Lecture 14

Theorem:- There exists a measurable set which is not a Borel set. That is, $\mathcal{B} \subsetneqq \mathcal{M}$.

proof:- For $x \in [0,1]$, the binary expansion of x is

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}, \quad \text{where } \varepsilon_n = 0 \text{ or } 1 \quad \forall n \geq 1.$$

Define a map $f : [0,1] \to \mathbb{R}$ as

$$f(x) = f\left(\sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}\right) := \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n}, \quad \varepsilon_n \in \{0,1\}$$

$2\varepsilon_n \in \{0,2\}$.

$\in P$, the Cantor set.

EXERCISE: Let $x \in [0,1]$ have the expansion to the base l, $x = 0.x_1x_2x_3\ldots\ldots$ Show that the map $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x_n$ (n^{th} decimal place value) is a measurable function of x, for each n.

Hint:- try to write x_n as a function of x.

$x_1 = [x\,l]$

$x = 0.x_1x_2\cdots$ base 10

$x_1 = [x \times 10]$

$= [x_1 . x_2x_3 \cdots]$

$= x_1$

$x \mapsto x$

$x \mapsto 10x$

$x \mapsto [10x]$

First show: $g(x) = [x]$ is a measurable function

By above excercise, $x \mapsto \varepsilon_n$ is a measurable function, $\forall n \geq 1$.

$x \mapsto \frac{2\varepsilon_n}{3^n}$ also a measurable function $\forall n \geq 1$.

Then $x \longmapsto \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n}$ is also a measurable function

$f(x) = \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n}$ is measurable.

Such function f is called the Cantor function.

Note that f is injective.

If $f(x) = f(y)$. & $x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}$, $y = \sum_{n=1}^{\infty} \frac{\varepsilon_n'}{2^n}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n} = \sum_{n=1}^{\infty} \frac{2\varepsilon_n'}{3^n}$

$\Rightarrow \varepsilon_n = \varepsilon_n' \quad \forall n \geq 1$

$\Rightarrow x = y$.

$\therefore f$ is injective.

Thus $f : [0,1] \to \mathbb{R}$ injective & measurable.

Suppose $\mathcal{B} = \mathcal{M}$

Since f is measurable, $f^{-1}(B)$ is measurable for any $B \in \mathcal{B} = \mathcal{M}$

Let V be a non-measurable set in $[0,1]$.

& $B = f(V) \subseteq \text{Im}(f) \subseteq P$

$\Rightarrow$ $B \subseteq P$ & $m^*(P) = 0$

$\Rightarrow$ $m^*(B) = 0$

$\Rightarrow$ B is measurable.

$\Rightarrow$ $B \in \mathcal{M} = \mathcal{B}$

$\Rightarrow$ B is a Borel set

$\because f$ is injective.

$\Rightarrow$ $f^{-1}(B) = f^{-1}(f(V)) = V$ is measurable

($\because$ f is measurable).

$\Rightarrow$ V is measurable

Which is a contradiction.

$\therefore$ $\mathcal{B} \neq \mathcal{M}$

$\Rightarrow$ $\mathcal{B} \subsetneq \mathcal{M}$.

Problems

① Suppose f is a measurable real-valued function & g is a continuous function defined on $\mathbb{R}$.

Then $g \circ f$ is measurable.

Sol:- For $\alpha \in \mathbb{R}$

$$\{x \in E \mid (g\circ f)(x) > \alpha\} = (g\circ f)^{-1}((\alpha, \infty))$$

$$= f^{-1}\Big(\underbrace{g^{-1}((\alpha,\infty))}_{\text{is an open set (since } g \text{ is continuous) which is Borel}}\Big)$$

$$\in \mathcal{M} \quad (\because f \text{ is measurable}).$$

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② Let $f: D \to \mathbb{R}$ be any function.

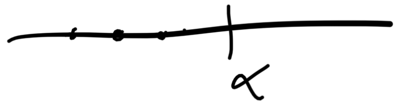
Show that if $\{x \in D \mid f(x) < r\}$ is measurable for every $r \in \mathbb{Q}$, then f is measurable.

Sol:- For $\alpha \in \mathbb{R}$, To show:

$$\{x \in D \mid f(x) < \alpha\} \text{ is measurable.}$$

Let $I = \{r \in \mathbb{Q} \mid r < \alpha\}$. Then I is countable.

$$\{x \in D \mid f(x) < \alpha\} = \bigcup_{r \in I} \underline{\{x \in D \mid f(x) < r\}} \in \mathcal{M}$$



Pf:- Let $x \in$ LHS.

$\Rightarrow f(x) < \alpha$

$\Rightarrow$ there exists $r \in \mathbb{Q}$ such that

$$f(x) < r < \alpha$$

$\Rightarrow \quad x \in \{x \in D \mid f(x) < r\} \quad \exists\, r \in I.$

$\Rightarrow \quad x \in$ RHS.

$x \in$ RHS, $\Rightarrow \quad x \in \{x \in D \mid f(x) < r\}$ for some $r \in I$.

$\Rightarrow \quad f(x) < r \;\&\; r < \alpha$

$\Rightarrow \quad f(x) < \alpha$

$\Rightarrow \quad x \in \{x \in D \mid f(x) < \alpha\} =$ LHS.

Remark :-
$$\sum_{n=1}^{\infty} f_n = \lim_{k\to\infty} \Big(\underbrace{\sum_{n=1}^{k} f_n}_{\text{is measurable } g_k} \Big)$$

Each g_k is measurable $\forall k \geq 1$, then

$$g_k \to g = \sum_{n=1}^{\infty} f_n$$
measurable.

Theorem :- The Cantor set is compact.

proof :- To show: P is closed & bounded.

We have $P \subseteq [0,1]$, there P is bounded.

& $P = \bigcap_{n=1}^{\infty} P_n$ & each P_n is a finite union of closed intervals. (sets)

thus each P_n is closed.

& hence P is closed.

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(3) For any $\varepsilon > 0$, Construct an open $U \subseteq \mathbb{R}$ such that $U \supseteq \mathbb{Q}$ & $m^*(U) \leq \varepsilon$.

Sol:- Let $Q = \{r_1, r_2, \cdots\}$ & $m^*(Q) = 0$

$Q \in \mathcal{M}$.

Let $\varepsilon > 0$.

& $I_n = \left(r_n - \frac{\varepsilon}{2^{n+1}}, r_n + \frac{\varepsilon}{2^{n+1}}\right) \quad \forall n \geq 1$

$r_n \in$ Open set

Let $V = \bigcup_{n=1}^{\infty} I_n$ an open set.

& $U \supseteq Q$

$$m^*(U) = m^*\left(\bigcup_{n=1}^{\infty} I_n\right) \leq \sum_{n=1}^{\infty} m^*(I_n)$$

$$= \sum_{n=1}^{\infty} l(I_n)$$

$$= \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$$

$$= \varepsilon.$$

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4 (a) Show that Q is an F_σ-set.

(b) Show that there exists a G_δ-set G such that $G \supseteq Q$ & $m^*(G) = 0$.

(c) Show that the set of all irrational numbers is a G_δ-set.

Sol:- (a) $\mathbb{Q} = \{r_1, r_2, \ldots\}$

$$= \bigcup_{n=1}^{\infty} \underbrace{\{r_n\}}_{\text{Closed set}} \quad F_\sigma\text{-set}.$$

(b) Since $\mathbb{Q}$ is measurable, there exists a G_δ-set G such that $G \supseteq \mathbb{Q}$ & $m^*(G) = m^*(\mathbb{Q}) = 0$.

(c) $$\mathbb{Q}^c = \left(\bigcup_{n=1}^{\infty} \{r_n\}\right)^c$$

$$= \bigcap_{n=1}^{\infty} \underbrace{\{r_n\}^c}_{\text{Open sets}} \quad G_\delta\text{-set}.$$