Lecture 27

Proof:— Given (i) $\{f_n\}$ is bounded by M.

$$\Rightarrow \quad |f_n(x)| \leq M \quad \forall x, \quad \forall n \geq 1.$$

(ii) $f_n(x) \to f(x)$ p.w a.e.

(iii) Each f_n is supported on a set E of finite measure.

It is clear that f is supported on a set E of finite measure.

$\because \{f_n(x)\} \to f(x)$ $\forall x \notin E$.
$\phantom{\because \{}\overset{\shortparallel}{0} \qquad \overset{\shortparallel}{0}$

Also.

$$|f(x)| = |f(x) - f_n(x) + f_n(x)| \quad \forall n \geq 1$$

$$\leq |f(x) - f_n(x)| + |f_n(x)| \quad \forall n \geq 1$$

$$\leq |f(x) - f_n(x)| + M \quad \forall n \geq 1.$$

$$\Rightarrow \quad |f(x)| \leq \underbrace{\lim_{n\to\infty}\left(|f(x) - f_n(x)|\right)}_{=\,0 \text{ a.e}} + M$$

$\Rightarrow \quad |f(x)| \le M \quad$ a.e $\quad \forall x \in E$.

$\therefore$ f is bounded a.e.

To show: $\int_E |f_n - f| \longrightarrow 0$ as $n \to \infty$.

Let $\varepsilon > 0$.

By Egorov's theorem (Littlewood 3rd principle), there exists a measurable set $A_\varepsilon \subseteq E$ such that $m(E \setminus A_\varepsilon) \le \varepsilon$ &

$f_n \longrightarrow f$ uniformly on A_ε.

$\therefore$ there exists $n_0 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall x \in A_\varepsilon, \ \forall n \ge n_0.$$

Now
$$\int_E^{\mathcal{L}} |f_n(x) - f(x)|\, dx = \int_{A_\varepsilon \cup (E \setminus A_\varepsilon)} |f_n(x) - f(x)|\, dx$$

$$= \int_{A_\varepsilon} |f_n(x) - f(x)|\,dx + \int_{E \setminus A_\varepsilon} |f_n(x) - f(x)|\,dx$$

$$< \int_{A_\varepsilon} \varepsilon\,dx + \int_{\underline{E \setminus A_\varepsilon}} \left(|f_n(x)| + \underline{|f(x)|}\right) dx \quad \forall n \geq n_0$$

$$< \varepsilon\, m(A_\varepsilon) + 2M \int_{\underline{E \setminus A_\varepsilon}} dx. \quad \left(\because |a-b| \leq |a| + |b|\right) \quad \forall n \geq n_0$$

$$< \varepsilon\, m(A_\varepsilon) + 2M\, m(E \setminus A_\varepsilon)$$

$$< \varepsilon\, m(A_\varepsilon) + 2M\varepsilon$$

$$= \varepsilon\,(m(A_\varepsilon) + 2M) \quad \Bigg\} \forall n \geq n_0.$$

$$< \varepsilon\,(m(E) + 2M)$$

$$\therefore \int_E |f_n(x) - f(x)|\,dx < \varepsilon\,(m(E) + 2M) \quad \forall n \geq n_0.$$

$$\Rightarrow \int_E |f_n(x) - f(x)|\,dx \longrightarrow 0 \quad \text{as } n \to \infty.$$

Now $\quad f_n - f \leq |f_n - f| \qquad \forall n \geq 1$

$$\Rightarrow \int_E (f_n - f) \leq \int_E |f_n - f|$$

$$\Rightarrow \left(\int_E f_n - \int_E f \right) \leq \int_E |f_n - f|$$

$$\Rightarrow \left| \int_E f_n - \int_E f \right| \leq \left| \int_E |f_n - f| \right| = \int_E |f_n - f|$$

$$\Rightarrow \quad 0 \leq \left| \int_E f_n - \int_E f \right| \leq \int_E |f_n - f|$$

As $n \to \infty$, $\quad \left| \int_E f_n - \int_E f \right| \longrightarrow 0$

$$\Rightarrow \quad \int_E f_n \longrightarrow \int_E f \quad \text{as } n \to \infty.$$

<u>Corollary</u>:- Suppose $f \geq 0$ is bounded & supported on a set E of finite measure & $\int_E f = 0$.

Then $f = 0$ a.e.

<u>proof</u>:- For each $k \geq 1$, define

$$E_k = \left\{ x \in E \,/\, f(x) \geq \frac{1}{k} \right\}.$$

$E_1 \subseteq E_2 \subseteq \cdots$

$$\{x \in E / f(x) > 0\} = \bigcup_{k=1}^{\infty} E_k$$

By definition of E_k, we have $\frac{1}{k} \chi_{E_k} \leq f$. $\forall\, k \geq 1$.

$$\left(\because \; \frac{1}{k} \chi_{E_k}(x) = \begin{cases} 1/k \text{ if } x \in E_k \\ 0 \quad \text{otherwise.} \end{cases} \right.$$

$$\Rightarrow \text{if } x \in E_k, \; = \frac{1}{k} \leq f(x)$$

$$\left. \text{if } x \notin E_k, \; \frac{1}{k} \chi_{E_k} = 0 \leq f(x). \right)$$

$$\therefore \int_E \frac{1}{k}\chi_{E_k} \le \int_E f. \qquad \left(\because f \le g \quad \int f \le \int g\right)$$

$$\Rightarrow \frac{1}{k} m(E_k) \le \int_E f = 0$$

$$\Rightarrow 0 \le \frac{1}{k} m(E_k) \le 0. \qquad \forall k \ge 1$$

$$\Rightarrow m(E_k) = 0 \qquad \forall k \ge 1.$$

$$\therefore \{x \in E / f(x) > 0\} = \bigcup_{k=1}^{\infty} E_k$$

$$\Rightarrow m(\{x \in E / f(x) > 0\}) = m\left(\bigcup_{k=1}^{\infty} E_k\right)$$

$$\le \sum_{k=1}^{\infty} m(E_k)$$

$$= \sum_{k=1}^{\infty} 0$$

$$= 0.$$

$$\Rightarrow m(\{x \in E / f(x) > 0\}) = 0.$$

$$\Rightarrow f = 0 \text{ a.e. } (\because f \ge 0).$$

Theorem:- Suppose $f:[a,b] \longrightarrow \mathbb{R}$ is a Riemann integrable function. Then f is measurable & f is Lebesgue integrable & $\overset{\mathcal{R}}{\int\limits_{[a,b]}} f = \overset{\mathcal{L}}{\int\limits_{[a,b]}} f$.

proof:- Given f is Riemann integrable.

$\Rightarrow$ f is bounded $\qquad \left(\overset{\mathcal{R}}{\int\limits_{[a,b]}} f < \infty \right)$.

Say $|f(x)| \leq M$ $\forall x \in [a,b]$, for some $M > 0$.

By definition of Riemann integrability we have

$$\overline{\int_a^b} f = \inf_{\substack{\psi \text{ step function} \\ \text{with } \psi \geq f}} \left(\int \psi \right)$$

& $$\underline{\int_a^b} f = \sup_{\substack{\varphi \text{ step function} \\ \text{with } \varphi \leq f}} \left(\int \varphi \right).$$

& $$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

Then we can construct sequences of step functions $\{\varphi_k\}$ & $\{\psi_k\}$ such that

$$|\varphi_k(x)| \le M \quad \& \quad |\psi_k(x)| \le M$$

$$\forall x \in [a, b], \ \forall k \ge 1$$

with

$$\varphi_1(x) \le \varphi_2(x) \le \dots \le f \le \dots \le \psi_2(x) \le \psi_1(x)$$

&

$$\lim_{k\to\infty} \overset{R}{\int_{[a,b]}} \varphi_k(x)\, dx = \underline{\int_a^b} f(x)\, dx$$

$$= \overline{\int_a^b} f(x)\, dx = \lim_{k\to\infty} \overset{R}{\int_{[a,b]}} \psi_k(x)\, dx$$

$$= \overset{R}{\int_{[a,b]}} f$$

we have

$$\overset{R}{\int_{[a,b]}} \varphi_k = \overset{L}{\int_{[a,b]}} \varphi_k, \quad \overset{R}{\int_{[a,b]}} \psi_k = \overset{L}{\int_{[a,b]}} \psi_k.$$

$$\forall k \ge 1$$

$\Big(\because \ \varphi = \sum_{i=1}^{r} c_i \chi_{[a_i, b_i]}$. Then $\int^{\mathcal{L}} \varphi = \sum_{i=1}^{r} c_i \, m([a_i, b_i])$

step function

$$= \sum_{i=1}^{r} c_i (b_i - a_i) \checkmark$$

$$\int^{\mathcal{R}} \varphi = \int^{\mathcal{R}} \sum_{i=1}^{r} c_i \chi_{[a_i, b_i]}(x) \, dx$$

$$= \sum_{i=1}^{r} c_i \int^{\mathcal{R}} \chi_{[a_i, b_i]}(x) \, dx$$

$$= \sum_{i=1}^{r} c_i \int^{\mathcal{R}}_{[a_i, b_i]} 1 \, dx$$

$$= \sum_{i=1}^{r} c_i (b_i - a_i)$$

$$= \int^{\mathcal{L}} \varphi \Big)$$

$$\therefore \lim_{k \to \infty} \int^{\mathcal{L}}_{[a,b]} \varphi_k(x) \, dx = \lim_{k \to \infty} \int^{\mathcal{L}}_{[a,b]} \varphi_k(x) \, dx = \int^{\mathcal{R}}_{[a,b]} f(x) \, dx.$$

Let $\tilde{\varphi}(x) = \lim\limits_{k\to\infty} \varphi_k(x)$,

$\tilde{\psi}(x) = \lim\limits_{k\to\infty} \psi_k(x)$

Then $\tilde{\varphi} \le f \le \tilde{\psi} \longrightarrow (*)$.

& $\tilde{\varphi}$, $\tilde{\psi}$ are measurable.

$\therefore$ By Bounded Convergence Theorem, we have

$$\lim_{k\to\infty} \overset{L}{\int\limits_{[a,b]}} \varphi_k(x)\,dx = \overset{L}{\int\limits_{[a,b]}} \left(\lim_{k\to\infty} \varphi_k(x)\right) dx$$

$$= \overset{L}{\int\limits_{[a,b]}} \tilde{\varphi}(x)\,dx$$

& $$\lim_{k\to\infty} \overset{L}{\int\limits_{[a,b]}} \psi_k(x)\,dx = \overset{L}{\int\limits_{[a,b]}} \tilde{\psi}(x)\,dx.$$

Thus we have

$$\overset{L}{\int\limits_{[a,b]}} \tilde{\varphi}(x)\,dx = \overset{L}{\int\limits_{[a,b]}} \tilde{\psi}(x)\,dx = \overset{R}{\int\limits_{[a,b]}} f(x)\,dx.$$

$$\Rightarrow \quad \int_{[a,b]}^{\mathcal{L}} \underbrace{\left(\tilde{\psi}(x) - \tilde{\varphi}(x)\right)}_{\geq 0} dx = 0$$

$$\& \qquad \tilde{\psi} \geq \tilde{\varphi} \quad (\text{by } (*)) .$$

$$\Rightarrow \qquad \tilde{\psi} - \tilde{\varphi} = 0 \quad a.e$$

by above Corollary

$$\Rightarrow \quad \tilde{\psi} = \tilde{\varphi} \quad a.e$$

But $\tilde{\varphi} \leq f \leq \tilde{\psi}$, therefore

$$\tilde{\varphi} = \tilde{\psi} = f \quad a.e \quad , \quad \tilde{\varphi}, \tilde{\psi} \text{ measurable}$$

$$\Rightarrow \quad f \text{ is measurable}$$

Also
$$\lim_{k\to\infty} \int_{[a,b]}^{\mathcal{L}} \varphi_k = \int_{[a,b]}^{\mathcal{L}} \underbrace{\left(\lim_{k\to\infty} \varphi_k\right)}_{= \tilde{\varphi}}$$

$$= \int_{[a,b]}^{\mathcal{L}} f$$

But $\displaystyle \mathcal{L}\int_{[a,b]} \tilde{\varphi} = \mathcal{R}\int_{[a,b]} f$.

$$\therefore \quad \mathcal{L}\int_{[a,b]} f = \mathcal{R}\int_{[a,b]} f < \infty$$

$\therefore$ f is Lebesgue integrable.