Group Theory

Lecture 10

02/02/2022

$$(\mathbb{Z}/n\mathbb{Z})^\times = \{\bar{a} \in \mathbb{Z}/n\mathbb{Z} \mid \gcd(a,n)=1\}.$$

$$\mathbb{Z}/6\mathbb{Z} = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}.$$

$$(\mathbb{Z}/6\mathbb{Z})^\times = \{\bar{1}, \bar{5}\}.$$

First isomorphism Thm:

Let $\phi : G \to G'$ be a surjective gp homo. Then $G/\ker\phi \cong G'$

Remark. If ϕ is not surjective then $G/\ker\phi \cong \operatorname{Im}\phi$.

Example 1. $\det : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^\times$

$$A \longmapsto \det A.$$

det is a surjective gp homo.

and $\ker(\det) = SL_n(\mathbb{R})$.

By 1st isomorphism Thm,

$$GL_n(\mathbb{R}) / SL_n(\mathbb{R}) \cong \mathbb{R}^\times$$

Example 2. $\phi : \mathbb{C}^\times \longrightarrow \mathbb{R}^\times_{>0}$.

$$\phi(z) = |z|.$$

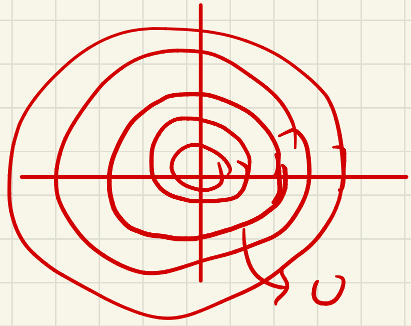
$$\ker \phi = \{ z \in \mathbb{C}^\times \mid \phi(z) = 1 \}$$
$$= \{ z \in \mathbb{C}^\times \mid |z| = 1 \} = U$$

unit circle

Note that ϕ is a surjective gp homo

By 1st isomorphism Thm,

$$\mathbb{C}^\times / U \cong \mathbb{R}^\times_{>0}$$



Example 3. $\phi : S_n \longrightarrow \{1, -1\}$

$$\phi(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Note ϕ is a surjective gp homo.

$$\ker \phi = \{\sigma \in S_n \mid \phi(\sigma) = 1\}$$

$= A_n$. $\rightsquigarrow$ gp of even permutations

By 1st isomophism Thm,

$$S_n / A_n \cong \{1, -1\}.$$

$\underline{\underline{\sigma A_n}} = \boxed{A_n}$ if σ is even permutation

$[S_n : A_n] = \left| S_n / A_n \right| = 2.$

$\boxed{\sigma A_n}$ if σ is odd.

Set of all odd perm.

Example 4. $\phi : O_n(\mathbb{R}) \longrightarrow \{1,-1\}$.

$$\phi(A) = \det A.$$

ϕ is a surjective gp homo.

$$\ker \phi = \{ A \in O_n(\mathbb{R}) \mid \phi(A) = 1 \}$$
$$= SO_n(\mathbb{R}).$$

By 1st isomorphism Thm,

$$O_n(\mathbb{R}) / SO_n(\mathbb{R}) \cong \{1,-1\} \cong \mathbb{Z}/2\mathbb{Z}.$$

Defn. For a non-empty subset A of G define the normalizer as

$$N_G(A) = \{ g \in G \mid gAg^{-1} = A \}.$$

Remark. If A is a normal subgp then $N_G(A) = G$. Note that $A \subseteq N_G(A)$.

2nd Isomorphism Thm:

Let G be a gp. Let A, B be subgps of G and assume $A \leq N_G(B)$.
Then $A \cap B$ is a normal subgp of A.
and $AB/B \cong A/A\cap B$.

Pf: First we want to show AB is a subgp of G.

Let $h_1 k_1$, $h_2 k_2 \in AB$. where $h_1, h_2 \in A$
$k_1, k_2 \in B$.

$h_1 k_1 h_2 k_2$.

$= \underline{h_1} h_2 h_2^{-1} k_1 h_2 k_2$..

$gBg^{-1} = B$.
$\forall g \in A$.

$= h_1 h_2 k_3 k_2$ where $h_2^{-1} k_1 h_2 = k_3$.

$\in AB$.

Check that AB is a subgp of G.

Next WTS $A \cap B \trianglelefteq A$.

Let $x \in A \cap B$. WTS $gxg^{-1} \in A \cap B$ for all $g \in A$.

$gxg^{-1} \in A$ (since $g \in A$ & $x \in A$).

$gxg^{-1} \in B$ (since $A \leq N_G(B)$).

$\therefore$ $gxg^{-1} \in A \cap B$ $\forall g \in A$.

Hence $A \cap B$ is a normal subgp of A.

$$\phi : AB \longrightarrow A/A \cap B.$$

$$\phi(ab) = a(A \cap B).$$

WTS ϕ is well defined.

Let $ab = a_1 b_1$ WTS $\varphi(ab) = \varphi(a_1 b_1)$.

$$a_1^{-1} a = b_1 b^{-1} \in A \cap B.$$

$\therefore$ $a_1^{-1} a \in A \cap B \Rightarrow a(A \cap B) = a_1(A \cap B)$

$\Rightarrow \varphi(ab) = \varphi(a_1 b_1)$.

WTS. ϕ is a gp homo.

$$\phi((ab)\cdot(a_1 b_1)) = \phi(a b a_1 b_1) = \phi(a a_1 a_1^{-1} b a_1 b_1)$$

$$= \phi(a a_1 b_2 b_1)$$

[where $a_1^{-1} b a_1 = b_2$]

$$= a a_1 (A \cap B).$$

$$\phi(ab)\cdot\phi(a_1 b_1) = a(A \cap B)\cdot a_1(A \cap B)$$

$$= a a_1 (A \cap B).$$

$$\therefore \phi(ab)\,\phi(a_1 b_1) = \phi((ab)\cdot(a_1 b_1))$$

Thus ϕ is a gp homo.

Note that ϕ is surjective as

$$\phi(a) = a(A \cap B).$$

$$\ker\phi = \{ ab \in AB \mid \phi(ab) = 1\cdot(A \cap B) \}.$$

$$= \{ ab \in AB \mid a(A \cap B) = A \cap B \}$$

$$= \{ ab \in AB \mid a \in A \cap B \}.$$

<u>WTS</u> $\ker\phi = B$.

$$\ker\phi = \{ab \in AB \mid a \in A\cap B\}.$$

$$B \subseteq \{ab \in AB \mid a \in A \cap B\}.$$

(because $1 \in A$)

and $\ker\phi \subseteq B$ $[\because A\cap B \subseteq B]$.

$\therefore$ $\ker\phi = B$.

Hence by 1st isomorphism Thm

$$AB/B \cong A/A\cap B.$$

Third isomorphism Thm :

Let G be a gp and let H & K be normal subgps of G s.t $H \subseteq K$. Then K/H is a normal subgp of G/H.

$$\frac{G/H}{K/H} \cong G/K.$$

<u>Pf</u>: WTS K/H is a normal subgp of G/H.

$$gH\,kH\,g^{-1}H = \underbrace{gkg^{-1}}_{\in K}H \in K/H.$$

[as $K \trianglelefteq G$.]

Thus K/H is a normal subgp of G/H.

Define $\phi : G/H \longrightarrow G/K$.

$$\phi(gH) = gK.$$

WTS. Φ is well defined.

Let $g_1 H = g_2 H$ WTS $\Phi(g_1 H) = \Phi(g_2 H)$

$\boxed{\text{WTS } g_1 K = g_2 K}$

$g_2^{-1} g_1 \in H \leq K.$

$\therefore g_2^{-1} g_1 \in K \Rightarrow g_1 K = g_2 K$

$\therefore$ Φ is well defined.

check Φ is a gp homo.

Note that Φ is surjective.

$$\ker \Phi = \{ gH \in G/H \mid \Phi(gH) = K \}.$$

$$= \{ gH \in G/H \mid gK = K \}.$$

$$= \{ gH \in G/H \mid g \in K \}$$

$$= K/H.$$

$\therefore$ By 1st isomorphism Thm, $\quad G/H \big/ K/H \cong G/K.$

$\underline{\underline{Q}}$ G is a gp. H is a normal subgp of G.

G/H. How does the subgps of G/H looks like?

what is the relation between the subgps of G and subgps of G/H?

<u>Propn</u>: Let $f : G \to \tilde{G}$ be a surjective gp homo and $K = \ker f$. Then there is a bijective correspondence

$$\left\{ \begin{array}{l} \text{subgps of } G \\ \text{containing } K \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{subgps of} \\ \tilde{G} \end{array} \right\}.$$

$$\begin{array}{ccc} \cup\!\!\!\!\cup & & \\ H & \longmapsto & f(H). \\ & & \\ f^{-1}(\tilde{H}) & \longleftarrow & \tilde{H} \end{array}$$

$$\left\{ \begin{array}{l} \text{normal subgps of } G \\ \text{containing } K \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{normal} \\ \text{subgps of } \tilde{G} \end{array} \right\}.$$

Pf: If H is a subgp of G containing K then WTS $f(H)$ is a subgp of $\tilde{G}$.

Let $a, b \in f(H)$ i.e $a = f(h_1)$.

& $b = f(h_2)$

where $h_1, h_2 \in H$.

Then
$$
\begin{aligned}
ab^{-1} &= f(h_1)\, f(h_2)^{-1} \\
&= f(h_1)\, f(h_2^{-1}) \\
&= f(h_1 h_2^{-1}). \in f(H).
\end{aligned}
$$

[Since H is a subgp so $h_1 h_2^{-1} \in H$].

$\therefore$ $f(H)$ is a subgp of $\tilde{G}$.

WTS $f^{-1}(f(H)) = H$.

$f^{-1}(f(H)) = \{g \in G \mid f(g) \in f(H)\} \supset H$ is clear, WTS $f^{-1}(f(H)) \subseteq H$.

Let $g \in f^{-1}(f(H))$

$\Rightarrow f(g) \in f(H)$.

$\Rightarrow f(g) = f(h)$ for some $h \in H$.

$\Rightarrow f(h^{-1}g) = 1$

$\Rightarrow h^{-1}g \in K \subseteq H$.

$\Rightarrow g \in H$.

$\therefore \boxed{f^{-1}(f(H)) = H}$

Let $\tilde{H}$ is a subgp of G,

Check. $f^{-1}(\tilde{H})$ is a subgp of G containing K.

WTS. $f(f^{-1}(\tilde{H})) = \tilde{H}$.

Note that $f(f^{-1}(\tilde{H})) \subset \tilde{H}$.

is clear.

<u>WTS</u> $\tilde{H} \subset f(f^{-1}(\tilde{H}))$.

Let $h \in \tilde{H}$ then $\exists\, g \in G$ s.t

$$f(g) = h \in \tilde{H}.$$

$$g \in f^{-1}(\tilde{H}).$$

Thus $h \in f(f^{-1}(\tilde{H}))$.

Hence $f(f^{-1}(\tilde{H})) = \tilde{H}$.