Lecture - 6 (23-01-2024)

Base for the topology:

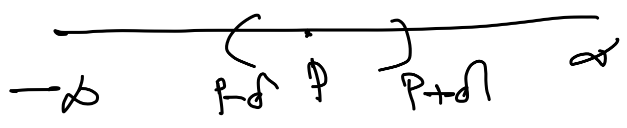
Let (X, T) be a topological space. A class $\mathcal{B}$ of open subsets of X, i.e., $\mathcal{B} \subseteq T$, is a base for the topology T if every open set $G \in T$ is union of members of $\mathcal{B}$.

or

For every $p \in G \in T$, $\exists$ $B_p \in \mathcal{B}$ such that $p \in B_p \subset G$.

Ex: Let $(\mathbb{R}, U)$ be a usual topological space.

Let G be an open subset of $\mathbb{R}$ containing a point $p \in \mathbb{R}$.



- Then by definition of an open set, $\exists$ an interval (a, b) such that

$$p \in (a, b) \subset G$$

or $$G = \bigcup_{a, b \in R} (a, b)$$

$\therefore$ $B = \{ (a, b) \,/\, a, b \in R, \; a < b \}$ is a base for the usual topology U on R.

Illy For the usual topology U on the plane R^2, open discs form a base for U.

For each p belongs to an open set G in R^2, there exists an open disc D_p centered at p with radius some $\delta > 0$ such that $p \in D_p \subset G$.

$\therefore\ \mathcal{B} = \{ D_p \mid$ where D_p is a open disc, $p \in \mathbb{R}^2 \}$

form a base for the usual topology U on $\mathbb{R}^2$.

Open rectangles form a base for U on $\mathbb{R}^2$.

Any rectangle $B \in \mathcal{B}$ whose vertices lie on the boundary of an open disc D_p satisfy $p \in B \subset D_p \subset G \Rightarrow p \in B \subset G$.

Open triangles form a base for U on $\mathbb{R}^2$.

|||ly Discs whose vertices lie on the boundary of an open disc form base for U on $\mathbb{R}^2$.

Ex : Let (X, D) be a discrete topological space and $B = \{\{p\} / p \in X\}$ be a class of singleton subsets of X. Then $B \subset D$.

$\because \forall p \in X$, $\{p\}$ is open set and every set $G \in D$ is union of singleton subsets of X, so the class $B = \{\{p\} / p \in X\}$ form a base for the discrete topology D on X.

Any other class B^* of subsets of X is a base for D iff B^* is a super class of B.

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Given a class $\mathcal{B}$ of subsets of a nonempty set X, when will the $\mathcal{B}$ be a base for some topology on X?

Since X is an open set in every topology, so it is necessary that

$$X = \cup\{B \mid B \in \mathcal{B}\}.$$

Ex: $X = \{a, b, c\}$

$\mathcal{B} = \{\{a, b\}, \{b, c\}\}$

$\{a, b\} \cap \{b, c\} = \{b\}$

Then $\mathcal{B}$ cannot be a base for any topology on X, because $\{a, b\}$ and $\{b, c\}$ are open and their intersection $\{a, b\} \cap \{b, c\} = \{b\}$ must be open and it must be union of members of $\mathcal{B}$.

So we have the following theorem.

Theorem: Let $\mathcal{B}$ be a class of subsets of a non empty set X. Then $\mathcal{B}$ is a base for some topology on X iff $\mathcal{B}$ satisfies the following conditions;

(i) $X = \cup\{B \mid B \in \mathcal{B}\}$.

(ii) For any $B, B^* \in \mathcal{B}$, $B \cap B^*$ is union of members of $\mathcal{B}$. That is for any $p \in B \cap B^*$, $\exists\ B_p \in \mathcal{B} \ni$

$$p \in B_p \subset B \cap B^*.$$

Proof: Suppose $\mathcal{B}$ is a base for some topology T on X.

$\because X \in T \Rightarrow X$ is union of members of $\mathcal{B}$ by definition of a base, which proves (i).

If $B, B^* \in \mathcal{B}$, then since $\mathcal{B} \subset T$, implies B, B^* are open sets. So $B \cap B^*$ is also an open set.

$\because$ $\mathcal{B}$ is a base for T, for any $p \in B \cap B^*$ $\exists$ $B_p \in \mathcal{B}$ $\ni$ $p \in B_p \subset B \cap B^*$, which proves (ii).

Conversely suppose that $\mathcal{B}$ be a class of subsets of X satisfying (i) and (ii).

Let T be a class of subsets of X which are union of members of $\mathcal{B}$ including the empty set ϕ. That is

$$T = \{\phi, G \;/\; G \text{ is union of members of } \mathcal{B}\}$$

Claim: T is a topology on X.

$\because$ $\mathcal{B}$ satisfies (i) and (ii), $\mathcal{B} \subset T$ will be a base for this topology T.

$\because$ by (i), $X = \cup\{B \in B \in \mathcal{B}\}$

$\Rightarrow X \in T$

Also $\phi \in T$.

Now let $\{G_i\}$ be a class of members of T. Since each $G_i \in T$ is union of members of B, so there exists some class

$$\{B_{ij} \mid B_{ij} \in B\} \text{ such that}$$

$$G_i = \bigcup_j B_{ij}.$$

$$\therefore \bigcup_i G_i = \bigcup_i \{\bigcup_j B_{ij}\}$$

$= \text{union of members of } B.$

$$\Rightarrow \bigcup_i G_i \in T.$$

Illy for $G, H \in T$ $\exists$ subclasses

$\{B_i \mid B_i \in B, i \in I\}$ and

$\{B_j \mid B_j \in B, j \in J\}$ where I and J are index sets, such that

$$G = \bigcup_i B_i \quad \text{and} \quad H = \bigcup_j B_j.$$

Now

$$G \cap H = \left(\bigcup_i B_i\right) \cap \left(\bigcup_j B_j\right)$$

$$= \bigcup_{i,j} \{ B_i \cap B_j \mid i \in I, \; j \in J \}.$$

= union of members of $\mathcal{B}$.

$\Rightarrow$ $G \cap H \in \mathcal{T}$.

$\therefore$ $\mathcal{T}$ is a topology on X for which $\mathcal{B}$ is a base which satisfy (i) and (ii)

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Ex: $X = \mathbb{R}$.

$$\mathcal{B} = \{ (a, b] \mid a < b, \; a, b \in \mathbb{R} \}.$$

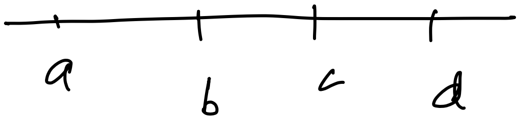
Then clearly $\mathbb{R}$ is union of members of $\mathcal{B}$.

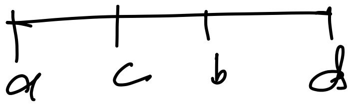
[$(a, b) = \bigcup \{ (a, b - \frac{1}{n}] \mid n \in \mathbb{N} \}$

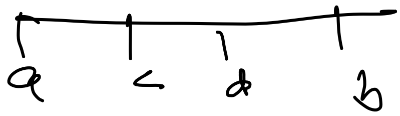
$\mathbb{R}$ is union of open sets and hence union of open intervals]

Now for any $(a, b]$, $(c, d] \in \mathcal{B}$,

$$(a,b] \cap (c,d] = \begin{cases} \phi & b < c \\ (c, b]. & a < c < b < d \\ (c, d] & \end{cases}$$







Thus $\mathcal{B}$ consisting of open-closed intervals form a base for the topology $\mathcal{T}$ on $\mathbb{R}$. This topology $\mathcal{T}$ is called upper-limit topology on $\mathbb{R}$.

Also $\mathcal{T}$ is not equal to usual topology $\mathcal{U}$ on $\mathbb{R}$, i.e., $\mathcal{U} \neq \mathcal{T}$.

Also $(a, b) = \cup \{ (a, b - \frac{1}{n}] \mid n \in \mathbb{N} \}$

$\implies \mathcal{U} \subset \mathcal{T}$.

$\therefore$ $(a,b] \in U$ and by (*) $(a,b]$ is union of member of open-closed intervals.

$\therefore$ $(a,b] \in T$.

Also open intervals form a base for the usual topology U on $\mathbb{R}$.

$\therefore$ $U \subset T$.

$\therefore$ usual topology U on $\mathbb{R}$ is coarser than the upper-limit topology T on $\mathbb{R}$.

Illy The class of closed-open intervals

$$B^* = \{ [a,b) \mid a<b,\ a,b \in \mathbb{R} \}$$

is a base for the topology T^* on $\mathbb{R}$ is called lower-limit topology on $\mathbb{R}$.

Clearly $T^* \neq U$.

Problem: Let (X, T) be a topological space and B be a base for T on X. If B^* is such that $B \subset B^* \subset T$, then show that B^* is also a base for T.

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[Attendance:
65, 08, 11, 40, 10, 19, 62, 17, 27]