Assignment 2
Mathematical Methods

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1) i) $2y'' + 18y = 6\tan(3t)$

$\Rightarrow y'' + \cancel{8}\, 9y = 3\tan(3t)$

Corresponding homogeneous ODE : $y'' + 9y = 0$

Characteristic eq$\underline{\underline{n}}$: $\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i$

So, $y_1 = e^{0.t}\sin(3t) = \sin 3t$

$y_2 = e^{0.t}\cos 3(t) = \cos 3t$ $\Big\}$ LI sol$\underline{\underline{ns}}$ of $y'' + 9y = 0$

Gen. sol$\underline{\underline{n}}$ of homogeneous part

$y_c = c_1 y_1 + c_2 y_2 = c_1 \sin(3t) + c_2 \cos(3t)$

Particular integral, $y_p = u(t).y_1(t) + v(t)y_2(t)$

$\Rightarrow y_p = u(t)\sin(3t) + v(t)\cos(3t)$

By method of variation of parameters,

$u'y_1 + v'y_2 = 0 \Rightarrow u'\sin(3t) + v'(\cos(3t)) = 0 \rightarrow (1)$

& $u'y_1' + v'y_2' = 3\tan(3(t)) \Rightarrow 3u'\cos(3t) - 3v'\sin(3t)$

$= 3\tan(3t)$

$\Rightarrow u'\cos(3t) - v'\sin(3t) = \tan(3t)$

$\rightarrow (2)$

Solving (1) & (2)

$v' = -u'\tan(3t)$

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$\therefore$ ② $\Rightarrow$ $u' \cos(3t) + u' \tan(3t) \sin(3t) = \tan(3t)$

$$u' = \frac{\sin(3t)}{\cos(3t)} \times \frac{1}{\cos(3t) + \frac{\sin^2(3t)}{\cos(3t)}}$$

$$\Rightarrow u' = \sin(3t) \Rightarrow u = \int \sin(3t)\,dt = -\frac{\cos(3t)}{3}$$

$$\therefore v' = \frac{-\sin^2(3t)}{\cos(3t)} = \frac{\cos^2(3t) - 1}{\cos(3t)} = \cos(3t) - \sec(3t)$$

$$v = \int \cos(3t) - \sec(3t)\,dt$$

$$= \frac{\sin(3t)}{3} - \frac{1}{3} \ln |\sec(3t) + \tan(3t)|$$

$\therefore$ Particular integral $\Rightarrow$

$$y_p = u \sin(3t) + v \cos(3t)$$

$$= -\cancel{\frac{\cos(3t)\sin(3t)}{3}} + \cancel{\frac{\sin(3t)\cos(3t)}{3}}$$

$$-\frac{1}{3} \cos(3t) \ln |\sec(3t) + \tan(3t)|$$

$$\Rightarrow y_p = \frac{-1}{3} \cos(3t) \ln |\sec(3t) + \tan(3t)|$$

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General sol^n $\Rightarrow$ $y = y_c + y_p$

$$\Rightarrow y = c_1 \sin(3t) + c_2 \cos(3t) - \frac{\cos(3t)}{3} \ln|\sec(3t) + \tan(3t)|$$

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1) ii) $y'' - 3y' + 2y = e^{3t}$

Homogeneous part :- $y'' - 3y' + 2y = 0$

Characteristic eqn :- $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1, 2$

$\left.\begin{array}{l} y_1 = e^t \\ y_2 = e^{2t} \end{array}\right\}$ LI solutions

Soln of homogeneous part:-

$y_c = c_1 e^t + c_2 e^{2t}$

Particular integral $\Rightarrow y_p = u(t)\, y_1(t) + v(t)\, y_2(t)$

$y_p = u(t) e^t + v(t) e^{2t}$

Acc. to method of variation of prameters :-

$u'y_1 + v'y_2 = 0 \Rightarrow u'\cancel{e^t} + v'e^{2t} = 0$

$\Rightarrow u' + v'e^t = 0 \longrightarrow (1)$

$u'y_1' + v'y_2' = e^{3t} \Rightarrow u'\cancel{e^t} + 2v'e^{2t} = e^{3t}$

$\Rightarrow u' + 2v'e^t = e^{2t} \longrightarrow (2)$

from (1) $v'e^t = -u'$

(2) :- $u' - 2u' = e^{2t} \Rightarrow u' = -e^{2t} \Rightarrow u = \int -e^{2t} dt$

$\Rightarrow u = \frac{-e^{2t}}{2}$

$\therefore v' = e^t \Rightarrow v = e^t$

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Particular integral :- $y_p = u(t)e^t + v(t)e^{2t}$

$$\Rightarrow y_p = \frac{-e^{2t}}{2} \cdot e^t + e^t \cdot e^{2t}$$

$$\Rightarrow y_p = \frac{e^{3t}}{2}$$

Gen. $sol^{\underline{n}}$:- $y = y_c + y_p$

$$\Rightarrow y = c_1 e^t + c_2 e^{2t} + \frac{e^{3t}}{2}$$

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2) $ty'' - (t+1)y' + y = t^2$

$\Rightarrow y'' - \frac{(t+1)}{t}y' + \frac{y}{t} = t$

Solutions of the homogeneous part :- $y_1 = e^t$;

$y_2 = t+1$

Particular integral :- $y_p = u(t)y_1(t) + v(t)y_2(t)$

$\Rightarrow y_p = u(t)e^t + v(t)(t+1)$

By method of variation of parameters

$u'y_1 + v'y_2 = 0 \Rightarrow u'e^t + v'(t+1) = 0 \rightarrow$ (1)

$u'y_1' + v'y_2' = t \Rightarrow u'e^t + v' = t \Rightarrow v' = t - u'e^t \rightarrow$ (2)

using $v' = t - u'e^t$ in (1)

$\Rightarrow u'e^t + (t - u'e^t)(t+1) = 0$

$\Rightarrow \cancel{u'e^t} + t^2 + \cancel{t} - u'\cancel{t}e^t - \cancel{u'e^t} = 0$

$\Rightarrow u'e^t = t+1 \Rightarrow u' = \dfrac{t+1}{e^t}$

$\Rightarrow u = \int (t+1)e^{-t}dt = -(t+1)e^{-t} + \int e^{-t}dt = -(t+2)e^{-t}$

~~now, $v' = t + (t+2)(e^{-t})e^{t} = 2t+2$~~

~~$\Rightarrow v = \int 2t+2\, dt = \frac{2t^2}{2} + 2t = t^2 + 2t$~~

Using $u'e^t = t - v'$ in (1)

$\Rightarrow t - v' + v'(t+1) = 0 \Rightarrow v' = -1$

$\Rightarrow v = \int -1\, dt = -t$

Particular integral :- ~~$y_p = -(t+2)e^{-t}e^{t}$~~

$y_p = -(t+2) - t(t+1) = -t^2 - 2t - 2$

Gen. Sol$^{\underline{n}}$:- $y = y_c + y_p$

$\Rightarrow y = c_1e^t + c_2(t+1) - t^2 - 2t - 2$

3) $y''' + a(x)y'' + b(x)y' + c(x)y = r(x) \longrightarrow (1)$

Let the above eq$^{\underline{n}}$ be considered

Suppose the corresponding homogeneous ODE of (1) has 3 L.I sol$^{\underline{n}s}$ $y_1(x)$, $y_2(x)$ & $y_3(x)$

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Let the particular integral of 1 be

$$y_p = u(x)\,y_1(x) + v(x)\,y_2(x) + w(x)\,y_3(x)$$

(OR)

$$y_p = u y_1 + v y_2 + w y_3$$

Then,

$$y'_p = u y_1' + v y_2' + w y_3' + (u' y_1 + v' y_2 + w' y_3)$$

Let us choose u, v, w such that

$$u' y_1 + v' y_2 + w' y_3 = 0 \longrightarrow 2$$

Then,

$$y'_p = u y_1' + v y_2' + w y_3'$$

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$\Rightarrow y''_p = u y_1'' + v y_2'' + w y_3'' + (u' y_1' + v' y_2' + w' y_3')$

Again we set

$u' y_1' + v' y_2' + w' y_3' = 0 \quad \rightarrow (3)$

we get

$y''_p = u y_1'' + v y_2'' + w y_3''$

$\Rightarrow y'''_p = u y_1''' + v y_2''' + w y_3''' + u' y_1'' + v' y_2'' + w' y_2''$

Putting expressions of y_p, y_p', y_p'', y_p''' in (1) we get

$u(y_1''' + a(x) y_1'' + b(x) y_1' + c(x) y_1) + v(y_2''' + a(x) y_2'' + b(x) y_2' + c(x) y_2)$
$+ w(y_3''' + a(x) y_3'' + b(x) y_3' + c(x) y_3) + u'.y_1'' + v'.y_2'' + w' y_3'' = r(x)$
$\rightarrow (4)$

Now, as y_1, y_2 & y_3 are sol^ns^ of homogeneous part of (1)

$y_i''' + a(x) y_i'' + b(x) y_i' + c(x) y_i = 0$, for $i = 1, 2, 3$

Thus (4) $\Rightarrow$ becomes $u'.y_1'' + v' y_2'' + w' y_3'' = r(x) \rightarrow (5)$

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Solving for $u'(x)$, $v'(x)$ & $w'(x)$ using (2), (3) & (5)

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r(x) \end{bmatrix}$$

By cramer's rule, $u' = \dfrac{W_1}{W}$, $v' = \dfrac{W_2}{W}$, $w' = \dfrac{W_3}{W}$

where W = determinant of coefficient matrix
$= W(y_1, y_2, y_3)$ i.e Wronskian of y_1, y_2, y_3

&

W_i = determinat obtained from W by replacing i^{th} column by $[0\ 0\ r(x)]^T$; for $i = 1, 2, 3$

Then, $u = \displaystyle\int \frac{W_1}{W(y_1, y_2, y_3)}\, dx$, $v = \displaystyle\int \frac{W_2}{W(y_1, y_2, y_3)}\, dx$,

$w = \displaystyle\int \frac{W}{W(y_1, y_2, y_3)}\, dx$

$\therefore$ Particular integral of (1)

$\Rightarrow y_p = u\, y_1 + v\, y_2 + w\, y_3$, where u, v, w are obtained from integrals above

Hence proved.