Lecture 32

(5) Let $f, g : [a,b] \to \mathbb{R}$ be continuous &

$\int_a^b f = \int_a^b g$. Show that there exists a point

$c \in (a,b)$ such that $f(c) = g(c)$.

Sol:- Let $h = f - g : [a,b] \to \mathbb{R}$,

Then $\int_a^b h = 0$ & h is continuous on $[a,b]$

Let $F(x) = \int_a^x h(t)\,dt$, $\forall x \in [a,b]$.

Then $F(a) = F(b) = 0$

Also F is continuous on $[a,b]$ & differentiable on (a,b)

& $F'(x) = h(x)$.

$\therefore$ By Rolle's theorem there exists $c \in (a,b)$

such that $F'(c) = 0$.

$\Rightarrow h(c) = 0 \Rightarrow f(c) = g(c)$.

⑥ Let $f:[0,1] \to \mathbb{R}$ be Continuous &

$$\int_0^x f(t)\,dt = \int_x^1 f(t)\,dt, \qquad \forall x \in [0,1].$$

Show that $f(x)=0$ $\forall x \in [0,1]$. That is,

$f \equiv 0$.

Sol:- From the given assumption we have

$$\int_0^1 f(t)\,dt = 0$$

Let $F(x) = \int_0^x f(t)\,dt$. $\forall x \in [0,1]$.

$$F(0) = 0 = F(1)$$

& F is Continuous on $[0,1]$ & differentiable on $(0,1)$, $F'(x) = f(x)$.

$F(1)=0 \Rightarrow$ for any $c \in (0,1)$, we have

$$\int_0^c f + \int_c^1 f = 0$$

$$\Rightarrow \quad 2\int_0^c f = 0 \quad \left(\because \int_0^c f = \int_c^1 f\right)$$

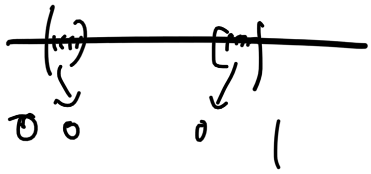
$\Rightarrow \displaystyle\int_0^c f = 0$, for any $c \in (0,1)$.

$\Rightarrow F(c) = 0$ for all $c \in (0,1)$.

$\Rightarrow F(x) \equiv 0$ on $(0,1)$.

$\Rightarrow f(x) = 0 \quad \forall x \in (0,1)$.

& $f(0) = 0 = f(1)$



$\therefore f \equiv 0$ on $[0,1]$.

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(7) Let $\{f_n\}$ be a sequence of Riemann-integrable functions & $f_n \to f$ p.w. Does f Riemann-integrable?

Sol:- Ans: No. Let $\mathbb{Q} = \{r_1, r_2, \dots\}$.

Let $f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, \dots, r_n\} \\ 0 & \text{otherwise.} \end{cases}$

$\forall n \geq 1$.

Then $$f_n(x) \longrightarrow f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

$$= \chi_{\mathbb{Q}}(x).$$

as $n \to \infty$.

By Lebesgue criterion, each f_n is Riemann-integrable because f_n is continuous a.e.

But $f = \chi_{\mathbb{Q}}$ is not Riemann-integrable.

<u>Theorem</u> (<u>Dominated Convergence theorem</u>) :-

Suppose $\{f_n\}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ a.e, as $n \to \infty$. If $|f_n(x)| \le g(x)$, where g is Lebesgue-integrable, then

$\left(\int_{\mathbb{R}^d} g < \infty\right)$

$$\int_{\mathbb{R}^d} |f_n - f| \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

& Consequently, $\int_{\mathbb{R}^d} f_n \longrightarrow \int_{\mathbb{R}^d} f$ as $n \to \infty$.

proof:- For each $N \geq 0$, let

$\boxed{E_N \subseteq B(0,N)}$

$$E_N = \{ x \in \mathbb{R}^d \mid |x| \leq N, \ g(x) \leq N \}.$$

Given $\varepsilon > 0$.

$\boxed{\lim_{N \to \infty} E_N = \mathbb{R}^d}$

Let $g_N(x) = g(x) \chi_{E_N}(x) \quad \forall N \geq 0$.

Then $g_N \geq 0$, measurable & $g_N(x) \leq g_{N+1}(x) \quad \forall N \geq 0$.

& $\lim_{N \to \infty} g_N(x) = g(x)$. $\quad (\because E_N \subseteq E_{N+1})$

$\therefore$ By Monotone Convergence theorem, we have

$$\lim_{N \to \infty} \left(\int g_N \right) = \int g$$

In particular, there exists some $N \in \mathbb{N}$ such that

$$0 \leq \left(\int g - \int g_N\right) < \varepsilon$$

$$\begin{aligned}\int g - \int g_N &= \int (g - g_N) \\ &= \int (g - g\chi_{E_N}) \\ &= \int g(1 - \chi_{E_N}) \\ &= \int g\chi_{E_N^c} \\ &= \int_{E_N^c} g\end{aligned}$$

$$\therefore \quad 0 \leq \int_{E_N^c} g < \varepsilon$$

Now

$$|f_n \chi_{E_N}| \leq |g\chi_{E_N}| = g\chi_{E_N}, \quad \forall n \geq 1$$

$$\leq N$$

$\therefore \{f_n \chi_{E_N}\}$ is bounded & also supported

on the set E_N of finite measure.

& $\quad f_n \chi_{E_N} \longrightarrow f \quad$ a.e

$\therefore$ By Bounded Convergence theorem,

$$\int_{E_N} |f_n - f| \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

$\Rightarrow$ for n sufficiently large we have

$$\int_{E_N} |f_n - f| < \varepsilon \quad \text{①}$$

Now for n sufficiently large, we have

$$\int_{\mathbb{R}^d} |f_n - f| = \int_{E_N \cup E_N^c} |f_n - f|$$

$$= \int_{E_N} |f_n - f| + \int_{E_N^c} |f_n - f|$$

$$\leq \varepsilon + \int_{E_N^c} |f_n - f|$$

$$< \varepsilon + \int_{E_N^c} 2g$$

$$= \varepsilon + 2\int_{E_N^c} g$$

$$< \varepsilon + 2\varepsilon = 3\varepsilon.$$

$|f_n - f| \leq |f_n| + |f|$

$\leq g + g.$

$= 2g.$ a.e

$|f_n| \leq g$

$\lim_{n\to\infty} |f_n(x)| \leq g$

$\|$

$|f(x)|$ a.e

$\therefore$ for n sufficiently large,

$$\int |f_n - f| < 3\varepsilon$$

$$\Rightarrow \quad \int |f_n - f| \longrightarrow 0 \text{ as } n \to \infty.$$

Complex valued functions.

Let $f: \mathbb{R}^d \to \mathbb{C}$ be a function. Write

$$f(x) = u(x) + i\, v(x),$$

where u, v are real & imaginary parts of f respectively. That is u, v are real valued functions

Definition: We say that f is Lebesgue measurable if both u & v are measurable.

Definition: We say that f is Lebesgue integrable if the function,

$$|f(x)| = |u(x) + i\, v(x)| = \sqrt{u(x)^2 + v(x)^2}$$

is Lebesgue Integrable.

$$\left(\text{i.e., } \int_{\mathbb{R}^d} |f| < \infty \right).$$

$(u, v: \mathbb{R}^d \to \mathbb{R})$