

Poisson Process & related distributions

Poisson Process $N(t) \sim PP(\lambda)$

(i) $N(t)$ has indep increment

(ii) $N(t)$ " stationary "

$$(iii) P(N(h)=1) = \lambda h + o(h)$$

$h \rightarrow \text{small}$

$$P(N(h) \geq 2) = o(h)$$

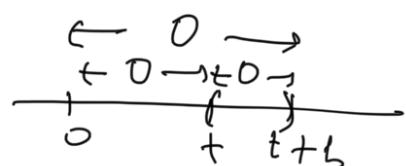
$$P(N(h)=0) = 1 - \lambda h + o(h)$$

$$P_n(t) = P(N(t)=n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, n=0,1,2,\dots$$

Set $h \rightarrow \text{small}$

$$P_0(t+h) = P(N(t+h)=0)$$

$$= P(\{N(t)=0\} \cap \{N(t, t+h]=0\})$$



$$= P(N(t)=0) P(N(t, t+h]=0) \quad \begin{aligned} N(t, t+h] &= N(t+h) - N(t) \\ &\quad] \text{ indep increment} \end{aligned}$$

$$= P(N(t)=0) P(N(h)=0)$$

$$= \underbrace{\left(1 - \lambda h + o(h)\right)}_{\cancel{\lambda}} P_0(t)$$

$$\Rightarrow \frac{P_0(t+h) - P_0(t)}{h} = -\frac{\lambda \cancel{\lambda} P_0(t)}{h} + P_0(t) \frac{o(h)}{h}$$

Take limit $\hbar \rightarrow 0$

$$\frac{d P_0(t)}{dt} = -\lambda P_0(t)$$

$$P_0(t) = C e^{-\lambda t}$$

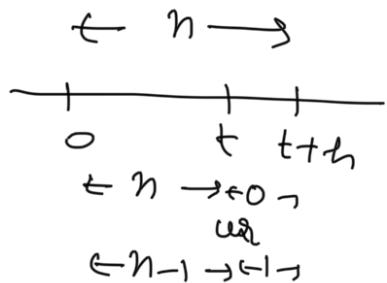
$$P_0(0) = 1$$

$$\Rightarrow C = 1$$

$$P_n(t+\hbar) = P(N(t+\hbar) = n)$$

$$= P(\underbrace{N(t)=n}_{\text{or } N(t)=n-1} \cap \underbrace{N(t, t+\hbar) = 0})$$

$$= \underbrace{P(N(t)=n)}_{\text{or } N(t)=n-1} \underbrace{(P(N(t, t+\hbar) = 0))}_{\text{or } N(t+\hbar)=0}$$



$$= P_n(t) \times \left(\underbrace{1 - \lambda \hbar + o(\hbar)}_{=} \right) + P_{n-1}(t) \times \left(\underbrace{\lambda \hbar + o(\hbar)}_{=} \right)$$

$$\Rightarrow \frac{P_n(t+\hbar) - P_n(t)}{\hbar} = -\lambda (P_n(t) - P_{n-1}(t)) + \cancel{o(\hbar)}$$

Limit $\hbar \rightarrow 0$

$$\Rightarrow \frac{d P_n(t)}{dt} = -\lambda (P_n(t) - P_{n-1}(t))$$

$$P_0(t) = e^{-\lambda t}$$

assume $P_{n-1}(t) \leq \frac{e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!}$

$$\frac{dP_n(t)}{dt} = -\lambda \left(P_n(t) - \frac{e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!} \right)$$

$$e^{\lambda t} \frac{dP_n(t)}{dt} + \lambda e^{\lambda t} P_n(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}$$

$$\frac{d(e^{\lambda t} P_n(t))}{dt} = \frac{\lambda^n}{(n-1)!} t^{n-1}$$

$$e^{\lambda t} P_n(t) = \frac{\lambda^n}{(n-1)!} \left(\int t^{n-1} dt \right) \rightarrow t^n$$

$$e^{\lambda t} P_n(t) = \frac{(\lambda t)^n}{n!}$$

$$\Rightarrow P_n(t) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

— X —

Interarrival and Waiting time dist:

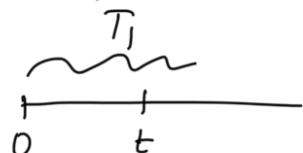
$N(t)$ = count # of event occurring in $[0, t]$

$$N(t) \sim \text{P.P. } (\lambda) \quad P(N(t)=n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \quad n=0, 1, 2, \dots$$

T_1 : time of the first event

T_n : time elapsed between $(n-1)^{\text{st}}$ and n^{th} event.

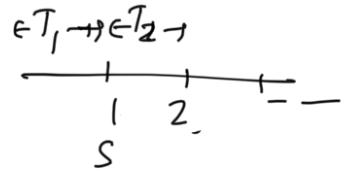
$$T_1 > t \Leftrightarrow N(t) = 0$$



$$P(T_1 > t) = P(N(t) = 0)$$

$$= e^{-\lambda t}$$

$$\Rightarrow T_1 \sim \exp(\lambda)$$



$$T_2 > t | T_1 = s \equiv N(s, t+s] = 0$$

$$P(T_2 > t | T_1 = s) = P(N(s, t+s] = 0)$$

$$= P(N(t) = 0)$$

$$= e^{-\lambda t}$$

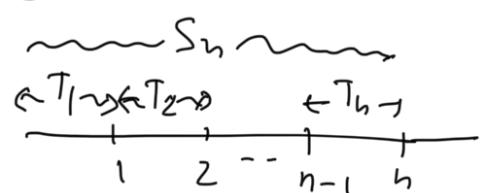
$$\rightarrow P(X > t) = E[P(X > t | Y)]$$

$$P(T_2 > t | T_1) = e^{-\lambda t}$$

$$P(T_2 > t) \leq E(P(T_2 > t | T_1)) = e^{-\lambda t}$$

$$T_2 \sim \exp(\lambda)$$

$$\text{midup } T_h \sim \exp(\lambda)$$

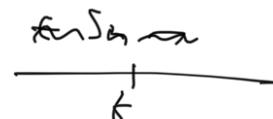


$$S_n = T_1 + T_2 + \dots + T_n \sim \text{Gamma}(n, \lambda)$$

$$M_{T_h}(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$M_{S_n}(t) = \prod_{i=1}^n M_{T_i}(t) = \left(1 - \frac{t}{\lambda}\right)^{-n}$$

$$S_n > t \equiv N(t) \leq n-1$$



$$D/\infty \sim 1 = D/N(1, 1, \dots)$$

$$P(S_n \leq t) = P(N(t) \leq n-1)$$

$$= \sum_{i=0}^{n-1} P(N(t)=i)$$

$(d\lambda \rightarrow 0)$

$$F_{S_n}(t) = P(S_n \leq t) = 1 - e^{-\lambda t} \sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!}$$

bdf of S_n

$$f_{S_n}(t) = \frac{d}{dt} F_{S_n}(t) = \frac{\lambda^n}{(n-1)!} e^{-\lambda t} t^{n-1}, \quad t \geq 0, \lambda > 0, n \geq 1$$

$$E(S_n) = \frac{n}{\lambda}, \quad V(S_n) = \frac{n}{\lambda^2}.$$

P1

$$M_{X_i}(t) = e^{\lambda_i (e^t - 1)}, \quad i=1,2$$

$$\text{Pois}(\lambda_1) \sim X_1, \quad X_2 \sim \text{Pois}(\lambda_2) \quad M_{X_1 + X_2}(t) = M_{X_1}(t) M_{X_2}(t) \\ = e^{(\lambda_1 + \lambda_2)(e^t - 1)}$$

$$X_1 + X_2 \sim \text{Pois}(\lambda_1 + \lambda_2)$$

P2 If $N(t) \sim P.P(\lambda)$

$$N(t) = N_1(t) + N_2(t)$$

indep. $\begin{cases} \xrightarrow{\text{then}} N_1(t) \sim P.P(\lambda p) \\ \xleftarrow{} N_2(t) \sim P.P(\lambda v) \end{cases}$

Set $P(N_1 = n, N_2 = m)$

$$= \sum_i P(N_1 = n, N_2 = m | N = i) P(N = i)$$

$$= P(N_1 = n, N_2 = m | N = n+m) P(N = n+m)$$

$$\begin{aligned}
 &= {}_{n+m} C_n p^n q^m \times \frac{e^{-\lambda t} (\lambda t)^{n+m}}{(n+m)!} \\
 &= \frac{(n+m)!}{n! m!} \frac{e^{-\lambda(p+q)t} (\lambda p t)^n (\lambda q t)^m}{(n+m)!} \\
 &= \frac{e^{-\lambda p t} (\lambda p t)^n}{n!} \times \frac{e^{-\lambda q t} (\lambda q t)^m}{m!}
 \end{aligned}$$

X Lim n->inf and p is small such that np=c then Bin(n,p)=P(c)

Binomial distribution also arises in context of PP;

$$N(t) \sim P.P.(\lambda)$$

Then for $0 < u < t$ and $0 \leq k \leq n$,

$$[N(u) | N(t)=n] \sim \text{Bin}\left(n, \frac{u}{t}\right)$$

Set For $0 \leq k \leq n$

$$P(N(u) \leq k | N(t)=n) = \frac{P(N(u) \leq k, N(t)=n)}{P(N(t)=n)}$$

$$= \frac{P(N(0,u]=k, N(u,t]=n-k)}{P(N(t)=n)}$$

$$\begin{aligned}
 &= \frac{\frac{e^{-\lambda u} (\lambda u)^k}{k!} \times \frac{e^{-\lambda(t-u)} (\lambda(t-u))^{n-k}}{(n-k)!}}{\frac{e^{-\lambda t} (\lambda t)^n}{n!}} \\
 &\quad \underset{k}{\dots} \quad \underset{n-k}{\dots}
 \end{aligned}$$

$$= \frac{n!}{k!(n-k)!} \cdot \frac{u^k (t-u)^{n-k}}{t^n}$$

$$= \binom{n}{k} \left(\frac{u}{t}\right)^k \left(1 - \frac{u}{t}\right)^{n-k}, \quad k = 0, 1, \dots, n.$$

—x—

indep $X \sim \text{Gamma}(\alpha, \lambda)$ $U = \frac{X}{X+Y}, V = X+Y$

$Y \sim \text{Gamma}(\beta, \lambda)$ *(show, indep)* $V = \text{Gamma}(\alpha+\beta, \lambda)$

, then $U \sim \text{Beta}(\alpha, \beta)$, i.e., $f_U(u) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1} (1-u)^{\beta-1}$

scl (Hist) $u = \frac{u}{u+y}, u = u+y, 0 < u < 1$

$$\mathcal{J} = \frac{\partial f_{U,V}(u,y)}{\partial(u,y)} = u \quad | \mathcal{J}| \leq u$$

$$f_{U,V}(u,y) = f_{X,Y}(u,y) |\mathcal{J}|$$

—x—

indep $S_m = \sum_{i=1}^m T_i, S_n = S_m + (S_n - S_m) \sim \text{Gamma}(m, \lambda)$

$$S_n - S_m \sim \text{Gamma}(n-m, \lambda)$$

$$U = \frac{S_m}{S_n} = \frac{S_m}{S_m + (S_n - S_m)} \sim \text{Beta}(m, n-m)$$

U and S_n are indep.

—x—

~~★~~ Example: Suppose customer streams into a drug store at a constant average rate of 15 per hr.

The pharmacist takes its break at $\sim 1 \text{ hr}$.

The pharmacy opens at 8am and closes at 8pm.

at 8 pm. Given that the 100th customer on a particular day walked in at 2pm, we want to know what is the prob. that 50th customer came before noon. $m=50$ $n=100$

Set

$$P(S_m < 4 \mid S_n = 6)$$

$$= P\left(\frac{S_m}{S_n} < \frac{4}{6} \mid S_n = 6\right)$$

$$= P\left(\frac{S_m}{S_n} < \frac{4}{6}\right)$$

$\left| \begin{array}{l} S_m \text{ and } S_n \text{ are indep} \\ \frac{S_m}{S_n} \sim \text{beta}(S_0, 100-S_0) \end{array} \right.$

C.L.T.
 $\approx P(Z < \frac{\frac{4}{6} - \frac{1}{2}}{\sqrt{0.0025}})$

$$\approx P(Z < 3.33)$$

$$\approx 0.9997$$

$\frac{S_m}{S_n} \sim \text{beta}(S_0, 100-S_0)$
 $\equiv \text{beta}(S_0, S_0)$

$$\sim \text{beta}(\alpha, \beta)$$

$$\text{mean} = \frac{\alpha}{\alpha+\beta} = \frac{1}{2}$$

$$\text{var} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = 0.0025$$

$$Z \sim N(0, 1)$$

— X —

→ $N(t) \sim \text{P.P.}(\lambda)$ and one event take place in interval $[0, t]$. Let Y r.v. describing the time of occurrence of this Poisson event, has a continuous uniform dist on the interval $(0, t]$

Set

$$Y = [T_1 \mid N(t)=1]$$

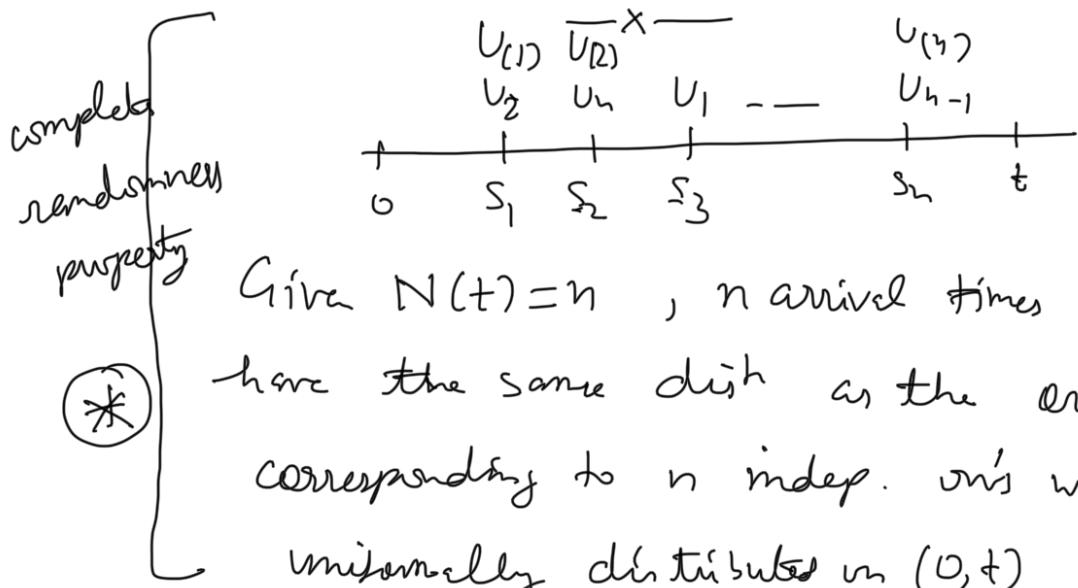
For exact

$$P(Y \leq x) = P(T_1 \leq x | N(t)=1)$$

$$= \frac{P(N[0,x] = 1, N(x,t] = 0)}{P(N(t) = 1)}$$

$$= \frac{e^{-\lambda x} \lambda x \times e^{-\lambda(t-x)}}{e^{-\lambda t} \lambda t}$$

$$= \frac{x}{t}, \quad 0 < x < t$$



y_1, \dots, y_n are i.i.d. and are

$y_{(1)}, \dots, y_{(n)}$ are order statistics corresponding to

$$y_1 < y_2 < \dots < y_n$$

$$f_{y_{(1)}, \dots, y_{(n)}}(y_1, \dots, y_n) = n! \prod_{i=1}^n f(y_i)$$

$$\int f_{y_{(1)}, y_{(2)}}(y_1, y_2)$$

$y_1 < y_2$

$$\text{U}_i \quad \text{III} \quad Y_i \sim U(0, t) \quad \left[= f_{Y_1, Y_2}(y_1, y_2) + f_{Y_2, Y_1}(y_2, y_1) \right]$$

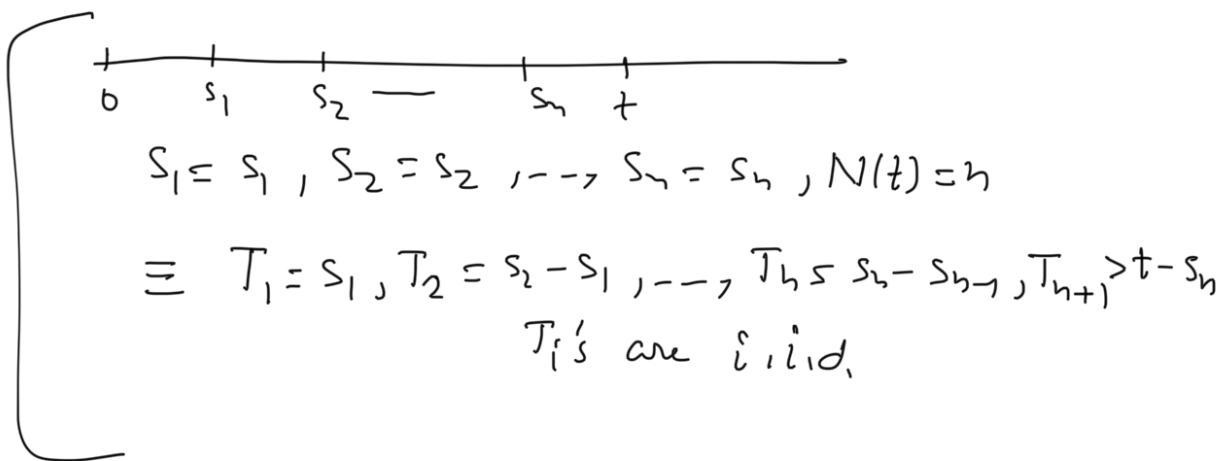
~~proof~~

$$f(y_1, \dots, y_n) = \frac{n!}{t^n}, \quad 0 < y_1 < y_2 < \dots < y_n < t$$

$$[S_1, S_2, \dots, S_n | N(t)=n]$$

density $S_1 < S_2 < S_3 < \dots < S_n < t$

$$f(S_1, S_2, \dots, S_n | t) = \frac{f(S_1, \dots, S_n, t)}{P(N(t)=n)}$$



$$= \frac{\lambda e^{-\lambda s_1} \lambda e^{-\lambda(s_2-s_1)} \dots \lambda e^{-\lambda(s_n-s_{n-1})} e^{-\lambda(t-s_n)}}{\frac{e^{-\lambda t} (\lambda t)^n}{n!}}$$

$$= \frac{n!}{t^n}$$

Ex.

For $U(0, 1)$ dist, the var's $V_i = \frac{U(i)}{U(j)}$ and

$V_2 = U_{(j)}$, $1 \leq i \leq j \leq n$ are statistically indep., with

$$V_1 \sim \text{beta}(i, j-i)$$

$$V_2 \sim \text{beta}(j, n-j+1)$$

—x—

Q

Example 1 b/w 9 AM - 5 PM, you recieve emails at the avg. contant rate of one per 10min. You left for lunch at 12 noon, and when you returned at 1pm, you found 9 emails. Assume that emails arrive according to a P-P.

$$\lambda = 6 \text{ per hr}$$

- 1) What is the prob. that 5 email arrived before 12:30 pm.

$$\text{Sol } [S_5 | N(1) = 9] = U_{(5)} \sim \text{beta}(5, 9-5+1) \\ \equiv \text{beta}(5, 5)$$

$$P(S_5 \leq \frac{1}{2} | N(1) = 9) = P(U_{(5)} \leq \frac{1}{2})$$

$$\stackrel{\text{CLT}}{=} P(Z \leq 0) = \frac{1}{2}$$

- 2) What is the expected gap b/w the times that the 3rd and 7th email arrived.

$$E(S_7 - S_3 | N(1) = 9)$$

$$= E(S_7 | N(1) = 9) - E(S_3 | N(1) = 9)$$

$$= E(U_{(7)}) - E(U_{(3)})$$

$$= \frac{7}{10} - \frac{3}{10} = 0.4 \text{ hrs} \\ = 24 \text{ min} \quad \begin{cases} U_{(1)} \sim \text{beta}(7, 3) \\ U_{(2)} \sim \text{beta}(3, 7) \end{cases}$$

Compound P.P. $\xrightarrow{\quad X \quad}$

$$N(t) \left\{ \begin{array}{l} \boxed{B_{11}} Y_1 \\ \vdots \\ \boxed{B_{NN(t)}} Y_{N(t)} \end{array} \right. \quad \text{form } X(t)$$

$$X(t) = \sum_{i=1}^{N(t)} Y_i \quad \text{compound P.P.}$$

indep. $\left\langle Y_i \right\rangle$ are i.i.d.
 $N(t) \sim \text{P.P.}(\lambda)$

$$E(X|t) = E\left(\sum_{i=1}^{N(t)} Y_i\right)$$

$$= E\left(E\left(\sum_{i=1}^{N(t)} Y_i \mid N(t)\right)\right)$$

$$\boxed{E\left(\sum_{i=1}^{N(t)} Y_i \mid N(t) = n\right) \leq E\left(\sum_{i=1}^n Y_i\right)}$$

$$= n E(Y_1)$$

$$\leq E(N(t) E(Y_1))$$

$$= E(X_1) \underbrace{E(N(t))}_{\lambda t}$$

$$= \lambda t E(Y_1)$$

$$\text{Var}(X(t)) = E(\underbrace{\text{Var}(X(t)|N(t))}_{+ \text{Var}(E(X(t)|N(t)))})$$

$$\text{Var}\left(\sum_{i=1}^{N(t)} X_i | N(t)\right) \quad N(t) E(Y_1)$$

$$\downarrow \quad \quad \quad \downarrow$$

$$N(t) \text{Var}(Y_1)$$

$$= (\underbrace{E(N(t))}_{\lambda t} \text{Var}(Y_1) + E^2(Y_1) \underbrace{\text{Var}(N(t))}_{\lambda t})$$

$$= \lambda t [\text{Var}(Y_1) + E^2(X_1)]$$

$$= \lambda t [E(Y_1^2) - \cancel{E^2(X_1)} + \cancel{E^2(Y_1)}]$$

$$= \lambda t E(Y_1^2)$$

Example: ① Customers arrive at the ATM in accordance with a P.P. with a rate 12 per hr. The amt of money withdrawn on each transaction is $\dots, 11, \dots, 4, \dots, 1$

..... as a r.v. with mean \$30 and sd \$50. The machine is in use for 15 hr daily. Approximate the prob. that the total daily withdrawal is less than \$6000.

$$\text{Sol} \quad \text{iid. } Y_i \quad E(Y_i) = 30, V(Y_i) = (50)^2$$

$$E(Y_i^2) = (50)^2 + (30)^2$$

$$X(15) = \sum_{i=1}^{N(15)} Y_i \quad \lambda = 12$$

$$E(X(15)) = 12 \times 15 \times 30$$

$$V(X(15)) = 12 \times 15 \times [(50)^2 + (30)^2]$$

$$P(X(15) \leq 6000)$$

$$\stackrel{CLT}{=} P\left(Z \leq \frac{6000 - 5400}{\sqrt{612000}}\right)$$

$$= P(Z \leq 0.767)$$

$$= \Phi(0.767) = 0.78$$

- ② Suppose that families migrate to an area at a Poisson rate $\lambda = 2$ per week. If the # of people in each family is indep and takes on values 1, 2, 3, 4 with resp. prob $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$. Find the expected value and var. of the

individual migrating to the area during a fixed time week period.

So

$$E(X_1) = 1 \times \frac{1}{6} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} + 4 \times \frac{1}{6}$$

$$E(X_1^2) = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{3} + 3^2 \times \frac{1}{3} + 4^2 \times \frac{1}{6}$$

$$E(X(5)) = 2 \times 5 \times \frac{5}{2} = 25$$

$$V(X(5)) = 2 \times 5 \times \frac{43}{6} = \frac{215}{3}$$

$$P(X(5) \geq 15) \stackrel{CLT}{=} P\left(Z \geq \frac{15 - 25}{\sqrt{\frac{215}{3}}}\right)$$

$$= 1 - \Phi\left(\frac{15 - 25}{\sqrt{\frac{215}{3}}}\right)$$

— X —