ANT Assignmen 2

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Q. Stability of Implicit Scheme

$$\Rightarrow \frac{u_i^{n+1} - u_i^n}{\delta t} = \gamma \left[\frac{u_{i+1}^{n+1} - u_i^{n+1} + u_{i-1}^{n+1}}{(\delta x)^2} \right]$$

$$\Rightarrow u_i^{n+1} - u_i^n = r\left[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right] \quad \text{where } r = \frac{\gamma \delta t}{(\delta x)^2}$$

$$\Rightarrow \xi_j^{n+1} - \xi_j^n = r\left(\xi_{j+1}^{n+1} - 2\xi_i^{n+1} + \xi_{i-1}^{n+1}\right]$$

Put $\xi_j^n = A^n e^{i\theta j}$

$$\Rightarrow A^{n+1} e^{i\theta j} - A^n e^{i\theta j} = r\left(A^{n+1} e^{i\theta(j+1)} - 2A^{n+1} e^{i\theta j} + A^{n+1} e^{i\theta(j-1)}\right)$$

Dividing by $A^n e^{i\theta j}$ & putting $\frac{A^{n+1}}{A^n} = \xi$

$$\Rightarrow \xi - 1 = r\left(\xi e^{i\theta} - 2\xi + \xi e^{-i\theta}\right)$$

$$\Rightarrow \xi - 1 = r\xi\left(e^{i\theta} - 2 + e^{-i\theta}\right) = r\xi(2\cos\theta - 2)$$

$$\Rightarrow \xi\left(1 + 2r(1 - \cos\theta)\right) = 1$$

$$\rightarrow \xi = \left(\frac{1}{1 + 2r(1-\cos\theta)}\right)$$

$$\Rightarrow \xi = \cancel{\frac{1}{1+2r}} = \frac{1}{1 + 2r\left(2\sin^2\frac{\theta}{2}\right)} = \frac{1}{1 + 4r\sin^2\frac{\theta}{2}}$$

for stability, $|\xi| \leq 1$

For any $r \geq 0$, we always have $1 + 4r\sin^2\frac{\theta}{2} \geq 1$

Hence $|\xi| \leq 1$ is satisfied unconditionally.

Thus, Implicit scheme is unconditionally stable

Q) Stability of Crank Nicolson Scheme

$$\Rightarrow u_i^{n+1} - u_i^n = \frac{r}{2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

$$\Rightarrow \xi_j^{n+1} - \xi_j^n = \frac{r}{2}\left(\xi_{j+1}^{n+1} - 2\xi_j^{n+1} + \xi_{j-1}^{n+1} + \xi_{j+1}^n - 2\xi_j^n + \xi_{j-1}^n\right)$$

$$\Rightarrow A^{n+1}e^{i\theta j} - A^n e^{i\theta j} = \frac{r}{2}\left(A^{n+1}\left(e^{i\theta(j+1)} - 2e^{i\theta j} + e^{i\theta(j-1)}\right) + A^n\left(e^{i\theta(j+1)} - 2e^{i\theta j} + e^{i\theta(j-1)}\right)\right)$$

$$\Rightarrow (A^{n+1} - A^n)e^{i\theta j} = \frac{r}{2}(A^{n+1} + A^n)\left(e^{i\theta(j+1)} - 2e^{i\theta j} + e^{i\theta(j-1)}\right)$$

$$\Rightarrow \xi - 1 = \frac{r}{2}(\xi + 1)\left(e^{i\theta} - 2 + e^{-i\theta}\right)$$

$$\Rightarrow \frac{\xi - 1}{\xi + 1} = \frac{r}{2}(2\cos\theta - 2) = r(\cos\theta - 1) = \frac{r\cos\theta - r}{1}$$

$$\Rightarrow \frac{\xi}{-1} = \frac{r(\cos\theta - 1) + 1}{r(\cos\theta - 1) - 1} = \frac{1 - 2r\sin^2\frac{\theta}{2}}{-1 - 2r\sin^2\frac{\theta}{2}}$$

$$\Rightarrow \xi = \frac{1 - 2r\sin^2\frac{\theta}{2}}{1 + 2r\sin^2\frac{\theta}{2}}$$

For any $r \geq 0$

$1 - 2r\sin^2\frac{\theta}{2} \geq -1 - 2r\sin^2\frac{\theta}{2}$ & $1 - 2r\sin^2\frac{\theta}{2} \leq 1 + 2r\sin^2\frac{\theta}{2}$

$$\Rightarrow \frac{1 - 2r\sin^2\frac{\theta}{2}}{1 + 2r\sin^2\frac{\theta}{2}} \geq -1 \quad \& \quad \frac{1 - 2r\sin^2\frac{\theta}{2}}{1 + 2r\sin^2\frac{\theta}{2}} \leq 1$$

$\Rightarrow -1 \leq \xi \leq 1 \Rightarrow |\xi| \leq 1$ is satisfied unconditionally

Thus crank nicolson is unconditionally stable

Q). $u_t + u u_x = \nu u_{xx}$;

$\nu = 1$, $u(x,0) = \sin \pi x$; $0 < x < 1$

$u(0,t) = u(1,t) = 0$, $t > 0$

Find the Discretized eq^{n}

$\Rightarrow$ Using Crank Nicolson.

$$\frac{u_j^{n+1} - u_j^{n}}{\delta t} + \frac{1}{2}\left(u_j^{n+1} \times \left(\frac{u_j^{n+1} - u_{j-1}^{n+1}}{2\delta x}\right)\right] - \frac{1}{2} \times \left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\delta x)^2}\right]$$

$$= \frac{1}{2} \times \left[\frac{u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n}}{(\delta x)^2}\right] - \frac{1}{2}\left[u_j^{n} \times \left(\frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\delta x}\right)\right]$$

Applying Newton's linearization on the above eq^{n} iteratively, we substitute

$$\left(u_j^{n+1}\right)^{k+1} = \left(u_j^{n+1}\right)^{(k)} + \Delta u_j^{n+1} \quad \text{at } (k+1)^{th} \text{ iteration}$$

Simplifying this, we get a tridiagonal system

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & \\ a_2 & b_2 & c_2 & 0 & \cdots & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdots & 0 & a_{m-1} & b_{m-1} \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ \vdots \\ u_{m-1}^{n+1} \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_{n-1} \end{bmatrix}$$

where

$$a_j = \frac{-\left(u_j^{n+1}\right)^{(k)}}{4\delta x} - \frac{1}{2(\delta x)^2}, \quad b_j = \frac{1}{\delta t} + \frac{\left(u_{j+1}^{n+1}\right)^{(k)} - \left(u_{j-1}^{n+1}\right)^{(k)}}{4\delta x} + \frac{1}{(\delta x)^2}$$

$$c_j = \frac{\left(u_j^{n+1}\right)^{(k)}}{4\delta x} - \frac{1}{2(\delta x)^2}$$

and

$$d_j = \frac{\left(u_j^{n+1}\right)^{(k)} + \Delta u_j^{n+1} - u_j^{n}}{\delta t} + \frac{1}{2}\frac{\left(\left(u_j^{n+1}\right)^{(k)} + \Delta u_j^{n+1}\right)}{2\delta x}\left(u_{j+1}^{n+1(k)} - u_{j-1}^{(n+1)(k)} + \Delta u_{j+1}^{n+1} - \Delta u_{j-1}^{n+1}\right)$$

$$- \frac{1}{2}\left(\frac{u_{j+1}^{n+1(k)} - 2u_j^{(n+1)(k)} + u_{j-1}^{(n+1)(k)} + \Delta u_{j+1}^{n+1} - 2u_j^{n+1} + \Delta u_{j-1}^{n+1}}{\delta x^2}\right)$$

$$- \frac{u_j^{n+1(k)} - u_j^{n}}{\delta t} - \frac{u_j^{n+1(k)}}{2}\left(\frac{u_j^{n+1(k)} - u_{j-1}^{n+1(k)}}{2\delta x}\right) + \frac{1}{2}\left(\frac{u_{j+1}^{n+1(k)} - 2u_j^{n+1(k)} + u_{j-1}^{n+1(k)}}{(\delta x)^2}\right)$$

Q) $\nabla^2 u = -10(x^2+y^2+10)$; $0 < x, y < 3$

$u = 0$ on the boundary & $\delta x = \delta y = 1$

$\Rightarrow$ Using Central Difference Scheme

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{1} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{1} = -10(j^2 + i^2 + 10)$$

$i, j = 1, 2$

$x_i = 0 + 1 \times \delta x = i$, $y_j = j$, $u_{0j} = u_{i,0} = 0$

- At $i=1; j=1$

$u_{21} - u_{11} + u_{12} - 2u_{11} = -120$

- $i=2, j=1$

$u_{3,1} - 2u_{2,1} + u_{1,1} + u_{2,2} - 2u_{2,1} = -150$

- $i=1, j=2$

$u_{2,2} - 2u_{1,2} + u_{1,3} - 2u_{1,2} + u_{1,1} = -150$

- $i=2, j=2$

$u_{3,2} - 2u_{2,2} + u_{1,2} + u_{2,3} - 2u_{2,2} + u_{2,1} = -180$

Using BC. $u=0$, $u_{0j} = u_{i0} = 0$ & $u_{3j} = u_{i3} = 0$

the eq$\underline{n}$s reduce to

$u_{2,1} - 4u_{1,1} + u_{1,2} = -120$, $-4u_{2,1} + u_{1,1} + u_{2,2} = -150$

$u_{2,2} - 4u_{1,2} + u_{1,1} = -150$, $u_{1,2} - 4u_{2,2} + u_{2,1} = -180$

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & 0 & -4 & 1 \\ 1 & -4 & 0 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} -120 \\ -150 \\ -150 \\ -180 \end{bmatrix}$$

$\Rightarrow$ $u_{1,1} = 67.5$

$u_{1,2} = 75$

$u_{2,1} = 75$

$u_{2,2} = 82.5$

Q) $\nabla^2 u - 2\frac{\partial u}{\partial x} = -2$, $0 < x < 1$, $0 < y < 1$

$u = 0$ on boundary, $h = \frac{1}{3}$

$$\Rightarrow \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{1/9} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{1/9} - 2\,\frac{u_{i+1,j} - u_{i-1,j}}{2\times\frac{1}{3}} = -2$$

$$\Rightarrow 9u_{i-1,j} - 18u_{i,j} + 9u_{i+1,j} + 9u_{i,j-1} - 18u_{i,j} + 9u_{i,j+1} - 3u_{i+1,j} + 3u_{i-1,j} = -2$$

$$\Rightarrow 12u_{i-1,j} - 36u_{i,j} + 6u_{i+1,j} + 9u_{i,j-1} + 9u_{i,j+1} = -2$$

BC's are $u_{i,3} = u_{3,j} = 0$

at $i=1, j=1 \Rightarrow -36u_{1,1} + 6u_{2,1} + 9u_{1,2} = -2$

$i=1, j=2 \Rightarrow -36u_{1,2} + 6u_{2,2} + 9u_{1,1} = -2$

$i=2, j=1 \Rightarrow 12u_{1,1} - 36u_{2,1} + 9u_{2,2} = -2$

$i=2, j=2 \Rightarrow 12u_{1,2} - 36u_{2,2} + 9u_{2,1} = -2$

Solving the above eq$\underline{\underline{n}}^s$ simultaneously, we get

$u_{1,1} = \frac{22}{219}$

$u_{1,2} = \frac{22}{219}$

$u_{2,1} = \frac{26}{219}$

$u_{2,2} = \frac{26}{219}$

9) $\frac{\partial u}{\partial t} = \nabla^2 u \; ; \; -1 < x, y < 1 \; , \; t > 0$

where $u(x,y,0) = \cos\frac{\pi x}{2} \cos\frac{\pi y}{2}$ and $u = 0$

on $x = \pm 1$, $y \pm 1$, $\delta x, \delta y = \frac{1}{2}$, $r = \frac{1}{6}$

$\Rightarrow r = \frac{\delta t}{(\delta x)^2} = \frac{1}{6} \Rightarrow \delta t = \frac{1}{6} \times \frac{1}{4} = \frac{1}{24}$

Step 1 :- $12 u_{i,j}^{n+0.5} - 12 u_{i,j}^{n} = u_{i+1,j}^{n+0.5} - 2u_{i,j}^{n+0.5} + u_{i-1,j}^{n+0.5} + u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}$

$\Rightarrow -u_{i-1,j}^{n+0.5} + 14 u_{i,j}^{n+0.5} - u_{i+1,j}^{n+0.5} = u_{i,j-1}^{n} + 10 u_{i,j}^{n} + u_{i,j+1}^{n}$

$\hookrightarrow$ Tridiagonal system

for $n = 0$, $j = 1$ we have

$$\begin{bmatrix} 14 & -1 & 0 \\ -1 & 14 & -1 \\ 0 & -1 & 14 \end{bmatrix} \begin{bmatrix} u_{1,1}^{0.5} \\ u_{2,1}^{0.5} \\ u_{3,1}^{0.5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$u_{1,1}^{0.5} = u_{2,1}^{0.5} = u_{3,1}^{0.5} = 0$

for $n = 0$, $j = 3$, we have

$$\begin{bmatrix} 14 & -1 & 0 \\ -1 & 14 & -1 \\ 0 & -1 & 14 \end{bmatrix} \begin{bmatrix} u_{1,2}^{0.5} \\ u_{2,2}^{0.5} \\ u_{3,2}^{0.5} \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

$u_{1,2}^{0.5} = 0.0515464$

$u_{2,2}^{0.5} = 0.7216495$

$u_{3,2}^{0.5} = 0.0515464$

for $n=0,\ j=3$, we get the same system as in ~~Fee~~ $n=0,\ j=0$.

Hence, $u_{1,3}^{0.5} = u_{2,3}^{0.5},\ u_{3,3}^{0.5} = 0$

So, at $t = \frac{\delta t}{2}$, we have

x\y	-1	-0.5	0	0.5	1
-1	0	0	0	0	0
-0.5	0	0	$u_{1,2}^{0.5}$	0	0
0	0	0	$u_{2,2}^{0.5}$	0	0
0.5	0	0	$u_{3,2}^{0.5}$	0	0
1	0	0	0	0	0
		0	0	0	0

Step 2: $12u_{i,j}^{n+1} - 12u_{i,j}^{n+0.5} = u_{i+1,j}^{n+0.5} - 2u_{i,j}^{n+0.5} + u_{i-1,j}^{n+0.5} + u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}$

$$\Rightarrow -u_{i,j-1}^{n+1} + 14u_{i,j}^{n+1} - u_{i,j+1}^{n+1} = u_{i-1,j}^{n+0.5} + 10u_{i,j}^{n+0.5} + u_{i+1,j}^{n+0.5}$$

For $n=0,\ i=1$, we get

$$\begin{bmatrix} 14 & -1 & 0 \\ -1 & 14 & -1 \\ 0 & -1 & 14 \end{bmatrix} \begin{bmatrix} u_{1,1}^{1} \\ u_{1,2}^{1} \\ u_{1,3}^{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1.2371135 \\ 0 \end{bmatrix}$$

$u_{1,1}^{1} = 0.0063769$

$u_{1,2}^{1} = 0.0892762$

$u_{1,3}^{1} = 0.0063769$

For $n=0$, $i=2$, we have

$$\begin{bmatrix} 14 & -1 & 0 \\ -1 & 14 & -1 \\ 0 & -1 & 14 \end{bmatrix} \begin{bmatrix} u'_{2,1} \\ u'_{2,2} \\ u'_{2,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 7.3195878 \\ 0 \end{bmatrix}$$

$u'_{2,1} = 0.03773$

$u'_{2,2} = 0.52822$

$u'_{2,3} = 0.03773$.

For $n=0$, $i=3$ we get the table same as $n=0$, $i=1$

hence

$u'_{3,1} = 0.0063769$

$u'_{3,2} = 0.0892762$

$u'_{3,3} = 0.0063769$

At $t = \delta t$, we have

$x \backslash y$	-1	-0.5	0	0.5	1
-1	0	0	0	0	0
-0.5	0	0.0063769	0.0892762	0.0063769	0
0	0	0.03773	0.52822	0.03773	0
0.5	0	0.0063769	0.0892762	0.0063769	0
1	0	0	0	0	0

Q) Find the Truncation Error of Crank Nicolson scheme and prove that it is consistent

⇒ PDE : $u_t = \gamma u_{xx}$

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \frac{\gamma}{2}\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\delta x)^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\delta x)^2}\right]$$

$\delta t = k$ & $\delta x = h$, Then

$$TE = \frac{u_i^{n+1} - u_i^n}{k} - \frac{\gamma}{2} \times \left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}\right)$$

Using Taylor series Expansion

$$u_i^{n+1} = u(x_i, t_{n+1}) = u\left(x_i, t_{n+\frac{1}{2}} + \frac{k}{2}\right)$$

$$\Rightarrow u_i^{n+1} = u(x_i, t_{n+\frac{1}{2}}) + \left(\frac{k}{2} u_t + \frac{\left(\frac{k}{2}\right)^2}{2!} u_{tt} + \frac{\left(\frac{k}{3}\right)^3}{3!} u_{ttt}\right)\Bigg|_{(x_i, t_{n+\frac{1}{2}})} \to (1)$$

Similarly

$$u_i^n = u(x_i, t_{n+\frac{1}{2}}) + \left[-\frac{k}{2} u_t + \frac{\left(\frac{k}{2}\right)^2}{2!} u_{tt} - \frac{\left(\frac{k}{3}\right)^3}{3!} u_{ttt}\right]\Bigg|_{(x_i, t_{n+\frac{1}{2}})} \to (2)$$

(1) - (2)

$$\frac{u_i^{n+1} - u_i^n}{k} = u_t(x_i, t_{n+\frac{1}{2}}) + \frac{k^2}{12} u_{ttt}(x_i, t_{n+\frac{1}{2}}) \to (3)$$

Now

$$u_{i+1}^n = u(x_i, t_n) + \left[h u_x + \frac{h^2}{2!} u_{xx} + \frac{h^3}{3!} u_{xxx}\right]\Bigg|_{(x_i, t_n)} \to (4)$$

and

$$u_{i-1}^n = u(x_i, t_n) + \left[-h u_x + \frac{h^2}{2!} u_{xx} - \frac{h^3}{3!} u_{xxx}\right]\Bigg|_{(x_i, t_n)} \to (5)$$

(4) + (5)

$$u_{i+1}^n - 2u_i^n + u_{i-1}^n = h^2 u_{xx}(x_i, t_n) \to (6)$$

Similarly, we get

$$u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} = h^2 u_{xx}(x_i, t_{n+1}) \longrightarrow (7)$$

putting (3), (6) and (7) in expression of Truncation ~~erro er~~ error expression

$$TE = \frac{u_i^{n+1} - u_i^n}{k} - \frac{\nu}{2}\left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}\right)$$

$$= \left[u_t(x_i, t_{n+\frac{1}{2}}) + \frac{k^2}{12} u_{ttt}(x_i, t_{n+\frac{1}{2}}) + O(k^4)\right]$$

$$- \frac{\nu}{2}\left[u_{xx}(x_i, t_{n+1}) + \frac{h^2}{12} u_{xxxx}(x_i, t_{n+1}) + O(h^4)\right.$$

$$\left. + u_{xx}(x_i, t_n) + \frac{h^2}{12} u_{xxxx}(x_i, t_n) + O(h^4)\right]$$

$$= \left[u_t - \nu u_{xx}\right]\Big|_{(x_i, t_{n+\frac{1}{2}})} + \left[\frac{k^2}{12} u_{ttt} - \nu \frac{h^2}{12} u_{xxxx} - \nu \frac{k^2}{8} \frac{\partial^2 u}{\partial t^2 \partial x^2}\right]\Bigg|_{(x_i, t_{n+\frac{1}{2}})}$$

$$+ O(h^4 + k^4)$$

Given PDE :- ~~Ex~~ $u_t = \nu u_{xx}$

Hence $\left[u_t - \nu u_{xx}\right]_{(x_i, t_{n+\frac{1}{2}})} = 0$

$$\therefore TE = \left[\frac{k^2}{12} u_{ttt} - \frac{\nu h^2}{12} u_{xxxx} - \nu \frac{k^2}{8} \frac{\partial^2 u}{\partial t^2 \partial x^2}\right]\Bigg|_{(x_i, t_{n+\frac{1}{2}})} + O(h^4 + k^4)$$

$$= O(h^2 + k^2) = O(\delta t^2, \delta x^2)$$

As $h \to 0$, $k \to 0$, $TE \to 0$

Hence Crank Nicolson is consistant and its $TE = O(\delta t^2, \delta x^2)$

Q) $y'' - (y')^2 - y^2 + y + 1 = 0$, $y(0) = 0.5$
Convert to linear BVP using quasi linearization $y(\pi) = -0.5$

$\rightarrow$ $F(x, y, y', y'') \Rightarrow y'' - (y')^2 - y^2 + y + 1 = 0$

$$\frac{\partial F}{\partial y} = -2y + 1 \ , \ \frac{\partial F}{\partial y'} = -2y' \ , \ \frac{\partial F}{\partial y''} = 1$$

Using Quasi-linearization we get

$$F(x, y^{(k)}, y'^{(k)}, y''^{(k)}) + \left(y^{(k+1)} - y^{(k)}\right)\left(2y^{(k)} + 1\right) + \left(y'^{(k+1)} - y'^{(k)}\right)\left(-2y'^{(k)}\right) + \left(y''^{(k+1)} - y''^{(k)}\right)(1) = 0$$

$$\Rightarrow y''^{(k+1)} - 2y'^{(k)} y'^{(k+1)} + \left(1 - 2y^{(k)}\right) y^{(k+1)} = -(y'^{(k)})^2 - (y^{(k)})^2 - 1$$

$$\Rightarrow y''^{(k+1)} - 2y'^{(k)} y'^{(k+1)} + \left(1 - 2y^{(k)}\right) y^{(k+1)} + \left[(y'^{(k)})^2 + (y^{(k)})^2 + 1\right] = 0$$

the above is at iteration $(k+1)$.
The BC's are $y^{k+1}(0) = 0.5$, $y^{k+1}(\pi) = -0.5$

We can discretize the above eq$^{\underline{n}}$ to get a tridiagonal system and solve as above is ~~a~~ a linear BVP.

Q) Solve $3yy'' + (y')^2 = 0$, $y(0) = 0$, $y(1) = 0$, $h = \frac{1}{3}$ using Newtons linearization

$\Rightarrow$ On Discretization, we get

$$12(y_i)(y_{i+1} - 2y_i + y_{i-1}) + y^2_{i+1} + y^2_{i-1} - 2y_{i+1} y_{i-1} = 0$$

Then $F_i \Rightarrow y^2_{i-1} - 24y_i^2 + y^2_{i+1} + 2y_i y_{i-1} + 12 y_i y_{i+1} - 2y_{i+1} y_{i-1} = 0$

$$\frac{\partial F_i}{\partial y_{i-1}} = 2y_{i-1} + 12y_i - 2y_{i+1}$$

$$\frac{\partial F_i}{\partial y_i} = 12y_{i-1} - 48y_i + 12y_{i+1}$$

$$\frac{\partial F_i}{\partial y_{i+1}} = y_{i-1} + 12y_i + 2y_{i+1}$$

Applying Newtons linearization at $(k+1)^{th}$ iteration

$$\left.\frac{\partial F_i}{\partial y_{i-1}}\right|^{(k)} \Delta y_{i-1} + \left.\frac{\partial F_i}{\partial y_i}\right|^{(k)} \Delta y_i + \left.\frac{\partial F_i}{\partial y_{i+1}}\right|^{(k)} \Delta y_{i+1} = -F_i^{(k)}$$

$$\Rightarrow (2y_{i-1} + 12y_i - 2y_{i+1})\Big|^{(k)} \Delta y_{i-1} + (12y_{i-1} - 48y_i + 12y_{i+1})\Big|^{(k)} \Delta y_i$$
$$+ (-2y_{i-1} + 12y_i + 2y_{i+1})\Big|^{(k)} \Delta y_{i+1} = \Big(-y^2_{i-1} + 24y_i^2 - y^2_{i+1}$$
$$- 12y_i y_{i-1} - 12y_i y_{i+1}$$
$$+ 2y_{i+1} y_{i-1}\Big)\Big|^{(k)}$$

Initial guess :- $y^{(0)}(x) = x$ (acc. to BC's)

$$\Rightarrow y_i^{(0)} = x_i, \quad \Delta y_0 = 0, \quad \Delta y_3 = 0$$

$$\Rightarrow y_0^{(0)} = 0, \quad y_1^{(0)} = \frac{1}{3}, \quad y_2^{(0)} = \frac{2}{3}, \quad y_3^{(0)} = 1$$

At $i = 1$, $k = 0$, we get

~~$7\Delta y_1\ 7\cancel{12}\Delta y_1$~~

$$(-16 + 8)\Delta y_1 + \left(4 + \frac{4}{3}\right)\Delta y_2 = \frac{-4}{9} \Rightarrow -8\Delta y_1 + \frac{16}{3}\Delta y_2 = -\frac{4}{9} \rightarrow (1)$$

At $i = 2$, $k = 0$, we get

$$\frac{20}{3}\Delta y_1 - 16\Delta y_2 = -\frac{4}{9} \rightarrow (2)$$

solving ① & ② we get

$\Delta y_1 = \frac{4}{39}$, $\Delta y_2 = \frac{11}{156}$

$\therefore\ y_1^{(1)} = y_1^{(0)} + \Delta y_1 = 0.4359$

$y_2^{(1)} = y_2^{(0)} + \Delta y_2 = 0.7372$

$\therefore\ y_1 \cong y\left(\frac{1}{3}\right) \cong 0.4359$ & $y_2 \cong y\left(\frac{2}{3}\right) \cong (0.7372)$