Lecture 24

Lemma 2:— Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Let $\varepsilon > 0$ be given. Assume that $\omega_f(x) < \varepsilon$ $\forall x \in [a,b]$.

Then there exists a $\delta > 0$ (depends only on ε) such that every closed subinterval $T \subseteq [a,b]$, we have

$$\Omega_f(T) < \varepsilon \quad \text{whenever the length}(T) < \delta$$

proof:— Given $\omega_f(x) = \lim\limits_{\delta \to 0^+} \Omega_f(B(x,\delta)) < \varepsilon$

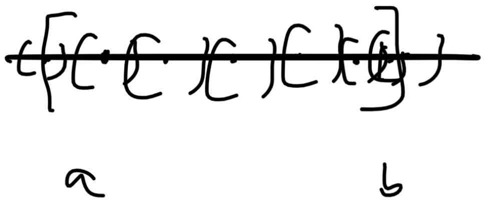
$$= \inf_{\delta > 0} \left(\Omega_f(B(x,\delta))\right) < \varepsilon$$

$\Rightarrow$ there exists $\delta_x > 0$ such that

$$\Omega_f(B(x,\delta_x)) < \varepsilon \quad \forall$$

Now $\left\{ B(x, \delta_x/2) \;\middle|\; x \in [a,b] \right\}$ is an open cover of $[a,b]$

i.e. $[a,b] \subseteq \bigcup_{x \in [a,b]} B(x, \frac{\delta_x}{2})$.



Since $[a,b]$ is a compact set, there exists finitely many $\delta_{x_1}, \dots, \delta_{x_k} > 0$ such that

$$[a,b] \subseteq \bigcup_{i=1}^{k} B(x_i, \frac{\delta_{x_i}}{2})$$

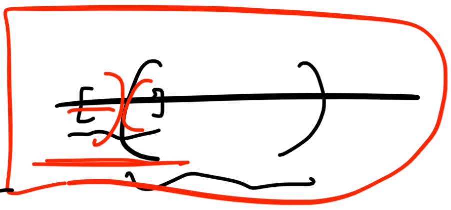
Choose $\delta = \min \left\{ \frac{\delta_{x_1}}{2}, \frac{\delta_{x_2}}{2}, \dots, \frac{\delta_{x_k}}{2} \right\} > 0$.

For any $T = [c,d] \subseteq [a,b]$ with $\text{length}(T) = d - c < \delta \leq \frac{\delta_{x_i}}{2}$ $\forall i$

& $T \subseteq \bigcup_{i=1}^{k} B(x_i, \frac{\delta_{x_i}}{2})$ $\qquad l(T) < \frac{\delta_{x_i}}{2}$ $\forall i$

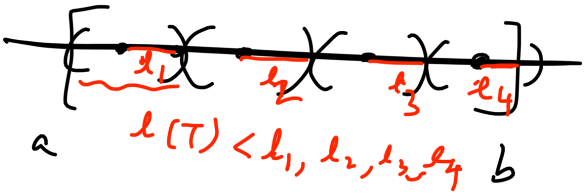
$\Rightarrow$ $T \cap B(x_{i_0}, \frac{\delta_{x_{i_0}}}{2}) \neq \emptyset$ for some x_{i_0}.

& $\text{length}(T) < \frac{\delta_{x_{i_0}}}{2}$



$\Rightarrow$ $T \subseteq B(x_{i_0}, \frac{\delta_{x_{i_0}}}{2})$

$\Rightarrow$ $\Omega_f(T) \leq \Omega_f(B(x_{i_0}, \frac{\delta_{x_{i_0}}}{2}))$



$< \varepsilon$ $\qquad\qquad T \subseteq [\ ,\ )$

Thus $\Omega_f[T) < \varepsilon$, whenever $l[T) < \delta$.

<u>Lemma 3</u>:— Let $f : [a,b] \to \mathbb{R}$ be bounded.

For each $\varepsilon > 0$, define $J_\varepsilon = \{ x \mid x \in [a,b],\ \underline{\omega_f(x) \geq \varepsilon} \}$.

Then J_ε is a closed set.

<u>proof</u>:— To show: $\overline{J_\varepsilon} = J_\varepsilon$.

$$\supseteq \checkmark$$

Let $x \in \overline{J_\varepsilon}$. To show: $x \in J_\varepsilon$.

Suppose $x \notin J_\varepsilon$.

$$\Rightarrow \quad \omega_f(x) < \varepsilon.$$

$$\omega_f(x) = \inf_{\delta > 0} \left(\Omega_f[x, \delta) \right)$$

$\Rightarrow$ there exists $\delta > 0$ such that

$$\Omega_f(\underline{B[x, \delta]}) < \varepsilon$$

$\Rightarrow \quad B(x,\delta) \cap J_\varepsilon = \phi$

$\Rightarrow \quad x$ is not a limit of J_ε.

$\Rightarrow \quad x \notin \overline{J_\varepsilon}$.

This is a Contradiction.

$\therefore \quad x \in J_\varepsilon$.

$\therefore \quad \overline{J_\varepsilon} \subseteq J_\varepsilon$. & hence $\overline{J_\varepsilon} = J_\varepsilon$.

Theorem (Lebesgue's Criterion for Riemann integration):-

Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable on $[a,b]$ $\Longleftrightarrow$ f is Continuous a.e on $[a,b]$

proof:-

$\boxed{\Rightarrow}$: Assume f is R-integrable on $[a,b]$.

To show: f is continuous a.e on $[a,b]$.

Suppose f is not Continuous a.e on $[a,b]$.

Let $D = \{ x \in [a,b] \,/\, f \text{ is not continuous at } x \}$.

Then by our supposition, we have $m(D) > 0$.

By Lemma 1, we have

$$D = \{ x \in [a,b] \,/\, \omega_f(x) > 0 \}.$$

Write $D = \bigcup_{r=1}^{\infty} D_r$,

where $D_r = \{ x \in [a,b] \,/\, \omega_f(x) \geq \frac{1}{r} \}$. $\forall r \geq 1$

By Lemma 3, D_r is a closed set, $\forall r \geq 1$.

Since $m(D) > 0$, there is some $r \geq 1$ such that

$m(D_r) > 0$

Thus $m(D_r) > 0$, then there is some $\varepsilon > 0$ such that for any countable collection of open intervals $\{ I_{r,n} \}_{n \geq 1}$ such that

$$D_r \subseteq \bigcup_{n=1}^{\infty} I_{r,n} \quad \& \quad \sum_{n=1}^{\infty} l(I_{r,n}) \geq \varepsilon.$$

(By using the def. of m:

$$m(D_r) = \inf\left(\left\{ \underline{\sum_{n=1}^{\infty} l(I_n)} \;\middle|\; D_r \subseteq \bigcup_{n=1}^{\infty} I_n \right\}\right) > 0$$

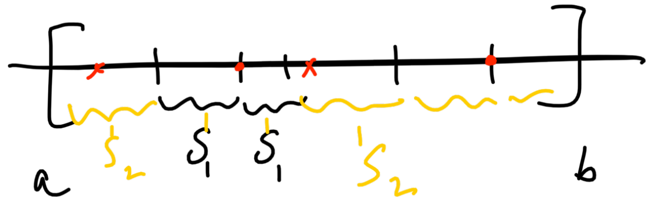
For any partition $P = \{ a = x_0 < x_1 < \cdots < x_n = b \}$ of $[a, b]$, we have

$$U(P, f) - L(P, f) = \sum_{k=1}^{n} \left(M_k(f) - m_k(f)\right)(x_k - x_{k-1}),$$

where $M_k(f) = \sup_{x \in [x_{k-1}, x_k]} (f(x))$.

$m_k(f) = \inf_{x \in [x_{k-1}, x_k]} (f(x))$.

$$= S_1 + S_2,$$



where

S_1 = the sum of those terms coming from subintervals $[x_{k-1}, x_k]$'s containing points of D in their interior.

& $S_2 =$ the remaining terms other than S_1.

Let $A = \left\{ x \in D_r \;\middle|\; \begin{array}{l} x \text{ is not an interior pt. of} \\ \text{any of the subintervals} \\ [x_{k-1}, x_k] \end{array} \right\}$

Then A is a finite set.

& $$D_r \setminus A \subseteq \bigcup_{\substack{k \\ (x_{k-1}, x_k) \\ \text{contains a point of } D_r \text{ in its interior.}}} (x_{k-1}, x_k)$$

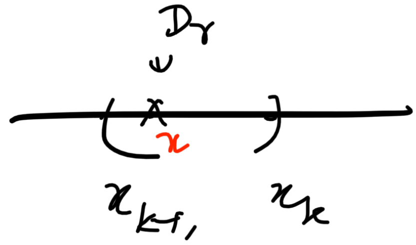
is an open covering of $D_r \setminus A$.

& $m(D_r) = m(D_r \setminus A)$ $\quad (\because m(A) = 0)$

Then by the choice of $\varepsilon > 0$, we have

$$\sum_{\substack{k \\ (x_{k-1}, x_k) \text{ contains pts of } D_r \text{ in its interior.}}} \text{length}(x_{k-1}, x_k) \geq \varepsilon \quad ✓$$

if (x_{k-1}, x_k) Contains a point of D_γ in its interior, then $\underline{\underline{M_k}}(f) - \underline{\underline{m_k}}(f) \geq \frac{1}{\gamma}$.



$$\therefore S_1 = \sum_{\substack{k \\ (x_{k-1}, x_k) \text{ Contains a pt. of } D_\gamma \\ \text{in its interior.}}}^{n} \left(M_k(f) - m_k(f)\right)\left(x_k - x_{k-1}\right)$$

$$\omega_f(x) \geq \frac{1}{\gamma}.$$
$$\omega_f(x) = \inf_{\delta > 0}\left(\Omega_f\left(B(x, \delta)\right)\right)$$

$$\geq \sum_{\substack{k \\ ''}} \frac{(x_k - x_{k-1})}{\gamma}$$

$$\geq \frac{\varepsilon}{\gamma}.$$

Thus for any partition P of $[a, b]$, we have

$$U(P, f) - L(P, f) = S_1 + S_2$$
$$\geq S_1$$
$$\geq \frac{\varepsilon}{\gamma}.$$

$\Rightarrow$ f is not satisfying the Riemann's Condition.

$\Rightarrow$ f is not R-integrable

$\underline{\underline{\Rightarrow\Leftarrow}}$. $\therefore m(D) = 0$.

$\Rightarrow$ f is Contin. a.e.