Renewal Theory:

→ Interarrival times for P.P. are IID expo, r.v's

→ Generalizing, counting process, for which interarrival times are IID with an arbitrary distn is called renewal process.

interarrival time X_n time btw $(n-1)^{st}$ and n^{th} renewal / event

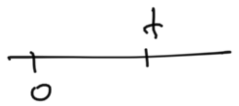
$$N(t) = \# \text{ of events by time } t = \sup\{n : S_n \le t\}$$

S_n = time for nth event / renewal

$$S_n = \sum_{i=1}^{n} X_i \qquad 0 < E(X_n) = \mu < \infty$$

$$X_n \overset{cdf}{\sim} F(.) \qquad \text{Assume } F(0) = P(X_n = 0) < 1$$

Since interarrival times are IID, it follows that at each renewal the process probabilistically starts over.



$$S_n > t \equiv N(t) \le n-1$$

$$S_n \le t \equiv N(t) \ge n$$

Q. Whether an infinite number of renewals can occur in a finite time? No

Sol

SLLN w.p 1 $\frac{S_n}{n} \to \mu$

$0 \le \mu < \infty$ $\quad \therefore \quad S_n \to \infty \quad$ as $n \to \infty$

$\therefore$ $S_n \le t$ for atmost a finite number of values of n

$$N(t) = \sup\{n : S_n \le t\} < \infty$$

$$\Rightarrow \quad N(t) = \max\{n : S_n \le t\}$$

$X_n \sim \text{IID} \overset{\text{cdf}}{F(\cdot)}$

$S_n = \sum_{i=1}^{n} X_i \sim F_n$, where F_n is n fold convolution of F with itself

$m(t) = E[N(t)]$ renewal function or mean value function.

Prop 1 : $m(t) = \sum_{n=1}^{\infty} F_n(t)$

Sol

$$I_n = \begin{cases} 1 & \text{if } n\text{th renewal occured in } [0,t] \\ 0 & \text{o.w.} \end{cases}$$

$$N(t) = \sum_{n=1}^{\infty} I_n$$

$$E(N(t)) = \sum_{n=1}^{\infty} E(I_n) = \sum_{n=1}^{\infty} P(S_n \le t)$$

$\to F_n(t)$

Prop 2 $m(t) < \infty$ for all $0 \le t < \infty$ (i.e, $N(t)$ has finite expectation.

Example $m(t) = E(N(t)) = \lambda t$

$$N(t) \sim P.P(\lambda)$$

i.i.d $F(x) = 1 - e^{-\lambda x}$, $x \ge 0$, $\lambda > 0$

$$X_n \overset{iid}{\sim} \exp(\lambda)$$

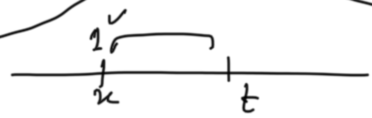
—×—

$\longrightarrow \quad m(t) = E(N(t)) = E(E(N(t)|X_1))$

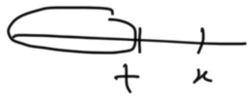
cdf bdt

$X_n \sim F, f$

$$= \int_0^\infty E(N(t)|X_1=x)\, f(x)\, dx$$



$$E(N(t)|X_1=x) = \begin{cases} 1+E(N(t-x)) & , \ x<t \\ 0 & , \ x>t \end{cases}$$



$$m(t) = \int_0^t \left(1+m(t-x)\right) f(x)\, dx + \int_t^\infty 0 \cdot f(x)\, dx$$

$$\boxed{m(t) = F(t) + \int_0^t m(t-x)\, f(x)\, dx}$$

Fundamental renewal equation

Example interarrival dist^n $X_i \sim U(0,1)$

pdf $f(x) = \begin{cases} 1 & , \ 0<x<1 \\ 0 & \ \text{o.w.} \end{cases}$; $F(x) = x, 0<x\le 1$

For $t \le 1$

$$m(t) = t + \int_0^t m(t-x)\, dx = t + \int_0^t m(y)\, dy$$

$y = t-x$
$dy = -dx$

$$m'(t) = 1 + m(t) = h(t) \ (\text{say})$$

$$h'(t) = m'(t) = h(t)$$

$$\frac{h'(t)}{h(t)} = 1 \Rightarrow \ln h(t) = t + c$$

$$\Rightarrow h(t) = k e^t$$

$$\Rightarrow m(t) = k e^t - 1$$

$$\Rightarrow m(t) = e^{t} - 1, \quad 0 \le t \le 1$$

$m(0) = E(N(0)) = 0$

$k e^{0} - 1 = 0 \Rightarrow k = 1$

— x —

We know $N(t) < \infty$, $\forall t$

$$N(\infty) = \lim_{t \to \infty} N(t)$$: the total # of renewals that occurs

Prop. $N(\infty) = \infty$ with prob 1.

Sol

$$P(N(\infty) < \infty) = P(X_n = \infty \text{ for some } n)$$

$$= P\left(\bigcup_{n=1}^{\infty} \{X_n = \infty\}\right)$$

$$\le \sum_{n=1}^{\infty} P(X_n = \infty)$$

$$= \sum_{n=1}^{\infty} P(\text{no renewal occurs})$$

$$= \sum_{n=1}^{\infty} 0$$

$P(X_1 = 0) < 1$

$\because F(0) < 1$

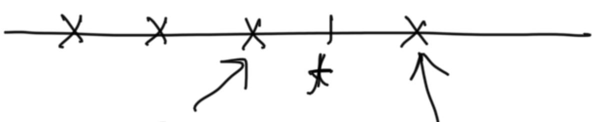
$P(\text{some renewal occurs}) \le 1$

$\therefore N(t) \to \infty$ as $t \to \infty$.

— x —

$$\lim_{t \to \infty} \frac{N(t)}{t}$$

$N(t)$ $\quad S_n$



$S_{N(t)}$

time for last renewal prior to or at time t

$S_{N(t)+1}$

time for the first renewal after time t

$$S_{N(t)} \le t < S_{N(t)+1}$$

$$\frac{S_{N(t)}}{} \le \frac{t}{} < \frac{S_{N(t)+1}}{}$$

$N(t) \qquad N(t) \qquad \overline{N(t)}$

$$\frac{S_{N(t)}}{N(t)} = \frac{\sum_{i=1}^{N(t)} X_i}{N(t)} = \mu \quad \text{as } t \to \infty$$

(SLLN) $\quad N(t) \to \infty$ as $t \to \infty$

$$\frac{S_{N(t)+1}}{N(t)+1} \times \frac{N(t)+1}{N(t)} = \mu \quad \text{as } t \to \infty$$

as $t \to \infty$

$$\mu \leq \frac{t}{N(t)} < \mu$$

$$\therefore \lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu} = \lambda \qquad \mu = E(X_n)$$

rate of renewal process

Example: ① 'A' has a radio that works on a single battery. As soon as the battery in use fails, 'A' immediately replaces it with a new battery. If the lifetime of battery (in hrs) is distributed U(30, 60), then at what rate does 'A' have to change batteries?

Sol

$$X_i \sim U(30, 60)$$

$$\mu = E(X_i) = \frac{60+30}{2} = 45$$

rate of renewal process $\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu} = \frac{1}{45}$

in long run, 'A' will have to replace one battery every 45 hrs

② (contd) ① Suppose 'A' does not keep any surplus batteries on hand, and so each time a failure occurs she must must go and buy a new battery. If the amt of time it takes to get a new battery is $U(0,1)$, then what is the av. rate that A changes batteries?

$$\mu = E(U_1) + E(U_2) = \frac{91}{2}$$

$$U_1 \sim U(30,60), \quad U_2 \sim U(0,1)$$

$$E(U_1) = \frac{90}{2} = 45, \quad E(U_2) = \frac{1}{2}$$

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{2}{91}$$

— x —

Prop. $E(S_{N(t)+1}) = \mu(m(t)+1)$

Sol. $g(t) = E(S_{N(t)+1}) = E\left[E\left(S_{N(t)+1} \mid X_1\right)\right]$

$$= \int_0^\infty E(S_{N(t)+1} \mid X_1 = x) f(x)\, dx$$

$$E(S_{N(t)+1} \mid X_1 = x) = \begin{cases} x, & x > t \\ x + g(t-x) & x < t \end{cases}$$

$$g(t) = \int_0^t \left(x + g(t-x)\right) f(x)\, dx + \int_t^\infty x f(x)\, dx$$

$$= \int_0^t g(t-x) f(x)\,dx + \underbrace{\int_0^\infty x f(x)\,dx}_{\mu}$$

$$g(t) = \int_0^t g(t-x) f(x)\,dx + \mu$$

Let $g(t) = \mu(g_1(t)+1)$ ✓

$$\mu(g_1(t)+1) = \int_0^t \mu\left(g_1(t-x)+1\right) f(x)\,dx + \mu$$

$$g_1(t) = \int_0^t g_1(t-x) f(x)\,dx + F(t)$$

Using Fundamental renewal equation

$$g_1(t) = m(t) = \frac{g(t)}{\mu} - 1$$

$$g(t) = E(S_{N(t)+1}) = \mu(m(t)+1)$$

—x—

<u>Elementary Renewal thm</u>

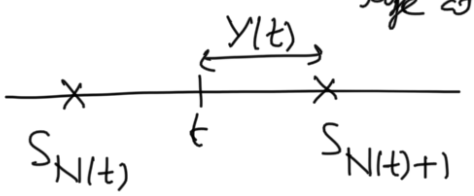
$$\frac{m(t)}{t} \to \frac{1}{\mu} \quad \text{as } t \to \infty$$

as before, $\frac{1}{\mu}$ is interpreted as 0 when $\mu = \infty$.

—x—

Let $Y(t)$ time from t until next renewal (excess or residual life at t)

$$S_{N(t)+1} = t + Y(t)$$



$$E(S_{N(t)+1}) = t + E(Y(t))$$

$$\mu(m(t)+1) = t + E[Y(t)]$$

$$\frac{m(t)}{t} + \frac{1}{t} = \frac{1}{\mu} + \frac{E(Y(t))}{t\mu}$$

$$\lim_{t\to\infty} \frac{E(Y(t))}{t\mu} = \underbrace{\lim_{t\to\infty} \frac{m(t)}{t}}_{\frac{1}{\mu}} - \frac{1}{\mu} = 0$$

$$\frac{E(Y(t))}{t} \to 0 \quad \text{as } t\to\infty$$

Renewal Reward Process :

Renewal process $\{N(t), t\geq 0\}$ interarrival times $X_n, n\geq 1$

$$X_n \longrightarrow R_n$$

↳ the reward earned at the time of n^{th} renewal ; $R_n, n\geq 1$, IID

$$R(t) = \sum_{n=1}^{N(t)} R_n \to \text{total reward earned by time } t$$

Let $E(R) = E(R_n)$; $E(X) = E(X_n)$

Prop If $E(R) < \infty$, $E(X) < \infty$, then

(a) with prob 1, $\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E(R)}{E(X)}$

b) $\lim_{t\to\infty} \frac{E(R(t))}{t} = \frac{E(R)}{E(X)}$

Sol(a) $$\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \times \frac{N(t)}{t}$$

↘ ↘ 1

$E(R)$ as $t\to\infty$ $\qquad \overline{E(X)}$ as $t\to\infty$

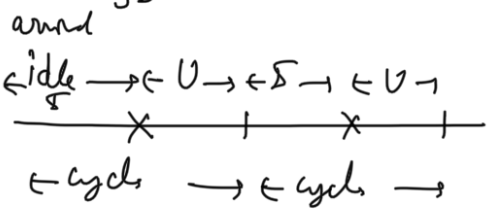
$$\frac{R(t)}{t} \to \frac{E(R)}{E(X)} \text{ as } t\to\infty.$$

Example (1) In an $M/G/1/1$ queue (Poisson arrivals to a single server, and a system capacity of 1), the service time dist $G \equiv U(10, 20)$ min. Arrivals are at the rate of 2 per hour (But customers arriving to a full system never enter the system). What is the long-run av % of time the server is idle?

Sol $\quad M/G/1/1$

$$G \equiv U(10,20) \qquad E(U) = \frac{30}{2} = \int_{10}^{20} x \times \frac{1}{10}\, dx$$

$$\lambda = 2 \text{ per hr} = \frac{2}{60} \text{ per min} = \frac{1}{30} \text{ per min} \qquad E(I) = 30 \text{ min}$$



$$E(X) = E(I) + E(U) = 30 + \frac{30}{2} = 45$$

$$P(\text{idle}) = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{E(R)}{E(X)} = \frac{E(I)}{E(X)} = \frac{30}{45} = \frac{2}{3}$$

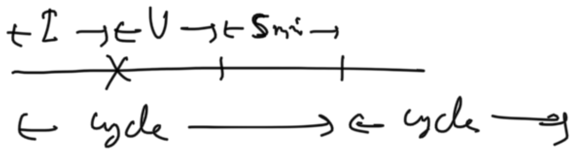
(2) Prof Ramesh works in a busy office where students arrive ~ P.P. with mean interarrival time of 20 mins. It takes Prof Ramesh an amount of time X to serve a student, $X \sim U(2,6)$ min. Immediately upon completion

of service, Prof Ramesh takes a coffee break, which lasts for a deterministic length of time of length 5 mins. While Prof Ramesh is serving a student or while he is on coffee break, any arriving student to the office turns around and go home.

(a) What fraction of time is Prof working to serve students?

$$\frac{4}{20+4+5}$$

$E(I)=20$, $E(U)=4$, 5



(b) On the av., how many student does Prof. serve.

av. 1 in 29 mins

(c) What fraction of student that shows up at the office actually end up being served?

$$\frac{20}{29}$$

20*29

(d) CTMC ?

Not a CTMC.

Since X and Y are not memoryless