Lecture-20 (26-03-2024)

Theorem: A topological space (X, T) is compact space iff every family of closed subsets of X with FIP has non empty intersection.

Proof: We prove this theorem by contrapositive argument.

Let $\{F_i\}$ be a family of closed subsets of X with FIP such that $\bigcap_i F_i = \varphi$.

Then

$$X = \varphi^c = \left(\bigcap_i F_i\right)^c$$

$$= \bigcup_i F_i^c$$

$\Rightarrow \{F_i^c\}$ is an open cover for X.

Since $\{F_i\}$ has FIP, implies that

$\bigcap_{i=1}^{m} F_i \neq \phi$, for every m.

$\Rightarrow X = \phi^c \neq \left(\bigcap_{i=1}^{m} F_i\right)^c$

$= \bigcup_{i=1}^{m} F_i^c$

$\Rightarrow X \neq \bigcup_{i=1}^{m} F_i^c, \forall m \in N.$

$\Rightarrow X$ is not a compact space.

Conversely suppose that X is not a compact space. Then there is an open cover say $\{G_i\}$ of X with no finite subcover.

Now $\{G_i^c\}$ is a class of closed subsets of X.

$\because \{G_i\}$ has no finite subcover so $X \neq \bigcup_{i=1}^{m} G_i$, for all $m \in N$.

$$\Rightarrow \varphi = X^c \neq \left(\bigcup_{i=1}^{m} G_i\right)^c$$

$$= \bigcap_{i=1}^{m} G_i^c, \quad \forall m \in \mathbb{N}.$$

$\Rightarrow \{G_i^c\}$ is a FIP

But $X = \bigcup_i G_i$

$$\Rightarrow \varphi = X^c = \left(\bigcup_i G_i\right)^c$$

$$= \bigcap_i G_i^c$$

$$\Rightarrow \varphi = \bigcap_i G_i^c$$

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Theorem: Every compact subset of a Hausdorff space is closed.

Proof: Let A be a compact subset of a Hausdorff space (X, T).

We prove A is a closed set.

i.e., we prove A^c is an open set.

Let $p \in A^c$. Then $p \notin A$.

Now for any point $a \in A$, we have $p \neq a$.

Now p, a are two distinct points in a Hausdorff space X.

$\therefore$ There exist two open sets G_a and H_a with

$a \in H_a$, $p \in G_a$ and $G_a \cap H_a = \varphi$.

If we do this for every $a \in A$, we obtain a class of open sets $\{G_a\}$ and $\{H_a\}$ with

$a \in H$, $p \in G_a$, $G_a \cap H_a = \varphi$, $\forall a \in A$.

Then clearly

$A \subseteq \cup \{H_a \mid a \in A\}$, $p \in G_a$ $\forall a \in A$.

$\Rightarrow \{H_a \mid a \in A\}$ is an open cover for the compact set A.

$\therefore$ There exists a finite subcover say $H_{a_1}, H_{a_2}, \ldots H_{a_m}$ with

$$A \subseteq \bigcup_{i=1}^{m} H_{a_i}.$$

Denote $H = \bigcup_{i=1}^{m} H_{a_i}$, $G = \bigcap_{i=1}^{m} G_{a_i}$

Then H and G are open sets with $A \subseteq H$ and $P \in G$.

Now

$$G \cap H = G \cap \left(\bigcup_{i=1}^{m} H_{a_i}\right)$$

$$= \bigcup_{i=1}^{m} (G \cap H_{a_i})$$

$$= \bigcup_{i=1}^{m} \varphi$$

$$= \varphi.$$

$\left[\because G_{a_i} \cap H_{a_i} = \varphi \;\; \forall a_i \in A. \;\; \because G \subset G_{a_i} \; \forall i, \; G \cap H_{a_i} \subset G_{a_i} \cap H_{a_i} = \varphi\right]$

Thus we have open sets G and H such that $P \in G$, $A \subset H$, $G \cap H = \varphi$.

This implies

$$A \cap G \subset H \cap G = \varphi$$

$$\Rightarrow \quad A \cap G = \varphi .$$

$$\Rightarrow \quad G \subset A^c \text{ and } P \in G$$

Thus for every $P \in A^c$, there exists an open set G such that

$$P \in G \subset A^c .$$

$\Rightarrow$ P is an interior point of A^c.

Since P is an arbitrary point of A^c, it follows that A^c is an open set.

$\Rightarrow$ A is a closed set.

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Theorem: Let A and B be disjoint compact subsets of a Hausdorff Space (X, T). Then there exist disjoint open sets G and H such that $A \subseteq G$ and $B \subseteq H$.

Proof: Let $a \in A$.

Then $a \notin B$ ($\because A \cap B = \varphi$).

Since B is a Compact Set and $a \notin B$.

Then by proof of the previous theorem, there exist open sets G_a and H_a such that $a \in G_a$, $B \subseteq H_a$, $G_a \cap H_a = \varphi$.

If we do this for every $a \in A$, we get class of open sets $\{G_a\}$ and $\{H_a\}$ such that

$$A \subseteq \bigcup_a G_a \quad \text{and} \quad B \subseteq H_a$$

with $G_a \cap H_a = \varphi$.

$\Rightarrow$ $\{G_a \mid a \in A\}$ is an open Cover for the compact set A.

So there exist finite number of sets $G_{a_1}, G_{a_2}, \ldots G_{a_m}$ such that

$$A \subseteq \bigcup_{j=1}^{m} G_{a_j}.$$

Also $B \subseteq H_{a_1} \cap H_{a_2} \cap \dots \cap H_{a_m}$

$[\because B \subseteq H_{a_i}\ \forall i]$

Now let $G := \bigcup_{i=1}^{m} G_{a_i}$

$H := \bigcap_{i=1}^{m} H_{a_i}$.

Then $A \subseteq G$ and $B \subseteq H$, and

G and H are open sets.

Now we prove $G \cap H = \varphi$.

Since $G_{a_i} \cap H_{a_i} = \varphi \quad \forall i$

$\Rightarrow G_{a_i} \cap H \subseteq G_{a_i} \cap H_{a_i} = \varphi$

$\Rightarrow G_{a_i} \cap H = \varphi, \ \forall i$

$$\therefore G \cap H = \left(\bigcup_{i=1}^{M} G_{a_i}\right) \cap H = \bigcup_{i=1}^{M} (G_{a_i} \cap H)$$

$$= \bigcup_{i=1}^{M} \varphi$$

$$= \varphi.$$

Thus we proved that there exist two disjoint open sets G and H with $A \subseteq G$ and $B \subseteq H$. This completes the proof.

Corollary :— Every Compact Hausdorff Space is a normal Space.

Proof: Let (X, T) be both Compact and Hausdorff Space.

Claim: (X, T) is a normal Space.

So let F_1 and F_2 be two disjoint closed subsets of X.

Since every closed subset of a Compact space is Compact, it follows that both F_1 and F_2 are compact sets.

Thus F_1 and F_2 are disjoint compact subsets of a Hausdorff Space X.

Then by previous theorem, there exist disjoint open sets G and H such that

$F_1 \subseteq G$ and $F_2 \subseteq H$

$\Rightarrow$ X is a normal space

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Theorem: Let (X, T) be a compact space and (Y, T^*) be a Hausdorff space. Let $f : (X, T) \longrightarrow (Y, T^*)$ be one-one continuous map. Then X and $f(X)$ are homeomorphic.

Proof: $f : X \longrightarrow f(X)$ is onto, 1-1 and continuous.

So we prove only $f^{-1} : f(X) \longrightarrow X$ is continuous.

Let F be any closed subset of X.

$\because$ X is a compact space and F is a closed subset of X, it follows that F is also a compact set.

Since $f : X \longrightarrow Y$ is a continuous map, it follows that $f(F)$ is also a compact set in $f(X)$.

Since (Y, τ^*) is a Hausdorff Space and every subspace of a Hausdorff Space is also a Hausdorff Space, it follows that $f(X)$ is also a Hausdorff Space w.r.t relative topology.

Thus $f(F)$ is a compact subset of a Hausdorff Space $f(X)$.

$\therefore$ $f(F)$ is a closed subset of $f(X)$. [by one of the previous theorem].

$\because$ $(f^{-1})^{-1}(F) = f(F)$ is closed.

$\Rightarrow$ $f^{-1} : f(X) \longrightarrow X$ is continuous.

$\therefore$ $f : X \longrightarrow f(X)$ is homeomorphism

$\therefore$ $X \cong f(X)$.

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Sequentially Compact Sets :

A subset B of a topological space (X, T) is said to be sequentially compact set, if every sequence in B has a subsequence which converges to a point in B.

Ex: Let B be any finite subset of a topological space (X, T). Then B is sequentially compact set.

For if $S = \{s_i\}$ is a sequence in B, then at least one of the elements in S say a_0 must appear infinitely many times.

$\therefore \{a_0, a_0, a_0, \ldots\}$ is subsequence of S converging to $a_0 \in B$.

Ex: Let $(\mathbb{R}, U)$ be usual topological space. Then $A = (0,1)$ is not a compact set.

Now consider the sequence

$\{s_n\} = \{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \}$ in $A = (0,1)$.

Then every subsequence of $\{s_n\}$ is converging to $0 \notin A = (0,1)$.

$\Rightarrow$ A is also not a sequentially compact set.

Note :- In general, there exist compact sets which are not sequentially compact sets and vice versa.

Countably Compact Sets :-

A subset B of a topological space (X, T) is said to be a countably compact set if every infinite subset S of B has a limit point in B.

Ex: let $B = [a, b]$ in $(\mathbb{R}, U)$.

let B be an infinite sub set of $[a, b]$

Then B is a bounded set.

Thus B is bounded infinite set of real numbers.

$\therefore$ By Bolzano–Weierstrass theorem, B has an accumulation point say p.

Since $[a, b]$ is closed, $p \in [a, b]$.

$\therefore$ $[a, b]$ is countably compact set,

Now let $C = (0, 1)$ in $(\mathbb{R}, U)$.

$\because \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ is infinite sub set of $(0, 1)$ and its limit point $0 \notin (0, 1)$, so $(0, 1)$ is not countably compact set.

In general, we have

Compact $\longrightarrow$ Countably compact $\longleftarrow$ Sequentially compact.

[Attendance:
65, 11, 06, 60, 19]

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