Lecture 7

We proved that $\mathcal{M}$ is a σ-algebra.

where $\mathcal{M}$ = the class of all Lebesgue measurable subsets of $\mathbb{R}$.

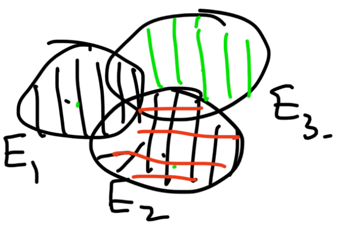
__Remark:-__

$$\bigcup_{i=1}^{n}\left(E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) = \bigcup_{i=1}^{n} E_i$$

$\forall\, n \geq 2$

__n = 3:__

$$\text{LHS} = \bigcup_{i=1}^{3}\left(E_i \cap \left(\bigcap_{j<i} E_j^c\right)\right)$$



$$= E_1 \cap (\cap \phi^c) \cup (E_2 \cap (E_1^c)) \cup (E_3 \cap (E_1^c \cap E_2^c))$$

$$= (E_1 \cap \mathbb{R}) \cup (E_2 \cap E_1^c) \cup (E_3 \cap (E_1^c \cap E_2^c))$$

$$= (E_1 \cup E_2 \cap E_1^c) \cup \quad ''$$

$$= (E_1 \cup E_2) \cup [E_3 \cap (E_1^c \cap E_2^c)]$$

$$= E_1 \cup E_2 \cup E_3.$$

$$= \text{RHS}.$$

for any $n \geq 2$, similar way we can prove.

<u>Examples:-</u> ϕ, $\mathbb{R}$, any finite set, $\mathbb{R}\setminus$ any finite set, $\mathbb{Q}$, $\mathbb{Q}^c$, are measurable.

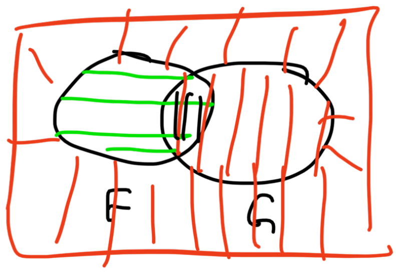
<u>Proposition:-</u> Suppose $F \in \mathcal{M}$ & $m^*(F \Delta G) = 0$, where $G \subseteq \mathbb{R}$. Then $G \in \mathcal{M}$.

<u>proof:-</u> $F \Delta G = (F \setminus G) \cup (G \setminus F)$.

$$0 = m^*(F \Delta G) = m^*\big((F\setminus G) \cup (G \setminus F)\big) \geq m^*(F\setminus G),\ m^*(G\setminus F).$$

$$\Rightarrow \quad m^*(F\setminus G) = m^*(G \setminus F) = 0.$$

$\therefore$ $F\setminus G$, $G \setminus F \in \mathcal{M}$.



$$F \cap G = F \cap (F \setminus G)^c$$

we have F, $F\setminus G \in \mathcal{M}$

$\therefore$ $(F\setminus G)^c \in \mathcal{M}$.

$$(F\cap G)^c = F^c \cup (F \setminus G)$$

F^c, $F\setminus G \in \mathcal{M}$ $\Rightarrow$ $F^c \cup (F\setminus G) \in \mathcal{M}$

$\Rightarrow$ $(F \cap G)^c \in \mathcal{M}$

$\Rightarrow$ $F \cap G \in \mathcal{M}$

Now $G = \underset{\mathcal{M}}{\underset{\cap}{(F \cap G)}} \cup \underset{\mathcal{M}}{\underset{\cap}{(G \setminus F)}} \in \mathcal{M}$.

$\therefore G \in \mathcal{M}$.

<u>Theorem</u>:- Measurable sets are Countably additive.

Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of disjoint measurable sets in $\mathbb{R}$. (ie, $E_i \cap E_j = \phi, \ \forall i \neq j$).

Then $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i)$.

<u>proof</u>:- We have, for any $A \subseteq \mathbb{R}$,

$$m^*(A) = m^*(A \cap E_1) + \sum_{i=2}^{\infty} m^*\left(A \cap E_i \cap \left(\bigcap_{j<i} E_j^c\right)\right)$$

$$+ \ m^*\Big(A \cap \underbrace{\left(\bigcap_{j=1}^{\infty} E_j^c\right)}_{\left(\bigcup_{j=1}^{\infty} E_j\right)^c}\Big) \longrightarrow (\S).$$

Take $A = \bigcup_{i=1}^{\infty} E_i$ in the equation $(\S)$, then

we get

$$m^*\left(\dot{\bigcup_{i=1}^{\infty}} E_i\right) = m^*\left(\left(\dot{\bigcup_{i=1}^{\infty}} E_i\right) \cap E_1\right) + \sum_{i=2}^{\infty} m^*\left(\left(\dot{\bigcup_{k=1}^{\infty}} E_k\right) \cap E_i \cap \left(\bigcap_{j<i} E_j^c\right)\right)$$

$$+\ m^*\left(\underbrace{\left(\dot{\bigcup_{i=1}^{\infty}} E_i\right) \cap \left(\dot{\bigcup_{j=1}^{\infty}} E_j\right)^c}_{\phi}\right)$$

$$\Rightarrow m^*\left(\dot{\bigcup_{i=1}^{\infty}} E_i\right) = m^*\left(\underbrace{\left(\dot{\bigcup_{i=1}^{\infty}} E_i\right) \cap E_1}_{E_1}\right) + \sum_{i=2}^{\infty} m^*\left(\left(\dot{\bigcup_{k=1}^{\infty}} E_k\right) \cap E_i \cap \left(\bigcap_{j<i} E_j^c\right)\right)$$

$$\Rightarrow m^*\left(\dot{\bigcup_{i=1}^{\infty}} E_i\right) = m^*(E_1) + \sum_{i=2}^{\infty} m^*\left(E_i \cap \left(\bigcap_{j<i} E_j^c\right)\right)$$

E_1 E_2 $\vdots$ E_i

Consider

$$E_i \cap \left(\bigcap_{j<i} E_j^c\right) = E_i \cap E_1^c \cap E_2^c \cap \cdots \cap E_{i-1}^c.$$

$$= E_i \cap E_2^c \cap \cdots \cap E_{i-1}^c$$

$$= E_i$$

$$\boxed{\begin{array}{c} E_i \cap E_j = \phi \\ j<i \\ \Rightarrow E_i \subseteq E_j^c \\ j<i \\ j=1,2,\ldots,i-1 \end{array}}$$

$$\therefore\ m^*\left(\dot{\bigcup_{i=1}^{\infty}} E_i\right) = m^*(E_1) + \sum_{i=2}^{\infty} m^*(E_i)$$

$$= \sum_{i=1}^{\infty} m^*(E_i).$$

Theorem:- Every interval is measurable.

proof:- Let $I = [a, \infty)$, for any $a \in \mathbb{R}$.

To show: I is measurable.

That is to show, for any $A \subseteq \mathbb{R}$,

$$m^*(A) \geq m^*(A \cap I) + m^*(A \cap I^c).$$

Let $A_1 = A \cap I^c = A \cap (-\infty, a)$

$A_2 = A \cap I = A \cap [a, \infty)$.

Let $\varepsilon > 0$. Then there exists intervals $\{I_n\}$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ &

$$\boxed{m^*(A) + \varepsilon \geq \sum_{n=1}^{\infty} l(I_n)}$$

Write $I_n' = I_n \cap (-\infty, a)$

$I_n'' = I_n \cap [a, \infty)$

Then $l(I_n) = l(I_n' \cup I_n'')$

$= l(I_n') + l(I_n'')$. ✓

$$A_1 = A \cap I^c = A \cap (-\infty, a) \subseteq \bigcup_{n=1}^{\infty} (I_n \cap (-\infty, a))$$

$$= \bigcup_{n=1}^{\infty} I_n'$$

$$\therefore \quad A_1 \subseteq \bigcup_{n=1}^{\infty} I_n' \quad \checkmark$$

$$A_2 = A \cap I = A \cap [a, \infty) \subseteq \bigcup_{n=1}^{\infty} I_n \cap [a, \infty)$$

$$= \bigcup_{n=1}^{\infty} I_n''.$$

$$\therefore \quad A_2 \subseteq \bigcup_{n=1}^{\infty} I_n'' \quad \checkmark$$

Now

$$m^*(A_1) + m^*(A_2) \leq m^*\left(\bigcup_{n=1}^{\infty} I_n'\right) + m^*\left(\bigcup_{n=1}^{\infty} I_n''\right)$$

$$\leq \sum_{n=1}^{\infty} m^*(I_n') + \sum_{n=1}^{\infty} m^*(I_n'').$$

$$= \sum_{n=1}^{\infty} \left(m^*(I_n') + m^*(I_n'') \right)$$

$$= \sum_{n=1}^{\infty} l(I_n)$$

$$\leq m^*(A) + \varepsilon.$$

Thus $m^*(A_1) + m^*(A_2) \leq m^*(A) + \varepsilon$

$\forall \, \varepsilon > 0.$

$$\Rightarrow \quad m^*(A_1) + m^*(A_2) \leq m^*(A).$$

$$\Rightarrow \quad m^*(A \cap I^c) + m^*(A \cap I) \leq m^*(A).$$

$$\therefore \quad I \in \mathcal{M}.$$

$\therefore [a,\infty) \in \mathcal{M}.$

$\Rightarrow \quad [a,\infty)^c = (-\infty, a) \in \mathcal{M}.$

$$(a,b) = (-\infty, b) \cap (a,\infty)$$

$$(a,\infty) = \bigcup_{n=1}^{\infty} \underbrace{\left[a+\tfrac{1}{n}, \infty\right)}_{\in \mathcal{M}} \in \mathcal{M}$$

a

$$\therefore \quad (a,b) = \underbrace{(-\infty,b)}_{\in \mathcal{M}} \cap \underbrace{(a,\infty)}_{\in \mathcal{M}} \in \mathcal{M}.$$

$$[a,b] = \underbrace{(-\infty,b]}_{\in \mathcal{M}} \cap \underbrace{[a,\infty)}_{\in \mathcal{M}} \in \mathcal{M}.$$

$$[a,b) = \underbrace{(-\infty,b)}_{\in \mathcal{M}} \cap \underbrace{[a,\infty)}_{\in \mathcal{M}} \in \mathcal{M}.$$

$$(a,b] = \underbrace{(-\infty,b]}_{\in \mathcal{M}} \cap \underbrace{(a,\infty)}_{\in \mathcal{M}} \in \mathcal{M}.$$

$\therefore$ any interval is measurable.

Proposition:- Countable intersection of measurable sets is measurable.

proof:- Suppose $\{E_i\}_{i=1}^{\infty}$ be a sequence of measurable sets.

E_i^c are measurable $\forall i$

$$\Rightarrow \bigcup_{i=1}^{\infty} E_i^c = \left(\bigcap_{i=1}^{\infty} E_i\right)^c \in \mathcal{M}$$

$$\Rightarrow \bigcap_{i=1}^{\infty} E_i \in \mathcal{M}.$$

Examples:- measurable sets: $[0,1]$, $(2,3]$, $(-2,1)$, $[3,10)$, $[0,1] \cup (2,3)$, $[0,1] \cup (3,5] \cup [6,7)$

- $m^*([0,1] \cup (2,3)) = m^*([0,1]) + m^*((2,3))$

$= 1+1$

$= 2.$

- $m^*([0,1] \cup (3,5] \cup [6,7)) = m^*([0,1]) + m^*((3,5]) + m^*([6,7))$

$= 1+2+1$

$= 4.$