Keerti P. Charantimath 19M420059

MTI Assignment

Ans: The set is similar to cantor set. In cantor set it contains all ternary expansion except digit 1 in it. Here we have the decimal ~~set~~ expansion not containing 5.
let us take E_5^c such that it is the set of $x \in \mathbb{R}$ on interval $[0,1]$ st. its decimal expansion does not contain 5.

i.e $x = 0.d_1d_2\ldots$ st. $d_i \neq 5 \ \forall i \in \mathbb{N}$

E_5 is compliment of E_5^c. E_5 has at least one i st $d_i = 5$

Subsets of E_5 :-

- $E_{5,1} \rightarrow$ set of $x \in [0,1]$ st. $x = 0.5d_2\ldots$ i.e $d_1 = 5$.
  i.e $x \in [0.5, 0.6)$

$$m^*(E_{5,1}) = m^*([0.5, 0.6)) = 0.1 = 10^{-1}$$

- $E_{5,2} \longrightarrow$ set of $x \in [0,1]$ st $x = 0.d_1 5\ldots$ i.e $d_2 = 5$
  i.e $x \in \bigcup_{d_i} [0.d_i5, 0.d_i6)$, $d_i \in \{0,1,2\ldots 9\}/\{5\}$

$$m^*(E_{5,2}) = m^*\left(\bigcup_{d_i} [0.d_i5, 0.d_i6)\right) = 0.01 \times 9 = 9 \times 10^{-2}$$

⋮

- $E_{5,i} \rightarrow$ set of $x \in [0,1]$ st. $x = 0.d_1d_2\ldots$ st. $d_i = 5$
  i.e $x \in \bigcup_{d_1,d_2\ldots d_{i-1}} [0.d_1d_2\ldots d_{i-1}5\ldots, 0.d_1d_2\ldots d_{i-1}6]$
  $d_1, d_2 \ldots d_{i-1} \in \{0,1,2\ldots 9\}/\{5\}$

$$m^*(E_{5,i}) = 9^{i-1}\, 10^{-(i)}$$

$$\therefore m^*(E_5) = m^*\left(\bigcup_{i=1}^{\infty} E_{5,i}\right) = \sum_{i=1}^{\infty} m^*(E_{5,i}) = \sum_{i=1}^{\infty} 9^{i-1}\,10^{-i}$$

$$= \frac{\frac{1}{10}}{1-\frac{9}{10}} = \frac{\frac{1}{10}}{\frac{1}{10}} = 1$$

$$m^*(E_5) = 1$$

$$\therefore m^*(E_5^c) = m^*([0,1]) - m^*(E_5) = 1 - 1 = 0$$

Ans2:- (a) Consider vitali set V s.t. it is contained in $[0,a]$

consider $E_1, E_2 \in \mathbb{R}$ s.t. $E_1 = V$ & $E_2 = [0,a] \setminus V$

$M^*(E_1) = M^*(V) = \varepsilon > 0$

$M^*(E_2) = M^*([0,a] \setminus V) = a$

$E_1 \cap E_2 = \phi$ clearly.

Now, $M^*(E_1 \cup E_2) = M^*([0,a]) = a < a + \varepsilon = M^*(E_1) + M^*(E_2)$

$M^*(E_1 \cup E_2) \neq M^*(E_1) + M^*(E_2)$

(b) No, it is countably subadditive

Ans3:- Let Ω be an ∞ set. Then, F is the collection of all subsets of Ω that are either finite or have finite compliments. Let $A \in F$, so if A is finite $A^c \in F$. If A^c is finite both $A, A^c \in F$.

Assume $A, B \in F$. If one of these sets is already finite, then clearly their intersections is finite and $\in F$. If, neither A, B are finite, then A^c, B^c are finite, $A^c \cup B^c$ is finite.

So $A^c \cup B^c \in F \Rightarrow (A \cap B)^c \in F \Rightarrow A \cap B \in F$

Let $\{x_n\}_n^\infty$ be an infinite sequence of distict elements of Ω. Being finite, each singleton set $\{x_n\} \in F$. But the countable union of the odd numbered elements $\cup_x$, odd $\{x_n\}$ does not belong to F, since neither it or its compliment is finite set. Thus, F is not a sigma algebra

The necessary & sufficient conditions for F of subsets of X to be σ- algebra are:-

a) $\phi \in F$

b) if $A \in F \Rightarrow A^c \in F$

c) if $E_1, E_2, \ldots$ is a countable collection of sets in F then $\bigcup_{n=1}^{\infty} E_n \in F$.

Here the first 2 conditions are already satisfied. For the 3rd to be true

Ans 4) a) let $E = \bigcap_{k=1}^{\infty} E_k \Rightarrow E_n = E \cup \bigcup_{k=n}^{\infty} (E_k \setminus E_{k+1})$

Using additive property

$$m(E_n) = m(E) + \sum_{k=n}^{\infty} m(E_k \mid E_{k+1})$$

for $n=1$

$$m(E) = m(E_1) - \sum_{k=1}^{\infty} m(E_k \setminus E_{k+1})$$

$$\Rightarrow m(E_1) = m(E) + \sum_{k=1}^{\infty} m(E_k \setminus E_{k+1})$$

Thus, series converges to 0, i.e

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k \setminus E_{k+1}) = 0$$

since $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$

$$\lim_{k\to\infty} m(E_n) = m(\bigcap_{k=1}^{\infty} E_k)$$

b) <u>Case 1</u>:- Assume $\exists\, m \in N$ st $m(E_m = \infty)$

$E_m \subseteq \bigcup_{k=1}^{\infty} E_k \Rightarrow m(\bigcup_{k=1}^{\infty} E_k) = \infty$

since $E_1 \subseteq E_2 \subseteq \dots$, $\therefore$ $m(E_k) = \infty \;\forall\, k > m$

Thus, $\lim_{k\to\infty} m(E_k) = \infty = m(\bigcup_{k=1}^{\infty} E_k)$

<u>Case 2</u>:- Asume there is no $m \in \mathbb{N}$ st $m(E_m) = \infty$

So, $0 \leq m(E_k) < \infty \;\forall\; k \in \mathbb{N}$

Now since

$E_{k+1} = E_k \cup (E_{k+1} \setminus E_k)$, $k \in \mathbb{N}$

we have

$m(E_{k+1} \setminus E_k) = m(E_{k+1}) - m(E_k)$

$$\Rightarrow \bigcup_{k=1}^{\infty} E_k = E_1 \cup (E_2 \setminus E_1) \cup \dots \cup (E_{k+1} \setminus E_k) \cup \dots$$

Since each element is pairwise disjoint on right side,

we have,

$$m(\bigcup_{k=1}^{\infty} E_k) = m(E_1) + \sum_{k=1}^{\infty} (m(E_{k+1}) - m(E_k))$$

n^{th} potential sums on the series on right

$$m(E_1) + \sum_{k=1}^{n-1} [m(E_{k+1}) - m(E_k)] - m(E_n)$$

$$\therefore \lim_{k\to\infty} m(E_k) = m(\bigcup_{k=1}^{\infty} E_k)$$

Ans5) Consider $a \in \mathbb{R}$ and f to be monotonically increasing
Let $\{f > a\}$ be an interval. It can be of the for (x, ∞) or an interval of for $[x, \infty)$, both for which are Borel sets. Thus, the given fn$^{\underline{n}}$ is Borel measurable.
Similarly if f is monotonically decreasing, we can say that the interval $\{f > a\}$ is of form $(-\infty, x)$ or $(-\infty, x]$ both for which are Borel sets, and hence f is Borel measurable

Ans6) $g(x) = \begin{cases} f(x) & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$

Let f be measurable. Let $A = \{x \in E : f(x) \neq g(x)\}$
here $m(A) = 0 \Rightarrow A \in F$, Then

$$\{x \in E : g(x) > a\} = \{x \in A : g(x) > a\} \cup \{x \in E \setminus A : g(x) > a\}$$

$$= \underbrace{\underbrace{\{x \in A : g(x) > a\}}_{\in F} \cup \underbrace{\{x \in E \setminus A : g(x) > a\}}_{(f/(E \setminus A))^{-1}(a, \infty) \in F}}_{\in F}$$

Thus $g(x)$ is measurable.
Converse:- $g(x)$ is measurable. Let $f = g$ on N^c where N is a set of measure 0. Then

$$\{f < b\} = \{\{g < b\} \cap N^c\} \cup \{\{f < b\} \cap N\}$$

$\{g < b\}$ is ~~not~~ measurable as g is measurable,
$\{f < b\} \cap N$ is measurable as it is a subset of N is already measurable with measure zero.
N^c is already measurable
Hence $(\{g < b\} \cap N^c) \cup (\{f < b\} \cap N)$ is measurable.
Thus $\{f < b\}$ is measurable $\Rightarrow$ f is measurable

Ans7) We have $f: \mathbb{R} \to \mathbb{R}$, f is a Lebesgue measurable function & $g: \mathbb{R} \to \mathbb{R}$, g is a continuous function. Thus, g is also Borel measurable.

$(g = (\mathbb{R}, B_R) \to (\mathbb{R}, B_R))$ for the σ algebras B_R, B_R.

Now, we have to show that $g \circ f$ is measurable,

i.e $g \circ f : (R, \mathcal{L}_R) \to (R, B_H)$ is measurable.

By measurability of g, we can say that, since $B \in B_R$, $B' = g^{-1}(B) \in B_R$. By measurability of $f \to f^{-1}(B') \in \mathcal{L}$

i.e $(g \circ f)^{-1}(B) \in \mathcal{L}$. So $g \circ f$ is measurable.

With similar justification, we can say that $f \circ g$ is not measurable.

Ans 8) <u>Case 1</u> :- $m^*(A) = \infty$

Take $U = R \implies U$ is a Borel set and $m^*(A) = m^*(U)$

<u>Case 2</u> :- $m^*(A) < \infty$

There exists $k \in N$ s.t. $m^*(A) + \frac{1}{k}$ is not the lowerbound for the set $\{\sum (b_i - a_i) : A \subseteq \cup (a_i, b_i)\}$.

So $\exists$ a set $U_k := \cup (a_i, b_i)$ st. $A \subset U_k$ &

$m^*(U_k) = \sum (b_i - a_i) < m^*(A) + \frac{1}{k}$

Take $U = \bigcap_k U_k$ then $A \subset U$, so $m^*(A) \leq m^*(U)$

and also, $m^*(U) \leq m^*(U_k) \leq m^*(A) + \frac{1}{k}$ $\forall k \in \mathbb{N}$

$\therefore m^*(A) = m^*(U)$ & U is Borel set

Ans 9) Assume function $f(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$ & a point $a \in \mathbb{R}$. Now consider the set $\{x : f(x) > a\}$. If we show that $\{x : f(x) > a\} = \bigcup_{n \in N} \{x : f_n(x) > a\}$ we can s.t. f is measurable and so will be the $\limsup(f_n)$

Take a point $y \in \{x : f(x) > a\}$. Then $f(y) > a$.

If $f_n(y) < a$ $\forall n$, then a is an upper bound for $\{f_n(y_n)\}_{n \in \mathbb{N}}$ and so since $f(y)$ is the least upper bound for that set, we would have $f(y) \leq a$, However, this does not hold true as $f(y) > a$ by definition of y.

$\therefore$ We must have some $m \in N$ s.t. $f_m(y) > a \Rightarrow$ $y \in \{\cancel{x}\, x : f_m(x) \not> a\}$. Then $y \in \{x : f_m(x) > a\}$,

$y \in \bigcup_{n \in N} \{x : f_n(x) > a\}$

This proves that $\{x : f(x) > a\} \subseteq \bigcup_{n \in N} \{x : f_n(x) > a\} \rightarrow$ 1

In order to prove the converse, lets take $z \in \bigcup_{n \in N} \{x : f_n(x) > a\}$

Then, $z \in \{x : f_k(x) > a\}$ for some $k \in N \rightarrow f_k(z) > a$

However, since $f(y)$ is supremum, $f(z) \geq f_k(z) > a$

& $\bigcup_{n \in N} \{x : f_n(x) > a\} \subseteq \{x : f(x) > a\} \rightarrow$ 2

From 1 & 2, we get $\{x : f(x) > a\} = \bigcup_{n \in N} \{x : f_n(x) > a\}$

Since a is arbitrary, f is measurable.

This implies $\sup\{f_n(x) : n \in \mathbb{N}\}$ is measurable.

Moreover since $\lim_{n \to \infty} \sup f_n = \inf_{n \in N} \sup_{k \geq n} f_k$,

$\limsup(f_n)$ is measurable.

Using similar produre for $\liminf (f_n)$, we can show that

$\{\inf_{n \in N} f_n > b\} = \bigcap_{n \in \mathbb{N}} \{f_n > b\}$ for any $b \in \mathbb{R}$

so, the infimum is measurable. Now, for lim inf,

$$\lim_{n \to \infty} \inf f_n = \sup_{n \in N} \inf_{k \geq n} f_n$$

Thus, liminf is also measurable

Ans 10) a) i) Every measurable set is "nearly" a finite union of intervals.

Let E be a measurable set of finite outer measure. Then for each $\varepsilon > 0$, $\exists$ a finite disjoint collection of open intervals $\{I_k\}_{k=1}^{n}$ for which $O = \bigcup_{k=1}^{n} I_k$, then $m(E \setminus O) + m(O \setminus E) = m(E \Delta O) < \varepsilon$.

ii) Every measurable function is nearly continuous.
let f be a real valued measurable fun^{n} on E. Then for each $\varepsilon > 0$, $\exists$ a continuous fun^{n} g on $\mathbb{R}$ & closed set F contained in E for which $g = f|_F$ & $m(E \setminus F) < \varepsilon$

iii) Every convergent sequence of measurable functions is "nearly" uniformly convergent.
Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable fun^{n}s on E that converges pointwise on E to the real valued function f. Then for each $\varepsilon > 0$, there is a closed set F contained in E for which
$\{f_n\} \to f$ uniformly on F and $m(E \setminus F) < \varepsilon$

b) f is a measurable function differentiable almost every where.
Lets define sequence of measurable fun^{n}s $\{f_n\}$ st

$$f_n = \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}}$$

so, f_n is measurable.

Now, the derivative of original fun^{n} can be written as

$$\frac{df}{dx} = f'(x) = \lim_{n \to \infty} \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}} = \lim_{n \to \infty} f_n$$

Thus the derivative of f is measurable too.