Lecture 12 (26-02-2024).

Metric :— Let X be a nonempty set. A real valued function d defined on $X \times X$ is called a metric or a distance function on X if d satisfies the following:

For any $a, b, c \in X$,

(i) $d(a, b) \geq 0$ and $d(a, b) = 0 \Leftrightarrow a = b$.

(ii) $d(a, b) = d(b, a)$

(iii) $d(a, c) \leq d(a, b) + d(b, c)$.

iv) If $a \neq b$, $d(a, b) > 0$,

In this case (X, d) is said to be a metric space.

Ex: $X = \mathbb{R}$, For any $a, b \in X$

let $d(a, b) = |a - b|$.

Then d is a metric on $\mathbb{R}$.

(ii) $X = \mathbb{R}^2$, for $\left.\begin{array}{l} p = (a_1, a_2) \\ q = (b_1, b_2) \end{array}\right\} \in \mathbb{R}^2$

define $d : X \times X \longrightarrow \mathbb{R}$

by $d(p, q) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$.

Then d is a metric on $\mathbb{R}^2$.

We also define

$$d_1(p, q) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$$

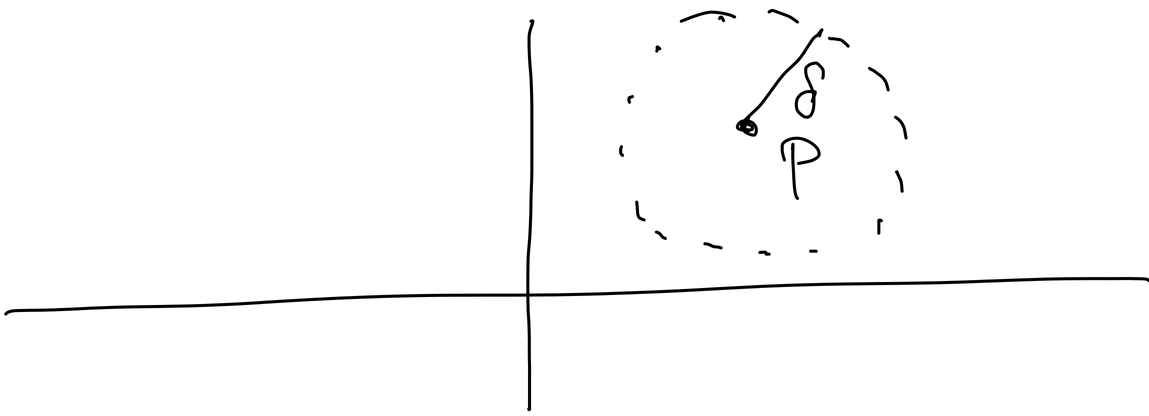
$$d_2(p, q) = |a_1 - b_1| + |a_2 - b_2|.$$

Then d_1 and d_2 are also metrics on $\mathbb{R}^2$.

Open sphere :— Let (X, d) be a metric space. For any point $p \in X$, and a real number $\delta > 0$, we define

$$\underset{\substack{\shortparallel \\ B_d(p,\delta)}}{S_d(p, \delta)} = \{x \in X \mid d(x, p) < \delta\}$$

For simplicity we write $S_d(P,\delta)$ as $S(P,\delta)$. It is called an open sphere with center P and radius δ.



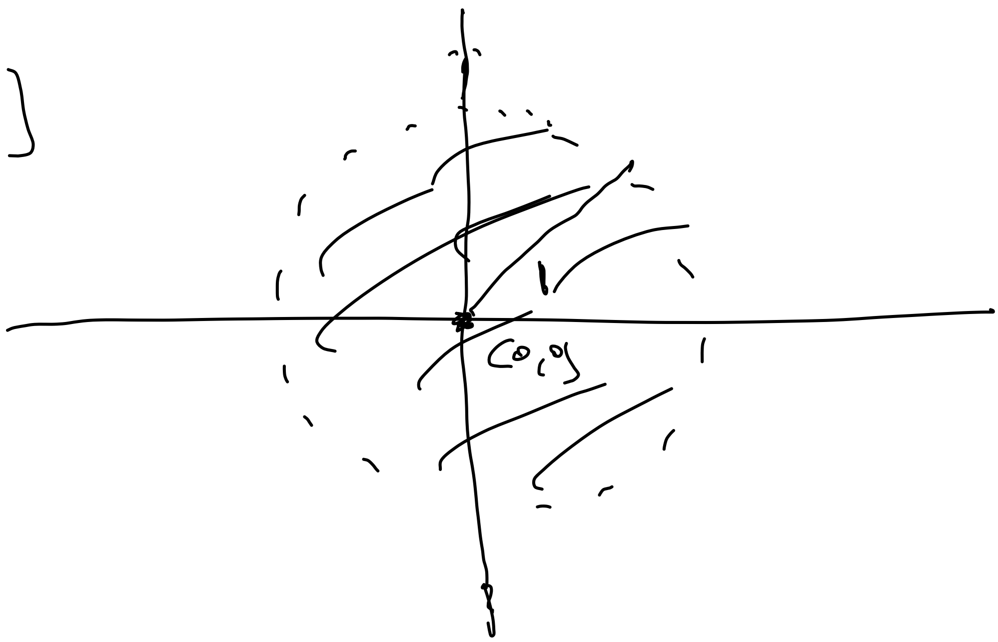
Ex: $P = (0,0) \in \mathbb{R}^2$, $\delta = 1$.

If $d(p,q) = \sqrt{(a_1-b_1)^2+(a_2-b_2)^2}$

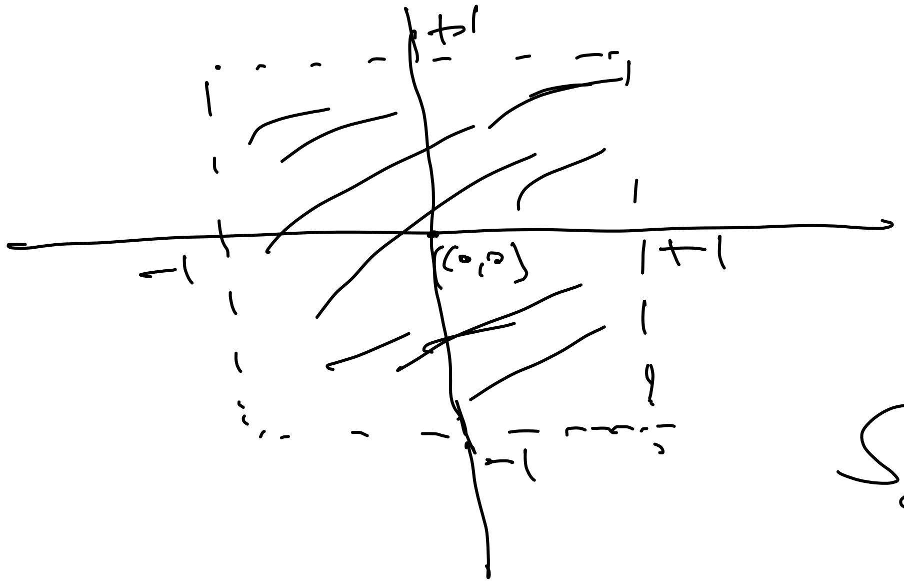
for $p = (a_1,a_2)$, $q = (b_1,b_2) \in \mathbb{R}^2$.

Then

$S_d(P,\delta)$

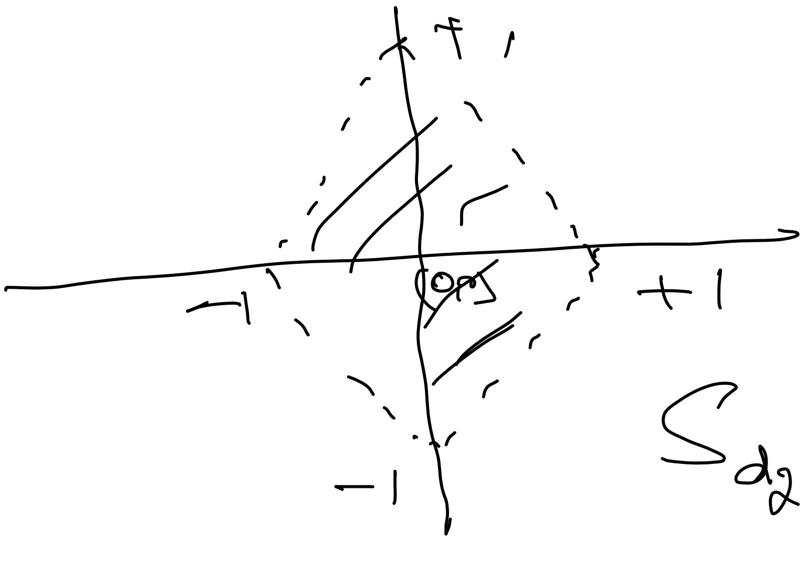


If $d(p,q) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$



$S_{d_1}(p,1)$

If $d_2(p,q) = |a_1 - b_1| + |a_2 - b_2|$.



$S_{d_2}(p,1)$.

Lemma :- let $S = S(p,\delta)$ be an open sphere in a metric space (X,d). Then for any $q \in S$, there exists an open sphere S_1 with center at q $\ni$ $S_1 \subset S$.

Proof: let $q \in S = S(p, \delta)$

$$\Rightarrow d(p,q) < \delta$$

So let $\epsilon = \delta - d(p,q) > 0$

let $S_1 = S(q, \epsilon)$

Then for any $x \in S_1$, we have

$$d(x,q) < \epsilon = \delta - d(p,q)$$

Now by Δle inequality, we have

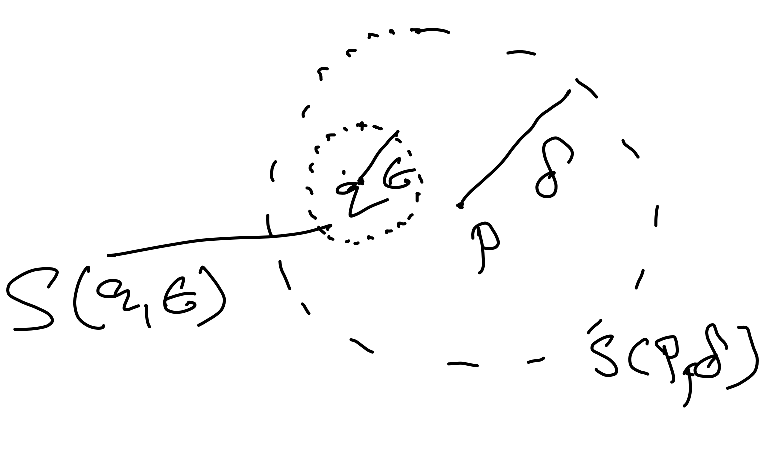
$$d(x,p) \leq d(x,q) + d(q,p)$$

$$< \delta - d(p,q) + d(p,q)$$

$$= \delta$$

$$\Rightarrow x \in S(p,\delta)$$

$$\Rightarrow S_1 = S(q, \epsilon) \subset S(p, \delta).$$



Note :- (1) If $\delta_1, \delta_2 \in R$ with $0 < \delta_1 \leq \delta_2$. Then for any point P in a metric space (X, d), we have $S(P, \delta_1) \subseteq S(P, \delta_2)$.

$\because x \in S(P, \delta_1) \Rightarrow d(P, x) < \delta_1 \leq \delta_2$

$\Rightarrow x \in S(P, \delta_2)$

$\therefore S(P, \delta_1) \subseteq S(P, \delta_2)$.

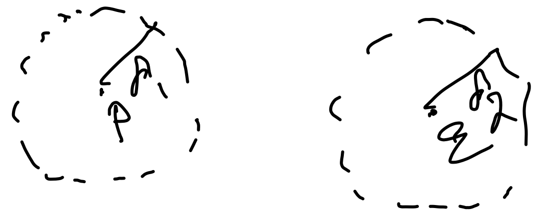
(2) If S and S_1 are any two open spheres with same center, then one of them is contained in the other.

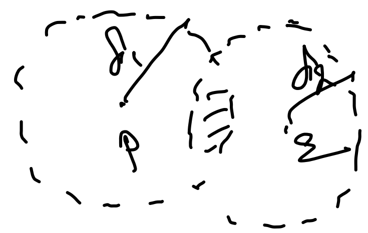
$\because S = S(P, \delta_1)$, $S_1 = S(P, \delta_2)$.

$\because \delta_1, \delta_2 \in R^+$, so either $\delta_1 \leq \delta_2$ or $\delta_2 \leq \delta_1$

$\Rightarrow$ by previous note (1), either $S \subseteq S_1$ or $S_1 \subseteq S$

(3) In general, intersection of any two open spheres need not be an open sphere.





However, we can show that for every point 'x' in the intersection of two open spheres, there exists an open sphere with center 'x' and contained in the intersection.

Lemma :- Let S_1 and S_2 be any two open spheres in a metric space (X, d) and let $p \in S_1 \cap S_2$. Then there exists an open sphere S_p with center p such that $p \in S_p \subset S_1 \cap S_2$.

Proof : Given that $P \in S_1 \cap S_2$.

$\Rightarrow$ $P \in S_1$ and $P \in S_2$.

$\Rightarrow$ $\exists$ open sphere S_1^* and S_2^* with center P such that

$$S_1^* \subset S_1 \quad \text{and} \quad S_2^* \subset S_2$$

Now since S_1^* and S_2^* are the open sphere with same center P, one of these say S_1^* is contained in S_2^* (or $S_2^* \subset S_1$).

Hence

$$P \in S_1^* \subset S_1 \quad \text{and} \quad P \in S_1^* \subset S_2^* \subset S_2$$

$$\Rightarrow \quad P \in S_1^* \subset S_1 \cap S_2$$

Hence we ~~take~~ here $S_p = S_1^*$.

Def: Let (X, d) be a metric space. The topology $\mathcal{T}$ on X generated by the class of open spheres

$$\{ S(p,\delta) \mid p \in X, \delta \in \mathbb{R}, \delta > 0 \}$$

in X is called metric topology on X or topology induced by the metric d.

Hence all the concepts defined for a topological space also defined for metric space (X, d).

[Attendance : 32, 27, 06, 60, 53, 26
41, 10, 17, 16, 62, 65].