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NM Assignment 1

(Part 2)

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1) $x\left(\frac{d^2y}{dx^2}\right) + 2\left(\frac{dy}{dx}\right) + \left(\frac{xy}{2}\right) = 0$

Substitute, $z = y\sqrt{x} \Rightarrow y = \frac{z}{\sqrt{x}} = z \cdot x^{-1/2}$

$\therefore \frac{dy}{dx} = -\frac{1}{2}x^{-3/2}z$, $\frac{d^2y}{dx^2} = x^{-1/2}\frac{d^2z}{dx^2} - x^{-3/2}\frac{dz}{dx} + \frac{3}{4}x^{-5/2}z$

by substituting these in the given eq^n,

$x\left(x^{-1/2}\frac{d^2z}{dx^2} - x^{-3/2}\frac{dz}{dx} + \frac{3}{4}x^{-5/2}z\right) +$

$2\left(x^{-1/2}\frac{dz}{dx} - \frac{1}{2}x^{-3/2}z\right) + \frac{x}{2}\cdot\frac{z}{\sqrt{x}} = 0$

$\Rightarrow x^{1/2}\frac{d^2z}{dx^2} + x^{-1/2}\frac{dz}{dx} - \frac{1}{4}x^{-3/2}z + \frac{1}{2}x^{1/2}z = 0$

Multiplying x^3 on both sides

$x^2\left(\frac{d^2z}{dx^2}\right) + x\left(\frac{dz}{dx}\right) + \left(\frac{x^2}{2} - \frac{1}{4}\right)z = 0$

let $u = \frac{x}{\sqrt{2}}$, $\frac{dz}{dx} = \frac{dz}{du}\cdot\frac{du}{dx} = \frac{1}{\sqrt{2}}\frac{dz}{du}$ &

$\frac{d^2z}{dx^2} = \frac{1}{2}\frac{d^2z}{du^2}$

Substituting these above,

$u^2\frac{d^2z}{du^2} + u\frac{dz}{du} + \left(u^2 - \frac{1}{4}\right)z = 0$

Above eq^n is Bessel's eq^n of order $1/2$, & as $1/2$ is not an integer, solution is given by

$z = A\,J_{1/2}(u) + B\,J_{-1/2}(u)$

now, $y = \frac{z}{\sqrt{x}}$, $z = y\sqrt{x}$, & $u = \frac{x}{\sqrt{2}}$,

$$y\sqrt{x} = AJ_{1/2}(x/\sqrt{2}) + BJ_{-1/2}(x/\sqrt{2})$$

$$\Rightarrow y = \frac{A}{\sqrt{x}} J_{1/2}\left(\frac{x}{\sqrt{2}}\right) + \frac{B}{\sqrt{x}} J_{-1/2}\left(\frac{x}{\sqrt{2}}\right),$$ (A & B are arbitrary constants)

Solution of given eq$^{\underline{n}}$

2) $$\int_0^1 \frac{u J_0(xu)\,du}{(1-u^2)^{1/2}} = \frac{\sin x}{x}$$

now,

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \dots$$

$$\Rightarrow J(xu) = 1 - \frac{x^2u^2}{2^2} + \frac{x^4u^4}{2^2.4^2} - \frac{x^6u^6}{2^2.4^2.6^2} + \dots$$

$$\therefore \int_0^1 \frac{u J(xu)}{(1-u^2)^{1/2}}\,du = \int_0^1 \frac{u}{\sqrt{1-u^2}}\left[1 - \frac{x^2u^2}{2^2} + \frac{x^4u^4}{2^2.4^2} - \frac{x^6u^6}{2^2.4^2.6^2} + \dots\right]du$$

let $u = \sin\theta \Rightarrow du = \cos\theta\,d\theta$, when $u=0, \theta=0$
$u=1, \theta=\pi/2$

$$\int_0^{\pi/2} \frac{\sin\theta}{\cancel{\cos\theta}}\left[1 - \frac{x^2}{2^2}\sin^2\theta + \frac{x^4}{2^2.4^2}\sin^4\theta - \dots\right]\cancel{\cos\theta}\,d\theta$$

$$= \int_0^{\pi/2} \sin\theta\,d\theta - \frac{x^2}{2}\int_0^{\pi/2}\sin^3\theta\,d\theta + \frac{x^4}{2^2.4^2}\int_0^{\pi/2} \sin^5\theta\,d\theta \dots$$

Using wallis formula,

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{(n-1)(n-3)\dots 4.2}{n(n-2)(n-4)\dots 5.3} \quad \text{for odd } n \geq 3$$

hence, the above eq$^{\underline{n}}$ reduces to

$$-\cos\theta\Big|_0^{\pi/2} - \frac{x^2}{4}\cdot\frac{2}{3} + \frac{x^4}{2^2.4^2}\cdot\frac{4.2}{5.3} - \frac{x^6}{2^2.4^2.6^2}\cdot\frac{6.4.2}{7.5.3} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}$$

$$= \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\dots\right) = \frac{1}{x}\sin x = \frac{\sin x}{x}$$

hence proved

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3) $J_n(x) = \dfrac{x^n}{2^{n-1}\Gamma n} \int_0^{\pi/2} \sin\theta \cos^{2n-1}\theta \, J_0(x\sin\theta)\, d\theta$, $n > -\frac{1}{2}$

$J_0(x) = 1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2.4^2} - \dfrac{x^6}{2^2.4^2.6^2} + \dots$

$\Rightarrow J_0(x\sin\theta) = 1 - \dfrac{x^2\sin^2\theta}{2^2} + \dfrac{x^4\sin^4\theta}{2^2.4^2} - \dfrac{x^6\sin^6\theta}{2^2.4^2.6^2} + \dots$

The given expression becomes,

$\dfrac{x^n}{2^{n-1}\Gamma n} \int_0^{\pi/2} \sin\theta \cos^{2n-1}\theta \left[1 - \dfrac{x^2\sin^2\theta}{2^2} + \dfrac{x^4\sin^4\theta}{2^2.4^2} + \dots\right] d\theta$

$= \dfrac{x^n}{2^{n-1}\Gamma n}\left[\int_0^{\pi/2} \sin\theta\cos^{2n-1}\theta\, d\theta - \dfrac{x^2}{2^2}\int_0^{\pi/2}\sin^3\theta\cos^{2n-1}\theta\, d\theta + \dfrac{x^4}{2^2.4^2}\int_0^{\pi/2}\sin^5\theta\cos^{2n-1}\theta\, d\theta\right]$

Multiplying & Dividing by 2 ~~on both sides~~

$= \dfrac{x^n}{2^n\Gamma n}\left[2\int_0^{\pi/2}\sin\theta\cos^{2n-1}\theta\, d\theta - \dfrac{x^2 . 2}{2^2}\int_0^{\pi/2}\sin^3\theta\cos^{2n-1}\theta\, d\theta + \dfrac{x^4 . 2}{2^2.4^2}\int_0^{\pi/2}\sin^5\theta\cos^{2n-1}\theta\, d\theta\right]$

$= \dfrac{x^n}{2^n\Gamma n}\left[\beta(1,n) - \dfrac{x^2}{2^2}\beta(2,n) + \dfrac{x^4}{2^2.4^2}\beta(3,n) - \dots\right]$

$= \dfrac{x^n}{2^n \cancel{\Gamma n}}\left[\dfrac{\Gamma 1}{\Gamma(n+1)} - \dfrac{x^2}{2^2}\dfrac{\Gamma 2}{\Gamma n+2} + \dfrac{x^4}{2^2.4^2}\dfrac{\Gamma 3}{\Gamma n+3} - \dots\right]$

$= \dfrac{x^n}{2^n \cancel{\Gamma n}}\left((-1)^0\dfrac{1}{0!\,\Gamma n+1} + (-1)^1\left(\dfrac{x}{2}\right)^2\dfrac{1}{1!\,\Gamma n+2} + (-1)^2\left(\dfrac{x}{2}\right)^4\dfrac{\Gamma 3}{(\Gamma 3)^2\,\Gamma n+3} + (-1)^3\left(\dfrac{x}{2}\right)^6\dfrac{\Gamma 4}{(\Gamma 4)^2\,\Gamma n+4}\right)$

$= \sum_{r=0}^{\infty}(-1)^r\left(\dfrac{x}{2}\right)^{n+2r} . \dfrac{1}{r!\,\Gamma n+r+1} = J_n(x)$

hence proved

4) $\int_0^x t.[J_n(t)]^2 dt = \frac{x^2}{2}\left[J_n^2(x) - J_{n-1}(x).J_{n+1}(x)\right]$

$LHS = \int_0^x t.[J_n(t)]^2 dt = \int_0^x [J_n(t)]^2 \, t.dt$

Integration by parts

$= \left[J_n(t)^2 \times \frac{t^2}{2}\right]_0^x - \int_0^x \frac{t^2}{2}.2J_n(t)J_n'(t)dt$

$= \frac{x^2}{2}.J_n^2(x) - \int_0^x t^2 J_n(t).J_n'(t)dt$

We solve for $I = \int_0^x t^2 J_n(t).J_n'(t)dt$

Using Recurrance formula $\leftarrow = \int_0^x \left(t^2 J_n(t).J_{n-1}(t) - t^2 J_n(t) J_{n+1}(t)\right) dt$

$= \int_0^x (1/2)\left(t^2 J_n(t) J_{n-1}(t) - t^2 J_n(t) J_{n+1}(t)\right) dt$

using $t^2 = t^{n+1}.t^{1-n}$ $\leftarrow = \frac{1}{2}\int_0^x \left(t^{n+1} J_n(t)\right)\left(t^{-(n-1)} J_{n-1}(t) - t^{n+1} J_{n+1}(t).t^{-(n-1)} J_n(t)\right) dt$

We know,

$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$ & $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

$\therefore t^{n+1}.J_n(t) = \frac{d}{dt}\left[t^{n+1} J_{n+1}\right]$, $-t^{-(n-1)} J_n(t) = \frac{d}{dt}\left[t^{-(n-1)} J_{n-1}(t)\right]$

Using the above in I, we get

$I = \frac{1}{2}\int_0^x \left[\frac{d}{dt}\left\{t^{n+1} J_{n+1}(t)\right\}.t^{-(n-1)} J_{n-1}(t) + \left(t^{n+1} J_{n+1}(t)\right)\frac{d}{dt}\left\{t^{-(n-1)} J_{n-1}(t)\right\}\right]$

$\Rightarrow I = \frac{1}{2}\int_0^x \frac{d}{dt}\left(t^{n+1} J_{n+1}(t).t^{-(n-1)} J_{n-1}(t)\right) dt \longrightarrow$ uv rule

$\Rightarrow I = \frac{1}{2}\int_0^x \frac{d}{dt}\left(t^2 J_{n+1}(t).J_{n-1}(t)\right) dt$

$\Rightarrow I = \frac{1}{2}\int_0^x d\left(t^2 J_{n+1}(t).J_{n-1}(t)\right) = \frac{1}{2}\left(t^2 J_{n+1}(t).J_{n-1}(t)\right)\Big|_0^x$

$= \frac{1}{2}x^2 J_{n+1}(x).J_{n-1}(x)$

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now,

$$LHS = \frac{x^2}{2} \times J_n^2(x) - J = \frac{x^2}{2} J_n^2(x) - \frac{x^2}{2} J_{n-1}(x) J_{n+1}(x) = RHS$$

hence proved

5) By recurrence formula, $J_n(x) = \frac{x}{2n}\left(J_{n-1}(x) + J_{n+1}(x)\right)$

$$\Rightarrow J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

For $n = 1, 2, 3,$

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

$$J_3(x) = 4/x \; J_2(x) - J_1(x)$$

$$J_4(x) = 6/x \; J_3(x) - J_2(x)$$

$$\therefore J_3(x) = \frac{4}{x}\left(\frac{2}{x} J_1(x) - J_0(x)\right) - J_1(x)$$

$$= \left(\frac{8}{x^2} \cancel{84} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

now

$$J_4(x) = \frac{6}{x}\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{6}{x} \cdot \frac{4}{x} J_0(x)$$

$$- \frac{2}{x} J_1(x) + J_0(x)$$

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

~~hence proved~~ Ans