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Assignment 2 MM - Part 2
Keerti P. Charantimath 19MA20059

1) i) $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

$n=0, \quad J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \Gamma(0+r+1)} \left(\frac{x}{2}\right)^{2r}$

$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, r!} \left(\frac{x}{2}\right)^{2r}$

Now,

$J_0'(x) = \frac{d}{dx}\left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,r!}\left(\frac{x}{2}\right)^{2r}\right) = \sum_{r=1}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2r-1} \cdot r$

For $r = m+1$,

$J_0'(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+1)!\, m!}\left(\frac{x}{2}\right)^{2(m+1)-1} = -\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma_{m+2}}\left(\frac{x}{2}\right)^{2m+1}$

$= -J_1(x)$

$\therefore J_0'(x) = -J_1(x)$ hence proved

ii) $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

$n=1, \quad J_1'(x) = \frac{1}{2}\left[J_0 - J_2\right]$

Also, $J_0'(x) = -J_1(x) \Rightarrow J_1'(x) = -J_0''(x)$

$\therefore \quad -J_0''(x) = \frac{1}{2}\left[J_0 - J_2\right] \Rightarrow 2J_0''(x) = J_2(x) - J_0(x)$

hence proved

iii) $J_{n+1}(x) = \frac{n}{x} J_n(x) - J_n'(x)$

$n=1, \quad J_2(x) = \frac{1}{x} J_1(x) - J_1'(x)$

Also, $J_1'(x) = -J_0''(x)$

$\therefore \quad J_2(x) = \frac{-1}{x} J_0'(x) + J_0''(x)$

$\Rightarrow J_2(x) = J_0''(x) - \frac{J_0'(x)}{x}$ hence proved

2) $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

$\Rightarrow J_n(xy) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \Gamma(n+r+1)} \left(\frac{xy}{2}\right)^{n+2r}$

$\Rightarrow y^{n+1} J_n(xy) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} y^{2(n+r)+1}$

now, $x \int_0^1 J_n(xy) \, y^{n+1} dy = x \int_0^1 \sum_{r=0}^{\infty} \left\{ \frac{(-1)^r}{r! \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} \cdot y^{2n+2r+1} \right\} dy$

$= x \sum_{r=0}^{\infty} \frac{-1^r}{r! \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} \int_0^1 y^{2n+2r+1} dy$

$= x \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} \frac{1}{2(n+r+1)}$

$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \Gamma(n+r+2)} \left(\frac{x}{2}\right)^{n+1+2r}$

$= J_{n+1}(x)$

$\therefore J_{n+1}(x) = x \int_0^1 J_n(xy) \, y^{n+1} dy$ hence proved

3) $\frac{d}{dx} [x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

$\Rightarrow \int -x^{-n} J_{n+1}(x) = x^{-n} J_n(x)$

$\therefore \int J_3(x) = \int x^2 [x^{-2} J_3(x)] dx$

$= x^2 [-x^{-2} J_2(x)] + \int 2x [x^{-2} J_2(x)] dx$

$= -J_2(x) + 2 \int x^{-1} J_2(x) dx$

$= -J_2(x) + 2 [-x^{-1} J_1(x)] + c$

$\therefore \int J_3(x) = -J_2(x) + 2(-x^{-1} J_1(x)) + c$

Also,

$J_n(x) = \frac{x}{2n} [J_{n-1}(x) + J_{n+1}(x)] \Rightarrow J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$

$n=1, \quad J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$

$\int J_3(x) dx = c - J_2(x) - \frac{2}{x} J_1(x) = c - \left(\frac{2}{x} J_1(x) - J_0(x)\right) - \frac{2}{x} J_1(x)$

$\int J_3(x) dx = c + J_0(x) - \frac{4}{x} J_1(x)$, c is constant

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4) i) $J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\frac{x}{2}\right)^{2r}$

$\Rightarrow J_0'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \frac{d}{dx}\left(\frac{x}{2}\right)^{2r} = \sum_{r=1}^{\infty} \frac{(-1)^r}{(r!)^2} r . \left(\frac{x}{2}\right)^{2r-1}$

put $r = m+1$

$J_0'(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{m!\, m+1!} \frac{x}{2}^{(2m)+2-1} = -\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma m+2} \left(\frac{x}{2}\right)^{2m+1} = -J_1(x)$

$\therefore J_0'(x) = -J_1(x)$

4) ii) $I = \int_a^b J_0(x).J_1(x)\,dx = \int_a^b J_0(x) -\frac{d}{dx}[J_0(x)]dx$

$= -\int_a^b J_0(x)\, d[J_0(x)] = \left[\frac{\{J_0(x)\}^2}{2}\right]_b^a$

$I = \frac{J_0^2(a) - J_0^2(b)}{2}$

$\therefore \int_a^b [J_0(x) \cdot J_1(x)]\,dx = \frac{1}{2}\left(J_0^2(a) - J_0^2(b)\right)$

5. $J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\frac{x}{2}\right)^{2r} \Rightarrow J_0(bx) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\frac{xb}{2}\right)^{2r}$

$\int_0^{\infty} e^{-ax} J_0(bx)\,dx = \int_0^{\infty} e^{-ax}\left[\sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{xb}{2}\right)^{2r}\right] dx = I$

Let $z = ax$, $dx/dz = 1/a$, when $x=0 \to z=0$, $x=\infty \to z=\infty$

$I = \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(r!)^2} \times \left(\frac{b}{2}\right)^{2r} . \int_0^{\infty} e^{-z} . \left(\frac{z}{a}\right)^{2r} . \frac{1}{a}\,dz\right)$

$= \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(r!)^2} . \left(\frac{b}{a}\right)^{2r} . \frac{1}{2^{2r}} . \frac{1}{a} . \Gamma 2r+1\right)$

$= \frac{1}{a}\sum_{r=0}^{\infty}\left[(-1)^r . \frac{(2r)!}{2^r} . \frac{1}{2r . r! . r!} . \left(\frac{b^2}{a^2}\right)^r\right]$

$= \frac{1}{a}\sum_{r=0}^{\infty}\left[(-1)^r . \frac{1\times3\times5\times\ldots\times(2r-1)}{2\times4\times\ldots\times2r} \times \left(\frac{b^2}{a^2}\right)^r\right] = \frac{1}{a}\left(1+\frac{b^2}{a^2}\right)^{-1/2}$

$= \frac{1}{a}\sqrt{\frac{a^2}{b^2+a^2}} = \frac{1}{a}\frac{|a|}{\sqrt{a^2+b^2}} = \frac{1}{\sqrt{a^2+b^2}}$ (as $a>0$)

$\therefore \int_0^{\infty} e^{-ax} J_0(bx)\,dx = \frac{1}{\sqrt{a^2+b^2}}$ when $a>0$, hence proved