Lecture 2

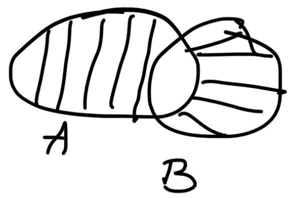
Let $A, B \subseteq X$.

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

$$A \triangle B := (A \setminus B) \cup (B \setminus A)$$

the symmetric difference of A & B.

$$A \triangle B = A \cup B \setminus (A \cap B).$$



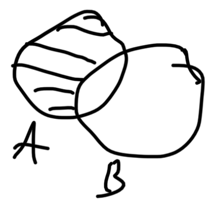
Proposition:- Let $A, B, C, D \subseteq X$.

1. $A \triangle B = B \triangle A$.
2. $(A \triangle B) \triangle C = A \triangle (B \triangle C)$
3. $(A \triangle B) \triangle (C \triangle D) = (A \triangle C) \triangle (B \triangle D)$
4. $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

proof:-

1.
$$\begin{aligned} A \triangle B &= (A \setminus B) \cup (B \setminus A) \\ &= (B \setminus A) \cup (A \setminus B) \\ &= B \triangle A. \end{aligned}$$

2. Consider



$$\begin{aligned} (A \triangle B)^c &= \big((A \setminus B) \cup (B \setminus A)\big)^c \\ &= \big((A \cap B^c) \cup (B \cap A^c)\big)^c \qquad (\because A \setminus B = A \cap B^c) \\ &= (A \cap B^c)^c \cap (B \cap A^c)^c \qquad \big((S \cup T)^c = S^c \cap T^c\big) \end{aligned}$$

$$= (A^c \cup B) \cap (B^c \cup A)$$

$\big(S \cap (T \cup W) = S\cap T \cup S \cap W\big)$

$$= \big((A^c \cup B) \cap B^c\big) \cup \big((A^c \cup B) \cap A\big)$$

$$= \big((A^c \cap B^c) \cup \underbrace{(B \cap B^c)}_{=\emptyset}\big) \cup \big(\underbrace{(A^c \cap A)}_{=\emptyset} \cup (B \cap A)\big)$$

$$\therefore \quad \boxed{(A \Delta B)^c = (A^c \cap B^c) \cup (B \cap A)} \longrightarrow (*)$$

Now

$$(A \Delta B) \Delta C = \big((A \Delta B) \setminus C\big) \cup \big(C \setminus (A \Delta B)\big)$$

$$= \big((A \Delta B) \cap C^c\big) \cup \big(C \cap (A \Delta B)^c\big)$$

$$= \Big[\big((A \cap B^c) \cup (B \cap A^c)\big) \cap C^c\Big] \cup \Big[C \cap \big((A^c \cap B^c) \cup (A \cap B)\big)\Big]$$

(by $(*)$)

$$= (A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap A^c \cap B^c) \cup (C \cap A \cap B)$$

By symmetry, this is equal to $A \Delta (B \Delta C)$.

(3) $(A \Delta B) \Delta (C \Delta D) = \big((A \Delta B) \Delta C\big) \Delta D$ (by (2))

$$= \big(A \Delta (B \Delta C)\big) \Delta D$$

,,

$$= \left((B \Delta C) \Delta A\right) \Delta D \quad (\text{by } (1))$$

$$= (B \Delta C) \Delta (A \Delta D) \quad (\text{by } (2)).$$

$$= (A \Delta D) \Delta (B \Delta C) \quad (\text{by } (1))$$

$$= \text{RHS}.$$

(4) $A \cap (B \Delta C) = A \cap \left((B \cap C^c) \cup (C \cap B^c)\right)$

$$= (A \cap B \cap C^c) \cup (A \cap C \cap B^c) \rightarrow (*)$$

Now $(A \cap B) \Delta (A \cap C) = \left((A \cap B) \cap (A \cap C)^c\right) \cup$

$$\left((A \cap C) \cap (A \cap B)^c\right)$$

$$= \left(A \cap B \cap (A^c \cup C^c)\right) \cup \left((A \cap C) \cap (A^c \cup B^c)\right)$$

$$= \underbrace{(A \cap B \cap A^c)}_{= \phi} \cup (A \cap B \cap C^c) \cup \underbrace{(A \cap C \cap A^c)}_{= \phi}$$

$$\cup (A \cap C \cap B^c)$$

$$= (A \cap B \cap C^c) \cup (A \cap C \cap B^c)$$

$$= A \cap (B \Delta C) \quad (\text{by } (*)).$$

Remark:- Suppose $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$ Then

$$\bigcup_{i=1}^{\infty}(E_1 \setminus E_i) = E_1 \setminus \left(\bigcap_{i=1}^{\infty} E_i\right)$$

Definition:- An equivalence relation R on a set E is a subset of $E \times E$ with the following conditions:

[i] $(x,x) \in R$, $\forall x \in E$ (reflexive)

(ii) $(x,y) \in R \Rightarrow (y,x) \in R$, $\forall x,y \in E$
(xRy) (symmetric)

(iii) if $(x,y) \in R$, $(y,z) \in R$, then $(x,z) \in R$, $\forall x,y,z \in E$.
(transitive).

We write $\underline{x \sim y}$ if $(x,y) \in R$.

An equlance relation on E, partitions E into disjoint equivalence classes. Let $x \in E$.

equivalence class of x, denote $[x]$ defined as

$$[x] := \{ y \in E \mid (x,y) \in R \} = \{ y \in E \mid x \sim y \}.$$

&

$$E = \bigcup_{x \in E} [x]$$

Any two equivalence classes are disjoint. i.e,

if $x \notin [y]$ (or $y \notin [x]$), then $[x] \cap [y] = \phi$.

Example:-

① Let $E = \mathbb{Z}$, Define $a, b \in E$,

$a \sim b$ if $a - b$ is divisible by n

i.e, $n \mid a-b$. (n fixed +ve integer).

$\sim$ is an equivalence relation:

- $a \sim a$? if and only if $a - a = 0$ is divisible by n.

$n \mid 0$,

$0 = n \cdot (0)$.

$\therefore a \sim a$

- Suppose $a \sim b \Rightarrow n \mid a-b$

$\Rightarrow n \mid b-a$

$\Rightarrow b \sim a$.

- Suppose $a \sim b$ & $b \sim c$.

$\Rightarrow \quad n \mid a-b$ & $n \mid b-c$.

Now $\quad a-c = \underline{(a-b)} + \underline{(b-c)}$

$\Rightarrow \quad n \mid (a-b)+(b-c)$

$\Rightarrow \quad n \mid a-c$

$\Rightarrow \quad a \sim c$.

$\therefore$ $\sim$ is an equivalence relation on E.

$a \in E = \mathbb{Z}$, $[a] = \{ b \in \mathbb{Z} \mid a \sim b \}$.

$\qquad = \{ b \in \mathbb{Z} \mid n \mid a-b \}$

$$\left.\begin{aligned}
[0] &= \{ b \in \mathbb{Z} \mid n \mid -b \} \\
[1] &= \{ b \in \mathbb{Z} \mid n \mid 1-b \} \\
&\vdots \\
[n-1] &= \{ b \in \mathbb{Z} \mid n \mid n-1-b \}
\end{aligned}\right\}.$$

$[n] = \{ b \in \mathbb{Z} \mid n \mid n-b \} = \{ b \in \mathbb{Z} \mid n \mid -b \} = [0]$

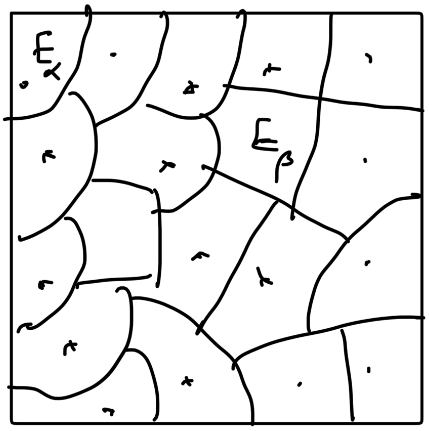
$[n+1] = [1]$

$\vdots$

$$\therefore\ E = \mathbb{Z} = \bigcup_{a \in E} [a] = \bigcup_{i=0}^{n-1} [i]$$

Axiom of choice :-

Suppose $\{E_\alpha\}_{\alpha \in A}$ is a non-empty collection of non-empty disjoint subsets of a set X.
Then there exists a set $V \subseteq X$ containing just one element from each E_α.



$V \cap E_\alpha$ = singleton set $\forall \alpha \in A$.

Example :-

(I) $A = \mathbb{N}$: $\{E_\alpha\}_{\alpha \in A} = \{E_\alpha\}_{\alpha \in \mathbb{N}}$ X

$= \{E_1, E_2, E_3, \dots\}$

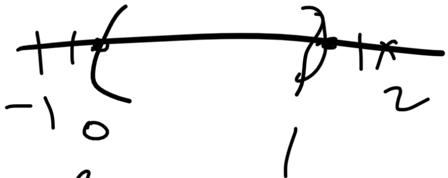
(I) $A = [0,1]$.

$\{E_\alpha\}_{\alpha \in A} = \{E_\alpha \mid \alpha \in [0,1]\}$

Recall:- Let $\underset{\neq \phi}{A} \subseteq \mathbb{R}$.

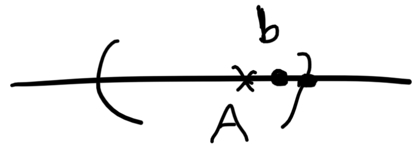
Def:- The supremum or the least upperbound of A is denoted as sup(A), defined as the least upper bound of A.

- An element $a \in \mathbb{R}$ is called an upperbound of A, if $x \leq a, \; \forall x \in A$.



- An element $a \in \mathbb{R}$ is called a lower bound of A, if $x \geq a, \; \forall x \in A$.

Def:- sup(A) = the least upperbound of A.
This means that if we take any element b in the strictly smaller than sup(A) $(b < \sup(A))$, then b can not be an upper bound of A.



$$\boxed{\text{For any } \varepsilon > 0, \text{ then there exists } a \in A \text{ such that } a > \sup(A) - \varepsilon.}$$

$\sup(A) - \varepsilon < \sup(A)$

Def:— The infimum of A is the greatest lower bound of A & denote by $\inf(A)$.

That is, any number strictly bigger than $\inf(A)$ can not be a lower bound of A.

$$\boxed{\text{For } \varepsilon > 0, \text{ there exists } a \in A \text{ such that } a < \inf(A) + \varepsilon.}$$

$$\boxed{A \subseteq [\inf(A), \sup(A)] \subseteq \text{open},}$$

Theorem (Heine–Borel Theorem):—

Suppose A is a closed and bounded subset of $\mathbb{R}$.

Suppose $A \subseteq \bigcup_{\alpha \in I} G_\alpha$, where G_α are open sets & I is some index set. Then there exists a finite subcollection of the sets, say $\{G_1, \ldots, G_n\}$ such that

$$A \subseteq \bigcup_{i=1}^{n} G_i .$$

Theorem:– (Lindelöf's theorem):

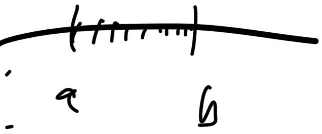
Suppose $\mathcal{G} = \{ I_\alpha / \alpha \in A \}$ is a collection of open intervals in $\mathbb{R}$. Then there exists a subcollection of $\mathcal{G}$, at most countable in number, say $\{I_1, I_2, \ldots\}$ such that

$$\bigcup_{\alpha \in A} I_\alpha = \bigcup_{i=1}^{\infty} I_i$$

<u>Theorem</u>:– Every non-empty open set G in $\mathbb{R}$ is the union of disjoint open intervals at most countable in number.

<u>proof</u>:– For $a, b \in G$, define $a \sim b$ if the closed interval $[a,b]$ or ($[b,a]$ if $b < a$) lies in G.

Check that $\sim$ is an equivalence relation on G.



$\therefore$ G is the union of disjoint equivalence classes.

Denote $C(a)$ the equivalence class of G containing a.

($b \in C(a)$, then $a \sim b$)

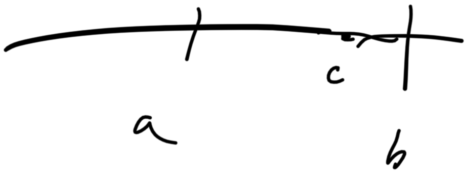
Infact $C(a)$ is an interval:

$$C(a) = \{ b \in G \mid a \sim b \}$$

$$= \{ b \in G \mid [a,b] \text{ or } [b,a] \text{ lies in } G \}.$$

is an interval.

Suppose $C(a)$ is not an interval. Then for any $b \in C(a)$, there exists a point $c \in G$ lies between a & b & $c \notin C(a)$.



$\Rightarrow$ $b \in C(a)$ $\Rightarrow$ $[a, b)$ or $[b, a)$ lies in G

$\Rightarrow$ $[a, c]$ lies in G or $[b, c]$

$\Rightarrow$ $c \in C(a) = C(b)$

This is a contradiction.

$\therefore$ $C(a)$ is an interval.

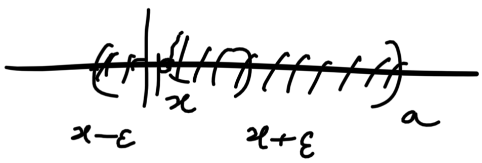
$C(a)$ is an open interval: if $x \in C(a) \subseteq G$ open

$\Rightarrow$ there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq G$

$x \sim a$

$[a, x) \subseteq G$

$\Rightarrow$ $(x-\varepsilon, x+\varepsilon) \subseteq C(a)$



$(\because$ $b \in C(a)$, $\forall$ $b \in (x-\varepsilon, x+\varepsilon))$.

$(x-\varepsilon, a] \subseteq G$

for $b \forall$

$[b, a] \subseteq G$

$b \sim a$

$b \in C(a)$

$\therefore$ $C(a)$ is an open interval.

$\therefore \quad G = \bigcup C(a)$ $\Big| \ \forall\, b \in (x-\varepsilon, x+\varepsilon)$

$\quad = $ disjoint union of open intervals.

$\quad = $ disjoint union of at most countable open intervals (by Lindelöf thm).