Date 6/2/23

ch-4

Method of successive Approximation (iterative method)

~~Picard~~ ~~$y = f'(x,y)$~~

Iterative scheme : consider the Fredholm IE of 2nd kind

$$g(s) = f(s) + \lambda \int_a^b K(s,t)\, g(t)\, dt \quad \text{——} (1)$$

where the functions $f(s)$ and $K(s,t)$ are L_2-functions

now take

$$g_0(s) = f(s) \qquad 0^{th} \text{ order approximation} \quad \text{——} (2)$$

$$g_1(s) = f(s) + \lambda \int_a^b K(s,t)\, g_0(t)\, dt \qquad 1^{st} \text{ order approximation} \quad \text{——} (3)$$

now, continuing this way, the $(n+1)^{th}$ approximation can be written as

Imp

$$g_{n+1}(s) = f(s) + \lambda \int_a^b K(s,t)\, g_n(t)\, dt \quad \text{——} (4)$$

If this sequence $\{g_n(s)\}$ of function tend to a limit uniformly as $n \to \infty$, then this limit is the required solution.

$\boxed{K_1 = K}$

let us write 1st & 2nd approximation

$$g_1(s) = f(s) + \lambda \int_a^b K(s,t)\, g_0(t)\, dt \quad \text{——} (5)$$

$$g_2(s) = f(s) + \lambda \int_a^b K(s,t)\, g_1(t)\, dt$$

or

$$g_2(s) = f(s) + \lambda \int_a^b K(s,t) \left[f(t) + \lambda \int_a^b K(t,x)\, g_0(x)\, dx \right] dt$$

$$= f(s) + \lambda \int_a^b K(s,t)\, f(t)\, dt$$

$$+ \lambda^2 \int_a^b K(s,t) \left[\int_a^b K(t,x)\, f(x)\, dx \right] dt \quad \text{——} (6)$$

let $K_2(s,t) = \int_a^b K(s,x)\, K(x,t)\, dx$ — (7)

(6) can be written as

$$g_2(s) = f(s) + \lambda \int_a^b K(s,t) f(t) dt + \lambda^2 \int_a^b K_2(s,t) f(t) dt$$ — (8)

Similarly,

$$g_3(s) = f(s) + \lambda \int_a^b K(s,t) f(t) dt + \lambda^2 \int_a^b K_2(s,t) f(t) dt$$
$$+ \lambda^3 \int_a^b K_3(s,t) f(t) dt$$ — (9)

where $K_3(s,t) = \int_a^b K(s,x)\, K_2(x,t)\, dx$ — (10)

In general we may write

Imp

$$K_m(s,t) = \int_a^b K(s,x)\, K_{m-1}(x,t)\, dx$$ — (11)

and n^{th} approximation soln can be written as

Imp

$$g_n(s) = f(s) + \sum_{m=1}^{n} \lambda^m \int_a^b K_m(s,t) f(t) dt$$ — (12)

we call the expression $K_m(s,t)$ with iterate for kernel where

$$K_1(s,t) = K(s,t)$$

Now passing limit $n \to \infty$ in (12), we get the so called neuman series

$$g(s) = \lim_{n\to\infty} g_n(s) = f(s) + \sum_{m=1}^{\infty} \lambda^m \int_a^b K_m(s,t)\, f(t) dt$$

Proof $K_m(s,t) = \int_a^b K_{m-1}(s,x)\, K(x,t)\, dx$

Sol try to prove for $m=2$ & $m=3$ first.

Next we try to find condition under which the series (13) converges uniformly for this consider

$$\int_a^b K_m(s,t) f(t)\, dt$$

and apply schwarz inequality.

$$\left| \int_a^b K_m(s,t) f(t)\, dt \right|^2 \leq \left(\int_a^b |K_m(s,t)|^2 dt \right) \left(\int_a^b |f(t)|^2 dt \right) \quad \text{— (14)}$$

Let 'D' be the norm of f.

$$D^2 = \int_a^b |f(t)|^2 dt \quad \text{— (15)}$$

and let

$$\int_a^b |K_m(s,t)|^2 dt \leq C_m^2 \quad \text{— (16)}$$

so, eq (14) becomes

$$\left| \int_a^b K_m(s,t) f(t)\, dt \right|^2 \leq C_m^2 D^2 \quad \text{— (17)}$$

here $D^2 = \int_a^b |f(t)|^2 dt$

and $\int_a^b |K_m(s,t)|^2 dt \leq C_m^2$

and we get

$$\left| \int_a^b K_m(s,t) f(t)\, dt \right|^2 \leq C_m^2 D^2$$

Date 8/2/23

Next we find relation b/w c_m^2 and c_1^2 for this, we consider $K_m(s,t)$

$$K_m(s,t) = \int_a^b K(s,x)\, K_{m-1}(x,t)\, dx$$

This may be written as

$$K_m(s,t) = \int_a^b K_{m-1}(s,x)\, K(x,t)\, dx$$

This can be seen as below

~~$K_3(s,t) = \int_a K_2(s,x)\, K(x,t)\, dx$~~

$$K_3(s,t) = \int_a^b K(s,\tau_1)\, K_2(\tau_1,t)\, d\tau_1$$

$$= \int_a^b K(s,\tau_1) \int_a^b K(\tau_1,\tau_2)\, K(\tau_2,t)\, d\tau_2\, d\tau_1$$

$$= \int_a^b K(\tau_2,t) \left(\int_a^b K(s,\tau_1)\, K(\tau_1,\tau_2)\, d\tau_1 \right) d\tau_2$$

$$K_3(s,t) = \int_a^b K(\tau_2,t)\, K_2(s,\tau_2)\, d\tau_2$$

$$= \int_a^b K_2(s,\tau_2)\, K(\tau_2,t)\, d\tau_2$$

$$= \int_a^b K_2(s,x)\, K(x,t)\, dx$$

In general

$$K_m(s,t) = \int_a^b K_r(s,x)\, K_{m-r}(x,t)\, dx$$

$$K_m(s,t) = \int_a^b K_{m-1}(s,x)\, K(x,t)\, dx$$

$m-1 = r$

$1 = m-r$

$$K_m(s,t) = \int_a^b K_r(s,x)\, K_{m-r}(x,t)\, dx$$

CamScanner

$$K_m(s,t) = \int_a^b K_{m-1}(s,x)\, K(x,t)\, dx$$

Apply Schwarz inequality

$$|K_m(s,t)|^2 \le \left[\int_a^b |K_{m-1}(s,x)|^2\, dx\right]\left[\int_a^b |K(x,t)|^2 dx\right]$$

$$\Rightarrow \int_a^b |K_m(s,t)|^2 dt \le \int_a^b |K_{m-1}(s,x)|^2 dx \int_a^b\int_a^b |K(x,t)|^2 dx\, dt \quad \text{— (A)}$$

no change in first term as not a function of t.

Assumption Important

i) $B^2 = \int_a^b \int_a^b |K(x,t)|^2 dx\, dt$

(ii) $\int_a^b |K_m(s,t)|^2 dt \le C_m^2$

(iii) $D^2 = \int_a^b |f(t)|^2 dt$

Taking supremium over s in (A)

$$C_m^2 \le B^2 C_{m-1}^2 \le B^2 B^2 C_{m-2}^2 \le B^2 B^2 B^2 C_{m-3}^2 = B^6 C_{m-3}^2$$

$m=1$ ~~$C_1^2 \le B^2 C_0^2 \le B^2$~~

$$\le B^{2(m-1)} C^2_{m-(m-1)} = B^{2(m-1)} C_1^2$$

so, $g_n(s) = f(s) + \sum_{m=1}^{n} \lambda^m \int_a^b K_m(s,t) f(t)\, dt$

and the general term of the above partial sum

$$\left|\lambda^m \int_a^b K_m(s,t) f(t)\, dt\right|^2 \le C_1^2 D^2 B^{2(m-1)} |\lambda|^{2m}$$

(from (4))

or $\left|\lambda^m \int_a^b K_m(s,t) f(t)\, dt\right| \le C_1 D B^{(m-1)} |\lambda|^m$

now from weierstrass M-test if $\boxed{|\lambda| B < 1}$ (Imp) — (21)
the infinite series on the r.h.s of (13) converges absolutely and uniformly. Thus the limit $g(s) = \lim_{n\to\infty} g_n(s)$ is solution of integral eq (1).

Next, we will show that solution is unique.

Proof
Uniqueness

Let $g_1(s)$ and $g_2(s)$ be two solutions then

$$g_1(s) = f(s) + \lambda \int_a^b K(s,t)\, g_1(t)\,dt \quad \text{—(22)}$$

$$g_2(s) = f(s) + \lambda \int_a^b K(s,t)\, g_2(t)\,dt \quad \text{—(23)}$$

now $Q(s) = g_1(s) - g_2(s)$ from (22) & (23)

$$Q(s) = \lambda \int_a^b K(s,t)\, Q(t)\, dt \quad \text{—(24)}$$

Applying schwartz inequality

$$|Q(s)|^2 \le |\lambda|^2 \left(\int_a^b |K(s,t)|^2 dt\right)\left(\int_a^b |Q(t)|^2 dt\right)$$

$$\Rightarrow \int_a^b |Q(s)|^2 ds \le |\lambda|^2 \int_a^b \int_a^b |K(s,t)|^2 dt\, ds \int_a^b |Q(t)|^2 dt$$

or $$\left(1 - |\lambda|^2 B^2\right) \int_a^b |Q(s)|^2 ds \le 0$$

as $|\lambda| B < 1$ we get $Q(s) = 0$ or $g_1(s) = g_2(s)$

note $B = \int_a^b \int_a^b |K(s,t)|^2 dt\, ds$ Imp. Hence unique solution

Estimation of the error : Next we estimate the error for neglecting terms after the n^{th} term in the neumann series.

Let

$$g(s) = f(s) + \sum_{m=1}^{n} \lambda^m \int_a^b K_m(s,t)\, f(t)\,dt + R_n(s)$$

or $$R_n(s) = \sum_{m=n+1}^{\infty} \lambda^m \int_a^b K_m(s,t)\, f(t)\,dt$$

$$|R_n(s)| \leq \left[\sum_{m\geq n+1} |\lambda|^m \left| \int_a^b K_m(s,t) f(t)\,dt \right| \right] \quad \text{— (25)}$$

from (19)

$$\left| \int K_m(s,t)\, f(t)\,dt \right|^2 \leq C_1^2 D^2 B^{2(m-1)}$$

now from (25)

$$|R_n(s)| \leq \sum_{m\geq n+1} |\lambda|^m C_1 D B^{m-1} \quad \text{— (26)}$$

or $$|R_n(s)| \leq |\lambda|^{n+1} C_1 D B^n + |\lambda|^{n+2} C_1 D B^{n+1} + \cdots$$

$$= \frac{|\lambda|^{n+1} D C_1 B^n}{1-|\lambda| B} \quad \left(\because \frac{a}{1-r}\right) \text{ sum of GP } \infty$$

$$|R_n(s)| \leq \frac{|\lambda|^{n+1} D C_1 B^n}{1-|\lambda| B} \quad \text{— (27)}$$

Date 9/2/23

now from

$$g(s) = f(s) + \sum_{m=1}^{\infty} \lambda^m \int_a^b K_m(s,t)\, f(t)\,dt$$

$$= f(s) + \lambda \int_a^b \left[\sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t) \right] f(t)\,dt$$

Take $$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t) \quad \text{— (28)}$$

gamma

then

$$g(s) = f(s) + \lambda \int_a^b \Gamma(s,t;\lambda) f(t)\, dt \qquad - (29)$$

The series (28) is ~~caught~~ convergent for $|\lambda| B < 1$. Hence the resolvent kernels is any analytic function of λ, regular at least inside the circle $|\lambda| < \frac{1}{B}$.

Uniqueness of Resolvent kernel: As the solution is unique, it is easy to prove that resolvent kernel is also unique. For this take for $\lambda = \lambda_0$, $\Gamma_1(s,t;\lambda_0)$ and $\Gamma_2(s,t;\lambda_0)$ or two resolvent kernels but solution is unique.

$$f(s) + \lambda_0 \int_a^b \Gamma_1(s,t;\lambda_0) f(t)\, dt = f(s) + \lambda_0 \int_a^b \Gamma_2(s,t;\lambda_0) f(t)\, dt$$

Take $\phi(s,t;\lambda_0) = \Gamma_1(s,t;\lambda_0) - \Gamma_2(s,t;\lambda_0)$

then we get

$$\int_a^b \phi(s,t;\lambda) f(t)\, dt = 0 \qquad - (30)$$

This is true for any arbitary function f so take
$f(t) = \phi^*$

$$\phi(s,t;\lambda_0) = 0 \Rightarrow \Gamma_1 = \Gamma_2$$

Imp

This establish the following basic theorem

Theorem: To each L_2-kernel $K(s,t)$ there corresponds a unique resolvent kernel $\Gamma(s,t;\lambda)$ which is an analytic function of λ, regular at least inside the circle $|\lambda| < B^{-1}$, and represented by the power series

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t)$$

furthermore if $f(s)$ is also an L_2- function, then the uniqueness L_2- Solution of the fredholm IE

$$g(s) = f(s) + \lambda \int_a^b K(s,t)\, g(t)\, dt$$

valid in the circle $|\lambda| < \frac{1}{B}$ is given by

$$g(s) = f(s) + \lambda \int_a^b \Gamma(s,t;\lambda) f(t)\, dt$$

Example Solve the IE

$$g(s) = f(s) + \lambda \int_0^1 \underbrace{e^{s-t}}_{K(s,t)} g(t)\, dt \quad \text{---} (1)$$

procedure

$$g_0(s) = f(s)$$

$$g_1(s) = f(s) + \lambda \int_a^b K(s,t)\, g_0(t)\, dt$$

$$g_2(s) = f(s) + \lambda \int_a^b K(s,t)\, g_1(t)\, dt$$

$$\vdots$$

$$g_n(s) = f(s) + \lambda \int_a^b K(s,t)\, g_{n-1}(t)\, dt$$

Simplifying $g_n(s)$ we get

$$g_n(s) = f(s) + \sum_{m=1}^{n} \lambda^{m} \int_a^b K_m(s,t) f(t)\, dt$$

and $g(s) = \lim_{n\to\infty} g_n(s)$ or

where

$$K_m(s,t) = \int_a^b K(s,x) K_{m-1}(x,t)\, dx$$

with $K_1(s,t) = K(s,t)$

equilibrium (2) can be written as

$$g(s) = f(s) + \lambda \int_a^b \left(\sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t) \right) f(t)\, dt \quad \text{---} (3)$$

by method of successive approx

Take $\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t)$ — (4)

then from (3)

$$g(s) = f(s) + \lambda \int_a^b \Gamma(s,t;\lambda) f(t)dt \quad \text{— (5)}$$

now for the ans

$K_1(s,t) = e^{s-t}$

$K_2(s,t) = \int_0^1 (e^{s-x})(e^{x-t})dt = \int_0^1 (e^{s-t})dt$

$= \int_0^1 \cancel{e^{(s-t)}}dt =$

so, $K_1(s,t) = K_2(s,t) = K_3(s,t)$ — — $K_m(s,t)$

$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t) = \sum_{m=1}^{\infty} \lambda^{m-1} e^{s-t} = e^{s-t} \sum_{m=1}^{\infty} \lambda^{m-1}$

$= e^{s-t}(1+\lambda+\lambda^2 —)$

$= e^{s-t}\left(\frac{1}{1-\lambda}\right)$ if $(|\lambda|<1)$

So, $\lambda = 1$ is a simple pole of the kernel. so, we cannot talk about solution for $\lambda=1$

then solution is

$$g(s) = f(s) + \frac{\lambda}{1-\lambda} \int_0^1 e^{s-t} f(t)dt$$

Example-2 Solve F.I.

Book	RP Kanwal	Assignment
This chapter	Exercises	1 to 6
Prev. chapter		1 to 5

$$g(s) = 1 + \lambda \int_0^1 (1-3st)\, g(t)\, dt$$

and evaluate the resolvent kernel.

Sol

$$g_0(s) = f(s) = 1$$

$$g_1(s) = 1 + \lambda \int_0^1 (1-3st) \cdot 1 \cdot dt$$

$$= 1 + \lambda \left(t - \frac{3st^2}{2} \right)\Big|_0^1$$

$$= 1 + \lambda \left(1 - \frac{3s}{2} \right)$$

$$g_2 = 1 + \lambda \int_0^1 (1-3st) \left[1 + \lambda\left(1 - \frac{3s}{2}\right) \right] dt$$

$$= 1 + \lambda \left(\int_0^1 (1 - 3st) + \lambda\left(1 - \frac{3s}{2} - 3st + \frac{9s^2t}{2}\right) dt \right)$$

$$= 1 + \lambda \left[t - \frac{3st^2}{2} + \lambda\left(t - \frac{3st}{2} - \frac{3st^2}{2} + \frac{9}{2}\frac{s^2t^2}{2} \right) \right]_0^1$$

$$= 1 + \lambda \left[1 - \frac{3s}{2} + \lambda\left(1 - \frac{3s}{2} - \frac{3s}{2} + \frac{9s^2}{4}\right) \right]$$

$$= 1 + \lambda \left[1 - \frac{3s}{2} + \lambda\left(1 - 3s + \frac{9}{4}s^2\right) \right]$$

$$g_2 = 1 + \lambda - \frac{3s}{2}\lambda + \lambda^2 - 3\lambda^2 s + \frac{9\lambda^2 s^2}{4}$$

Ans.

$$g(s) = \frac{4 + 2\lambda(2-3s)}{4-\lambda^2}$$

as

$$g(s) = \left[1 + \lambda\left(1 - \frac{3}{2}s\right)\right] \left[1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{16} + \cdots \right]$$

$$= \frac{1 + \lambda\left(1 - \frac{3}{2}s\right)}{1 - \frac{\lambda^2}{4}}$$

valid for $|\lambda| < 2$

CamScanner

Date
1/3/23

Solve the IE Method of successive approximation

$$g(s) = 1 + \lambda \int_0^{\pi} \sin(s+t)\, g(t)\, dt$$

Sol

$$K_1(s,t) = \sin(s+t)$$

$$K_2(s,t) = \int_0^{\pi} \sin(s+x)\sin(x+t)\, dx$$

$$= \int_0^{\pi} K(s,x)\, K_1(x,t)\, dx$$

$$= \int_0^{\pi} (\sin s \cos x + \cos s \sin x)(\sin x \cos t + \cos x \sin t)\, dx$$

$$K_3(s,t) = \int_0^{\pi} K(s,x)\, K_2(x,t)\, dx$$

$$K_2 = \frac{\pi}{2} \cos(s-t)$$

$$K_3 = \int_0^{\pi} (\sin s \cos x + \cos s \sin x)\left(\frac{\pi}{2} \cos(x-t)\right) dx$$

$$= \left(\frac{\pi}{2}\right)^2 \sin(s+t)$$

$$K_4 = \int_0^{\pi} \sin(s+x) \left(\frac{\pi}{2}\right)^2 \sin(x+t)\, dx$$

$$= \left(\frac{\pi}{2}\right)^2 \left(\frac{\pi}{2}\right) \cos(s-t)$$

$$K_4 = \left(\frac{\pi}{2}\right)^3 \cos(s-t)$$

$$K_5 = \left(\frac{\pi}{2}\right)^4 \sin(s+t)$$

$$\vdots$$

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t) = K_1(s,t) + \lambda K_2(s,t) + \lambda^2 K_3(s,t) + \cdots$$

$$= \sin(s+t) + \lambda\frac{\pi}{2}\cos(s-t) + \lambda^2\left(\frac{\pi}{2}\right)^2\sin(s+t) + \lambda^3\left(\frac{\pi}{2}\right)^3\cos(s-t)$$

$$+ - - -$$

$$= \left[1 + \left(\frac{\lambda\pi}{2}\right)^2 + \left(\frac{\lambda\pi}{2}\right)^4 - - - -\right]\sin(s+t) +$$

$$+ \left[\frac{\lambda\pi}{2} + \left(\frac{\lambda\pi}{2}\right)^3 + \left(\frac{\lambda\pi}{2}\right)^5 + - -\right]\cos(s-t)$$

$$= \frac{\sin(s+t)}{1-\left(\frac{\lambda\pi}{2}\right)^2} + \frac{\frac{\lambda\pi}{2}\cos(s-t)}{1-\left(\frac{\lambda\pi}{2}\right)^2}$$

$$\left|\frac{\lambda\pi}{2}\right| < 1$$

$$\boxed{\lambda < \frac{2}{\pi}}$$

$$\Gamma(s,t;\lambda) = \frac{\sin(s+t) + \frac{\lambda\pi}{2}\cos(s-t)}{1-\left(\frac{\lambda\pi}{2}\right)^2}$$

$$g(s) = 1 + \lambda\int_0^{\pi}\Gamma(s,t;\lambda)\,f(t)\,dt \quad , \quad f(t) = 1$$

$$= 1 + \frac{\lambda}{1-\left(\frac{\lambda\pi}{2}\right)^2}\int_0^{\pi}\Big[\underbrace{\sin(s+t)}_{2\sin s} + \frac{\lambda\pi}{2}\underbrace{\cos(s-t)}_{2\cos s}\Big]dt$$

$$\cancel{= 1 + \frac{\lambda}{1-\left(\frac{\lambda\pi}{2}\right)^2}\left[2\sin s + \frac{\lambda\pi}{2}2\cos s\right]}$$

$$\int_0^{\pi}\sin(s+t)\,dt = 2\cos s$$

and $\int_0^{\pi}\cos(s-t) = 2\sin s$

So, $$g(s) = 1 + \frac{\lambda}{1-\left(\frac{\lambda\pi}{2}\right)^2}\left[2\cos s + \frac{\lambda\pi}{2}(2\sin s)\right]$$

$$g(s) = 1 + \frac{\lambda}{1-\left(\frac{\lambda\pi}{2}\right)^2}\left(2\cos s + \lambda\pi \sin s\right)$$

$$g(s) = 1 + \frac{2\lambda\cos s + \lambda^2\pi \sin s}{1-\left(\frac{\lambda\pi}{2}\right)^2}$$

note $|\lambda B| < 1$, see B from notes

$$B^2 = \int_0^{\pi}\int_0^{\pi} \sin^2(s+t)\,ds\,dt = \frac{\pi^2}{2}$$

$$\left|\lambda \frac{\pi}{\sqrt{2}}\right| < 1 \qquad \text{so,}$$

so, the interval of convergence of series lies b/w $-\frac{\sqrt{2}}{\pi}$ and $\frac{\sqrt{2}}{\pi}$

Solving for Volterra Equation of 2nd kind :

$$g(s) = f(s) + \lambda\int_a^s K(s,t)\, g(t)\,dt \qquad \text{— (1)}$$

The formula corresponding to fredholm IE is given by

$$g(s) = f(s) + \sum_{m=1}^{\infty} \lambda^m \int_a^s K_m(s,t)\, f(t)\,dt \qquad \text{— (2)}$$

$$g_0(s) = f(s)$$

$$g_1(s) = f(s) + \lambda\int_a^s K(s,t)\, g_0(t)\,dt$$

$$g_2(s) = f(s) + \lambda\int_a^s K(s,t)\, g_1(t)\,dt$$

$$\vdots$$

$$g_m(s) = f(s) + \lambda\int_a^s K(s,t)\, g_{m-1}(t)\,dt$$

So,
$$g_1(s) = f(s) + \lambda \int_a^s K(s,t) f(s) dt$$
$$g_2(s) = f(s) + \lambda \int_a^s K(s,t) \left[f(t) + \cancel{\lambda \int_a^s K(t,x) f(x) dx} + \lambda \int_a^s K(t,x) g_0(x) dx \right] dt$$

$$\Rightarrow g_2(s) = f(s) + \lambda \int_a^s K(s,t) f(t) dt + \lambda^2 \int_a^s \left(\int_a^s K(s,t) K(t,x) g_0(x) dx \right) dt$$

complete the derivation from here

$$K_m(s,t) = \int_t^s K(s,x) K_{m-1}(x,t) dx \quad — (4)$$

with $K_1(s,t) = K(s,t)$. The resolvent kernel $\Gamma(s,t;\lambda)$ given by
$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t) \quad — (5)$$

~~$g(s) = f(s)$~~

and it is an entire function (analytic function in the entire λ-plane) of λ for any given (s,t).

Q! Find the neumann series for the solution of the IE (volterra

$$g(s) = (1+s) + \lambda \int_0^s (s-t)\, g(t)\, dt$$

Sol

$$K_1(s,t) = (s-t)$$

$$K_2 = \int_t^s K(s,x)\, K_1(x,t)\, dx$$

$$= \int_t^s (s-x)(x-t)\, dt$$

$$= \int_t^s (sx - st - x^2 + xt)\, dx \quad = \frac{(x-t)^3}{3!}\Big|_t^s$$

$$= \cancel{sxt\Big|_t^s - \frac{st^2}{2}\Big|_t^s - x^2t\Big|_t^s + \frac{xt^2}{2}\Big|_t^s}$$

$$= \cancel{(sx)(s-t) - \frac{s}{2}(s^2-t^2) - x^2(s-t)}$$

$$\cancel{+ \frac{x}{2}(s^2-t^2)}$$

$$= \frac{(s-t)^3}{3!} = \frac{(s-t)^3}{6}$$

$$K_3 = \frac{(s-t)^5}{5!}$$

now from

$$g(s) = f(s) + \sum_{m=1}^{\infty} \lambda^m \int_0^s K_m(s,t)\, f(t)\, dt$$

$$= 1 + s + \sum_{m=1}^{\infty} \lambda^m \int_0^s K_m(s,t)(1+t)\, dt$$

neumann series solution $\Rightarrow$ $g(s) = 1 + s + \lambda\left(\frac{s^2}{2!} + \frac{s^3}{3!}\right) + \lambda^2\left(\frac{s^4}{4!} + \frac{s^5}{5!}\right) + \cdots$

note if $\lambda = 1$

$g(s) = e^s$

$$\lambda \int_0^s (s-t)(1+t)\, dt = \lambda \int_0^s (s + st - t - t^2)\, dt$$

$$= \lambda\left[st\Big|_0^s + \frac{st^2}{2}\Big|_0^s - \frac{t^2}{2}\Big|_0^s - \frac{t^3}{3}\Big|_0^s \right]$$

$$= \lambda\left[s^2 + \frac{s^3}{2} - \frac{s^2}{2} - \frac{s^3}{3} \right] = \frac{\lambda s^2}{2} + \frac{s^3}{6}$$

Date 2/3/23

Q2 Solve the IE

$$g(s) = f(s) + \lambda \int_{t}^{s} e^{s-t} g(t)\,dt$$

$$K_1(s,t) = e^{s-t} = (e^s)(e^{-t})$$

$$K_2(s,t) = \int_t^s (e^{s-x})(e^{x-t})\,dx$$

$$= \int_t^s e^{s-t}\,dx$$

$$K_2(s,t) = (s-t)(e^{s-t})$$

$$K_3(s,t) = \int_t^s (e^{s-x})((x-t)e^{x-t})\,dx$$

$$= \int_t^s (x-t)e^{s-t}\,dx$$

$$K_3 = \frac{(s-t)^2}{2!} e^{s-t}$$

$$= \left| \frac{x^2}{2} - tx \right|_t^s$$

$$\left(\frac{s^2}{2} - st\right) + \left(\frac{t^2}{2}\right)$$

$$= \frac{(s-t)^2}{2} \qquad \frac{s^2 - 2st + t^2}{2}$$

$$K_4 = \frac{(s-t)^3}{3!} e^{s-t}$$

So

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t)$$

$$= (e^{s-t}) + \lambda (s-t) e^{s-t} + \lambda^2 \frac{(s-t)^2}{2!} e^{s-t} + \lambda^3 \frac{(s-t)^3}{3!} e^{s-t}$$

$$= (e^{s-t}) \left[1 + \lambda(s-t) + \lambda^2\frac{(s-t)^2}{2!} + \lambda^3\frac{(s-t)^3}{3!} - \cdots \right]$$

$$= (e^{s-t}) \left[e^{\lambda(s-t)} \right]$$

$$\boxed{\Gamma = e^{(s-t)(1+\lambda)}}$$

$$\Gamma(s,t;\lambda) = \begin{cases} e^{(\lambda+1)(s-t)} & ; \quad t \le s \\ 0 & ; \quad t > s \end{cases}$$

Solution is

$$g(s) = f(s) + \lambda \int_0^s \underbrace{e^{(\lambda+1)(s-t)}}_{\Gamma(s,t;\lambda)} f(t)\, dt \quad ; \quad t \le s$$

$$g(s) = f(s) \quad ; \quad t > s$$

Some Results about Resolvent Kernel

$|\lambda B| < 1$

1. The series for the resolvent kernel $\Gamma(s,t;\lambda)$

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t) \quad \text{— (1)}$$

can be proved to be absolutely and uniformly convergent for all values of s and t in the circle $|\lambda| \le \frac{1}{B}$.

In addition to the assumptions, take following assumptions

(i) $D^2 = \int_a^b |f(t)|^2 dt$

(ii) C_m^2 denote the upper bound of the integral

$$C_m^2 = \int_a^b |K_m(s,t)|^2 dt$$

(iii) $|\lambda B| < 1$, where

$$B^2 = \int_a^b \int_a^b |K(s,t)|^2 ds\, dt$$

(iv) $\int_a^b |K(s,t)|^2 ds \le E^2$ (E = constant)

Proof

$$K_m(s,t) = \int_a^b K(x,t) K_{m-1}(s,x)\, dx = \int_a^b K(x,t) K_{m-1}(s,x)\, dx$$

Cauchy - Schwartz inequality

$$|K_m(s,t)|^2 \le \int_a^b |K_{m-1}(s,x)|^2 dx \cdot \int_a^b |K(x,t)|^2 dx$$

$$|K_m(s,t)|^2 \le C_{m-1}^2 \cdot E^2 \quad \text{— (2)}$$

But $C_m^2 \le B^{2(m-1)} C_1^2$ — (3) (we have proved it pr..

method of successive approximation

Thus

$$|K_m(s,t)|^2 \le B^{2(m-1)} C_1^2 E^2$$

or $$|K_m(s,t)| \le B^{m-1} C_1 E \quad — (4)$$

Thus the series is dominated by the geometric series with general term as $C_1 E(\lambda^{m-1} B^{m-1})$ and as $|\lambda|B < 1$ we get that the series (1) is uniformly and absolutely convergent.

$|\lambda B| < 1$ Important

2. The resolvent kernel satisfies the following IE

To prove $$\Gamma(s,t;\lambda) = K(s,t) + \lambda \int_a^b \Gamma(s,x;\lambda) K(x,t)\,dx \quad — (5)$$

Given and $$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t) \quad — (6)$$

and $$K_m(s,t) = \int_a^b K_{m-1}(s,x) K(x,t)\,dt \quad — (7)$$

Proof

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} \int_a^b K_{m-1}(s,x) K(x,t)\,dt$$

$$= \int_a^b K_0(s,x) K(x,t)\,dt + \lambda \int_a^b K_1(s,x) K(x,t)\,dt + \left[\lambda^2 \int_a^b K_2(s,x) K(x,t)\,dt + \cdots\right]$$

$$= \sum_{m=1}^{\infty} \lambda^{m-1} K_{m-1}(s,x) \int_a^b K(x,t)\,dt$$

$=$

prev)

Sir

$$\Gamma(s,t;\lambda) = K_1(s,t) + \sum_{m=2} \lambda^{m-1} K_m(s,t)$$

or. $$\Gamma(s,t;\lambda) = K(s,t) + \lambda \sum_{m=2}^{\infty} \lambda^{m-2} K_m(s,t)$$

$$= K(s,t) + \lambda \sum_{m=1}^{\infty} \lambda^{m-1} K_{m+1}(s,t)$$

See proof in Book

Problem — Book Problems — Exercise 1 to 6

Q1 Solve the following Fredholm IE by Method of successive Appro

(i) $$g(s) = e^s - \frac{1}{2}e + \frac{1}{2} + \frac{1}{2}\int_0^1 g(t)dt$$

(ii) $$g(s) = (\sin s) - \frac{s}{4} + \frac{1}{4}\int_0^{\pi/2} st\,g(t)dt$$

Sol (i) $K_1 = 1$

$$K_2 = \int_t^s (1)(1)\,dx = x\Big|_t^s = (s-t)$$

$$K_3 = \int_t^s (1)(x-t)\,dx$$

$$= \frac{x^2}{2} - tx\Big|_t^s = \left(\frac{s^2}{2} - st\right) - \left(\frac{t^2}{2} - t^2\right)$$

$$= \frac{s^2}{2} - st + \frac{t^2}{2} =$$

$$= \left(\frac{s-t}{\sqrt{2}}\right)^2 = \frac{(s-t)^2}{2}$$

5)

7)

) dt

Date 6/3/23

$\frac{\partial}{\partial}\cancel{f(x)dx} = \cancel{f(a)} + \cancel{f'(a)(x) + \frac{f''(x)(b-a)^2}{2!}}$

Non-Separable Kernels

Consider the Fredholm IE

$$g(s) = f(s) + \lambda \int_a^b K(s,t)g(t)\,dt \quad \text{—— (1)}$$

Here we will reduce eq (1) to system of algebraic equations as it is done for separable kernels but we will not put the ~~resist~~ restriction that $K(s,t)$ is separable.

Divide the interval (a,b) into n equal parts,

$$s_1 = t_1 = a, \quad s_2 = t_2 = a+h, \ldots$$
$$s_n = t_n = a+(n-1)h$$

where $h = \frac{b-a}{n}$

Now the integral

$$\int_a^b K(s,t)g(t)\,dt \cong h \sum_{j=1}^{n} K(s, s_j)\, g(s_j) \quad \text{—— (2)}$$

for $\forall\ s \in (a,b)$

Writing $f(s_i) = f_i$, $g(s_i) = g_i$ and $K(s_i, s_j) = K_{ij}$

and ~~the eqn~~

So, equation (1) takes form

$$g(s) \simeq f(s) + \lambda h \sum_{j=1}^{n} K(s, s_j)\, g(s_j) \quad \forall\ s \in (a,b)$$

and the equation (3) at $s = s_i$, we get the system of n-linear equations of n unknowns $g_1, g_2, \ldots g_n$

$$g_i - \lambda h \sum_{j=1}^{n} K_{ij} g_j = f_i \quad , i = 1(1)n \quad \text{—— (4)}$$

So, In matrix form

$$g_1 - \lambda h\,(K_{11} g_1 + K_{12} g_2 + K_{13} g_3 - \cdots K_{1n} g_n) = f_1$$
$$g_2 - \lambda h\,(K_{21} g_1 + K_{22} g_2 + \cdots\cdots - K_{2n} g_n) = f_2$$

so $(1-\lambda hk_{11})g_1 - \lambda h k_{12} g_2 - \lambda h k_{13} g_3 - - \cdot$

so
$$\begin{bmatrix} 1-\lambda h k_{11} & -\lambda h k_{12} & -\lambda h k_{13} & - & -\lambda h k_{1n} \\ -\lambda h k_{21} & 1-\lambda h k_{22} & -\lambda h k_{23} & - & -\lambda h k_{2n} \\ \vdots & & & & \\ -\lambda h k_{n1} & - & & - & 1-\lambda h k_{nn} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

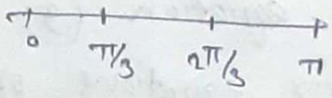
$$\underbrace{}_{D_n(\lambda)}$$

From the system of algebraic eqn (4) solution I E (1) can be obtained at discrete points $s = s_i$, $i = 1(1)n$ and actual solution can be obtained by interpolation. With the help of this algebraic system, we can also determine approximation for the eigen values of the kernel. The resolvent determinant of the algebraic system is $D_n(\lambda)$. To obtain approximate eigen values set $D_n(\lambda) = 0$.

Example

$$g(s) - \lambda \int_0^{\pi} \sin(s+t)\, g(t)\, dt = 0$$

Take $n = 3$ $\quad h = \frac{\pi}{3}$



$k(s,t) = \sin(s+t)$

~~$k_{11} = \sin(0+0)$~~

$s_1 = t_1 = 0$, $\quad s_2 = t_2 = \pi/3$, $\quad s_3 = t_2 = 2\pi/3$

$$K_{ij} = \begin{vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{vmatrix}$$

$K_{ij} = K(s_i, s_j) = \sin(s_i + s_j)$

$K_{11} = \sin(s_1 + s_1) = 0$

$K_{12} = \sin(s_1 + s_2) = \sin(0 + \pi/3) = \sin \pi/3 = 0.866025$

$K_{13} = \sin(s_1 + s_3) = \sin(0 + 2\pi/3) = \sin(2\pi/3) = 0.866025$

$K_{21} = \sin(s_2 + s_1) = 0.866025 = \sin(\pi/3)$

$K_{22} = \sin(s_2 + s_2) = \sin(2\pi/3) = 0.866025$

$K_{23} = \sin(s_2 + s_3) = \sin(\pi) = 0$

$K_{31} = \sin(s_3 + s_1) = \sin(2\pi/3) = 0.866025$

$K_{32} = \sin(s_3 + s_2) = \sin(2\pi/3 + \pi/3) = \sin \pi = 0$

$K_{33} = \sin(s_3 + s_3) = \sin(4\pi/3) = \sin(\pi + \pi/3)$

$= -0.866025$

$$K_{ij} = \begin{vmatrix} 0 & 0.866 & 0.866 \\ 0.866 & 0.866 & 0 \\ 0.866 & 0 & -0.866 \end{vmatrix}$$

$h = \pi/3$

$$D_n(\lambda) = \begin{vmatrix} 1 & -0.907\lambda & -0.907\lambda \\ -0.907\lambda & (1 - 0.907\lambda) & 0 \\ -0.907\lambda & 0 & (1 + 0.907\lambda) \end{vmatrix}$$

$0.866 \times \frac{\pi}{3}$

$= 0.906873$

For eigen values $D_n(\lambda) = 0$

$D_n(\lambda) = 1 - 3(0.907\lambda)^2$

$\lambda = \frac{1}{(\sqrt{3})(0.907)}$ and $\frac{-1}{(\sqrt{3})(0.907)}$

CamScanner

$$\boxed{\lambda = \pm\ 0.63634935}$$

The exact eigen values are $\lambda = \pm\sqrt{\frac{2}{\pi}} = \pm\ 0.6364$ (given just for comparison).

Classical Fredholm theory

Fredholm's first Theorem: The inhomogeneous FIE

$$g(s) = f(s) + \lambda \int_a^b K(s,t)\, g(t)\, dt \qquad \text{--- (1)}$$

where $f(s)$ and $g(s)$ are integrable, has unique solution

$$g(s) = f(s) + \lambda \int_a^b \Gamma(s,t;\lambda)\, f(t)\, dt \qquad \text{--- (2)}$$

where the resolvent kernel $\Gamma(s,t;\lambda)$

$$\Gamma(s,t;\lambda) = \frac{D(s,t;\lambda)}{D(\lambda)} \qquad \text{--- (3)}$$

where

$$D(s,t;\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \lambda^n B_n(s,t) \qquad \text{--- (4)}$$

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} c_n \lambda^n \qquad \text{--- (5)}$$

$$B_0(s,t) = K(s,t)$$

$$B_n(s,t) = \int_a^b \cdots \int_a^b \begin{vmatrix} K(s,t) & K(s,y_1) & \cdots & K(s,y_n) \\ K(y_1,t) & K(y_1,y_1) & \cdots & K(y_1,y_n) \\ \vdots & \vdots & & \vdots \\ K(y_n,t) & K(y_n,y_1) & \cdots & K(y_n,y_n) \end{vmatrix} dy_1 \cdots dy_n \qquad \text{--- (6)}$$

$$\boxed{c_0 = 1} \qquad \text{--- (7)}$$

$$C_n = \int_a^b \cdots \int_a^b \begin{vmatrix} K(y_1,y_1) & K(y_1,y_2) & \cdots & K(y_1,y_n) \\ K(y_2,y_1) & K(y_2,y_2) & \cdots & K(y_2,y_n) \\ \vdots & & & \\ K(y_n,y_1) & \cdots & \cdots & K(y_n,y_n) \end{vmatrix} dy_1\, dy_2 \cdots dy_n \quad \text{--- (8)}$$

$D(\lambda)$ is called Fredholm determinant and $D(s,t;\lambda)$ the first fredholm minor. If the integral

$$\int_a^b \int_a^b |K(s,t)|^2 \, ds\, dt < \infty \quad \text{--- (9)}$$

Then the series (5) for the fredholm determinant converges for all λ, so that $D(\lambda)$ is an entire function of λ. Thus (a) - (5) defines the resolvent on the whole complex plane λ, with an exception of zeros of $D(\lambda)$ which are poles of the resolvent

Note1: for practical computation of the resolvent, one can use following recurrent relationships:

$$B_0(s,t) = K(s,t) \quad , \quad c_0 = 1 \quad \text{--- (10)}$$

$$c_n = \int_a^b B_{n-1}(s,s)\, ds \quad , \quad n = 1, 2, \quad \text{--- (11)}$$

$$B_n(s,t) = c_n K(s,t) - n \int_a^b K(s,x)\, B_{n-1}(x,t)\, dx \quad \text{--- (12)}$$

$$n = 1, 2, 3, \ldots$$

so
$c_0 = 1$ is given and B_0 is $K(s,t)$

now
$$c_1 = \int_a^b B_0(s,s)\, ds$$

then
$$B_1 = c_1 K(s,t) - \int_a^b K(s,x)\, B_0(x,t)\, dx$$

now repeat
$$c_2 = \int_a^b B_1(s,s)\, ds$$

and
$$B_2 = c_2 - 2 \int_a^b K(s,x)\, B_1(x,t)\, dx$$

Note 2: The solution of homogeneous IE

$$g(s) = \lambda \int_a^b K(s,t)\, g(t)\, dt$$

is identically zero.

eg-1 Evaluate the resolvent for the IE

$$g(s) = f(s) + \lambda \int_0^1 (s+t)\, g(t)\, dt$$

Sol

$C_0 = 1, \quad B_0 = s+t$

$$C_1 = \int_0^1 (2s)\, ds = \frac{2s^2}{2}\Big|_0^1 = 1$$

$$B_1 = 1 - \int_0^1 (s+x)(x+t)\, dx$$

$$= 1 - \int_0^1 (sx + st + x^2 + xt)\, dx$$

$$= 1 - \left(\frac{sx^2}{2} + stx + \frac{x^3}{3} + \frac{x^2}{2}t\right)\Big|_0^1$$

$$= 1 - \left[\frac{s}{2} + st + \frac{1}{3} + \frac{t}{2}\right]$$

$$= \frac{2}{3} - \frac{s}{2} - st - \frac{t}{2} \quad ? \qquad \text{Sir } \left(\frac{1}{3} - \frac{s}{2} - st - \frac{t}{2}\right)$$

$$C_2 = \int_a^b \left(\frac{2}{3} - \frac{s}{2} - s^2 - \frac{s}{2}\right) ds$$

$$= \int_0^1 \left(\frac{2}{3} - s - s^2\right) ds$$

$$= \left(\frac{2s}{3} - \frac{s^2}{2} - \frac{s^3}{3}\right)\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{2} - \frac{1}{3} = \quad \text{Sir: } -\frac{1}{6}$$

$$B_2 = C_2 - 2\int_0^1 (s+x)\left(\frac{1}{3} - \frac{x}{2} - xt - \frac{t}{2}\right) dx$$

$$= -\frac{1}{6} - 2\left[\int_0^1 \frac{s}{3} - \frac{xs}{2} - xts - \frac{st}{2} + \frac{x}{3} - \frac{x^2}{2} - x^2t - \frac{xt}{2}\right] dx$$

$$B_2 = -\frac{1}{6}K(s,t) - 2\left[\frac{sx}{3} - \frac{x^2}{4}s - \frac{x^2 t s}{2} - \frac{st x}{2} + \frac{x^2}{6} - \frac{x^3}{6} - \frac{x^3 t}{3} - \frac{x^2 t}{4}\right]_0^1$$

$$B_2 = 0 \quad \text{sir}$$

So, $C_3 = 0 \qquad B_3 = 0$ ...

So,
$$D(s,t;\lambda) = \frac{-1}{1!}(\lambda)B_1(s,t) + \frac{1}{2!}(\lambda^2)B_2(s,t)$$

$$= (s+t) - \frac{\lambda}{1}\left(\frac{1}{3} - \frac{s}{2} - \frac{t}{2} - st\right) \quad \text{sir} \quad \boxed{(s+t) - \left[\frac{1}{2}(s+t) - st - \frac{1}{3}\right]\lambda}$$

$$D(\lambda) = 1 - \frac{1}{1!}C_1\lambda + \frac{1}{2!}C_2\lambda^2$$

$$= 1 - \lambda + \frac{1}{2}\left(-\frac{1}{6}\right)\lambda^2$$

$$= \boxed{1 - \lambda - \frac{\lambda^2}{12}} \quad \text{sir}$$

So,
$$\Gamma(s,t;\lambda) = \frac{-\lambda\left(\frac{-1}{3} + \frac{s}{2} + \frac{t}{2} - st\right) + (s+t)}{1 - \lambda - \frac{\lambda^2}{12}}$$

$$\cancel{\Gamma(s,t;\lambda) = \frac{\left(\frac{1}{3} - \frac{(s+t)}{2} - st\right)}{\left(1 + \frac{\lambda}{12}\right)}}$$

$$= \frac{(s+t) - \lambda\left[\frac{(s+t)}{2} - \frac{1}{3} - st\right]}{1 - \lambda - \frac{\lambda^2}{12}}$$

eg 2 $$g(s) = 1 + \lambda \int_0^{\pi} \sin(s+t)\, g(t)\, dt$$

sol

$$C_0 = 1 \quad , \qquad B_0 = \sin(s+t)$$

$$c_1 = \int_a^b \sin(2s)\, ds = \frac{\cos 2s}{2}\Big|_0^{\pi} = \frac{(\cos 2\pi - \cos 0)}{2}$$

$$= 0$$

$$B_1 = -\int_a^b \sin(s+x)\sin(x+t)\, dx$$

$$= -\int_a^b (\sin s\cos x + \cos s\sin x)(\sin x\cos t + \cos x\sin t)\, dx$$

$$= -\int_0^{\pi} \sin s\cos t\,\sin x\cos x\, dx - \int_0^{\pi} \sin s\sin t\cos^2 x\, dx$$

$$- \int_0^{\pi} \cos s\cos t\sin^2 x\, dx - \int_0^{\pi} \cos s\sin t\,\sin x\cos x\, dx$$

$$= -(\sin s\cos t + \cos s\sin t)\int_0^{\pi} \sin x\cos x\, dx$$

$$- \sin s\sin t\left(\int_0^{\pi} \cos^2 x\, dx\right) - (\cos s\sin t)\int_0^{\pi} \sin^2 x\, dx$$

$$=$$

Fredholm 2nd Theorem

If λ_0 is a zero of multiplicity m of the function $D(\lambda)$, then the homogeneous equation

$$g(s) = \lambda_0 \int_a^b K(s,t)\, g(t)\, dt$$

possesses at least one and at most m linearly independent solutions not identically zero. Any other solution of this equation is linear combination of these solution.

Date 13/3/23

Fredholm 3rd Theorem : For an inhomogeneous equation

$$g(s) = f(s) + \lambda_0 \int_a^b K(s,t)\, g(t)\, dt$$

to possess a solution in the case $D(\lambda_0) = 0$, it is necessary and sufficient that the given function $f(s)$ be orthogonal to all the eigen functions of the transposed homogeneous equations corresponding to the eigen value λ_0.

Prob : Find the resolvent kernels for the following integral equations

1) $g(s) = f(s) + \lambda \int_0^1 |s-t|\, g(t)\, dt$

2) $g(s) = f(s) + \lambda \int_0^1 \exp(-(s-t))\, g(t)\, dt$

3) $g(s) = f(s) + \lambda \int_0^{2\pi} \cos(s+t)\, g(t)\, dt$

Applications to ODE

Consider the IVP

$$y'' + A(s)\,y' + B(s)\,y = F(s) \quad — (1)$$

$$y(a) = q_0, \qquad y'(a) = q_1 \quad — (2)$$

where the functions A, B and f are defined and continous in the closed interval $a \le s \le b$.

Integrate (1) from $a \to s$

$$y'(s) - q_1 = -A(s)y(s) + q_0 A(a) - \int_a^s [B(t) - A'(t)]\, y(t)\,dt + \int_a^s F(t)\,dt \quad — (3)$$

we are using integration by parts in $\int A(s)\,y'\,ds$

8-r

$$y'(s) - y'(a) + \int_0^s A(s)\,y'(s)\,ds + \int_a^s B(s)\,y(s)\,ds = \int_a^s F(s)\,ds$$

$$\Rightarrow y'(s) - y'(a) + A(s)y(s)\Big|_a^s - \int_a^s A'(s)\,y(s)\,ds + \int_a^s B(s)\,y(s)\,ds = \int_a^s F(s)\,ds$$

$$\Rightarrow y'(s) - q_1 + A(s)y(s) - A(a)q_0 + \int_a^s (B(s) - A'(s))\,y(s)\,ds = \int_a^s F(s)\,ds$$

$$\Rightarrow y'(s) - q_1 = -A(s)y(s) + A(a)q_0 - \int_a^s [B(s) - A'(s)]\,y(s)\,ds + \int_a^s F(s)\,ds$$

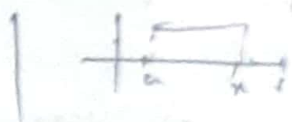
~~now~~ now integrate eqn (3) again from a to s

$$y(s) - q_0 - q_1(s-a) = -\int_a^s A(s)y(s)\,ds + q_0 A(a)(s-a) - \int_a^s\int_a^x [B(t) - A'(t)]y(t)\,dt\,dx + \int_a^s\left(\int_a^x F(t)\,dt\right)dx$$

now change the order of integral in last 2 terms

$$y(s) - q_0 - q_1(s-a) = -\int_a^s A(s)y(s)\,ds + q_0 A(a)(s-a) - \int_a^s\int_a^x [B(t) - A'(t)]y(t)\,dt\,dx + \int_a^s F(t)\left(\int_t^s dx\right)dt$$

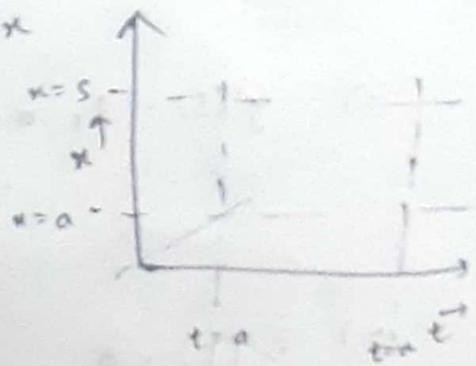
$$+ \cancel{\left(\int_a F(t)\,dt\right)(s-a)}$$

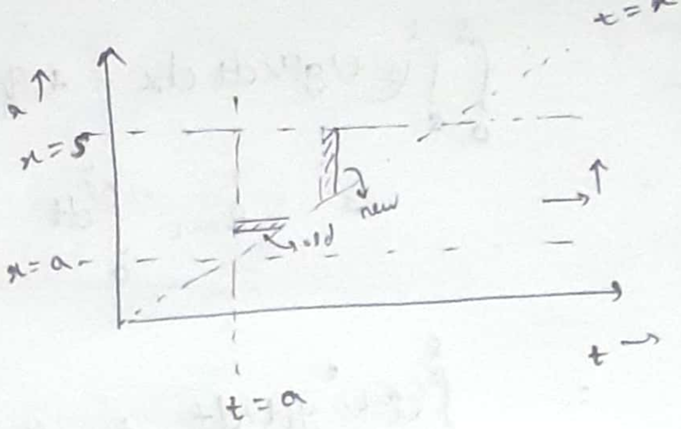


as for the term $\int_a^s \int_a^x (F(t)\,dt)\,dx$

$$= \int_{x=a}^{x=s}\int_{t=a}^{t=x} (F(t)\,dt)\,dx$$

$$= \int_{t=a}^{t=s} F(t)\int_{x=t}^{x=s} dx\,dt$$





~~F(a) - F(a) eq.~~
~~∫F(x)dx - F(a)(s-a)~~

now

Date 15/3/23

$$y(s) = q_0 + [A(a)q_0 + q_1](s-a) + \int_a^s (s-t)F(t)\,dt - \int_a^s \{A(t) + (s-t)[B(t) - A'(t)]\}\, y(t)\,dt \qquad \text{— (6)}$$

now set

$$K(s,t) = -\{A(t) + (s-t)[B(t) - A'(t)]\} \qquad \text{— (7)}$$

and

$$f(s) = \int_a^s (s-t)F(t)\,dt + [A(a)q_0 + q_1](s-a) + q_0 \qquad \text{— (8)}$$

now do this for a n-order ode

$$\frac{d^n y}{dx^n} = y^n(x) = g(s)$$

$$\frac{d^{n-1}y}{dx^{n-1}} = \int_a^s y^n(x)\,dx = \int_a^s g(x)\,dx + q_{n-1}$$

integral constant

$$\Rightarrow \frac{d^{n-2}y}{dx^{n-2}} = \int_a^s \int_a^x \left(g(t)\,dt\right)dx + q_{n-1}\int_a^s dt + q_{n-2}$$

$$= \int_a^s \int_a^x g(t)\,dt\,dx + q_{n-1}(s-a) + q_{n-2}$$

$$= \int_a^s (s-t)g(t)\,dt + q_{n-1}(s-a) + q_{n-2}$$

$$\Rightarrow \frac{d^{n-3}y}{dx^{n-3}} = \int_a^s \int_a^s (s-t) g(t)\,dt\,dx + q_{n-1}(s-a)\int_a^s dt + q_{n-2}\int_a^s dt$$

$$= \int_a^s \frac{(s-t)^2}{2} g(t)\,dt + q_{n-1}\frac{(s-a)^2}{2!} + q_{n-2}(s-a) + q_{n-3}$$

and

$$\frac{d^{n-4}y}{dx^{n-4}} = \int_a^s \int_a^s \frac{(s-t)^2}{2} g(t)\,dt\,dx + q_{n-1}(s-a)^2\int_a^s dt + q_{n-2}(s-a)\int_a^s dt + (q_{n-3})\int_a^s dt$$

$$= \int_a^s \frac{(s-t)^3}{3!} g(t)\,dt + \frac{q_{n-1}(s-a)^3}{3!} + \frac{q_{n-2}(s-a)^2}{2!} + q_{n-3}(s-a) + q_{n-4}$$

$\Rightarrow$ now in general now

$$\frac{dy}{dx} = \int_a^s \frac{(s-t)^{n-2}}{(n-2)!} g(t)\,dt + \frac{q_{n-1}(s-a)^{n-2}}{(n-2)!} + \frac{q_{n-2}(s-a)^{n-3}}{(n-3)!} + \frac{q_{n-3}(s-a)^{n-4}}{(n-4)!} + \cdots + \frac{(s-a)q_2}{1!} + q_1$$

now Next consider linear differential equation of order n

$$\frac{d^n y}{ds^n} + A_1(s)\frac{d^{n-1}y}{ds^{n-1}} + \text{---} + A_{n-1}(s)\frac{dy}{ds} + A_n(s)y = F(s) \quad \text{---} (10)$$

with initial conditions

$$y(a) = q_0, \quad y'(a) = q_1, \quad \text{---}, \quad y^{n-1}(a) = q_{n-1} \quad \text{---} (11)$$

where the functions $A_1, A_2,$ —— A_n and F are defined and continuous in $a \leq s \leq b$.

so, putting the values found previously by multiplying with corresponding $A_i(s)$ and then adding.

$$g(s) + A_1(s)\left[\int_a^s g(t)dt + q_{n-1}\right] + A_2(s)\left[\int_a^s (s-t)g(t)dt + q_{n-1}(s-a) + q_{n-2}\right]$$

$$+ A_3(s)\left[\int_a^s \frac{(s-t)^2}{2!}g(t)\,dt + q_{n-1}\frac{(s-a)^2}{2!} + q_{n-2}(s-a) + q_{n-3}\right]$$

$$+ A_4(s)\left[\int_a^s \frac{(s-t)^3}{3!}g(t)dt + q_{n-1}\frac{(s-a)^3}{3!} + q_{n-2}\frac{(s-a)^2}{2!} + q_{n-3}(s-a) + q_{n-4}\right]$$

$$A_{n-1}(s)\left[\int_a^s \frac{(s-t)^{n-2}}{(n-2)!}g(t)dt + q_{n-}\right.$$

$$\left.\right]$$

$$+ A_n(s)y = F(s)$$

now arranging in the form and compare

$$g(s) = f(s) + \int_a^s K(s,t)\,g(t)dt$$

$$K(s,t) = -\sum_{k=1}^{n} A_k(s) \frac{(s-t)^{k-1}}{\lfloor (k-1)!} \quad \text{--- (19)}$$

and

$$f(s) = F(s) - q_{n-1} A_1(s) - \left[(s-a)q_{n-1} + q_{n-2}\right] A_2(s)$$

$$- - - \left\{ \left[\frac{(s-a)^{n-1}}{\lfloor(n-1)!}\right] q_{n-1} + \text{---} . (s-a)q_1 + q_0 \right\} A_n(s)$$

$$\text{--- (20)}$$

Imp for exam

Boundry Value Problems consider the BVP

$$y''(s) + A(s)\, y'(s) + B(s)\, y = F(s) \quad \text{--- (1)}$$

with Boundry conditions (BCs) :

$$y(a) = y_0 \quad \text{and} \quad y(b) = y_1 \quad \text{--- (2)}$$

Integrate from a to s.

$$y'(s) - y'(a) + \underbrace{\int_a^s A(s)\, y'(s)\, ds}_{\text{integration by parts}} + \int_a^s B(s)\, y\, ds = \int_a^s F(s)\, ds$$

let $y'(a) = c$

$$y'(s) - c + \left[A(s)\, y(s) \Big|_a^s - \int_a^s y(s) A'(s) ds \right]$$

$$+ \int_a^s B(s)\, y\, ds = \int_a^s F(s) ds$$

$$\Rightarrow \quad y'(s) - c + \left[A(s) y(s) - A(a) y(a)\right] + \left[\int_a^s y(s)\, (B(s) - A'(s)\right.$$

$$= \int_a^s F(s) ds$$

sir

$$y'(s) = C + \int_a^s F(t)\,dt - A(s)y(s) + A(a)y_0$$

$$+ \int_a^s [A'(t) - B(t)]\, y(t)\,dt \quad — (4)$$

Integrating again from a to s

$$y(s) - \underbrace{y(a)}_{y_0} = \int_a^s C\,dx + \int_a^s \left(\int_a^s F(t)dt\right)dx - \int_a^s A(s)y(s)\,dx + A(a)y_0 \int_a^s dx$$

$$+ \int_a^s \left[\int_a^{x} [A'(t) - B(t)]\, y(t)\,dt\right] dx$$

now change the order of integral

sir

$$y(s) = y_0 + C(s-a) + \int_a^s (s-t)\,F(t)\,dt + A(a)\,y_0\,(s-a)$$

$$- \int_a^s \left\{ A(t) - (s-t)\,[A'(t) - B(t)] \right\} y(t)\,dt \quad — (5)$$

miss

now using $y(b) = y_1$, find the unknown C.

$$y_1 = y(b) = y_0 + C(b-a) + A(a)\,y_0\,(b-a) + \int_a^b (b-t)\,F(t)\,dt$$

$$+ A(a)\,y_0\,(b-a) - \int_a^b \left\{ A(t) - (b-t)\,[A'(t) - B(t)] \right\} y(t)\,dt$$

there will be 2 integrals $\int_a^b$ and $\int_a^s$

$\int_a^s + \int_s^b$ then combine the two terms of $\int_a^s$

ds]

Date 20/3/23

$y_1 = \cancel{y_0 + (b-a)\,(C + A(a)y_0)} + \int_a^s \cancel{\left(F(t) + A(t) - b(t)\right)(b-t)}$

Sir

$$y'(s) = C + \int_a^s F(t)\,dt - A(t)\,y'(t)\Big|_a^s + \int_a^s A'(t)\,y(t)\,dt - \int_a^s B(t)\,y(t)\,dt \quad \text{—③}$$

$$y(s) = y_0 + C\int_a^s dt + \int_a^s\int_a^x F(t)\,dt\,dx - \int_a^s A(t)\,y(t) + A(a)\,y_0(s-a) + \int_a^s\int_a^x \left[A'(t) - B(t)\right] y(t)\,dt\,dx \quad \text{—⑤}$$

now

$$y(s) = y_0 + C(s-a) - \int_a^s A(t)\,y(t) + A(a)\,y_0(s-a) + \int_a^s\int_a^x \left\{\left[A'(t) - B(t)\right] y(t) + F(t)\right\} dt\,dx$$

now perform change of integral

$$y(s) = y_0 + C(s-a) - \int_a^s A(t)\,y(t) + A(a)\,y_0(s-a) + \int_a^s \left\{\left[A'(t) - B(t)\right] y(t) + F(t)\right\} dt$$

~~now~~ using BC $y(b) = y_1$

$$y_1 = y_0 + c(b-a) - \int_a^s A(t)y(t) + A(a)\, y_0(s-a)$$

$$+ \int_a^b \{ [A'(t) - B(t)]\, y(t) + F(t) \}\, dt$$

Sir

$$y(s) = \underbrace{y_0 + \left(\frac{s-a}{b-a}\right) \{ y_1 - y_0 - \int_a^b (b-t)\, F(t)\,dt}_{f(s)}$$

$$+ \int_a^b \{ A(t) - (b-t)[A'(t) - B(t)] \} y(t)\,dt \}$$

$$+ \underbrace{\int_a^b (s-t)\, F(t)\,dt}_{\text{included in } f(s)} - \int_a^s \{ A(t) - (s-t)[A' - B] \} y(t)\,dt$$

now break $\int_a^b$ to $\int_a^s + \int_s^b$

$-b+t$
$+s-$

$-b+t$
$s-t$
~~$-b+t \quad s+t$~~

so, Integral ones $= \cancel{\int_a^s [A'(t) - B(t)](s-b)\, y(t)\,dt}$

$$= \int_s^b \{ A(t) - (b-t)[A'(t) - B(t)] \} y(t)\,dt$$

$$+ \int_a^s \{ [A' - B](s-b) + A(t) \} y(t)\,dt$$

so, $$k(s,t) = \begin{cases} A(t) + (s-b)[A' - B] & ; \ t < s \\ A(t) - (b-t)[A' - B] & ; \ t > s \end{cases}$$

$\underline{\text{sir}}$

$$y(s) = f(s) + \frac{s-a}{b-a}\int_s^b \{A(t) - (b-t)[A'(t) - B(t)]\}y(t)\,dt$$

$$+ \int_a^s \left[\left\{\frac{s-a}{b-a} - 1\right\} A(t) - \{A'(t) - B(t)\}\left\{(b-t)\frac{s-a}{s-b} - (s-t)\right\}\right] y(t)\,dt$$

$$K(s,t) = \begin{cases} \frac{s-a}{b-a}\left(A(t) - (b-t)[A'(t) - B(t)]\right) & ; \quad t > s \\ \left(\frac{s-a}{b-a} - 1\right)A(t) - \{A'(t) - B(t)\}\left\{(b-t)\frac{s-a}{s-b} - (s-t)\right\} & ; \quad t < s \end{cases}$$

$\underline{\text{sir}}$

$$K(s,t) = \begin{cases} \left(\frac{s-a}{b-a} - 1\right)A(t) - \{A'(t) - B(t)\}\frac{(t-a)(b-s)}{b-a} & ; \quad s > t \\ \frac{s-a}{b-a}\left(A(t) - (b-t)[A'(t) - B(t)]\right) & ; \quad s < t \end{cases}$$

If boundary conditions $y(0) = y(1) = 0$, then kernel simplify $a = 0$, $b = 1$, $y_0 = 0$, $y_1 = 0$

$$k(s,t) = \begin{cases} s\{A(t) - (1-t)[A'(t) - B(t)]\} & ; \quad s < t \\ (s-1)A(t) - t(1-s)\{A'(t) - B(t)\} & ; \quad s > t \end{cases}$$

If $A(t)$ and $B(t)$ are constant functions

$$K(s,t) = \begin{cases} B(1-t)s + As & ; s < t \\ Bt(1-s) + As - A & ; s > t \end{cases}$$

note The kernel $K(s,t)$ is symmetric and discontinous at $s=t$ unless $A=0$

Next consider 2^{nd} order differential equation

$$y'' + \lambda P(s) y = Q(s) \qquad \text{—— (1)}$$

with B.C.

$$y(a) = 0, \quad y(b) = 0 \qquad \text{—— (2)}$$

then corresponding Fredholm integral eqn can be written as

$$y(s) = f(s) + \int_a^b K(s,t)\, y(t)\, dt \qquad \text{—— (3)}$$

where

$$f(s) = \int_a^s (s-t) Q(t) - \frac{s-a}{b-a} \int_a^b (b-t) Q(t) dt \qquad \text{—— (4)}$$

and kernel

$$K(s,t) = \begin{cases} \lambda P(t) \dfrac{(s-a)(b-t)}{b-a} & ; s < t \\ \lambda P(t) \dfrac{(t-a)(b-s)}{b-a} & ; s > t \end{cases} \qquad \text{—— (5)}$$

$K(s,t)$ is continous at $s=t$

then e

CamScanner

Date 22/3/23

now check the differentiation

$$\frac{\partial K}{\partial s} = \begin{cases} \lambda p(t) \frac{(b-t)}{(b-a)} (1) & ; \quad s < t \\ \lambda p(t) \left(\frac{t-a}{b-a}\right)(-1) & ; \quad s > t \end{cases} \quad — (6)$$

Check at $s = t$, it is not continous. Now find the jump of the function at $s = t$

$$\underset{\text{from right side}}{\left.\frac{\partial K}{\partial s}\right|_{t^+}} - \underset{\text{from left side}}{\left.\frac{\partial K}{\partial s}\right|_{t^-}} = -\lambda\, p$$

more

$$\frac{\partial^2 K}{\partial s^2} = \begin{cases} 0 & ; \quad s < t \\ 0 & ; \quad s > t \end{cases}$$

next consider $y'' + \lambda y = 0$ — (8)

$y(0) = 0, \quad y(l) = 0$ — (9)

then

$$K(s,t) = \begin{cases} \frac{\lambda s}{l}(l-t) & ; \quad s < t \\ \frac{\lambda t}{l}(l-s) & ; \quad s > t \end{cases} \quad — (10)$$

K is continous at $s = t$

not a change of variable

$$y'' = f(x, y) \qquad a \le x \le b$$

$$y(a) = A, \quad y(b) = B$$

then it can be written as

$$y'' = F \qquad [0, 1]$$

$$y(0) = \quad , \quad y(1) =$$

now

$$\frac{\partial K}{\partial s} = \begin{cases} \frac{\lambda}{l}(l-t) = \lambda\left(1 - \frac{t}{l}\right) & ; \quad s < t \\ -\frac{\lambda t}{l} & ; \quad s > t \end{cases}$$

So, jump $= -\lambda$ (find and verify jump)

$$\left.\frac{\partial K}{\partial s}\right|_{t^+} - \left.\frac{\partial K}{\partial s}\right|_{t^-} = -\frac{\lambda s}{l} - \lambda\left(1 - \frac{s}{l}\right)$$

$$= -\lambda \qquad \text{--- (12)}$$

Green's function Lecture

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ \frac{\partial y_1}{\partial x} & \frac{\partial y_2}{\partial x} \end{vmatrix}$$

Variation of parameters

Consider the equation non-homogeneous differential equation

$$y'' + P(x)y' + Q(x)y = R(x) \quad \text{--- (1)}$$

Let $y_1(x)$ and $y_2(x)$ are two linearly independent Solution of homogeneous equation

$$y'' + Py' + Qy = 0 \quad \text{--- (2)}$$

Then soln of inhomogeneous eq(1) can be written as

$$\boxed{y = y_1 \int \frac{-y_2}{w(y_1,y_2)} R(x)\,dx + y_2 \int \frac{y_1}{W(y_1,y_2)} R(x)\,dx + c_1 y_1(x) + c_2 y_2(x)} \quad \text{--- (*)}$$

where

$$w(y_1,y_2) = y_1 y_2' - y_2 y_1' = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Consider the BVP

$$y'' + \lambda y = 0 \quad \text{--- (3)}$$

$$y(0) = 0, \quad y'(1) + \underline{v_2}\, y(1) = 0 \quad \text{--- (4)}$$

(mixed BC)

now consider the non-homogeneous problem

$$y'' = f(x)$$

integrating $y'' = 0$, twice we get

$$y' - c_1 = 0 \quad , \quad y - c_1 x - c_2 = 0$$

$$y(x) = \underset{y_2}{c_2} + \underset{y_1}{c_1 x} \quad — (6)$$

Then from (*) [take $y_1 = 1$, $y_2 = x$]

$$w(y_1, y_2) = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1$$

$$y(x) = c_2 + c_1 x - \int_0^x t\, f(t)\,dt + x\int_0^x f(t)\,dt \quad — (7)$$

$1, \cancel{x, x^2, x^3}, x^n$

first we consider simple boundry conditions
$y(0) = 0$, $y(1) = 0$

then from (7)
$c_2 = 0$ and

$$0 = c_1 - \int_0^1 t\, f(t)\,dt + \int_0^1 f(t)\,dt$$

$$\Rightarrow c_1 = \int_0^1 t\, f(t) - \int_0^1 f(t)\,dt$$

$$= \int_0^1 (t-1)\, f(t)\,dt$$

$$c_1 = \int_0^x (t-1)\, f(t)\,dt + \int_x^1 (t-1)\, f(t)\,dt$$

using it back in (7)

$$y(x) = x\int_0^x (t-1) f(t)\,dt + x\int_x^1 (t-1) f(t)\,dt - \int_0^x t f(t)\,dt$$

$$+ x\int_0^x f(t)\,dt$$

$$= x\int_0^x t f(t)\,dt + x\int_x^1 (t-1) f(t)\,dt - \int_0^x t f(t)\,dt$$

$$= (x-1)\int_0^x t f(t)\,dt + x\int_x^1 (t-1) f(t)\,dt$$

$$y(x) = \int_0^x (x-1) t f(t)\,dt + \int_x^1 x(t-1) f(t)\,dt \quad (10)$$

now for

$$\boxed{y'' = -f(x)} \quad \text{or} \quad -y'' = f(x) \quad \text{or} \quad y'' + f(x) = 0$$

$$y(s) = \int_0^s (1-s)\cdot t f(t)\,dt + \int_s^1 (1-t) s f(t)\,dt$$

Take

$$G(s,t) = \begin{cases} (1-t)s & ; \quad s < t \\ (1-s)t & ; \quad s > t \end{cases} \quad \text{--- } (11)$$

green's function

$$\boxed{y(s) = \int_0^1 G(s,t) f(t)\,dt \quad \text{--- } (12)}$$

Next we see some properties of Green's function given in (11).

This is Green's function corresponding to

$$y'' = -f(x) \quad \text{with BC} \quad y(0)=0, \; y(1)=0$$

1.: $G(s,t)$ satisfy the homogeneous differential equation for $0 \le s < t$ and also for $t < s \le 1$

obvious

2. $\lim_{s \to t^+} G(s,t) = \lim_{s \to t^-} G(t,s)$

3. $\boxed{\dfrac{\partial G}{\partial s}} = \begin{cases} 1-t & ; \; s<t \\ -t & ; \; s>t \end{cases}$

$$\lim_{s \to t^+} \frac{\partial G}{\partial s} - \lim_{s \to t^-} \frac{\partial G}{\partial s} = -t-(1-t) = -t-1+t$$
$$= -1$$

4. $G(0,t) = G(1,t) = 0$

$G(0,t) = (1-t).0 = 0$

$G(1,t) = (1-1)t = 0$

5. $G(s,t) = G(t,s)$ i.e. G is symmetric

Date 27/3/23

I EVM

till last class - CT syllabus
After midsem 6 April - CT class time 10-11
8th April - Assignment
or
9th April

① Degenerate kernel
② generate kernel
③ Application of ODE (learn particular solution variation method)

eg ①. Find the Greens function for the BVP

$$-(y''+y) = f(s), \quad y(0)=0, \quad y(1)=0$$

and write the solution in the following form.

$$y(s) = \int_0^1 G(s,t)\, f(t)\,dt$$

Sol

Ans

$$G(s,t) = \begin{cases} \dfrac{\sin t \, \sin(1-s)}{\sin 1} & ; \ 0 \le s < t \\[2ex] \dfrac{\sin s \, \sin(1-t)}{\sin 1} & ; \ 1 \ge s > t \end{cases}$$

eg ② It is possible to show that the Sturm - Liouville problem

$$L[y] = -(p(x)\, y')' + q(x)\, y = f(x) \quad \text{—(i)}$$

$$a_1 y(0) + a_2 y'(0) = 0, \quad b_1 y(1) + b_2 y'(1) = 0 \quad \text{—(ii)}$$

has a Green's function solution

$$y(x) = \int_0^1 G(x,s)\, f(s)\,ds \quad \text{—(iii)}$$

provided $\lambda = 0$ is not an eigen value of $L[y] = \lambda y$ subject to the boundary conditions (ii). Further $G(x,s)$ is given by

$$G(x,s) = \begin{cases} \dfrac{-y_1(s)\, y_2(x)}{p(x)\, W(y_1, y_2)(x)}, & 0 \le s < x \\[2ex] \dfrac{-y_1(x)\, y_2(s)}{p(x)\, W(y_1, y_2)(x)}, & x \le s \le 1 \end{cases} \qquad \text{---(iv)}$$

where y_1 is a solution of $L[y]=0$ satisfying the B.C. at $x=0$, y_2 is a soln of $L[y]=0$ satisfying the BC at $x=1$ and $W(y_1, y_2)$ is the Wronskian of y_1 and y_2.

Prove all of them

(a) Verify that the green's functions obtained in previous two problems are given by the formulae (iv).

(b) Show that $p(x)\, W(y_1, y_2)(x)$ is a constant, by showing that its ~~ds~~ derivative is zero.

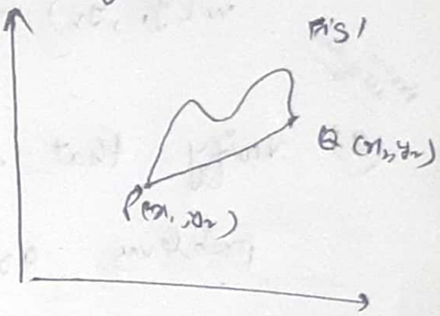
(c) Using (iv) and the result of part (b) show that $G(x,s) = G(s,x)$.

Calculus of Variation

Prob 1: Suppose that two points P & Q are given in a plane. There are infinite many curves joining these points and we can ask which of these curves is the shortest.

⇒ The intuitive answer is of course a straight line.

Fig 1

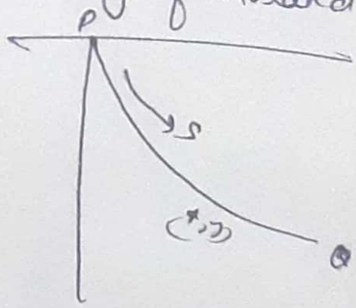


Prob 2: If we ask which curve will generate the surface of revolution of smallest area (surface area) when revolved about x-axis.

⇒ In this case answer is far from clear.

Prob 3. If we think of a typical curve as a frictionless wire in a vertical plane, then a non-trivial problem is that of finding the curve down which a bead will slide from P to Q in shortest time.

All these three problems can be mathematically formulated as follows:



i) length of ~~$\sqrt{(x_1-x_2)^2+(y_2-y_1)^2}$~~ of curve in Fig 1 is given by

$$\boxed{J_1 = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx}$$

(ii) Surface area is formulated by revolving about x-axis

$$I_2 = \int_{x_1}^{x_2} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

(iii) for time formulation is

$$v = \frac{ds}{dt} = \sqrt{2gy}$$

$$\Rightarrow dt = \frac{ds}{v}$$

$$I_3 = \int_{x_1}^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\, dx$$

The optimal values can be found by minimizing these functions.

Date 29/3/23

last class behind FODS 27/3/23

Defination Admissible function : A function $y(x)$ that have continous second derivation and satisfy given BC $y(x_1) = y_1$ and $y(x_2) = y_2$ is called admissible function.

$\bar{y} = y(x) + \alpha\eta(x)$, $\eta(x_1) = \eta(x_2) = 0$

$\alpha\eta(x)$

$$I = \int_{x_1}^{x_2} f(x, y, y')\,dx$$

$$I(\alpha) = \int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}')\,dx$$

$$= \int_{x_1}^{x_2} f\left(x,\ y(x) + \alpha\eta(x),\ y'(x) + \alpha\eta'(x)\right)dx$$

Given

$I'(\alpha) = 0$ at $\alpha = 0$

$\boxed{I'(0) = 0}$ → no need to write this line. (sir leave)

$$\frac{d}{d\alpha} I(\alpha) = \int_{x_1}^{x_2} \frac{\partial}{\partial\alpha}\left[f\{x, y(x) + \alpha y(x), y'(x) + \alpha\eta'(x)\}\right]dx$$

$$\frac{\partial f}{\partial\alpha} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha}{\Rightarrow 0} + \frac{\partial f}{\partial\bar{y}}\underbrace{\frac{\partial\bar{y}}{\partial\alpha}}_{\eta(x)} + \frac{\partial f}{\partial\bar{y}'}\underbrace{\frac{\partial\bar{y}'}{\partial\alpha}}_{\eta'(x)}$$

$$\frac{dI}{d\alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right] dx$$

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right] dx = 0$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right] dx$$

taking

$$\int_{x_1}^{x_2} \underset{\mathrm{I}}{\frac{\partial f}{\partial y'}} \underset{\mathrm{II}}{\eta'(x)}\, dx = \left. \eta(x) \frac{\partial f}{\partial y'} \right|_{x_1}^{x_2} \to 0 - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \eta(x)\, dx$$

$$= - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

by

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2} \eta(x) \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right] dx = 0 \quad \text{— (8)}$$

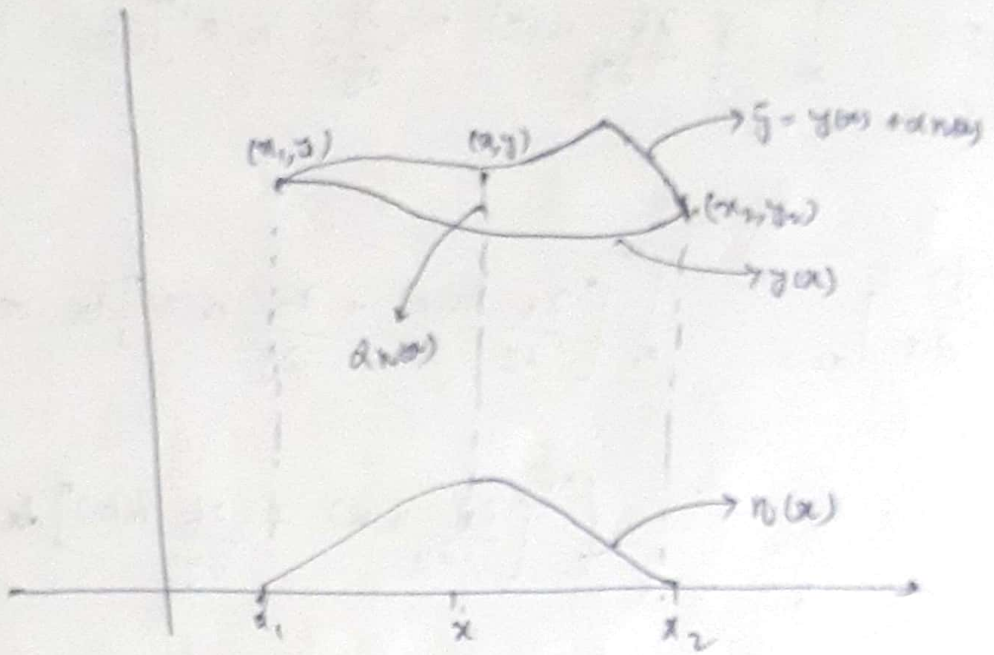
that means

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

As $\eta(x)$ is arbitary with $\eta(x_1) = \eta(x_2) = 0$ thus

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad \text{— (9)}$$

This is Euler equations.



The vertical deviation of the curve in this family from the minimizing curve $y(x)$ is $\alpha\,\eta(x)$ as shown in the fig.

Stationary function or stationary curve :

Any admissible solution of Euler's equation is called stationary function or stationary curve and corresponding value of integral as stationary value.

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

↳ function of (x, y, y')

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) + \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right)\frac{dy}{dx}}_{2} + \underbrace{\frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right)\frac{dy'}{dx}}\right) - \frac{\partial f}{\partial y} = 0$$

$$f_{y'y'}\frac{d^2y}{dx^2} + f_{yy}\frac{dy}{dx} + (f_{y'x} - f_y) = 0 \quad \text{—— (10)}$$

special cases

case – A : If x and y are missing from the function f, then

$$f_{y'y'}\frac{d^2y}{dx^2} = 0 \; ; \qquad [\text{from } 10]$$

and if $f_{y'y'} \neq 0$, then $\frac{d^2y}{dx^2} = 0$ and $y = c_1 x + c_2$

so the extemal functions are straight lines.

Case - B : If y is missing from the function f, then Eulers equation becomes

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad [\text{from } 9]$$

and this can be integrated at once to yield to first order equation

$$\frac{\partial f}{\partial y'} = c \quad \text{for the extoremals (minimal or maximal)}$$

Case - C : If x is missing from the function f,

then $\frac{\partial f}{\partial x} = 0$

then

$$y'\left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y}\right] - \frac{\partial f}{\partial x} = 0$$

What is given, what to prove

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial f}{\partial y'}y' - f\right) = 0$$

Proof

$$= \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) y' + y'' \frac{\partial f}{\partial y'} - \frac{d}{dx} f(x,y,y')$$

$$\Downarrow$$

$$-\left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\right]$$

$$= \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) y' - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} y'$$

$$\Rightarrow y'\left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y}\right] - \frac{\partial f}{\partial x} = 0$$

So, we have proved they are equal

now

$$\boxed{\frac{\partial f}{\partial y'} y' - f = \text{constant}}$$

Date
30/3/23

Isoperimetric Problems : The ancient Greek's proposed the problem of finding the closed plane curve of given length that encloses the largest area.

The area integral in parametric form

$$\boxed{A = \frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) dt} \quad \text{—— (1)}$$

where the curve is parametrically represented by $x = x(t)$, $y = y(t)$ and is traversed once counter-clockwise

The length of the curve

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad \text{—— (2)}$$

Then the problem is to maximize (1) subject to the side condition that (2) must have a constant value.

Integral side condition : Here we want to find the differential eqⁿ that must be satisfied by a function $y(x)$ that gives a stationary value to the integral.

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx \quad \leftarrow (3)$$

where y is subject to the side condition

$$J = \int_{x_1}^{x_2} g(x, y, y')\, dx = c \quad \text{—— (4)}$$

Let $\bar{y}(x) = y(x) + \alpha_1 \eta_1(x) + \alpha_2 \eta_2(x)$ —— (5)

be a two parametric family of neighbouring function where $\eta_1(x)$ and $\eta_2(x)$ have continous second derivative and

$$\eta_1(x_1) = \eta_1(x_2) = 0 \quad \text{and} \quad \eta_2(x_1) = \eta_2(x_2) = 0 \quad \text{—— (6)}$$

The paramters α_1 and α_2 are not independent, but are related to the condition that

$$J(\alpha_1, \alpha_2) = \int_{x_1}^{x_2} g(x, \bar{y}, \bar{y}')\,dx = C \quad — (7)$$

our problem is now reduced to find necessary conditions for the function

$$I(\alpha_1, \alpha_2) = \int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}')\,dx \quad — (8)$$

to have stationary value at $\alpha_1 = \alpha_2 = 0$, where α_1, α_2 satisfy (7)
Here, we use method of Langrange multipliers.

Let $K(\alpha_1, \alpha_2, \lambda) = I(\alpha_1, \alpha_2) + \lambda J(\alpha_1, \alpha_2)$

$$= \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')\,dx \quad — (9)$$

now From necessary condition

$$\left.\frac{\partial K}{\partial \alpha_1}\right|_{\alpha_1 = 0} = 0 \quad \text{and} \quad \left.\frac{\partial K}{\partial \alpha_2}\right|_{\alpha_2 = 0} = 0$$

$$\frac{\partial K}{\partial \alpha_1} = \frac{\partial K}{\partial \alpha_2} = 0 \quad \text{when} \quad \alpha_1 = \alpha_2 = 0 \quad — (10)$$

By differentiating (9) with respect to α_1 & α_2 we get

$$\frac{\partial K}{\partial \alpha_i} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \bar{y}} \eta_i(x) + \frac{\partial F}{\partial \bar{y}'} \eta_i'(x)\right] dx \quad — (11)$$

$i = 1, 2$

setting $\alpha_1 = \alpha_2 = 0$, we get

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \eta_i(x) + \frac{\partial F}{\partial y'} \eta_i'(x)\right] dx = 0 \quad — (12)$$

$i = 1, 2$

As $\frac{\partial K}{\partial \alpha_1} = \frac{\partial K}{\partial \alpha_2} = 0$ for $\alpha_1 = \alpha_2 = 0$

Now, integrating 2nd term by parts and using

$\eta_i(x_1) = \eta_i(x_2) = 0 \qquad i=1,2$, we get

$$\int_{x_1}^{x_2} \eta_i(x)\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]dx = 0 \quad \text{—— (13)}$$

As $\eta_1(x)$ and $\eta_2(x)$ both are arbitary we get only

we get

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{—— (14)}$$

Same equation for both ~~η_i, $i=1,2$~~ η_1 and η_2

The solution of (14) involves 3 undertermined parameters. 2 const of integration and the lagrange multiplier λ. The stationary function is thus selected from these extremals by imposing two boundary conditions and giving the integral J its prescribed value c.

Ex Maximize $\int_0^1 y\,dx$

Such that the perimeter being fixed

$$\int_0^1 \sqrt{1+y'^2}\,dx = 7. \qquad y(0)=0,\ y(1)=0$$

$$F = y + \lambda\sqrt{1+y'^2}$$

Sol

$$\cancel{\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)} \Rightarrow \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) - 1 = 0$$

$$\Rightarrow \frac{\lambda y'}{\sqrt{1+y'^2}} = x - \alpha.$$

$$\lambda y' = \alpha \sqrt{1+y'^2} \quad \cancel{\ } \; -\alpha \sqrt{1+y'^2}$$

$$\Rightarrow \frac{y'}{\sqrt{1+y'^2}} = \frac{x-\alpha}{\lambda}$$

$$\Rightarrow \frac{y'^2}{1+y'^2} = \left(\frac{x-\alpha}{\lambda}\right)^2$$

$$\Rightarrow y'^2\left[1 - \left(\frac{x-\alpha}{\lambda}\right)^2\right] = \left(\frac{x-\alpha}{\lambda}\right)^2$$

$$\Rightarrow y' = \sqrt{\frac{\left(\frac{x-\alpha}{\lambda}\right)^2}{1-\left(\frac{x-\alpha}{\lambda}\right)^2}}$$

Put $\frac{x-\alpha}{\lambda} = t$, $dx = \lambda dt$

$$\Rightarrow y' = \lambda \sqrt{\frac{t^2}{1-t^2}}$$

$$y' = \frac{\lambda t}{\sqrt{1-t^2}}$$

$$(y-b) = -\lambda \sqrt{1-t^2}$$

$$y-b = -\lambda \sqrt{1-\left(\frac{x-\alpha}{\lambda}\right)^2}$$

$$\Rightarrow (y-b) = -\lambda$$

$1-t^2 = p$

$\cancel{1-2t\,dt = dp}$

$\cancel{t\,dt}$

$t\,dt = \frac{dp}{-2}$

$-\frac{1}{2}\int\frac{dp}{\sqrt{p}}$

$= -\frac{1}{2} \cdot 2\sqrt{p}$

$-\sqrt{1-t^2}$

Answer :

$(y-b)^2 = (1-t^2)\lambda^2$

$(y-b)^2 = \left(1-\left(\frac{x-a}{\lambda}\right)^2\right)\lambda^2$

3 Apr, 2023

$I = \int_a^b f(t, x(t), x'(t), y(t), y'(t))\,dt$

$f(\underline{v, v, w}, \overline{r, s})$ real valued function

Exercise

$\frac{d}{dt}\left(\frac{\partial f}{\partial w}\right) - \frac{\partial f}{\partial v} = 0$

* Verify these

$\frac{d}{dt}\left(\frac{\partial f}{\partial s}\right) - \frac{\partial f}{\partial r} = 0$

Green's theorem $\rightarrow$ The Area A enclosed by a curve C is given by

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$$= \frac{1}{2}\int_0^1 (x(t)y'(t) - y(t)x'(t))\,dt \quad \text{—} ①$$

The perimeter $P = \int_0^1 \sqrt{x'^2(t) + y'^2(t)}\,dt$ — ②

Maximize A s.t $P = \int_0^1 \sqrt{x'^2(t) + y'^2(t)}\,dt = K$

Using Lagrange multiplier

$F(t, x, x', y, y') = \frac{1}{2}(x(t)y'(t) - y(t)x'(t)) + \lambda\sqrt{x'^2(t) + y'^2(t)}$

7:48 Thu, Apr 6 48%

levm

$$F(t, x(t), x'(t), y(t), y'(t)) = \frac{1}{2}\left(x(t)y'(t) - y(t)x'(t)\right) + \lambda\sqrt{x'^2(t)+y'^2(t)}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial x'(t)}\right) - \frac{\partial F}{\partial x(t)} = 0 \qquad \frac{d}{dt}\left(\frac{-y(t)}{2} + \lambda\frac{x'(t)}{\sqrt{x'^2(t)+y'^2(t)}}\right) - \frac{1}{2}y'(t)$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial y'(t)}\right) - \frac{\partial F}{\partial y(t)} = 0 \qquad \frac{d}{dt}\left(\frac{x(t)}{2} + \lambda\frac{y'(t)}{\sqrt{x'^2(t)+y'^2(t)}}\right) + \frac{1}{2}x'(t) = 0$$

$$\frac{-y(t)}{\cancel{2}} + \frac{\lambda x'(t)}{\sqrt{x'^2(t)+y'^2(t)}} = \cancel{\frac{1}{2}y(t)} + C$$

$$\frac{x(t)}{\cancel{2}} + \frac{\lambda y'(t)}{\sqrt{x'^2(t)+y'^2(t)}} = \cancel{\frac{1}{2}x(t)} + A$$

* Problems

Find extremals using Euler equation

1. $\int_0^b (12xy + y'^2)dx \rightarrow y(x) = x^3 + c_1x + c_2$

2. $\int_0^b (3x + \sqrt{y'})dx \longrightarrow y(x) = c_1x + c_2$

3. $\int_0^b (x + y^2 + 3y')dx \Rightarrow y(x) = 0$

7:48 Thu, Apr 6 48%

levm

3. $\int_a^b (x+y^2+3y')dx \Rightarrow$ $y(x)=0$

4. $\int_a^b (y^2)dx$ $y(x)=0$

5 $\int_a^b (y^2+xyy')dx$ $y(x)=x$

6. $\int_a^b (y+xy')dx \longrightarrow ?$

7 $\int_a^b y'dx \longrightarrow ?$

8. $\int_a^b (y^2-y'^2)\,dx$ $y(x) = c_1\cos x + c_2\sin x$

9. $\int_a^b \sqrt{1+y'^2}\,dx$ $y(x) = c_1x + c_2$

10. $\int_a^b (xy'+y'^2)dx$ $y(x) = -\frac{x^2}{4} + c_1x + c_2$

11. $\int_a^b (1+x)y'^2dx$ $y = c_1 + c_2\ln(1+x)$

12. $\int_a^b (2ye^x + y^2 + y'^2)dx$ Euler eqn $2e^x + 2y - 2y'' = 0$

13. $\int_a^b (2y + y'^2)\,dx$ $y = \frac{x^2}{2} + c_1x + c_2$

14. $\int_a^b (2y'^3)dx$ $y = c$

15 $\int^b$

11. $\int_a^b (1+x)y'^2 \, dx$ $y = c_1 + c_2 \ln(1+x)$

12. $\int_a^b (2y e^x + y^2 + y'^2) \, dx$ Euler eqn
$2e^x + 2y - 2y'' = 0$

13. $\int_a^b (2y + y'^2) \, dx$ $y = \frac{x^2}{2} + c_1 x + c_2$

14. $\int_a^b (2y'^3) \, dx$ $y = c_1$

15. $\int_a^b (4xy' + y'^2) \, dx$ $y = -x^2 + c_1 x + c_2$