Lecture 9

$$= m(E_1) - \lim(m(E_i)).$$

Consider $\lim(E_1 \setminus E_i) = \bigcup_{i=1}^{\infty}(E_1 \setminus E_i)$

$$= E_1 \setminus \bigcap_{i=1}^{\infty} E_i$$

$$= E_1 - \lim(E_i)$$

$\therefore\ m(\lim(E_1 \setminus E_i)) = m(E_1 \setminus \lim(E_i))$

$$= m(E_1) - m(\lim(E_i))$$

$\therefore$ By (*), $\quad m(E_1) - m(\lim(E_i)) = m(E_1) - \lim(m(E_i))$

$$\Rightarrow \quad m(\lim(E_i)) = \lim(m(E_i)).$$

Proposition:- (i) Every non-empty open set has +ve measure. That is, If $U \subseteq \mathbb{R}$ is any non-empty open set, then $m(U) > 0$.

(ii) Let $\mathbb{Q} = \{q_1, q_2, \ldots\}$ & $G = \bigcup_{n=1}^{\infty}\left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)$.

Then for any closed set $F \subseteq \mathbb{R}$, $m(G \Delta F) > 0$.

Proof:-

(i) We know any open set U in $\mathbb{R}$ is the union of disjoint open intervals at most countable in number.

Say $U = \dot{\bigcup}_{j=1}^{\infty} U_j$, U_j is an open interval.

We know, $U, U_j \in \mathcal{M} \quad \forall j$

$$\therefore m(U) = m\left(\dot{\bigcup}_{j=1}^{\infty} U_j\right) = \sum_{j=1}^{\infty} m(U_j)$$

$$= \sum_{j=1}^{\infty} l(U_j) > 0$$

$[\because \exists U_{j_0}$ st $l(U_{j_0}) > 0)$.
open

$\therefore m(U) > 0.$

$\boxed{m(U) = m^*(U)}$

(ii). Given

$$G = \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right).$$ open set $\therefore \in \mathcal{M}$.

Let $F \subseteq \mathbb{R}$ closed set.

Consider $G \setminus F = G \cap F^c$

To show: $m(G \Delta F) > 0$.

$G \Delta F = \underbrace{(G \setminus F)}_{\in \mathcal{M}} \cup \underbrace{(F \setminus G)}_{\in \mathcal{M}}$

$$m(G \triangle F) = m(G \setminus F) + m(F \setminus G)$$

If $m(G \setminus F) > 0$, then $m(G \triangle F) > 0$, we are done.

Assume $m(G \setminus F) = 0$.

$G \setminus F = G \cap F^c$ is an open set.

$\therefore$ By (i), $G \setminus F = \phi$

$$\Rightarrow \underline{G \subseteq F}.$$

But $\mathbb{Q} \subseteq G \subseteq F$

$\Rightarrow \mathbb{R} = \overline{\mathbb{Q}} \subseteq \overline{G} \subseteq \overline{F} = F$ [$\because F$ is a closed set]

$\Rightarrow m(F) = m(\mathbb{R}) = +\infty$.

Also
$$m(G) = m\left(\bigcup_{n=1}^{\infty}\left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)\right)$$
$$\leq \sum_{n=1}^{\infty} m\left(\left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)\right)$$
$$= \sum_{n=1}^{\infty} \frac{2}{n^2} < \infty$$

Thus $m(G) < \infty$ & $m(F) = \infty$, $G \subseteq F$

Then $m(F \setminus G) = \infty$

$$F = G \,\dot\cup\, F \setminus G$$

$$\Rightarrow \; m(F \setminus G) > 0.$$

$$m(F) = m(G) + m(F \setminus G)$$
$$\underset{\infty}{m(F)} = \underset{<\infty}{m(G)} + \underline{\underline{m(F\setminus G)}}$$

$$\Rightarrow \; m(G \,\Delta\, F) > 0.$$

<u>Proposition</u>:- The Cantor set has measure zero.

Thus, there exists an uncountable set of measure zero.

<u>proof</u>:-

Cantor set $P = \bigcap_{n=1}^{\infty} P_n$, $P_n = \dot{\bigcup}_{r=1}^{2^{n-1}} J_{n,r}$.

& $\ell(J_{n,r}) = \frac{1}{3^n}$.

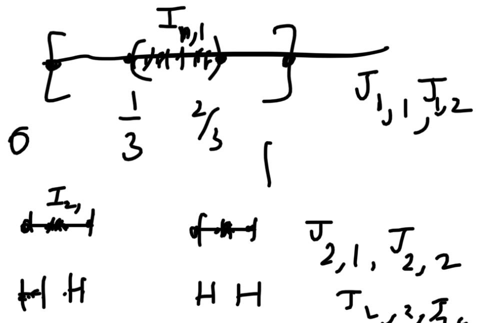
Now $P^c = [0,1] \setminus P$

$$= [0,1] \setminus \left(\bigcap_{n=1}^{\infty} P_n \right)$$

$$= \bigcup_{n=1}^{\infty} \left([0,1] \setminus P_n \right)$$

$$= \bigcup_{n=1}^{\infty} \underbrace{\left([0,1] \setminus \bigcup_{r=1}^{2^{n-1}} J_{n,r} \right)}_{\shortparallel}$$

$$= \dot{\bigcup}_{n=1}^{\infty} \underline{\left(\dot{\bigcup}_{r=1}^{2^{n-1}} I_{n,r} \right)}$$

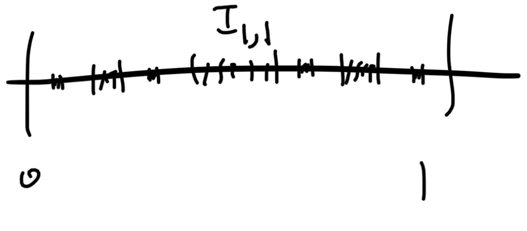


$n=1$: $I_{1,1}$ ✓

$n=2$: $I_{2,1}$, $I_{2,2}$ ✓

$\vdots$

$$m(P^c) = m\left(\dot{\bigcup}_{n=1}^{\infty} \dot{\bigcup}_{r=1}^{2^{n-1}} I_{n,r} \right)$$



$$= \sum_{n=1}^{\infty} \sum_{r=1}^{2^{n-1}} l(I_{n,r})$$

$$= \sum_{n=1}^{\infty} \sum_{r=1}^{2^{n-1}} \frac{1}{3^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{3^n} \left(\sum_{r=1}^{2^{n-1}} 1 \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot 2^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

$$= \frac{1}{2} \cdot \frac{(2/3)}{1 - 2/3} = 1$$

$$\therefore \quad m(P^c) = 1$$

$$\Rightarrow m(P) = m([0,1]) - m(P^c)$$

$$= 1 - 1$$

$$= 0.$$

The following result states that the measurable sets are those which can be approximated closely in terms of m^* by open or closed sets.

Theorem:- Let $E \subseteq \mathbb{R}$. Then the following are equivalent:

(i) E is measurable

(ii) Given $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}$, such that $E \subseteq U$ & $m^*(U \setminus E) \leq \varepsilon$.

(iii) There exists a G_δ-set $G \subseteq \mathbb{R}$ such that $E \subseteq G$ & $m^*(G \setminus E) = 0$.

(iv) Given $\varepsilon > 0$, there exists a closed set $F \subseteq \mathbb{R}$ such that $F \subseteq E$ & $m^*(E \setminus F) \leq \varepsilon$.

(v) There exists an F_σ-set $F \subseteq \mathbb{R}$ such that $F \subseteq E$ & $m^*(E \setminus F) = 0$.

(**Def:-** A non-negative countably additive set function (for example $m : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$) satisfying one of the above conditions (ii) to (v) is said to be a <u>regular measure</u>)

The above theorem says the Lebesgue measure is a regular measure.

proof:-

(i) $\Rightarrow$ (ii): Assume E is measurable. Let $\varepsilon > 0$.

To show: There exists an open set $U \subseteq \mathbb{R}$ such that $E \subseteq U$ & $m^*(U \setminus E) \leq \varepsilon$.

Suppose $m(E) < \infty$.

We already prove that $m^*(U) \leq m^*(E) + \varepsilon$

$\exists\ U \subseteq \mathbb{R}$ open set, $E \subseteq U$.

But $m^*(U \setminus E) = m^*(U) - m^*(E)$, if $m^*(E) < \infty$.

$\therefore$ $m^*(U \setminus E) \leq \varepsilon$, as required.

Suppose $m(E) = \infty$.

Let $\mathbb{R} = \dot{\bigcup}_{n=1}^{\infty} I_n$, a disjoint union of finite left closed, right open intervals.

Then $E = E \cap \mathbb{R} = \dot{\bigcup}_{n=1}^{\infty} E \cap I_n$

Let $E_n = E \cap I_n$. Then $E = \dot{\bigcup}_{n=1}^{\infty} E_n$.

& $m(E_n) \leq m(I_n) < \infty$. $\forall n$ ✓

& E_n measurable & $m(E_n) < \infty$,

Then there exists (by above) an open set $U_n \subseteq \mathbb{R}$ such that $E_n \subseteq U_n$ &

$$m^*(U_n \setminus E_n) \leq \frac{\varepsilon}{2^n}.$$

Let $U = \bigcup_{n=1}^{\infty} U_n$, U open set.

$$U \setminus E = \left(\bigcup_{n=1}^{\infty} U_n\right) \setminus \left(\bigcup_{n=1}^{\infty} E_n\right)$$

$$\subseteq \bigcup_{n=1}^{\infty} (U_n \setminus E_n) \quad \text{(check it!)}$$

$$\therefore\ m^*(U \setminus E) \leq m^*\left(\bigcup_{n=1}^{\infty} (U_n \setminus E_n)\right)$$

$$\leq \sum_{n=1}^{\infty} m^*(U_n \setminus E_n).$$

$$\leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$$

$\therefore\ m(U \setminus E) \leq \varepsilon$.

(ii) $\Rightarrow$ (iii): Assume (ii).

To show: There exists a G_δ-set G such that $E \subseteq G$ & $m^*(G \setminus E) = 0$.

Let $\varepsilon = \frac{1}{n} > 0$.

Then there exists (by (ii)) an open set $U_n \subseteq \mathbb{R}$ such that

$$E \subseteq U_n \quad \& \quad m(U_n \setminus E) \leq \frac{1}{n}$$

$\forall n \geq 1$

Let $G = \bigcap_{n=1}^{\infty} U_n$ G_δ-set.

& $E \subseteq G$

$m^*(G \setminus E) \leq m^*(U_n \setminus E)$ $\quad (\because G \setminus E \subseteq U_n \setminus E)$

$\leq \frac{1}{n} \quad \forall n \geq 1$

$\therefore \quad m^*(G \setminus E) = 0$.

(iii) $\Rightarrow$ (i): Assume (iii).

To show: E is measurable.

We have $E = G \setminus (G \setminus E)$

$\because G$ is a Borel set, G is measurable

& $m^*(G \setminus E) = 0. \Rightarrow G \setminus E$ is measurable.

Thus $G, G \setminus E \in \mathcal{M}$

$\Rightarrow E = G \setminus (G \setminus E) \in \mathcal{M}$

$\therefore E \in \mathcal{M}$.

(i) $\Rightarrow$ (iv): Assume E is measurable.

Let $\varepsilon > 0$.

$\therefore$ $E^c \in \mathcal{M}$

There exists an open set $U \subseteq \mathbb{R}$ such that

$E^c \subseteq U$ & $m^*(U \setminus E^c) \leq \varepsilon$ (by (ii)).

Now $U \setminus E^c = U \cap (E^c)^c$

$= U \cap E$

$= E \cap (U^c)^c$

$= E \setminus U^c$.

Let $F = U^c$. Then F is a closed set.

& $m^*(E \setminus F) = m^*(E \setminus U^c)$

$= m^*(U \setminus E^c)$

$\leq \varepsilon$.

& $E^c \subseteq U \Rightarrow E \supseteq U^c = F$.

Thus $\underset{\text{closed}}{F} \subseteq E$ & $m(E \setminus F) \leq \varepsilon$.

This proves (iv).

(iv) $\Rightarrow$ (v): Assume (iv).

For each n, let F_n be a closed set such that $F_n \subseteq E$ & $m^*(E \setminus F_n) \leq \frac{1}{n}$.

Let $F = \bigcup_{n=1}^{\infty} F_n$, F_σ-set.

& $F \subseteq E$,

$\boxed{\begin{array}{l} F_n \subseteq F \\ E \setminus F_n \supseteq E \setminus F \end{array}}$

$m^*(E \setminus F) \leq m^*(E \setminus F_n) \leq \frac{1}{n}$. $\forall n$

$\Rightarrow \quad m^*(E \setminus F) = 0.$

(v) $\Rightarrow$ (i): Assume (v): $\exists\, F$ F_σ-set, $E \supseteq F$, $m(E \setminus F) = 0$.

To show: $E \in \mathcal{M}$.

We have $E = \underset{\overset{\cap}{\mathcal{M}}}{F} \cup \underset{\overset{\cap}{\mathcal{M}}}{(E \setminus F)} \in \mathcal{M}$.

Measurable functions

Definition:- Let $E \subseteq \mathbb{R}$ be a measurable set.

An extended real valued function,

$f: E \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ is said to be a

Lebesgue measurable function or a measurable function, if for each $\alpha \in \mathbb{R}$, the set

$$\{x \in E \mid f(x) > \alpha\} = f^{-1}((\alpha, \infty))$$

is measurable.

Example:- (1) Let $E \subseteq \mathbb{R}$ be measurable.

$$\chi_E : \mathbb{R} \to \mathbb{R}, \quad \chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

For any $\alpha \in \mathbb{R}$,

$$\chi_E^{-1}((\alpha, \infty)) = \begin{cases} \phi \in \mathcal{M}, & \text{if } \alpha \geq 1 \\ E \in \mathcal{M} & \text{if } 0 \leq \alpha \lneq 1 \\ \mathbb{R} \in \mathcal{M} & \text{if } \alpha < 0. \end{cases}$$

$$\| $$

$$\{x \in E \mid \chi_E(x) \not> \alpha\}. \in \mathcal{M}$$

$\therefore$ χ_E is a measurable function.

Check that χ_E is measurable if and only if E is measurable.