Lecture 16

Theorem:-

1. Every open set in $\mathbb{R}^d$ is measurable
2. If $m^*(E)=0$, $E \subseteq \mathbb{R}^d$, then E is measurable.
3. A countable union of measurable sets is measurable.

proof:-

1. Directly follows from the def.
2. Let $E \subseteq \mathbb{R}^d$ & $m^*(E)=0$.

$$\Rightarrow \quad m^*(E) = \inf_{U \supseteq E} \left(m^*(U) \right) = 0$$

For any $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}^d$ such that $E \subseteq U$ & $m^*(U) \leq m^*(E) + \varepsilon$
$= \varepsilon$

Now $\quad U \setminus E \subseteq U$

$$\Rightarrow m^*(U \setminus E) \leq m^*(U) \leq \varepsilon$$

$\therefore$ E is measurable.

③ Let $\{E_j\}_{j=1}^{\infty}$ be a countable collection of measurable sets in $\mathbb{R}^d$.

<u>To show:</u> $\bigcup_{j=1}^{\infty} E_j$ is measurable.

Let $E = \bigcup_{j=1}^{\infty} E_j$. & $\varepsilon > 0$.

E_j is measurable $\Rightarrow$ there exists an open set $U_j \subseteq \mathbb{R}^d$ such that $E_j \subseteq U_j$ & $m^*(U_j \setminus E_j) \leq \frac{\varepsilon}{2^j}$.

Let $U = \bigcup_{j=1}^{\infty} U_j$. Then U is open & $E \subseteq U$

Now $U \setminus E = \left(\bigcup_{j=1}^{\infty} U_j\right) \setminus \left(\bigcup_{j=1}^{\infty} E_j\right)$

$\subseteq \bigcup_{j=1}^{\infty} (U_j \setminus E_j)$. (check it!)

$$\therefore m^*(U \setminus E) \leq m^*\left(\bigcup_{j=1}^{\infty} (U_j \setminus E_j)\right)$$

$$\leq \sum_{j=1}^{\infty} m^*(U_j \setminus E_j)$$

$$\leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$$

$\therefore$ E is measurable.

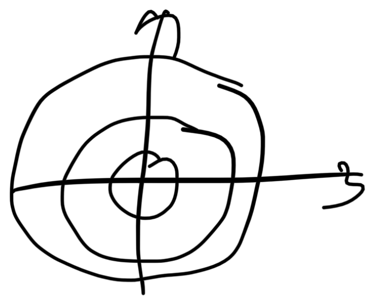
Proposition:- Every closed in $\mathbb{R}^d$ is measurable.

proof:- Let $F \subseteq \mathbb{R}^d$ be a closed set.

We have $\mathbb{R}^d = \bigcup_{k=1}^{\infty} B_k$, where

$$B_k = \overline{B(\underline{0}, k)} = \left\{ \underline{x} \in \mathbb{R}^d \,/\, d(\underline{x}, \underline{0}) \le k \right\}$$

= the closed ball around $\underline{0}$ with radius k.



$$\therefore F = \bigcup_{k=1}^{\infty} F \cap B_k$$

where $F \cap B_k$ is closed and bounded.

That is each $F \cap B_k$ is compact.

By above theorem, (3) it is enough to show that $F \cap B_k$ is measurable.

Now Assume F is compact.

To show: F is measurable.

Note that $m^*(F) < \infty$.

Let $\varepsilon > 0$. & we have $m^*(F) = \inf_{\substack{U \supseteq F \\ \text{open}}} \left(m^*(U) \right)$

There exists an open set U such that

$$F \subseteq U \quad \& \quad m^*(U) \leq m^*(F) + \varepsilon. \quad \rightarrow (*)$$

Now $U \setminus F = U \cap F^c$ is an open set.

& write $U \setminus F = \bigcup_{j=1}^{\infty} Q_j$, where Q_j are almost disjoint closed cubes.

For a fixed $N \in \mathbb{N}$, let $K = \bigcup_{j=1}^{N} Q_j \subseteq U$

Then K is a compact set.

$\boxed{K \subseteq U \setminus F}$

& $K \cup F \subseteq U$

$$\therefore m^*(U) \geq m^*(K \cup F) = m^*(K) + m^*(F)$$
$$= m^*(F) + \sum_{j=1}^{N} m^*(Q_j)$$

$$\Rightarrow \sum_{j=1}^{N} m^*(Q_j) \leq m^*(U) - m^*(F)$$
$$\leq \varepsilon \quad (\text{by } (*))$$

True for any $N \in \mathbb{N}$, therefore

$$\sum_{j=1}^{\infty} m^*(Q_j) \leq \varepsilon$$

& $$m^*(U \setminus F) = m^*\left(\bigcup_{j=1}^{\infty} Q_j\right)$$

$$\leq \sum_{j=1}^{\infty} m^*(Q_j)$$

$$\leq \varepsilon$$

$\therefore$ F is a measurable set.

Theorem:- Let $E_1, E_2, \ldots$ be a sequence of disjoint measurable sets in $\mathbb{R}^d$. Then

$$m^*\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} m^*(E_j).$$

Proposition:- The complement of a measurable set is measurable.

proof:- Let $E \subseteq \mathbb{R}^d$ be a measurable set.

To show: E^c is measurable.

For each $n \in \mathbb{N}$, there exists an open set $U_n \subseteq \mathbb{R}^d$ such that $E \subseteq U_n$ &

$$m^*(U_n \setminus E) \leq \frac{1}{n}.$$

U_n^c is a closed set & hence by above theorem it is measurable.

$$\Rightarrow \quad S := \bigcup_{n=1}^{\infty} U_n^c \text{ is measurable.}$$

& $S \subseteq E^c$. $\left(\begin{array}{l} E \subseteq U_n \quad \forall n \\ U_n^c \subseteq E^c \quad \forall n \\ \Rightarrow \bigcup_{n=1}^{\infty} U_n^c \subseteq E^c \end{array} \right)$

Then $E^c \setminus S \subseteq U_n \setminus E \quad \forall n$

$$\Rightarrow m^*(E^c \setminus S) \leq m^*(U_n \setminus E) \leq \frac{1}{n} \quad \forall n$$

$$\therefore \quad m^*(E^c \setminus S) \leq \frac{1}{n} \quad \forall n$$

$$\Rightarrow m^*(E^c \setminus S) = 0.$$

$$\Rightarrow \quad E^c \setminus S \text{ is measurable.}$$

Now $E^c = \underbrace{(E^c \setminus S)}_{\text{measurable}} \cup S$ is measurable.

Proposition:– For $E \subseteq \mathbb{R}^d$, $m^*(E + x) = m^*(E)$ & if E is measurable, then $E + x$ is measurable.

Theorem:– Let $E \subseteq \mathbb{R}^d$. Then for any $\varepsilon > 0$, the following statements are equivalent:

(i) E is measurable.

(i.e., there exists an open set $U \supseteq E$ & $m^*(U \setminus E) \leq \varepsilon$)

$U = \bigcap_{i=1}^{\infty} U_i$ – open

$U_i = U$

G_δ-set

(ii) There exists a closed set $F \subseteq E$ such that $m^*(E \setminus F) \leq \varepsilon$.

(iii) If $m^*(E) < \infty$, then there exists a compact set $K \subseteq \mathbb{R}^d$ with $K \subseteq E$ & $m^*(E \setminus K) \leq \varepsilon$.

(iv) If $m^*(E) < \infty$, then there exists a finite union $F = \bigcup_{j=1}^{N} Q_j$ of closed cubes such that $m^*(E \Delta F) \leq \varepsilon$.

Recall:-

σ-algebra on $\mathbb{R}^d$

- A family $\mathcal{M}$ of subsets of $\mathbb{R}^d$ is said to be a σ-algebra if

(i) $\mathbb{R}^d \in \mathcal{M}$

(ii) if $E \in \mathcal{M}$ then $E^c \in \mathcal{M}$

(iii) if $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

- $\mathcal{B}_{\mathbb{R}^d}$ = Borel σ-algebra = the smallest σ-algebra containing all open sets

= the σ-algebra generated by all open sets.

Let $\mathcal{M}_{\mathbb{R}^d}$ = the collection of all measurable subsets of $\mathbb{R}^d$

Then $\mathcal{M}_{\mathbb{R}^d}$ is a σ-algebra.

- $\mathcal{B}_{\mathbb{R}^d} \subsetneqq \mathcal{M}_{\mathbb{R}^d}$.

- There exists a non-measurable set in $\mathbb{R}^d$!

<u>Def</u>:- Let $E \subseteq \mathbb{R}^d$ be a measurable set. A function $f: E \to \mathbb{R} \cup \{\pm\infty\}$ is said to be a <u>measurable function</u>, if for all $\alpha \in \mathbb{R}$,

$$\{ \underline{x} \in E \mid f(\underline{x}) > \alpha \} = f^{-1}((\alpha, +\infty])$$

is measurable.

<u>Examples</u>:-

① Any continuous function defined on a measurable set is measurable.

② $\chi_E : \mathbb{R}^d \to \mathbb{R}$, $\quad \chi_E(\underline{x}) = \begin{cases} 1 & \text{if } \underline{x} \in E \\ 0 & \text{if } \underline{x} \notin E. \end{cases}$

Let $E \subseteq \mathbb{R}^d$

Then

χ_E is measurable $\iff$ E is measurable.