Lecture 4

Proposition:– Let $x \in P$, the Cantor set. Then the ternary expansion of $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$, where $a_i \in \{0,2\}$ $\forall i$.

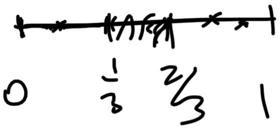
& Conversely, if $x \in [0,1]$ with ternary expansion $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$, where $a_i \in \{0,2\}$, then $x \in P$.

proof:– Let $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \in P = \bigcap_{n=1}^{\infty} P_n$

$$\frac{1}{3}x = \sum_{i=1}^{\infty} \frac{b_i}{3^i}, \quad \text{where } b_1 = 0,\ b_i = a_{i-1},\ \forall i \geq 2$$

$$\frac{1}{3}x + \frac{2}{3} = \sum_{i=1}^{\infty} \frac{b_i}{3^i}, \quad \text{where } b_1 = 2,\ b_i = a_{i-1},\ \forall i \geq 2.$$

$\frac{1}{3} = 0.0\bar{2} = 0.1$



$x \in P_1$ if and only if x has ternary expansion where either $a_1 = 0$ or $a_1 = 2$

$0.x_1x_2x_3\cdots$
0 or 2.

Since it is of the form $\frac{1}{3}y$ or $\frac{2}{3} + \frac{1}{3}y$, for some $y \in [0,1]$

Repeat this argument we get that

$x \in P_n$ iff x has ternary expansion where $a_i = 0$ or 2, for all $1 \le i \le n$.

<u>Proposition</u>:- Cantor set is uncountable.

<u>proof</u>:- We know from above proposition, $x \in P$ has ternary expansion

$$x = 0.x_1x_2x_3\cdots\cdots, \quad \text{where } x_n \in \{0, 2\} \quad \forall n \ge 1.$$

Suppose P is countable say

$$P = \{ x^{(1)}, x^{(2)}, \cdots\cdots, x^{(n)}, \cdots\cdots \}. \qquad \underline{x^{(n)}_n}$$

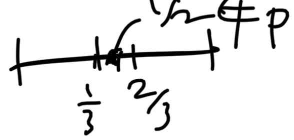
Construct $x = 0.x_1x_2\cdots\cdots$ such that

$$\left.\begin{array}{l} \text{if } x^{(n)}_n = 0, \text{ then take } x_n = 2 \ \& \\ \text{if } x^{(n)}_n = 2, \text{ then take } x_n = 0. \end{array}\right\} \forall n \ge 1.$$

Then $x \ne x^{(n)}$ $\forall n \ge 1$.

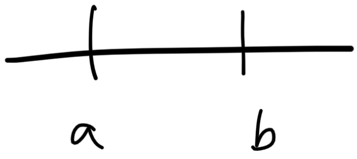
& $x \in P$. This is a contradiction.

$\therefore$ P is uncountable.



Remark:- $I = (a,b)$ or $[a,b)$ or $(a,b]$ or $[a,b]$ $\quad a<b$.

$\text{length}(I) = l(I) = b-a$

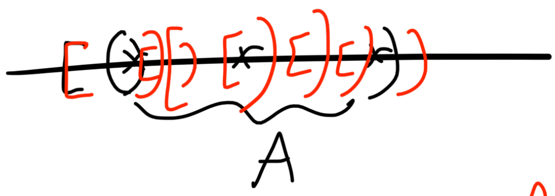


Qn:- For any $A \subseteq \mathbb{R}$, Can we assign a non-ve number to A so that it Coincides with the length if A is a finite interval?

Definition:- The <u>Lebesgue outer measure</u> or <u>outer measure</u> of $A \subseteq \mathbb{R}$ is defined as

$$m^*(A) := \inf\left(\sum_n l(I_n)\right),$$ where infimum is taken over all finite or Countable collection of the intervals $\{I_n\}$ of the form $I_n = [a_n, b_n)$ such that $A \subseteq \bigcup_n I_n$.

$$m^*(A) := \inf\left\{ \sum_n l(I_n) \;\middle|\; \begin{array}{l} A \subseteq \bigcup_n I_n,\ I_n\text{'s are finite or countable collection of intervals} \\ I_n = [a_n, b_n)\ \forall n \end{array} \right\}$$



$\inf\left(\sum_1 \ell(I_n)\right)$

Remark:- If $A = [a,b]$, then $m^*(A) = \text{length}(A) = b-a$.

$$m^*(A) = \inf_{A \subseteq \cup I_n} \left(\sum_1 \ell(I_n) \right)$$

$$= \inf_{A \subseteq I_1} \left(\sum_1 \ell(I_1) \right)$$

$$= \inf_{A \subseteq I_1} \left(\ell(I_1) \right)$$

$$= \ell(I_1) = b-a$$

Take $I_1 = [a,b]$

$\{I_1\}$

$A \subseteq I_1$

Properties of m^*:

Theorem:- Let $A \subseteq \mathbb{R}$. Then

(i) $m^*(A) \geq 0$.

(ii) $m^*(\phi) = 0$.

(iii) if $A \subseteq B \subseteq \mathbb{R}$, then $m^*(A) \leq m^*(B)$.

(iv) $m^*(\{x\}) = 0$, $\forall x \in \mathbb{R}$.

Proof:-

(i) $$m^*(A) = \inf_{A \subseteq \bigcup I_n} \Big(\underbrace{\sum_n l(I_n)}_{\geq 0} \Big) \geq 0$$

(ii) We have $\phi = [a, a)$

$$m^*(\phi) = m^*([a,a))$$
$$= \text{length}([a,a))$$
$$= a - a$$
$$= 0.$$

(iii) Let $A \subseteq B \subseteq \mathbb{R}$.

To show: $m^*(A) \leq m^*(B)$.

Let $\{I_n\}_n$ be a collation of intervals such that $B \subseteq \bigcup_n I_n$

Then $A \subseteq \bigcup_n I_n$.

$$m^*(B) = \inf_{B \subseteq \bigcup I_n} \Big(\sum_n l(I_n) \Big)$$

$$\geq \inf_{A \subseteq \bigcup I_n'} \Big(\sum l(I_n') \Big) = m^*(A)$$

$$\boxed{\begin{array}{c} A_1 \subseteq A_2 \\ \inf(A_1) \geq \inf(A_2) \end{array}}$$

(iv) Let $x \in \mathbb{R}$

$$\{x\} \subseteq \left[x, x+\frac{1}{n}\right) = I_n \text{ (say)}. \quad \forall n \geq 1.$$

$$\Rightarrow \quad m^*(\{x\}) \leq m^*(I_n) \quad (\text{by (iii)})$$

$$m^*(I_n) = x+\frac{1}{n}-x = \frac{1}{n}.$$

$$\Rightarrow \quad m^*(\{x\}) \leq \frac{1}{n} \quad \forall n \geq 1.$$

$$\Rightarrow 0 \leq m^*(\{x\}) \leq 0.$$

$$\therefore \quad m^*(\{x\}) = 0.$$

$\{x\} = [x, x]$ ✓

$\boxed{[\,,\,)}$

$(\,,\,)$ ✗

$(\,,\,]$ ✗

$[\,,\,]$ ✗

Proposition:- Outer measure is translation invariant.

That is, if $A \subseteq \mathbb{R}$, then $m^*(A+x) = m^*(A)$, for any $x \in \mathbb{R}$.

where $A+x := \{a+x \mid a \in A\}$.

proof:-

Let $\varepsilon > 0$.

$$m^*(A) = \inf_{A \subseteq \bigcup I_n} \left(\sum_n l(I_n)\right)$$

Then by the property of infimum

$[\,\,)$ $[\,\,)$

0 A 1 2 3

$A+2$

$A+2$

$= [2, 3)$

$m^*(A) = 1$

$m^*(A+2)$

$= 3-2 = 1$

$= m^*(A)$

$m^*(A) + \varepsilon \geq \sum_n l(I_n)$, for some $\{I_n\}$ such that $A \subseteq \bigcup_n I_n$.

$\longrightarrow (*)$

We have $A+x \subseteq \bigcup_n \underbrace{(I_n + x)}_{= I_n'}$.

$\therefore \quad m^*(A+x) \leq \sum_n l(I_n + x)$ (by def. of m^*)

$= \sum_n l(I_n)$

$\leq m^*(A) + \varepsilon$ (by $(*)$)

$\therefore \quad m^*(A+x) \leq m^*(A) + \varepsilon, \quad \forall\, \varepsilon > 0.$

$\Rightarrow \quad \underline{m^*(A+x) \leq m^*(A)}.$

Also $\quad A = (A+x) - x$

$\therefore \quad m^*(A) = m^*((A+x)-x)$

$\leq m^*(A+x)$ (by above argument)

$\therefore \quad m^*(A) = m^*(A+x).$

__Remark:-__ Let $A \subseteq \mathbb{R}$.

- $m^*(A) = \inf\left(\left\{ \sum_n l(I_n) \,\middle/\, \begin{array}{l} \{I_n\} \text{ collection of } \overset{([,))}{\text{intervals}} \\ \text{such that } A \subseteq \bigcup_n I_n \end{array} \right\}\right)$

- $m^*([a,b)) = b-a.$

<u>Theorem</u>:- The outer measure of any interval is equal to its length.

<u>proof</u>:-

<u>Case 1</u>: Let $I = [a,b]$ $(a \leq b.)$

<u>To show</u>: $m^*(I) = l(I) = b-a$

Given $\varepsilon > 0$.

We have $[a,b] \subseteq [a, b+\varepsilon)$

$$m^*([a,b]) \leq m^*([a, b+\varepsilon)) = \text{length}([a, b+\varepsilon))$$

$$= b+\varepsilon-a$$

$\therefore$ $m^*([a,b]) \leq b-a+\varepsilon,$ $\forall \varepsilon > 0.$

$\Rightarrow$ $m^*([a,b]) \leq b-a = l([a,b]).$

<u>To prove</u>: $m^*([a,b]) \geq b-a.$

a, b are finite nos.

$[a,\infty)$ is not a finite interval

$(-\infty, b]$