Assignment 3 – MM

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1 i) $\frac{d^4y}{dx^4} = 0$; $y(0) = y'(0) = y''(1) = y'''(1) = 0$

Gen soln :- $y = A + Bx + Cx^2 + Dx^3$

Using the BC's :- $y(0) = 0 \Rightarrow A = 0$

$y'(0) = 0 \Rightarrow B = 0$

$y''(1) = 0 \Rightarrow 2C + 6D = 0 \Rightarrow C + 3D = 0$

$y'''(1) = 0 \Rightarrow 6D = 0 \Rightarrow D = 0$

$\therefore A = B = C = D = 0$

The only trivial solution is $y(x) \equiv 0$

$\therefore$ The green's function for the BVP will be unique & is given by

$$G(x,t) = \begin{cases} a_1 + a_2 x + a_3 x^2 + a_4 x^3 ; & 0 \le x \le t \\ b_1 + b_2 x + b_3 x^2 + b_4 x^3 ; & t \le x \le 1 \end{cases}$$

$G(x,t)$ will satisfy the following properties :-

(a) Continuity of G, $\frac{\partial G}{\partial x}$, $\frac{\partial^2 G}{\partial x^2}$ at $x = t$:

$a_1 + a_2 t + a_3 t^2 + a_4 t^3$

$= b_1 + b_2 t + b_3 t^2 + b_4 t^3$

$\Rightarrow (b_1 - a_1) + (b_2 - a_2) t + (b_3 - a_3) t^2 + (b_4 - a_4) t^3 = 0$

Define : $c_i = b_i - a_i$ $(i = 1, 2, 3, 4)$

Then, we get : $c_1 + c_2 t + c_3 t^2 + c_4 t^3 = 0$

From the continuity of $\frac{\partial G}{\partial x}$ & $\frac{\partial^2 G}{\partial x^2}$ at $x = t$, we get

$c_2 + 2c_3 t + 3c_4 t^2 = 0$

& $2c_3 + 6c_4 t = 0$

DOMS Page No.
Date / /

19MA20059

(b) $\frac{\partial^3 G}{\partial x^3}$ has a jump discontinuity at $x=t$:

$$\left(\frac{\partial^3 G}{\partial x^3}\right)_{x=t^+} - \left(\frac{\partial^3 G}{\partial x^3}\right)_{x=t^-} = \frac{-1}{P_0(t)}$$

[$P_0(x)$ = coeff of y'''' in $y''''=0$ $\Rightarrow P_0(x)=1 \Rightarrow \therefore P_0(t)=1$]

$$\Rightarrow 6b_4 - 6a_4 = -\frac{1}{1} \Rightarrow 6c_4 = -1$$

So we have the following system of eq$^{\underline{ns}}$ as:-

$$c_1 + c_2 t + c_3 t^2 + c_4 t^3 = 0$$
$$c_2 + 2c_3 t + 3c_4 t^2 = 0$$
$$2c_3 + 6c_4 t = 0$$
$$6c_4 = -1$$

We get:- $c_1 = t^3/6$, $c_2 = -t^2/2$, $c_3 = t/2$, $c_4 = -1/6$

(c) As $G(x,t)$ satisfies the given BC's,

$\therefore$ $G(0,t)=0 \Rightarrow a_1 = 0$

$G'(0,t) = 0 \Rightarrow a_2 = 0$

$G''(1,t) = 0 \Rightarrow 2b_3 + 6b_4 = 0 \Rightarrow b_3 + 3b_4 = 0$

$G'''(1,t) = 0 \Rightarrow 6b_4 = 0 \Rightarrow b_4 = 0$

$\therefore a_1 = a_2 = b_3 = b_4 = 0$

Now $b_1 = c_1 + a_1 = t^3/6 + 0 = t^3/6$

$b_2 = c_2 + a_2 = -t^2/2 + 0 = -t^2/2$

$a_3 = b_3 - c_3 = 0 - t/2 = -t/2$

$a_4 = b_4 - c_4 = 0 - (-1/6) = 1/6$

Thus,
$$G(x,t) = \begin{cases} \left(-\frac{t}{2}\right)x^2 + \left(\frac{1}{6}\right)x^3 & ; \ 0 \le x < t \\ \left(\frac{t^3}{6}\right) - \left(\frac{t^2}{2}\right)x & ; \ t < x \le 1 \end{cases}$$

19MA20059

1) ii) $\frac{\partial^3 y}{\partial x^3} = 0$; $y(0) = y'(1) = 0$; $y'(0) = y(1)$

<u>Gen Sol^n</u> :- $y = A + Bx + Cx^2$

<u>Using BC's</u> $\rightarrow$ $y'(1) = 0 \Rightarrow B + 2C = 0$ | $y'(0) = y(1)$

$y(0) = 0 \Rightarrow A = 0$ | $\Rightarrow \cancel{B} = A + \cancel{B} + C \Rightarrow C = 0$

$A = B = C = 0$

$y(x) \equiv 0$ is the only solution of BVP. Thus, Green's fun^n is unique & will be of the form

$$G(x,t) = \begin{cases} a_1 + a_2 x + a_3 x^2 & ; \quad 0 \leq x < t \\ b_1 + b_2 x + b_3 x^2 & ; \quad t < x \leq 1 \end{cases}$$

(a) Continuity of G, $\frac{\partial G}{\partial x}$ at $x = t$

Let $c_i = b_i - a_i$ for $i = 1, 2, 3$

$c_1 + c_2 t + c_3 t^2 = 0$ & $c_2 + 2c_3 t = 0$

(b) Discontinuity of $\frac{\partial^2 G}{\partial x^2}$ at $x = t$

$2c_3 = \frac{-1}{1}$

$\Rightarrow 2c_3 = -1$

$\therefore c_1 = -t^2/2$, $c_2 = t$, $c_3 = -1/2$

(c) $G(x,t)$ satisfies the BC's, thus

$G(0,t) = 0 \Rightarrow a_1 = 0$, $G'(1,t) = 0 \Rightarrow b_2 + 2b_3 = 0$

$G'(0,t) = G(1,t) \Longrightarrow a_2 = b_1 + b_2 + b_3$

$\Rightarrow b_2 - c_2 = b_1 + b_2 + b_3$

$\Rightarrow b_1 + b_3 = -t$

Now, $b_1 = c_1 + a_1 = -t^2/2 + 0 = -t^2/2$

$\therefore b_3 = +t^2/2 - t$

19MA20059

$a_3 = b_3 - c_3 = (t^2/2) - t - (-1/2) = (t^2/2) - t + (1/2)$

$b_2 = -2b_3 = -t^2 + 2t$

$a_2 = b_2 - c_2 = t - t^2$

$$\therefore \quad G(x,t) = \begin{cases} (t-t^2)x + \left(\frac{t^2}{2} - t + \frac{1}{2}\right)x^2 ; & 0 \leq x < t \\ (2x - x^2)t + \left(\frac{x^2}{2} - x - 1/2\right)t^2 ; & t < x \leq 1 \end{cases}$$

1) iii) $y''' = 0$; $y(0) = y(1) = 0$; $y'(0) = y'(1)$

Gen Sol$\underline{^n}$:- $y = A + Bx + Cx^2$

Using BC's :- $y(0) = 0 \Rightarrow A = 0$ | $y(1) = 0 \Rightarrow A + B + C = 0 \Rightarrow B + C = 0$

$y'(0) = y'(1)$

$\Rightarrow B = B + 2C \Rightarrow C = 0$

$\therefore A = B = C = 0$

$y \equiv 0$ is the only trivial sol$\underline{^n}$. Hence, the unique Green's function is of the form

$$G(x,t) = \begin{cases} a_1 + a_2 x + a_3 x^2 ; & 0 \leq x < t \\ b_1 + b_2 x + b_3 x^2 ; & t < x \leq 1 \end{cases}$$

Let $c_i = b_i - a_i$, for $i = 1, 2, 3$

We know that G & $\partial G/\partial x$ are continuous at $x = t$

& $\partial^2 G/\partial x^2$ is ~~discontinuo~~ discontinuous at $x = t$.

From these conditions, we get :- $c_1 = -t^2/2$, $c_2 = t$, $c_3 = -1/2$

(just like in previous ~~question)~~ Q2 question)

DOMS Page No.
Date / /

19MA20059

Now, $G(x,t)$ satisfies BC's, thus :

$G(0,t)=0 \Rightarrow a_1=0$; $G(1,t)=0 \Rightarrow b_1+b_2+b_3=0$

$G'(0,t)=G'(1,t) \Rightarrow a_2=b_2+2b_3 \Rightarrow b_2-c_2=2b_3+b_2 \Rightarrow b_3=-t/2$

$b_1 = c_1+a_1 = \cancel{0}-t^2/2+a_1 = -t^2/2$

$b_2 = \cancel{c_2+a_2}-b_1-b_3 = t^2/2+t/2$

$a_2 = b_2-c_2 = (t^2/2)+(t/2)-t = (t^2/2)-(t/2)$

$a_3 = b_3-c_3 = -t/2-(-1/2) = -t/2+1/2$

hence, $G(x,t) = \begin{cases} \left(\frac{t^2}{2}-\frac{t}{2}\right)x + \left(-\frac{t}{2}+1/2\right)x^2 ; & 0\le x<t \\ \left(-\frac{x^2}{2}+\frac{x}{2}\right)t + \left(\frac{x}{2}-\frac{1}{2}\right)t^2 ; & \cancel{0}\, t<x\le 1 \end{cases}$

1) iv) $y''+y=0$; $y(0)=y(1)$; $y'(0)=y'(1)$

Gen Sol^n :- $y = A\cos x + B\sin x$

Using BC's :- $y(0)=y(1) \Rightarrow A = \cancel{B}A(\cos(1)) + B(\sin(1))$

$y'(0)=y'(1) \Rightarrow B = -A\sin(1)+B\cos(1)$

$\therefore A=B=0$

$y(x)\equiv 0$ is the only trivial sol^n possible. $\therefore$ The unique green's function for this BVP is :-

$G(x,t) = \begin{cases} a_1\cos x + a_2\sin x ; & 0\le x<t \\ b_1\cos x + b_2\sin x ; & t<x\le 0 \end{cases}$

a) continuity of G at $x=t$:

Let $c_i = b_i - a_i$ $(i=1,2)$

$\therefore c_1\cos t + c_2\sin t = 0$

b) Jump discontinuity ~~at~~ of $\frac{\partial G}{\partial x}$ at $x=t$:

$\left(\frac{\partial G}{\partial x}\right)_{x=t^+} - \left(\frac{\partial G}{\partial x}\right)_{x=t^-} = -1/1$

$\Rightarrow (-b_1\sin t + b_2\cos t) - (-a_1\sin t + a_2\cos t) = -1$

$\Rightarrow -c_1\sin t + c_2\cos t = -1$

$\therefore c_1 = \sin t$; $c_2 = -\cos t$

19MA20059

c) $G(x,t)$ satisfies the BC's,

$G(0,t) = G(1,t) \Rightarrow a_1 = b_1 \cos 1 + b_2 \sin 1$

$G'(0,t) = G'(1,t) \Rightarrow a_2 = -b_1 \sin 1 + b_2 \cos 1$

$\therefore b_1(1-\cos 1) - b_2 \sin 1 = \sin t$

$b_1(\sin 1) + b_2(1-\cos 1) = -\cos t$

$b_1 = \frac{-1}{2\sin(1/2)} \cdot \cos(t+1/2)$, $b_2 = \frac{-1}{2\sin(\frac{1}{2})} \cdot \sin(t+\frac{1}{2})$

Then, $a_1 = b_1 - c_1 = \frac{-1}{2\sin(1/2)} \cos(t+1/2) - \sin t$

$= \frac{-\cos(t+\frac{1}{2}) - \cos(t-\frac{1}{2}) + \cos(t+\frac{1}{2})}{2\sin(\frac{1}{2})} = \frac{-\cos(t-\frac{1}{2})}{2\sin(\frac{1}{2})}$

$a_2 = b_2 - c_2 = \frac{-\sin(t+\frac{1}{2})}{2\sin(\frac{1}{2})} + \cos t = \frac{-\sin(t-\frac{1}{2})}{2\sin(\frac{1}{2})}$

$$G(x,t) = \begin{cases} \frac{-1}{2\sin(\frac{1}{2})} \times \cos(x-t+\frac{1}{2}) ; & 0 \le x < t \\ \frac{-1}{2\sin(\frac{1}{2})} \times \cos(t-x+\frac{1}{2}) ; & t < x \le 1 \end{cases}$$

2) i) $y'' + \pi^2 y = \cos \pi x$; $y(0) = y(1)$; $y'(0) = y'(1)$

Corresponding homogeneous ODE $\Rightarrow$ ~~$y'' + x^2 y = 0$~~ $y'' + \pi^2 y = 0$

Gen Sol^n :- $y(x) = A\cos(\pi x) + B\sin(\pi x)$

Using BC's :- $y(0) = y(1) \Rightarrow A = -A \Rightarrow A = 0$

$y'(0) = y'(1) \Rightarrow B = -B \Rightarrow B = 0$

Trivial sol^n $y \equiv 0$ is possible. Green's function will be :

$$G(x,t) = \begin{cases} a_1\cos(\pi x) + a_2 \sin(\pi x) ; & 0 \le x < t \\ b_1\cos(\pi x) + b_2 \sin(\pi x) ; & t < x \le 1 \end{cases}$$

Let $c_i = b_i - a_i$ for $i = \{1, 2\}$

19MA20069

a) Continuity of G at $x = t$:-

$c_1 \cos(\pi t) + c_2 \sin(\pi t) = 0$

b) Jump discontinuity of $\frac{\partial G}{\partial x}$ at $x = t$:-

$$\left(\frac{\partial G}{\partial x}\right)_{x=t^+} - \left(\frac{\partial G}{\partial x}\right)_{x=t^-} = \frac{-1}{P_0(t)} = \frac{-1}{1}$$

$\Rightarrow \pi[-c_1 \sin(\pi t) + c_2 \cos(\pi t)] = -1$

$\Rightarrow c_2 \cos(\pi t) - c_1 \sin(\pi t) = -1/\pi$

from (a) & (b) we get $\quad c_1 = \frac{\sin \pi t}{\pi}, \quad c_2 = \frac{-\cos \pi t}{\pi}$

c) $G(x,t)$ satisfies BC's

$G'(0,t) = G'(1,t) \Rightarrow a_2\pi = -b_2\pi \Rightarrow a_2 = -b_2 \Rightarrow 2a_2 = -c_2 = \frac{\cos(\pi t)}{\pi}$

$a_2 = \frac{\cos \pi t}{2\pi}, \quad b_2 = \frac{-\cos \pi t}{2\pi}$

$G(0,t) = G(1,t) \Rightarrow a_1 = -b_1 \Rightarrow 2a_1 = -c_1 = \frac{-\sin \pi t}{\pi}$

$a_1 = \frac{-\sin \pi t}{2\pi}, \quad b_1 = \frac{\sin \pi t}{2\pi}$

$$\therefore G(x,t) = \begin{cases} \frac{1}{2\pi} \sin(\pi(x-t)) \; ; & 0 \leq x < t \\ \frac{1}{2\pi} \sin(\pi(t-x)) \; ; & t < x \leq 1 \end{cases}$$

Now, The particular sol^n of BVP is calculated as

$$y = \int_0^1 G(x,t)\, \phi(t)\, dt \qquad (\text{here } \phi(t) = -\cos \pi t)$$

$$\Rightarrow y = -\int_0^x G(x,t) \cos(\pi t)\, dt - \int_x^1 G(x,t) \cos(\pi t)\, dt$$

$$\Rightarrow y = -\int_0^x \frac{1}{2\pi} \sin(-\pi x + \pi t) \cos(\pi t)\, dt - \int_0^1 \frac{1}{2\pi} \sin(\pi t + \pi x) \cos(\pi t)\, dt$$

$$\Rightarrow y = \frac{x \sin \pi x}{4\pi} + \frac{(x-1) \sin \pi x}{4\pi}$$

$$\Rightarrow \boxed{y = \frac{(2x-1) \sin(\pi x)}{4\pi}}$$

19MA20059

2) ii) Given BVP:- $y'' + y = x^2$; $y(0) = y(\pi/2) = 0$

Corresponding homogeneous ODE:- $y'' + y = 0$

Gen. Sol$^{\underline{n}}$:- $y = A\cos x + B\sin x$

Using BC's:- $y(0) = 0 \Rightarrow A = 0 \mid y(\pi/2) = 0 \Rightarrow B = 0$

Only $y \equiv 0$ is the trivial sol$^{\underline{n}}$ possible. Green's fun$^{\underline{n}}$:-

$$G(x,t) = \begin{cases} a_1\cos(x) + a_2\sin x \ ; & 0 \le x < t \\ b_1\cos(x) + b_2\sin x \ ; & t < x \le \frac{\pi}{2} \end{cases}$$

Let $c_i = b_i - a_i$ for $i = \{1, 2\}$

a) Continuity of G at $x = t$

$c_1\cos(x) + c_2\sin x = 0 \mid_{x=t}$

b) discontinuity of $\frac{\partial G}{\partial x}$ at $x = t$

$-c_1\sin x + c_2\cos x = 0 \mid_{x=t}$

from (a) & (b), $c_1 = \sin t$, $c_2 = -\cos t$

c) $G(x,t)$ satisfies BC's

$G(0,t) = 0 \Rightarrow a_1 = 0 \Rightarrow b_1 = c_1 = \sin t$

$G(\pi/2, t) = 0 \Rightarrow b_2 = 0 \Rightarrow a_1 = -c_1 = \cos t$

$$\therefore G(x,t) = \begin{cases} \cos t.\sin x \ ; & 0 \le x < t \\ \sin t.\cos x \ ; & t < x \le \pi/2 \end{cases}$$

Sol$^{\underline{n}}$ of BVP is calculated as:-

$y = \int_0^{\pi/2} G(x,t)\,\phi(t)\,dt$, here $\phi(t) = -t^2$

$= -\int_0^x \cos x.\sin t.t^2\,dt - \int_x^{\pi/2} \sin x.\cos t.t^2\,dt$

$= -\cancel{x\sin(2x)} + (x^2-2)\cos^2 x + 2\cos x + (x^2-2)\sin^2 x + \cancel{x\sin(2x)}$
$-\frac{(\pi^2-8)}{4}\sin x$

$$\boxed{y = (x^2-2) + 2\cos x - \frac{(\pi^2-8)}{4}\sin x}$$

19MA20059

2) iii) $y'' - y = 2\sinh(1)$; $y(0) = y(1) = 0$.

Corresponding homogeneous ODE:- $y'' - y = 0$

Gen sol^n :- $y = Ae^{x} + Be^{-x}$

Using BC's:- $y(0) = 0 \Rightarrow A + B = 0$

$y(1) = 0 \Rightarrow A(e) + \frac{B}{e} = 0 \Rightarrow Ae^2 + B = 0$

$\therefore A = B = 0$

Only trivial sol^n $y \equiv 0$ is possible. Thus, Green's fun^c is of the form:-

$$G(x,t) = \begin{cases} a_1e^{x} + a_2e^{-x} & ; \ 0 \le x < t \\ b_1e^{x} + b_2e^{-x} & ; \ t < x \le 1 \end{cases}$$

Let $c_i = b_i - a_i$ for $i = \{1, 2\}$

(a) Continuity of G at $x = t$:-

$c_1e^{x} + c_2e^{-x} = 0 \,\Big|_{x=t}$

(b) Discontinuity of $\frac{\partial G}{\partial x}$ at $x = t$:-

$c_1e^{t} - c_2e^{-t} = -1 \,\Big|_{x=t}$

from (a) & (b), $c_1 = \frac{-e^{-t}}{2}$, $c_2 = \frac{e^{t}}{2}$

(c) $G(x,t)$ will satisfy the BC's:-

$G(0,t) = 0 \Rightarrow a_1 + a_2 = 0 \Rightarrow b_1 - c_1 + b_2 - c_2 = 0$

$\Rightarrow b_1 + b_2 = (e^{t} - e^{-t})/2 = \sinh(t)$

$G(1,t) = 0 \Rightarrow b_1e + \frac{b_2}{e} = 0 \Rightarrow b_1e^2 + b_2 = 0 \Rightarrow b_1 = \frac{-\sinh(t)}{2e\sinh(1)}$

& $b_2 = \frac{e\sinh(t)}{2\sinh(1)}$

$a_1 = b_1 - c_1 = \frac{-\frac{1}{e}\sinh(t)}{2\sinh(1)} + \frac{e^{-t}}{2}$

$a_2 = \frac{e\sinh(t)}{2\sinh(1)} - \frac{e^{t}}{2}$

$\therefore a_1e^{x} + a_2e^{-x} = \frac{\sinh(1-x) \cdot \sinh(t) + \sinh(x-t)}{\sinh(1)}$

$= \frac{\sinh(x) \cdot \sinh(1-t)}{\sinh(1)}$

19MA20059

Similarly, $b_1 e^{x} + b_2 e^{-x} = \dfrac{\sinh(t)\cdot\sinh(1-x)}{\sinh(1)}$

$$\therefore G(x,t) = \begin{cases} \dfrac{\sinh(x)\cdot\sinh(1-t)}{\sinh(1)} ; & 0 \le x < t \\ \dfrac{\sinh(t)\cdot\sinh(1-x)}{\sinh(1)} ; & t < x \le 1 \end{cases}$$

Solution of the given BVP is calculated as :-

$y = \int_0^1 G(x,t)\cdot\phi(t)\,dt$, here $\phi(t) = -2\sinh(1)$

$= -\int_0^x 2\sinh(t)\cdot\sinh(1-x)\,dt - 2\int_x^1 \sinh(x)\sinh(1-t)\,dt$

$= -2\sinh(1-x)\cdot[\cosh(x)-1] - 2\sinh(x)[\cosh(1-x)-1]$

$y = -2\sinh(1) + \cancel{4\sinh}\ 4\sinh\left(\tfrac{1}{2}\right)\cdot\cosh\left(x-\tfrac{1}{2}\right)$

2) iv) $y'' - y = -2e^{x}$, $y(0) = y'(0)$; ~~$y(l) + y'(l) = 0$~~

$y(l) + y'(l) = 0$

Corresponding Homogeneous ODE :- $y'' - y = 0$

Gen. Sol^n :- $y = Ae^{x} + Be^{-x}$

Using BC's :- $y(0) = y'(0) \Rightarrow A+B = A-B \Rightarrow B=0$

$y(l) + y'(l) = 0 \Rightarrow 2Ae^{l} = 0 \Rightarrow A = 0$

$\therefore$ Only trivial sol^n $y \equiv 0$ is possible. $\therefore$ Green's fun^n is of the form

$$G(x,t) = \begin{cases} a_1 e^{x} + a_2 e^{-x} ; & 0 \le x < t \\ b_1 e^{x} + b_2 e^{-x} ; & t < x \le l \end{cases}$$

Let $c_i = b_i - a_i$ for $i = \{1, 2\}$

As G is continuous at $x=t$ & $\frac{\partial G}{\partial x}$ is discontinuous at $x=t$, we get $c_1 = \dfrac{-e^{-t}}{2}$, $c_2 = \dfrac{e^{t}}{2}$

As $G(x,t)$ satisfies the BC's,

$G(0,t) = G'(0,t) \Rightarrow a_1 + a_2 = a_1 - a_2 \Rightarrow a_2 = 0$

$b_2 = c_2 + a_2 = \frac{+e^{+t}}{2} + 0 = \frac{+e^{+t}}{2}$

$G(l,t) + G'(l,t) = 0 \Rightarrow 2b_1 e^{l} = 0 \Rightarrow b_1 = 0$

$a_1 = b_1 - c_1 = \frac{e^{-t}}{2}$

$\therefore G(x,t) = \begin{cases} (e^{-t}e^{x})/2 \ ; & 0 \le x < t \\ (e^{t}e^{-x})/2 \ ; & t < x \le l \end{cases}$

$\therefore$ Solution of BVP is given by:-

$y = \int_0^l G(x,t)\,\phi(t)\,dt$, here $\phi(t) = 2e^{t}$

$= \int_0^x \frac{e^{t-x}}{2} \cdot 2e^{t}\,dt + \int_x^l \frac{e^{x-t}}{2} \cdot 2e^{t}\,dt$

$= \left. \frac{e^{2t-x}}{2} \right|_0^x + e^{x}(l-x)$

$= \frac{e^{x} - e^{-x}}{2} + e^{x}(l-x)$

$$\boxed{y = \sinh(x) + (l-x)e^{x}}$$