NSOPDE ASSIGNMENT - 5

GADDAM YOGESH
19MA20015

1) $u_t = u_{xx}$ $x \in (0,1)$ $t \in (0, \infty)$

$u(x,0) = 2x$

$u_x(0,t) = 0$ $u_x(1,t) = 1$ $h = k = 0.5$

using derivative at $(x_m, t_n + \frac{k}{2}) \mid (m, n+\frac{1}{2})$ as

$$u_t \Big|_{(m,n+\frac{1}{2})} \approx \frac{u_m^{n+1} - u_m^n}{k}$$

$$u_{xx}\Big|_{m,n+\frac{1}{2}} \approx \frac{1}{2}\left\{ u_{xx}\Big|_{(m,n)} + u_{xx}\Big|_{(m,n+1)} \right\}$$

$$\approx \frac{1}{2}\left[\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n + u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}}{h^2}\right]$$

from $u_t = u_{xx}$

$$\Rightarrow \frac{u_m^{n+1} - u_m^n}{k} = \frac{1}{2h^2}\left[u_{m+1}^n - 2u_m^n + u_{m-1}^n + u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}\right]$$

$$\left[\lambda = \frac{k}{h^2}\right]$$

$$\Rightarrow 2u_m^{n+1} - 2u_m^n = \lambda\left[u_{m+1}^n - 2u_m^n + u_{m-1}^n + u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}\right]$$

$$\Rightarrow \boxed{-\lambda u_{m+1}^{n+1} + (2+2\lambda)u_m^{n+1} - \lambda u_{m-1}^{n+1} = \lambda u_{m-1}^n + (2-2\lambda)u_m^n + \lambda u_{m+1}^n}$$

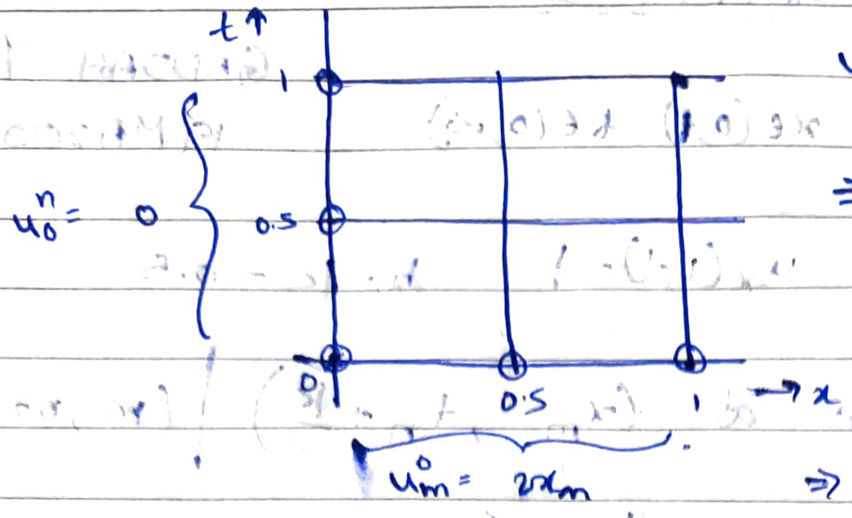
Crank - Nicolson method.

given $h = k = 1/2 \Rightarrow \lambda = \frac{1/2}{1/4} = 2$

$u(x,0) = 2x$

$u_x(0,t) = 0$

$u_x(1,t) = 1$



$u_x(0,t) = 0$

$$\Rightarrow \frac{u_1^n - u_{-1}^n}{2h} = 0$$

$$\Rightarrow \boxed{u_1^n = u_{-1}^n}$$

$u_x(1,t) = 1$

$$\Rightarrow \frac{u_3^n - u_1^n}{2h} = 1$$

$$\Rightarrow \boxed{u_3^n = 1 + u_1^n}$$

for $n=0$

a) $m=1 \Rightarrow -2u_{-1}^1 + 6u_0^1 - 2u_1^1 = 2 + 0 + 2$

$$6u_0^1 - 4u_1^1 = 4$$

$$\Rightarrow 3u_0^1 - 2u_1^1 = 2$$

b) $m=2 \Rightarrow -2u_0^1 + 6u_1^1 - 2u_2^1 = 0 - 2(1) + 2(2)$

$$\Rightarrow -2u_0^1 + 6u_1^1 - 2u_2^1 = 2$$

$$\Rightarrow -u_0^1 + 3u_1^1 - u_2^1 = 1$$

c) $m=3 \Rightarrow -2u_1^1 + 6u_2^1 - 2u_3^1 = 2(1) - 2(2) + 2(2)$

$$\Rightarrow -2u_1^1 + 6u_2^1 - 2 - 2u_1^1 = 2$$

$$\Rightarrow -4u_1^1 + 6u_2^1 = 4$$

$$\Rightarrow -2u_1^1 + 3u_2^1 = 2$$

$$3u_0^1 - 2u_1^1 + 0u_2^1 = 2$$

$$-u_0^1 + 3u_1^1 - u_2^1 = 1$$

$$0u_0^1 - 2u_1^1 + 3u_2^1 = 2$$

$$\Rightarrow u_0^1 = \frac{8}{5}, \quad u_1^1 = \frac{7}{5}, \quad u_2^1 = \frac{8}{5}.$$

$$\boxed{\therefore u(0.5, 0.5) = u_1^1 \approx \frac{7}{5} = 1.4}$$

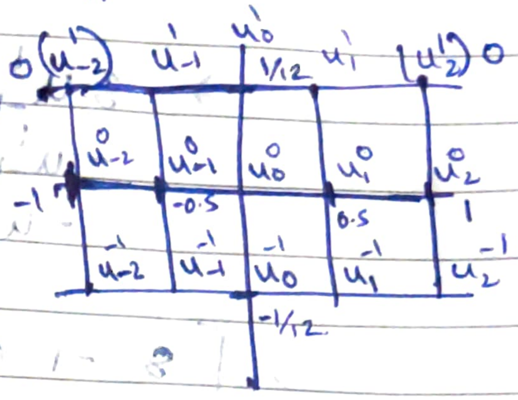
2) $u_t = u_{xx}$ $\quad h = 1/2 \quad \lambda = 1/3 \quad \lambda = \frac{k}{h^2} \Rightarrow k = 1/12$

$u(x,0) = \cos\frac{\pi x}{2} \quad -1 \le x \le 1,\ t=0$

$u(-1,t) = u(1,t) = 0$

$\Rightarrow u_{-2}^n = u_2^n = 0 \quad t>0\ \&\ n>0.$

$u_m^0 = \cos\frac{\pi x_m}{2}$



for $\lambda = 1/3$

$\Rightarrow -u_{m-1}^{n+1} + 8u_m^{n+1} - u_{m+1}^{n+1} = u_{m-1}^n + 4u_m^n + u_{m+1}^n$

for $n=0$

(a) $m=-1 \Rightarrow -u_{-2}^1 + 8u_{-1}^1 - u_0^1 = u_{-2}^0 + 4u_{-1}^0 + u_0^0$

$= \cos\left(\frac{\pi(-1)}{2}\right) + 4\cos\left(\frac{\pi(-0.5)}{2}\right) + 1$

$= 0 + \frac{4}{\sqrt{2}} + 1 = 1 + 2\sqrt{2}$

$\Rightarrow -u_{-2}^1 + 8u_{-1}^1 - u_0^1 = 1 + 2\sqrt{2}$

(b) $m=0 \Rightarrow -u_{-1}^1 + 8u_0^1 - u_1^1 = u_{-1}^0 + 4u_0^0 + u_1^0$

$= \cos\frac{\pi(-0.5)}{2} + 4(1) + \cos\frac{\pi 0.5}{2}$

$= 4 + \frac{2}{\sqrt{2}} = \sqrt{2} + 4$

$\Rightarrow -u_{-1}^1 + 8u_0^1 - u_1^1 = 4 + \sqrt{2}$

(c) $m=1 \Rightarrow -u_0^1 + 8u_1^1 - u_2^1 = u_0^0 + 4u_1^0 + u_2^0$

$= 1 + 4\cos\frac{\pi(0.5)}{2} + \cos\frac{\pi}{2}$

$= 1 + \frac{4}{\sqrt{2}} = 1 + 2\sqrt{2}$

$\Rightarrow -u_0^1 + 8u_1^1 - u_2^1 = 1 + 2\sqrt{2}$

from the equations, using: $u_{-2}^n = u_2^n = 0$.

$$\Rightarrow \quad 8u_{-1}^1 - u_0^1 = 1+2\sqrt{2}$$
$$-u_{-1}^1 + 8u_0^1 - u_1^1 = 4+\sqrt{2}$$
$$-u_0^1 + 8u_1^1 = 1+2\sqrt{2}$$

$$\Rightarrow \begin{bmatrix} 8 & -1 & 0 \\ -1 & 8 & -1 \\ 0 & -1 & 8 \end{bmatrix} \begin{bmatrix} u_{-1}^1 \\ u_0^1 \\ u_1^1 \end{bmatrix} = \begin{bmatrix} 1+2\sqrt{2} \\ 4+\sqrt{2} \\ 1+2\sqrt{2} \end{bmatrix}$$

$\Rightarrow u_{-1}^1 = 0.5813166 2194$ $\qquad u_{-2}^1 = 0$

$u_0^1 = 0.82210585078$ $\qquad u_2^1 = 0$

$u_1^1 = 0.5813166 2194$

$\therefore u(-1,1) = 0$

$u(-0.5,1) = 0.5813$

$u(0,1) = 0.8221$

$u(0.5,1) = 0.5813$

$u(1,1) = 0$

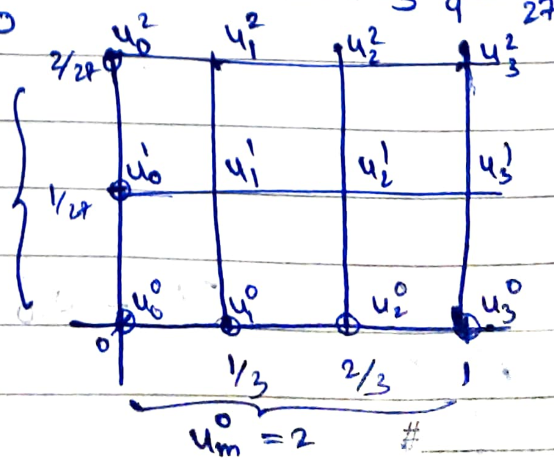
3) $u_t = u_{xx}$ $\quad 0<x<1$

$u(x,0) = 2$ $\quad 0 \le x \le 1$ $\qquad h = 1/3$ $\qquad \lambda = 1/3$

$u(0,t) = 2$ $\quad t \ge 0$

$K = \lambda h^2 = \frac{1}{3} \cdot \frac{1}{9} = \frac{1}{27}$

$u_x(1,t) = -u(1,t)$ $\quad t \ge 0$



$\Rightarrow u(1,t) = u_3^n$

$\Rightarrow \dfrac{u_4^n - u_2^n}{2h} = -u_3^n$

$\Rightarrow \boxed{u_4^n = u_2^n - \frac{2}{3}u_3^n}$

Crank - Nicolson method for $\lambda = 1/3$

$\Rightarrow -u_{m-1}^{n+1} + 8u_m^{n+1} - u_{m+1}^{n+1} = u_{m-1}^n + 4u_m^n + u_{m+1}^n$

for $n=0$

$m=1 \Rightarrow -u_0^1 + 8u_1^1 - u_2^1 = u_0^0 + 4u_1^0 + u_2^0$

$\cancel{-2} + 8u_1^1 - u_2^1 = \cancel{2}\,4 + 4(2) + 2$

$\Rightarrow 8u_1^1 - u_2^1 = \cancel{12}\,14 \quad -(1)$

$m=2 \Rightarrow -u_1^1 + 8u_2^1 - u_3^1 = u_1^0 + 4u_2^0 + u_3^0$

$= 2 + 4(2) + 2 = 12$

$\Rightarrow -u_1^1 + 8u_2^1 - u_3^1 = 12 \quad -(2)$

$m=3 \Rightarrow -u_2^1 + 8u_3^1 - u_4^1 = u_2^0 + 4u_3^0 + u_4^0$

$-u_2^1 + 8u_3^1 - u_2^1 + \frac{2}{3}u_3^1 = 2 + 4(2) + u_2^0 - \frac{2}{3}u_3^0$

$-2u_2^1 + \frac{26}{3}u_3^1 = 10 + 2 - \frac{4}{3}$

$\Rightarrow -6u_2^1 + 26u_3^1 = 32$

$\Rightarrow -3u_2^1 + 13u_3^1 = 16 \quad -(3)$

$$\Rightarrow \begin{bmatrix} 8 & -1 & 0 \\ -1 & 8 & -1 \\ 0 & -3 & 13 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = \begin{bmatrix} \cancel{12}\,14 \\ 12 \\ 16 \end{bmatrix}$$

$\Rightarrow \cancel{u_0^1 = 2}$

$\Rightarrow \cancel{u_1^1 = 1.486792452 83}$

$\cancel{u_2^1 = 1.8943396 2264}$

$\cancel{u_3^1 = 1.6679245283}$

$\Rightarrow u_0^1 = 2$

$u_1^1 = 1.99496855346 \cong 1.9950$

$u_2^1 = 1.95974842767 \cong 1.9597$

$u_3^1 = 1.68301886792 \cong 1.6830$

for n=1

$m=1 \Rightarrow -u_0^2 + 8u_1^2 - u_2^2 = u_0^1 + 4u_1^1 + u_2^1$

$= \cancel{9.84450\ 843576}$

$-2 + 8u_1^2 - u_2^2 = 11.93962264151$

$\Rightarrow 8u_1^2 - u_2^2 = 13.93962264151$

$m=2 \Rightarrow -u_1^2 + 8u_2^2 - u_3^2 = u_1^1 + 4u_2^1 + u_3^1$

$= 11.51698113206$

$-u_1^2 + 8u_2^2 - u_3^2 = 11.51698113206$

$m=3 \Rightarrow -u_2^2 + 8u_3^2 - u_4^2 = u_2^1 + 4u_3^1 + u_4^1$

$-u_2^2 + 8u_3^2 - u_2^2 + \frac{2}{3}u_3^2 = u_2^1 + 4u_3^1 + u_2^1 - \frac{2}{3}u_3^1$

$\Rightarrow -2u_2^2 + \frac{26}{3}u_3^2 = 2u_2^1 + \frac{10}{3}u_3^1$

$\Rightarrow -3u_2^2 + 13u_3^2 = 3u_2^1 + 5u_3^1$

$-3u_2^2 + 13u_3^2 = 14.29433962261$

$$\begin{bmatrix} 8 & -1 & 0 \\ -1 & 8 & -1 \\ 0 & -3 & 13 \end{bmatrix} \begin{bmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{bmatrix} = \begin{bmatrix} 13.93962264151 \\ 11.51698113206 \\ 14.29433962261 \end{bmatrix}$$

$\Rightarrow u_0^2 = 2$

$u_1^2 = 1.97725406432 \approx 1.977$

$u_2^2 = 1.87840987302 \approx 1.8784$

$u_3^2 = 1.53304378782 \approx 1.583$

4) $u_{tt} = u_{xx}$ $\quad 0 < x < 1$, $t > 0$. $\quad c = 1$

$u(0,t) = -\sin t \;(= u_0^n)$ $\quad h = \frac{1}{5}$ $\quad k = 1$

$u(1,t) = \sin(1-t) \;(= u_5^n)$

$u(x,0) = \sin x \;(= u_m^0)$ $\quad r = \frac{ck}{h} = \frac{1 \cdot 1}{1/5} = 5$

$u_t(x,0) = -\cos x$

$r = 5$

<u>explicit method</u> $\Rightarrow$

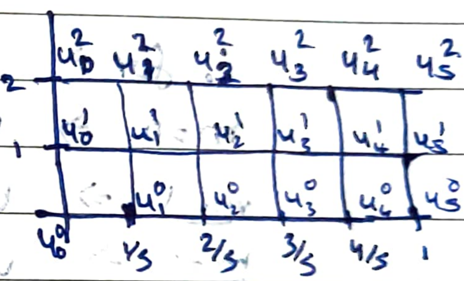
$$u_m^{n+1} = r^2 u_{m-1}^n + 2(1-r^2)u_m^n + r^2 u_{m+1}^n - u_m^{n-1}$$

$$\Rightarrow u_m^{n+1} = 25 u_{m-1}^n - 48 u_m^n + 25 u_{m+1}^n - u_m^{n-1}$$

$u_0^n = -\sin(t_n)$ $\quad u_5^n = \sin(1-t_n)$

$u_m^0 = \sin(x_m)$ $\quad t_n = n$

$x_m = \frac{m}{5}$



$\Rightarrow u_0^n = -\sin(n)$ $\quad u_5^n = \sin(1-n)$

$u_m^0 = \sin(m/5)$, $\quad u_t(x,0) = -\cos x$

$\Rightarrow \dfrac{u_m^1 - u_m^{-1}}{2} = -\cos x_m = -\cos(m/5)$

<u>for $n=0$</u>

$\Rightarrow \cancel{u_m^1 = 25u_{m-1}^0 - 48u_m^0 + 25u^0}$

$$\Rightarrow u_m^1 = 25 u_{m-1}^0 - 48 u_m^0 + 25 u_{m+1}^0 - u_m^{-1}$$

$$2u_m^1 = 25\left(u_{m-1}^0 + u_{m+1}^0\right) - 48 u_m^0 - 2\cos(m/5)$$

$$\Rightarrow u_m^1 = \frac{25}{2}\left(u_{m-1}^0 + u_{m+1}^0\right) - 24 u_m^0 - \cos\left(\frac{m}{5}\right)$$

<u>$m=0$</u> $\Rightarrow u_0^1 = -\sin(1)$

<u>$m=1$</u> $\Rightarrow u_1^1 = \frac{25}{2}\left(u_0^0 + u_2^0\right) - 24 u_1^0 - \cos(1/5)$

$$= \frac{25}{2}\left(0 + \sin(2/5)\right) - 24\sin(1/5) - \cos(1/5)$$

$$= -0.88040123806$$

#

$m=0 \Rightarrow u_0^1 = -0.84147098481 \cong -0.8415$

$m=1 \Rightarrow u_1^1 = -0.88040123806 \cong -0.8804$

$m=2 \Rightarrow u_2^1 = -0.72570865703 \cong -0.7257$

$m=3 \Rightarrow u_3^1 = -0.54207456129 \cong -0.5420$

$m=4 \Rightarrow u_4^1 = -0.3368346634 \cong -0.3368$

$m=5 \Rightarrow u_5^1 = 0$

for $n=1$

$$u_m^2 = 25\left(u_{m-1}^1 + u_{m+1}^1\right) - 48u_m^1 - u_m^0$$

$$u_m^2 = 25\left(u_{m-1}^1 + u_{m+1}^1\right) - 48u_m^1 - \sin\left(\frac{m}{5}\right)$$

$\therefore$ for $\Rightarrow$

$m=0 \Rightarrow u_0^2 = -0.9092974268 \cong -0.9093$

$m=1 \Rightarrow u_1^2 = 2.881224053 \cong 2.8812$

$m=2 \Rightarrow u_2^2 = -1.117537791 \cong -1.1175$

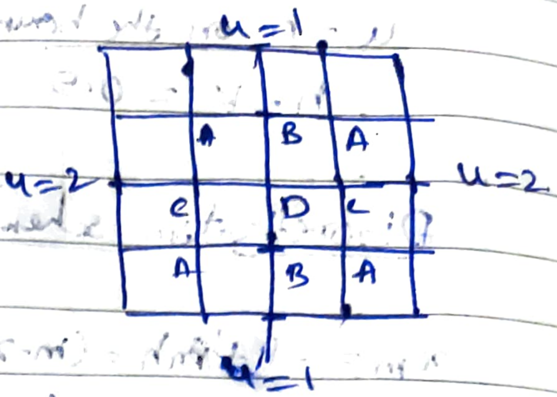
$m=3 \Rightarrow u_3^2 = -1.108521541 \cong -1.1085$

$m=4 \Rightarrow u_4^2 = 1.89884372 \cong 1.8988$

$m=5 \Rightarrow u_5^2 = -0.84147098481 \cong -0.8415$

5) $u_{xx}+u_{yy}=x^2+y^2$ $\quad x\in(-1,1)$ $\quad y\in(-1,1)$

$h=k=1/2$



from symmetry

$u_1^1=u_3^1=u_1^3=u_3^3=A$

$u_1^3=u_3^2=C \qquad u_2^1=u_2^3=B \qquad u_2^2=D$

$x_m=(m-2)/2 \qquad y_n=(n-2)/2$

$$h^2f(x,y)=\frac{1}{4}\times\frac{(m-2)^2+(n-2)^2}{4}=\frac{(m-2)^2+(n-2)^2}{16}$$

$$u_m^n=\frac{1}{4}\left[u_{m-1}^n+u_{m+1}^n+u_m^{n-1}+u_m^{n+1}-\frac{(m-2)^2+(n-2)^2}{16}\right]$$

at $m=1, n=1$

$\Rightarrow 4A=B+C+3-\frac{1}{8} \Rightarrow 4A-B-C=\frac{23}{8}$

at $m=2$ $n=1$

$\Rightarrow 4B=2A+D+1-\frac{1}{16} \Rightarrow -2A+4B-D=\frac{15}{16}$

at $m=1$ $n=2$

$\Rightarrow 4C=2A+D+2-\frac{1}{16} \Rightarrow -2A+4C-D=\frac{31}{16}$

at $m=2, n=2$

$\Rightarrow 4D=2C+2B \Rightarrow -B-C+2D=0$

using gauss siedel iterations for A, B, CD with initial values 1,1,1,1.

Iteration	A	B	C	D
1	1.2188	1.0938	1.3438	1.2188
2	1.3281	1.2031	1.4531	1.3281
3	1.3828	1.2578	1.5078	1.3828
4	1.4102	1.2852	1.5352	1.4102
5	1.4238	1.2988	1.5488	1.4238
6	1.4307	1.3057	1.5557	1.4307
7	1.4341	1.3091	1.5591	1.4341
8	1.4358	1.3108	1.5608	1.4358
9	1.4366	1.3116	1.5616	1.4366
10	1.4371	1.3121	1.5621	1.4371
11	1.4373	1.3123	1.5623	1.4373
12	1.4374	1.3124	1.5624	1.4374
13	1.4374	1.3124	1.5624	1.4374
14	1.4375	1.3125	1.5625	1.4375
15	1.4375	1.3125	1.5625	1.4375

$\Rightarrow \quad u_1^1 = u_3^1 = u_1^3 = u_3^3 = A = 1.4375$

$u_2^1 = u_2^3 = B = 1.3125$

$u_2^2 = D = 1.4375$

$u_1^2 = u_3^2 = C = 1.5625$

$\Rightarrow \quad u_1^1 = 1.4375 \qquad u_1^2 = 1.5625 \qquad u_1^3 = 1.4375$

$u_2^1 = 1.3125 \qquad u_2^2 = 1.4375 \qquad u_2^3 = 1.3125$

$u_3^1 = 1.4375 \qquad u_3^2 = 1.5625 \qquad u_3^3 = 1.4375$

6) $u_m^{n+1} = 2(1-p^2)u_m^n + p^2(u_{m-1}^n + u_{m+1}^n) - u_m^{n-1}$

$u_{xx} = u_{tt} \quad x \in (0,1)$

$u(x,0) = \frac{x^2}{16} \quad u_t(x,0) = 0 \quad u_x(0,t) = \frac{t}{5} \quad u(1,t) = \frac{(1+t)^2}{10}$

$h = 1/2 \quad k = 0.1 \quad x_m = \frac{m}{2} \quad y_n = \frac{n}{10}$

$c = 1, \quad P = \frac{ck}{h} = \frac{1 \cdot 01}{1/2} = 0.2$

$u_m^{n+1} = 1.92 u_m^n + 0.04(u_{m-1}^n + u_{m+1}^n) - u_m^{n-1}$

$u(x,0) = \frac{x^2}{10} \Rightarrow u_m^0 = \frac{m^2}{40} \quad u(1,t) = \frac{(1+t)^2}{10} \Rightarrow u_1^n = \frac{(n+10)^2}{1000}$

$\Rightarrow u_t(x,0) = 0 \Rightarrow u_m^1 - u_m^{-1} = 0 \Rightarrow u_m^1 = u_m^{-1}$

$u_x(0,t) = \frac{t}{5} \Rightarrow \frac{u_1^n - u_{-1}^n}{2h} = \frac{n}{50} \Rightarrow u_{-1}^n = u_1^n - \frac{n}{50}$

for $n=0$

$u_m^1 = 1.92\, u_m^0 + 0.04(u_{m-1}^0 + u_{m+1}^0) - \cancel{u_m^{-1}} \to u_m^1$

$\Rightarrow u_m^1 = 0.96\, u_m^0 + 0.02(u_{m-1}^0 + u_{m+1}^0)$

$\Rightarrow u_0^1 = 0.001 \quad u_1^1 = 0.026 \quad u_2^1 = 0.121$

for $n=1$

$u_m^2 = 1.92\, u_m^1 + 0.04(u_{m-1}^1 + u_{m+1}^1) - \cancel{u_m^0} \to \frac{m^2}{40}$

$\Rightarrow u_0^2 = 0.0032 \quad u_1^2 = 0.0298 \quad u_2^2 = 0.144$

$\therefore u_0^2 = u(0, 0.2) = 0.0032$

$u_1^2 = u(0.5, 0.2) = 0.0298$

$u_2^2 = u(1, 0.2) = 0.144.$

7) $\delta_t^2 u_m^n = r^2 \delta_x^2 \left[\theta u_m^{n+1} + (1-2\theta) u_m^n + \theta u_m^{n-1} \right]$

$\theta = 1/2 \qquad u_{tt} = u_{xx} \Rightarrow c^2 = 1 \Rightarrow c = 1$

$u(x,0) = \sin x \qquad u_t(x,0) = -\frac{1}{5}\cos x \qquad h = k = 0.25$

$u(0,t) = -\sin\left(\frac{t}{5}\right) \quad u(1,t) = \sin\left(1-\frac{t}{5}\right) \qquad r = \frac{ck}{h} = 1$

$\delta_t^2 u_m^n = u_m^{n+1} - 2u_m^n + u_m^{n-1}$

simplifying RHS $\Rightarrow$

$\Rightarrow r^2 \delta_x^2 \left[\theta u_m^{n+1} + (1-2\theta) u_m^n + \theta u_m^{n-1} \right]$

$= \theta\left[\delta_x^2 u_m^{n+1}\right] + (1-2\theta)\left[\delta_x^2 u_m^n\right] + \theta\left[\delta_x^2 u_m^{n-1}\right]$

$= \theta\left(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}\right) + \cancel{(1-2\theta)}\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right)$

$+ \theta\left(u_{m-1}^{n-1} - 2u_m^{n-1} + u_{m+1}^{n-1}\right)$

$= \frac{1}{2}\left[u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1} + u_{m-1}^{n-1} - 2u_m^{n-1} + u_{m+1}^{n-1} \right]$

$\left(\because \delta_x^2 u_m^n = u_{m+1}^n - 2u_m^n + u_{m-1}^n\right)$

$\Rightarrow 2u_m^{n+1} - 4u_m^n + 2u_m^{n-1} = u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}$

$+ u_{m-1}^{n-1} - 2u_m^{n-1} + u_{m+1}^{n-1}$

$\Rightarrow \boxed{4u_m^{n+1} - u_{m+1}^{n+1} - u_{m-1}^{n+1} = 4u_m^n - 4u_m^{n-1} + u_{m-1}^{n-1} + u_{m+1}^{n-1}}$

$u_t(x,0) = -\frac{1}{5}\cos x \Rightarrow \frac{u_m^1 - u_m^{-1}}{2k} = -\frac{1}{5}\cos x_m = -\frac{1}{5}\cos\frac{m}{4}$

$\Rightarrow \quad u_m^{-1} = u_m^1 + \frac{1}{10}\cos\left(\frac{m}{4}\right)$

#

for $n=0$

$$\Rightarrow 4u_1^1 - u_0^1 - u_2^1 = 4u_1^0 - 4u_1^{-1} + u_0^{-1} + u_2^{-1}$$

$$\Rightarrow 4u_1^1 - u_2^1 - u_0^1 = 4\sin\tfrac{1}{4} - (4u_1^1 - u_0^1 - u_2^{-1})$$
$$+ \tfrac{1}{10}\left(\cos(0) + \cos\tfrac{2}{4} - 4\cos\tfrac{1}{4}\right)$$

(a) $\Rightarrow 4u_1^1 - u_2^1 = 2\sin\frac{1}{4} + \frac{1}{20}\left[\cos 0 + \cos\frac{2}{4} - 4\cos\frac{1}{4}\right] - \sin\left(\frac{1}{20}\right)$

(b) $-u_1^0 + 4u_2^1 - u_3^1 = 2\sin\frac{1}{2} + \frac{1}{20}\left[\cos\frac{1}{4} + \cos\frac{3}{4} - 4\cos\frac{2}{4}\right]$

(c) $-u_2^1 + 4u_3^1 = 2\sin\frac{3}{4} + \frac{1}{20}\left[\cos\frac{3}{4} + \cos 1 - 4\cos\frac{3}{4}\right] + \sin\frac{19}{20}$

$$\Rightarrow \begin{aligned} 4u_1^1 - u_2^1 + 0u_3^1 &= 0.3449253 \\ -u_1^1 + 4u_2^1 - u_3^1 &= 0.8683646 \\ 0u_1^1 - u_2^1 + 4u_3^1 &= 2.10124949 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = \begin{bmatrix} 0.3449253 \\ 0.8683646 \\ 2.10124949 \end{bmatrix}$$

~~$\therefore u_1^1 = u(0.25, 0.25)$~~

$$\begin{aligned} \therefore\ u_0^1 &= u(0, 0.25) = -0.04997916927 \\ u_1^1 &= u(0.25, 0.25) = 0.1919\,3908162 \\ u_2^1 &= u(0.5, 0.25) = 0.422830\,95749 \\ u_3^1 &= u(0.75, 0.25) = 0.63\,102011299 \\ u_4^1 &= u(1, 0.25) = 0.81341550479. \end{aligned}$$

8) $u_{tt} = \frac{1}{25} u_{xx}$. $\qquad c = \frac{1}{5}$

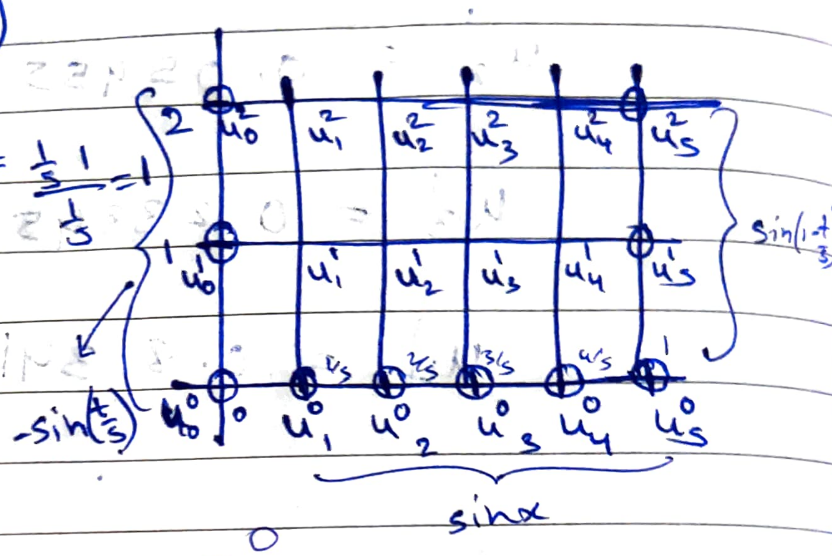
$u(0,t) = -\sin\left(\frac{t}{5}\right)$ $\qquad u(x,0) = \sin x$

$u_t(x,0) = \left(-\frac{1}{5}\cos x\right)$

$u(1,t) = \sin\left(1 - \frac{t}{5}\right)$

$h = \frac{1}{5} \quad k = 1 \quad r = \frac{\frac{1}{5} \cdot 1}{\frac{1}{5}} = 1$

$u_m^1 - u_m^{-1} = -\frac{2}{5}\cos x_m$



$$u_m^{n+1} = r^2 u_{m-1}^n + 2(1 - r^2) u_m^n + r^2 u_{m+1}^n - u_m^{n-1}$$

$$u_m^{n+1} = u_{m-1}^n + u_{m+1}^n - u_m^{n-1}$$

$n = 0$

$$u_m^1 = u_{m-1}^0 + u_{m+1}^0 \underbrace{- u_m^{-1}}$$

$$\Rightarrow -u_m^1 - \frac{2}{5}\cos x_m$$

$$\Rightarrow u_m^1 = \frac{u_{m-1}^0 + u_{m+1}^0}{2} - \frac{1}{5}\cos x_m$$

$\boxed{\text{for } m = 1, 2, 3, 4}$

$u_0^1 = -\sin\left(\frac{1}{5}\right) = -0.1986693308$

$u_5^1 = \sin\left(1 - \frac{1}{5}\right) = 0.7173560909$

$$u_1^1 = \frac{u_0^0 + u_2^0}{2} - \frac{1}{5}\cos(1/5)$$

$$= \frac{0 + \sin(2/5)}{2} - \frac{1}{5}\cos(1/5)$$

$$= -0.0013041444l$$

$$u_2^1 = \frac{u_1^0 + u_3^0}{2} - \frac{1}{5}\cos(2/5)$$

$$= \frac{\sin(1/5) + \sin(3/5)}{2} - \frac{1}{5}\cos(2/5)$$

$$= 0.19744370329$$

$$u_3^1 = \frac{u_2^0 + u_4^0}{2} - \frac{1}{5}\cos(3/5)$$

$$= \frac{\sin(2/5) + \sin(4/5)}{2} - \frac{1}{5}\cos(3/5)$$

$$= 0.38832009362$$

$$u_4^1 = \frac{u_3^0 + u_5^0}{2} - \frac{1}{5}\cos(4/5)$$

$$= \frac{\sin(3/5) + \sin(1)}{2} - \frac{1}{5}\cos(4/5)$$

$$= 0.56371538723$$

for $n=1$

$$u_m^2 = u_{m-1}^1 + u_{m+1}^1 - u_m^0$$

$$= u_{m-1}^1 + u_{m+1}^1 - \sin x_m$$

$$\rightarrow u_0^2 = -\sin\left(\frac{2}{5}\right) = -0.389418342 31$$

$$\rightarrow u_5^2 = \sin\left(1-\frac{2}{5}\right) = 0.5646424734$$

for $m = 1, 2, 3, 4$

$$\Rightarrow u_1^2 = u_0^1 + u_2^1 - \sin(x_1)$$

$$= -0.1998949583$$

$$\Rightarrow u_2^2 = u_1^1 + u_3^1 - \sin(x_2)$$

$$= -0.0024023931$$

$$\Rightarrow u_3^2 = u_2^1 + u_4^1 - \sin(x_3)$$

$$= 0.1965166 1712$$

$$\Rightarrow u_4^2 = u_3^1 + u_5^1 - \sin(x_4)$$

$$= 0.38832009362$$

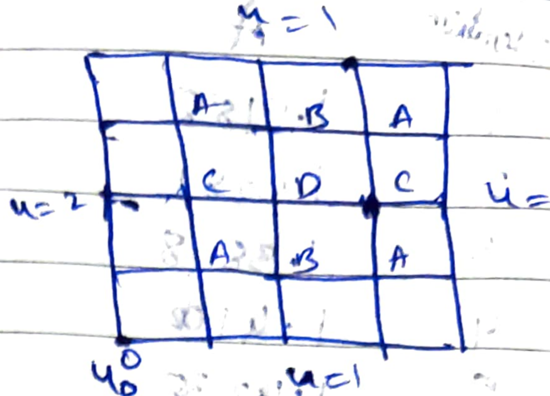
$u_0^2 = -0.3894$	$u_3^2 = 0.1965$
$u_1^2 = -0.1999$	$u_4^2 = 0.3883$
$u_2^2 = -0.0024$	$u_5^2 = 0.5646$

9) $u_{xx} + u_{yy} = 8xy \quad x \in (-1,1) \quad y \in (-1,1)$

at $x = -1, 1 \quad u = 2$

at $y = -1, 1 \quad u = 1$

$h = k = 0.5$



By symmetry ⇒

$u_1^1 = u_3^1 = u_1^3 = u_3^3 = A$

$u_2^1 = u_2^3 = B \qquad u_2^2 = D \qquad x_m = \frac{m}{2} - 1 = (m-2)/2$

$u_1^2 = u_3^2 = C$

$y_n = (n-2)/2$

$$u_m^n = \frac{1}{4}\left[u_{m-1}^n + u_{m+1}^n + u_m^{n+1} + u_m^{n-1} - \frac{(m-2)(n-2)}{2}\right]$$

at $m=1, n=1$

$\Rightarrow 4A = B + C + 1 + 2 - \frac{1}{2} \Rightarrow 4A - B - C = 5/2$

at $m=2 \; n=1$

$\Rightarrow 4B = 2A + 1 + D \Rightarrow -2A + 4B - D = 1$

at $m=2, n=2$

$\Rightarrow 4D = 2B + 2C \Rightarrow -B + 2D - C = 0$

at $m=1, n=2$

$\Rightarrow 4C = 2A + D + 2 \Rightarrow -2A - D + 4C = 2$

Taking initial values as $A = B = C = D = 1$

& applying Gauss-Siedel Iterative method

Iteration	A	B	C	D
1	1.125	1.0625	1.3125	1.1875
2	1.2188	1.1563	1.4063	1.2813
3	1.2656	1.2031	1.4531	1.3281
4	1.2891	1.2266	1.4766	1.3516
5	1.3008	1.2383	1.4883	1.3633
6	1.3066	1.2441	1.4941	1.3691
7	1.3096	1.2471	1.4971	1.3721
8	1.3110	1.2485	1.4985	1.3735
9	1.3118	1.2493	1.4993	1.3743
10	1.3121	1.2496	1.4996	1.3746
11	1.3123	1.2498	1.4998	1.3748
12	1.3124	1.2499	1.4999	1.3749
13	1.3125	1.250	1.5	1.375
14	1.3125	1.250	1.5	1.375

$\therefore \; u_1^1 = u_3^1 = u_1^3 = u_3^3 = A = 1.3125$

$u_2^1 = u_2^3 = B = 1.25$

$u_2^2 = D = 1.375$

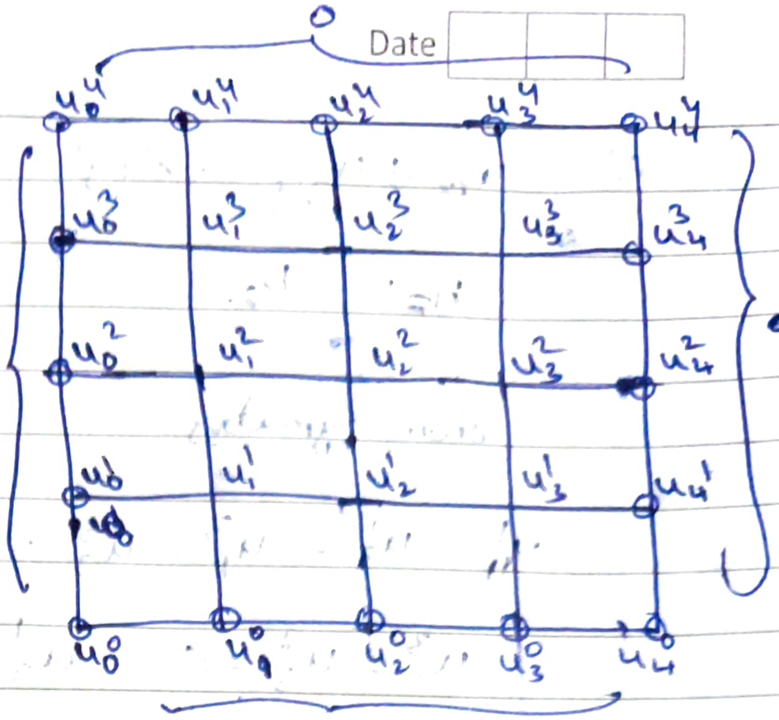
$u_1^2 = u_3^2 = C = 1.5$

$\therefore \; u_1^1 = 1.3125 \qquad u_1^2 = 1.5 \qquad u_1^3 = 1.3125$

$u_2^1 = 1.25 \qquad u_2^2 = 1.375 \qquad u_2^3 = 1.25$

$u_3^1 = 1.3125 \qquad u_3^2 = 1.5 \qquad u_3^3 = 1.3125$

Date



10) $u_{xx} + u_{yy} = -2$

$u = 0$ on the boundary

$h = k = 0.5$

<u>Discretization scheme</u>

$x_m = -1 + mh = (m-2)/2$

$y_n = -1 + nk = (n-2)/2$

$$u_m^n = \frac{1}{4}\left[u_{m-1}^n + u_{m+1}^n + u_m^{n+1} + u_m^{n-1} - h^2(-2)\right] \quad -(1)$$

<u>From symmetry</u> →

$u_1^1 = u_3^1 = u_1^3 = u_3^3 = A \qquad u_{2\,2} = B$

$u_1^2 = u_2^1 = u_3^2 = u_2^3 = C$

<u>From (1)</u>

$m=1,\ n=1 \Rightarrow 4A = 2C + 1/2 \Rightarrow 8A - 2C = 1$

$m=2\ n=1 \Rightarrow 4C = 2A + B + 1/2 \Rightarrow -4A - 2B + 4C = 1$

$m=2,\ n=2 \Rightarrow 4B = 4C + 1/2 \Rightarrow 0A + 8B - 8C = 1$

By solving we get $\Rightarrow C = \frac{8B-1}{8} \quad A = \frac{2C+1}{8} = \frac{8B+3}{32}$

$\Rightarrow -\cancel{4}\left(\frac{8B+3}{\cancel{32}\,8}\right) - 2B + \frac{8B-1}{2} = 1$

$\Rightarrow -8B - 3 - 16B + 32B - 4 = 8 \Rightarrow 8B = 15 \Rightarrow B = \frac{15}{8}$

$\Rightarrow C = \frac{7}{4} \qquad A = 9/16$

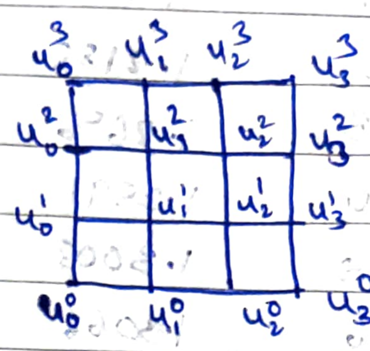
$\Rightarrow u_2^2 = B = 15/8 = 1.875$

$$\boxed{\therefore u(0,0) = 1.875}$$

11) $u_{xx} + u_{yy} = 0$

$u(x,y) = e^{3x} \cdot \cos 3y$ on boundary

$h = k = 1/3 \quad x, y \in (0,1)$



$x_m = 0 + mh = m/3$

$y_n = n/3$

$\Rightarrow$ on boundary $u_m^n = e^m \cos n$

Using $u_1^1 = u_1^2 = u_2^1 = u_2^2 = 0$ as initial guess for gauss-Siedel iterations:-

Standard 5pt formula -

$$\Rightarrow u_m^n = \frac{1}{4}\left[u_{m-1}^n + u_{m+1}^n + u_m^{n-1} + u_m^{n+1}\right]$$

Iteration	u_1^1	u_2^1	u_1^2	u_2^2
1	0.8146	4.7640	−0.5731	−2.8707
2	1.8624	4.3082	−1.0289	−3.0986
3	1.6345	4.1943	−1.1428	−3.1555
4	1.5775	4.1658	−1.1713	−3.1698
5	1.5633	4.1587	−1.1784	−3.1733
6	1.5597	4.1569	−1.1802	−3.1742
7	1.5588	4.1565	−1.1807	−3.1745
8	1.5586	4.1564	−1.1808	−3.1745
9	1.5585	4.1563	−1.1808	−3.1745
10	1.5585	4.1563	−1.1808	−3.1745

$\therefore u_1^1 = 1.5585 \qquad u_1^2 = -1.1808$

$u_2^1 = 4.1563 \qquad u_2^2 = -3.1745$