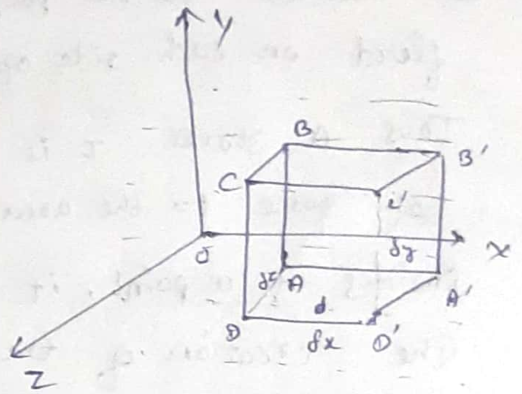
Components of stress vector (fluid present in 3-D space)

Consider a rectangular parallelopiped $\delta x \delta y \delta z$ at a point P within the flow having P at its center. Let the mean stresses on the surfaces to be the stresses at the middle point of the related surfaces.

The co-ordinates of the middle points of the surfaces $\perp^{lar}$ to x, y, z axes are $\left(x \pm \frac{\delta x}{2}, y, z\right)$ $\left(x, y \pm \frac{\delta y}{2}, z\right)$ and $\left(x, y, z \pm \frac{\delta z}{2}\right)$



$R := \delta x \delta y \delta z$

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Stress at a point and Cauchy postulate

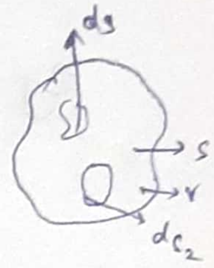
Body force: The lateral body force acting on a material body $\mathcal{B}$ occupying a configuration B of a volume V at a time t, is expressed as

$$\vec{f}^{(b)} = \vec{F}^{(b)} = \int \rho\, \vec{b} \cdot d^3 x \, ,$$ where

$\rho = \rho(x,t)$ is the density at a point x in $\mathcal{B}$ at time t and $\vec{b}$ is a vector with physical direction of force per unit mass.

2) Surface force (contact force / attraction force)

We postulate that the total surface force acting on a oriented surface S in the configuration B of a material body B is given by



$$\vec{f}^{(s)} = \vec{F}^{(s)} = \int_S \vec{S} \cdot ds$$

where $\vec{S}$ is the force per unit area

case

1) $ds_1 \neq ds_2$ $\qquad S$ is a function of $\vec{x}$

$S = S(t, x)$

2) Let us take $ds_1 = ds_2$

$$f_1^{ds_1} = \int_S \vec{S}\, ds_1$$

$$f_2^{ds_2} = \int_S \vec{S}\, ds_2$$

The vector S is called stress vector or traction vector which depends on t, $\vec{x}$ and also on the orientation of the surface element upon which acts i.e.

$$\vec{S} = \vec{S}(t, x, \hat{n})$$

Cauchy stress postulate

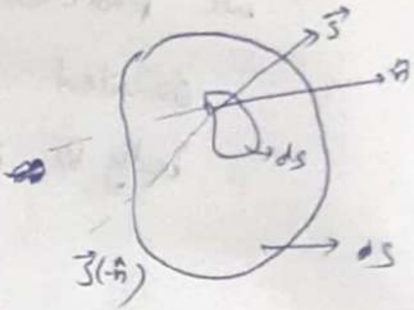
now when ds is taken on the boundary surface of a material, it is conventional to choose the unit normal $\hat{n}$ to ds as the external normal to the surface. The vector $\vec{S} = \vec{S}(t, x, n)$ is interpreted as the external surface force per unit area.

when ds is considered inside the material $\hat{n}$ has two possible direction $\hat{n}$ is directed upwards then $\vec{S} = \vec{S}(t, x, \hat{n})$

$= S(\hat{n})$

$\hat{n}$ is taken downwards

$$\vec{s} = \vec{s}(t, x, -\hat{n})$$
$$= \vec{s}(-\hat{n}) \quad — \quad (1)$$



By newton's 3rd law

$$\vec{s}(-\hat{n}) = -\vec{s}(\hat{n})$$

Cauchy's reciprocal law

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Fluid Mechanics

Stress at a point:

$\vec{s}(\hat{n}) = -s(-\hat{n})$

The total resultant force acting on a material body B in the configuration B of volume V with boundary S is defined as the vector sum of the total body force and total contact force/surface force.

$$\bar{f}^{(r)} = \bar{F}^{(r)} = \int_V \rho \bar{b} \cdot dv + \int_S \bar{s} \cdot ds$$

rho

Let us take $\hat{n} = \hat{e}_1$

$\bar{s}^{(\hat{e}_1)} = \sigma_{11} \hat{e}_1 + \sigma_{12} \hat{e}_2 + \sigma_{13} \hat{e}_3 \rightarrow$ resulting along co-ordinate axes

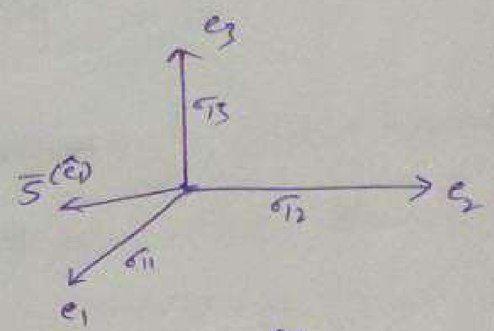
$= \sum_k e_{1k} \hat{e}_k = \sigma_{1k} \hat{e}_k$

$\bar{s}^{(\hat{e}_2)} = \sigma_{2k} \hat{e}_k$ $\Rightarrow$ $\bar{s}^{(\hat{e}_i)} = \sigma_{ik} \hat{e}_k \quad \forall\, i = 1,2,3$

$\bar{s}^{(\hat{e}_3)} = \sigma_{3k} \hat{e}_k$ $\Rightarrow$ $\bar{s}^{(\hat{e}_i)} \cdot \hat{e}_k = \sigma_{ik} \quad \forall\, i, k\ 1,2,3$

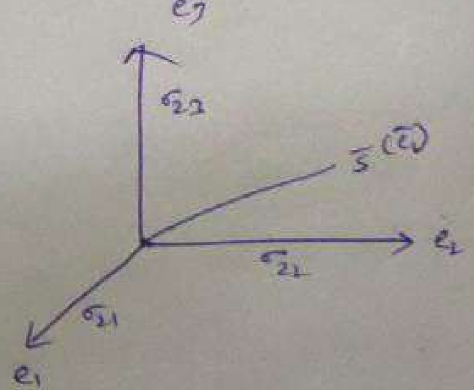
where $\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$

stress matrix



σ_{ij} - stress components at point x

and



For any arbitrary normal $\bar{n}$ we obtain

$$s^{(n)} = \sigma_{kj} n_k \hat{e}_j \Rightarrow s^{(n)} \cdot \hat{e}_i = \sigma_{kj} n_k \hat{e}_j \cdot \hat{e}_i$$

$$s_i^{(n)} = \sigma_{ki} n_k \} \text{ of a tensor of order 2}$$

$\sigma_{ki} \to$ a tensor of order cauchy stress tensor

$$\vec{s}^{(n)} = \sigma \cdot \vec{n} \quad (3')$$

Ex1: the stress matrix at a point $P(x_1, x_2, x_3)$ is given by.

$$\sigma = \begin{bmatrix} x_3 x_1 & x_3^2 & 0 \\ x_3^2 & 0 & -x_2 \\ 0 & -x_2 & 0 \end{bmatrix}$$

Find the stress vector at the point $Q(1, 0, -1)$ on the surface $x_2^2 + x_3^2 = x_1$

Sol: Let $f(x, y, z) = x_2^2 + x_3^2 - x_1 \Rightarrow \nabla f \ (-1, 2x_2, 2x_3)$

$$\nabla f \big|_{(1,0,-1)} = (-1, 0, -2) \Rightarrow \hat{n} = \frac{(-1, 0, -2)}{\sqrt{5}}$$

$$\bar{s}^{(\hat{n})} = \left. \begin{bmatrix} x_3 x_1 & x_3^2 & 0 \\ x_3^2 & 0 & -x_2 \\ 0 & -x_2 & 0 \end{bmatrix}_{(1,0,-1)} \right| \cdot \frac{(-1, 0, -2)}{\sqrt{5}} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{(-1, 0, -2)}{\sqrt{5}}$$

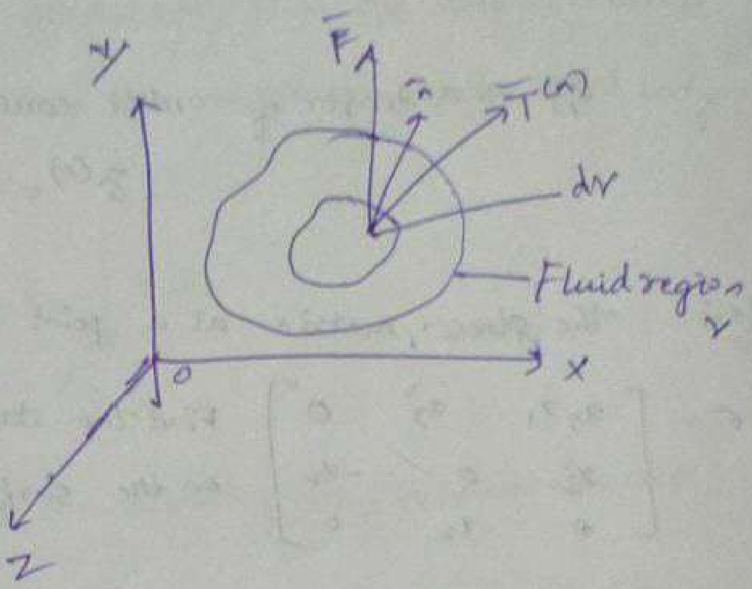
$$= \frac{(1, -1, 0)}{\sqrt{5}}$$

Cauchy's First law of motion for continuum:

The principle states that if a continuous body is in equilibrium, then the components of stress tensor satisfie

$$\sigma_{ij} + F_i = 0$$

$$\sigma_{ij} = -F_i$$



Proof:

By the statement if the continuum is in equilibrium then the resultant force acting on V must be zero.

$$\int_S T^{(\hat{n})} \, ds + \int_V F \, dV = 0.$$

$$\Rightarrow \int_S \bar{T}_i^{(\hat{n})} \, ds + \int_V \bar{F}_i \, dV = 0$$

$$\Rightarrow \int_S \sigma_{ji} n_j \, ds + \int_V F_i \, dV = 0.$$

$\rightarrow$ By Gauss Divergence theorem $\int_V (\sigma_{ji,j}) \, dV + \int_V F_i \, dV = 0.$

$$\int_V (\sigma_{ji,j} + F_i) \, dV = 0.$$

$\sigma_{ji,j}$ > putting a comma implies doing a Partial derivative.

V is arbitrary, $\sigma_{ji,j} + F_i = 0 \quad \forall \ i,j = 1,2,3$

Ex: For the fluids $\sigma_{ij} = -p\,\delta_{ij}$ then $F_i = \sigma_{ij,j} = -\nabla p$

?

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Principle stresses :-

Given the state of stress at any point, there exists three mutually perpendicular planes where shear stress vanishes and consequently the resultant stress components are normal. These planes are called principal planes & these stresses are called principal stresses. The directions of their normals are principal normals.

We denote principal stresses by $\sigma_1, \sigma_2, \sigma_3$. or $\sigma_x, \sigma_y, \sigma_z$

To determine the principal stresses

$$\det \begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{bmatrix} = 0$$

This gives 3 roots which are $\sigma = \sigma_1, \sigma_2, \sigma_3$ are principal stresses at a point.

Exam
eg

$$\sigma_{ij} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

find eigenvalues

Then determine the principal stresses acting at the point P corresponding to σ.

Sol

$$\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 0-\lambda & 2 \\ 1 & 2 & 0-\lambda \end{bmatrix}$$

$\lambda = \sigma$

$\Rightarrow (3-\lambda)(\lambda^2-4) - 1(-\lambda-2) + 1(2+\lambda)$

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$\Rightarrow 3\lambda^2 - 3\lambda^2 - 6\lambda + 8 = 0$

$\Rightarrow (\lambda-1)(\lambda-4)(\lambda+2) = 0$

$\Rightarrow \lambda = 1, 4, -2$

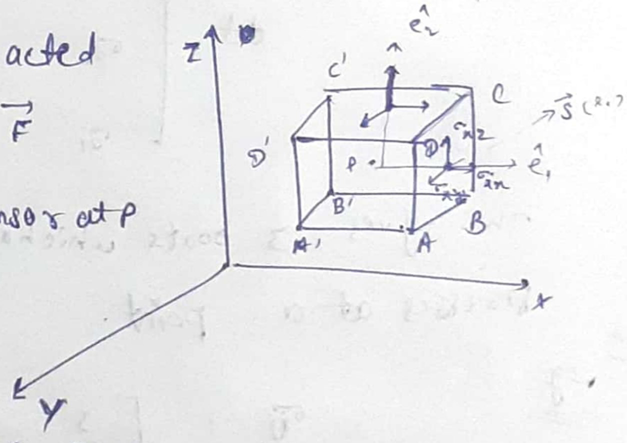
so, $\sigma_p = (1, -2, 4)$ components of principal stress

Symmetric property of stress tensor

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

we want to prove $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$, $\sigma_{zx} = \sigma_{xz}$

Suppose the fluid element is acted upon by an external force $\vec{F}$ and let $\vec{S}$ be the stress tensor at P



Since the fluid is in equilibrium, the moment about origin must be zero.

i.e. $\int_S \vec{r} \times \vec{T} ds + \int_V \vec{r} \times \vec{F} dv = 0$

~~$\Rightarrow \int_S \epsilon_{ijk} x_j S_k^{(n)} ds + \int_V \epsilon_{ijk} x_j F_k dv = 0$~~

~~$\Rightarrow \int_S \epsilon_{ijk} x_j \sigma_{mk} n_m ds + \int_V \epsilon_{ijk} x_j F_k dv = 0$~~

~~By gauss inversion theorem~~

$$\int_V \epsilon_{ijk} (x_j \sigma_{mk})_{,m} \, dv + \int_V \epsilon_{ijk} x_j F_k \, dv = 0$$

$$\Rightarrow \int_V \epsilon_{ijk} (x_{j,m} \sigma_{mk} + x_j \sigma_{mk,m}) \, dv + \int_V \epsilon_{ijk} x_j F_k \, dv = 0$$

$$\Rightarrow \int_V \epsilon_{ijk} x_{j,m} \sigma_{mk} \, dv + \int_V \epsilon_{ijk} x_j \underbrace{(\sigma_{mk,m} + F_k)}_{=0} \, dv = 0$$

$$\Rightarrow \int_V \epsilon_{ijk} x_{j,m} \sigma_{mk} \, dv = 0 \qquad \frac{\partial x_j}{\partial x_m} = \delta^j_m$$

$$\Rightarrow \int_V \epsilon_{ijk} \sigma_{jk} \, dv = 0$$

$$\Rightarrow \epsilon_{ijk} \sigma_{jk} = 0 \quad \text{at } i, j, k$$

$i = 1$, $\epsilon_{1jk} \sigma_{jk} = 0 \Rightarrow \sigma_{23} = \sigma_{32}$

$i = 2 \qquad \Rightarrow \sigma_{13} = \sigma_{31}$

$i = 3 \qquad \Rightarrow \sigma_{12} = \sigma_{21}$

Strain

The strain at a point P in a fluid region can be defined as a non-dimensional deformation which measures the change of relative positions of the parts of the fluid element under the cause.

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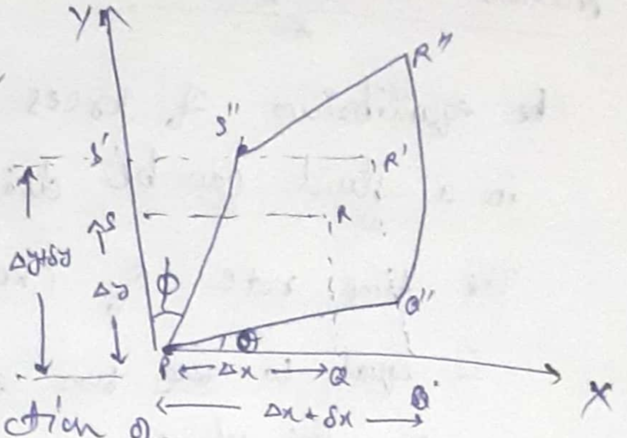
Stress Analysis : when the various elements of a system undergo relative displacement under the action of applied (external) force, it is said to be strained. Stress is a non-dimensional deformation which measures the change of relative positions of the parts of a body under any cause. In general, under the influence of an external force, the displacement consists of translation, rotation and distortion. The rigid body translation and rotation of the system as a whole do not produce any relative displacement and consequently they do not produce any strain.

I) Normal strain : It is defined as the ratio of the change in length to the original length of a straight line element. It is also known as extension or compression. Let the rectangle PQRS is deformed into PQ'R'S'.

Such that $PQ = \Delta x$, $PS = \Delta y$, $PQ' = \Delta x + \delta x$, $PS' = \Delta y + \delta y$

The normal strain in the direction of X-axis

$$\xi_{xx} = \frac{\delta x}{\Delta x} \quad —(1)$$



The normal strain in the direction of Y-axis

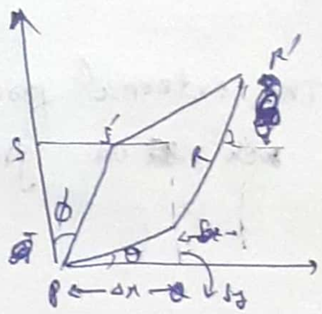
$$\xi_{yy} = \frac{\delta y}{\Delta y} \quad —(2)$$

Shearing stress : It is defined in terms of the change in angle between the linear elements from the unstrained state to strained state.

The rectangle $PQ'R'S'$ is deformed into $PQ''R''S''$ such that $QQ'' = \delta y$ and $S''S' = \delta x$.

The shearing stress is defined as

$$\xi_{xy} = \theta + \phi = \frac{\delta y}{\Delta x} + \frac{\delta x}{\Delta y}$$



Now for fluid element

The normal strain, $\xi_{xx} = \frac{\partial u}{\partial x}$, $\xi_{yy} = \frac{\partial v}{\partial y}$

The shearing strain $\xi_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$ or $\frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$

$\theta = \tan\theta$, $\theta << 1$

$= \frac{\delta y}{\Delta x}$

Since, in fluid there is no resistance to the deformation of the shape but there is a resistance to the time rate of change of change of shape.

Navier - Stokes equation

The equilibrium of forces acting on a volume element in a fluid can be described by momentum theorem.

The time rate of change of momentum of a closed volume is equal to the sum of the external forces $\vec{F}$ acting on the volume.

then,

$$\frac{D}{Dt}\int_{v} \rho \vec{q}\, dv = \vec{F} \quad \text{——} \quad (1)$$

ρ = density
q = velocity
$\rho dv = dm$
$\int_{v(t)} \vec{q}\cdot dm$ = total momentum

$$\Rightarrow \int_{v} \frac{d}{dt}(\rho\vec{q})\, dv = \vec{F}$$

$$\Rightarrow \int_{v}\left[\frac{\partial}{\partial t} + \vec{q}\,\nabla\right](\rho\vec{q})\, dv = \vec{F}$$

$$\left[\because \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{q}\,\nabla\right.$$

$$\Rightarrow \int_{v} \rho\left[\frac{\partial\vec{q}}{\partial t} + \vec{q}(\nabla\vec{q})\right] dv = \vec{F} \quad \text{——} \quad (2)$$

ρ is taken constant, as mass & volume are constant

The external forces are the volume forces / body forces such as gravity

$$\vec{F}_g = \int_{v} \rho\vec{g}\, dv$$

and the surface force

s -> force/area

sigma->stress vector

$$\vec{F}_{surf} = -\int_{A} \vec{s}\, dA = -\int_{A} \vec{\sigma}\cdot\hat{n}\, dA$$

(surface force $\vec{s}$ $-\hat{n}$)

By gauss divergence theorem

$$= -\int_{v} \vec{\nabla}\vec{\sigma}\, dv$$

From (2) we can have

$$\int_V \rho\left[\frac{\partial \vec{q}}{\partial t} + \vec{q}\,(\vec{\nabla}\vec{q})\right] dv = \int_V \rho\vec{g}\, dv - \int_V \vec{\nabla}.\vec{\sigma}\, dv$$

$$\Rightarrow \int_V \left[\rho\left(\frac{\partial q}{\partial t} + \vec{q}\,(\vec{\nabla}\vec{q})\right) - \rho\vec{g} + \vec{\nabla}.\vec{\sigma}\right] dv = 0$$

so the entire integrand must be zero

$$\boxed{\rho\left(\underbrace{\frac{\partial \vec{q}}{\partial t} + \vec{q}\,(\vec{\nabla}.\vec{q})}_{Dq/Dt}\right) = \rho\vec{g} - \vec{\nabla}.\vec{\sigma}} \quad \text{--- (3)}$$

3×1 vector, 3×3

vector form of Navier-Stokes equation

we know
$$\sigma = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

Putting the value of σ in (iii) and equating componentwise

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} - \frac{\partial \tau_{xz}}{\partial z} \quad \text{--- (iva)}$$

Similarly for other two components i.e. along y & z axis :-

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial \tau_{yx}}{\partial x} - \frac{\partial \tau_{yy}}{\partial y} - \frac{\partial \tau_{yz}}{\partial z} \quad \text{--- (ivb)}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial \tau_{zx}}{\partial x} - \frac{\partial \tau_{zy}}{\partial y} - \frac{\partial \tau_{zz}}{\partial z} \quad \text{--- (ivc)}$$

In Stokes' law stresses are assumed proportional to strain and in fluid mechanics, the stresses are assumed to be proportional to the time rate of change of ~~time~~ strain. Valid for both newtonian & non-newtonian fluids.

$\lambda = 1, 4, -2$ $\quad a - 1 = -3$

$a = -2$

For the derivation of the relation for dependence of stresses on the time rate of change of strain, it is assumed that $\tau_{xx}, \tau_{yy}, \tau_{zz}$ causes elongations and shearing stresses causes angular displacement. Then,

rate of change of strain $\quad \xi_{xx} = \frac{\partial u}{\partial x}, \quad \xi_{yy} = \frac{\partial v}{\partial y}, \quad \xi_{zz} = \frac{\partial w}{\partial z}$

The sum of the components of the time rate of change of strain yields the relative change in volume per time interval dt.

$$\xi = \xi_{xx} + \xi_{yy} + \xi_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \vec{\nabla} \cdot \vec{q} \quad \text{--- (V)}$$

The angular displacement per unit time (shearing strain)

$$\xi_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \xi_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\xi_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad \text{--- (VI)}$$

In the state of rest (or ~~inviscous~~ inviscid fluid) the shearing ~~strain~~ stress vanishes, the normal stresses $\tau_{xx}, \tau_{yy}, \tau_{zz}$ are solely given by hydrostatic pressure P. This behaviour of the flow can be expressed by the following hypothesis, introduced by Stoke's.

$$\left.\begin{aligned} \tau_{xx} &= P - 2\mu\,\xi_{xx} - \bar{\lambda}\xi \\ \tau_{yy} &= P - 2\mu\,\xi_{yy} - \bar{\lambda}\xi \\ \tau_{zz} &= P - 2\mu\,\xi_{zz} - \bar{\lambda}\xi \end{aligned}\right\} \text{(VII)} \quad \text{and} \quad \begin{aligned} \tau_{xy} &= -\mu\,\xi_{xy} \\ \tau_{yz} &= -\mu\,\xi_{yz} \\ \tau_{xz} &= -\mu\,\xi_{xz} \end{aligned}$$

$$\sigma = P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \bar{\lambda}\begin{bmatrix} \vec{\nabla}\cdot q & 0 & 0 \\ 0 & \vec{\nabla}\cdot q & 0 \\ 0 & 0 & \vec{\nabla}\cdot q \end{bmatrix} - 2\mu\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial x} \end{bmatrix}$$

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Fluid Mechanics:

$$\sigma = p\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix} - \lambda\begin{bmatrix}\nabla\cdot\bar{q} & & \\ & \nabla\cdot\bar{q} & \\ & & \nabla\cdot\bar{q}\end{bmatrix} - 2\mu\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ & & \\ & & \end{bmatrix}$$

The coefficient λ is generally split into two parts on $\lambda = \beta - \frac{2}{3}\mu$

Then the normal stresses,

$$\tau_{xx} = p - 2\mu\frac{\partial u}{\partial x} - \left(\beta - \frac{2\mu}{3}\right)\nabla\cdot\bar{q}$$

$$\tau_{yy} = p - 2\mu\frac{\partial v}{\partial y} - \left(\beta - \frac{2\mu}{3}\right)\nabla\cdot\bar{q}$$

$$\tau_{zz} = p - 2\mu\frac{\partial w}{\partial z} - \left(\beta - \frac{2\mu}{3}\right)\bar{\nabla}\cdot\bar{q}$$

Their mean value is $\sigma = \frac{1}{3}(\tau_{xx}+\tau_{yy}+\tau_{zz})$

$$= p - \beta\,\bar{\nabla}\cdot\bar{q} = p - \hat{\mu}\,\bar{\nabla}\cdot\bar{q}$$

where $\hat{\mu}$ is the volume viscosity which takes into account the molecular degrees of freedom. For incompressibility fluids, $\bar{\nabla}\cdot\bar{q} = 0$, $\sigma \to p$ hydrostatic pressure. If the stress-strain relations are put into (iva - ivc) then the NS-equation reduce to

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} + \lambda(\bar{\nabla}\cdot\bar{q})\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]$$

Du/Dt

$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right] + \rho g_x \quad \text{---} (**a)$$

$$\rho\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y} + \lambda(\bar{\nabla}\cdot\bar{q})\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]$$

$$+ \rho g_y \quad \text{---} (**b)$$

$$\rho\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z} + \lambda(\nabla\cdot\vec{q})\right] + \rho g_z \quad \text{—(**c)}$$

If ... case of incompression

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u + \rho g_x$$

$$\rho\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \mu\nabla^2 v + \rho g_y$$

$$\rho\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \mu\nabla^2 w + \rho g_z$$

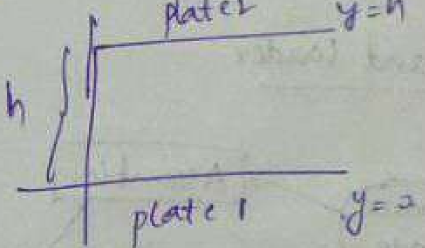
$$\rightarrow \boxed{\rho\frac{D\vec{q}}{Dt} = -\nabla\vec{p} + \mu\nabla^2\vec{q} + \rho F_q}$$

Laminar Flow:

(Couette flow)?

Steady laminar flow between two parallel plates. Consider a steady laminar flow of viscous incompressible fluid between two infinite parallel plates at a distance h apart let x be direction of the flow, y be the direction $\perp^r$ to the flow and the width of plates parallel to the z-direction.

$v = 0$, $w = 0$, $\frac{\partial u}{\partial t} = 0$ $\quad v_t = w_t = 0$.



Then the continuity eq^n

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0. \quad \Rightarrow u = f(y) + c$$

For our two dimensional flow, in the absence of body forces, then the Navier stokes eq^n reduces to

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} \quad \text{—(I)}$$

$$0 = -\frac{\partial p}{\partial y} \quad \text{—(II)} \quad \Rightarrow p \text{ is a function of } x$$

From (I) $\frac{dp}{dx} = \mu\frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{d}{dx}\left(\frac{dp}{dx}\right) = 0 \Rightarrow \boxed{\frac{dp}{dx} = \text{const} = c}$ —(III)

From ① & ③ ~~$\frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial y^2}$~~ $\mu \frac{d^2 u}{dy^2} = C$ $\Rightarrow$ $\boxed{u(y) = \frac{C}{2\mu} y^2 + Dy + e}$ — (iv)

(i) For the plane Couette flow we take $C = 0$

(ii) $u(0) = 0$ & $u(h) = U$, ie the plate $y=0$ is kept at rest & the plate $y = h$ is allowed to move with velocity U.

From iv $\quad u(0) = 0 \quad u(h) = U \quad C = 0.$

$\Downarrow \qquad \Downarrow$

$e = 0. \qquad Dh = U \Rightarrow D = U/h$

$$\Rightarrow \boxed{u(y) = Uy/h}$$

$$\text{Shearing stress} = \mu \frac{\partial u}{\partial y} = \frac{\mu U}{h}$$

Date 7/11/22

Plane Poiseuelle flow

consider a steady laminar flow of a viscous incompressible fluid between two infinite parallel plates separated by a distance h. Let x axis be taken in the middle of the channel parallel to the direction of flow. y be the direction $\perp^{lar}$ to the flow, width of the plates $\parallel$ to z-direction. Since the plates are extended upto infinity, the flow may be treated as 2-D. By the given condition, the y and z component of velocity i.e. $v=0$, $w=0$.

since

$$\vec{\nabla}.\vec{q}=0$$

$$\Rightarrow \partial_x u + \partial_y v + \partial_z w = 0$$

$$\Rightarrow \partial_x u = 0 \Rightarrow u = u(y) \quad , v=0, \; w=0$$

i.e. $\vec{q} = (u(y), 0, 0)$

In the body forces are absent, then NS eq^n reduces to

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \quad \text{—} ①$$

$$0 = -\frac{\partial p}{\partial y} \Rightarrow p = p(x) \quad \text{—} ②$$

From ①

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}$$

Diff w.r.t x

$$\frac{1}{\mu}\frac{d^2P}{dx^2} = 0$$

$$\Rightarrow \int d\left(\frac{dP}{dx}\right) = \text{const}$$

$$\frac{dP}{dx} = c \text{ (say)}$$

From (1) $\boxed{\frac{d^2u}{dy^2} = \frac{c}{\mu}} \Rightarrow u(y) = \frac{c}{2\mu}y^2 + Dy + E$ —— (3)

For plane poiseuille flow, the plates are kept at rest and the fluid is in motion by a pressure gradient P. Let the two planes at $y = h/2$ & $y = -h/2$. Then from no-slip condition $u(y) = 0$ at $y = h/2$

$u(y) = 0$ only $\forall y$

$$0 = \frac{ch^2}{8\mu} - \frac{Dh}{2} + E \qquad 0 = \frac{ch^2}{8\mu} + \frac{Dh}{2} + E$$

Comparing the last two relation, taking

$E = -\frac{ch^2}{8a}$, then $D = 0$

so

$$u(y) = \frac{c}{2\mu}y^2 - \frac{ch^2}{8\mu} = \frac{c}{2a}\left(y^2 - \frac{h^2}{4}\right)$$

$$\Rightarrow u(y) = -\frac{ch^2}{8\mu}\left(1 - 4\left(\frac{y}{h}\right)^2\right)$$

The max velocity is attained at $y=0$,

i.e. $$U_{max} = \frac{c}{2\mu}\left(y^2 - \frac{h^2}{4}\right)\Big|_{y=0}$$

$$= -\frac{ch^2}{8\mu}$$

The avg. velocity

$$V_{ave} = \frac{1}{h}\int_{-h/2}^{h/2} u(y)\, dy$$

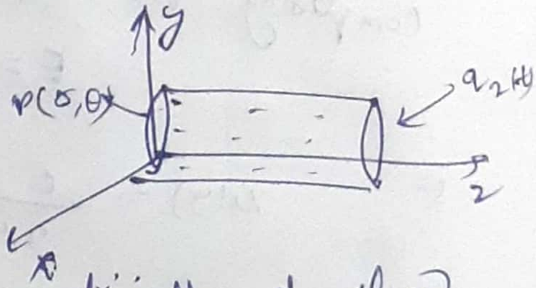
$$= \frac{1}{h}\int_{-h/2}^{h/2} \frac{c}{2\mu}\left(y^2 - \frac{h^2}{4}\right) dy$$

Shearing stress $$\tau_{yx} = \mu\frac{du}{dy} = \mu\frac{c}{2\mu}\cdot 2y = cy$$

Hagen - poiseuille flow (flow through a circular pipe)

consider a steady laminar flow without any body force for an incompressible fluid, through an infinite circular pipe of radius r and axis of symmetry on z-axis.

For the given flow, we consider the eq^n of motion in cylindrical coordinates i.e. (r, θ, z), where z is a direction of flow along the pipe



This gives $q_r = 0$, $q_\theta = 0$ & $q_z \neq 0$

Due to symmetry, q_z is independent of θ.
Then from eq of continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0$$

$$\Rightarrow \frac{\partial}{\partial z}(\rho q_z) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}(q_z) = 0 \Rightarrow q_z = q_z(r) \quad — (1)$$

The NS in cyclindrical coordinates will reduce to

$$0 = -\frac{\partial p}{\partial r}, \qquad 0 = -\frac{\partial p}{\partial t}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta}, \qquad + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r}\right)$$

From first two relation, p is indep of r and θ
i.e. $p = p(z)$, then from (3)

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r}\right) = \frac{dp}{dz} = c$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r}\right) = \frac{cr}{\mu}$$

$$\Rightarrow r\frac{\partial q_z}{\partial r} = \frac{cr^2}{2\mu} + \phi$$

$$\Rightarrow \frac{\partial q_z}{\partial r} = \frac{cr}{2\mu} + \frac{D}{r}$$

$$\Rightarrow q_z(r) = \frac{cr^2}{4\mu} + D\log r + E$$

To keep $q_z(0)$ finite, $B = 0$, then

$$q_z(r) = \frac{Cr^2}{4\mu} + E$$ constant

(again parabolic)

we know $r = a$, $q_z(r) = 0$) then

$$E = -\frac{Ca^2}{4\mu}$$

The sol $q_z(r) = \frac{C}{4\mu}(r^2 - a^2)$

Date 9/11/22

Problem sheet-2

Q1 $\vec{q} = -x\hat{i} + (y+t)\hat{j}$ → (a) streamlines, (b) stream function

Sol $u = -x, \quad v = y+t$

$$\frac{dx}{u} = \frac{dy}{v} \Rightarrow \frac{dx}{-x} = \frac{dy}{y+t}$$

$$\Rightarrow -\log x = \log(y+t) + c$$

$$\Rightarrow x(y+t) = e^c = c'$$

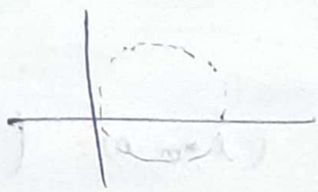
at $t = 2$

(b) $$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

$$= (y+t)dx + x\,dy = d(x(y+t))$$

(2) stream velocity $= U$, circulation $= K$, $q_{max} = ?$

$$w = \frac{iK}{2\pi}\log z$$



The required complex potential

$$w = w_1 + w_2$$

$$w_2 = Uz + \frac{Ua^2}{z}$$

$$\phi(r,\theta) = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{K\theta}{2\pi}$$

The velocity $$\vec{q} = \left(-\frac{\partial\phi}{\partial r}, -\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

$$\Rightarrow |\vec{q}|^2 = \left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2 = U^2\left(1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right) + \frac{UK}{\pi r}\left(1 + \frac{a^2}{r^2}\right) + \frac{K^2}{4\pi^2 r^2}$$

At $r=a$, $|\vec{q}|^2 = \left(2U\sin\theta + \frac{K}{2\pi a}\right)^2$

for $\theta = \pi/2$ $\quad |q|^2 = q_{max}^2 = \left(2U + \frac{K}{2\pi a}\right)^2$

So $\boxed{q_{max} = 2U + \frac{K}{2\pi a}}$

(3) $\sigma = \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ $\quad$ plane: $\frac{x}{4} + \frac{y}{2} + \frac{z}{6} = 1$

Sol $\vec{S} = \sigma.\hat{n}$

The given plane $\frac{x}{4} + \frac{y}{2} + \frac{z}{6} = 1 \Rightarrow 3x + 6y + 2z = 12$ — ①

The direction ratio of the normal $\perp^{lar}$ to the plane ① is 3, 6 and 2.

Direction cosines of the normal $= \underset{l}{\frac{3}{\sqrt{3^2+6^2+2^2}}}, \underset{m}{\frac{6}{\sqrt{3^2+6^2+2^2}}}$ and $\underset{n}{\frac{2}{\sqrt{3^2+6^2+2^2}}}$

So $(l, m, n) = \left(\frac{3}{7}, \frac{6}{7}, \frac{2}{7}\right)$

required, unit normal is $= \left(\frac{3}{7}, \frac{6}{7}, \frac{2}{7}\right)$

So, $\vec{S} = \sigma.\hat{n} = \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3/7 \\ 6/7 \\ 2/7 \end{bmatrix}$

(4) Determine Principal stresses

(e) $\sigma = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\sigma_1 = ?$ $\sigma_2 = ?$ $\sigma_3 = ?$

Sol let λ be the principal stress

$\det(\sigma - \lambda I) = 0$

$\Rightarrow \det\begin{bmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix} = 0$

$\Rightarrow$ solve the cubic eqn thus formed

$\lambda = -1, 2, -1$
$\quad\ \ \sigma_2 \ \ \sigma_1 \ \ \sigma_3$

(5) (a) $u(x,y) = \dfrac{ax-by}{x^2+y^2}$, $v(x,y) = \dfrac{ay+bx}{x^2+y^2}$, $w = 0$

- nature of motion $\rightarrow$ rotation or irrotation, compressible or not. i.e. divergence of q is 0 or not then curl zero or not

$\vec{\nabla}\cdot\vec{q} = \dfrac{\partial}{\partial x}\left(\dfrac{ax-by}{x^2+y^2}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{ay+bx}{x^2+y^2}\right) + \dfrac{\partial}{\partial z}(w) = 0$

it comes out to be zero so, it is incompressible flow.

then $\text{curl}(\vec{q}) = \vec{\nabla}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \vec{0}$

upon calculation this also comes out to be $\vec{0}$, hence flow is irrotational.

(b) $\exists\ \phi$ such that $\vec{q} = -\nabla\phi \Rightarrow \dfrac{\partial\phi}{\partial x} = -\dfrac{(ax-by)}{x^2+y^2}$

$$\frac{\partial\phi}{\partial y} = -\left(\frac{ay+bx}{x^2+y^2}\right) , \quad \frac{\partial\phi}{\partial z} = 0$$

simple partial integration no need of any special method to integrate

$\phi(x,y) =$

(c) pressure at any point (x,y) find using Euler equation

$$\rho \frac{dq}{dt} + \nabla P = \vec{F} \qquad \text{external force} = 0$$

$$\Rightarrow \frac{\partial P}{\partial x} = \_\_\_\_ , \quad \frac{\partial P}{\partial y} = \_\_\_\_$$

(6)

$$w = -\frac{m}{2\pi}\log(z-f) + \frac{m}{2\pi}\log(z) - \frac{m}{2\pi}\log\left(\frac{a^2}{z}-f\right)$$

find in terms of Real & imaginary part $\quad f(z) + \bar{f}\left(\frac{a^2}{z}\right)$

$\phi + i\psi = \dots$ Milne ~~...~~ Thomson theorem

find ϕ then find q

$$\vec{q} = \left(-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}\right)$$

then relate this $\vec{q}$ with euler equation to find resultant pressure.

(7) notes

(8) notes

(9) notes

(10) notes

(11)

(12) notes deriving eulers equation

ii) The complex potential of an undisturbed / uniform flow is

$$w_1 = (\alpha - i\beta) z$$

Then the complex potential due to cylinder

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$$

$$= (\alpha - i\beta) z + (\alpha + i\beta)\left(\frac{a^2}{z}\right)$$

using Blascius theorem

$$X - iY = -\frac{i\rho}{2} \int_C \left(\frac{dw}{dz}\right)^2 dz$$

$$\frac{dw}{dz} = (\alpha - i\beta) - (\alpha + i\beta)\frac{a^2}{z^2}$$

So,

$$X - iY = -\frac{i\rho}{2} \int_C \underbrace{\left[(\alpha - i\beta) - (\alpha + i\beta)\frac{a^2}{z^2}\right]^2}_{\text{analytic function}} dz$$

calculate residue to find the integral. hence it comes out to be zero.

$$X - iY = 0$$

$$\text{so } X = 0, \quad Y = 0$$

$$X - iY = \frac{i\rho}{2} \int_C \left[(\alpha - i\beta)^2 - 2(\alpha - i\beta)(\alpha + i\beta)\frac{a^2}{z^2} + \frac{a^4}{z^4}\right] dz$$

First term = 0

pole $z = 0$

Residue at $z = 0$ $= \frac{1}{(k-1)!} \lim_{z \to 0} \frac{d^{k-1}}{dz^{k-1}} \left(z^k f(z)\right)$

Cauchy-Residual theorem

Problem Sheet-3

① (a) $\vec{q} = \left(\frac{x + l r}{r(r+x)}, \frac{y + m r}{r(r+y)}, \frac{z + n r}{r(r+z)} \right)$

$l, m, n = ?$

sol $\vec{\nabla} \cdot \vec{q} = 0$ equation of continuity of flow

$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ —— ①

$u = \frac{x + l(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}\,(x + \sqrt{x^2+y^2+z^2})}$ $\left| \frac{du}{dx} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} \right.$

$v = \ldots$

$w = \text{—}$

see page 78
dell r/ dell x = x/r

Then find l, m & n using ①

(b) $\phi(x, y, z) = \frac{a}{2}(x^2 + y^2 - 2z^2)$

sol $\nabla^2 \phi = 0$ laplace equation

$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$

The eqn of stream lines

$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

$\Rightarrow \frac{dx}{\left(-\frac{\partial \phi}{\partial x}\right)} = \frac{dy}{\left(-\frac{\partial \phi}{\partial y}\right)} = \frac{dz}{\left(-\frac{\partial \phi}{\partial z}\right)}$

take two fractions at a time

$$\frac{dx}{-ax} = \frac{dy}{-ay} = \frac{dz}{2az}$$

taking 1st & 2nd fraction

$$\frac{dx}{x} = \frac{dy}{y},$$

$$\Rightarrow \log\frac{x}{y} = \log c \Rightarrow \boxed{x = c_1 y}$$

2nd & 3rd

$$\boxed{y^2 z = c_2}$$ c_1 & c_2 are arbitrary constants.

(c) $u = -\alpha y, \quad v = \alpha x, \quad w = 0$

streamlines $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

surface orthogonal to this is given by

$$u\,dx + v\,dy + w\,dz = 0$$

$$\Rightarrow -\alpha y\,dx + \alpha x\,dy + 0\,dz = 0$$

$$\Rightarrow -\alpha(y\,dx - x\,dy) = 0$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

$$\Rightarrow \frac{y}{x} = \text{constant}$$

To see if velocity potential exist, check if curl is zero or not

$\vec{\nabla} \times \vec{q} \neq \vec{0}$ that means velocity potential does not exists.

Q2 (a) similar to a problem in prev. assignment refer page 143

2 (b) $\underset{\text{vorticity}}{w} = \frac{1}{2}(\vec{\nabla} \times \vec{q}) = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ A(x,y) & -A(x,y) & 0 \end{vmatrix}$

2 (c) speed is same

$\vec{q}$ = constant

$\Rightarrow$ u = constant, v = constant, w = constant

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\Rightarrow \frac{dx}{c_1} = \frac{dy}{c_2} = \frac{dz}{c_3}$$

$\Rightarrow$ $x = c_4 y$ & $y = c_5 z$

are straight lines.

3 (a) same as 2 (a) $\rightarrow$ here body force is not zero in euler equation.

(b) use some vector identity to prove refer page 142

(c) $\rho = \rho(p)$ and $F = -\nabla V$ (conservative)

Then from Eulers equations

$$\rho \frac{d\vec{q}}{dt} + \nabla p = \rho F$$

refer page 931

$$\Rightarrow \frac{d\vec{q}}{dt} + \frac{1}{\rho} \nabla p = -\nabla V$$

$$\Rightarrow \vec{q}\,(\nabla q) = -\nabla V - \frac{1}{\rho}\nabla p$$

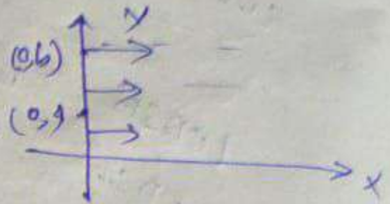
10/11/22

Examples

1. If fluid fills the region of xy plane on the x axis, which is a rigid boundary and if there is a source m at (0,a) equal sink at (0,b) and if the pressure on the -ve side be same as the pressure at infinity. Then determine the resultant pressure at rigid boundary.

Sol: The complex potential $w = -m\log(z-ai) + m\log(z-bi)$

$$\frac{dw}{dz} = \frac{-m}{z-ai} + \frac{m}{z-bi}$$



$$\Rightarrow |q|^2 = \left|\frac{dw}{dz}\right|^2 = \left(\frac{-m}{z-ai} + \frac{m}{z-bi}\right)^2$$

at the rigid boundary $y=0$,

$$|q|^2 = \left|\left(\frac{-m}{x-ai} + \frac{m}{x-bi}\right)^2\right| = \frac{(-mx + mbi + mx - mai)^2}{(x-ai)^2(x-bi)^2}$$

$$= \left|\frac{-m^2(b-a)}{(x-ai)^2(x-bi)^2}\right| = \frac{m^2(b-a)}{(x^2+a^2)(x^2+b^2)}$$

Bernoulli's Eqn: $\frac{q^2}{2} + \frac{p}{\rho} = \frac{\partial\phi}{\partial t}$

$$\Rightarrow \frac{p}{\rho} = \frac{\partial\phi}{\partial t} - \frac{q^2}{2}$$

Initially p_0 be the pressure at infinite where $q=0$

$$p = \rho p_0 - \frac{q^2}{2}$$

Show that for an incompressible steady flow, with constant viscosity the velocity $u(y) = \frac{yU}{h} + \frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\frac{y}{h}\left(1-\frac{y}{h}\right)$, $v = 0, w = 0$ satisfy the eq^n of motion when $F = 0$, $\frac{dp}{dx} = \text{const}$, $p = p(x)$.

Soln: we are given

$$u(y) = y\frac{U}{h} + \frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\frac{y}{h}\left(1-\frac{y}{h}\right), \quad v=0,\ w=0.$$

The general NS eq^n for viscous fluid,

$$\underbrace{\frac{\partial \vec{q}}{\partial t}}_{0} + (\bar{q}\cdot\bar{\nabla})\bar{q} = \underbrace{F}_{0} - \frac{1}{\rho}\nabla p + \underbrace{\mu}_{\text{constant}}\nabla^2\bar{q}$$

Then $(\bar{q}\cdot\bar{\nabla})\bar{q} = -\frac{1}{\rho}\nabla p + \mu\nabla^2\bar{q}$ — (1)

Given that $p = p(x)$ $\quad \nabla p = \hat{i}\frac{\partial p}{\partial x}$

$$\nabla^2\bar{q} = \hat{i}\nabla^2 u, \quad (\bar{q}\cdot\bar{\nabla})\bar{q} = u\frac{\partial u}{\partial x}\hat{i}$$

$$\Rightarrow \underbrace{u\frac{\partial u}{\partial x}}_{0} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\nabla^2 u$$

$$\Rightarrow -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial y^2}$$

$-\frac{y^2}{h}$

$-2y$

$\frac{h^2}{2\mu}\left(\frac{dp}{dx}\right) \times \frac{h/2}{h^2}$

$\left(\frac{\partial p}{\partial x}\right)\frac{1}{\mu}$