Cutting plane method for pure integer soln.
(R.E. Gomory)

- when applied for the soln of problem with rational data the optimal soln. is obtained in a finite no. of iterations.
- introduces new constraints (or cuts) to the problem which slice away non-integer optimal soln. exhibited by the extreme pts of the feasible region of the associated LPP and at the same time leave all feasible & integer solns untouched.
- the new LPP is then solved as usual.
  - if the new optimal soln. obtained does not satisfy the integer requirement, then another new constraint is added and the process is repeated iteratively until one of the related LPP gives an all integer optimal soln.
- As these are hyperplanes which cut off a portion of the feasible region of the related LPP, the constraints are called cutting planes.

Consider the pure LPP:

$$\text{Max } z = cx$$

$$\text{s.t. } Ax = b \quad \text{——— (1)}$$
$$x \geq 0$$

and x_j are integer for $j = 1, 2, \ldots, n$. ——— (2)

If the constraint (2) is ignored, then the problem is the related LPP.

let $A = [B|R]$, B being the basis matrix consisting of the first m columns of A and R contains the remaining $(n-m)$ columns of A.

For the optimal soln. of the problem, we have

$$x^* = [x_B^*, x_R^*]$$

when $x_R^* = 0$ (non-basic).

$$\therefore Ax = b \Rightarrow \quad [B|R]\begin{bmatrix} x_B^* \\ x_R^* \end{bmatrix} = b$$

$$A = [\overbrace{a_1, \ldots, a_m}^{B}, \overbrace{a_{m+1}, a_n}^{R}]$$

$$\Rightarrow \quad Bx_B^* + R x_R^* = b.$$

$$B^{-1}a_j = y_j$$

$$B^{-1}b = y_0.$$

$$\Rightarrow \quad x_B^* = B^{-1}b - B^{-1}R\, x_R^*$$

$$= y_0 - \sum_{j=m+1}^{n} y_j x_j$$

when x_j are non-basic variables of the current soln.

Suppose the i-th variable of x_B^* is non-integer.

So $\quad x_{Bi} = y_{i0} - \sum_{j=m+1}^{n} y_{ij} x_j \quad$ — (3)

when x_j are all zero in the current soln.

As x_{Bi} is not integer, y_{i0} is non-integral.

let $y_{i0} = d_{i0} + f_{i0}$, $\quad d_{i0} \to$ integral part of y_{i0} & $0 < f_{i0} < 1$.

Similarly, let $y_{ij} = d_{ij} + f_{ij}$, $\quad d_{ij} \to$ integral part of y_{ij} & $0 \le f_{ij} < 1$.

$\therefore$ (3)

$\therefore$ (3) $\Rightarrow$ $x_{B_i} = (d_{i0} + f_{i0}) - \sum_{j=m+1}^{n} (d_{ij} + f_{ij}) x_j$

$= \left(d_{i0} - \sum_{j=m+1}^{n} d_{ij} x_j\right) + \left(f_{i0} - \sum_{j=m+1}^{n} f_{ij} x_j\right)$ — (4)

Now the soln. x_B^* is unique with all non-basic $x_j = 0$.
If we wish to change this s.m., then one of the non-basic x_j's must become positive, being feasible.

Hence, for an all integer soln., the quantity $\left(d_{i0} - \sum_{j=m+1} d_{ij} x_j\right)$ of (4) must be an integer, since each x_j must be an integer and d_{ij} is an integer by defn.

$\Rightarrow \left(f_{i0} - \sum_{j=m+1}^{n} f_{ij} x_j\right)$ must be an integer.

$f_{ij} \geq 0$ and each $x_j \geq 0 \Rightarrow \sum_{j=m+1}^{n} f_{ij} x_j \geq 0$.

Also $0 < f_{i0} < 1$

Hence, $\boxed{f_{i0} - \sum f_{ij} x_j \leq 0}$ — (5)

This is the cutting plane constraint of Gomory in the case of all integer problem.

This constraint is added to the optimal table of the related LPP and solved by dual simplex method.

Note. For the current s.m., all non-basic $x_j = 0$ and $f_{i0} \geq 0$.
Hence, the present s.m. is sliced away by the constraint (5).
Moreover, every integer feasible soln. of the original IPP will satisfy this constraint.

Note 2 It has been assumed that all variables, including the slack variables, are to be integers in any feasible soln.
This can be done by clearing fractions from the constraint

co-efficient before introducing the slack variables.

e.g. $\frac{3}{11}x_1 + \frac{2}{9}x_2 \le 1$ is converted to $27x_1 + 22x_2 \le 99$

Before applying the simplex method to solve the initial problem, all necessary constraints are similarly modified.

Note 3 If more than one basic variable is non-integer, then that variable whose fractional part is the largest, is used to form the cutting plane.

Note 4 If some y_{ij} is negative, even then a non-negative fractional part of it ~~as required by~~ may be obtained.

e.g. $-3\frac{2}{3}$ can be written as $-4+\frac{1}{3}$

whereby the non-negative fraction part $\frac{1}{3}$ has been separated.

Branch and bound algm. (Land & Doig)

Consider the problem

$$\text{Max } z = cx$$

$$\text{s.t. } Ax = b$$

$$x \geq 0, \; x_j = \text{integer}, \; j \in I, \; j = 1, 2, \dots, n.$$

where I is the set of subscripts for those variables which are required to be integers.

If $I = \{1, 2, \dots, n\}$, then the problem is a pure integer programming problem.

Algm.

Step 1: Solve the given LPP.

If the optimal soln. is s.t. all $x_j, j \in I$ are integers, then this is the desired optimal soln.

If at least one $x_j, j \in I$ is non-integer, then proceed to the next step.

Step 2: Of the ~~int~~ non-integral $x_j, j \in I$, choose one and for that suppose

$$x_j = w_j + f_j$$

where ~~f_j~~ w_j is an integer and $0 < f_j < 1$

Any feasible integer soln. of the problem must be such that either

$$x_j \leq w_j \text{ or } x_j \geq w_j + 1.$$

If we add these constraints individually to the constraints of the given problem, then we get two subproblems.

These two subproblems are then solved.

... in slack variables.

Step 3: for each sol^n. of the above 2 subproblems,

i) If the sol^n. is feasible integer sol^n., then accept that sol^n. which will give the larger value for the objective fun^n. as the required optimal sol^n.

Select another non-integer $x_j, j \in I$, if any exists and proceed as in step 2.

ii) If the sub-problem gives no feasible sol^n., then select another non-integer $x_j, j \in I$, if any exists and proceed as in step 2.

iii) If the sol^n. is not feasible integer sol^n., then repeat the process of Step 2 on the constraint set for this related sub-problem.

Step 4. When all possibilities are exhausted in step 3, the algm. terminates.

- The algm. generates a tree of sol^n. Each non-terminal node has 2 branches issuing from it, corresponding to $x_j \le \omega_j$ and $x_j \ge \omega_j + 1$.
- for sol^n of the subproblems, graphical method of sol^n. is applied if the no. of variables is 2, otherwise, sol^n. is obtained by simplex method.

Example: (Branch and bound).

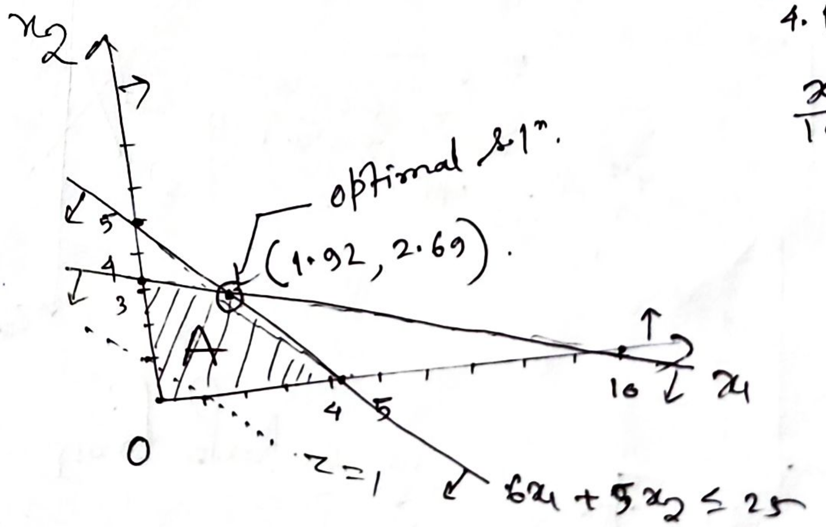
Solve the following all integer programming problem using branch and bound method.

problem A
$$\begin{cases} \text{Max} \quad z = 2x_1 + 3x_2 \\ \text{s.t.} \quad 6x_1 + 5x_2 \le 25 \\ \qquad x_1 + 3x_2 \le 10 \\ \text{and } x_1, x_2 \ge 0 \text{ and integers.} \end{cases}$$

Sol^n: Step 1 Relaxing the integer condition, let us find the optimal sol^n to the given LPP by graphical method.

$\frac{x_1}{4.16} + \frac{x_2}{5} \le 1$

$\frac{x_1}{10} + \frac{x_2}{3.33} \le 1$



$6x_1 + 5x_2 = 25$
$6x_1 + 18x_2 = 60$
$-13x_2 = -35$
$x_2 = \frac{35}{13} = 2.69$
$\Rightarrow x_1 = 10 - 3x_2 = 1.92$

optimal sol^n is $x_1 = 1.92$, $x_2 = 2.69$

$z_{max} = 11.91$.

Step 2 Rounding off, lower bound of z is $z_L = 11$ $(x_1 = 1, x_2 = 3)$

$z = 1$
$\Rightarrow x_1 + x_2 = 1$

x_1, x_2 not integers.

x_2 has more fractional value than x_1.

The new constraints to be added are $x_2 \le 2$ and $x_2 \ge 3$

Subproblem B

$\text{Max } Z = 2x_1 + 3x_2$

s.t. $6x_1 + 5x_2 \le 25$ — (1)

$x_1 + 3x_2 \le 10$. — (2)

$x_2 \le 2$ — (3)

and $x_1, x_2 \ge 0$ and integers

$\boxed{\text{opt. sol}^n.\ x_1 = 2.5,\ x_2 = 2}$ $Z_2 = 11$ Max

The corresponding graph is

Subproblem C

$\text{Max } Z = 2x_1 + 3x_2$

s.t. $6x_1 + 5x_2 \le 25$

$x_1 + 3x_2 \le 10$

$x_2 \ge 3$ — (4).

and $x_1, x_2 \ge 0$ and integers

$\boxed{\text{opt. sol}^n.\ x_1 = 1, x_2 = 3}$ $Z_3 = 11$ Max

$x_2 = 3$

$x_1 + 3x_2 = 10$

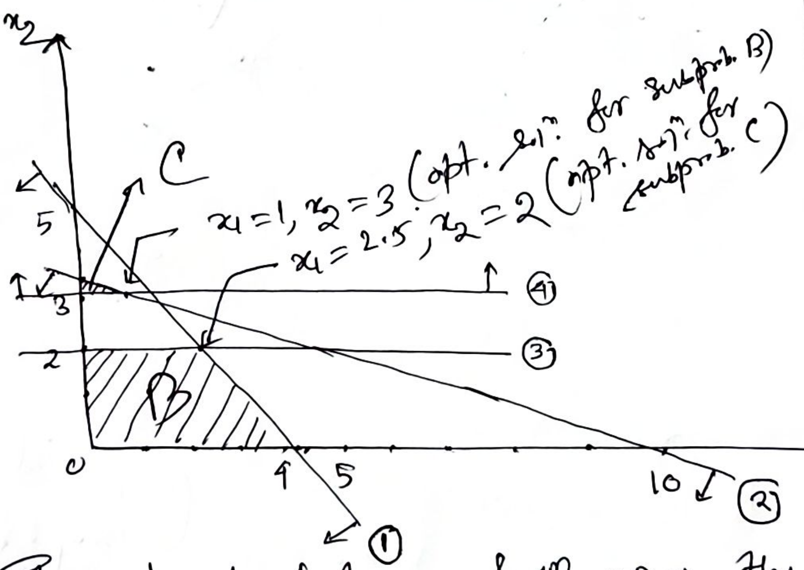
$\Rightarrow x_1 = 10 - 9$
$= 1$

$x_2 = 2$

$6x_1 + 5x_2 = 25$

$x_1 = \frac{25 - 10}{6}$

$= \frac{15}{6} = \frac{5}{2} = 2.5$



Rule: The best integer sol^n gives the lower bound.

Here best int. sol^n is $\underbrace{x_1 = 1, x_2 = 3}_{\text{both integer, so no need to branch subproblem C further.}}$, $Z_3 = 11$ → cannot be less than 11

$\underset{11}{Z_L} < \underset{11}{Z_3} < \underset{11.991}{Z_U}$ ⇒ New upper bound is $Z_U = 11$.

Step 1 Subproblem B needs further branching as with variable x_1 as x_1 is not an integer. ($x_1 = 2.5$)

The new constraints to be added to Subproblem B are $x_1 \le 2$ and $x_1 \ge 3$.

Subproblem D

Max $Z = 2x_1 + 3x_2$

s.t. $6x_1 + 5x_2 \le 25$ — ①

$x_1 + 3x_2 \le 10$ — ②

$x_2 \le 2$ — ③

$x_1 \le 2$ — ④

and $x_1, x_2 \ge 0$ and integers

max $\boxed{x_1 = 2, x_2 = 2, Z_4 = 10}$

Subproblem E.

Max $Z = 2x_1 + 3x_2$

s.t. $6x_1 + 5x_2 \le 25$

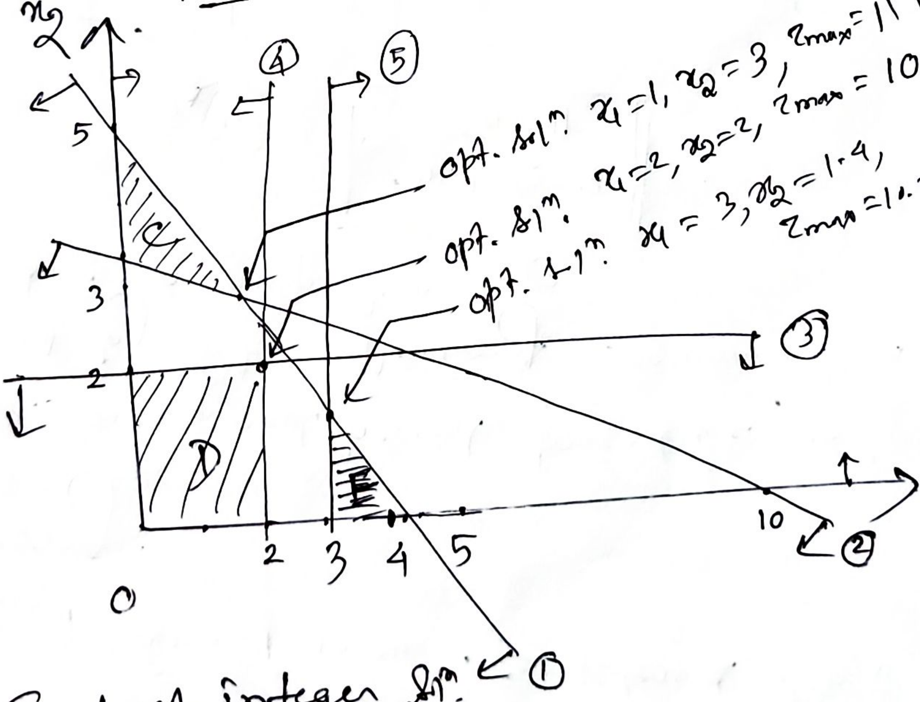
$x_1 + 3x_2 \le 10$

$x_2 \le 2$

$x_1 \ge 3$ — ⑤.

and $x_1, x_2 \ge 0$ and integers.

$\boxed{x_1 = 3, x_2 = 1.4, \text{Max } Z_5 = 10.2}$



$x_1 + 3x_2 = 10$

$x_1 = 2$

$\Rightarrow x_2 = \frac{8}{3}$

$6x_1 + 5x_2 = 25$

$x_1 = 2$

$\Rightarrow x_2 = \frac{25-12}{5} = \frac{13}{5}$

$x_2 = 2$

$x_1 = 2$

$x_1 = 3$

$6x_1 + 5x_2 = 25$

$\Rightarrow x_2 = \frac{25-18}{5} = \frac{7}{5} = 1.4$

$1.4 \times 3 = 4.2$

$6 + 4.2 = 10.2$

Rule The best integer sol^n gives the lower bound

Here best int. sol^n is $x_1 = 2, x_2 = 2$, Max $Z_4 = 10$

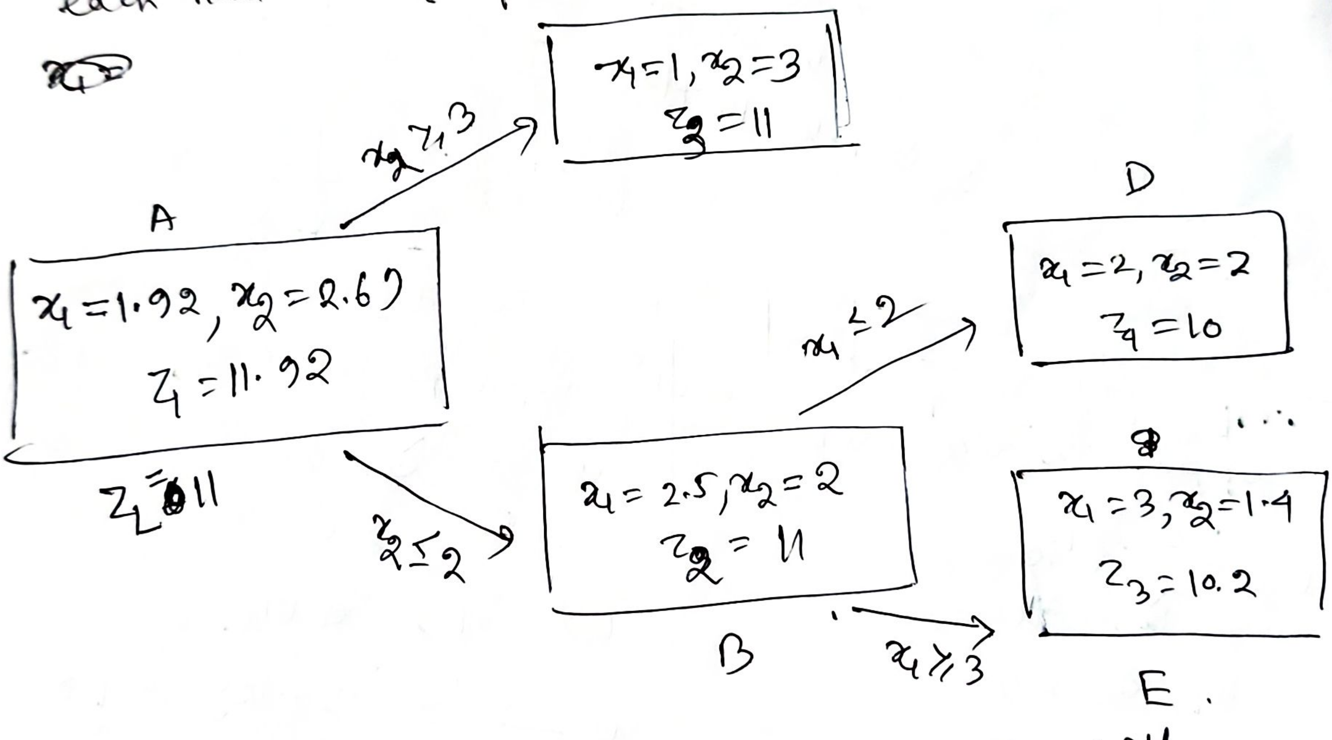
both integer, no need to further branching.

Since sol^n of subproblem E is not integer (x_2 is not integer), we can further branch with variable x_2.

But $Z_5 = 10.2 < Z_L = 11$

So subproblem E is not considered for further branching.

The entire branch and bound procedure can be represented by the following enumeration tree where each node is a subproblem.



Note: Subproblem with non-integer sol^n with max. objective value has to be branched first.

Note. If the no. of variables is more than 2, Simplex method is used to get the sol^n.

Sensitivity Analysis

- Input parameters of LPP →
  - Co-efficient matrix A
  - requirement vector b
  - Cost vector c

  → not const. in a practical situation ↓ neither ~~cost~~ const. nor known exactly for a model for a specific period of time.

- How sensitive the optimal sol^n. is to small discrete changes in these parameters? → sensitivity analysis.

- Sensitiveness → fulfilment of the condition of optimality + determining the limits of variations of these parameters for the sol^n. to remain optimal feasible.

- Post optimal analysis → ~~investi~~ Sensitivity analysis + investigation of how changes in the input parameters affect the optimal sol^n.

- Sensitivity analysis → reduces the additional computational effort which arises in solving the problem anew.

Changes in the objective fun.

$$C_k \to C_k + \delta_k$$

i) When C_k is associated with the non-basic variable x_k

let $x_B^* \to$ optimal basic f.s. to the LPP

$$\text{Max } z = cx$$

$$\text{s.t. } Ax = b, x \geq 0$$

and $B \to$ optimal basis matrix.

Let $C_B \to$ cost vector corresponding to x_B^*.

C_B not changed

$z_j - c_j = C_B y_j - c_j \geq 0$ for optimal soln $\forall j$

For basis vector a_j, $z_j - c_j$ remains unchanged

For non-basic variable a_k, if C_k is changed to $C_k + \delta_k$ then for optimality

$$z_k - (C_k + \delta_k) \geq 0 \text{ for optimality.}$$

Hence $\delta_k \leq z_k - C_k$.

- If the cost C_k of any non-basic variable x_k is increased by more than $(z_k - C_k)$, then the resulting $(z_k - C_k)$ will be -ve and a few more iteration will be necessary to determine the new optimal soln.
- Cost of any non-basic variables can be reduced without limit, without affecting the optimality of x_B^*.

Ex

ii) When c_k is associated with a basic variable x_k.

$$c_k \to c_k + \delta_k .$$
$$c_B \to c_B'$$

$$c_B = (c_1, c_2, \ldots, c_k, \ldots, c_m)$$
$$c_B' = (c_1, c_2, \ldots, c_k + \delta_k, \ldots, c_m)$$
$$= (c_1, c_2, \ldots, c_k, \ldots, c_m) + (0, 0, \ldots, \delta_k, \ldots 0)$$
$$= c_B + \delta_k e_k$$

$$z_j = c_B B^{-1} a_j, \quad y_j = B^{-1} a_j$$
$$z_j' = c_B' B^{-1} a_j = (c_B + \delta_k e_k) B^{-1} a_j$$
$$= c_B B^{-1} a_j + \delta_k e_k y_j$$
$$= z_j + \delta_k y_{kj}$$

For optimal soln,

$$z_j' - c_j \geq 0 \Rightarrow z_j + \delta_k y_{kj} - c_j \geq 0$$
$$\Rightarrow -(z_j - c_j) \leq \delta_k y_{kj}$$

Hence, the obtained soln. will remain optimal (maximal) feasible, if

$$-\frac{(z_j - c_j)}{y_{kj}} \leq \delta_k \quad \text{for } y_{kj} > 0$$

and $$-\frac{(z_j - c_j)}{y_{kj}} \geq \delta_k \quad \text{for } y_{kj} < 0.$$

Combining,

$$\text{Min}\left\{-\frac{(z_j - c_j)}{y_{kj}}, \ y_{kj} < 0\right\} \geq \delta_k \geq \text{Max}\left\{-\frac{(z_j - c_j)}{y_{kj}}, \ y_{kj} > 0\right\}$$

for all j for which a_j is non-basic.

- If δ_k lies in the range as given earlier, then the soln. remains optimal.
- If δ_k ~~the~~ falls outside this range, then at least one $(z_j - c_j)$ will be -ve and the soln. will ~~not~~ be negative and the soln. will no longer remain optimal.
- If no $y_{kj} > 0$, then there is no lower bound of δ_k and if no $y_{kj} < 0$, then there is no upper bound of δ_k.

Variations of the requirement vector

$$z_j - c_j = c_B y_j - c_j = c_B B^{-1} a_j - c_j$$

$\Rightarrow$ factor determining the optimality condition does not depend on b, the requirement vector.

$b \to b+d$, ~~does not change~~ d +ve or -ve

$\rightarrow$ does not change the optimality of the soln.

only to check the feasibility of the new soln.

$$x_B = B^{-1} b \rightarrow x_B = B^{-1}(b+d).$$

- Consider ~~the~~ x_B^* be an optimal soln. of the LPP

$$\text{Max } z = cx$$
$$\text{s.t. } Ax = b,\ x \geq 0.$$

- Let the i-th component of b is changed by an amount d_i (+ve or -ve)

i.e. $\bar{b}_i = b_i + d_i \quad (i = 1, 2, \ldots, m)$

$\bar{b} = b + d$, $d = (0, 0, \ldots, 0, d_i, \ldots, 0)$

$$\bar{x}_B = B^{-1}\bar{b} = B^{-1}(b+d) = B^{-1}b + B^{-1}d = x_B^* + B^{-1}d.$$

- $\bar{x}_B$ will be optimal if it is feasible.
- if $\bar{x}_B$ is not feasible, then one or more $\bar{x}_{B_i}$ will be negative & we can use the dual simplex method to find the new optimal solⁿ.

- let $B^{-1} = [y_1, y_2, \cdots, y_m]$

$$\& \ \bar{x}_B = x_B^* + B^{-1}d = x_B^* + \sum_{j=1}^{m} d_j y_j$$

$$\Rightarrow \bar{x}_{B_i} = x_{B_i}^* + \sum_{j=1}^{m} d_j y_{ij}$$

- when only one component of b, say b_k, is changed, then the i-th basic variable of the new problem is given by

$$\bar{x}_{B_i} = x_{B_i}^* + d_k y_{ik}.$$

when y_{ik} is the (i,k)-th element of B^{-1}.

- for feasibility of the new solⁿ, we must have

$$x_{B_i} + y_{ik} d_k \geq 0$$

$$\text{or } d_k \geq -\frac{x_{Bi}}{y_{ik}} \quad \text{if } y_{ik} > 0$$

$$\text{and } d_k \leq -\frac{x_{Bi}}{y_{ik}} \quad \text{if } y_{ik} < 0.$$

- Thus if we choose d_k s.t.

$$\max_{y_{ik}>0} \left\{ -\frac{x_{Bi}}{y_{ik}} \right\} \leq d_k \leq \min_{y_{ik}<0} \left\{ -\frac{x_{Bi}}{y_{ik}} \right\}$$

then the current solⁿ will remain feasible.

$\bar{z} = c_B B^{-1} \bar{b} = c_B \bar{x}_B$. ($z$ changed to $\bar{z}$ when b_k is changed to $(b_k + d_k$

Addition of a variable

New problem is $\max \; z = cx + c_{n+1}x_{n+1}$

$$\text{s.t. } [A, a_{n+1}] \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = b$$

$$x \geq 0, x_{n+1} \geq 0.$$

$x_{n+1} \rightarrow$ new variable introduced with cost vector c_{n+1} and associated activity vector a_{n+1}.

Let $x_B^* \rightarrow$ optimal soln of the original problem.

- it will main ~~&~~ basic feasible soln. to the newly constructed problem as the new variable x_{n+1}, being non-basic, is equal to zero.

- But ~~is~~ this soln will remain optimal for the new problem if

$z_{n+1} - c_{n+1} \geq 0$ as ~~for~~ $z_j - c_j \geq 0$ for $j = 1, 2, \ldots, n.$ are not changed by the addition of new variable x_{n+1} which is non-basic.

Now $z_{n+1} - c_{n+1} = c_B B^{-1} a_{n+1} - c_{n+1}$

if it is negative, we need to calculate

$y_{n+1} = B^{-1} a_{n+1}$ &

& proceed with simplex or the revised simplex method so that a_{n+1} is brought in the basis.

Addition of a constraint

The addition of new constraint to the constraint set of the given problem means the addition of a new variable (slack or surplus) associated with the new constraint.

- Let the constraint set be

$$Ax = b, \; x \geq 0$$

- Let us add a new constraint to this in the form

$$\sum_{j=1}^{m} a_{n+1,j}\, x_j + x_{n+1} = b_{m+1} \quad \text{— (1)}$$

where x_{n+1} is a slack or surplus variable (& hence $C_{n+1} = 0$) added to the new constraint.

- Thus the addition of a new constraint means addition of a new variable.

- b_{m+1} is not necessarily positive as the added constraint may be either of the '$\leq$' type or '$\geq$' type.

- If the current optimal soln. satisfies (1), then it is still the optimal soln. since the objective fun. is not changed thereby.

Otherwise, we have to find the new optimal soln. for the new set of $(m+1)$ equations consisting of $(m+1)$ basic vectors.

Change in the co-efficient matrix

i). Suppose we replace the vector a_k, a non-basic vector of A.

- $\bar{a}_k = a_k + \alpha$.

  where α is an m-component vector.

- a_k non-basic $\Rightarrow$ B remains unchanged

$$\Rightarrow x_B = B^{-1}b, \quad z_j - c_j = c_B B^{-1} a_j - c_j, \quad j \neq k$$

remain unchanged.

- Only change will be in y_k

$$\bar{y}_k = B^{-1}\bar{a}_j = B^{-1}(a_k + \alpha)$$
$$= B^{-1}a_k + B^{-1}\alpha$$
$$= y_k + B^{-1}\alpha$$

$$\bar{z}_k - c_k = c_B B^{-1}\bar{a}_k - c_k$$
$$= c_B B^{-1}(a_k + \alpha) - c_k$$
$$= c_B B^{-1} a_k + c_B B^{-1}\alpha - c_k$$
$$= z_k - c_k + c_B B^{-1}\alpha$$

Now if $\bar{z}_k - c_k \geq 0$, the present soln. remains optimal, but if $\bar{z}_k - c_k < 0$, then y_k is to be computed and the optimal soln. is to be found through few more iterations.

- For a single change ~~δ_{lk} in the a~~ δ_{lk} in the a_{lk} in the lth ~~component~~ element, we have the new element $a_{lk} + \delta_{lk}$

ay

individually

Subproble

and hence for the optimality of this soln.,
we must have

$$c_B B^{-1}(a_k + \delta_{lk} e_l) - c_k \geq 0,$$ e_l is the unit vector having 1 in the l-th position

$$\Rightarrow \quad z_k - c_k + \delta_{lk} \sum_{i=1}^{m} y_{il} c_{Bi} \geq 0$$

$\delta_{kl} c_B B^{-1} e_l = \delta_l$
$= \delta_{kl} c_B \delta_l$ (l-th column of B^{-1})
$= \delta_{kl} \sum_{i=1}^{m} c_{Bi} y_{il}$, $\delta_l = \begin{pmatrix} y_{1l} \\ y_{ml} \end{pmatrix}$

where y_{il} is the element of B^{-1} in the i-th row and l-th column.

Hence, optimality and feasibility will be maintained if

$$\underset{l}{\text{Min}} \frac{-(z_k - c_k)}{\left(\sum_{i=1}^{m} y_{il} c_{Bi}\right) < 0} \geq \delta_{lk} \geq \underset{l}{\text{Max}} \frac{-(z_k - c_k)}{\left(\sum_{i=1}^{m} y_{il} c_{Bi}\right) > 0}$$

for a non-existent denominator, the corresponding bound will not exist.

ii) If the change in elements is desired in a vector a_k which is basic, we first remove this vector from the basis before giving any change to the elements of a_k and then we make the desired change in the presently obtained non-basic vector a_k.

In this case, we can otherwise resolve the problem from the beginning or recompute B^{-1}, all y_j and $(z_j - c_j)$.

Initial basic soln.

- The dual simplex method can be used only to any problem that forms an optimal soln. but infeasible soln. in the initial table (i.e. $z_j - c_j \geq 0 \ \forall j$ for a maximization prob.)
- Because of the apparent difficulty associated with finding an initial basic soln with optimality condition satisfied, the dual simplex method has never been used as a general purpose LP algm.

- If in a problem, each constraint has a slack or surplus variable with it and $c_j \leq 0 \ \forall j$, then these slack or ~~surf~~ surplus variables will afford an initial basic soln. which will not be necessarily feasible.

  Then $c_B = 0$ & $z_j - c_j = c_B y_B - c_j = -c_j > 0$

  In this case, dual simplex method can directly be applied.

- However, there are methods for finding an initial basic soln. with $z_j - c_j \geq 0 \ \forall j$ when it is not apparent.
- One such method is the Artificial Constraint Method.
  1. Find an initial basis for the primal problem. Suppose for this basis, one or more of the basic variables is non-positive $(x_{B_r} < 0)$ and one or more of $z_j - c_j < 0$.
  2. Add to the constraint set a new constraint

  $$\Sigma x_j \leq M$$

  where summation is over all the j's for which $z_j - c_j < 0$ and M is sufficiently large +ve no.

- Adding slack variable x_M, we get
$$\sum x_j + x_M = M.$$
- if for $j = p$, $(z_p - c_p)$ has the largest absolute value (i.e. $z_p - c_p$ is most -ve ~~value~~), ~~write~~ we have
$$x_p = M - \left(\sum_{j \neq p} x_j + x_M\right)$$
- This value of x_p is then substituted in the original objective fun. and the set of constraints.
- Thus a ~~modification~~ modified but equivalent problem is obtained in which all $z_j - c_j \geq 0$.
- Then the dual simplex method is applied to the modified problem.

Example: (Artificial constraint method)

Use the artificial constraint method to find the initial basic soln. of the following problem and then apply the dual simplex algm. to solve it.

$$\text{Max } z = 2x_1 - 3x_2 - 2x_3$$
$$\text{s.t. } x_1 - 2x_2 - 3x_3 = 8$$
$$2x_2 + x_3 \leq 10$$
$$x_2 - 2x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0.$$

Soln. We use slack variables to put the constraints as
$$\text{Max } z = 2x_1 - 3x_2 - 2x_3 + 0.x_4 + 0.x_5$$
$$\text{s.t. } x_1 - 2x_2 - 3x_3 = 8$$
$$2x_2 + x_3 + x_4 = 10$$
$$-x_2 + 2x_3 + x_5 = -4$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

have basis $B = [a_1, a_4, a_5] = I_3 \Rightarrow B^{-1} = I_3$

$\therefore x_B = B^{-1}b = [8, 10, -4]$. (at least one component -ve)

So $x_{B3} = -4 < 0$.

Furthermore, $z_j - c_j = c_B y_j - c_j = c_B a_j - c_j$ | $z_1 - c_1 = 0 = z_2 - c_2 = z_3 - c_3$

$z_2 - c_2 =$

$c_B = [c_1, c_4, c_5]$ $\quad y_j = B^{-1}a_j = I_3 a_j = a_j$

$= [2, 0, 0]$

c_B	B	x_B	c_j / b	2 / a_1	-3 / a_2	-2 / a_3	0 / a_4	0 / a_5
2	a_1	x_1	8	1	-2	-3	0	0
0	a_4	x_4	10	0	2	1	1	0
0	a_5	x_5	-4	0	-1	-2	0	1
			$z_j - c_j$	0	-1	-4	0	0

most -ve.

$j = p = 3$.

-4
-6

Add new constraint

$$x_2 + x_3 \le M$$

Add artificial variable x_M

$$x_2 + x_3 + x_M = M$$

$$\Rightarrow x_3 = M - x_2 - x_M.$$

Substitute x_3 in the original objective function and the set of constraints

The augmented problem then becomes,

$$\text{Max } Z = 2x_1 - 3x_2 - 2M + 2x_2 + 2x_M$$
$$= 2x_M + 2x_1 - x_2 - 2M.$$

s.t.
$$3x_M + x_1 + x_2 = 8 + 3M$$
$$-x_M + x_2 + x_4 = 10 - M$$
$$-2x_M + x_2 + x_5 = -4 + 2M.$$
$$x_M + x_2 + x_3 = M.$$

$x_1 - 2x_2 - 3M + 3x_2 + 3x_M = 8$

$2x_2 + M - x_2 - x_M + x_4 = 10$

$-x_2 + 2M - 2x_2 - 2x_M + x_5 = -4$

			c_j	2	2	-1	0	0	0	Max ratio
C_B	B	x_B	b	a_M	a_1	a_2	a_3	a_4	a_5	$\frac{z_j-c_j}{y_{rj}}, y_{rj}<0$
2	a_1	x_1	$3M+8$	3	1	1	0	0	0	
0	a_4	x_4	$-M+10$	-1	0	1	0	1	0	$\max\left\{-\frac{4}{2}, -\frac{3}{3}\right\}$
0	a_5	x_5	$-2M-4$	-2	0	$\boxed{-3}$	0	0	1	$=-1$
0	a_3	x_3	M	1	0	1	1	0	0	
			z_j-c_j	4	0	3	0	0	0	

• all $z_j - c_j \geq 0$ so we can apply dual simplex
• Most -ve in b col. gives leaving variable x_5, $r=5$
• Max ratio determines the entering vector a_2

2	a_1	x_1	$\frac{7M+20}{3}$	$7/3$	1	0	0	0	$1/3$
0	a_4	x_4	$\frac{-5M+26}{3}$	$\boxed{-5/3}$	0	0	0	1	$1/3$
-1	a_2	x_2	$\frac{2M+4}{3}$	$\frac{2}{3}$	0	1	0	0	$-1/3$
0	a_3	x_3	$\frac{M-4}{3}$	$1/3$	0	0	1	0	$1/3$
			z_j-c_j	2	0	0	0	0	1

2	a_1	x_1	$94/5$	0	1	0	0	$7/5$	$4/5$
2	a_M	x_M	$\frac{5M-26}{5}$	1	0	0	0	$-3/5$	$-1/5$
-1	a_2	x_2	$24/5$	0	0	1	0	$2/5$	$-1/5$
0	a_3	x_3	$2/5$	0	0	0	1	$1/5$	$2/5$
			z_j-c_j	0	0	0	0	$6/5$	$7/5$

all $x_B \geq 0$, all $z_j - c_j \geq 0$.

We have optional soln.

Eliminate the extra row & extra col. cor. to aM.
In the final table, we restore the original prob.

C_B	B	b	a_1	a_2	a_3	a_4	a_5
		c_j	2	-3	-2	0	0
2	a_1	$94/5$	1	0	0	$7/5$	$4/5$
-3	a_2	$24/5$	0	1	0	$2/5$	$-1/5$
-2	a_3	$2/5$	0	0	1	$1/5$	$2/5$
		z_j-c_j	0	0	0	$6/5$	$7/5$

The op. soln $x_1 = \frac{94}{5}$, $x_2 = \frac{24}{5}$, $x_3 = 2/5$.

$\frac{94}{5} - \frac{72}{5} - \frac{2.2}{5} = 112/5$ Zmax

$\frac{14}{5} - \frac{6}{5} - \frac{2}{5}$

$\frac{8}{5} - \frac{2}{5} + \frac{1}{5}$

$\frac{7}{5}$

$\frac{7M+20}{3} + \left(\frac{-5M+26}{3}\right)\frac{7}{3.5}$

$\frac{49M+140-15M+78}{21}$

$\frac{M-4}{3} + \frac{1}{3}\cdot\frac{-5M+26}{3}\cdot\frac{3}{5}$

$\frac{5M-20-5M+26}{15}$

$\frac{6}{15}$

$\frac{2M+4}{3} + \frac{2}{3}\cdot\left(\frac{-5M+26}{3}\right)\frac{3}{5}$

$\frac{10M+20-10M+52}{15}$

$3 - \frac{2}{3} = \frac{7}{3}$

$-1 - \frac{2}{3}$

$1 - \frac{2}{3}$

$3M+8 - \frac{2M+4}{3}$

$\frac{9M+24-2M-4}{3} = \frac{7M+20}{3}$

$-M+10 - \frac{2M+4}{3}$

$\frac{-3M+30-2M-4}{3} = \frac{-5M+26}{3}$

$M - \frac{2M+4}{3}$

$\frac{14}{3} - \frac{2}{3} - 2$

$\frac{2}{3} + \frac{1}{3}$

$+\frac{7}{3}\cdot\frac{3}{5}$

$\frac{2}{3}\cdot\frac{3}{5}$

$\frac{1}{3}\cdot\frac{3}{5}$

$\frac{1}{3} + \frac{7}{3}\cdot\frac{3}{5}$

$-\frac{1}{3} + \frac{2}{3}\cdot\frac{3}{5}$

$\frac{1}{3} + \frac{1}{3}\cdot\frac{3}{5}$