Partial Differential Equation (PDE) Part B

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Note that in Part A, I show some examples where we verify CI of given PDE. In this Part, in Sec 1, we will discuss method of finding CI of 1st order PDE by Charpit's method. Before going to details of this method, I'll quote two results, without proof, as these two results will be used in the discussion of Charpit's method. Interested reader may see Ref 1 for proofs of two results stated below.

Result 1: Lagrange's Theorem for 1st order linear PDE of $n \geq 2$ independent variables x_j and one dependent variable $z(x_1, x_2, \dots, x_n)$:

For convenience, below we write the PDE by using the symbols $p_j \equiv \partial z/\partial x_j$:

$$P_1 p_1 + P_2 p_2 + \cdots + P_n p_n = R, \tag{A.1}$$

where P_js, R are functions of $x_1, x_2, \dots, x_n, z$. Then general integral is given by

$$\phi(u_1, u_2, \dots u_n) = 0, \tag{A.2}$$

where $u_j(x_1, x_2, \dots, x_n, z) = c_j, j = 1,2, \dots n$ are n independent solutions of following AE

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n}. \tag{A.3}$$

Result 2: Pfaffian Differential equation

$$P\mathrm{dx} + \mathrm{Qdy} + \mathrm{Rdz} = 0, \tag{A.4}$$

is said to be integrable if ∃ a relation

$$\phi(x, y, z; c) = 0, \tag{A.5}$$

which satisfy (A.4). Equation (A.5) is one-parameter family of surface in space.

Condition of Integrability of (A.4):

$$P(Q_z - R_y) + Q(R_x - P_z) + R(P_y - Q_x) = 0. \tag{A.6}$$

- Compatible Systems of 1st order PDEs

We will consider two following PDEs (linear or nonlinear) of two independent variables x, y and one dependent variable (solution) $z(x, y)$, by denoting partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ by p,q respectively:

$$f(x,y,z,p,q)=0,\ \ g(x,y,z,p,q)=0, \qquad (1.1)$$

where f,g are assumed to be functionally independent, i.e. to say, $\exists$ no relation between f,g which is independent of p,q. This assumption is natural because otherwise two PDEs would be automatically “Compatible”.

By “Compatible”, we mean every solution $z=z(x,y)$ of one PDE will be solution of other. The assumption made above mathematically transpires to following relation

$$f_p g_q - f_q g_p \neq 0\ , \qquad (1.2)$$

where suffix denote the variable w.r.t. which partial differentiation taken.

Let us now derive the condition of compatibility of two PDEs in (1.1) [Ref 1]. We begin by assuming the existence of same solution $z=z(x,y)$ for both PDEs. So with the substitution of solution z in both PDEs, we can view them as two non-differential relations for 2 variables p,q, so that under assumption (1.2), these two relations can be solved for p,q:

$$p=\phi(x,y,z), q=\psi(x,y,z) \qquad (1.3)$$

Note $p\equiv\frac{\partial z}{\partial x}, q\equiv\frac{\partial z}{\partial y}$, so that from well-known identity, we can write

$$p\mathrm{d}x+q\mathrm{d}y=\mathrm{d}z \Longrightarrow \phi\mathrm{d}x+\psi\mathrm{d}y-\mathrm{d}z=0. \qquad (1.4)$$

Equation (1.4) is Pfaffian differential equation (DE), which must be integrable so that solution of (1.4) must exist as $z=z(x,y)$, which is solution of both PDEs, we assumed at beginning. The condition of integrability of Paffian DE (1.4) is given by (A.6) for $P=\phi, Q=\psi, R=-1$, so that condition of compatibility of two PDEs in (1.1) is

$$\phi\psi_z-\psi\phi_z-\left(\phi_y-\psi_x\right)=0 \Longrightarrow \psi_x+\phi\psi_z=\phi_y+\psi\phi_z. \qquad (1.5)$$

Our work is not finished yet ! Because compatibility condition will be a differential relation between 7 quantities x,y,z,p,q,f,g, so that we will have to eliminate unknown functions ϕ,ψ using given two relations in (1.1) with assumed solution $z=z(x,y)$ and p,q from (1.3) are substituted into (1.1), i.e. f,g in (1.1) now becomes functions of 5 variables x,y,z,p,q, where p,q are functions of x,y,z by

(1.3), and x, y, z will now be considered independent. Apparently, reader may worry at this point that how z may be independent of x, y! The answer to this riddle is as follows: Note that we are here not solving PDEs, in which case z is of course some function of x, y. But here at the beginning we assume that some solution $z = z(x, y)$ exist for both PDEs, that means that we substitute z in (1.1) making z independent of x, y when we will differentiate f, g partially w.r.t. z, for otherwise again p, q will arise as a differentiation of z w.r.t x, y, but p, q are already fixed by (1.3).

Hence, considering x, y, z as independent variables, we will now differentiate f, g partially w.r.t x, y, z. At the first step, differentiate w.r.t x, z to get following pair of two relations respectively, using chain rule:

$$f_x + f_p\phi_x + f_q\psi_x = 0\,, f_z + f_p\phi_z + f_q\psi_z = 0, \tag{1.6}$$
$$g_x + g_p\phi_x + g_q\psi_x = 0\,, g_z + g_p\phi_z + g_q\psi_z = 0. \tag{1.7}$$

Notice that partial differentiation of f w.r.t. p, q are denoted by suffix f_p, f_q (not as f_ϕ, f_ψ), but partial differentiation of $p \equiv \phi, q \equiv \psi$ w.r.t x, z are denoted by ϕ_x, ϕ_z (not as p_x, p_z). This is because we are to eliminate two unknown functions ϕ, ψ to get a relation between x, y, z, p, q, f, g, where symbolical meaning of p, q are what was in the system (1.1) as partial derivative of z w.r.t. x, y respectively.

Add 1[st] relation with ϕ times 2[nd] for each pair (1.6) & (1.7), to get following two linear algebraic relations for $\psi_x + \phi\psi_z$ and $\phi_x + \phi\phi_z$:

$$f_q(\psi_x + \phi\psi_z) + f_p(\phi_x + \phi\phi_z) + (f_x + \phi f_z) = 0,$$
$$g_q(\psi_x + \phi\psi_z) + g_p(\phi_x + \phi\phi_z) + (g_x + \phi g_z) = 0.$$

By cross-multiplication, we may get expressions for both. However we need one of them, which is LHS of compatibility condition (1.5), the expression is

$$\psi_x + \phi\psi_z = \frac{1}{f_p g_q - f_q g_p}\left[\left(f_x g_p - f_p g_x\right) + \phi\left(f_z g_p - f_p g_z\right)\right]. \tag{1.8}$$

Note that ϕ in the RHS of (1.8) is not unknown function, because it is actually p.

Now we will find the similar expression for $\phi_y + \psi\phi_z$, which is the RHS of compatibility condition (1.5). To get that expression, we will now differentiate f, g partially w.r.t. y, z, and proceed as before. The detailed steps are skipped here because readers at this point should be able to produce these steps. Finally, we get

$$\phi_y + \psi\phi_z = -\frac{1}{f_p g_q - f_q g_p}\left[\left(f_y g_q - f_q g_y\right) + \psi\left(f_z g_q - f_q g_z\right)\right]. \qquad (1.9)$$

Equating RHS of (1.8) and (1.9), our condition of compatibility in the final form is

$$\left(f_x g_p - f_p g_x\right) + \left(f_y g_q - f_q g_y\right) + p\left(f_z g_p - f_p g_z\right) + q\left(f_z g_q - f_q g_z\right) = 0 \quad (1.10)$$

- Charpit's Method

This method is used to find solution of given non-linear PDE

$$f(x, y, z, p, q) = 0. \qquad (2.1)$$

The main point of Charpit's method is to find a compatible partner PDE

$$g(x, y, z, p, q, a) = 0, \qquad (2.2)$$

where f, g are not functionally dependent, i.e. $\begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix} \neq 0$, and a is an arbitrary parameter. Clearly compatibility condition (1.10) holds.

Solving (2.1) and (2.2) for p, q, we get $p = p(x, y, z, a), q = q(x, y, z, a)$, and consequently the DE

$$\mathrm{d}z = p(x, y, z, a)\mathrm{d}x + q(x, y, z, a)\mathrm{d}y$$

must have solution $F(x, y, z, a, b) = 0$, which is actually Complete Integral (CI) of given PDE (2.1). Note that the solution of a PDE of n independent variables and one dependent variable, containing exactly n arbitrary constants, is called CI.

Hence, our goal in Charpit's method is to find compatible partner g. This is actually now straightforward, because compatibility condition (1.10) is actually linear 1$^{\text{st}}$ order PDE of 5 independent variables x, y, z, p, q and one dependent variable g. Just by regrouping terms of (1.10), we get

$$f_p g_x + f_q g_y + \left(p f_p + q f_q\right) g_z + (-f_x - p f_z) g_p + \left(-f_y - q f_z\right) g_q = 0. \qquad (2.3)$$

This can be solved by Lagrange's method [See (A.1)-(A.3)]. We will find integrals of following equations, known as Charpit's auxiliary equation (AE):

$$\frac{\mathrm{d}x}{f_p} = \frac{\mathrm{d}y}{f_q} = \frac{\mathrm{d}z}{p f_p + q f_q} = \frac{\mathrm{d}p}{-(f_x + p f_z)} = \frac{\mathrm{d}q}{-\left(f_y + q f_z\right)}. \qquad (2.4)$$

In practice, we choose suitable pair of fractions from (2.4) involving at least one of p, q, and solving that ODE, we get either $p = p(x, y, z, a)$ or $q = q(x, y, z, a)$ or some relation involving p, q. In either case we can use given PDE (2.1) to get $p = p(x, y, z, a),\ q = q(x, y, z, a)$.

Last step is to substitute these expressions for p, q into the identity

$$\mathrm{d}z = p\mathrm{d}x + q\mathrm{d}y, \qquad (2.5)$$

and solve this DE to get CI of given PDE (2.1) in the form

$$F(x, y, z, a, b) = 0. \qquad (2.6)$$

Note that here neither we find general solution (GS) of (2.3) nor we find explicitly the expression for the compatible partner g, as our focus here is to find p, q in terms of x, y, z, sufficient to get CI of given PDE. However, compatible partner g can be found from (2.3) by noticing another fraction $\mathrm{d}g/0$ in AE (2.4) [not written there] so that GS of (2.3) is $\varphi(u_1, u_2, u_3, u_4, u_5) = 0$, where $u_j(x, y, z, p, q; g) = c_j$ are solutions of AE (2.4), φ arbitrary function.

This outlines Charpit's method.

Problem 1. Find CI of $(p^2 + q^2)y = qz$

Solution: Rewrite given PDE as $f \equiv (p^2 + q^2)y - qz = 0$, and by formula (2.4), Charpit's AE is

$$\frac{\mathrm{d}x}{2py} = \frac{\mathrm{d}y}{2qy - z} = \frac{\mathrm{d}z}{2p^2y + 2q^2y - q} = \frac{\mathrm{dp}}{pq} = \frac{\mathrm{d}q}{q^2}$$

From AE, we need one relation between p, q. By inspection, we see that last two fractions yield a simple 1st order ODE $p\mathrm{d}p + q\mathrm{d}q = 0$, solving which we have $p^2 + q^2 = c^2$. Substituting this relation into given PDE, we get

$q = \frac{c^2 y}{z}$, $p = \frac{c}{z}\sqrt{z^2 - c^2y^2}$. [we ignore $\pm$ sign due to arbitrariness of c.]

Then we will solve the DE $p\mathrm{d}x + q\mathrm{d}y = \mathrm{d}z$, which after substituting for p, q from above, becomes after little regrouping

$\mathrm{d}(z^2 - c^2y^2) = 2c\sqrt{z^2 - c^2y^2}\,\mathrm{d}x$. This ODE is easily integrable to obtain

$$z^2 = (cx + d)^2 + c^2y^2.$$

- Special Types of 1st order PDEs

1. Type I: Only p, q present, i.e. given PDE : $f(p,q) = 0$

In this case, it can be easily seen that from AE, we will get either $\mathrm{d}p = 0$ or $\mathrm{d}q = 0$ which means that we can consider either $p = a$ or $q = b$. Substituting either of them in given PDE will definitely give other as constant also. So we will have to solve a simple DE $\mathrm{d}z = a\mathrm{d}x + Q(a)\mathrm{d}y$, for $p = a$. Hence, CI of this special Type PDE is $z = ax + yQ(a) + b$.

Example: $p^2 - q^2 = 4$

Solution: From AE, $\mathrm{d}p = 0 \Longrightarrow p = a, q = \sqrt{a^2 - 4}$. CI: $z = ax + y\sqrt{a^2 - 4} + b$

2. Type II : Reducible to Type I by transformation

Example: $x^2p^2 + y^2q^2 = z^2$

Solution: Rewrite as $(px/z)^2 + (qy/z)^2 = 1$. So if we put $X = \ln x\,, Y = \ln y\,, z = \ln z$, and calculate $P \equiv \frac{\partial Z}{\partial X} = \frac{xp}{z}\,, Q \equiv \frac{\partial Z}{\partial Y} = \frac{yq}{z}$, so that given PDE now transforms to above form: $P^2 + Q^2 = 1$. So the CI is

$$Z = aX + Y\sqrt{1 - a^2} + b \Longrightarrow \ln z = a \ln x + \sqrt{1 - a^2} \ln y + b$$

3. Type III: Independent variables absent: $f(z, p, q) = 0$

The AE of this Type will always contain simple ODE: $q\mathrm{d}p = p\mathrm{d}q \Longrightarrow p = aq$, substituting into given PDE, $q = Q(a, z)$, so that the DE: $\mathrm{d}z = Q(a, z)(a\mathrm{d}x + \mathrm{d}y)$ is easily integrable.

Example: $p^2z^2 + q^2 = 1$

Solution: From AE, $p = aq \Longrightarrow q = 1/\sqrt{1 + a^2z^2}$, ignoring '–' sign.

Then solve $\sqrt{1 + a^2z^2}\mathrm{d}z = a\mathrm{d}x + \mathrm{d}y$ to get CI:

$$az\sqrt{1 + a^2z^2} + \ln\left(az + \sqrt{1 + a^2z^2}\right) = 2a(ax + y + b)$$

4. Type IV: Separable equation and z absent: $f(x,p) = g(y,q)$

From AE of this Type, we will always get $f_p \mathrm{d}p + f_x \mathrm{d}x = 0$, $g_q \mathrm{d}q + g_y \mathrm{d}y = 0$. From these two, we get $f(x,p) = a, g(y,q) = a$, since actually lhs of 1st & 2nd are actually $\mathrm{d}f, \mathrm{d}g$. Note that we can't take two arbitrary constants, because solving the next DE $\mathrm{d}z = p\mathrm{d}x + q\mathrm{d}y$, 2nd arbitrary constant will appear.

Solving $f(x,p) = a, g(y,q) = a$ for p,q, we must get $p = P(a,x), q = Q(a,y)$, so that the next DE is easily integrable to obtain CI.

Example: $p^2 y(1+x^2) = qx^2$

Solution: Rewrite PDE as $f(x,p) \equiv \frac{p^2(1+x^2)}{x^2} = \frac{q}{y} \equiv g(y,q)$, so that from AE, we will get $\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a \Longrightarrow q = ay, p = x\sqrt{\frac{a}{1+x^2}}$. Then the next DE is

$\mathrm{d}z = x\sqrt{\frac{a}{1+x^2}}\,\mathrm{d}x + ay\,\mathrm{d}y$, which can be integrated easily to get CI:

$$z = \sqrt{a(1+x^2)} + \frac{y^2}{2} + b.$$

5. Type V: Clairaut's Equation: $z = px + qy + f(p,q)$

For this special from, we always get from AE: $\frac{\mathrm{d}p}{0} = \frac{\mathrm{d}q}{0} \Longrightarrow p = a, q = b$, so that next DE $\mathrm{d}z = a\mathrm{d}x + b\mathrm{d}y$. Thus Claurаut's equation always has CI, obtained by replacing p,q in given PDE simply by a,b in the form: $z = ax + by + f(a,b)$. Note that from DE, we have $z = ax + by + c$, and from given PDE, by substituting for p,q, we have $z = ax + by + f(a,b)$, so that $c = f(a,b)$.

- Higher order linear PDE with constant coefficients

We will consider two independent variables x,y and one dependent variable z, and we will denote here $D \equiv \frac{\partial}{\partial x}, D' \equiv \frac{\partial}{\partial y}$. Note that in the previous topics p,q were used for two partial derivatives $Dz, D'z$. The reason for adopting symbol for the operators here is that it helps expressing formulae in compact forms.

A General linear 2nd order PDE with constant coefficients may be written as

$$[F(D,D')]z \equiv \left[aD^2 + bD'^2 + 2hDD' + 2gD + 2fD' + c\right]z = f(x,y) \quad (3.1)$$

where a, b, h, g, f, c are constants.

<u>A word of caution about symbols</u>: Observe p, q were merely functions so that p^2 means $(\partial z/\partial x)^2$, whereas D, D' are differential operators, so that D^2 means $\partial^2 z/\partial x^2$. Point is that this topic covers linear 2nd order PDE, so that we will not need to express non-linear terms like $(\partial z/\partial x)^2$ by the operators.

GS of PDE (3.1) is $z(x,y) =$ CF+PI, where CF stands for "Complementary Function" and PI stands for "Particular Integral".

a) <u>Method of finding CF</u>

CF is GS of homogeneous part of PDE (3.1), i.e. we will have to find GS of the homogeneous PDE $F(D,D') = 0$.

We will discuss the case when $F(D,D')$ is factorable into two linear expressions in D, D', i.e. $F(D,D') = (\alpha_1 D + \beta_1 D' + \gamma_1)(\alpha_2 D + \beta_2 D' + \gamma_2)$.

Note that corresponding to each linear factor, $\exists$ a solution u_j of homogeneous part, and so $\mathbf{CF} = \boldsymbol{u_1} + \boldsymbol{u_2}$ (linear superposition).

CF of 2nd order PDE contains exactly two arbitrary functions.

Case 1) $\alpha_j \neq 0$ and no repeated factors, i.e. two factors are distinct

Corresponding to each factor $(\alpha_j D + \beta_j D' + \gamma_j), u_j = e^{-\frac{\gamma_j x}{\alpha_j}} \phi_j(\beta_j x - \alpha_j y)$, where ϕ_j are arbitrary functions, so that

$$\text{CF} = e^{-\frac{\gamma_1 x}{\alpha_1}} \phi_1(\beta_1 x - \alpha_1 y) + e^{-\frac{\gamma_2 x}{\alpha_2}} \phi_2(\beta_2 x - \alpha_2 y). \quad (3.2)$$

Case 2) $\alpha_j \neq 0$ and repeated factors, i.e. $F(D,D') = (\alpha D + \beta D' + \gamma)^2$

$$\text{CF} = e^{-\gamma x/\alpha}[\phi_1(\beta x - \alpha y) + x\phi_2(\beta x - \alpha y)] \quad (3.3)$$

Case 3) $\alpha_j = 0$ and no repeated factors, i.e. $F(D,D') = (\beta_1 D' + \gamma_1)(\beta_2 D' + \gamma_2)$

$$\text{CF} = e^{-\frac{\gamma_1 y}{\beta_1}} \phi_1(x) + e^{-\frac{\gamma_2 y}{\beta_2}} \phi_2(x) \quad (3.4)$$

Case 4) $\alpha_j = 0$ and repeated factors, i.e. $F(D, D') = (\beta D' + \gamma)^2$

$$\mathrm{CF} = \mathrm{e}^{-\gamma y/\beta}[\phi_1(x) + y\phi_2(x)] \tag{3.5}$$

Example: Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$

Solution: Note that given PDE is homogeneous, i.e. RHs = 0. In our notation, rewrite given PDE: $D^2 - {D'}^2 = 0 \Longrightarrow [(D + D')(D - D')]z = 0$, so that by formula (3.2), GS : $z = \phi_1(x - y) + \phi_2(x + y)$, where ϕ_1, ϕ_2 are arbitrary functions.

Note: I skip proofs of above formulae. However, interested reader may prove by noticing that each linear factor actually corresponds to a linear 1st order PDE, which can be solved by Lagrange's method.

b) Method of finding PI of linear PDE $F(D, D')z = f(x, y)$

General formula for finding PI

PI is any solution of given PDE, and so mathematically it may be defined as

$$\mathrm{PI} = [F(D, D')]^{-1} f(x, y),$$

where the action of the inverse operator F^{-1} on $f(x, y)$ may be computed corresponding to each linear factor of $F(D, D') = (\alpha D + \beta D' + \gamma)(\alpha' D + \beta' D' + \gamma')$ by following general formula:

$$(\alpha D + \beta D' + \gamma)^{-1} f(x, y) = \frac{\mathrm{e}^{-\gamma x/\alpha}}{\alpha} \int \mathrm{e}^{\gamma x/\alpha} f\left(x, \frac{\beta x + c}{\alpha}\right) \mathrm{d}x \Bigg|_{c = \alpha y - \beta x} \tag{3.6a}$$

where $\alpha \neq 0, \beta, \gamma \in \mathbb{R}$ and for $\beta \neq 0, \alpha, \gamma \in \mathbb{R}$,

$$(\alpha D + \beta D' + \gamma)^{-1} f(x, y) = \frac{\mathrm{e}^{-\gamma y/\beta}}{\beta} \int \mathrm{e}^{\gamma y/\beta} f\left(\frac{\alpha y - c}{\beta}, y\right) \mathrm{d}y \Bigg|_{c = \alpha y - \beta x} \tag{3.6b}$$

In particular, for homogeneous $F(D, D')$, say $F(D, D') = (D - aD')(D - bD')$,

putting $\alpha = 1, \beta = a, \gamma = 0$ in (3.6a)

$$[D - aD']^{-1} f(x, y) = \int f(x, c - ax) \mathrm{d}x \Bigg|_{c = y + ax} \tag{3.6c}$$

i.e. after integration w.r.t. x, we have to replace parameter c by $\alpha y - \beta x$. Since we are considering only the case when $F(D, D')$ is factorable, above formula suffices to find PI, provided the integrand is analytically integrable. Formula (3.6) may be proved using Lagrange's method to 1st order PDE $[\alpha D + \beta D' + \gamma]\phi = f(x, y)$. Note that in Textbook and References 1-3, only formula (3.6c) is provided. I introduce generalized formulae (3.6a), (3.6b).

Example: $4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16\ln(x + 2y)$

Solution: Given PDE: $F(D, D') \equiv 4D^2 - 4DD' + {D'}^2 = 16\ln(x + 2y)$

$$F = (2D - D')^2$$

Thus, by formula (3.3), CF = $\phi_1(x + 2y) + x\phi_2(x + 2y)$

PI will be evaluated by formula (3.6) as follows

$$\text{PI} = 4^{-1}\left(D - \frac{D'}{2}\right)^{-1}\left(D - \frac{D'}{2}\right)^{-1} 16\ln(x + 2y)$$

$$= \left[4\left(D - \frac{D'}{2}\right)^{-1} \int \ln\left[x + 2\left(c - \frac{x}{2}\right)\right] \mathrm{d}x\right]_{c=y+x/2}$$

$$= 4\left(D - \frac{D'}{2}\right)^{-1} x\ln 2c = 4\ln 2c \int x\mathrm{d}x = 2x^2\ln(2y + x)$$

Hence, GS is

$$z = \phi_1(x + 2y) + x\phi_2(x + 2y) + 2x^2\ln(2y + x).$$

Note that in finding PI of above problem, we have to use formula (3.6) twice, and after final integration, we replace c by $y + x/2$.

<u>Finding PI when RHS $f(x, y)$ assumes special forms</u>

- $f(x, y)$ is a general polynomial in x, y, i.e. contains terms like $x^m y^n$

We expand $[F(D, D')]^{-1}$, using binomial theorem, as an infinite series of ascending powers of D/D' or D'/D, so that we will end up finite number of terms to be considered only, because a polynomial of degree k is exactly k-times differentiable.

Note the interpretation of action of operator D is partial differentiation, and that of inverse operator D^{-1} is partial integration, i.e. integration w.r.t x only, treating y as constant. Similar meaning is for $', D'^{-1}$.

Example: Find PI of PDE $(D^2 - D')z = 2y - x^2$

Solution: Note that here LHS is not factorable, but we do not need to worry about that, because here we are to find only PI.

$$\mathrm{PI} = [D^2 - D']^{-1}(2y - x^2) = D^{-2}[1 - D'/D^2]^{-1}(2y - x^2)$$

$$= D^{-2}[1 + D'/D^2 - \;\ldots](2y - x^2) = D^{-2}[(2y - x^2) + D^{-2}(2)]$$

$$= D^{-2}[2y - x^2 + \textstyle\int 2x\mathrm{d}x] = D^{-1}\int(2y)\mathrm{d}x == \int(2xy)\mathrm{d}x = x^2 y.$$

- $f(x, y) = \mathrm{e}^{ax+by}$
  $\mathrm{PI} = [F(D, D']^{-1}\mathrm{e}^{ax+by} = \mathrm{e}^{ax+by}/F(a, b)$, provided $F(a, b) \neq 0$ (3.7)
  If $F(a, b) = 0$, use the formula below taking $\phi(x, y) = 1$.

Example: Find PI of $(D^2 + DD' + 2D - 3)z = \mathrm{e}^{2x+y}$

Solution: $\mathrm{PI} = \mathrm{e}^{2x+y}/(4 + 2 + 4 - 3) = \mathrm{e}^{2x+y}/7$

- $f(x, y) = \mathrm{e}^{ax+by}\phi(x, y)$
  $\mathrm{PI} = [F(D, D']^{-1}\mathrm{e}^{ax+by}\phi(x, y) = \mathrm{e}^{ax+by}[F(D + a, D' + b]^{-1}\phi(x, y)$(3.8)

Example: Find PI of $(D^2 + DD' - 2D - 2)z = \mathrm{e}^{2x+y}$

Solution: Notice here $F(2,1) = 0$, and so formula (3.7) can't be applied. We will apply formula (3.8) by taking $\phi(x, y) = 1$.

$$\mathrm{PI} = \mathrm{e}^{2x+y}[(D + 2)^2 + (D + 2)(D' + 1) - 2(D + 2) - 2]^{-1}(1)$$

$$= \mathrm{e}^{2x+y}[D^2 + DD' + 3D + 2D']^{-1}(1)$$

$$= \frac{\mathrm{e}^{2x+y}}{2}\left[1 + \frac{1}{2}\left(\frac{D^2}{D'} + D + 3\frac{D}{D'}\right)\right]^{-1} D'^{-1}(1)$$

$$= \frac{e^{2x+y}}{2}\left[1 - \frac{1}{2}\left(\frac{D^2}{D'} + D + 3\frac{D}{D'}\right) + \cdots\right] y = \frac{y e^{2x+y}}{2} \qquad \left[D'^{-1}(1) = \int \mathrm{d}y\right]$$

- $f(x, y) = \sin(ax + by)$ or $\cos(ax + by)$

Three methods are available.

Method 1. This is the most straightforward. Idea is that since sine or cos function interchange among themselves by the operation of differentiation or integration, we can always assume PI as

$$\mathrm{PI} = c_1 \cos(ax + by) + c_2 \sin(ax + by), \tag{3.9}$$

where two unknowns c_1, c_2 will be computed by substituting this solution into given PDE.

Method 2: We can use Euler's formula $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ to convert sin or cos term in RHS into exponential function, so that PI can be obtained using formulae (3.7) and/or (3.8).

Method 3: Since $\sin(ax + by)$ and $\cos(ax + by)$ always remains shape-invariant with changed sign under twice operations of differentiation and integration, we can always remove D^2, D'^2, DD' in $F^{-1}(D, D')$ by the assignments $D^2 \to -a^2, D'^2 \to -b^2, DD' \to -ab$ provided F doesn't vanish [If vanishes, then use formula (3.8) after converting sin/cos function to exponential function using Euler formula]. The remaining 1st degree terms D, D' in F^{-1}, if any, are then converted to 2nd degree terms by taking out conjugate factor and then again 2nd degree terms are removed by the assignments. The process will continue until any operator remains in F^{-1}. Then final form of operator $F^{-1}(D, D')$ contains no D, D' in inverse form, and hence their operation on sin/cosine functions means simply partial differentiation. Below I've demonstrated the procedure for sufficiently general form of $F(D, D')$.

$$F^{-1}(D, D') = (kD + lD' + m)^{-1} = (kD + m - lD')\left[(kD + m)^2 - l^2 D'^2\right]^{-1}$$

$$= (kD + m - lD')\left[k^2 D^2 - l^2 D'^2 + 2mkD\right]^{-1} = (kD + m - lD')[2mkD - C]^{-1}$$

$$= (kD + m - lD')(2mkD - C)[4m^2k^2D^2 - C^2]^{-1} \qquad [C = a^2k^2 - b^2l^2]$$

$$= [2mk(kD^2 - lDD') + C(kD - lD' + m)][4m^2k^2D^2 - C^2]^{-1}$$

$= [2mk(kD^2 - lDD') + C(kD - lD' + m)][-4m^2k^2a^2 - C^2]^{-1}$

$= K^{-1}[2mk(-a^2k + lab) + C(kD - lD' + m)]$ $\qquad [-K = 4m^2k^2a^2 + C^2]$

Notice that now there is no inverse operator. So rest part is just operating D, D' which is merely partial differentiation on RHS function w.r.t x, y respectively.

Example: Find PI of the PDE $2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial y} = 5\cos(3x - 2y)$

Solution: $F^{-1}(D, D') = \left[2DD' + D'^2 - 3D'\right]^{-1}$

<u>**Using Method 1**</u>, Assume PI $= c_1 \cos(3x - 2y) + c_2 \sin(3x - 2y) = z$

Calculate $Dz, D'z$ etc, and substituting into given PDE, we get after regrouping for sin and cos terms,

$$(8c_1 + 6c_2 - 5)\cos(3x - 2y) + (8c_2 - 6c_1)\sin(3x - 2y) = 0$$

Hence, for the assumed PI to be a solution of given PDE, the coefficients of sin and cos term must separately vanish. Solving those two equations for c_1, c_2, we get PI as follows

$$\text{PI} = \frac{1}{10}[4\cos(3x - 2y) + 3\sin(3x - 2y)]$$

<u>**Using Method 2**</u>, By Euler's formula,

$\cos(3x - 2y) = \frac{1}{2}\left(\mathrm{e}^{i(3x-2y)} + \mathrm{e}^{-i(3x-2y)}\right)$, $F(D, D') = 2DD' + D'^2 - 3D'$.

PI $= \frac{5}{2}\left[(8 + 6i)^{-1}\mathrm{e}^{i(3x-2y)} + (8 - 6i)^{-1}\mathrm{e}^{-i(3x-2y)}\right] = \frac{1}{40}\left[(8 - 6i)\mathrm{e}^{i(3x-2y)} + (8 + 6i)\mathrm{e}^{-i(3x-2y)}\right] = \frac{1}{10}[4\cos(3x - 2y) + 3\sin(3x - 2y)]$

<u>**Using Method 3**</u>, PI$= \left[2DD' + D'^2 - 3D'\right]^{-1} 5\cos(3x - 2y)$

Replace D'^2, DD' by $-4, 6$ in $F(D, D')$ to get PI as

PI $= 5[12 - 4 - 3D']^{-1}\cos(3x - 2y) = 5(8 - 3D')^{-1}\cos(3x - 2y)$

$= 5(8 + 3.D')(64 - 9D'^2)^{-1}\cos(3x - 2y)$

$= \dfrac{1}{10}[4\cos(3x - 2y) + 3\sin(3x - 2y)]$

- $f(x,y) = \sin(ax+by)\,\phi(x,y)$ or $\cos(ax+by)\,\phi(x,y)$

  In this case, always convert sin or cos function to exponential using Euler's formula (given in discussion of Method 2 above), and then use formula (3.8) to compute PI, and finally give answer in sin or cos by reversing Euler's formula, i.e. by $\cos\theta \pm i\sin\theta = \exp(\pm i\theta)$.

- Classification of 2nd order PDE and Canonical Forms

As before, we will restrict ourselves to 2 independent variables x, y and 1 dependent variable z. Recall the symbols p, q, which were used previously to denote $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$. Here we will use 3 more symbols r, s, t to denote respectively $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$, which, in terms of operators D, D', used in previous topic, are $Dz, DD'z, D'z$ respectively.

General form of a 2nd order PDE :

$$Rr + Ss + Tt + f(x,y,z,p,q) = 0, \tag{4.1a}$$

where R, S, T may be functions of x, y, z in general. Note that the 2nd order operator of PDE (4.1) is linear, since 2nd order partial derivatives r, s, t in it are of power at most one, which may be expressed as

$$L \equiv R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x \partial y} + T\frac{\partial^2}{\partial y^2} \tag{4.1b}$$

Note also that 2nd order PDE (4.1a) may be semi-linear, quasi-linear or non-linear, since non-linear terms like zq, z^2, p^2, pq etc. may present in $f(x,y,z,p,q)$. If no non-linear term is present f, then PDE (4.1a) will be linear 2nd order. However, 2nd order operator L, given by (4.1b), is always linear w.r.t. r, s, t, since terms like r^2, rs, t^3 etc. will not appear in PDE (4.1a) when L operates on z.

PDE (4.1) will be linear if coefficient functions R, S, T are independent of z, and the non-homogeneous (w.r.t. D, D') term f contains terms in z, p, q of degree at most one. We assume that $\forall x, y$, any two of $R(x,y), S(x,y), T(x,y)$ are not simultaneously zero. and also for $S(x,y) = 0$, R, T are not simultaneously equal to unity, so that PDE (4.1) will not be automatically in any of three canonical forms, to be discussed below.

We will focus, in this topic, on linear 2nd order PDE (4.1a), which is generalization of those discussed in previous Section, because here coefficients are variables.

PDE (4.1) is classified into three categories: Hyperbolic, Parabolic and Elliptic.

Canonical forms:

- Hyperbolic PDE: $\frac{\partial^2 z}{\partial x \partial y} = F(x,y,z,p,q)$ (4.2)
- Parabolic PDE: $\frac{\partial^2 z}{\partial x^2} = F(x,y,z,p,q)$ or $\frac{\partial^2 z}{\partial y^2} = F(x,y,z,p,q)$ (4.3)
- Elliptic PDE: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = F(x,y,z,p,q)$ (4.4)

We now seek conditions between R, S, T in PDE (4.1) which classifies into three categories (4.2)-(4.4). To get those conditions, we need canonical reduction of (4.1) by the transformations of independent variables $x, y \to \xi = \xi(x,y), \eta = \eta(x,y)$, and we will denote changed dependent variable by ζ, i.e $z(x(\xi,\eta), y(\xi,\eta)) \equiv \zeta(\xi,\eta)$. Note that ξ, η must be functionally independent, so that we have following constraint in choosing the transformations:

$$\frac{\xi_x}{\xi_y} \neq \frac{\eta_x}{\eta_y} \tag{4.5a}$$

Using chain rule, we can compute p, q, r, s, t as follows:

$$\left.\begin{aligned}
p \equiv \zeta_x = \xi_x \zeta_\xi + \eta_x \zeta_\eta\,, \qquad q \equiv \zeta_y = \xi_y \zeta_\xi + \eta_y \zeta_\eta\,,\\
r \equiv \zeta_{xx} = {\xi_x}^2 \zeta_{\xi\xi} + 2\xi_x \eta_x \zeta_{\xi\eta} + {\eta_x}^2 \zeta_{\eta\eta} + \xi_{xx}\zeta_\xi + \eta_{xx}\zeta_\eta\\
s \equiv \zeta_{xy} = \xi_x \xi_y \zeta_{\xi\xi} + (\xi_x \eta_y + \xi_y \eta_x)\zeta_{\xi\eta} + \eta_x \eta_y \zeta_{\eta\eta} + \xi_{xy}\zeta_\xi + \eta_{xy}\zeta_\eta\\
t \equiv \zeta_{yy} = {\xi_y}^2 \zeta_{\xi\xi} + 2\xi_y \eta_y \zeta_{\xi\eta} + {\eta_y}^2 \zeta_{\eta\eta} + \xi_{yy}\zeta_\xi + \eta_{yy}\zeta_\eta
\end{aligned}\right\} \tag{4.5b}$$

Now, substitute p, q, r, s, t from (4.5b) into PDE (4.1a), and collect the coefficients of 1st and 2nd order partial derivatives:

$$\begin{aligned}
&(R{\xi_x}^2 + S\xi_x\xi_y + T{\xi_y}^2)\zeta_{\xi\xi} + 2\left[(R\xi_x\eta_x + T\xi_y\eta_y) + \frac{1}{2}S(\xi_x\eta_y + \xi_y\eta_x)\right]\zeta_{\xi\eta}\\
&\quad + (R{\eta_x}^2 + S\eta_x\eta_y + T{\eta_y}^2)\zeta_{\eta\eta} + (R\xi_{xx} + S\xi_{xy} + T\xi_{yy})\zeta_\xi +\\
&(R\eta_{xx} + S\eta_{xy} + T\eta_{yy})\zeta_\eta + F(\xi,\eta,\zeta,\zeta_\xi,\zeta_\eta) = 0\,,
\end{aligned} \tag{4.5c}$$

where $F(\xi,\eta,\zeta,\zeta_\xi,\zeta_\eta) \equiv f(x(\xi,\eta),y(\xi,\eta),z(x,y),p,q,)$. Note that here we are not interested about exact form of F as this contributes only to 1st order derivative terms. However, given $f(x,y,z,p,q)$, one can always compute F , wherein there will be 1st order terms, in general.

I now express coefficients by three functions:

$$\left.\begin{aligned} A(u,v) &= Ru^2 + Suv + Tv^2 , \\ B(u_1,u_2;v_1,v_2) &= Ru_1v_1 + Tu_2v_2 + \frac{1}{2}S(u_1v_2 + u_2v_1), \\ C(u) &= Ru_{xx} + Su_{xy} + Tu_{yy}, \end{aligned}\right\} \tag{4.5d}$$

so that transformed PDE (4.5c) may be written in compact forms as follows

$$A(\xi_x,\xi_y)\frac{\partial^2\zeta}{\partial\xi^2} + 2B(\xi_x,\xi_y;\eta_x,\eta_y)\frac{\partial^2\zeta}{\partial\xi\partial\eta} + A(\eta_x,\eta_y)\frac{\partial^2\zeta}{\partial\eta^2} + \tilde{F}(\xi,\eta,\zeta,\zeta_\xi,\zeta_\eta) = 0, \tag{4.6a}$$

$$\tilde{F}(\xi,\eta,\zeta,\zeta_\xi,\zeta_\eta) = F(\xi,\eta,\zeta,\zeta_\xi,\zeta_\eta) + C(\xi)\frac{\partial\zeta}{\partial\xi} + C(\eta)\frac{\partial\zeta}{\partial\eta}, \tag{4.6b}$$

In above equation, $\tilde{F}(\xi,\eta,\zeta,\zeta_\xi,\zeta_\eta)$ contains only 1st order terms. Our focus, here, lies in 2nd order terms. Note that in all Books, expression is written in terms of two functions A, B. I introduce a third function C which makes expression more compact [see (4.5d) & (4.6)].

Since we are interested to choose such a transformations $\xi = \xi(x,y), \eta = \eta(x,y)$, which will reduce PDE (4.1a) to one of the canonical forms given by (4.2)-(4.4), we need to remove all but one 2nd order partial derivative terms from (4.6). This means that for our choice of ξ,η, either of A or B or both must vanish.

Clearly we are to consider roots of the following quadratic equation

$$R(x,y)\lambda^2 + S(x,y)\lambda + T(x,y) = 0 \Longrightarrow \lambda = \lambda_1(x,y), \lambda_2(x,y)$$

Note that the two roots will be real & distinct ($\lambda_1 \neq \lambda_2$), real & coincident ($\lambda_1 = \lambda_2 = \lambda$) and complex conjugate ($\lambda = \alpha \pm i\beta$) according as the discriminant $S^2 - 4RT >=< 0$.

Hence, we have the following formula for classification of PDE (4.1):

$S^2 - 4RT > 0 \Longrightarrow$ Hyperbolic, $S^2 - 4RT = 0 \Longrightarrow$ Parabolic, $S^2 - 4RT < 0 \Longrightarrow$ Elliptic

This statement will be explained below in the process of canonical reduction.

Canonical Reduction

Look here the three different situations.

- $S^2 - 4RT > 0 \Longrightarrow \lambda = \lambda_1, \lambda_2 = \left[-S \pm \sqrt{S^2 - 4RT}\right]/2R \in \mathbb{R}, \lambda_1 \neq \lambda_2$

Clearly we can choose two suitable real transformation, not compromising the constraint (4.5), and yet capable of removing both of $\zeta_{\xi\xi}, \zeta_{\eta\eta}$:

$$\xi_x/\xi_y = \lambda_1(x,y),\ \ \eta_x/\eta_y = \lambda_2(x,y) \tag{4.7}$$

Note that two PDEs (4.7) are 1st order linear, and hence can be solved by Lagrange's method. We will do it, but before that notice that by the choice (4.7), we have $A(\xi_x, \xi_y) = A(\eta_x, \eta_y) = 0, 2B(\xi_x, \xi_y; \eta_x, \eta_y) \neq 0$, and so transformed PDE (4.6) reduces to canonical form:

$$\frac{\partial^2 \zeta}{\partial \xi \partial \eta} = \tilde{F}(\xi, \eta, \zeta, \zeta_\xi, \zeta_\eta) \tag{4.8}$$

Note again at this point we are not interested about exact form of $\tilde{F}$. What matters here is that PDE (4.8) is in canonical form, and according to classification, given in (4.2)-(4.4), it is elliptic. Hence we conclude that PDE (4.1a) is Elliptic PDE provided $S^2 - 4RT > 0$.

Before going to two other situations, let us look into solutions of linear PDE (4.7). Consideration of one of them is sufficient for understanding. Lagrange's AE for 1st one is $\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{-\lambda_1(x,y)} = \frac{\mathrm{d}\xi}{0} \Longrightarrow \xi = c_1$, and from ODE $\frac{\mathrm{d}y}{\mathrm{d}x} = -\lambda_1(x,y) \Longrightarrow \phi(x,y) = c_1$, so that 1st transformation is $\xi = \phi(x,y)$. Similarly, we can find $\eta = \varphi(x,y)$ from 2nd linear PDE of (4.7).

Equations (4.7) are called Characteristic Equations for PDE (4.1a), and the solutions $\phi(x,y) = c_1$ and $\varphi(x,y) = c_2$ are called Characteristic Curves.

- $S^2 - 4RT = 0 \Longrightarrow \lambda_1 = \lambda_2 = -S/2R \in \mathbb{R}$

Clearly, here we can choose one real transformation definitively for removing $\frac{\partial^2 \zeta}{\partial \xi^2}$

$$\xi_x/\xi_y = \lambda(x,y), \tag{4.9}$$

but other transformation remains arbitrary. However, this one transformation (4.9) serves our purpose of getting canonical form, because it not only makes $A(\xi_x, \xi_y) = 0$, but also make $B(\xi_x, \xi_y; \eta_x, \eta_y) = 0$, even though η_x, η_y remains arbitrary. This is due to the fact that $A(\xi_x,\xi_y)A(\eta_x,\eta_y)-B^2(\xi_x,\xi_y;\eta_x,\eta_y)$ always has a factor S^2-4RT, which is zero for the present situation. In fact, it is quite straightforward to derive the following:

$$A(\xi_x,\xi_y)A(\eta_x,\eta_y)-B^2(\xi_x,\xi_y;\eta_x,\eta_y)=\frac{1}{4}(4RT-S^2)(\xi_x\eta_y-\xi_y\eta_x)^2, \quad (4.10)$$

the derivation of which is left for readers as an exercise. Note that RHS of identity (4.10) will be identically zero only for the present situation ($S^2-4RT=0$). Note that 2nd factor can never be zero [see constraint (4.5a)].

Coming back to canonical reduction, see that transformed PDE (4.6) reduces to the canonical form:

$$\frac{\partial^2\zeta}{\partial\eta^2}=\tilde{F}(\xi,\eta,\zeta,\zeta_\xi,\zeta_\eta). \quad (4.11)$$

Again, at this point, we are not interested about exact form of $\tilde{F}$, for which we need to choose 2nd transformation $\eta(x,y)$, which will be functionally independent to 1st transformation. Comparing PDE (4.11) with classification scheme, given by (4.2)-(4.4), we see that it is canonical form of parabolic PDE.

Hence, PDE (4.1a) will be parabolic PDE, provided $S^2-4RT=0$.

- $S^2-4RT<0 \Longrightarrow \lambda=\alpha\pm i\beta\in\mathbb{C}$

Notice that here we do not have real transformations, but nevertheless we can proceed with two distinct complex transformations

$$\frac{\xi_x}{\xi_y}=\alpha+i\beta,\ \frac{\eta_x}{\eta_y}=\alpha-i\beta \implies \xi(x,y)=\bar{\eta}(x,y)\in\mathbb{C}\,. \quad (4.12)$$

So, even if ξ,η are complex (in fact, both are complex conjugate of each other), these two transformations, as in the case for 1st situation, will remove both of $\frac{\partial^2\zeta}{\partial\xi^2},\frac{\partial^2\zeta}{\partial\eta^2}$, because of the simple fact that $\lambda=\alpha\pm i\beta$ are two distinct roots of $R\lambda^2+S\lambda+T=0$, making $A(\xi_x,\xi_y)=A(\eta_x,\eta_y)=0$; thereby transformed PDE (4.6) becomes

$$\frac{\partial^2\zeta}{\partial\xi\partial\eta}=\tilde{F}(\xi,\eta,\zeta,\zeta_\xi,\zeta_\eta)\,. \quad (4.13)$$

Note that we can't say it is hyperbolic, because the independent variables ξ,η here are complex, whereas those in the classification scheme (4.2)-(4.4) are real.

However, it is not difficult to make a 2nd transformation to get back to real variables. Note that $\xi = \bar{\eta} \in \mathbb{C}$, so the legitimate transformations are

$$2\alpha(x,y) = \eta(x,y) + \xi(x,y),\ 2i\beta(x,y) = \eta(x,y) - \xi(x,y), \qquad (4.14)$$

where these two transformations are automatically functionally independent. Let us now calculate 2nd order derivative term, where we will continue the same symbol for dependent variable, i.e. $\zeta(\xi(\alpha,\beta),\eta(\alpha,\beta)) \equiv \zeta(\alpha,\beta)$. Note that $\zeta = \zeta(\alpha,\beta)$, where α,β are functions of ξ,η by (4.14). Hence, using chain rule, $2\zeta_\xi(\alpha,\beta) = \zeta_\alpha + i\zeta_\beta, 2\zeta_\eta(\alpha,\beta) = \zeta_\alpha - i\zeta_\beta \Longrightarrow 4\zeta_{\xi\eta} = \zeta_{\alpha\alpha} + \zeta_{\beta\beta}$, so that transformed PDE (4.13), after 2nd transformation (4.14), reduces to the canonical form:

$$\frac{\partial^2\zeta}{\partial\alpha^2} + \frac{\partial^2\zeta}{\partial\beta^2} = G\left(\alpha,\beta,\zeta,\zeta_\alpha,\zeta_\beta\right), \qquad (4.15)$$

where $G\left(\alpha,\beta,\zeta,\zeta_\alpha,\zeta_\beta\right) \equiv \tilde{F}\left(\xi(\alpha,\beta),\eta(\alpha,\beta),\zeta,\zeta_\xi,\zeta_\eta\right)$.

As before, at this point, we are not interested about the exact form of G, what matters here is that PDE (4.15), according to classification in (4.2)-(4.4), is elliptic. Hence we conclude that PDE (4.1a) is elliptic, provided $S^2 - 4RT < 0$.

Examples: For the following linear 2nd order PDEs, first determine the nature of PDE, and then reduce it to canonical forms:

i) $\frac{\partial^2 z}{\partial x^2} = x^2\frac{\partial^2 z}{\partial y^2}$, ii) $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2 z}{\partial y^2} = 0$, iii)) $\frac{\partial^2 z}{\partial x^2} + x^2\frac{\partial^2 z}{\partial y^2} = 0$

Solution: We first express given PDEs in terms of $r \equiv \frac{\partial^2 z}{\partial x^2}, s \equiv \frac{\partial^2 z}{\partial x\partial y}, t \equiv \frac{\partial^2 z}{\partial y^2}$.

i) Given PDE: $r - x^2 t = 0 \Longrightarrow R = 1, S = 0, T = -x^2 \Longrightarrow S^2 - 4RT = 4x^2 > 0$

Hence, given PDE is hyperbolic. Next to reduce to canonical form, find the roots of the quadratic equation $R\lambda^2 + S\lambda + T = 0 \Longrightarrow \lambda^2 - x^2 = 0 \Longrightarrow \lambda = \pm x \in \mathbb{R}$.

So, choose two transformations x,y→ ξ,η :

$$\frac{\xi_x}{\xi_y} = -x\,, \frac{\eta_x}{\eta_y} = x \ \Longrightarrow \xi,\eta = y \mp \frac{x^2}{2} \Longrightarrow \xi_x = -x, \xi_y = 1; \eta_x = x, \eta_y = 1$$

Next we know that both of $A\left(\xi_x,\xi_y\right), A\left(\eta_x,\eta_y\right)$ vanish in this case, so we need to compute only $B\left(\xi_x,\xi_y;\eta_x,\eta_y\right)$ by the formula:

$$B(u_1,u_2;v_1,v_2) = Ru_1v_1 + Tu_2v_2 + \frac{1}{2}S(u_1v_2 + u_2v_1) = u_1v_1 - x^2u_2v_2,$$

$$\Longrightarrow B\left(\xi_x,\xi_y;\eta_x,\eta_y\right) = B(-x,1;x,1)) = -2x^2 \neq 0$$

In the given PDE, there is no non-homogeneous f term, so no contribution of 1st order terms from that. Only contribution is from 2nd order terms, so we calculate C by the formula: $C(u) = Ru_{xx} + Su_{xy} + Tu_{yy} = u_{xx} - x^2u_{yy}$.

Hence, $C(\xi) = \xi_{xx} - x^2\xi_{yy} = -1$, $C(\eta) = \eta_{xx} - x^2\eta_{yy} = 1$

Hence, given PDE in canonical form is

$$2B\frac{\partial^2\zeta}{\partial\xi\partial\eta} + C(\xi)\frac{\partial\zeta}{\partial\xi} + C(\eta)\frac{\partial\zeta}{\partial\eta} = 0 \Longrightarrow -4x^2\frac{\partial^2\zeta}{\partial\xi\partial\eta} = \frac{\partial\zeta}{\partial\xi} - \frac{\partial\zeta}{\partial\eta}$$

From $\xi = y - \frac{x^2}{2}$, $\eta = y + \frac{x^2}{2} \Longrightarrow -4x^2 = 4(\xi - \eta) \Longrightarrow \frac{\partial^2\zeta}{\partial\xi\partial\eta} = \frac{1}{4(\xi-\eta)}\left(\frac{\partial\zeta}{\partial\xi} - \frac{\partial\zeta}{\partial\eta}\right)$.

Solution ii) $r + 2s + t = 0 \Longrightarrow R = 1, S = 2, T = 1 \Longrightarrow S^2 - 4RT = 0$

Hence, given PDE is parabolic. To reduce it to canonical form, since here we get one root $\lambda = -S/2R = -1$, we can choose one transformation $\xi_x/\xi_y = -1$, which will make $A\left(\xi_x,\xi_y\right) = 0$, and also $B\left(\xi_x,\xi_y;\eta_x,\eta_y\right) = 0$, whatever may be the 2nd transformation η. The 1st transformation ξ may be explicitly computed from Lagrange's AE :

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{d\xi}{0} \Longrightarrow \xi = c_1, y - x = c_1 \Longrightarrow \xi = y - x \Longrightarrow \xi_x = -1, \xi_y = 1$$

We will have to choose arbitrarily 2nd transformation η, which has to be functionally independent of ξ. A simple choice is $\eta(x,y) = y + x \Longrightarrow \eta_x = \eta_y = 1$.

Then given PDE becomes

$$A\left(\eta_x,\eta_y\right)\zeta_{\eta\eta} + C(\xi)\zeta_\xi + C(\eta)\zeta_\eta = 0$$

$$A\left(\eta_x,\eta_y\right) = R{\eta_x}^2 + S\eta_x\eta_y + T{\eta_y}^2 = 1.1 + 2.1.1 + 1.1 = 4$$

$$C(\xi) = R\xi_{xx} + S\xi_{xy} + T\xi_{yy} = 0, C(\eta) = R\eta_{xx} + S\eta_{xy} + T\eta_{yy} = 0$$

Hence, final form of canonical form for given PDE is

$$\frac{\partial^2\zeta}{\partial\eta^2} = 0$$

Solution iii) Given PDE: $r + x^2 t = 0 \Longrightarrow R = 1, S = 0, T = x^2$

$$\Longrightarrow S^2 - 4RT = -x^2 < 0 \Longrightarrow \text{PDE is elliptic.}$$

Roots are $\lambda = \pm ix \Longrightarrow \xi_x/\xi_y = -ix, \eta_x/\eta_y = ix$

AE: $\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{\mp ix} \Longrightarrow \xi, \eta = y \pm i\frac{x^2}{2} \Longrightarrow \xi_x = ix, \xi_y = 1, \eta_x = -ix, \eta_y = 1$

$$A(\xi_x, \xi_y) = A(\eta_x, \eta_y) = 0$$

$$B(\xi_x, \xi_y; \eta_x, \eta_y) = R\xi_x\eta_x + T\xi_y\eta_y + \frac{1}{2}S(\xi_x\eta_y + \xi_y\eta_y) = 2x^2 = 2i(\eta - \xi)$$

Given PDE becomes $2B\frac{\partial^2\zeta}{\partial\xi\partial\eta} + C(\xi)\frac{\partial\zeta}{\partial\xi} + C(\eta)\frac{\partial\zeta}{\partial\eta} = 0$, $C(\xi) = i, C(\eta) = -i$

$$\Longrightarrow 4(\eta - \xi)\frac{\partial^2\zeta}{\partial\xi\partial\eta} + \left(\frac{\partial\zeta}{\partial\xi} - \frac{\partial\zeta}{\partial\eta}\right) = 0$$

To get back to real variables, we use 2nd transformations:

$$2\alpha(x, y) = \eta(x, y) + \xi(x, y), 2i\beta(x, y) = \eta(x, y) - \xi(x, y)$$

$\Longrightarrow 2\zeta_\xi(\alpha, \beta) = \zeta_\alpha + i\zeta_\beta,\ 2\zeta_\eta(\alpha, \beta) = \zeta_\alpha - i\zeta_\beta, 4\zeta_{\xi\eta} = \zeta_{\alpha\alpha} + \zeta_{\beta\beta},$

Hence, final form is $2\beta\left(\frac{\partial^2\zeta}{\partial\alpha^2} + \frac{\partial^2\zeta}{\partial\beta^2}\right) + \frac{\partial\zeta}{\partial\beta} = 0$

- Hyperbolic Equation:1D Wave Equation, Vibration of string, Characteristics

1D wave equation is

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2}, \quad x \in D, t \geq 0, \tag{5.1}$$

where 1D domain D may be finite, semi-infinite or infinite, i.e. $[a, b], [0, \infty)$ or $(-\infty, \infty)$. Note that in equation (5.1), t denotes time variable $(0 < t < \infty)$, and x denotes spatial variable, and so it is called 1D, i.e. space dimension is one. Since time variable is present, PDE (5.1) is a dynamical equation, i.e. the solution $\psi(x, t)$ yields wave-form at position x and at time t. Since for infinite/semi-infinite string wave is travelling w.r.t time, its solution is called "Travelling Wave Solution" [This interpretation will be discussed below]. The quantity $c > 0$ is speed of wave.

Wave Equation (5.1) is hyperbolic equation, since comparing with standard form $Rr + Ss + Tt = f(x, t, \psi, p, q)$, we see that $R = c^2, S = 0, T = -1$, so that $S^2 - 4RT = 4c^2 > 0$.

Hence, it has two real characteristics. Roots of quadratic $R\lambda^2 + S\lambda + T = 0 \Longrightarrow c^2\lambda^2 - 1 = 0 \Longrightarrow \lambda = \pm 1/c$. Thus, two characteristic curves are

$$\xi = x - ct = k_1, \eta = x + ct = k_2. \qquad (5.2)$$

- D'Alembert's Solution of 1D Wave Equation

Vibration of an infinite string

Problem 1.1D Wave Equation:

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2}, \quad x \in (-\infty, +\infty), t \geq 0 \qquad (6.1a)$$

Initial Conditions (IC):

$$\psi(x, 0) = f(x),\ \psi_t(x, 0) = g(x), \quad x \in (-\infty, +\infty), \qquad (6.1b)$$

Solution: Along two characteristic directions [see (5.2)] $\xi = x - ct = k_1, \eta = x + ct = k_2$, we reduce PDE to the canonical form

$$\frac{\partial^2\psi}{\partial\xi\partial\eta} = 0\ . \qquad (6.2)$$

Integrating (6.2) partially w.r.t. ξ (treating η as constant), we get

$$\frac{\partial\psi}{\partial\eta} = h(\eta), \qquad (6.3)$$

where h is an arbitrary function of single variable $\eta = x + ct$. Integrating (6.3) now partially w.r.t η, we have

$$\psi(x, t) = \phi_1(x + ct) + \phi_1(\xi) = \phi_1(x + ct) + \phi_2(x - ct), \qquad (6.4)$$

where $\phi_1'(u) = h(u)$ and ϕ_2 are two arbitrary functions of single variable $x + ct$ & $x - ct$ respestively. Differentiating (6.4) w.r.t. t

$$\psi_t(x, t) = c[\phi_1'(x + ct) - \phi_2{}'(x - ct)] \qquad (6.5)$$

Using IC,

$$\psi(x, 0) = f(x) = \phi_1(x) + \phi_2(x), \psi_t(x, 0) = g(x) = c[\phi_1'(x) - \phi_2{}'(x)] \qquad (6.6)$$

Integrating 2nd equation of (6.6) w.r.t x,

$$\phi_1(x) - \phi_2(x) = \frac{1}{c}\int_{x_0}^{x} g(u)\mathrm{d}u, \quad [x_0 \text{ arbitrary pt in } u\text{-line}] \tag{6.7}$$

Adding and subtracting (6.6) & (6.7), we get ϕ_1 & ϕ_2:

$$\phi_1(x+ct) = \frac{1}{2}\left[f(x+ct) + \frac{1}{c}\int_{x_0}^{x+ct} g(u)\mathrm{d}u\right], \tag{6.8a}$$

$$\phi_2(x-ct) = \frac{1}{2}\left[f(x-ct) - \frac{1}{c}\int_{x_0}^{x-ct} g(u)\mathrm{d}u\right]. \tag{6.8b}$$

Note that both of the arguments $x+ct$ and $x-ct$ run from $-\infty$ to $+\infty$, as for infinite string $x \in (-\infty,+\infty), t \in (0,\infty)$, and both of given functions f & g are defined on $(-\infty,+\infty)$ according to IC (6.1b).

Hence, solution of PDE (6.1) is

$$\psi(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\left[\int_a^{x+ct} g(u)\mathrm{d}u - \int_a^{x-ct} g(u)\mathrm{d}u\right], \tag{6.8}$$

where a is some arbitrary point in u −axis such that $a \le x \pm ct$. Hence, final form of solution is

$$\psi(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\left[\int_{x-ct}^{x+ct} g(u)\mathrm{d}u\right]. \tag{6.9}$$

Solution (6.9) is called D'Alembert's solution for infinite string.

Vibration of a semi-infinite string

Problem 2. 1D Wave Equation:

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2}, \quad x \in [0,+\infty), t \ge 0 \tag{6.10a}$$

Initial Conditions (IC):

$$\psi(x,0) = f(x),\ \psi_t(x,0) = g(x), \quad x \in [0,+\infty), \tag{6.10b}$$

Boundary Conditions (BC):

$$\psi(0,t) = 0\ , 0 \le t < \infty\ . \tag{6.10c}$$

Solution: Note here we are dealing with entirely different problem, since one argument $x+ct \in (0,+\infty)$ making $\phi_1(x+ct)$ in (6.8a) meaningful, but the other argument $x-ct \in (-\infty,+\infty)$ making $\phi_2(x-ct)$ in (6.8b) meaningful only for

$x \geq ct$ due to the fact that given functions f and g in Problem (6.10) are defined only for non-negative argument.

Hence, D'Alembert's solution for semi-infinite string is of the form, as before,

$$\psi(x,t) = \phi_1(x+ct) + \phi_2(x-ct), \qquad (6.11)$$

where $\phi_1(x+ct)$, as before, is given by (6.8a), but (6.8b) only gives $\phi_2(x-ct)$ for $x \geq ct$.

Hence, we will have to find $\phi_2(x-ct)$ for $x < ct$.

To do this, we use BC (6.10c) in (6.11), $\psi(0,t) = 0 = \phi_1(ct) + \phi_2(-ct)$, i.e.

$$\phi_2(-ct) = -\phi_1(ct) \Longrightarrow \phi_2(v) = -\phi_1(-v), v < 0\ . \qquad (6.12a)$$

But from (6.8a),

$$\phi_1(-v) = \frac{1}{2}\left[f(-v) + \frac{1}{c}\int_{x_0}^{-v} g(u)\mathrm{d}u\right]. \qquad (6.12b)$$

Putting $v = x - ct < 0$ in (6.12a), and using (6.12b),

$$\phi_2(x-ct) = -\frac{1}{2}\left[f(ct-x) + \frac{1}{c}\int_{x_0}^{ct-x} g(u)\mathrm{d}u\right],\ x < ct \qquad (6.13)$$

Hence, D'Alembert's solution for semi-infinite string problem (6.10) is given by

(6.9) for $x \geq ct$, and by (6.4),(6.8a) and (6.13) for $x < ct$, i.e.

$$\psi(x,t) = \begin{cases} \dfrac{1}{2}[f(x+ct) + f(x-ct)] + \dfrac{1}{2c}\left[\displaystyle\int_{x-ct}^{x+ct} g(u)\mathrm{d}u\right], x \geq ct \\[2ex] \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c}\left[\int_{ct-x}^{x+ct} g(u)\mathrm{d}u\right], \quad x < ct \end{cases} \qquad (6.14)$$

Note that the wave (6.14), for any particular time, is continuous in x.

<u>Alternative method to find solution for semi-infinite problem (6.10)</u>

In this approach, we convert semi-infinite string problem to an infinite string problem by taking following odd extensions of given functions f, g as

$$F(x) = \begin{cases} f(x), \ \ 0 < x < \infty \\ -f(-x), -\infty < x < 0 \end{cases} \qquad G(x) = \begin{cases} g(x), \ \ 0 < x < \infty \\ -g(-x), -\infty < x < 0 \end{cases} \qquad (6.15)$$

Then, we consider following Infinite-string Problem:

$$\frac{\partial^2 \tilde{\psi}}{\partial t^2} = c^2 \frac{\partial^2 \tilde{\psi}}{\partial x^2}, \quad x \in (-\infty, +\infty), t \geq 0 \tag{6.16a}$$

Initial Conditions (IC):

$$\tilde{\psi}(x,0) = F(x),\ \tilde{\psi}_t(x,0) = G(x), \quad x \in (-\infty, +\infty), \tag{6.16b}$$

where $\tilde{\psi}(x,t) = \psi(x,t)$ for $0 \leq x < \infty$, $t \geq 0$. Hence, we need $\tilde{\psi}(x,t)$ for $x > 0$ only.

For the infinite-string Problem (6.16), we already derive D'Alembert's solution

$$\tilde{\psi}(x,t) = \frac{1}{2}[F(x+ct) + F(x-ct)] + \frac{1}{2c}\left[\int_{x-ct}^{x+ct} G(u)\mathrm{d}u\right]. \tag{6.17}$$

Note that at $t = 0$ and for $x \geq 0$, equation (6.17) implies that $\tilde{\psi}(x,0) = f(x) = \psi(x,0)$.

Now, at the end $x = 0$, from (6.17), we get

$$\tilde{\psi}(0,t) = \psi(0,t) = \frac{1}{2}[F(ct) + F(-ct)] + \frac{1}{2c}\left[\int_{-ct}^{+ct} G(u)\mathrm{d}u\right] = 0, \tag{6.18}$$

because according to the construction (6.15), both of F and G are odd.

Equation (6.18) shows that BC (6.10c) of semi-infinite Problem is satisfied,

Differentiating (6.17) partially w.r.t x, we get

$$\tilde{\psi}_t(x,t) = \frac{c}{2}[F'(x+ct) - F'(x-ct)] + \frac{1}{2}[G(x+ct) + G(x-ct]$$

$$\Longrightarrow \tilde{\psi}_t(0,t) = \psi_t(0,t) = \frac{c}{2}[F'(ct) - F'(-ct)] + \frac{1}{2}[G(ct) + G(-ct] = 0,$$

since F' is even and G is odd function. Also, we see given IC is satisfied:

$$\tilde{\psi}_t(x,0) = G(x) \Longrightarrow \text{For } x \geq 0, \tilde{\psi}_t(x,0) = \psi_t(x,0) = g(x)$$

Thus, solution (6.17) for $ct \le x < \infty$, $t \ge 0$ becomes

$$\tilde{\psi}(x,t) = \psi(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\left[\int_{x-ct}^{x+ct} G(u)\mathrm{d}u\right], x \ge ct$$

And, for $0 \le x \le ct$,

$$\tilde{\psi}(x,t) = \psi(x,t) = \frac{1}{2}[f(x+ct) - f(ct-x)]$$
$$+\tfrac{1}{2c}\left[-\int_{x-ct}^{0} g(-u)\mathrm{d}u + \int_{0}^{x+ct} g(u)\mathrm{d}u\right]$$
$$= \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(u)\mathrm{d}u$$

Hence, we got same solution (6.14), using an alternative approach.

Interpretation of D'Alembert's solution as Progressive Wave

Consider infinite string. Since both ends are free, it is expected that wave would propagate in both directions with speed c. I'll show that this is actually the case. For simplicity suppose external force is absent, so that $g(x) = 0$. Then D'Alembert's solution may be split as

$$2\psi(x,t) = \psi_L(x,t) + \psi_R(x,t), \psi_L = f(x+ct), \psi_R = f(x-ct)$$

Let us analyze $\psi_L(x,t)$. Similar will be applied for other component.

Suppose at $t = 0$, solution-profile $\psi_L(x,0) = f(x)$ has nodes at $x = a, b$ in ψx-plane. After time $1/c$, $\psi_L(x, 1/c)$ moves towards left along x-axis with same profile with new nodes at $x = a-1, b-1$, because of the following fact:

$$\psi_L\left(x, \tfrac{1}{c}\right) = f(x+1) \Longrightarrow \psi_L\left(x-1, \tfrac{1}{c}\right) = \psi_L(x,0)$$

We can proceed with time passes by as $t = 1/c, 2/c, 3/c, \ldots$, the initial wave-profile $\psi_L(x,0)$ keep on moving left with new nodes at $x = a-2, b-2; a-3. b-3; \ldots$, and after time $t = 1$, the wave profile travels the distance of c in the left of initial position implying that speed of wave is c.

Along with similar analyze on $\psi_R(x,t) = f(x - ct)$, we will see that the initial wave-profile $\psi_R(x,0) = f(x)$ moves towards right at speed c.

Thus, for an infinite string, there will be mixture of left-moving and right-moving waves. Since waves travels with time, this solution is also called "Travelling Wave solution". The interpretation of the characteristics as the direction of disturbance is given in Ref 2.

*************END OF PART B************