Closed Graph Theorem:

Let X and Y be Banach spaces. Then every closed operator $A : X \longrightarrow Y$ is a continuous operator.

Proof: Let X and Y be Banach spaces and $A : X \longrightarrow Y$ be a closed operator.

Claim: A is a continuous operator.

Let $B_0 = \{ x \in X \,/\, \|x\| < 1 \}$

We show that

$$B_0 \subseteq \{ x \in X \,/\, \|Ax\| \leq c \}$$

for some $c > 0$, so that A is continuous operator.

For each $\alpha > 0$, let

$$V_\alpha = \{ x \in X \,/\, \|Ax\| \leq \alpha \}$$

Then $X = \bigcup_{j=1}^{\infty} \overline{V_j}$.

Since X is a Banach space, by Baire-Category theorem, there is some $k \geq 0$ such that $\overline{V_k}^{\circ} \neq \varphi$.

Then there is some $x_0 \in X$ and $r > 0$ such that

$$B(x_0, r) \subset \overline{V_k}.$$

Let $x \in B_0 = \{x \in X \mid \|x\| < 1\}$ and set $u = x_0 + r x$

Then $\|u - x_0\| = \|r x\| = r \|x\| < r$

$$\Rightarrow \quad u \in B(x_0, r) \subset \overline{V_k}$$

Now $u, x_0 \in \overline{V_k}$, implies there exist sequency $\{u_n\}$ & $\{v_n\}$ in V_k

such that $u_n \to u$, $v_n \to x_0$.

$\because$ $u_n, v_n \in V_k \Rightarrow \|Au_n\| \leq k$

$\|Av_n\| \leq k$.

Also, since $u = x_0 + \gamma x$, implies

$$x = \frac{1}{\gamma}(u - x_0)$$

$$= \frac{1}{\gamma} \lim_{n \to \infty} (u_n - v_n) \quad \text{——} (1)$$

and

$$\left\| A\left(\frac{u_n - v_n}{\gamma}\right)\right\| \leq \frac{1}{\gamma}\left[\|Au_n\| + \|Av_n\|\right]$$

$$\leq \frac{1}{\gamma}[k + k]$$

$$= \frac{2k}{\gamma}.$$

$$\Rightarrow \frac{u_n - v_n}{\gamma} \in V_{\frac{2k}{\gamma}}.$$

Thus $\frac{u_n - v_n}{\gamma} \in V_{\frac{2k}{\gamma}}$, $\frac{u_n - v_n}{\gamma} \to x$

$$\Rightarrow x \in \overline{V_{\frac{2k}{\gamma}}} \quad \longleftarrow (*)$$

Since $\circledast$ is true for any $x \in B_0$, we see that

$$B_0 \subseteq \overline{V_{\frac{2k}{r}}}.$$

Denote $W = V_{\frac{2k}{r}}$.

Let $x \in B_0$, and $0 < \epsilon < 1$.

Since $B_0 \subseteq \overline{W} \Rightarrow \exists\, x_1 \in W$

$\ni \|x - x_1\| < \epsilon$

$\Rightarrow \|\epsilon^{-1}(x - x_1)\| < 1$

$\Rightarrow \epsilon^{-1}(x - x_1) \in B_0 \subset \overline{W}$

$\Rightarrow \exists\, x_2 \in W$ such that

$\|\epsilon^{-1}(x - x_1) - x_2\| < \epsilon$

$\Rightarrow \|x - x_1 - \epsilon x_2\| < \epsilon^2$

$\Rightarrow \|x - (x_1 + \epsilon x_2)\| < \epsilon^2$

$$\Rightarrow \left\| \epsilon^{-2}\left[x-(x_1+\epsilon x_2)\right]\right\| < 1$$

$$\Rightarrow \epsilon^{-2}\left[x-(x_1+\epsilon x_2)\right] \in B_0 \subset \overline{W}$$

Continuing as above, after obtaining $x_1, x_2, \ldots x_n \in W$ such that

$$\left\| x - (x_1 + \epsilon x_2 + \cdots + \epsilon^{n-1} x_n \right\| < \epsilon^n$$

$$\Rightarrow \left\| \epsilon^{-n} \left[x - \sum_{j=0}^{n-1} \epsilon^j x_{j+1}\right]\right\| < 1$$

$$\Rightarrow \epsilon^{-n} \left[x - \sum_{j=0}^{n-1} \epsilon^j x_{j+1}\right] \in B_0 \subset \overline{W}$$

$$\Rightarrow \exists\, x_{n+1} \in W \text{ such that}$$

$$\left\| \epsilon^{-n} \left[x - \sum_{j=0}^{n-1} \epsilon^j x_{j+1}\right] - x_{n+1} \right\| < \epsilon$$

$$\Rightarrow \left\| x - \sum_{j=0}^{n} \epsilon^j x_{j+1} \right\| < \epsilon^{n+1}$$

Thus we obtained a sequence $\{x_n\}$ in W such that

$$\left\| x - \sum_{j=0}^{n-1} \epsilon^j x_{j+1} \right\| < \epsilon^n \quad \forall n \in \mathbb{N}$$

—— $(**)$

Denote $s_n = \sum_{j=1}^{n} \epsilon^{j-1} x_j, \ \forall n \in \mathbb{N}$

Then by $(**)$ $s_n \longrightarrow x$ as $n \to \infty$.

Also $x_j \in W = V_{\frac{2k}{\gamma}} \quad \forall j$

$$\Rightarrow \|Ax_j\| \leq \frac{2k}{\gamma} \quad \forall j$$

Now for any $n > m$, $n, m \in \mathbb{N}$, we have

$$\|As_n - As_m\| = \left\| A\sum_{j=1}^{n} \epsilon^{j-1} x_j - A\sum_{j=1}^{m} \epsilon^{j-1} x_j \right\|$$

$$= \left\| \sum_{j=1}^{n} \epsilon^{j-1} Ax_j - \sum_{j=1}^{m} \epsilon^{j-1} Ax_j \right\|$$

$$= \left\| \sum_{j=m+1}^{n} \epsilon^{j-1} Ax_j \right\|$$

$$\leq \sum_{j=m+1}^{n} \epsilon^{j-1} \|Ax_j\|$$

$$\leq \sum_{j=m+1}^{n} \epsilon^{j-1} \frac{2k}{\gamma}$$

$$= \frac{2k}{\gamma} \cdot \sum_{j=m+1}^{n} \epsilon^{j-1}$$

$$< \frac{2k}{\gamma} \cdot \epsilon^{m} \left[1+\epsilon+\epsilon^2+\cdots\right]$$

$$= \frac{2k}{\gamma} \frac{\epsilon^{m}}{1-\epsilon} \longrightarrow 0$$

as $m \to \infty$

This implies $\{As_n\}$ is a Cauchy sequence in Y. Since Y is a Banach space, $As_n \longrightarrow y \in Y$.

Thus $s_n \to x \in X$, $As_n \to y \in Y$ and A is a closed operator, implies

$$Ax = y = \lim_{n\to\infty} A f_n$$

Mà

$$\|Af_n\| = \left\| A\left(\sum_{j=1}^{n} \epsilon^{j-1} x_j\right)\right\|$$

$$= \left\| \sum_{j=1}^{n} \epsilon^{j-1} A x_j \right\|$$

$$\leq \sum_{j=1}^{n} \epsilon^{j-1} \|Ax_j\|$$

$$\leq \frac{2k}{\gamma} \sum_{j=1}^{n} \epsilon^{j-1}$$

$$\leq \frac{2k}{\gamma} \cdot \sum_{j=1}^{\infty} \epsilon^{j-1}$$

$$= \frac{2k}{\gamma} \cdot \frac{1}{1-\epsilon}$$

$$\Rightarrow \|Af_n\| \leq \frac{2k}{\gamma} \cdot \frac{1}{1-\epsilon}, \ \forall n$$

$$\therefore \|Ax\| = \|\lim_{n\to\infty} Ay_n\|$$

$$= \lim_{n\to\infty} \|Ay_n\|$$

$$\le \lim_{n\to\infty} \frac{2k}{\gamma} \cdot \frac{1}{1-\epsilon}$$

$$= \frac{2k}{\gamma} \cdot \frac{1}{1-\epsilon}.$$

Thus for all $x \in B_0$, we have

$$\|Ax\| \le \frac{2k}{\gamma(1-\epsilon)}$$

$$\Rightarrow x \in V_c, \quad c = \frac{2k}{\gamma(1-\epsilon)}$$

$$\therefore B_0 \subset V_c = \{x \in X \mid \|Ax\| \le c\}$$

$\therefore$ A is continuous

—//—

Continuity of a Projection operator

A linear operator $A : X \longrightarrow X$ is said to be Projection operator if $P^2 = P$, i.e., $Px = x, \forall x \in R(P)$

$P : X \longrightarrow R(P)$

$x \in R(P)$, $\quad Px = x$

Now for any $y \in X$

$$Py \in R(P)$$

$$P(Py) = Py$$

$$\Rightarrow P^2 y = Py \quad \forall y \in X$$

$$\Rightarrow P^2 = P$$

We write

$$X = \underline{R(P)} + \underline{N(P)}$$

and $R(P) \cap N(P) = \{0\}$

Here P is a projection on $R(P)$ along $N(P)$.

Suppose X is a n.l.s and

$P : X \to X$ is a continuous projection operator.

Then $N(P)$, the null space of P is a closed set in X.

Consider

$I - P : X \to X$

for any $x \in R(P)$,

$$(I-P)x = Ix - Px$$
$$= x - x = 0$$

$\Rightarrow \quad x \in N(I-P)$

$\therefore \quad \underline{\underline{R(P) \subseteq N(I-P)}}.$

$$(I-P)^2 = (I-P)(I-P)$$

$$= I-P-(I-P)P$$

$$= I-P-P+P^2$$

$$= I-P-P+P \quad [\because P^2=P]$$

$$= I-P$$

$\therefore \; I-P : X \longrightarrow R(I-P)$ is a projection operator and since I and P are continuous, implies $I-P$ also continuous.

$\because \quad R(P) = N(I-P)$, it follows that $R(P)$ is also closed subspace of X.

$$\left[x \in N(I-P) \Rightarrow (I-P)x = 0 \right.$$
$$\Rightarrow Px - Ix = 0$$
$$Px = Ix = x$$
$$\left. \overline{\qquad\qquad} \atop x \in R(P) \right]$$

Thus if $P : X \longrightarrow X$ is a continuous projection operator, both $R(P)$ and $N(P)$ are closed subspaces of X.

Corollary :— Let X be a Banach space and $P : X \longrightarrow X$ be a projection operator. If $R(P)$ and $N(P)$ are closed subspaces of X, then P is continuous.

Proof: Suppose $R(P)$ and $N(P)$ are

Closed subspace of X.

To prove $P : X \longrightarrow X$ is continuous, it is enough to prove that P is a closed operator by closed graph theorem.

Let $\{x_n\}$ be a sequence in X such that $x_n \longrightarrow x$ & $Px_n \longrightarrow y$.

$\because$ $R(P)$ is a closed subspace of X and $Px_n \longrightarrow y \implies y \in R(P)$

$\implies Py = y$ [by definition of P]

Also

$x_n - Px_n \longrightarrow x - y$

and

$x_n - Px_n = (I-P)x_n \in R(I-P)$
$= N(P)$

Thus $\{(I-P)x_n\}$ is a sequence in a closed subspace $N(P)$ and $(I-P)x_n \longrightarrow x-y$

$\Rightarrow$ $x-y \in N(P)$

$$P(x-y) = 0$$

$\Rightarrow$ $Px = Py = y$

$\therefore$ P is a closed operator.

$\Rightarrow$ P is continuous by closed graph theorem

—//—