Lecture-4 (16-01-2024)

* $p \in \bar{A} = A \cup A' \iff$ either $p \in A$ or $p \in A'$.

Theorem: In a topological space (X, T), for $A, B \subset X$, we have

(i) $\bar{\phi} = \phi$, (ii) $A \subseteq \bar{A}$

(iii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$, (iv) $\overline{(\bar{A})} = \bar{A}$.

Proof: Since ϕ and $\bar{A}$ are closed sets, we have $\bar{\phi} = \phi$ and $\overline{(\bar{A})} = \bar{A}$.

Clearly by definition of a closure of a set, we have $A \subseteq \bar{A}$.

(iii) Since $A \subseteq \bar{A}$ and $B \subseteq \bar{B}$ we have

$$A \cup B \subseteq \bar{A} \cup \bar{B}$$

Also

$$A, B \subseteq A \cup B$$

$$\Rightarrow \bar{A}, \bar{B} \subseteq \overline{A \cup B}$$

$$\left[\because F_1 \subseteq F_2 \Rightarrow F_1' \subseteq F_2', \quad F_1 \cup F_1' \subseteq F_2 \cup F_2' \Rightarrow \bar{F_1} \subseteq \bar{F_2} \right]$$

$$\Rightarrow \bar{A} \cup \bar{B} \subseteq \overline{A \cup B} \quad \text{—— } (*)$$

Since $A \cup B \subset \bar{A} \cup \bar{B}$ and $\bar{A} \cup \bar{B}$ is a closed set containing $A \cup B$, then by definition of closure of a set, we have

$$A \cup B \subseteq \overline{A \cup B} \subseteq \bar{A} \cup \bar{B} \quad \text{—— } (**)$$

Combining $(*)$ and $(**)$, we get

$$\overline{A \cup B} = \bar{A} \cup \bar{B}.$$

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Dense Set:—

A subset A of a topological space (X, T) is said to be dense in a subset B of X if $B \subseteq \bar{A}$.

In Particular, A is dense in X or A is a dense subset of X, if

$$\bar{A} = X \qquad [\because X \subseteq \bar{A} \subseteq X$$

Ex: $X = \{a, b, c, d, e\}$

$T = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

Closed sets: ϕ, X, $\{b, c, d, e\}$, $\{a, b, e\}$, $\{b, e\}$, $\{a\}$.

Now

$\overline{\{a, c\}} = X \Rightarrow \{a, c\}$ is dense in X.

$\overline{\{b, d\}} = X \cap \{b, c, d, e\} = \{b, c, d, e\}$.

$\Rightarrow \{b, d\}$ is not dense in X.

Interior of a set:

Let A be a subset of a topological space (X, T). A point $p \in A$ is called interior point of a set A if there exists an open set G contained in A and containing p. That is there exists $G \in T \ni p \in G \subseteq A$.

The set of all interior points of a set A is denoted by A° or $\text{int}(A)$ and it is called interior of A.

Exterior of a set :—

The exterior of a set A in a topological space (X, T) is denoted by $\text{ext}(A)$ and is defined as $\text{int}(A^c)$, interior of complement of A.

That is $\text{ext}(A) = \text{int}(A^c)$.

Boundary points :—

The boundary of a set A is denoted by $b(A)$ and it is the set of points which do not belong to the interior or exterior of the set A.

Ex: $X = \{a, b, c, d, e\}$.

$T = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

Let $A = \{b, c, d\} \subset X$.

For $c, d \in A$, $\exists$ an open set $G = \{c, d\}$ such that

$$c, d \in G \subset A$$

$$\Rightarrow \quad c, d \in A^{\circ}.$$

The point $b \in A$ is not interior point of A, as there is no open set G such that

$$b \in G \subset A.$$

$$\therefore \quad \text{Int}(A) = \{c, d\}.$$

Now

$$A^{c} = \{a, e\}$$

$$\because \quad a \in \{a\} \subset A^{c} \Rightarrow a \in \text{int}(A^{c})$$

and $e \notin \text{int}(A^{c})$ as there is no open set G $\ni$ $e \in G \subset A^{c}$.

$$\therefore \quad \text{ext}(A) = \text{int}(A^{c}) = \{a\}.$$

$\therefore$ Boundary of a set A is

$$b(A) = \{b, e\}.$$

Ex: Let $(\mathbb{R}, U)$ be a usual topological space.

$$U = \{G \subseteq \mathbb{R} \mid G \text{ is open set in } \mathbb{R}\}.$$

Let $A = \mathbb{Q} \subset \mathbb{R}$.

Clearly
$$\text{Int}(\mathbb{Q}) = \varphi$$
$$\text{ext}(\mathbb{Q}) = \varphi$$
$$b(\mathbb{Q}) = \mathbb{R}.$$

Def: A subset A of a topological space (X, T) is said to be nowhere dense in X if $\text{int}(\overline{A}) = \varphi$.

Ex: Let $(\mathbb{R}, U)$ be usual topological space.

$$A = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\ldots\} \subset \mathbb{R}$$

$$A' = \{0\}.$$

$$\therefore \overline{A} = A \cup A' = \{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\}.$$

Clearly $Int(\bar{A}) = \phi$

$\Rightarrow$ A is nowhere dense in $\mathbb{R}$.

Now let $B = \{x \in \mathbb{Q} \mid 0 < x < 1\}$.

$$Int(B) = \phi$$

$$\bar{B} = [0, 1]$$

$$Int(\bar{B}) = Int([0,1]) = (0,1) \neq \phi$$

$\therefore$ B is not nowhere dense in $\mathbb{R}$.

- Theorem: - The interior of a set A in a topological space (X, T) is the union of all open subsets of A. Furthermore

(i) A^o is open set

(ii) A^o is the largest open subset of A.

(iii) A is an open set iff $A = A^o$.

Proof: Let $\{G_i\}$ be the class of open subsets of A.

If $x \in A^0 \Rightarrow \exists\, G_{i_0} \in \{G_i\}$ such that

$$x \in G_{i_0} \subset A$$

$$\Rightarrow x \in \bigcup_i G_i$$

$$\Rightarrow A^0 \subseteq \bigcup_i G_i \text{ —— (1)}$$

On the other hand, if $y \in \bigcup_i G_i$, then there exists some G_{j_0} such that

$$y \in G_{j_0}$$

$$\therefore y \in G_{j_0} \subset A. \qquad [\because G_{j_0} \subset A]$$

$$\Rightarrow y \in A^0.$$

$$\therefore \bigcup_i G_i \subseteq A^0 \text{ —— (2)}$$

From (1) and (2) we have

$$A^0 = \bigcup_i G_i.$$

Since arbitrary union of open sets is an open set, it follows that A° is an open set.

Now let G be any open subset of A. Then $G \in \{G_i\}$

$$\Rightarrow G \subseteq \bigcup_i G_i = A^\circ$$

Thus A° is the largest open subset of A.

Now suppose A is an open set, then

$$A \subseteq A^\circ \subseteq A$$

$$\Rightarrow A^\circ = A.$$

Conversely if $A = A^\circ$, and since A° is an open set implies A is an open set

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Theorem: let A be any subset of a topological space (X,T). Then

$$\overline{A} = A^{\circ} \cup b(A).$$

Proof: Since $A \subseteq X$, we can write

$$X = A^{\circ} \cup b(A) \cup ext(A).$$

$$\Rightarrow ext(A) = X - (A^{\circ} \cup b(A))$$

$$= [A^{\circ} \cup b(A)]^{c}.$$

Hence it is sufficient to prove that

$$(\overline{A})^{c} = ext(A).$$

Let $p \in ext(A) = int(A^{c})$

$\Rightarrow \exists$ an open set G such that

$$p \in G \subset A^{c}.$$

$$\Rightarrow G \cap A = \phi$$

Thus G is an open set containing p and does not contain any point of A.

$\Rightarrow$ p is not a limit point of A.

$\therefore$ $p \notin A'$.

Also since $p \in A^c \Rightarrow p \notin A$.

$\therefore$ $p \notin A \cup A' = \overline{A}$.

$\Rightarrow$ $p \in (\overline{A})^c$.

$\therefore$ $\text{ext}(A) \subseteq (\overline{A})^c$ ——— (1)

Now assume that

$p \in (\overline{A})^c = (A \cup A')^c$

$\Rightarrow$ $p \notin A \cup A'$

$\Rightarrow$ $p \notin A$ and $p \notin A'$

$\Rightarrow$ $\exists$ an open set G such that

$p \in G$ and $(G - \{p\}) \cap A = \varphi$

$\therefore p \notin A \Rightarrow G \cap A = \varphi$
and $p \in A^c$.

$\Rightarrow p \in G \subset A^c$

$\Rightarrow p \in \text{Int}(A^c) = \text{ext}(A)$

$\therefore (\bar{A})^c \subseteq \text{ext}(A)$ —— (2)

$\therefore$ From (1) and (2) we have

$(\bar{A})^c = \text{ext}(A)$.

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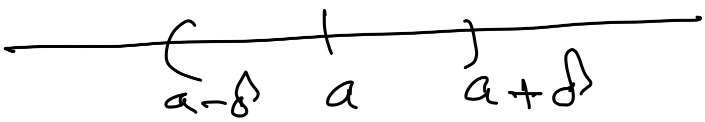
Neighborhood (nbd) :

Let (X, T) be a topological space and $p \in X$. A subset N of X is said to be neighborhood of point $p \in X$ if N is superset of an open set G containing p.

i.e., $p \in G \subset N$.

- The class of all neighborhoods of a point p is denoted by N_p.

Ex: Let $(\mathbb{R}, U)$ be usual topological space.
Let $a \in \mathbb{R}$.



$\because \ a \in (a-\delta, a+\delta) \subset [a-\delta, a+\delta]$

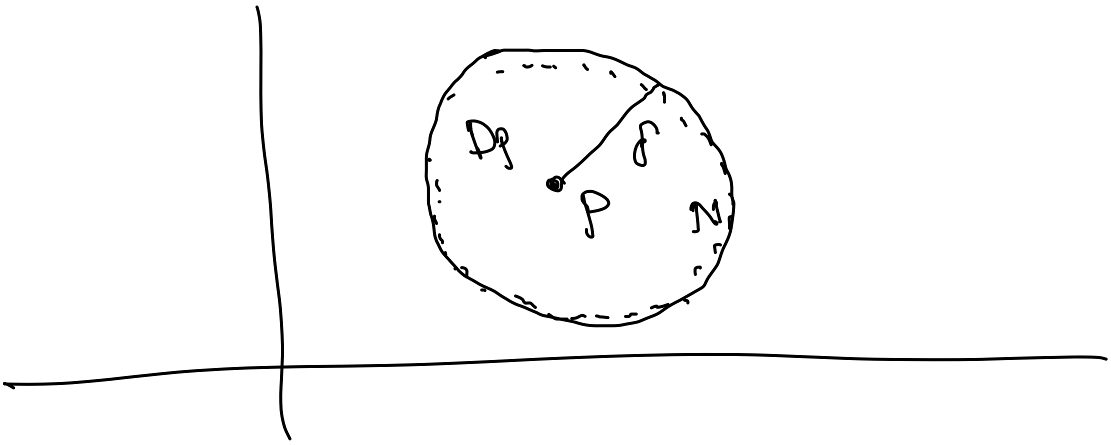
$\therefore \ N = [a-\delta, a+\delta], \delta > 0$ is a nbd of a point $a \in \mathbb{R}$.

Ex: $(\mathbb{R}^2, U)$

For any $p \in \mathbb{R}^2$, the closed disc

$$N = \{ q \in \mathbb{R}^2 / d(p, q) \leq \delta \neq 0 \}$$

with center p and radius δ is a nbd of $p \in \mathbb{R}^2$.



$p \in D_p \subset N.$

Ex: $X = \{a, b, c, d, e\}$

$T = \{X, \phi, \{a\}, \{a, b\}, \{a, c, d\},$
$\{a, b, c, d\}, \{a, b, e\}\}.$

Find neighborhoods of e and c.

Now for
$e \in X$:—

The open sets containing e are
X and $\{a, b, e\}$.

Super sets of $\{a, b, e\}$ are
$\{a, b, e\}$, $\{a, b, c, e\}$, $\{a, b, d, e\}$
and X
and super set of X is itself X.

$\therefore$ $N_e = \{ \{a, b, e\}, \{a, b, c, e\}, \{a, b, d, e\}, X \}$.

Illy

$N_c = \{ \{a, c, d\}, \{a, b, c, d\}, X, \{a, c, d, e\} \}$.

Ex: $(X, \mathcal{I})$ indiscrete topological space.

$p \in X$. Then $N_p = \{ X \}$.

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[Attendance:
64, 27, 06, 60, 23, 10, 41,
63, 61, 65]