Keerti P Charantimath
19MA20059

Assignment 1 (Functional Analysis)

1) a) The properties of a metric ~~d~~space are

- $\forall\ x,y \in X\ d(x,y) \geq 0$ & $d(x,y)=0$ iff $x=y$
- $\forall\ x,y \in X,\ d(x,y) = d(y,x)$
- $\forall\ x,y,z \in X,\ d(x,y) + d(y,z) \geq d(x,z)$

let us assume at we have a bounded metric d on a linear space $X \neq \{0\}$ induced by a norm $\|\cdot\|$

which means

$$d(x,y) = \|x-y\| < M \quad \text{for some } M < \infty$$

let α be any constant such that $\alpha \in K$ ~~and~~

$d(x,y) < M$ is true $\forall\ x,y \in X$

let us take any $x = x_1$ & $y = y_1$ such that $x_1 \neq y_1$

For this case let us take

$|\alpha| > \frac{M}{d(x_1, y_1)}$ and $\alpha \neq 0$

now, for $x = \alpha x_1$, $y = \alpha y_1$

$d(\alpha x_1, \alpha y_1) = \|\alpha x_1 - \alpha y_1\|$

$= \|\alpha (x_1 - y_1)\|$

$= |\alpha| \, \|x_1 - y_1\|$

$= |\alpha| d(x_1, y_1)$

now, $|\alpha| d(x_1, y_1) > M$

$d(\alpha x_1, \alpha y_1) > M$

but $d(\alpha x_1, \alpha y_1)$ has to $< M$ to satisfy the bounded condition. This is a contradiction.

Hence, our assumption is wrong.

Hence proved that any bounded metric on a linear space $X \neq \{0\}$ cannot be induced by a norm

1) b)(i) $d(x+a, y+a)$

$= \| (x+a) - (y+a) \|$

$= \| x + \cancel{a} - y - \cancel{a} \|$

$= \| x - y \|$

$= d(x, y)$

hence proved

1) b) ii) $d(\beta x, \beta y)$

$= \| \beta x - \beta y \|$

$= \| \beta (x - y) \|$

$= |\beta| \, \| x - y \|$

$= |\beta| \, d(x, y)$

hence proved

1) c) let us assume that the metric

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{3^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}$$

be induced by norm $\| \cdot \|$

i.e

$d(x, y) = \| x - y \|$

let $\alpha \in K$ be some scalar

now

$$d(\alpha x, \alpha y) = \|\alpha x - \alpha y\|$$
$$= |\alpha| \, \|x - y\| = |\alpha| \, d(x,y) \rightarrow (1)$$

$$d(\alpha x, \alpha y) = \sum_{i=1}^{\infty} \frac{1}{3i} \frac{|\alpha \xi_i - \alpha \varsigma_i|}{1 + |\alpha \xi_i - \alpha \varsigma_i|}$$

$$= \sum_{i=1}^{\infty} \frac{|\alpha|}{3i} \frac{|\xi_i - \varsigma_i|}{1 + |\alpha| \, |\xi_i - \varsigma_i|}$$

$$= |\alpha| \sum_{i=1}^{\infty} \frac{1}{3i} \frac{|\xi_i - \varsigma_i|}{1 + |\xi_i - \varsigma_i|}$$

$$\neq |\alpha| \, d(x,y) \rightarrow (2)$$

but according to our assumption statement (2) has to be in accordance with statement (1)

Thus our assumption is wrong.

Hence the given metric cannot be induced by norm.

FEBRUARY - 2020

M	T	W	T	F	S	S	M	T	W	T	F	S	S
					1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19	20	21	22	23
24	25	26	27	28	29								

9

2) a) Given → $\|\cdot\|$ is a norm

10 To prove → $\|x\|_\alpha = \|\alpha x\|$ is a norm for some $\alpha \neq 0 \in K$

11

Now $\|x\|_\alpha = \|\alpha x\|$

12 $= |\alpha| \, \|x\|$

1 as $\|x\| \geq 0 \;\; \forall \, x \in X$ and $|\alpha| \geq 0$ then $|\alpha| \|x\| \geq 0$

2 as $\alpha \neq 0$, $|\alpha| > 0$

hence ~~if~~ for $|\alpha| \|x\| = 0$

3 $\Rightarrow \|x\| = 0$

$\Rightarrow x = 0$ as $\|\cdot\|$ is a norm

4

Let $\beta \in K$ be some constant

5 $\|\beta x\|_\alpha = \|\beta \alpha x\|$

$= |\beta| \, \|\alpha x\| = |\beta| \, \|x\|_\alpha$

6

Now consider triangle inequality for $\|\cdot\|$

$\|x+y\| \leq \|x\| + \|y\|$

$|\alpha| \|x+y\| \leq |\alpha| (\|x\| + \|y\|)$

$\|\alpha x + \alpha y\| \leq \|\alpha x\| + \|\alpha y\|$

 $\|x+y\|_\alpha \leq \|x\|_\alpha + \|y\|_\alpha$

hence $\|\cdot\|_\alpha$ is a norm

2) b) $n(f) = \sum_{i=1}^{j} |f(t_i)|$ where

$t_1, t_2 \dots t_j$ are distinct pts in $[a,b]$ and f is some K valued polynomial of degree atmost n on $[a,b]$ i.e $f \in P_n[a,b]$

To prove:- $n(f) = \sum_{i=1}^{j} |f(t_i)|$ is a norm iff $j \geq n+1$

for $n(f)$ to be a norm it has to satisfy the following conditions

→ $n(\alpha f) = |\alpha|\, n(f)$ → ①

$n(\alpha f) = \sum_{i=1}^{j} |\alpha f(t_i)| = |\alpha| \sum_{i=1}^{j} |f(t_i)|$ → ②

① = ② hence this condition is satisfied for all j

→ $n(f_1 + f_2) \leq n(f_1) + n(f_2)$

we know that

$|f_1(t_i) + f_2(t_i)| \leq |f_1(t_i)| + |f_2(t_i)|$ $\forall\, t_i$

$\sum_{i=1}^{j} |f_1(t_i) + f_2(t_i)| \leq \sum_{i=1}^{j} |f_1(t_i)| + \sum_{i=1}^{j} |f_2(t_i)|$

$n(f_1 + f_2) \leq n(f_1) + n(f_2)$

hence proved this condition $\forall\, j$

$\rightarrow$ $n(f) \geq 0$ & $n(f) = 0$ iff ~~f = 0~~ $f(x) = 0$ $\forall$ $x \in [a,b]$

now

$|f(t_i)| \geq 0$ $\forall$ t_i

$\sum_{i=1}^{j} |f(t_i)| \geq 0$ hence $n(f) \geq 0$

But f has n roots at max

if $j \leq n$, we can choose

$t_1, t_2 \ldots t_j$ to be roots of f and

$f \neq 0$ of degree $\geq j$

in that case $n(f) = 0$ even if

$f(x) \neq 0$ at all ~~pts~~ $x \in [a,b]$

Thus $j > n$ for $n(f) = 0$ only

at $f(x) = 0$

hence $n(f)$ is a norm iff

$j \geq n+1$

09 SUNDAY

3) $x \in C'[0,1]$

$$\|x'\|_2 = \left(\int_0^1 |x'(t)|^2 \, dt \right)^{1/2}$$

let
$\|x'\|_2 = 0$
The above is satisfied if
$x = c$ where c is some constance in $[0,1]$

But $\|x'\|_2 = 0$ even when $x \neq 0$

Thus $\|x'\|_2$ is not a norm

4) We have a norm $\|\cdot\|$ on vector space X
We need to prove that normed linear space $(X, \|\cdot\|)$ is a continuous map into $\mathbb{R}$
We know $\|\cdot\|$ maps X into $\mathbb{R}$. To prove continuity let us consider $x, y \in X$ such that
$\|x - y\| < \varepsilon$ for some $\varepsilon > 0$

we know that
$\big| \|x\| - \|y\| \big| \leq \|x - y\|$
$\Rightarrow \big| \|x\| - \|y\| \big| < \varepsilon$
hence if $\|x - y\| < \varepsilon$ then $\big| \|x\| - \|y\| \big| < \varepsilon$
hence $(X, \|\cdot\|)$ is continuous map into $\mathbb{R}$

5) 9 For $\|\cdot\|_\infty$ norm

10 Let $\{x_n\}$ be a cauchy sequence in $C[a,b]$

11 Then there exist $\varepsilon \in K$ and function n_0 where

$\max_t |x_n(t) - x_m(t)| < \varepsilon$ ~~for~~ ~~all~~ $n, m > n_0$

12

Then for every $t \in [a,b]$

1

$|x_n(t) - x_m(t)| < \varepsilon$

2

This is a cauchy sequence in

3 K. We know that K is complete

Hence,

4 $x_n(t)$ converges to some $x(t)$

5 The above is true for all t

Hence

6 $\max_t |x_n(t) - x(t)| < \varepsilon$

~~no~~ $\|x_n \cancel{(t)} - x \cancel{(t)}\| < \varepsilon$

$\lim_{n \to \infty} \|x_n - x\|_\infty \to 0$

now, $x_n \in C[a,b]$

as ~~x~~ $x_n - x$ has to be defined for

 all $t \in [a,b]$

$x_n - x \in C[a,b]$

9 hence

$x_n - (x_n - x) = x \in C[a,b]$

10 Thus,

$C[a,b]$ is a banach space wrt $\|\cdot\|_\infty$

11

12 For $\|\cdot\|_1$ norm

1 Let $\{x_n\}$ be a cauchy sequence in $C[a,b]$ which converges to
2 x which is discontinuous at $\alpha \in [a,b]$

3

$$\|x_n - x\|_1 = \int_0^1 |x_n(t) - x(t)|\,dt < \varepsilon$$

4

$$= \int_0^\alpha |x_n(t) - x(t)|\,dt +$$

5

$$\int_\alpha^1 |x_n(t) - x(t)|\,dt < \varepsilon$$

6

$\lim_{n\to\infty} \|x_n - x\|_1 \to 0$

now $x_n \in C[a,b]$ but $x_n - x \notin C[a,b]$ as x is discontinuous at α

Thus $C[a,b]$ is not a banach space wrt $\|\cdot\|_1$

7) We have a nls $(X, \|\cdot\|)$ and $M \subseteq X$ is a subspace of X st
$M = \{x : \|x\| \leq 1\}$

We have M to be compact ~~b/c~~
~~closed and bounded~~

Claim: X is a finite dimensional space.

Suppose X is not a finite dimensional space.

Then we have
$\{y_1, y_2, \ldots\} \subseteq X$ s.t the set is linearly independendent

Let $Z_n = \text{span}(y_1, y_2 \ldots y_n)$
as y_{n+1} is Linearly independent wrt $\{y_1, y_2, \ldots y_n)$
$Z_{n+1} \neq Z_n$

Then, we can find x_n st $\|x_n\| = 1$
Acc. to Riezs ~~theo~~ lemma
$\text{dist}(Z_{n+1}, x_n) \geq \frac{1}{2}$ $(r = \frac{1}{2})$

~~Thus~~ Similarly, we can find
9 a sequence $\{x_n\}$ st
$dist(x_i, z_{i+1}) \geq \frac{1}{2}$ for all i

10

We have obtained a sequence in M
11 which is not cauchy.
Hence, no subsequence of $\{x_n\}$ is
12 convergent.
Thus, ~~x~~ M is not compact.
1 But this is a contradiction.
$\Rightarrow$ Our assumption is wrong.
2
hence, X is finite dimensional
3
Conversely:-
4
We have X as a finite dimensional
5 space.

Claim: M is compact

as X is finite, let $\{y_1, y_2 \dots y_m\}$
be the basis of X

We know that M is closed ~~an~~ d
bounded.

9 Let $\{x_n\}$ be a sequence in M

10 $x_n = \sum_{j=1}^{m} k_{nj} y_j$ as $x_n \in X \ \forall n$

11 As M is bounded by 1
$\|x\| \leq 1 \quad \forall x \in M$

12 $\therefore \|x_n\| \leq 1 \quad \forall n$

1 Thus $\{k_{nj}\}$ is bounded for all $j = 1, 2 \dots m$
Ac. to Bolzano-weistrass theorem,
2 every bounded sequence in k has a convergent sub sequence. We apply this
3 to sub sequence of subsequence and thus we get
~~Thus, we have a sub sequence~~
4 ~~$\{k_{npj}\}$ st $\{k_{npj}\} \to k_{pj} \ \forall j$~~
$\exists$ a subsequence $\{k_{npj}\}$ st $\{k_{npj}\} \to k_{pj} \ \forall j$
5 Thus, by the lemma that
$\{x_n\}$ converges to x iff $\{k_{nj}\}$ converges
6 to k_j,
we can find $\{x_{np}\}$ which converges

16 SUNDAY to x_p in X. Now, as
M is closed, $x_p \in M$.

Thus, M is a compact space

8) let S be a proper subspace of nls $(V, \|\cdot\|)$

Suppose there is a non-empty open set $A \subseteq S$. let $a_0 \in A \subseteq S$.

Now,
$A - a_0 := \{x - a_0 : x \in A\}$ is also an open set in V that contains the zero element.

As S is sub-space. & $A \subseteq S \Rightarrow A - a_0 \subseteq S$

let us define $f_x : K \to V$ st
$f_x(\lambda) = \lambda x$ for every $x \in V$
The above function is clearly continuous as $\lambda \in K$ is continuous

Take any vector $v \in V$
$f_v^{-1}(A - a_0)$ is an open set that contains 0
$\lambda \in f_v^{-1}(A - a_0) \Rightarrow \lambda v \in A - a_0$

Thus we can multipey v with an appropriate λ st it belongs to $A - a_0$

Thus $\lambda v \in S \rightarrow v \in S$
Thus $V = S \Rightarrow S$ is not proper
Hence, Subset of V has non-empty interior if it is V itself

12) Given: • $f \in C^1[a,b]$ st

$f : [a,b] \to \mathbb{K}$ is continuously differentiable

• $\|f\|_* = \|f\|_\infty + \|f'\|_\infty$

To prove:- $(C^1[a,b], \|\cdot\|_*)$ is a Banach space

Consider a cauchy sequence $\{x_n\}$ in $C^1[a,b]$ wrt $\|\cdot\|_*$

$\exists$ n_0 ~~an~~ for which we have an ε st

$\|x_n - x_m\|_* < \varepsilon, \quad n, m > n_0$

$$\max_{t\in[a,b]} |x_n(t) - x_m(t)| + \max_{s\in[a,b]} |x_n'(s) - x_m'(s)| < \varepsilon$$

$$\max_{t\in[a,b]} |x_n(t) - x_m(t)| < \varepsilon \ \& \ \max_{s\in[a,b]} |x_n'(s) - x_m'(s)| < \varepsilon$$

~~for each t~~ $\cancel{|x_n(t) - x_m(t)| < \varepsilon}$ | ~~for each s~~ $\cancel{|x_n'(s) - x_m'(s)| < \varepsilon}$

$\therefore$ Each of the above sequences are cauchy sequences in ~~$\mathbb{K}$~~ $C[a,b]$ and this $C[a,b]$ is complete hence, say

9 $x_n(t) \longrightarrow x(t)$ $*$

$x'_n(t) \longrightarrow y(t)$ $\quad \forall\, t \in [a,b]$

uniformly and $x, y \in C[a,b]$

10

We know that

11 if $f_n(t) \longrightarrow g$ $\quad$ then $g' = h$

$f'_n(t) \longrightarrow h$

12

thus $x' = y$

1

$\| x_n - x_m \|_* < \varepsilon$

2 $\lim\limits_{m \to \infty} \| x_n - x \|_* = \| x_n - x \|_\infty + \| x'_n - x' \|_\infty < \varepsilon$

3 $\therefore$ $x_n \to x$ wrt $\| \cdot \|_*$

4 $x_n \in C'[a,b]$

$x_n - x \in C'[a,b]$

5

Thus, $x_n - (x_n - x) = x \in C'[a,b]$

6

Thus $(C'[a,b], \| \cdot \|_*)$ is a banach space

13) we are given $x \in \mathbb{K}^n$

To prove : $\|x\|_p \to \|x\|_\infty$ as $p \to \infty$

where

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \quad \&$$

$$\|x\|_\infty = \max_i |x_i|$$

Proof:-

We know that

$$|x_i| \le \max_i |x_i| \quad \forall\, i = \{1, 2 \ldots n\}$$

$$|x_i|^p \le \max_i |x_i|^p$$

$$\sum_{i=1}^{n} |x_i|^p \le \sum_{i=1}^{n} \left(\max_i |x_i|^p\right)$$

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n} \left(\max_i |x_i|^p\right)\right)^{1/p}$$

$$= n^{1/p} \max |x_i| \longrightarrow (1)$$

Also, we know that

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \ge \left(\max_i |x_i|^p\right)^{1/p} = \max_i |x_i| \longrightarrow (2)$$

for (1).

$$\lim_{p\to\infty} \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \le \max_{n} \cancel{\|x_i\|} \lim_{p\to\infty} n^{1/p} \|x\|_\infty$$

$$= \|x\|_\infty$$

for (2)

$$\lim_{p\to\infty} \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \ge \lim_{p\to\infty} \|x\|_\infty = \|x\|_\infty$$

By the above statements, we prove that

$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$$

hence $\|x\|_p \to \|x\|_\infty$ as $p \to \infty$

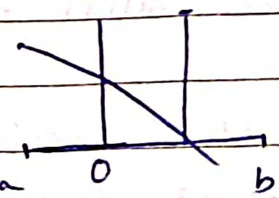
14) We have $C[a,b]$ as the linear space and

$$\|x\|_\infty = \max_{t\in[a,b]} |x(t)|$$

and

$$\|x\|_1 = \int_a^b |x(t)|\, dt$$

~~Let us assume~~
Let ~~us consider~~ a ~~function of the form~~



9 ~~let~~

Claim:- There exists no $c > 0$ st

10 $\|x\|_\infty \le c\|x\|_1$

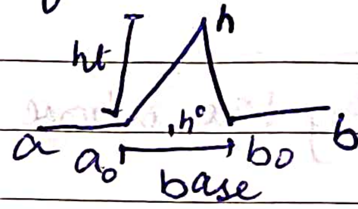
11 Let us assume we have a $c > 0$

St $\|x\|_\infty \le c\|x\|_1$

12

Let us consider a function of the

1 following form



$|h^0 - h| = ht$, $|a^0 - b^0| =$ base

2 $\|x\|_\infty = ht$

$\|x\|_1 = \int_a^b |x|\,dt = \frac{1}{2}\,\text{base} \times ht$

3

$$x(t) = \begin{cases} 0, & x \in [a, a_0] \cup [b_0, b] \\ \frac{h\,x(t)}{h^0 - a^0}, & x \in [a_0, h_0] \\ \frac{h\,x(t)}{h^0 - b^0}, & x \in [h_0, b^0] \end{cases}$$

Now, ~~for~~

4

$\|x\|_\infty \le c\|x\|_1$

5 $\cancel{ht} \le c \times \frac{1}{2}\,\text{base} \times \cancel{ht}$

$\frac{2}{\text{base}} \le c$

6

23 SUNDAY But, for any ~~d~~ $c > 0$, we can always find a small enough base such that $\frac{2}{\text{base}} > c$.

This is a contradiction to our assumption.

2020 Hence, our assumption is [illegible]g.

There ~~that~~ exist no $c>0$ st
9 $\|x\|_\infty \leq c\|x\|_1 \quad \forall x$ in $C[a,b]$

10 b) let $x \in c_0$, $x = (x(1), x(2) \dots\dots)$
let $x_n \in c_{00}$, $x_n (x(1), x(2) \dots x(n), 0, 0 \dots)$

11

Then
12 $\|x_n - x\|_\infty = \sup\{|x(j)|, j>n\}$
$\to 0$ as $x \in c_0$

1
$\therefore$ c_0 is dense in c_{00} wrt $\|\cdot\|_\infty$
2
Thus c_{00} is the closure of c_0
3 wrt $(l^\infty, \|\cdot\|_\infty)$
[Part (iii) proved]
4
Now, let $x \in l^p$, then
5 $\|x_n - x\|_p^p = \sum_{j=n+1}^{\infty} |x_j|^p \to 0$ as $n \to \infty$

6 $\therefore$ c_0 is dense in l^p wrt $\|\cdot\|_p$

Thus ~~that~~ l^1 is the closure of c_0
wrt $(l^1, \|\cdot\|_\infty)$ and
l^2 is the closure of c_0 wrt $(l^2, \|\cdot\|_2)$

[Part (i) and (ii) proved)

9 ii) Let $1 \le p < r < \infty$

10 Now $x = (x(1), x(2) \ldots) \in \ell^p$

st $\|x\|_p \le 1$

11 i.e $\left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{1/p} \le 1$

12 $\Rightarrow \quad |x(i)| \le 1$

1 as $p < r \Rightarrow |x(i)|^r \le |x(i)|^p$

2 $\Rightarrow \left(\sum_{i=1}^{\infty} |x_i|^r\right) \le \left(\sum_{i=1}^{\infty} |x_i|^p\right)$

3 $\Rightarrow \cancel{\|x(i)\|} \; \|x\|_r^r \le \|x\|_p^p$

4 $\Rightarrow \|x\|_r^r \le \|x\|_p^p \le 1$

5 $\Rightarrow \|x\|_r \le 1 \rightarrow (1)$

6 Now, for any $0 \ne x \in \ell^p$

let $y = \frac{x}{\|x\|_p} \Rightarrow \|y\|_p = 1 \Rightarrow \|y\|_p \le 1$

Then by (1), $\|y\|_r \le 1$

$\Rightarrow \left\|\frac{x}{\|x\|_p}\right\|_r \le 1$

$\Rightarrow \frac{\|x\|_r}{\|x\|_p} \le 1 \Rightarrow \|x\|_r \le \|x\|_p$

Now, if $x \in \ell^p$, $\|x\|_p < \infty$

9

$\Rightarrow \|x\|_r \le \|x\|_p < \infty$

10 $\Rightarrow x \in \ell^r$

11 Thus $\ell^p \subset \ell^r$ for $1 \le p < r < \infty$

12 Thus, the equality holds only when $p = r$

1

Hence, $\ell^p \subseteq \ell^r$ for $1 \le p \le r < \infty$

2

3

Also, if $x \in \ell^p$

4

$\Rightarrow \left(\sum_{j=1}^{\infty} |x(j)|^p \right)^{1/p} < \infty$

5

$\Rightarrow \sum_{j=1}^{\infty} |x(j)|^p < \infty$

6

$\Rightarrow |x(j)| \le \sum_{j=1}^{\infty} |x(j)|^p < \infty$

$\Rightarrow \max_j |x(j)| < \infty$

$\Rightarrow x \in \ell^\infty$

Thus $\ell^p \subset \ell^r$ even when $r \to \infty$

Thus $\ell^p \subseteq \ell^r$ for $1 \le p \le r \le \infty$

15) We have $t_1, t_2 \cdots t_n$ distinct pts in $[a, b]$,

for $x \in C[a,b]$

Now,

$$n_p(x) = \begin{cases} \left(\sum_{j=1}^{n} |x(t_j)|^p\right)^{1/p}, & 1 \leq p < \infty \\ \max_j |x(t_j)|, & p = \infty \end{cases}$$

$$\& \; n_p(\alpha x) = \begin{cases} \left(\sum_{j=1}^{n} |\alpha x(t_j)|^p\right)^{1/p}, & p \in [1, \infty) \\ \max_j |\alpha x(t_j)|, & p = \infty \end{cases}$$

$$= |\alpha| \, n_p(x)$$

Thus the condition of $\|\alpha x\| = |\alpha| \, \|x\|$ is satisfisfied

$$|x(t_j) + y(t_j)| \leq |x(t_j)| + |y(t_j)|$$

$$\left(\sum_{j=1}^{n} |x(t_j) + y(t_j)|^p\right)^{1/p} \leq \left(\sum_{j=1}^{n} |x(t_j)|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |y(t_j)|\right)^{1/p}$$

also $\max_j |x(t_j) + y(t_j)| \leq \max_j |x(t_j)| + \max_j |y(t_j)|$

Thus, $n_p(x+y) \leq n_p(x) + n_p(y)$

Hence, the condition $\|x+y\| \leq \|x\| + \|y\|$ is satisfied

Now, we know that

$\left(\sum_{j=1}^{n} |x(t_j)|^p\right)^{1/p} \geq 0$ is always true

Also $\max_j |x(t_j)|$ is true always.

Hence $\|x\| \geq 0$ condition is satisfied
Thus, n_p is a seminorm.

For a seminorm to be a norm,
The seminorm should only be
zero ~~at~~ when $x = 0$

~~This~~ In our case, let us consider
x to be a polynomial of degree n
with $t_1, t_2, \ldots, t_n$ as its roots

$n_p(x) = 0$ for any p in ~~in~~ this
condition.

But, ~~x~~ we have found a non-zero
x such that $n_p(x) = 0$.

Thus, n_p is not a norm

9) let Banach space X be ∞ dim^n
Then $X = \text{span}(e_1, e_2, \ldots\ldots)$
let $X_n = \text{span}(e_1, e_2 \ldots e_n)$
$\Rightarrow$ x_n is a proper closed ~~subsa~~ subspace of X
$\Rightarrow$ $\text{int}(X_n) = \phi$ (from Q 8)

Now $x \in X$, $\exists\, n \in \mathbb{N}$ st
$x = \sum_{j=1}^{n} \alpha_j e_j \Rightarrow x \in X_n$ as well

$\Rightarrow \bigcup_{n=1}^{\infty} X_n = 0$

$\Rightarrow$ X is a countable union of X_0 where X is a dense set

Hence, belongs to 1st category acc. to Baire's Category theorm

But as X is a banach space, it belong to 2nd category.

Hence our assumption is wrong and this X is finite dimensional

10) An absolutely convergent series converges in a complete space.

as $(C_{00}, \|\cdot\|_2)$ is not complete, $\exists$ abs. conv. series which does not converge.

where as $(\ell^2, \|\cdot\|_2)$ is complete, hence $\nexists$ abs conv. series which does not converge.

Eg:- $\{e_n\}$ be a series in C_{00} st. $e_n = (k, 0, 0 \ldots, \underset{n^{th} \text{ position}}{l}, 0, 0, \ldots)$

where k & l are abitrary.

Then $\left(\sum_n \left|\frac{e_n}{n}\right|^2\right)^{1/2} = \left\|\frac{e_n}{n^2}\right\| < \infty$

(absolutely convergent)

but is not convergent in C_{00}, as it ~~go~~ converges to $\frac{1}{n^2} \notin C_{00}$