Lecture 10

Proposition:- Every continuous function defined on a measurable set is measurable.

proof:- Let $f : E \to \mathbb{R}$ be a continuous function, where E is measurable.

Let $\alpha \in \mathbb{R}$,

$f^{-1}(\underbrace{(\alpha, \infty)}_{\text{open}}) = U$, an open set because f is continuous

But every open set is a Borel set & hence U is measurable.

$\therefore f$ is measurable.

Examples:- Constant function, $f(x) = e^x \sin x$, measurable.

Theorem:- Let $E \subseteq \mathbb{R}$ be a measurable set & $f : E \to \mathbb{R} \cup \{\pm\infty\}$ any function. Then the following statements are equivalent.

(i) f is measurable

(ii) $\forall \alpha \in \mathbb{R}$, $\{x \in E \mid f(x) \geq \alpha\}$ is measurable

(iii) $\forall \alpha \in \mathbb{R}$, $\{x \in E \mid f(x) < \alpha\}$ is measurable

(iv) $\forall \alpha \in \mathbb{R}$, $\{x \in E \mid f(x) \leq \alpha\}$ is measurable.

<u>proof</u>:- <u>(i) $\Rightarrow$ (ii)</u>: Assume f is measurable.

Then for any $\alpha \in \mathbb{R}$,

$\{x \in E \mid f(x) > \alpha\}$ is measurable.

For any $\alpha \in \mathbb{R}$,

$$\{x \in E \mid f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \underbrace{\{x \in E \mid f(x) > \alpha - \tfrac{1}{n}\}}_{\in \mathcal{M}}$$

$$\in \mathcal{M}$$

This proves (ii).

<u>(ii) $\Rightarrow$ (iii)</u>: Assume (ii).

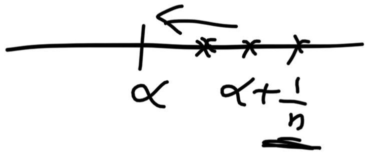
For $\alpha \in \mathbb{R}$,

$$\{x \in E \mid f(x) < \alpha\} = \{x \in E \mid f(x) \geq \alpha\}^c \in \mathcal{M}$$

(iii) $\Rightarrow$ (iv): Assume (iii). For $\alpha \in \mathbb{R}$,

$$\{x \in E \mid f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \underbrace{\{x \in E \mid f(x) < \alpha + \tfrac{1}{n}\}}_{\in \mathcal{M}}$$

$$\in \mathcal{M}$$



(iv) $\Rightarrow$ (i): Assume (iv).

To show: f is measurable.

For $\alpha \in \mathbb{R}$,

$$\{x \in E \mid f(x) > \alpha\} = \underbrace{\{x \in E \mid f(x) \leq \alpha\}}_{\in \mathcal{M}}{}^{c}$$

$$\in \mathcal{M}.$$

$\therefore$ f is measurable.

Remark:

$$\{x \in E \mid f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \underbrace{\{x \in E \mid f(x) < \alpha + \tfrac{1}{n}\}}$$

proof: Let $x \in LHS$.

$$\Rightarrow f(x) \leq \alpha < \alpha + \frac{1}{n}, \quad \forall n \geq 1$$

$$\Rightarrow \quad f(x) < \alpha + \tfrac{1}{n} \quad \forall n \geq 1$$

$$\Rightarrow \quad x \in \{ y \in E \mid f(y) < \alpha + \tfrac{1}{n} \}, \ \forall n \geq 1$$

$$\Rightarrow \quad x \in \bigcap_{n=1}^{\infty} \{ y \in E \mid f(y) < \alpha + \tfrac{1}{n} \}$$

$$\Rightarrow \quad x \in RHS.$$

Let $x \in RHS.$

$$\Rightarrow \quad f(x) < \alpha + \tfrac{1}{n}, \quad \forall n \geq 1$$

$$\Rightarrow \quad \underbrace{\lim_{n \to \infty} f(x)}_{\shortparallel} \leq \lim_{n \to \infty} \left(\alpha + \tfrac{1}{n} \right)$$

$$\Rightarrow \quad f(x) \leq \alpha$$

$$\Rightarrow \quad x \in \{ x \in E \mid f(x) \leq \alpha \}$$

$$\Rightarrow \quad x \in LHS.$$

$$f^{-1}((a,b)) = \{ x \in E \mid f(x) \in (a,b) \}.$$

$$a < f(x) < b$$

$$\overset{\uparrow}{\mathcal{M}} \quad = \underbrace{\{ x \in E \mid f(x) > a \}}_{\overset{\uparrow}{\mathcal{M}}} \cap \underbrace{\{ x \in E \mid f(x) < b \}}_{\overset{\uparrow}{\mathcal{M}}}$$

$$= f^{-1}((a,\infty)) \cap f^{-1}((-\infty, b))$$

(a, ∞)

Suppose $f^{-1}((a,b))$ $a,b \in \mathbb{R}$ measurable

$$\Downarrow \quad ?$$

f is measurable.

<u>proposition:-</u> Suppose $f: E \to \mathbb{R} \cup \{\pm\infty\}$ is measurable, where $E \subseteq \mathbb{R}$ measurable.

Then for any $\alpha \in \mathbb{R} \cup \{\pm\infty\}$,

$$\{x \in E \mid f(x) = \alpha\} \text{ is measurable.}$$

<u>proof:-</u> For $\alpha \in \mathbb{R}$,

$$\{x \in E \mid f(x) = \alpha\} = \underbrace{\{x \in E \mid f(x) \leq \alpha\}}_{\in \mathcal{M}} \cap \underbrace{\{x \in E \mid f(x) \geq \alpha\}}_{\in \mathcal{M}}.$$

$$\in \mathcal{M}$$

Suppose $\alpha = +\infty$,

$$\{x \in E \mid f(x) = +\infty\} = \bigcap_{n=1}^{\infty} \underbrace{\{x \in E \mid f(x) > n\}}_{\in \mathcal{M}}$$

$$\in \mathcal{M}$$

||ly for $\alpha = -\infty$.

Theorem:- Let $f, g : E \to \mathbb{R}$, be measurable functions, $E \subseteq \mathbb{R}$ measurable & $c \in \mathbb{R}$. Then $f+c$, cf, $f+g$, $f-g$, fg, f^2 are all measurable functions.

proof:- For $\alpha \in \mathbb{R}$, for $f+c$

$$\underbrace{\{x \in E \mid f(x)+c > \alpha\}}_{(f+c)^{-1}((\alpha,\infty))} = \{x \in E \mid f(x) > \alpha - c\} \in \mathcal{M} \quad (\because f \text{ is measurable})$$

$\therefore$ $f+c$ is measurable.

For cf, if $c=0$, then $cf \equiv 0$ which is measurable.

Suppose $c \neq 0$. say $c > 0$,

$$(cf)^{-1}((\alpha,\infty)) = \{x \in E \mid (cf)(x) > \alpha\}$$

$$= \{x \in E \mid cf(x) > \alpha\}$$

$$= \{x \in E \mid f(x) > \tfrac{\alpha}{c}\}$$

$$= f^{-1}\left(\left(\tfrac{\alpha}{c}, \infty\right)\right) \in \mathcal{M}$$

$\therefore$ cf is measurable if $c>0$, ||ly for $c<0$.

To show: $f+g$ is measurable.

For any $\alpha \in \mathbb{R}$,

$$(f+g)^{-1}\big((\alpha,\infty)\big) = \{x \in E \mid (f+g)(x) > \alpha\}$$

$$= \{x \in E \mid f(x)+g(x) > \alpha\}$$

$$= A \text{ (say)}.$$

Suppose $x \in A$.

$$\Rightarrow \quad f(x)+g(x) > \alpha$$

$$\Rightarrow \quad f(x) > \alpha - g(x)$$

Let $r_i \in \mathbb{Q}$ be a rational number such that $f(x) > r_i > \alpha - g(x)$

where $\mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$

$$\Rightarrow \quad f(x) > r_i \quad \& \quad r_i > \alpha - g(x)$$

$$\Rightarrow \quad f(x) > r_i \quad \& \quad g(x) > \alpha - r_i$$

$$\Rightarrow \quad x \in \bigcup_{i=1}^{\infty}\Big[\underbrace{\{x \in E \mid f(x) > r_i\}}_{\in \mathcal{M}} \underset{\mathcal{M}}{\cap} \underbrace{\{x \in E \mid g(x) > \alpha - r_i\}}_{\in \mathcal{M}}\Big]$$

$$= \underset{\in \mathcal{M}}{B} \text{ (say)}$$

$\therefore A \subseteq B$.

$(*) \Rightarrow (**)$

The reverse inclusion is also true $B \subseteq A$. (check it).

$\therefore A = B \in \mathcal{M}$

$\therefore (f+g)^{-1}((\alpha, \infty)) \in \mathcal{M}, \ \forall \alpha \in \mathbb{R}$

$\therefore f+g$ is a measurable function.

- $f-g = f+(-g)$ also measurable.

- $fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$

- For f^2, & any $\alpha \in \mathbb{R}$

$$(f^2)^{-1}((\alpha,\infty)) = \{ x \in E \,/\, \underbrace{f^2(x)}_{=(f(x))^2} > \alpha \}$$

$$= \begin{cases} E \in \mathcal{M} & \text{if } \alpha < 0 \\ \underbrace{\{x \in E \,/\, f(x) > \sqrt{\alpha}\}}_{\in \mathcal{M}} \cup \underbrace{\{x \in E \,/\, f(x) < -\sqrt{\alpha}\}}_{\in \mathcal{M}} & \text{if } \alpha \geq 0 \end{cases}$$

$\in \mathcal{M}$

$\therefore f^2$ is measurable. $\Rightarrow$ fg is also measurable.

Theorem:- Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Then

(i) $\sup\limits_{1\le i\le n}(f_i) = \max\limits_{1\le i\le n}(f_i) = \max\{f_1, f_2, \dots, f_n\}$ is measurable $\forall n \ge 1$

(ii) $\inf\limits_{1\le i\le n}(f_i) = \min\limits_{1\le i\le n}(f_i)$ is measurable, $\forall n \ge 1$.

(iii) $\sup\limits_n(f_n) : E \to \mathbb{R}$, $\sup\limits_n(f_n)(x) := \sup\limits_n(f_n(x))$ $\forall x \in E$,

$\sup(f_n)$ is measurable.

(iv) $\inf(f_n) : E \to \mathbb{R}$, $\inf(f_n)(x) = \inf\limits_n(f_n(x))$, $\forall x \in E$

$\inf(f_n)$ is measurable.

(v) $\limsup(f_n)$ is measurable, where

$\limsup(f_n) : E \to \mathbb{R}$, defined as

$$\begin{aligned}
\limsup(f_n)(x) &:= \inf_n\left(\sup_{i\ge n}(f_i(x))\right) \\
&= \inf_n\left(\sup(\{f_n(x), f_{n+1}(x), \dots\})\right) \\
&= \inf\Big(\{\sup(\{f_1(x), f_2(x), \dots\}), \\
&\qquad \sup(\{f_2(x), f_3(x), \dots\}), \dots \\
&\qquad \dots\dots\}\Big) \quad \forall x \in E
\end{aligned}$$

(vi) $\liminf_n (f_n)$ is measurable, where

$\liminf_n (f_n) : E \to \bar{\mathbb{R}}$, defined as

$$\liminf_n (f_n)(x) := \sup_n \left(\inf_{i \geq n} (f_i(x)) \right) \quad \forall x \in E.$$

Remark:- $S \subseteq \mathbb{R}$

$\sup(S) =$ the least upper bound of S

$= \alpha$ (say).

$\Rightarrow \quad x \leq \alpha, \quad \forall x \in S$, α is least such

$\Rightarrow \quad -x \geq -\alpha, \quad \forall x \in S$, $-\alpha$ is the largest such

$\Rightarrow \quad \inf(-S) = -\alpha$

$-S := \{-x \mid x \in S\}$

$\Rightarrow \quad \inf(-S) = -\sup(S)$

$\Rightarrow \quad \boxed{\sup(S) = -\inf(-S).}$

||ly $\quad \boxed{\inf(S) = -\sup(-S).}$ ✓

Take $S = \{ f_1(x), f_2(x), \ldots \}$.

Then $\inf(f_n(x)) = -\sup(-f_n(x)) \ \forall x \in E$

$\therefore$ $$\boxed{\inf_n(f_n) = -\sup_n(-f_n)}$$

Why check that
$$\liminf(f_n) = -\limsup(-f_n).$$

proof the theorem:-

Given $f_n : E \to \mathbb{R}$ is measurable, $\forall n \in \mathbb{N}$.

(i) To show $\sup\limits_{1\le j\le n}(f_i)$ is measurable.

For $\alpha \in \mathbb{R}$,

$$\left(\sup_{1\le i\le n}(f_i)\right)^{-1}((\alpha,\infty)) = \left\{x \in E \,\middle|\, \sup_{1\le j\le n}(f_i)(x) \not\geq \alpha\right\}$$

$$= \left\{x \in E \,\middle|\, \sup_{1\le j\le n}(f_i(x)) > \alpha\right\}$$

$$= \left\{x \in E \,\middle|\, \max\{f_1(x), \dots, f_n(x)\} > \alpha\right\}$$

$$= \bigcup_{i=1}^{n} \left\{x \in E \,\middle|\, f_i(x) > \alpha\right\}$$

$$= \bigcup_{i=1}^{n} f_i^{-1}\big((\alpha,\infty)\big) \in \mathcal{M}$$

$\therefore \sup\limits_{1 \le i \le n}(f_i)$ is measurable.

(ii) $\inf\limits_{1\le i\le n}(f_i) = -\sup\limits_{1\le i\le n}(-f_i)$ is measurable.
(by (i))

(iii) <u>To show</u>: $\sup\limits_{n}(f_n)$ is measurable.

For $\alpha \in \mathbb{R}$,

$$\big(\sup(f_n)\big)^{-1}\big((\alpha,\infty)\big) = \{x \in E \mid \sup(f_n)(x) > \alpha\}$$

$$= \{x \in E \mid \sup_n (f_n(x)) > \alpha\}$$

$$= \bigcup_{n=1}^{\infty} \{x \in E \mid f_n(x) > \alpha\}$$

$$= \bigcup_{n=1}^{\infty} f_n^{-1}\big((\alpha,\infty)\big) \in \mathcal{M}$$

(iv) $\inf\limits_{n}(f_n) = -\sup\limits_{n}(-f_n)$ is measurable.

(v) $\limsup\limits_{n}(f_n) = \inf\limits_{n}\Big(\sup\limits_{i\ge n}(f_i)\Big)$

By (iii), we get $\sup_{i \geq n} (f_i)$ is measurable

& by (iv), $\inf_n \left(\sup_{i \geq n} (f_i) \right)$ is measurable
$= \limsup_n (f_n)$

(vi) $\liminf_n (f_n) = -\limsup (-f_n)$ is measurable by (v).

<u>Definition</u>:- Let $E \subseteq \mathbb{R}$ be measurable. We say that a function $f : E \to \mathbb{R} \cup \{\pm\infty\}$ is <u>Borel measurable</u> or a <u>Borel function</u>, if for each $\alpha \in \mathbb{R} \cup \{\pm\infty\}$,

$$\{x \in E \mid f(x) > \alpha\} \text{ is a Borel set.}$$
$$= f^{-1}[(\alpha, \infty)]$$

<u>Example</u>:- Any continuous function defined on a measurable set is Borel measurable.

proof:- For $\alpha \in \mathbb{R}$,

$$f^{-1}((\alpha,\infty)) = \text{open set}$$

$=$ Countable union of open intervals

$\in \mathcal{B}$, which is a Borel set.

$\therefore$ f is Borel measurable.

Theorem:- Let $f : E \to \mathbb{R}$, be a function, where $E \subseteq \mathbb{R}$ is measurable.

Then the following statements are equivalent:

(i) f is Borel measurable.

(ii) $\forall \alpha \in \mathbb{R}$, $\{x \in E \mid f(x) \geq \alpha\}$ is a Borel set.

(iii) $\forall \alpha \in \mathbb{R}$, $\{x \in E \mid f(x) < \alpha\}$ is a Borel set.

(iv) $\forall \alpha \in \mathbb{R}$, $\{x \in E \mid f(x) \leq \alpha\}$ is a Borel set.

proof:- EXERCISE.

Theorem:- Suppose $f, g : E \to \mathbb{R}$ are Borel measurable functions, & $c \in \mathbb{R}$. Then $f+c$, cf, $f \pm g$, fg, f^2 are Borel measurable functions.

Note:- $\mathcal{B} \subsetneqq \mathcal{M}$. Let $A \in \mathcal{M}$ & $A \notin \mathcal{B}$.
(yet to prove)

$\Rightarrow$ A is measurable but A is not a Borel set.

Look at χ_A is measurable but not Borel measurable.

$$\chi_A^{-1}\left(\underline{\left(\tfrac{1}{2}, \infty\right)}\right) = \left\{ x \in \mathbb{R} \;\middle|\; \chi_A(x) > \tfrac{1}{2} \right\}$$

$$= A \notin \mathcal{B}$$

$\therefore$ χ_A is not Borel measurable.

$$\boxed{\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}}$$