Lecture - 24 (09-04-2024)

Problem: Let $\mathcal{A} = \{A_i\}$ be a class of connected subsets of a topological space (X, T) with the nonempty intersection, i.e., $\bigcap_i A_i \neq \phi$. Then $B = \bigcup_i A_i$ is a connected set.

Sol: Given that each A_i connected set and $\bigcap_i A_i \neq \phi$.

— This implies no two members of $\mathcal{A} = \{A_i\}$ are not disjoint sets and hence not separated sets.

Hence by previous theorem $B = \bigcup_i A_i$ is connected set.

Problem: Let A be a connected subset of a topological space (X, T) and B is any subset of X such that $A \subset B \subset \bar{A}$. Then B is also connected set. In particular $\bar{A}$ is connected set.

Sol: Given that A is connected set and B is a subset of X with $A \subset B \subset \bar{A}$.

Claim: B is connected set.

Suppose B is disconnected set and let $G \cup H$ be disconnection of B.

$$\therefore B = (B \cap G) \cup (B \cap H)$$
$$\subseteq G \cup H$$

$$\varphi = (B \cap G) \cap (B \cap H), \quad B \cap H \neq \varphi$$
$$B \cap G \neq \varphi,$$
$$G, H \in \mathcal{T}.$$

Since $A \subset B \subseteq G \cup H$ and A is connected set, we have

either $A \cap G = \varphi$ or $A \cap H = \varphi$

[by one of the previous problems]

Suppose $A \cap H = \varphi$

$$\Rightarrow A \subset H^c$$

Thus H^c is a closed superset of the set A.

$\therefore A \subset B \subset \bar{A} \subset H^c$

$\Rightarrow B \cap H = \varphi$, this is contradiction to our assumption that $G \cup H$ is disconnection of B.

$\therefore B$ is a connected set.

$\because A \subset \bar{A} \subseteq \bar{A}$,

applying the above proof, we can say that $\bar{A}$ is connected set

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Theorem: Let (X, T) be a topological space. Then the following are equivalent:

(i) X is disconnected

(ii) There exists a non-empty proper subset of X which is both open and closed set.

Proof: (i) $\Longrightarrow$ (ii)

Suppose X is disconnected and $G \cup H$ be disconnection of X.

$\therefore$ $X = G \cup H$, $G \neq \phi$, $H \neq \phi$,
$G \cap H = \phi$,
$G, H \in \mathcal{T}$.

Then G is a non empty proper open sub-set of X.

Also $G = X - H$
$= H^c$ is a closed set,
since H is an open set.

Thus there exists a proper sub-set G of X which is both open and closed.

Now we prove
(ii) $\Longrightarrow$ (i)

Assume that there exists a proper sub-set say A of X which is both open and closed.
Then A^c is also nonempty and open sub-set of X.

Hence $X = A \cup A^c$,

$A \neq \varphi$, $A^c \neq \varphi$, $A \cap A^c = \varphi$

and A, A^c are open sets.

$\therefore$ X is disconnected space.

H.W

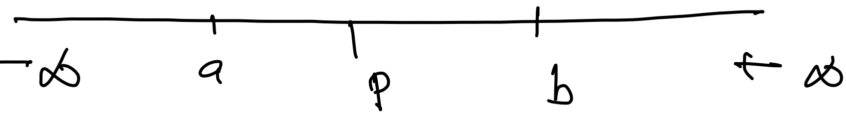
Theorem: Let A be any subset of a topological space (X, T) and T_A be the relative topology on A. Then A is T-connected iff A is T_A-connected.

Theorem: Let E be a subset of the real line R containing atleast two points. Then E is connected iff E is an interval.

Proof:

[We say a set I is an interval if $a, b \in I$ with $a < p < b$, then $p \in I$]

Suppose E is not an interval.
Then there exist $a, b \in E$, $p \notin E$ such that $a < p < b$.



Set $G := (-\infty, p)$

$H := (p, \infty)$

Then $a \in G$ and $b \in H$

Hence $a \in E \cap G \Rightarrow E \cap G \neq \varphi$

$b \in E \cap H \Rightarrow E \cap H \neq \varphi$

and $(E \cap G) \cap (E \cap H) = \varphi$,

and

$$E = (E \cap G) \cup (E \cap H)$$

$\therefore$ E is disconnected set.

Now Suppose E is an interval.
We prove E is connected set.

Assume E is disconnected and let $G \cup H$ be disconnection of E.

Then $E = (E \cap G) \cup (E \cap H)$,

$\varphi = (E \cap G) \cap (E \cap H)$,

$E \cap G \neq \varphi$, $E \cap H \neq \varphi$, G, H are open sets.

Denote

$$A := E \cap G \quad , \quad B := E \cap H$$

Then $E = A \cup B$, $A \neq \varphi$, $B \neq \varphi$.

Let $a \in A = E \cap G$ and $b \in B = E \cap H$.

and let

$$p = \sup \{ A \cap [a, b] \}$$

$$= \sup \{ (E \cap G) \cap [a, b] \}.$$

$\because$ $[a, b]$ is a closed set $p \in [a, b]$.

Now since $a, b \in E$ and $p \in [a, b]$ and E is an interval, implies $p \in E$.

Again since $E = A \cup B$ and

$p \in E = A \cup B$

$\Rightarrow$ either $p \in A$ or $p \in B$

Suppose $p \in A = E \cap G$.

Then $p < b$ [$\because b \in B = E \cap H$

and $p \in G$ $A \cap B = \varphi$]

Since G is an open set, $p \in G$, there exists $\delta > 0$ such that $(p, p+\delta) \subset G$.

Thus implies $p + \delta < b$

Now $p < p+\delta < b$

$\Rightarrow p+\delta \in [p, b] \subset [a, b]$ — $(*)$

Also $p < p+\delta < b$, $p, b \in E$, E is an interval

$\Rightarrow p+\delta \in E$.

Thus $p+\delta \in E \cap G$ — $(**)$

From $(*)$ and $(**)$ we have

$p+\delta \in (E \cap G) \cap [a, b] = A \cap [a, b]$.

This is contradiction to

$p = \sup\{A \cap [a, b]\}$

$\therefore \; p \notin A$.

On the other hand, suppose $p \in B = E \cap H$.

Then $p \in H$ and since H is an open set there exists $\delta^* > 0$ such that $[p - \delta^*, p] \subset H$ and $a < p - \delta^*$.

$\because \; a < p - \delta^* < p$, $a, p \in E$, E is an interval, implies $p - \delta^* \in E$.

$\Rightarrow [p - \delta^*, p] \subset E$

$\Rightarrow [p - \delta^*, p] \subset E \cap H = B$

Accordingly $[p - \delta^*, p] \cap A = \varphi$ $[\because A \cap B = \varphi]$

But then $p - \delta^*$ is an upper bound for the set $A \cap [a, b]$, which is impossible, since $p = \sup\{A \cap [a, b]\}$

Again our assumption that $P \in B = E \cap H$ is wrong.

$\therefore P \notin B.$

Thus we have

$E = A \cup B$, and $P \notin A$, $P \notin B$, but $P \in E$. This is contradiction

$\therefore E$ is connected set

Components :—

A component E of a topological (X, T) is a maximal connected subset of X. That is E is connected set and E is not a proper subset of any connected subset of X.

Clear the Component $E \neq \varphi$

Problem: Every Component is a Closed set.

Sol: Let E be a Component in a topological space (X, T).

Then E is Connected.

Then $\bar{E}$ is also Connected and $E \subseteq \bar{E}$

But E is a Component

$\therefore E = \bar{E}$

$\therefore E$ is a closed set.

Problem: Let (X, T) be a topological space and $P \in X$. Let $\mathcal{A}_p = \{A_i\}$ be the class of connected subsets of X containing $P \in X$. Furthermore, let $C_p = \bigcup_i A_i$. Then

(i) C_p is Connected set

(ii) If B is any connected subset of X containing $p \in X$, then $B \subset C_p$

(iii) C_p is a component.

Proof: Since each $A_i \in \mathcal{A}_p$ containg $p \in X$ implies $p \in \bigcap_i A_i \neq \varphi$ and A_i is connected set, implies $C_p = \bigcup_i A_i$ is a connected set.

If B is any connected set containing $p \in X$, then $B \in \mathcal{A}_p = \{A_i\}$

$\Rightarrow \quad B \subset \bigcup_i A_i = C_p$.

Now suppose D is any connected subset of X with $C_p \subset D$

$\therefore \; p \in C_p \subset D$

$\Rightarrow p \in D$

Thus D is a connected set containing $p \in X$.

$\therefore \; D \subset \mathcal{A}_p = \{A_i\}$

$\Rightarrow \; D \subset \bigcup_i A_i = C_p$

$\Rightarrow \; D \subset C_p$

$\therefore \; C_p = D$

$\Rightarrow$ C_p is a component.

Theorem: The components of a topological space $(X, \mathcal{T})$ form a partition of X. Every connected subset of X is contained in some component of X.

Proof: Consider the class

$$\mathcal{C} = \{ C_p \mid p \in X \}$$ be

the class of components of X.

If D is a component, then D contains some point $p_0 \in X$. Then

$$D \subset C_{p_0}.$$

But D is a component, so $D = C_{p_0}$.

$\therefore$ $\mathcal{C}$ consists of all the components of X.

Claim: the class $\mathcal{C}$ is partition of X.

Clearly

$$X = \cup \{ C_p \mid p \in X \},$$

Hence we only need to show that distinct components are disjoint.

i.e., if $C_p \cap C_q \neq \phi$

$$\Rightarrow C_p = C_q .$$

Let $a \in C_p \cap C_q \Rightarrow a \in C_p$ and $a \in C_q$

$\Rightarrow a \in C_p$ and C_p is a connected set

$$\Rightarrow C_p \subseteq C_a$$

lllly $C_q \subseteq C_a$

But C_p and C_q are components

$$\therefore C_a = C_p = C_q .$$

If E is a non empty connected subset of X containing $p_0 \in X$, then $E \subseteq C_{p_0}$.

If $E = \phi$, then E is contained in every component.

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Attendence

[11, 27, 06, 60]