The Transportation Model

Formulations

The Transportation Model

The transportation model is a special class of LPPs that deals with transporting(=shipping) a commodity from **sources** (e.g. factories) to **destinations** (e.g. warehouses). The objective is to determine the shipping schedule that minimizes the total shipping cost while satisfying supply and demand limits. We assume that the shipping cost is proportional to the number of units shipped on a given route.

We assume that there are m sources 1,2, …, m and n destinations 1, 2, …, n. The cost of shipping one unit from Source i to Destination j is c_{ij}.

We assume that the availability at source i is a_i (i=1, 2, …, m) and the demand at the destination j is b_j (j=1, 2, …, n). We make an important assumption: the problem is a **balanced** one. That is

$$\boxed{\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j}$$

That is, total availability equals total demand.

We can always meet this condition by introducing a dummy source (if the total demand is more than the total supply) or a dummy destination (if the total supply is more than the total demand).

Let x_{ij} be the amount of commodity to be shipped from the source i to the destination j.

Thus the problem becomes the LPP

Minimize $$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i \quad (i = 1, 2, \ldots, m)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad (j = 1, 2, \ldots, n)$$

$$x_{ij} \geq 0$$

Thus there are $m \times n$ decision variables x_{ij} and $m+n$ constraints. Since the sum of the first m constraints equals the sum of the last n constraints (because the problem is a balanced one), one of the constraints is redundant and we can show that the other $m+n-1$ constraints are linearly independent. **Thus any BFS will have only $m+n-1$ nonzero variables.**

Though we can solve the above LPP by Simplex method, we solve it by a special algorithm called the transportation algorithm. We present the data in an $m \times n$ tableau as explained below.

Destination

Source		1	2	.	.	n	Supply
	1	c_{11}	c_{12}			c_{1n}	a_1
	2	c_{21}	c_{22}			c_{2n}	a_2
	.						
	.						
	m	c_{m1}	c_{m2}			c_{mn}	a_m
Demand		b_1	b_2			b_n	

Destination

Source		Denver	Miami	Supply
S o u r c e	Los Angeles	80	215	1000
	Detroit	100	108	1300
	New Orleans	102	68	1200
	Dummy	0	0	200
	Demand	2300	1400	

We write inside the (i,j) cell the amount to be shipped from source i to destination j. A blank inside a cell indicates no amount was shipped.

Destination

Source	Denver	Miami	Supply
Los Angeles	80	M	1000
Detroit	100	108	1300
New Orleans	102	68	1200
Dummy	200	300	200
Demand	2300	1400	

Note: M indicates a very "big" positive number.
In a software it is denoted by "infinity".

Destination

Distribution Area

Source

		1	2	3	Supply
Refinery	1	12	18	M	6
	2	30	10	8	5
	3	20	25	12	8
Demand		4	8	7	

The problem is a balanced one. M indicates a very "big" positive number.

The total cost will be 10*

$$\sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij} x_{ij}$$

Destination

Distribution Area

Source Refinery

		1	2	3	Dummy	Supply
Refinery	1	12	18	M	15	6
	2	30	10	8	22	5
	3	20	25	12	0	8
Demand		4	8	4	3	

M indicates a very "big" positive number.

The total cost will be 10* $\sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij} x_{ij}$

Destination
Retailer

Source / Orchard

	1	2	3	4	Dummy	Supply
1	1	2	3	2	0	350
2	2	4	1	2	0	400
3	1	3	5	3	M	250
Demand	150	150	400	100	200	

Determination of Starting Basic Feasible Solution:

Phase- I Solution

Determination of the starting Solution(Phase-I):

In any transportation model we determine a starting BFS and then iteratively move towards the optimal solution which has the least shipping cost.

There are three methods to determine a starting BFS. As mentioned earlier, any BFS will have only m+n-1 basic variables (which may assume non-zero =positive values) and the remaining variables will all be non-basic and so have zero values. In any transportation tableau, we only indicate the values of basic variables. The cells corresponding to non-basic variables will be blank.

Degenerate BFS:

If in a cell we find a zero mentioned, it means that that cell corresponds to a basic variable which assumes a value of zero. In simplex language, we say that we have a degenerate BFS.

NORTH-WEST Corner Method for determining a starting BFS

The method starts at the north-west corner cell (i.e. cell (1,1)).

Step 1. We allocate as much as possible to the selected cell and adjust the associated amounts of supply and demand by subtracting the allocated amount.

Step 2. Cross out the row (column) with zero supply (zero demand) to indicate that no further assignments can be made to that row(column).

If both a row and a column are simultaneously satisfied then

if exactly one row or column is left uncrossed make the obvious allocations and stop. Else cross out one only (either the row or the column) and leave a zero supply(demand) in the uncrossed out row(column).

Step 3. If no further allocation is to be made, stop. Else move to the cell to the right (if a column has just been crossed out) or to the cell below if a row has just been crossed out. Go to Step 1.

Consider the transportation tableau:

Destination

		1	2	3	4	Supply
Source	1	3 3	7 2	6	4	~~5~~ ~~2~~
	2	2	4 1	3 1	2	~~2~~ ~~1~~
	3	4	3	8 1	5 2	~~3~~ ~~2~~
Demand		~~3~~	~~3~~ ~~1~~	~~2~~ ~~1~~	~~2~~	

Total shipping cost = 48

2. LEAST COST Method (LCM): Phase-I

In this method we start assigning as much as possible to the cell with the least unit transportation cost (ties are broken arbitrarily) and the associated amounts of supply and demand are adjusted by subtracting the allocated amount.

Cross out the row (column) with zero supply (zero demand) to indicate that no further assignments can be made to that row(column).

If both a row and a column are simultaneously satisfied then

if exactly one row or column is left uncrossed make the obvious allocations and stop. Else cross out one only (either the row or the column) and leave a zero supply(demand) in the uncrossed out row(column).

Next look for the uncrossed out cell with the smallest unit cost and repeat the process until no further allocations are to be made.

Consider the transportation tableau:

Destination

		1	2	3	4	Supply
Source	1	3 / 1	7 / 0	6 / 2	4 / 2	~~5~~ ~~4~~ ~~2~~
	2	2 / 2	4	3	2	~~2~~
	3	4	3 / 3	8	5	~~3~~
Demand		~~3~~ ~~1~~	~~3~~ ~~0~~	~~2~~	~~2~~	

Total shipping cost = 36

Transportation Problem: Phase-II Solution known as MODI Method: MOdified(MO) DIstribution(DI)

Iterative computations of the Transportation algorithm

After determining the starting BFS by any one of the three methods discussed earlier, we use the following algorithm to determine the optimum solution.

(Use either NWCR or LCM for Phase-I Solution)

Step1: Use the Simplex optimality condition to determine the entering variable as a current non-basic variable that can improve the solution. If the optimality condition is satisfied by all non-basic variables, the current solution is optimal and we stop. Otherwise we go to Step 2.

Step 2. Determine the leaving variable using the Simplex feasibility condition. Change the basis and go to Step 1.

The determination of the entering variable from among the current non-basic variables is done by the **method of multipliers (see Class Note)**.

In the method of multipliers, we associate with each row a dual variable (also called a multiplier) u_i and with each column we associate a dual variable (also called a multiplier) v_j.

Noting that each row corresponds to a constraint and each column corresponds to a constraint we recall from duality theory that

At any simplex iteration ,

$$\begin{pmatrix} \text{Primal z-equation} \\ \text{coefficient of} \\ \text{variable xij} \end{pmatrix} = \begin{pmatrix} \text{Left hand side} \\ \text{of corresponding} \\ \text{dual constraint} \end{pmatrix} - \begin{pmatrix} \text{Right hand side} \\ \text{of corresponding} \\ \text{dual} \end{pmatrix}$$

constraint

That is $$"z_{ij} - c_{ij}" = u_i + v_j - c_{ij}$$

(Verify this by taking m=3 and n=4 !)

Since there are m+n-1 basic variables and since

$$z_{ij} - c_{ij} = 0$$

for all such basic variables, we have m+n-1 equations

$$u_i + v_j = c_{ij}$$

to determine the m+n variables u_i, v_j

We arbitrarily choose one of them and equate to zero and determine the remaining m+n-1 of them. Then we calculate

$$z_{ij} - c_{ij} = u_i + v_j - c_{ij}$$

for all non-basic variables x_{ij}. Then the entering variable is that one for which $u_i + v_j - c_{ij}$ is most positive.

If we consider a maximization type Transportation Problem instead of most positive element, most negative element is selected for the entering variable.

We do this on the transportation tableau itself (and NOT separately) as the following example shows.

Destination

Starting Tableau

Total Cost =48

Source		$v_1=3$	$v_2=7$	$v_3=6$	$v_4=3$	Supply
	$u_1=0$	3 3	7 2	6 0	4 -1	5
	$u_2=-3$	2 -2	4 1	3 1	2 -2	2
	$u_3=2$	4 1	3 6	8 1	5 2	3
Demand		3	3	2	2	

Thus x_{32} enters the basis.

Determining the leaving variable

We first construct a **closed loop** that starts and ends at the entering variable cell. The loop consists of connected horizontal and vertical segments only (no diagonals are allowed). Except for the entering variable cell, each vertex (or corner) of the closed loop must correspond to a basic variable cell. The loop can **cross itself** and **bypass one or more basic variables**. The amount θ to be allocated to the entering variable cell is such that it satisfies all the demand and supply restrictions and must be non-negative. Usually θ is the minimum of the amounts allocated to the basic cells adjacent to the entering variable cell. Having decided about the amount θ to be allocated to the entering cell, for the supply and demand limits to remain satisfied, we must alternate between subtracting and adding the amount θ at the successive corners of the loop. In this process one of the basic variables will drop to zero. In simplex language, we say it leaves the basis. We repeat this process till optimality is reached. We illustrate with a numerical example.

Destination

Starting Tableau

Total Cost =48

Source	$v_1=3$	$v_2=7$	$v_3=6$	$v_4=3$	Supply
$u_1=0$	3 **3**	7 **2**	6 0	4 -1	5
$u_2=-3$	2 -2	4 **1**	3 **1**	2 -2	2
$u_3=2$	4 1	3 **θ** 6	8 **1**	5 **2**	3
Demand	3	3	2	2	

Thus x_{32} enters the basis.

Thus θ will become 1 and in the process both the basic variables x_{22} and x_{33} will become simultaneously zero. Since only one of them should leave the basis we make x_{22} leave the basis and keep x_{33} in the basis but with value zero. Thus the transportation cost reduces by 6 (as x_{23} increases by 1) and we say one iteration is over. The resulting new tableau is on the next slide.

Destination

Start of Iteration 2

Total Cost =42

Source		$v_1=3$	$v_2=7$	$v_3=12$	$v_4=9$	Supply
	$u_1=0$	3 **3**	7 **2**	6 **θ** 6	4 5	5
	$u_2=-9$	2 -8	4 -6	3 **2**	2 -2	2
	$u_3=-4$	4 -5	3 **1**	8 **0**	5 **2**	3
	Demand	3	3	2	2	

Thus x_{13} enters the basis.

Thus θ will become 0 and x_{32} leaves the basis. Again the BFS is degenerate . But the transportation cost remains the same and we say the second iteration is over. The resulting new tableau is on the next slide.

Destination

Start of Iteration 3

Total Cost =42

Source	v_1=3	v_2=7	v_3=6	v_4=9	Supply
u_1=0	3 **3**	7 **2**	6 **0**	4 **θ** 5	5
u_2= -3	2 -2	4 0	3 **2**	2 4	2
u_3= -4	4 -5	3 **1**	8 -6	5 **2**	3
Demand	3	3	2	2	

Thus x_{14} enters the basis.

Destination

Start of Iteration 4

Total Cost =32

	$v_1=3$	$v_2=7$	$v_3=6$	$v_4=4$	Supply
$u_1=0$	3 (3)	7 (0)	6 (0)	4 (2)	5
$u_2=-3$	2 -2	4 0	3 (2)	2 -1	2
$u_3=-4$	4 -5	3 (3)	8 -6	5 -5	3
Demand	3	3	2	2	

Source

Thus this is the **optimal** tableau. Alt Opt solutions exist.

	D_1	D_2	D_3	D_4	
S_1	3 [3]	2 [7]	[6]	[4]	5
S_2	[2]	− 1 [4]	1 + [3]	[2]	2
S_3	[4]	+ [3]	1 − [8]	2 [5]	3
	3	3	2	2	

$u_1 + v_1 = 3$

$u_1 + v_2 = 7$

$u_2 + v_2 = 4$

$u_2 + v_3 = 3$

$u_3 + v_3 = 8$

$u_3 + v_4 = 5$

$\boxed{Z = 48}$

x_{32} enters to the basis.

$u_1 = 0$, $v_1 = 3$

$u_2 = -3$, $v_2 = 7$

$u_3 = 2$, $v_3 = 6$

$v_4 = 3$

$u_1 + v_3 = 6 \leq 6$ ✓

$u_1 + v_4 = 3 \leq 4$ ✓

$u_2 + v_1 = 0 \leq 2$ ✓

$u_2 + v_4 = 0 \leq 2$ ✓

$u_3 + v_1 = 5 > 4$ ×

$\boxed{u_3 + v_2 = 7 > 3}$ ×

	v_1	v_2	v_3	v_4	
u_1	(3) 3	− (2) 7	(+) 6	4	5
u_2	2	4	(2) 3	2	2
u_3	4	+ (1) 3	(0) − 8	(2) 5	3
	3	3	2	2	

$u_1+v_1=3$

$u_1+v_2=7$

$u_2+v_3=3$

$u_3+v_2=3$

$u_3+v_3=8$

$u_3+v_4=5$

$\therefore \boxed{Z=42}$

x_{13} enters to the basis.

$u_1=0 \quad v_1=3$

$v_2=7$

$u_2=-9 \quad v_3=12$

$u_3=-4 \quad v_4=9$

$u_1+v_3=12 > 6$ ×

$u_1+v_4=9 > 4$ ×

$u_2+v_1=-6 \leq 2$

$u_2+v_2=-2 \leq 4$

$u_2+v_4=0 \leq 2$

$u_3+v_1=-1 \leq 4$

	v_1	v_2	v_3	v_4	
u_1	3 / 3	− 2 / 7	0 / 6	+ / 4	5
u_2	— / 2	— / 4	2 / 3	— / 2	2
u_3	— / 4	+ 1 / 3	— / 8	− 2 / 5	3
	3	3	2	2	

$u_1+v_1=3$

$u_1+v_2=7$

$u_1+v_3=6$

$u_2+v_3=3$

$u_3+v_2=3$

$u_3+v_4=5$

$u_1=0$

$u_2=-3$

$u_3=-4$

$v_1=3$

$v_2=7$

$v_3=6$

$v_4=9$

$\boxed{Z=42}$

$\boxed{x_{14} \text{ enters}}$

$u_1+v_4=9>4$ ✗

$u_2+v_1=0\le 2$

$u_2+v_2=4\le 4$

$u_2+v_4=6>2$ ✗

$u_3+v_1=-1\le 4$ ✓

$u_3+v_3=2\le 8$ ✓

	v_1	v_2	v_3	v_4	
u_1	3 $\boxed{3}$	0 $\boxed{7}$	0 $\boxed{6}$	2 $\boxed{4}$	5
u_2	$\boxed{2}$	$\boxed{4}$	2 $\boxed{3}$	$\boxed{2}$	2
u_3	$\boxed{4}$	3 $\boxed{3}$	$\boxed{8}$	$\boxed{5}$	3
	3	3	2	2	

$u_1 + v_1 = 3$ $\quad u_1 = 0$ $\quad v_1 = 3$

$u_1 + v_2 = 7$ $\quad u_2 = -3$ $\quad v_2 = 7$

$u_1 + v_3 = 6$ $\quad u_3 = -4$ $\quad v_3 = 6$

$u_1 + v_4 = 4$ $\quad v_4 = 4$

$u_2 + v_3 = 3$

$u_3 + v_2 = 3$

$$\boxed{x_{11} = 3,\ x_{14} = 2,\ x_{23} = 2,\ x_{32} = 3}$$

$u_2 + v_1 = 0 \le 2$ $\quad u_3 + v_3 = 2 \le 8$

$u_2 + v_2 = 4 \le 4$ $\quad u_3 + v_4 = 0 \le 5$

$u_2 + v_4 = 1 \le 2$

$u_3 + v_1 = -1 \le 4$ $\quad$ Soln is optimal

$Z = 9 + 0 + 0 + 8 + 6 + 9 = 32$