Graph theory aug 3-25

Graph Theory And Algorithms (Indian Institute of Technology Kharagpur)

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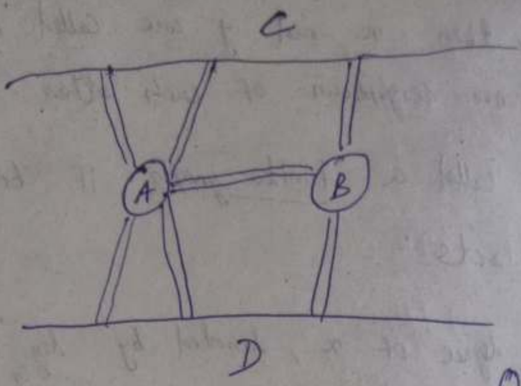
Konigsberg seven bridge problem



(1) start at any of the land areas A, B, C, D

(2) pass through every bridge exactly once.

(3) Come back to your place of origin.

Q: Is it possible?

Ans: No.

In 1736, Euler proved that it is not possible.

1852 four color conjecture

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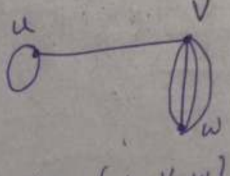
Graphs

two categories 1/ Graphs (undirected)

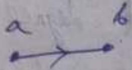
2/ Directed graphs / Digraphs

Defn A graph G is a pair (V, E), where V is a nonempty set and E is a multiset of 1-subsets and 2-subsets of V.

- An element of V is called a vertex and an element of E is called an edge.
- An edge of the form $\{x\}$ is called a loop.
- If an edge $\{x, y\}$ appears more than once then it is called a parallel edge or multiple edge.



$\{u, v\} = \{v, u\}$



$(a, b) \neq (b, a)$

$V = \{u, v, w\}$

$E = \{\{u\}, \{u, v\}, \{v, w\}, \{v, w\}, \{v, w\}, \{v, w\}\}$

If $e = \{x, y\}$ is an edge then

- we say x and y are adjacent or x and y are neighbours of each other.

• x and y are end vertices of e.

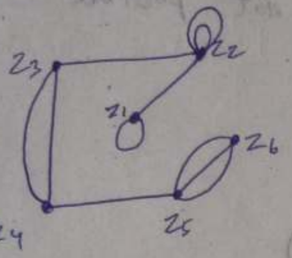
• e is incident on x and y.

If $\{x,y\} \notin E(G)$, then x and y are called non-adjacent or x and y are non-neighbour of each other.

• A graph $G=(V,E)$ is called a finite graph if both V and E are finite sets.

• For $x \in V(G)$, the degree of x, denoted by $\deg_G x$ or $\deg x$ is the no of edges ~~inc~~ incident on x where loops are counted twice.

ex



$\deg z_1 = 3$ $\quad$ $\deg z_4 = 3$

$\deg z_2 = 6$ $\quad$ $\deg z_5 = 4$

$\deg z_3 = 3$ $\quad$ $\deg z_6 = 3$

total $= 22 = 2 \times |E(G)| = 2 \cdot e$

edges $= 11$.

$|E(G)| = 11$

Theorem (First theorem of Graph theory) proof by induction

For every graph G, we get $\sum_{x \in V(G)} \deg x = 2 \cdot e$,

where e is the total no of edges in G. i.e. $e = |E(G)|$

Corollary: In every graph there are even no of vertices of odd degree.

Proof Let G be a graph and let S be the set of all odd degree vertices in G.

To show that $|S|=k$ is even.

$$\sum_{x \in V(G)} \deg x = 2 \cdot e$$

$$\Rightarrow \sum_{x \in S} \deg x + \underbrace{\sum_{x \in V(G)-S} \deg x}_{\text{even}} = 2 \cdot e$$

$$\Rightarrow \sum_{x \in S} \underbrace{\deg x}_{\text{odd}} = \underbrace{2 \cdot e - 2 \cdot r}_{\text{even}}$$

$\Rightarrow |S|$ is an even integer.

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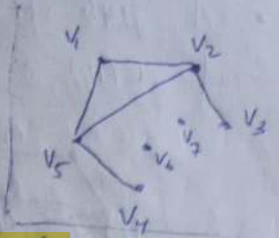
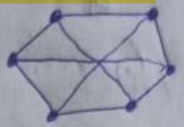
Isolated vertices – degree of vertex 0.



- Degree zero vertices are called isolated vertices (v_6, v_7)

- A degree 1 vertex is called a **pendant vertex**. (v_4, v_3).

$\deg x = 3, \ \forall x \in V(H)$

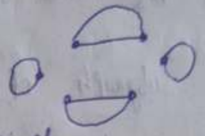


H

$\therefore$ H is a 3-regular graph.

- A graph G is called **K-regular**, $k \geq 0$, if $\deg x = k, \ \forall x \in V(G)$

← 2-regular graph →



simple

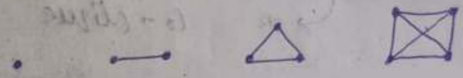
not simple

- **Simple graph** → no parallel edges, no self loops

A graph G is called simple if G has no self loops and parallel edges.

- A simple graph G on n vertices can have at most ${}^nC_2$ $= \frac{n(n-1)}{2}$ no of edges.

- A simple graph on n vertices and $\frac{n(n-1)}{2}$ no of edges is called a **complete graph**, denoted by K_n.



K_1 K_2 K_3 K_4

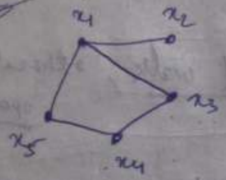
Let G be a simple graph. The **complement of G**, denoted by $\bar{G}$ or G^c, is a graph with $V(\bar{G}) = V(G)$ and

$$E(\bar{G}) = \{\{x,y\} : \{x,y\} \notin E(G)\}$$

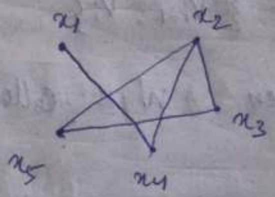
$$= \binom{V(G)}{2} - E(G)$$

→ set of all 2-subsets of $V(G)$

ex

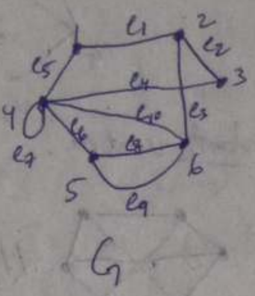


G



$\bar{G}$

- A graph H is called a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.



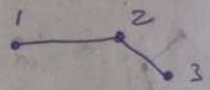
H_1, $V(H_1) = \{4,5,6\}$
$E(H_1) = \{e_6, e_8\}$
Subgraph

H_2, $V(H_2) = \{4,5,1\}$
$E(H_2) = \{e_5, e_6, e_{10}\}$
Not subgraph.

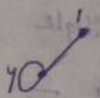
H_3, $V(H_3) = \{4,5,6\}$
$E(H_3) = \{e_6, e_7, e_8, e_9, e_{10}\}$
Subgraph. (Induced subgraph)

H_1 and H_3 are subgraphs of G.

Induced subgraph of G on $\{1,2,3\}$



on $\{1,4\}$



- A subgraph H of G is called an induced subgraph if H contains all the edges of G whose end vertices are in ~~and~~ $V(H)$.

$\overline{K_n} \rightarrow$ **null graph** $E(\overline{K_n}) = \phi$.

Induced subgraph of G on $\{1,3,6\}$ is a null graph; so $\{1,3,6\}$ is called an independent set.

- $S \subseteq V(G)$ is called an **independent set** or co-clique if S induces a null graph.
- $S \subseteq V(G)$ is called an **Clique**, if S induces a complete graph.

Walk, Trail, Path, Cycle

- Let $u, v \in V(G)$. An u-v **walk** in G is an alternating sequence of vertices and edges i.e.

$u = v_0\, e_1\, v_1\, e_2\, v_2 \ldots, e_k\, v_k = v$.

such that $e_i = \{v_{i-1}, v_i\}$

if $u = v$, then W is called a **closed walk**, otherwise **open walk**.

$W: 1\ e_5\ 4\ e_6\ 5\ e_8\ 6\ e_9\ 5\ e_6\ 4\ e_7\ 4$

length of W = 6.

- A walk W is called a Trail if all the edges in W are distinct (i.e. no repetation of edges in W)
  $W_1: 4\ e_6\ 5\ e_8\ 6\ e_9\ 5$
  note that vertices may repeat.

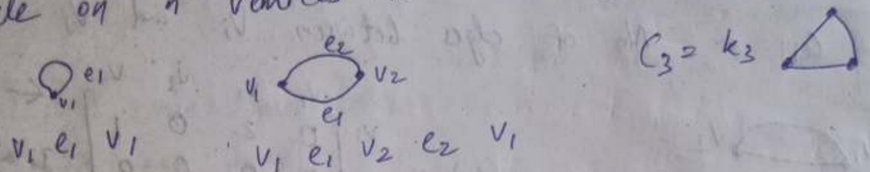
- A walk W is called a path if all the vertices in W are distinct ($\Rightarrow$ edges are also distinct)

proof by contradiction

- A path on n vertices is denoted by P_n

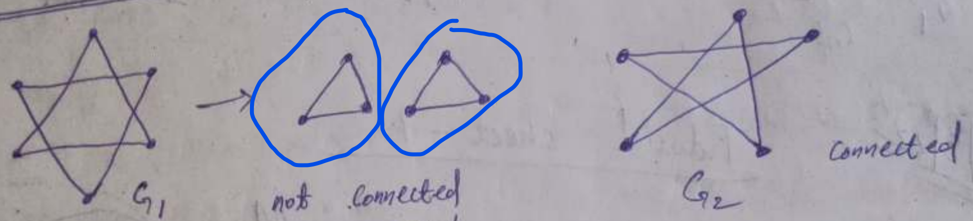
P_1 P_2 P_3

- A cycle is a closed trail in which all the vertices are distinct except the end vertices.
  A cycle on n vertices is denoted by C_n.



Connected Graphs

components of G1



- A Graph G is called connected if for every pair of vertices $x, y \in V(G)$, $x \neq y$, $\exists$ a x-y path in G. otherwise G is disconnected.

- A maximal connected subgraph of G is called a (connected) component of G.

Proposition - Every u-v walk contains a u-v path.

Let W be a u-v walk, i.e. $W: u = v_0\, e_1\, v_1\, e_2\, v_2 \cdots e_k\, v_k = v$
$: v_0\ v_1\ v_2 \cdots v_k$

1) If all the vertices are distinct, then W itself is a u-v path. Return.

2) otherwise let $v_i = v_j$, $i \neq j$, $i < j$,

$W: v_0\, e_1 \ldots v_i\, e_{i+1}\, v_{i+1}\, e_{i+2}\, v_{i+2} \ldots e_j\, v_j\, e_{j+1}\, v_{j+1} \ldots e_k\, v_k$

W_1 : $u = v_0 \; e_1 \cdots v_j \; e_{j+1} \; v_{j+1} \cdots v_k$

Go to ①.

Since G is finite, after finite no of steps we get a u-v path contained in W.

Adjacency matrix of a graph G

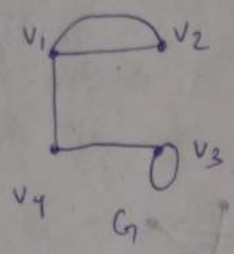
Let $|V(G)| = n$.

$V(G) = \{x_1, x_2, \cdots, x_n\}$.

Adjacency matrix of G, denoted by $A(G)$, is an $n \times n$ matrix

$A(G) = (a_{ij})_{n \times n}$, where the rows and columns of G are indexed by vertices in G.

a_{ij} = No of edges between v_i and v_j.



$$A(G) = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 2 & 0 & 1 \\ v_2 & 2 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 1 & 1 \\ v_4 & 1 & 0 & 1 & 0 \end{array}$$

← symmetric matrix

In a simple graph, row/col sum of a vertex is equal to degree of that vertex.

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Tutorial sheet - 1

① a/ $|V(G)| = mn$

b/ Yes it is regular.

degree of each edge $= m-1 + n-1$

$= m+n-2$

c/ $\sum_{x \in V(G)} \deg x = 2e$, $e = |E(G)|$

$\Rightarrow 2e = m \cdot n \cdot (m+n-2)$

$\Rightarrow e = |E(G)| = \dfrac{mn(m+n-2)}{2}$.

② n persons as n vertices. adjacent if friends. Simple graph

$\deg x \longrightarrow$ no of friends of x.

To show there exists two vertices x and z, s.t $\deg x = \deg z$.

[In every simple graph of n vertices, there exists two vertices with same degree.]

possible degrees of a simple graph with n vertices : $0, 1, 2, \cdots, n-1$

[Pigeon-hole principle

m holes, n pigeons

if $n > m$, then atleast two pigeons share one hole.]

if a vertex have degree 0, then degree $n-1$ can not exist.

possible degree set : $\{1, 2, \cdots, n-1\}$, $\{0, 1, \cdots, n-2\}$

use pigeon hole principle.

(3) $x, y \in V(G)$, $\deg x, \deg y$ odd. ???

(4) NO. $15 \times 3 = 45$ (odd). Sum of degrees should be a even number.

(5) $\delta(G) \to$ min degree in G

$\Delta(G) \to$ max degree in G

G simple, $|V(G)| = n$ $\quad \deg x \geq \frac{n-1}{2}$, $\forall x \in V(G)$.

To prove G is connected.

$x, y \in V(G)$

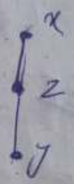
i) If x, y are adjacent $\overset{x}{\bullet}\!\xrightarrow[l]{}\!\overset{y}{\bullet}$, l is an x-y path.

ii) If x, y are not adjacent $x \not\sim y$

$N(x) \to$ the set of all neighbours of x

$N(y) \to$ " " " y

$y \notin N(x)$, $x \notin N(y)$



To show $|N(x) \cap N(y)| \geq 1$.

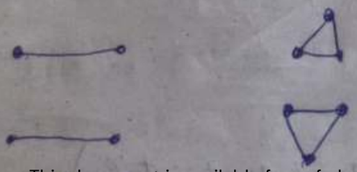
$|A \cup B| = |A| + |B| - |A \cap B|$

$= |N(x)| + |N(y)| - |N(x) \cup N(y)|$

$\geq \frac{n-1}{2} + \frac{n-1}{2} - (n-2)$ $\qquad |N(x) \cup N(y)| \leq n-2$ because $x, y \notin N(x) \cup N(y)$

$\geq n-1-n+2 = \underline{1}$ $\qquad$ total n vertices.

(6)



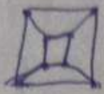
$4-3 = 1$ $\qquad 6-2 = 2$

7/ i/ True   ii/ False

8/ $|V(G)| = n$ $|E(G)| = n-1$

$\frac{\sum \deg x}{n \text{ terms}} = 2(n-1)$ $n-1$ = no. of edges

$\exists\ z \in V(G)$, s.t. $\deg z < 2$.

11/ 

Cube

12/ Petersen graph ??

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⑨ G is a simple graph on 6 vertices.

Every simple graph on 6 vertices contains K_3 or $\overline{K_3}$ as an induced subgraph.

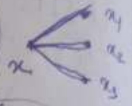
$\Leftrightarrow$

G contains K_3 or $\overline{G}$ contains K_3.

G 

$x \in V(G)$, $|V(G)| = 6$.

x has atleast 3 neighbours either in G or $\overline{G}$.



Case I: Let x be having 3 neighbours in G,

(i) $\{x_1, x_2, x_3\}$ contains an edge. K_3 is contained in G

(ii) $\{x_1, x_2, x_3\}$ is an independent set. $K_3 \subset \overline{G}$.

Case II: Let x be having 3 neighbour in $\overline{G}$

Rest is same as Case I.

⑩ G is simple graph. $|V(G)| = n$,

G be having K components say $G_1, G_2, \cdots, G_K$.

To show that $|E(G)| \leq \frac{(n-k)(n-k+1)}{2}$

Let $|V(G_i)| = n_i$, $i = 1, 2, \cdots, K$, $n_i \geq 1$.

$\sum_{i=1}^{K} n_i = n$

$n_i = n - (n_1 + n_2 + \cdots + n_{i-1} + n_{i+1} + \cdots + n_K)$

$$n_i \leq n-(k-1) = n-k+1$$

$$|E(G)| \leq \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} \leq \sum_{i=1}^{k} \frac{(n-k+1)(n_i-1)}{2}$$

$$= \frac{n-k+1}{2} \sum_{i=1}^{k} (n_i - 1) \qquad (\because \sum n_i = n)$$

$$= \frac{n-k+1}{2}(n-k)$$

Incidence matrix

(in general not a square matrix)

$|V(G)| = n$, $|E(G)| = m$

Incidence matrix of $G = I(G) =$

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{array} \begin{array}{c} e_1 \quad e_2 \quad \cdots \quad e_j \quad \cdots \quad e_m \\ \left[\begin{array}{ccccc} & & & & \\ & & & & \\ & & & a_{ij} & \\ & & & & \\ & & & & \end{array} \right]_{n \times m} \end{array}$$

$$a_{ij} = \begin{cases} 1, & \text{if } x_i \text{ is an end vertex of } e_j \\ 0 & \text{otherwise} \end{cases}$$

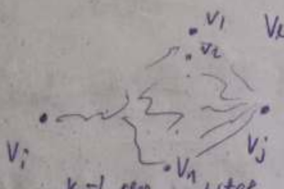
Thm Let G be a loop free graph and let A be the adjacency matrix of G. For $k \geq 1$, the $(i,j)^{th}$ entry of A^k is the number of $v_i - v_j$ walks of length k in G where $V(G) = \{v_1, v_2, \cdots, v_n\}$.

Proof If $k=1$, $A^k = A$. the result is true.

assume that the result is true up to $k-1$.

$A^{k-1} = (a_{ij}^{k-1})_{n\times n}$, $A^k = (a_{ij}^{k})_{n\times n}$, $A = (a_{ij})_{n\times n}$

$A^k = A^{k-1} \cdot A$; $a_{ij}^{k} = \sum_{l=1}^{n} a_{il}^{k-1} a_{lj}$

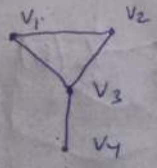


$v_l, l \in \{1, 2, \cdots, n\}$

$\underline{k-1 \text{ step}}$ $\underline{1 \text{ step}}$

$a_{ij}^{k-1} \longrightarrow$ the no of $v_i - v_j$ walk of length $k-1$ in G.

Ex Let G be the graph



find total no of walks of length 3 between v_1 and v_3. Also list all these walks.

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$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

4 walks of length 3 between V_1 to V_3

$w_1 : V_1 \; V_2 \; V_1 \; V_3$

$w_2 : V_1 \; V_3 \; V_1 \; V_3$

$w_3 : V_1 \; V_3 \; V_2 \; V_3$

$w_4 : V_1 \; V_3 \; V_4 \; V_3$

Distance concept in a Graph (loop Free)

Let $x, y \in V(G)$. The distance between x and y in G, denoted by $d_G(x,y)$ or $d(x,y)$ is the length of a shortest x-y path in G. If there is no x-y path in G then we take $d(x,y) = \infty$.

$$d : V(G) \times V(G) \longrightarrow Z^+ \cup \{0, \infty\}$$

Lemma: d is a metric.

i.e. (i) $d(x,y) \geq 0$, equality iff $x = y$.

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,y) \leq d(x,z) + d(z,y) \qquad z \in V(G)$.

(triangle inequality)

Let P be a shortest x-y path i.e. $d(x,y) = \ell(P)$.

$P_1 \rightarrow$ shortest x-z path

$P_2 \rightarrow$ shortest y-z Path

$P_1 \cup P_2 \rightarrow$ an x-y walk.

$\Rightarrow P_1 \cup P_2$ Contains an x-y Path, say P'.

$$\ell(P) \leq \ell(P') \leq \ell(P_1) + \ell(P_2)$$

Eccentricity of a vertex in G

Let $x \in V(G)$. Eccentricity of x denoted by $e(x)$, is

$$e(x) = \max_{y \in V(G)} d(x, y)$$

Minium of

$$\min_{x \in V(G)} e(x) = \text{radius of } G \ (rad(G))$$

$$\max_{x \in V(G)} e(x) = \text{diameter of } G \ (diam(G))$$

A vertex with minimum eccentricity is called a central vertex
" " maximum " " peripheral "

The subgraph induced by all the central vertices is called the center of the graph.

The subgraph induced by all the peripheral vertices is called the periphery of the graph

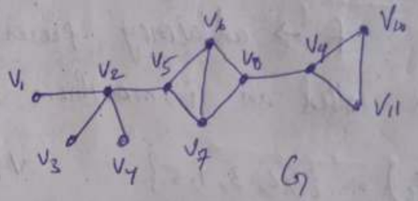
$6 = e(v_{11}) = e(v_{10})$

$e(v_1) = 6 = e(v_3) = e(v_4)$

$e(v_2) = 5 = e(v_9)$

$e(v_5) = 4 = e(v_8)$

$e(v_6) = 3 = e(v_7)$



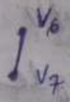
$$rad(G) = \min_{x \in V(G)} e(x) = 3$$

$$diam(G) = \max_{x \in V(G)} e(x) = 6$$

Central vertices : V_6, V_7

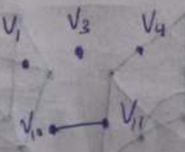
Peripheral vertices : $V_1, V_3, V_4, V_{10}, V_{11}$

Center of G is



which is K_2

Periphery of G



Lemma - For every loop free graph G, we have

$$rad(G) \leq diam(G) \leq 2 \cdot rad(G).$$

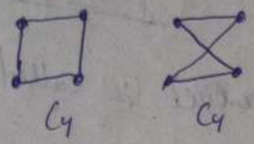
Proof Let $diam(G) = d(x,y)$: $x,y \in V(G)$.

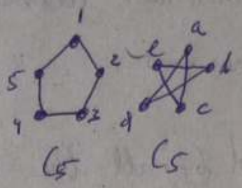
Let z be a central vertex.

$d(x,y) \leq d(x,z) + d(z,y)$ $\quad$ $d(x,z) \leq rad(G)$

$\leq 2 \cdot rad(G)$ $\quad$ $d(z,y) \leq rad(G)$

Isomorphic Graph





Graphs G_1 and G_2 are said to be isomorphic if $\exists$ a bijection $f: V(G_1) \to V(G_2)$ s.t $\{x,y\} \in E(G_1)$ iff $\{f(x), f(y)\} \in E(G_2)$.

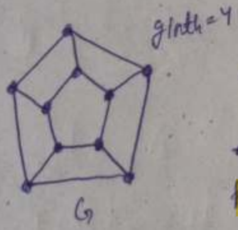
$f \to$ adjacency preserving bijection mapping.

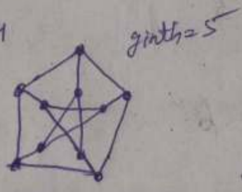
f is called an isomorphism between G_1 and G_2.

$V(C_5) = \{1, 2, 3, 4, 5\}$ $\quad$ $V(G) = \{a, b, c, d, e\}$

$F(1) = a$, $f(2) = c$, $f(3) = e$, $f(4) = b$, $f(5) = d$

$1 \sim 2$ $\quad$ $f(1) \sim f(2)$



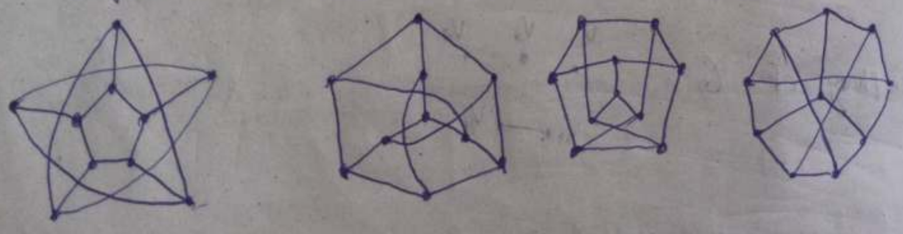


Petersen Graph

not isomorphs

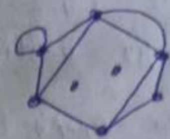
Def Length of a smallest cycle present in a graph is called the girth of the graph.

Length of a Longest cycle is called circumference of G



isomorphs of petersen graph

degree sequence of Graph



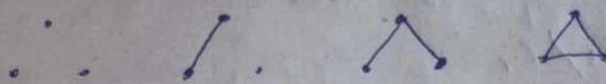
4, 4, 4, 3, 3, 2, 0, 0

non increasing sequence of degrees of the vertices.

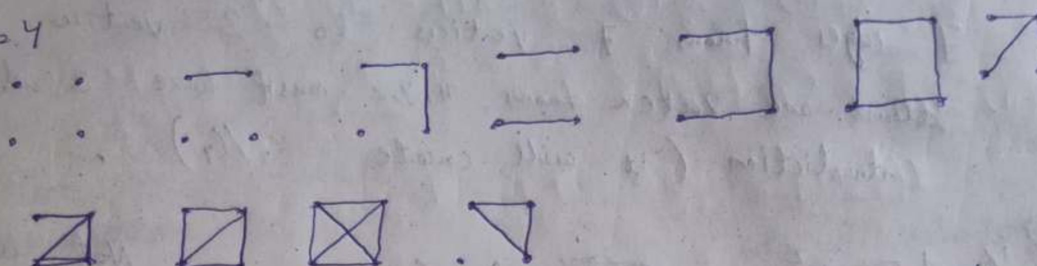
G_1   G_2

3, 2, 2, 1, 1, 1   3, 2, 2, 1, 1, 1

17/8/22 (1) $n = 3$

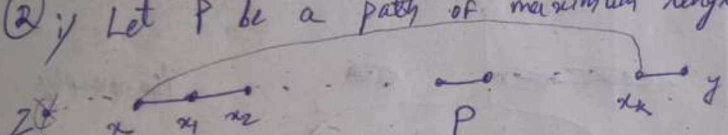


$n = 4$



upto isomorphism these are simple graphs on 3/4 vertices.

(2) i/ Let P be a path of maximum length in G.



claim: All the neighbours of x lie on P.

deg $x \geq K$.

Let z be a neighbour of x not on P.

so z - y can be a path of length > x-y path

so every neighbour of x lies on P.

minimum no of vertices in P = K + 1 (k neighbours, x itself)

$\Rightarrow \ell(P) \geq K$.

ii/ $x - x_k$

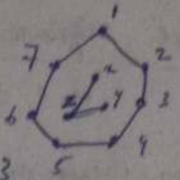
min no of vertices in P = k + 1

so cycle of length k+1 exists.

(3) $S = \{1, 2, 3, 4, 5\}$

$V(G) = \{ \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \ldots, \{4,5\} \}$

(4) Suppose $\exists$ a C_7 in the petersen graph.



Total 10 vertices, 7 in cycle

In petersen graph every vertex has degree 3.

x, y, z can not form C_3. (Girth of Petersen graph is 5)

vertices $1, 2, 3, \dots, 7$ have degree 2 each from the C_7

So every vertex $1, 2, 3, \dots, 7$ must have one more edge

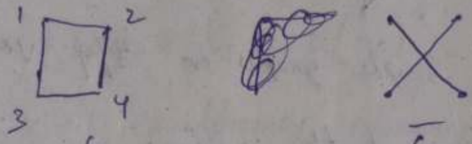
They can not have a edge between them else it will create C_3 / C_4.

7 edges from 7 vertices to 3 vertices x, y, z.

atleast one vertex from x, y, z must have 3 adjacencies

Contradiction (it will create C_3 / C_4).

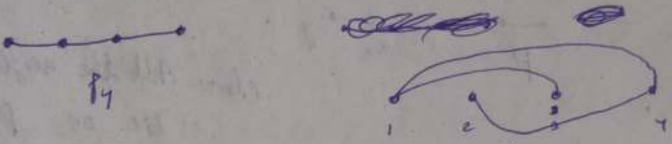
(5) i/



G $\qquad$ $\bar{G}$

Not isomorphs

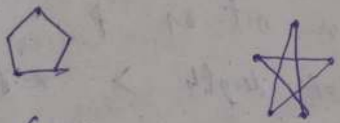
ii/



P_4

isomorphs

iii/



C_5

isomorphs

(6) Let $G \simeq \bar{G}$

$|E(G)| = |E(\bar{G})|$

$|E(G)| + |E(\bar{G})| = \frac{n(n-1)}{2}$

$\Rightarrow |E(G)| = \frac{n(n-1)}{4}$ should be an integer.

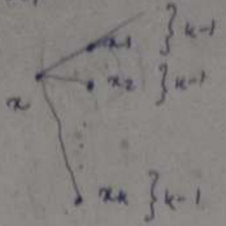
$\Rightarrow 4 \mid n(n-1)$

$\Rightarrow 4 \mid n$ or $4 \mid n-1$

$n \equiv 0$ or $1 \pmod 4$

24/8/20

(7) girth of $G = 5$, G is k regular.



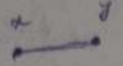
x_1 can not be adjacent to any $x_i, \ldots, x_k$.
else girth = 3

each x_i is not adjacent to x_j , $i \neq j$.

min no of vertices

$$\underset{x}{1} + \underset{x_i}{k} + \underset{\text{adjacency of } x_i\text{'s}}{k(k-1)} = k^2+1$$

(8)

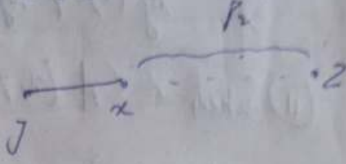


to show $|d(x,z) - d(y,z)| \leq 1$.

let $d(x,z) = l_1$, $d(y,z) = l_2$.

$x-z$ shortest path $\rightarrow P_1$, $l(P_1) = l_1$

$y-z$ shortest path $\rightarrow P_2$, $l(P_2) = l_2$



case-I $\{x,y\} \in P_2$

claim $l(P_2') = l(P_1)$

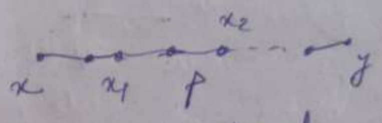
let ~~[illegible]~~ $l(P_2') \neq l(P_1)$

w.l.o.g $l(P_1) > l(P_2')$ then a path P_2' exists

so P_1 is not shortest path $\Rightarrow l(P_2') = l(P_1)$

$\Rightarrow l(P_2) = l(P_1) + 1$

(9) diam $G = d$



$l(P) = d$

no of vertices $= d+1$

G contains a path of length d.

select alternate vertices.

$\{x, x_1, x_2, \ldots\}$

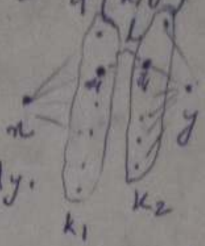
no of vertices $= \left\lceil \frac{d+1}{2} \right\rceil$.

This set is independent.

(10) G is simple connected graph $|V(G)| = n$.

$x, y \in V(G)$. $d(x,y) > 2$:

to prove $\deg(x) + \deg(y) \leq n+1 - d(x,y)$



$d(x,y) > 2$. $d(x,y) = l$

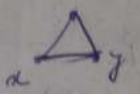
so no common adjacency between x and y.

$n \geq k_1 + k_2 + (l+1-2) \longrightarrow$ $l+1$ vertices in x-y path, already considered x, y,

$\Rightarrow k_1 + k_2 \leq n+1-l$

$\Rightarrow \deg(x) + \deg(y) \leq n+1 - d(x,y)$

(ii) when $d(x,y) \leq 2$.



Result is not true when $d(x,y) \leq 2$.

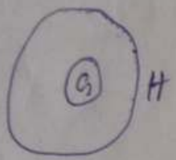
$2+2 \leq 3+1-1$

$4 \leq 3 \Rightarrow \Leftarrow$

$2+2 \leq 4+1-2$

$4 \leq 3 \Rightarrow \Leftarrow$

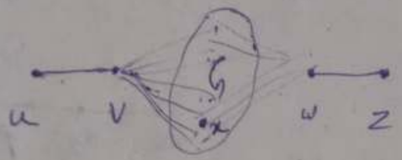
(11)



$C(H) = G$

add 4 vertices u, v, w, z

$u \sim v$, $w \sim z$



$H = G \cup \{\{u,v\}, \{w,z\}\} \cup \{\{v,x\}, \{w,x\} \mid x \in V(G)\}$

$e(u) = 4 = e(z)$

$e(v) = 3 = e(w)$

$x \in V(G)$, $e(x) = 2$. $d(x,y) \leq 2$ $x, y \in V(G)$

$d(x,v) = d(x,w) = 1$

central vertices $= V(G)$. $d(x,u) = d(x,z) = 2$

center of H = graph induced by central vertices

$= G$.

Graph operations

(1) Disjoint union. $(G_1 \cup G_2 \cup \cdots \cup G_k)$

$G_1, G_2, \ldots, G_k$ graphs.

$V(G_i) \cap V(G_j) = \phi$.

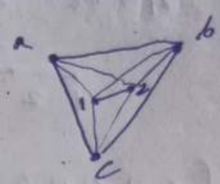
(2) Join of Graphs G_1 and G_2 $(G_1 \vee G_2)$

$V(G_1) \cap V(G_2) = \phi$

$V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$

$E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{\{x,y\} \mid x \in V(G_1), y \in V(G_2)\}$

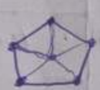
ex $K_2 \vee K_3$



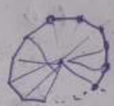
$K_1 \vee \overline{K_5}$

25/3/22 $G_1 \vee G_2 \simeq G_2 \vee G_1$

$K_1 \vee C_5$

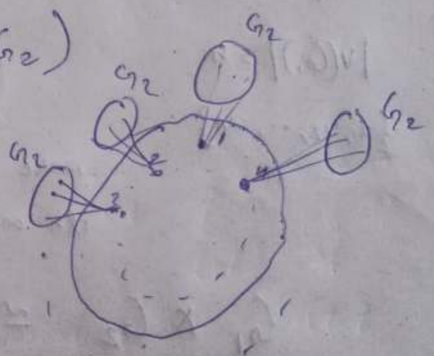


$K_1 \vee C_n$



wheel graph

(3) Corona of G_1 and G_2 $(G_1 \circ G_2)$

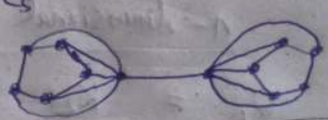


$V(G_1) \cap V(G_2) = \phi$.

$|V(G_1)| = n$

ex $C_5 \circ K_2$

$K_2 \circ C_5$



$G_1 \circ G_2 \not\simeq G_2 \circ G_1$

$nG_2 \rightarrow$ disjoint union of n copies of G_2

$V(G_1 \circ G_2) = V(G_1) \cup V(nG_2)$

where $n = |V(G_1)|$, $V(G_1) = \{x_1, x_2, \ldots, x_n\}$

$E(G_1 \circ G_2) = E(G_1) \cup E(nG_2) \cup \bigcup_{i=1}^{n} \ldots$

$|V(G_2)| = m$. $G_2^i \simeq G_2$

(4) Cartesian product of G_1 and G_2 $(G_1 \times G_2)$ ✓ $(G_1 \square G_2)$

$V(G_1) \cap V(G_2) = \phi$.

$V(G_1 \times G_2) = V(G_1) \times V(G_2)$

$= \{(x,y) \mid x \in V(G_1),\ y \in V(G_2)\}$

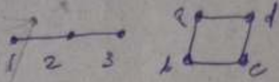
$(x_1, y_1), (x_2, y_2) \in V(G_1 \times G_2)$

$(x_1, y_1) \sim (x_2, y_2)$ if

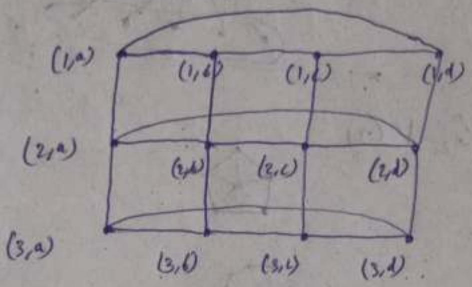
either (i) $x_1 = x_2$ and $y_1 \sim y_2$

or (ii) $x_1 \sim x_2$ and $y_1 = y_2$

ex $P_3 \times C_4$



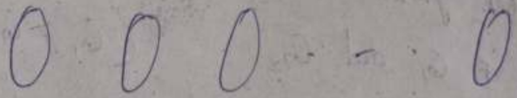
$V(P_3 \times C_4) = \{(1,a), (1,b), (1,c), (1,d), (2,a), (2,b), (2,c), (2,d), (3,a), (3,b), (3,c), (3,d)\}$



$G_1 \times G_2$

$|V(G_1)| = n$

$\{x_1, x_2, \cdots, x_n\}$



n copies of G_2

make adjacent corresponding pairs of vertices in ith and jth copy of G_2 if $x_i \sim x_j$ in G_1

$G_1 \times G_2 \cong G_2 \times G_1$

---

n-dimensional hypercubes or cubes (Q_n)

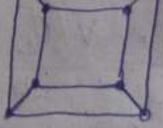
$Q_1 = K_2$ 

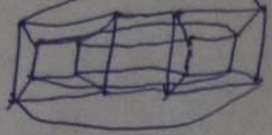
$Q_2 = Q_1 \times K_2$

$\vdots$

$\boxed{Q_n = Q_{n-1} \times K_2}$

Q_2 

Q_3 

$Q_4 =$ 

Ex Let G_n be a graph with

$V(G_n) = \{(a_1, a_2, \cdots, a_n) : a_i = 0 \text{ or } 1\}$

$(a_1, a_2, \cdots, a_n) \sim (b_1, b_2, \cdots, b_n)$ iff they differ in exactly one position

Prove that $G_n \simeq Q_n$.

$V(G_3) = \{000, 001, 010, 011, 100, 101, 110, 111\}$