Lecture-9 (05-02-2024)

Let (X, T_1) and (Y, T_2) be any two topological space.

Let

$$\mathcal{B} = \{ G \times H \mid G \in T_1, H \in T_2 \}$$

$$= \{ G \times H \mid G \text{ is open in } X, H \text{ is open } Y \}.$$

Then $\mathcal{B} \subseteq X \times Y$.

Then $\mathcal{B}$ is a basis for a topology on $X \times Y$.

$\because X \in T_1$ and $Y \in T_2$

$\therefore X \times Y \in \mathcal{B}$

Also for any $G_1 \times H_1, G_2 \times H_2 \in \mathcal{B}$ where $G_1, G_2 \in T_1$ and $H_1, H_2 \in T_2$.

Then

$$(G_1 \times H_1) \cap (G_2 \times H_2) = (G_1 \cap G_2) \times (H_1 \cap H_2) \in \mathcal{B}$$

[$\because G_1 \cap G_2 \in T_1$ and $H_1 \cap H_2 \in T_2$].

$\therefore \mathcal{B}$ is a base for a topology T on $X \times Y$.

Then $(X \times Y, T)$ is called a Product topological space.

Theorem: If $\mathcal{B}$ is basis for the topology of X and $\mathcal{C}$ is a topology of Y, then

$$\mathcal{D} = \{ B \times C \mid B \in \mathcal{B},\ C \in \mathcal{C} \}$$

is a basis for the topology on $X \times Y$.

Proof: let W be any open set in the Product of $X \times Y$.

We know that

$$\mathcal{B}^* = \{ G \times H \mid G \text{ is open in } X,\ H \text{ is open in } Y \}$$

is a basis for the product topology on $X \times Y$.

Now let $(x, y) \in W$

$\because \mathcal{B}^*$ is a base for $X \times Y$,

so $\exists\ G \times H \in \mathcal{B} \ni$

$(x, y) \in G \times H \subset W$, where G is open in X, H is open in Y.

$\because (x, y) \in G \times H \Rightarrow x \in G$ and $y \in H$.

$\because x \in G$ and G is open in X, $\mathcal{B}$ is a base for X,

$\Rightarrow \exists\ B \in \mathcal{B} \ni x \in B \subset G$.

Il///ly $y \in H$ and H is open in Y, $\mathcal{C}$ is a base for Y, so $\exists\ C \in \mathcal{C}$ such that $y \in C \subseteq H$.

$\therefore$ $x \in B$ and $y \in C \Rightarrow (x, y) \in B \times C$

Then

$$(x, y) \in B \times C \subseteq G \times H \subseteq W$$

Then given any $(x, y) \in W$, the open set in $X \times Y$, there exists an element $B \times C \in \mathcal{D}$ such that

$$(x, y) \in B \times C \subset W$$

$\therefore$ $\mathcal{D}$ is a base for the product topology on $X \times Y$.

Ex: Let $(\mathbb{R}, U)$ be usual topological space.

Then $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

$\{G \times H \mid G \in U, H \in U\}$ is a base for the product topology on $\mathbb{R} \times \mathbb{R}$.

Now

$$\mathcal{B} = \{ (a,b) \mid a<b,\ a,b \in \mathbb{R} \}$$

is a base for $\mathcal{U}$ on $\mathbb{R}$.

$$\mathcal{C} = \{ (c,d) \mid c<d,\ c,d \in \mathbb{R} \}$$

is a base for $\mathcal{U}$ on $\mathbb{R}$.

Then

$$\mathcal{B} \times \mathcal{C} = \{ B \times C \mid B \in \mathcal{B},\ C \in \mathcal{C} \}$$

$$= \{ (a,b) \times (c,d) \mid (a,b) \in \mathcal{B},\ (c,d) \in \mathcal{C} \}$$

is a base for a topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. This topology is the usual topology on $\mathbb{R}^2$.

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Let $\pi_1 : X \times Y \longrightarrow X$

$\pi_2 : X \times Y \longrightarrow Y$

be the mappings defined by

$\pi_1(x, y) = x$, $\pi_2(x, y) = y$.

Here π_1 and π_2 are projections of $X \times Y$ onto X and Y, respectively.

Let G be any open set in X

H " " " in Y.

Then

$\pi_1^{-1}(G) = G \times Y$

$\pi_2^{-1}(H) = X \times H$.

Consider

$S = \{\pi_1^{-1}(G) \mid G \text{ is open in } X\}$
$\cup \{\pi_2^{-1}(H) \mid H \text{ is open in } Y\}$,

Then $\mathcal{S}$ is a subset of $X \times Y$ and $\mathcal{S}$ is a subbase for the product topology on $X \times Y$.

Note that $\pi_1^{-1}(G) \cap \pi_2^{-1}(H)$

$$= (G \times Y) \cap (X \times H)$$

$$= G \times H$$

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Theorem: Let X_1 be a subspace of $(X, \mathcal{T})$ and Y_1 be a subspace of $(Y, \mathcal{T}^*)$. Then prove that the product topology on $X_1 \times Y_1$ is same as the relative topology on $X_1 \times Y_1$ w.r.t the product topology on $X \times Y$.

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Continuity

We know that a function $f:\mathbb{R}\to\mathbb{R}$ is continuous at a point $x_0\in\mathbb{R}$ if for given any $\epsilon>0$, $\exists\ \delta>0$ such that

$$|x-x_0|<\delta \implies |f(x)-f(x_0)|<\epsilon.$$

Now

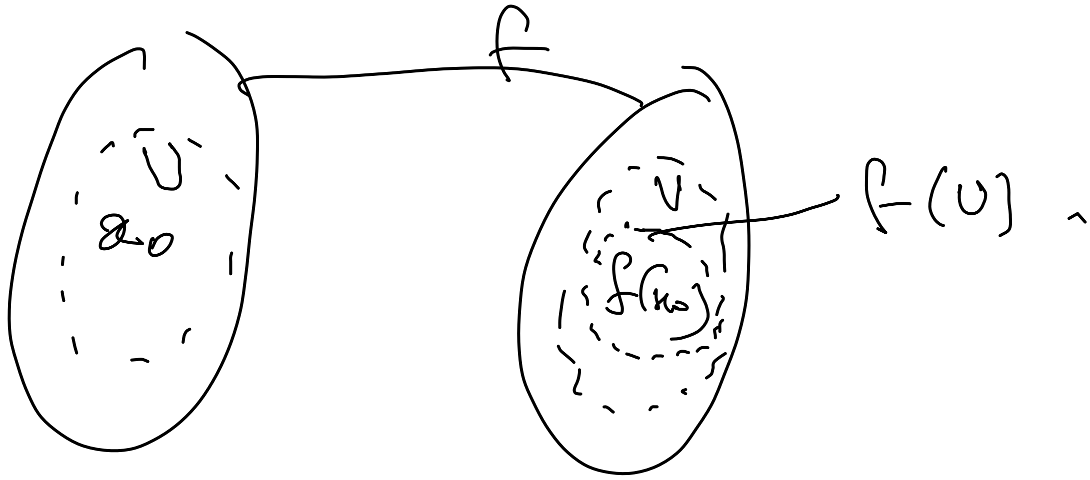
$$|x-x_0|<\delta \implies |f(x)-f(x_0)|<\epsilon$$

$$\iff -\delta<x-x_0<\delta \implies -\epsilon<f(x)-f(x_0)<\epsilon$$

$$\iff x_0-\delta<x<x_0+\delta \implies f(x_0)-\epsilon<f(x)<f(x_0)+\epsilon$$

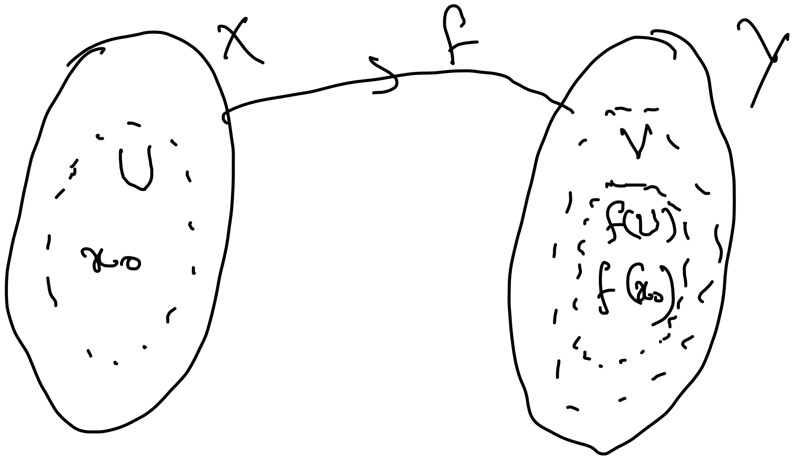
$$\iff x\in \underbrace{(x_0-\delta,\ x_0+\delta)}_{U} \implies f(x)\in\underbrace{(f(x_0)-\epsilon,\ f(x_0)+\epsilon)}_{V}$$

Then $f(U)\subset V$



[Attendance: morning 15, 63, 43, 19, 42, 17, 38, 06, 32, 27, 57, 01, 09, 60, 39, 10, 23]

Def: Let (X, T) and (Y, T') be any two topological spaces and $f : (X, T) \longrightarrow (Y, T')$ be a mapping. We say f is continuous at $p \in X$ if for each open set V containing $f(p)$, there exists an open set U containing p such that $f(U) \subseteq V$, i.e., $\forall y \in U \Rightarrow f(y) \in V$.



If $f : (X, T) \longrightarrow (Y, T')$ is continuous at every point $p \in X$, we say f is a continuous function from X into Y or f is T—T' continuous function.

Theorem: Let (X, T) and (Y, T') be any two topological spaces. Then $f : (X, T) \longrightarrow (Y, T')$ is continuous iff for each open set V in Y, $f^{-1}(V)$ is open in X.

Proof: Let $f : (X, T) \longrightarrow (Y, T')$ be continuous.

Let V be any open subset of Y.

We prove $f^{-1}(V)$ is open in X.

If $f^{-1}(V) = \phi$, then clearly $f^{-1}(V)$ is an open subset of X,

Suppose $f^{-1}(V) \neq \phi$.

Let $x \in f^{-1}(V)$, then $f(x) \in V$.

$\because$ f is $T-T'$ continuous and V is an open subset of Y containing $f(x)$ implies there exists an open subset U of X containing 'x' such that

$f(U) \subset V$.

$\Rightarrow x \in U \subset f^{-1}(V)$

$\Rightarrow x$ is an interior point of $f^{-1}(V)$.

Since x is arbitrary point of $f^{-1}(V)$, it follows that $f^{-1}(V)$ is an open set.

[$\because f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$]

Conversely assume that $f^{-1}(V)$ is an open subset of X for every open subset V of Y.

We prove $f : X \longrightarrow Y$ is T-T' continuous.

Let $x \in X$ and V be any open set in Y such that $f(x) \in V$.

Let $U = f^{-1}(V)$. Then U is an open subset of X by assumption.

Also $x \in U$ $[\because f(x) \in V$
$\Rightarrow x \in f^{-1}(V) = U]$

[We know for any set A, we have
$f(f^{-1}(A)) \subset A$]

$\therefore$ We have

$$f(f^{-1}(V)) \subset V \Rightarrow f(U) \subset V.$$

$\Rightarrow$ f is continuous at $x \in X$.

Since x is an arbitrary point of X, it follows that f is $T - T'$ continuous.

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$*$ $f : (X, T) \longrightarrow (Y, T^*)$ is continuous

$\Leftrightarrow$ $\forall\, H \in T^* \Rightarrow f^{-1}(H) \in T$.

Ex: Let (X, D) be a discrete topological space and (Y, T) be any topological space. Then $f : (X, D) \longrightarrow (Y, T)$ is $D - T$ continuous.

Since for any $H \in T \Rightarrow f^{-1}(H) \in D$.

Theorem: $f : (X, T) \longrightarrow (Y, T')$ is continuous iff inverse image of a closed in Y is a closed set in X.

Proof: let $f : X \longrightarrow Y$ is continuous, and G be any closed set in Y.

Then G^c is an open set in Y.

$\Rightarrow$ $f^{-1}(G^c)$ is an open set in X.

$$\left[\because f^{-1}(G^c) = f^{-1}(Y - G) \right.$$
$$= f^{-1}(Y) - f^{-1}(G)$$
$$= X - f^{-1}(G)$$
$$\left. = [f^{-1}(G)]^c \right]$$

$\therefore$ $[f^{-1}(G)]^c$ is open in X

$\Rightarrow$ $f^{-1}(G)$ is closed in X.

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Now let (X, T) and (Y, T^*) be any two topological spaces and $f : X \longrightarrow Y$ be a function.

Let $\underline{B}^*$ be a base for T^* on Y.

Suppose for each $B \in \underline{B}^*$, $f^{-1}(B)$ is open in X.

Now let H be any open subset of Y.

Then $H = \bigcup_i B_i$, $B_i \in \underline{B}^*$.

$$\Rightarrow f^{-1}(H) = f^{-1}(\bigcup_i B_i)$$

$$= \bigcup_i f^{-1}(B_i) \text{ which is open in } X.$$

$$\Big[\, x \in f^{-1}(\bigcup_i B_i) \Leftrightarrow f(x) \in \bigcup_i B_i$$

$$\Leftrightarrow f(x) \in B_i \text{ atleast for some } i$$

$$\Leftrightarrow x \in f^{-1}(B_i)$$

$$\Leftrightarrow x \in \bigcup_i f^{-1}(B_i) \,\Big]$$

$\therefore$ $f : X \longrightarrow Y$ is continuous.

Theorem : Let $f : (X, T) \longrightarrow (Y, T^*)$ be a function and S be a sub-base for T^* on Y. Then $f : X \longrightarrow Y$ is continuous iff inverse of every member of S is an open subset of X. i.e., $s \in S \Rightarrow f^{-1}(s) \in T$.

Proof:

[Attendance: ^{5pm} 11, 65, 17, 19, 62, 23, 51, 60, 06, 32, 27, 40].