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Assignment 3 MM - Part 2
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1) $J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2r} \Rightarrow J_0\left[\sqrt{x(t-x)}\right] = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \cdot \frac{(x(t-x))^r}{2^{2r}}$

$$\int_0^t J_0\left(\sqrt{x(t-x)}\right)dx = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2 \cdot 2^{2r}} \int_0^t x^r (t-x)^r dx = I$$

let $x = tz$, $\frac{dx}{dz} = t$, when $x=0 \rightarrow z=0$, $x=t \rightarrow z=1$

$$I = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2 \cdot 2^{2r}} \underbrace{\int_0^1 t^r . z^r . t^r (1-z)^r t \, dz}_{t^{2r+1} \int_0^1 z^r (1-z)^r dz}$$

$$I = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2 \cdot 2^{2r}} t^{2r+1} . \underbrace{\beta(r+1, r+1)}_{\frac{\Gamma(r+1)\Gamma(r+1)}{\Gamma(2r+2)}}$$

$$I = \sum_{r=0}^{\infty} \frac{(-1)^r}{\cancel{(r!)^2} \cdot 2^{2r}} . 2 t^{2r+1} . \frac{\cancel{r! . r!}}{(2r+1)!} = 2 \sum_{n=0}^{\infty} \frac{(-1)^r \left(\frac{t}{2}\right)^{2r+1}}{(2r+1)!}$$

$\therefore I = 2\sin(t/2)$

$\therefore \int_0^t J_0\left(\sqrt{x(t-x)}\right)dx = 2\sin\left(\frac{t}{2}\right)$, hence proved

2) $e^{x/2(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n \Rightarrow e^{\left(\frac{x+y}{2}\right)\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x+y) t^n$

now, $e^{\left(\frac{x+y}{2}\right)\left(t-\frac{1}{t}\right)} = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} . e^{\frac{y}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n . \sum_{m=-\infty}^{\infty} J_m(y) t^m$

$$= \sum_{n,s=-\infty}^{\infty} J_n(x) J_s(y) t^{n+s}$$

$\therefore$ ~~for a fixed r,~~ coefficient of t^n is $\sum_{r=-\infty}^{\infty} J_r(x) J_{n-r}(y)$

$$\therefore J_n(x+y) = \sum_{r=-\infty}^{\infty} J_r(x) J_{n-r}(y)$$

hence proved

3) ${}_2F_1(a-1, b-1; c; x) - {}_2F_1(a, b-1; c; x)$

$$= \sum_{r=0}^{\infty}\left(\frac{(a-1)_r \times (b-1)_r}{(c)_r} \times \frac{x^r}{r!}\right) - \left(\frac{(a)_r \cdot (b-1)_r}{(c)_r} \cdot \frac{x^r}{r!}\right)$$

$$= \sum_{r=1}^{\infty}\left[\frac{(a-1)_r - (a)_r}{(c)_r}\right] \cdot (b-1)_r \cdot \frac{x^r}{r!} = I$$

as $(a-1)_0 - (a)_0 = 1 - 1 = 0$

Now, we know, $(\alpha)_{n+1} = \alpha(\alpha+1)_n$

$\therefore (a-1)_r = (a-1)(a)_{r-1}$

Also, $(a)_r = (a+r-1)(a)_{r-1}$

$\therefore (a-1)_r - (a)_r = (a-1-a-r+1)\,a_{r-1} = (a-1)_r - (a)_r$

$= -r \times (a)_{r-1}$

$$\therefore I = \sum_{r=1}^{\infty} \frac{(-r)\cdot(a)_{r-1}\cdot(b-1)\cdot(b)_{r-1}}{c\cdot(c+1)_{r-1}} \cdot \frac{x^r}{r!}$$

$$= \frac{(1-b)(x)}{c}\sum_{r=1}^{\infty} \frac{(a)_{r-1}(b)_{r-1}}{(c+1)_{r-1}} \cdot \frac{x^{r-1}}{(r-1)!}$$

put $r = m+1$

$$I = \frac{(1-b)(x)}{c}\sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c+1)_m} \frac{x^m}{m!} = \frac{x}{c}(1-b)\,{}_2F_1(a, b; c+1; x)$$

$\therefore {}_2F_1(a-1, b-1; c; x) - {}_2F_1(a, b-1; c; x) = \frac{x}{c}(1-b)\,{}_2F_1(a, b; c+1; x)$

hence proved.

4) i) $x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0$

$\gamma = \frac{3}{2}$, $\alpha+\beta = 1$, $\alpha\beta = -2 \Rightarrow \gamma = \frac{3}{2}$, $\alpha = 2$, $\beta = -1$

solution of above eq$^{\underline{n}}$ is

$y = A\,{}_2F_1(2, -1; 3/2; x) + B\,x^{-1/2}\,{}_2F_1(3/2, -3/2; 1/2; x)$

now, ${}_2F_1(2, -1; 3/2; x) = 1 + \frac{(2)(-1)}{3/2}\frac{x}{1!} + 0 \ldots = 1 - (4/3)x$

$\therefore y = A\left(1 - \frac{4}{3}x\right) + \frac{B}{\sqrt{x}}\,{}_2F_1\left(\frac{3}{2}, -\frac{3}{2}; \frac{1}{2}; x\right)$ where

A & B are arbitrary constants

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4ii) $x(x-1)y'' + \left(\frac{3}{2} - 2x\right)y' - \left(\frac{y}{4}\right) = 0$

$\gamma = \frac{3}{2}, \ \alpha+\beta = 1, \ \alpha\beta = 1/4 \Rightarrow \gamma = 3/2, \alpha = 1/2, \beta = 1/2$

Solution of the above eq$^{\underline{n}}$ is :-

$y = A \ {}_2F_1\left(\frac{1}{2}; \frac{1}{2}; 3/2; x\right) + B x^{-1/2} \ {}_2F_1\left(\cancel{0}, 0; 1/2; x\right)$ 0

$y = A \ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x\right) + \frac{B}{\sqrt{x}}$ where A & B are arbitrary constants

5) Legendre equation: $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$

let $x^2 = t \Rightarrow \frac{dt}{dx} = 2x, \ \frac{dy}{dx} = 2x\frac{dy}{dt}, \ \frac{d^2y}{dx^2} = 2\frac{dy}{dt} + 4x^2\frac{d^2y}{dt^2}$

∴ Legendre equation becomes

$(1-t)\left[2\frac{dy}{dt} + 4t\frac{d^2y}{dt^2}\right] - 4t\frac{dy}{dt} + n(n+1)y = 0$

$\Rightarrow t(1-t)\frac{d^2y}{dt^2} + \left(\frac{1}{2} - \frac{3}{2}t\right)\frac{dy}{dt} + \frac{(n+1)n}{4}y = 0$

$\gamma = 1/2, \ \alpha+\beta = \frac{1}{2}, \ \alpha\beta = \frac{(-n)(n+1)}{4} \Rightarrow \gamma = 1/2, \alpha = \frac{n+1}{2}, \beta = \frac{-n}{2}$

Solution of Legendre's equation is given by

$y = A \ {}_2F_1\left(\frac{n+1}{2}, \frac{-n}{2}; \frac{1}{2}; t\right) + B t^{1/2} \ {}_2F_1\left(\frac{n}{2}+1, \frac{1-n}{2}; \frac{3}{2}; t\right)$

$y = A \ {}_2F_1\left(\frac{n+1}{2}, \frac{-n}{2}; \frac{1}{2}; x^2\right) + Bx \ {}_2F_1\left(\frac{n}{2}+1, \frac{1-n}{2}; \frac{3}{2}; x^2\right)$

↳ Solution of Legendre's equation in terms of hypergeometric series