Lecture 31

Problems

① Let $f_n(x) = \dfrac{nx}{1+n^2x^2}$ $\forall\, x \in [0,1]$, $\forall\, n \geq 1$.

Then show that $\{f_n\}$ is uniformly bounded on $[a,b]$

& evaluate $\displaystyle \lim_{n\to\infty} \int_{[0,1]} \frac{nx}{1+n^2x^2}\,dx$.

Also show that $\{f_n\}$ does not converge uniformly on $[0,1]$.

Sol:- We have $(1-nx)^2 \geq 0$ $\forall\, n$, $\forall\, x \in [0,1]$.

$$\Rightarrow \quad 1+n^2x^2-2nx \geq 0$$

$$\Rightarrow \quad 1+n^2x^2 \geq 2nx \geq 0$$

$$\Rightarrow \quad 0 \leq f_n(x) = \frac{nx}{1+n^2x^2} \leq \frac{1}{2} \quad \forall\, n \geq 1,\ \forall\, x \in [0,1].$$

$\therefore$ $\{f_n\}$ is uniformly bounded.

Each $f_n(x) = \dfrac{nx}{1+n^2x^2}$ is Continuous on $[0,1]$.

$\Rightarrow$ each is f_n is Riemann-integrable & hence Lebesgue integrable.

$$\therefore \overset{L}{\int_{[0,1]}} f_n(x)\,dx = \overset{R}{\int_{[0,1]}} f_n(x)\,dx$$

$$= \int_0^1 \frac{nx}{1+n^2x^2}\,dx$$

Set
$t = 1+n^2x^2$
$dt = 2n^2x\,dx$

$$= \int_{t=1}^{1+n^2} \frac{dt}{2n(t)}.$$

$$= \frac{1}{2n} \ln(t)\Big|_{t=1}^{1+n^2}$$

$$= \frac{1}{2n}\left(\ln(1+n^2)\right)$$

$$\therefore \lim_{n\to\infty} \int_{[0,1]} f_n(x)\,dx = \lim_{n\to\infty} \frac{\ln(1+n^2)}{2n}$$

$$= \lim_{n\to\infty} \frac{1}{1+n^2} \frac{2n}{2}$$

$$= 0$$

$$\therefore \lim_{n\to\infty} \int_{[0,1]} \frac{nx}{1+n^2x^2}\, dx = 0.$$

Also $\displaystyle\lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0 \quad \forall x \in [0,1].$

$\therefore$ $f_n \to 0 = f$ p.w on $[0,1]$.

chose $x_n = \frac{1}{n}$ $\forall n \geq 1$ $\{x_n\}$ is a seq. & $x_n \to 0$ as $n \to \infty$

Now $f_n(x_n) = f_n\left(\frac{1}{n}\right)$

$$= \frac{n\left(\frac{1}{n}\right)}{1+n^2\frac{1}{n^2}} = \frac{1}{2} \quad \forall n \geq 1$$

$\therefore$ $f_n(x_n) \to \frac{1}{2}$ as $n \to \infty$

$\therefore$ $\left.\begin{array}{l} f_n(x_n) \not\to f(0) = 0 \\ \text{where } x_n \to 0 \end{array}\right\}$ as $n \to \infty$.

$\therefore$ if $f_n \to f$ uniformly, then

$f_n(x_n) \to f_n(x)$ where

$x_n \to x$ as $n \to \infty$.

$\therefore$ $f_n \not\to 0$ uniformly on $[0,1]$.

---

② Let $\{f_n\}$ be a sequence of non-negative measurable functions defined on $\mathbb{R}$ such that $f_n \to f$ a.e, p.w on $\mathbb{R}$.

Suppose $\lim\limits_{n \to \infty} \int_{\mathbb{R}} f_n(x)\,dx = \int_{\mathbb{R}} f(x)\,dx < \infty$.

Then show that for each measurable set $E \subseteq \mathbb{R}$, we have $\lim\limits_{n \to \infty} \int_E f_n = \int_E f$.

<u>proof</u>:- Let $E \subseteq \mathbb{R}$ be a measurable set.

Let $g_n = f_n - f_n \chi_E \geq 0 \quad \forall n \geq 1$

Given that $f_n \to f$ a.e. (p.w) as $n \to \infty$.

Then $g_n \longrightarrow f - f\chi_E$ a.e. as $n \to \infty$.

$\therefore$ By Fatou's Lemma,

$$\int_{\mathbb{R}} \lim_{n\to\infty} g_n \leq \liminf_{n\to\infty} \left(\int_{\mathbb{R}} g_n \right)$$

$$\Rightarrow \int_{\mathbb{R}} (f - f\chi_E) \leq \liminf_{n\to\infty} \left(\int_{\mathbb{R}} (f_n - f_n \chi_E) \right)$$

$$\Rightarrow \int_{\mathbb{R}} f - \int f\chi_E \leq \liminf_{n\to\infty} \left(\int_{\mathbb{R}} f_n - \int_{\mathbb{R}} f_n \chi_E \right)$$

$$\Rightarrow \int_{\mathbb{R}} f - \int_E f \leq \liminf_{n\to\infty} \left(\int_{\mathbb{R}} f_n \right) - \limsup \left(\int_E f_n \right)$$

$$\| $$

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f \quad \text{(given)}$$

$$\Rightarrow \cancel{\int_{\mathbb{R}} f} - \int_E f \leq \cancel{\int_{\mathbb{R}} f} - \limsup_{n\to\infty}\left(\int_E f_n\right)$$

$$\Rightarrow \boxed{\int_E f \geq \limsup_{n\to\infty}\left(\int_E f_n\right)}$$

Let $h_n = f_n + f_n \chi_E \geq 0. \quad \forall\, n \geq 1$

By Fatou's lemma,

$$\int_{\mathbb{R}} \lim_{n\to\infty} h_n \leq \liminf_{n\to\infty}\left(\int_{\mathbb{R}} h_n\right)$$

$$\Rightarrow \int_{\mathbb{R}} \left(f + f\chi_E\right) \leq \liminf_{n\to\infty}\left[\int_{\mathbb{R}} \left(f_n + f_n \chi_E\right)\right]$$

$$\Rightarrow \int_{\mathbb{R}} f\chi_E \leq \liminf_{n\to\infty}\left(\int_{\mathbb{R}} f_n \chi_E\right)$$

$$\Rightarrow \int_E f \leq \liminf_{n\to\infty}\left(\int_E f_n\right) \leq \limsup_{n\to\infty}\left(\int_E f_n\right) \leq \int_E f$$

$$\Rightarrow \quad \lim_{n\to\infty} \int_E f_n = \int_E f.$$

---

③ Let f_n, f be a real valued measurable function defined on a measurable set $D \subseteq \mathbb{R}$; $\forall\, n \geq 1$.

Suppose $\lim_{n\to\infty} \left(\int_D |f_n - f|^p \right) = 0$ for some +ve real number p.

Show that $f_n \xrightarrow{m} f$ on D.

proof:- Let $\varepsilon > 0$. & $A_n = \{ x \in D \,/\, |f_n(x) - f(x)| \geq \varepsilon \}$ $\forall\, n \geq 1$

To show: $m(A_n) \longrightarrow 0$ as $n \to \infty$.

Now $$\int_D |f_n - f|^p = \int_{A_n} |f_n - f|^p + \underbrace{\int_{D \setminus A_n} |f_n - f|^p}$$

$$\geq \int_{A_n} |f_n - f|^p$$

$$\geq \int_{A_n} \varepsilon^p = \varepsilon^p\, m(A_n).$$

$$\Rightarrow \quad \int_D |f_n - f|^p \geq \varepsilon^p\, m(A_n), \quad \forall\, n \geq 1$$

$$\Rightarrow \quad \underset{\text{given}}{0} = \lim_{n\to\infty} \int_D |f_n - f|^p \geq \varepsilon^p \cdot \underbrace{\left(\lim_{n\to\infty} m(A_n)\right)}_{\geq 0}$$

$$\Rightarrow \quad \lim_{n\to\infty} m(A_n) = 0.$$

$\therefore \quad f_n \xrightarrow{m} f$ as $n \to \infty$.

---

(4) Let f_n, f be real valued measurable functions defined on $\underbrace{D \subseteq \mathbb{R}}_{\text{measurable set}}$.

Suppose there exists a sequence of +ve nos $\{\varepsilon_n\}$ such that

(i) $\sum_{n=1}^{\infty} \varepsilon_n < \infty$

(ii) $\int_D |f_n - f|^p < \varepsilon_n \quad \forall\, n \geq 1$, where $p \in (0, \infty)$.

Show that the sequence $\{f_n\}$ converges to f a.e on D.

<u>Sol:-</u> We have $|f_n - f|^p \geq 0$, measurable $\forall\, n \geq 1$

& $\left\{ \sum_{k=1}^{n} |f_k - f|^p \right\}_{n \geq 1}$ is an increasing sequence.

$\therefore$ By Monotone Convergence theorem,

$$\int_D \lim_{n\to\infty} \left(\sum_{k=1}^{n} |f_k - f|^p \right) = \lim_{n\to\infty} \int_D \left(\sum_{k=1}^{n} |f_k - f|^p \right)$$

$$\Rightarrow \underbrace{\int_D \left(\sum_{k=1}^{\infty} |f_k - f|^p \right)} = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \left(\int_D |f_k - f|^p \right) \right)$$

$$= \sum_{k=1}^{\infty} \left(\int_D |f_k - f|^p \right)$$

$$\leq \sum_{k=1}^{\infty} \varepsilon_k < \infty$$

$\Rightarrow \sum_{k=1}^{\infty} |f_k - f|^p$ is finite a.e on D.

$\Rightarrow \lim_{n\to\infty} |f_n - f|^p = 0$ a.e on D

$\Rightarrow \lim_{n\to\infty} |f_n - f| = 0$ a.e on D

$\Rightarrow f_n \to f$ a.e on D.
