Lecture 5

Let $\varepsilon > 0$. Then by the property of infimum, there exists a collection of intervals $\{I_n\}_n$ such that $[a,b] \subseteq \bigcup_n I_n$

& $\boxed{m^*([a,b]) + \varepsilon \geq \sum_n l(I_n)} \longrightarrow (**)$

where $I_n = [a_n, b_n)$ (say) $\forall n \geq 1$

Let $I_n' = \left(a_n - \frac{\varepsilon}{2^n}, b_n\right)$ open. $\quad I_n \subseteq I_n' \;\; \forall n.$

$\therefore \; [a,b] \subseteq \bigcup_n I_n \subseteq \bigcup_n I_n'$, open

$\boxed{\begin{aligned} l(I_n') &= b_n - a_n + \frac{\varepsilon}{2^n} \\ &= l(I_n) + \frac{\varepsilon}{2^n} \end{aligned}}$

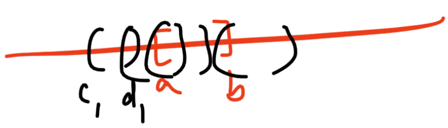
$\therefore$ By Heine-Borel Thm, there exists a finite sub collection of $\{I_n'\}$, say $J_1, \dots, J_N$ such that

$[a,b] \subseteq \bigcup_{k=1}^{N} J_k$, let $J_k = (c_k, d_k)$ $\forall$ $k = 1, \dots, N.$

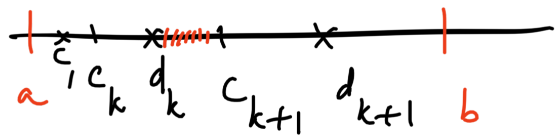
We may assume no J_k is contained in any other. $c_k < d_k$

Suppose $c_1 < c_2 < \dots < c_N$.

$\left([a,b] \subseteq \bigcup_{k=1}^{N} (c_k, d_k) \right.$



If $d_k < c_{k+1}$ for some k,



Then $[d_k, c_{k+1}] \subseteq [a,b]$ not covered by $\bigcup_{j=1}^{N} (c_j, d_j)$.

which is a contradiction to $[a,b] \subseteq \bigcup_{j=1}^{N} (c_j, d_j)$

$\therefore \quad d_k \geq c_{k+1} \quad \forall\, k = 1, \ldots, N-1.$

Now,
$$d_N - c_1 = \sum_{k=1}^{N} (d_k - c_k) - \sum_{k=1}^{N-1} \underbrace{(d_k - c_{k+1})}_{\geq 0}$$

$$\leq \sum_{k=1}^{N} (d_k - c_k) = \sum_{k=1}^{N} \ell(J_k). \longrightarrow (\S)$$

From $(**)$,
$$m^*([a,b]) \geq \sum_{n=1}^{\infty} \ell(I_n) - \varepsilon$$

$$\geq \sum_{n=1}^{\infty} \left(\ell(I_n') - \frac{\varepsilon}{2^n}\right) - \varepsilon$$

$$= \sum_{n=1}^{\infty} \ell(I_n') - \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} - \varepsilon$$

$$= \sum_{n=1}^{\infty} \ell(I_n') - 2\varepsilon.$$

$$\geq \sum_{k=1}^{N} \ell(J_k) - 2\varepsilon$$

$$\left(\because \{J_k\}_{k=1}^{N} \subset \{I_n'\}_{n=1}^{\infty}\right)$$

$$\geq d_N - c_1 - 2\varepsilon \quad (\text{from } (\S))$$

$$> b - a - 2\varepsilon$$

$$= \ell([a,b]) - 2\varepsilon \quad \left(\because [a,b] \subseteq \bigcup_{k=1}^{N} J_k\right)$$

$$\therefore \; m^*([a,b]) \geq \ell([a,b]) - 2\varepsilon, \quad \forall \varepsilon > 0.$$

$$\Rightarrow \; m^*([a,b]) \geq \ell([a,b]) = b-a.$$

Thus $m^*([a,b]) = b-a$.

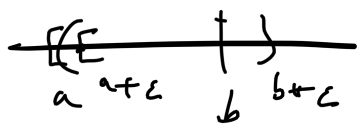
Case 2: Suppose $I = (a,b]$, $a,b \in \mathbb{R}$,

$a < b$.

Let $0 < \varepsilon < b-a$.

Let $I' = [a+\varepsilon, b]$

Then $I' \subseteq I$



$$\underset{\substack{\| \\ b-a-\varepsilon}}{m^*(I')} \leq m^*(I)$$

(by case 1)

$$\therefore \; m^*(I) \geq b-a-\varepsilon \quad \forall \varepsilon > 0.$$

$\therefore \quad m^*(I) \geq b-a.$

Also $\quad I \subseteq I'' = [a, b+\varepsilon)$

$$m^*(I) \leq m^*\big([a, b+\varepsilon)\big) = b-a+\varepsilon$$

Thus $\quad m^*(I) \leq b-a+\varepsilon \quad \forall \varepsilon > 0.$

$\therefore \quad m^*(I) \leq b-a.$

Hence $\quad m^*(I) = b-a.$

EXERCISE: For $I = (a,b)$.

Case 3: Suppose I is an infinite interval,

$[a, \infty)$ or (a, ∞) or $(-\infty, a]$ or $(-\infty, a)$

say $I = (-\infty, a]$

For any $M > 0$, there exists k such that the interval $I_M = [k, k+M) \subseteq I$

$$\therefore \quad m^*(I_M) \leq m^*(I)$$

$$\Rightarrow \quad \underset{M}{\overset{m^*(I_M)}{\|}} \leq m^*(I).$$

Since $M > 0$ is arbitrary, we get $m^*(I) = +\infty.$
$= l(I)$

Theorem:- Outer measure is countable subadditive.

That is, for any sequence of sets $\{E_i\}$ of real numbers, $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$.

proof:- Let $\varepsilon > 0$.

For each i, there exists a sequence of intervals $\{I_{i,j}\}_{j=1}^{\infty}$ such that

$$E_i \subseteq \bigcup_{j=1}^{\infty} I_{i,j} \quad \&$$

$$m^*(E_i) + \frac{\varepsilon}{2^i} \geq \sum_{j=1}^{\infty} l(I_{i,j}) \longrightarrow (*)$$

$$\therefore \bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} I_{i,j}$$

& $\{I_{i,j}\}_{i,j}$ is a countable collection of intervals covering $\bigcup_{i=1}^{\infty} E_i$.

$$\therefore m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} l(I_{i,j}) \quad (\text{by def of } m^*)$$

$$\leq \sum_{i=1}^{\infty} \left(m^*(E_i) + \frac{\varepsilon}{2^i} \right)$$

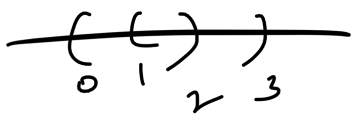
$$= \sum_{i=1}^{\infty} m^*(E_i) + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i}$$

$$= \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon$$

$$\therefore \quad m^*\left(\bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon, \quad \forall \varepsilon > 0.$$

$$\Rightarrow \quad m^*\left(\bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} m^*(E_i).$$

Examples:- ① $m^*((0,2)) = 2$

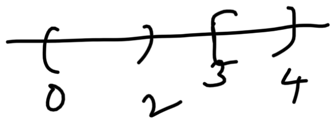


② $m^*((0,2) \cup (1,3)) = m^*((0,3))$

$= 3-0$

$= 3$

③ $m^*((0,2) \cup [3,4)) = ?$



$\leq m^*((0,2)) + m^*([3,4))$ (Subadditive)

$= 2+1$

$\therefore m^*((0,2) \cup [3,4)) \leq 3.$?

Proposition:- For any $A \subseteq \mathbb{R}$ & $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}$ such that $A \subseteq U$ &

$$m^*(U) \leq m^*(A) + \varepsilon.$$

proof:- Let $\varepsilon > 0$.

There exists $\{I_n\}_{n=1}^{\infty}$ intervals such that

$$m^*(A) + \frac{\varepsilon}{2} \geq \sum_{n=1}^{\infty} l(I_n) \longrightarrow (*)$$

Let $I_n = [a_n, b_n)$.

Set $I_n' = \left(a_n - \frac{\varepsilon}{2^{n+1}}, b_n\right)$ open set

so that $I_n \subseteq I_n'$ $\forall n \geq 1$

Let $U := \bigcup_{n=1}^{\infty} I_n'$. Then U is an open set.

& $$A \subseteq \bigcup_{n=1}^{\infty} I_n \subseteq \bigcup_{n=1}^{\infty} I_n' = U$$

$\therefore A \subseteq U$

& $$m^*(U) = m^*\left(\bigcup_{n=1}^{\infty} I_n'\right)$$

$$\leq \sum_{n=1}^{\infty} m^*(I_n') \quad \text{(by subadditive property)}$$

$$= \sum_{n=1}^{\infty}\left(l(I_n) + \frac{\varepsilon}{2^{n+1}}\right)$$

$$= \sum_{n=1}^{\infty} l(I_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} l(I_n) + \frac{\varepsilon}{2}$$

$$\leq m^*(A) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad (\text{by } (*))$$

$\therefore$ $m^*(U) \leq m^*(A) + \varepsilon$, as required.
