Recurrence Relations

Discrete Mathematics & its applications
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NR 224.

Example: • $\{a_r\}$, $a_r = 3^r, r \geq 0.$

Generating fun. for a seq. $\{a_r\}$

$$A(z) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots$$

$$\bullet\ A(z) = 1 + 3x + 3^2 x^2 + 3^3 x^3 + \cdots$$

$$= \frac{1}{1-3z}$$

$$\bullet\ \boxed{a_r = 3a_{r-1}, \quad a_0 = 1}$$

boundary conditn

Completely specifies the sequence $\{a_n\}$.

(discrete)
Numeric fun. $f : N \to R$.

Example: 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Not so obvious

$$a_r = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{r+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{r+1}$$

(general expression)

$$A(z) = \frac{1}{1-z-z^2}$$ (generating fun.)

$$\boxed{a_r = a_{r-1} + a_{r-2}, \quad a_0 = 1, a_1 = 1}$$

boundary conditions.

in many discrete computational problems

• recurrence relation / difference eqn.
⇒ easier to get than to obtain a general expression or a closed formed expression for its generating fun.

- Numeric fn^ns. can be described by a recurrence relation together with an appropriate set of boundary condition also referred to as the soln. of the recurrence relation.

- No general method of solving all recurrence relations.

Linear recurrence relations with const. coefficients

- $C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r)$, — (1).

- C_i constants.

- k-th order.

e.g. 1) $2a_r + 3a_{r-1} = 2^r$ (1st. order)

2) $3a_r - 5a_{r-1} + 2a_{r-2} = r^2 + 5$ (2nd. order)

3) $a_r + 7a_{r-2} = 0$ (2nd. order).

Given $a_3 = 3$, $a_4 = 6$,

$a_5 = \frac{1}{3}[5^2 + 5 + 5 \cdot \times 6 - 2 \times 3]$
$= 18$

$a_6 = \cdots$

$\vdots$

- For k-th order eqn, if k consecutive values $a_{m-k}, a_{m-k+1}, \ldots, a_{m-1}$ are known for some m, then the numeric fn. can be determined uniquely. Fewer that k values of the numeric fn. is not sufficient to determine it uniquely.

Homogeneous Solutions (RHS of eqn ① $f(m)=0$).

(Method of characteristic roots)

$$a = a^{(h)} + a^{(p)}$$

$a^{(h)}$ → hom. soln.

$a^{(p)}$ → particular soln.

② — $C_0 a_r^{(h)} + C_1 a_{r-1}^{(h)} + \cdots + C_k a_{r-k}^{(h)} = 0$

③ — $C_0 a_r^{(p)} + C_1 a_{r-1}^{(p)} + \cdots + C_k a_{r-k}^{(p)} = f(r)$

- particular soln. alone will not, in general, satisfy the boundary conditions, while we can adjust the homogeneous soln. so that the total soln. will satisfy the difference eqn. as well as the boundary conditions.

- many numeric fn's satisfy the same difference eqn. but only one of them satisfies the given boundary conditions at the same time.

- $\boxed{a_r^{(h)} = A\alpha_1^r}$ → a homogeneous soln. of a linear difference equn. with const-coefficient

α_1 → characteristic root

A → const. determined by the boundary conditions.

① ⟹ $C_0 A\alpha_1^r + C_1 A\alpha^{r-1} + C_2 A\alpha^{r-2} + \cdots + C_k A\alpha^{r-k} = 0$

⟹ $C_0\alpha^k + C_1\alpha^{k-1} + C_2\alpha^{k-2} + \cdots + C_{k-1}\alpha + C_k = 0$

→ characteristic equn. of the difference eqn. ①.

- A characteristic eqn of k-th degree has k characteristic roots

<u>case of distinct roots</u>

- $a_r^{(h)} = A_1\alpha_1^r + A_2\alpha_2^r + \cdots + A_k\alpha_k^r$ (Why?)

is a homogeneous soln to the difference eqn, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the distinct characteristic roots and $A_1, A_2, \ldots, A_k$ are constants which are to be determined by the boundary conditions.

Example: The recurrence relation for the fibonacci sequn of nos. $a_r = a_{r-1} + a_{r-2}$.

Corresponding characteristic eqn is

$$\alpha^2 = \alpha + 1$$

$$\alpha_1 = \frac{1+\sqrt{5}}{2}, \quad \alpha_2 = \frac{1-\sqrt{5}}{2}$$

$$\frac{\alpha^r - \beta^r}{\frac{\alpha-\beta}{\sqrt{5}}}$$

$$\therefore a_r^{(h)} = A_1\left(\frac{1+\sqrt{5}}{2}\right)^r + A_2\left(\frac{1-\sqrt{5}}{2}\right)^r$$

is a homogeneous soln. where the two consts. A_1 and A_2 are to be determined from the boundary conditions $a_0 = 1$ and $a_1 = 1$.

$$a_r^{(h)} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{r+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{r+1}\right]$$

$$A_1 + A_2 = 1$$
$$A_1 - A_2 = \frac{1}{\sqrt{5}}$$
$$A_1 = \frac{1+\frac{1}{\sqrt{5}}}{2} = \frac{\sqrt{5}+1}{2\sqrt{5}}$$

$$A_1 + A_2 = 1.$$
$$A_1\left(\frac{1+\sqrt{5}}{2}\right) + A_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$$
$$\frac{1}{2}(A_1 + A_2) + \frac{\sqrt{5}}{2}(A_1 - A_2) = 1$$
$$\Rightarrow A_1 - A_2 = \left(1 - \frac{1}{2}\right) \times \frac{2}{\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$A_1 + A_2 = 1$$
$$A_1\left(\frac{1+\sqrt{5}}{2}\right) + \cancel{A_2}\left(\frac{1-\sqrt{5}}{2}\right) = 1$$
$$A_1 = \left[1 - \frac{1-\sqrt{5}}{2}\right] \Big/ \left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = \frac{1+\sqrt{5}}{2\sqrt{5}}, \quad A_2 = 1 - \frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{-1+\sqrt{5}}{2\sqrt{5}}$$

Example;

$$a_n = 5a_{n-1} - 6a_{n-2}, \quad a_0 = 1, \; a_1 = 1.$$

Ch. equn

$$\alpha^2 - 5\alpha + 6 = 0.$$

$$\alpha = 3, 2$$

$$a_n^{(h)} = A_1 3^n + A_2 2^n, \quad A_1, A_2 \text{ consts.}$$

$$1 = A_1 + A_2 \qquad A_1 + A_2 - 1 = 0$$

$$1 = 3A_1 + 2A_2. \qquad 3A_1 + 2A_2 - 1 = 0.$$

$$\frac{A_1}{-1+2} = \frac{1}{-1+3} = \frac{1}{2-3}$$

$$\Rightarrow A_1 = -1$$
$$A_2 = 2$$

$$\boxed{\therefore a_n^{(h)} = -3^n + 2 \cdot 2^n.}$$

Case of multiple root

Let α_1 be a ~~multiple~~ root of multiplicity m.

We shall show that the corresponding homogeneous soln is

$$(A_1 r^{m-1} + A_2 r^{m-2} + \cdots + A_{m-2} r^2 + A_{m-1} r + A_m)\alpha_1^r$$

When A_i's are constants to be determined by the boundary conditions in the difference eqn.

(1) $C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r)$

- α_1 has multiplicity m.
- $A_m \alpha_1^r$ is a homogeneous sol^n. of the difference equ^n. (1).

<u>claim</u> $A_{m-1} r \alpha_1^r$ is also a homogeneous sol^n. of (1).

- α_1 is not only a root of the equ^n.

$$C_0\alpha^r + C_1\alpha^{r-1} + C_2\alpha^{r-2} + \dots + C_k\alpha^{r-k} = 0,$$

but also a root of the derivative equ^n. of (1'),

$$C_0 r\alpha^{r-1} + C_1(r-1)\alpha^{r-2} + C_2(r-2)\alpha^{r-3} + \dots + C_k(r-k)\alpha^{r-k-1} = 0$$

multiplying by $A_{m-1}\alpha$ & replacing α by α_1

$$C_0 A_{m-1} r\alpha_1^r + C_1 A_{m-1}(r-1)\alpha_1^{r-1} + C_2 A_{m-1}(r-2)\alpha_1^{r-2} + \dots + C_k A_{m-1}(r-k)\alpha_1^{r-k} = 0.$$

$\Rightarrow$ $A_{m-1} r \alpha_1^r$ is indeep a homogeneous sol^n.

- Infact, α_1 satisfies the 2^{nd}, 3^{rd}, .., $(m-1)$th derivative equ^n. of (1), enables us to prove that $A_{m-2} r^2 \alpha_1^n$, $A_{m-3} r^3 \alpha_1^n$, .., $A_1 r^{m-1} \alpha_1^r$ are also homogeneous sol^{ns}. in a similar manner.

- Let equ^n (1) has characteristic roots ~~of~~ $\alpha_1, \alpha_2, \ldots, \alpha_q$, with α_i having multiplicity m_i.
- Then

$$\left.\begin{array}{l} \alpha_1^r, r\alpha_1^r, r^2\alpha_1^r, \ldots, r^{m_1-1}\alpha_1^r \\ \alpha_2^r, r\alpha_2^r, r^2\alpha_2^r, \ldots, r^{m_2-1}\alpha_2^r \\ \vdots \\ \alpha_q^r, r\alpha_q^r, r^2\alpha_q^r, \ldots, r^{m_q-1}\alpha_q^r \end{array}\right\} \begin{array}{l}\text{basic sol}^{ns}. \\ b_1, b_2, \ldots, b_p \text{ say.}\end{array}$$

must all be sol^{ns}. of the recurrence (1).

- Since weighted sum of a sol^n. is a sol^n., for any Constants $\lambda_1, \lambda_2, \ldots, \lambda_p$,

$$a_r = \lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_p b_p$$

is also a sol^n of the recurrence (1).

Unique sol^n.
$a_0, a_1, \ldots, a_{p-1}$ initial conditions
→ a system of p equ^{ns} in $\lambda_1, \ldots, \lambda_p$

Example:

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}.$$

$$a_0 = 1, a_1 = 2, a_2 = 0.$$

Sol^n. Characteristic equ^n.

$$x^3 - 7x^2 + 16x - 12 = 0.$$

$$(x-2)^2(x-3) = 0.$$

$$\alpha_1 = 2, \alpha_2 = 3.$$

put $a_n = x^n$

$$x^n = 7x^{n-1} - 16x^{n-2} + 12x^{n-3}$$

$$8 - 28 + 32 - 12$$

$$+40 - 40$$

General sol^n. is $a_n = \lambda_1 2^n + \lambda_2 n 2^n + \lambda_3 3^n$; $\lambda_1, \lambda_2, \lambda_3$ consts.

8

$$a_n = \lambda_1 2^n + \lambda_2 n 2^n + \lambda_3 3^n$$

$$a_0 = 1, \; a_1 = 2, \; a_2 = 0$$

$1 = \lambda_1 + \lambda_3$ $\qquad\qquad \lambda_3 = 1 - \lambda_1$

$2 = 2\lambda_1 + 2\lambda_2 + 3\lambda_3 \Rightarrow 2 = 2\lambda_1 + 2\lambda_2 + 3 - 3\lambda_1$

$0 = 4\lambda_1 + 8\lambda_2 + 9\lambda_3 \qquad = -\lambda_1 + 2\lambda_2 + 3$

$\Downarrow$ $\qquad$ or $-\lambda_1 + 2\lambda_2 + 1 = 0.$

$0 = 4\lambda_1 + 8\lambda_2 + 9 - 9\lambda_1 \qquad -5\lambda_1 + 8\lambda_2 + 9 = 0$

$= -5\lambda_1 + 8\lambda_2 + 9.$

$$\frac{\lambda_1}{18-8} = \frac{\lambda_2}{-5+9} = \frac{1}{-8+10}$$

$$\Rightarrow \lambda_1 = \frac{10}{2} = 5$$

$$\lambda_2 = \frac{4}{2} = 2$$

$$\lambda_3 = 1 - \lambda_1 = 1 - 5 = -4.$$

$\therefore$ The unique sol^n. with the given initial Conditions is

$$a_n = 5.2^n + 2n.2^n - 4.3^n.$$

- $$\boxed{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_p a_{n-p},} \quad n \geq p. \quad (1)$$

$c_1, c_2, \dots, c_p$ consts, $c_p \neq 0$.

→ linear homogeneous ~~eqn.~~ recurrence relation with constant coefficients.

- unique solⁿ. if $a_0, a_1, \dots, a_{p-1}$ (1st. p terms) are specified.

↓ initial / boundary conditions.

~~Theorem) Suppose (1) has characteristic roots~~

- Substitute $\boxed{a_k = x^k}$ & solve for x.

↳ $x^n - c_1 x^{n-1} - c_2 x^{n-2} - \dots - c_p x^{n-p} = 0$.

or $x^p - c_1 x^{p-1} - c_2 x^{p-2} - \dots - c_p = 0$ — (2)

↳ Characteristic eqn. of (1)

Case I : roots all distinct — $\alpha_1, \alpha_2, \dots, \alpha_p$

Case II : multiple roots.

Theorem (Distinct roots)

- $a_n = \lambda_1 \alpha_1^n + \lambda_2 \alpha_2^n + \dots + \lambda_p \alpha_p^n$ — (3) is a solⁿ. of (1). ↳ the general

$\lambda_1, \lambda_2, \dots, \lambda_p$ are consts.

$\alpha_1, \alpha_2, \dots, \alpha_p$ all distinct roots of (2).

- Every solⁿ. of (1) has the form (3).

$$a_n = \lambda_1\alpha_1^n + \lambda_2\alpha_2^n + \cdots + \lambda_p\alpha_p^n$$

$$\left.\begin{aligned}
\lambda_1 + \lambda_2 + \cdots + \lambda_p &= a_0 \\
\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_p\alpha_p &= a_1 \\
\lambda_1\alpha_1^2 + \lambda_2\alpha_2^2 + \cdots + \lambda_p\alpha_p^2 &= a_2 \\
&\vdots \\
\lambda_1\alpha_1^{p-1} + \lambda_2\alpha_2^{p-1} + \cdots + \lambda_p\alpha_p^{p-1} &= a_{p-1}
\end{aligned}\right\}$$

p linear eqns. in p unknowns $\lambda_1, \lambda_2, \dots, \lambda_p$.

coefficient matrix

$$\begin{bmatrix}
1 & 1 & & 1 \\
\alpha_1 & \alpha_2 & & \alpha_p \\
\alpha_1^2 & \alpha_2^2 & \cdots & \alpha_p^2 \\
\vdots & \vdots & & \vdots \\
\alpha_1^{p-1} & \alpha_2^{p-1} & & \alpha_p^{p-1}
\end{bmatrix}$$

determinant of this matrix

$\rightarrow$ Vandermonde determinant.

$$= \prod_{1 \le i < j \le p} (\alpha_i - \alpha_j) \neq 0$$ as $\alpha_1, \alpha_2, \dots, \alpha_p$ are all distinct.

So there is a unique soln. of the system.

Theorem (multiple roots).

$a_n = \lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_p b_p$ is the general sol^n. of (1) for some constants $\lambda_1, \lambda_2, \ldots, \lambda_p$ where $b_1, b_2, \ldots, b_p$ are the basic sol^{ns}. of (1).

$\alpha_1, \alpha_2, \ldots, \alpha_q$ Characteristic roots

$\downarrow$

$m_1, m_2, \ldots, m_q$ multiplicity

$(x-\alpha_1)^{m_1} \cdots (x-\alpha_q)^{m_q}$

$m_1 + m_2 + \cdots + m_q = p$

$$\boxed{\alpha_i^n, n\alpha_i^n, n^2\alpha_i^n, \ldots, n^{m_i - 1}\alpha_i^n}$$

$1 \le i \le q$

Basic Sol^{ns}.

Particular Solutions

- No general procedure
- by the method of inspection. (in some special cases)

[First set up the general ~~sol^n~~ form the particular sol^n according to the form of $f(r)$.
Then determine the exact sol^n according to the given difference eq^n.

Example: $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$ — (1)

We assume that the general form of the particular sol^n. is

$a_r^{(p)} = P_1 r^2 + P_2 r + P_3$ — (2)

where P_1, P_2, P_3 are constants to be determined.

Substituting (2) in (1),

$$P_1 r^2 + P_2 r + P_3 + 5P_1(r-1)^2 + 5P_2(r-1) + 5P_3 + 6P_1(r-2)^2 + 6P_2(r-2) + 6P_3 = 3r^2$$

or. $12P_1 r^2 + (P_2 - 10P_1 + 5P_2 - 24P_1 + 6P_2) r$
$+ (P_3 + 5P_1 \cancel{0} - 5P_2 + 5P_3 + 24P_1 - 12P_2 + 6P_3) = 3r^2$

or $12P_1 r^2 + (12P_2 - 34P_1) r + (12P_3 - 17P_2 + 29P_1) = 3r^2$

$\Rightarrow \quad 12P_1 = 1 \quad \Rightarrow P_1 = 3/12 = \frac{1}{4}$

$12P_2 - 34P_1 = 0 \quad \Rightarrow P_2 = \frac{\overset{17}{\cancel{34}}}{\underset{2}{\cancel{4}} \times 12} = \frac{17}{24}$

$12P_3 - 17P_2 + 29P_1 = 0. \quad \Rightarrow P_3 = \dfrac{17 \times \frac{17}{24} - 29 \times \frac{1}{4}}{12}$

$= \frac{115}{288}$

$\therefore a_r^{(P)} = \frac{1}{4} r^2 + \frac{17}{24} r + \frac{115}{288}.$

General Rule

When $f(r) = F_1 r^t + F_2 r^{t-1} + \cdots + F_t r + F_{t+1}$,
the corresponding particular sol^n. will be of the form

$P_1 r^t + P_2 r^{t-1} + \cdots + P_t r + P_{t+1}.$

Exam

~~Particular~~

Example: $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1$ — ①

Particular sol^n. is of the form

$$a_r^{(p)} = P_1 r^2 + P_2 r + P_3 \quad \text{— ②}$$

Substituting ② in ①, we obtain

$$P_1 r^2 + P_2 r + P_3 + 5\left[P_1 (r-1)^2 + P_2 (r-1) + P_3\right] + 6\left[P_1 (r-2)^2 + P_2 (r-2) + P_3\right]$$

$$= 3r^2 - 2r + 1.$$

or $12P_1 r^2 - (34P_1 - 12P_2) r + (29P_1 - 17P_2 + 12P_3) = 3r^2 - 2r + 1$

$$\Rightarrow \quad 12P_1 = 3$$

$$34P_1 - 12P_2 = 2$$

$$29P_1 - 17P_2 + 12P_3 = 1.$$

$$\Rightarrow \quad P_1 = \frac{1}{4}, \quad P_2 = \frac{13}{24}, \quad P_3 = \frac{71}{288}.$$

$\therefore$ the Particular sol^n. is

$$a_r^{(p)} = \frac{1}{4} r^2 + \frac{13}{24} r + \frac{71}{288}$$

Example: $a_r - 5a_{r-1} + 6a_{r-2} = 1.$

P. sol^n. $\quad a_r^{(p)} = P.$

$P - 5P + 6P = 1 \Rightarrow 2P = 1 \Rightarrow P = 1/2.$

$$\therefore \quad a_r^{(p)} = \frac{1}{2}$$

Example:

$$a_r + 5a_{r-1} + 6a_{r-2} = 42.4^r \quad \text{--- (1)}$$

We assume that the general form of the particular soln. is $P4^r$.

Then (1) $\Rightarrow$

$$P4^r + 5P4^{r-1} + 6P4^{r-2} = 42.4^r$$

$$\Rightarrow 16P + 20P + 6P = 42 \times 16$$

$$\cancel{\Rightarrow \frac{25}{15}0P = \frac{42 \times 16}{7}}$$

$$\Rightarrow \cancel{42}P = \cancel{42} \times 16$$

$$P = 16.$$

$x^2 + 5x + 6 = 0.$

4 is not a root.

$$a_r^{(P)} = 16.4^r.$$

General Rule. When $f(r)$ is of the form β^r, the corresponding particular soln. is of the form $P\beta^r$, if β is not a characteristic root of the difference eqn.

When $f(r) = (F_1 r^t + F_2 r^{t-1} + \dots + F_t r + F_{t+1})\beta^r$, the corresponding particular soln. is of the form $(P_1 r^t + P_2 r^{t-1} + \dots + P_t r + P_{t+1})\beta^r$. If β is not a characteristic root of the difference eqn.

Example:

$$a_r + a_{r-1} = 3r\,2^r \quad \text{—} (1)$$

The general form of the particular sol^n is

$$a_r^{(p)} = (P_1 r + P_2)\,2^r \quad \text{—} (2).$$

Substituting (2) into (1), we obtain

$$(P_1 r + P_2)\,2^r + \frac{[P_1(r-1) + P_2]\,2^{r-1} \times 2}{2} = 3r\,2^r$$

$$\Rightarrow \frac{3}{2} P_1 r\,2^r + \left(-\frac{1}{2}P_1 + \frac{3}{2}P_2\right)2^r = 3r\,2^r$$

$$\Rightarrow \quad \frac{3}{2}P_1 = 3$$

$$-\frac{1}{2}P_1 + \frac{3}{2}P_2 = 0.$$

$$\Rightarrow \quad P_1 = 2, \; P_2 = 2/3.$$

& the particular sol^n is $\quad a_r^{(p)} = \left(2r + \frac{2}{3}\right)2^r$.

General Rule

When $f(r) = \left(F_1 r^t + F_2 r^{t-1} + \cdots + F_t r + F_{t+1}\right)\beta^r$

when β is a characteristic root of multiplicity $m-1$, then the corresponding particular sol^n is of the form

$$r^{m-1}\left(P_1 r^t + P_2 r^{t-1} + \cdots + P_t r + P_{t+1}\right)\beta^r$$

Example: $a_r - 2a_{r-1} = 3.2^r$ —(1)

$x - 2 = 0$
$\Rightarrow x = 2$

As 2 is a characteristic root of multiplicity 1, the general form of the particular sol^n is

$$a_r^{(p)} = Pr2^r \quad \text{—(2)}$$

(1),(2) $\Rightarrow$ $Pr.2^r - 2P(r-1)2^{r-1} = 3.2^r$

$$2P\cancel{r} - 2P(\cancel{r}-1) = 6.$$

$\cancel{\Rightarrow P = r + r - 1\; r2 + r}$

$\Rightarrow P = 3.$

$\therefore a_r^{(p)} = 3r2^r.$

Example: $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)2^r$ —(1)

As 2 is a characteristic root of multiplicity 2, the general form of the particular sol^n is

$$a_r^{(p)} = r^2(P_1 r + P_2)2^r \quad \text{—(2)}.$$

$r^2(P_1r+P_2)2^r - 2(r^2-2r+1)[P_1r+(P_2-P_1)]2^r + (r^2-4r+4)[P_1r+(P_2-2P_1)]2^r$

(1),(2) $\Rightarrow$ $6P_1 r 2^r = r2^r$

Check $2^r\{-2(P_2-P_1) + 4(P_2-2P_1)\} = (-6P_1 + 2P_2)2^r = 2^r.$

yielding $P_1 = 1/6$, $P_2 = 1$.

$\therefore a_r^{(p)} = r^2\left(\frac{r}{6}+1\right)2^r.$

$r^2(P_1r+P_2)2^r - 2(r-1)^2[P_1(r-1)+P_2]2^{r-1} + 4(r-2)^2[P_1(r-2)+P_2]2^{r-2} = (r+1)2^r$

Example: $a_r = a_{r-1} + 7$

ch. eqn $x - 1 = 0$

$x = 1$

ch. root of mul. 1

$a_r - a_{r-1} = 7 \cdot 1^r$

P. sol^n $\rightarrow$ $r^1 P \cdot 1^r = Pr$

$\therefore Pr - P(r-1) = 7 \Rightarrow P = 7 \Rightarrow a_r^{(p)} = 7$

$a_r - a_{r-1} = 7$

P. sol^n $\rightarrow$ P

$P - P = 7$

(absurd)

Example: $a_r - 2a_{r-1} + a_{r-2} = 7$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$1 \rightarrow$ ch. root of mul. 2

$a_r^{(p)} = Pr^2$

$Pr^2 - 2P(r-1)^2 + P(r-2)^2 = 7$

~~$Pr^2 - 2P(r^2 - 2r + 1) + r - 2 = 7$~~

~~$-Pr^2 + 4Pr$~~

~~$-Pr^2 + (1 + 4r)P - 2 - 2P = 7$~~

$r^2[P - 2P + P] + r[4P - 4P] - 2P + 4P = 7$

$\Rightarrow 2P = 7$

$\Rightarrow P = 7/2$

Example: $a_r - 5a_{r-1} + 6a_{r-1} = 2^r + r$

The general form of the particular sol^n is

$P_1 r 2^r + P_2 r + P_3$.

P. sol^n + P. sol^n

$x^2 - 5x + 6 = 0$

$x^2 - 3x - 2x + 6 = 0$

$x = 3, 2$

$a_r^{(p)} = -r\, 2^{r+1} + \frac{1}{2} r + \frac{7}{4}$.

Solution by the method of generating fun^ns.

$$a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r, \; r \geq 2$$

$a_0 = 1, a_1 = 1$ (boundary conditions).

$$\sum_{r=2}^{\infty} a_r z^r - \sum_{r=2}^{\infty} 5a_{r-1} z^r + \sum_{r=2}^{\infty} 6a_{r-2} z^r = \sum_{r=2}^{\infty} 2^r z^r + \sum_{r=2}^{\infty} r z^r$$

$2z^2 + 3z^3 + 4z^4 + \ldots = z^2[2z + 3z^2 + 4z^3 + \ldots]$

$2^2 z^2 + 2^3 z^3 + \ldots$

$2^2 z^2 \left[\frac{1}{1-2z}\right]$

$a_1 z^2 + a_2 z^3 + \ldots$

$z(A(z) - a_0)$

$$(A(z) - a_0 - a_1 z) - 5z(A(z) - a_0) + 6z^2 A(z) = \frac{4z^2}{1-2z} + z\left[\frac{1}{(1-z)^2} - 1\right]$$

$$\Rightarrow A(z) = \frac{1 - 8z + 27z^2 - 35z^3 + 14z^4}{(1-z)^2 (1-2z)^2 (1-3z)}$$ (Verified)

$$= \frac{5/4}{1-z} + \frac{1/2}{(1-z)^2} - \frac{3}{1-2z} - \frac{2}{(1-2z)^2} + \frac{17/4}{1-3z}$$

$$\therefore a_r = \frac{5}{4} + \frac{1}{2}(r+1) - 3.2^r - 2(r+1)2^r + \frac{17}{4}3^r$$

$$= \frac{7}{4} + \frac{r}{2} - r2^{r+1} - 5.2^r + \frac{17}{4}3^r$$

Example: Suppose we toss a coin r times. There are 2^r possible sequences of outcomes. We want to know the no. of sequences of outcomes in which heads never appear on successive tosses. Find a_r.

Let a_r denote this no.

$a_r \to$ To obtain a seq. of r heads & tails with no consecutive heads

$\to a_{r-1}$

$+$

$\to a_{r-2}$

$\boxed{\;\;r-2\;\;}$ $\;\overset{r-1}{T}\;\overset{r}{H}$

no consecutive heads

$\boxed{\;\;r-1\;\;}$ $\;\overset{r}{T}$

no consecutive heads

Simultaneous equ^{ns} for Generating functions.

- alphabet $\{0,1,2,3\}$
- a_k = # of codewords of length k with an even no. of 0's and an even no. of 3's.

$a_k = ?$

let b_k = # of k-digit codewords from the alphabet $\{0,1,2,3\}$ with an even no. of 0's and an odd no. of 3's.

c_k = # of k-digit codewords from the alphabet $\{0,1,2,3\}$ with an odd no. of 0's and an even no. of 3's.

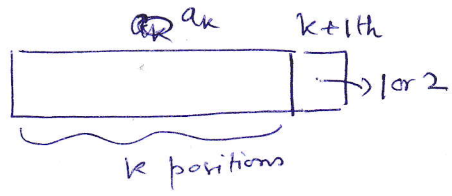
d_k = # of k digit codewords from the alphabet $\{0,1,2,3\}$ with an odd no. of 0's & odd no. of 3's.

$$4^k = a_k + b_k + c_k + d_k$$

$a_k \leftarrow$ even 0 even 3
$b_k \leftarrow$ even 0 odd 3.
$c_k \leftarrow$ odd 0 even 3.
$d_k \leftarrow$ odd 0 odd 3.

$$a_{k+1} = 2a_k + b_k + c_k$$

Similarly
$$\begin{cases} b_{k+1} = 2b_k + a_k + d_k \\ c_{k+1} = 2c_k + a_k + d_k \end{cases}$$



$$b_{k+1} = b_k - c_k + 4^k.$$

$$c_{k+1} = c_k - b_k + 4^k.$$

b_k 3
c_k 0

initial conditions

$a_1 = 2$ (1, 2 → with no 0's, no 3's)
(no 0's → even, no 3's → even)
$b_1 = 1$ (3 itself)
$c_1 = 1$ (0 itself).

The method of generating fun^{s} can be applied to solve a system of recurrences.

Illustration

$$\left.\begin{aligned} a_{k+1} &= 2a_k + b_k + c_k \\ b_{k+1} &= b_k - c_k + 4^k \\ c_{k+1} &= c_k - b_k + 4^k \end{aligned}\right\} \quad (1)$$

$a_1 = 2$
$b_1 = 1$
$c_1 = 1$

We first choose a_0, b_0, c_0 so that (1) holds.

$k = 0$

$$\left.\begin{aligned} 2 &= 2a_0 + b_0 + c_0 \\ 1 &= b_0 - c_0 + 1 \end{aligned}\right\} \Rightarrow 3 = 2a_0 + 2b_0 + 1 \Rightarrow a_0 + b_0 = 1.$$

$$1 = c_0 - b_0 + 1 \longrightarrow c_0 - b_0 = 0 \Rightarrow c_0 = b_0 = 0 \text{ (say)}$$

$$2 = 2a_0 + (1 - a_0) + (1 - a_0) \qquad \therefore a_0 = 1.$$

$$\begin{aligned} 2a_0 + b_0 + c_0 &= 2 \\ b_0 - c_0 &= 0 \\ -b_0 + c_0 &= 0. \end{aligned} \qquad \cancel{\begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix}}$$

with $a_0 = 1$, $b_0 = c_0 = 0$, (1) holds for $k \geq 0$.

$$\sum_{k\geq 0}^{\infty} a_{k+1} x^k = \sum 2a_k x^k + \sum b_k x^k + \sum c_k x^k$$

$$\sum_{k\geq 0}^{\infty} b_{k+1} x^k = \sum b_k x^k - \sum c_k x^k + \sum 4^k x^k$$

$$\sum_{k\geq 0}^{\infty} c_{k+1} x^k = \sum c_k x^k - \sum b_k x^k + \sum 4^k x^k$$

Let $A(x) = \sum_{k\geq 0}^{\infty} a_k x^k$, $B(x) = \sum_{k\geq 0}^{\infty} b_k x^k$, $C(x) = \sum_{k\geq 0}^{\infty} c_k x^k$

$$\frac{1}{x}\left[A(x) - a_0\right] = 2A(x) + B(x) + C(x)$$

$$\frac{1}{x}\left[B(x) - a_0\right] = B(x) - C(x) + \frac{1}{1-4x}$$

$$\frac{1}{x}\left[C(x) - c_0\right] = C(x) - B(x) + \frac{1}{1-4x}.$$

$$A(x) = \frac{1}{1-2x}\left[x B(x) + x C(x) + 1\right]$$
$$B(x) = \frac{1}{1-x}\left[-x C(x) + \frac{x}{1-4x}\right]$$
$$C(x) = \frac{1}{1-x}\left[-x B(x) + \frac{x}{1-4x}\right]$$

$$\Rightarrow B(x) = C(x) = \frac{x}{1-4x} = \sum_{k=0}^{\infty} 4^k x^{k+1}$$

$\Downarrow$

$$\begin{cases} b_k = c_k = 4^{k-1} & \text{for } k > 0 \\ b_k = c_k = 0 & \text{for } k = 0 \end{cases}$$

$$A(x) = \frac{2x^2 - 4x + 1}{(1-2x)(1-4x)}$$
$$= \frac{1-3x}{1-4x} + \frac{x}{1-2x}$$
$$= 1 + \frac{x}{1-4x} + \frac{x}{1-2x}$$
$$= 1 + \sum_{k=0}^{\infty} 4^k x^{k+1} + \sum_{k=0}^{\infty} 2^k x^{k+1}$$

Thus, $\begin{cases} a_k = 4^{k-1} + 2^{k-1} & \text{for } k > 0 \\ a_0 = 1 \end{cases}$

$$\cancel{A(x) = \frac{1-2x-2x+2x^2}{(1-2x)(1-4x)} = \frac{1}{1-4x} + \frac{2x^2-2x}{(1-2x)(1-4x)}}$$

$$A(x) = \frac{2x^2-4x+1}{(1-2x)(1-4x)} = \frac{Ax+B}{1-2x} + \frac{Cx+D}{1-4x}$$

$$(1-x)B(x) + x C(x) = \frac{x}{1-4x}$$
$$x B(x) + (1-x) C(x) = \frac{x}{1-4x}$$

$$(1-2x)B(x) - (1-2x)C(x) = 0$$
$$B(x) = C(x)$$

$$(1-x)B(x) + x B(x) = \frac{x}{1-4x}$$
$$B(x) = \frac{x}{(1-4x)(1-x+x)}$$
$$= \frac{x}{1-4x}.$$

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