MA 51002: Measure Theory and Integration
Assignment - 2 (Spring 2022)
Lebesgue integration

Note: Unless otherwise stated, $\int f$ will denote the Lebesgue integral of a measurable function f.

1. (i) Show that Monotone Convergence Theorem and Fatou's Lemma imply each other.
   (ii) Give an example when the inequality is strict in Fatou's Lemma.
   (iii) Show that Fatou's Lemma is not true for any sequence of measurable functions.
   (iv) Deduce Bounded Convergence Theorem from Fatou's Lemma.

2. (i) Let $f : [0,1] \to \mathbb{R}$ defined by
   $$f(x) = \begin{cases} 1, & \text{if } x \text{ is a Cantor point} \\ 0, & \text{otherwise} \end{cases}$$
   Then show that f is Riemann integrable on $[0,1]$ and hence find the Riemann integral $\int_{[0,1]}^{\mathcal{R}} f(x)\, dx$.

   (ii) Show that $\int_1^\infty dx/x = \infty$ (don't use the log function).

3. Let $f : [0,1] \to \mathbb{R}$ be defined as : $f(x) = 0$ for $x \in \mathbb{Q}$ and $f(x) = n$ if $x \in \mathbb{Q}^c$ and n is the number of zeros immediately after the decimal point in the decimal representation of x. Show that f is measurable and find $\int_0^1 f\, dx$.

4. Let $f : [0,1] \to \mathbb{R}$ be defined as : $f(x) = 0$ if x is a Cantor point and $f(x) = p$ if x belongs to any one of the 2^{p-1} many complementary open intervals each of length 3^{-p} deleted at the stage p in order to construct Cantor set. Show that f is measurable and find $\int_0^1 f\, dx$.

5. The function f is defined on $(0,1)$ by
   $$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \cap (0,1) \\ [1/x]^{-1}, & x \in \mathbb{Q}^c \cap (0,1), \end{cases}$$
   where $[x]$ denotes the 'box' function. Show that $\int_0^1 f\, dx = \infty$.

6. Let $f_n(x) = \min(f(x), n)$, where $f \geq 0$. Show that $\int f_n\, dx \uparrow \int f\, dx$.

7. (i) Let $E_1, E_2, \cdots$ be a sequence of measurable sets. Show that if $m(E_n) < 2^{-n}$ for all n, then $\chi_{E_n} \to 0$ a.e. Show that $m(E_n) \to 0$ is not sufficient to ensure that $\chi_{E_n} \to 0$ a.e.

   (ii) Let f be an integrable function over a set S. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that $\int_E f < \epsilon$ whenever $E \subset S$ is a measurable set and $m(E) < \delta$.

8. (i) Show that Dominated Convergence Theorem implies Bounded Convergence Theorem.
   (ii) Give an example of a sequence $\{f_n\}$ which satisfies the conditions of the Dominated Convergence Theorem but does not satisfy the conditions of Bounded Convergence Theorem.
   (iii) Show that without the domination condition, the conclusion of Dominated Convergence Theorem may not hold good.
   (iv) Does there exist a sequence of Lebesgue integrable functions which converges pointwise to a measurable non-Lebesgue integrable function?

9. Let $f_n(x) = \frac{n^{3/2}x}{1+n^2x^2}$ for $x \in [0,1]$.
   (i) Show that $f_n(x) \to 0$ for all $x \in [0,1]$.
   (ii) Show that the sequence $\{f_n\}$ is not uniformly bounded.
   (iii) Show that $f_n(x) \leq \frac{1}{\sqrt{x}}$.
   (iv) Explain why the conditions of the Dominated Convergence Theorem are satisfied and make a conclusion concerning the limit of $\int f_n$.