Lecture 1

Measure Theory & Integration.

$\mathbb{R}$, $\mathbb{C}$

Let X be a non-empty set. &

$d: X \times X \to \mathbb{R}$ be a function. $\left(X \times X := \{ (x,y) / x, y \in X \} \right)$

Then X together with d, (X, d) is called a <u>metric space</u> if the following conditions hold:

(i) $d(x,y) \geq 0, \quad \forall x, y \in X$

& $d(x,y) = 0 \iff x = y.$

$(x,y) \in X \times X$ equivalently say $x, y \in X$.

(ii) $d(x,y) = d(y,x), \quad \forall x, y \in X.$

* (iii) $d(x,z) \leq d(x,y) + d(y,z), \quad \forall x, y, z \in X$

(triangular inequality)

d is called the distance function.

& (X, d) is a <u>metric space</u>.

Examples:- ① $X = \mathbb{R}$, $d(x,y) = |x-y|$, $\forall x, y \in \mathbb{R}$

Euclidean distance.
(metric)

② $X = \mathbb{R}^2$, $d(\underset{\substack{\| \\ (x_1, x_2)}}{\underline{x}}, \underset{\substack{\| \\ (y_1, y_2)}}{\underline{y}}) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

(X, d) metric space.

Definition Let (X, d) be a metric space & $x \in X$.
Then the open ball / ball around x is defined as

$$B(x, \varepsilon) \doteq \{ y \in X \mid d(x,y) < \varepsilon \} \quad (\text{where } \varepsilon > 0)$$

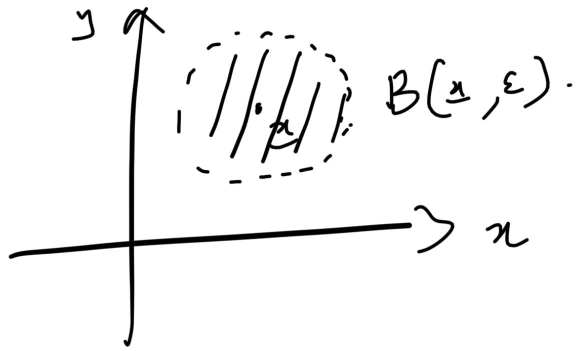
Also called an ε-neighbourhood of x.

$$\overline{B(x, \varepsilon)} = \{ y \in X \mid d(x,y) \leq \varepsilon \}.$$

Examples:- ① $X = \mathbb{R}$, $B(x, \varepsilon) = \{ y \in \mathbb{R} \mid |y - x| < \varepsilon \}$

$= (x-\varepsilon, x+\varepsilon)$
open interval.

② $X = \mathbb{R}^2$, $B(\underline{x}, \varepsilon) =$ open disk around $\underline{x}$
$\underline{x}$ $\in$



<u>Definition</u>:- Let (X, d) be a metric space.
A subset $A \subseteq X$ is called an <u>open set</u>

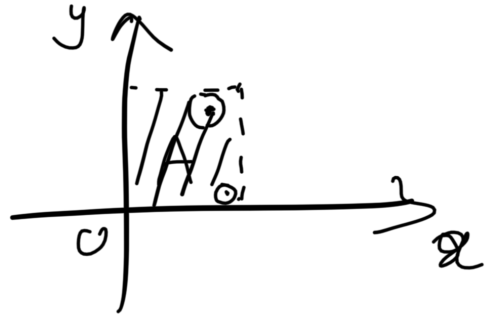
if for any point $x \in A$, there exists $\varepsilon > 0$
such that the open ball around x, $B(x, \varepsilon) \subseteq A$.

<u>Definition</u>:- A subset $A \subseteq X$, is called a <u>closed set</u>
if $A^c = X \setminus A$ is an open set.

<u>Examples</u>:- ① $X = \mathbb{R}$, $A = (-1, 1) \cup (0, 2)$ open set.

<u>Remark</u>:- $\emptyset, X$ are both open & closed sets in X.

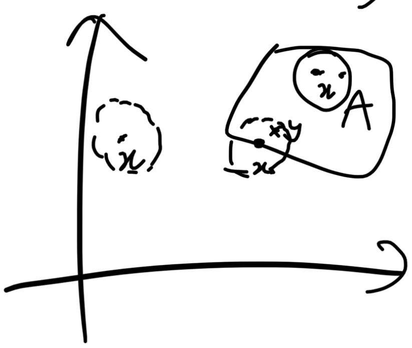
② $X = \mathbb{R}^2$, $A = \{ (x,y) \in \mathbb{R}^2 \mid 0 \lneq x \lneq 1,\ 0 \lneq y \lneq 1 \}$
$\subseteq \mathbb{R}^2$
is an open set.



$$B = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 1 \le x \le 2 \\ 0 \le y \le 1 \end{array} \right\}$$ is a closed set.

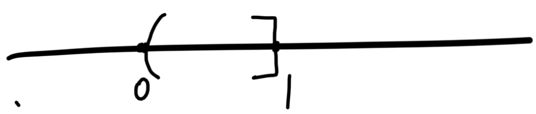
Definition:- A point $x \in X$ is called a <u>limit point</u> of a subset A of X, if given $\varepsilon > 0$, there exists $y \in A$, $y \neq x$ such that $d(x,y) < \varepsilon$. (ie, there exists $y \in A$ such that $y \in B(x,\varepsilon)$.)

(ie $\big(B(x,\varepsilon) \setminus \{x\}\big) \cap A \neq \phi$)



Examples:-

① $X = \mathbb{R}$, $A = [0,1] \subseteq X$



0 is a limit point of A

2 is not a limit point of A.

$\frac{1}{2}$ is a limit point of A.

$A' = [0,1]$.

Def:- Let $A \subseteq X$. $A' := \{$set of all limit points of $A\}$

Def:- Let $A \subseteq X$. Then the <u>closure</u> of A,

$\overline{A}$ = the intersection of all closed sets containing A.

$$= \bigcap_{\substack{V \subseteq X \\ \text{closed} \\ \& \ A \subseteq V}} V$$

Remark:- $\overline{A} = A \cup A'$.

Def:- A subset $A \subseteq X$ is called dense in X, if $\overline{A} = X$.

Examples:- $X = \mathbb{R}$, $A = \mathbb{Q}$, $\overline{\mathbb{Q}} = \mathbb{R}$,
$\therefore$ $\mathbb{Q}$ is dense in $\mathbb{R}$.

Def:- A subset $A \subseteq X$ is called a G_δ-set, if

$$A = \bigcap_{i=1}^{\infty} G_i, \quad \text{where } G_i \text{ are open sets.}$$

Def:- A subset $A \subseteq X$ is called an F_σ-set,

if $A = \bigcup_{i=1}^{\infty} F_i$, where F_i's are closed sets.

Remark: ① Suppose $I_1, \ldots, I_n, \ldots$ open intervals (sets) in $\mathbb{R}$.
Then $\bigcup_{j=1}^{\infty} I_j$ is an open set.

② More generally, in a metric space, arbitrary union of open sets is an open set.

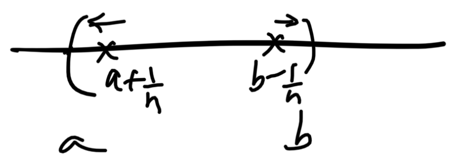
③ In a metric space, an arbitrary intersection of closed sets is a closed set.

<u>Examples:-</u>

① Any open interval is an F_σ-set.

$$(a,b) = \bigcup_{n=1}^{\infty} \left[a+\tfrac{1}{n},\, b-\tfrac{1}{n}\right]$$

closed sets



② Any closed interval is a G_δ-set.

$$[a,b] = \bigcap_{n=1}^{\infty} \left(a-\tfrac{1}{n},\, b+\tfrac{1}{n}\right)$$

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③ $(0,1) \cup (3,4)$ F_σ-set but this is not an interval.

Reference Books:-

1. Measure Theory & Integration by G. de Barra

2. Real Analysis : Measure Theory, Integration & Hilbert space by Elian M. Stein & R. Shakarchi.