Crank Nicolson method

If the forward difference approximation for time derivative in the one dimensional heat equation (6.4.1) is replaced with the backward difference and as usual central difference approximation for space derivative term are used then equation (6.4.1) can be written as

$$\frac{u_i^n - u_i^{n-1}}{\Delta t} = C\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2}$$

.... (6.4.4)

where i=1,2,3..........N-1 and n=1,2,3.................

or

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = C\frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{\Delta x^2}$$

.... (6.4.5)

for i=1,2,3..........N-1 and n=0,1,2,3.................

$$u_i^{n+1} - u_i^n = r\left(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right)$$

$$-ru_{i-1}^{n+1} + (1+2r)u_i^{n+1} - ru_{i+1}^{n+1} = u_i^n$$

.... (6.4.6)

for i=1,2,3..........N-1 and n=0,1,2,3.................

since there are three unknown terms in each of the equations (6.4.6) the scheme so obtained is an implicit method. The main drawback of having more than one unknown coefficient in any equation, unlike FTCS method, is value of the dependent variable at any typical node say (i, n) can't be obtained from the single finite difference equation of the node (i, n) but one has to be generate a system of equations for each time level separately by varying **i**. Then for each time level there will be system of equations equivalent to the number of unknowns in that time level(say N-1 in the present case). This linear system of algebriac equations in (N-1) unknowns has to be solved to obtain the solution for each time level . This process has to be repeated until the desired time level is reached. The scheme(6.4.6) is called fully implicit method.

Schemes (6.4.2) and (6.4.5) are two different methods to solve the one dimensional heat equation (6.4.1). Crank-Nicolson scheme is then obtained by taking average of these two schemes that is

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{C}{2}\left[\frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{\Delta x^2} + \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2}\right]$$

for **i=1, 2, 3..........N-1** and **n=0, 1, 2, 3.................**

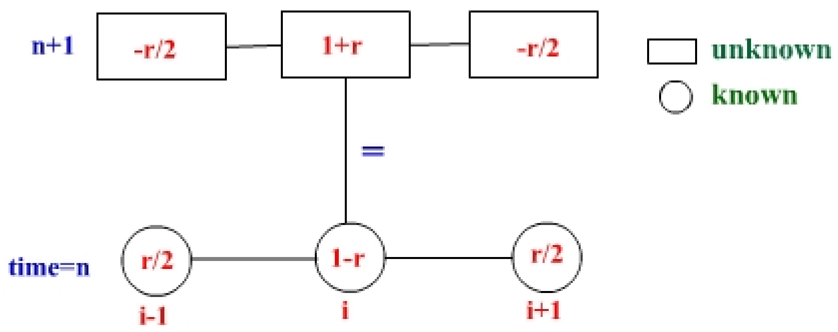
$$\Rightarrow -\frac{r}{2}u_{i-1}^{n+1} + (1+r)u_i^{n+1} - \frac{r}{2}u_{i+1}^{n+1} = \frac{r}{2}u_{i-1}^{n} + (1-r)u_i^{n} + \frac{r}{2}u_{i+1}^{n}$$

.... (6.4.7)

for **i=1, 2, 3..........N-1** and **n=0, 1, 2, 3.................**

Since more than one unknown is involved for each i in equation (6.4.7) Crank - Nicholson scheme is also an implicit scheme hence one has to solve a system of linear algebraic equations for every time level to get the field variable u.

The sketch for the Crank-Nicolson scheme is



The linear algebraic system of equations generated in Crank-Nicolson method for any time level $\mathbf{t^{n+1}}$ are sparse because the finite difference equation obtained at any space node, say **i** and at time level $\mathbf{t^{n+1}}$ has only three unknown coefficients involving space nodes '**i-1**' , '**i**' and '**i+1**' at $\mathbf{t^{n+1}}$ time level, so in matrix notation these equations can be written as **AU=B** , where **U** is the unknown vector of order N-1 at any time level $\mathbf{t^{n+1}}$, **B** is the known vector of order N-1 which has the values of **U** at **nth** time level and **A** is the coefficient square matrix of order N-1 x N-1 with a tri-diagonal structure

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & . & . & . & 0 \\ a_{21} & a_{22} & a_{23} & 0 & . & . & . & 0 \\ 0 & a_{32} & a_{33} & a_{34} & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & a_{N-2,N-3} & a_{N-2,N-2} & a_{N-2,N-1} \\ 0 & 0 & . & . & . & 0 & a_{N-1,N-2} & a_{N-1,N-1} \end{bmatrix}$$

A matrix of this structure that has only three non-zero terms at diagonal and one before and one after the principal diagonal for any row is called a tri-diagonal matrix and the system of equations with tridiagonal coefficient matrix is called tridiagonal system. Though direct solvers like Gauss elimination and LU decomposition can be used to solve these systems there are some special schemes available to solve the tridiogonal systems. One of them is

Thomas algorithm which exploits the tridiogonal nature of the coefficient matrix. Thomas algorithm is similar to Gauss elimination however, the novelty in the method is the forward elimination and back substitution parts of Gauss elimination are used only for the non-zero positions of the system **AU=B**.

Thomas Algorithm

Let us call the three non-zero diagonals of **A** as a, b and c where b has the elements of the principal diagonal, a has the elements of the diagonal before the principle diagonal with zero as the first element and c has the elements of the diagonal which lie after the principal diagonal with a zero as the last element then the order of a, b and c is equal to the number of unknowns for any time level with the known vector **B**. Then the Thomas algorithm can be written as

Do i=2 to N (if **N** is the number of unknowns)

$$b_i = b_i - a_i \frac{b_{i-1}}{a_{i-1}}$$

$$d_i = d_i - a_i \frac{d_{i-1}}{a_{i-1}}$$

end do

$$u_N = \frac{d_N}{b_N}$$

Do i = N-1 to 1

$$u_i = \frac{d_i - c_i u_{i+1}}{b_i}$$

end do

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