## TOPICS COVERED

1. Solution of Non-linear BVP
2. Newton's method for linearisation
3. Quasi-linear form of BVP

## NON-LINEAR BVP

$$y'' = f(x, y, y')$$

$$y(a) = y_a \quad , \quad y(b) = b$$

$$x_i = a + i\,\delta x \quad , \qquad a < x < b$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right) = 0$$

$$\Rightarrow F_i(y_{i-1}, y_i, y_{i+1}) = 0$$

TASK: To solve $(N-1)$ non linear algebraic equations involving $(N-1)$ variables $y_1, y_2, \ldots, y_{N-1}$.

We solve the n/ linear system (*) iteratively using the NEWTON-RAPHSON METHOD.

At any kth iteration, let $y_i^{(k)}$ be the approximation and $\Delta y_i$ is the error.

$$\underbrace{y_i}_{\text{exact value}} = \underbrace{y_i^{(k)}}_{\text{approximate value}} + \underbrace{\Delta y_i}_{\text{error}}$$

which satisfy (*), i.e.

$$F_i\left(y_{i-1}^{(k)} + \Delta y_{i-1}, \; y_i^{(k)} + \Delta y_i, \; y_{i+1}^{(k)} + \Delta y_{i+1}\right) = 0$$

We expand using the Taylor Series (for 3 variables)

$$F_i\left(y_{i-1}^{(k)}, y_i^{(k)}, y_{i+1}^{(k)}\right) + \frac{\partial F_i}{\partial y_{i-1}}\bigg|^{(k)} \Delta y_{i-1} + \frac{\partial F_i}{\partial y_i}\bigg|^{(k)} \Delta y_{i-1} + \frac{\partial F_i}{\partial y_{i+1}}\bigg|^{(k)} \Delta y_{i+1}$$

$$+ \cdots$$

$$y_i = y_i^{(k)} + \Delta y_i$$

$$\Delta y_0 = 0 = \Delta y_N \quad \because \quad y_0 \text{ and } y_N \text{ are prescribed.}$$

NEGLECTING higher order ($\geq 2$) derivative terms, we get

$$F_i(y_{i-1}, y_i, y_{i+1}) + \frac{\partial F_i}{\partial y_{i-1}}\bigg|^{(k)} \Delta y_{i-1} + \frac{\partial F_i}{\partial y_i}\bigg|^{(k)} \Delta y_i + \frac{\partial F_i}{\partial y_{i+1}}\bigg|^{(k)} \Delta y_{i+1}$$

which involves unknowns as $\Delta y_1, \Delta y_2, \ldots \Delta y_{N-1}$
appearing linearly in the system of $(N-1)$ eq$^n$.

Which forms an $(N-1) \times (N-1)$ tri-diagonal system,

$$AX = b \quad , \quad X^T = [\Delta y_1 \quad \Delta y_2 \quad \ldots \quad \Delta y_{N-1}]$$

Solving this gives us an approximate value of the error term

We use these to get the approximation for the $(k+1)^{th}$ iteration -

$$y_i^{(k+1)} = y_i^{(k)} + \Delta y_i$$

$$i = 1, 2, \ldots, N-1$$

Which generates a sequence of iterates

$$y_i^{(k)} \quad , \quad k = 0, 1, \ldots$$

initial assumption

To start the method, $y_i^{(0)}$ needs to be prescribed.

The procedure is repeated till the convergence criterie is achieved.

$$|\Delta y_i| < \varepsilon_i \;,\; |y_i^{(k+1)} - y_i^{(k)}| < \varepsilon$$

$$\forall \; i, k \geq K$$

$$\Rightarrow |\Delta y_i| < \varepsilon \quad \forall \; i \;, \underset{\substack{\uparrow \\ \text{a finite number}}}{k \geq K}$$

$$\max_{1 \leq i \leq N-1} |\Delta y_i| < \varepsilon$$ , $\varepsilon$ is an arbitrarily small, positive number.

<u>STOP</u>

## PROs of this method

* Quadratic rate of convergence. ( in practice we see that convergence occurs in 4-5 iterations of N-R
If it seems to be taking longe change $y_i^{(0)}$ guess)

**Exercise 1.**

$$y y'' + 1 + (y')^2 = 0$$

$$y(0) = 1 \;,\; y(1) = 2$$

Write as $a_i \Delta y_{i-1} + b_i \Delta y_i + c_i \Delta y_{i+1} = d_i$

$$i = 1, 2, \ldots, N-1$$

At $i$, $y_i y_i'' + 1 + (y_i')^2 = 0$

$$y_i \left( \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) + 1 + \left( \frac{y_{i+1} - y_{i-1}}{2h} \right)^2 = 0$$

$$\underbrace{\frac{y_i y_{i+1} - 2y_i^2 + y_i y_{i-1}}{h^2} + 1 + \frac{y_{i+1}^2 - 2y_{i+1} y_{i-1} + y_{i-1}^2}{4h^2}}_{F_i} = 0$$

$$\frac{\partial F}{\partial y_{i-1}} = \frac{y_i}{h^2} - \frac{\cancel{2} y_{i+1}}{2 \cancel{4} h^2} + \frac{\cancel{2} y_{i-1}}{2 \cancel{4} h^2} = \frac{y_i}{h^2}$$

$$F_i(y_{i-1}, y_i, y_{i+1}) + \left.\frac{\partial F_i}{\partial y_{i-1}}\right|^{(k)} \Delta y_{i-1} + \left.\frac{\partial F_i}{\partial y_i}\right|^{(k)} \Delta y_i + \left.\frac{\partial F_i}{\partial y_{i+1}}\right|^{(k)} \Delta y_{i+1}$$

$$\frac{y_{i+1}}{h^2} - \frac{4y_i}{h^2} + \frac{y_{i-1}}{h^2}$$

Solve for $h = 1/4 = 0.25$

Consider the problem:

- $$y'' - (y')^2 - y^2 + y + 1 = 0$$

$$\text{B.C.} \begin{cases} y(0) = 0.5 \\ y(\pi) = -0.5 \end{cases}$$

Solve iteratively by linearisation technique and obtain the ensuing tri-diagonal system at each iteration.

$$\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) - \left(\frac{y_{i+1} - y_{i-1}}{2h}\right)^2 - y_i^2 + y_i + 1 = 0$$

$$\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) - \frac{y_{i+1}^2 - 2y_{i+1}y_{i-1} + y_{i-1}^2}{4h^2} - y_i^2 + y_i + 1 = 0$$

$$\boxed{\frac{\partial F}{\partial y_{i-1}} = \frac{1}{h^2} - \frac{y_{i-1} - y_{i+1}}{2h^2} \quad ; \quad \frac{\partial F}{\partial y_i} = -\frac{2}{h^2} - 2y_i + 1 \qquad \frac{\partial F}{\partial y_{i+1}} = \frac{1}{h^2} - \frac{y_{i+1} - y_{i-1}}{2h^2}}$$

$$\overbrace{\left(\frac{1}{h^2} - \frac{y_{i-1} - y_{i+1}}{2h^2}\right)}^{a_i} \Delta y_{i-1} + \overbrace{\left(-\frac{2}{h^2} - 2y_i + 1\right)}^{b_i} \Delta y_i + \overbrace{\left(\frac{1}{h^2} - \frac{y_{i+1} - y_{i-1}}{2h^2}\right)}^{c_i} \Delta y_{i+1}$$

$$= -\underbrace{\left(\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) - \frac{y_{i+1}^2 - 2y_{i+1}y_{i-1} + y_{i-1}^2}{4h^2} - y_i^2 + y_i + 1\right)}_{d_i}$$

$$y_i^{(k+1)} = y_i^{(k)} + \Delta y_i$$

$$\max_{1 \le i \le N-1} |\Delta y_i| < \varepsilon << 1$$

[Takeup this problem in the lab class, be careful about storage]

## Alternative (illustrated by example) (HT)

Q. $3yy'' + (y')^2 = 0$

B.C. $\begin{cases} (y')^2 = 0 \\ y(0) = 0, \quad y(1) = 2 \end{cases}$

Ans. Discretising -

$$3y_i \left( \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) + \left( \frac{y_{i+1} - y_{i-1}}{2h} \right)^2 = 0$$

$$y_i = y_i^{(k)} + \Delta y_i$$

Replace all $y_j$ with $y_j^{(k)} + \Delta y_j$ and expand.

Dropping the square and higher orders terms of $\Delta y_i$s, we get

$$\Delta y_{i-1} [\cdots] + \Delta y_i [\cdots] + \Delta y_{i+1} [\cdots] = d_i$$

$$i = 1, 2, \ldots, N-1$$

$AX = d$ system.

Q. $y'' + 2yy' = 4 + 4x^3$

B.C.s $\begin{cases} y(1) = 2 \\ y(2) = 4.5 \end{cases}$

Ans.

$$\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + \cancel{2}y_i\left(\frac{y_{i+1} - y_{i-1}}{\cancel{2}h}\right) - 4x_i^3 - 4 = 0$$

$$a_i = \frac{\partial F}{\partial y_{i-1}} = \frac{1}{h^2} - \frac{y_i^{(k)}}{h}$$

$$b_i = \frac{\partial F}{\partial y_i} = \frac{y_{i+1}^{(k)} - y_{i-1}^{(k)}}{h} - \frac{2}{h^2}$$

$$c_i = \frac{\partial F}{\partial y_{i+1}} = \frac{1}{h^2} + \frac{y_i^{(k)}}{h}$$

$$d_i = 4 + 4x_i - y_i\left(\frac{y_{i+1} - y_{i-1}}{h}\right) - \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right)$$

Fair initial approximation:

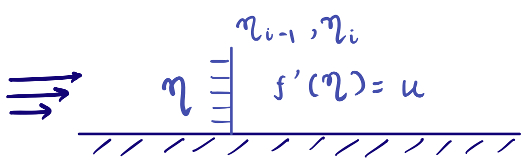
$$y - 2 = \frac{2.5}{1}(x - 1)$$

$$\boxed{y^{(0)}(x) = 2.5x - 0.5}$$

→ Ultimate solution should not depend on initial approx.

Q. $f''' + ff'' + 1 - (f')^2 = 0$

$f(0) = 0, \quad f'(0) = 0, \quad f'(10) = 1$




modeling of fluid velocity

$$F = f'$$

$$F'' + fF' + 1 - F^2 = 0$$

$$f(0) = 0, \quad F(0) = 0, \quad F(10) = 1$$

Steps:

1. Discretise eq$^n$
2. Solve discretised equation

$$f_i^{(k+1)} = f_i^{(k)} + \Delta f_i, \quad F_i^{(k+1)} = F_i^{(k)} + \Delta F_i$$

$$X_i = \begin{pmatrix} \Delta f_i \\ \Delta F_i \end{pmatrix} \rightarrow k \geq 0$$

Get the block tri-diagonal equation

$$\bar{A}_i X_{i-1} +$$

Ans. Given:

$$F = f'$$

$$F = \frac{df}{d\eta} \Rightarrow F\, d\eta = df$$

$$\Rightarrow \int_{\eta_{i-1}}^{\eta_i} F\, d\eta = \int_{\eta_{i-1}}^{\eta_i} df$$

$$\Rightarrow \frac{\delta\eta_i}{2}(F_i + F_{i-1}) = f_i - f_{i-1}$$

$$f_i - f_{i-1} - \frac{\delta\eta}{2}(F_i + F_{i-1}) = f_i - \frac{h}{2}(F_i + F_{i-1}) - f_{i-1} = 0$$

$$\Delta f_i - \frac{h}{2}(\Delta F_i + \Delta F_{i-1}) - \Delta f_{i-1} = 0$$

$$F'' + fF' + 1 - F^2 = 0$$

$$f(0) = 0, \quad F(0) = 0, \quad F(10) = 1$$

$$\frac{F_{i+1}-2F_i+F_{i-1}}{h^2} + f_i\,\frac{F_{i+1}-F_{i-1}}{2h} - F_i^2 + 1 = 0 \longrightarrow P$$

$$\frac{\partial P}{\partial F_{i-1}} = a_i = \frac{1}{h^2} - \frac{f_i}{2h} \; ; \quad b_i = -\frac{2}{h^2} - 2F_i = \frac{\partial P}{\partial F_i}$$

$$\frac{\partial P}{\partial f_i} = \frac{F_{i+1}-F_{i-1}}{2h}$$

$$c_i = \frac{1}{h^2} + \frac{f_i}{2h} = \frac{\partial P}{\partial F_{i+1}}$$

$d_i =$ "

$$A_i = \begin{pmatrix} -1 & -\frac{h}{2} \\ 0 & \frac{1}{h^2} - \frac{f_i^{(k)}}{2h} \end{pmatrix} \text{ for } \begin{bmatrix} \Delta f_{i-1} \\ \Delta F_{i-1} \end{bmatrix}$$

$$B_i = \begin{pmatrix} 1 & -h/2 \\ \frac{F_i^{(k)} - F_{i-1}^{(k)}}{2h} & -\frac{2}{h^2} - 2f_i^{(k)} \end{pmatrix} \text{ for } \begin{bmatrix} \Delta f_i \\ \Delta F_i \end{bmatrix}$$

$$C_i = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{h^2} + \frac{f_i^{(k)}}{2h} \end{pmatrix} \text{ for } \begin{bmatrix} \Delta f_{i+1} \\ \Delta F_{i+1} \end{bmatrix}$$

Solve using Block Tridiagonal method

HT

Q. $F''' + (2F+4)F' = 0$

B.C. $\left\{ F(0) = 0, \quad F''(0) = -0.1, \quad F'(0.087) = 0 \right.$

Another imp. BVP - modelling flow through orifice

Use fictitious point to handle $F''(0) = -0.1 = z'(0)$
as $z = F'$

$\Longrightarrow f'$

## NEW TECHNIQUE

Currently — Newton Linearisation technique

Solving algebraic system iteratively

↳ Not always good.

NEW

## QUASI-LINEARISATION TECHNIQUE [Original paper: 1957, Bellman & Kolaba]

Consider a BVP

$$F(y'', y', y) = 0, \quad a < x < b$$

$$y(a) = y_a, \quad y(b) = y_b$$

→ $x$ is the variable
→ $F$ is a $f^n$ of $f^n$s.

Let $y^{(k)}, y'^{(k)}, y''^{(k)}$ be the approximate form of $y, y', y''$ respectively at any iteration $k$.

We expand $F$ by Taylor Series about the approximate form $y^{(k)}, y'^{(k)}, y''^{(k)}$ by treating $F$ as a function of $y, y', y''$, we get

$$F(y^{(k)}, y'^{(k)}, y''^{(k)}) + (y - y^{(k)}) \left.\frac{\partial F}{\partial y}\right|^{(k)}$$

$$+ (y' - y'^{(k)}) \left.\frac{\partial F}{\partial y'}\right|^{(k)} + (y'' - y''^{(k)}) \left.\frac{\partial F}{\partial y''}\right|^{(k)} + \cdots$$

$$+ (y' - y') \frac{\partial F}{\partial y'}\Big| + (y'' - y'') \frac{\partial F}{\partial y''}\Big| + \dots$$

Neglecting the square and higher order terms, we get the modified form of $y, y', y''$ denoted by $y^{(k+1)}$, $y'^{(k+1)}$, $y''^{(k+1)}$, satisfying -

$$\left(y^{(k+1)} - y^{(k)}\right) \frac{\partial F}{\partial y}\Big|^{(k)} + \left(y'^{(k+1)} - y'^{(k)}\right) \frac{\partial F}{\partial y'}\Big|^{(k)}$$

$$+ \left(y''^{(k+1)} - y''^{(k)}\right) \frac{\partial F}{\partial y''}\Big|^{(k)} = -F\left(y^{(k)}, y'^{(k)}, y''^{(k)}\right)$$

... QUASI LINEAR FORM

we have reduced the given non-linear equation to a linear equation of $y^{(k+1)}$, which is referred to as the quasi-linear form of the given BVP.

ADVANTAGES : Reduction of non-linear system to linear system.

But it does not end here, we also have to solve this obtained linear system.

Q. Find the QUASILINEAR FORM of

$$3yy'' + (y')^2 = 0$$

$$y(0) = 0$$

$$y(1) = 1$$

$$\left(y^{(k+1)} - y^{(k)}\right)\left(3y''^{(k)}\right) + \left(y'^{(k+1)} - y'^{(k)}\right)\left(2y'^{(k)}\right) + \left(y''^{(k+1)} - y''^{(k)}\right)\left(3y^{(k)}\right) = 0$$

$\longrightarrow$ RHS is wrong ?

$$3y^{(k)}\, y''^{(k+1)} + 2y'^{(k)}\, y'^{(k+1)} + 3y''^{(k)}\, y^{(k+1)} = 3y^{(k)} y''^{(k)} + 2\left(y'^{(k)}\right)^2 + 3y^{(k)} y''^{(k)}$$

(check ?)

(Try all given eq^ns w/ both linearisation techniques)