LINEAR ALGEBRA

1) Linear Algebra done right second edition by Sheldon Axler [Springer]

2) An Introduction to Linear Algebra by V. Krishna Murthy [east-west press Pvt Ltd]

3) Linear Algebra, 2nd edition by Hoffmann & Ray Kunze. [Prentice Hall, Inc, Englewood, Cliffs, New Jersey]

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Field

Suppose F is a non empty set with two binary operations [addition "+" & multiplication "·"] satisfying the following:

(a) $(F,+)$ is a abelian group

i.e $a,b \in F \Rightarrow a+b \in F$

$a+(b+c) = (a+b)+c$

$\exists\ 0 \in F \ni a+0 = 0+a = a$

* for every $a \in F \ \exists\ -a \in F$

$\ni a+(-a) = (-a)+a = 0$

* $a+b = b+a$.

(b) $(F,\cdot)$ is abelian group.

$\Rightarrow \forall\ a,b,c \in F$ we have

$a.b \in F$ $\quad ab$ unique

$a.b = b.a$

$a.(b \cdot c) = (a.b)\cdot c$

$\exists$ a non zero element '1' in F for $a.1 = a = 1.a \ \forall\ a \in F$

* $\forall$ non-zero element $a \in F$

$\exists\ a^{-1} \in F \ni a.a^{-1} = 1 = a^{-1}.a$.

(c) Multiplication is distributive w.r.t addition i.e

$a \cdot (b+c) = a \cdot b + a \cdot c$

Ex: $F = \mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$

If F is a field we denote it by $(F, +, \cdot)$

* Binary operation on a set –

Let G be a non-empty set, then

$G \times G = \{(a,b) \mid a, b \in G\}$

If $f : G \times G \to G$ then f is said to be binary operation on G, if $(a,b) \in G \times G$

$\Rightarrow f(a,b) \in G$.

* Binary operations:

$f = +, \times, \ldots \circ, *,$ etc.

If $G = \mathbb{N}$

for $a, b \in \mathbb{N} \Rightarrow a+b \in \mathbb{N}$

"+" is a binary operation on $\mathbb{N}$.

for $4, 3 \in N \Rightarrow 3 - 4 = -1 \notin N$

$\therefore$ "$-$" is not a binary operation.

* Algebriac structure:

A non empty set G with one or more binary operations is called an algebriac structure.

If "$*$" is a binary operation on G then $(G, *)$ is an algebriac structure

Ex: $(N, +)$, $(I, +)$ $(I, -)$ $(R, +, \cdot)$

* Internal Composition:

If $a * b \in G$ $\forall a, b \in G$ & $a * b$ is unique, we say "$*$" is an internal Composition on G.

* External Composition:

let V & F be any two sets. If $a \circ \alpha \in V$, $\forall a \in F$ & $\alpha \in V$ & $a \circ \alpha$ is unique, then "$\circ$" is called external composition in V over F.

Vector Space:

let $(F, +, \cdot)$ be a field. The elements of F will be called scalars. let V be a non empty set whose elements are called vectors.

We say "V" is a vector space over the field F, if

(i) There is defined an internal composition "+" in V which satisfies the foll conditions:

(a) $\forall u, v \in V \Rightarrow u+v \in V$

(associative) (b) $\forall u, v, w \in V \Rightarrow (u+v)+w = u+(v+w)$

(c) $\exists\ 0 \in V \ni$

(identity element) $v+0 = 0+v = v \quad \forall v \in V$

(d) To every $v \in V\ \exists\ -v \in V$

(inverse) $\ni\ v+(-v) = 0 = (-v)+v$

(commutative) (e) $v+u = u+v, \quad \forall u, v \in V$

i.e $\boxed{(V, +) \text{ is abelian group}}$

(ii) there exists an external composition "." in V over F called scalar multiplication i.e $\forall a \in F, v \in V$

$\Rightarrow a.v \in V$

(iii) The two compositions "+" and "." satisfies the following

$\forall a, b \in F;\ u, v \in V$

(a) $a.(u+v) = a.u + a.v$

(b) $(a+b).u = a.u + b.u$

(c) $(ab)u = a(bu)$

(d) $1.u = u$

if V is a vector space over a field F, we denote it by $V(F)$

Ex: (i) $V = F$ then $F(F)$ is a vector space

(ii) $V_n = \{(a_1, a_2, \ldots a_n) \mid a_1, a_2 \ldots a_n \in F\}$

then V_n is a vector space over F

$C(R) \Rightarrow$ vector space

$R(C) \Rightarrow$ not a vector space.

P: set of all polynomials in x.

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$.

$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$

$(f+g)(x) = c_0 + c_1 x + c_2 x^2 + \cdots$

where $c_i = a_i + b_i$

$i = 0, 1, 2, \ldots$

$(\alpha f)(x) = \alpha f(x)$

$= \alpha a_0 + (\alpha a_1) x + (\alpha a_2) x^2 + \cdots$

$\therefore$ The set of polynomials is a vector space over F.

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* Let S be a non empty set & F be any field

$V = \{ f \mid f = S \to F \}$

for $f, g \in V$ define.

$(f+g)(x) = f(x) + g(x)$

$(cf)(x) = c f(x)$, $\forall c \in F$

Then $V(F)$ is a vector space

ex: $V = \{(x,y) \mid x, y \in F\}$

$F = \mathbb{R}$,

Define $\forall\ (x,y), (x_1, y_1) \in V$

$(x,y) + (x_1, y_1) = (x + x_1, 0)$

$c(x,y) = (cx, 0) \quad \forall\ c \in \mathbb{R}$

V is not a vector space over $\mathbb{R}$

if (x_1, y_1) is additive identity in V,

then we must have

$(x,y) + (x_1, y_1) = (x,y) \quad \forall\ (x,y) \in V$

$(x + x_1, 0) = (x,y) \quad \forall\ (x,y) \in V$

So if $y \neq 0$, above equality doesn't hold

$\therefore \nexists$ ~~no~~ $(x_1, y_1) \in V \ni$

$(x,y) + (x_1, y_1) = (x,y) \quad \forall\ (x,y) \in V$

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8) $V = \{(x,y) \mid x, y \in \mathbb{R}\}$. are the foll vector spaces?

(i) $(x,y) + (x_1, y_1) = (x + x_1, y + y_1)$

$c(x,y) = (|c|x, |c|y)$

(ii) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$

$c(x, y) = (0, cy)$

(iii) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$

$c(x, y) = (c^2 x, c^2 y)$

Lemma: In a vector space $V(F)$

(a) $\alpha 0 = 0 \quad \forall \alpha \in F$ ["0" vector]

(b) $0u = 0 \quad \forall u \in V$

$0 \downarrow$ number; $0 \downarrow$ vector

(c) $(-1)u = -u \quad \forall u \in V$

$(-1) \downarrow$ number

Proof:

(a) consider

$\alpha \cdot 0 = \alpha(0 + 0) = \alpha 0 + \alpha 0$

$0 \downarrow$ vector

add on both sides with $-(\alpha 0)$

$0 = \underbrace{\alpha 0 + 0}_{\text{vector}} \Rightarrow 0 + \alpha 0$

$0 \downarrow$ vector

(b) $\underset{\text{number}}{0}u = (0+0)u = 0u + 0u$

add on both sides with $(-0u)$

$$0 = [0u + 0u] + [-0u]$$

$$\underset{\text{vector}}{0} = \underset{\text{numbers}}{0}u + \underset{\text{vector}}{0}$$

$$\boxed{\underset{\text{num}}{0} \cdot u = \underset{\text{vector}}{0}}$$

~~(c) $(-1)u = -1u + 1u$~~
~~$= (-1+1)u$~~
~~$= 0u$ num~~
~~$= 0$ vector~~

(c) $(-1)u + u = -1u + 1 \cdot u$

$$= (-1+1)u$$

$$= 0 \cdot u$$

$$= 0$$

So by the uniqueness of the additive inverse

we got $\underset{\text{number}}{(-1)}u = \underset{\text{additive inverse of } u}{-u}$

HW
1) Get some eg of a vector space } 5 each
2) Get some eg of a non vector space

Sub-Space:
let W be a non-empty subset of a vector space V(F) we say W is a sub-space of V if W itself is a vector space over the same field F.

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* Theorem:
A non empty subset W of a vector space V(F) is a subspace of V iff the foll conditions are satisfied.

(i) $u, v \in W \Rightarrow u+v \in W$

(ii) $u \in W$, $\alpha \in F \Rightarrow \alpha u \in W$

Proof:
If W is a subspace of V, then W itself is a vector space.
So, (i) & (ii) are satisfied.

Conversely let (i) & (ii) hold
$\because -1 \in F$ & for any $u \in W$
$-1 \cdot u = -u \in W$ by (ii)

iii) for any $u \in W$, $-u \in W$

$\Rightarrow u + (-u) \in W$ by (i)

$\Rightarrow 0 \in W$

since every element of W is also a element of V, other remaining properties of vector space are satisfied.

$\therefore W$ is a subspace of V.

* A non-empty ~~subspace~~ subset W of a vector space $V(F)$ is a subspace of V iff $au + bv \in W$ $\forall u, v \in W$, $a, b \in F$

or

$$\begin{cases} u - v \in W, & \forall u, v \in W \\ au \in W, & \forall u \in W \quad a \in F \end{cases}$$

Ex: Let P be the set of all polynomials in x &

$W = \{ p \in P \mid p(x_0) = 0 \}$

$\because$ let $p, q \in W$ then $p(x_0) = 0$ $q(x_0) = 0$

Now for any $a, b \in F$ we have

$(ap + bq)(x_0) = (ap)(x_0) + (bq)(x_0)$

$\Rightarrow \quad ap(x_0) + b\,q(x_0) = 0 + 0 = 0 \in W$

$\Rightarrow \quad ap + bq \in W$

$\therefore$ W is a subspace of P

eg:- consider the eqn

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots \alpha_n x_n = 0,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$

& $x_1, x_2, \ldots x_n$ are unknowns

The soln of the above eqn. is a n-tupple vector $(x_1, x_2, \ldots, x_n) \in V_n$

Let.

$$W = \{ (x_1, x_2, \ldots x_n) \mid \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0 \}$$

Then W is a subspace of V_n

* $V_0 = \{0\}$ & V are always subspaces of a vector space V

- These are called trivial subspaces of V

Any other subspaces of V are called non-trivial subspaces or proper subspaces of V.

Def - Let $u_1, u_2, \ldots u_n$ be n vectors of a vector space V and $\alpha_1, \alpha_2 \ldots \alpha_n$ be scalars then $\alpha_1 u_1 + \alpha_2 u_2 \ldots + \alpha_n u_n$ is called linear combination of $u_1, u_2 \ldots u_n$ since $u_1, u_2 \ldots u_n$ are finite, $\alpha_1 u_1 + \alpha_2 u_2 + \ldots \alpha_n u_n$ is called finite linear combination.

Span :

The span of a subset S of a vector space V is the set of all finite linear combinations of elements of S.

i.e if $S = \{u_1, u_2, \ldots u_n\}$

Then

$$\text{span } S = \{\alpha_1 u_1 + \alpha_2 u_2 + \ldots \alpha_n u_n \mid \alpha_1, \alpha_2 \ldots \alpha_n \in F\}$$

It is also denoted by $[S]$

so if $S = \{u_1, u_2 \ldots u_n\}$ then

$$[S] = [u_1, u_2 \ldots u_n].$$

Theorem : If S is a non empty subset of a vector space V, then span S is a subspace of V, containing S.

<u>Proof</u> Let $u \in S$, then

$u = 1 \cdot u \in [S]$

$\Rightarrow$ ~~[illegible]~~ $S \subset [S]$

Let $u, v \in [S]$, then

$u = \sum_{i=1}^{n} \alpha_i u_i$, $\quad u_1, u_2 \ldots u_n \in S$

$\alpha_1, \alpha_2 \ldots \alpha_n \in F$

$v = \sum_{j=1}^{n} \beta_j u_j \quad u_1, u_2 \ldots u_n \in S$

$\beta_1, \beta_2 \ldots \beta_n \in F$

$au + bv = \sum_{i=1}^{n} (a\alpha_i + b\beta_i) u_i$

$\in [S]$

$\therefore$ $[S]$ is a subspace of V

Let T be any other subspace of V containing S.

claim: $[S] \subseteq T$

Let $\alpha_1 u_1 + \alpha_2 u_2 + \ldots \alpha_n u_n \in [S]$

where $u_1, u_2 \ldots u_n \in S$ $\quad \alpha_1 \alpha_2 \ldots \alpha_n \in F$

$\because S \subset T$

$\Rightarrow u_1, u_2 \ldots u_n \in T$

& T is a subspace

$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \ldots \alpha_n u_n \in T$

$\Rightarrow [S] \subseteq T$

$[u_1, u_2 \in W \Rightarrow \alpha_1 u_1 + \alpha_2 u_2 \in W$

$w_1, u_3 \in W \Rightarrow 1 . w_1 + \alpha_3 u_3 \in W$

$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 \in W$

$\alpha_1 u_1 + \alpha_2 u_2 + \ldots \alpha_n u_n \in W$

Ex

$U_3 = \{ (\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in R \}$

be a vector space.

let $S = \{ (1,0,0), (0,1,0) \}$ then

$[S] = [(1,0,0), (0,1,0)]$

$= \{ \alpha (1,0,0) + \beta (0,1,0) \mid \alpha, \beta \in R \}$

$= \{ (\alpha, \beta, 0) \mid \alpha, \beta \in R$

$=$

Eg - $V_2 = \{ (x, y) \mid x, y \in \mathbb{R} \}$

then show that

$(3,7) \in [(1,2), (0,1)]$

& $(3,7) \notin [(1,2), (2,4)]$

Sol: $(3,7) = \alpha (1,2) + \beta (0,1)$

$\Rightarrow (3,7) = (\alpha, 2\alpha + \beta)$

$\Rightarrow \alpha = 3 \quad \beta + 2\alpha = 7 \Rightarrow \beta = 1$

$\Rightarrow (3,7) \in [(1,2), (0,1)]$

If $(3,7) \in [(1,2), (2,4)]$

$\Rightarrow (3,7) = \alpha (1,2) + \beta (2,4)$

$= (\alpha + 2\beta, 2\alpha + 4\beta)$

$\Rightarrow \alpha + 2\beta = 3$

$2\alpha + 4\beta = 7 \Rightarrow \alpha + 2\beta = 7/2$

$\therefore$ No soln.

$\Rightarrow (3,7) \notin [(1,2), (2,4)]$

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* ST in $V_2 = \{ (x, y) \mid x, y \in \mathbb{C} \}$

$(1+i, 1-i) \in [(1+i, 1), (1, 1-i)]$

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1) Let U & W be two subspaces of a vector space $V(F)$.

then $U \cap W$ is also a subspace of V

since for any $u, v \in U \cap W$

$\Rightarrow u, v \in U$ & $u, v \in W$

$\because$ U & W both are subspaces of V, for any $a, b \in F$,

$au + bv \in U$ & $au + bv \in W$

$\Rightarrow au + bv \in U \cap W$

If $U_1, U_2, \dots U_n$ are n subspaces of V,

then $\bigcap_{i=1}^{n} U_i$ is also a subspace of $V(F)$

Ex let W be the set of all vectors $(x_1, x_2, \dots x_n) \in V_n$

satisfying the foll three equations:

$$\sum_{i=1}^{n} \alpha_i x_i = 0 \quad -(1)$$

$$\sum_{i=1}^{n} \beta_i x_i = 0 \quad -(2)$$

$$\sum_{i=1}^{n} \gamma_i x_i = 0 \quad -(3)$$

let

$W_1 = \{(x_1, x_2, \ldots x_n) \in V_n \mid \sum_{i=1}^{n} \alpha_i x_i = 0\}$

$W_2 = \{(x_1, x_2, \ldots x_n) \in V_n \mid \sum_{i=1}^{n} \alpha_i x_i = 0\}$

$W_3 = \{(x_1, x_2, \ldots x_n) \in V_n \mid \sum_{i=1}^{n} \gamma_i x_i = 0\}$

Then clearly $W = W_1 \cap W_2 \cap W_3$

is the solution set which satisfies all the three eqn

<u>Note:</u> the union of two subspaces need not be a subspace.

eg: $U = \{(x, 0) \mid x \in R\}$

$W = \{(0, y) \mid y \in R\}$

Then U & W are subspaces of $V_2(R)$

Now $U \cup W = \{(x, 0), (0, y) \mid x, y \in R\}$

$\because (1, 0), (0, 1) \in U \cup W$

$\Rightarrow (1, 0) + (0, 1) = (1, 1) \notin U \cup W$

$\therefore$ $U \cup W$ is not a subspace of $V_2(\mathbb{R})$

* Theorem. The union of two subspaces is also a subspace iff one is contained in another

[W_1, W_2 subspaces, if $W_1 \subset W_2 \Rightarrow$

$W_1 \cup W_2 = W_2$.

if $W_2 \subset W_1 \Rightarrow W_1 \cup W_2 = W_1$

conversely assume $W_1 \cup W_2$ is a subspace & prove either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

* Theorem:

If U & W are two subspaces of a vector space V, then $U+W$ is also a subspace of V & $U+W = [U \cup W]$

Proof: Since each vector of $U+W$ is a linear combination of vectors of $U \cup W$ we see that

$U+W \subseteq [U \cup W]$

Now let $v \in [U \cup W]$

Here v must be of the form $u+w$ where $u \in U$ $w \in W$

$\Rightarrow v = u + w \in U + W$

$\Rightarrow [U \cup W] \subseteq U + W$

$\therefore U + W = [U \cup W]$

* $U + W$ is the smallest subspace of V containing $U \cup W$.

Ex. $V = V_3 = \{(x, y, z) \mid x, y, z \in F\}$

$U = \{\alpha(1, 0, 0) \mid \alpha \in F\}$

$= x$-axis

$W = \{\beta(0, 1, 0) \mid \beta \in F\}$

$= y$-axis

Then

$U + W = x$-axis $+$ y axis

$= \{\alpha(1, 0, 0) + \beta(0, 1, 0) \mid \alpha, \beta \in F\}$

$= \{(\alpha, \beta, 0) \mid \alpha, \beta \in F\}$

$= xy$-plane.

On the other hand,

$[U \cup W] = \{ \alpha u + \beta w \mid u \in U, w \in W \}$

$= \{ \alpha(1,0,0) + \beta(0,1,0) \mid \alpha, \beta \in F \}$

$= U + W$

* $[x\text{-axis} \cup y\text{-axis}] = xy\text{-plane}$

$= x\text{axis} + y\text{ axis}$

* Direct Sum :

Let U & W be two subspaces of a vector space $V(F)$. Then $U+W$ is also a subspace of V.

If, in addition $U \cap W = \{0_V\}$, then $U+W$ is called direct sum of U & W & it is denoted by $U \oplus W$.

Ex: $U = xy$ plane

$W = yz$ plane

then $U+W =$ ~~xyz space~~ xyz space $= V_3$

& (xy plane) $\cap$ (yz plane)

= y axis $\neq \{0_v\}$

$\therefore$ U+W is not a direct sum.

U + W

xy plane + z-axis

= xyz space = V_3

(xy plane) $\cap$ (z-axis) = $\{0_v\}$ singleton zero vector

$\therefore$ U+W is a direct sum

(a, b, c) $\in V_3$, can be written uniquely

(a,b,c) = (a,b,0) + (0,0,c)

~~(a,b,c) =~~

* In direct sum every vector in a vector space V can be written uniquely as sum of vectors in its subspaces.

Ex: U = xy - plane

W = yz - plane.

$U \cap W$ = y axis

$U + W = V_3$

for any vector $(a,b,c) \in V_3$, we have

$(a,b,c) = (a,b,0) + (0,0,c)$

$(a,b,c) = (a,0,0) + (0,b,c)$

xy plane : yz plane

Thus (a,b,c) has two ~~sepre~~ expressions as seem of the elements in xy plane & yz-plane

$\therefore$ xy-plane + yz-plane is not a direct sum of V_3

* **Theorem:**

Let U & W be two subspaces of a vector space V; & $Z = U + W$. then $Z = U \oplus W$ iff any vector $z \in Z$ can be written uniquely as the sum

$z = u + w$; $u \in U$ $w \in W$

Proof: suppose $Z = U \oplus W$ & $z \in Z$

if possible $z = u + w$ and $z = u' + w'$

$u, u' \in U$

$w, w' \in W$

Then $u + w = u' + w'$

$\Rightarrow u - u' = w' - w$

Now, $u, u' \in U \Rightarrow u - u' \in U$

$w', w \in W \Rightarrow w' - w \in W$

but $u - u' = w' - w$ & $\in U \cap W$

$= \{0_v\}$

$\Rightarrow u - u' = 0_v$ & $w' - w = 0_v$

$\Rightarrow u = u'$, $w' = w$

Thus every element $z \in Z$, has unique expression as sum of elements in U & W.

Conversely. suppose

$z = u + w$, $u \in W$ $w \in W$

be the unique expression for every $z \in Z$.

claim: $U \cap W = \{0_v\}$ then $U \oplus W = Z$

If $U \cap W \neq \{0_v\}$ then $0 \neq v \in U \cap W$

$\Rightarrow v \in U$ & $v \in W$

$v \in U$

$\Rightarrow v = \underset{U+W}{v + 0}$ & $v = \underset{\in U+W}{0 + v}$

$\Rightarrow$ $v \in U+W$ has two expression, which is contradiction.

$\therefore U \cap W = \{0\}$

Hence $Z = U \oplus W$

$v \in U$

$v = v + 0$
$\quad \in U \quad \in W$

$v \in W$

$v = 0 + v$
$\quad \in U \quad \in W$

$v = v_1 + v_2$

$\boxed{\begin{array}{l} v = u + 0 \\ v = 0 + u \end{array}}$ ×

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Def: If $u_1, u_2, \ldots u_n$ are n vectors in a vector space V, then the linear combination $\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$ is called trivial linear combination if all the scalars are zero. otherwise, the linear combination.

i.e $0u_1 + 0u_2 + \ldots + 0u_n$ is trivial LC & $0u_1 + 0u_2 + \ldots + 1u_n$ is non trivial LC

* Trivial combination of any set of vectors is always a zero vector.

$\therefore 0u_1 + 0u_2 + \ldots + 0u_n$

$= 0 + 0 + \ldots + 0$

$= 0$

The question is whether a non-trivial L.C of vector gives a zero vector?

Ex: $\{(1,0,0), (2,0,0), (0,0,1)\}$ in V_3

$\therefore 1(1,0,0) + (-\frac{1}{2})(2,0,0) + 0(0,0,1)$

$= 0$

This is non trivial LC

Ex: $\{(1,0,0), (0,1,0), (0,0,1)\}$

$\alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1)$

$= (0,0,0)$

$\Rightarrow \alpha = 0 \quad \beta = 0 \quad \gamma = 0$

This is trivial LC

In the first example we have

$(1,0,0) = \frac{1}{2}(2,0,0) + 0(0,0,1)$

i.e $(1,0,0)$ depends on $(2,0,0)$ & $(0,0,1)$

In this case we say the above three vectors $(1,0,0)$ $(2,0,0)$ $(0,0,1)$ are linearly dependant.

Def: let $V(F)$ be a vector space. A finite set of vectors $\{u_1, u_2 \ldots u_n\}$ is said to be linearly independent (L.I) if non-trivial L.C of $u_1, u_2 \ldots u_n$ equals the zero vector i.e $\alpha_1 u_1 + \alpha_2 u_2 + \ldots \alpha_n u_n = 0$

$\Rightarrow \alpha_1 = 0 \quad \alpha_2 = 0 \quad \ldots \quad \alpha_n = 0$

Linearly dependant:

The set of vectors $\{u_1, u_2 \ldots u_n\}$ are said to be linearly dependant (L.D)

If $\alpha_1 u_1 + \alpha_2 u_2 \ldots + \alpha_n u_n = 0$

$\Rightarrow$ atleast one α_i is non zero

Ex: $C[-1,1]$ is a V.S let $S = \{x, |x|\}$

Discuss the L.I or L.D.

Consider $\alpha x + \beta |x| = 0, \quad \forall x \in (-1,1)$

Let $x = \frac{1}{2}$, then

$$\frac{\alpha}{2} + \frac{\beta}{2} = 0 \quad -(1)$$

for

$x = -\frac{1}{2}$, then

$$-\frac{\alpha}{2} + \frac{\beta}{2} \quad -(2)$$

solving these we get $\alpha = \beta = 0$

$\therefore \{x, |x|\}$ is L.I on $[-1, 1]$

Ex-2

$\{x, |x|\}$ on $[0, 1]$

$$\alpha x + \beta |x| = 0 \quad , \forall x \in [0, 1]$$

$$\alpha x + \beta x = 0$$

$$(\alpha + \beta) x = 0$$

$$\alpha + \beta = 0 \Rightarrow \alpha = -\beta$$

$\therefore \{x, |x|\}$ are LD on $[0, 1]$

Ex: $C(-\infty, \infty)$ is a V.S

check the LI or LD of $\{e^x, e^{2x}\}$.

consider $\alpha e^x + \beta e^{2x} = 0 \quad x \in (-\infty, \infty) -(1)$

Differentiating, we get

$$\alpha e^{x} + 2\beta e^{2x} = 0 \quad -(2)$$

solving (1) & (2), we get

$\beta = 0$, & hence $\alpha = 0$

$\therefore \{e^{x}, e^{2x}\}$ are LI

Def: Given a vector $v \neq 0$, the set of scalar multiples of v is called the line through v.

Def: Two vectors v_1 & v_2 are collinear if one of them lies in the line through the other

clear zero vector '0' is collinear with any non-zero vector $[\vec{0} = 0\vec{v}]$

Def: Given two vectors v_1 & v_2 which are not collinear, this span $[v_1, v_2]$ is called plane through v_1 & v_2

Def: Three vectors v_1, v_2 & v_3 are coplanar if one of them lies in the plane through the other two i.e $v_1 \in [v_2, v_3]$ or $v_2 \in [v_1, v_3]$ or $v_3 \in [v_1, v_2]$

Ex: The vectors v and $2v$ of a vector space $V(F)$ are collinear

(2) $\sin x$, $2\sin x$ are collinear in $F(I)$.

(3) $\{\sin x, \cos x\}$,

$[\sin x, \cos x] = \{\alpha \sin x + \beta \cos x \mid \alpha, \beta \in F\}$

(4) $\{\sin x, \cos x, \tan x\}$ in $F(I)$ are not coplanar

(5) $\{\sin^2 x, \cos^2 x, \cos 2x\}$ coplanar

$\because \cos 2x = \cos^2 x - \sin^2 x$

Theorem: Let V be a vector space. Then

(a) the set $\{v\}$ is L.D iff $v = 0$

(b) The set $\{v_1, v_2\}$ is L.D iff v_1, v_2 are collinear

(c) The set $\{v_1, v_2, v_3\}$ is L.D iff v_1, v_2 & v_3 are coplanar.

Proof :

(a). $\{v\}$ is L.D $\Leftrightarrow \alpha v = 0, \quad \alpha \neq 0.$

$\Leftrightarrow \alpha^{-1}(\alpha v) = 0$

$\Leftrightarrow (\alpha^{-1}\alpha)\, v = 0$

$\Leftrightarrow 1 \cdot v = 0$

$\Leftrightarrow v = 0$

(b) suppose $\{v_1, v_2\}$ is L.D.

$\Rightarrow \exists$ scalars α, β, atleast are say $\alpha \neq 0 \ni \alpha v_1 + \beta v_2 = 0$

$\Rightarrow v_1 = \left(-\beta/\alpha\right) v_2$

$\Rightarrow \{v_1, v_2\}$ are collinear

conversely, suppose $\{v_1, v_2\}$ are collinear, then

$v_1 = \beta v_2, \quad \beta \in F$

$\Rightarrow 1 \cdot v_1 + (-\beta) v_2 = 0$

$\Rightarrow \{v_1, v_2\}$ is L.D

(c) suppose $\{v_1, v_2, v_3\}$ is L.D Then $\exists$ scalars α, β, γ, with atleast one of them say $\alpha \neq 0$ such that

$\alpha v_1 + \beta v_2 + \gamma v_3 = 0$

$\Rightarrow$ $v_1 = \left(\frac{-\beta}{\alpha}\right) v_2 + \left(\frac{-\gamma}{\alpha}\right) v_3$

$\Rightarrow$ $v_1 \in [v_2 ; v_3]$

$\Rightarrow$ v_1 lies in the plane through v_2 & v_3

$\therefore$ v_1, v_2 & v_3 are coplanar.

Then one of the vector say $v_1 \in [v_2, v_3]$

$\Rightarrow$ $v_1 = \alpha v_2 + \beta v_3$

$\Rightarrow$ $1 \cdot v_1 + (-\alpha) v_2 + (-\beta) v_3 = 0$

$\Rightarrow$ $\{v_1, v_2, v_3\}$ is L.D

Ex: $\{(1,1,1), (1,-1,1) (3,-1,3)\}$

Show that one of the vector, is plane through the other two vectors.

Consider.

$\alpha(1,1,1) + \beta(1,-1,1) + \gamma(3,-1,3) = (0,0,0)$

solving we get

$\alpha = 1, \beta = 2, \gamma = -1$

$(1,-1,1) \in [(1,1,1), (3,-1,3)]$

Theorem: In a vector space V if the set $\{v_1, v_2, \dots v_n\}$ is L.I &

& $v \notin [v_1, v_2, \dots v_n]$, then the set $\{v, v_1, v_2, \dots v_n\}$ is LI.

Proof: consider

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n + \beta v = 0$$

Then clearly $\beta = 0$, otherwise if $\beta \neq 0$

$$v = \left(-\frac{\alpha_1}{\beta}\right) v_1 + \left(-\frac{\alpha_2}{\beta}\right) v_2 + \dots + \left(-\frac{\alpha_n}{\beta}\right) v_n$$

$$\in [v_1, v_2, \dots v_n]$$

which is contradiction $\therefore \beta = 0$

Then from (1) we have

$$\sum_{i=1}^{n} \alpha_i v_i + 0 v = 0$$

$$\Rightarrow \alpha_1 = 0 \quad \alpha_2 = 0 \quad \dots \quad \alpha_n = 0$$

$\therefore \{v_1, v_2, \dots v_n, v\}$ is L.I

Theorem: (a) Any subset of a L.I set is also a L.I

(b) Any superset of a L.D set is also a L.D

Sol: (a) Let $S\{u_1, u_2, \ldots u_n\}$ be L.I

Let $S_1 = \{u_1, u_2 \ldots u_K\}$ be a subset of S

$\therefore$ Consider $\sum_{i=1}^{K} \alpha_i u_i = 0$

$\Rightarrow \sum_{i=1}^{K} \alpha_i u_i + 0 u_{K+1} + \ldots 0 u_K = 0$

$\Rightarrow \alpha_1 = 0, \alpha_2 \cancel{=} \ldots \alpha_K = 0,$

$\therefore \{u_1, u_2 \ldots u_K\}$ is LD

$\therefore S_1$ is L.I

(b) Let $S = \{v_1, v_2 \ldots v_n\}$ be L.D

$S' = \{v_1, v_2, \ldots v_n, w_1, w_2 \ldots w_m\}$

be any superset of S

Then clearly

$\sum_{i=1}^{n} \alpha_i v_i + 0w_1 + 0w_2 + \cdots 0w_n = 0$

$\Rightarrow \sum_{i=1}^{n} \alpha_i v_i = 0$

$\because \{v_1, v_2 \ldots v_n\}$ is L.D, atleast one $\alpha_i \neq 0$

$\Rightarrow S'$ is L.D

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* **Theorem:** Suppose $\{v_1, v_2, \ldots v_n\}$ be an ordered set of vectors in a vector space V with $v_1 \neq 0$. Then the set $\{v_1, v_2 \ldots v_n\}$ is L.D iff one of the vectors $v_2, v_3 \ldots v_n$ ..... Say v_k belongs to span of $v_1, v_2, \ldots v_{k-1}$, i.e $v_k \in [v_1, v_2 \ldots v_{k-1}]$, $k = 2$ to n

Proof: Suppose $v_k \in [v_1, v_2 \ldots v_{k-1}]$

Then $v_k = \sum_{i=1}^{k-1} \alpha_i v_i$

$\Rightarrow 1 \cdot v_k + \sum_{i=1}^{k-1} (-\alpha_i) v_i = 0$

$\Rightarrow \{ v_1, v_2 \ldots v_k \}$ is L.D

Conversely suppose that

$\{ v_1, v_2 \ldots v_n \}$ is L.D

Consider the sets :

$S_1 = \{ v_1 \}$ $S_2 = \{ v_1, v_2 \}$

$\vdots$

$S_i = \{ v_1, v_2 \ldots v_i \}$

$\vdots$

$S_n = \{ v_1, v_2, \ldots v_n \}$

clearly $S_1 = \{ v_1 \}$ is L.I

[$\because \alpha v_1 = 0$ if $\alpha \neq 0$

$\Rightarrow \alpha^{-1} (\alpha v) = 0$

$\Rightarrow v = 0$ which is not true

$\therefore \alpha = 0$]

& by assumption S_n is L.D.

so we go down the list & choose the first L.D set say S_k.

Then S_k is L.D implies S_{k-1} is L.I

Here $2 \leq k \leq n$.

$\because$ S_K is L.D, we have

$$\sum_{i=1}^{K} \alpha_i v_i = 0$$, with atleast one

$\alpha_i \neq 0$

This α_i must be α_K.

otherwise,

if $\alpha_K = 0$ from ⊛

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_{K-1} v_{K-1} + 0 v_K = 0$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_K v_{K-1} = 0$$

$$\Rightarrow \alpha_1 = 0 \quad \alpha_2 = 0 \quad \cdots \quad \alpha_K = 0$$

($\because S_{K-1}$ is L.I)

$$\Rightarrow \alpha_1 = 0 \quad \alpha_2 = 0 \quad \cdots \quad \alpha_{K-1} = 0 \quad \alpha_K = 0$$

$\Rightarrow S_K$ is L.I

contradiction to S_K is L.D

$\therefore \alpha_K \neq 0$

$$\therefore \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_{K-1} v_{K-1} + \alpha_K v_K = 0$$

$$\Rightarrow v_K = \left(-\frac{\alpha_1}{\alpha_K}\right) v_1 + \left(\frac{-\alpha_2}{\alpha_K}\right) v_2 + \cdots + \left(\frac{-\alpha_{K-1}}{\alpha_K}\right) v_{K-1}$$

$$\Rightarrow v_K \in [v_1, v_2 \cdots v_{K-1}] \quad 2 \leq K \leq n$$