Completion of $B(X, Y)$:

Theorem: Let X and Y be n.l.s. If Y is a Banach space, then $BL(X, Y)$ is also a Banach space. In particular $X' = BL(X, K)$ is a Banach space.

Proof: Given that Y is a Banach space.

Let $\{A_n\}$ be a Cauchy sequence in $BL(X, Y)$.

Then given $\epsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ such that

$$\|A_n - A_m\| < \epsilon \quad \forall\, n, m \geq n_0.$$

Then for each $x \in X$, we have

$$\|A_n x - A_m x\| = \|(A_n - A_m)x\|$$
$$\leq \|A_n - A_m\| \|x\|$$
$$< \epsilon \|x\|, \quad \forall\, n, m \geq n_0.$$

$\Rightarrow \{A_n x\}$ is a Cauchy sequence in Y for each $x \in X$.

Since Y is a Banach space, $\{A_n x\}$ converges in Y.

Also $\{A_n\}$ is a Cauchy sequence in $BL(X, Y)$, $\{\|A_n\|\}$ is bounded.

[$\because A_n = A_n - A_{n_0} + A_{n_0}, \quad \forall\, n \geq n_0$

$$\|A_n\| \leq \|A_n - A_{n_0}\| + \|A_{n_0}\|$$
$$< \epsilon + \|A_{n_0}\| < \infty \;]$$

Define $A : X \longrightarrow Y$ by

$$Ax = \lim_{n\to\infty} A_n x, \quad x \in X.$$

Then A is linear and

$$\|A\| \leq \limsup_{n} \|A_n\| < \infty$$

$$\Rightarrow A \in BL(X, Y)$$

Now for any $x \in X$, $m, n \geq n_0$ and for fixed m, we have

$$\|(A - A_m)x\| = \lim_{n\to\infty} \|(A_n - A_m)x\|$$

$$\leq \left(\limsup_{n} \|A_n - A_m\|\right) \|x\|$$

$$< \epsilon \|x\|$$

$$\Rightarrow \|A - A_m\| < \epsilon \quad \forall\, m \geq n_0$$

$\therefore A_n \longrightarrow A \in BL(X, Y)$.

$\therefore BL(X, Y)$ is a Banach space.

—— // ——

Let X and Y be n.l.s

Let $\mathcal{A} = \{A_i \,/\, A_i \in BL(X, Y)\}$ be a family of bounded operators from X into Y.

We say $\mathcal{A}$ is Pointwise bounded on X iff for each $x \in X$, $\exists$ $M_x > 0$ such that

$$\|Ax\| \leq M_x \|x\|, \quad \forall A \in \mathcal{A}.$$

We say $\mathcal{A}$ is uniformly bounded if $\{ \|A\| / A \in \mathcal{A} \}$ is a bounded set.

Clearly uniformly bounded implies pointwise bounded.

Converse need not be true.

Ex: $X = c_{00}$, with $\|\cdot\|_\infty$.

For $x = (x(1), x(2) \ldots) \in X$,

Define $f_n : X \longrightarrow K$ by

$$f_n(x) = \sum_{j=1}^{n} x(j)$$

Then $\|f_n\| = n$

$$\Big[x_n = (\underbrace{1, 1, 1 \ldots 1}_{n \text{ times}}, 0, 0 \ldots) \in X$$

$\|x_n\|_\infty = 1$, $\quad |f_n(x)| = n$

$\Rightarrow \|f_n\| = n \Big]$

$\Rightarrow \{ \|f_n\| \,/\, n = 1, 2, 3 \ldots \}$ is not uniformly bounded.

But $\{ f_n \mid n = 1, 2, 3 \ldots \}$ is pointwise bounded.

[$f : X \longrightarrow K$ by

$$f(x) = \lim_{n \to \infty} f_n(x), \quad x \in X$$

$$\|f\| \leq \underline{\left(\lim \sup_n \|f_n\|\right)} \|x\| \longrightarrow (*)$$

$$\leq n \|x\|$$

$\Rightarrow$ f is unbounded.

If $\{ \|f_n\| \}$ is uniformly bounded

$\exists\, M > 0 \ni \|f_n\| \leq M$

Then by (*) $\|fx\| \leq M \|x\|$

$$\Rightarrow \|f\| \leq M$$

which is not true.

Note :—

If X is a finite dimentional n.l.s, then pointwise bounded also implies uniformly bounded.

Let $\{u_1, u_2, \ldots u_m\}$ be a basis for X and $\{f_1, f_2, \ldots f_m\}$ be its dual base.

i.e., $f_i(u_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

Now for any $x \in X$, we have

$$x = \sum_{j=1}^{m} \alpha_j u_j$$

$$\Rightarrow f_i(x) = \sum_{j=1}^{m} \alpha_j \underbrace{f_i(u_j)}_{\delta_{ij}}$$

$$= \alpha_i$$

$$\therefore \; x = \sum_{i=1}^{m} f_i(x) u_i$$

Now let $\mathcal{A} = \{A \mid A \in BL(X, Y)\}$ be a family of bounded operators, which are pointwise bounded.

Now for any $A \in \mathcal{A}$,

$$Ax = A\left(\sum_{i=1}^{m} f_i(x) \cdot u_i\right)$$

$$= \sum_{i=1}^{n} f_i(x) A u_i$$

$$\|Ax\| = \left\| \sum_{i=1}^{m} f_i(x) A u_i \right\|$$

$$\leq \sum_{i=1}^{m} |f_i(x)| \, \|A u_i\|$$

$$\leq \left(\sum_{i=1}^{M} \|f_i\| \, \|x\| \, \|A u_i\| \right) \quad \text{---} (*)$$

$\because \mathcal{A}$ is Pointwise bounded

$\exists\ \beta_1, \beta_2 - - \beta_M \geq 0 \ \ni$

$$\|A u_i\| \leq \beta_i \quad \forall A \in \mathcal{A}.$$

Let $\beta = \max\{\beta_1, \beta_2 - - \beta_M\}$.

$$\alpha = \sum_{i=1}^{M} \|f_i\|.$$

Then from $\circledast$ we have

$$\|Ax\| \leq \left(\sum_{i=1}^{M} \|f_i\| \ \|x\| \beta_j\right)$$

$$\leq \beta \cdot \alpha \|x\|$$

$\forall x \in X.$

$\forall A \in \mathcal{A}$

$\Rightarrow \|A\| \leq \alpha\beta \quad \forall A \in \mathcal{A}.$

$\therefore \mathcal{A}$ is Uniformly bounded.

Uniform Boundedness Principle

Let X be a Banach Space and Y be a n.l.s, and $\mathcal{A} \subseteq BL(X,Y)$.
If $\mathcal{A}$ is Pointwise bounded, then $\mathcal{A}$ is uniformly bounded.

Proof: Suppose $\mathcal{A} = \{A_i \mid A_i \in BL(X,Y)\}$ is Pointwise bounded.

Then for each $x \in X$, $\exists\, M_x > 0$ such that

$$\|Ax\| \leq M_x \|x\|, \quad \forall A \in \mathcal{A}.$$

For each $n \in \mathbb{N}$, let

$$E_n = \{x \in X \mid \|Ax\| \leq n, \forall A \in \mathcal{A}\}$$

Claim: E_n is closed

Let $x \in \overline{E_n}$.

Then there exists a sequence $\{x_k\}$ in E_n such that

$$x_k \longrightarrow x.$$

$\because x_k \in E_n \Rightarrow \|Ax_k\| \leq n, \forall A \in \mathcal{A}, \forall k.$

Also $x_k \longrightarrow x$

$$\Rightarrow Ax_k \longrightarrow Ax, \quad \forall A \in \mathcal{A}.$$

$$\therefore \|Ax_k\| \longrightarrow \|Ax\|.$$

$$\therefore \|Ax\| = \lim_{k \to \infty} \|Ax_k\|$$

$$\leq \lim_{k \to \infty} n$$

$$\leq n, \quad \forall A \in \mathcal{A}$$

$\Rightarrow x \in E_n \Rightarrow \overline{E_n} \subseteq E_n \subseteq \overline{E_n}$

$\therefore E_n$ is closed.

Claim: $X = \bigcup_{n=1}^{\infty} E_n$

Suppose $X \neq \bigcup_{n=1}^{\infty} E_n$.

Then there exists $x \in X$ and $x \notin \bigcup_{n=1}^{\infty} E_n$.

$\Rightarrow x \notin E_n \ \forall \ n$

$\Rightarrow \|Ax\| > n, \ \forall \ A \in \mathcal{A}$,

which is contradiction to $\mathcal{A}$ is pointwise bounded.

$\therefore X = \bigcup_{n=1}^{\infty} E_n$.

[Baire-Catagory theorem: let X be a complete metric space. If $\{X_n\}$ is a sequence of subsets of X

such that $X = \bigcup_{n=1}^{\infty} X_n$, then there exists some j such that interior of $\overline{X_j} \neq \phi$]

$\therefore$ By Baire-Catagory theorem, there exists some $k \in \mathbb{N}$ such that interior of $E_k \neq \phi$ [$\because \overline{E_k} = E_k$]

So let $u \in E_k$ and $r > 0$ such that

$$B(u, r) \subset E_k.$$

$\because u \in E_k \Rightarrow \|Au\| \leq k, \forall A \in \mathcal{A}$.

Now for any $x \in X$, we have

$$\left\| u + \frac{rx}{2\|x\|} - u \right\| = \left\| \frac{rx}{2\|x\|} \right\|$$
$$= \frac{r}{2} < r$$

$$\Rightarrow u + \frac{rx}{2\|x\|} \in B(u, r) \subset E_k$$

$$\Rightarrow \left\| A\left(u + \frac{rx}{2\|x\|}\right)\right\| \leq k \quad \forall A \in \mathcal{A}.$$

Consider for any $A \in \mathcal{A}$

$$\left\| A\left(\frac{rx}{2\|x\|}\right)\right\| = \left\| A\left(u + \frac{rx}{2\|x\|} - u\right)\right\|$$

$$= \left\| A\left(u + \frac{rx}{2\|x\|}\right) - Au\right\|$$

$$\leq \left\| A\left(u + \frac{rx}{2\|x\|}\right)\right\| + \|Au\|$$

$$\leq k + k$$

$$= 2k.$$

$$\Rightarrow \|Ax\| \leq \frac{4k}{r}\|x\|, \ \forall A \in \mathcal{A}.$$

$$\Rightarrow \|A\| \leq \frac{4k}{r}, \ \forall A \in \mathcal{A}.$$

$\therefore$ $\mathcal{A}$ is uniformly bounded

—//—

Corollary (Banach-Steinhauss Theorem):-

Let X be a Banach space and Y be a n.l.s, and $\{A_n\}$ be a sequence of operators in $BL(X,Y)$ such that for every $x \in X$, $\{A_n x\}$ converges in Y. Let $A : X \to Y$ be defined by

$$Ax = \lim_{n\to\infty} A_n x, \quad x \in X.$$

Then $\{\|A_n\|\}$ is bounded and $A \in BL(X,Y)$.

Proof: For each $x \in X$, $\{A_n x\}$ converges in Y.

$\therefore \{A_n x \mid n = 1, 2, \ldots\}$ is a bounded set in Y.

$\therefore \{A_n \mid n = 1, 2, 3, \ldots\}$ is pointwise bounded set.

$\because X$ is a Banach space, by uniform boundedness principle,

$\{A_n \mid n \in \mathbb{N}\}$ is uniformly bounded set.

$\therefore \{\|A_n\| \mid n \in \mathbb{N}\}$ is a bounded set.

So let $\|A_n\| \leq c$ (say)

$$\because \quad Ax = \lim_{n \to \infty} A_n x, \quad \forall x \in X$$

$$\Rightarrow \|Ax\| \leq \left(\limsup_{n} \|A_n\|\right) \|x\|$$

$$\leq c \, \|x\|, \quad \forall x \in X$$

$$\Rightarrow \|Ax\| \leq c \, \|x\|, \quad \forall x \in X$$

$$\Rightarrow A \in BL(X, Y).$$

—//—

Theorem: Let X and Y be Banach spaces and $\{A_n\}$ be a sequence in $BL(X, Y)$. Then $\{A_n x\}$ converges in Y for every $x \in X$ iff $\{\|A_n\|\}$ is bounded and there exists a dense subset D of X such that $\{A_n u\}$ converges for every $u \in D$.

Proof: Suppose $\{\|A_n\|\}$ is bounded and $\{A_n u\}$ converges for every $u \in D$, where $\overline{D} = X$.

Claim: $\{A_n x\}$ converges in Y for every $x \in X$.

$\because x \in X = \overline{D} \implies \exists\, u \in D$

such that $\|x-u\| < \epsilon$ ——(1)

Now for any $m, n \in \mathbb{N}$, consider

$$\|A_n x - A_m x\| = \|A_n x - A_n u + A_n u - A_m u + A_m u - A_m x\|$$

$$\leq \|A_n x - A_n u\| + \|A_n u - A_m u\| + \|A_m x - A_m u\|$$

$$\leq \|A_n\| \|x-u\| + \|A_n u - A_m u\| + \|A_m\| \|x-u\|$$

$$\leq (\|A_n\| + \|A_m\|) \|x-u\| + \|A_n u - A_m u\|$$ ——(1)

$\because \{A_n u\}$ cgs for every $u \in D$

$\Rightarrow \{A_n u\}$ is a Cauchy sequence

$\therefore \exists\ n_0 \in \mathbb{N} \ni \forall\ n, m \geq n_0,$

$\|A_n u - A_m u\| < \epsilon$ ——(2)

Also since $\{\|A_n\|\}$ is bounded set, There exists $C > 0$ such that

$$\|A_n\| \leq C \quad \forall \ n \in \mathbb{N}.$$

— (3)

From (1), (2) & (3), we get

$$\|A_n x - A_m x\| \leq (C + C)\epsilon + \epsilon$$

$$= (2C + 1)\epsilon \quad \forall \ n, m \geq n_0$$

$\Rightarrow$ $\{A_n x\}$ is a Cauchy sequence in Y.

$\because$ Y is a Banach space,

$\{A_n x\}$ converges in Y.

Conversely suppose that $\{A_n x\}$ converges in Y for every $x \in X$.

$\because$ X is a Banach space by uniform bounded principle, $\{\|A_n\|\}$ is a bounded set.

Also since $D \subset X$, In particular $\{A_n u\}$ converges for every $u \in D \subset X$.

Corollary :— Let X be a Banach space, Y be a n.l.s and $\{A_n\}$ be a sequence in $BL(X, Y)$ such that $\{A_n x\}$ converges in Y for every $x \in X$. Let $A : X \to Y$ be defined by $Ax = \lim_{n \to \infty} A_n x$, $\forall x \in X$.

Then for every totally bounded subset $S \subseteq X$,

$$\sup_{x \in S} \|A_n x - Ax\| \longrightarrow 0 \text{ as } n \to \infty$$

i.e., $\|(A_n - A)|_S\| \longrightarrow 0$

Proof: [We say a set S is a totally bounded set, if for all $\epsilon > 0$,

$$S \subseteq \bigcup_{i=1}^{n} B(x_i, \epsilon),$$

certain $x_1, x_2 \ldots x_n \in S$]

Given that S is a totally bounded subset of X and $\epsilon > 0$ be given. Then there exist $x_1, x_2, \ldots x_k \in S$ such that

$$S \subseteq \bigcup_{i=1}^{k} B(x_i, \epsilon)$$

$$= \bigcup_{i=1}^{k} \{x \in X \mid \|x - x_i\| < \epsilon\}$$

Let $x \in S$, then $\exists$ $j \in \{1, 2, \ldots k\}$

such that $x \in B(x_j, \epsilon)$

$$\Rightarrow \|x - x_j\| < \epsilon.$$

$\because \{A_n x\}$ converges for every $x \in X$ implies $\{A_n x_j\}$ also converges for $x_j \in S \subseteq X$.

$$\therefore \quad Ax_j = \lim_{n \to \infty} A_n x_j$$

$\Rightarrow \exists \; n_0 \in \mathbb{N} \ni$

$$\|Ax_j - A_n x_j\| < \epsilon \quad \forall \; n \geq n_0.$$

Also since $\{A_n x\}$ cgs for every $x \in X$ and X is a Banach space implies $\{\|A_n\|\}$ is a bounded set, by Uniform boundedness Principle.

$$\therefore \quad \|A_n\| \leq C, \qquad C > 0, \; \forall \; n.$$

$\Rightarrow \|A\| \leq C \qquad [\because \|Ax\| \leq \left(\lim_{n \to \infty} \|A_n\|\right) \|x\| \leq C\|x\|.$

Now consider for $x \in S$,

$$\|A_n x - Ax\| \le \|A_n x - A_n x_j\| + \|A_n x_j - A x_j\| + \|A x_j - Ax\|$$

$$\le \|A_n\| \, \|x - x_j\| + \|A\| \, \|x - x_j\| + \|A_n x_j - A x_j\|$$

$$\le C\epsilon + \|A\|\epsilon + \epsilon$$

$$\le (C + \|A\| + 1)\epsilon$$

$$\le (C + C + 1)\epsilon$$

$$= (2C+1)\epsilon$$

$\therefore$ for any $x \in S$, we have

$$\|A_n x - Ax\| < (2C+1)\epsilon$$

$$\sup_{x \in S} \|A_n x - Ax\| < (2C+1)\epsilon$$

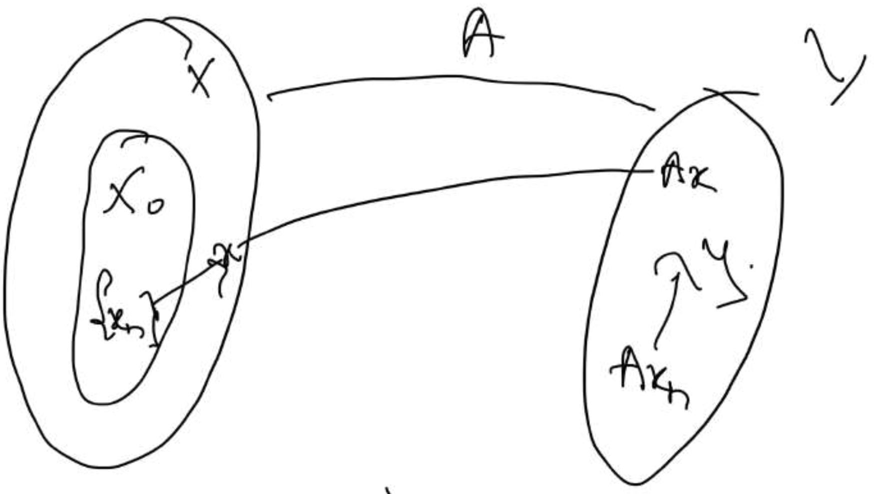
$$\Rightarrow \|(A_n - A)|_S\| < (2c+1)\epsilon \longrightarrow 0 \text{ as } n \to \infty.$$

—//—

Closed operator :—

Let X and Y be n.l.s and X_0 be a subspace of X.

A linear operator $A : X_0 \to Y$ is said to be closed operator if for every sequence $\{x_n\}$ in X_0 such that $x_n \to x \in X$ and $Ax_n \to y \in Y$ implies $x \in X_0$ and $Ax = y$.



$\Rightarrow x \in X_0 \quad Ax = y$

Ex: $X = Y = C[0,1]$ with norm $\|\cdot\|_\infty$.

$X_0 = C'[0,1] \subset C[0,1] = X$

Define $A : C'[0,1] \subseteq C[0,1] \longrightarrow C[0,1]$

by $Ax = x'$, $\forall x \in C'[0,1]$.

Claim: A is a closed operator.

Let $\{x_n\}$ be a sequence in $X_0 = C'[0,1]$ such that $x_n \longrightarrow x \in C[0,1]$ and

$Ax_n \longrightarrow y \in C[0\ 1]$.

Then for each $t \in [0\ 1]$, we have

$$\int_0^t y(\tau)\, d\tau = \lim_{n \to \infty} \int_0^t Ax_n(\tau)\, d\tau$$

$$= \lim_{n \to \infty} \int_0^t x_n'(\tau)\, d\tau$$

$$= \lim_{n \to \infty} [x_n(\tau)]_0^t$$

$$= \lim_{n \to \infty} [x_n(t) - x_n(0)]$$

$$= x(t) - x(0)$$

$$\therefore \quad x(t) = x(0) + \int_0^t y(\tau)\, d\tau$$

$\forall\, t \in [0\ 1]$.

$$\Longrightarrow x'(t) = y(t), \quad \forall\, t \in [0\ 1]$$

$\Rightarrow \quad Ax(t) = y(t), \quad \forall\, t \in [0,1]$

$\Rightarrow \quad Ax = y$

$\because \quad x' = y \in C[0,1]$

$\Rightarrow \quad x' \in C[0,1]$

$\Rightarrow \quad x \in C'[0,1]$

Thus $x \in C'[0,1]$ & $Ax = y$

$\therefore$ A is a closed operator.

Let $x_n(t) = t^n, \quad \forall\, t \in [0,1]$

Then $Ax_n(t) = nt^{n-1}$

$\Rightarrow \quad \|Ax_n\|_\infty = n$

$\Rightarrow$ A is unbounded operator.

* A closed operator need not be a bounded operator.

Problem: Is every bounded operator, a closed operator?

Let $A : X_0 \subseteq X \to Y$ be a linear map, where X, Y are n.l.s, and X_0 is a subspace of X.

- Then
$$G(A) = \{ (x, Ax) / x \in X_0 \}$$ is called graph of the operator A.

- Then $G(A)$ is a subspace of the product n.l.s $X \times Y$.

- The norm on $X \times Y$ is given by
$$\|(x,y)\| = \|x\|_X + \|y\|_Y, \quad \forall (x,y) \in X \times Y.$$

Theorem: Let X and Y be n.l.s and X_0 be a subspace of X.

A linear operator $A : X_0 \subset X \to Y$ is a closed linear operator iff its graph $G(A) = \{(x, Ax) \mid x \in X_0\}$ is a closed subspace of $X \times Y$.

Proof: Suppose $A : X_0 \subset X \to Y$ be a closed operator.

Claim: $G(A)$ is a closed.

Let $(x, y) \in \overline{G(A)}$

$\Rightarrow \exists$ a sequence $\{(x_n, Ax_n)\}$ in $G(A)$ such that

$$(x_n, Ax_n) \longrightarrow (x, y)$$

$\Rightarrow (x_n - x, Ax_n - y) \longrightarrow (0, 0)$

$\Rightarrow \|(x_n - x, Ax_n - y)\| \longrightarrow 0$

$\Rightarrow \|x_n - x\|_X + \|Ax_n - y\|_Y \longrightarrow 0$

$\Rightarrow x_n \to x \in X$ and $Ax_n \to y \in Y$

$\because$ A is closed operator, $x_n \in X_0$
$\to x \in X$

$\Rightarrow x \in X_0$ and $Ax = y$

$\therefore (x, y) = (x, Ax) \in G(A)$

$\therefore G(A)$ is a closed subspace of $X \times Y$.

Conversely $G(A)$ be a closed subspace $X \times Y$.

Claim: A is a closed operator.

Let $\{x_n\}$ be a sequence in X_0 such that $x_n \longrightarrow x \in X$, $Ax_n \to y$ in Y

$\Rightarrow \|x_n - x\|_X \longrightarrow 0$ & $\|Ax_n - y\|_Y \to 0$

$\Rightarrow \|x_n - x\|_X + \|Ax_n - y\|_Y \longrightarrow 0$

$\Rightarrow \|(x_n - x, Ax_n - y)\| \longrightarrow 0$

$\Rightarrow \|(x_n, Ax_n) - (x, y)\| \longrightarrow 0$

$\Rightarrow (x_n, Ax_n) \longrightarrow (x, y)$

Thus $\{(x_n, Ax_n)\}$ is a sequence in $G(A) \to (x_n, Ax_n) \longrightarrow (x, y)$.

$\because$ $G(A)$ is a closed subspace

implies $(x, y) \in G(A)$.

$$= \{(x, Ax) \mid x \in X_0\}$$

$\therefore$ $x \in X_0$ & $y = Ax$

$\therefore$ A is a closed operator