Primal Simplex Method, Dual Simplex Method and Primal-Dual Simplex Method (Combined)

By Prof M P Biswal
E-mail: mpbiswal@maths.iitkgp.ac.in

1. Linear Programming Problem–Primal (LPP-Primal)

$$\max: \quad Z = c^T x \tag{1.1}$$

subject to

$$\begin{aligned} Ax &\leq b, \quad b \geq 0, &(1.2)\\ x &\geq 0. &(1.3)\end{aligned}$$

where

$$\begin{aligned} c &= (n \times 1) \text{ column vector containing the coefficient of } x_j\\ A &= (m \times n) \text{ coefficient matrix of the constraints}\\ b &= (m \times 1) \text{right-hand-side column vector}\\ x &= (n \times 1) \text{ column vector of the decision variables}\end{aligned}$$

After adding slack variables (Basic variables) to all the constraints the problem can be rewritten as:

$$\max: \quad Z = c^T x + c_B^T x_s \tag{1.4}$$

subject to

$$\begin{aligned} Ax + x_s &= b &(1.5)\\ x, x_s &\geq 0. &(1.6)\end{aligned}$$

where $\boldsymbol{A} = [a_{ij}]_{m\times n}$, $x = \begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}$, $x_s = \begin{bmatrix} x_{B,1}\\ x_{B,2}\\ \vdots\\ x_{B,m}\end{bmatrix}$, $c_B = \begin{bmatrix} c_{B,1}\\ c_{B,2}\\ \vdots\\ c_{B,m}\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ \vdots\\ 0\end{bmatrix}$, $b = \begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_m\end{bmatrix}$, $c = \begin{bmatrix} c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix}$.

The basis matrix, denotes as $\boldsymbol{B}_{m\times m}$, is composed of m linearly independent columns from $\boldsymbol{A} = [a_{ij}]_{m\times(n+m)}$. Thus $\boldsymbol{B}$ is an $(m \times m)$ identity matrix (non-singular matrix).

Table 1: Initial Simplex Table (Condensed Tableau)

Coefficient of the Basic Variables	Basic Variable Labels	Non-Basic Variable Labels				x_B
		c_1	c_2	$\dots$	c_n	
		x_1	x_2	$\dots$	x_n	
$c_{B,1}$	$x_{B,1}$	$a_{1,1}$	$a_{1,2}$	$\dots$	$a_{1,n}$	b_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$c_{B,m}$	$x_{B,m}$	$a_{m,1}$	$a_{m,2}$	$\dots$	$a_{m,n}$	b_m
$\dots$	Indicator Row	$z_1 - c_1$	$z_2 - c_2$	$\dots$	$z_n - c_n$	Z

Normally, the initial columns in $\boldsymbol{B}$ are those associated with the slacks and/or artificial variables. Presently, only slack variables are included in the converted constraints.

A basic solution designated as $\boldsymbol{X_B}$ is given by

$$X_B = B^{-1}b, \quad |B| \neq 0, \tag{1.7}$$

where

$$X_B = \begin{pmatrix} x_{B,1} \\ x_{B,2} \\ \vdots \\ x_{n,m} \end{pmatrix}$$

If all $x_{B,i} \geq 0$, then $\boldsymbol{X_B}$ is a basic feasible solution.

Given a basic feasible solution, $\boldsymbol{X_B}$, the value of the objective function (Z) is given as:

$$Z = c_B^T X_B \tag{1.8}$$

where

$$c_B^T = (c_{B,1} \quad c_{B,2} \quad \dots \quad c_{B,m})$$

We shall define Y as:

$$Y = c_B^T B^{-1}, \text{(for the non-basic variables)} \tag{1.9}$$

where $B^{-1} = I_{m \times m}$, $Y = \big(y_1, \quad \dots \quad y_m\big)_{1 \times m}$

$$\text{and} \quad z_j - c_j = YP_j - c_j \quad \text{(for all the non-basic variables)} \tag{1.10}$$

1.1. ***Steps of the Simplex Algorithm (Condensed Tableau)***

Step 1. We begin our search with a basic feasible solution, $\boldsymbol{X_B} = \boldsymbol{B^{-1}b}$ (where $\boldsymbol{X_B}$ is normally composed of slacks variables for primal simplex method).

Step 2. Examine $z_j - c_j$ for all $\boldsymbol{P_j}$ (columns of the non-basic variables) not in the basis. If all $z_j - c_j \geq 0$, go to Step 6.

Step 3. If, for any $\boldsymbol{P_j}$ for which $z_j - c_j$ is negative, there are no positive elements in P_j, then the problem is unbounded and we Stop. Otherwise, we select the associated variable (i.e. associated vector) with the most negative $z_j - c_j$ as an entering variable to enter the basis.

Step 4. Use $\dfrac{x_{B,r}}{a_{r,j}} = \min\limits_i \left\{ \dfrac{x_{B,i}}{a_{i,j}},\ a_{i,j} > 0 \right\}$
to determine the departing variable (departing vector) which leave the basis.

Step 5. Establish new Simplex tableau, and new basis matrix B_{new}. Then find the new basis feasible solution and the new objective function value. Return to Step 2.

Step 6. If any variable in the basis is both an artificial variable and has a positive value, the problem is infeasible. Otherwise, we have obtained the optimal solution. Note that if any $z_j - c_j$ equals to zero for an $\boldsymbol{P_j}$, not in the basis, an alternative optimal solution exist. To find an alternate optimal solution, one must complete one more iteration.

2. Steps of the Simplex Algorithm (Extended Tableau)

Step 1. ***Check all the possible improvement***. Examine the $z_j - c_j$ values in the indicator row. If these are all non-negative, go to Step 2. If, however, any $z_j - c_j$ is negative, we go to Step 3.

Step 2. ***Check for optimality or infeasibility***. If all $z_j - c_j \geq 0$ and no artificial variable is in the basis at a positive value, the solution is optimal. Otherwise (if an artificial is in a the basis at a positive value), the problem is (mathematically) infeasible. In either case, we are finished.

Step 3. ***Check for unboundedness***. If, for any $z_j - c_j < 0$, there are no positive elements in the associated y_j vector (the column directly above $z_j - c_j$ in the tableau), the problem is unbounded. Otherwise, an improvement is possible and we go to step 4.

Step 4. ***Determining the entering variable***. Select, as the entering variable, the (non-basic) variable withe most negative $z_j - c_j$ value. Designate this variable as x_j and its corresponding column as j'. Ties in the selection of j' may be broken arbitrarily. Go to Step 5.

Step 5. ***Determining the departing variable***. We use the relationship of $\frac{x_{B,r}}{a_{r,j}} = \min_i \left\{ \frac{x_{B,i}}{a_{i,j}},\ a_{i,j} > 0 \right\}$ to determine the departing variable (vector). This is accomplished in the tableau by taking the ratio

$$\frac{x_{B,i}}{a_{i,j'}}, \quad (a_{i,j'} > 0). \tag{2.1}$$

For each row, designate the row having the minimum ratio of $\frac{x_{B,i}}{a_{i,j'}}$ as row i'. The basis variable associated with row i' is the departing variable.

Step 6. ***Establishment of a new Simplex Tableau***.

- Set up a new tableau with all $y_j, z_j - c_j, z$ and basic feasible solution $(\boldsymbol{X_B})$ value empty. Replace the departing basic variables row heading $(\boldsymbol{x_{B,i}})$ with the entering variable label (x_j'). Replace $\boldsymbol{c_{B,i}}$ with c_j'.
- Row i' of the new tableau is obtained by dividing row i' of the preceding tableau by $a_{i',j'}$ (the element at the intersection of the entering variable column and departing variable row).
- Column j' of the new tableau consist of all zeros elements except for a 1 at $a_{i',j'}$.
- The remaining elements of the tableau are computed as follows. Let $\hat{x}_{B,i}, \hat{z}, \hat{z}_j - \hat{c}_j$ and $\hat{a}_{i',j'}$ represent the new set of elements to be computed and let $x_{B,i}, z, z_j - c_j$ and $a_{i',j'}$ represent the value for these elements from the preceding tableau Then, for those elements not in row i' or column j':

$$\begin{aligned}
\hat{a}_{i,j} &= a_{i,j} - \frac{(a_{i',j})(a_{i,j'})}{a_{i',j'}} && (2.2)\\
\hat{x}_{B,i} &= x_{B,i} - \frac{(x_{B,i'})(a_{i,j'})}{a_{i',j'}} && (2.3)\\
\hat{z}_j - \hat{c}_j &= (z_j - c_j) - \frac{(z_{j'} - c_{j'})(a_{i,j'})}{a_{i',j'}} && (2.4)\\
\hat{z} &= z - \frac{(z_{j'} - c_{j'})(x_{B,i'})}{a_{i',j'}} && (2.5)
\end{aligned}$$

- Return to Step 1.

Notes: Please see the Numerical Examples for the Condensed and Extended Simplex Tableau.

3. Two-Phase Simplex Algorithm

Step 1. Establish the problem formulation in a form suitable for the implementation of the simplex algorithm (i.e., convert the objective function to a maximization form and convert all the constraints by adding proper, the slack, surplus, or artificial variables).

Step 2. The artificial objective function of Phase-I is constructed by changing all the coefficient of the variables in the original objective function as follows:

- The coefficient of any artificial variables will be -1.
- The coefficient of all other variables in the objective will be zero.

Step 3. *Phase-I.* Employ the simplex algorithm which is provided earlier on the problem constructed in Step 1 and 2. However, we may terminate the process (i.e., Phase-I) as soon as the value of Z (the value of the artificial objective) is *zero*. If the simplex process ends with either $Z = 0$ or all $z_j - c_j \geq 0$ and there are no artificial variables in the basis at a positive value, we go to Step 4 (i.e., Phase-II). Otherwise, the problem is (mathematically) infeasible and we stop.

<u>Phase-II</u>

Step 4. Assign the actual objective function coefficient (the original c_j's) to each variable except for the artificial variables. Any artificial variable in the basis at a zero level are given c_j value of 0 in Phase-II. Any artificial not in the basis may be dropped from the consideration by striking out their entire associated column in the tableau.

Step 5. The first tableau of Phase-II is the final tableau of Phase-I except for the objective function coefficients and the indicator row values. We recompute the indicator row values (all $z_j - c_j$) and objective function Z value.

Step 6. If no artificial variable were in the basis (at zero values) at the end of Phase-I, we simply use the simplex algorithm and proceed as usual manner. If, however, there are artificial variables in the basis, go to Step 7.

Step 7. We must take sure that the artificial variables in the basis do not ever becomes positive form zero in Phase-II. This is accomplished by modifying the departing variable rule of the simplex algorithm as follows:

- Determine the entering variable and its associated column (j') is the usual manner.
- Examine the $a_{i,j'}$ values for each artificial variable. If any of these are negative, let an artificial with a negative $a_{i,j'}$ depart. Otherwise, employ the usual departing variable rule.

4. Duality Theory (Primal-Dual Linear programming Problem)

Primal LPP:

$$(P) \qquad \max: \quad Z = c^T x \tag{4.1}$$

subject to

$$\begin{aligned} Ax &\leq b, \quad b \geq 0, && (4.2)\\ x &\geq 0. && (4.3) \end{aligned}$$

where $\boldsymbol{A} = [a_{ij}]_{m\times n}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.

Dual LPP:

$$(D) \qquad \min: \quad z = b^T y \tag{4.4}$$

subject to

$$\begin{aligned} A^T y &\geq c, \quad c \geq 0, && (4.5)\\ y &\geq 0. && (4.6) \end{aligned}$$

where $\boldsymbol{A}^T = [a_{ji}]_{n\times m}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$.

Relationship 1. The dual of the dual is primal.

Relationship 2. A $m \times n$ primal LPP gives an $n \times m$ dual LPP.

Relationship 3. For each primal constraint, there is a related dual variable, and vice versa.

Relationship 4. For each primal variable, there is a related dual constraint, and vice versa.

Relationship 5. In a general LPP, an unrestricted variable in one problem gives an associated equality constraint in the other, and vice versa.

Relationship 6. Given the canonical form of the LPP with Z the objective function value if the maximizing primal, z the objective function value of the minimizing dual, x_0 a feasible solution to the primal, and y_0 a feasible solution to the dual:

1. $Z \leq z$ (Weak Duality). $\longrightarrow$ True for all BFS of Z & z
2. $Z^* = z^*$ (Strong Duality).

3. If $c^T x_0 = b^T y_0$, then $x_0 = x^*$ and $y_0 = y^*$, $z^* = Z^*$ where Z^* = maximum value of the primal objective function and z^* = minimum value of dual objective function.

Relationship 7. If one problem has an optimal solution, the other has an optimal solution.

Relationship 8. If the primal is unbounded, the dual is infeasible.

Relationship 9. If the primal is infeasible, the dual may be either unbounded or infeasible.

5. Dual Simplex Algorithm: Condensed Simplex Tableau

Step 1. To employ the algorithm, the problem must be dual feasible and primal infeasible. That is, all $z_s - c_s \geq 0$ and one or more $x_{B,i} < 0$. If these conditions are met, go to Step 2.

Step 2. Select the row associated with the most negative $x_{B,i}$ element. The basic variable associated with this row is departing variable. Denote those row as row $i^{'}$.

Step 3. Determine the column ratios for only those columns having a negative element in row $i^{'}$ (i.e., $a_{i^{'},s} < 0$).the column ratio is given by

$$\Phi_s \quad = \quad \min_s \left\{ \left| \frac{z_s - c_s}{a_{i',s}} \right| \right\} \tag{5.1}$$

where $a_{i',s} < 0$ and $z_s - c_s \geq 0$. Designate the column associated with the minimum Φ_s as column $s^{'}$. The non-basic variable associated with column $s^{'}$ is the new entering variable.

Step 4. Using the same procedure as with the original simplex algorithm, exchange the departing variable for the entering variable and establish the new simplex tableau.

Step 5. If all $x_{B,i}$ are now positive, we stop, having found the optimal feasible solution. If not, return to Step 2.

6. Primal-Dual Simplex Algorithm (Combined) → IMP

Step 1. ***Problem Form.*** All the constraint must be converted to Type-I form ($\leq$) and the objective function must be of the maximization form.

Step 2. Add the slack variable to each constraint and establish the condensed simplex tableau for the problem. (Note that the initial basic solution will always consist of strictly slack variables).

Step 3. Evaluate the impact (i.e., the numerical change in value) on the objective function by both primal and dual simplex method as follows:

- ***Primal Simplex Impact=PI.*** If a primal simplex pivot is possible[a], designate the associated pivot row and column as $i^{'}$ and $s^{'}$, respectively. The primal impact is then

$$PI = \left| \frac{(z_{s'} - c_{s'})(x_{B,i'})}{a_{i',s'}} \right| \tag{6.1}$$

- ***Dual Simplex Impact=DI.*** If a dual simplex pivot is possible[b], designate the associated pivot row and column as $i^{'}$ and $s^{'}$, respectively. Then the dual impact is then given by:

$$DI = \left| \frac{(z_{s'} - c_{s'})(x_{B,i'})}{a_{i',s'}} \right| \tag{6.2}$$

Step 4. Select either the primal or dual simplex pivot according to which has the largest impact value in Step 3. If neither a dual simplex nor a primal simplex pivot is possible, we terminate the process. Otherwise, return to Step 3.

[a] The conditions for a primal simplex pivot are $z_{s'} - c_{s'} \leq 0, x_{B,i'} \geq 0$, and $a_{i',s'} > 0$.

[b] The conditions for a dual simplex pivot are $z_{s'} - c_{s'} \geq 0, x_{B,i'} \leq 0$, and $a_{i',s'} < 0$.

(i) If $PI > DI$, then proceed with primal simplex method.

(ii) If $PI < DI$, then proceed with dual simplex method.

(iii) If $PI = DI$, then select any method.

$\boxed{\text{Simplex Method: Extended Tableau}}$ Numerical Example (b1):

$$\max : Z = 2x_1 + 8x_2$$

Subject to

$$x_1 + x_2 \leq 10$$

$$x_1 + 4x_2 \leq 16$$

$$x_1, x_2 \geq 0$$

$\boxed{\text{Simplex Method: Extended Tableau}}$ Numerical Example (b1):

$$\max : Z = 2x_1 + 8x_2 + 0s_1 + 0s_2$$

Subject to

$$x_1 + x_2 + s_1 = 10$$

$$x_1 + 4x_2 + s_2 = 16$$

$$x_1, x_2 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

$\boxed{\text{Simplex Method: Extended Tableau}}$ Numerical Example (b1):

Table 0:

SIMP	CV	2	8	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	XB
0	s_1	1	1	1	0	10
0	s_2	1	* 4	0	1	16
*	*	-2	- 8	0	0	0

$\boxed{\text{Simplex Method: Extended Tableau}}$ Numerical Example (b1):

Table 1:

SIMP	CV	2	8	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	XB
0	s_1	*3/4	0	1	-1/4	6
8	x_2	1/4	1	0	1/4	4
*	*	0	0	0	2	32

$$Optimal \quad Solution:$$

$$x_1^* = 0, x_2^* = 4, Z^* = 32,$$

It has alternate optimal solutions.

$\boxed{\text{Simplex Method: Extended Tableau}}$ Numerical Example (b1):

Table 2:

SIMP	CV	2	8	0	0	b
CB	BV/V	x_1	x_2	s_1	s_2	XB
2	x_1	1	0	4/3	-1/3	8
8	x_2	0	1	-1/3	1/3	2
*	*	0	0	0	2	32

$$Alternate \quad Optimal \quad Solution:$$

$$x_1^* = 0, x_2^* = 4, Z^* = 32,$$

$$x_1^* = 8, x_2^* = 2, Z^* = 32,$$

$\boxed{\text{Simplex Method: Condensed Tableau}}$ Numerical Example (b2):

$$\max : Z = 2x_1 + 8x_2$$

Subject to

$$x_1 + x_2 \leq 10$$

$$x_1 + 4x_2 \leq 16$$

$$x_1, x_2 \geq 0$$

$\boxed{\text{Simplex Method: Condensed Tableau}}$ Numerical Example (b2):

$$\max : Z = 2x_1 + 8x_2 + 0s_1 + 0s_2$$

Subject to

$$x_1 + x_2 + s_1 = 10$$

$$x_1 + 4x_2 + s_2 = 16$$

$$x_1, x_2 \geq 0$$

Slack variables(Basic variables) :

$$s_1, s_2 \geq 0$$

$\boxed{\text{Simplex Method: Condensed Tableau}}$ Numerical Example (b2):

Table 0:

SIMP	CN	2	8	b
CB	BV/NV	x_1	x_2	XB
0	s_1	1	1	10
0	s_2	1	* 4	16
*	*	-2	-8	0

$\boxed{\text{Simplex Method: Condensed Tableau}}$ Numerical Example (b2):

Table 1:

SIMP	CN	2	0	b
CB	BV/NV	x_1	s_2	XB
0	s_1	*3/4	-1/4	6
8	x_2	1/4	1/4	4
*	*	0	2	32

$\boxed{\text{Simplex Method: Condensed Tableau}}$ Numerical Example (b2):

Table 2:

SIMP	CN	0	0	b
CB	BV/NV	s_1	s_2	XB
2	x_1	4/3	-1/3	8
8	x_2	-1/3	1/3	2
*	*	0	2	32

$$Alternate\ \ Optimal\ \ Solution :$$

$$x_1^* = 8, x_2^* = 2, Z^* = 32,$$

$$x_1^* = 0, x_2^* = 4, Z^* = 32,$$