Duality Theory of LPP

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Linear Programming Models

General form of a linear programming problem is given by:

$$\text{(I)} \qquad \max : f = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad i = 1, 2, ..., m$$

$$x_j \geq 0, \qquad j = 1, 2, ..., n$$

$$\text{(II)} \qquad \max : f = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, 2, \ldots, m$$

$$x_j \geq 0, \qquad j = 1, 2, \ldots, n.$$

(III) $$\max : f = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad i = 1, 2, \ldots, m$$

$$x_j \quad \text{is} \quad \text{free}, \qquad j = 1, 2, \ldots, n$$

(IV) $$\max : f = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, 2, \ldots, m$$

$$x_j \quad \text{is} \quad \text{free}, \qquad j = 1, 2, \ldots, n$$

For these models (I)-(IV), it is assumed that all a_{ij}, b_i, c_j are deterministic real number for all i and j. Since the objective function and the constraints of the models are linear, we may apply the following Linear programming methods to find the optimal solution:

- Primal Simplex Method
- Dual Simplex Method
- Charne's Penalty Method (Big-M Method)
- Two-Phase Simplex Method
- Revised Simplex Method
- Interior Point Methods(Projective and Scaling) of Karmarkar (1984).

Primal(P) and Dual(D) LPP: Type-I

$$(P) \qquad \max : f = c^T X$$

$$\text{Subject to} \quad AX \leq b, X \geq 0$$

$$\text{where } c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \; b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m \times n}$$

m = no. of constraints

n = no. of variables given initially

In expanded form, it can be written as:

$$(P) \qquad \max : f = \sum_{j=1}^{n} c_j x_j$$

Subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i, \quad i = 1, 2, ..., m \\ x_j &\geq 0. \quad j = 1, 2, ..., n \end{aligned}$$

Dual of the Primal LPP:

$$(D) \qquad \min : f^{'} = b^T Y$$

$$\text{s. t.} \quad A^T Y \geq c, Y \geq 0$$

$$\text{where } Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

In expanded form, it can be written as:

$$(D) \qquad \min: f' = \sum_{i=1}^{m} b_i y_i$$

s. t.

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i &\geq c_j, \quad j = 1, 2, ..., n \\ y_i &\geq 0. \quad i = 1, 2, ..., m \end{aligned}$$

How to find Dual of a LPP ?
We will use Lagrange Multipliers Method to establish the Dual. Details are given below:

$$(P) \qquad \max : f = \sum_{j=1}^{n} c_j x_j$$

$$\Rightarrow \quad \min : -f = -\sum_{j=1}^{n} c_j x_j$$

s. t.

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, ..., m \tag{1}$$

$$x_j \geq 0. \;\; j = 1, 2, ..., n$$

Add a slack variable

$$
\begin{aligned}
s_i^2 &\geq 0, \quad i = 1, 2, ..., m \\
\Rightarrow \quad \sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i &= 0, \quad i = 1, 2, ..., m \qquad (2) \\
\text{where} \quad x_j &\geq 0, \quad j = 1, 2, ..., n \\
\Rightarrow \quad - x_j &\leq 0, \quad j = 1, 2, ..., n \\
-x_j + t_j^2 &= 0, \quad j = 1, 2, ..., n \\
\text{Another slack variable} \quad t_j^2 &\geq 0, \quad j = 1, 2, ..., n
\end{aligned}
$$

Let $L(\ldots)$ be the Lagrange Function.

$$L(X, S, T, \lambda, \mu) = -\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i \right) + \sum_{j=1}^{n} \mu_j \left(-x_j + t_j^2 \right)$$

$$\begin{aligned} \text{where} \quad \lambda_1, \lambda_2, \ldots, \lambda_m &\geq 0 \\ \mu_1, \mu_2, \ldots, \mu_n &\geq 0 \\ x_1, x_2, \ldots, x_n &\geq 0 \\ s_1^2, s_2^2, \ldots, s_m^2 &\geq 0 \\ t_1^2, t_2^2, \ldots, t_n^2 &\geq 0 \end{aligned}$$

All the Lagrange multipliers $\lambda_i, \forall\ i$ and $\mu_j, \forall\ j$ are non-negative. Total number of variables are $2m+3n$. There are $m+n$ number of Lagrange multipliers. Using the Necessary Conditions for the minimum we have:

$$
\begin{aligned}
\frac{\partial L}{\partial x_j} &= 0, \quad j = 1, 2, ..., n \\
\Rightarrow \quad -c_j + \sum_{i=1}^{m} \lambda_i a_{ij} - \mu_j &= 0, \quad j = 1, 2, ..., n \\
\Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} - c_j &= \mu_j, \text{ but } \ \mu_j \geq 0 \\
\Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} - c_j &\geq 0, \quad j = 1, 2, ..., n \\
\Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} &\geq c_j, \quad j = 1, 2, ..., n \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \lambda_i} &= 0, \quad i = 1, 2, \ldots, m \\
\sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i &= 0, \ \text{but} \ \ s_i^2 \geq 0 \\
\Rightarrow \quad \sum_{j=1}^{n} a_{ij} x_j &\leq b_i, \quad i = 1, 2, \ldots, m \qquad (4) \\
\frac{\partial L}{\partial s_i} &= 0, \quad i = 1, 2, \ldots, m \\
\Rightarrow \quad 2 s_i \lambda_i &= 0, \quad i = 1, 2, \ldots, m \\
\Rightarrow \quad s_i \lambda_i &= 0, \quad i = 1, 2, \ldots, m \\
\Rightarrow \quad s_i^2 \lambda_i &= 0, \quad i = 1, 2, \ldots, m \qquad (5)
\end{aligned}
$$

$$\Rightarrow \quad \lambda_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) = 0, \quad i = 1, 2, ..., m$$

$$\Rightarrow \quad \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) = 0, \quad i = 1, 2, ..., m$$

$$\frac{\partial L}{\partial \mu_j} = 0, \quad j = 1, 2, ..., n$$

$$\Rightarrow \quad - x_j + t_j^2 = 0, \quad j = 1, 2, ..., n$$

$$\Rightarrow \quad x_j = t_j^2, \quad j = 1, 2, ..., n$$

$$\frac{\partial L}{\partial t_j} = 0, \quad j = 1, 2, \ldots, n$$

$$2\mu_j t_j = 0, \quad j = 1, 2, \ldots, n$$

$$\Rightarrow \quad \mu_j t_j = 0, \quad j = 1, 2, \ldots, n$$

$$\text{So} \quad \mu_j t_j^2 = 0, \quad j = 1, 2, \ldots, n$$

From the last equation $t_j^2 = x_j$

$$\mu_j x_j = 0, \quad j = 1, 2, \ldots, n \tag{6}$$

where

$$\sum_{i=1}^{m} \lambda_i a_{ij} - c_j = \mu_j$$

From the last equation , we know that

$$c_j \leq \sum_{i=1}^{m} \lambda_i a_{ij}$$

Multiply both side by $x_j (\geq 0)$

$$
\begin{aligned}
c_j x_j &\leq \left(\sum_{i=1}^{m} a_{ij} \lambda_i \right) x_j, \quad \forall\, j \\
\Rightarrow \quad \sum_{j=1}^{n} c_j x_j &\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \lambda_i \right) x_j,
\end{aligned}
$$

We can interchange the summation notation

$$
\Rightarrow \quad \sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \lambda_i,
$$

$$\Rightarrow \quad \sum_{j=1}^{n} c_j x_j \quad \leq \quad \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j \right), \text{ but } \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \forall\, i,$$

$$\Rightarrow \quad \sum_{j=1}^{n} c_j x_j \quad \leq \quad \sum_{i=1}^{m} b_i \lambda_i \tag{7}$$

where $\lambda_1, \lambda_2, ..., \lambda_m$ are multipliers called the dual variable ($\lambda_1, \lambda_2, ..., \lambda_m \geq 0$)

$$
\begin{aligned}
\text{Let} \quad y_i &= \lambda_i, \quad i = 1, 2, ..., m \\
y_i &\geq 0, \quad i = 1, 2, ..., m \\
\sum_{j=1}^{n} c_j x_j &\leq \sum_{i=1}^{m} b_i y_i, \quad \lambda_i = y_i \\
\text{Now} \quad \Rightarrow \quad c^T X &\leq b^T Y \\
\Rightarrow \quad f &\leq f' && (8) \\
\text{Also} \quad \max: \quad c^T X &\leq \min: \quad b^T Y && (9) \\
\max: \quad f &\leq \min: f' && (10) \\
\max: \quad f &= \min: f' && (11)
\end{aligned}
$$

(12)

This is called Strong Duality.

Also we have

$$\min: \; b^T Y = \sum_{i=1}^{m} b_i y_i$$

s. to

$$\begin{aligned}
\sum_{i=1}^{m} a_{ij}\lambda_i &\geq c_j \quad j = 1, 2, ..., n, \\
\text{Since} \;\; \lambda_i &= y_i \geq 0 \\
\sum_{i=1}^{m} a_{ij} y_i &\geq c_j \quad j = 1, 2, ..., n
\end{aligned}$$

Finally we have Dual LPP

$$(D) \qquad \min : f' = \sum_{i=1}^{m} b_i y_i$$

s. to

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i &\geq c_j, \quad j = 1, 2, ..., n \\ y_i &\geq 0. \quad i = 1, 2, ..., m \end{aligned}$$

m+n Pairs of Complementary Conditions:

$$(\sum_{j=1}^{n} a_{ij} x_j - b_i) y_i = 0, \quad i = 1, 2, ..., m$$

where

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad y_i \geq 0, \qquad i = 1, 2, ..., m$$

$$(\sum_{i=1}^{m} a_{ij} y_i - c_j) x_j = 0, \quad j = 1, 2, ..., n$$

where

$$\sum_{j=1}^{m} a_{ij} y_i \geq c_j, \qquad x_j \geq 0, \qquad j = 1, 2, ..., n$$

Numerical Example:(a) Type-I Primal LPP:

$$\max : Z = X_1 + 3X_2$$

Subject to

$$X_1 + X_2 \leq 10$$
$$X_1 + 2X_2 \leq 11$$
$$X_1 + 4X_2 \leq 16$$
$$X_1, X_2 \geq 0$$

Optimal Solution:

$$X^* = (6, 5/2), Z^* = 13.5 = 27/2$$

DUAL LPP:

$$\min : z = 10Y_1 + 11Y_2 + 16Y_3$$

Subject to

$$Y_1 + Y_2 + Y_3 \geq 1$$

$$Y_1 + 2Y_2 + 4Y_3 \geq 3$$

$$Y_1, Y_2, Y_3 \geq 0$$

Optimal Solution:

$$Y^* = (0, 1/2, 1/2), Z^* = 27/2$$

1. Please Check $X_1 = 6, X_2 = 2.5$ is a feasible solution of the LPP.
2. Please Check $X_1 = 6, X_2 = 2.5$ is an Optimal solution of the LPP using Duality Theory.

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* Note

- first we find all feasable solutions for P
- using the complimentary conditions, we check which solutions are feasible in D
- among these solutions, the set of variables which give equal D & P values are the solution set

We have five pairs of complimentary conditions:

$$\boxed{\begin{gathered}
(X_1 + X_2 - 10)\,Y_1 = 0 \\
(X_1 + 2X_2 - 11)\,Y_2 = 0 \\
(X_1 + 4X_2 - 16)\,Y_3 = 0 \\
(Y_1 + Y_2 + Y_3 - 1)X_1 = 0 \\
(Y_1 + 2Y_2 + 4Y_3 - 3)X_2 = 0
\end{gathered}}$$

used to calculate x_i's from y_i's & vice versa

$$X_1 = 6, X_2 = 2.5, X_1 + X_2 < 10, Y_1 = 0$$

$$(Y_1 + Y_2 + Y_3 - 1) = 0, (Y_1 + 2Y_2 + 4Y_3 - 3) = 0, \ Y_2 = Y_3 = 1/2$$

$$\text{min} : z = 13.5, \text{max} : Z = 13.5$$

Given solution is an optimal solution.

Numerical Example(b) : Type -I
Primal LPP:

$$\max : Z = 6X_1 + 9X_2 + 6X_3$$

Subject to

$$X_1 + X_2 + X_3 \leq 20$$

$$3X_1 + 3X_2 + 4X_3 \leq 48$$

$$X_1, X_2, X_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,16,0), Z=144.

DUAL LPP:

$$\min : z = 20Y_1 + 48Y_2$$

Subject to

$$Y_1 + 3Y_2 \geq 6$$

$$Y_1 + 3Y_2 \geq 9$$

$$Y_1 + 4Y_2 \geq 6$$

$$Y_1, Y_2 \geq 0.$$

Optimal Solution:

$$Y^* = (0, 3), z^* = 144$$

Numerical Example(c): Type-I
Primal LPP:

$$\max : Z = 3X_1 + X_2 + 2X_3 + X_4$$

Subject to

$$2X_1 + X_2 + X_3 + X_4 \leq 600$$
$$X_1 + X_2 + X_3 + 2X_4 \leq 400$$
$$X_1, X_2, X_3, X_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(200,0,200,0), Z=1000.

DUAL LPP:

$$\min : z = 600Y_1 + 400Y_2$$

Subject to

$$2Y_1 + Y_2 \geq 3$$

$$Y_1 + Y_2 \geq 1$$

$$Y_1 + Y_2 \geq 2$$

$$Y_1 + 2Y_2 \geq 1$$

$$Y_1, Y_2 \geq 0.$$

Optimal Solution:

$$Y^* = (1, 1), z^* = 1000$$

Numerical Example:(d) : Type-I
Primal LPP:

$$\min : Z = 2X_1 + 3X_2 + 2X_3 + X_4 + X_5$$

Subject to

$$3X_1 - 3X_2 + 4X_3 + 2X_4 - X_5 \geq 150$$

$$X_1 + X_2 + X_3 + X_4 + X_5 \geq 300$$

$$X_1, X_2, X_3, X_4, X_5 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,0,0,300,0), Z=300.

DUAL LPP:

$$\max : z = 150Y_1 + 300Y_2$$

Subject to

$$3Y_1 + Y_2 \leq 2$$

$$-3Y_1 + Y_2 \leq 3$$

$$4Y_1 + Y_2 \leq 2$$

$$2Y_1 + Y_2 \leq 1$$

$$-Y_1 + Y_2 \leq 1$$

$$Y_1, Y_2 \geq 0.$$

Optimal Solution:

$$Y^* = (0, 1), z^* = 300$$

Primal(P) and Dual(D) LPP: Type-II

equality constraints

$$(P) \qquad \max : f = c^T X$$

$$\text{Subject to } AX = b, X \geq 0$$

$$\text{where } c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m \times n}$$

In expanded form, it can be written as:

$$(P) \qquad \max : f = \sum_{j=1}^{n} c_j x_j$$

Subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &= b_i, \quad i = 1, 2, ..., m \\ x_j &\geq 0. \quad j = 1, 2, ..., n \end{aligned}$$

Dual of the Primal LPP:

$$(D) \qquad \min : f' = b^T Y$$

$$\text{s. t.} \quad A^T Y \geq c, Y \quad \textit{is free}.$$

$$\text{where } Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

In expanded form, it can be written as:

$$(D) \qquad \min : f' = \sum_{i=1}^{m} b_i y_i$$

s. t.

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad j = 1, 2, ..., n$$

$$y_i \quad is \quad free, \quad i = 1, 2, ..., m$$

How to find Dual of a LPP ?
We will use Lagrange Multipliers Method to establish the Dual. Details are given below:

$$(P) \qquad \max : f = \sum_{j=1}^{n} c_j x_j$$

$$\Rightarrow \quad \min : -f = -\sum_{j=1}^{n} c_j x_j$$

s. t.

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, ..., m \tag{13}$$

$$x_j \geq 0. \quad j = 1, 2, ..., n$$

Slack variable are not needed in an equation.

$$\begin{aligned} \Rightarrow \quad \sum_{j=1}^{n} a_{ij}x_j - b_i &= 0, \quad i = 1, 2, \ldots, m \qquad (14) \\ \text{where} \quad x_j &\geq 0, \quad j = 1, 2, \ldots, n \\ \Rightarrow \quad -x_j &\leq 0, \quad j = 1, 2, \ldots, n \\ -x_j + t_j^2 &= 0, \quad j = 1, 2, \ldots, n \\ \text{Another slack variable} \quad t_j^2 &\geq 0, \quad j = 1, 2, \ldots, n \end{aligned}$$

Let $L(\ldots)$ be the Lagrange Function.

$$L(X, T, \lambda, \mu) = -\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) + \sum_{j=1}^{n} \mu_j \left(-x_j + t_j^2 \right)$$

$$\begin{aligned} \text{where} \quad \lambda_1, \lambda_2, \ldots, \lambda_m \quad & \textit{are free}. \\ \mu_1, \mu_2, \ldots, \mu_n \quad & \geq \quad 0 \\ x_1, x_2, \ldots, x_n \quad & \geq \quad 0 \\ t_1^2, t_2^2, \ldots, t_n^2 \quad & \geq \quad 0 \end{aligned}$$

All the Lagrange multipliers $\lambda_i, \forall\ i$ are free and $\mu_j, \forall\ j$ are non-negative. Total number of variables are $m + 3n$. There are $m + n$ number of Lagrange multipliers. Using the Necessary Conditions for the minimum we have:

$$
\begin{aligned}
\frac{\partial L}{\partial x_j} &= 0, \quad j = 1, 2, \dots, n \\
\Rightarrow \quad -c_j + \sum_{i=1}^{m} \lambda_i a_{ij} - \mu_j &= 0, \quad j = 1, 2, \dots, n \\
\Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} - c_j &= \mu_j, \ \text{but} \ \ \mu_j \geq 0 \\
\Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} - c_j &\geq 0, \quad j = 1, 2, \dots, n \\
\Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} &\geq c_j, \quad j = 1, 2, \dots, n \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \lambda_i} &= 0, \quad i = 1, 2, ..., m \\
\sum_{j=1}^{n} a_{ij} x_j - b_i &= 0, \\
\Rightarrow \quad \sum_{j=1}^{n} a_{ij} x_j &= b_i, \quad i = 1, 2, ..., m
\end{aligned}
$$

$$\Rightarrow \quad \lambda_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) = 0, \quad i = 1, 2, \ldots, m$$

$$\Rightarrow \quad \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) = 0, \quad i = 1, 2, \ldots, m$$

$$\frac{\partial L}{\partial \mu_j} = 0, \quad j = 1, 2, \ldots, n$$

$$\Rightarrow \quad - x_j + t_j^2 = 0, \quad j = 1, 2, \ldots, n$$

$$\Rightarrow \quad x_j = t_j^2, \quad j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial t_j} = 0, \quad j = 1, 2, ..., n$$

$$2\mu_j t_j = 0, \quad j = 1, 2, ..., n$$

$$\Rightarrow \quad \mu_j t_j = 0, \quad j = 1, 2, ..., n$$

So $\mu_j t_j^2 = 0, \quad j = 1, 2, ..., n$

From the last equation $t_j^2 = x_j$

$$\mu_j x_j = 0, \quad j = 1, 2, ..., n \tag{16}$$

where

$$\sum_{i=1}^{m} \lambda_i a_{ij} - c_j = \mu_j$$

From the last equation, we know that

$$c_j \leq \sum_{i=1}^{m} \lambda_i a_{ij}$$

Multiply both side by $x_j (\geq 0)$

$$
\begin{aligned}
c_j x_j &\leq \left(\sum_{i=1}^{m} a_{ij}\lambda_i\right) x_j, \quad \forall\, j \\
\Rightarrow \quad \sum_{j=1}^{n} c_j x_j &\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij}\lambda_i\right) x_j,
\end{aligned}
$$

We can interchange the summation notation

$$
\Rightarrow \quad \sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j\right) \lambda_i,
$$

$$\Rightarrow \quad \sum_{j=1}^{n} c_j x_j \quad \leq \quad \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j \right), \text{ but } \quad \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad \forall\ i,$$

$$\Rightarrow \quad \sum_{j=1}^{n} c_j x_j \quad \leq \quad \sum_{i=1}^{m} b_i \lambda_i \tag{17}$$

where $\lambda_1, \lambda_2, ..., \lambda_m$ are multipliers called the dual variable.

$$
\begin{aligned}
\text{Let} \quad y_i &= \lambda_i, \quad i = 1, 2, ..., m \\
\sum_{j=1}^{n} c_j x_j &\leq \sum_{i=1}^{m} b_i y_i, \quad \lambda_i = y_i \\
\text{Now} \quad \Rightarrow \quad c^T X &\leq b^T Y \\
\Rightarrow \quad f &\leq f' \qquad (18) \\
\text{Also} \quad \max: \quad c^T X &\leq \min: \quad b^T Y \qquad (19) \\
\max: \quad f &\leq \min: f' \qquad (20) \\
\max: \quad f &= \min: f' \qquad (21) \\
& \qquad (22)
\end{aligned}
$$

This is called Strong Duality.

Also we have

$$\min:\ b^T Y = \sum_{i=1}^{m} b_i y_i$$

s. to

$$\begin{aligned} \sum_{i=1}^{m} a_{ij}\lambda_i &\geq c_j \quad j = 1, 2, \ldots, n, \\ \text{Since } \lambda_i &= y_i \geq 0 \\ \sum_{i=1}^{m} a_{ij} y_i &\geq c_j \quad j = 1, 2, \ldots, n \end{aligned}$$

Finally we have Dual LPP

$$(D) \qquad \min : f' = \sum_{i=1}^{m} b_i y_i$$

s. to

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad j = 1, 2, ..., n$$

y_i *are free*

m+n Pairs of Complementary Conditions:

$$(\sum_{j=1}^{n} a_{ij}x_j - b_i)y_i = 0, \quad i = 1, 2, ..., m$$

where

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad y_i \quad is \quad free, \qquad i = 1, 2, ..., m$$

$$(\sum_{i=1}^{m} a_{ij}y_i - c_j)x_j = 0, \quad j = 1, 2, ..., n$$

where

$$\sum_{j=1}^{m} a_{ij}y_i \geq c_j, \qquad x_j \geq 0, \qquad j = 1, 2, ..., n$$

Numerical Example (a): Type-II

Primal LPP:

$$\max : Z = 2X_1 + 9X_2 + 11X_3$$

Subject to

$$X_1 + 4X_2 + 5X_3 = 100$$

$$X_1 + 3X_2 + 4X_3 = 90$$

$$X_1, X_2, X_3 \quad \geq 0$$

Optimal Solution:

$$X^* = \underbrace{(60, 10, 0), (50, 0, 10)}, Z^* = 210$$

∞ solutions present

extreme pts of all optimal solutions

as BFS occurs at extreme pts only

DUAL LPP:

$$\min : z = 100Y_1 + 90Y_2$$

Subject to

$$Y_1 + Y_2 \geq 2$$

$$4Y_1 + 3Y_2 \geq 9$$

$$5Y_1 + 4Y_2 \geq 11$$

$$Y_1, Y_2 \quad \textit{are} \quad \textit{free}.$$

as y_1 & y_2 are free
replace y_1 with
$y_3 - y_4$, & y_2 with
$y_5 - y_6$ st
$y_3, y_4, y_5, y_6 \geq 0$
Then apply
Big M or 2phase
Simplex to
solve

Optimal Solution:

$$Y^* = (3, -1), z^* = 210$$

Numerical Example(b) : Type-II
Primal LPP:

$$\max : Z = 3X_1 + 9X_2 + 15X_3$$

Subject to

$$X_1 + 2X_2 + 4X_3 = 8$$
$$2X_1 + X_2 + 5X_3 = 7$$
$$X_1, X_2, X_3 \quad \geq 0$$

Optimal Solution:

$$X^* = (0, 2, 1), (2, 3, 0), Z^* = 33$$

DUAL LPP:

$$\min : z = 8Y_1 + 7Y_2$$

Subject to

$$Y_1 + 2Y_2 \geq 3$$

$$2Y_1 + Y_2 \geq 9$$

$$4Y_1 + 5Y_2 \geq 15$$

$$Y_1, Y_2 \quad are \quad free.$$

Optimal Solution:

$$Y^* = (5, -1), z^* = 33$$

Numerical Example(c) : Type-II
Primal LPP:

$$\max : Z = 9X_1 + 16X_2 + 24X_3$$

Subject to

$$2X_1 + 3X_2 + 5X_3 = 20$$
$$3X_1 + 2X_2 + 6X_3 = 15$$
$$X_1, X_2, X_3 \quad \geq 0$$

Optimal Solution:

$$X^* = (1, 6, 0), (0, 45/8, 5/8), Z^* = 105$$

DUAL LPP:

$$\min : z = 20Y_1 + 15Y_2$$

Subject to

$$2Y_1 + 3Y_2 \geq 9$$

$$3Y_1 + 2Y_2 \geq 16$$

$$5Y_1 + 6Y_2 \geq 24$$

$$Y_1, Y_2 \quad are \quad free.$$

Optimal Solution:

$$Y^* = (6, -1), z^* = 105$$

Numerical Example(d) : Type-II
Primal LPP:

$$\max : Z = 15X_1 + 20X_2 + 36X_3$$

Subject to

$$2X_1 + 3X_2 + 5X_3 = 20$$
$$3X_1 + 2X_2 + 6X_3 = 15$$
$$X_1, X_2, X_3 \quad \geq 0$$

Optimal Solution:

$$X^* = (1, 6, 0), (0, 45/8, 5/8), Z^* = 135$$

DUAL LPP:

$$\min : z = 20Y_1 + 15Y_2$$

Subject to

$$2Y_1 + 3Y_2 \geq 15$$

$$3Y_1 + 2Y_2 \geq 20$$

$$5Y_1 + 6Y_2 \geq 36$$

$$Y_1, Y_2 \quad are \quad free.$$

Optimal Solution:

$$Y^* = (6, 1), z^* = 135$$

Primal(P) and Dual(D) LPP: Type-III

$$(P) \qquad \max : f = c^T X$$

$$\text{Subject to} \quad AX \leq b, X \quad is \quad free.$$

$$\text{where } c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m \times n}$$

In expanded form, it can be written as:

$$(P) \qquad \max : f = \sum_{j=1}^{n} c_j x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \quad \leq \quad b_i, \quad i = 1, 2, ..., m$$

$$x_j \quad \text{is free} \quad , j = 1, 2, ..., n$$

Dual of the Primal LPP:

$$(D) \qquad \min : f' = b^T Y$$

$$\text{s. t.} \;\; A^T Y = c, Y \geq 0$$

$$\text{where } Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

In expanded form, it can be written as:

$$(D) \qquad \min : f' = \sum_{i=1}^{m} b_i y_i$$

s. t.

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i &= c_j, \quad j = 1, 2, ..., n \\ y_i &\geq 0. \quad i = 1, 2, ..., m \end{aligned}$$

How to find Dual of a LPP ?
We will use Lagrange Multipliers Method to establish the Dual. Details are given below:

$$
\begin{aligned}
(P) \qquad \max : f &= \sum_{j=1}^{n} c_j x_j \\
\Rightarrow \quad \min : -f &= -\sum_{j=1}^{n} c_j x_j
\end{aligned}
$$

s. t.

$$
\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, ..., m \tag{23}
$$

$$
x_j \quad is \quad free \quad , j = 1, 2, ..., n
$$

Add a slack variable to (31)

$$\begin{aligned} s_i^2 &\geq 0, \quad i = 1, 2, ..., m \\ \Rightarrow \quad \sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i &= 0, \quad i = 1, 2, ..., m \end{aligned}$$

Let $L(\ldots)$ be the Lagrange Function.

$$L(X, S, \lambda) = -\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i \right)$$

$$\begin{aligned} \text{where} \quad \lambda_1, \lambda_2, \ldots, \lambda_m &\geq 0 \\ x_1, x_2, \ldots, x_n \textit{ are free.} & \\ s_1^2, s_2^2, \ldots, s_m^2 &\geq 0 \end{aligned}$$

All the Lagrange multipliers $\lambda_i, \forall\, i$ are non-negative. Total number of variables are $2m + n$. There are m number of Lagrange multipliers. Using the Necessary Conditions for the minimum we have:

$$\begin{aligned} \frac{\partial L}{\partial x_j} &= 0, \quad j = 1, 2, ..., n \\ \Rightarrow \quad -c_j + \sum_{i=1}^{m} \lambda_i a_{ij} &= 0, \quad j = 1, 2, ..., n \\ \Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} - c_j &= 0, \\ \Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} &= c_j, \quad j = 1, 2, ..., n \end{aligned} \tag{24}$$

$$
\begin{aligned}
\frac{\partial L}{\partial \lambda_i} &= 0, \quad i = 1, 2, ..., m \\
\sum_{j=1}^{n} a_{ij} x_j + s_i^2 - b_i &= 0, \;\; \text{but} \;\; s_i^2 \geq 0 \\
\Rightarrow \quad \sum_{j=1}^{n} a_{ij} x_j &\leq b_i, \quad i = 1, 2, ..., m \\
\frac{\partial L}{\partial s_i} &= 0, \quad i = 1, 2, ..., m \\
\Rightarrow \quad 2 s_i \lambda_i &= 0, \quad i = 1, 2, ..., m \\
\Rightarrow \quad s_i \lambda_i &= 0, \quad i = 1, 2, ..., m \\
\Rightarrow \quad s_i^2 \lambda_i &= 0, \quad i = 1, 2, ..., m
\end{aligned}
$$

$$\Rightarrow \quad \lambda_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) = 0, \quad i = 1, 2, \ldots, m$$

$$\Rightarrow \quad \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) = 0, \quad i = 1, 2, \ldots, m$$

We know

$$\sum_{i=1}^{m} \lambda_i a_{ij} - c_j = 0$$

From the last equation, we may write

$$c_j = \sum_{i=1}^{m} \lambda_i a_{ij}$$

Multiply both side by x_j

$$\begin{aligned} c_j x_j &= \left(\sum_{i=1}^{m} a_{ij}\lambda_i\right) x_j, \quad \forall\, j \\ \Rightarrow \quad \sum_{j=1}^{n} c_j x_j &= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij}\lambda_i\right) x_j, \end{aligned}$$

We can interchange the summation notation

$$\Rightarrow \quad \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j\right) \lambda_i,$$

$$\begin{aligned}
\Rightarrow \quad \sum_{j=1}^{n} c_j x_j \quad &= \quad \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j \right), \text{ but } \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad \forall\, i, \\
\Rightarrow \quad \sum_{j=1}^{n} c_j x_j \quad &\le \quad \sum_{i=1}^{m} b_i \lambda_i \qquad (25)
\end{aligned}$$

where $\lambda_1, \lambda_2, ..., \lambda_m$ are multipliers called the dual variable $(\lambda_1, \lambda_2, ..., \lambda_m \ge 0)$

$$
\begin{aligned}
\text{Let} \quad y_i &= \lambda_i, \quad i = 1, 2, ..., m \\
y_i &\geq 0, \quad i = 1, 2, ..., m \\
\sum_{j=1}^{n} c_j x_j &\leq \sum_{i=1}^{m} b_i y_i, \quad \lambda_i = y_i \\
\text{Now} \quad \Rightarrow \quad c^T X &\leq b^T Y \\
\Rightarrow \quad f &\leq f' && (26) \\
\text{Also} \quad \max: \quad c^T X &\leq \min: \quad b^T Y && (27) \\
\max: \quad f &\leq \min: f' && (28) \\
\max: \quad f &= \min: f' && (29) \\
& && (30)
\end{aligned}
$$

This is called Strong Duality.

Also we have

$$\min: \; b^T Y = \sum_{i=1}^{m} b_i y_i$$

s. to

$$\begin{aligned} \sum_{i=1}^{m} a_{ij}\lambda_i &= c_j \quad j = 1, 2, ..., n, \\ \text{Since} \;\; \lambda_i &= y_i \geq 0 \\ \sum_{i=1}^{m} a_{ij} y_i &= c_j \quad j = 1, 2, ..., n \end{aligned}$$

Finally we have Dual LPP

$$(D) \qquad \min : f' = \sum_{i=1}^{m} b_i y_i$$

s. to

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i &= c_j, \quad j = 1, 2, ..., n \\ y_i &\geq 0. \quad i = 1, 2, ..., m \end{aligned}$$

m+n Pairs of Complementary Conditions:

$$(\sum_{j=1}^{n} a_{ij}x_j - b_i)y_i = 0, \quad i = 1, 2, ..., m$$

where

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \qquad y_i \geq 0, \qquad i = 1, 2, ..., m$$

$$(\sum_{i=1}^{m} a_{ij}y_i - c_j)x_j = 0, \quad j = 1, 2, ..., n$$

where

$$\sum_{j=1}^{m} a_{ij}y_i = c_j, \qquad x_j \quad is \quad free, \qquad j = 1, 2, ..., n$$

Numerical Example(a) : Type-III
Primal LPP:

$$\max : Z = 8X_1 + 21X_2 + 29X_3$$

Subject to

$$X_1 + 3X_2 + 4X_3 \leq 7$$

$$X_1 + 2X_2 + 3X_3 \leq 8$$

$$X_1, X_2, X_3 \quad \textit{are} \quad \textit{free.}$$

as x_1, x_2, x_3 are free, we have $=$ constraints in D

Optimal Solution:

$$X^* = (0, -11, 10), (10, -1, 0), (11, 0, -1), Z^* = 59$$

DUAL LPP:

$$\min : z = 7Y_1 + 8Y_2$$

Subject to

$$Y_1 + Y_2 = 8$$

$$3Y_1 + 2Y_2 = 21$$

$$4Y_1 + 3Y_2 = 29$$

$$Y_1, Y_2 \geq 0$$

Optimal Solution:

$$Y^* = (5, 3), z^* = 59$$

Numerical Example(b) : Type-III
Primal LPP:

$$\max : Z = 7X_1 + 11X_2 + 25X_3$$

Subject to

$$X_1 + 2X_2 + 4X_3 \leq 8$$

$$2X_1 + X_2 + 5X_3 \leq 7$$

$$X_1, X_2, X_3 \quad \textit{are} \quad \textit{free}$$

Optimal Solution:

$$X^* = (0, 2, 1), (2, 3, 0), (-4, 0, 3), Z^* = 47$$

DUAL LPP:

$$\min : z = 8Y_1 + 7Y_2$$

Subject to

$$Y_1 + 2Y_2 = 7$$

$$2Y_1 + Y_2 \overset{=}{\cancel{\geq}} 11$$

$$4Y_1 + 5Y_2 \overset{=}{\cancel{\geq}} 25$$

$$Y_1, Y_2 \geq 0$$

Optimal Solution:

$$Y^* = (5, 1), z^* = 47$$

Numerical Example(c) : Type-III

Primal LPP:

$$\max : Z = 15X_1 + 20X_2 + 36X_3$$

Subject to

$$2X_1 + 3X_2 + 5X_3 \overset{=}{\cancel{\leq}} 20$$

$$3X_1 + 2X_2 + 6X_3 \overset{=}{\cancel{\leq}} 15$$

$$X_1, X_2, X_3 \quad \textit{are} \quad \textit{free}$$

Optimal Solution:

$$X^* = (1, 6, 0), (0, 45/8, 5/8), (-15, 0, 10), Z^* = 135$$

DUAL LPP:

$$\min : z = 20Y_1 + 15Y_2$$

Subject to

$$2Y_1 + 3Y_2 \geq 15$$

$$3Y_1 + 2Y_2 \geq 20$$

$$5Y_1 + 6Y_2 \geq 36$$

$$Y_1, Y_2 \geq 0.$$

Optimal Solution:

$$Y^* = (6, 1), z^* = 135$$

Primal(P) and Dual(D) LPP: Type-IV

$$(P) \qquad \max : f = c^T X$$

$$\text{Subject to} \quad AX = b, X \quad is \quad free.$$

$$\text{where } c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \ b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m \times n}$$

In expanded form, it can be written as:

$$(P) \qquad \max : f = \sum_{j=1}^{n} c_j x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \quad = \quad b_i, \quad i = 1, 2, ..., m$$

$$x_j \quad \textit{is free} \quad , j = 1, 2, ..., n$$

Dual of the Primal LPP:

$$(D) \qquad \min : f^{'} = b^T Y$$

$$\text{s. t. } A^T Y = c, Y \;\; is \;\; free.$$

$$\text{where } Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

In expanded form, it can be written as:

$$(D) \qquad \min : f' = \sum_{i=1}^{m} b_i y_i$$

s. t.

$$\sum_{i=1}^{m} a_{ij} y_i = c_j, \quad j = 1, 2, ..., n$$

$$y_i \quad is \quad free, \qquad i = 1, 2, ..., m$$

How to find Dual of a LPP ?
We will use Lagrange Multipliers Method to establish the Dual. Details are given below:

$$
\begin{aligned}
(P) \qquad \max : f &= \sum_{j=1}^{n} c_j x_j \\
\Rightarrow \quad \min : -f &= -\sum_{j=1}^{n} c_j x_j
\end{aligned}
$$

s. t.

$$
\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, \ldots, m \tag{31}
$$

$$
x_j \quad is \quad free \quad , j = 1, 2, \ldots, n
$$

Let $L(\ldots)$ be the Lagrange Function.

$$L(X,\lambda) = -\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right)$$

$$\text{where} \quad \lambda_1, \lambda_2, \ldots, \lambda_m \quad \textit{are} \quad \textit{free},$$

$$x_1, x_2, \ldots, x_n \ \textit{are free}.$$

All the Lagrange multipliers $\lambda_i, \forall\ i$ are free. Total number of variables are $m+n$. There are m number of Lagrange multipliers. Using the Necessary Conditions for the minimum we have:

$$
\begin{aligned}
\frac{\partial L}{\partial x_j} &= 0, \quad j = 1, 2, \dots, n \\
\Rightarrow \quad -c_j + \sum_{i=1}^{m} \lambda_i a_{ij} &= 0, \quad j = 1, 2, \dots, n \\
\Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} - c_j &= 0, \\
\Rightarrow \quad \sum_{i=1}^{m} \lambda_i a_{ij} &= c_j, \quad j = 1, 2, \dots, n
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \lambda_i} &= 0, \quad i = 1, 2, \ldots, m \\
\Rightarrow \quad \sum_{j=1}^{n} a_{ij} x_j &= b_i, \quad i = 1, 2, \ldots, m
\end{aligned}
$$

$$
\Rightarrow \quad \lambda_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) = 0, \quad i = 1, 2, \ldots, m
$$

$$
\Rightarrow \quad \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) = 0, \quad i = 1, 2, \ldots, m
$$

We know that

$$\sum_{i=1}^{m} \lambda_i a_{ij} - c_j = 0$$

From the last equation, we may write

$$c_j = \sum_{i=1}^{m} \lambda_i a_{ij}$$

Multiply both side by x_j

$$\begin{aligned} c_j x_j &= \left(\sum_{i=1}^{m} a_{ij}\lambda_i\right) x_j, \quad \forall\, j \\ \Rightarrow \quad \sum_{j=1}^{n} c_j x_j &= \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\lambda_i\right) x_j, \end{aligned}$$

We can interchange the summation notation

$$\Rightarrow \quad \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij} x_j\right) \lambda_i,$$

$$\Rightarrow \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j \right), \text{ but } \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad \forall\, i,$$

$$\Rightarrow \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i \lambda_i \tag{33}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are multipliers called the dual variable ($\lambda_1, \lambda_2, \ldots, \lambda_m$ *are free*)

$$
\begin{aligned}
\text{Let} \quad y_i &= \lambda_i, \quad i = 1, 2, \ldots, m \\
y_i \quad is \quad free, \quad i = 1, 2, \ldots, m \\
\sum_{j=1}^{n} c_j x_j &= \sum_{i=1}^{m} b_i y_i, \\
\text{Now} \quad \Rightarrow \quad c^T X &= b^T Y \\
\Rightarrow \quad f &= f' && (34) \\
\text{Also} \quad \max : \quad c^T X &= \max : \quad b^T Y && (35) \\
\max : \quad f &\leq \min : f' && (36) \\
\max : \quad f &= \min : f' && (37) \\
& && (38)
\end{aligned}
$$

This is called Strong Duality.

Also we have

$$\min: \; b^T Y = \sum_{i=1}^{m} b_i y_i$$

s. to

$$\begin{aligned} \sum_{i=1}^{m} a_{ij}\lambda_i &= c_j \quad j = 1, 2, ..., n, \\ \text{Since } \lambda_i &= y_i \\ \sum_{i=1}^{m} a_{ij} y_i &= c_j \quad j = 1, 2, ..., n \end{aligned}$$

Finally we have Dual LPP

$$(D) \qquad \min : f^{'} = \sum_{i=1}^{m} b_i y_i$$

s. to

$$\sum_{i=1}^{m} a_{ij} y_i \quad = \quad c_j, \quad j = 1, 2, ..., n$$

$$y_i \quad is \quad free. \quad i = 1, 2, ..., m$$

m+n Pairs of Complementary Conditions:

$$(\sum_{j=1}^{n} a_{ij}x_j - b_i)y_i = 0, \quad i = 1, 2, ..., m$$

where

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad y_i \quad is \quad free, \quad i = 1, 2, ..., m$$

$$(\sum_{i=1}^{m} a_{ij}y_i - c_j)x_j = 0, \quad j = 1, 2, ..., n$$

where

$$\sum_{j=1}^{m} a_{ij}y_i = c_j, \qquad x_j \quad is \quad free, \ j = 1, 2, ..., n$$

Numerical Example(a) : Type-IV
Primal LPP:

$$\max : Z = 2X_1 + 9X_2 + 11X_3$$

Subject to

$$X_1 + 4X_2 + 5X_3 = 10$$

$$X_1 + 3X_2 + 4X_3 = 9$$

$$X_1, X_2, X_3 \quad \textit{are} \quad \textit{free}$$

y_1, y_2 are free in D

= constraints in D

Optimal Solution:

$$X^* = (6, 1, 0), (5, 0, 1), (0, -5, 6), Z^* = 21$$

DUAL LPP:

$$\min : z = 10Y_1 + 9Y_2$$

Subject to

$$Y_1 + Y_2 = 2$$

$$4Y_1 + 3Y_2 = 9$$

$$5Y_1 + 4Y_2 = 11$$

$$Y_1, Y_2, Y_3 \quad are \quad free.$$

Optimal Solution:

$$Y^* = (3, -1), z^* = 21$$

Numerical Example(b) : Type-IV
Primal LPP:

$$\max : Z = 3X_1 + 9X_2 + 15X_3$$

Subject to

$$X_1 + 2X_2 + 4X_3 = 8$$
$$2X_1 + X_2 + 5X_3 = 7$$
$$X_1, X_2, X_3 \quad \textit{are} \quad \textit{free}.$$

Optimal Solution:

$$X^* = (0, 2, 1), (2, 3, 0), (-4, 0, 3), Z^* = 33$$

DUAL LPP:

$$\min : z = 8Y_1 + 7Y_2$$

Subject to

$$Y_1 + 2Y_2 = 3$$

$$2Y_1 + Y_2 = 9$$

$$4Y_1 + 5Y_2 = 15$$

$$Y_1, Y_2 \quad are \quad free.$$

Optimal Solution:

$$Y^* = (5, -1), z^* = 33$$

Numerical Example(c) : Type-IV
Primal LPP:

$$\max : Z = 9X_1 + 16X_2 + 24X_3$$

Subject to

$$2X_1 + 3X_2 + 5X_3 = 20$$

$$3X_1 + 2X_2 + 6X_3 = 15$$

$$X_1, X_2, X_3 \quad are \quad free.$$

Optimal Solution:

$$X^* = (1, 6, 0), (0, 45/8, 5/8), (-15, 0, 10), Z^* = 105$$

DUAL LPP:

$$\min : z = 20Y_1 + 15Y_2$$

Subject to

$$2Y_1 + 3Y_2 = 9$$

$$3Y_1 + 2Y_2 = 16$$

$$5Y_1 + 6Y_2 = 24$$

$$Y_1, Y_2 \quad \textit{are} \quad \textit{free}.$$

Optimal Solution:

$$Y^* = (6, -1), z^* = 105$$

Numerical Example(d) : Type-IV
Primal LPP:

$$\max : Z = 15X_1 + 20X_2 + 36X_3$$

Subject to

$$2X_1 + 3X_2 + 5X_3 = 20$$

$$3X_1 + 2X_2 + 6X_3 = 15$$

$$X_1, X_2, X_3 \quad are \;\; free.$$

Optimal Solution:

$$X^* = (1, 6, 0), (0, 45/8, 5/8), (-15, 0, 10), Z^* = 135$$

DUAL LPP:

$$\min : z = 20Y_1 + 15Y_2$$

Subject to

$$2Y_1 + 3Y_2 = 15$$

$$3Y_1 + 2Y_2 = 20$$

$$5Y_1 + 6Y_2 = 36$$

$$Y_1, Y_2 \quad are \quad free.$$

Optimal Solution:

$$Y^* = (6, 1), z^* = 135$$

Numerical Example:1
Primal LPP:

$$\max : Z = X_1 + 3X_2$$

Subject to

$$X_1 + X_2 \leq 10$$

$$X_1 + 2X_2 \leq 11$$

$$X_1 + 4X_2 \leq 16$$

$$X_1, X_2 \geq 0$$

DUAL LPP:

$$\min : z = 10Y_1 + 11Y_2 + 16Y_3$$

Subject to

$$Y_1 + Y_2 + Y_3 \geq 1$$

$$Y_1 + 2Y_2 + 4Y_3 \geq 3$$

$$Y_1, Y_2, Y_3 \geq 0$$

1. Please Check $X_1 = 6, X_2 = 2.5$ is a feasible solution of the LPP.
2. Please Check $X_1 = 6, X_2 = 2.5$ is an Optimal solution of the LPP using Duality Theory.

We have five pairs of complimentary conditions:

$$(X_1 + X_2 - 10)Y_1 = 0$$

$$(X_1 + 2X_2 - 11)Y_2 = 0$$

$$(X_1 + 4X_2 - 16)Y_3 = 0$$

$$(Y_1 + Y_2 + Y_3 - 1)X_1 = 0$$

$$(Y_1 + 2Y_2 + 4Y_3 - 3)X_2 = 0$$

$$X_1 = 6, X_2 = 2.5, X_1 + X_2 < 10, Y_1 = 0$$

$$(Y_1+Y_2+Y_3-1) = 0, (Y_1+2Y_2+4Y_3-3) = 0, Y_2 = Y_3 = 1/2$$

$$\text{min} : z = 13.5, \text{max} : Z = 13.5$$

Given solution is an optimal solution.

Numerical Example:2

For all the LPP find the Primal and the Dual Vriables for the Optimal Solution.

Primal LPP:

$$\min : Z = 2X_1 + 6X_2$$

Subject to

$$X_1 + X_2 \geq 10$$

$$X_1 + 2X_2 \geq 11$$

$$X_1 + 4X_2 \geq 16$$

$$X_1, X_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory. Check X=(8,2), Z=28.

Numerical Example:3
Primal LPP:

$$\max : Z = X_1 + 4X_2$$

Subject to

$$X_1 + X_2 \leq 10$$

$$X_1 + 2X_2 \leq 11$$

$$X_1 + 4X_2 \leq 16$$

$$X_1, X_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory. Check X=(0,4), Z=16.

Numerical Example:4
Primal LPP:

$$\min : Z = X_1 + 4X_2$$

Subject to

$$X_1 + X_2 \geq 10$$

$$X_1 + 2X_2 \geq 11$$

$$X_1 + 4X_2 \geq 16$$

$$X_1, X_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory. Check X=(8,2), Z=16.

Numerical Example: 5
Primal LPP:

$$\max : Z = X_1 + 6X_2$$

Subject to

$$X_1 + X_2 \leq 10$$

$$X_1 + 2X_2 \leq 12$$

$$X_1 + 4X_2 \leq 16$$

$$X_1 + 6X_2 \leq 20$$

$$X_1, X_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory. Check X=(8,2),(0,10/3), Z=20.

Numerical Example: 6
Primal LPP:

$$\min : Z = X_1 + 8X_2$$

Subject to

$$X_1 + X_2 \geq 10$$

$$X_1 + 2X_2 \geq 12$$

$$X_1 + 4X_2 \geq 16$$

$$X_1 + 6X_2 \geq 20$$

$$X_1, X_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(20,0), Z=20.

Numerical Example: 7
Primal LPP:

$$\max : Z = X_1 + 10X_2$$

Subject to

$$X_1 + X_2 \leq 10$$
$$X_1 + 2X_2 \leq 12$$
$$X_1 + 4X_2 \leq 16$$
$$X_1 + 6X_2 \leq 20$$
$$X_1, X_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,10/3), Z=100/3.

Numerical Example: 8
Primal LPP:

$$\min : Z = X_1 + 12X_2$$

Subject to

$$X_1 + X_2 \geq 10$$

$$X_1 + 2X_2 \geq 12$$

$$X_1 + 4X_2 \geq 16$$

$$X_1 + 6X_2 \geq 20$$

$$X_1, X_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(20,0), Z=20.

Numerical Example: 9
Primal LPP:

$$\max : Z = 6X_1 + 6X_2$$

Subject to

$$X_1 + X_2 \leq 10$$

$$X_1 + 2X_2 \leq 12$$

$$X_1 + 4X_2 \leq 16$$

$$X_1 + 8X_2 \leq 24$$

$$X_1, X_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(8,2),(10,0), Z=60.

Numerical Example: 10
Primal LPP:

$$\max : Z = 20X_1 + 50X_2$$

Subject to

$$3X_1 + 2X_2 \leq 25$$

$$2X_1 + 5X_2 \leq 30$$

$$2X_1 + 3X_2 \leq 20$$

$$X_1, X_2 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(5/2, 5),(0,6), Z=300.

Numerical Example: 11
Primal LPP:

$$\max : Z = 6X_1 + 6X_2 + 8X_3$$

Subject to

$$X_1 + X_2 + X_3 \leq 12$$

$$3X_1 + 3X_2 + 4X_3 \leq 36$$

$$X_1, X_2, X_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(6,6,0),(0,0,9), Z=72.

Numerical Example: 12
Primal LPP:

$$\min : Z = 6X_1 + 6X_2 + 8X_3$$

Subject to

$$X_1 + X_2 + X_3 \geq 12$$

$$3X_1 + 2X_2 + 3X_3 \geq 30$$

$$X_1, X_2, X_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(6,6,0),(12,0,0), Z=72.

Numerical Example: 13
Primal LPP:

$$\min : Z = 8X_1 + 8X_2 + 6X_3$$

Subject to

$$X_1 + X_2 + X_3 \geq 18$$

$$4X_1 + 4X_2 + 3X_3 \geq 60$$

$$X_1, X_2, X_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(6,0,12),(0,6,12), Z=120.

Numerical Example: 14
Primal LPP:

$$\max : Z = 6X_1 + 6X_2 + 8X_3$$

Subject to

$$X_1 + X_2 + X_3 \leq 12$$

$$3X_1 + 2X_2 + 4X_3 \leq 40$$

$$X_1, X_2, X_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,4,8), Z=88.

Numerical Example: 15
Primal LPP:

$$\max : Z = 6X_1 + 9X_2 + 6X_3$$

Subject to

$$X_1 + X_2 + X_3 \leq 20$$

$$3X_1 + 3X_2 + 4X_3 \leq 48$$

$$X_1, X_2, X_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,16,0), Z=144.

Numerical Example: 16
Primal LPP:

$$\max : Z = X_1 + X_2 + X_3 + 3X_4$$

Subject to

$$X_1 - X_2 + X_3 + 5X_4 = 5$$

$$2X_1 + 3X_2 - 2X_3 + 4X_4 = 6$$

$$X_1, X_2, X_3, X_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,16,21,0), Z=37.

Numerical Example: 17
Primal LPP:

$$\min : Z = X_1 + X_2 + X_3 + 3X_4$$

Subject to

$$X_1 - X_2 + X_3 + 5X_4 = 10$$

$$2X_1 + 3X_2 - 2X_3 + 4X_4 = 12$$

$$X_1, X_2, X_3, X_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(10/3,0,0,4/3), Z=22/3.

Numerical Example: 18
Primal LPP:

$$\max : Z = X_1 + 2X_2 + X_3$$

Subject to

$$4X_1 + X_2 + X_3 \leq 6$$

$$2X_1 + X_2 - X_3 \leq 2$$

$$2X_1 - X_2 + 5X_3 \leq 6$$

$$X_1, X_2, X_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,4,2), Z=10.

Numerical Example: 19
Primal LPP:

$$\min : Z = X_1 + 2X_2 + X_3$$

Subject to

$$4X_1 + X_2 + X_3 \geq 18$$

$$2X_1 + X_2 - X_3 \geq 6$$

$$2X_1 - X_2 + 5X_3 \geq 18$$

$$X_1, X_2, X_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(4,0,2), Z=6.

Numerical Example: 20
Primal LPP:

$$\max : Z = X_1 + 3X_2 + 4X_3$$

Subject to

$$2X_1 + X_2 + X_3 \leq 9$$

$$X_1 + 4X_2 + 3X_3 \leq 12$$

$$X_1, X_2, X_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,0,4), Z=16.

Numerical Example: 21
Primal LPP:

$$\min : Z = X_1 + 3X_2 + 4X_3$$

Subject to

$$2X_1 + X_2 + X_3 = 63$$

$$X_1 + 4X_2 + 3X_3 = 84$$

$$X_1, X_2, X_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(24,15,0), Z=69.

Numerical Example: 22
Primal LPP:

$$\max : Z = 2X_1 + 3X_2 + 2X_3 + X_4 + X_5$$

Subject to

$$3X_1 - 3X_2 + 4X_3 + 2X_4 - X_5 = 10$$

$$X_1 + X_2 + X_3 + X_4 + X_5 = 20$$

$$X_1, X_2, X_3, X_4, X_5 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,10,10,0,0), Z=50.

Numerical Example: 23
Primal LPP:

$$\max : Z = 2X_1 + 3X_2 + 2X_3 + X_4 + X_5$$

Subject to

$$3X_1 - 3X_2 + 4X_3 + 2X_4 - X_5 \leq 10$$

$$X_1 + X_2 + X_3 + X_4 + X_5 \leq 20$$

$$X_1, X_2, X_3, X_4, X_5 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,20,0,0,0), Z=60.

Numerical Example: 24
Primal LPP:

$$\min : Z = 2X_1 + 3X_2 + 2X_3 + X_4 + X_5$$

Subject to

$$3X_1 - 3X_2 + 4X_3 + 2X_4 - X_5 \geq 10$$

$$X_1 + X_2 + X_3 + X_4 + X_5 \geq 20$$

$$X_1, X_2, X_3, X_4, X_5 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,0,0,20,0), Z=20.

Numerical Example: 25
Primal LPP:

$$\max : Z = X_1 + 2X_2 + 3X_3 + 4X_4$$

Subject to

$$20X_1 + 9X_2 + 6X_3 + X_4 \leq 20$$

$$10X_1 + 4X_2 + 2X_3 + X_4 \leq 10$$

$$X_1, X_2, X_3, X_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,0,0,10), Z=40.

Numerical Example: 26
Primal LPP:

$$\min : Z = X_1 + 2X_2 + 3X_3 + 4X_4$$

Subject to

$$20X_1 + 9X_2 + 6X_3 + X_4 \geq 20$$

$$10X_1 + 4X_2 + 2X_3 + X_4 \geq 10$$

$$X_1, X_2, X_3, X_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(1,0,0,0), Z=1.

Numerical Example: 27
Primal LPP:

$$\max : Z = 9X_1 + 8X_2 + 6X_3 + 5X_4$$

Subject to

$$7X_1 + 10X_2 + 4X_3 + 9X_4 \leq 120$$

$$3X_1 + 4X_2 + X_3 + X_4 \leq 30$$

$$X_1, X_2, X_3, X_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,0,30,0), Z=180.

Numerical Example: 28
Primal LPP:

$$\min : Z = 9X_1 + 8X_2 + 6X_3 + 5X_4$$

Subject to

$$7X_1 + 10X_2 + 4X_3 + 9X_4 \geq 120$$

$$3X_1 + 4X_2 + X_3 + X_4 \geq 35$$

$$X_1, X_2, X_3, X_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,15/2,0,5), Z=85.

Numerical Example: 29
Primal LPP:

$$\max : Z = 5X_1 + 6X_2 + 4X_3 + 2X_4$$

Subject to

$$10X_1 + 5X_2 + 5X_3 + 3X_4 \leq 50$$

$$12X_1 + 4X_2 + 6X_3 + X_4 \leq 48$$

$$X_1, X_2, X_3, X_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,10,0,0), Z=60.

Numerical Example: 30
Primal LPP:

$$\min : Z = 5X_1 + 6X_2 + 4X_3 + 2X_4$$

Subject to

$$10X_1 + 5X_2 + 5X_3 + 3X_4 \geq 50$$

$$12X_1 + 4X_2 + 6X_3 + X_4 \geq 48$$

$$X_1, X_2, X_3, X_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(5,0,0,0), Z=25.

Numerical Example: 31
Primal LPP:

$$\min : Z = 3X_1 + X_2 + 2X_3 + X_4$$

Subject to

$$X_1 + X_2 - X_3 + X_4 \geq 6$$

$$X_1 - X_2 + X_3 + X_4 \geq 4$$

$$X_1, X_2, X_3, X_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,1,0,5), Z=6.

Numerical Example: 32
Primal LPP:

$$\max : Z = 3X_1 + X_2 + 2X_3 + X_4$$

Subject to

$$2X_1 + X_2 + X_3 + X_4 \leq 6$$

$$X_1 + X_2 + X_3 + 2X_4 \leq 4$$

$$X_1, X_2, X_3, X_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(2,0,2,0), Z=10.

Numerical Example: 33
Primal LPP:

$$\min : Z = 8X_1 + 3X_2 + 8X_3 + 6X_4$$

Subject to

$$4X_1 + 3X_2 - X_3 + 3X_4 \geq 10$$
$$X_1 - X_2 + X_3 + X_4 \geq 15$$
$$X_1, X_2, X_3, X_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,0,0,15), Z=90.

Numerical Example: 34
Primal LPP:

$$\max : Z = 8X_1 + 3X_2 + 8X_3 + 6X_4$$

Subject to

$$4X_1 + 3X_2 - X_3 + 3X_4 \leq 15$$

$$X_1 - X_2 + X_3 + X_4 \leq 5$$

$$X_1, X_2, X_3, X_4 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,10,15,0), Z=150.

Numerical Example: 35
Primal LPP:

$$\min : Z = X_1 + 2X_2 + X_3$$

Subject to

$$2X_1 + X_2 - X_3 \geq 6$$

$$X_1 + 4X_2 + 5X_3 \geq 14$$

$$X_1, X_2, X_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(4,0,2), Z=6.

Numerical Example: 36
Primal LPP:

$$\max : Z = X_1 + 2X_2 + X_3$$

Subject to

$$2X_1 + X_2 - X_3 \leq 2$$

$$X_1 + 4X_2 + 5X_3 \leq 26$$

$$X_1, X_2, X_3 \geq 0$$

1. Find an Optimal solution of the LPP and check it using Duality Theory.Check X=(0,4,2), Z=10.