = 0) \Rightarrow (u, v, w) \times (dx, dy, dz) = 0 \Rightarrow \cancel{(u\,dx, v\,dy, w\,dz) = 0}$

- Matrial (Spacial or total), local & Convection derivative

$$\underset{\substack{\text{Material}\\\text{(lagrangian)}}}{\frac{D}{Dt}} = \underset{\substack{\text{local}\\\text{(Eulerian)}}}{\frac{\partial}{\partial t}} + \underset{\text{convection}}{\vec{q}\cdot\vec{\nabla}}$$

$\vec{q}\cdot\vec{\nabla} = \vec{q}\,\frac{\partial}{\partial \vec{r}}$

$$\boxed{\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}}$$

Proof :- $\underset{t}{\vec{r}}$ —— $\underset{t+\delta t}{\vec{r} + \delta\vec{r}}$, $\vec{F}$ → fluid prop, $\vec{q}$ → velocity

- $\frac{D\vec{F}}{Dt} = \frac{d\vec{F}}{dt} = \lim_{\delta t \to 0} \frac{\vec{F}(\vec{r} + \vec{q}\,\delta t, t + \delta t) - \vec{F}(\vec{r}, t)}{\delta t}$

$\left(\vec{q} = \frac{\delta\vec{r}}{\delta t} \Rightarrow \delta\vec{r} = \vec{q}\,\delta t\right)$

~~$\vec{F}(\vec{r} + \vec{q}\,\delta t, t + \delta t) = \vec{F}(\vec{r}, t) + \ldots$~~

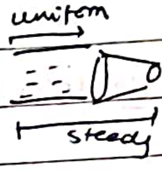
~~$\vec{F}(\vec{r} + q\,\delta t, t + \delta t) = \vec{F}(\vec{r}, t) + \frac{\partial \vec{F}}{\partial \vec{r}}\,\vec{q}\,\delta t + \frac{\partial^2 \vec{F}}{\partial \vec{r}^2}(\vec{q}\,\delta t)^2 \ldots + \frac{\partial \vec{F}}{\partial t}\delta t + \frac{\partial^2 \vec{F}}{\partial t^2}(\delta t)^2 + \ldots$~~ (using taylors)

- $\vec{F}(\vec{r}+\vec{q}\,\delta t,\ t+\delta t) = \vec{F}(\vec{r},t) + \frac{\partial \vec{F}}{\partial \vec{r}}\vec{q}\,\delta t + \frac{\partial^2 \vec{F}}{\partial \vec{r}^2}(\vec{q}\,\delta t)^2 + \dots +$
$\frac{\partial \vec{F}}{\partial t}\delta t + \frac{\partial^2 \vec{F}}{\partial t^2}(\delta t)^2 + \dots$ (using taylor expansion)

- Ignore higher terms to get $\lim_{\delta t \to 0} \frac{1}{\delta t}\left(\vec{q}\frac{\partial \vec{F}}{\partial \vec{r}} + \frac{\partial \vec{F}}{\partial t}\right)\delta t$

$$\frac{\partial \vec{F}}{\partial \vec{r}} = \frac{\partial \vec{F}}{\partial x}\hat{i} + \frac{\partial \vec{F}}{\partial y}\hat{j} + \frac{\partial \vec{F}}{\partial z}\hat{k} = \nabla \vec{F}$$

$\hookrightarrow \frac{\partial \vec{F}}{\partial \vec{r}} = \frac{\partial \vec{F}}{\partial x}\cdot\frac{\partial x}{\partial \vec{r}} + \frac{\partial \vec{F}}{\partial y}\cdot\frac{\partial y}{\partial \vec{r}} + \frac{\partial \vec{F}}{\partial z}\frac{\partial z}{\partial \vec{r}}$



- Steady $\rightarrow \frac{\partial}{\partial t} = 0$, Uniform $\rightarrow \vec{\nabla} = 0$

- Bernoulli's Theorem :- $P_0 + \rho g h + \frac{1}{2}\rho v^2 = \text{const}$

- Acceleration :- $\frac{D\vec{q}}{Dt}$

- Irrotational fluid :- curl $\vec{q} = \vec{0} \Rightarrow \vec{\nabla}\times\vec{q} = \vec{0}$
For rotational, Ω = vorticity = curl $\vec{q}$

- Normal $\vec{n} = \vec{\nabla} f$ (Normal of a function)

- If $\vec{\nabla}\times\vec{q} = 0$, $\exists$ velocity potential 'ϕ' st
$\vec{q} = -\vec{\nabla}\phi$

- Incompressible :- $\vec{\nabla}\cdot\vec{q} = 0$ or $\frac{\partial \rho}{\partial t} = 0$

- (ϕ is Harmonic)
Incompressible & Irrotational :- $\vec{\nabla}\cdot\vec{q} = -\vec{\nabla}^2\phi = 0$

- Conservation of Mass = $A_1 u_1 = A_2 u_2$ (A → crossectional area, u → velocity)

- Eqn of Continuity :- $\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{q}) = 0}$

Proof :- $m = \rho v \rightarrow \log m = \log \rho + \log v \Rightarrow \frac{1}{m}\frac{dm}{dt} = \frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{v}\frac{dv}{dt}$ (steady flow: $\frac{dm}{dt}=0$)

$\Rightarrow \frac{1}{\rho}\frac{d\rho}{dt} + \vec{\nabla}\cdot\vec{q} = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{q}) = 0$
↳ using linear strain rate

Proof 1:- • $\delta V \rightarrow$ small vol, $\delta S \rightarrow$ small SA of δV

Normal of $\vec{q}$ wrt $\delta V = \vec{q} \cdot \hat{n}$

• Rate of mass flow $\delta m = \rho (\vec{q} \cdot \hat{n}) \delta s$

$$M = \int_S \rho (\vec{q} \cdot \hat{n}) \delta s \quad \text{(over total surface)}$$

Gauss divergence thm $\Rightarrow \oint q \, ds = \int_V \vec{\nabla} \cdot \vec{q} \, dv$ where $q = \vec{q} \cdot \hat{n}$

$$M = \int_V \vec{\nabla} \cdot (\rho \vec{q}) \, dv$$

concened with inner ~~volume~~ flow, so normal $= -\hat{n}$

$$M = -\int_V \vec{\nabla} \cdot (\rho \vec{q}) \, dv$$

$M = \frac{d}{dt} \int_V \rho \, dV$ acc. to ~~differentiation~~ definition

$$\frac{d}{dt} \int_V \rho \, dv = -\int_V \vec{\nabla} \cdot (\rho \vec{q}) \, dv \Rightarrow \int_V \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{q} \right) dv = 0$$

(v arbitrary)

• Condition of (incompressibility) & (incompress + irrotation) can be derived from eqn of continuity

• $\vec{\nabla} \cdot \vec{F} \rightarrow$ divergence. $\vec{\nabla} \times \vec{F} \rightarrow$ curl

• ∇^2 (Laplacian) $= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \vec{\nabla} \cdot (\vec{\nabla} f)$

• $\nabla^2 \vec{F} = \nabla^2 F_1 \hat{i} + \nabla^2 F_2 \hat{j}$ (on vector)

$\nabla^2 (uv) = (\nabla^2 u) v + u (\nabla^2 v) + 2 (\nabla u \cdot \nabla v)$

• Derive $\frac{D}{Dt}$ using eqn of continuity using taylor ρ

$$\rho_1 (x_1, y_1, z_1, t_1) \longrightarrow \rho (x_2, y_2, z_2, t_2)$$

$$\rho_2 = \rho_1 + \left(\frac{\partial \rho}{\partial x} \right)_p (x_2 - x_1) + \left(\frac{\partial \rho}{\partial y} \right)_p (y_2 - y_1) + \left(\frac{\partial \rho}{\partial z} \right)_p (z_2 - z_1) + \left(\frac{\partial \rho}{\partial t} \right)_p (t_2 - t_1)$$

(taylor expansion)

$$\Rightarrow \frac{\rho_2 - \rho_1}{t_2 - t_1} \text{ at } t_2 \rightarrow t_1 = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} u + \frac{\partial \rho}{\partial y} v + \frac{\partial \rho}{\partial z} w = \frac{\partial \rho}{\partial t} + \vec{\nabla} \rho \cdot q$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{\nabla} \rho \cdot \vec{q}$$

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Q. Page 14 ? Sakshi

Q Page 11? Kalan

Eq^n of continuity in polar coordinates:-

? 3D : $\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\sin\theta q_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho q_\phi) = 0$

in 3D cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\cdot\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0$$

- Incompressable :- $\frac{\partial \rho}{\partial t} = 0$

- Stream fun^n $\psi \rightarrow u = -\frac{\partial\psi}{\partial y}$, $v = +\frac{\partial\psi}{\partial x}$

$\Rightarrow d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy \Rightarrow \boxed{\psi = \int +v\,dx - u\,dy + c}$

- Relation b/w ψ & $\phi \rightarrow \nabla^2\phi = \nabla^2\psi = 0 \rightarrow$ elliptic PDE (Laplace Eq^n)

Cauchy - Rieman $\leftarrow \left(\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}\right) = \left(+\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}\right) = (u, +v)$

$w = \phi + i\psi$ = complex potential where ϕ & ψ are complex conjugates
↳ only for incompressible & irrotational

- $q = -\vec{\nabla}\phi \rightarrow (u, v) = \left(-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}\right)$

- Eq^n of Continuity in Lagrangian (q independent of t)

$\delta x_0, \delta y_0, \delta z_0$ $\rho_{R_0, t_0} \longrightarrow \rho_{R,t}$ $\delta x, \delta y, \delta z$

Conservation of mass

$\int_{R_0} \underbrace{\rho_0 \delta x_0 \delta y_0 \delta z_0}_{mass} = \int_R \underbrace{\rho\, \delta x \delta y \delta z}_{mass}$ | $dx\,dy\,dz = J\,dx_0\,dy_0\,dz_0$ where $J = \frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)}$

$\int(\rho_0 - J\rho)\delta x_0 \delta y_0 \delta z_0 = 0 \Rightarrow \boxed{\rho_0 = J\rho}$ as x_0, y_0, z_0 are arbitrary

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- $$\boxed{\frac{DJ}{Dt} = J\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}$$

- Lagrangian to Euler :- • Differentiate wrt t • J is non zero zero

- Euler to Lagrangian :- Opp. of previous

- ϕ exist ⇒ irrotational ⇒ can be used to prove incomp.
  ψ " → rotational or irr ⇒ rotational flow along fixed axis (motion in 2D)

Q streamlines ⇒ ? (only streamline or with path) ?

Q Momentum Operation ? $P = \lim_{\delta s \to 0} \frac{\delta F}{\delta s}$ → Force / Surface

- ~~Lagrangian~~ Euler Eq^n :- $\boxed{\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla} P}$ → $\vec{F}$ → external force/mass, P → internal pressure

Proof :- • consider incompressible fluid → ρ constant
• External Force $= \int_V \vec{F}\rho\, dv$ → ①
Force due to pressure $= \iint P(-\hat{n})\,ds$ → ②
$= -\int_V \vec{\nabla} P\, dv$ (using Gauss Divergence)
$\boxed{① + ② = \frac{dP}{dt}}$
$P = \int_V \vec{q}\rho\, dv$
$\int_V \rho\frac{d\vec{q}}{dt}\, dv = \int_V (\vec{F}\rho - \vec{\nabla} P)\, dv$ (V arbitrary)

After wards • Bernoulli's Eq^n of a 1D inviscous, incompressable, irrotational flow in a conservative force field
Pre :- irrotation ⇒ $\vec{q} = -\vec{\nabla}\phi$, $\vec{F}$ is conservative ⇒ $\vec{F} = \vec{\nabla} V$ (scalar field)
Proof :- • consider Euler's eq^n ⇒ $\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla} P$ | $\vec{F} = (X, Y, Z)$
• resolve along x axis ⇒ $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial P}{\partial x}$

now → $\frac{\partial u}{\partial y} = -\frac{\partial^2\phi}{\partial y\partial x} = \frac{\partial v}{\partial x}$, ||ly $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$ & $\frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}$

→ $\frac{\partial}{\partial t}\left(\frac{-\partial\phi}{\partial x}\right) + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} = X - \frac{1}{\rho}\frac{\partial P}{\partial x}$ ①

||ly for y & z axis ② ③ | $\frac{\partial}{\partial t}\left(\frac{-\partial\phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial x}(u^2 + v^2 + w^2) = X - \frac{1}{\rho}\frac{\partial P}{\partial x}$

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1. Streamlines :- $d\psi = 0 \Rightarrow \psi(x,y,t) = c_1$, ~~$\phi(x,y,t) = c_2$~~

→ Continuation :- ① dx + ② dy + ③ dz

$\Rightarrow$ ~~ns~~ $-\cancel{\partial} d\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2} d(\underbrace{u^2+v^2+w^2}_{q^2}) = \underbrace{X dx + Y dy + Z dz}_{-dV} - \frac{1}{\rho} dP$

~~⇒ -~~ Integrate on both sides

$$\boxed{-\frac{\partial\phi}{\partial t} + \frac{1}{2}q^2 = -V - \int \frac{dP}{\rho}}$$ → Bernoulli's / Pressure eq^n

• complex potential :- $w(z) = w(x,y) = \phi(x,y) + i\psi(x,y)$

Q. ~~Derive C-R eq^n from complex potential.~~

~~$\frac{\partial w}{\partial x} = i\frac{\partial w}{\partial y}$ → for CR eq^n~~ ~~$w' = \frac{\partial w}{\partial x} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}$~~ $\phi = \phi(x+iy)$
~~$w' = i\frac{\partial w}{\partial y} =$~~ $\psi = \psi(x+iy)$

$W'(x+iy) = \left[\frac{\partial w}{\partial x} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}\right] = \left[i\frac{\partial w}{\partial y} = i\frac{\partial\phi}{\partial y} + i^2\frac{\partial\psi}{\partial y}\right]$ → solve to get CR

• CR in Polar form $(z = re^{i\theta})$

$w(z) = f(z) = f(re^{i\theta}) = \phi + i\psi$ $\quad \left(\begin{matrix}\phi = \phi(r,\theta) \\ \psi = \psi(r,\theta)\end{matrix}\right)$

~~$\frac{\partial w}{\partial r} = f'(re^{i\theta})e^{i\theta}$ / = / $\frac{\partial w}{}$~~

$\frac{\partial\phi}{\partial r} + i\frac{\partial\psi}{\partial r} = f'(re^{i\theta})e^{i\theta}$, $\quad \frac{\partial\phi}{\partial\theta} + i\frac{\partial\phi}{\partial\theta} = f'(re^{i\theta})\,ri\,e^{i\theta}$

Equate $f'(re^{i\theta})e^{i\theta}$ $\quad \boxed{\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad \frac{\partial\phi}{\partial\theta} = -r\frac{\partial\psi}{\partial r}}$ → CR

• $w = \phi + i\psi, \quad z = x+iy$

$\frac{dw}{dz} = \frac{dw}{dx}\cdot\cancel{\frac{dx}{dz}}^{1} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = -u + iv$?

$\left|\frac{dw}{dz}\right| = \sqrt{u^2+v^2} = |\vec{q}|$ → magnitude of velocity from complex potential

left :- ~~Bernoulli~~, cylindrical or $\vec{\nabla}$ in other coordinate systems
Residual part

Sources & Sink

- **Strength of source:-** Source at origin, mass of fluid/time coming out of origin ('m') → flow across the source = $2\pi m$
  $\parallel^{ly}$ **strength of sink** (Sink is source with $-m$ strength)

- No source/sink $\approx$ streamlines don't intersect $\Rightarrow \psi = \text{const}\ \forall t > 0$

- with source/sink  " intersect at only source & sink

- $q_r = -\dfrac{\partial \phi}{\partial r} = -\dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}$ (using complex CR in polar) → ①
  ↳ radial velocity

$q_r = \frac{m}{r}$, $\phi = \frac{m}{2\pi}$

- $2\pi r q_r = 2\pi m \Rightarrow \boxed{q_r = \frac{m}{r}}$ → ② , $q_\theta = 0$

Using ① & ② → $\psi(r,\theta) = -m\theta$ streamlines from source ↳ θ = const (straight)
→ $\phi(r,\theta) = -m \log r$ q_r const along circle
→ equi-velocity potential at concentric circles (at r const) (r measured from centre)

- $\boxed{w = \phi + i\psi = -m \log z}$ → source at origin
  $w = -m \log(z - z')$ → " " z'

- $w = m \log z$ → sink at origin
  $w = m \log(z - z')$ → " " z'

$z = re^{i\theta}$

- Volume flow rate per unit arc across circle of radius r
  $N = \int_0^{2\pi} \vec{q} \cdot \hat{n}\, ds = \int_0^{2\pi} (q_r, q_\theta) \cdot \vec{n}\, ds = \int_0^{2\pi} q_r\, r\, d\theta = 2\pi m = m'$

- Doublet :- Source + Sink

- ϕ at origin in doublet   Source → $P(r,\theta)$
  sink → $Q(r+\delta r, \theta + \delta\theta)$

Physical Interpretation

$\phi(r,\theta) = -m \log r + m \log(r + \delta r) = m \log\left(1 + \frac{\delta r}{r}\right)$

$\boxed{\phi(r,\theta) = \frac{m\,\delta r}{r}}$ + higher powers chegg

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- Strength of doublet $\mu = m\,\delta s$ ($\delta s = \delta r/\cos\theta$)
- $\phi(r,\theta) = \frac{\mu}{r}\cos\theta$

Proof :- $\phi(r,\theta) = \frac{m\,\delta r}{r} = \frac{m}{r}\,\delta s\cos\theta = \frac{\mu}{r}\cos\theta$

- $\psi(r,\theta) = \frac{-\mu}{r}\sin\theta$

Proof :- $\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r} = \frac{-\mu\cos\theta}{r^2}$

$$\psi = \frac{-\mu\sin\theta}{r} \quad \left[\text{as } \frac{\partial\phi}{\partial\theta} \text{ is zero} \Rightarrow \frac{\partial\psi}{\partial r} = 0\right]$$

- $w = \frac{\mu}{r}e^{-i\theta} = \frac{\mu}{z}$
- Doublet located at $z_1 \rightarrow w = \frac{\mu}{z - z_1}$

" makes angle α

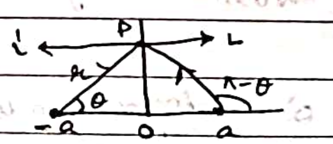
$$w = \frac{\mu}{re^{i(\theta-\alpha)}} = \frac{\mu e^{i\alpha}}{re^{i\theta}} = \frac{\mu e^{i\alpha}}{z}$$

Doublet at z_1 & makes angle $\alpha \Rightarrow w = \frac{\mu e^{i\alpha}}{z - z_1}$

- Multiple doublets $\rightarrow w = \Sigma w_i$
- $\boxed{\log z = \underbrace{\frac{1}{2}\log(x^2+y^2)}_{\log r} + i\tan^{-1}\left(\frac{y}{x}\right)}$

Q. Pg 431 ? How to draw streamlines from from ~~to~~ potential

- Image :- wrt a surface, flow accross it is 0 i.e there are equivalent source - sink systems on either side of surface.
- Image wrt line :-

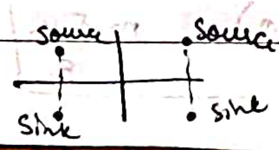


$-a,\ a \rightarrow$ source

$q_r = \frac{m}{r}$

at P $\perp$ to OP

$\frac{m}{r}\cos\theta + \frac{m}{r}\cos(\pi-\theta) = 0$



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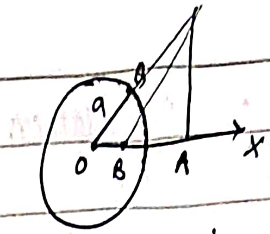
- **Image wrt circular surface :-**



A → source
B → img " , O → img sink

A → sink , O → " source
B → img "

$|OA| = f \quad |OQ| = a$

$|OB||OA| = a^2 \Rightarrow |OB| = \frac{a^2}{f}$

N, B/ source

Take sources at A, B only
$w = f(z) = -m\log(z-f) - m\log\left(z - \frac{a^2}{f}\right)$
- open z & apply log formula, Take ϕ as real part
- take $-\frac{\partial \phi}{\partial r}\Big|_{r=a}$ to get $\frac{m}{a}$, hence we need a sink at origin
  $= q_r$ at a

- **Milne - Thompson Theorem :-**

let $f(z)$ is complex potential & if no singularities of flow & within $|z|=a$, then the flow out cylinder $|z|=a$ is $\boxed{w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)}$ for $|z| \geq a$

Consider source at A which is f away from (0,0)
- $w = -m\log(z-f) \qquad \bar{w} = -m(\log z - f) = -m\log(z-f)$
- Apply Milne Thompson to get
  new $w = w + \bar{w}\left(\frac{a^2}{z}\right)$
  $= -m\log(z-f) - m\log\left(z - \frac{a^2}{f}\right) + m\log z + \text{const}$

Show $\frac{\partial \phi}{\partial r}\Big|_{a=r}$ is zero ↳ Image wrt circular surface

- Potential line → $\phi = \text{constant} \Rightarrow d\phi = 0$

Q. Pg 36:- last part (x-x wala)

Q. Blacius thm Q.

- **Blacius Thm :-** • steady, irrotational & incompressible
  - No external force → $w = f(z)$
  - Force on a fixed cylinder (X, Y)
    $F = X - iY = \frac{1}{2} i\rho \oint_C \left(\frac{dw}{dz}\right)^2 dz$
  - couple of moment $= \text{Re}\left(-\frac{1}{2} i\rho \oint z \left(\frac{dw}{dz}\right)^2 dz\right)$

- <u>Circulation</u> → flow around closed curve

$$\Gamma = \oint_C \vec{q}\cdot d\vec{r} = \oint_C (u\,dx + v\,dy + w\,dz)$$

Γ → circulation, c → curve

- Kevin Circulation theorem
  - ↳ external forces → conservative & derivable by <u>single valued</u> potential function
    - single valued: same on curve
  - ↳ density → f^n of pressure only
  - ⇒ Circulation in any closed circuit moving with the fluid is constant at all time

<u>Proof</u> :- $\frac{d\Gamma}{dt} = \oint_c \frac{d}{dt}(\vec{q}\cdot d\vec{r}) = \oint_c \frac{dq}{dt}\cdot dr + \vec{q}\cdot \frac{d\,d\vec{r}}{dt}$ ⟶ $\vec{q}\cdot d\vec{q}$

Applying Euler

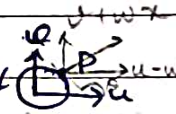
$$= \oint_c \left(\vec{F} - \frac{1}{\rho}\nabla P\right)\cdot d\vec{r} + \vec{q}\cdot d\vec{q} = 0$$

(all singleton value)

- <u>Flow around a cylinder</u>

$$\boxed{\psi(x,y) = Vx - Uy + \frac{1}{2}w(x^2+y^2) + C}$$ where $z = x+iy$

<u>Proof</u> :- fluid → ∞ mass → rest at ∞



$\vec{q} = 0 \Rightarrow u, v = 0 \Rightarrow \frac{\partial\psi}{\partial x} = 0$, $\frac{\partial\psi}{\partial y} = 0 \Rightarrow \psi$ const

$\frac{dx}{ds} = \cos\theta$, $\frac{dy}{ds} = \sin\theta$

normal velocity at $P = \underbrace{(u - wy)\cos(90-\theta)}_{\sin\theta = \frac{dx}{ds}} + \underbrace{(v+wx)\cos(\theta-180)}_{-\cos\theta = \frac{dy}{ds}}$

= " component of velocity of fluid

$= \frac{-\partial\psi}{\partial s}$ → integrate

- Motion of cylinder moving with <u>u velocity in x-axis</u> with irrotational flow (∞ fluid, at rest at ∞)

$$\boxed{W = u\left(\frac{a^2}{z}\right)}$$ $z = re^{i\theta}$

<u>Proof</u> :- $\vec{q} = -\vec{\nabla}\phi \Rightarrow \vec{\nabla}^2\phi = 0$ (irrotational)

$$\boxed{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0}$$

• Solution can be of form $Ar^n \cos n\theta$ or $Br^n \sin n\theta$

• normal velocity of cylinder → $u\cos\theta = -\frac{\partial\phi}{\partial r}\Big|_{r=a}$ (~~normal~~ velocity of fluid in normal dir)

$\boxed{q_r = q_\theta = 0} \Rightarrow \boxed{-\frac{\partial\phi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = 0}$, $r \to \infty$

• ∴ ϕ is of form $Ar\cos\theta + \frac{A_1}{r}\cos\theta$, $-\frac{\partial\phi}{\partial r} = 0 \Rightarrow A = 0$

$-\frac{\partial\phi}{\partial r}\Big|_{r=a} = u\cos\theta \Rightarrow A_1 = ua^2 \Rightarrow \phi = \frac{ua^2}{r}\cos\theta$

• CR eq^n $\boxed{\frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}} = q_\theta$, $\psi = -\frac{ua^2}{r}\sin\theta \Rightarrow w = \frac{ua^2}{z}$

$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r} = -q_r$

• Flow past a fixed cylinder in stream

$\boxed{w = u(z + a^2/z)}$

Proof: • $\psi(x,y) = Vx + C_1$ (y = const, flow only along x axis, $w=0$ (fixed)

$\Rightarrow \phi(x,y) = -Vy + C_2$ (using CR)

• $w = -\frac{ua^2}{z}$ (flow from $-$ve x for moving cylinder)

• $\phi = ux + \frac{ua^2}{r}\cos\theta$, $\psi = uy - u\frac{a^2}{r}\sin\theta$ → to make fixed $\Rightarrow w = (z + \frac{a^2}{z})$

⇒ $\frac{dw}{dz} = -u + iv \Rightarrow \left|\frac{dw}{dz}\right| = |\vec{q}| = 2|u|\sin\theta$ → $\theta = 0$ $q_{min} = 0$, $\theta = \pi/2$ $q_{max} = 2|u|$

• Flow past a cylinder with circulation

$\alpha = \oint_C \vec{q}\cdot d\vec{r} = q_\theta(2\pi r) = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}2\pi r \Rightarrow \phi = -\frac{\alpha\theta}{2\pi} \Rightarrow \psi = \frac{\alpha}{2\pi}\log r$

→ $w_{cir} = \frac{\alpha i}{2\pi}\log z$ ⇒ $\boxed{w = u\left(z + \frac{a^2}{z}\right) + \frac{\alpha i}{2\pi}\log z}$ as $w_{cir} = \phi + i\psi$

~~$\frac{dw}{dz}$~~

$\frac{dw}{dz}\Big|_{r=a} = u(1 - e^{-2i\theta}) + \frac{i\alpha}{2\pi a}e^{-i\theta} \Rightarrow |\vec{q}| = \left|2u\sin\theta + \frac{\alpha}{2\pi a}\right|$

Stagnation pt → $|\vec{q}| = 0 \Rightarrow \left|\frac{\alpha}{4\pi au}\right| \le 1$ → <1 2 stagnation pts; $=1$ 1 " ($\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$)

_/_/_

Aerofoils

→ Incompressible → Aerofoil ~~is the~~ is a cylinder with ~~of~~ type cross-sect. → flow 2D irrotational

- Kutta - Jowkowski Mapping $\Rightarrow \zeta = z + \frac{a^2}{z}$

- Kutta - Jowkowski Theorem → ▭ placed in → u uniform then ↑ thrust is $\boxed{k\rho u}$ per unit length (k is circulation around cylinder)

Proof :- Disturbance $w_1 = Az + \frac{B}{z^2}$ (fixed cylinder in xy plane making $\angle\alpha$)

$w_2 = u e^{-i\alpha} z$ (stream makes α with cylinder)

$w_3 = \frac{ik}{2\pi} \log z$ (circulation)

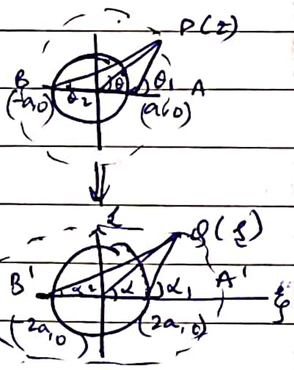
$w = w_1 + w_2 + w_3$, apply Blasius th$^{\underline{m}}$

$X - iY = \frac{-i\rho}{2} \int_c \left(\frac{dw}{dz}\right)^2 dz$ → sum of residual at $z=0$

$= -k\rho u (\sin\alpha + i\cos\alpha)$

Thrust $= \sqrt{X^2+Y^2} = k\rho u$

Jowkowski Transformation $\zeta = z + \frac{a^2}{z}$



- A → A' , B → B' (substitute $a, -a$ resp.)
- $\boxed{\angle A'QB' = \alpha_1 - \alpha_2 = 2\angle APB}$

Proof :- $\zeta - 2a = \frac{(z-a)^2}{z} \Rightarrow A'Q\, e^{i\alpha_1} = \frac{(AP e^{i\theta_1})^2}{OP\, e^{i\theta}}$

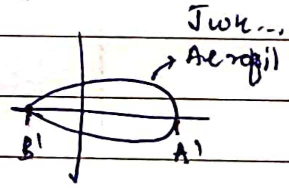
Real part $\Rightarrow A'Q = \frac{AP^2}{OP}$, img $\Rightarrow \alpha_1 = 2\theta_1 - \theta$

||^{ly} $B'Q = \frac{AP^2}{OP}$, $\alpha_2 = 2\theta_2 - \theta$

Also,

$\boxed{A'Q + B'Q = \frac{AP^2 + BP^2}{OP}}$ → const when P lies on any circle

ellipse



when P lies on original circle → st line

- ~~$d\zeta$~~ $\frac{d\zeta}{dz} = 0$ at $\pm a$

now $|\vec{q}'| = \left|\frac{dw}{d\zeta}\right| = |\vec{q}| \left|\frac{dz}{d\zeta}\right|$ → at $\pm a$, $\vec{q}' = \infty$ unless $\vec{q} = 0$

__/__/__

- **Flow around aerofoil** (same assumptions as cylinder)

Stagnation pts $\Rightarrow z = \pm a \Rightarrow \zeta = \pm 2a$

$\left(\frac{dw}{d\zeta}\right)^2 d\zeta = \frac{(dw/dz)^2}{1-\frac{a^2}{z^2}} dz$ using $\zeta = z + \frac{a^2}{z}$

- $w = Ue^{i\alpha}(z-a) + \frac{Ua^2}{z-a}e^{-i\alpha} + \frac{i\kappa}{2\pi}\log(z-a)$

$\Rightarrow$ obtain $\frac{dw}{dz}$

- $X - iY = \frac{i\rho}{2}\int_C \left(\frac{dw}{d\zeta}\right)^2 d\zeta$ using sum of residuals

$= \rho\kappa U(\sin\alpha - i\cos\alpha)$

thrust $= \sqrt{X^2+Y^2} = \rho\kappa U$

Motion of Viscous fluid

→ Tensor :- $\vec{a} = a_i e_i = a_i' e_i' \Rightarrow \vec{a}\cdot e_i' = a_i'$

$= [a_1 e_1 + \dots].e_i'$

$e_i e_j' = \cos(e_i \,\&\, e_j')\,|e_i'|\,|e_j'| = \alpha_{ij}$

$a_i' = \alpha_{ij} a_j$, $\|^{ly}$ $a_i = \alpha_{ji} a_j'$ → Tensor representation (order 1)

→ Order 2 transform

$a_i' b_j' = (\alpha_{ip} a_p)(\alpha_{jq} b_q)$

$$\vec{a}\vec{b} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & \dots & \\ a_3b_1 & \dots & \end{bmatrix}$$

Stress at a pt in fluid

pt mass fluid ⇒ forces zero, hence take

δS with $\hat{n}$ as normal, and δF force and δG moment

$\tau = \lim_{\delta S \to 0} \frac{\delta F}{\delta S}$ (δF is proportional to δS when δS is small)

↓

depends on $\hat{n}$ dir, and stress vector itself

$\tau_{\alpha\beta}$ → $\alpha \to \hat{n}$ dir

↘ $\beta \to$ direction of τ considered