Group Theory

Lecture 8

24/01/2022

Linear Algebra:

$V = \mathbb{R}^2$.

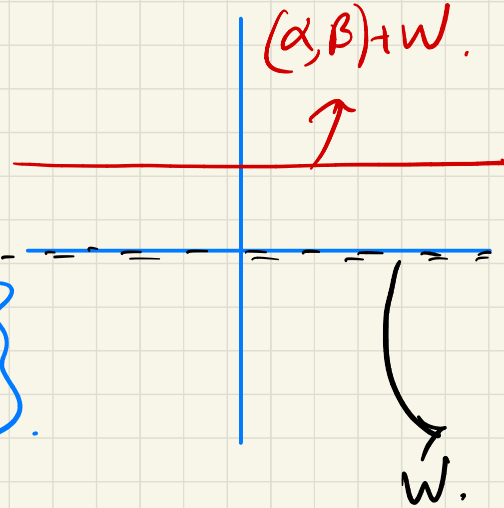
$W = \{(x, 0) \mid x \in \mathbb{R}\}$ is a subsp of V.

$(\alpha_1, \beta_1) \sim (\alpha_2, \beta_2)$ iff $(\alpha_1 - \alpha_2, \beta_1 - \beta_2) \in W$.

Say $(\alpha, \beta) \in \mathbb{R}^2$

$(\alpha, \beta) + W$

$= \{(\alpha + x, \beta) \mid x \in \mathbb{R}\}$.



$(\alpha, \beta) = (1, 2)$.

$(1,2) + W = (3, 2) + W$.

$\mathbb{R}^2 = \bigsqcup \{(\alpha, \beta) + W \mid (\alpha, \beta) \in \mathbb{R}^2\}$.

$V/W = \{(\alpha, \beta) + W \mid (\alpha, \beta) \in \mathbb{R}^2\} \cong \mathbb{R}$.

↳ This has a vector sp structure quotient space.

Construction of rationals from integers

$$\underline{\mathbb{Z}} \times \mathbb{Z}\setminus\{0\} = \left\{ (a,b) \;\middle|\; \begin{array}{l} a \in \mathbb{Z} \\ b \in \mathbb{Z}\setminus\{0\} \end{array} \right\}$$

$(a,b) \sim (c,d)$ if $ad - bc = 0$.

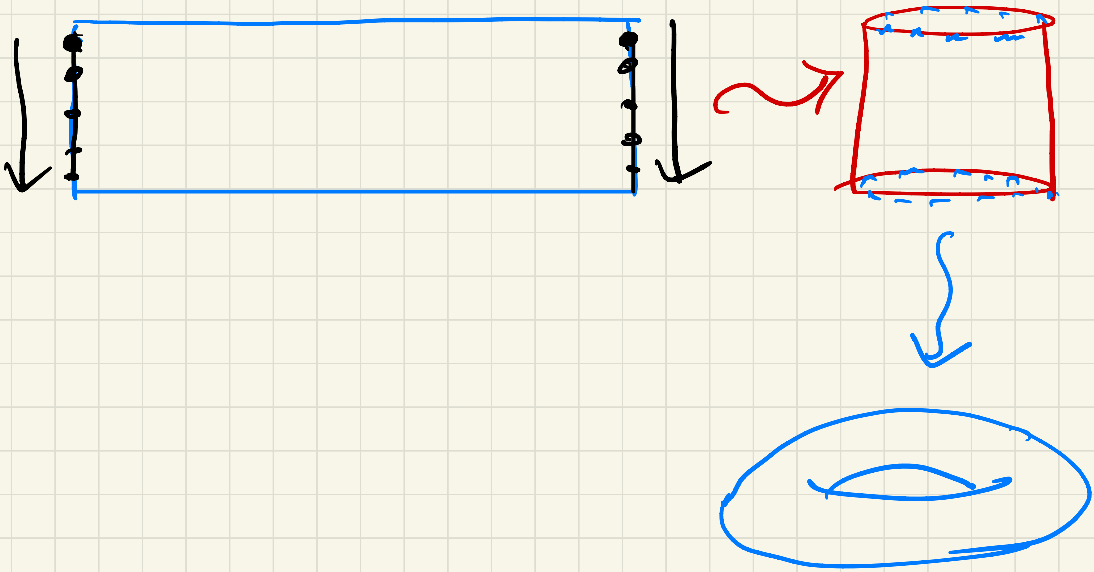
$\mathbb{Q}$ = Equivalance class of $\sim$.

The equivalance class of (a,b) is written as $\frac{a}{b}$.

$$\frac{a}{b} = \frac{c}{d} \quad \text{iff} \quad ad - bc = 0.$$

$$\frac{1}{2} = \frac{2}{4}.$$

$$\underline{\underline{(1,2)}} \sim (2,4)$$



Möbius strip.

Quotient Group:

Q. Let G be a gp and H is a subgp of G. Consider the set $G/H = \{aH \mid a \in G\}$ is the set of all left cosets of H. Does G/H has a gp structure?

Let $a_1 H$ & $a_2 H$ are two left cosets.

$$a_1 H \cdot a_2 H = a_1 a_2 H.$$

$$\underset{a_1 h_1}{\Downarrow} \cdot \underset{a_2 h_2}{\Downarrow} \overset{?}{=} a_1 a_2 h. \in a_1 a_2 H.$$

If G is abelian then

$$a_1 h_1 \cdot a_2 h_2 = a_1 a_2 h_1 h_2 \in a_1 a_2 H.$$

If H is a normal subgp

$$a_1 h_1 a_2 h_2 = a_1 a_2 a_2^{-1} h_1 a_2 h_2.$$

$$= a_1 a_2 h' h_2$$

Since H is a normal subgp

$$a_2^{-1} h_1 a_2 = h' \in H.$$

Lemma. Let $H \trianglelefteq G$. Then the product of two left cosets is again a left coset i.e $aH \cdot bH = abH$.

Pf: We have seen earlier that

$$aH \cdot bH = abH.$$

Now we show that it is well defined.

Let $aH = a'H$ and $bH = b'H$.

WTS $abH = a'b'H$

ETS, $(a'b')^{-1}ab \in H$.

$\rightarrow a'^{-1}a \in H$ say $a'^{-1}a = h_1$ for some $h_1 \in H$

$b'^{-1}b \in H$ say $b'^{-1}b = h_2$ for some $h_2 \in H$.

$(a'b'^{-1})ab = b'^{-1}a'^{-1}ab = b'^{-1}h_1 b$

$$b'^{-1} b = h_2.$$

$$= b'^{-1} h_1 b.$$

$$= \underbrace{b'^{-1} h_1 b'}\, \underbrace{b'^{-1} b}$$

$$= h_3\, h_2 \in H.$$