After Midterm

Saathi

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How to solve primal using dual or vice versa?

Q- Apply simplex method to solve the following LPP-

$$\text{Max } Z = 30x_1 + 23x_2 + 29x_3$$

s.t. $6x_1 + 5x_2 + 3x_3 \le 26$

$4x_1 + 2x_2 + 5x_3 \le 7$

$x_1, x_2, x_3 \ge 0$

From the final table find the optimal solution of the dual problem.

Dual →

$$\text{min } w = 26v_1 + 7v_2$$

s.t. $6v_1 + 4v_2 \ge 30$

$5v_1 + 2v_2 \ge 23$

$3v_1 + 5v_2 \ge 29$

$v_1, v_2 \ge 0$

Std. form- (Primal)

$$\text{Max } Z = 30x_1 + 23x_2 + 29x_3 + 0.x_4 + 0.x_5$$

s.t. $6x_1 + 5x_2 + 3x_3 + x_4 = 26$

$4x_1 + 2x_2 + 5x_3 + x_5 = 7$

$x_j \ge 0, \; j = 1, 2, 3, 4, 5$

		c_j		30	23	29	0	0	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	
0	a_4	x_4	26	6	5	3	1	0	52/12
0	a_5	x_5	7	$\boxed{4}$	2	5	0	1	21/12 →
		$z_j - c_j$		−30	−23	−29	0	0	
				↑					

0	a_4	x_4	31/2	0	2	−3/2	1	−3/2	31/4
30	a_1	x_1	7/4	1	$\boxed{1/2}$	5/4	0	1/4	14/4 →
		$z_j - c_j$		0	−8	17/2	0	1/2	
					↑				

final Table →

0	a_4	x_4	17/2	−4	0	−13/2	1	−5/2
23	a_2	x_2	7/2	2	1	5/2	0	1/2
		$z_j - c_j$		16	0	57/2	0	23/2

Optimal soln for primal -

$x_1 = 0, \; x_2 = 7/2, \; x_3 = 0$ $\qquad Z_{max} = \frac{161}{2}$

Optimal soln for dual -

$v_1 = 0, \; v_2 = \frac{23}{2}$ $\qquad Z_{max} = \frac{161}{2}$

c_j

$C_B B^{-1} a_j$

Q- By solving the dual of the following problem, show that the given problem has no feasible solution-

min. $z = x_1 - x_2$

s.t. $2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq +2$

$x_1, x_2 \geq 0$

Dual → max $z = 2v_1 + 2v_2$

s.t. $2v_1 - v_2 \leq 1$

$-v_1 + v_2 \geq 1$

$v_1, v_2 \geq 0$

Std. form-

max $w = 2v_1 + 2v_2 + 0 \cdot v_3 + 0 \cdot v_4 - M v_5$

s.t. $2v_1 - v_2 + v_3 = 1$

$-v_1 + v_2 - v_4 + v_5 = 1$

$v_1, v_2, v_3, v_4, v_5 \geq 0$

			c_j	2	2	0	0	-M	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	Min. Ratio
0	a_3	v_3	1	2	-1	1	0	0	
-M	a_5	v_5	1	-1	1	0	-1	1	
		$z_j - c_j$ →		M-2	-M-2	0	M ↑	0	

~~Cannot find departing vector. dual is unbounded.~~

Final →

2	a_1	v_1	2	1	0	1	-1	1
1	a_2	v_2	3	0	1	1	-2	2.
		$z_j - c_j$		0	0	3	-4	4+M.

Unable to decide leaving variable so this problem has unbounded solution so primal has infeasible solution

Primal	Dual	Conclusion
f.s	f.s	both have optimal f.s.
No f.s	f.s.	dual obj. func. is unbounded
f.s.	No. f.s.	primal obj. func. is unbounded
No. f.s.	No. f.s.	func. is unbounded No. soln exists

Duality Theorem

Theorem If any of the constraints in the primal problem is a perfect equality, then the corresponding dual variable is unrestricted in sign.

Theorem If any variable of the primal problem is unrestricted in sign then the corresponding constraint of the dual is an equality.

Theorem Dual of the dual is the primal

Theorem If x is any f.s. to the primal problem and v is any f.s. to the dual problem then

$$cx \leq b^T v$$

max $z = cx$ min $w = b^T v$

$Ax \leq b$ s.t. $A^T v \geq c^T$

$x \geq 0$ $v \geq 0$

Theorem If x^* is a f.s. of the primal problem and v^* is the f.s. to the dual problem such that

$$cx^* = b^T v^*$$

then both x^* and v^* are optimal solution to the respective problem.

Theorem Fundamental Theorem of Duality — A f.s. x^* to the primal problem is optimal iff there exists a f.s. v^*

to the dual problem such that

$$cx^* = b^T v^*$$

If a finite opt. soln exists for primal then there exists a finite opt soln for the dual and conversely.

<u>Revised Simplex Method</u>

Solving LPP with less labour -

Example → Use the revised simplex method to solve the LPP -

$$\text{Max } Z = x_1 + x_2$$
$$\text{s.t. } 3x_1 + 2x_2 \leq 6$$
$$x_1 + 4x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

$y_j = B^{-1} a_j$, $x_B = B^{-1} b$, $z_j - c_j = C_B y_j - c_j$

$= C_B B^{-1} a_j - c_j$

→ only transformed quantities are needed

If a_K is the entering vector.

y_K not all y_j, $j = 1, 2, \ldots, n$

only X_B, Z, $C_B B^{-1}$, B^{-1}

→ are transformed not all y_j

$$\text{Max } Z = x_1 + x_2 + 0 \cdot x_3 + 0 \cdot x_4$$
$$\text{S.t. } 3x_1 + 2x_2 + x_3 = 6$$
$$x_1 + 4x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$

→ max $Z = cx$

$Ax \leq b$

$x \geq 0$

Rewrite

$$z - x_1 - x_2 = 0$$
$$3x_1 + 2x_2 + x_3 = 6$$
$$x_1 + 4x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$

$\rightarrow A^* x^* = b^*$
$x^* \geq 0$

$$A^* = \left[\begin{array}{c|c} 1 & -C \\ \hline 0 & B \end{array}\right]$$

$$a_0^* = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad a_1^* = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \quad a_2^* = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$$

$$a_3^* = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad a_4^* = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad b^* = \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix}$$

Initial basis -

$$B^* = (a_0^*, a_3^*, a_4^*) \; \cancel{(\beta_0^*, \beta_1^*, \beta_2^*)}$$

$$(B^*)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\beta_0^*, \beta_1^*, \beta_2^*)$$

$$x_B^* = (B^*)^{-1} b^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix}$$

Determine the entering vector
(Compute $z_j - c_j$ correspond to non-basic vectors)

$$z_1 - c_1 = (1 \; 0 \; 0) \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = -1$$

first row of $(B^*)^{-1} \times a_j$

$$z_2 - c_2 = (1 \; 0 \; 0) \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = -1$$

$\text{Min} \{-1, -1\} \xRightarrow{\text{tie}}$ Take $k = 1$ $\quad a_k^*$ entering

Determining departing vector

Compute: $y_1^* = (B^*)^{-1} a_1^* = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

2nd & 3rd col. of $(B^*)^{-1}$

y_k column

Table 1

Basis (B^*)	β_1^*	β_2^*	x_B^*	y_1^*	Min Ratio
a_0^*	0	0	0	-1	
a_3^*	1	0	6	3	$6/3 = 2 \rightarrow$
a_4^*	0	1	4	1	$4/1 = 4$
				$\uparrow$	

a_3^* is the leaving vector.

$\bar{\beta}_1^* = \begin{pmatrix} 1/3 \\ 1/3 \\ -1/3 \end{pmatrix}$ $\quad \bar{\beta}_2^* = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ $\quad x_B^* = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

$B^* = (a_0^*, a_1^*, a_4^*)$

$(B^*)^{-1} = (\bar{\beta}_0^*, \bar{\beta}_1^*, \bar{\beta}_2^*)$

Find entering vector $\rightarrow$

$z_2 - c_2 = (\text{first row}(B^*)^{-1}) \times a_2^* = (1 \quad 1/3 \quad 0) \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = -1/3$

~~$z_3 - c_3 =$~~

$z_3 - c_3 = (1 \quad 1/3 \quad 0) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1/3$

a_2^* is entering vector.

(See case for artificial variable)

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-determining departing vector

mpute. $y_2^* = (B^*)^{-1} a_3^* = \begin{pmatrix} 1 & 1/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & -1/3 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \\ 10/3 \end{pmatrix}$

Table 2

Basis (B^*)	β_1^*	β_2^*	x_B^*	y_2^* entering vector	Min Ratio
a_0^*	1/3	0	2	-1/3	
a_1^*	1/3	0	2	2/3	3
a_4^*	-1/3	1	2	10/3	3/5 →

a_4^* is the leaving vector

$\bar{\beta}_1^* = \begin{pmatrix} 3/10 \\ 2/5 \\ -1/10 \end{pmatrix}$ $\bar{\beta}_2^* = \begin{pmatrix} 1/10 \\ -1/5 \\ 3/10 \end{pmatrix}$ $\bar{x}_B^* = \begin{pmatrix} 11/5 \\ 8/5 \\ 3/5 \end{pmatrix}$

$B^* = (a_0^*, a_1^*, a_2^*)$

$(B^*)^{-1} = (\bar{\beta}_0^*, \bar{\beta}_1^*, \bar{\beta}_2^*)$

$z_3 - c_3 =$ (first row of $(B^*)^{-1}$) $\times a_3^* = \begin{pmatrix} 1 & 3/10 & 1/10 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{3}{10} >$

$z_4 - c_4 = \begin{pmatrix} 1 & \frac{3}{10} & \frac{1}{10} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{10} > 0$

Optimal condition reached. Optimal solution

Table 3 -

Basis B^*	$\hat{\beta}_1^*$	$\hat{\beta}_2^*$	$\hat{x}_B^*$	y_k^*
a_0^*	3/10	1/10	11/5	
a_1^*	2/5	-1/5	8/5	
a_2^*	-1/10	3/10	3/5	

$x_1^* = \frac{8}{5}$ $x_2^* = \frac{3}{5}$ $z^*_{max} = \frac{11}{5}$

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Revised Simplex Method (Artificial Variable)

Computation of Inverse by Partitioning

$$M_{n\times n} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$a_{l\times l}$, $b_{l\times m}$
$c_{m\times l}$, $d_{m\times m}$

$n = l + m$

$$M^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$MM^{-1} = I_n$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_l & 0 \\ 0 & I_m \end{bmatrix}$$

$aA + bC = I_l$, $aB + bD = 0$
$cA + dC = 0$, $cB + dD = I_m$

If d has inverse, we get

$A = (a - bd^{-1}c)^{-1}$ $\qquad B = -Abd^{-1}$
$C = -d^{-1}cA$ $\qquad D = d^{-1} - d^{-1}cB$

$$M = \begin{bmatrix} I & \phi \\ 0 & R \end{bmatrix}$$ & R^{-1} exists

then $$M^{-1} = \begin{bmatrix} I & -\phi R^{-1} \\ 0 & R^{-1} \end{bmatrix}$$

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Max $Z = CX$

$AX = b$ $\qquad A_{m \times n}$

$x \geq 0$

Rewrite

$Z - CX = 0$ $\qquad A^* x^* = b^*$

$AX = b$ $\qquad x^* = \begin{bmatrix} Z \\ 2 \end{bmatrix}$

$X \geq 0$

$b^* = \begin{bmatrix} 0 \\ b \end{bmatrix}$

$$B^*_{(m+1)\times(m+1)} = \left[\begin{array}{c|c} 1 & -C_B \\ \hline 0 & B \end{array}\right]$$

$$(B^*)^{-1} = \begin{bmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{bmatrix}$$

$$x_B^* = (B^*)^{-1} b^* = \begin{bmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{pmatrix} C_B B^{-1} b \\ B^{-1} b \end{pmatrix} = \begin{pmatrix} C_B x_B \\ x_B \end{pmatrix}$$

$$= \begin{pmatrix} Z \\ x_B \end{pmatrix}$$

$j \neq 0$

$$y_j^* = (B^*)^{-1} a_j^*$$

$$= \begin{bmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} -c_j \\ a_j \end{bmatrix}$$

$$= \begin{bmatrix} -c_j + c_B B^{-1} a_j \\ B^{-1} a_j \end{bmatrix}$$

$$= \begin{bmatrix} z_j - c_j \\ y_j \end{bmatrix}$$

Q- Solve the following LPP by revised simplex method

$$\text{Min } z = x_1 + 2x_2$$
$$\text{s.t. } 2x_1 + 5x_2 \geq 6$$
$$x_1 + x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

Phase I

$$\text{Max } z' = -z = -x_1 - 2x_2$$
$$\text{Max } z_a = -x_{a_1} - x_{a_2}$$
$$\text{s.t. } 2x_1 + 5x_2 - x_3 + x_{a_1} = 6$$
$$x_1 + x_2 - x_4 + x_{a_2} = 2$$
$$x_1, x_2, x_3, x_4 \geq 0$$
$$x_{a_1}, x_{a_2} \geq 0$$

z' $\quad \alpha_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ x1 $\quad \alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ x2 $\quad \alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 5 \\ 1 \end{pmatrix}$ x3 $\quad \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

x4 $\quad \alpha_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ | za $\quad \alpha_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ xa1 $\quad \alpha_6 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ xa2 $\quad \alpha_7 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

$$d = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 2 \end{pmatrix}$$

Initial basis.

$$S = \begin{array}{c} \begin{matrix} \alpha_0 & \alpha_5 & \alpha_6 & \alpha_7 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad S^{-1} = \begin{array}{c} \begin{matrix} g_0 & g_1 & g_2 & g_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$x_s = S^{-1}d = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -8 \\ 6 \\ 2 \end{pmatrix}$$

Initial Table

Basis (S)	g_1	g_2	g_3	x_s	η_k	min ratio
α_0	0	0	0	0	2	
α_5	1	-1	-1	8	-6	pivot ~~table~~ element
α_6	0	1	0	6	$\boxed{5}$	6/5 →
α_7	0	0	1	2	1	2/1

Determining entering vector Compute $z_j - c_j$ for all non-basic vectors $j = 1,2,3,4$

$$z_1 - c_1 = \text{2nd row of } S^{-1} \times \alpha_1 = (0 \;\; 1 \;\; -1 \;\; -1)\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} = -3$$

$z_2 - c_2 = -6$

$z_3 - c_3 = 1$

$z_4 - c_4 = 1$

Compute $\eta_2 = S^{-1}\alpha_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 5 \\ 1 \end{pmatrix}$

Modified Table

Basis (S)	$\bar{g}_1$	$\bar{g}_2$	$\bar{g}_3$	x_B	η_K
α_0	0	$-2/5$	0	$-12/5$	
α_5	1	$1/5$	-1	$-9/5$	
α_2	0	$1/5$	0	$6/5$	
α_7	0	$-1/5$	1	$4/5$	

artificial variable

Determining the entering vector

$z_j - c_j \quad j = 1, 3, 4, 6$

$z_1 - c_1 = 2^{nd}$ row of $(S)^{-1} \times \alpha_1$

$$= \left(0, 1, \tfrac{1}{5}, -1\right) \times \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} = -\frac{3}{5}$$

$z_3 - c_3 = -1/5$

$z_4 - c_4 = 1$

$z_6 - c_6 = 6/5$

Determining the leaving vector

$$\eta_1 = S^{-1}\alpha_1 = \begin{pmatrix} 1 & 0 & -2/5 & 0 \\ 0 & 1 & 1/5 & -1 \\ 0 & 0 & 1/5 & 0 \\ 0 & 0 & -1/5 & 1 \end{pmatrix} = \begin{pmatrix} 1/5 \\ -3/5 \\ 2/5 \\ 3/5 \end{pmatrix}$$

Table

Basis (S)	$\hat{g}_1$	$\hat{g}_2$	$\hat{g}_3$	x_0	η_k
α_6	0	$-1/3$	$-1/3$	$-8/3$	
α_5	1	0	0	0	
α_2	0	$1/3$	$-2/3$	$2/3$	
α_1	0	$-1/3$	$5/3$	$4/3$	

Non-basic variables x_3, x_4, x_6, x_7

$$z_3 - c_3 = (2^{nd} \text{ row of } S^{-1}) \times \alpha_3$$

$$= (0, 1, 0, 0) \times \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = 0$$

$z_4 - c_4 = 0$

$z_6 - c_6 = 1$

$z_7 - c_7 = 1$

all $z_j - c_j \geq 0$

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Duality Theorem

Theorem - The dual of the dual is the primal.

Proof -

Primal
$$\left.\begin{aligned} \text{Max } z &= cx \\ \text{s.t. } Ax &\le b \\ x &\ge 0 \end{aligned}\right\} - (1)$$

Dual
$$\left.\begin{aligned} \text{Min } w &= b^T v \\ \text{s.t. } A^T v &\ge c^T \\ v &\ge 0 \end{aligned}\right\} - (2)$$

$\downarrow$

Rewrite:
$$\left.\begin{aligned} \text{let } w_1 &= -w \\ \text{Max } w_1 &= -b^T v \\ \text{s.t. } -A^T v &\le -c^T \\ v &\ge 0 \end{aligned}\right\} - (3)$$

Dual
$$\begin{aligned} \min Z_1 &= -(c^T)^T x \\ \text{s.t. } -(A^T)^T x &\ge -(b^T)^T \\ x &\ge 0. \end{aligned}$$

$$\left.\begin{aligned} \max Z_1 &= cx \\ \text{s.t. } A\cdot x &\le b \\ x &\ge 0 \end{aligned}\right\} \text{Same as } (1)$$

Theorem (Weak Duality Theorem)

If x is any f.s. to the primal problem (1) and v is any f.s. to the associated dual problem (2).

then $cx \le b^T v$

$z \le w$

Proof
$$Ax \le b$$
$$v^T(Ax) \le v^T b$$
$$(v^T A)x \le v^T b \quad - (3)$$
$$(A^T v)^T x \le (b^T v)^T$$

$$A^T v \geq c^T$$
$$x^T(A^T v) \geq x^T c^T$$
$$x^T(v^T A)^T \geq (cx)^T$$
$$(v^T A x)^T \geq c x^T$$
$$v^T A x \geq cx \quad \text{— (4)}$$

(3), (4) $\rightarrow$

$$cx \leq v^T A x \leq v^T b \leq b^T v$$

Theorem (Strong Duality Theorem)

If x^* is a f.s. to the primal (1) & v^* is the f.s. to the associated dual (2) s.t.

$$cx^* = b^T v^*$$

then both x^* and v^* are optimal solution to the respective problems.

Proof - Given $cx^* = b^T v^*$

Now $cx \leq \underbrace{b^T v^*}_{= cx^*}$ for any f.s. x of (1) and any f.s. v such as v^* of (2)

$cx \leq cx^*$ for any f.s. x of (1)

$\Rightarrow x^*$ is optimal f.s. to (1).

Other part.. $\rightarrow$ Same way.

(Gauthi)

Theorem (Fundamental Theorem of Duality)

If a finite optimal f.s. exists for the primal then $\exists$ a finite optimal f.s. for the dual & conversely.

Restate - A f.s. x^* to the primal is optimal iff $\exists$ a f.s. v^* to the associated dual s.t.

$$cx^* = b^T v^*$$

Proof -

① $\Rightarrow$

$$\text{Max } z = cx + 0 \cdot x_s$$
$$Ax + I x_s = b$$
$$x, \quad x_s \geq 0$$

$x_s \to$ vector of slack variables

let $x_B^* \to$ optimal b.f.s. of ③

$B \to$ corresponding basis

$C_B \to$ associated cost vector

x_B^* optimal f.s. $\Rightarrow$ $z_j - c_j \geq 0 \quad \forall j$

$$z_j = \sum_{i=1}^{m} c_{Bi}\, y_{ij}$$

$$= C_B y_j$$
$$= C_B B^{-1} a_j$$

let $v^{*T} = C_B B^{-1}$

Along with slack variables we get

$$C_B B^{-1} (A, I) \geq (C, 0)$$

$$(v^*)^T A \geq C \Rightarrow A^T v^* \geq C^T$$

$$(v^*)^T \geq 0$$

v^* satisfies the constraints of the dual ②

Then v^* is a f.s to the dual ②

Claim - $(v^*)^T = C_B B^{-1}$ is an optimal soln of ②.

Proof $\quad z_{max} = C_B x_B^* = C_B B^{-1} b = (v^*)^T b$
$= b^T v^*$
$= W_{min}$

$x^*, v^* \to$ f.s of the primal & the associated dual respectively with

$$cx^* = b^T v^*$$

$\Rightarrow x^*$ is a finite optimal f.s. of the primal & v^* is a finite optimal f.s. of the dual by the strong duality theorem.

Theorem -

If the primal an unbounded objective function then the dual has no f.s.

Proof - Suppose (Use contradiction)

primal u.o.f. $\rightarrow$ dual has finite optimal soln f.o.s. $\longrightarrow$ dual of dual f.o.s. $\rightarrow$ primal

Theorem -

If the dual has no f.s. and a primal has a f.s then primal obj. func is unbounded

Proof - Suppose dual has no f.s.

But primal a f.s. $\longrightarrow x^0$

value of obj. func $= cx^0$

Claim - Obj. func of the primal is unbounded

x^0 cannot be optimal soln of the primal as by the Fundamental Th. of Duality the dual will have a f.s.. The primal has no optimal soln

$\Rightarrow$ obj func of primal is unbounded.

2^{nd} classtest & assignment → 13^{th} Nov.

Date 17/10/2023

Complementary Slackness Theorem

For any part of optimal solution to a LPP and its associated dual

(a) the product of the j^{th} variable of the primal and the j^{th} surplus variable of the dual is zero,

for each $j = 1, 2, \ldots, n$

(b) the product of the i^{th} variable of the dual and the i^{th} slack variable of the primal is zero,

for each $i = 1, 2, \ldots, n$.

$$x_j (v_s)_j = 0 \qquad v_i (x_s)_i = 0$$

Proof -

Primal ① $\begin{cases} \text{Max } z = cx \\ \text{s.t. } Ax \le b \\ x \ge 0 \end{cases}$ ③ $\begin{cases} \text{max } z = cx \\ v^T \times \text{ s.t. } \underset{m\times n}{A} x + \underset{n\times 1}{x_s} = \underset{m\times 1}{b} \\ x, x_s \ge 0. \end{cases}$

x_s - vector of slack varia

Dual ② $\begin{cases} \text{Min } w = b^T v \\ \text{s.t. } A^T v \ge c^T \\ v \ge 0 \end{cases}$ ④ $\begin{cases} \text{min } w = b^T v \\ x^T \times \text{ s.t. } A^T v - v_s = c^T \\ v, v_s \ge 0, \end{cases}$ v_s = vector of surplus variable

$$v^T A x + v^T x_s = v^T b$$
$$x^T A^T v + v^T x_s = b^T v \quad -⑤$$
$$x^T A^T v - x^T v_s = x^T c^T$$
$$x^T A^T v - x^T v_s = cx \quad -⑥$$

(x^0, x_s^0) ↓ feasible optimal solution of the primal ③ , (v^0, v_s^0) ↓ feasible optimal solution of the dual ④

$$(v^0)^T x_s^0 = 0$$
$$(x^0)^T v_s^0 = 0$$

$$b^T v = cx.$$

$$\cancel{x^T A^T v} + v^T x_s = \cancel{x^T A^T v} - x^T v_s$$

$$x^T v_s + v^T x_s = 0$$

Saathi

Theorem - If (x, x_s), (v, v_s) are feasible solution of the primal ① and the associated dual ② under conditions where complementary slackness holds then (x, x_s) and (v, v_s) are also their respective optimal solution.

Proof - Complementary slackness holds

$$\Rightarrow v^T x_s + x^T v_s = 0$$

$$v^T x_s = -x^T v_s = -v_s^T x$$

Add $v^T A x \Rightarrow$

$$v^T A x + v^T x_s = v^T A x - v_s^T x$$

$$v^T (Ax + x_s) = x(A^T v - v_s)$$

$$v^T b = x^T c^T = (cx)^T$$

$$cx = b^T v$$

Result follows by the Fundamental Theorem of Duality

Saathi

Assignment Problem (Hungarian Method)

Based on the work of two Hungarian mathematician Konig and Egervary

Q- Find the optimal assignment for a problem with the following cost matrix -

	M_1	M_2	M_3	M_4	M_5
J_1	8	4	2	6	1
J_2	0	9	5	5	4
J_3	3	8	9	2	6
J_4	4	3	1	0	3
J_5	9	5	8	9	5

$\longrightarrow$

	M_1	M_2	M_3	M_4	M_5
J_1	7	3	1	5	0
J_2	0	9	5	5	4
J_3	1	6	7	0	4
J_4	4	3	1	0	3
J_5	4	0	3	4	0

	M_1	M_2	M_3	M_4	M_5
J_1	7	3	0	5	0
J_2	0	9	4	5	4
J_3	1	6	6	0	4
J_4	4	3	0	0	3
J_5	4	0	2	4	0

	M_1	M_2	M_3	M_4	M_5
× J_1	7	3	0	5	$\boxed{0}$
J_2	$\boxed{0}$	9	4	5	4
J_3	1	6	6	$\boxed{0}$	4
J_4	4	3	$\boxed{0}$	0	3
× J_5	4	$\boxed{0}$	2	4	0
	×		×	×	

First check rowwise for single 0 & then col wise

$J_1 \rightarrow M_5$

$J_2 \rightarrow M_1$

$J_3 \rightarrow M_4$

$J_4 \rightarrow M_3$

$J_5 \rightarrow M_2$

Min Cost $= 1 + 0 + 2 + 1 + 5$

$= 9$

Q- The head of the department has five jobs A, B, C, D, E and five sub ordinates V, W, X, Y, Z. The no. of hours each men could take to perform each job is as follows-

	V	W	X	Y	Z
A	3	5	10	15	8
B	4	7	15	18	8
C	8	12	20	20	12
D	5	5	8	10	6
E	10	10	15	25	10

→

	V	W	X	Y	Z
A	0	2	7	12	5
B	0	3	11	14	4
C	0	4	12	12	4
D	0	0	3	5	1
E	0	0	5	15	0

↓

	V	W	X	Y	Z
A	0	2	4	7	5
B	0	3	3	8	9
C	0	4	9	7	4
D	0	0	0	0	1
E	0	0	2	18	0

Draw min no. of lines (horizontal & vertical) to cover all zeros ←

min among uncovered = 2

Add 2 at the intersection

Subtract 2 from all uncovered costs.

↓

	V	W	X	Y	Z
A	0	0	2	5	3
B	0	1	6	7	2
C	0	2	7	5	2
D	2	0	0	0	1
E	2	0	2	10	0

min no of lines = 4 < m

min among uncovered cost = 2.

←

	V	W	X	Y	Z
A	0	0	0	3	3
B	0	1	4	5	2
C	0	2	5	3	2
D	4	2	0	0	3
E	2	0	0	8	0

4 < m

	V	W	X	Y	Z
A	1	0	0	3	3
B	0	0	3	4	1
C	0	1	4	2	1
D	5	2	0	0	3
E	3	0	0	8	0

→

	V	W	X	Y	Z
A	1	0	$\boxed{0}$	3	3
B	0	$\boxed{0}$	3	4	7
C	$\boxed{0}$	1	4	2	1
D	5	2	0	$\boxed{0}$	3
E	3	0	0	8	$\boxed{0}$

Variation of Assignment Problems

(i) Max problem → min problem

$$\begin{array}{c|c} 3 & 9 \\ \hline 6 & 4 \end{array} \longrightarrow \begin{array}{c|c} -3 & -9 \\ \hline -6 & -4 \end{array} \quad \text{or} \quad \begin{array}{c|c} 6 & 0 \\ \hline 3 & 5 \end{array}$$

(ii) Unbalanced → Dummy row / column with 0 cost

(iii) Impossible Assignment → Put a large +ve M for min problem

travelloing salesman?

Transportation Problem

Q- For the following problem ~~obtained~~ obtain the different starting solution by adapting-

(i) the North-West Corner Method

(ii) the Vogel's approximation method

(iii) the Matrix mining method (least cost entry method)

destination

~~NWC~~

	D_1	D_2	D_3	a_i
O_1	5	1	8	12
O_2	2	4	0	14
O_3	3	6	7	4
b_j	9	10	11	

supply

source

matrix entries - cost

demand

North - West Corner Method -

	D_1	D_2	D_3	a_i
O_1	$\boxed{9}$ 5	$\boxed{3}$ 1	8	$\cancel{12}$ $\cancel{3}$
O_2	2	$\boxed{7}$ 4	$\boxed{7}$ 0	$\cancel{14}$ $\cancel{7}$
O_3	3	6	$\boxed{4}$ 7	$\cancel{4}$
b_j	$\cancel{9}$	$\cancel{10}$ $\cancel{7}$	$\cancel{11}$	

Initial b.f.s.

$x_{11} = 9$, $x_{12} = 3$, $x_{22} = 7$,

$x_{23} = 7$, $x_{33} = 4$

$$\text{Total Cost} = \sum_i \sum_j c_{ij} x_{ij}$$

$$= 5 \times 9 + 1 \times 3 + 4 \times 7 + 0 \times 7 + 7 \times 4$$

$$= 104$$

VAM -

	D_1	D_2	D_3	a_i	
O_1	5	1	8	12	(4)
O_2	2	4	$\boxed{11}$ 0	$\cancel{14}$ 3	(2)
O_3	3	6	7	4	(3)
b_j	9	10	$\cancel{11}$		
	(1)	(3)	(7)		

	D_1	D_2	a_i	
O_1	$\boxed{2}$ 5	$\boxed{10}$ 1	$\cancel{12}$ 2	(4) ✓
O_2	$\boxed{3}$ 2	4	3	(2)
O_3	$\boxed{4}$ 3	6	4	(3)
b_j	9	$\cancel{10}$		
	(1)	(3)		

	D_1	D_2	D_3	a_i
O_1	$\boxed{2}$ 5	$\boxed{10}$ 1	8	12
O_2	$\boxed{3}$ 2	4	$\boxed{11}$ 0	14
O_3	$\boxed{4}$ 3	6	7	4
b_j	9	10	11	

$x_{11} = 2$, $x_{12} = 10$, $x_{21} = 3$, $x_{23} = 11$

$x_{31} = 4$

Total cost $= 10 + 10 + 6 + 0 + 12 = 38$

Matrix Minima Method.

	D_1	D_2	D_3	D_4	a_i
O_1	2	$\boxed{30}$ 1	3	4	~~30~~
O_2	3	2	1	4	50
O_3	5	2	3	8	20
b_j	20	~~40~~ 10	30	10	

	D_1	D_2	D_3	D_4	a_i
O_2	3	2	$\boxed{30}$ 1	4	~~50~~ 20
O_3	5	2	3	8	20
b_j	20	10	~~30~~	10	

	D_1	D_2	D_4	a_i
O_2	3	$\boxed{10}$ 2	4	~~20~~ 10
O_3	5	2	8	20
b_j	20	~~10~~	10	

	D_1	D_4	a_i
O_2	3	$\boxed{10}$ 4	~~10~~
O_3	5	8	20
b_j	20	~~10~~	

	D_1	a_i
O_3	$\boxed{20}$ 5	~~20~~
b_j	~~20~~	

	D_1	D_2	D_3	D_4	a_i
O_1	2	$\boxed{30}$ 1	3	4	30
O_2	3	$\boxed{10}$ 2	$\boxed{30}$ 1	$\boxed{10}$ 4	50
O_3	$\boxed{20}$ 5	2	3	8	20
b_j	20	40	30	10	

Total Cost $= 30 + 20 + 30 + 40 + 100$

$= 220$

Ex: cannot have more than $(m+n-1)$ quantities

MODI Method???