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Keerti P. Charantimath — MM Assignment 5
19MA20059 — Part 2

Q1. Suppose A^{ij} be a skew-symmetric contravariant tensor of the type (2,0) wrt i & j in x^i coordinate system, i.e $A^{ji} = -A^{ij}$

We know,

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k} \times \frac{\partial \bar{x}^j}{\partial \bar{x}^l} \times A^{kl} = \frac{\partial \bar{x}^i}{\partial x^k} \times \frac{\partial \bar{x}^j}{\partial x^l} \times (-A^{lk}) = -\bar{A}^{ji}$$

Thus, $\bar{A}^{ij} = -\bar{A}^{ji} \Rightarrow \bar{A}^{ij}$ is skew symmetric wrt i,j in $\bar{x}^i$ coordinate system.

Similarly, let A_{ij} be a skew-symmetric covarient tensor of type (0,2) wrt i & j in x^i coordinate system, i.e $A_{ji} = -A_{ij}$

$$\bar{A}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \times \frac{\partial x^l}{\partial \bar{x}^j} \times A_{kl} = \frac{\partial x^k}{\partial \bar{x}^i} \times \frac{\partial x^l}{\partial \bar{x}^j} \times (-A_{lk}) = -\bar{A}_{ji}$$

Thus, $\bar{A}_{ij} = -\bar{A}_{ji} \Rightarrow \bar{A}_{ij}$ is skew-symmetric wrt i,j in $\bar{x}^i$ system

Hence proved that a skew-symmetric tensor is skew-symmetric in every system

Q2 ~~Consider~~ ~~&~~ Consider $A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$ of the (r,s) type. Suppose its components vanish in x^i coordinate system, i.e $A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} = 0$

We know,

$$\bar{A}^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} = \frac{\partial \bar{x}^{i_1}}{\partial x^{k_1}} \times \dots \times \frac{\partial \bar{x}^{i_r}}{\partial x^{k_r}} \times \frac{\partial x^{l_1}}{\partial \bar{x}^{j_1}} \times \dots \times \frac{\partial x^{l_s}}{\partial \bar{x}^{j_s}} \times A^{k_1 k_2 \dots k_r}_{l_1 l_2 \dots l_s} = 0$$

Thus, components of tensor vanish identically in any other coordinate system also. Hence Proved.

Q3. Contraction is the process of getting a tensor of lower rank (reduced by 2) by putting a covariant index equal to a contravarient index, & performing the summation acc. to summation convention.

Consider A^{ijk}_{lm} of order 5, type (3,2) putting $l =$ contravariant index i, we get law of transformation as:- $\bar{A}^{ijk}_{im} = \frac{\partial \bar{x}^i}{\partial x^p} \times \frac{\partial \bar{x}^j}{\partial x^q} \times \frac{\partial \bar{x}^k}{\partial x^h} \times \frac{\partial x^s}{\partial \bar{x}^i} \times \frac{\partial x^t}{\partial \bar{x}^m} \times A^{pqr}_{st}$

$$= \frac{\partial \bar{x}^j}{\partial x^q} \times \frac{\partial \bar{x}^k}{\partial x^h} \times \frac{\partial x^t}{\partial \bar{x}^m} \times \left(\delta^s_p \times A^{pqr}_{st}\right) \quad \left[\frac{\partial x^s}{\partial \bar{x}^p} = \delta^s_p\right]$$

A^{pqs}_{pt}

A^{ijk}_{im} is of rank 3 & type (2,1) whereas A^{ijk}_{lm} is of rank 5 & type (3,2)

Thus by contraction, we get a tensor of rank reduced by 2.

now, ~~cond~~ consider $a_{ij} \times a^{ij}$

let $d = |a_{ij}|$ be the determinant with elements a_{ij} & $d \neq 0$

Then, the reciprocal tensor of a_{ij} is defined as

$$a^{ij} = \frac{\text{cofactor of } a_{ij} \text{ in } |a_{ij}|}{d} = \frac{B_{ij}}{d}$$

We know, that sum of an element multiplied by its cofactor over any row/column gives the determinant

Hence, $a_{ij} \times B_{ij} = d \Rightarrow a_{ij} \times \frac{B_{ij}}{d} = 1 \Rightarrow a_{ij} \times a^{ij} = 1$

Also, $\delta^j_j = 1$

Hence proved that $a_{ij} \times a^{ij} = \delta^j_j$

Q4 $c_{ij} A^i A^j$ is an invariant, for A^i being an arbitrary contravariant vector.

So, $\bar{c}_{ij} \bar{A}^i \bar{A}^j = c_{kl} A^k A^l \Rightarrow \bar{c}_{ij} \times \frac{\partial \bar{x}^i}{\partial x^k} A^k \times \frac{\partial \bar{x}^j}{\partial x^l} A^l = c_{kl} A^k A^l$

$$\Rightarrow \cancel{\bar{c}_{ij} \bar{A}^i \bar{A}^j} \underbrace{\left(\bar{c}_{ij} \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} - c_{kl}\right)}_{B_{kl}} A^k A^l = 0 \Rightarrow B_{kl} A^k A^l = 0$$

As k, l are dummy indices, we can ~~then~~ interchange, $B_{lk} A^l A^k = 0$

Hence $(B_{kl} + B_{lk}) A^l A^k = 0$, As A^i is arbitrary contravariant vector, we must have $(B_{kl} + B_{lk}) = 0$

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$$\Rightarrow \bar{C}_{ij}\frac{\partial \bar{x}^i}{\partial x^k}\times\frac{\partial \bar{x}^j}{\partial x^l} - C_{kl} + \bar{C}_{ji}\times\frac{\partial \bar{x}^j}{\partial x^l}\times\frac{\partial \bar{x}^i}{\partial x^k} - C_{lk} = 0$$

$$\Rightarrow \bar{C}_{ij} + \bar{C}_{ji} = \left(\frac{\partial x^k}{\partial \bar{x}^i}\times\frac{\partial x^l}{\partial \bar{x}^j}\right)(C_{kl} + C_{lk})$$

Thus $(C_{ij} + C_{ji})$ is a covariant vector of order 2. Hence proved

Q5 let $g = |g_{ij}|$ be the determinant.

We know, $g_{ik}\times g^{il} = \delta^l_k \Rightarrow g_{ik}\times g^{ik} = \delta^k_k = 1$

Also, ~~g~~ $g^{ik} = \frac{G_{ik}}{g} \rightarrow$ cofactor of g_{ik} in $|g_{ik}|$

$\Rightarrow g\times g^{ik} = G_{ik} \Rightarrow g\times g_{ik}\times g^{ik} = g_{ik}G_{ik} \Rightarrow g = g_{ik}\times G_{ik}$

differentiating partially wrt g_{ik},

$$\frac{\partial g}{\partial g_{ik}} = G_{ik}$$

now, $\frac{\partial g}{\partial x^j} = \frac{\partial g}{\partial g_{ik}}\times\frac{\partial g_{ik}}{\partial x^j} = G_{ik}\times\frac{\partial g_{ik}}{\partial x^j} = g\times g^{ik}\times\frac{\partial g_{ik}}{\partial x^j}$

$$\Rightarrow \frac{1}{g}\times\frac{\partial g}{\partial x^j} = g^{ik}\frac{\partial g_{ik}}{\partial x^j} = g^{ik}\times\{[jk, i] + [ij, k]\} = \left\{ {k \atop jk} \right\} + \left\{ {i \atop ij} \right\}$$

$$\Rightarrow \frac{1}{g}\times\frac{\partial g}{\partial x^j} = \left\{ {i \atop ji} \right\}\left\{ {i \atop ji} \right\} = 2\left\{ {i \atop ji} \right\}$$ [as k was dummy index]

$$\Rightarrow \frac{1}{2g}\frac{\partial g}{\partial x^j} = \left\{ {i \atop ji} \right\} \Rightarrow \frac{\partial \log(\sqrt{g})}{\partial x^j} = \left\{ {i \atop ji} \right\}$$

hence proved that $\left\{ {i \atop ij} \right\} = \frac{\partial \log(\sqrt{g})}{\partial x^j}$

Q6 For each pair (ij), or for each independent g_{ij}, there are n distinct Christoffel symbols of each kind due to another free index k in each christoffel symbol. Since g_{ij} is a symmetric tensor of rank 2, it has $\frac{n(n+1)}{2}$ independent components at max. Hence, independent components of Christoffel's symbol of each kind is $= n\times\frac{n(n+1)}{2} = \frac{n^2(n+1)}{2}$

let the symbols in x^i coordinate system be transformed twice to $\bar{x}^i$ coordinate system

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According to law of transformation of Christoffel's symbols of second kind

$$\overline{\left\{ {k \atop ij} \right\}} = \frac{\partial \bar{x}^k}{\partial x^s} \times \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} + \frac{\partial \bar{x}^k}{\partial x^s} \times \frac{\partial x^p}{\partial \bar{x}^i} \times \frac{\partial x^q}{\partial \bar{x}^j} \left\{ {s \atop pq} \right\} \rightarrow (1)$$

$$\overline{\overline{\left\{ {r \atop uv} \right\}}} = \overline{\left\{ {k \atop ij} \right\}} \times \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} \times \frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^r}{\partial \bar{x}^k} + \frac{\partial^2 \bar{x}^k}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^r}{\partial \bar{x}^k}$$

from (1) we get

$$\overline{\overline{\left\{ {r \atop uv} \right\}}} = \left\{ {s \atop pq} \right\} \times \frac{\partial x^p}{\partial \bar{\bar{x}}^u} \times \frac{\partial x^q}{\partial \bar{\bar{x}}^v} + \frac{\partial^2 \bar{x}^k}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^r}{\partial \bar{x}^k} + \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} \times \frac{\partial \bar{\bar{x}}^r}{\partial x^s} \times \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} \times \frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^v} \rightarrow (2)$$

we know: $\frac{\partial x^s}{\partial \bar{x}^i} \times \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} = \frac{\partial x^s}{\partial \bar{\bar{x}}^u}$, differentiating partially wrt $\bar{\bar{x}}^v$

$$\frac{\partial}{\partial \bar{\bar{x}}^v}\left(\frac{\partial x^s}{\partial \bar{x}^i}\right)\frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} + \frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial}{\partial \bar{\bar{x}}^v}\left(\frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u}\right) = \frac{\partial^2 x^s}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v}$$

$$\Rightarrow \frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} \times \frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^v} \times \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} + \frac{\partial x^s}{\partial \bar{x}^i} \times \frac{\partial^2 \bar{x}^i}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v} = \frac{\partial^2 x^s}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v}$$

Multiplying by $\frac{\partial \bar{\bar{x}}^r}{\partial x^s}$ & replacing i with k as it is dummy index then

$$\frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} \times \frac{\partial \bar{x}^j}{\partial \bar{\bar{x}}^v} \times \frac{\partial \bar{x}^i}{\partial \bar{\bar{x}}^u} \times \frac{\partial \bar{\bar{x}}^r}{\partial x^s} + \frac{\partial \bar{\bar{x}}^r}{\partial \bar{x}^k} \times \frac{\partial^2 \bar{x}^k}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v} = \frac{\partial^2 x^s}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^r}{\partial x^s} \rightarrow (3)$$

Using (2) & (3) $\Rightarrow$ $\overline{\overline{\left\{ {r \atop uv} \right\}}} = \left\{ {s \atop pq} \right\} \times \frac{\partial x^p}{\partial \bar{\bar{x}}^u} \times \frac{\partial x^q}{\partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^r}{\partial x^s} + \frac{\partial^2 x^s}{\partial \bar{\bar{x}}^u \partial \bar{\bar{x}}^v} \times \frac{\partial \bar{\bar{x}}^r}{\partial x^s}$

The above equation is similar to (1). Thus we made direct transformation from x^i to $\bar{\bar{x}}^i$ coordinate system and we get the same law of transformation. Hence christoffel symbol of second kind possess group property upon transformation.

We know that $[ij,m] = g_{km}\left\{ {k \atop ij} \right\}$, ($g_{km} \rightarrow$ fundamental tensor). As $\left\{ {k \atop ij} \right\}$ possesses group property upon transformation & $[ij,m]$ is a product of $\left\{ {k \atop ij} \right\}$ & g_{km}, & both follow transitive property, we can say that $[ij,m]$ also follows transitive property. ($[ij,m] \rightarrow$ christoffel symbol of first kind)

(g_{km} follows transitive property under transformation as proved earlier)

Hence we can conclude that the laws of transformations of Christoffel symbols possess group properties

Hence proved.