Date

NSOPDE ASSIGNMENT - 2

GADDAM YOGESH
19MA20015

1. $y' = \lambda y \quad \lambda < 0$

$u_{n+1} = u_n + \frac{h}{4}(K_1 + 3K_2)$

$K_1 = f(t_n, u_n) \quad K_2 = f\left(t_n + \frac{h}{3}, u_n + \frac{h}{3}(K_1 + K_2)\right)$

$K_1 = \lambda u_n$

$K_2 = \lambda\left(u_n + \frac{h}{3}(K_1 + K_2)\right) = \lambda u_n + \frac{h}{3}\lambda(\lambda u_n) + \frac{h}{3}\lambda K_2$

$\Rightarrow K_2\left(1 - \frac{\lambda h}{3}\right) = \lambda u_n\left(1 + \frac{\lambda h}{3}\right)$

$\Rightarrow \quad K_2 = \lambda u_n \frac{(3+\lambda h)}{(3-\lambda h)}$

$u_{n+1} = u_n + \frac{h}{4}\left[\lambda u_n + 3\lambda u_n \frac{(3+\lambda h)}{(3-\lambda h)}\right]$

$u_{n+1} = u_n + \frac{\lambda h}{4} u_n + \frac{3\lambda h u_n}{4} \frac{(3+\lambda h)}{(3-\lambda h)}$

$\frac{u_{n+1}}{u_n} = 1 + \frac{\lambda h}{4}\left[1 + \frac{3(3+\lambda h)}{(3-\lambda h)}\right]$

$= 1 + \frac{\lambda h}{4}\left[\frac{3 - \lambda h + 9 + 3\lambda h}{(3-\lambda h)}\right]$

$\frac{u_{n+1}}{u_n} = 1 + \frac{\lambda h(6+\lambda h)}{2(3-\lambda h)}$

for absolute stability, $\left|1 + \frac{\lambda h(6+\lambda h)}{2(3-\lambda h)}\right| \le 1$

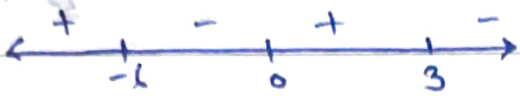
$\Rightarrow$

$-1 \le 1 + \frac{\lambda h(6+\lambda h)}{2(3-\lambda h)} \le 1$

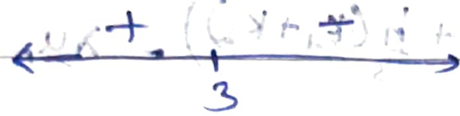
#

$$-2 \le \frac{\lambda h(6+\lambda h)}{2(3-\lambda h)} \le 0$$

let $\lambda h = x \Rightarrow -4 \le \dfrac{x(6+x)}{(3-x)} \le 0$

$\Rightarrow$  $\dfrac{x(6+x)}{(3-x)} \le 0$

$x \in [-6,0] \cup [3,\infty)$

$\Rightarrow$  $\dfrac{x(6+x)}{3-x} + 4 \ge 0$

$\Rightarrow x < 3.$

$\Rightarrow$

$\therefore \; x \in [-6;0]$

$\Rightarrow \lambda h \in [-6;0]$ is the region of absolute stability.

2. $y' = \lambda y \; [\lambda < 0]$

$u_{n+1} = u_n + \frac{1}{4}(3K_1 + K_2)$ $\quad K_1 = hf\left(t_n + \frac{h}{3}, u_n + \frac{K_1}{3}\right)$

$K_2 = hf(t_n + h, u_n + K_1)$

$K_1 = h\lambda\left(u_n + \frac{K_1}{3}\right) = u_n\lambda h + \frac{\lambda h K_1}{3}$

$\Rightarrow K_1 = \dfrac{3u_n\lambda h}{(3-\lambda h)}$

$K_2 = \lambda h(u_n + K_1) = \lambda h\left(u_n + \dfrac{3u_n\lambda h}{(3-\lambda h)}\right)$

$$K_2 = u_n\lambda h\left[\frac{3-\lambda h+3\lambda h}{3-\lambda h}\right] = u_n\lambda h\left[\frac{3+2\lambda h}{3-\lambda h}\right]$$

$$u_{n+1} = u_n + \frac{1}{4}\left[\frac{9u_n\lambda h}{(3-\lambda h)} + \frac{u_n\lambda h[3+2\lambda h]}{(3-\lambda h)}\right]$$

$$= u_n + \frac{u_n\lambda h}{4(3-\lambda h)}\left[9+3+2\lambda h\right]$$

$$\frac{u_{n+1}}{u_n} = 1 + \frac{\lambda h}{2(3-\lambda h)}(6+\lambda h)$$

let $\lambda h = x$ $\Rightarrow$ $\left|1+\frac{\lambda h(6+\lambda h)}{2(3-\lambda h)}\right| \leq 0$

$\Rightarrow$ $-4 \leq \frac{\lambda h(6+\lambda h)}{(3-\lambda h)} \leq 0$ $\Rightarrow$ $-4 \leq \frac{x(6+x)}{(3-x)} \leq 0$

$\Rightarrow$ $x \in [-6,0]$

$\therefore$ $\lambda h \in [-6,0]$ is the region of absolute stability.

given $y' = t^2+y^2$ $\quad y(1)=2$ $\quad h=0.1$

$$y(1.1) = y_1 = y_0 + \frac{1}{4}(3K_1+K_2)$$

$$K_1 = (0.1)\left[\left(1+\frac{0.1}{3}\right)^2 + \left(2+\frac{K_1}{3}\right)^2\right]$$

$$F(K_1) = \frac{K_1^2}{9} - \frac{26K_1}{3} + 4 + \frac{(3.1)^2}{9} = 0$$

$$= K_1^2 - 78K_1 + 36 + (3.1)^2 = 0$$

using NR Method to compute $K_1$

$$\Rightarrow F(K_1) = K_1^2 - 78K_1 + 36 + (3.1)^2$$

$$F'(K_1) = 2K_1 - 78 = 2(K_1 - 39)$$

$$K_1^{(s+1)} = K_1^{(s)} - \frac{F(K_1^{(s)})}{F'(K_1^{(s)})} \qquad \text{let } K_1^0 = 0$$

$$\Rightarrow K_1^1 = 0 - \frac{(36 + (3.1)^2)}{-78} = 0.5847435897$$

$$K_1^2 = K_1^1 - \frac{F(K_1^1)}{F'(K_1^1)} = 0.5891939709$$

$$K_1^3 = K_1^2 - \frac{F(K_1^2)}{F'(K_1^2)} = 0.5891942287$$

$$K_1^4 = K_1^3 - \frac{F(K_1^3)}{F'(K_1^3)} = 0.5891942287$$

$$\therefore K_1 = 0.5891942287$$

$$K_2 = h\left[(1+0.1)^2 + (2+K_1)^2\right]$$

$$= 0.1\left[(1.1)^2 + (2+K_1)^2\right] = 0.7913926754$$

$$y(1.1) = 2 + \frac{1}{4}(3K_1 + K_2) \Rightarrow 2.63974384$$

$$\therefore y(1.1) = 2.63974384$$

3. $y' = x + y$ $\quad [x(0) = 0 \quad y(0) = 1] \Rightarrow x_0 = 0 \quad y_0 = 1$

$h = 0.1$

using $4^{th}$ order classical R-K method to compute unknown values $\Rightarrow$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_n, y_n) \qquad k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

for $x = 0.1$

$k_1 = 0.1 \quad k_2 = 0.11 \quad k_3 = 0.1105 \quad k_4 = 0.12105$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\Rightarrow y(0.1) = y_1 = 1.110341667$$

for $x = 0.2$

$k_1 = 0.1210341667 \qquad k_2 = 0.1320858750$

$k_3 = 0.1326384604 \qquad k_4 = 0.1442980127$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\Rightarrow y(0.2) = y_2 = 1.242805141\mathbf{7}$$

for $x = 0.3$

$k_1 = 0.1442805142 \qquad k_2 = 0.1564945399$

$k_3 = 0.1571052412 \qquad k_4 = 0.1699910383$

$$\Rightarrow y(0.3) = y_3 = 1.399716994\mathbf{1}$$

using Milne's 4th order P-C method

Predictor-

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$y_4^{(P)} = 1.58364162408$$

Corrector

$$y_4^{(C)} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4\right)$$

$$\Rightarrow y_4^{(C)} = 1.58364896651$$

for a 4th order method, it is accurate upto 5th decimal

$$\boxed{\therefore\ y(0.4) \cong 1.58364}$$

4. $y' = x^3 - y^2 - 2$ $x_0 = 0$ $y_0 = 1$ $h = 0.1$

using taylor's series to find $y(-0.1)$, $y(0.1)$, $y(0.2)$

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n'''$$

$$y' = x^3 - y^2 - 2$$
$$y'' = 3x^2 - 2yy'$$
$$y''' = 6x - 2(y')^2 - 2yy''$$

$y(-0.1) = 1.335 \Rightarrow y_{-1}$

$y(0.1) = 0.725 \Rightarrow y_1$

$y(0.2) = 0.48807453344 \Rightarrow y_2$

using Milne's predictor - corrector method.

$$y_{n+1}^{(p)} = y_{n-3} + \frac{4h}{3}\left[2f_n - f_{n-1} + 2f_{n-2}\right]$$

$$y_{n+1}^{(c)} = y_{n-1} + \frac{h}{3}\left[f_{n+1}^{(p)} + 4f_n + f_{n-1}\right]$$

for $x = 0.3$

$y_3^{(p)} = 0.27689222$

$y_3^{(c)} \Rightarrow y_4^{(1)} = 0.27516129$

$y_4^{(2)} = 0.275193314212$

$\boxed{\therefore y(0.3) = 0.275193314212}$

for $x = 0.4$

$y_4^{(p)} = 0.0778005626$

$y_4^{(c)} \Rightarrow y_5^{(1)} = 0.07583471$

$y_5^{(2)} = 0.075844776$

$\boxed{\therefore y(0.4) = 0.075844776}$

for $x = 0.5$

$y_5^{(p)} = -0.11436 0946$

$y_5^{(c)} = y_5^{(1)} = -0.114934172$

$y_5^{(2)} = -0.114938553$

$\boxed{\therefore y(0.5) = -0.114938553}$

for $x = 0.6$

$y_6^{(p)} = -0.3028763789$

$y_6^{(c)} \Rightarrow y_6^{(1)} = -0.303166224$

$y_6^{(2)} = -0.3031720795 1$

$\therefore \boxed{y(0.6) = -0.3031720795 1}$

5. $\frac{dy}{dx} = y + x \qquad y(0) = 1 \quad h = 0.1$

Using taylor's method to find the missing values

$\Rightarrow y_{n+1} = y_n + h y_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n''' + \frac{h^4}{24} y_n^{iv}$

$y' = y + x$

$y'' = y' + 1 = y + x + 1$

$y''' = y'' = y + x + 1$

$y^{iv} = y' + 1 = y + x + 1$

$\Rightarrow y(0.1) = y_1 = 1.1103416667$

using Milne's method $\Rightarrow$

$y_{n+1} = y_{n-1} + \frac{h}{3} [f_{n+1} + 4 f_n + f_{n-1}]$

$$\Rightarrow \boxed{\begin{aligned} y_2 &= y(0.2) = 1.242805747 \\ y_3 &= y(0.3) = 1.399717747 \end{aligned}}$$

6. $y' = 2 - xy^2$ $y(0) = 10$ $y(1) = 1.6505$ $h = 0.2$

using 4th order R-K method to find unknown values

$\Rightarrow$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h f(x_n, y_n)$$
$$k_2 = h f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$
$$k_3 = h f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$$
$$k_4 = h f(x_n + h, y_n + k_3)$$

for $x = 0.2$

$k_1 = 0.4$ $k_2 = -1.6808$ $k_3 = -1.277965443 2$

$k_4 = -2.64295547 24$

$y_1 = y(0.2) \approxeq 8.6399189402$

for $x = 0.4$

$k_1 = -2.585927 9717 8$ $k_2 = -2.83866 4826$

$k_3 = -2.7282121873$ $k_4 = -2.39586 2138$

$y_2 = y(0.4) \approxeq 5.9539949 1732.$

for $x = 0.6$

$k_1 = -2.43600 44 880 4$ $k_2 = -1.84286 2683$

$k_3 = -2.13261928$ $k_4 = -1.3523494$

$y_3 = y(0.6) \approxeq 3.8673636387.$

for $x = 0.8$

$K_1 = -1.39478018162$ $\quad K_2 = -1.00682252$

$K_3 = -1.842645$ $\quad K_4 = -0.75184328$

$y_4 = y(0.8) \simeq 2.77923069231$

using Milne's method ⇒

$$y_5^{(p)} = y_1 + \frac{4h}{3}\left(2f_2 - f_3 + 2f_4\right)$$

$$y_5^{(c)} = y_3 + \frac{h}{3}\left(f_3 + 4f_4 + f_5^{(p)}\right)$$

| $x$ | $y$ | $f$ |
|---|---|---|
| 0 | 10 | 2 |
| 0.2 | 8.63399189402 | −12.9296398586 |
| 0.4 | 5.95399491732 | −12.1800221902 |
| 0.6 | 3.8673636387 | −6.97390090836 |
| 0.8 | 2.77923069231 | −4.17929859286 |
| 1 | 1.7746547648 | −1.14939953423 |

$y_5^{(p)} = 1.7746547648$

$y_5^{(c)} \Rightarrow$ $\quad y_5^{(0)} = 2.21133064608$

$y_5^{(1)} = 2.09529 1733$

$y_5^{(2)} = 2.12860748$

$y_5^{(3)} \approx 2.11922596942$

$y.$

$$\boxed{\therefore\ y(1) \simeq y_5 = 2.11922596942}$$

7. $y' = -y \qquad y(0) = 1 \qquad x = 0.5$ to $0.8$ ; $h = 0.1$

using $4^{th}$ order classical R-K method to find unknowns initial values.

$\Rightarrow \quad y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$K_1 = h f(x_n ; y_n)$

$K_2 = h f(x_n + \frac{h}{2}, y_n + \frac{K_1}{2})$

$K_3 = h f(x_n + \frac{h}{2}, y_n + \frac{K_2}{2})$

$K_4 = h f(x_n + h, y_n + K_3)$

for $x = 0.1$

$K_1 = -0.1 \quad K_2 = -0.095 \quad K_3 = -0.09525$

$K_4 = -0.090475$

$y_1 = y(0.1) \cong 0.9048375.$

for $x = 0.2$

$K_1 = -0.09048375 \quad K_2 = -0.0859595$

$K_3 = -0.08618577 \quad K_4 = -0.081865172.$

$y_2 = y(0.2) \cong 0.818730923$

for $x = 0.3$

$K_1 = -0.0818730923 \quad K_2 = -0.07777943$

$K_3 = -0.0779841 2 \quad K_4 = -0.07407468$

$y_3 = y(0.3) \cong 0.74081844154$

for $x = 0.4$

$K_1 = -0.074081844 15 \quad K_2 = -0.0703777$

$K_3 = -0.07056295 \quad K_4 = -0.0670255$

$y_4 = y(0.4) \cong 0.67032030 6597.$

Using Milne's methods

$$y_{n+1}^{(p)} = y_{n-3} + \frac{4h}{3}\left[2f_n - f_{n-1} + 2f_{n-2}\right]$$

$$y_{n+1}^{(c)} = y_{n-1} + \frac{h}{3}\left[f_{n+1}^{(p)} + 4f_n + f_{n-1}\right]$$

| $x$ | $y$ | $f$ |
|---|---|---|
| 0 | 1 | |
| 0.1 | 0.9048375 | -0.9048375 |
| 0.2 | 0.818730923 | -0.818730923 |
| 0.3 | 0.74081844 | -0.74081844 |
| 0.4 | 0.67032032 | -0.67032032 |

for $x = 0.5$

$y_5^{(p)} = 0.606532965867$

$y_5^{(c)} = 0.60653068647$

$$\boxed{\therefore\ y_5 = y(0.5) = 0.60653068647}$$

for $x = 0.6$

$y_6^{(p)} = 0.548813862608$

$y_6^{(c)} = 0.54881173638$

$$\boxed{\therefore\ y_6 = y(0.6) = 0.54881173638}$$

for $x = 0.7$

$y_7^{(p)} = 0.4965847321827$

$y_7^{(c)} = 0.4965851880 1$

$\boxed{\therefore\ y_7 = y(0.7) = 0.4965851880 1}$

for $x = 0.8$

$y_8^{(p)} = 0.44933096499$

$y_8^{(c)} = 0.44932895460 4$

$\boxed{\therefore\ y_8 = y(0.8) = 0.44932895460 4}$

8. $\frac{dy}{dx} = x - y^2 \quad y(0) = 1$

given $y(0.1) = 0.9117$, $y(0.2) = 0.8494$, $y(0.3) = 0.8061$

using 5th order Adams Moulton method

$$\Rightarrow y_{n+4} = y_{n+3} + \frac{h}{720}\Big[251 f_{n+4} + 646 f_{n+3} - 264 f_{n+2} + 106 f_{n+1} - 19 f_n\Big]$$

$$y_4 = y_3 + \frac{h}{720}\left[251 f_4 + 646 f_3 - 264 f_2 + 106 f_1 - 19 f_0\right]$$

$f_0 = -1 \quad f_1 = -0.7311962$

$f_2 = -0.5214803 \quad f_3 = -0.3497972.$

$f_4 = 0.4 - y_4^2$

$\Rightarrow$ By substituting in the equation, we get

$$25.1 y_4^2 + 720 y_4 - 575.75149535 = 0$$

Using Newton-Raphson method $\Rightarrow$

$$g(y_4) = 25.1 y_4^2 + 720 y_4 - 575.75149535$$

$$g'(y_4) = 50.2 y_4 + 720$$

$$y_4^{s+1} = y_4^s - \frac{g(y_4^s)}{g'(y_4^s)}$$

Let $y_4^0 = 0$

$\Rightarrow$ $y_4^1 = 0.799654858$

$y_4^2 = 0.7785462106$

$y_4^3 = 0.7785254688$

} accurate upto 4 decimal places

$$\Rightarrow \boxed{y(0.4) = 0.7785254688}$$

9. $\frac{dy}{dx} = \frac{xy}{2}$

$y(0) = 1$ $\quad y(0.1) = 1.0025$ $\quad y(0.2) = 1.0101$ $\quad y(0.3) = 1.0228$

4th order Adams-Bashforths formula.

$$\Rightarrow y_{n+1} = y_n + \frac{h}{24}\left(55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right)$$

Date

$$y_4 = y_3 + \frac{h}{24}\left(55 f_3 - 59 f_2 + 37 f_1 - 9 f_0\right)$$

$f_0 = 0 \quad f_1 = 0.05012 \quad f_2 = 0.10101$

$f_3 = 0.15342$

$$y_4 = 1.0228 + \frac{0.1}{24}\left(55 f_3 - 59 f_2 + 37 f_1 - 9 f_0\right)$$

$$= 1.040854729$$

$$\boxed{\therefore\ y(0.4) \simeq 1.040854729}$$

10. $$u_{n+1} = u_n + \frac{h}{2}\left[5u_n' + \alpha u_{n-1}'\right]$$

$$u_{n+1} - u_n = h\left[\frac{5}{2} u_n' + \frac{\alpha}{2} u_{n-1}'\right]$$

$\alpha_1 = 1 \quad \alpha_0 = -1 \quad \alpha_{-1} = 0 \;\Big|\; \beta_0 = \frac{5}{2} \quad \beta_{-1} = \frac{\alpha}{2} \quad \beta_1 = 0$

$$C_0 = \Sigma \alpha_j = 1 + -1 + 0 = 0$$

$$C_1 = \Sigma\left(j\alpha_j - \beta_j\right) = (1 + 0 + 0) - \left(\frac{5}{2} + \frac{\alpha}{2} + 0\right) = 0$$

$$\Rightarrow 1 - \frac{5}{2} - \frac{\alpha}{2} = 0$$

$$\Rightarrow 1 - \frac{5}{2} = \frac{\alpha}{2} \Rightarrow \alpha = -3$$

$$\Rightarrow \boxed{\alpha = -3}$$