Lecture-7 (29-01-2024)

Lemma:- Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively on a nonempty set X. Then the following are equivalent:

(i) $\mathcal{T}'$ is finer than $\mathcal{T}$.

(ii) For each $x \in X$ and each $B \in \mathcal{B}$ containing x, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

$[x \in B \in \mathcal{B} \Rightarrow \exists\, B' \in \mathcal{B}' \ni x \in B' \subset B]$.

Proof: (ii) $\Longrightarrow$ (i)

Let $G \in \mathcal{T}$, we prove $G \in \mathcal{T}'$, so that $\mathcal{T}'$ becomes finer than $\mathcal{T}$.

Let $x \in G$, then there exists $B \in \mathcal{B}$ such that

$$x \in B \subset G \longrightarrow (1)$$

Then by (ii), since $x \in B$, $\exists\, B' \in \mathcal{B}'$

$\ni \quad x \in B' \subset B$ —— (2)

By combining (1) & (2) we have

for each $x \in G$ $\exists$ $B' \in \mathcal{B}'$ $\ni$

$x \in B' \subset B \subset G$

$\Rightarrow x \in B' \subset G$

$\Rightarrow$ Since x is arbitary element of G, it follows that G is union of members of $\mathcal{B}'$.

$\therefore G \in \mathcal{T}'$.

$\therefore \mathcal{T} \subset \mathcal{T}'$.

$\Rightarrow \mathcal{T}'$ is finer than $\mathcal{T}$.

Conversely assume (i) holds

i.e., $\mathcal{T} \subset \mathcal{T}'$,

we prove (ii) of the theorem statement.

let $x \in X$ and $B \in \mathcal{B}$ $\ni$ $x \in B$.

Now $B \in \mathcal{B} \subset \mathcal{T}$, but $\mathcal{T} \subset \mathcal{T}'$

$\Rightarrow B \in \mathcal{T}'$

Thus B is an open set w.r.t $\mathcal{T}'$ containing $x \in X$. But $\mathcal{B}'$ is a base for $\mathcal{T}'$ on X.

$\therefore$ By definition of a base, there exists $B' \in \mathcal{B}'$ $\ni$

$$x \in B' \subset B$$

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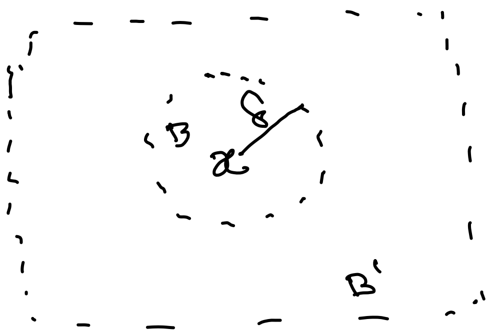
Ex: let $X = \mathbb{R}^2$.

$\mathcal{B}$: Collection of all circular regions in $\mathbb{R}^2$.

$\mathcal{B}'$: Collection of all rectangular regions in $\mathbb{R}^2$.

Let T and T' be the topologies generated by $\mathcal{B}$ and $\mathcal{B}'$, respectively.



$\mathcal{B}'$

$\mathcal{B}$

$\therefore$ The topologies generated by $\mathcal{B}$ and $\mathcal{B}'$ are same on $\mathbb{R}^2$.

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Def: Let $\mathcal{A} = \{ (a,b), (a,b) - K \}$, where $a, b \in R$, $a < b$, $K = \{ \frac{1}{n} \mid n \in N \}$. Let R_K be the topology generated by $\mathcal{A}$ on $\mathbb{R}$. This topology is called K-topology on $\mathbb{R}$.

Lemma : The topology of $\mathbb{R}_l$, the lower limit topology and $\mathbb{R}_K$, the K-topology are strictly finer than the usual (standard) topology on $\mathbb{R}$, but not comparable with one another.

Proof :

Let U be the usual topology on $\mathbb{R}$

$\mathbb{R}_l$: T' " topology on $\mathbb{R}$

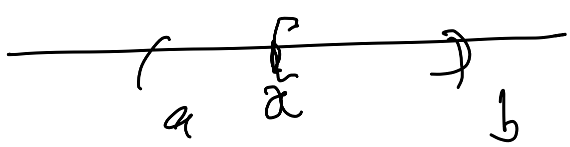
$\mathbb{R}_K$: T'' " K-topology on $\mathbb{R}$.

Let (a, b) be any basis element of $(\mathbb{R}, U)$

and $x \in (a, b)$.

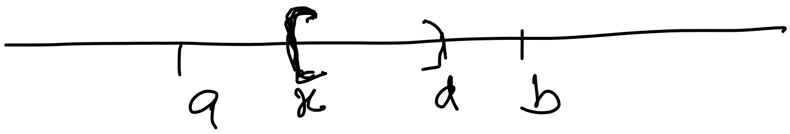
Then $[x, b)$ is a basis element of $\mathbb{R}_l$.

and $x \in [x, b) \subset (a, b)$



$\Rightarrow$ $\mathbb{R}_l$ is finer than usual topology on $\mathbb{R}$.

On the other hand, given any basis element $[x, d)$ of $\mathcal{T}'$, there is no open interval (a, b) that contains x and lies in $[x, d)$.



$x \in (a, b) \not\subset [x, d)$

$\therefore$ $\mathcal{T}'$ is strictly finer than $\mathcal{U}$.

$[\because \mathcal{U} \subsetneq \mathcal{T}']$

Ily Given any basis element $(a, b]$ in $\mathcal{U}$ with $x \in (a, b]$, the same interval $(a, b]$ is also a basis element in the K-topology $\mathcal{T}''$.

$\therefore$ $x \in (a, b] \subset (a, b)$

$\therefore$ $\mathcal{U} \subset \mathcal{T}''$.

On the other hand, for the basis element $B'' = (a, b) - K$ of $\mathcal{T}''$ with $0 \in B''$, there is

no open interval (a,b) with
$0 \in (a,b) \subset (a,b) - K$.

$\therefore \quad U \not\subseteq T''$.

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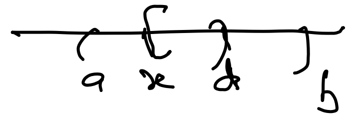
Attendance: 55, 44, 06, 27, 32, 40, 17, 09, 57, 38, 60, 26, 10, 65, 11, 63, 19, 62, 61].

But R_l and R_K are not comparable.

Since given any basis element $[x, d)$ in T', there is no open interval either (a,b) or $(a,b) - K$ in T'' such that

$$x \in (a,b) \subset [x, d)$$



or

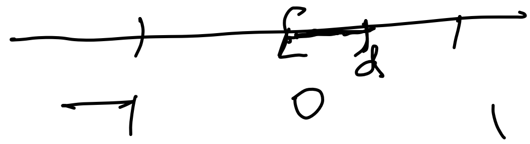
$$x \in (a,b) - K \subset [x, d).$$

OR

Given any basis element $B'' = (-1, 1) - K$ for T'' with $0 \in B''$, there is no basis element of the form $[0, d)$ containing '0'.

and

$$0 \in [0, d) \subset (-1, 1) - K,$$



$\therefore$ T' and T'' are not comparable.

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Subbase:

Let (X, T) be a topological space. A class S of open subsets of X, i.e., $S \subset T$, is said to be a subbase for the topology T on X if finite intersections of members of S form a base for the topology T.

$S = \{ s_1, s_2, s_3 \ldots \}$

$\mathcal{B} = \{ B \mid B = \bigcap_i s_{ik}, \; s_{ik} \in S \}$

if $\mathcal{B}$ is a base for T

then subbase.

Ex: (R, U) be usual topological space

$\because (a,b) = (-\infty, b) \cap (a, \infty) \longrightarrow (*)$

$\forall \, a, b \in R$

$a < b$.

$\because \{(a,b) \,/\, a, b \in R, \; a < b\}$ form a base for the usual topology U on $\mathbb{R}$,

by $(*)$, it follows that

$S = \{ (a, \infty), (-\infty, b) \,/\, a, b \in \mathbb{R} \}$

form a subbase for U on $\mathbb{R}$.

Ex: $(\mathbb{R}^2, U)$

The intersection of a verticle and horizantal infinite open strips in a plane $\mathbb{R}^2$ is an open rectangle and open rectangles form a base for the usual topology U on $\mathbb{R}^2$. Hence the class S of all infinite open strips form a subbase for U on $\mathbb{R}^2$.

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Theorem: Let X be a non-empty set and S be a collection of subsets of X such that $X = \bigcup\{s \mid s \in S\}$.

Then there is a unique topology T on X such that S is a subbase for T on X.

Proof: let

$\mathcal{B} = \{ B \in P(X) \mid$ B is the intersection of finite number of members of S $\}$,

clearly $S \subset \mathcal{B}$.

and let

$T = \{ U \in P(X) \mid U = \varphi$, or there is a subcollection $\mathcal{B}'$ of $\mathcal{B}$ such that $U = \cup \{ B \mid B \in \mathcal{B}' \} \}$

clearly $S \subset T$.

Claim: T is the unique topology on X for which S is a sub-base.

Clearly $\varphi \in T$.

Also $X = \cup \{ s \mid s \in S \}$,

and $S \subset \mathcal{B} \subset T$

$\Rightarrow X \in T$.

Next let $\{ U_\alpha \}$ be an arbitary collection of sets in T.

Now for each α, there is a subcollection $\mathcal{B}_\alpha$ of $\mathcal{B}$ such that

$$U_\alpha = \bigcup \{ B_\alpha \mid B_\alpha \in \mathcal{B}_\alpha \}.$$

$$\therefore \bigcup_\alpha U_\alpha = \bigcup_\alpha \left\{ \bigcup \{ B_\alpha \mid B_\alpha \in \mathcal{B}_\alpha \} \right\}$$

$$\in \mathcal{T}$$ (by definition of $\mathcal{T}$)

Now let $U_1, U_2 \in \mathcal{T}$,

Then $\exists$ subclasses $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathcal{B}$ such that

$$U_1 = \bigcup_i \{ B_{i_1} \mid B_{i_1} \in \mathcal{B}_1 \}$$

$$U_2 = \bigcup_j \{ B_{j_2} \mid B_{j_2} \in \mathcal{B}_2 \}.$$

Then

$$U_1 \cap U_2 = \bigcup_i \{ B_{i_1} \mid B_{i_1} \in \mathcal{B}_1 \} \cap \bigcup_j \{ B_{j_2} \mid B_{j_2} \in \mathcal{B}_2 \}$$

$$= \bigcup_{i,j} \{ B_{i_1} \cap B_{j_2} \mid B_{i_1} \in \mathcal{B}_1, B_{j_2} \in \mathcal{B}_2 \}$$

$\longrightarrow (*)$

$\because$ B_{i_1} and B_{j_2} are intersection of finite number of members of S, it follows that $B_{i_1} \cap B_{j_2}$ is also intersection of finite number of members of S.

$\therefore$ By $\circledast$ it follows that

$$U_1 \cap U_2 \in T$$

$\therefore$ T is a topology on X, for which S is subbase.

Claim: T is unique.

Suppose T^* is another topology on X for which S is a subbase.

let $G \in T \Rightarrow G = \bigcup_i (S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k})$

$S_{ij} \in S$.

$\because$ S is also a sub-base for T^*,

$S \subset T^*$

$\Rightarrow s_{ij} \in T^* \ \forall \ i,j$

$\Rightarrow \bigcup_i (s_{i_1} \cap s_{i_2} \cdots \cap s_{i_k}) \in T^*$

$\Rightarrow G \in T^*$ [$\because T^*$ is a topology on X]

$\therefore T \subseteq T^*$

By interchanging the roles of T and T^* in the above we can show that

$$T^* \subseteq T$$

$$\therefore T = T^*$$

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[5PM Attendance
11, 65, 17, 40, 27, 06, 57, 09, 60]