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ANT Test 1

Q1) $y^{iv} + 81y = 81x^2$, $y(0) = y(1) = y''(0) = y''(1) = 0$ $h = 0.25$

i) $p = y''$ $\Rightarrow$ $p_i = \dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \Rightarrow \boxed{16y_{i+1} - 32y_i + 16y_{i-1} - p_i = 0}$

$p'' + 81y = 81x^2$ $\Rightarrow$ $\dfrac{p_{i+1} - 2p_i + p_{i-1}}{h^2} + 81y_i = 81x_i^2$

$\Rightarrow \boxed{16p_{i+1} - 32p_i + 16p_{i-1} + 81y_i = 81x_i^2}$

option B

ii) y_4, y_0 → known | p_4, p_0 → known

y_1, y_2, y_3 → unknown | p_1, p_2, p_3 → unknown

$\boxed{\text{6 unknowns}}$ --- op

iii) $16y_{i+1} - 32y_i + 16y_{i-1} - p_i = 0$

$16p_{i+1} - 32p_i + 16p_{i-1} + 81y_i = 81x_i^2$

$$\underbrace{\begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}}_{A_i} \begin{bmatrix} y_{i-1} \\ p_{i-1} \end{bmatrix} + \underbrace{\begin{bmatrix} -32 & -1 \\ 81 & -32 \end{bmatrix}}_{B_i} \begin{bmatrix} y_i \\ p_i \end{bmatrix} + \underbrace{\begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}}_{C_i} \begin{bmatrix} y_{i+1} \\ p_{i+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 81x_i^2 \end{bmatrix}$$

option A

iv) All options wrong, option D the closest

v) Option D

Q2) $3yy'' + y'^2 = 0 \quad y(0)=0, \; y(1)=1, \; h = 0.25$

Initial guess $y^{(0)}(x) = x$

$F \equiv 3yy'' + y'^2$

Discretizing F

$$F_i \equiv 3y_i\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + \left(\frac{y_{i+1} - y_{i-1}}{2h}\right)^2 = \frac{3}{h^2}(y_i y_{i+1} - 2y_i^2 + y_i y_{i-1}) + \frac{1}{4h^2}(y_{i+1}^2 - 2y_{i+1}y_{i-1} + y_{i-1}^2)$$

$$\frac{\partial F_i}{\partial y_{i-1}} = \frac{3y_i}{h^2} - \frac{2y_{i+1}}{4h^2} + \frac{2y_{i-1}}{4h^2}$$

$$= \frac{1}{h^2}\left(3y_i - \frac{y_{i+1}}{2} + \frac{y_{i-1}}{2}\right)$$

$$\frac{\partial F}{\partial y_i} = \frac{3}{h^2}(y_{i+1} - 4y_i + y_{i-1})$$

$$\frac{\partial F}{\partial y_{i+1}} = \frac{3y_i}{h^2} + \frac{1}{4h^2}(-2y_{i-1} + 2y_{i+1})$$

$$= \frac{1}{h^2}\left(3y_i - \frac{y_{i-1}}{2} + \frac{y_{i+1}}{2}\right)$$

$y(0)=0, \; y(1)=1, \; y^0(0.25)=0.25, \; y^0(0.5)=0.5, \; y^0(0.75)=0.75$

$$\frac{\partial F_i}{\partial y_{i-1}}\Delta y_{i-1} + \frac{\partial F_i}{\partial y_i}\Delta y_i + \frac{\partial F_i}{\partial y_{i+1}}\Delta y_{i+1} = -F_i$$

$y_0^0 = 0, \; y_1^0 = 0.25, \; y_2^0 = 0.5, \; y_3^0 = 0.75, \; y_4^0 = 1$

$i = 1$ as $\Delta y_0 = 0$, ignoring the term

$$-24\Delta y_1 + 16\Delta y_2 = -1$$

$i = 2$

$$20\Delta y_1 - 48\Delta y_2 + 28\Delta y_3 = -1$$

$i = 3$ as $\Delta y_4 = 0$, ignoring the term

$$32\Delta y_2 - 72\Delta y_3 = -1$$

Solving the equations

$\Delta y_1 = \frac{13}{120} = 0.10833 \Rightarrow y_1^1 = y_1^0 + \Delta y_1 = 0.25 + \Delta y_1 = 0.3588$

$\Delta y_2 = \frac{1}{10} = 0.1 \Rightarrow y_2^1 = y_2^0 + \Delta y_2 = 0.5 + \Delta y_2 = 0.6$

$\Delta y_3 = \frac{7}{120} = 0.05833 \rightarrow y_3^1 = y_3^0 + \Delta y_3 = 0.75 + 0.05833 = 0.80833$

i) 0.3583
ii) 0.6
iii) 0.8083

Q5) $y'' + 2xy' + 2y = 4x \quad y(0) = 1,\ y(0.5) = 1.279,\ h = 0.1$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{2x}{2h}(y_{i+1} - y_{i-1}) + 2y_i = 4x_i$$

using Central difference scheme

$$y_{i+1}\left(\frac{1}{h^2} + \frac{x_i}{h}\right) + y_i\left(-\frac{2}{h^2} + 2\right) + y_{i-1}\left(\frac{1}{h^2} - \frac{x_i}{h}\right) = 4x_i \rightarrow (1)$$

$y(0) = 1,\ y(0.5) = 1.279$

y_1, y_2, y_3, y_4 unknown

$i = 1$

$$y_2\left(\frac{1}{h^2} + \frac{x_1}{h}\right) + y_1\left(\frac{-2}{h^2} + 2\right) = 4x_1 - 1\left(\frac{1}{h^2} - \frac{x_1}{h}\right)$$

$i = n-1$

$$y_{n-1}\left(\frac{-2}{h^2} + 2\right) + y_{n-2}\left(\frac{1}{h^2} - \frac{x_{n-1}}{h}\right) = 4x_{n-1} - y(0.5)\left(\frac{1}{h^2} + \frac{x_{n-1}}{h}\right)$$

$n = 5$ here

for all other i follow (1)

we get

$$\underbrace{\begin{bmatrix} -198 & 101 & 0 & 0 \\ 98 & -198 & 162 & 0 \\ 0 & 97 & -198 & 103 \\ 0 & 0 & 96 & -198 \end{bmatrix}}_{\text{coefficient matrix}} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -98.6 \\ 0.8 \\ 1.2 \\ -131.42 \end{bmatrix}}_{\text{RHS vector}}$$

(i) Option A (ii) Option C

Q3) $y'' + 2y' + y = 30x$ $\quad y(0)=0$ & $y(1)=0$ $\quad h=0.5$

$2y_i' = 30x_i - y_i - M_i \Rightarrow y_i' = \frac{30}{2}x_i - \frac{y_i}{2} - \frac{M_i}{2}$

$y'_k = \frac{-y_k}{h} - \frac{M_k h}{3} - \frac{M_{k+1}}{6}h + \frac{y_{k+1}}{h} = p'_k(x_k)$ for $k=0,1$

$y'_k = \frac{-y_{k-1}}{h} + \frac{y_k}{h} + \frac{M_k h}{3} + \frac{M_{k-1}}{6}h = p'_{k-1}(x_k)$ for $k=1,2$

$y'_0 = -\frac{M_0}{6} - \frac{M_1}{12} + 2y_1 = -\frac{M_0}{2} \Rightarrow \frac{M_0}{3} - \frac{M_1}{12} = -2y_1$

$\Rightarrow \boxed{4M_0 - M_1 = -24y_1}$ (i) option D

$y'_2 = -2y_1 + \frac{M_2}{6} + \frac{M_1}{12} = \frac{30 - M_2}{2} \Rightarrow 2\frac{M_2}{3} + \frac{M_1}{12} = 15 + 2y_1$

$\Rightarrow \boxed{8M_2 + M_1 = 180 + 24y_1}$

(ii) option A

now,

$4M_0 - M_1 = -24y_1$

$8M_2 + M_1 = 24y_1 + 180$

$15 - \frac{y_1}{2} - M_1 = -2y_1 - \frac{M_1}{6} - \frac{M_2}{12} \Rightarrow 90 + 18y_1 = 4M_1 - M_2$

$15 - \frac{y_1}{2} - M_1 = 2y_1 + \frac{M_1}{6} + \frac{M_0}{12} \Rightarrow 90 - 30y_1 = 8M_1 + M_0$

$720 + 144y_1 = 32M_1 - 8M_2$, $\quad 900 + 168y_1 = 33M_1$

$360 - 120y_1 = 32M_1 + 4M_0$

$-24y_1 = 4M_0 - M_1$

$360 - 96y_1 = 33M = 900 + 168y_1$

$-540 = 264y_1$

$\boxed{y_1 = -2.045}$ → (iii) Option B

(Q4) $F_i = y_i'' + 2y_i y_i' - 4 - 4x^3 = 0$, $y(1)=2$, $y(2)=4.5$, $h=0.25$

Initial guess = $y^{(0)}(x) = -0.5 + 2.5x$, $n = 4$

$$\frac{\partial F_i}{\partial y_i} = 2y_i', \quad \frac{\partial F_i}{\partial y_i'} = 2y_i, \quad \frac{\partial F_i}{\partial y_i''} = 1$$

For k^{th} iteration

$$\left(y^{(k+1)} - y^{(k)}\right) 2y'^{(k)} + \left(y'^{(k+1)} - y'^{(k)}\right) 2y^{(k)} + \left(y''^{(k+1)} - y''^{(k)}\right) = -y''^{(k)} - 2y^{(k)}y'^{(k)} + 4 + 4x^3$$

$$\Rightarrow y''^{(k+1)} + y'^{(k+1)}(2y^{(k)}) + y^{(k+1)}(2y'^{(k)}) - 2y^{(k)}y'^{(k)} - \cancel{2y^{(k)}y'^{(k)}} - \cancel{y''^{(k)}} + \cancel{y''^{(k)}} + \cancel{2y^{(k)}y'^{(k)}} - 4 - 4x^3 = 0$$

$$\rightarrow \boxed{y''^{(k+1)} + 2y^{(k)}y'^{(k+1)} + 2y'^{(k)}y^{(k+1)} = 2y^{(k)}y'^{(k)} + 4 + 4x^3}$$

↳ option B (i)

On Discretizing

$$\frac{y_{i+1}^{(k+1)} - 2y_i^{(k+1)} + y_{i-1}^{(k+1)}}{h^2} + \frac{\cancel{2}y_i^{(k)}}{\cancel{2}h}\left(y_{i+1}^{(k+1)} - y_{i-1}^{(k+1)} + 2y_i'^{(k)} y_i^{(k+1)}\right) = 2y_i^{(k)}y_i'^{(k)} + 4 + 4x_i^3$$

$$\rightarrow y_{i-1}^{(k+1)}\left[\frac{1}{h^2} - \frac{y_i^{(k)}}{h}\right] + y_i^{(k)}\left[-\frac{2}{h^2} + 2y_i'^{(k)}\right] + y_{i+1}\left[\frac{1}{h^2} + \frac{y_i^{(k)}}{h}\right] = 2y_i^{(k)}y_i'^{(k)} + 4 + 4x_i^3$$

$$A_i y_{i-1}^{(k)} + B_i y_i^{(k)} + C_i y_{i+1}^{(k)} = D_i$$

$A_i = \frac{1}{h^2} - \frac{y_i^{(k)}}{h}$, $\boxed{B_i = -\frac{2}{h^2} + \frac{y_{i+1}^{(k)} - y_{i-1}^{(k)}}{h}}$ → (ii) option C

$C_i = \frac{1}{h^2} + \frac{y_i^{(k)}}{h}$, $D_i = \frac{y_i^{(k)}}{h}\left(y_{i+1}^{(k)} - y_{i-1}^{(k)}\right) + 4 + 4x^3$ (iii) → option A

$$\begin{bmatrix} -6.75 & 6.625 & 0 \\ 0.75 & -6.75 & 7.25 \\ 0 & 0.125 & -6.75 \end{bmatrix}$$

↳ coefficient matrix

at $k=0, i=1$, $B_1 = -\frac{2}{h^2} + \frac{y_2^{(0)} + y_0^{(0)}}{h}$

$y_2^{(0)} = -0.5 + 2.5(1 + 2\times 0.25) = 2.5 \times 1.5 - 0.5 = 3.25$

$y_0^{(0)} = 2$

$y_2^{(0)} - y_0^{(0)} = 1.25 \Rightarrow \frac{y_2^{(0)} - y_0^{(0)}}{0.25} = 5 \Rightarrow B_1 = -\frac{2}{(0.25)^2} + 5 = -27$

$C_1 = \frac{1}{h^2} + \frac{y_1^0}{h} = 16 + 10.5 = 26.5$

$y_1^{(0)} = -0.5 + 2.5 \times 1.5 = 2.625$
$y_2^{(0)} = -0.5 + 2.5(1+2h) = 3.25$
$y_3^{(0)} = -0.5 + 2.5(1+3h) = 3.875$

$k=0, \ i=1$

$D_1 = \frac{y_1^{(0)}}{h}(y_2^{(0)} - y_0^{(0)}) + 4 + 4(x_0+h)^3$
$= 2.625 \times 4 \times (3.25-2) + 4 + 4(1.25)^3$
$= 24.9375 \times h = 6.234375$

$D_2 = \frac{y_2^{(0)}}{h}(y_3^{(0)} - y_1^{(0)}) + 4 + 4 \times (1.5)^3 = 33.75 \times h = 8.4375$

$D_3 = 1 + (1.75)^3 + 3.875 \times 1.25 = 11.203125$

$D_2^* = D_2, \quad D_1^* = D_1 - a_0 y_0 = 3.4846, \quad D_3^* = D_3 - C_3 y_4 =$ -24.2344

(iv) → option C

$$\begin{bmatrix} D_1^* \\ D_2^* \\ D_3^* \end{bmatrix} = \begin{bmatrix} 3.4864 \\ 8.4375 \\ -24.2344 \end{bmatrix}$$

(iii) option A

$$\text{coeff matrix} = \begin{bmatrix} -6.75 & 6.625 & 0 \\ 0.75 & -6.75 & 7.25 \\ 0 & 0.125 & -6.75 \end{bmatrix}$$