Number of distinit labeled trees of n vertices

Cayley's Theorem ([1889])

If $n \geq 2$, then there are n^{n-2} distinct labeled trees of n vertices.

- Let $N(d_1, d_2, \ldots, d_n)$ count the no. of labeled trees with n vertices with the vertex labeled i having degree $d_i + 1$.

Theorem. If $n \geq 2$ and all d_i are non-negative, then

$$N(d_1, d_2, \ldots, d_n) = \begin{cases} 0 & \text{if } \sum_{i=1}^{n} d_i \neq n-2 \\ C(n-2; d_1, d_2, \ldots, d_n) & \text{if } \sum_{i=1}^{n} d_i = n-2 \end{cases}$$

when $C(n-2; d_1, d_2, \ldots, d_n) = \dfrac{(n-2)!}{d_1!\, d_2! \cdots d_n!}$

is the multinomial coefficient.

proof. (Exercise)

Illustration

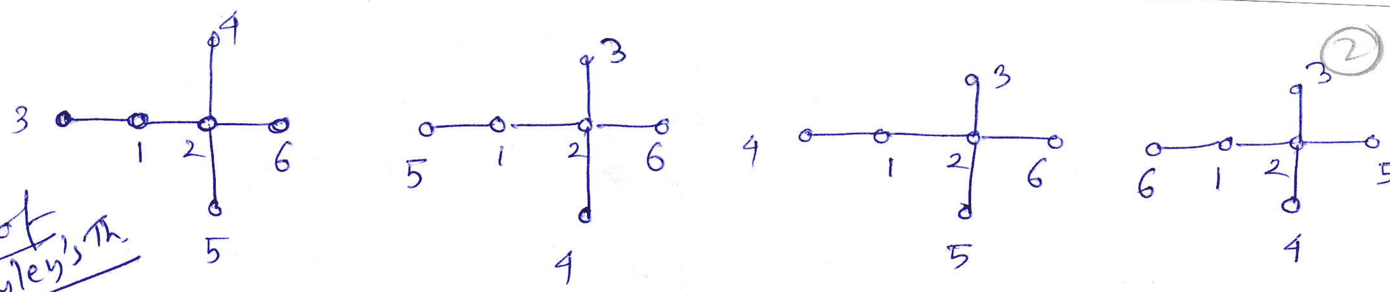
$$N(1, 3, 0, 0, 0, 0) = C(6-2; 1, 3, 0, 0, 0, 0)$$
$$= \frac{4!}{1!\, 3!\, 0!\, 0!\, 0!\, 0!}$$
$$= 4.$$

$\Rightarrow$ There are 4 labeled trees of six vertices with

vertex 1 having degree 2

Vertex 2 having degree 4

Vertex 3-6 having degree 1 each.



proof of Cayley's Th.

The number $T(n)$ of labeled trees of n vertices is

$$T(n) = \sum \left\{ N(d_1, d_2, \ldots, d_n) \mid d_i \geq 0, \sum_{i=1}^{n} d_i = n-2 \right\}$$

$$= \sum \left\{ C(n-2; d_1, d_2, \ldots, d_n) : d_i \geq 0, \sum_{i=1}^{n} d_i = n-2 \right\}$$

Multinomial expansion (Generalization of binomial expansion)

If p is a positive integer, then

$$(a_1 + a_2 + \cdots + a_k)^p = \sum \left\{ C(p; d_1, d_2, \ldots, d_k) a_1^{d_1} a_2^{d_2} \cdots a_k^{d_k} \mid d_i \geq 0, \sum_{i=1}^{k} d_i = p \right\}.$$

Taking $k = n$, $p = n-2$ and $a_1 = a_2 = \cdots = a_k = 1$, we have

$$T(n) \cancel{n^{n-2}} = (\underbrace{1 + 1 + \cdots + 1}_{n \text{ terms.}})^{n-2} = n^{n-2}$$

Alternative ~~method~~ proof of Cayley's Theorem

(Using Prüfer code).

- To prove a set has a certain no. of elements, we find a bijection between that set & some other set with a known no. of elements.
- Find a bijection between the set of Prüfer sequences & the set of spanning trees.

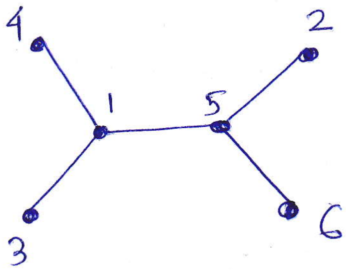
- A Prüfer seq. is a seq. of $n-2$ nos., each being one of the nos. $1, 2, \ldots, n$.
- There are n^{n-2} Prüfer seq. for any given n.

Algm. to encode any tree into a Prüfer seq.

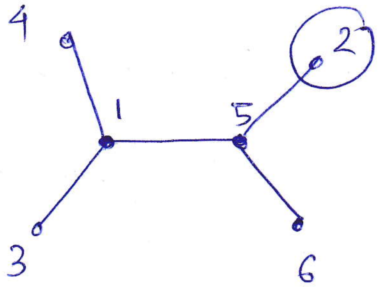
1. Take a tree $T \in T_n$, whose vertices are labeled from 1 to n in any manner.
2. Take the vertex with ~~the~~ the smallest label whose degree is equal to 1.
   (Any tree must have at least two pendant vertices (vertices of degree 2)).
   Delet it from the tree and write down the value of its only neighbor.
3. Repeat this process with the new, smaller tree. Continue until one vertex remains.

- This algm. gives a seq. of $n-1$ terms with the last term n because even if initially degree of vertex n is 1, there will always be another vertex of degree 1 with smaller label.
- As we already know the no. of vertices on the graph by the length of ~~the~~ our seq., we can drop the last term as it is redundant.
- So we have a seq. of $n-2$ elements encoded from our tree.

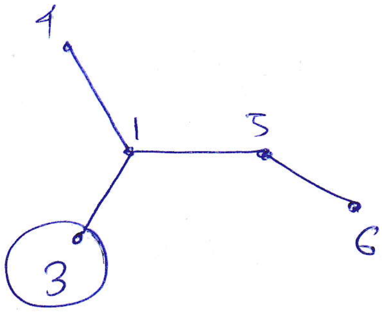
Example: (Prüfer encoding of a tree)



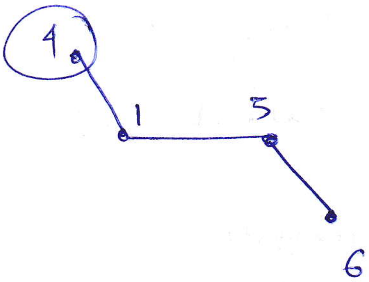
Seq. ....



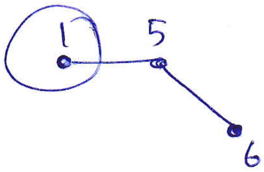
Seq. 5, ...



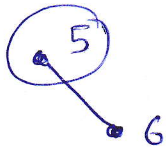
Seq. 5, 1, ...



seq. 5, 1, 1, ...

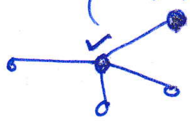


Seq. 5, 1, 1, 5, ...



Seq. 5, 1, 1, 5, 6

Prüfer seq. $P = \{5, 1, 1, 5\}$ each neighbour of a deg 1 vertex



in P means it is neighbor of a deg 1 vertex

if a is the # of times a vertex appear in P, then deg. of that vertex is 1+a.

does not have any vertex of degree 1 as a vertex of deg 1 will never be written as the neighbors of other deg 1 vertices (except when vertex n has deg 1, but vertex n will never appear in P)

- no vertex of deg 1 in P
- deg of a vertex in P = 1+(# of times the vertex appears in P)

in P means it is neighbour of a deg 1 vertex

Reconstructing an encoded tree from a Prüfer seq.

1. Find the smallest no. from 1 to n that is not in the Sequence P and attach the vertex with that number to the vertex with the first no. in P.
   (We know that $n = 2 +$ no. of elements in P).
2. Remove the first no. of P from the seq. Repeat this process considering only the nos. whose vertices have not yet attained their correct degree.
3. Do this until there are no nos. left in P. Remember to attach the last no. in P to the vertex n.

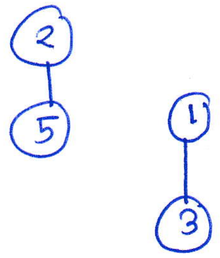
6, 2 pendant vertices
3, 4

<u>Illustation</u> $P = \{5, 1, 1, 5\}$

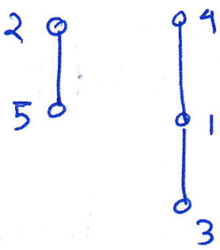
\# of nodes in the tree
$n = 2 + 4 = 6$.

$P = \{5, 1, 1, 5\}$

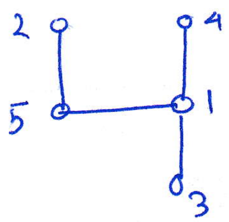
$P = \{1, 1, 5\}$



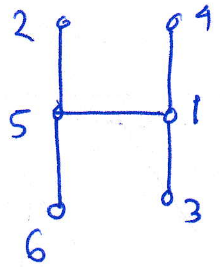
$P = \{1, 5\}$



$P = \{5\}$



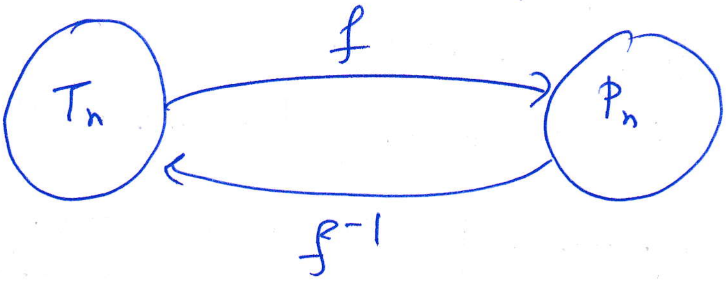
$P =$



- No ambiguity on how to encode the tree or decode the sequence.
- for every tree, there is exactly one corresponding Prüfer code seq. & for each Prüfer seq, there is exactly one correspondi[ng] tree.

• $T_n \to$ set of spanning trees on n vertices

$P_n \to$ set of Prüfer seq^s. with $n-2$ terms.



• $f: T_n \to P_n$ encoding fu^n.

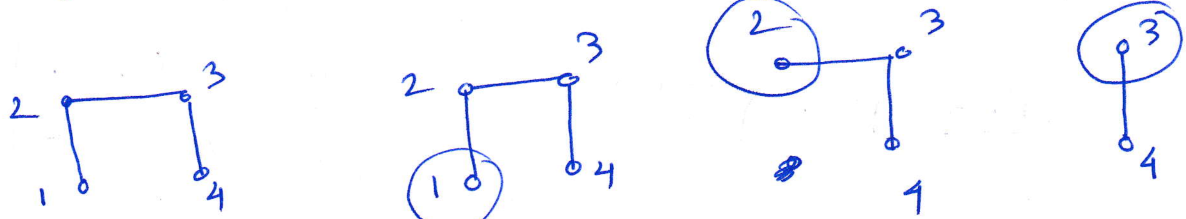
• $f^{-1}: P_n \to T_n$ decoding fu^n.

& $f^{-1} \circ f$ = identity map

• f is bijection fu^n. between T_n & P_n.

$\therefore |T_n| = |P_n| = n^{n-2}$.

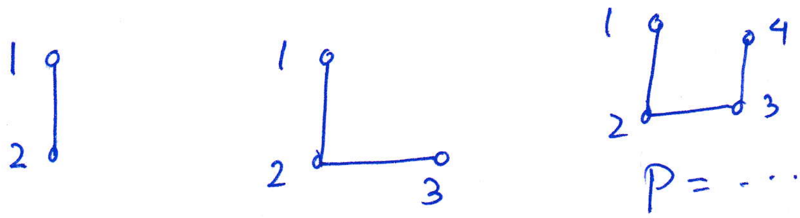
Example;



P = ... P = 2 P = 2,3 P = 2,3,

• P = {2,3} → 1,4 pendant vertex

Tree reconstruction.



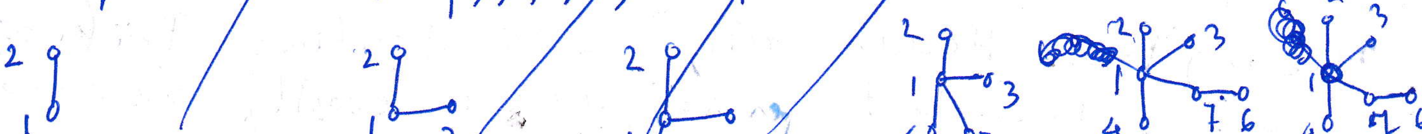
P = {2,3} P = {3} P = ...

Example: P = {1,1,1,1,6,5}. deg 5 → 1; deg 2 → 6; deg 2 → 5

$n = 6 + 2 = 8$.

→ 2,3,4,7,8 pendant vertices.

P = {1,1,1,1,6,5} P = {1,1,1,6,5} P = {1,1,6,5} P = {1,6,5} P = {6,5} P = {5}



Example,

Draw the tree whose Prüfer code is 1, 1, 1, 1, 6 5.

Sol^n.

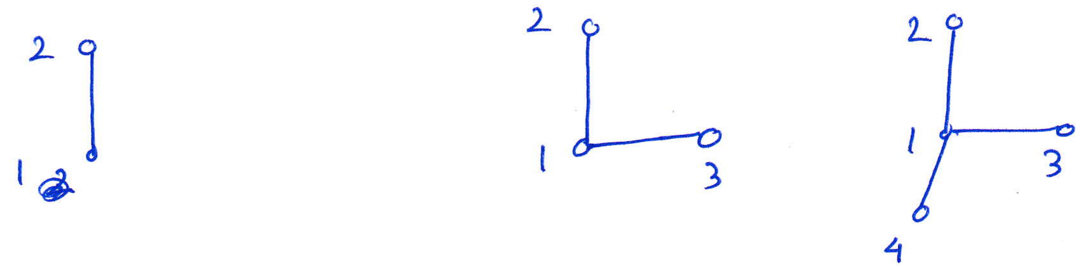
$P = \{1, 1, 1, 1, 6, 5\}$ $\Rightarrow$ $n = 8$ vertices.

deg of vertex ~~1~~ → 5

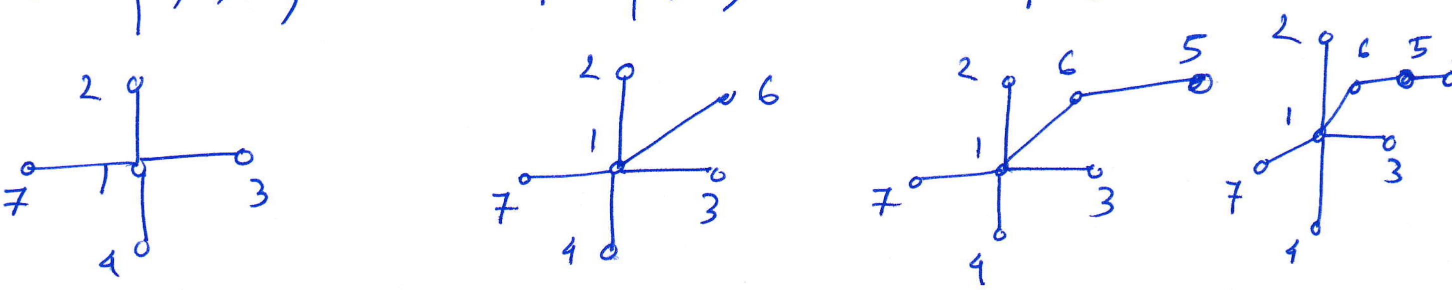
deg of vertex ~~2~~, ~~3~~, ~~4~~, ~~7~~, 8 → 1 each

deg of vertex 5, 6 → 2 each.

$P = \{1, 1, 1, 1, 6, 5\}$ $P = \{1, 1, 1, 6, 5\}$ $P = \{1, 1, 6, 5\}$



$P = \{1, 6, 5\}$ $P = \{6, 5\}$ $P = \{5\}$ $P = -$



Trees → connected simple graph with no circuits.

- T → tree with n vertices
  Then # of edges $= n-1$
- G → a graph with n vertices & e edges.
  Then G is a tree iff G is connected & $e = n-1$.

- T → tree with more than one vertex.
  Then there are at least two pendant vertices (vertices of deg. 1)

proof. T connected $\Rightarrow$ every vertex has deg. ≥ 1.
(no isolated vertex)

$$\sum_{u \in V(T)} \deg(u) = 2e = 2(n-1) = 2n-2$$

Sum of degrees of all the vertices in a graph $= 2 \times$ # of edges of the graph

Assumption no self loop or multiple edges.

if $n-1$ vertices have deg. at least 2, then

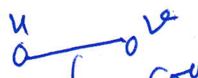
(sum of deg. of all the vertices in T) $\geq 2(n-1)+1 = 2n-1 > 2n-2$

$(\rightarrow\leftarrow)$

$\therefore$ No more than $n-2$ vertices can have deg. ≥ 2

$\Rightarrow \exists$ at least two pendant ~~vertex~~ vertices.

- deg of a vertex u in a graph $G = (V, E)$
  → # of neighbours of u in G.
- $\sum_{u \in V(G)} \deg u = 2|E(G)|$ as each edge counted twice on LHS.



counted once in deg u & once in deg v.

Chromatic Polynomials

- Suppose G is a graph and $P(G,x)$ counts the no. of ways to ~~c~~ color G in at most x colors.
- $P(G,x) = 0$ if it is not possible to color G using x colors.

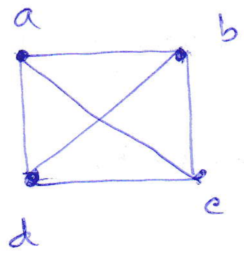
[• Coloring G means vertex colouring so that adjacent vertices get different colors.
- G is k-colorable if G can be colored in this way using at most k colors.
- $\chi(G)$ = smallest such k = chromatic number.]

Example: 1)

K_2

$$P(K_2,x) = x(x-1) = x^2 - x$$

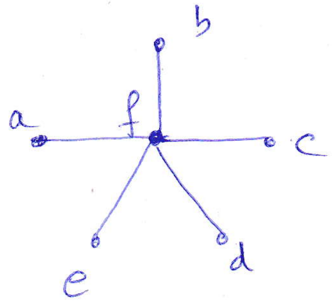
2)



K_4

$$P(K_4,x) = x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x$$

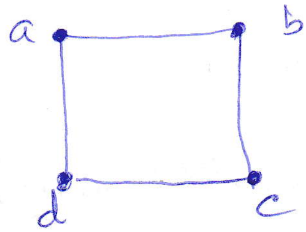
3)



G

$$P(G,x) = x(x-1)^5 = x^6 - 5x^5 + 10x^4 - 10x^3 + 5x^2 - x$$

4)



Z_4

$$P(Z_4,x) = x\underbrace{(x-1)}_{b,d \text{ get the same color}}(x-1) + x\underbrace{(x-1)(x-2)}_{b,d \text{ get different colors}}(x-2)$$

$$= x^4 - 4x^3 + 6x^2 - 3x.$$

Theorem (a) If G is K_n, then

$$P(G,x) = x(x-1)(x-2)\cdots(x-n+1) \left(= \underset{say}{x^{(n)}} \right).$$

(b) If G is I_n, then $P(G,x) = x^n$.

[$I_n \to$ empty graph with n vertices & no edges]

- $\chi(G)$ = Chromatic no.

  = Smallest +ve integer x such that $P(G,x) \neq 0$.

  $\to$ Can be estimated by finding the roots of a polynomial

- Problem of characterizing the chromatic polynomials is not yet solved.
- We can learn a great deal about a graph by knowing its chromatic polynomial.

Theorem (Read [1968])

Suppose that G is a graph of n vertices and

$$P(G,x) = a_p x^p + a_{p-1} x^{p-1} + \cdots + a_1 x + a_0.$$

Then (i) $\deg P(G,x) = n$ i.e. $p = n$

(ii) $a_n = 1$

(iii) $a_0 = 0$

(iv) Either $P(G,x) = x^n$ or the sum of the Co-efficients in $P(G,x)$ is 0.

Theorem (Whitney [1932])

$P(G,x)$ is the sum of consecutive powers of x and the coefficients of these powers alternate in sign.

That is, for some I,

$$P(G,x) = x^n - \alpha_{n-1}x^{n-1} + \alpha_{n-2}x^{n-2} \mp \cdots \pm \alpha_0,$$

with $\alpha_i > 0$ for $i \geq I$ and $\alpha_i = 0$ for $i < I$.

Theorem (Read [1968])

In $P(G,x)$, the absolute value of the co-efficient of x^{n-1} is the no. of edges of G.

Unfortunately, the properties of chromatic polynomials that we have listed in the above Theorems, do not characterize chromatic polynomials.

e.g. $P(x) = x^4 - 4x^3 + 3x^2$.

co-efficient of x^n is 1

const. term 0

Sum of the coefficient 0

However, $P(x)$ is not a chromatic polynomial for any graph.

If it were, then # of edges would by 4.

Check that no graph with 4 vertices & 4 edges has ~~this~~ $p(x)$ as its chromatic poly.

Reduction Theorem

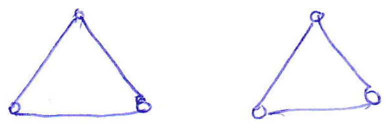
Theorem (The two-piece Theorem)

Let the vertex set of G be partitioned into disjoint sets W_1 and W_2, and let G_1 and G_2 be the subgraphs generated by W_1 and W_2, respectively. Suppose in G, no edge joins a vertex of W_1 to a vertex of W_2. Then

$$P(G,x) = P(G_1,x)\,P(G_2,x).$$

proof If x colors are available, there are $P(G_1,x)$ coloring of G_1. For each of these coloring, there are $P(G_2,x)$ coloring of G_2. This is because a coloring of G_1 does not affect a coloring of G_2, since there are no edges joining the two pieces.

e.g.



G_1 G_2

$$G = G_1 \cup G_2$$

$$P(G,x) = P(G_1,x)\,P(G_2,x)$$

$$= x^{(3)}\,x^{(3)}$$

$$= [x(x-1)(x-2)]^2$$

- Let α be an edge in G joining vertices a, b.

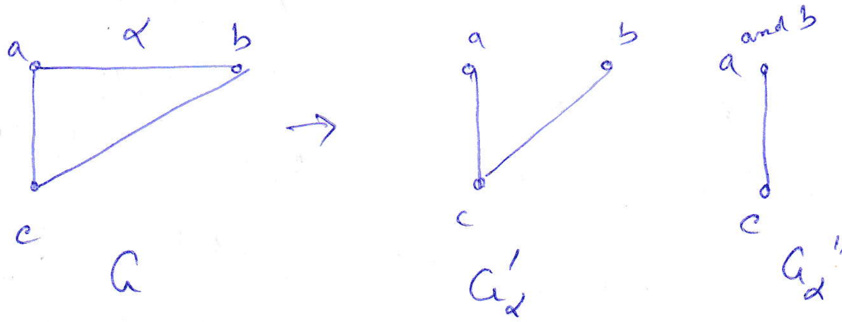
G_α' = the graph obtained from G by deleting edge α, but retaining the vertices a, b.

G_α'' = the graph obtained from G by contracting the edge α.

(i.e. by identifying the two vertices a and b. In this case, the new combined vertex is joined to all those vertices to which either a or b were joined

If both a and b were joined to a vertex c, only one of the edges from the combined vertex is included). (5)

e.g.



Theorem (Fundamental Reduction Theorem).

$$P(G,x) = P(G_\alpha',x) - P(G_\alpha'',x).$$

proof. Suppose that we use up to x colors to color G_α' when edge α joins vertices a and b in G.

Then either a and b receive different colors

or a and b receive the same color.

# of ways of coloring G_α' so that a and b receive different colors $=$ # of ways of coloring G

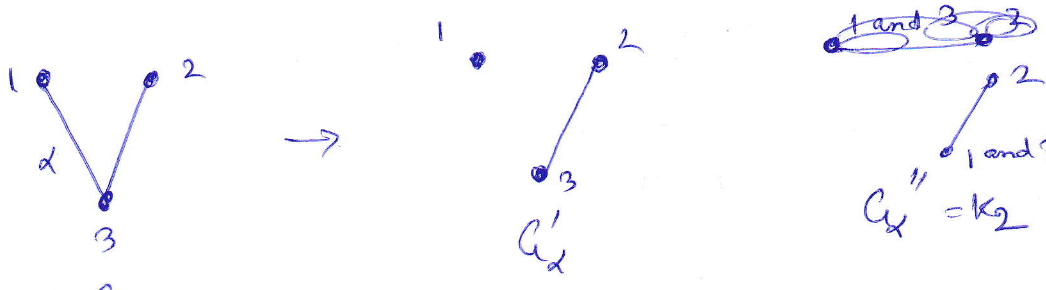
$= P(G,x)$.

# of ways of coloring G_α' so that a and b receive the same color $=$ # of ways of coloring G_α''

$= P(G_\alpha'', x)$. → [In G_α'', a & b would get the same color, and moreover, the joint vertex a and b is forced to get a different color from that given to vertex c iff one of a and b, hence both, is forced to get a different color from that given to vertex c.]

$\therefore P(G_\alpha',x) = P(G,x) + P(G_\alpha'',x)$

Example



G $\qquad$ G'_α $\qquad$ $G''_\alpha = K_2$

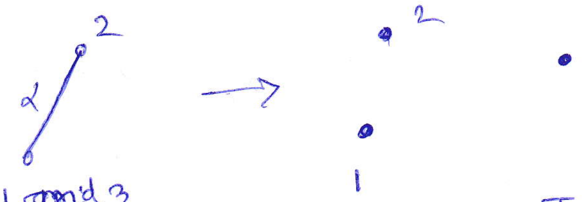
$P(G'_\alpha, x) = P(K_1, x)P(K_2, x)$
$= x.x(x-1) = x^2(x-1)$

, $P(G''_\alpha, x) = P(K_2, x)$
$= x(x-1)$

$\therefore P(G, x) = x^2(x-1) - x(x-1) = x(x-1)(x-1) = x(x-1)^2$

(by the fundamental Reduction Theorem)

Again
$P(G''_\alpha, x) = P(K_2, x)$.



G''_α $\qquad$ I_2 $\qquad$ I_1

$= P(I_2, x) - P(I_1, x)$
$= x^2 - x$

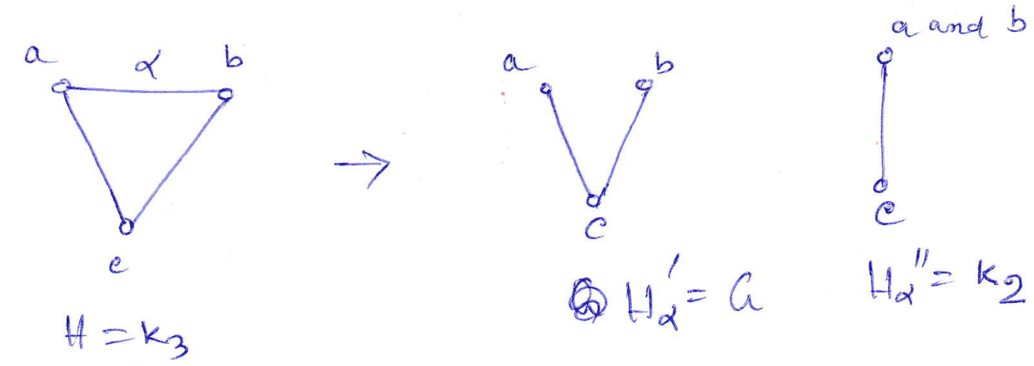
$P(G'_\alpha, x) = P(K_1, x)P(K_2, x)$
$= P(I_1, x)[P(I_2, x) - P(I_1, x)]$

~~$= P(I_1, x)[P(I_2, x)$~~

$\therefore P(G, x) = P(G'_\alpha, x) - P(G''_\alpha, x)$
$= P(I_1, x)[P(I_2, x) - P(I_1, x)] - [P(I_2, x) - P(I_1, x)]$
$= x[x^2 - x] - [x^2 - x]$
$= (x^2 - x)(x-1)$
$= x(x-1)^2$

We have reduced $P(G, x)$ to an expression that requires only knowledge of the polynomials $P(I_k, x)$ for different values of k.
$P(I_k, x) = x^k$

Example:



$$\therefore P(K_3, x) = P(G, x) - P(K_2, x)$$
$$= x(x-1)^2 - x(x-1)$$
$$= x(x-1)[x-1-1]$$
$$= x(x-1)(x-2)$$
$$= x^{(3)}.$$

Note: Each application of the Fundamental Reduction Theorem reduces the # of edges in each graph that remains. Hence, by repeated uses of the Fundamental Reduction Theorem, we must eventually end up with graphs with no edges, namely the graphs of the form I_k. This implies that $P(G,x)$ is always a polynomial in x as $P(I_k, x) = x^k$ and $P(G,x)$ can eventually be reduced to an expression that is sum, product or diff of terms of the form x^k for some k.

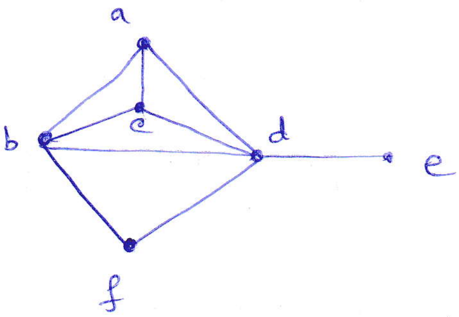
The proof may be formalized by arguing by induction on the # of edges in the graph.

Clique number and chromatic number

Suppose G is a graph.

- A clique in G is a collection of vertices, each joined to the other by an edge.

e.g.



Cliques

$\{a,b,c\}$

$\{d,e\}$

$\{a,b,c,d\}$

- The clique number of G, $\omega(G)$, is the size of the largest clique of G.

e.g. $\omega(G) = 4$ in the above example.

- Since all vertices in a clique must receive different colors,

$\chi(G) \geq \omega(G)$.

Chromatic number and Independence number

Suppose G is a graph and W is a subset of the vertex set of G

- W is said to be an independent set of G if no two vertices of W are joined by an edge.
- The independence number $\alpha(G)$ is the size of a largest independent set of G.

e.g. independence sets

$\{a,f\}$, $\{e,c,f\}$

in the above example.

$\alpha(G) = 3$ as there is no independent set with 4 vertices

Let us Suppose that the vertices of G have been colored. As there can be no edges between vertices of the same color, all the vertices of a given color define an independent set.

Thus, a coloring of $n = |V(G)|$ vertices of G in $\chi(G)$ Colors partitions the vertices into $k = \chi(G)$ "Color classes", each defining an independent set.

The average size of such an independent set is

$$\frac{n}{k} = \frac{|V(G)|}{\chi(G)}.$$

Thus, by application of the pigeonhole principle, there is at least one independent set of size at least $\frac{n}{k}$, i.e.

$$\alpha(G) \geq \frac{|V(G)|}{\chi(G)}$$ (by an application of the pigeonhole principle)

or $\alpha(G)\chi(G) \geq |V(G)|$.

26
26
156
52×
676

__Pigeonhole principle and its generalizations__

If there are "many" pigeons and "few" pigeonholes, then there must be two or more pigeons occupying the same pigeonhole. (it doesnot identify such pigeons)

__Theorem__ (Pigeonhole principle)

If $k+1$ pigeons are placed into k pigeonholes, then at least one pigeonhole will contain two or more pigeons.

Example: If there are 13 people in a room, then at least two of them are sure to have a birthday in the same month.

Example: If there are 677 people chosen from the telephone book, then there will be at least two whose first and last names begin with the same letter.

$$\begin{array}{r} 26 \\ \times 26 \\ \hline 156 \\ 52\times \\ \hline 676 \end{array}$$

Example: (Manufacturing Personal computers).

A manufactures of personal computers (PCs) makes at least one PC every day over a period of 30 days, does not start a new PC on a day when it is impossible to finish it, and averages no more than $1\frac{1}{2}$ PCs per day. Then there must be a period of consecutive days during which exactly 14 PCs are made. Why?

let a_i = # of PCs made through the end of the i-th day.

$a_1 < a_2 < \cdots < a_{30}$.

$a_1 \geq 1$

$a_{30} \leq \overset{15}{\cancel{30}} \times \frac{3}{\cancel{2}} = 45.$

(as at least one PC is made each day, and at most 45 PCs in 30 days).

Also, $a_1+14 < a_2+14 < \cdots < a_{30}+14 \leq 45+14 = 59$

Now consider the following 60 nos, each between 1 and 59

$\underbrace{a_1, a_2, \cdots, a_{30}}_{\text{all distinct}}, \underbrace{a_1+14, a_2+14, \cdots, a_{30}+14}_{\text{all distinct}}$. $\exists\, i, j\, (i \neq j)$ so that $a_i = a_j + 14$.

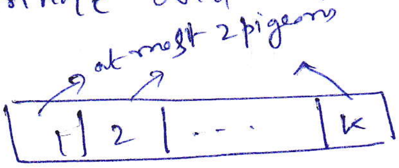
$\Rightarrow$ between days i & j the manufacturer makes exactly 14 PCs.

So by the pigeonhole principle, two of these must be equal

Generalizations and applications of the pigeonhole principle

• Stronger versions of the pigeonhole principle.

- if $2k+1$ pigeons are placed into k pigeonholes, at least one pigeonhole will contain more than two pigeons.



$\rightarrow$ $2k$ pigeons.

- if $3k+1$ pigeons are placed into k pigeonholes, then at least one pigeonhole will contain more than three p

- Speaking generally, we have the following theorem.

<u>Theorem</u> If m pigeons are placed into k pigeonholes, then at least one pigeonhole will contain more that $\left\lfloor \frac{m-1}{k} \right\rfloor$ pigeons.

<u>proof</u>. If the largest no. of pigeons in a pigeonhole is at most $\left\lfloor \frac{m-1}{k} \right\rfloor$, then the total no. of pigeons is at most

$$k \left\lfloor \frac{m-1}{k} \right\rfloor \leq m-1 < m$$

<u>Example</u>

- If there are 40 people in a room, a group of more than 3 will have a common birth month, for $\left\lfloor \frac{39}{12} \right\rfloor = 3$.

$\underbrace{1 \cdots 12}_{\hookrightarrow 1}\ \underbrace{13 \cdots 24}_{\hookrightarrow 1}\ \underbrace{25 \cdots 36}_{\hookrightarrow 1}\ 37$-

Corollary: The average value of a set of numbers is between the smallest and the largest of the numbers.

Corollary: Given a set of numbers, there is always a number in the set whose value is at least as large (at least as small) as the average value of the numbers in the set.

Application longest increasing/decreasing subsequence of a sequence of nos.

$x_1, x_2, \ldots, x_p \rightarrow$ a sequence of numbers.

$x_{i_1}, x_{i_2}, \ldots, x_{i_q}$ with $1 \le i_1 < i_2 < \cdots < i_q \le p \rightarrow$ a subsequence of $x_1, \ldots, x_p$

e.g. 9, 6, 14, 8, 17

- longest increasing subsequence is 9, 14, 17
- longest decreasing subsequence is 14, 8.

$5 = 2^2+1$ length seq. has $2+1$ length inc/dec. subseq.

i	x_i	t_i	
1	9	3	9, 14, 17
2	6	3	6, 14, 17
3	14	2	14, 17
4	8	2	8, 17
5	17	1	17

$t_i \rightarrow$ # of terms in the longest inc. seq beginning at x_i

Theorem (Erdős and Szekeres [1935])

Given a sequence of n^2+1 distinct integers, either there is an increasing subsequence of $n+1$ terms or a decreasing subsequence of $n+1$ terms.

proof. let the sequence be $x_1, x_2, \ldots, x_{n^2+1}$.

let t_i be the no. of terms in the longest increasing subsequence beginning at x_i.

(n^2+1 is required, otherwise conclusion ~~fail~~ can fail for a seq. of fewer than n^2+1 integers.

e.g. $\underbrace{3, 2, 1, 6, 5, 4, 9, 8, 7}_{9 = 3^2 \text{ elements}}$

longest inc/dec. subseq.)

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If any t_i is at least $n+1$, then the theorem is proved.
Assume that each t_i is between 1 and n.
We therefore have n^2+1 pigeons (the n^2+1 many t_i's) to be placed into n pigeonholes (the nos. $1,2,\ldots,n$).
Then there is a pigeonhole containing at least

$$\left\lfloor \frac{(n^2+1)-1}{n} \right\rfloor + 1 = n+1 \text{ pigeons.}$$

i.e. there are at least $n+1$ many t_i's that are equal.

Claim: the x_i's associated with these t_i's form a decreasing subsequence.

proof

Suppose $t_i = t_j$ with $i < j$

Subclaim

We shall show that $x_i > x_j$

if not, then $x_i \leq x_j$ i.e. $x_i < x_j$ as all n^2+1 nos. are distinct

Then x_i followed by the longest increasing subsequence beginning at x_j forms an increasing subsequence of length $t_j + 1$.

Thus, $t_i \geq \underbrace{t_j + 1}_{= t_i + 1}$, which is a contradiction.

Illustration

$10, 3, 2, 1, 6, 5, 4, 9, 8, 7 \rightarrow 10 = 3^2+1$, $n=3$

i	x_i	t_i	Sample subseq.
1	10	1 ✓	10
2	3	3	3, 6, 9
3	2	3	2, 6, 9
4	1	3	1, 6, 9
5	6	2	6, 9

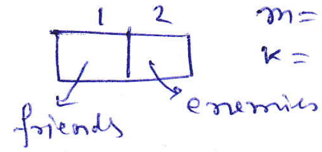
i	x_i	t_i	Sample subseq.
6	5	2	5, 9
7	4	2	4, 9
8	9	1 ✓	9
9	8	1 ✓	8
10	7	1 ✓	7

no ∄ $n+1 = 4$ length subseq. (no $t_i = 4$) $\Rightarrow$ at least $n+1 = 4$ many t_i's are same corresponding x_i $\rightarrow$ 10, 9, 8, 7 $\rightarrow$ ∃ longest dec. subseq.

Theorem. Suppose $p_1, p_2, \ldots, p_k$ are positive integers. If $p_1+p_2+\cdots+p_k-k+1$ pigeons are put into k pigeonholes then either the first pigeonhole contains at least p_1 pigeons, or the second pigeonhole contains at least p_2 pigeons, or, .., or the $k$the pigeonhole contains at least p_k pigeons.

Ramsey Numbers

Theorem Assume that among 6 persons, each pair of persons are either friends or enemies. Then either there are 3 persons who are mutual friends or 3 persons who are mutual enemies.



$n=5$
$k=2$

$\geq \left\lfloor \frac{5-1}{2} \right\rfloor + 1 = 3$

proof. Let a be a person.
Then by the pigeonhole principle, of the remaining 5 people,
- either 3 or more persons are friends of a ~~or 3 or more~~
- or 3 or more persons are enimies of a.

Suppose that b, c, d are friends of a.
- if any two of these persons are friends, then these two persons & a form a group of 3 mutual friends
- if none of these ~~person~~ are friends, then b, c, d form a group of 3 mutual enemies of a.

The argument is similar if we suppose that b, c, d are enemies of a.

- The above theorem is the simplest result in Ramsey theory stated below.

Theorem. Suppose S is any set of 6 elements. If we divide the 2-element subsets of S into two classes X and Y, then

1. either there is a 3-element subset of S all of whose ~~two~~ 2-element subsets are in X,
2. or there is a 3-element subset of S all of whose 2-element subsets are in Y.

Generalizing these conclusion

Suppose p and q are integers with $p, q \geq 2$.
We say that a positive integer N has (p,q) Ramsey property if the following holds:

Given any set S of N elements, if we divide 2-element subsets of S into two classes X and Y, then

1. either there is a p-element subset of S all of whose 2-element subsets are in X,
2. or there is a q-element subset of S all of whose 2-element subsets are in Y.

- 6 has $(3,3)$ Ramsey property.
- 3, 4, 5 do not have $(3,3)$ Ramsey property.

For instance, consider $N = 3$ & $S = \{a,b,c\}$.
Let the division of 2-element subsets of S be
$X = \{\{a,b\}, \{b,c\}\}$, $Y = \{\{a,c\}\}$.
Then is no 3-element subset of S all of whose 2-element subsets are in X.
Also no 3-element subset of S all of whose 2-element subsets are in Y.

- Let $N=4$, $S=\{a,b,c,d\}$

  $X=\{\{a,b\},\{c,d\}\}$, $Y=\{\{a,c\},\{a,d\},\{b,c\},\{b,d\}\}$

  - no 3-element subset of S all of whose 2-element subsets are in X
  - no 3-element subset of S all of whose 2-element subsets are in Y.

- If a number N has the (p,q) Ramsey property and $M>N$, then the number M has the (p,q) Ramsey property.

Theorem (Ramsey Theorem)

If p and q are integers with $p,q \geq 2$, there is a positive integer N which has the (p,q) Ramsey property.

Ramsey Number $R(p,q) \to$ the smallest number that has (p,q) Ramsey property.

- $R(3,3) \leq 6$ as ~~stated~~ by the 1st. Theorem.

  In fact, $R(3,3)=6$.

- finding a Ramsey Number $\to$ an optimization problem $\to$ a difficult problem in general.

Only known Ramsey nos.

p	q	$R(p,q)$
2	n	n
3	3	6
3	4	9
3	5	14
3	6	18
3	7	23

p	q	$R(p,q)$
3	8	28
3	9	36
4	4	18
4	5	25.

- $R(p,q)=R(q,p)$.

Applications of Ramsey Theory

- Confusion graphs for noisy channel
- design of packet - switched networks
- information retrieval
- dicision making.