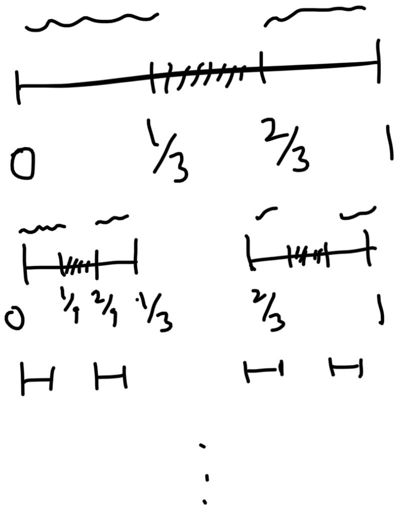
Lecture 3

The Cantor set



From the interval $[0,1]$ first remove $\left(\frac{1}{3}, \frac{2}{3}\right)$,

then $\left(\frac{1}{9}, \frac{2}{9}\right)$, & $\left(\frac{7}{9}, \frac{8}{9}\right)$ etc

removing at each stage the open interval containing the middle thirds of the closed intervals left at the previous stage.

At the n^{th} stage we get the closed intervals $J_{n,1}, \dots, J_{n,2^n}$ each of length $\frac{1}{3^n}$. write

$$P_n = \bigcup_{k=1}^{2^{n-1}} J_{n,k} \text{. Then}$$

$$P = \bigcap_{n=1}^{\infty} P_n$$ is called the <u>Cantor set</u> or <u>Cantor ternary set</u>.

$P_1 = [0,1]$

$P_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$P_3 = [0, \frac{1}{3^2}] \cup [\frac{2}{3^2}, \frac{1}{3}] \cup [\frac{2}{3}, 7/3^2] \cup [8/3^2, 1]$

$\vdots$

$$\boxed{P_n = \frac{1}{3} P_{n-1} \cup \left(\frac{2}{3} + \frac{1}{3} P_{n-1}\right)}$$

$$\frac{1}{3} P_{n-1} = \left\{ \frac{a}{3} \;\middle|\; a \in P_{n-1} \right\}.$$

$$\frac{2}{3} + \frac{1}{3} P_{n-1} = \left\{ \frac{2}{3} + \frac{a}{3} \;\middle|\; a \in P_{n-1} \right\}.$$

<u>Proposition</u>:- Let $x \in P$, the Cantor set. Then the ternary expansion of $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$, where $a_i \in \{0,2\}$ $\forall i$.

& Conversely, if $x \in [0,1]$ with ternary expansion $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$, where $a_i \in \{0,2\}$, then $x \in P$.