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19MA20059 — Part 2

Q1 Let $A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$ be a general mixed tensor of rank $(r+s)$, type (r, s) in a coordinate system x^i $(i = 1, 2, \dots n)$

Let x^i be transformed to $\bar{x}^i$, then $\bar{x}^i$ transformed to $\bar{\bar{x}}^i$

For first transformation:-

$$\bar{A}^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s} = \frac{\partial \bar{x}^{p_1}}{\partial x^{m_1}} \times \frac{\partial \bar{x}^{p_2}}{\partial x^{m_2}} \times \dots \times \frac{\partial \bar{x}^{p_r}}{\partial x^{m_r}} \times \frac{\partial x^{n_1}}{\partial \bar{x}^{q_1}} \times \frac{\partial x^{n_2}}{\partial \bar{x}^{q_2}} \times \dots \times \frac{\partial x^{n_s}}{\partial \bar{x}^{q_s}} \times A^{m_1 m_2 \dots m_r}_{n_1 n_2 \dots n_s} \quad (1)$$

For second transformation:- $\bar{\bar{A}}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r}$

$$\cancel{\bar{\bar{A}}^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s}} = \frac{\partial \bar{\bar{x}}^{i_1}}{\partial \bar{x}^{p_1}} \times \frac{\partial \bar{\bar{x}}^{i_2}}{\partial \bar{x}^{p_2}} \times \dots \times \frac{\partial \bar{\bar{x}}^{i_r}}{\partial \bar{x}^{p_r}} \times \frac{\partial \bar{x}^{q_1}}{\partial \bar{\bar{x}}^{j_1}} \times \frac{\partial \bar{x}^{q_2}}{\partial \bar{\bar{x}}^{j_2}} \times \dots \times \frac{\partial \bar{x}^{q_s}}{\partial \bar{\bar{x}}^{j_s}} \times \bar{A}^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s} \quad (2)$$

Substituting $\bar{A}^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s}$ from (1) in (2) we get

$$\bar{\bar{A}}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r} = \frac{\partial \bar{\bar{x}}^{i_1}}{\partial x^{m_1}} \times \frac{\partial \bar{\bar{x}}^{i_2}}{\partial x^{m_2}} \times \dots \times \frac{\partial \bar{\bar{x}}^{i_r}}{\partial x^{m_r}} \times \frac{\partial x^{n_1}}{\partial \bar{\bar{x}}^{j_1}} \times \dots \times \frac{\partial x^{n_s}}{\partial \bar{\bar{x}}^{j_s}} \times A^{m_1 m_2 \dots m_r}_{n_1 n_2 \dots n_s}$$

The above equation is the law of transformation of mixed tensor from x^i to $\bar{\bar{x}}^i$

Hence, eq^{ns} of transformation of a mixed tensor posses the group property. Hence Proved.

Q2 Let $A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$ be of type (r, s) & $B^{k_1 k_2 \dots k_{r'}}_{l_1 l_2 \dots l_{s'}}$ be of type (r', s')

We know,

$$\bar{A}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}} \times \frac{\partial \bar{x}^{i_2}}{\partial x^{p_2}} \times \dots \times \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} \times \frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}} \times \frac{\partial x^{q_2}}{\partial \bar{x}^{j_2}} \times \dots \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} \times A^{p_1 p_2 \dots p_r}_{q_1 q_2 \dots q_s}$$

$$\bar{B}^{k_1 k_2 \dots k_{r'}}_{l_1 l_2 \dots l_{s'}} = \frac{\partial \bar{x}^{k_1}}{\partial x^{m_1}} \times \frac{\partial \bar{x}^{k_2}}{\partial x^{m_2}} \times \dots \frac{\partial \bar{x}^{k_{r'}}}{\partial x^{m_{r'}}} \times \frac{\partial x^{n_1}}{\partial \bar{x}^{l_1}} \times \frac{\partial x^{n_2}}{\partial \bar{x}^{l_2}} \times \dots \frac{\partial x^{n_{s'}}}{\partial \bar{x}^{l_{s'}}} \times B^{m_1 m_2 \dots m_{r'}}_{n_1 n_2 \dots n_{s'}}$$

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Multiplying the above two eq^{n}s, we get

$$\bar{C}^{i_1 i_2 \ldots i_r k_1 \ldots k_{r'}}_{j_1 j_2 \ldots j_s l_1 \ldots l_{s'}} = \bar{A}^{i_1 \ldots i_r}_{j_1 \ldots j_s} \times \bar{B}^{k_1 \ldots k_{r'}}_{l_1 \ldots l_{s'}}$$

Let $C^{p_1 \ldots p_r m_1 \ldots m_{r'}}_{q_1 \ldots q_s n_1 \ldots n_{s'}} = A^{p_1 \ldots p_r}_{q_1 \ldots q_s} \times B^{m_1 \ldots m_{r'}}_{n_1 \ldots n_{s'}}$

Thus, $$\bar{C}^{i_1 i_2 \ldots i_r k_1 \ldots k_{r'}}_{j_1 j_2 \ldots j_s l_1 \ldots l_{s'}} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}} \times \frac{\partial \bar{x}^{i_2}}{\partial x^{p_2}} \times \ldots \times \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} \times \frac{\partial \bar{x}^{k_1}}{\partial x^{m_1}} \times \ldots \times \frac{\partial \bar{x}^{k_{r'}}}{\partial x^{m_{r'}}} \times$$
$$\frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}} \times \ldots \times \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} \times \frac{\partial x^{n_1}}{\partial \bar{x}^{l_1}} \times \ldots \frac{\partial x^{n_{s'}}}{\partial \bar{x}^{l_{s'}}} \times C^{p_1 \ldots p_r m_1 \ldots m_{r'}}_{q_1 \ldots q_s n_1 \ldots n_{s'}}$$

The above eq^{n} is the law of transformation of a mixed tensor of rank $r+r'+s+s'$ & type $(r+r', s+s')$
Thus, product of tensors of types (r, s) & (r', s') is a tensor of type $(r+r', s+s')$.
Hence Proved.

Q3 We know that product of tensors of types (r, s) & (r', s') is a tensor of type $(r+r', s+s')$. We also know that vector is a tensor of rank 1, thus it is either $(1,0)$ or $(0,1)$ dependeng on weather it is contravarient or covarient

Case 1 :- Consider 2 contravariant vectors (type $(1,0)$)
Open product of vectors is of type $(1+1, 0+0)$ i.e $(2,0)$
Thus, rank of product is $2+0=2$.

Case 2 :- Consider 2 covariant vectors (type $(0,1)$)
Open product of vectors is of type $(0+0, 1+1)$ i.e $(0,2)$
Thus rank of product is $2+0=2$

Case 3 :- Consider vectors of 2 different types
Open product of vectors is of type $(1+0, 0+1)$ i.e $(1,1)$
Thus rank of product is $1+1=2$

Hence proved that the product of 2 vectors is of rank = 2

But the converse is not true. Not every tensor of rank 2 can be expressed as product of 2 ~~vectors~~ tensors.

Consider e_1, e_2 and let us build a tensor : $e_1 \otimes e_1 + e_2 \otimes e_2$

Let us suppose, vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ & $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ are such that

$(a_1 e_1 + a_2 e_2) \otimes (b_1 e_1 + b_2 e_2) = e_1 \otimes e_1 + e_2 \otimes e_2$

$\Rightarrow a_1 b_1 (e_1 \otimes e_1) + a_2 b_2 (e_2 \otimes e_2) + a_1 b_2 (e_1 \otimes e_2) + a_2 b_1 (e_2 \otimes e_1)$
$= e_1 \otimes e_1 + e_2 \otimes e_2$

$a_1 b_1 = 1 = a_2 b_2 \Rightarrow a_1, b_1, a_2, b_2 \neq 0$ from above
but again, $a_1 b_2 \neq 0$ has to be zero along with $a_2 b_1$ to satisfy the RHS
It is not possible. Thus, $e_1 \otimes e_2 + e_2 \otimes e_2$ is a counter example. Hence proved that the converse is not true

Q4 Let $A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$ be of rank $(r+s)$ & $B^{k_1 k_2 \dots k_m}_{l_1 l_2 \dots l_n}$ be of rank $(m+n)$
We know

$$\bar{A}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}} \times \dots \times \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} \times \frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}} \times \dots \times \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} \times A^{p_1 \dots p_r}_{q_1 \dots q_s}$$

$$\bar{B}^{k_1 \dots k_m}_{l_1 \dots l_n} = \frac{\partial \bar{x}^{k_1}}{\partial x^{d_1}} \times \dots \times \frac{\partial \bar{x}^{k_m}}{\partial x^{d_m}} \times \frac{\partial x^{b_1}}{\partial \bar{x}^{l_1}} \times \dots \times \frac{\partial x^{b_n}}{\partial \bar{x}^{l_n}} \times B^{d_1 \dots d_m}_{b_1 \dots b_m}$$

Let, $$\bar{C}^{i_1 \dots i_r k_1 \dots k_m}_{j_1 \dots j_s l_1 \dots l_n} = \bar{A}^{i_1 \dots i_r}_{j_1 \dots j_s} \times \bar{B}^{k_1 \dots k_m}_{l_1 \dots l_n} = \frac{\partial \bar{x}^{i_1}}{\partial x^{p_1}} \times \dots \times \frac{\partial \bar{x}^{i_r}}{\partial x^{p_r}} \times \frac{\partial \bar{x}^{k_1}}{\partial x^{d_1}} \times \dots \times \frac{\partial \bar{x}^{k_m}}{\partial x^{d_m}} \times$$

$$\frac{\partial x^{q_1}}{\partial \bar{x}^{j_1}} \times \dots \times \frac{\partial x^{q_s}}{\partial \bar{x}^{j_s}} \times \frac{\partial x^{b_1}}{\partial \bar{x}^{l_1}} \times \dots \times \frac{\partial x^{b_n}}{\partial \bar{x}^{l_n}} \times C^{p_1 \dots p_r d_1 \dots d_m}_{q_1 \dots q_s b_1 \dots b_n}$$

where $C^{p_1 \dots p_r d_1 \dots d_m}_{q_1 \dots q_s b_1 \dots b_n} = A^{p_1 \dots p_r}_{q_1 \dots q_s} \times B^{d_1 \dots d_m}_{b_1 \dots b_n}$

The above equation is the law of transformation of a mixed tensor of rank $(r+s+m+n)$ which is the sum of ranks of tensors whose outer product is taken. Hence proved.

Q5 $A^p_r \rightarrow$ rank 2, type (1,1)
$B^{rs}_t \rightarrow$ rank 3, type (2,1)

We know,

$$\bar{A}^i_j = \frac{\partial \bar{x}^i}{\partial x^p} \times \frac{\partial x^r}{\partial \bar{x}^j} \times A^p_r \qquad \bar{B}^{kl}_m = \frac{\partial \bar{x}^k}{\partial x^q} \times \frac{\partial \bar{x}^l}{\partial x^s} \times \frac{\partial x^t}{\partial \bar{x}^m} \times B^{qs}_t$$

Now, $k=j$

Multiplying $\rightarrow$ $\bar{A}^i_j \bar{B}^{jl}_m = \frac{\partial \bar{x}^i}{\partial x^p} \times \frac{\partial x^r}{\partial \bar{x}^j} \times \frac{\partial \bar{x}^j}{\partial x^q} \times \frac{\partial \bar{x}^l}{\partial x^s} \times \frac{\partial x^t}{\partial \bar{x}^m} \times A^p_r \times B^{qs}_t$

$$= \frac{\partial \bar{x}^i}{\partial x^p} \times \frac{\partial \bar{x}^l}{\partial x^s} \times \frac{\partial x^t}{\partial \bar{x}^m} \times \delta^r_q \times A^p_r \times B^{qs}_t$$

$\left[\text{as } \frac{\partial x^r}{\partial x^q} = \delta^r_q \text{, now } \delta^r_q B^{qs}_t = B^{rs}_t\right]$

$$= \frac{\partial \bar{x}^i}{\partial x^p} \times \frac{\partial \bar{x}^l}{\partial x^s} \times \frac{\partial x^t}{\partial \bar{x}^m} \times A^p_r \times B^{rs}_t$$

The above equation is the law of transformation of a mixed tensor of rank 3. Hence proved.

Q6 We know g_{ij} can be written as

$$g_{ij} = \underbrace{\frac{1}{2}(g_{ij}+g_{ji})}_{\text{symmetric}} + \underbrace{\frac{1}{2}(g_{ij}-g_{ji})}_{\text{skew-symmetric}} = A_{ij} + B_{ij}$$

Therefore, $g_{ij}\,dx^i dx^j = (A_{ij}+B_{ij})dx^i dx^j \Rightarrow (g_{ij}-A_{ij})dx^i dx^j = B_{ij}dx^i dx^j \rightarrow (1)$

Interchanging dummy indices in $B_{ij}dx^i dx^j$, we have:

$B_{ij}dx^i dx^j = B_{ji}dx^i dx^j \Rightarrow B_{ij}dx^i dx^j = -B_{ij}dx^i dx^j$ (as B_{ij} is skew-symmetric)

$\Rightarrow 2B_{ij}dx^i dx^j = 0 \Rightarrow B_{ij}dx^i dx^j = 0 \rightarrow (2)$

from (1) & (2)

$(g_{ij}-A_{ij})dx^i dx^j = 0 \Rightarrow g_{ij} = A_{ij} \Rightarrow g_{ij}$ is symmetric

Hence, g_{ij} is a covarient symmetric tensor of rank 2, i.e it will have n^2 components out of which n components $(g_{11}, \dots, g_{nn})$ are independent at max. As g_{ij} is symmetric, half of the remaining (n^2-n) components are independent at max. (As $g_{12}=g_{21}$, $g_{23}=g_{32}$, ...)

Thus, total no. of independent components of g_{ij} of the metric tensor cannot ~~excede~~ exceed $n+\frac{n^2-n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$

Hence, proved