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Reducing a feasible solⁿ (f.s.) to a Basic Feasible solⁿ (b.f.s.)

Theorem - If there is a f.s. to a set of m simultaneous linear equation $Ax=b$ in n unknowns and $r(A)=m$, then there is a b.f.s.

$A_{m \times n}$

Example-

$x_1 = 2$, $x_2 = 3$, $x_3 = 1$ is a f.s. of the LPP.

$$\text{Max } z = x_1 + 2x_2 + 4x_3$$

s.t.
$$2x_1 + x_2 + 4x_3 = 11$$

$Ax=b$
$$3x_1 + x_2 + 5x_3 = 14$$
$$x_1, x_2, x_3 \geq 0$$

Find a b.f.s.

$$A = [a_1 \quad a_2 \quad a_3] = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 5 \end{bmatrix} \qquad b = \begin{bmatrix} 11 \\ 14 \end{bmatrix}$$

$$x_1 a_1 + x_2 a_2 + x_3 a_3 = b$$
$$2a_1 + 3a_2 + a_3 = b \quad - (1)$$

a_1, a_2, a_3 are linearly dependent

$$\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0$$
$$\lambda_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2\lambda_1 + \lambda_2 + 4\lambda_3 = 0$$
$$3\lambda_1 + \lambda_2 + 5\lambda_3 = 0$$
$$\frac{\lambda_1}{5-4} = \frac{\lambda_2}{12-10} = \frac{\lambda_3}{2-3} = k \text{ (say)}$$

$$\left.\begin{aligned} \lambda_1 &= k \\ \lambda_2 &= 2k \\ \lambda_3 &= -k \end{aligned}\right\} \quad \text{Taking } k=1 \quad \lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = -1$$

$$a_1 + 2a_2 - a_3 = 0 \quad - (2)$$

(2, 3, 1) is not a b.f.s as the b.f.s. of this problem cannot have more than 2 positive variable.

To determine the departing variable

$$\min_j \left\{ \frac{x_j}{\lambda_j} \;\middle|\; \lambda_j > 0 \right\}$$

$$= \min \left\{ \frac{2}{1}, \frac{3}{2} \right\} = \frac{3}{2} \Rightarrow j = 2$$

a_2 is the departing ~~depending~~ vector and ~~the~~ make $x_2 = 0$.

① x_2 − ② x_3

$$4a_1 + 6a_2 + 2a_3 = 2b$$
$$3a_1 + 6a_2 - 3a_3 = 0$$
$$- \quad - \quad +$$

$$a_1 + 5a_3 = 2b$$

$$\frac{1}{2}a_1 + 0 \cdot a_2 + \frac{5}{2}a_3 = b$$

$$\left(\frac{1}{2}, 0, \frac{5}{2}\right)$$

Instead we can calculate max ratio as

$$\max_j \left\{ \frac{x_j}{\lambda_j} \;\middle|\; \lambda_j < 0 \right\} = \left\{ \frac{x_3}{\lambda_3} \right\} \Rightarrow j = 3$$

a_3 departing vector

make $x_3 = 0$

$$\text{new } x_j = \text{old } x_j - \left(\frac{x_3}{\lambda_3}\right) \lambda_j$$

Max ratio

New $x_1 = 2 - (-1) \cdot 1 = 3$

New $x_2 = 3 - (-1) 2 = 5$

New $x_3 = 1 - (-1)(-1) = 0$

~~b.f.~~ Another b.f.s. (3, 5, 0)

Example -

$$x_1 + 2x_2 + 4x_3 + x_4 = 7$$
$$2x_1 - x_2 + 3x_3 - 2x_4 = 4$$

f.s. $\rightarrow$ $x_1 = 1,\ x_2 = 1,\ x_3 = 1,\ x_4 = 0$

Reduce it to a b.f.s.

$$x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4 = b$$
$$a_1 + a_2 + a_3 = b$$
$$\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0$$

Example -

Show that $x_1 = 5,\ x_2 = 0,\ x_3 = -1$ is a b.f.s. of the system of eq^n

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5$$

Find the basic sol^n if there is any.

$$A = (a_1, a_2, a_3) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$x_2 = 0 \qquad (a_1, a_3) = \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}$$

$$\det \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix} = 5 - 2 = 3 \neq 0$$

a_1, a_3 linearly independent

So the given solution is basic

Other basic sol^n.

$$B = \begin{array}{cc} a_1 & a_3 \end{array} \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix} \qquad B^{-1} = \frac{1}{3}\begin{bmatrix} 5 & -1 \\ -2 & 1 \end{bmatrix}$$

$$x_B = B^{-1} b = \frac{1}{3}\begin{pmatrix} 5 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 15 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

checked.

For non-basic variable x_2,

$Y_2 = B^{-1} a_2$

$= \frac{1}{3}\begin{pmatrix} 5 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 9 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \rightarrow$ the

atleast one component of Y_2 is $-ve$

a_2 can enter the basis replacing a_1.

New Basis

$$\bar{B} = \begin{array}{cc} a_2 & a_3 \\ \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} \end{array} \qquad (\bar{B})^{-1} = \frac{1}{9}\begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}$$

$$x_{\bar{B}} = (\bar{B})^{-1} b = \frac{1}{9}\begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

$x_1 = 0$, $x_2 = 5/3$, $x_3 = 2/3$

Other basic solution-

for Non-basic variable x_1

$$Y_1 = (\bar{B})^{-1} a_1 = \frac{1}{9}\begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$$

a_1 enter replacing a_3 in the basis

$$B' = (a_2, a_1) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \Rightarrow (B')^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$x_B' = (B')^{-1} b$$

$$= \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$x_1 = 2 \quad x_2 = 1 \quad x_3 = 0$

Q- Apply the simplex process to solve the LPP (without using simplex table)

Max $Z = 2x_1 - 3x_2$

s.t. $2x_1 + 5x_2 \geq 10$

$3x_1 + 8x_2 \leq 24$

$x_1, x_2 \geq 0.$

Max $Z = 2x_1 - 3x_2$

s.t. $\left|\begin{array}{l} 2x_1 + 5x_2 - x_3 \qquad = 10 \\ 3x_1 + 8x_2 \qquad + x_4 = 24 \end{array}\right.$

$Ax = b$

$x_1, x_2, x_3, x_4 \geq 0$

$A = (a_1, a_2, a_3, a_4) = \begin{pmatrix} 2 & 5 & -1 & 0 \\ 3 & 8 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 10 \\ 24 \end{pmatrix}$

$x = (x_1, x_2, x_3, x_4)^T$

a_1, a_4 linearly independent

Take basis $B = (a_1, a_4) = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$

$B^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$

Basic feasible solution (BFS)

$x_B = B^{-1} b = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 10 \\ 24 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}$

$x_1 = 5, \; x_2 = 0, \; x_3 = 0, \; x_4 = 9 \qquad -(1)$

$C_B = (2, 0) \qquad Z_B = C_B x_B = (2, 0) \begin{pmatrix} 5 \\ 9 \end{pmatrix} = 10$

for non-basic vector, compute

$$Y_2 = B^{-1}a_2 = \frac{1}{2}\begin{pmatrix}1 & 0\\ -3 & 2\end{pmatrix}\begin{pmatrix}5\\ 8\end{pmatrix} = \begin{pmatrix}5/2\\ 1/2\end{pmatrix}$$

$$Y_3 = B^{-1}a_3 = \frac{1}{2}\begin{pmatrix}1 & 0\\ -3 & 2\end{pmatrix}\begin{pmatrix}-1\\ 0\end{pmatrix} = \begin{pmatrix}-1/2\\ 3/2\end{pmatrix}$$

Compute $z_2 - c_2$, $z_3 - c_3$

$$z_2 - c_2 = C_B Y_2 - c_2 = (2, 0)\begin{pmatrix}5/2\\ 1/2\end{pmatrix} - (-3)$$

$$= 5 + 3 = 8$$

$$z_3 - c_3 = C_B Y_3 - c_3 = (2, 0)\begin{pmatrix}-1/2\\ 3/2\end{pmatrix} - 0$$

$$= -1 < 0$$

① is not an optimal solution

a_3 is the entering vector in the basis

<u>Leaving variable determination</u>

find the min ratio,

$$\min\left\{\frac{x_{Bi}}{y_{j3}} \;\middle|\; y_{j3} > 0\right. \qquad = \frac{\cancel{4}^{8}\times 2}{\cancel{3}} = 16 \qquad r = 4$$

x_4 is leaving

<u>New basis</u>

$$B = (a_1 . a_3) = \begin{pmatrix}2 & -1\\ 3 & 0\end{pmatrix}$$

$$B^{-1} = \frac{1}{3}\begin{pmatrix}0 & 1\\ -3 & 2\end{pmatrix}$$

$$x_B = B^{-1}b = \frac{1}{3}\begin{pmatrix}0 & 1\\ -3 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 2 & 4\end{pmatrix} = \begin{pmatrix}8\\ 6\end{pmatrix} = \begin{pmatrix}x_1\\ x_3\end{pmatrix}$$

$x_1 = 8$, $x_2 = 0$, $x_3 = 6$, $x_4 = 0$ —②

$C_B = (2, 0)$

$$z = z_B = C_B X_B = (2 \cdot 0)\begin{pmatrix}8\\ 6\end{pmatrix} = 16$$

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For non-basic variables x_2, x_4 we compute

$$Y_2 = B^{-1}a_2 = \begin{pmatrix} 8/3 \\ 1/3 \end{pmatrix}$$

$$Y_4 = B^{-1}a_4 = \begin{pmatrix} 1/3 \\ 3/2 \end{pmatrix}$$

check

Compute -

$$z_2 - c_2 = c_B Y_2 - (-3) = 25/3 > 0$$

$$z_4 - c_4 = c_B Y_4 - 0 = 2/3 > 0$$

Optimality is reached

② is the optimal solution

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Q- Max $Z = 3x_1 + 4x_2$ attains unbounded solⁿ.

s.t. $x_1 - x_2 \geq 0$

$-x_1 + 3x_2 \leq 3$

$x_1, x_2 \geq 0$

Q- Max $Z = 6x_1 + 10x_2$ attains an alternate optimal solⁿ

s.t. $3x_1 + 5x_2 \leq 10$

$5x_1 + 3x_2 \leq 15$

$x_1, x_2 \geq 0$

$z_j - c_j = 0$ for some non-basic variable.

Improving a BFS (Simplex Method)

Max $Z = Cx$

$Ax = b$ $\quad A_{m\times n}$ $\quad m<n$

$x \geq 0$ $\quad$ Rank of $A = m$

Basis $B = [\beta_1, \beta_2, \beta_3, \dots, \beta_m]$

$$\begin{array}{c|cccc} i \downarrow \; j \rightarrow & a_1 & a_2 & a_j & a_n \\ \hline & \multicolumn{4}{c}{\begin{pmatrix} y_{11} & y_{12} & y_{1j} & y_{1n} \\ y_{21} & y_{22} & y_{2j} & y_{2n} \\ \vdots & \vdots & & \vdots \\ y_{m1} & y_{m2} & y_{mj} & y_{mn} \end{pmatrix}} \end{array}$$

① – $\boxed{a_j = \sum_{i=1}^{m} y_{ij}\beta_i} = y_{1j}\beta_1 + y_{2j}\beta_2 + \dots + y_{mj}\beta_m$

$y_{rj} \neq 0 \Rightarrow \beta_r$ can be replaced by a_j

$$\boxed{\beta_r = \frac{a_j}{y_{ij}} - \sum_{\substack{i=1 \\ i\neq r}}^{m} \frac{y_{ij}\beta_i}{y_{ij}}} \quad – ②$$

x_B is a BFS $\Rightarrow Bx_B = b$

$$\boxed{\sum_{i=1}^{m} \beta_i x_{Bi} = b} \quad – ③$$

$C = (C_B \mid C_R)$

$$\boxed{Z_B = C_B x_B = \sum_{i=1}^{m} C_{Bi} x_{Bi}} \quad – ④$$

② , ③ ⇒

$$\sum_{i=1}^{m} x_{Bi}\beta_i + x_{Br}\left[\frac{a_j}{y_{ij}} - \sum_{\substack{i=1\\ i\neq r}}^{m} \frac{y_{ij}\beta_i}{y_{rj}}\right] = b$$

$$\sum_{\substack{i=1\\ i\neq r}}^{m} \left[x_{Bi} - \frac{y_{ij}}{y_{rj}} x_{Br}\right]\beta_i + x_{Br}\frac{a_j}{y_{rj}} = b \quad -⑤$$

New Basic solution -

$$\left.\begin{array}{l} ⑦ - \; 0 \leq x_{Bi} - \dfrac{y_{ij}}{y_{rj}} x_{Br} \quad i=1,2,\dots,m, \; i\neq r \\ \text{and} \\ ⑧ - \; \leq 0 \leftarrow \dfrac{x_{Br}}{y_{rj}} \end{array}\right\} -⑥$$

corresponds to the new basis -

$(\beta_1, \beta_2, \dots, \beta_{r-1}, a_j, \beta_{r+1}, \dots, \beta_m)$

$y_{ij} > 0$

$x_{Br} = 0 \quad \rightarrow$ ⑥ f.s. as ⑦, ⑧ satisfied.

$x_{Br} \neq 0 \quad$ ⑧ satisfied and for ⑦ to satisfy we need

$$⑨ - \frac{x_{Bi}}{y_{ij}} - \frac{x_{Br}}{y_{ij}} \geq 0, \quad y_{ij} \geq 0$$

$$\frac{x_{Br}}{y_{rj}} \leq \frac{x_{Bi}}{y_{ij}} \qquad y_{ij} > 0$$

$$\frac{x_{Br}}{y_{rj}} = \min \left\{ \frac{x_{Bi}}{y_{ij}} \,\middle|\, y_{ij} > 0 \right\}$$

(min ratio)

a_j is the entering vector

$$z' = \sum_{i=1}^{m} c_{Bi} \left(x_{Bi} - \frac{y_{ij}}{y_{rj}} x_{Br} \right) + c_j \left(\frac{x_{Br}}{y_{rj}} \right)$$

$$= \sum_{i=1}^{m} c_{Bi} x_{Bi} - \frac{x_{Br}}{y_{rj}} \underbrace{\left(\sum_{i=1}^{m} c_{Bi} y_{ij} - c_j \right)}_{z_j - c_j}$$

$$= z_B - \frac{x_{Br}}{y_{rj}} (z_j - c_j)$$

$$z_k - c_k = \min \{ z_j - c_j \mid z_j - c_j < 0 \}$$

↳ choice of EV {Entering variable}

Alternative Optima -

Theorem - If there is an optimal BFS to a LPP and $z_j - c_j = 0$ for some non-basic vector a_j and $y_{ij} > 0$ for atleast one i then there exists an alternative Basic Optimal solⁿ.

Theorem - If there is an optimal BFS to a LPP and $z_j - c_j = 0$ for some non-basic vector a_j and $y_{ij} < 0 \; \forall \; i = 1, 2, \ldots, m$ then there exists an alternative non-basic optimal solution.

Proof -

$$a_j = \sum_{i=1}^{m} y_{ij} \beta_i$$

$$Ax = b$$

$$B = (\beta_1, \beta_2, \ldots, \beta_m)$$

$$Bx_B = b$$

$$x_B = B^{-1} b \qquad \theta > 0$$

$$\sum_{i=1}^{m} x_{Bi} \beta_i + \theta a_j - \theta a_j = b$$

$$\sum_{i=1}^{m} (x_{Bi} - y_{ij} \theta) \beta_i + \theta a_j = b$$

$\to \beta_i$

Non basic f.s. $\leftarrow \begin{cases} x_{Bi} - \theta y_{ij} & i = 1, 2, \ldots, m \\ \theta \to a_j \end{cases}$

$\hookrightarrow$ a new f.s. with $(m+1)$ positive variables remaining zero

$$Z' = \sum_{i=1}^{m} c_{Bi} (x_{Bi} - \theta y_{ij}) + c_j \theta$$

$$= \sum_{i=1}^{m} c_{Bi} x_{Bi} - \theta \sum_{i=1}^{m} c_{Bi} y_{ij} + c_j \theta$$

$$= Z_B - \theta (z_j - c_j)$$

$$= Z_B$$

$z_j - c_j = 0$ for some non-basic variable for the existence of alternative option $\to$ not a necessary condition

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$z_j - c_j > 0$ for a non-basic vector a_j

$\beta_r \rightarrow$ leaving vector $\quad x_{\beta r} = 0 \,,\; y_{rj} > 0$
$\quad\quad \rightarrow$ degenerate case.

$$\frac{x_{\beta r}}{y_{rj}} = 0$$

$$z' = z_B - \frac{x_{\beta r}}{y_{rj}} (z_j - c_j)$$

$$= z_B$$

Two-Phase Method

Phase I-

$$\text{Max } z^* = 0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n$$
$$-1 \cdot x_{a_1} - 1 \cdot x_{a_2} - \cdots - 1 \cdot x_{am}.$$

→ no artificial variable
→ artificial variable at zero level.

Phase II-

Careful about artificial variable,
should always remain at zero level

Case I-

$a_k \quad EV$

$a_r \quad LV$

$$\bar{x}_{Bi} = x_{Bi} - \frac{x_{Br}}{y_{rk}} y_{ik}$$