Linear Optimization

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Optimization is an act of obtaining best results under given restrictions. In several engineering design problems, engineers have to take many technological and managerial decisions at several stages. The objective of such decisions is to either minimize the effort required or to maximize the desired benefit.

The optimum seeking methods are known as Optimization Techniques. It is a part of Operations Research (OR). OR is a branch of Mathematics concerned with some techniques for finding best solutions.

SOME APPLICATIONS:

1. Optimal Design of Solar Systems,
2. Electrical Network Design,
3. Energy Model and Planning,
4. Optimal Design of Components of a System,

5. Planning and Analysis of Existing Operations,
6. Optimal Design of Motors, Generators and Transformers,
7. Design of Aircraft for Minimum Weight,
8. Optimal Design of Bridge and Building.

Optimization Techniques are divided into two different types, namely **Linear Models** and **Non-Linear Models**. At first we shall discuss about all the Linear Models. Later we shall discuss about Non-Linear Models. Mathematical statement of a linear model is stated as follows:

Find $x_1, x_2, x_3, \ldots, x_n$ so as to

$$\max : Z = \sum_{j=1}^{n} c_j x_j \qquad (1)$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \;\; i = 1, 2, 3, \ldots, m \qquad (2)$$

$$x_j \geq 0, \;\; j = 1, 2, 3, \ldots, n \qquad (3)$$

Linear Models are known as Linear Programming Problem (LPP).

(LPP-I): $$\max : Z = \sum_{j=1}^{n} c_j x_j \tag{4}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j (\leq, =, \geq) b_i, \;\; i = 1, 2, 3, \ldots, m \tag{5}$$

(Generally ≤ (as ≥ will give unbounded results)

$$x_j \geq 0, \;\; j = 1, 2, 3, \ldots, n \tag{6}$$

(LPP-II): $$\min : Z = \sum_{j=1}^{n} c_j x_j \quad (7)$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j (\leq, =, \geq) b_i, \;\; i = 1, 2, 3, \ldots, m \quad (8)$$

↳ Generally $\geq$ / (as $\leq$ will give 0, 0 ... as the optimal solution which is meaningless)

$$x_j \geq 0, \;\; j = 1, 2, 3, \ldots, n \quad (9)$$

After introducing slack, surplus and artificial variables a LPP can be put in standard form.

(1.) Add a slack variable x_{n+i}, for

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, b_i \geq 0$$

Actual variables present in optimization function are not Basic.

$$\Rightarrow \sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i, \qquad x_{n+i} \geq 0$$

(2.) Subtract a surplus variable x_{n+i} and add an artificial variable x_{n+i+1}, where x_{n+i}, $x_{n+i+1} \geq 0$ for

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i, b_i \geq 0$$

$$\Rightarrow \sum_{j=1}^{n} a_{ij}x_j - x_{n+i} + x_{n+i+1} = b_i$$

Suppose all $x_j = 0$, we get $\boxed{-x_{n+i} = b_i}$ negative $\neq$ +ve hence artificial variable is added

slack & artificial variables are basic variables

This is not the case with surplus variable

(3.) Add an artificial variable x_{n+i}, for

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, b_i \geq 0$$

All slack & artificial & surplus variables are ≥ 0

$$\Rightarrow \sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i, \quad x_{n+i} \geq 0.$$

After introducing slack, surplus and artificial variables a LPP can be put in standard form.

(LPP-I): $$\max : Z = \sum_{j=1}^{N} c_j x_j \quad (10)$$

subject to

$$\sum_{j=1}^{N} a_{ij} x_j = b_i, \;\; i = 1, 2, 3, \ldots, m \quad (11)$$

$$x_j \geq 0, \;\; j = 1, 2, 3, \ldots, N \quad (12)$$

(LPP-II): $$\min : Z = \sum_{j=1}^{N} c_j x_j \quad (13)$$

subject to

$$\sum_{j=1}^{N} a_{ij} x_j = b_i, \; i = 1, 2, 3, \ldots, m \quad (14)$$

$$x_j \geq 0, \; j = 1, 2, 3, \ldots, N \quad (15)$$

SOLUTION PROCEDURES:

A LPP can be solved by the following methods:

1. Graphical Method
   (Only for 2-variable problems),
2. Simplex Method,

3. Big-M Method/ Charne's Penalty Method,
4. Two-Phase Simplex Method,
5. Revised Simplex Method,
6. Dual Simplex Method,
7. Primal-Dual Simplex Method,
8. Interior Point Method.

BASIC SOLUTION:

Given a system $AX = b$ of m linear equations in n variables $(n > m)$, the system is consistent and the solutions are infinite

$$if \quad r(A) = m, m < n$$

i.e. Rank of A is m where $m < n$.

We may select any m variables out of n variables. Set the remaining $(n-m)$ variables to zero. The system $AX=b$ becomes $BX_B=b$ where $|B|\neq 0$.

If it has a solution then $X_B=B^{-1}b$.
X_B is called basic solution. Maximum possible basic solutions: $\binom{n}{m}=\binom{n}{n-m}$. $= {}^{n}C_{m}$

EXAMPLE:

Find the Basic Solutions:

$$
\begin{aligned}
x_1 + x_2 + x_3 \qquad &= 10 \\
x_1 + 4x_2 \qquad + x_4 &= 16
\end{aligned}
$$

Sl.	Non-Basic Variables	Basic Variables
1.	$x_1 = 0, x_2 = 0$	$x_3 = 10, x_4 = 6$
2.	$x_1 = 0, x_3 = 0$	$x_2 = 10, x_4 = -24$
3.	$x_1 = 0, x_4 = 0$	$x_2 = 4, x_3 = 6$

4.	$x_2 = 0, x_3 = 0$	$x_1 = 10, x_4 = 6$
5.	$x_2 = 0, x_4 = 0$	$x_1 = 16, x_3 = -6$
6.	$x_3 = 0, x_4 = 0$	$x_1 = 8, x_2 = 2$

There are six Basic Solutions. Only four are Basic Feasible Solutions. Sl. No. (2) and (5) are not Basic Feasible Solutions (B.F.S.). as the solutions have negative values for the variables

Some Definitions and Theorems:

Point in n-dimensional space:

A point $X = (x_1, x_2, x_3, \ldots, x_n)^T$ has n co-ordinates $x_i, i = 1, 2, 3, \ldots, n$. Each of them are real numbers.

Line Segment in n-dimensions:

Let X_1 be the coordinates of A and X_2 be the coordinates of B. The line segment joining these two points is given by $X(\lambda)$ i.e.

$$L = \{X(\lambda) | X(\lambda) = \lambda X_1 + (1-\lambda) X_2, 0 \leq \lambda \leq 1\}$$

Hyper-plane:

A hyper-plane H is defined as:

$$H = \{X \mid C^T X = b\}$$

$$\Rightarrow c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = b$$

A hyper-plane has $(n-1)$-dimensions in an n-dimensional space. In 2-dimensional space hyper-plane is a line.

In 3-dimensional space it is a plane.

A hyper-plane divides the n-dimensional space into two closed half spaces as:

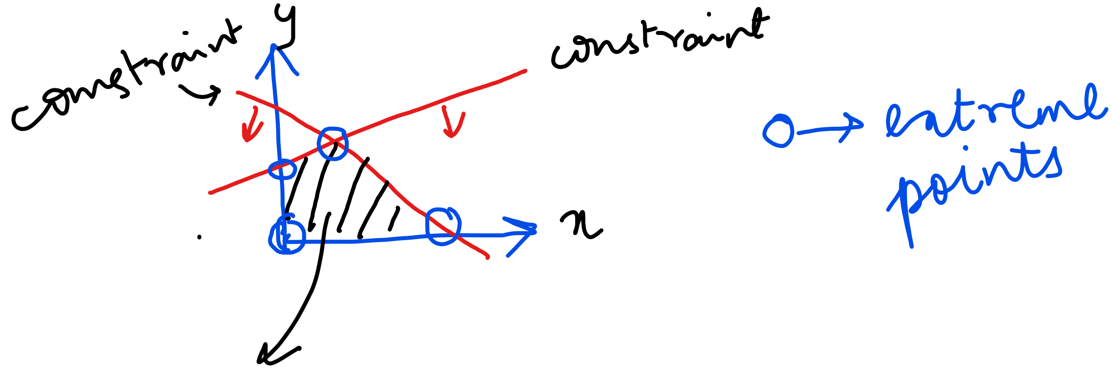
(i) $c_1x_1 + c_2x_2 + \ldots + c_nx_n \leq b$

(ii) $c_1x_1 + c_2x_2 + \ldots + c_nx_n \geq b$

Convex Set: A convex set S is a collection of points such that if X_1 and X_2 are any two points in the set, the line segment joining them is also in the set S.

$$Let \quad X = \lambda X_1 + (1 - \lambda) X_2, 0 \leq \lambda \leq 1$$

If X_1, $X_2 \in S$, then $X \in S$.

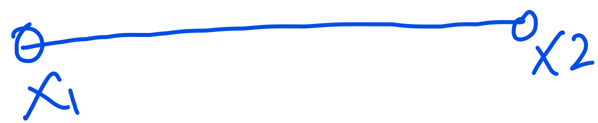


Convex Polyhedron and Polytope:

A convex polyhedron is a set S (a set of points) which is common to one or more half spaces. A convex polyhedron that is bounded is called a convex polytope.

Extreme Point: It is a point in the convex set S which does not lie on a line segment joining two other points of the set.

x_1, x_2 are extreme points



$s = \lambda X_1 + (1-\lambda) X_2$

$\lambda \in [0, 1]$

Feasible Solution: In a LPP any solution X which satisfy $AX = b$ and $X \geq 0$ is called a feasible solution.

Basic Solution: This is a solution in which $(n - m)$ variables are set equal to zero in $AX = b$. It has m equations and n unknowns $n > m$.
Basis: The collection of variables which are not set equal to zero to obtain the basic solution is the basis.

Basic Feasible Solution (B.F.S.):
The basic solution which satisfy the conditions $X \geq 0$ is called B.F.S.

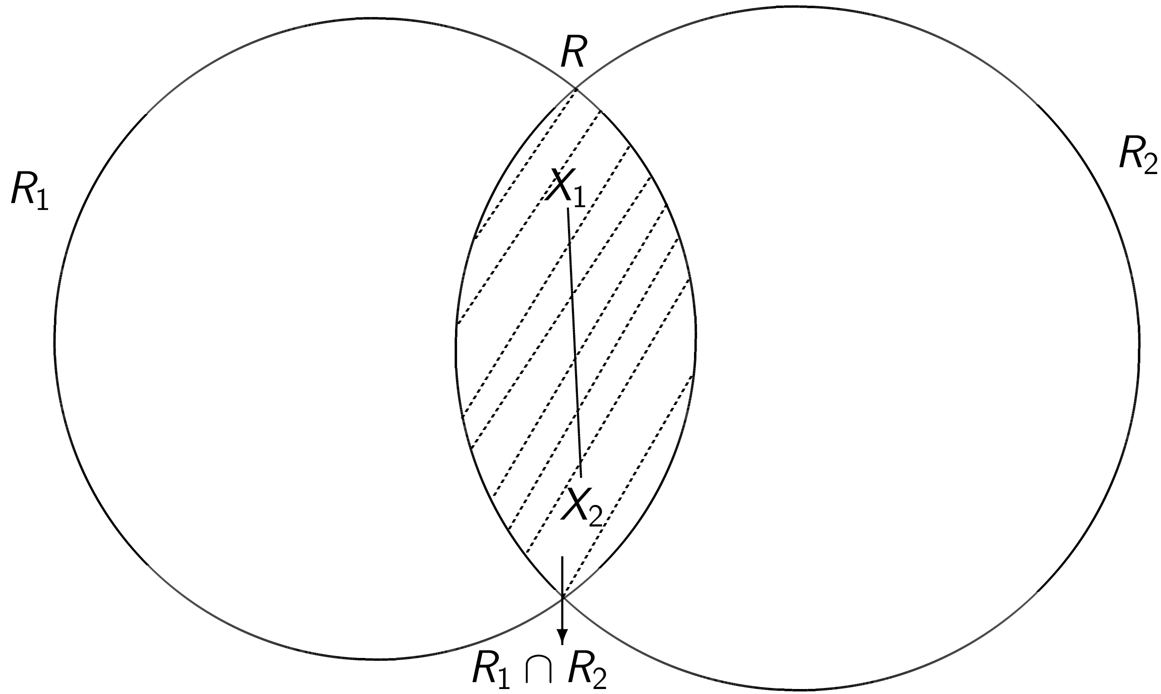
Non-Degenerate B.F.S.: It is a B.F.S. which has exactly m positive x_i out of n.

Optimal Solution: B.F.S. which optimizes(Max / Min) the objective function is called an optimal solution.

Theorem 1: The intersection of any number of convex sets is also convex.

Proof: Let R_1, R_2, $\ldots$, R_k be convex sets and their intersection be R i.e.

$$R = \bigcap_{i=1}^{k} R_i$$



R
R_1
R_2
X_1
X_2
$R_1 \cap R_2$

Let X_1 and $X_2 \in R$. Then $\lambda X_1 + (1-\lambda)X_2 \in R$, where $0 \leq \lambda \leq 1$, $X = \lambda X_1 + (1-\lambda)X_2$.
Thus $X \in R_i, i = 1, 2, \ldots, k$.
Hence
$$X \in R = \bigcap_{i=1}^{k} R_i$$

Theorem 2: The feasible region of a LPP forms a convex set.

Proof: The feasible region of LPP is defined as:

$$S = \{X | AX = b, X \geq 0\}$$

Let the points X_1 and X_2 be in the feasible set S so that $AX_1 = b, X_1 \geq 0$;

$AX_2 = b, X_2 \geq 0.$

Let $X_\lambda = \lambda X_1 + (1-\lambda)X_2$. Now we have:

$$A[\lambda X_1 + (1-\lambda)X_2] = \lambda b + (1-\lambda)b = b$$

$$\Rightarrow AX_\lambda = b.$$

Thus the point X_λ satisfies the constraints if $0 \leq \lambda \leq 1$ i.e. $\lambda \geq 0, 1-\lambda \geq 0$, $X_\lambda \geq 0.$

Theorem 3: In general a LPP has either one optimal solution or no optimal solution or infinite number of optimal solutions.

Any local minimum/maximum solution is a global minimum/maximum solution of a LPP.

$$(LPP - I) \ \max : Z = C^T X \tag{16}$$

subject to

$$AX = b \tag{17}$$

$$X \geq 0 \tag{18}$$

$X*$ is a maximizing point of the LPP.

$$(LPP-II) \quad \min: Z = C^T X \tag{19}$$

subject to

$$AX = b \tag{20}$$

$$X \geq 0 \tag{21}$$

X^* is a minimizing point of the LPP.

Theorem 4: Every B.F.S. is an extreme point of the convex set of the feasible region.

Proof:

Let $X = (x_1, x_2, x_3, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_n)^T$ be a BFS of the LPP where $x_1, x_2, x_3, \ldots, x_m$ are basic variables. Now $x_1 = \bar{b}_1, x_2 = \bar{b}_2, x_3 = \bar{b}_3, \ldots, x_m = \bar{b}_m$, $x_1, x_2, \ldots, x_m \geq 0$.

This feasible region forms a convex set. To show X is an extreme point, we must show that there do not exist feasible solutions Y and Z such that

$$X = \lambda Y + (1 - \lambda)Z, 0 \leq \lambda \leq 1$$

Let $Y = (y_1, y_2, y_3, \ldots, y_m, y_{m+1}, y_{m+2}, \ldots, y_n)^T$
and $Z = (z_1, z_2, z_3, \ldots, z_m, z_{m+1}, z_{m+2}, \ldots, z_n)^T$

Last $(n-m)$ components gives:
$\lambda y_j + (1-\lambda) z_j = 0, j = m+1, m+2, \ldots, n$.
Since $\lambda \geq 0, 1-\lambda \geq 0$, $y_j \geq 0, z_j \geq 0$, it gives
$y_j = z_j = 0$, $j = m+1, m+2, \ldots, n$.
This shows that $Y = Z = X$. So, X is an extreme point by contradiction.

Theorem 5: Let S be a closed bounded convex polyhedron with p number of extreme points $X_i, i = 1, 2, \ldots, p$. Then any vector $X \in S$ can be written as:

$$X = \sum_{i=1}^{p} \lambda_i X_i, \quad \sum_{i=1}^{p} \lambda_i = 1, \quad \lambda_i \geq 0$$

Theorem 6: Let S be a closed convex polyhedron. Then the minimum of a linear function over S is attained at an extreme point of S.

Proof: Suppose X^* minimizes the objective function $Z = C^T X$ over S and minimum does not occur at an extreme point.

From the definition of minimum $C^T X^* < C^T X_i, i = 1, 2, \ldots, p$ with p number of extreme points.

For $0 \leq \lambda_i \leq 1$, $\lambda_i C^T X^* < \lambda_i C^T X_i$,
$i = 1, 2, \ldots, p$

$$\sum_{i=1}^{p} \lambda_i C^T X^* < \sum_{i=1}^{p} C^T \lambda_i X_i, i = 1, 2, \ldots, p.$$

Now taking

$$\lambda_i = \lambda_i^*, X^* = \sum_{i=1}^{p} \lambda_i^* X_i, \lambda_i^* \geq 0, \sum_{i=1}^{p} \lambda_i^* = 1$$

$$Thus \ \sum_{i=1}^{p} \lambda_i^* C^T X^* = C^T X^* < \sum_{i=1}^{p} \lambda_i^* C^T X_i$$

$$\Rightarrow C^T X^* < C^T\left(\sum_{i=1}^{p} \lambda_i^* X_i\right)$$
$$\Rightarrow C^T X^* < C^T X^*.$$

which is a contradiction.
Hence minimum occurs at an extreme point only. Similarly, maximum occurs at an extreme point only.

Graphical Methods for a LPP (Only for 2-Variable Problems):

Step 1: Define the coordinate system and plot the axes. Associate each axis with a variable.

Step 2: Plot all the constraints. A constraint represents either a line or a region.

Step 3: Identify the solution space (feasible region). Feasible region is the intersection of all the constraints. If there is no feasible region the problem is infeasible.

Step 4: Identify the extreme points of the feasible region.

Step 5: For each extreme point determine the value of the objective function. The point that maximizes/minimizes this value is optimal.

EXAMPLE:

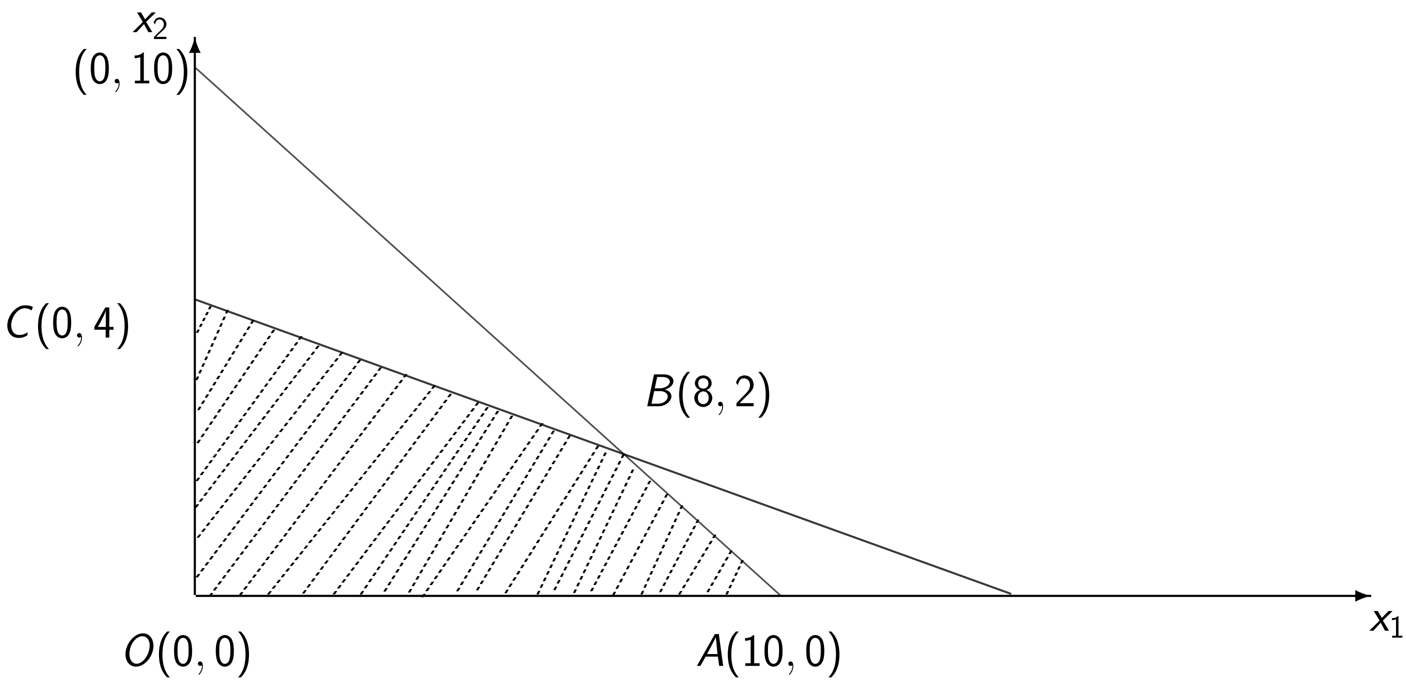
$$\max : Z = x_1 + 3x_2$$

subject to

$$x_1 + x_2 \leq 10$$

$$x_1 + 4x_2 \leq 16$$

$$x_1, x_2 \geq 0$$



x_2
$(0, 10)$
$C(0, 4)$
$B(8, 2)$
$O(0, 0)$
$A(10, 0)$
x_1

Extreme points of the Feasible region are O, A, B, and C.

At $O\ \ (0,0), Z = 0$

$A\ \ (10,0), Z = 10$

$B\ \ (8,2), Z = 14$

$C\ \ (0,4), Z = 12$

Refer to previous Basic Solution problem.
We observe that, the basic feasible solutions lie on these extreme points

Maximum value of the objective function is 14. Maximizing point is $B\ \ (8,2)$.

We apply Simplex Method to solve a standard LPP in the form:

$$\max : z = \sum_{j=1}^{n} c_j x_j + d$$

$$subject\ \ to : \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, 2, \ldots, m$$

$$x_1, x_2 \ldots, x_n \geq 0$$

It is assumed that ~~$b_1, b_2, \ldots, b_m \geq 0$~~. $b_1, b_2 \ldots b_m \geq 0$

This problem can be reformulated as:

$$\max: z = \sum_{j=1}^{n} c_j x_j + d$$

subject to

$$-\sum_{j=1}^{n} a_{ij} x_j + b_i = z_i, i = 1, 2, \ldots, m$$

$$x_1, x_2 \ldots, x_n \geq 0$$

$$z_1, z_2 \ldots, z_m \geq 0 \quad (Slack Variables)$$

To solve the problem, we present the problem in a tabular form called Simplex Tableau.

$-x_1$	$-x_2$	$\ldots$	$-x_v$	$\ldots$	$-x_n$	1	
a_{11}	a_{12}	$\ldots$	a_{1v}	$\ldots$	a_{1n}	b_1	$= z_1$
a_{21}	a_{22}	$\ldots$	a_{2v}	$\ldots$	a_{2n}	b_2	$= z_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$-x_1$	$-x_2$	$\ldots$	$-x_v$	$\ldots$	$-x_n$	1	
a_{u1}	a_{u2}	$\ldots$	a_{uv}	$\ldots$	a_{un}	b_u	$= z_u$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{m1}	a_{m2}	$\ldots$	a_{mv}	$\ldots$	a_{mn}	b_m	$= z_m$
$-c_1$	$-c_2$	$\ldots$	$-c_v$	$\ldots$	$-c_n$	d	$= z$

const-
-raints

objective function

Simplex Tableau:

The point $x_1 = x_2 = \ldots = x_n = 0$ becomes an extreme point. The value of the non-basic variables: $x_1, x_2 \ldots, x_n$ are zero. The values of the basic variables $z_1 = b_1, z_2 = b_2, \ldots, z_m = b_m$. The value of the objective function $z = d$ at $x_1 = x_2 = \ldots = x_n = 0$.

Steps of the Simplex Algorithm:

Step 1: Select the most negative element in the last row of the simplex tableau. If no negative element exists, then the maximum value of the LPP is d and a maximizing point is

all the column header variables = 0

imp

$\cancel{x_1 = x_2 = \dots = x_n = 0}$. Stop the method.

While selecting the most negative element of last row, ignore 'd' as its sign | value won't matter during optimization

Step 2: Suppose Step 1 gives the element $-c_v$ at the bottom of the v-th column. Form all positive ratios of the element in the last column to corresponding elements in the v-th column. That is form ratios b_i/a_{iv} for which $a_{iv} > 0$. The element say a_{uv} which produces the smallest ratio b_i/a_{uv} is called pivotal element.

imp

If the elements of the v-th column are all negative or zero the problem is called unbounded.

Stop else go to Step 3.

Step 3: Form a new Simplex Tableau using the following rules:

(a) Interchange the role of x_v and z_u.

That is relabel the row and column of the pivotal element while keeping other labels unchanged.

(b) Replace the pivotal element ($p > 0$) by its reciprocal $1/p$ i.e. a_{uv} by $1/a_{uv}$.
(c) Replace the other elements of the row of the pivotal element by the (row elements/pivotal element).
(d) Replace the other elements of the column of the pivotal element by the (negative of the column elements/ pivotal element).

(e) Replace all other elements (say s) of the Tableau by the elements of the form:

$$s^* = \frac{ps - qr}{p}$$

$$\begin{array}{|ccc|}
\hline
s & & q \\
 & & \\
r & & p \\
\hline
\end{array}$$

where p is the pivotal element and q and r are the Tableau elements for which p, q, r, s form a rectangle. (Step 3: leads to a new Tableau that presents an equivalent LPP)

Step 4: Go to Step 1.

EXAMPLE-1:

Method when only slack variables are present

$$\max : z = x_1 + 3x_2$$

(not applicable when artificial / surplus variables are present)

subject to

$$x_1 + x_2 \leq 100$$

$$x_1 + 2x_2 \leq 110$$

$$x_1 + 4x_2 \leq 160$$

$$x_1, x_2 \geq 0$$

Note:- RHS of constraints must be positive, if not make them positive

Adding slack variables $z_1, z_2, z_3 \geq 0$, we express the constraints as:

$$x_1 + x_2 + z_1 = 100 \Rightarrow -x_1 - x_2 + 100 = z_1$$

$$x_1 + 2x_2 + z_2 = 110 \Rightarrow -x_1 - 2x_2 + 110 = z_2$$

$$x_1 + 4x_2 + z_3 = 160 \Rightarrow -x_1 - 4x_2 + 160 = z_3$$

Now the problem can be put in Tabular form with $z = x_1 + 3x_2$, $d = 0$.

Initial Simplex Tableau:

$-x_1$	$-x_2$	1	
1	1	100	$= z_1$
1	2	110	$= z_2$
1	4 *	160	$= z_3$
−1	−3 *	0	$= z$

$\frac{160}{4} = 40$

(gives the smallest ratio)

4 is the pivotal element

most negative

(1) z_3 & x_2 exchanged without -ve sign

(2) new pivot = $1/\text{pivot}$

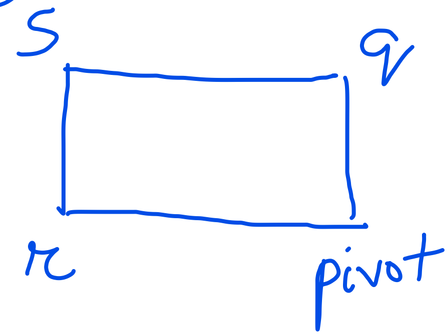
Table-1

$-x_1$	$-z_3$	1	
$\frac{3}{4}$	$-\frac{1}{4}$	60	$= z_1$
$\frac{2}{4}*$	$-\frac{2}{4}$	30	$= z_2$
$\frac{1}{4}$	$\frac{1}{4}$	40	$= x_2$
$-\frac{1}{4}*$	$\frac{3}{4}$	120	$= z$

(3) pivotal row divided by pivot

(4) pivotal column divided by (-pivot)

(5)



new s $= \dfrac{(\text{pivot} \times s) - qr}{\text{pivot}}$

for all other elements of the table

Repeating all the steps in Table 1, we get table 2

Table-2(OPTIMAL TABLEAU)

in the last row, if there is any 0 (except for in '1' column), there are ∞ solution, else there is a single solution

	$-z_2$	$-z_3$	1
$= z_1$	$-\frac{3}{2}$	$-\frac{1}{2}$	15
$= x_1$	2	-1	60
$= x_2$	$-\frac{1}{2}$	$\frac{1}{2}$	25
$= z$	$\frac{1}{2}$	$\frac{1}{2}$	135

as the last row (except for '1' column) has no -ve elements

where $z^* = 135$, $x_1^* = 60$, $x_2^* = 25$,
$z_1^* = 15$, $z_2^* = 0$, $z_3^* = 0$.

Method when artificial & surplus variables are present

EXAMPLE-2:

$$\max : z = 2x_1 + 3x_2$$

subject to

$$2x_1 + 3x_2 = 12$$

$$2x_1 + x_2 \geq 8$$

$$x_1, x_2 \geq 0$$

We express

$$2x_1 + 3x_2 = 12$$

as:

$$2x_1 + 3x_2 + z_1 = 12, z_1 \geq 0.$$

z_1 is an artificial variable and

$$2x_1 + x_2 \geq 8$$

as:

$$2x_1 + x_2 - x_3 + z_2 = 8$$

where $z_2 \geq 0$ is an artificial variable and x_3 is a surplus variable.

We reformulate the new objective function as:

$$\left.\begin{aligned}&\text{We reformulate the new objective function as:}\\ &\max : z = 2x_1 + 3x_2 - M(z_1 + z_2)\end{aligned}\right\}\text{extra step}$$

$$= 2x_1 + 3x_2 - M(20 - 4x_1 - 4x_2 + x_3)$$

$$= x_1(2 + 4M) + x_2(3 + 4M) - Mx_3 - 20M$$

where M is a very large positive number.

This method is called Big-M method.

To solve the problem we transform the problem into a Tabular form.

Initial Simplex Tableau:

$-x_1$	$-x_2$	$-x_3$	1	
2	3_*	0	12	$= z_1$
2	1	-1	8	$= z_2$
$-2-4M$	$-3-4M_*$	M	$-20M$	$= z$

Table-1

$-x_1$	$-z_1$	$-x_3$	1	
$\frac{2}{3}$	$\frac{1}{3}$	0	4	$= x_2$
$\frac{4}{3}*$	$-\frac{1}{3}$	- 1	4	$= z_2$
$\frac{-4M}{3}*$	$\frac{3+4M}{3}$	M	$12 - 4M$	$= z$

In final solution, the value of the artificial variables (z_1, z_2) has to be zero. Only then is the solution feasable, else it is an infeasable solution.

Table -2(OPTIMAL)

$-z_2$	$-z_1$	$-x_3$	1	
$-\frac{2}{4}$	$\frac{1}{2}$	$\frac{1}{2}*$	2	$= x_2$
$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{3}{4}$	3	$= x_1$
M	$M+1$	$0*$	12	$= z$

Optimal: $z^* = 12,\ x_1^* = 3,\ x_2^* = 2$

Since there is a zero in the last row of the Tableau further iteration is possible.

Table- 3 (ALTERNATE OPTIMAL SOLUTION)

$-z_2$	$-z_1$	$-x_2$	1	
-1	1	2	4	$=x_3$
0	$\frac{1}{2}$	$\frac{3}{2}$	6	$=x_1$
M	$M+1$	0	12	$=z$

This is also an optimal Tableau.
where $z^* = 12$, $x_1^* = 6$, $x_2^* = 0$
So this LPP has several optimal solutions:
$X^* = \lambda X^1 + (1 - \lambda)X^2$, where $0 \leq \lambda \leq 1$

$$X^* = \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

$$\Rightarrow X^* = \begin{pmatrix} 6 - 3\lambda \\ 2\lambda \end{pmatrix}, 0 \leq \lambda \leq 1$$

This problem also can be solved by Two-Phase Simplex Method.

This method is used to avoid the Big-M method in presence of artificial variables.

Two-Phase Simplex Method:

Phase-I:

In this Phase, an artificial objective function f is used where we minimize the sum of artificial variables to zero. We try to drive out the all the artificial variables from the basis to make them zero. If they can not be removed i.e. all can not be made zero, we conclude that the problem is infeasible.

* If all the artificial variables are zero in Phase-I, we go to Phase-II.

Phase-II:

In this Phase-II, we replace the artificial objective function f by the original objective function z using the last Tableau of Phase-I.* Then we apply usual Simplex Method until an Optimal solution X^* is reached.

*→ We change the coefficients according to original objective function

* → below the artificial variables, we fill all zeros as they are insignificant in conditions as well.

* → Then using inner product, compute the last row

If an artificial variable is there in the basis at zero value at the end of Phase-I, we modify the departing variable rule. An artificial variable must not become positive from zero. So, we allow an artificial variable with negative y_{ij} value to depart. It is an important point to note.

→ Acc. to new condensed table

* if in II[nd] phase table, '0' is present in the last row of artificial variable column, ignore it as it does not imply the presence of multiple solutions.

Example: Two-Phase Simplex Method:

$$\max : z = 5x_1 + 4x_2$$

subject to

$$x_1 + x_2 \geq 2$$

$$5x_1 + 4x_2 \leq 20$$

$$x_1, x_2 \geq 0$$

Introduce surplus and artificial variables:

$$x_1 + x_2 \geq 2$$

as:

$$x_1 + x_2 - x_3 + z_1 = 2, \;\; x_3 \geq 0, \;\; z_1 \geq 0$$

z_1 is an artificial variable (basic variable) and x_3 is a surplus variable.
It can be written as:

$$z_1 = 2 - x_1 - x_2 + x_3$$

Introduce slack variable:

$$5x_1 + 4x_2 \leq 20$$

as:

$$5x_1 + 4x_2 + z_2 = 20, \;\; z_2 \geq 0$$

z_2 is a slack variable.
It can be written as:

$$z_2 = 20 - 5x_1 - 4x_2$$

where z_2 is a basic variable.

For Phase-I method we formulate an artificial objective function f for minimum. i.e. Phase-I objective function

$$\hookrightarrow \min : f = z_1 = \sum \text{artificial variables}$$

It is equivalent to:

$$\max : -f = -z_1$$

$$\max : -f = x_1 + x_2 - x_3 - 2$$

Phase-I Problem:

$$\max: -f = x_1 + x_2 - x_3 - 2$$

subject to

$$-x_1 - x_2 + x_3 + 2 = z_1$$

$$-5x_1 - 4x_2 + 20 = z_2$$

$$x_1,\ x_2,\ x_3,\ z_1,\ z_2 \geq 0$$

We start with Phase-I procedure with an artificial objective function $-f$.

Phase-I: Initial Simplex Tableau:

$-x_1$	$-x_2$	$-x_3$	1	
1_*	1	-1	2	$= z_1$
5	4	0	20	$= z_2$
-1_*	-1	1	-2	$= -f$

Phase-I: Table-1:

$-z_1$	$-x_2$	$-x_3$	1	
1	1	-1	2	$=x_1$
-5	-1	5	10	$=z_2$
1_*	0	0	0	$=-f$

Optimal Phase-I Solution:

$z_1 = 0$ (Artificial variable)

$f = 0$ (Artificial Objective Function)

Phase-II: Formulation

Set z_1 column elements to zero. Then we replace the artificial objective function with the original objective function z.

$$\begin{aligned} z &= 5x_1 + 4x_2 \\ &= 5(-x_2 + x_3 + 2) + 4x_2 \\ &= -x_2 + 5x_3 + 10 \end{aligned}$$

Phase-II: Initial Simplex Tableau

$-z_1$	$-x_2$	$-x_3$	1	
0	1	-1	2	$= x_1$
0	-1	5_*	10	$= z_2$
0	1	-5_*	10	$= z$

There is a negative element in the last row of the Simplex Tableu.

Phase-II: Optimal Simplex Tableau

$-z_1$	$-x_2$	$-z_2$	1	
0	$4/5_*$	$1/5$	4	$=x_1$
0	$-1/5$	$1/5$	2	$=x_3$
0	0_*	1	20	$=z$

Optimal: $x_1^* = 4, \quad , x_2^* = 0, \quad z^* = 20$

Phase-II: Alternate Optimal Solution

$-z_1$	$-x_1$	$-z_2$	1	
0	5/4	1/4	5	$= x_2$
0	1/4	1/4	3	$= x_3$
0	0 $*$	1	20	$= z$

Optimal: $x_1^* = 0, \quad , x_2^* = 5, \quad z^* = 20$

This problem has Infinite number of optimal solutions:
$X^* = \lambda X^1 + (1-\lambda)X^2$, where $0 \le \lambda \le 1$

$$X^* = \lambda \begin{pmatrix} 4 \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$$\Rightarrow X^* = \begin{pmatrix} 4\lambda \\ 5-5\lambda \end{pmatrix}, 0 \le \lambda \le 1$$

Duality Theory for LPP:

Primal Program (P):

$$\max : z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \;\; i = 1, 2, 3, \ldots, m$$

$$x_j \geq 0, \;\; j = 1, 2, 3, \ldots, n$$

With respect to the above Primal Problem (P) we find a Dual Problem (D) as:
Dual Program (D):

$$\min : z' = \sum_{i=1}^{m} b_i y_i$$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \; j = 1, 2, 3, \ldots, n$$

$$y_i \geq 0, \; i = 1, 2, 3, \ldots, m$$

Example-1: Primal Program (P):

$$\max : z = x_1 + 3x_2$$

Dual is minimization problem

Subject to

D obj fun coeffs

$$x_1 + x_2 \leq 100$$

$$x_1 + 2x_2 \leq 110$$

$$x_1 + 4x_2 \leq 160$$

$$x_1, x_2 \geq 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}^T = \text{constraint coeffs of D}$$

Note :-
optimal solution of z & z' is the same value

RHS values of constraints

Dual Program (D):

$$\min : z' = 100y_1 + 110y_2 + 160y_3$$

Subject to

as P constraints are $\leq$

coeff of x_1, x_2 in 1st constraint

$$y_1 + 1y_2 + 1y_3 \geq 1$$

$$1y_1 + 2y_2 + 4y_3 \geq 3$$

$$y_1, y_2, y_3 \geq 0$$

coeffs of x_1, x_2 in obj fun of P

Example-2: Primal Program (P):

$$\max : z = x_1 + 3x_2$$

subject to

$$x_1 + x_2 \leq 100$$

$$x_1 + 2x_2 = 110$$

$$x_1 + 4x_2 = 160$$

$$x_1, x_2 \geq 0$$

Dual Program (D):

$$\min : z' = 100y_1 + 110y_2 + 160y_3$$

Subject to

Free variables can be +ve, -ve or 0

$$y_1 + y_2 + y_3 \geq 1$$

$$y_1 + 2y_2 + 4y_3 \geq 3$$

$$y_1 \geq 0, \quad y_2, \quad y_3 \quad \textit{are} \quad \textit{free}.$$

as last 2 constraints are = type

Example-3: Primal Program (P):

$$\min : z = x_1 + 3x_2$$

Subject to

$$x_1 + x_2 \geq 100$$

$$x_1 + 2x_2 \geq 110$$

$$x_1 + 4x_2 \geq 160$$

$$x_1, x_2 \geq 0$$

Dual Program (D):

$$\max : z' = 100y_1 + 110y_2 + 160y_3$$

Subject to

$$y_1 + y_2 + y_3 \leq 1$$

$$y_1 + 2y_2 + 4y_3 \leq 3$$

$$y_1, y_2, y_3 \geq 0$$

Example-4: Primal Program (P):

$$\min: z = x_1 + 3x_2$$

subject to

$$x_1 + x_2 \geq 100$$

$$x_1 + 2x_2 \geq 110$$

$$x_1 + 4x_2 = 160$$

$$x_1, x_2 \geq 0$$

To solve minimization problem, find dual D and solve using simplex method

Dual Program (D):

$$\max : z' = 100y_1 + 110y_2 + 160y_3$$

Subject to

$$y_1 + y_3 + y_3 \leq 1$$

$$y_1 + 2y_2 + 4y_3 \leq 3$$

$$y_1 \; , \; y_2 \geq 0, \; y_3 \text{ is free.}$$

Free variables need to be replaced by diff. of 2 non -ve variables for simplex to work

To handle y_3, let $y_3 = y_4 - y_5$ st. $y_4, y_5 \geq 0$ replace y_3 everywhere

as last constraint is = type

Example-5: Primal Program (P):

$$\max : z = 10x_1 + 20x_2 + 30x_3$$

subject to

$$x_1 + x_2 + x_3 = 60$$

$$x_1 + 5x_2 + 10x_3 = 410$$

$$x_1, x_2, x_3 \geq 0$$

→ solve using BFS method

Dual Program (D):

$$\min : z' = 60y_1 + 410y_2$$

Subject to

$$y_1 + y_2 \geq 10$$
$$y_1 + 5y_2 \geq 20$$
$$y_1 + 10y_2 \geq 30$$
$$y_1, \quad y_2, \quad \textit{are} \quad \textit{free}.$$

all constraints are converted to $\geq$ as D is a minimization problem

Theorem 1: LPP Primal (P) is consistent and has a maximum value M_P if and only if its Dual (D) is consistent and has a minimum value M_D. Moreover $M_P = M_D$.

Imp

Theorem 2: If X satisfies the constraints of the Primal Program (P) and Y satisfies the constraints of the Dual Program (D), then

optimum value of D $\geq$ optimum value of P

$$\sum_{i=1}^{m} b_i y_i \geq \sum_{j=1}^{n} c_j x_j$$

$M_D \geq M_P$

Equality holds if and only if

- Either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j, j = 1, 2, \ldots, n$
- Either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, \ldots, m$

To solve the dual program we may use Dual Simplex Method.

Some Discrete Models:

$$\max / \min : z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \; i = 1, 2, 3, \ldots, m$$

$$x_j = 0, 1, 2, 3, \ldots, \; \textit{for all} \;\; j$$

To Solve this discrete LPP we use two different methods:

1. Gomory Cutting Plane Method
2. Branch and Bound Method

Further if the decision variables are Binary $(0/1)$ additive algorithm may be used to solve the problem.

Text Book References:

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