Lecture-5 (22-01-2024)

Problem: A set G in a topological space (X, T) is open iff it is nbd of each of its points.

Sol: Suppose G is an open set.
Then each $p \in G$ belongs to open G such that

$$p \in G \subset G$$

$\Rightarrow$ G is a nbd of $p \in G$

$\because$ p is arbitary point of G, it follows that G is nbd of each of its points.

Conversely assume that G is nbd of each of its points.
So for each $p \in G$, $\exists$ an open set G_p such that

$$p \in G_p \subset G.$$

$\Rightarrow$ G is an open set $\because$ $G = \bigcup_{p \in G} G_p$

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* let (X, τ) be a topological space.

For any $p \in X$, $N_p \neq \varphi$.

$\because$ $p \in X \subseteq X$, $X \in \tau$ $\Big\} \Rightarrow X \in N_p$

$\therefore N_p \neq \varphi$.

* If $N, M \in N_p$, then $N \cap M \in N_p$.

$\because N, M \in N_p \Rightarrow p \in N$ and $p \in M$

$\Rightarrow p \in N \cap M$.

$\because p \in N \Rightarrow \exists\, G_p \in \tau \ni$

$p \in G_p \subset N$

$\because p \in M \Rightarrow \exists\, H_p \in \tau \ni$

$p \in H_p \subset M$.

$\Rightarrow p \in G_p \cap H_p \subset N \cap M$

and $G_p \cap H_p \in \tau$

$\therefore N \cap M$ is nbd of p.

* Every Superset of a Member of N_p also belongs to N_p in in a topological Space (X,T).

Sol: Suppose $N \in N_p$ and $M \subset X$ with $N \subset M$.

$\because$ $p \in N$, $\exists$ $G_p \in T$ $\ni$

$$p \in G_p \subset N \subset M$$

$$\Rightarrow p \in G_p \subset M$$

$$\Rightarrow M \in N_p$$

Convergent Sequence:

A sequence $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ of points in a topological Space (X,T) is said to converges to a point $p \in X$ or p is the limit of the Sequence $\{a_n\}$ if for each open set G containing p containing almost all terms terms of the sequence $\{a_n\}$.

- That is $\exists\ n_0 \in \mathbb{N} \ni$
$\forall\ n \geq n_0 \Rightarrow a_n \in G$.

[$a_n \to P$, if $\forall\ G \in T$ with $P \in G$
$\exists\ n_0 \ni a_n \in G,\ \forall\ n \geq n_0$].

Ex:- (1) let (X, I) be indiscrete topological space.
let $\{a_n\}$ be a sequence in X.

So for any point $P \in X$, X is the only open set containing P and X contains all the terms of the sequence $\{a_n\}$.

$\therefore$ The sequence $\{a_n\}$ converges to every point $P \in X$.

(2) Let (X, D) be a discrete topological space and $\{a_n\}$ be a sequence in X such that $a_n \longrightarrow p \in X$.

Since $\{p\}$ is an open set in a topological space (X, D), $\{p\}$ must contain almost all terms of the sequence $\{a_n\}$.

$$\therefore \{a_n\} = \{a_1, a_2, \ldots a_{n_0}, p, p, p, p, \ldots\}.$$

(3). Let (R, T) be a cofinite topological space. Let $\{a_n\} = \{a_1, a_2, a_3 \ldots\}$ be a sequence of distinct terms in R. Then the sequence $\{a_n\}$ converges to every real number p.

Sol: Let G be any open set containing $p \in R$.

$\Rightarrow G^c$ is a finite set by definition of T.

$\therefore \mathbb{R} = G \cup G^c$

Hence G contains almost all terms of the sequence $\{a_n\}$.

$\therefore a_n \longrightarrow p$, for every $p \in \mathbb{R}$.

Ex 4: let (X, T) be a cocountable topological space.

i.e., $T = \{A \subseteq X, \phi \mid A^c$ is countable$\}$.

let $\{a_n\}$ be a sequence in X such that $a_n \longrightarrow p \in X$

$\{a_1, a_2, a_3, a_4 \cdots \} \longrightarrow p \in X$.

let $A = \{a_n \mid a_n \neq p\}$.

i.e., A consists of the terms of the sequence $\{a_n\}$ different from p.

Then A is a countable set.

— Then A^c is an open set containing p.
$[\because X = A \cup A^c]$

— Thus A^c is open set containing p with $a_n \longrightarrow p$.

$\Rightarrow$ A^c contains almost all terms of the sequence $\{a_n\}$ [by definition of a cgt sequence]

So $\exists\, n_0 \in \mathbb{N} \ni$

$\forall\, n \geq n_0 \quad a_n \in A^c$.

$\Rightarrow$ A is a finite set.

$\therefore \{a_n\} = \{a_1, a_2, \ldots a_{n_0}, p, p, p, p \ldots\}$

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[Attendance
11, 65, 62, 27, 57, 60].