Let X and Y be any two metric spaces. A map $F: X \longrightarrow Y$ is said to be an open map iff for every open set E in X, its image $F(E)$ is an open set in Y.

* $F: X \longrightarrow Y$ is continuous iff for every open set E in Y, its inverse image $F^{-1}(E)$ is open in X.

Theorem: Let X and Y be any two n.l.s, and $F: X \longrightarrow Y$ be a linear map. Then F is an open map iff there exists some $\gamma > 0$, such that for every $y \in Y$, there exists some $x \in X$ such that

$$F(x) = y \quad \text{and} \quad \|x\| \leq \gamma \|y\|.$$

$$= \gamma \|F(x)\|$$

Proof: let $F: X \longrightarrow Y$ be a linear and open map.

Since $U_X(0,1)$ is open in X, so

$F(U_X(0,1))$ is open in Y.

Also $0 = F(0) \in F(U_X(0,1))$,
so there is some $\delta > 0$ such that

$$\overline{U_Y(0,\delta)} \subset F(U_X(0,1)) \quad (1)$$

$$\left[\overline{U_Y(0,\delta)} = \{y \in Y \mid \|y\| \le \delta\}\right].$$

Now consider $y \in Y$, $y \neq 0$.

Then $$\frac{\delta y}{\|y\|} \in \overline{U_Y(0,\delta)} \quad \left[\because \left\|\frac{\delta y}{\|y\|}\right\| = \delta\right]$$

So by (1), $\exists$ some $x_1 \in U_X(0,1)$
such that $F(x_1) = \frac{\delta y}{\|y\|}$.

So letting $x = \frac{\|y\|}{\delta} x_1$, we get

$$F(x) = F\left(\frac{\|y\|}{\delta} x_1\right)$$

$$= \frac{\|y\|}{\delta} \cdot F(x_1)$$

$$= \frac{\|y\|}{\delta} \cdot \frac{\delta y}{\|y\|}$$

$$= y.$$

Also

$$\|x\| = \left\| \frac{\|y\|}{\delta} x_1 \right\|$$

$$= \frac{\|y\|}{\delta} \cdot \|x_1\|$$

$$\leq \frac{1}{\delta} \|y\|. \quad [\because x_1 \in U_X(0,1) \quad \|x_1\| < 1]$$

So letting $\gamma = 1/\delta$, we have

$$\|x\| \leq \gamma \|y\|.$$

Conversely assume that for every $y \in Y$, $\exists$ some $x \in X$ such that

$$F(x) = y \quad \text{and} \quad \|x\| \leq \gamma \|y\|, \ \gamma > 0.$$

Claim: $F : X \longrightarrow Y$ is an open map.

Let E be any open set in X and $x_0 \in E$. Then

$$U_X(x_0, \delta) \subset E, \text{ for some } \delta > 0.$$

Let $y \in Y$ with $\|y - F(x_0)\| < \frac{\delta}{\gamma}$

i.e., $y \in U_Y\left(F(x_0), \frac{\delta}{\gamma}\right)$.

$\because$ $y - F(x_0) \in Y$, by assumption there exists some $x \in X$ such that

$$F(x) = y - F(x_0) \text{ and}$$

$$\|x\| \leq \gamma \|y - F(x_0)\|.$$

Hence

$$\|x\| \leq \gamma \, \|y - F(x_0)\| < \gamma \cdot \frac{\delta}{\gamma}$$
$$= \delta$$

$\Rightarrow \quad x \in U_X(0, \delta)$

Also

$F(x) = y - F(x_0)$

$\Rightarrow \quad y = F(x) + F(x_0) = F(x + x_0)$

and

$$\|x + x_0 - x_0\| = \|x\| < \delta$$

$\Rightarrow \quad x + x_0 \in U_X(x_0, \delta)$

$\Rightarrow \quad y = F(x + x_0) \subset F(U_X(x_0, \delta))$
$$\subset F(E)$$

$\Rightarrow \quad y \in F(E).$

$\Rightarrow \quad y \in U_Y\left(F(x_0), \frac{\delta}{\gamma}\right) \subset F(E)$

$\Rightarrow$ $F(E)$ is open in Y

$\Rightarrow$ F is open Map

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* Interior of a proper subspace of a n.l.s is empty.

Problem: Let X and Y be n.l.s and $F: X \longrightarrow Y$ be a linear map. If F is an open map, then F is surjective.

Corollary :— Let X and Y be n.l.s and $F: X \longrightarrow Y$ be bijective linear map. Then F is an open map iff $F^{-1}: Y \longrightarrow X$ is continuous.

Proof: $F: X \longrightarrow Y$ is an open map

$\Longleftrightarrow$ For $y \in Y$, $\exists$ $x \in X$ with $F(x) = y$ and $\|x\| \leq \gamma \|y\|$, $\gamma > 0$.

$\Longleftrightarrow$ $x = F^{-1}(y)$, $\|x\| \leq \gamma \|y\|$

$\|F^{-1}(y)\| \leq \gamma \|y\|$

$\iff \quad \|F^{-1}\| \leq ?$

$\iff \quad F^{-1}$ is continuous.

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