Lecture -10 (06-02-2024)

Theorem: Let $f : (X, T) \longrightarrow (Y, T^*)$ be a function and S be a subbase for T^* on Y. Then f is $T - T^*$ is continuous iff inverse of every member of S is an open subset of X. That is $\forall\ s \in S \Rightarrow f^{-1}(s) \in T$.

Proof: Let $f : (X, T) \longrightarrow (Y, T^*)$ is continuous. Then inverse image of every open subset of Y is an open subset of X.

$\because S \subset T^*$

$\Rightarrow \forall\ s \in S$, we have $f^{-1}(s) \in T$.

Conversely assume that $\forall s \in S \Rightarrow f^{-1}(s) \in T$.

let $G \in T^*$, then

$$G = \bigcup_i (s_{i_1} \cap s_{i_2} \cap \dots \cap s_{i_k}),$$

$$s_{i_1}, s_{i_2}, \dots s_{i_k} \in S.$$

$$\Rightarrow f^{-1}(G) = f^{-1}\Big(\bigcup_i (s_{i_1} \cap s_{i_2} \cap s_{i_3} \cap \dots \cap s_{i_k})\Big)$$

$$= \bigcup_i \big(f^{-1}(s_{i_1}) \cap f^{-1}(s_{i_2}) \dots \cap f^{-1}(s_{i_k})\big),$$

$\in T$ [$\because T$ is a topology on X].

$\therefore f : X \longrightarrow Y$ is Continuous

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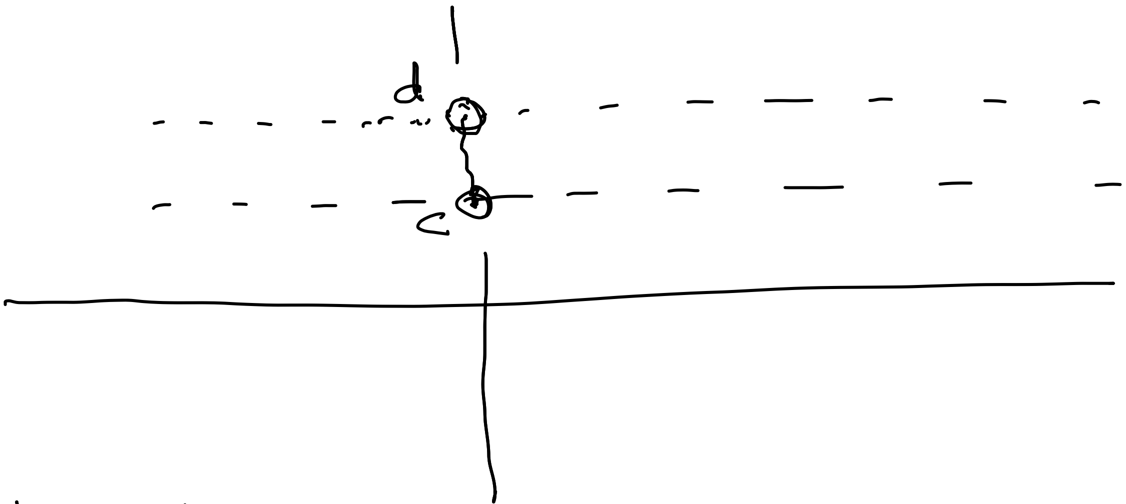
Ex: $(\mathbb{R}^2, U)$ and $(\mathbb{R}, U^*)$ be usual topological spaces.

Define $T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$T_1(x, y) = y, \quad \forall (x, y) \in \mathbb{R}^2.$$



Then the inverse image of an open interval (c, d) is an infinite open strip.

~~Hence~~ inverse image of every basis element of $(\mathbb{R}, U^*)$ is an open subset of $(\mathbb{R}^2, U)$

$\therefore \pi_1 : (\mathbb{R}^2, U) \longrightarrow (\mathbb{R}, U)$ is continuous.

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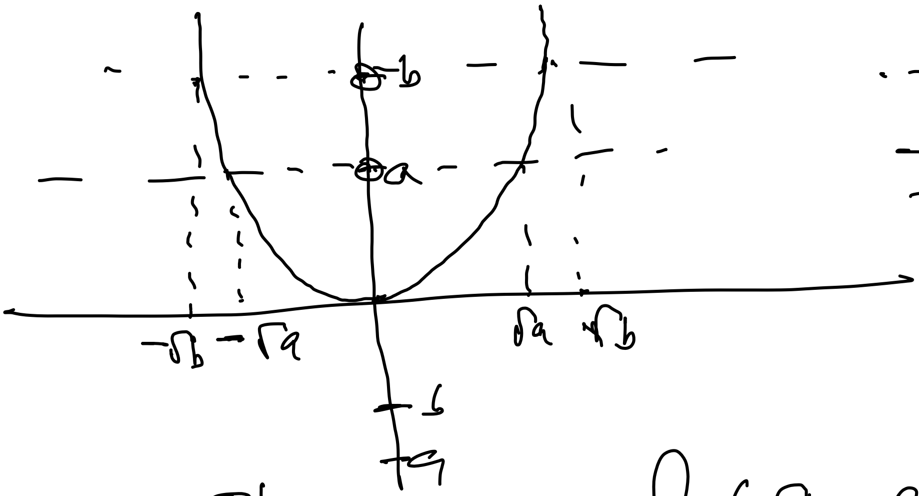
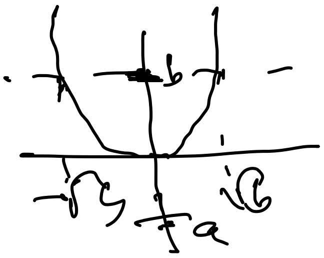
Ex. $f : (\mathbb{R}, U) \longrightarrow (\mathbb{R}, U)$ be defined by $f(x) = x^2$. Discuss the continuity of f.

Sol





Then $f^{-1}(a, b) = \begin{cases} (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b}), & 0 < a < b \\ \varphi & a < b < 0 \\ (-\sqrt{b}, \sqrt{b}), & a < 0 < b \end{cases}$

$\therefore$ $f : (\mathbb{R}, U) \longrightarrow (\mathbb{R}, U)$ is Continuous.

Problem
let $f : (\mathbb{R}, U) \longrightarrow (\mathbb{R}, U)$ be a function defined by $f(x) = |x|$. Discuss the Continuity of f.

Problem: let $f : (X, T) \longrightarrow (Y, T^*)$ be a constant map $f(x) = p$. Then discuss the continuity of f.

for any $H \in T^*$

$$f^{-1}(H) = \begin{cases} X & p \in H \\ \phi & p \notin H \end{cases}$$

$\therefore$ f is continuous.

Theorem: let $f: (X, T) \to (Y, T^*)$ be a map. Then f is continuous iff for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$.

Proof: Suppose $f: X \to Y$ is $T - T^*$ continuous.

Now for any subset A of X, we have

$$f(A) \subset \overline{f(A)}$$

$$\Rightarrow \quad A \subset f^{-1}(f(A)) \subset f^{-1}(\overline{f(A)})$$

$\because$ $\overline{f(A)}$ is a closed set and f is continuous, so $f^{-1}(\overline{f(A)})$ is closed in X.

Hence by definition of a closure of a set, we have

$$A \subset \overline{A} \subset f^{-1}(\overline{f(A)})$$

$$\Rightarrow \quad f(\overline{A}) \subset \overline{f(A)}.$$

Conversely assume that for any subset A of X, $f(\overline{A}) \subset \overline{f(A)}$.

Claim: $f : (X, \tau) \longrightarrow (Y, \tau^*)$ is continuous.

Let F be any closed subset of Y.

Denote $A := f^{-1}(F)$.

We prove A is closed.

Now

$$f(\overline{A}) = f\left[\overline{f^{-1}(F)}\right]$$

$$\subset \overline{f(f^{-1}(F))}$$

$$= \overline{F}$$

$$= F \quad [\because F \text{ is closed}]$$

$$\therefore f(\overline{A}) \subset F$$

$$\Rightarrow \overline{A} \subset f^{-1}(F) = A$$

Thus $\overline{A} \subset A$, but $A \subset \overline{A}$

$$\therefore A = \overline{A}$$

$\Rightarrow A = f^{-1}(F)$ is a closed set.

$\Rightarrow f$ is continuous

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Def: A function $f:(X,T) \to (Y,T^*)$ is continuous at a point $p \in X$ if the inverse image $f^{-1}(H)$ of every open subset H of Y containing $f(p)$ is superset of an open set $G \subset X$ containing p. That is Inverse image of every nbd of $f(p)$ is a nbd of p.

$$N \in \mathcal{N}_{f(p)} \Rightarrow f^{-1}(N) \in \mathcal{N}_p.$$

Sequentially Continuity at a point:

A function $f:(X,T) \longrightarrow (Y,T^*)$ is said to be sequentially continuous at a point $p \in X$ if for every sequence $\{a_n\}$ in X with $a_n \longrightarrow p$, the sequence $\{f(a_n)\}$ converges to $f(p)$ in Y, i.e., $a_n \longrightarrow p \Longrightarrow f(a_n) \longrightarrow f(p)$.

Theorem: If a function $f: (X, T) \to (Y, T^*)$ is continuous at $p \in X$, then it is sequentially continuous.

Proof: let f is $T - T^*$ continuous, and let $\{a_n\}$ be a sequence in X with $a_n \to p \in X$.

Claim: $f(a_n) \to f(p)$.

To prove this, we that any nbd N of $f(p)$ contains almost all terms of the sequence $\{f(a_n)\}$.

So let N be any nbd of $f(p)$.

$\because f$ is $T - T^*$ continuous, then

$M = f^{-1}(N)$ is a nbd of p.

Then M contains almost all terms of the sequence $\{a_n\}$.

So there exists $n_0 \in \mathbb{N}$ $\ni$ $\forall n \geq n_0$, $a_n \in M$.

$\Rightarrow \quad f(a_n) \in f(M) = N, \ \forall\, n \geq n_0$

Thus for all $n \geq n_0$, $f(a_n) \in N$ and N is nbd of $f(p)$.

$$\therefore \quad f(a_n) \longrightarrow f(p)$$

$\Rightarrow$ f is sequentially continuous

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Remark: Converse of the above theorem need not be true.

Let $(\mathbb{R}, T)$ be a cocountable topological space.

Let $\{a_n\}$ be a sequence in $(\mathbb{R}, T)$ such that $a_n \longrightarrow b \in \mathbb{R}$.

Then the sequence $\{a_n\}$ must be of the form $\{a_1, a_2, a_3, \ldots a_{n_0}, b, b, b, \ldots\ldots\}$

Let A be the set consisting of the terms of the sequence $\{a_n\}$ different from b. Then A is finite set.

$\Rightarrow$ A is countable set.

$\Rightarrow$ $A^c \in T$ and A^c contains the point b.

Hence $a_n \to b \Rightarrow A^c$ contains almost all terms of the sequence $\{a_n\}$.

Let $f : (R, T) \to (R, T^*)$ be a function, where (R, T^*) is any other topological space.

Then

$\{f(a_n)\} = \{f(a_1), f(a_2), \ldots f(a_{n_0}), f(b), f(b) \text{ ------ } \}$

$\Rightarrow$ $f(a_n) \to f(b)$ as $n \to \infty$.

$\Rightarrow$ f is sequentially continuous.

On the other hand, let $(\mathbb{R}, U)$ be a usual topological space.

Let $f : (\mathbb{R}, T) \longrightarrow (\mathbb{R}, U)$ be the map defined by $f(x) = x$.

Since $f^{-1}((0,1)) = (0,1)$ is not a T- open set

[$\because$ $(0,1)^c = (-\infty, 0] \cup [1, \infty)$ is uncountable set].

Open and Closed maps :

Let $f : (X, T) \longrightarrow (Y, T^*)$ be a map. We say

(i) f is open map if $G \in T \Rightarrow f(G) \in T^*$

i.e., image of an open set is an open set.

(ii) f is closed map if image of a closed set is a closed set.

Ex: Let $(\mathbb{R}, \mathcal{D})$ be a discrete topological space and $(\mathbb{R}, \mathcal{U})$ be usual topological space. Define $f : (\mathbb{R}, \mathcal{D}) \to (\mathbb{R}, \mathcal{U})$ by

$$f(x) = x, \quad \forall x \in \mathbb{R}.$$

and $g : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{D})$ by

$$g(x) = x, \quad \forall x \in \mathbb{R}.$$

Then for each $G \in \mathcal{U}$, $f^{-1}(G) \in \mathcal{D} = P(\mathbb{R})$.

$\therefore$ f is continuous.

Also $\{a\} \in \mathcal{D} \Rightarrow f(\{a\}) = \{a\} \notin \mathcal{U}$

$\therefore$ f is not an open map.

But $g(G) = G \in \mathcal{D} = P(\mathbb{R})$

$\Rightarrow g$ is an open map.

Also for any $\{a\} \in D$, $g^{-1}(\{a\}) = \{a\}$
$\notin U$

$\Rightarrow g$ is not continuous.

Illy check g is closed map and f is not a closed map.

Note :- let $f : (X, T_1) \to (Y, T_2)$ be a one-one function. Then f is open map iff f^{-1} is continuous.

Suppose $f^{-1} : (Y, T_2) \to (X, T_1)$ is continuous.

Claim: f is open map.

So let $G \in T_1$, we prove $f(G) \in T_2$.

$\because \; f^{-1} : (Y, T_2) \longrightarrow (X, T_1)$ is continuous,

we have $(f^{-1})^{-1}(G)$ is open in Y.

$\Rightarrow f(G)$ is open in Y

$\Rightarrow f$ is open map.

Conversely assume that $f : (X, T_1) \to (Y, T_2)$ is open map.

Claim: f^{-1} is continuous.

Let $G \in T_1$. Since f is open map implies $f(G) \in T_2$.

$$\Rightarrow (f^{-1})^{-1}(G) = f(G) \in T_2.$$

$\Rightarrow f^{-1} : (Y, T_2) \longrightarrow (X, T_1)$ is continuous

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Ily we can prove $f: (X, T_1) \to (Y, T_2)$ is a closed map iff f^{-1} is continuous, where f is 1–1 map.

[Attendance: 65, 17, 62, 19, 27, 06]