Let X and Y be n.l.s. The set of all bounded or continuous linear maps from X into Y is denoted by $BL(X, Y)$ or $B(X, Y)$.

If $Y = K$, field, then

$BL(X, K)$ or $B(X, K)$ denotes set of all bounded linear functionals on X.

If $X = Y$, then

$BL(X, X) = BL(X)$ or $B(X)$ denotes the set of all bounded or continuous linear operators on X.

3/ $A \in BL(X, Y)$, $A \neq 0$

then there exists some $\alpha > 0$

such that $\|Ax\|_Y \leq \alpha \|x\|_X \ \forall x \in X$.

$\|Ax\| \leq \alpha \|x\| \Rightarrow \frac{1}{\|x\|} \|Ax\| \leq \alpha \Rightarrow \left\| A\left(\frac{x}{\|x\|}\right) \right\| \leq \alpha$

* 3/ $\exists$ some $\gamma > 0$ such that

$\|Ax\| \geq \gamma \|x\|, \ \forall x \in X,$

we say A is bounded below.

Problem: Show that $BL(X, Y)$ is a linear space under the pointwise operations: For $x \in X$,

$(A_1 + A_2)(x) = A_1 x + A_2 x$

$(\alpha A_1)(x) = \alpha A_1 x, \ \forall \alpha \in K$

$A_1, A_2 \in BL(X, Y)$

Theorem: let X and Y be n.l.s.
For $A \in BL(X, Y)$, define
$$\|A\| = \sup\{\|Ax\|_Y \,/\, x \in X,\ \|x\|_X \le 1\}.$$
Then $\|\cdot\|$ is a norm on $BL(X,Y)$,
called operator norm.

Proof: if $X = \{0\}$, there is nothing to prove.

So let $X \neq \{0\}$.

$\because \|Ax\|_Y \geq 0 \quad \forall x \in X, \ \forall A \in BL(X,Y)$

$\Rightarrow \sup\{\|Ax\|_Y \,/\, x \in X, \|x\| \le 1\} \geq 0$

$\Rightarrow \|A\| \geq 0.$

Also if $\|A\| = 0$

$\Leftrightarrow \sup\{\|Ax\|_Y \,/\, x \in X, \|x\| \le 1\} = 0$

$\Leftrightarrow \quad \|Ax\|_Y = 0, \quad \forall x \in X, \|x\| \leq 1.$

$\Leftrightarrow \quad \|Ay\|_Y = 0 \quad , \forall \; y = \frac{x}{\|x\|}, \; \|y\| \leq 1,$

$\Leftrightarrow \quad \|A(\frac{x}{\|x\|})\|_Y = 0 \qquad x \in X, \; x \neq 0$

$\Leftrightarrow \quad \|Ax\|_Y = 0, \quad \forall \; x \in X$

$\Leftrightarrow \quad Ax = 0 \quad , \forall \; x \in X$

$\Leftrightarrow \quad A := 0$

Now for any $\alpha \in K$, $A \in B(X, Y)$

$$\|\alpha A\| = \sup\{\|(\alpha A)(x)\|_Y / x \in X, \|x\| \leq 1\}$$

$$= \sup\{\|\alpha (Ax)\|_Y / x \in X, \|x\| \leq 1\}$$

$$= \sup\{|\alpha| \|Ax\|_Y / x \in X, \|x\| \leq 1\}$$

$$= |\alpha| \sup\{\|Ax\|_Y / x \in X, \|x\| \leq 1\}$$

$$= |\alpha| \|A\|.$$

Now for any $A_1, A_2 \in BL(X, Y)$,

$$\|A_1+A_2\| = \sup\{\|(A_1+A_2)x\|_Y \,/\, x \in X, \|x\| \le 1\}$$

$$= \sup\{\|A_1x + A_2x\|_Y \,/\, x \in X, \|x\| \le 1\}$$

$$\le \sup\{\|A_1x\|_Y + \|A_2x\|_Y \,/\, x \in X, \|x\| \le 1\}$$

$$= \sup\{\|A_1x\|_Y \,/\, x \in X, \|x\| \le 1\}$$

$$+ \sup\{\|A_2x\| \,/\, x \in X, \|x\| \le 1\}$$

$$= \|A_1\| + \|A_2\|.$$

$\therefore$ $BL(X, Y)$ is a normed linear space with the norm

$$\|A\| = \sup\{\|Ax\|_Y \,/\, x \in X, \|x\| \le 1\}.$$

Now let

$$\alpha_0 = \inf\{\alpha \geq 0 \,/\, \|Ax\| \leq \alpha \|x\|, \forall x \in X\} \quad \text{—(i)}$$

$$\beta = \sup\{\|Ax\| \,/\, x \in X, \|x\| < 1\} \quad \text{—(ii)}$$

$$\gamma = \sup\{\|Ax\| \,/\, x \in X, \|x\| = 1\} \quad \text{—(iii)}$$

Since $\|A\| = \sup\{\|Ax\| \,/\, x \in X, \|x\| \leq 1\}$,

we have

$$\beta, \gamma \leq \|A\| \quad \text{—(1)}$$

Now consider $x \in X$ and $0 < r \leq 1$.

$\because$ A is a linear map,

$$\|Ax\| = \left\|A\left(\frac{rx}{\|x\|}\right)\right\| \cdot \frac{\|x\|}{r}.$$

$$\leq \sup\{\|Az\| \,/\, z \in X, \|z\| = r\} \cdot \frac{\|x\|}{r}. \quad \text{—(2)}$$

$\left[z = \frac{rx}{\|x\|} \right.$

$\left. \|z\| = r \right]$

If $r = 1$ in (2), we get

$$\|Ax\| \leq \sup\{\|Az\| \,/\, z \in X, \|z\| = 1\} \cdot \|x\|$$
$$= \gamma \|x\|$$

$\therefore \|Ax\| \leq \gamma \|x\|, \ \forall x \in X.$

$\Rightarrow \inf\{\alpha > 0 \,/\, \|Ax\| \leq \alpha \|x\|, x \in X\} \leq \gamma$

$\Rightarrow \alpha_0 \leq \gamma$

If $r < 1$, from (2) we get

$$\|Ax\| \leq \sup\{\|Az\| \,/\, z \in X, \|z\| < 1\} \cdot \frac{\|x\|}{r}$$
$$= \beta \cdot \frac{\|x\|}{r}.$$

letting $r \to 1$, we get

$$\|Ax\| \leq \beta \|x\|, \ \forall x \in X$$

$\Rightarrow \alpha_0 \leq \beta$

$\therefore \alpha_0 \leq \min\{\beta, \gamma\}$ — (3)

Consider $\alpha \geq 0$ such that

$\|Ax\| \leq \alpha \|x\|, \quad \forall\, x \in X, \; A \in BL(X, Y)$.

$\Rightarrow \sup\{\|Ax\| / x \in X, \|x\| \leq 1\} \leq \alpha \cdot 1$

$\Rightarrow \quad \|A\| \leq \alpha$

$\therefore$ Since α_0 is infimum of all such α's, we see that

$$\|A\| \leq \alpha_0 \quad \text{— (4)}$$

$\therefore$ From (1), (3) & (4) we get

$$\|A\| \leq \alpha_0 \leq \min\{\beta, \gamma\} \leq \|A\|.$$

$\therefore$ (i), (ii), (iii) are all equivalent to $\|A\|$.

$$* \quad \|A\| = \sup\{ \|Ax\| \;/\; x \in X, \|x\| = 1 \}$$

$$= \sup\left\{ \left\|A\left(\frac{x}{\|x\|}\right)\right\| \;/\; x \in X \right\}$$

$$= \sup\left\{ \frac{\|Ax\|_Y}{\|x\|_X} \;/\; \begin{array}{l} x \neq 0 \\ x \in X \end{array} \right\}$$

$$\geq \frac{\|Ax\|_Y}{\|x\|_X}$$

$$\therefore \boxed{\|Ax\|_Y \leq \|A\| \, \|x\|_X, \quad x \in X.}$$

Ex: $X = C[a,b]$ with $\|\cdot\|_\infty$ norm

For $k(\cdot,\cdot) \in C([a,b] \times [a,b])$,

let

$$Ax(s) = \int_a^b k(s,t)\, x(t)\, dt, \quad s \in [a,b]$$

$$\forall x \in C[a,b].$$

Claim : For $x \in C[a,b]$, $Ax \in C[a,b]$

For any $s, s_0 \in [a,b]$, $x \in C[a,b]$,

consider

$$|Ax(s) - Ax(s_0)| = \left| \int_a^b k(s,t)\, x(t)\, dt - \int_a^b k(s_0,t)\, x(t)\, dt \right|$$

$$= \left| \int_a^b \left[k(s,t) - k(s_0,t) \right] x(t)\, dt \right|$$

$$\leq \int_a^b |k(s,t) - k(s_0,t)|\, |x(t)|\, dt.$$

$$\leq \sup_{t \in [a,b]} |k(s,t) - k(s_0,t)| \int_a^b |x(t)|\, dt$$

$\because$ $k(s,t) \in C([a,b] \times [a,b])$, i.e.,

$k(s,t)$ is continuous on a compact set $[a,b] \times [a,b]$, it is uniformly continuous.

So given $\epsilon > 0$, $\exists$ $\delta = \delta(\epsilon) > 0$

such that $|s - s_0| < \delta \implies |k(s_0, t) - k(s, t)| < \epsilon$
$\forall\ t \in [a, b]$,

$\therefore$

$$|Ax(s) - Ax(s_0)| \leq \sup_{t \in [a,b]} |k(s, t) - k(s_0, t)| \cdot \sup_{t \in [a,b]} |x(t)| \cdot \int_a^b 1\, dt$$

$$< \epsilon\, \|x\|_\infty (b - a), \quad \forall\ x \in C[a, b]$$

$\implies$ Ax is continuous on $[a, b]$.

$\therefore$ $A : C[a, b] \longrightarrow C[a, b]$ is a map.

Clearly A is a linear map

For any $x, y \in C[a, b]$, $\alpha, \beta \in K$,
consider

$$A(\alpha x + \beta y)(s) = \int_a^b k(s, t)\, (\alpha x + \beta y)(t)\, dt$$

$$= \int_a^b k(s, t)\, \alpha x(t)\, dt + \int_a^b k(s, t)\, \beta y(t)\, dt$$

$$= \alpha \int_a^b k(s,t)\, x(t)\, dt + \beta \int_a^b k(s,t)\, y(t)\, dt$$

$$= \alpha Ax(s) + \beta Ay(s), \quad \forall s \in [a,b]$$

$$\forall x, y \in C[a,b]$$

$$\therefore A(\alpha x + \beta y) = \alpha Ax + \beta Ay.$$

Claim: A is a bounded linear map.

For any $x \in C[a,b]$, $s \in [a,b]$,

Consider

$$|Ax(s)| = \left| \int_a^b k(s,t)\, x(t)\, dt \right|$$

$$\leq \left(\sup_{s \in [a,b]} \int_a^b |k(s,t)|\, dt \right) \|x\|_\infty \quad \text{——— (1)}$$

$\because$ $s \longrightarrow \int_a^b |k(s,t)|$ is continuous on the interval $[a,b]$, we have

$$C = \sup_{s \in [a,b]} \int_a^b |k(s,t)|\, dt$$

$$\left[\int_a^b |k(s,t)|\,dt - \int_a^b |k(s_0,t)|\,dt = \int_a^b \left[|k(s,t)| - |k(s_0,t)|\right] dt \right.$$

$$\leq \int_a^b \left| k(s,t) - k(s_0,t) \right| dt$$

$$< \sup_{t \in [a,b]} |k(s,t) - k(s_0,t)| \cdot (b-a)$$

$$\left. < \epsilon\,(b-a), \qquad |s - s_0| < \delta \right]$$

$\therefore$ from (1), we have

$$|Ax(s)| \leq C\|x\|_\infty, \quad \forall\, s \in [a,b]$$

$$\Rightarrow \sup_{s \in [a,b]} |Ax(s)| \leq C\|x\|_\infty$$

$$\Rightarrow \|Ax\|_\infty \leq C\|x\|_\infty, \longleftarrow (*)$$

where $C = \sup_{s \in [a,b]} \int_a^b |k(s,t)|\,dt$.

$\therefore$ A is a bounded linear operator on a n.l.s $C[a,b]$, w.r.t $\|\cdot\|_\infty$.

Claim: $\|A\|_\infty = c = \sup_{s\in[a,b]} \int_a^b |k(s,t)|\,dt$.

$\because$ $s \longrightarrow \int_a^b |k(s,t)|$ is continuous on a compact interval $[a,b]$, $\exists$ $s_0 \in [a,b]$ such that

$$\sup_{a\le s\le b} \int_a^b |k(s,t)|\,dt = \int_a^b |k(s_0,t)|\,dt.$$

Now for any given any $\epsilon > 0$, we have

$$\int_a^b \left[|k(s_0,t)| - \epsilon \right] dt = \int_a^b \frac{|k(s_0,t)|^2 - \epsilon^2}{|k(s_0,t)| + \epsilon}\,dt$$

$$\le \int_a^b \frac{|k(s_0,t)|^2}{|k(s_0,t)| + \epsilon}\,dt$$

$$= \int_a^b k(s_0,t) \cdot \frac{\overline{k(s_0,t)}}{|k(s_0,t)| + \epsilon}\, dt$$

$$= \int_a^b k(s_0,t) \cdot x_\epsilon(t)\, dt \,,$$

where $x_\epsilon(t) = \dfrac{\overline{k(s_0,t)}}{|k(s_0,t)| + \epsilon}$, $\forall\, t \in [a,b]$.

$$\therefore \left| \int_a^b \left[|k(s_0,t)| - \epsilon \right] dt \right| \le \left| \underbrace{\int_a^b k(s_0,t)\, x_\epsilon(t)\, dt} \right|$$

$$= |A x_\epsilon(s_0)|$$

$$\le \|A x_\epsilon\|_\infty$$

$$\le \|A\|_\infty \cdot \|x_\epsilon\|_\infty$$

$$\left[\because |x_\epsilon(t)| = \frac{|\overline{k(s_0,t)}|}{|k(s_0,t)| + \epsilon} < 1 \right.$$

$$\Rightarrow \|x_\epsilon\|_\infty < 1$$

$\therefore \|Ax_\epsilon\|_\infty \le \|A\| \, \|x_\epsilon\|_\infty]$

$$\therefore \int_a^b \left[|k(s_0, t)| - \epsilon \right] dt \le \|A\|_\infty$$

$$\Rightarrow \int_a^b |k(s_0, t)| \, dt \le \|A\|_\infty + \epsilon (b-a).$$

Letting $\epsilon \to 0$, we get

$$\int_a^b |k(s_0, t)| \, dt \le \|A\|_\infty$$

$$\therefore \sup_{s \in [a,b]} \int_a^b |k(s,t)| \, dt = \int_a^b |k(s_0, t)| \, dt$$

$$\le \|A\|_\infty. \quad \text{—— } (**)$$

Also from $(*)$, we have

$$\|Ax\|_\infty \le C \|x\|_\infty$$

$$\Rightarrow \|A\|_\infty \le C = \sup_{s \in [a,b]} \int_a^b |k(s,t)| \, dt$$

$\therefore$ from $\star\star$, we have

$$\|A\|_\infty = \sup_{s \in [a,b]} \int_a^b |k(s,t)|\, dt.$$

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Problem: $X = C[a,b]$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$

Define $Ax(s) = \int_a^b k(s,t)\, x(t)\, dt$, $x \in X$.

$$\alpha_{p,q} = \int_a^b \left(\int_a^b |k(s,t)|^q\, dt \right)^{p/q} ds < \infty$$

$$\beta = \sup_{s \in [a,b]} \int_a^b |k(s,t)|\, dt < \infty$$

$$\gamma = \sup_{t \in [a,b]} \int_a^b |k(s,t)|\, dt < \infty.$$

Then show that

$$\|A\| \le \min \left\{ \alpha_{p,q}^{1/p},\ \beta^{1/q} \gamma^{1/p} \right\}$$

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Suppose $A : X \longrightarrow Y$ be a linear map, from a n.l.s X into a n.l.s Y.

We know that if $A \in BL(X, Y)$, then the null space $N(A)$ is a closed subspace of X.

But converse need not be true.

Ex: $X = C^1[0,1]$, $Y = C[0,1]$ both with $\|\cdot\|_\infty$.

Let $A : X \longrightarrow Y$ be defined by

$$Ax = x^{(1)}, \quad x \in C^1[a,b].$$

Then

$$N(A) = \{x \in X \mid Ax = 0\}$$
$$= \{x \in X \mid x^{(1)} = 0\}.$$

$= \{x \in X \mid x = c\}$

= set of all constant functions, which is a closed subspace of X.

But A is unbounded

$\because \; x_n(t) = t^n, \; t \in [0,1],$

$\|x_n\|_\infty = 1$, but $\|Ax_n\|_\infty = n$.

However, such a situation will not arise for linear functionals.

Theorem: Let X be a n.l.s and $f : X \to K$ be a non-zero linear functional on X such that null space $N(f)$ is closed. Then f is continuous and for every

$x_0 \in X - N(f)$,

$$\|f\| = \frac{|f(x_0)|}{\operatorname{dist}(x_0, N(f))}.$$

Proof: Let $x_0 \in X \ni f(x_0) \neq 0$.

Then for any $x \in X$, we have

$$x = x - \frac{f(x)}{f(x_0)} \cdot x_0 + \frac{f(x)}{f(x_0)} \cdot x_0$$

$$= y + \alpha \cdot x_0,$$

where $y = x - \frac{f(x)}{f(x_0)} \cdot x_0$, $\alpha = \frac{f(x)}{f(x_0)}$.

Now $f(y) = f\left(x - \frac{f(x)}{f(x_0)} \cdot x_0\right)$

$$= f(x) - \frac{f(x)}{f(x_0)} \cdot f(x_0) = 0$$

$$\Rightarrow y \in N(f).$$

$$\therefore \operatorname{dist}(x, N(f)) = \operatorname{dist}(y + \alpha x_0, N(f))$$

$$= \operatorname{dist}(\alpha x_0, N(f))$$

$$= |\alpha| \operatorname{dist}(x_0, N(f)).$$

$$\left|\frac{f(x)}{f(x_0)}\right| = |\alpha| = \frac{\operatorname{dist}(x, N(f))}{\operatorname{dist}(x_0, N(f))}$$

$$\Rightarrow |f(x)| = \frac{|f(x_0)|}{\operatorname{dist}(x_0, N(f))} \cdot \operatorname{dist}(x, N(f))$$

$$\leq \underbrace{\frac{|f(x_0)|}{\operatorname{dist}(x_0, N(f))}}_{c} \cdot \|x\|.$$

$\Rightarrow \quad |f(x)| \leq C\|x\|, \quad \forall x \in X$

$\therefore f : X \longrightarrow K$ is Continuous.

[$\because$ $N(f)$ is closed $\Rightarrow$ dist $(x_0, N(f)) > 0$

& dist $(x, N(f)) \leq \|x - 0\| = \|x\|$].

$$\therefore \quad \|f\| \leq \frac{|f(x_0)|}{\text{dist}\,(x_0, N(f))} \quad \text{--- (1)}$$

Now for any $u \in N(f)$

$$\begin{aligned} |f(x_0)| &= |f(x_0) - f(u)| \\ &= |f(x_0 - u)| \\ &\leq \|f\| \, \|x_0 - u\|, \quad \forall u \in N(f). \end{aligned}$$

Since this is true for any $u \in N(f)$, it follows that

$$|f(x_0)| \leq \|f\| \inf_{u \in N(f)} \|x_0 - u\|$$

$$= \|f\| \operatorname{dist}(x_0, N(f))$$

$$\Rightarrow \frac{|f(x_0)|}{\operatorname{dist}(x_0, N(f))} \leq \|f\| \quad \text{---} \quad (2)$$

$\therefore$ From (1) & (2) we get the result

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Ex:- $X = C^1[0,1]$, $\|\cdot\|_\infty$.

and $f : X \to K$ be defined by

$$f(x) = x^{(1)}(1), \quad \forall x \in X$$

f is discontinuous & $N(f)$ is not closed.

OR

Let $x(t) = t$, $x_n(t) = t - \frac{t^n}{n}$,

$\forall n \in \mathbb{N}$.

$\forall t \in [0,1]$,

$\|x_n - x\|_\infty = \frac{1}{n} \longrightarrow 0$ as $n \to \infty$.

and $f(x_n) = x_n^{(1)}(1) = 1 - 1 = 0$

$\{x_n\}$ is a sequence in $N(f)$ with

$x_n \to x$, but $x \notin N(f)$

$\because f(x) = x^{(1)}(1) = 1 \neq 0$.

$\therefore N(f)$ is not closed.

Problem: $X = c_{00}$, with $\|\cdot\|_\infty$.

for $x \in X$, $f : X \to K$ be

defined by $f(x) = \sum_{j=1}^{\infty} x(j)$.

Then s.t $N(f)$ is not a closed subspace of $X = c_{00}$.

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Suppose X and Y be n.l.s and $\{A_n\}$ be a sequence of linear maps from X into Y.

If $\{A_n x\}$ converges for every $x \in X$, then a function

$A : X \longrightarrow Y$ defined by

$$Ax = \lim_{n \to \infty} A_n x, \quad x \in X \text{ is}$$

also a linear map from X into Y.

$\because$ for any $x, y \in X$, $\alpha, \beta \in K$

$$A(\alpha x + \beta y) = \lim_{n \to \infty} A_n(\alpha x + \beta y)$$

$$= \lim_{n\to\infty} \left(\alpha A_n x + \beta A_n y \right)$$

$$= \alpha \lim_{n\to\infty} A_n x + \beta \lim_{n\to\infty} A_n y$$

$$= \alpha Ax + \beta Ay \, .$$

If each A_n is bounded linear map, what can you say about the boundedness of A?

— The answer is negative

Ex: $X = c_{00}$, $\|\cdot\|_\infty$.

For each $n \in \mathbb{N}$, define

$$f_n : X \longrightarrow \mathbb{K} \text{ by}$$

$$f_n(x) = \sum_{j=1}^{n} x(j), \quad \forall\, x = (x(1), x(2), x(3), \ldots) \in X.$$

Then $\|f_n\| = n$

$[\because x_n = (\underbrace{1, 1 \ldots 1}_{n\text{ times}}, 0, 0 \ldots) \in c_{00},$

$f_n(x_n) = n \Rightarrow \|f_n\| = n.$

$\|x_n\|_\infty = 1$

Hence each f_n is bounded

So $\{f_n\}$ is a sequence of bounded linear functionals on $X = c_{00}$.

Now

$$f(x) := \sum_{j=1}^{\infty} x(j) = \lim_{n\to\infty} \sum_{j=1}^{n} x(j)$$

$$= \lim_{n\to\infty} f_n(x)$$

But f is discontinuous linear map.

Now by imposing the boundedness of $\{\|A_n\|\}$, we can show that

$Ax = \lim_{n\to\infty} A_n x$ is also bounded linear map.

Theorem: Let X and Y be n.l.s and $\{A_n\}$ be a sequence in $BL(X,Y)$ such that $\{A_n x\}$ converges in Y for each $x \in X$. If $\{\|A_n\|\}$ is bounded, then $A : X \longrightarrow Y$ defined by $Ax = \lim_{n\to\infty} A_n x$, is also belongs to $BL(X,Y)$, and

$$\|A\| \leq \liminf_{n\to\infty} \|A_n\|.$$

Proof:

$\because$

$$Ax = \lim_{n\to\infty} A_n x, \quad x \in X$$

$$\Rightarrow \quad \|Ax\| = \|\lim_{n\to\infty} A_n x\|$$

$$= \lim_{n\to\infty} \|A_n x\|$$

$$\leq \lim_{n\to\infty} \left(\|A_n\| \, \|x\|\right)$$

$$\leq \left(\liminf_{n\to\infty} \|A_n\|\right) \|x\| .$$

$$\therefore \quad \|A\| \leq \liminf_{n\to\infty} \|A_n\| .$$

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