# Tutorial 3 & 4 Discussion

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# Tut-3

**Q2** Suppose $S_3 = H \times K$.

$|S_3| = 6$ then $|H| = 2$ or $3$

and $|K| = 3$ or $2$.

$H \cong \mathbb{Z}/2\mathbb{Z}$, $K \cong \mathbb{Z}/3\mathbb{Z}$.

Note $H$ & $K$ both are abelian.

$H \times K$ is also abelian.

But $S_3$ is not abelian.

$\therefore$ $S_3$ can not be written as $H \times K$.

**Q3** $\mathbb{Z} \times \mathbb{Z} = \langle 1, 1 \rangle$.

$G_1 \times G_2$

$m, n$

$g_1^{\alpha}, g_2^{\beta}$ $G_1 = \langle g_1 \rangle$, $G_2 = \langle g_2 \rangle$

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$$|HK| = \frac{|H|.|K|}{|H\cap K|} = ab.$$

$HK \subseteq G.$ $\therefore$ $\underline{\underline{G = HK}}.$

$$\boxed{\text{NOT} \quad \underline{\underline{G \cong H\times K.}}}$$

$G = S_3$, $H = \langle (12) \rangle$, $K = \langle (123) \rangle$.

$H\cap K = \{1\}$, $G = HK$.

But $G \not\cong H\times K$.

**97.** $\phi : GL_n(\mathbb{R}) \longrightarrow \{1,-1\}$.

$$\phi(A) = \frac{\det A}{|\det A|}$$

$\ker \phi = H.$

By 1st isomorphism Thm, $GL_n(\mathbb{R})/H \cong \mathbb{Z}_2.$

Q8. $N = \langle x^{-1}y^{-1}xy \mid x, y \in G \rangle$.

$G/N$.

WTS. $gNg^{-1} \in N$. $\forall g \in G$.

$n \in N$.

$$gng^{-1} = \underbrace{(gng^{-1}n^{-1})}_{\in N} n \in N.$$

Let $a, b \in G$. WTS $G/N$ is abelian.

$$aN . bN = abN$$

WTS $abN = baN$. ✓

$(ba)^{-1}ab \in N$.

$a^{-1}b^{-1}ab \in N$.

Q9 $\quad G/_{M \cap N} \cong G/M \times G/N.$

$$\phi : G \longrightarrow G/M \times G/N.$$
$$g \longmapsto (gM, gN).$$

WTS. $\phi$ is surj gp hom.

$$\ker \phi = M \cap N.$$

$\phi$ is surjective is the difficult part.

Q12 $\quad (\mathbb{R}^{\times}_{>0}, \cdot) \times (\mathbb{R}/\mathbb{Z}, +) \longrightarrow (\mathbb{C}^{\times}, \cdot)$

$$(r, \theta + \mathbb{Z}) \longrightarrow r e^{2\pi i \theta}$$

$0 \leq \theta < 1.$

$2.5 \in \mathbb{R}.$

$\boxed{.5 + \mathbb{Z}.}$

$e^{2\pi i \theta} = e^{2\pi i (\theta + n)}$

when $n \in \mathbb{Z}.$

## Class equation of $S_4$:

$$\tau\,(a_1\ a_2\ \ a_k)\,\tau^{-1} = (\tau(a_1)\cdot\tau(a_2)\cdots\tau(a_k)).$$

**Defn.** If $\sigma \in S_n$ is the product of disjoint cycles of lengths $n_1, n_2, \dots, n_p$ (including 1 cycles) with $n_1 \le n_2 \le \cdots \le n_p$ then the integers $n_1, n_2, \dots, n_p$ is called the cycle type of $\sigma$.

$S_7 \ni (1\ 2)(3\ 4)(5\ 6\ 7) \rightsquigarrow 2, 2, 3.$ cycle type.

**Thm.** For every $\sigma \in S_n$ the conjugacy class $\mathcal{C}$ consists of all elts in $S_n$ which have the same cycle type as $\sigma$.

**Thm.** Two elts of $S_n$ are conjugates in $S_n$ iff they have the same cycle type. The number of conjugacy class of $S_n$ equals the number of partitions of $n$.

**Example** Describe conjugacy classes of $S_4$:

| cycle type | an elt of that cycle type | The conjugates | $\lvert C \rvert$ |
|---|---|---|---|
| $1+1+1+1$ | $(1)$ | $(1)$ | 1. |
| $1+1+2$ | $(12)$ | $(12), (13), (14), (23, (24), (34)$ | 6 |
| $1+3$ | $(123)$ | $(123), (124), (234), (134), (132), (142), (143), (243)$ | 8 |

| | | | |
|---|---|---|---|
| 2+2 | (12)(34) | (12)(34), (13)(24), (14)(23) | 3 |
| 4 | (1234) | (1234), (1243), (1324), (1342), (1423), (1432). | 6. |

$\therefore \quad |S_4| = 1 + 6 + 8 + 3 + 6.$

__Propn__. Let $G$ be a finite gp with $|G| > 1$ and $p$ be the smallest prime factor of $|G|$. Any subgp of $G$ with index $p$ is normal subgp.

__Pf__: Let $H$ be a subgp with index $p$.

$$G \times G/H \longrightarrow G/H.$$

$G/H$: Set of all left cosets of $H$.

$$(g, hH) \longrightarrow ghH.$$

check that it is a gp action

$$\left| G/H \right| = p.$$

Let $\phi : G \longrightarrow S_p$. $\quad \sigma_g : G/H \to G/H$

$$\phi(g) = \sigma_g. \qquad \sigma_g(hH) = ghH.$$

$\phi$ is a gp homo.

<u>WTS</u>. $\ker \phi = H$.

$\ker \phi \subseteq H$.

Let $g \in \ker \phi \Rightarrow \phi(g) = Id.$

$$\sigma_g = Id.$$

Note that $\qquad \sigma_g(hH) = hH$

$\ker \phi \subseteq H. \qquad \Rightarrow ghH = hH.$

$$G/\ker\phi \cong \operatorname{Im}\phi = \text{Subgp of } S_p.$$

$$\begin{array}{c} G \\ | \\ H \\ | \\ \ker\phi \end{array}$$

$$\therefore\ [G:\ker\phi] \mid p! \quad \text{—— (1)}$$

$$[G:\ker\phi] = [G:H]\,[H:\ker\phi]$$

$$= p\,[H:\ker\phi] \quad \text{—— (2)}$$

Combining (1) & (2) we get

$$[H:\ker\phi] \mid (p-1)!$$

$$\because\ [H:\ker\phi] \mid |G|.$$

But the smallest prime divisor of $|G|$ is $p$. Hence $[H:\ker\phi] = 1$.

$$\Rightarrow\ H = \ker\phi.$$

Defn. A gp $G$ is said to be simple if the only normal subgp of $G$ are the trivial gp and the whole gp.

Q2. Any gp of order $p^n$ where $n > 1$. is not simple.

$|Z(G)| \geq p$, $Z(G) \trianglelefteq G$.

If $G$ is abelian then any subgp of $G$ is a normal subgp.

If $G$ is not abelian then $Z(G) \neq G$.

$Z(G)$ is a proper normal subgp. of $G$. $\therefore$ $G$ is not simple.

# Tut 4.

**Q4.**

Let be a gp with three conjugacy classes:

$$|G| = 1 + x + y. \text{ where } 1 \le x \le y \quad (1)$$

$$\Rightarrow x \mid |G| \ \& \ y \mid |G|.$$

$$|G| - x = 1 + y.$$

Since $x \mid |G|$ $\therefore$ $x \mid (1+y)$.

Similary $y \mid (1+x)$.

$$\Rightarrow x \le 1+y \le 2+x.$$

$$y \le x+1. \quad (2)$$

From (1) & (2) we have

$$\therefore x \le y \le 1+x.$$

So either $y = x$ or $y = x+1$.

<u>case 1</u>. $y = x$ then $|G| = 1 + 2x$.

$|G| - 2x = 1$.

$\because$ $x \,\big|\, |G|$ $\Rightarrow$ $x \,\big|\, 1$.

$\therefore$ $x = 1$ $\therefore$ $x = y = 1$.

and $|G| = 3$.

$\therefore$ $G \cong \mathbb{Z}/3\mathbb{Z}$.

<u>case 2</u> $y = x + 1$. then $|G| = 2x + 2$.

$\because$ $x \,\big|\, |G|$ $\Rightarrow$ $x \,\big|\, 2$.

$\therefore$ $x = 1$ and $y = 2$.

$|G| = 4$.

But every gp of order 4 is abelian and so has 4 conjugacy class.

$\therefore$ $|G| = 4$ is not possible.

If $x = 2$ then $y = 3$

$\Rightarrow |G| = 6$

The only nonabelian gp of order 6 is $S_3$ which has exactly 3 conjugacy classes.

$$G \cong S_3.$$

(EX If $G$ be a gp with 2 conjugacy classes then $G \cong \mathbb{Z}/2\mathbb{Z}$.)

$$\underline{\underline{20 = 5 \cdot 2^2.}}$$

$n_5 \mid 4$ & $n_5 \equiv 1 \pmod 5$.

$$\underline{n_5 = 1}$$

§11

$30 = 5 \cdot 2 \cdot 3.$

$n_5 = 1, 6$

$n_2 = 1, 3, 5, 15$

$n_3 = 1, 10.$

If $n_5 = 6$,

$n_3 = 10.$

$4 \times 6$ – many of elt of order 5 in all Sylow 5 subgp.

$2 \times 10$ – elt of order 3 in all Sylow 3-subgp.

$4 \times 6 + 2 \times 10$

$= 44$ elts. $> 30 = |G|$

Therefore either $n_5 = 1$ or $n_3 = 1$.

Here there exist one Sylow subgp of order 5 or order 3.

Thus it will be normal subgp.