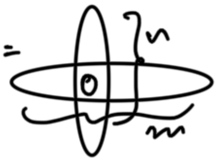
Tutorial 1 :

① $M_{n \times m} \rightarrow$ a) $n.m$ vertices

b) Deg =  $(n-1) + (m-1)$ = " $n+m-2$ – Regular "

c) Total no. of edges in regular graph = $\frac{\#\text{vertices} \times \text{degree}}{2}$

$$\left(\sum_{x \in V(G)} \deg(x) = 2e \right)$$

② Atleast 2 vertices w/ same degree

Given: Simple graph w/ n vertices 'G'

TS : $\exists\, x, y \in V(G)$ st $\deg(x) = \deg(y)$

Proof: Possible degrees for a simple graph of n-vertices: $0, 1, \ldots, n-1$

If a vertex has 0 degree $\Rightarrow$ $n-1$ can't exist $\{1, 2, \ldots, n-1\}$ / $\{0, 1, 2, \ldots, n-2\}$

Use $\boxed{\text{Pigeon-hole principle}}$ n pig, m holes
$n > m \Rightarrow$ atleast 2 pig. in same hole

③ T.S.: Exactly 2 vertices of odd degree $\Rightarrow$ $\exists$ path | G – connected – trivial / disconnected

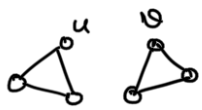
When G is disconnected graph, let $u \in$ connected component of G.

By Handshake lemma, no. of odd degree vertices is always even.

$\boxed{\sum_{x \in V(G)} \deg(x) = 2e}$ for undir. graphs

But v is the only other odd deg. vertex.

$\therefore$ u & v $\in$ same connected component.

④ ⓘ G disconnected :  No cycle necessary

G connected : v_1

$v \ldots v_2 = \cdots$ but graph is finite & simple
$\therefore$ cycle.

(ii) Every closed trail contains a cycle

$$\underbrace{v_0 e_0 \cdots v_i e_i v_{i+1} \cdots v_k e_k v_{k+1}}$$

cycle :) — No $\rightarrow \exists$ $\underline{v_i = v_j}$ for sure

cycle :) — repeat

(5) $G \rightarrow$ simple, $V(G) = n$, $\delta(G) = \frac{n-1}{2}$ (min deg; $\Delta(G)$ — max deg)

Proof by contradiction:
Let x, y be non-adjacent vertices.
$y \notin N(x)$, $x \notin N(y)$

T.S. $|N(x) \cap N(y)| \geq 1$

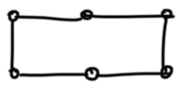
$|N(x) \cup N(y)| \leq n-2$
$\because x, y \notin N(x) \cup N(y)$

$$|N(x) \cap N(y)| = |N(x)| + |N(y)| - |N(x) \cup N(y)|$$
$$\geq \frac{n-1}{2} + \frac{n-1}{2} - (n-2)$$
$$\geq 1 \qquad \text{H.P.}$$

(6) Max edges $\frac{n(n-1)}{2} > \frac{n-2}{2}$ $\left[n^2 - n - (n-2) = n^2 - 2n + 2 = (n-1)^2 + 1 > 0 \right]$

Hence not all edges are connected w/ each other.

eg.



$\frac{6-2}{2} = 2$

(7) No pair of $\underline{\text{adjacent edges}}$ $\Rightarrow$ deg $= 1$
↳ have 1 vertex in common

T.S. G has vertex of deg atleast m

(8) $\sum_{x \in V(G)} deg(x) = 2e$ Avg. $deg(x) = \frac{2e}{n}$

Suppose $\nexists$ u s.t. $deg(u) \geq m$ or $deg(u) < m$ $\forall u \in V(G)$
$\Rightarrow \sum_{x \in V(G)} deg(x) \leq (m-1)n$ or $2e \leq (m-1)n$

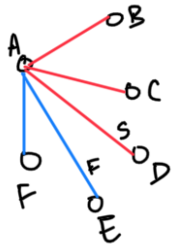
or $m \geq \frac{2e}{n} + 1$
- $\frac{2e}{n}$ is an integer — inequ. not possible
- $\frac{2e}{n}$ is not an integer — then $m = \frac{2e}{n} + c \in (0,1)$

$\rightarrow\leftarrow$

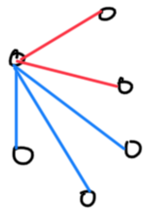
⑨ $\sum_{x \in V(G)} \deg(x) = 2(n-1)$ n vertices, n-1 edges

if $\deg(x) \geq 2 \ \forall x \in V(G)$, $\sum \deg(x) \geq 2n$ $\rightarrow\!\!\!\!\times\!\!\!\leftarrow$

⑩ Ⓘ

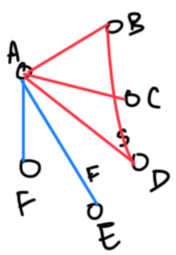


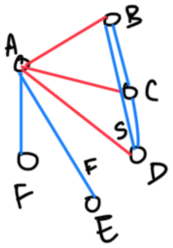
Ⓘⓘ



any 2 of $\{B, C, D\}$ strangers

all of them are frnds





In both the cases we get a blue or red Δ.

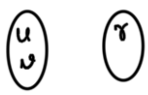
Result: Every of G of 6 vertices has a K_3 in G or $\overline{G}$

⑪ $G \rightarrow$ disconnected

Let $u, v \in V(G)$

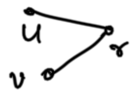
Case Ⓘ $(u,v) \notin E(G)$ then $(u,v) \in E(\overline{G})$ connected

Case Ⓘⓘ $(u,v) \in E(G) \Rightarrow$ u & v are in the same component of G



$\exists$ r $\in$ other component of G s.t. $(u,r) \notin E(G)$ & $(v,r) \notin E(G)$

$\Rightarrow (u,r)$ & $(v,r) \in E(\overline{G})$



Hence connected.

⑫ Given, G w/ $\delta(G) \geq k$

ⓘ T.S.: G contains path of len $\geq k$

Let n be the max possible len. Proof by contra: $\boxed{\text{Assume } n < k}$

Then the path $v_0 v_1 v_2 \ldots v_n$ has length $< k$.

Let $S = N(v_n) \setminus \{v_0, v_1, \dots, v_{n-1}\} \Rightarrow |S| \geq \delta(G) - n$
$= k - n > 0$

$\Rightarrow S \neq \phi$

Let $u \in S$, then the path $v_0 v_1 \dots v_n u$ has length $> n$ $\rightarrow\leftarrow$

(ii) $v_0 v_1 \dots v_n \to$ path of length n ; from above, $n \geq k$

remove k-1 vertices

Let $S = N(v_n) \setminus \{v_{n-k+1}, \dots, v_{n-1}\}$

$|S| \geq \delta(G) - (k-1) \quad \Rightarrow |S| \geq 1$
$= k - (k-1) = 1$

Let $u \in S$, then $v_0 v_1 \dots v_n u$ not a path $\because$ length $> n$ (n+1)

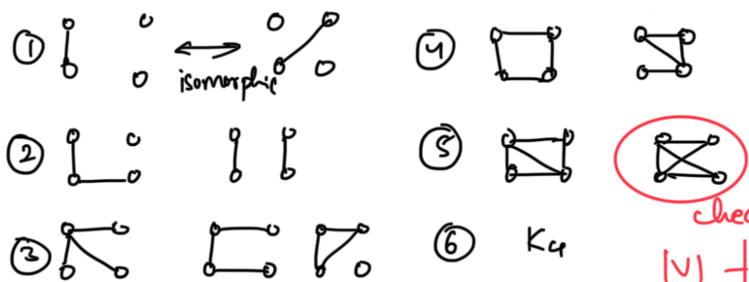
$\Rightarrow u = v_m$ for some $m \in [1, n-k]$

$\Rightarrow$ $\underline{v_m v_{m+1} \dots v_n u}$ is a cycle of length $\underline{n-m+1}$

$n-m+1 \geq n+1-(n-k) = k+1$ H.P.

(13) 4 vertices $\Rightarrow$ $^4C_2 = 6 \to$ #edges in complete

$0 \to 1$
$1 \to 1$
$2 \to 2$
$3 \to 3$
$4 \to 2$
$5 \to 1$
$6 \to 1$
} (11)



|V|
|E|
Deg. seq.
same cycle
↑
check for isomorphism

(14) (i) $V = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}$
$\{3,4\}, \{3,5\}, \{4,5\}\}$ $|V| = 10$

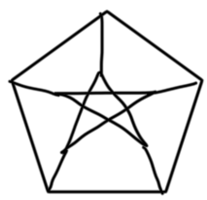
$|E| =$

$(\{\_,\_\}, \{\_,\_\})$

1 2 3 4 5

$\frac{^5C_4 \cdot {^4C_2}}{2} = 15$

(ii) Peterson graph



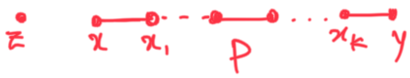
Tutorial-2:

(2) Let $G=(V,E)$ & $H=(V',E')$
$V=(v_1,v_2,\dots,v_n)$ & $V'=(\omega_1,\omega_2,\dots,\omega_n)$
Let $f: V\to V'$ $\quad$ ($xy\in E(G)$ iff $f(x)f(y)\in E'(H)$)

Now, $\forall\, i: 1$ to n, $\quad y\in N_G(v_i)$ iff $v_i y\in E(G)$
$\Rightarrow f(v_i)f(y)\in E'(H)$
$\Rightarrow f(y)\in N_H(f(v_i))$

$\Rightarrow \deg_G(v_i)=|N_G(v_i)| = |N_H(f(v_i))| = \deg_H(f(v_i))$ $\quad$ H.P.

(3) $P\to$ path of max. length in P. every nbr of x lies on P



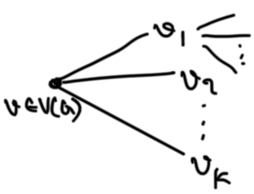
$l(zy) > l(P)$ $\rightarrow\leftarrow$
Hence, $\deg(x)\geq k \Rightarrow l(P)\geq k$
$P\to$ min $k+1$ vertices

(6) $G\simeq \bar{G} \Rightarrow |V(G)|=|V(\bar{G})|$ & $|E(G)|=|E(\bar{G})|$

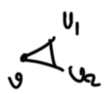
$\underbrace{|E(G)|+|E(\bar{G})|}_{\text{connected}} = {}^nC_2 = \frac{n(n-1)}{2} \Rightarrow E(G)=\frac{n(n-1)}{4}$

$\Rightarrow 4\,|\,n$ or $4\,|\,n-1 \Rightarrow n=0$ or $1 \pmod 4$

(7) $G\to$ k-regular connected; girth $=5$
T.S.: $|V(G)|\geq k^2+1$ $\quad$ ($=$ if $k=2$ or 3)
Proof:



No two v_i's ($i=1$ to k) can be nbrs
else a C_3 will be formed (but girth $=5$)



Also, each v_i has $k-1$ nbrs excluding v: $v_{i1}, v_{i2}, \dots v_{i(k-1)}$ all of which are distinct (else C_4)

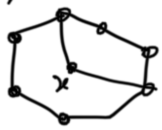
Hence it will have min of $1+k+k(k-1) = k^2+1$ vertices
$k=2$ or 3 then no more extensions.

⑧ Peterson graph : 10V & 15 E , girth = 5, 3-connected

suppose $\exists C_7$  $\to C_3$ ✗

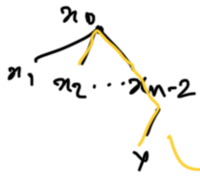
1–7 must have an edge to $x/y/z$ (not among themselves, else C_3)

Now, 7 edges from 7 vertices to 3 vertices x, y, z.
Say x is joined to 3 vertices (Pigeon-hole)
$\downarrow$
C_3 or C_4



⑨ T.P. $\exists x \in V(G)$ s.t. $\deg(x) = n-1$
Proof by contrapositive $p \to q$ or $\sim q \to \sim p$

Suppose $\Delta(G) \le n-2$

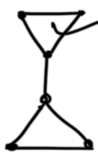


} n nodes

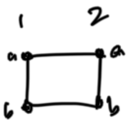
induced P_4 $\to\leftarrow$

Tutorial-3

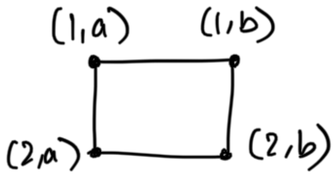
① i) False. eg. $|\vee| \to$ 

ii) False $|\cdot| \to$  $\to$ odd cycle
$\downarrow$
not bipartite

iii) Yes $|\times| \to \{(1,a), (1,b), (2,a), (2,b)\}$



or



even cycle

② $G_1 \cup G_2$ bipartite for $\overline{K_n}$ & $\overline{K_m}$

③ $G_1 \circ G_2$ bipartite for $\overline{K_n}$ & any bipartite graph

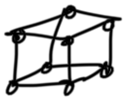
④ $Q_n \longrightarrow$ vertices 2^n

edges $n \cdot 2^{n-1}$

degree of reg. n

radius, diam. $\}$ n

girth 4 $(n \geq 2)$



Q_3 $(n=3)$

⑥ Q_n is a bipartite grp

sum of coords: even → V_1, odd → V_2

edges exist if coords differ in exactly 1 place ⇒ different parity

∴ Edge exists b/w V_1 & V_2.

⑦ Case I $\longrightarrow$ u & v differ in 2 bit posⁿ → no common nbrs

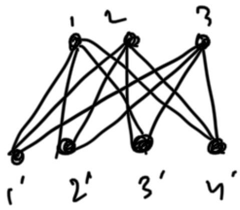
Case II $\longrightarrow$ >2 →

Tutorial - 5 :

① i)  Only true for regular graphs

ii) Eulerian → cycle, bipartite → even cycle → even #edges

② $K_{3,4}$ $K_{3,3}$ Petersson



✗

✓

Petersson ✗

Hamiltonian ↑ $K_{m,n}$ if $m=n$

$\left\{ \because v_1 \in A \Rightarrow v_2 \in B \ldots \Rightarrow v_n \in B \neq A \right.$ for it to be Hamiltonian

Q3

$K_{3,3}$ v/s $K_{3,4}$

④ Peterson

⑤ Case 1: $n = 2k$

$\delta \geq k$

Dirac's thm: $n \geq 3$ & $\delta \geq \frac{n}{2}$ → Hamiltonian

↓

a path that visits each vertex exactly once

Case 2: $n = 2k+1$

$\delta(G) \geq k + \frac{1}{2}$

or $\delta(G) \geq k+1$

$G' : \underbrace{G \cup v}_{\text{Hamiltonian}}$

↓

Hamiltonian cycle C in G'

$\deg(v) = n \Rightarrow v$ adj to two distinct vertices in C

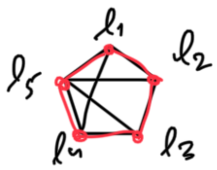
↓

removing v we get Hamiltonian path

⑥ $\underbrace{{}^{n-1}C_2}_{K_{n-1}} + \underbrace{1}_{\text{non-returnable}}$

⑦

G



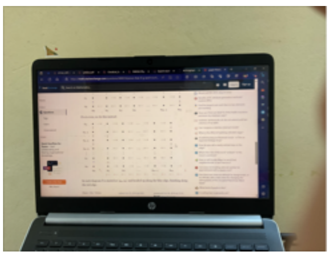
$L(G)$

To prove ↓ Eulerian ⟷ prove Hamiltonian

Eulerian ↓ closed trail passing all edges

Hamiltonian ↓ visits each vertex exactly once & returns to starting vertex

⑧ $G_1 \times G_2$



Prove: $G \to$ bipartite if all its connected components

disconnected

aro "