TOPICS COVERED

1) Consistency & stability Analysis
2) 'DuFort Frankel Scheme'
3) Brief discussion on Non-linear Scheme

[last week - Crank Nicolson]

Let U be the exact solution of the numerical scheme

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solving linear parabolic PDEs.

Some imp. things to analyse while solving

(1) CONSISTENCY T.E. $\longrightarrow 0$ as $h \to 0 \Rightarrow$ consistent scheme

& (2) STABILITY OF NUMERICAL SCHEMES

(1) CONSISTENCY

Let $F_{i,j}(U) = 0$ be the FINITE DIFFERENCE EQUATION (FDE)

for the PDE $L(u) = 0$

for a given linear IBVP.

$U \to$ numerical solution ; $u \to$ exact solution of PDE.

★T.E. : the residue by which the exact solution of the PDE fails to satisy the FDE

$$T.E. = F_{ij}(u)$$

H.T. Find the TE, check for consistency of –

(i) Implicit Scheme

(ii) Crank Nicolson scheme.

↳ $(j, n+\frac{1}{2})$

THEOREM LAX EQUIVALENCE THEOREM

The consistency and stabity of the numerical scheme for an IBVP provides a convergent solution.

(2) STABILITY of the numerical scheme.

$$U_j^{n+1} \longrightarrow U_j^n \quad , \quad n \gg 1$$

$\bar{U}_j^n$ " " solution obtained.

$U_j^n = \bar{U}_j^n + \varepsilon_j^n$, ε_j^n is the error, mostly the round-off error.

Test: if a small amt. of error is introduced at some step, how does it add up along the steps.

Due to the linearity of the problem, ε_j^n also satisfies the FDE.

→ A discrete distribution of ε_j^n is considered.

(outside of the grid points, ε_j^n does not exist).

→ ε_j^n remains bounded within the spatial domain.

One of the methods is

VON NEUMANN STABILITY ANALYSIS

(fourier Series method)

What do we do?

↳ Approximate the error distribution by a finite FOURIER SERIES:

$$\varepsilon(x,t) = \sum_{m=0}^{M} \left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right)$$

L is the interval over which x varies

$$\varepsilon(x,t) = \sum^{M} A_m(t)\, e^{im\pi x/L}$$

A_ms are arbitrary

$m=0$

complex coefficients
= amplitude for wave $A_m e^{\frac{im\pi}{L}}$

$\varepsilon(x,t)$ satisfies the linear FDE.

at $t=t_n$
Let $\bar{A}^n_m$ be the $\max_m \{|A_m(t_n)|\}$ ... A_m = amplitude

and denote the term at $t=t_n$ as

$$\varepsilon_m(x,t_n) = A_m(t_n)e^{im\pi x/L}$$

At $x=x_j$, $\varepsilon^n_j = A^n_m e^{im\pi x_j/L}$

$$x_j = j\delta x$$

$$\varepsilon^n_j = A^n_m e^{i\frac{m\pi\delta x}{L}j}$$, $\theta = \frac{m\pi\delta x}{L}$ is the phase angle and $|A^n_m|$ is the amplitude.

Let us define the ratio

$$\xi \stackrel{\text{def}}{=} \frac{\bar{A}^{n+1}}{\bar{A}^n}$$ as the amplification factor

In the explicit scheme,

$$U^{n+1}_j = U^n_j + r(U^n_{j+1} - 2U^n_j + U^n_{j-1})$$
$$\varepsilon^{n+1}_j = \varepsilon^n_j + r(\varepsilon^n_{j+1} - 2\varepsilon^n_j + \varepsilon^n_{j-1})$$

Taking: $\varepsilon^n_j = \bar{A}^n e^{i\theta j}$

$$\bar{A}^{n+1}e^{i\theta j} = \bar{A}^n e^{i\theta j} + r(\bar{A}^n e^{i\theta(j+1)} - 2\bar{A}^n e^{i\theta j} + \bar{A}^n e^{i\theta(j-1)})$$

$$\Rightarrow \xi = \frac{\bar{A}^{n+1}}{A^n} = 1 + r(e^{i\theta} - 2 + e^{-i\theta})$$

$$\Rightarrow \xi = 1 + 2r(\cos\theta - 1)$$

For stability, $|\xi| \leq 1$

$$\Rightarrow \quad -1 \leq 1 + 2r(\cos\theta - 1) \leq 1$$

$$\Rightarrow \quad -2 \leq 2r(\cos\theta - 1) \leq 0$$

$$\Rightarrow \quad -2 \leq 2r\left(1 - 2\sin^2\left(\tfrac{\theta}{2}\right) - 1\right) \leq 0$$

$$\Rightarrow \quad -2 \leq 2r\left(-2\sin^2\tfrac{\theta}{2}\right) \leq 0$$

$$\Rightarrow \quad 0 \leq 4r\sin^2\frac{\theta}{2} \leq 2$$

$$\Rightarrow \quad 0 \leq r\sin^2\frac{\theta}{2} \leq \frac{1}{2}$$

$$\Rightarrow \quad r \leq 1/2$$

STABILITY ANALYSIS

$$\varepsilon(x, t_n) = \sum_m A_m^n e^{im\pi x/L}$$

- $$\varepsilon_j^n = \bar{A}^n e^{i\frac{m\pi x_j}{L}} = \bar{A}^n e^{i\theta j} \; ; \; \theta = \frac{m\pi\delta x}{L}$$

  $\uparrow$ fourier coeff, has max amplitude

- Amplification factor $\xi = \dfrac{\bar{A}^{n+1}}{\bar{A}^n}$

$\rightarrow$ note: we take 1 term (that having the max. amplitude)

STABLE: $|\xi| \leq 1$

UNSTABLE: $|\xi| > 1$

For EXPLICIT : $|\xi| \le 1$ for $r \le 1/2$

IMPLICIT

$$U_j^{n+1} - U_j^n = r\left(U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}\right)$$

$$\varepsilon_j^{n+1} - \varepsilon_j^n = r\left(\varepsilon_{j+1}^{n+1} - 2\varepsilon_j^{n+1} + \varepsilon_{j-1}^{n+1}\right)$$

$$\xi = 1 + r\xi\left[e^{i\theta} - 2 + e^{-i\theta}\right]$$

$$\Rightarrow \xi\left[1 - r(2c\theta - 2)\right] = 1$$

$$\xi = \frac{1}{1-2r(\cos\theta - 1)}$$

$$= \frac{1}{1+4r\sin^2\left(\frac{\theta}{2}\right)} \le 1 \ \forall r>0$$

$\therefore$ IMPLICIT SCHEME is unconditionally stable.

H.T. Check for stability of Crank-Nicolson scheme.

- CTCS : Leapfrog Scheme

  3- time level

$$U_t = U_{xx}$$

disc. as : $$\frac{U_j^{n+1} - U_j^{n-1}}{2\delta t} = c\,\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\delta x^2}$$

cannot start from initial step, only in intermediate steps.

LEAPFROG SCHEME is UNCONDITIONALLY UNSTABLE.

Du Fort-Frankel

$$\frac{U_j^{n+1}-U_j^{n-1}}{2k} - \frac{1}{h^2}\left[U_{j+1}^n - 2\left\{\theta U_j^{n+1} + (1-\theta)U_j^{n-1}\right\} + U_{j-1}^n\right]=0$$

where

$0<\theta<1$ is a parameter, $k=\delta t$, $h=\delta x$.

Show that (i) if $k=rh$, the scheme is not consistent for the given FDE.

(ii) if $k=rh^2$,

scheme is not

$$\frac{U_j^{n+1}-U_j^{n-1}}{2k} - \frac{1}{h^2}\left[U_{j+1}^n - 2\left\{\theta U_j^{n+1} + (1-\theta)U_j^{n-1}\right\} + U_{j-1}^n\right]=0$$

$$T.E. = \frac{1}{2k}\Bigg[U_j^n + k\frac{\partial U}{\partial t}\Big|_j^n + \frac{k^2}{2}\frac{\partial^2 U}{\partial t^2}\Big|_j^n + \cdots$$

$$- U_j^n + k\frac{\partial U}{\partial t}\Big|_j^n - \frac{k^2}{2}\frac{\partial^2 U}{\partial t^2}\Big|_j^n + \cdots\Bigg]$$

$$-\frac{1}{h^2}\Bigg[U_j^n + h\frac{\partial U}{\partial x}\Big|_j^n + \frac{h^2}{2}\frac{\partial^2 U}{\partial x^2}\Big|_j^n + \cdots$$

$$+ U_j^n - h\frac{\partial U}{\partial x} + \frac{h^2}{2}\frac{\partial^2 U}{\partial x^2}\Big|_j^n + \cdots$$

$$-2\Big\{\theta\Big[U_j^n + k\frac{\partial U}{\partial t}\Big|_j^n + \frac{k^2}{2}\frac{\partial^2 U}{\partial t^2}\Big|_j^n + \cdots\Big]$$

$$+(1-\theta)\Big[U_j^n - k\frac{\partial U}{\partial t}\Big|^n + \frac{k^2}{2}\frac{\partial^2 U}{\partial t^2}\Big|^n + \cdots\Big]\Big\}\Bigg]$$

$$\left.\frac{\partial U}{\partial t}\right|_j \qquad \left.\frac{1}{2}\frac{\partial^2 U}{\partial t^2}\right|_j \qquad \Big]\Big\}\Big]$$

$$T.E. \approx \left.\frac{\partial U}{\partial t}\right|_j^n + \frac{k^2}{6}U_{ttt}\Big|_j^n + (1-2\theta)\frac{2k}{h^2}\left.\frac{\partial U}{\partial t}\right|_j^n - \left.\frac{\partial^2 U}{\partial x^2}\right|_j^n$$

$$+ \frac{k^2}{h^2}U_{tt} - \frac{h^2}{12}U_{xxxx} + O\left(\frac{k^3}{h^2}, k^4, h^4\right)$$

(redo calc.)

$h, k \to 0$ with $k = rh$, $\frac{k}{h} = r$

if $\theta \neq 1/2$, $T.E. \to \infty$ (unbounded)

$$TE = \left.\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right|_j^n + r^2 U_{tt}\Big|_j^n$$

if $\theta = 1/2$

The FDE is consistent with the PDE –

$$U_t - U_{xx} + rU_{tt} = 0$$

This is not the PDE we are concerned with.

If $k = rh^2$,

$$TE = (U_t - U_{xx})\Big|_j^n + 2r(1-2\theta)U_t\Big|_j^n$$

If $\theta = 1/2$, not only is the scheme consistent, it is so with our required PDE.

Dufort Frankel scheme

$$U_j^n = \frac{1}{2}\left(U_j^{n+1} + U_j^{n-1}\right)$$

H.T. Do consistency analysis for all schemes.

Q. $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, $t > 0$, $0 < x < 1$

$U(x, 0) = 1 \longrightarrow$ I.C.

$$\left.\begin{array}{ll} U_x = U & \text{at } x = 0 \\ U_x = -U & \text{at } x = 1 \end{array}\right\} t \geq 0, \text{ flux condition}$$

(Incompatibility)

Ans.

Solve by Crank-Nicolson method,
use first order FWD & BCKWRD.

Non-Linear Transport EQUATION

(no need to remember the name / terminology)

BURGER'S EQUATION

$$U_t + UU_x = \mu U_{xx}$$

- Use CN scheme followed by Newton's linearisation technique.

Get the reduced tri-diagonal system.