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Linear Algebra
TEST 2

Q1. Set of all non-invertible linear operators on V is NOT a sub-space of V

Q2. Linear map does NOT exist

Q3. Linear map is surjective

Q4. $S_1 S_2 \ldots S_n$ is a injective linear operator

Q5. $Tv = \alpha v$, $\alpha \in F$ & $v \in V$

Q6. $$[T]_B = \begin{bmatrix} 4.25 & 8.75 & 5.5 \\ -0.75 & 3.75 & -1.5 \\ -0.5 & -3.5 & 0 \end{bmatrix}$$

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Q1 Solution by counter example.
let us consider $\mathbb{R}^2$ with $B = \{e_1, e_2\}$ as basis where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Let $x_1, \cancel{x_2} \in \mathbb{R}^2$ and S & T be the linear operators on $\mathbb{R}^2$

now $x_1 = (a_1, b_1)$ ~~&/ x₂/~~ , $a_1, b_1 \in \mathbb{R}$

Let $T(x_1) = (a_1, 0) \in \mathbb{R}^2$
&
$S(x_1) = (0, b_1) \in \mathbb{R}^2$

We see that both T & S are non-invertible
For T & S to ~~blo~~ belong to ~~&~~ sub-space of V, they first need to clear the additive property. i.e

$(T+S)(x_1) = T(x_1) + S(x_1)$ should be non-invertible

but $(T+S)(x_1) = (a_1, b_1)$ which is invertible.
Thus, $T+S \notin$ set of non-invertible linear operations on V.
Thus,
set of non-invertible linear operations on V is <u>not</u> a subspace of V

Q2. Let $T: F^5 \longrightarrow F^2$ be a linear map with
$N(T) = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 = 3x_2 ; x_3 = x_4 = x_5\}$

Let us take an example of $x \in N(T)$

$\therefore$ x would be of the form
$x = (3x_2, x_2, x_3, x_3, x_3)$ for any $x \in N(T)$

$\therefore x = x_2(3,1,0,0,0) + x_3(0,0,1,1,1)$

as we see $(3,1,0,0,0)$ & $(0,0,1,1,1)$ are LI and span $N(T)$. Thus $\{(3,1,0,0,0), (0,0,1,1,1)\}$ form the basis of $N(T)$

Thus $N(T)$ has dim $= 2$ i.e Nullity $= 2$

From rank-nullity theorem, we can say that

$\dim(F^5) = 5 = \dim(N(T)) + \dim(R(T)) \leq \dim(N(T)) + \dim(F^2)$

~~$\Rightarrow$ ...~~

$5 \leq \dim(N(T)) + 2$
$5 - 2 \leq \dim(N(T)) \Rightarrow \dim(N(T)) \geq 3$

This is a contradiction.
Hence,
if $N(T)$ is of the given form, linear map does not exist.

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Q3. Let $T: F^4 \to F^2$ be a linear map with $N(T) = \{(x_1, x_2, x_3, x_4) \in F^4 \mid x_1 = 5x_2 \text{ \& } x_3 = 7x_4\}$

Let any $x \in N(T)$, s.t

$$x = (5x_2, x_2, 7x_4, x_4)$$
$$= x_2(5,1,0,0) + x_4(0,0,7,1)$$

Now, $(5,1,0,0)$ & $(0,0,7,1)$ are LI & span $N(T)$, thus they form basis of $N(T)$

$\therefore \dim(N(T)) = 2$

From rank-nullity theorem,

$$\dim(F^4) = 4 = \dim(N(T)) + \dim(R(T)) = 2 + \dim(R(T))$$

$$\Rightarrow 2 = \dim(R(T))$$

We know that $\dim(R(T)) \leq \dim(F^2)$, which is satisfied, thus T is a linear-map.

Not just ~~a~~ that, $\dim(R(T)) = \dim(F^2)$ holds, ~~thus~~ As $R(T) \subseteq F^2$ & dimensions are equal, $R(T) = F^2$. Thus, T is surjective

Q4. Let $S_1, S_2, \dots S_n$ be injective linear operators on V.
We know that a linear map is injection iff $N(T) = \{0\}$

Let us consider the linear operator ~~$S \neq S_1 ...$~~
$$S = S_1 S_2 S_3 \dots S_n$$
We need to check if S is injective or not ~~using~~ using induction

Let $n = 1$, then $S = S_1$ is injective as given
Let us assume that $n = k$ ~~is injective~~ s.t. $k < n$ &
~~$S \neq S_1 S_2$~~.
$$S = S_1 S_2 \dots S_k \text{ is injective } \text{ \& let}$$
$$Sx = (S_1 S_2 \dots S_k)x = 0 \longrightarrow (1)$$
Consider, $n = k+1$
$\therefore S = S_1 S_2 \dots S_{k+1}$
Now,
$$Sx = (S_1 S_2 \dots S_{k+1})x$$
$$0 = S_{k+1}(S_1 S_2 \dots S_k)x$$
~~by (1)~~

Now, by (1) $(S_1 S_2 \dots S_k)$ is injective, $\therefore$
$N(S_1 S_2 \dots S_k) = \{0\} \quad \therefore x = 0$

For $S = S_1 S_2 \dots S_{k+1}$ too this holds.

Thus $N(S_1 S_2 \dots S_{k+1}) = \{0\}$

Thus, by induction $S_1 S_2 \dots S_n$ is a injective linear operator

Q5. Given V is one-dimensional,
$T \in \mathcal{L}(V)$,

$\therefore Tv_1 = dv_1$ for $d \in F$ & $V = [v_1]$

now $dv_1 \in V$ as V is vector space

Let us consider, $v \in V$ ie $v = \alpha v_1, \alpha \in F$

$$Tv = T(\alpha v_1) = dv$$
$$= d\alpha v_1$$
$$Tv = \beta v_1 \in V$$

$\therefore$ T is a linear operation which is multiplication by $d \in F$

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Q6. Given $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map

$$T(x,y,z) = (3x+z,\ -2x+y,\ -x+2y+4z)$$

$$B = \{(1,0,1),\ (-1,2,1),\ (2,1,1)\} = \{b_1, b_2, b_3\}$$

$$T(b_1) = (4,-2,3)$$
$$T(b_2) = (-2,4,9)$$
$$T(b_3) = (7,-3,4)$$

now

$$(4,-2,3) = \alpha_1 b_1 + \beta_1 b_2 + \gamma_1 b_3$$
$$= (\alpha_1 - \beta_1 + 2\gamma_1,\ 2\beta_1 + \gamma_1,\ \alpha_1 + \beta_1 + \gamma_1)$$

$$\begin{aligned} 7 &= 2\alpha_1 + 3\gamma_1 \\ 1 &= -2\beta_1 + \gamma_1 \\ -2 &= 2\beta_1 + \gamma_1 \end{aligned} \quad \to \quad \begin{aligned} \alpha_1 &= 17/4 = 4.25 \\ \beta_1 &= -3/4 = -0.75 \\ \gamma_1 &= -1/2 = -0.5 \end{aligned}$$

$$(-2,4,9) = (\alpha_2 - \beta_2 + 2\gamma_2,\ 2\beta_2 + \gamma_2,\ \alpha_2 + \beta_2\ \gamma_2)$$

$$\alpha_2 = 8.75$$
$$\beta_2 = 3.75$$
$$\gamma_2 = -3.5$$

$$(7,-3,4) = (\alpha_3 - \beta_3 + 2\gamma_3,\ 2\alpha_3 + \beta_3,\ \alpha_3 + \beta_3 + \gamma_3)$$

$$\alpha_3 = 5.5$$
$$\beta_3 = -1.5$$
$$\gamma_3 = 0$$

$$[T]_B = \begin{bmatrix} 4.25 & 8.75 & 5.5 \\ -0.75 & 3.75 & -1.5 \\ -0.5 & -3.5 & 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$