Graph theory sep 1-9

Graph Theory And Algorithms (Indian Institute of Technology Kharagpur)

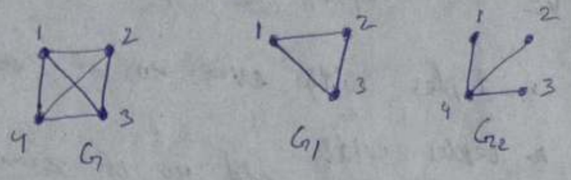
1/9/22

Decomposition of a Graph

Let G be a connected graph. and let $G_1, G_2, \ldots, G_k$ be subgraphs of G. We say that G decomposes into the subgraphs $G_1, G_2, \ldots, G_k$ if

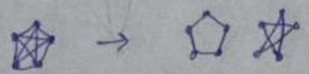
$$E(G) = \bigcup_{i=1}^{k} E(G_i), \quad E(G_i) \cap E(G_j) = \phi, \quad 1 \le i \ne j \le k$$

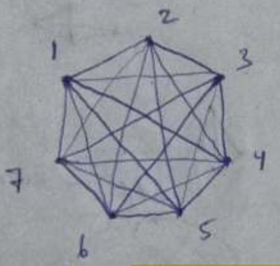
Here we also say that $\{G_1, G_2, \ldots, G_k\}$ is a decomposition of G.

ex  $\{G_1, G_2\}$ is a decomposition

K_4 can't be decomposed into two copies of K_3.

K_5 can be decomposed into two copies of C_5.





Can be decomposed into 7 copies of K_3.

Q/ for which value of n, K_n decomposes into triangles?

ans: n is odd and (n)(n-1) are divisible by 6

Theorem A connected graph G is Eulerian iff G decomposes into cycles.

Pf G is Eulerian iff degree of every vertex in G is even.

If G decomposes into cycles then degree of every vertex in G is even. Hence it is Eulerian.

Converse ...

Hamiltonian Graphs

Let $|V(G)| = n$. A cycle of length n in G is called a Hamiltonian cycle. A path P in G on n vertices ($P = P_n$) is called a Hamiltonian path.

A graph G is called Hamiltonial graph if G contains a Hamiltonian cycle

Q_3  is Hamiltonian

whether Petersen graph is Hamiltonian? NO.

<u>Travelling Salesman problem</u>:
weighted graph
least weight Hamiltonian cycle in a weighted Graph.

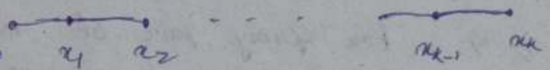
delta g is the minimum of all degrees of vertices

<u>Theorem (Dirac 1952)</u>
Let G be a simple graph on n vertices, $n \geq 3$. if $\delta(G) \geq \frac{n}{2}$ then G is Hamiltonian.

since $\delta(G) \geq \frac{n}{2}$, G is connected.

- The condition $\delta(G) \geq \frac{n}{2}$ is sufficient but may not be necessary condition
  ex - C_{10}.

Let $P: x_0, x_1, \ldots, x_k$ be a longest path in G.



$k+1 \leq n$.

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$S = \{0, 1, 2, \ldots, k-1\}$

All the neighbours of x_0 and x_k lie on path P.

Let $S_1 = \{i \in S : x_k \sim x_i\}$, $S_2 = \{j \in S : x_0 \sim x_{j+1}\}$

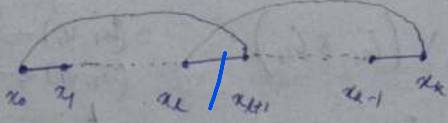
$|S| = k \leq n-1$, $|S_1| \geq \frac{n}{2}$, $|S_2| \geq \frac{n}{2}$

$S_1 \cap S_2 \neq \phi$ ($\because$ $|S| = n-1$, $|S_1| + |S_2| = n$, $|S| = |S_1 \cup S_2|$)

$|S_1 \cap S_2| = |S_1| + |S_2| - |S_1 \cup S_2| \geq \frac{n}{2} + \frac{n}{2} - (n-1) = 1$

$\Rightarrow \exists\, l \in S$ s.t $l \in S_1 \cap S_2$.

$\Rightarrow x_k \sim x_l$ and $x_0 \sim x_{l+1}$



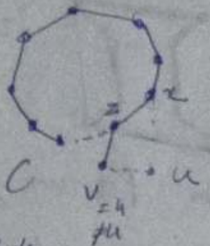
Now $C = P - \{x_l, x_{l+1}\} \cup \{\{x_0, x_{l+1}\}, \{x_k, x_l\}\}$ is a cycle on $k+1$ vertices.

Claim: C contains all the vertices of G.

Suppose $\exists\, u \in V(G)$ s.t. $u \notin C$.

V may or may not be equal to u.

Then we get a path longer than length k.



Since G is connected, for $x \in C$, $\exists$ a x-u path in G.

Then $\exists$ a vertex $v \notin V(C)$ s.t $v \sim z$, for some $z \in V(C)$.
(z may be equal to x and v may be equal to u).

Let $z = x_i$. Then $P' = C - \{x_i, x_{i+1}\} \cup \{v, x_i\}$ is a path with $L(P') = k+1$, which is longer than path P, and is a contradiction.

Hence the claim is true. i.e. C is a Hamiltonian cycle.

<u>Theorem (ore, 1960)</u>

Let G be a simple connected graph on n vertices, $n \geq 3$. If $\deg x + \deg y \geq n$, for every pair of non adjacent vertices x and y in G, then G is a Hamiltonian graph.

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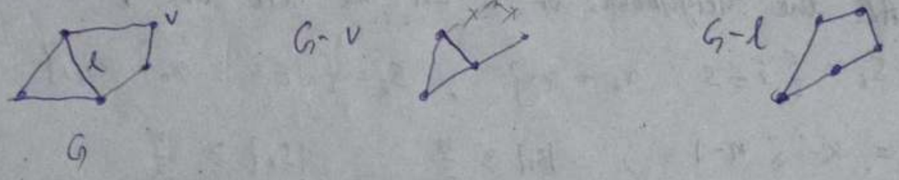


① petersen graph.

② $u_1, u_2, u_3, \dots, u_m, u_1$ Hamiltonian cycle in G_1

$v_1, v_2, v_3, \dots, v_n, v_1$ Hamiltonian cycle in G_2

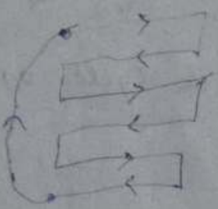
$V(G_1 \times G_2)$

$(u_1, v_1) \to (u_1, v_2) \to \dots (u_1, v_n)$

$(u_2, v_1) \leftarrow (u_2, v_2) \dots \leftarrow (u_2, v_n)$

$(u_m, v_1) \leftarrow (u_m, v_2) \leftarrow \dots (u_m, v_n)$

if m is even



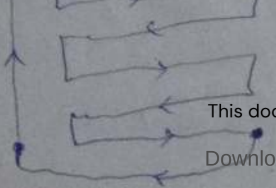
$(u_1, v_1) - (u_1, v_2) \dots - (u_1, v_n)$

$(u_2, v_1) \quad (u_2, v_2) - \dots - (u_2, v_n)$

if m is odd

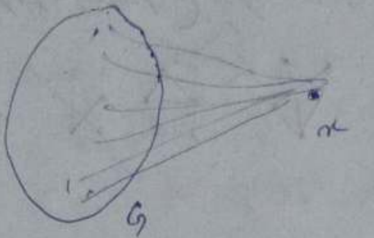
$(u_{m-1}, v_1) \quad (u_{m-1}, v_2) \quad (u_{m-1}, v_n)$

$(u_m, v_1) \quad (u_m, v_2)$



(3) $\delta(G) \geq \frac{n-1}{2} \rightarrow G$ contains a Hamiltonial path.

$\delta(G) \geq \frac{n}{2} \rightarrow G$ is Hamiltonian.

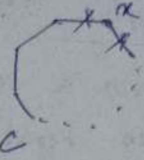


$\hat{G} = G \vee K_1$

$\delta_{\hat{G}}(v) = \frac{n-1}{2} + 1 = \frac{n+1}{2} \geq \frac{n}{2}$

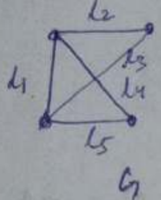
$\Rightarrow \hat{G}$ is Hamiltonian

Hamiltonian cycle



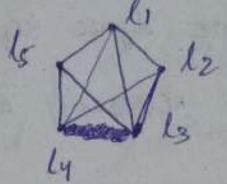
$C - x$ is a Hamiltonian path in G.

(4)



Line graph of G

$L(G)$



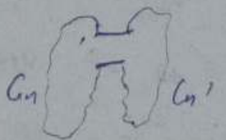
If G is Eulenian then prove that $L(G)$ is Hamiltonian.

??

(5) $Q_n = Q_{n-1} \times K_2$

Induction on n.

Assume that Q_{n-1} is Hamiltonian.

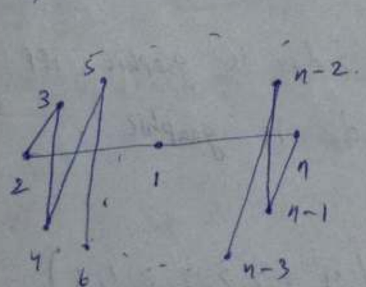


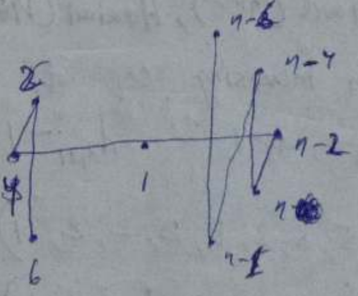
(6) K_n, n odd. ans: $\frac{n-1}{2}$

$|E(K_n)| = \frac{n(n-1)}{2}$

Every Hamiltonian cycle in K_n contains n edges.

$\Rightarrow K_n$ contains at most $\frac{n-1}{2}$ no of edge disjoint Hamiltonian cycles





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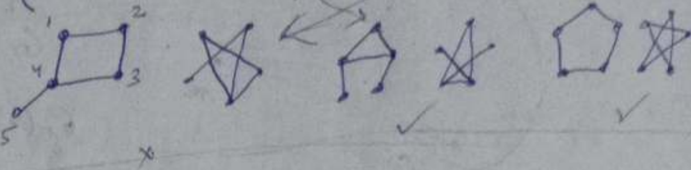
(3) $G \simeq \bar{G}$ $x \in V(G)$, $\deg x \neq 4, 0$

$|E(G)| = 5$ $3, 2, 1$

deg sequences $(3, 3, 2, 1, 1)$, $(3, 2, 2, 2, 1)$, $(2, 2, 2, 2, 2)$



Graphic sequences

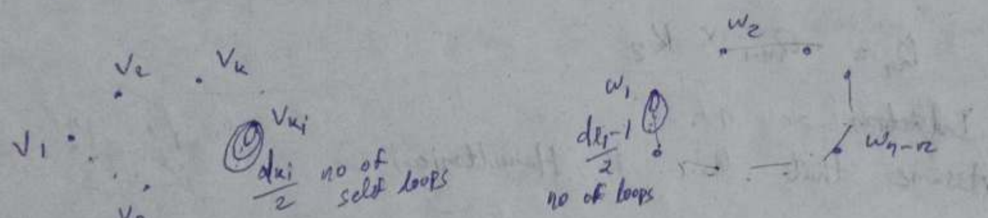
Q. Given a non increasing sequence $d: d_1 \geq d_2 \geq \cdots \geq d_n$ of non negative integers with $\sum_{i=1} d_i$ = even : Is there a graph G whose degree sequence is d ?

Case-I : G not necessarily a simple graph.

ans: Yes.

Suppose $d_{k_1}, d_{k_2}, \ldots, d_{k_{n}}$ be even and the rest in

$S = \{d_1, d_2, \ldots, d_n\} - \{d_{k_1}, d_{k_2}, \ldots, d_{k_n}\}$ be odd.



even no of vertices with odd degree.

Case-II : G is simple.

Ans: no, in general

ex $(4, 1, 1)$

Def A non increasing sequence $d: d_1 \geq d_2 \geq \cdots \geq d_n$ is said to be graphic if $\exists$ a simple graph G whose deg seq is d.

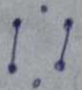
Theorem: (Havel (1955), Hakimi (1962))

A non increasing seq $d: d_1 \geq d_2 \geq \cdots \geq d_n$ is graphic iff $d': d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ is graphic.

ex $d: (5, 5, 3, 3, 3, 2, 2, 2, 1)$

$d': (4, 2, 2, 2, 1, 2, 2, 1) \rightarrow (4, 2, 2, 2, 2, 2, 1, 1)$

$d'': (1, 1, 1, 1, 2, 1, 1) \rightarrow (2, 1, 1, 1, 1, 1, 1)$

$d''' : (0,0,1,1,1,1) \to (1,1,1,1,0,0)$  ✓

Hence $5,5,3,3,3,2,2,2,1$ is graphic

Proof: Suppose d' is graphic. Then $\exists$ a simple graph G with degree sequence d'.

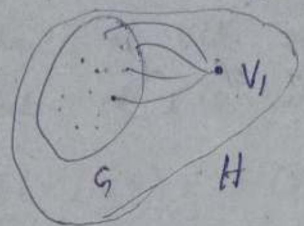
Suppose $\deg_G(v_2) = d_2 - 1$, $\deg_G(v_3) = d_3 - 1$, $\dots$ $\deg_G(v_{d_1+1}) = d_{d_1+1} - 1$,

$\dots$ $\deg_G(v_n) = d_n$.

Construct a simple graph H from G in the following way.

$V(H) = V(G) \cup \{v_1\}$.

$E(H) = E(G) \cup \{\{v_1, v_i\} : i = 2, 3, \dots, d_1+1\}$



H is a realization of d.

$\Rightarrow d$ is graphic.

Conversely, let d is graphic. To show that d' is graphic.

9/9/22 G be a realisation of d. G is a simple graph whose degree sequence is d.

Let $V(G) = \{v_1, v_2, \dots, v_n\}$ with $\deg v_i = d_i$, $i = 1, 2, \dots, n$

Let $S = \{v_2, v_3, \dots, v_{d_1+1}\}$.

If v_1 is adjacent with all the vertices in S, then we remove v_1 from G. $G - v_1$ is the realisation of d' and we are done.

If $\exists$ a vertex $v_r \in S$ s.t. $v_r \nsim v_1$.

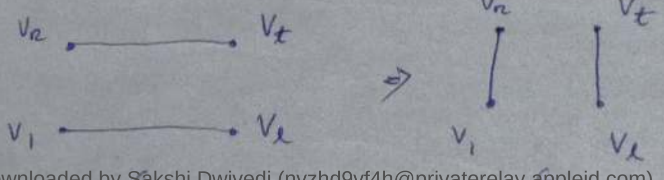
$\deg v_1 = d_1$. Then $\exists$ a vertex v_l in S^c s.t. $v_1 \sim v_l$

$\deg v_r = d_r$, $\deg v_l = d_l$ $d_r \geq d_l$. $l \in \{d_1+2, \dots, n\}$.

$\Rightarrow v_r$ has atleast d_l no of neighbours.

$v_l \sim v_1$ and $v_r \nsim v_1$.

$\Rightarrow \exists\, v_t \in V(G)$ s.t. $v_r \sim v_t$ and $v_l \nsim v_t$.



Let G_1 be the graph obtained from G as follows:

$V(G_1) = V(G)$

$E(G_1) = E(G) - \{\{v_n, v_t\}, \{v_1, v_2\}\} \cup \{\{v_1, v_n\}, \{v_2, v_t\}\}$.

degree of every vertex remain same.

G_1 is a simple graph with degree sequence d.

In G_1 $v_1 \sim v_n$.

If v_1 is not adjacent with atleast one vertex in S (in the graph G_1) then repeat the above process.

After finite no of steps, we get a simple graph G_k with degree sequence d and in which

$v_1 \sim v_2,\ v_1 \sim v_3, \ldots, v_1 \sim v_{d_1+1}$.

Now $H = G_k - v_1$ is a realisation of d'.

Ex Is $(5,5,5,4,2,1,1,1)$ graphic ?

d : $5, 5, 5, 4, 2, 1, 1, 1$

d' : $4, 4, 3, 1, 0, 1, 1 \rightarrow 4, 4, 3, 1, 1, 1, 0$

d'' : $3, 2, 0, 0, 1, 0 \rightarrow 3, 2, 1, 0, 0, 0$

d''' : $1, 0, -1, 0, 0$ ✗ not possible.

Algorithm (checking graphic)

Input : $d : d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$, $\sum d_i$ = even.

output : True if d is graphic, otherwise false.

1. $k = n$.
2. If $d_n < 0$ then return false
3. if $d_1 = 0$ then return true.
4. $k = k - 1$
5. $(d_1, d_2, \ldots, d_n)$ be a non increasing ordering of $(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_k)$

Go to 2.

Theorem (Erdos and Gallai, 1960)

A non increasing seq $d: d_1 \geq d_2 \geq \cdots \geq d_n$ of non negative integers is graphic iff $\sum d_i$ even and for each k, $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$.

Directed graphs or Digraphs

A digraph D is a pair $(V(D), E(D))$, where $V(D)$ is a non empty (finite) set and $E(D)$ is a multiset of ordered pairs of elements (not necessarily different) of $V(D)$.

Elements of $V(D)$ are called vertices.

Elements of $E(D)$ are called edges or arcs.

If $l = (u, v) \in E(D)$ then v is called head of l and u is called tail of l.

Out degree of $v \in V(G)$, denoted by $d^+(v)$ is

$$= |\{u \in V(G) : (v, u) \in E(G)\}|$$

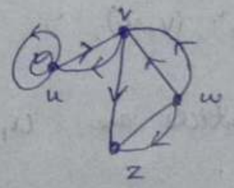
In degree of v is denoted by $d^-(v)$, given by

$$d^-(v) = |\{u \in V(G) : (u, v) \in E(G)\}|.$$

Theorem For every diagraph D, $\sum_{v \in V(D)} d^+(v) = \sum_{v \in V(D)} d^-(v) = |E(G)|$

Strongly connected

weakly connected



Let D be a diagraph. The graph obtained from D by removing all the directions from all the edges of D is called the underlying graph of D.

D is called weakly connected if the underlying graph of D is connected.

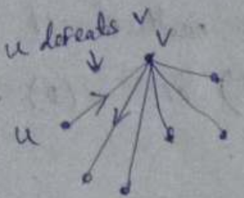
D is called strongly connected if for every pair of vertices u, v $\exists$ a $u-v$ directed path in D.

D_1, D_2 diagraphs, isomorphic iff $f : V(D_1) \to V(D_2)$ s.t $(u, v) \in E(D_1)$ iff $(f(u), f(v)) \in E(D_2)$.

A digraph D obtained from a graph G by giving direction to edges of G is called an orientation of G.

K_4  $\xrightarrow[\text{upto isomorphism}]{\text{no of orientations}}$?

An orientation of K_n is called a tournament.



Score of $v = d^+(v)$

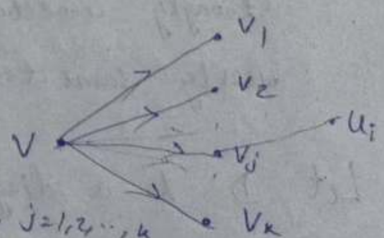
A vertex with max outdegree in a tournament is called a king.

Thm Let D be a tournament and v be a king in it. Then for every vertex $u \in V(D) - v$, $\exists$ a directed $v-u$ path of length at most two.

Pf Let $d^+(v) = k$ which is the max outdegree in D.

Let $(v, v_1), (v, v_2), \dots, (v, v_k) \in E(D)$.

Remaining vertices are $u_1, u_2, \dots, u_{n-k-1}$

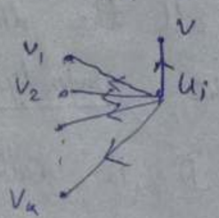


Claim $(v_j, u_i) \in E(D)$ for some j, $j = 1, 2, \dots, k$

If not,

then $d^+(u_i) \geq k+1$

which contradicts that k is the max outdegree.



$d^+(u_i) \geq k+1$