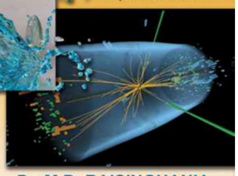
Revised Edition

FLUID DYNAMICS

WITH COMPLETE HYDRODYNAMICS AND BOUNDARY LAYER THEORY

For Honours, Postgraduate and M. Phil. Students of All Indian Universities, Engineering Students and Various Competitive Examinations



Dr. M.D. RAISINGHANIA

S. CHAND

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Dr. M.D. RAISINGHANIA
M.Sc., Ph.D.,
Formerly Reader & Head Department of Mathematics
S.D. College, Muzaffarnagar, U.P.

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FOREWORD

"Fluid dynamics" is one of the most important part of the recent interdisciplinary activities concerning engineering and technological developments. For this to be accomplished, Dr. M.D. Raisinghania has provided, in a single volume, nearly all the basic theory needed for a thorough grounding in the subject. The book is teachable, readable and most suitable for adoption for postgraduate and honours students of Indian universities. Numerous examples with illustrations and exercises at every step have made the book more useful for students, teachers and researchers.

I congratulate the author on his untiring efforts and I warmly recommend the book to scientists and engineers as well.

P.N. TANDON,
Professor and Head,
Department of Mathematics,
H.B.T.I. Kanpur, U.P.

Dedicated to the
memory of my parents

– M.D. Raisinghania

PREFACE TO THE ELEVENTH EDITION

Questions asked in recent papers of various university examinations have been inserted at appropriate places. Some articles solutions have been re-written. A miscellaneous set of problems is also provided at the end of this book. I hope that these changes will make the book more useful to the reader.

Suggestions for further improvement of the book will be gratefully received.

M.D. RAISINGHANIA

PREFACE TO THE NINTH REVISED EDITION

Due to recent changes in syllabi of various universities, a thorough revision of the book was overdue. Accordingly, each chapter of this book has been enlarged and rearranged. References to the latest papers of various universities and I.A.S. have been inserted at proper places. A set of objective problems has been provided at the end of chapters. An additional chapter, namely, Flow of a compressible viscous fluids has been added. Some matter of previous edition has been rearranged in form of two new chapters, namely, "The use of conformal representations. Aerofoils" and "Theory of very slow motion".

I hope that these changes will make the material more accessible and more attractive to the reader. All valuable suggestions for further improvement of the book will be highly appreciated.

M.D. RAISINGHANIA

PREFACE TO THE SECOND EDITION

It gives me great pleasure to inform the reader that the previous edition of this book has been improved, well-organised and made up-to-date in the light of the latest syllabi of the various universities and suggestions received from the learned teachers, students and engineers from various universities and institutions of India. The following major changes have been made in the present edition:

- Errors and omissions of previous edition have been corrected.
- A detailed index has been provided at the end of this book.
- More solved examples have been added so that reader may gain more confidence in technique of solving problems.
- Matter of the previous edition has been re-organised so that now each topic gets its proper place in the book.
- Almost all chapters have been rewritten so that in the present form, the reader will not find any difficulty in understanding the subject matter.

In view of the above large scale improvements it can be assured that this new edition will surely prove more useful to the reader.

I am really thankful to the Director, Shri R.K. Gupta and Manager, Shri T. N. Goel for showing keen interest throughout the publication of this book. I am also thankful to Mrs. Dipika Sen for bringing out the book in nice form at a short period.

Suggestions for further improvement of the book will be gratefully received.

M.D. RAISINGHANIA

CONTENTS

THE GREEK ALPHABET

alpha	α	A	nu	ν	N
beta	β	B	xi	ξ	Ξ
gamma	γ	Γ	omicron	o	O
delta	δ	Δ	pi	π	Π
epsilon	ε	E	rho	ρ	P
zeta	ζ	Z	sigma	σ	Σ
eta	η	H	tau	τ	T
theta	θ	Θ	upsilon	υ	Υ
iota	ι	I	phi	ϕ	Φ
kappa	κ	K	chi	χ	X
lambda	λ	Λ	psi	ψ	Ψ
mu	μ	M	omega	ω	Ω

CHAPTER

1 Introduction

1.1. General description.

Fluid dynamics or Hydrodynamics is that branch of science which is concerned with the study of the motion of fluids or that of bodies in contact with fluids. Fluids are classified as liquids and gases. The former are not sensibly compressible except under the action of heavy forces whereas the latter are easily compressible, and expand to fill any closed space.

It is well known that matter is made up of molecules or atoms which are always in a state of random motion. In fluid dynamics the study of individual molecule is neither necessary nor appropriate from the point of view of use of mathematical methods. Hence we consider the macroscopic (bulk) behaviour of fluid by supposing the fluid to be continuously distributed in a given space. This assumption is known as the *continuum hypothesis*. This continuum concept of matter allows us to subdivide a fluid element indefinitely. Furthermore, we define a *fluid particle* as the fluid contained within the physically infinitesimal volume.

1.2. Isotropy.

A fluid is said to be *isotropic* with respect to some property (pressure, density etc.) if that property is the same in all directions at a point. A fluid is said to be *anisotropic* with respect to a property if that property is not the same in all directions.

1.3. Some basic properties of the fluid.

(*i*) **Density, specific weight, specific volume.**

The density of a fluid is defined as the mass per unit volume. Mathematically, the density ρ at a point *P* may be defined as

$$\rho = \lim_{\delta v \to 0} \frac{\delta m}{\delta v},$$

where δv is the volume element around *P* and δm is the mass of the fluid within δv.

The specific weight γ of a fluid is defined as the weight per unit volume. Thus $\gamma = \rho g$, where g is the acceleration due to gravity.

The specific volume of a fluid is defined as the volume per unit mass and is clearly the reciprocal of the density.

(*ii*) **Pressure.**

When a fluid is contained in a vessel, it exerts a force at each point of the inner side of the vessel. Such a force per unit area is known as *pressure*. Mathematically, the pressure *p* at a point P may be defined as

$$p = \lim_{\delta S \to 0} \frac{\delta F}{\delta S}$$

where δS is an elementary area around P and δF is the normal force due to fluid on δS.

(*iii*) **Temperature.**

Suppose two bodies of different heat content are brought into contact while isolated from all other bodies. Then some thermal energy will move from one body into the other body. The body from where the thermal energy moves is said to be at a higher temperature while the body into which the energy flows is said to be at a lower temperature. When two bodies are in thermal equilibrium then they are said to have a common property, known as *temperature* T.

(*iv*) **Thermal conductivity.** **[Himanchal 2009]**

The well-known *Fourier's heat conduction law* states that the conductive heat flow per unit area (or heat flux) q_n is proportional to the temperature decrease per unit distance in a direction normal to the area through which the heat is flowing. Thus, mathematically

$$q_n \propto -\frac{\partial T}{\partial n} \qquad \text{so that} \qquad q_n = -k\frac{\partial T}{\partial n},$$

where k is said to be the *thermal conductivity*.

(*v*) **Specific heat.**

The specific heat C of a fluid is defined as the amount of heat required to raise the temperature of a unit mass of the fluid by one degree. Thus $C = \partial Q/\partial T$, where δQ is the amount of heat added to raise the temperature by δT. The value of the specific heat depends on two well-known processes — the constant volume process and the constant pressure process. The specific heats of the above processes are denoted and defined as

Specific heat at constant volume = $C_v = (\partial Q/\partial T)_v$

Specific heat at constant pressure = $C_p = (\partial Q/\partial T)_p$

Ratio of these two specific heats is denoted by γ. Thus, $\gamma = C_p / C_v$

(*vi*) **Incompressible and compressible fluids**

Gases are compressible and their density changes readily with temperature and pressure. Liquids, on the other hand, are rather difficult to compress and for most problems we can treat them incompressible. Only in such situations as sound propagation in liquids does one need to consider their compressibility.

The density of a fluid is a thermodynamic property which depends on the state of the fluid. The density ρ can be expressed as a function of pressure and temperature. Such a relation is known as an *equation of state*. For an ideal gas the equation of state may be expressed as

$$p = \rho RT, \qquad \ldots(1)$$

where R is the characteristic gas constant. The constant R has different values for various gases and its units has the form

R = (energy) / (mass × temperature) = J/kg–K

We also, have $$R = C_p - C_v, \qquad \ldots(2)$$

where C_p is specific heat at constant pressure and C_v is specific heat at constant volume.

(*vii*) **Compressibility and bulk modulus**

The compressibility of a fluid is expressed by the quantity bulk modulus K, which is defined as the ratio of volumetric stress to volumetric strain. If a small increase in pressure dp causes a change dv of the specific volume v, then by definition

$$K = \frac{dp}{-(dv/v)} = \frac{dp}{(d\rho/\rho)} = \rho\frac{dp}{d\rho}, \qquad \text{as} \qquad v \propto \frac{1}{\rho} \quad \Rightarrow \quad \frac{dv}{v} = -\frac{d\rho}{\rho}$$

The coefficient of compressibility β is defined as $\beta = 1/K$.

We now proceed to establish relationship between the bulk modulus and the local pressure for a perfect gas for two different processes of compression.

Relationship for isothermal process. We have $p/\rho = \text{const.} = C_1$ ...(1)

From the equation of state for a perfect gas, $p = \rho RT$ ...(2)

From (2), $dp/d\rho = RT = p/\rho$ ...(3)

By definition, $K = \rho(dp/d\rho) = \rho \times (p/\rho) = p$, by (3)

Relationship for isentropic process. We have $p/\rho^\gamma = \text{const.} = C_2$ ...(4)

where $\gamma = C_p/C_v$; C_v being specific heat at constant volume and C_p being specific heat at constant pressure.

From (2) and (4), $dp/d\rho = RT = \gamma\,\rho^{\gamma-1}C_2$

By definition, $K = \rho\,(dp/d\rho) = \gamma\,\rho^\gamma C_2 = \gamma p$, by (4)

Classification of fluids on the basis of density and viscosity:

In the following table, fluids have been classified on the basis of density and viscosity.

For complete understanding, read articles 1.4A, 1.4B, 1.4C and 1.4D.

S. No.	Type of fluid	Density	Viscosity
1	Ideal fluid	constant	zero
2	Incompressible fluid	constant	zero or non-zero
3	Inviscid fluid	constant or variable	zero
4	Real fluid	Variable	non-zero
5	Newtonian fluid	constant or variable	non-zero and $\tau = \mu\,(du/dy)$
6	Non-Newtonian fluid	constant or variable	$\tau \neq \mu\,(du/dy)$
7	Compressible fluid	variable	zero or non-zero

(*viii*) **The Mach number. Subsonic and supersonic flows**

In compressible flow there is a great distinction between flow involving velocities less than that of sound (subsonic flow) and flow involving velocities greater than that of sound (supersonic flow). Main difference between subsonic and supersonic flow will be discussed in chapter 20.

The Mach number M is defined as the ratio of the fluid speed to the local speed of sound. Thus,

$$M = V/c,$$

where V is the fluid velocity and c the local speed of sound.

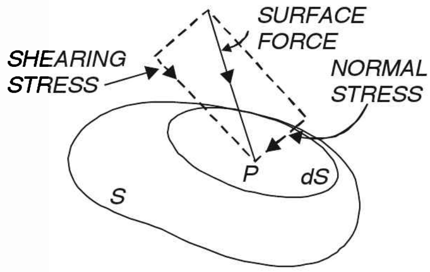
Classification of compressible fluids on the basis of Mach number

The Mach number gives us an important information about the type of compressible flow as shown in the following table: For complete understanding, read chapter 20.

S. No.	Type of flow	Mach number, M
1	Subsonic flow	$M < 1$
2	Sonic flow	$M = 1$
3	Supersonic flow	$1 < M < 6$
4	Hypersonic flow	$M > 6$
5	Transonic flow	$M < 1$ as well as $M > 1$

Remark. For flows around objects, where M is less than about 0.3, the flow is regarded as approximately incompressible.

1.4A. Viscous (or real) and inviscid (non-viscous, frictionless, perfect or ideal) fluids.



An infinitesimal fluid element is acted upon by two types of forces, namely, *body forces* and *surface forces*. The former is a type of force which is proportional to the mass (or possibly the volume) of the body on which it acts while the latter is one which acts on a surface element and is proportional to the surface area.

Suppose that the fluid element be enclosed by the surface S. Let P be an arbitrary point of S and let dS be the surface element around P. Then the surface force on dS is, in general, not in the direction of normal at P to dS. Hence the force may be resolved into components, one normal and the other tangential to the area dS. The normal force per unit area is said to be the *normal stress* or *pressure* while the tangential force per unit area is said to be the *shearing stress.*

A fluid is said to be *viscous* when normal as well as shearing stresses exist. On the other hand, a fluid is said to be *inviscid* when it does not exert any shearing stress, whether at rest or in motion. Clearly the pressure exerted by an inviscid fluid on any surface is always along the normal to the surface at that point. Due to shearing stress a viscous fluid produces resistance to the body moving through it as well as between the particles of the fluid itself. Water and air are treated inviscid fluids whereas syrup and heavy oil are treated as viscous fluids.

1.4B. Viscosity (or internal friction)

We know that the flow of water and air is much easier than syrup and heavy oil. This demonstrates the existence of a property in the fluid, which controls its rate of flow. This property of a fluid is known as *viscosity.* Thus, viscosity of a fluid is that property of fluid which exhibits a certain resistance to alteration of form. All existing fluids possess the property of viscosity in varying degrees.

To explain the nature of viscosity consider a fluid which is initially at rest between two parallel plates separated by a small distance h along the y-direction and extended infinitely in other directions. Suppose the upper plate is moving with velocity U in x-direction whereas the lower plate is kept at rest. By virtue of viscosity, the fluid will also be in motion.

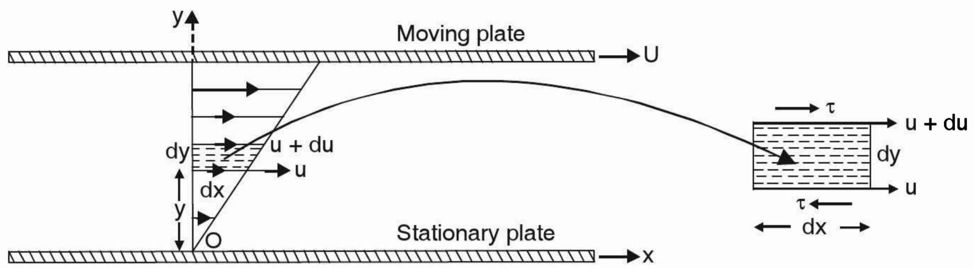


Figure showing fluid motion between a stationary plate and a moving plate. A small element shows the shear stress.

Fluid between the plates has a linear velocity profile (provided no pressure gradient exists along the plate in the direction of motion). It is a fact supported by experimental observations that the relative velocity between a solid surface and a fluid is zero for all fluids. Accordingly, the bottom layer of the fluid at $y = 0$ will be at rest and the upper layer at $y = h$ will be moving with the plate having a velocity U.

If we consider a small element of the fluid as shown in the figure, the shear stress τ on the top (which is numerically the same as the bottom in this case) is given by

$$\tau = \mu\ (du/dy), \qquad \ldots(1)$$

where μ is a constant of proportionality which is called the *coefficient of viscosity* or the *coefficient of dynamic viscosity*. (1) is known as *Newton's law of viscosity*.

The viscosity of a liquid decreases rapidly with increasing temperature whereas the viscosity of a gas increases with temperature. The viscosity of fluids also depends on pressure, but this dependence is usually of little importance compared to the temperature variation in problems of fluid dynamics.

1.4C. Newtonian and non-Newtonian fluids [Kanpur 2003]

The fluids that obeys Newton's law of viscosity (1) of Art. 1.4B is known as *Newtonian fluid*, for example, air and water. Viscous fluids such as tar and polymers do not obey Newton's law of viscosity and the relation between stress shear and rate of shear strain (refer chapter 13 for details) is non-linear. Such fluids are known as *non-Newtonian fluids*. The nature of relation between shear stress and rate of shear strain for Newtonian and non-Newtonian fluids is shown in the following figure.

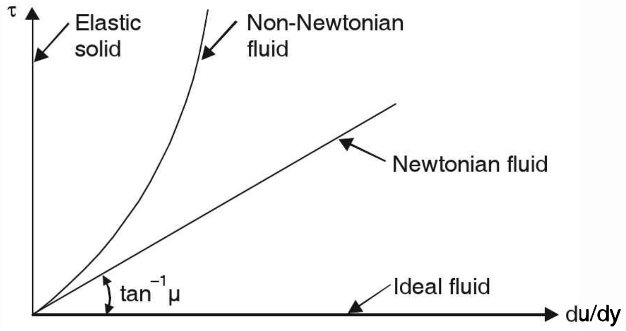


Figure showing classification of fluids

1.4D. Real and ideal fluids

The concept of an *ideal fluid* is based on theoretical consideration because all real fluids exhibit viscous property. From relation (1) of Art. 1.4B, it follows that τ vanishes either for $\mu = 0$ or for $du/dy = 0$. Hence, we see that a real fluid with small viscosity and small velocity gradient can be regarded as frictionless. For an *ideal fluid*, we shall take shear stress as zero and assume that, at the surface of contact with solid, an ideal fluid can have relative velocity in the tangential direction although normal velocity must be zero at the surface of contact.

1.5. Some important types of flows.

(*i*) **Laminar (streamline) and turbulent flows.**

A flow, in which each fluid particle traces out a definite curve and the curves traced out by any two different fluid particles do not intersect, is said to be *laminar*. On the other hand, a flow, in which each fluid particle does not trace out a definite curve and the curves traced out by fluid particles intersect, is said to be *turbulent*. The following figure illustrate laminar and turbulent flows.

Figure. Lines indicate the paths of particles (*ii*) **Steady and unsteady flows.**

A flow, in which properties and conditions (P, say) associated with the motion of the fluid are independent of the time so that the flow pattern remains unchanged with the time, is said to be *steady*. Mathematically, we may write $\partial P / \partial t = 0$. Here P may be velocity, density, pressure, temperature etc. On the other hand, a flow, in which properties and conditions associated with the motion of the fluid depend on the time so that the flow pattern varies with time, is said to be *unsteady*.

(*iii*) **Uniform and non-uniform flows.**

A flow, in which the fluid particles possess equal velocities at each section of the channel or *pipe is called uniform*. On the other hand, a flow, in which the fluid particles possess different velocities at each section of the channel or pipe is called *non-uniform*. These terms are usually used in connection with flow in channels.

(*iv*) **Rotational and Irrotational flows.**

A flow, in which the fluid particles go on rotating about their own axes, while flowing, is said to be *rotational*. On the other hand, a flow in which the fluid particles do not rotate about their own axes, while flowing, is said to be *irrotational*.

(*v*) **Barotropic Flow.**

The flow is said to be *barotropic* when the pressure is a function of the density.

1.6. Some useful results of vector analysis.

In the study of fluid dynamics the use of vector and tensor notations is becoming more frequent. Their use not only simplifies and condenses the proposed results but also makes physical concepts more easy to understand. It is the purpose of this article to present a summary of the properties of vectors. Throughout this book, bold face type is used to denote vector quantities.

Let **a, b, c** denote vector quantities such that $\mathbf{a} = \mathbf{i}a_1 + \mathbf{j}a_2 + \mathbf{k}a_3 = (a_1, a_2, a_3)$ and $|\mathbf{a}| = a = \sqrt{(a_1^2 + a_2^2 + a_3^2)}$ etc. Then, we have

(*i*) **Vector addition and substraction etc.**

$$\mathbf{a} + \mathbf{b} = \mathbf{i}\,(a_1 + b_1) + \mathbf{j}\,(a_2 + b_2) + \mathbf{k}\,(a_3 + b_3)$$

$$\mathbf{a} - \mathbf{b} = \mathbf{i}\,(a_1 - b_1) + \mathbf{j}\,(a_2 - b_2) + \mathbf{k}\,(a_3 - b_3)$$

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \qquad \text{(Cummutative Law)}$$

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} \qquad \text{(Associative Law)}$$

(*ii*) **Scalar (dot) product of two vectors**

$$\mathbf{a} \cdot \mathbf{b} = ab\cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

where θ is the angle between the vectors **a** and **b.**

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}, \qquad \mathbf{a} \cdot \mathbf{a} = \mathbf{a}^2 = a^2$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \qquad \text{(Distributive Law)}$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0, \qquad \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

(*iii*) **Vector (cross) product of two vectors.**

$$\mathbf{a} \times \mathbf{b} = ab\sin\theta\,\mathbf{n}$$

where θ is the angle measured from **a** to **b** and **n** is unit vector normal to the plane consisting of **a** and **b** such that **a, b** and **n** form a right handed system.

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$$

In general; $\mathbf{a}\times\mathbf{b} \neq \mathbf{b}\times\mathbf{a}.$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(*iv*) **Vector calculus.** Let **a, b** etc. be functions of scalar quantity t. Then we define

$$\frac{d\mathbf{a}}{dt} = \lim_{\delta t\to 0}\frac{\mathbf{a}(t+\delta t)-\mathbf{a}(t)}{\delta t}$$

Again, $\frac{d}{dt}(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b}$, $\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\cdot\mathbf{b}$

The vector operator ∇ (called del) is defined by

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right).$$

Let $\phi(x, y, z)$ and $\psi(x, y, z)$ be scalar point functions and let **a** (x, y, z) and **b** (x, y, z) be vector point functions. Then we define some basic terms, namely, gradient, divergence and curl as follows.

$$\text{grad}\,\phi = \nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x}+\mathbf{j}\frac{\partial\phi}{\partial y}+\mathbf{k}\frac{\partial\phi}{\partial z}, \qquad \text{div}\,\mathbf{a} = \nabla\cdot\mathbf{a} = \frac{\partial a_1}{\partial x}+\frac{\partial a_2}{\partial y}+\frac{\partial a_3}{\partial z}$$

$$\text{curl}\,\mathbf{a} = \nabla\times\mathbf{a} = \left(\frac{\partial a_3}{\partial y}-\frac{\partial a_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial a_1}{\partial z}-\frac{\partial a_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial a_2}{\partial x}-\frac{\partial a_1}{\partial y}\right)\mathbf{k}.$$

The divergence of grad ϕ is known as the Laplacian of ϕ and is given by

$$\text{div grad}\,\phi = \nabla\cdot(\nabla\phi) = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}.$$

Then the Laplacian operator, ∇^2, is defined as

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$$

Some important vector identities.

$$\text{grad}\,(\phi\psi) = \phi\,\text{grad}\,\psi + \psi\,\text{grad}\,\phi$$

$$\text{grad}\,(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\,\mathbf{a} + \mathbf{a}\times\text{curl}\,\mathbf{b} + \mathbf{b}\times\text{curl}\,\mathbf{a}$$

$$\text{div}\,(\phi\mathbf{a}) = \phi\,\text{div}\,\mathbf{a} + \textbf{grad}\,\phi\cdot\mathbf{a}, \qquad \text{div}\,(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\text{curl}\,\mathbf{a} - \mathbf{a}\,.\,\text{curl}\,\mathbf{b}$$

$$\text{curl}\,(\phi\,\mathbf{a}) = \phi\,\text{curl}\,\mathbf{a} + \text{grad}\,\phi\times\mathbf{a}$$

$$\text{curl}\,(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\,\text{div}\,\mathbf{b} - \mathbf{b}\,\text{div}\,\mathbf{a} + (\mathbf{b}\cdot\nabla)\,\mathbf{a} - (\mathbf{a}\cdot\nabla)\,\mathbf{b}$$

$$\text{curl curl}\,\mathbf{a} = \text{grad div}\,\mathbf{a} - \nabla^2\mathbf{a}, \qquad \text{curl grad}\,\phi = \mathbf{0}, \qquad \text{div curl}\,\mathbf{a} = 0$$

where $(\mathbf{a}\cdot\nabla) = a_1(\partial/\partial x) + a_2(\partial/\partial y) + a_3(\partial/\partial z)$

Some important integral theorems (without proof).

I. The Divergence theorem (Gauss's theorem). Let S denote a surface bounding a volume V and $\mathbf{n}$ the unit vector normal to the surface. Then,

$$\int_S \mathbf{a}\cdot\mathbf{n}\, dS = \int_V \operatorname{div}\mathbf{a}\, dV$$

II. Stokes's theorem. Let S be a surface bounded by a closed curve C, and $\mathbf{n}$ the unit vector normal to the surface. Then,

$$\int_C \mathbf{a}\cdot d\mathbf{r} = \int_S \operatorname{curl}\mathbf{a}\cdot\mathbf{n}\, dS$$

III. Green's theorem. Let ϕ and ψ be two scalar point functions and let S be a surface bounding a volume V. Then

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \int_S \left(\phi\frac{d\psi}{\partial n} - \psi\frac{\partial\psi}{\partial n}\right) dS.$$

In particular, if ϕ and ψ are harmonic so that $\nabla^2\phi = 0 = \nabla^2\psi$, then the above Green's theorem takes the form (known as *Green's reciprocal theorem*)

$$\int_S \phi\frac{\partial\psi}{\partial n} dS = \int_S \psi\frac{\partial\phi}{\partial n} dS.$$

1.7A. Orthogonal curvilinear coordinates.

Let the rectangular cartesian coordinates (x, y, z) of any point P in space be expressed in terms of three independent, single-valued and continuously differentiable scalar point functions u_1, u_2, u_3 as follows:

$$x = x(u_1, u_2, u_3), \qquad y = y(u_1, u_2, u_3), \qquad z = z(u_1, u_2, u_3). \quad \ldots(1)$$

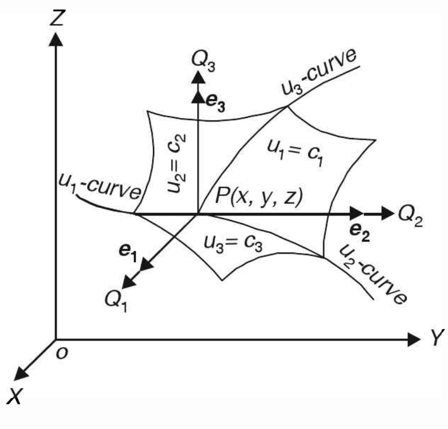
Suppose that the Jacobian of x, y, z with respect to u_1, u_2, u_3 does not vanish, *i.e.*, $\dfrac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} \neq 0$. Then the transformation (1) can be inverted, *i.e.*, u_1, u_2, u_3 can be expressed in terms of x, y, z giving

$$u_1 = u_1(x, y, z), \qquad u_2 = u_2(x, y, z), \qquad u_3 = u_3(x, y, z). \quad \ldots(2)$$

Due to the conditions imposed on these functions, with each point $P(x, y, z)$ in space there exists a unique triad of numbers u_1, u_2, u_3 and to each such triad there is a definite point in space.

The set (u_1, u_2, u_3) are called the *curvilinear coordinates* of P. The set of equations (1) and (2) define a '*transformation of coordinates*.'

The surfaces $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$, where c_1, c_2 and c_3 are constants, are called *coordinate surfaces*. The surface on which u_1 is constant is known as u_1*–surface*. Similarly, we have u_2*–surface* and u_3*–surface*. When taken in pairs these coordinate surfaces intersect each other in curves called *coordinate curves*: (*i*) u_1*–curve* is given by $u_2 = c_2$, $u_3 = c_3$ (*ii*) u_2*–curve* is given by $u_3 = c_3$, $u_1 = c_1$ (*iii*) u_3*–curve* is given by $u_1 = c_1$, $u_2 = c_2$. The *coordinate axes* are determined by the tangents PQ_1, PQ_2 and PQ_3 to the coordinate curves at the point P. Note carefully that the directions of these coordinate axes depend on the chosen point P. of space and consequently the unit vectors associated with them are not necessarily constant.



If at every point P (x, y, z), the coordinate axes are mutually perpendicular, then u_1, u_2, u_3 are called *orthogonal curvilinear coordinates* of P. In the present chapter we shall study such systems only.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form a right-handed system of *unit vectors tangent to the coordinate curves at* P, extending in the directions of increasing u_1, u_2, u_3 respectively. Then, we have

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1, \qquad \ldots (3A)$$

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_2 = 0, \qquad \ldots (3B)$$

$$\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_3 = \mathbf{0}, \qquad \ldots (3C)$$

and $$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \qquad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \qquad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2. \qquad \ldots (3D)$$

We now define three numbers h_1, h_2, h_3 known as *scalar factors* or *material coefficients* as follows:

$$h_1 = |\partial \mathbf{r}/\partial u_1|, \qquad h_2 = |\partial \mathbf{r}/\partial u_2|, \qquad h_3 = |\partial \mathbf{r}/\partial u_3|. \qquad \ldots (4)$$

where the position vector $\mathbf{r}$ of P (x, y, z) is given by

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = x\,(u_1, u_2, u_3)\,\mathbf{i} + y\,(u_1, u_2, u_3)\,\mathbf{j} + z\,(u_1, u_2, u_3)\,\mathbf{k} = \mathbf{r}\,(u_1, u_2, u_3).$$

Since the equation of u_1-curve is given by $u_2 = c_2$, $u_3 = c_3$, it follows that a tangent vector to u_1-curve at P is $\partial \mathbf{r}/\partial u_1$. Since $\mathbf{e}_1$ is the unit vector in this direction, we have

$$\partial \mathbf{r}/\partial u_1 = |\partial \mathbf{r}/\partial u_1|\,\mathbf{e}_1 = h_1 \mathbf{e}_1, \quad \text{using (4)},$$

Thus, $$\partial \mathbf{r}/\partial u_1 = h_1 \mathbf{e}_1, \qquad \partial \mathbf{r}/\partial u_2 = h_2 \mathbf{e}_2, \qquad \partial \mathbf{r}/\partial u_3 = h_3 \mathbf{e}_3. \qquad \ldots (5)$$

1.7B. Differential of an arc length.

Using $\mathbf{r} = \mathbf{r}\,(u_1, u_2, u_3)$ and relations (5) of Art. 1.7A, we have

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3$$

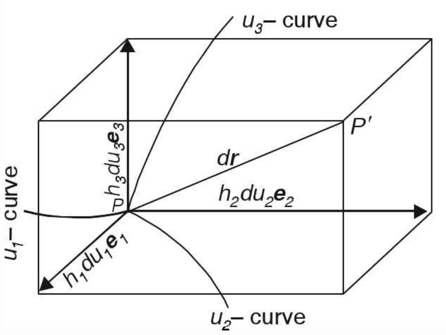
or $$d\mathbf{r} = h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3. \qquad \ldots (1)$$

Then the differential of an arc length, ds, is given by

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = (h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3) \cdot (h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3).$$

$$\therefore \quad (ds)^2 = h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_2^3 (du_3)^2, \quad \text{using (3A) and (3B) of Art. 1.7A.} \qquad \ldots (2)$$

which is known as *quadratic differential form.*



We are now in a position to give geometrical significance of the coefficients h_1, h_2, h_3. Suppose that an element of arc ds is directed along u_1-curve so that $du_2 = du_3 = 0$. Then the differential of arc length ds_1 along u_1-curve at P is given by $ds_1 = h_1 du_1$. Similarly, ds_2 and ds_3 can be calculated. Thus, we have

$$ds_1 = h_1 du_1, \quad ds_2 = h_2 du_2, \quad ds_3 = h_3 du_3. \qquad \ldots (3)$$

Consider an infinitesimal parallelepiped with one vertex at P as shown in the figure. Then, from (1) we see that

(*i*) lengths of the edges of parallelepiped are $h_1 du_1$, $h_2 du_2$ and $h_3 du_3$,

(*ii*) the areas of the faces of the parallelepiped are $h_2 h_3\, du_2 du_3$, $h_3 h_1 du_3 du_1$ and $h_1 h_2\, du_1 du_2$,

(*iii*) volume of the parallelepiped is $h_1 h_2 h_3 du_1 du_2 du_3$.

1.7C. Differential operators in terms of orthogonal curvilinear coordinates.

(*i*) **Gradient.** Let $\psi(x, y, z)$ be continuously differentiable scalar point function of orthogonal curvilinear coordinates u_1, u_2, u_3. Since grad ψ is a vector, we assume that

$$\text{grad } \psi = \nabla \psi = k_1 \mathbf{e}_1 + k_2 \mathbf{e}_2 + k_3 \mathbf{e}_3, \qquad \ldots (1)$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are unit vectors along the curvilinear coordinates axes, and k_1, k_2 and k_3 are unknown scalars, to be determined.

Now, $$dr = h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3. \qquad \ldots(2)$$

and $$d\psi = \nabla\psi \cdot d\mathbf{r}.$$

$\therefore$ $$d\psi = k_1 h_1 du_1 + k_2 h_2 du_2 + k_3 h_3 du_3 \qquad \ldots(3)$$

But $$d\psi = \frac{d\psi}{du_1} du_1 + \frac{d\psi}{du_2} du_2 + \frac{d\psi}{du_3} du_3 \qquad \ldots(4)$$

Comparing (3) and (4), we obtain

$$k_1 h_1 = \partial\psi/\partial u_1, \qquad k_2 h_2 = \partial\psi/\partial u_2, \qquad k_3 h_3 = \partial\psi/\partial u_3.$$

so that $$k_1 = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}, \qquad k_2 = \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}, \qquad k_3 = \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}.$$

$\therefore$ From (1), $$\text{grad } \psi = \nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}\mathbf{e}_3, \qquad \ldots(5)$$

which is the required expression. From (5), it follows that the components of grad ψ along the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are $(1/h_1) \times (\partial\psi/\partial u_1)$, $(1/h_2) \times (\partial\psi/\partial u_2)$, $(1/h_3) \times (\partial\psi/\partial u_3)$ respectively.

Rewriting (5), $$\nabla\psi = \left(\frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial u_3}\right)\psi.$$

$\therefore$ $$\nabla \equiv \frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial u_3}. \qquad \ldots(6)$$

Remark 1. Replacing ψ by u_1, u_2 and u_3, by turn, (5) gives

$$\nabla u_1 = \mathbf{e}_1/h_1, \qquad \nabla u_2 = \mathbf{e}_2/h_2, \qquad \nabla u_3 = \mathbf{e}_3/h_3, \qquad \ldots(7)$$

where vectors $\nabla u_1, \nabla u_2$ and ∇u_3 are along normals to the coordinate surfaces $u_1 = c_1$, $u_2 = c_2$ and $u_3 = c_3$ respectively.

Remark 2. Using (7), relation (5) may be rewritten as

$$\text{grad } \psi = \nabla\psi = \frac{\partial\psi}{\partial u_1}\nabla u_1 + \frac{\partial\psi}{\partial u_2}\nabla u_2 + \frac{\partial\psi}{\partial u_3}\nabla u_3 \qquad \ldots(8)$$

(*ii*) **Divergence.** Let $\mathbf{F}$ (u_1, u_2, u_3) be continuously differentiable vector point function of orthogonal curvilinear coordinates u_1, u_2, u_3. We assume that

$$\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F\mathbf{e}_3, \qquad \ldots(9)$$

where F_1, F_2, F_3 are the components $\mathbf{F}$ along $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ respectively.

Using relation 3(D) of Art. 1.7A, (9) may be rewritten as

$$\mathbf{F} = F_1\, \mathbf{e}_2 \times \mathbf{e}_3 + F_2\, \mathbf{e}_3 \times \mathbf{e}_1 + F_3\, \mathbf{e}_1 \times \mathbf{e}_2$$

or $$\mathbf{F} = F_1 h_2 h_3 \nabla u_2 \times \nabla u_3 + F_2 h_3 h_1 \nabla u_3 \times \nabla u_1 + F_3 h_1 h_2 \nabla u_1 \times \nabla u_2, \text{ using (7)}$$

$\therefore$ $$\text{div } \mathbf{F} = \nabla\cdot\mathbf{F} = \nabla\cdot(F_1 h_2 h_3 \nabla u_2 \times \nabla u_3) + \nabla\cdot(F_2 h_3 h_1 \nabla u_3 \times \nabla u_1) + \nabla\cdot(F_3 h_1 h_2 \nabla u_1 \times \nabla u_2). \qquad \ldots(10)$$

Recall the following vector identities

$$\nabla\cdot(\psi\mathbf{F}) = \psi\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla\psi, \qquad \ldots(11)$$

$$\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \text{curl } \mathbf{F}\cdot\mathbf{G} - \text{curl } \mathbf{G}\cdot\mathbf{F}. \qquad \ldots(12)$$

and $$\text{curl grad } \psi = 0. \qquad \ldots(13)$$

$\therefore$ $$\nabla\cdot(F_1 h_2 h_3 \nabla u_2 \times \nabla u_3) = F_1 h_2 h_3 \nabla\cdot(\nabla u_2 \times \nabla u_3) + (\nabla u_2 \times \nabla u_3)\cdot\nabla(F_1 h_2 h_3), \text{ by (11)}$$

$= F_1h_2h_3$ [curl $\nabla u_2 \cdot \nabla u_3$ – curl grad $u_3 \cdot \nabla u_2$] + $(\nabla u_2 \times \nabla u_3) \cdot \nabla (F_1h_2h_3)$, by (12)

$= (\nabla u_2 \times \nabla u_3) \cdot \nabla (F_1h_2h_3)$, by (13)

$$= (\nabla u_2 \times \nabla u_3) \cdot \left[\frac{\partial (F_1h_2h_3)}{\partial u_1} \nabla u_1 + \frac{\partial (F_1h_2h_3)}{\partial u_2} \nabla u_2 + \frac{\partial (F_1h_2h_3)}{\partial u_3} \nabla u_3\right], \text{ by (8)}$$

$$= (\nabla u_2 \times \nabla u_3) \cdot \nabla u_1 \frac{\partial (F_1h_2h_3)}{\partial u_1}, \quad \text{as} \quad \nabla u_2 \times \nabla u_3 \cdot \nabla u_2 = 0 \quad \text{and} \quad \nabla u_2 \times \nabla u_3 \cdot \nabla u_3 = 0$$

$$= \frac{1}{h_1h_2h_3} (\mathbf{e}_2 \times \mathbf{e}_3) \cdot \mathbf{e}_1 \frac{\partial (F_1h_2h_3)}{\partial u_1}, \text{ using (7)}$$

$$= \frac{1}{h_1h_2h_3} \frac{\partial (F_1h_2h_3)}{\partial u_1}, \text{ as } (\mathbf{e}_2 \times \mathbf{e}_3) \cdot \mathbf{e}_1 = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1.$$

By symmetry,
$$\nabla \cdot (F_2h_3h_1 \nabla u_3 \times \nabla u_1) = \frac{1}{h_1h_2h_3} \frac{\partial (F_2h_3h_1)}{\partial u_2}$$

and
$$\nabla (F_3h_1h_2 \nabla u_1 \times \nabla u_2) = \frac{1}{h_1h_2h_3} \frac{\partial (F_3h_1h_2)}{\partial u_3}$$

Substituting the above values in (1), we have

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{1}{h_1h_2h_3} \left[\frac{\partial (F_1h_2h_3)}{\partial u_1} + \frac{\partial (F_2h_3h_1)}{\partial u_2} + \frac{\partial (F_3h_1h_2)}{\partial u_3}\right]. \qquad \ldots(14)$$

(*ii*) **Curl.** Let $\mathbf{F}$ (u_1, u_2, u_3) be continuously differentiable vector point function of orthogonal curvilinear coordinates u_1, u_2, u_3. We assume that

$$\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3, \qquad \ldots(15)$$

where F_1, F_2, F_3 are the components of $\mathbf{F}$ along $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ respectively.

Using (7), (15) may be re-written as

$$\mathbf{F} = h_1F_1\nabla u_1 + h_2F_2\nabla u_2 + h_3F_3\nabla u_3. \qquad \ldots(16)$$

$$\therefore \qquad \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \nabla \times (h_1F_1\nabla u_1) + \nabla \times (h_2F_2\nabla u_2) + \nabla \times (h_3F_3\nabla u_3). \qquad \ldots(17)$$

Using the identity $\nabla \times (\psi \mathbf{F}) = \nabla \psi \times \mathbf{F} + \psi \text{ curl } \mathbf{F}$, we get

$$\nabla \times (h_1F_1\nabla u_1) = \nabla (h_1F_1) \times \nabla u_1 + h_1F_1 \text{ curl grad } u_1$$

$$= \nabla (h_1F_1) \times \nabla u_1, \text{ using identity curl grad } u_1 = \mathbf{0}$$

$$= \left[\frac{\partial (h_1F_1)}{\partial u_1} \nabla u_1 + \frac{\partial (h_1F_1)}{\partial u_2} \nabla u_2 + \frac{\partial (h_1F_1)}{\partial u_3} \nabla u_3\right] \times \nabla u_1, \text{ using (8)}$$

$$= \frac{\partial (h_1F_1)}{\partial u_2} \nabla u_2 \times \nabla u_1 + \frac{\partial (h_1F_1)}{\partial u_3} \nabla u_3 \times \nabla u_1, \quad \text{as } \nabla u_1 \times \nabla u_1 = \mathbf{0}$$

$$= \frac{1}{h_1h_2} \frac{\partial (h_1F_1)}{\partial u_2} \mathbf{e}_2 \times \mathbf{e}_1 + \frac{1}{h_3h_1} \frac{\partial (h_1F_1)}{\partial u_3} \mathbf{e}_3 \times \mathbf{e}_1, \text{ using (7)}$$

$$= -\frac{1}{h_1h_2} \frac{\partial (h_1F_1)}{\partial u_2} \mathbf{e}_3 + \frac{1}{h_3h_1} \frac{\partial (h_1F_1)}{\partial u_3} \mathbf{e}_2, \text{ using (3D) of Art 7.1A.}$$

$$\therefore \qquad \nabla \times (h_1 F_1 \nabla u_1) = \frac{\mathbf{e}_2}{h_3 h_1}\frac{\partial (h_1 F_1)}{\partial u_3} - \frac{\mathbf{e}_3}{h_1 h_2}\frac{\partial (h_1 F_1)}{\partial u_2}.$$

Proceeding similarly, we have

$$\nabla \times (h_2 F_2 \nabla u_2) = \frac{\mathbf{e}_3}{h_1 h_2}\frac{\partial (h_2 F_2)}{\partial u_1} - \frac{\mathbf{e}_1}{h_2 h_3}\frac{\partial (h_2 F_2)}{\partial u_3}$$

and

$$\nabla \times (h_3 F_3 \nabla u_3) = \frac{\mathbf{e}_1}{h_2 h_3}\frac{\partial (h_3 F_3)}{\partial u_2} - \frac{\mathbf{e}_2}{h_3 h_1}\frac{\partial (h_3 F_3)}{\partial u_1}.$$

Substituting the above values in (17), we have

curl $\mathbf{F} = \nabla \times \mathbf{F}$

$$= \frac{\mathbf{e}_2}{h_3 h_1}\frac{\partial (h_1 F_1)}{\partial u_3} - \frac{\mathbf{e}_3}{h_1 h_2}\frac{\partial (h_1 F_1)}{\partial u_2} + \frac{\mathbf{e}_3}{h_1 h_2}\frac{\partial (h_2 F_2)}{\partial u_1} - \frac{\mathbf{e}_1}{h_2 h_3}\frac{\partial (h_2 F_2)}{\partial u_3} + \frac{\mathbf{e}_1}{h_2 h_3}\frac{\partial (h_3 F_3)}{\partial u_2} - \frac{\mathbf{e}_2}{h_3 h_1}\frac{\partial (h_3 F_3)}{\partial u_1}$$

$$= \frac{\mathbf{e}_1}{h_2 h_3}\left[\frac{\partial (h_3 F_3)}{\partial u_2} - \frac{\partial (h_2 F_2)}{\partial u_3}\right] + \frac{\mathbf{e}_2}{h_3 h_1}\left[\frac{\partial (h_1 F_1)}{\partial u_3} - \frac{\partial (h_3 F_3)}{\partial u_1}\right] + \frac{\mathbf{e}_3}{h_1 h_2}\left[\frac{\partial (h_2 F_2)}{\partial u_1} - \frac{\partial (h_1 F_1)}{\partial u_2}\right] \qquad \ldots(18)$$

$$= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}. \qquad \ldots(19)$$

1.7D. Laplacian operator ∇^2.

We have

$$\nabla^2 \psi = \nabla \cdot (\nabla \psi) = \nabla \cdot \left(\frac{1}{h_1}\frac{\partial \psi}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial \psi}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial \psi}{\partial u_3}\mathbf{e}_3\right), \text{ using (5)}$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial u_3}\right)\right], \text{ using (14)} \qquad \ldots(20)$$

1.7E. Special orthogonal coordinate systems.

(*i*) **Cartesian coordinate.** These constitute a trivial case of orthogonal curvilinear coordinate systems. For this case, we have

$$u_1 = x, \quad u_2 = y, \quad u_3 = z; \qquad \mathbf{e}_1 = \mathbf{i}, \quad \mathbf{e}_2 = \mathbf{j}, \quad \mathbf{e}_3 = \mathbf{k}; \qquad h_1 = h_2 = h_2 = 1 \qquad \ldots(1)$$

Hence formulae (5), (20), (14) and (19) of Art. 1.7C give

$$\text{grad } \psi = \nabla \cdot \psi = \frac{\partial \psi}{\partial x}\mathbf{i} + \frac{\partial \psi}{\partial y}\mathbf{j} + \frac{\partial \psi}{\partial z}\mathbf{k}, \qquad \ldots(2)$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}, \qquad \ldots(3)$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \qquad \ldots(4)$$

and $$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}. \qquad \ldots(5)$$

where $$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}, \qquad \ldots(6)$$

F_x, F_y, F_z being the components of $\mathbf{F}$ along $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively.

(*i*) **Cylindrical coordinates or cylindrical polar coordinates.**

Let $P\ (x, y, z)$ be any point in space. Let ρ, ϕ, z respectively denote the projection OM of OP on xy-plane, the angle which OM makes with x-axis and perpendicular PM on xy-plane. Then cylindrical coordinates of P are (ρ, ϕ, z) and so here, we have

$$u_1 = \rho, \qquad u_2 = \phi, \qquad u_3 = z. \qquad \ldots(7)$$

Again, from the figure, we have

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z, \qquad \ldots(8)$$

where $$\rho \geq 0, \qquad 0 \leq \phi \leq 2\pi, \qquad -\infty < z < \infty. \qquad \ldots(9)$$

Expressing ρ, ϕ, z in terms of x, y, z, (8) yields

$$\rho = (x^2 + y^2)^{1/2}, \qquad \phi = \tan^{-1}(y/x), \qquad z = z. \qquad \ldots(10)$$

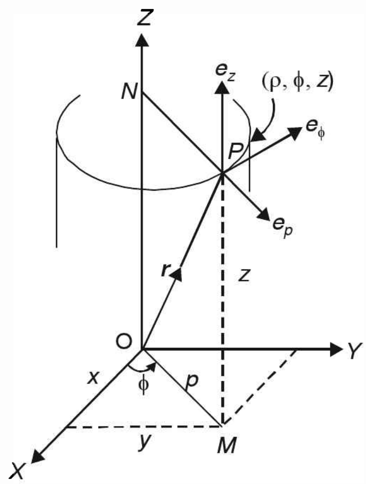
The coordinate surfaces are given by

(*i*) $\rho = c_1$, *i.e.*, $x^2 + y^2 = c_1^2$, *i.e.*, cylinders co-axial with z-axis.

(*ii*) $\phi = c_2$, *i.e.*, $y = (\tan c_2)\,x$, *i.e.*, planes through the z-axis.

(*iii*) $z = c_3$, *i.e.*, planes perpendicular to the z-axis.

The point P is the point of intersection of these surfaces.



The coordinate curves for ρ, ϕ and z are respectively straight lines perpendicular to the z-axis, horizontal circles with centres on the z-axis and lines parallel to the z-axis.

Let the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be denoted by $\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z$ respectively in cylindrical coordinates. Then $\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z$ extend respectively along NP in the direction of increasing ρ, perpendicular to the plane $ONPM$ in the direction of increasing ϕ and vetically upwards in the direction of increasing z.

Let $\mathbf{r}$ be the position vector of P. Then, we have

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

or $$\mathbf{r} = \rho\cos\phi\,\mathbf{i} + \rho\sin\phi\,\mathbf{j} + z\mathbf{k}, \text{ using (8).} \qquad \ldots(11)$$

$$\therefore \quad \frac{\partial \mathbf{r}}{\partial \rho} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}, \qquad \frac{\partial \mathbf{r}}{\partial \phi} = -\rho\sin\phi\,\mathbf{i} + \rho\cos\phi\,\mathbf{j}, \qquad \frac{\partial r}{\partial z} = \mathbf{k}. \qquad \ldots(12)$$

$$\therefore \quad h_1 = |\partial\mathbf{r}/\partial\rho| = 1, \qquad h_2 = |\partial\mathbf{r}/\partial\phi| = \rho, \qquad h_3 = |\partial\mathbf{r}/\partial z| = 1. \qquad \ldots(13)$$

But $$h_1\mathbf{e}_1 = \partial\mathbf{r}/\partial u_1, \qquad h_2\mathbf{e}_2 = \partial\mathbf{r}/\partial u_2, \qquad h_3\mathbf{e}_3 = \partial\mathbf{r}/\partial u_3.$$

Here, $$h_1\mathbf{e}_\rho = \partial\mathbf{r}/\partial\rho, \qquad h_2\mathbf{e}_\phi = \partial\mathbf{r}/\partial\phi, \qquad h_3\mathbf{e}_z = \partial\mathbf{r}/\partial z.$$

$$\therefore \quad \mathbf{e}_\rho = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}, \qquad \mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}, \qquad \mathbf{e}_z = \mathbf{k}. \qquad \ldots(14)$$

From (14), $$\mathbf{e}_\rho \cdot \mathbf{e}_\phi = \mathbf{e}_\phi \cdot \mathbf{e}_z = \mathbf{e}_z \cdot \mathbf{e}_\rho = 0, \qquad \ldots(15)$$

showing that $\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z$ are mutually perpendicular and hence cylindrical coordinates are orthogonal curvilinear coordinates.

Again, $$\mathbf{e}_\rho \times \mathbf{e}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} = \mathbf{k} = \mathbf{e}_z, \qquad \ldots(15A)$$

$$\mathbf{e}_\phi \times \mathbf{e}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j} = \mathbf{e}_\rho \qquad \ldots(15B)$$

and $$\mathbf{e}_z \times \mathbf{e}_\rho = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ \cos\phi & \sin\phi & 0 \end{vmatrix} = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j} = \mathbf{e}_\phi. \qquad \ldots(15C)$$

Moreover, we have

$$|\mathbf{e}_\rho| = \sqrt{(\cos^2\phi + \sin^2\phi)} = 1, \qquad |\mathbf{e}_\phi| = \sqrt{(\cos^2\phi + \sin^2\phi)} = 1, \qquad |\mathbf{e}_z| = 1 \qquad \ldots(15D)$$

Hence, it follows that $(\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z)$ forms an orthogonal right-handed basis.

Using formulae (5), (20), (14) and (19) of Art. 1.7C, we obtain

$$\text{grad}\,\psi = \nabla\psi = \frac{\partial\psi}{\partial\rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\mathbf{e}_\phi + \frac{\partial\psi}{\partial z}\mathbf{e}_z, \qquad \ldots(16)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}, \qquad \ldots(17)$$

$$\text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z} \qquad \ldots(18)$$

and $$\text{curl}\,\mathbf{F} = \nabla\times\mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} \qquad \ldots(19)$$

where $$\mathbf{F} = F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z, \qquad \ldots(20)$$

F_ρ, F_ϕ, F_z being the components of $\mathbf{F}$ along $\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z$ respectively.

The elementary arc-length in curvilinear coordinates is given by

$$ds = [h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_2^3(du_3)^2]^{1/2}. \qquad \ldots(21)$$

$\therefore$ For cylindrical coordinates, (21) reduces to

$$ds = [(d\rho)^2 + (\rho d\phi)^2 + (dz)^2]^{1/2}. \qquad \ldots(22)$$

The elementary areas in the coordinate planes u_1u_2, u_2u_3, u_3u_1 in curvilinear coordinates are $h_1h_2du_2du_2$, $h_2h_3du_2du_1$, $h_3h_1du_3du_1$ respectively. Hence the for cylindrical coordinates the elementary areas in $\rho\phi, \phi z, z\rho$ planes are $\rho\, d\rho\, d\phi, \rho\, d\phi\, dz, dz\, d\rho$ respectively.

The elementary volume in curvilinear coordinates is $h_1 h_2 h_3 \; du_1 du_2 du_3$ and hence the elementary volume in cylindrical coordinates reduces to $\rho \, d\rho \, d\phi \, dz$.

(*iii*) **Spherical coordinates or spherical polar coordinates**

Let $P(x, y, z)$ be any point in space. Let r, θ, ϕ respectively denote the distance OP of P from the origin, the angle which OP makes with the z-axis and the angle between the projection OM of OP on xy-plane and the x-axis. Then spherical coordinates of P are (r, θ, ϕ) and so here, we have

$$u_1 = r, \qquad u_2 = \theta, \qquad u_3 = \phi. \qquad \ldots(23)$$

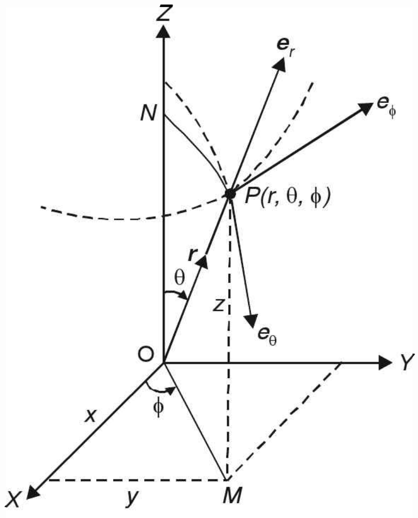
Again, from the figure we have

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta, \qquad \ldots(24)$$

where $\quad r \ge 0, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi \le 2\pi \qquad \ldots(25)$

Expressing r, θ, ϕ in terms of x, y, z (24) yields

$$\left.\begin{aligned} r &= (x^2 + y^2 + z^2)^{1/2} \\ \theta &= \tan^{-1}\{(x^2 + y^2)^{1/2} / z\} \\ \text{and} \qquad \phi &= \tan^{-1}(y/x) \end{aligned}\right\} \ldots(26)$$



The coordinate surfaces are given by:

(*i*) $r = c_1$, $\quad$ *i.e.*, $\quad x^2 + y^2 + z^2 = c_1^2$,

i.e., spheres concentric with O,

(*ii*) $\theta = c_2$, $\quad$ *i.e.*, $\quad x^2 + y^2 = (\tan^2 c_2) z^2$,

i.e., right circular cones with axis as z-axis and vertex at the origin O,

(*iii*) $\theta = c_3$, $\quad$ *i.e.*, $\quad y = (\tan c_3)\, x$,

i.e., planes through the z-axis.

The point P is the point of intersection of these surfaces.

The coordinate curves for r, θ and ϕ are respectively straight lines passing through the origin, vertical circles with centre at the origin and the horizontal circles with centre on the z-axis.

Let the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be denoted by $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$ respectively in spherical coordinates. Let these unit vectors extend respectively in the directions of r increasing, θ increasing and ϕ increasing.

Let $\mathbf{r}$ be the position vector of P. Then, we have

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

or $\quad \mathbf{r} = r\sin\theta(\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) + r\cos\theta\,\mathbf{k}$, using (24) $\qquad \ldots(27)$

$\therefore \quad \partial\mathbf{r}/\partial r = \sin\theta(\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) + \cos\theta\,\mathbf{k}, \qquad \ldots(28A)$

$\quad \partial\mathbf{r}/\partial\theta = r\cos\theta(\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) - r\sin\theta\,\mathbf{k} \qquad \ldots(28B)$

and $\quad \partial\mathbf{r}/\partial\phi = r\sin\theta(-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}). \qquad \ldots(28C)$

$\therefore \quad h_1 = |\partial\mathbf{r}/\partial r| = 1, \qquad h_2 = |\partial\mathbf{r}/\partial\theta| = r, \qquad h_3 = |\partial\mathbf{r}/\partial\phi| = r\sin\theta. \qquad \ldots(29)$

But $\quad h_1\mathbf{e}_1 = \partial\mathbf{r}/\partial u_1, \qquad h_2\mathbf{e}_2 = \partial\mathbf{r}/\partial u_2, \qquad h_3\mathbf{e}_3 = \partial\mathbf{r}/\partial u_3$

Here $h_1\mathbf{e}_r = \partial\mathbf{r}/\partial r,\qquad h_2\mathbf{e}_\theta = \partial\mathbf{r}/\partial\theta,\qquad h_3\mathbf{e}_\phi = \partial\mathbf{r}/\partial\phi.$

$\therefore$ and

$$\left.\begin{aligned}\mathbf{e}_r &= \sin\theta(\cos\phi\,\mathbf{i}+\sin\phi\,\mathbf{j})+\cos\theta\,\mathbf{k}\\ \mathbf{e}_\theta &= \cos\theta(\cos\phi\,\mathbf{i}+\sin\phi\,\mathbf{j})-\sin\theta\,\mathbf{k}\\ \mathbf{e}_\phi &= -\sin\phi\,\mathbf{i}+\cos\phi\,\mathbf{j}.\end{aligned}\right\} \qquad \ldots(30)$$

From (30), $$\mathbf{e}_r\cdot\mathbf{e}_\theta = \mathbf{e}_\theta\cdot\mathbf{e}_\phi = \mathbf{e}_\phi\cdot\mathbf{e}_r = 0, \qquad \ldots(31)$$

showing that $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$ are mutually perpendicular and hence spherical coordinates are orthogonal curvilinear coordinate.

Again, $$\mathbf{e}_r\times\mathbf{e}_\theta = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\end{vmatrix} = -\sin\phi\,\mathbf{i}+\cos\theta\,\mathbf{j}=\mathbf{e}_\phi, \qquad \ldots(31\text{A})$$

$$\mathbf{e}_\theta\times\mathbf{e}_\phi = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{vmatrix} = \sin\theta(\cos\phi\,\mathbf{i}+\sin\phi\,\mathbf{j})+\cos\theta\,\mathbf{k}=\mathbf{e}_r, \qquad \ldots(31\text{B})$$

and $$\mathbf{e}_\phi\times\mathbf{e}_r = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin\phi & \cos\phi & 0\\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\end{vmatrix} = \cos\theta(\cos\phi\,\mathbf{i}+\sin\phi\,\mathbf{j})-\sin\theta\,\mathbf{k}=\mathbf{e}_\theta. \qquad \ldots(31\text{C})$$

Moreover, $$|\mathbf{e}_r| = \sqrt{\sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\theta+\cos^2\theta} = 1, \qquad \ldots(31\text{D})$$

$$|\mathbf{e}_\theta| = \sqrt{\cos^2\theta\cos^2\phi+\cos^2\theta\sin^2\phi+\sin^2\theta} = 1, \qquad \ldots(31\text{E})$$

and $$|\mathbf{e}_\phi| = \sqrt{\sin^2\phi+\cos^2\phi} = 1. \qquad \ldots(31\text{F})$$

Hence from relation (31) and relation (31A) to (31F), it follows that $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ forms an orthogonal right handed basis.

Using formulae (5), (20), (14) and (19) of Art 1.7C, we obtain

$$\text{grad}\,\psi = \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\mathbf{e}_\phi, \qquad \ldots(32)$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}, \qquad \ldots(33)$$

$$\text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} \qquad \ldots(34)$$

and $$\text{curl}\,\mathbf{F} = \nabla\times\mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\mathbf{e}_r & r\,\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi\\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi\\ F_r & rF_\theta & r\sin\theta F_\phi\end{vmatrix} \qquad \ldots(35)$$

where $$\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi, \qquad \ldots(36)$$

F_r, F_θ, F_ϕ being the components of $\mathbf{F}$ along $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$ respectively.

The elementary arc-length in curvilinear coordinates is given by

$$ds = [h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2]^{1/2}.$$

$\therefore$ For spherical coordinates, we have

$$ds = [(dr)^2 + (rd\theta)^2 + (r \sin\theta\, d\phi)^2]^{1/2} \qquad \ldots(37)$$

The elementary areas in the coordinate planes u_1u_2, u_2u_3, u_3u_1 in curvilinear coordinates are $h_1h_2du_1du_2$, $h_2h_3du_2du_3$, $h_3h_1du_3du_1$ respectively. Hence for spherical coordinates the elementary areas in $r\theta, \theta\phi, \phi r$ planes are $r\,dr\,d\theta, r^2 \sin\theta\, d\theta\, d\phi, r \sin\theta\, d\phi\, dr$ respectively.

The elementary volume in curvilinear coordinates is $h_1h_2h_3du_1du_2du_3$ and hence the elementary volume in spherical coordinates reduces to $r^2 \sin\theta\, dr\, d\theta\, d\phi$.

1.8. Some useful results of cartesian tensor analysis.*

As will be discussed later on, stress and strain are examples of a tensor. In modern literature of fluid dynamics and continuum mechanics, there is a frequent applications of tensor notation. Essentially this is a shorthand notation the use of which makes the writing of many equations less cumbersome. The full advantage of tensor notation and the power of tensor can be realised only in curvilinear coordinates. However, we shall confine ourselves to cartesian coordinates in a three dimensional Euclidean space, and illustrate the main properties of cartesian tensor of second order (or rank). The material presented in this article is strictly confined to the applications in this book.

In what follows each of the suffixes *i, j, k, m, n* etc. will take values 1, 2, 3 only, *i.e.* x_i will denote either one or all (depending upon the context) of x_1, x_2, x_3 and a_{mn} will stand for either one or all of a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}.

Let OX_i $(i = 1, 2, 3)$ be a set of rectangular cartesian coordinate axes with origin O. Let x_i $(i = 1, 2, 3)$ be the cartesian coordinates of a point P in three dimensional space.

Summation Convention. According to it, a repeated suffix implies summation with respect to that suffix. Thus,

$$a_{ij}x_j = \sum_{j=1}^{3} a_{ij}x_j \qquad (i = 1, 2, 3) \qquad \ldots(1)$$

When written out fully, (1) stands for the following three summations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{aligned}\right\}, \qquad \ldots(2)$$

where i is allowed to take values 1, 2, 3. Another example of the summation convention is

$$a_{ij}\,\xi_i\,\eta_j = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\,\xi_i\,\eta_j \qquad \ldots(3)$$

When written out fully, (3) stands for the following sum

$$\left.\begin{aligned} a_{11}\xi_1\eta_1 + a_{12}\xi_1\eta_2 + a_{13}\xi_1\eta_3 \\ + a_{21}\xi_2\eta_1 + a_{22}\xi_2\eta_2 + a_{23}\xi_2\eta_3 \\ + a_{31}\xi_3\eta_1 + a_{32}\xi_3\eta_2 + a_{33}\xi_3\eta_3 \end{aligned}\right\} \qquad \ldots(4)$$

Now and

$$\left.\begin{aligned} a_{ij}x_j &= a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 \\ a_{ik}x_k &= a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 \end{aligned}\right\} \qquad \ldots(5)$$

* For more details, refer "A text book of cartesian tensors" by Shanti Narayan, published by S. Chand & Co., New Delhi.

Thus $$a_{ij}x_j = a_{ik}x_k, \qquad \ldots(6)$$
showing that both expressions in (6) have the same value and they do not depend on j and k. Hence the repeated suffix is a *dummy suffix* and it may be replaced by any other letter as and when required. However, no letter should be repeated more than once.

The Kronecker delta or substitution tensor. It is denoted and defined by

$$\delta_{ij} = \begin{cases} 1 \text{ when } i = j \text{ (not summed)} : \\ 0 \text{ when } i \neq j \end{cases} \qquad \ldots(7)$$

From (7), $$\delta_{ij} = \delta_{ji} \qquad \ldots(8)$$

and $$\delta_{ij}x_j = x_i \qquad \ldots(9)$$

In (9) the operation of δ_{ij} on x_j substitutes i for j. Therefore δ_{ij} is also known as *substitution tensor*.

Partial derivatives are denoted as follows:

$$\left.\begin{aligned} \partial\phi/\partial x_i &= \phi,_i & (i = 1, 2, 3), \\ \partial u_i/\partial x_j &= u_{i,j} & (i, j = 1, 2, 3). \end{aligned}\right\} \qquad \ldots(10)$$

where a comma preceding a suffix i stands for partial differentiation with respect to x_i. Also, we have

$$\phi,_{ii} = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_2^3} = \nabla^2\phi \qquad \ldots(11)$$

$$u_{i,i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}. \qquad \ldots(12)$$

$$\text{div } \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} = A_{i,i} \qquad \ldots(13)$$

$$\text{grad } \phi = \frac{\partial\phi}{\partial x_1}\hat{\boldsymbol{a}}_1 + \frac{\partial\phi}{\partial x_2}\hat{\boldsymbol{a}}_2 + \frac{\partial\phi}{\partial x_3}\hat{\boldsymbol{a}}_3 = \phi,_i \hat{\boldsymbol{a}}_i, \qquad \ldots(14)$$

where $\mathbf{A} = A_i\hat{\boldsymbol{a}}_i$, and $\hat{\boldsymbol{a}}_i$ is the unit vector in the positive x_i-axis.

Concept of a vector or tensor of the first order (or rank).

Let OX_i, OX_i' $(i = 1, 2, 3)$ be two sets of rectangular cartesian coordinate axes through O. Further, suppose the direction cosines of each axis of one system with respect to the three axes of the other are as indicated in the following table.

	OX_1	OX_2	OX_3
OX_1'	l_{11}	l_{12}	l_{13}
OX_2'	l_{21}	l_{22}	l_{23}
OX_3'	l_{31}	l_{32}	l_{33}

In the above table (l_{21}, l_{22}, l_{23}) are the direction cosines of OX_2' with respect to the frame OX_i (1, 2, 3) and (l_{13}, l_{23}, l_{33}) are the direction cosines of OX_3 with respect to the frame OX_i' (1, 2, 3). These direction cosines satisfy twelve relations, three each of the following four types.

$$l_{11}^2 + l_{12}^2 + l_{13}^2 = 1 \quad \ldots(15)$$

$$l_{11}l_{21} + l_{12}l_{22} + l_{13}l_{23} = 0 \quad \ldots(16)$$

$$l_{11}^2 + l_{21}^2 + l_{31}^2 = 1 \quad \ldots(17)$$

$$l_{11}l_{12} + l_{21}l_{22} + l_{31}l_{32} = 0 \quad \ldots(18)$$

Using summation convention, (15) and (16) may be put in the following compact form

$$l_{ij}\, l_{kj} = \delta_{ik} \quad \ldots(19)$$

Similarly, (17) and (18) may be expressed as

$$l_{ij}\, l_{ik} = \delta_{jk} \quad \ldots(20)$$

Let there be a vector with components ξ_i $(i = 1, 2, 3)$ with respect to the axes OX_i and ξ_i' $(i = 1, 2, 3)$ with respect to OX_i'. Then we know that

and $$\left.\begin{aligned} \xi_k' &= l_{ki}\,\xi_i \\ \xi_k &= l_{ik}\,\xi_i' \end{aligned}\right\} \quad \ldots(21)$$

The first of the above relations may be treated as the law of transformation of the set of numbers ξ_i, in the axes OX_i to the set ξ_i' in OX_i', where ξ_i and ξ_i' are the components of the same vector. Thus the second equation is regarded as giving the inverse transformation. Since a vector can be completely represented in terms of its components, any set of three quantities may be taken to define a vector provided the quantities can be transformed according to (21). In what follows we shall use the following notation and definition of vector.

$\{\xi_i\}$ *is said to be a vector* (*or tensor of first order*) *if its components can be transformed according to the transformation law* (21) *when the coordinate axes are changed by rotation.*

Suppose $\{a_i\}$ is a vector with components a_i in the coordinate axes OX_i and a_i' in OX_i'. The scalar product of $\{a_i\}$ and $\{\xi_i\}$ is defined as

$$\{a_i\}\cdot\{\xi_i\} = a_i\xi_i = a_1\xi_1 + a_2\xi_2 + a_3\xi_3 \quad \ldots(22)$$

Using (21), (22) may be rewritten as

$$a_i\xi_i = (l_{ki}a_k')(l_{ji}\xi_j') = l_{ki}l_{ji}a_k'\xi_j' = \delta_{jk}a_k'\xi_j' = a_j'\xi_j' = a_i'\xi_i' \quad \ldots(23)$$

Thus the quantity $a_i\xi_i$ is invariant under the transformation. This result is in accordance with the fact that scalar product of two vectors is always independent of the choice of the axes.

Concept of a tensor of second order (or rank).

We have just noted that the homogeneous linear expression in ξ_i, namely (22), may be employed as a criterion to test whether a_i are components of a tensor of first order (or vector). Generalizing (22), a tensor of second order may be defined by a homogeneous bilinear expression in ξ_i and η_i, where $\{\xi_i\}$ and $\{\eta_i\}$ are vectors. Let us consider the form

$$E = a_{ij}\xi_i\eta_j = a_{ij}\xi_j\eta_i$$

$$= a_{11}\xi_1\eta_1 + a_{12}\xi_1\eta_2 + a_{13}\xi_1\eta_3 + a_{21}\xi_2\eta_1 + a_{22}\xi_2\eta_2 + a_{23}\xi_2\eta_3 + a_{31}\xi_3\eta_1 + a_{32}\xi_3\eta_2 + a_{33}\xi_3\eta_3 \quad \ldots(24)$$

where a_{ij} $(i, j = 1, 2, 3)$ are nine quantities. These nine quantities will be denoted by $\{a_{ij}\}$. Let $\{\xi_i\}$ and $\{\eta_i\}$ be defined in the frame OX_i. Again let these be transformed to $\{\xi_i'\}$ and $\{\eta_i'\}$

respectively when the axes are changed by rotation to the frame OX_i'. Suppose $\{a_{ij}'\}$ be another set of nine quantities such that

$$a_{ij}'\xi_i'\eta_j' = E = a_{ij}\xi_i\eta_j \qquad \ldots(25)$$

Then a_{ij} are said to be transformed to a_{ij}' with the change of axes. Moreover the expression E is invariant under the transformation. $\{a_{ij}\}$ is said to be a *tensor of the second order*. We now formulate the law of transformation of $\{a_{ij}\}$. Using (21), we have

$$a_{ij}'\xi_i'\eta_j' = a_{ij}'l_{im}\xi_m l_{jn}\eta_n = (l_{im}l_{jn}a_{ij}')\,\xi_m\eta_n \qquad \ldots(26)$$

Again from (24), we have

$$a_{ij}'\ \xi_i'\ \eta_j' = a_{ij}\ \xi_i\ \eta_j = a_{mn}\ \xi_m\ \eta_n \qquad \ldots(27)$$

Comparing (26) and (27), we find

$$a_{mn} = l_{im}\ l_{jn}\ a_{ij}' \qquad \ldots(28)$$

Similarly, the inverse transformation can be shown as

$$a_{mn}' = l_{mi}\ l_{nj}\ a_{ij} \qquad \ldots(29)$$

The set of nine quantities a_{ij} $(i, j = 1, 2, 3)$ *obeying the transformation law* (28) *or* (29) *constitutes the components of a tensor of the of the second order.* (28) *or* (29) *may be taken as the definition of tensor of second order.*

Symmetric Tensor. The tensor $\{a_{ij}\}$ is said to be symmetric if $a_{ij} = a_{ji}$ $(i, j = 1, 2, 3)$. The transformation law (29) shows the symmetric property remains unchanged when the axes are changed by rotation. For a symmetric tensor, the bilinear form (24) may be further simplified. Taking one vector $\{\xi_i\}$ instead of two vectors $\{\xi_i\}$ and $\{\eta_i\}$, the following quadratic form is employed for a symmetric tensor.

$$E = a_{ij}\xi_i\xi_j = a_{11}\xi_1^2 + a_{22}\xi_2^2 + a_{33}\xi_3^2 + 2a_{12}\xi_1\xi_2 + 2a_{23}\xi_2\xi_3 + 2a_{31}\xi_3\xi_1 \qquad \ldots(30)$$

1.9. Units and dimensions.

To measure various physical quantities three fundamental units are frequently used in fluid dynamics. They are (*i*) length (*ii*) Mass and (*iii*) Time. All other units are derived from these fundamental units and they are called *derived units*. In some cases, temperature is also taken as the fourth fundamental unit.

The powers of the fundamental units in terms of which a physical quantity can be represented are known as *dimensions*.

There are three systems of units, namely, (*i*) MKS (*ii*) CGS and (*iii*) FPS.

MKS system. Here the unit of length is metre, the unit of mass is kilogram and the unit of time is second.

CGS system. Here the unit of length is centimetre, the unit of mass is gram and the unit of time is second.

FPS system. Here the unit of length is foot, the unit of mass is pound and the unit of time is second.

Note. In British gravitational units, force (pound force), length (foot), and time (second) are taken as the fundamental units. The unit of mass (slug) is defined as follows: The slug is defined as the mass which will be accelerated at the rate of 1 ft/sec^2 when acted upon by a force of one pound.

The following table gives a list of basic units and some derived units along with their dimensions.

Units and dimensions of some important quantities

Quantities	*Dimension*	*Name of Unit*	*Unit Symbol*
		Fundamental Units	
Mass	[M]	kilogram	kg
Length	[L]	metre	m
Time	[T]	second	s
Temperature	[θ]	degree kelvin (or centigrade)	K (or °C)
		Derived Units	
Acceleration	[LT^{-2}]	metre per second squared	m/s^2
Angular velocity	[T^{-1}]	radian per second	rad/s
Angular acceleration	[T^{-2}]	radian per second squared	rad/s^2
Area	[L^2]	square metre	m^2
Circulation	[L^2T^{-1}]	metre squared per second	m^2/s
Density	[ML^{-3}]	kilogram per cubic metre	kg/m^3
Dynamic viscosity	[$ML^{-1}T^{-1}$]	kilogram per metre per second	kg/m-s
Dissipation function	[$ML^{-1}T^{-3}$]	kilogram per metre per second cubed	$kg/m\text{-}s^3$
Energy, work, quantity of heat	[ML^2T^{-2}]	Joule	J($\equiv$ N-m)
Force	[MLT^{-2}]	Newton	N($\equiv$ $kg\text{-}m/s^2$)
Kinematic viscosity	[L^2T^{-1}]	metre squared per second	m^2/s
Momentum	[MLT^{-1}]	kilogram metre per second	kg-m/s
Mass flow rate	[MT^{-1}]	kilogram per second	kg/s
Pressure, stress	[$ML^{-1}T^{-2}$]	Newton per metre squared	N/m^2
Rate of strain	[T^{-1}]	per second	1/s
Specific heat	[$L^2T^{-2}\theta^{-1}$]	Joule per kilogram per degree Kelvin	J/kg-K
Torque	[ML^2T^{-2}]	Newton metre	N-m
Velocity	[LT^{-1}]	metre per second	m/s
Velocity potential	[L^2T^{-1}]	metre squared per second	m^2/s
Viscosity coefficient	[$ML^{-1}T^{-1}$]	Newton metre per second squared	$N\text{-}m/s^2$
Vorticity	[T^{-1}]	per second	1/s
Volume	[L^3]	cubic metre	m^3

1.10 Expressions of $\sin\theta$, $\sinh\theta$, $\cos\theta$ and $\cosh\theta$ as infinite products

$$\sin\theta = \theta \prod_{n=1}^{\infty}\left(1-\frac{\theta^2}{n^2\pi^2}\right), \qquad \sinh\theta = \theta \prod_{n=1}^{\infty}\left(1+\frac{\theta^2}{n^2\pi^2}\right)$$

$$\cos\theta = \prod_{n=1}^{\infty}\left\{1-\frac{4\theta^2}{(2n-1)^2\pi^2}\right\}, \qquad \cosh\theta = \prod_{n=1}^{\infty}\left\{1+\frac{4\theta^2}{(2n-1)^2\pi^2}\right\}$$

Exercises

1. Write down a note on continuum hypothesis **[Himanchal 1999, 2002)**
2. Define (*i*) Viscous fluid **[Kanpur 2003]**
   (*ii*) Non-viscous fluid **[Kanpur 2003]**
   (*iii*) Shearign stress **[Meerut 2004]**
   (*iv*) Laminar flow **[Meerut 1999, Kanpur 2001]**
   (*v*) Turbulent flow **[Meerut 1999, Kanpur 2001]**
   (*vi*) Steady flow **[Kanpur 2000; Meerut 2001, 02]**
   (*vii*) Unsteady flow **[Kanpur 2000; Meerut 2001, 02]**
   (*viii*) Uniform flow **[Garhwal 2000, Meerut 2001, 02]**
   (*ix*) Non-uniform flow **[Garhwal 2000, Meerut 2001, 02]**
3. What is hydrodynamics **[Kanpur 2006]**
4. Define viscosity of the fluid. **[GNDU Amritsar 2003, 04, Meerut 2001]**

CHAPTER

2 Kinematics of Fluids in Motion

2.1. Methods of describing fluid motion.

There are two methods for studying fluid motion mathematically. These are Lagrangian and Eulerian (flux) methods and refer to 'individual time-rate of change' and 'local time rate of change' respectively.

(I) Lagrangian method. **[Garhwal 2005; Meerut 2009, 10, 12]**

In this method we study the history of each fluid particle, *i.e.* any fluid particle is selected and is pursued on its onward course observing the changes in velocity, pressure and density at each point and at each instant. Let (x_0, y_0, z_0) be the coordinates of the chosen particle at a given time $t = t_0$. At a later time, $t = t$, let the coordinates of the same particle be (x, y, z). Since the chosen particle is any particle in the fluid, the coordinates (x, y, z) will be functions of t and also of their initial values (x_0, y_0, z_0), so that

$$x = f_1(x_0, y_0, z_0, t), \qquad y = f_2\,(x_0, y_0, z_0, t), \qquad z = f_3\,(x_0, y_0, z_0, t). \qquad ...(1)$$

Let u, v, w and a_x, a_y, a_z be the components of velocity and acceleration respectively. Then, we have

$$u = \partial x / \partial t, \qquad v = \partial y / \partial t \qquad w = \partial z / \partial t \qquad ...(2)$$

and
$$a_x = \partial^2 x / \partial t^2, \qquad a_y = \partial^2 y / \partial t^2, \qquad a_z = \partial^2 z / \partial t^2 \qquad ...(3)$$

Remark 1. The fundamental equations of motion in Lagrangian form are non-linear and hence it leads to many difficulties while solving a problem. In fact, the present method is employed with an advantage only in some one-dimensional (involving one space coordinate) problems. Hence we need to think about another method of describing fluid motion.

Remark 2. This method resembles that of dynamics of a particle in so far as (x, y, z) are dependent on t. However, in Lagrangian method of fluid dynamics (x, y, z) are dependent on four independent variables x_0, y_0, z_0, t.

(II) Eulerian method. **[Ranchi 2010, Agra 2005; Garhwal 2005; Meerut 2009, 2010, 12]**

In this method we select any point fixed in space occupied by the fluid and study the changes which take place in velocity, pressure and density as the fluid passes through this point. Let u, v, w be the components of velocity at the point (x, y, z) at time t. Then, we have

$$u = F_1\,(x, y, z, t), \qquad v = F_2\,(x\,, y, z, t), \qquad w = F_3\,(x, y, z, t). \qquad ...(4)$$

For a particular value of t, (4) exhibits the motion at all points in the fluid at that time. Again for a particular point (x, y, z), u, v, w are functions of t, which define the mode of variations of velocity at that point.

Remark 1. The point under consideration being fixed, x, y, z and t are independent variables and hence dx/dt d^2x/dt^2 etc. have no meaning in this method.

Remark 2. In Lagrangian method a particular fluid particle is identified and changes in velocity etc. are studied as that fluid particle moves onwards. On the other hand, in Eulerian method the individual fluid particles are not identified. Instead, a point in fluid is chosen and changes in velocity etc. are studied as the fluid passes through the chosen fixed point.

Relationship between the Lagrangian and Eulerian methods.

[Garhwal 2001, 05; Meerut 2005]

In order to establish relationship between the two methods, we investigate a relation between the particle parameters and space parameters.

(*i*) **Lagrangian to Eulerian.** Suppose $\phi\,(x_0, y_0, z_0, t)$ be some physical quantity involving Lagrangian description

$$\phi = \phi\,(x_0, y_0, z_0, t) \qquad ...(5)$$

Since Lagrangian description is given, (1) holds. Solving (1) for x_0, y_0, z_0 we have

$$x_0 = g_1\,(x, y, z, t), \qquad y_0 = g_2\,(x, y, z, t), \qquad z_0 = g_3(x, y, z, t) \qquad ...(6)$$

Using (6), (5) reduces to

$$\phi = \phi\,[g_1\,(x, y, z, t), g_2\,(x, y, z, t), g_3\,(x, y, z, t), t], \qquad ...(7)$$

which expresses ϕ in terms of Eulerian description.

(*ii*) **Eulerian to Lagrangian.** Suppose $\psi(x, y, z, t)$ be some physical quantity involving Eulerian description

$$\psi = \psi(x, y, z, t). \qquad ...(8)$$

Since Eulerian description is given, (4) holds. Again, (2) holds for the proposed Lagrangian description. Hence (2) and (4) yield

$$dx/dt = F_1\,(x, y, z, t), \qquad dy/dt = F_2\,(x, y, z, t), \qquad dz/dt = F_3\,(x, y, z, t) \qquad ...(9)$$

The integration of (9) involves three constants of integration which may be taken as initial coordinates x_0, y_0, z_0 of the fluid particle. Thus the integration of (9) leads to the well known equations of Lagrange (1). Using (1), (8) reduces to

$$\psi = \psi\,[f_1\,(x_0, y_0, z_0, t), f_2\,(x_0, y_0, z_0, t)\, f_3\,(x_0, y_0, z_0, t), t], \qquad ...(10)$$

which expresses ψ in terms of Lagrangian description.

2.2. Illustrative solved examples.

Ex. 1. *The velocity components for a two-dimensional fluid system can be given in the Eulerian system by $u = 2x + 2y + 3t$, $v = x + y + t/2$.*

Find the displacement of a fluid particle in the Lagrangian system.

[kanpur 2000, 05, Rajasthan 2003, Rohalkhand 2005]

Sol. Given $\quad u = 2x + 2y + 3t, \qquad v = x + y + t/2 \qquad ...(1)$

In terms of the displacement x and y, the velocity components u and v may be represented by

$$u = dx/dt, \qquad v = dy/dt \qquad ...(2)$$

From (1) and (2), we have

$$dx/dt = 2x + 2y + 3t, \qquad dy/dt = x + y + t/2 \qquad ...(3)$$

Let $D \equiv d/dt$. Then equations (3) become

$$(D-2)\,x - 2y = 3t \qquad ...(4)$$

$$-x + (D-1)\,y = t/2 \qquad ...(5)$$

Operating (5) by $(D-2)$, we have

$$-(D-2)\,x + (D-2)\,(D-1)\,y = (1/2) \times (D-2)\,t$$

or

$$-(D-2)\,x + (D^2 - 3D + 2)\,y = (1/2) - t \qquad ...(6)$$

Adding (4) and (6), we have

$$(D^2 - 3\,D)\,y = (1/2) + 2t \qquad ...(7)$$

Auxiliary equation of (7) is $D^2 - 3D = 0$. Solving for D, it gives D = 0, 3. Hence complementary function (C.F.) is given by

$$\text{C.F.} = c_1 + c_2 e^{3t}$$

Next, the particular integral (P.I.) is given by

$$P.I. = \frac{1}{D^2 - 3D}\left(\frac{1}{2} + 2t\right)$$

$$= \frac{1}{-3D(1 - D/3)}\left(\frac{1}{2} + 2t\right) = -\frac{1}{3D}\left(1 - \frac{1}{3}D\right)^{-1}\left(\frac{1}{2} + 2t\right)$$

$$= -\frac{1}{3D}\left(1 + \frac{1}{3}D + ...\right)\left(\frac{1}{2} + 2t\right) = \frac{1}{3D}\left(\frac{1}{2} + 2t + \frac{1}{3} \times 2\right)$$

$$= -\frac{1}{3}\cdot\frac{1}{D}\left(2t + \frac{7}{6}\right) = -\frac{1}{3}\left(2 \times \frac{t^2}{2} + \frac{7}{6} \times t\right) = -\frac{t^2}{3} - \frac{7t}{18}$$

Hence the general solution of (7) is

$$y = c_1 + c_2 e^{3t} - (t^2/3) - (7t/18) \qquad ...(8)$$

From (8), $\quad dy/dt = 3c_2 e^{3t} - (2t/3) - (7/18) \qquad ...(9)$

Re-writing the second equation of (3), we get

$$x = dy/dt - y - (t/2) \qquad ...(10)$$

Putting the values of y and dy/dt given by (8) and (9) in (10), we get

$$x = 3c_2 e^{3t} - \frac{2}{3}t - \frac{7}{18} - c_1 - c_2 e^{3t} + \frac{1}{3}t^2 + \frac{7}{18}t - \frac{1}{2}t$$

or $\quad x = -c_1 + 2c_2 e^{3t} + (t^2/3) - (7t/9) - (7/18) \qquad ...(11)$

We now use the following initial conditions :

$$x = x_0, \qquad y = y_0 \qquad \text{when} \qquad t = t_0 = 0 \qquad ...(12)$$

Using (12), (8) and (11) reduce to

$$y_0 = c_1 + c_2 \qquad \text{and} \qquad x_0 = -\,c_1 + 2c_2 - (7/18) \qquad ...(13)$$

Solving (13) for c_1 and c_2, we have

$$c_1 = \frac{2y_0 - x_0}{3} - \frac{7}{54} \qquad \text{and} \qquad c_2 = \frac{x_0 + y_0}{3} + \frac{7}{54} \qquad ...(14)$$

Using (14), (11) and (8) give

$$x = \frac{1}{3}x_0 - \frac{2}{3}y_0 + \frac{1}{3}\left(2x_0 + 2y_0 + \frac{7}{9}\right)e^{3t} - \frac{7}{9}t + \frac{1}{3}t^2 - \frac{7}{27} \qquad ....(15)$$

and $$y = -\frac{1}{3}x_0 + \frac{2}{3}y_0 + \frac{1}{3}\left(x_0 + y_0 + \frac{7}{18}\right)e^{3t} - \frac{7}{18}t + \frac{1}{3}t^2 - \frac{7}{54} \qquad ....(16)$$

(15) and 16 give the desired displacements x and y in the Langrangian system involving the initial positions x_0 and y_0 and the time, t.

Ex. 2. *For a two-dimensional flow the velocities at a point in a fluid may be expressed in the Eulerian coordinates by $u = x + y + 2t$ and $v = 2y + t$. Determine the Lagrange coordinates as functions of the initial positions x_0 and y_0 and the time t.* **[I.A.S. 1999]**

Sol. Given $\quad u = x + y + 2t \quad$ and $\quad v = 2y + t. \quad \ldots(i)$

In terms of the displacements x and y, we have

$$u = dx/dt \qquad \text{and} \qquad v = dy/dt. \qquad \ldots(2)$$

From (1) and (2), $\quad dx/dt = x + y + 2t \quad \ldots(3)$

and $\quad dy/dt = 2y + t \quad$ or $\quad dy/dt - 2y = t. \quad \ldots(4)$

Integrating factor (I.F.) of (4) $= e^{\int (-2)\,dt} = e^{-2t}$ and solution of (4) is

$$ye^{-2t} = c_1 + \int t\,(e^{-2t})\,dt, \quad c_1 \text{ being an arbitrary constant}$$

or
$$ye^{-2t} = c_1 + t\left(-\frac{1}{2}e^{-2t}\right) - \int 1\cdot\left(-\frac{1}{2}e^{-2t}\right)dt = c_1 - \frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} = c_1 - \frac{1}{4}(2t+1)e^{-2t}$$

or
$$y = c_1 e^{2t} - (2t + 1)/4. \qquad \ldots(5)$$

Substituting the above value of y in (3), we have

$$\frac{dx}{dt} = x + c_1 e^{2t} - \frac{1}{4}(2t+1) + 2t \qquad \text{or} \qquad \frac{dx}{dt} - x = c_1 e^{2t} + \frac{1}{4}(6t-1) \qquad \ldots(6)$$

I.F. of (6) $= e^{\int (-1)\,dt} = e^{-t}$ and solution of (6) is

$$xe^{-t} = c_2 + \int e^{-t}\left[c_1 e^{2t} + \frac{1}{4}(6t-1)\right]dt = c_2 + c_1 e^{t} + \int \frac{(6t-1)}{4}e^{-t}\,dt$$

or
$$xe^{-t} = c_2 + c_1 e^{t} + \frac{6t-1}{4}(-e^{-t}) - \int\left(\frac{6}{4}\right)(-e^{-t})\,dt$$

or
$$xe^{-t} = c_2 + c_1 e^{t} - \frac{1}{4}(6t-1)e^{-t} - \frac{6}{4}e^{-t} = c_2 + c_1 e^{t} - \frac{1}{4}e^{-t}(6t+5)$$

or
$$x = c_2 e^{t} + c_1 e^{2t} - (6t+5)/4. \qquad \ldots(7)$$

We now use the following initial conditions :

$$x = x_0, \qquad y = y_0 \qquad \text{when} \qquad t = t_0 = 0. \qquad \ldots(8)$$

Using (8), (5) and (7) reduce to

$$y_0 = c_1 - (1/4) \qquad \text{and} \qquad x_0 = c_2 + c_1 - (5/4). \qquad \ldots(9)$$

Solving (9) for c_1 and c_2, $\quad c_1 = y_0 + (1/4), \quad c_2 = x_0 - y_0 + 1. \quad \ldots(10)$

Using (10), (7) and (5) reduce to

$$x = (x_0 - y_0 + 1)e^{t} + (y_0 + 1/4)e^{2t} - (6t + 5)/4 \qquad \ldots(11)$$

and
$$y = (y_0 + 1/4)e^{2t} - (2t + 1)/4. \qquad \ldots(12)$$

(11) and (12) give the desired displacements x and y in the Lagrangian system involving the initial positions x_0, y_0 and the time t.

Ex. 3. *The velocity distribution of a certain two-dimensional flow is given by $u = Ay + B$ and $v = Ct$, where A, B, C are constants. Obtain the equation of the motion of fluid particles in Lagrangian method.*

Sol. Let $\mathbf{r}\,(x, y)$ be the position of the given particle at any time t. Then the path lines for the fluid particle are given by

$$\mathbf{q} = \frac{d\mathbf{r}}{dt} \qquad \Rightarrow \qquad u\mathbf{i} + v\mathbf{j} = \frac{d}{dt}(x\mathbf{i} + y\mathbf{j}).$$

$\Rightarrow \qquad u = dx/dt = Ay + B \qquad \ldots(1)$

and $\qquad v = dy/dt = Ct. \qquad \ldots(2)$

Integrating (2), $\quad y = (1/2) \times Ct^2 + c_1$ where c_1 is a constant of integration $\quad \ldots(3)$

Initially, let $y = y_0$, $t = 0$. Then (3) gives $c_1 = y_0$

So (3) gives $\qquad y = (1/2) \times Ct^2 + y_0. \qquad \ldots(4)$

Substituting the above value of y in (1), we get

$$\frac{dx}{dt} = A\left(\frac{1}{2}Ct^2 + y_0\right) + B \qquad \text{so that} \qquad x = A\left(\frac{1}{6}Ct^3 + y_0 t\right) + Bt + c_2, \qquad \ldots(5)$$

where c_2 is a constant of integration.

Initially, let $x = x_0$, $t = 0$. Then (5) gives $c_2 = x_0$.

So (5) gives
$$x = A\left(\frac{1}{6}Ct^3 + y_0 t\right) + Bt + x_0. \qquad \ldots(6)$$

The required equation of motion is given by (4) and (6).

Ex. 4. (*a*) *The velocities at a point in a fluid in the Eulerian system are given by*

$$u = x + y + z + t, \qquad v = 2(x + y + z) + t, \qquad w = 3\,(x + y + z) + t.$$

Obtain the displacements of a fluid particle in the Lagrangian system. **[Garhwal 2000]**

(*b*) *The velocity field at a point in fluid is given by*

$$q = [x + y + z + t, 2(x + y + z) + t, 3(x + y + z) + t].$$

Obtain the velocity of a fluid partifcle which is at (x_0, y_0, z_0) *initially.*

Sol. (*a*) In terms of the displacements x, y and z, the velocity components u, v and w, may also be represented by

$$u = dx/dt, \qquad v = dy/dt \qquad \text{and} \qquad w = dz/dt \qquad \ldots(1)$$

Using (1) and the given values of u, v, and w, we have

$$dx/dt = x + y + z + t \qquad \ldots(2)$$
$$dy/dt = 2\,(x + y + z) + t \qquad \ldots(3)$$
$$dz/dt = 3\,(x + y + z\,) + t \qquad \ldots(4)$$

Let $D \equiv d/dt$. Then (2) and (3) yield

$$(D - 1)\,x - y = z + t \qquad \ldots(5)$$
$$-\,2x + (D - 2)\,y = 2z + t \qquad \ldots(6)$$

Operating (5) by $(D - 2)$ and then adding the resulting equation to (6), we have

$$(D - 2)\,(D - 1)\,x - 2x = (D - 2)\,(z + t) + 2z + t$$

or $\qquad (D^2 - 3D)\,x = Dz + 1 - t \qquad \ldots(7)$

Next, multiplying both sides of (5) by 2, operating (6) by $(D - 1)$ and adding the resulting equations, we have

$$-\,2y + (D - 1)\,(D - 2)\,y = 2\,(z + t)\,(D - 1)\,(2z + t)$$

or $\qquad (D^2 - 3D)\,y = 2D\,z + 1 + t \qquad \ldots(8)$

Re-writing (4), we have $\qquad (D - 3)\,z = 3x + 3y + t$

or $\qquad (D^2 - 3D)\,(D - 3)\,z = 3(D^2 - 3D)x + 3(D^2 - 3D)y + (D^2 - 3D)t$

or $\qquad (D^3 - 6D^2 + 9D)z = 3Dz + 3 - 3t + 6Dz + 3 + 3t - 3$, using (7) and (8)

or $\qquad (D^3 - 6D^2)z = 3 \qquad \ldots(9)$

Auxiliary equation of (9) is $D^3 - 6D^2 = 0$. Solving for D, it gives $D = 0, 0, 6$. Hence, $\qquad$ C.F. $= c_1 + c_2 t + c_3 e^{6t}$, c_1, c_2 and c_3 being arbitrary constants.

Next, $$\text{P.I.} = \frac{1}{D^3 - 6D^2} 3 = 3 \times \frac{1}{-6D^2(1-D/6)} 1 = -\frac{1}{2D^2}\left(1 - \frac{D}{6}\right)^{-1} 1$$

$$= -\frac{1}{2D^2}\left(1 + \frac{D}{6} + \ldots\right) 1 = -\frac{1}{2D^2} 1 = -\frac{1}{2} \times \frac{t^2}{2}$$

Hence the general solution of (9) is

$z = c_1 + c_2 t + c_3 e^{6t} - (t^2/4)$, c_1, c_2 and c_3 being arbitrary constants ...(10)

Re-writing (3) and (4), we have

$$(D - 2)\, y - 2z = 2x + t \quad \ldots(11)$$

and $$-3y + (D - 3)\, z = 3x + t \quad \ldots(12)$$

As before, (11) and (12) give

$$(D^2 - 5D)y = 2Dx + 1 - t \quad \ldots(13)$$

and $$(D^2 - 5D)\, z = 3Dx + 1 + t \quad \ldots(14)$$

But from (2), $$(D - 1)x = y + z + t$$

$\Rightarrow$ $$(D^2 - 5D)\,(D - 1)x = (D^2 - 5D)y + (D^2 - 5D)\, z + (D^2 - 5D)\, t$$

or $$(D^3 - 6D^2 + 5D)x = 2Dx + 1 - t + 3Dx + 1 + t - 5, \text{ using (13) and (14)}$$

or $$(D^3 - 6D^2)x = -3 \quad \ldots(15)$$

As before, the general solution is $x = a_1 + a_2 t + a_3 e^{6t} + (1/4) \times t^2$ ...(16)

Re-writing (4) and (2), we have

$$(D - 3)\, z - 3x = 3y + t \quad \ldots(17)$$

$$-z + (D - 1)\, x = y + t \quad \ldots(18)$$

As before, (17) and (18) give

$$(D^2 - 4D)\, z = 3Dy + 1 + 2t \quad \ldots(19)$$

and $$(D^2 - 4D)x = Dy + 1 - 2t \quad \ldots(20)$$

But from (3), $$(D - 2)\, y = 2x + 2z + t$$

$\Rightarrow$ $$(D^2 - 4D)\,(D - 2)y = 2(D^2 - 4D)x + 2\,(D^2 - 4D)\, z + (D^2 - 4D)t$$

or $$(D^3 - 6D^2 + 8D)y = 2Dy + 2 - 4t + 6Dy + 2 + 4t - 4, \text{ using (19) and (20)}$$

or $$(D^3 - 6D^2)y = 0 \quad \ldots(21)$$

As before, the general solution is $y = b_1 + b_2 t + b_3 e^{6t}$ ...(22)

Also suppose $x = x_0$, $y = y_0$, $z = z_0$ when $t = t_0 = 0$ ...(23)

Using (23), (16), (22) and (10) give

$$x_0 = a_1 + a_3, \qquad y_0 = b_1 + b_3, \qquad z_0 = c_1 + c_3$$

so that $$a_1 = x_0 - a_3, \qquad b_1 = y_0 - b_3, \qquad c_1 = z_0 - c_3 \quad \ldots(24)$$

Using (24), (16), (22) and (10), we have

$$x = x_0 - a_3 + a_2\, t + a_3 e^{6t} + (1/4) \times t^2 \quad \ldots(25)$$

$$y = y_0 - b_3 + b_2 t + b_3 e^{6t} \quad \ldots(26)$$

$$z = z_0 - c_3 + c_2 t + c_3 e^{6t} - (1/4) \times t^2 \quad \ldots(27)$$

Substituting these values of x, y and z into (2), (3) and (4), we have

$$a_2 + 6a_3 e^{6t} + t/2 = x_0 + y_0 + z_0 - (a_3 + b_3 + c_3) + (a_2 + b_2 + c_2)t + (a_3 + b_3 + c_3)\, e^{6t} + t \quad \ldots(28)$$

$$b_2 + 6b_3 e^{6t} = 2(x_0 + y_0 + z_0) - 2(a_3 + b_3 + c_3) + 2(a_2 + b_2 + c_2)t + 2(a_3 + b_3 + c_3)\, e^{6t} + t \quad \ldots(29)$$

$$c_2 + 6c_3 e^{6t} - (t/2) = 3(x_0 + y_0 + z_0) - 3(a_3 + b_3 + c_3) + 3(a_2 + b_2 + c_2)t + 3(a_3 + b_3 + c_3)e^{6t} + t \quad \ldots(30)$$

(28), (29) and (30) are identities. So equating coefficients of t, e^{6t} and absolute terms, these identities give

$$x_0 + y_0 + z_0 - (a_3 + b_3 + c_3) = a_2 \qquad ...(31a)$$

$$a_3 + b_3 + c_3 = 6a_3 \qquad ...(31b)$$

$$a_2 + b_2 + c_2 + 1 = 1/2 \qquad ...(31c)$$

$$2(x_0 + y_0 + z_0) - 2(a_3 + b_3 + c_3) = b_2 \qquad ...(32a)$$

$$2(a_3 + b_3 + c_3) = 6b_3 \qquad ...(32b)$$

$$2(a_2 + b_2 + c_2) + 1 = 0 \qquad ...(32c)$$

$$3(x_0 + y_0 + z_0) - 3(a_3 + b_3 + c_3) = c_2 \qquad ...(33a)$$

$$3(a_3 + b_3 + c_3) = 6c_3 \qquad ...(33b)$$

$$3(a_2 + b_2 + c_2) + 1 = -(1/2) \qquad ...(33c)$$

From (31c) or (32c) or (33c), we have

$$a_2 + b_2 + c_2 = -(1/2) \qquad ...(34)$$

Adding (31a), (32a) and (33a), we get

$$6[(x_0 + y_0 + z_0) - (a_3 + b_3 + c_3)] = a_2 + b_2 + c_2$$

or $$6[(x_0 + y_0 + z_0) - (a_3 + b_3 + c_3)] = -(1/2), \text{ by (34)}$$

or $$a_3 + b_3 + c_3 = x_0 + y_0 + z_0 + (1/12) \qquad ...(35)$$

Using (35), (31b), (32b) and (33b) give

$$a_3 = (1/6) \times (x_0 + y_0 + z_0 + 1/12) \qquad ...(36)$$

$$b_3 = (1/3) \times (x_0 + y_0 + z_0 + 1/12) \qquad ...(37)$$

$$c_3 = (1/2) \times (x_0 + y_0 + z_0 + 1/12) \qquad ...(38)$$

Again, using (35), (31a), (32a) and (33a) give

$$a_2 = -1/12, \qquad b_2 = -1/6, \qquad \text{and} \qquad c_2 = -1/4 \qquad ...(39)$$

Substututiting the above values of a_2, b_2, c_2, a_3, b_3 and c_3 into (25), (26) and (27) and simplifying, we have

$$x = (5/6) \times x_0 - (1/6) \times y_0 - (1/6) \times z_0 + (1/6) \times (x_0 + y_0 + z_0 + 1/12)e^{6t} - t/12 + t^2/4 - (1/72) \;...(40)$$

$$y = -(1/3) \times x_0 + (2/3) \times y_0 - (1/3) \times z_0 + (1/3) \times (x_0 + y_0 + z_0 + 1/12)\, e^{6t} - t/6 - (1/36) \;...(41)$$

$$z = -(1/2) \times x_0 - (1/2) \times y_0 + (1/2) \times z_0 + (1/2) \times (x_0 + y_0 + z_0 + 1/12)\, e^{6t} - t/4 - t^2/4 - (1/24) \;...(42)$$

which give the desired displacements.

Part. (b) Let u', v', w' be the components of the velocity in Lagrangian system. Then using (40), (41) and (42), we have

$$u' = \partial x/\partial t = (x_0 + y_0 + z_0 + 1/12)e^{6t} - (1/12) + (t/2) \qquad ...(43)$$

$$v' = \partial y/\partial t = 2(x_0 + y_0 + z_0 + 1/12)e^{6t} - (1/6) \qquad ...(44)$$

$$w' = \partial z/\partial t = 3(x_0 + y_0 + z_0 + 1/12)e^{6t} - (1/4) - (t/2) \qquad ...(45)$$

The required velocity is $u'i + v'j + w'k$, where u', v' and w' are given by (43), (44) and (45) respectively.

EXERCISE 2(A)

1. Assuming that the velocity components for a two dimensional flow system can be given in the Eulerian system by $u = A\,(x + y) + Ct$, $v = B\,(x + y) + Et$, find the displacement of a fluid particle in the Lagrangian system. **[Lucknow 2000, 05, Rajasthan 1998]**

2. The velocity at a point in a fluid for a one-dimensional flow may be given in the Eulerian coordinates by $u = Ax + Bt$. Show that $x = f(x_0, t)$ in the Lagrange coordinates can be obtained from the Eulerian system. The initial position of the fluid particle is designated by x_0 and the initial time $t_0 = 0$ may be assumed.

3. Describe the Lagrange' s and Eulerian methods of describing the fluid flows and distinguish between them.

[Garhwal 2001, 05; Kanput 1998, Kurukshetra 2005; Meerut 1995; Rohilkhand 2000]

ANSWERS

1. $$x = -C_1 + \frac{A}{B}C_2 e^{(A+B)t} - \frac{A(E+C)t}{(A+B)^2} + \frac{(BC - AE)t^2}{2(A+B)} - \frac{E+C}{(A+B)^2}$$

$$y = -C_1 + C_2 e^{(A+B)t} - \frac{B(E+C)t}{(A+B)^2} + \frac{(BC - AE)t^2}{2(A+B)}$$

where $$C_1 = -\frac{B}{A+B}\left[x_0 - \frac{A}{B}y_0 + \frac{E+C}{(A+B)^2}\right], \qquad C_2 = \frac{B}{A+B}\left[x_0 + y_0 + \frac{E+C}{(A+B)^2}\right]$$

and $x = x_0$, $y = y_0$ when $t = t_0 = 0$.

3. $x = (x_0 - y_0 + 1)e^t + (y_0 + 1/4)\,e^{2t} - (6t + 5)/4$, $\qquad y = (y_0 + 1/4)\,e^{2t} - (2t + 1)/4$

where $x = x_0$, $y = y_0$ when $t = t_0 = 0$.

2.3. Velocity of a fluid particle.

Let the fluid particle be at P at any time t and let it be at Q at time $t + \delta t$ such that

$$\overrightarrow{OP} = \mathbf{r} \qquad \text{and} \qquad \overrightarrow{OQ} = \mathbf{r} + \delta\mathbf{r}.$$

Then in the interval δt the movement of the particle is $\overrightarrow{PQ} = \delta\mathbf{r}$ and hence the velocity of the fluid particle $\mathbf{q}$ at P is given by

$$\mathbf{q} = \lim_{\delta t \to 0} (\delta\mathbf{r} / \delta t) = d\mathbf{r} / dt,$$

assuming such a limit to exist uniquely. Taking the fluid as continuous, the above assumption is justified. Clearly $\mathbf{q}$ is a function of $\mathbf{r}$ and t and hence it can be expressed as $\mathbf{q} = f(\mathbf{r}, t)$. If u, v, w are the components of $\mathbf{q}$ along the axes, we have $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$.

2.4. Material, local and convective derivatives. (Meerut 2009, 2011)

Suppose a fluid particle moves from $P\,(x, y, z)$ at time t to $Q(x + \delta x, y + \delta y, z + \delta)$ at time $t + \delta t$. Further suppose $f\,(x, y, z, t)$ be a scalar function associated with some property of the fluid (*e.g.* the pressure or density etc.). Let the total change of f due to movement of the fluid particle from P to Q be δf. Then, we have

$$\delta f = (\partial f / \partial x)\delta x + (\partial f / \partial y)\delta y + (\partial f / \partial z)\delta z + (\partial f / \partial t)\delta t$$

or
$$\frac{\delta f}{\delta t}=\frac{\partial f}{\partial x}\frac{\delta x}{\delta t}+\frac{\partial f}{\partial y}\frac{\delta y}{\delta t}+\frac{\partial f}{\partial z}\frac{\delta z}{\delta t}+\frac{\partial f}{\partial t} \qquad ...(1)$$

Let
$$\left.\begin{array}{lll}\lim\limits_{\delta t\to 0}\dfrac{\delta f}{\delta f}=\dfrac{Df}{Dt} \text{ or } \dfrac{df}{dt}, & & \lim\limits_{\delta t\to 0}\dfrac{\delta x}{\delta t}=\dfrac{dx}{dt}=u, \\ \lim\limits_{\delta t\to 0}\dfrac{\delta y}{\delta t}=\dfrac{dy}{dt}=v & \text{and} & \lim\limits_{\delta t\to 0}\dfrac{\delta z}{\delta t}=\dfrac{dz}{dt}=w\end{array}\right\} \qquad ...(2)$$

where $\boldsymbol{q}=(u, v, w)$ is the velocity of the fluid particle at P. Making $\delta t \to 0$ and using (2), (1) reduces to

$$\frac{Df}{Dt}=u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}+\frac{\partial f}{\partial t} \qquad ...(3)$$

But
$$\mathbf{q}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k} \qquad ...(4)$$

and
$$\nabla=(\partial/\partial x)\mathbf{i}+(\partial/\partial y)\mathbf{j}+(\partial/\partial z)\mathbf{k} \qquad ...(5)$$

From (4) and (5),
$$\mathbf{q}\cdot\nabla=u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z} \qquad ...(6)$$

Using (6) and (3) reduces to
$$\frac{Df}{Dt}=\frac{\partial f}{\partial t}+(\mathbf{q}\cdot\nabla)f \qquad ...(7)$$

Again, suppose $\mathbf{g}$ (x, y, z, t) be a vector function associated with some property of the fluid (*e.g.* velocity etc.). Then proceeding as above, we have

$$\frac{D\mathbf{g}}{Dt}=\frac{\partial\mathbf{g}}{\partial t}+(\mathbf{q}\cdot\nabla)\mathbf{g} \qquad ...(8)$$

From (7) and (8), we have for both scalar and vector functions

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\mathbf{q}\cdot\nabla \qquad ...(9)$$

D/Dt is called the material (*or particle or substantial*) *derivative*. It is also spoken of as *differentiation following the motion of the fluid*. The first term on R.H.S. of (9), namely $\partial/\partial t$, is called the *local derivative* and it is associated with time variation at a fixed position. The second term on R.H.S. of (9), namely $\mathbf{q}\cdot\nabla$, is called the *convective derivative* and it is associated with the change of a physical quantity f or $\mathbf{g}$ due to motion of the fluid particle.

Note. The operator *D/Dt* signifies that we are calculating the rate of change of a physical quantity f or $\mathbf{g}$ associated with the same fluid particle as it moves about. The symbol *d/dt* is also used for the material derivative *D/Dt*.

2.5A. Acceleration of a fluid particle. [Kanpur 2004]

Suppose a fluid particle moves from P (x, y, z) at time t to $Q(x+\delta x, y+\delta y, z+\delta z)$ at time $t+\delta t$. Let

$$\mathbf{q}=(u, v, w)=u\mathbf{i}+v\mathbf{j}+w\mathbf{k} \qquad ...(1)$$

be the velocity of the fluid particle at P and let $\boldsymbol{q}+\delta\boldsymbol{q}$ be the velocity of the same fluid particle at Q. Then, we have

$$\delta \boldsymbol{q} = \frac{\partial \boldsymbol{q}}{\partial x}\delta x + \frac{\partial \boldsymbol{q}}{\partial y}\delta y + \frac{\partial \boldsymbol{q}}{\partial z}\delta z + \frac{\partial \boldsymbol{q}}{\partial t}\delta t \quad \text{or} \quad \frac{\delta \boldsymbol{q}}{\delta t} = \frac{\partial \boldsymbol{q}}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial \boldsymbol{q}}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial \boldsymbol{q}}{\partial z}\frac{\delta z}{\delta t} + \frac{\partial \boldsymbol{q}}{\partial t} \qquad \text{...(2)}$$

Let
$$\left.\begin{aligned} &\lim_{\delta t \to 0} \frac{\delta \mathbf{q}}{\delta t} = \frac{D\mathbf{q}}{Dt} \text{ or } \frac{d\mathbf{q}}{dt}, \qquad \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} = u, \\ &\lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt} = v \quad \text{and} \quad \lim_{\delta t \to 0} \frac{\delta z}{\delta t} = \frac{dz}{dt} = w \end{aligned}\right\} \qquad \text{...(3)}$$

Making $\delta t \to 0$ and using (3), (2) reduces to

$$\boldsymbol{a} = \frac{D\mathbf{q}}{Dt} = u\frac{\partial \mathbf{q}}{\partial x} + v\frac{\partial \mathbf{q}}{\partial y} + w\frac{\partial \mathbf{q}}{\partial z} + \frac{\partial \mathbf{q}}{\partial t} \qquad \text{...(4)}$$

Let $$\nabla = (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j} + (\partial/\partial z)\mathbf{k} \qquad \text{...(5)}$$

From (1) and (5), $$\mathbf{q}\cdot\nabla = u(\partial/\partial x) + v(\partial/\partial y) + w(\partial/\partial z) \qquad \text{...(6)}$$

Using (6), (4) may be re-written as

$$\boldsymbol{a} = \frac{D\mathbf{q}}{Dt} = (\mathbf{q}\cdot\nabla)\mathbf{q} + \frac{\partial \mathbf{q}}{\partial t}, \qquad \text{...(7)}$$

which shows that the acceleration $\boldsymbol{a}$ of a fluid particle of fixed identity can be expressed as the material derivative of the velocity vector $\mathbf{q}$.

(*i*) **Components of acceleration in cartesian coordinates** (x, y, z). **(Meerut 2010)**

Let $\boldsymbol{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$. Then (4) yields

$$a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} = u\frac{\partial}{\partial x}(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) + v\frac{\partial}{\partial y}(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) + w\frac{\partial}{\partial z}(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) + \frac{\partial}{\partial t}(u\mathbf{i} + v\mathbf{j} + w\mathbf{k})$$

$\therefore$
$$a_x = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t},$$

$$a_y = \frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t},$$

and
$$a_z = \frac{Dw}{Dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}.$$

(*ii*) **Components of acceleration** $(a_r, a_\theta \cdot a_z)$ **in cylindrical coordinates (r, θ, z) with velocity components** $(v_r, v_\theta \cdot v_z)$.

$$a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}$$

$$a_z = \frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}$$

(*iii*) **Components of acceleration $(a_r, a_\theta \cdot a_\phi)$ in spherical polar coordinates (r, 0, ϕ) with velocity components $(v_r, v_\theta \cdot v_\phi)$.**

$$a_r = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\theta^2}{r}$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r}$$

$$a_\phi = \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\phi v_\theta \cot\theta}{r}$$

2.5B. Acceleration in cartesian coordinates (an alternative proof).

Let $P(x, y, z)$ be any point within the fluid. Let u, v, w be components of velocity of the element of the fluid at P.

Let $$u = f(x, y, z, t) \qquad \ldots(1)$$

Let particle which is at $P(x, y, z)$ at time t move to $Q(x+u\delta t, y+v\delta t, z+w\delta t)$ after a short interval δt. If $u+\delta u$ be *x-component* of velocity at Q, then

$$u+\delta u = f(x+u\delta t,\ y+v\delta t,\ z+w\delta t,\ t+\delta t)$$

$$= f(x,y,z) + \left(u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t}\right)\delta t$$

+ terms containing higher power of δt, by Taylor's theorem

$$\therefore \quad u+\delta u = u + \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right)\delta t + \ldots, \text{ using (1)} \qquad \ldots(2)$$

Let a_x, a_y, a_z be the components of acceleration of the element of the fluid at P. Then,

$$a_x = \lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \lim_{\delta t \to 0} \frac{(u+\delta u) - u}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t})\delta t - u}{\delta t}, \quad \text{using (2)}$$

$$= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}, \quad = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u = \frac{Du}{Dt},$$

where $$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

which in known as *material or substantial derivative.*

$$\therefore \quad a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad \ldots(3)$$

Similarly, we have

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial y}{\partial z}, \qquad ...(4)$$

and

$$a_z = \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}, \qquad ...(5)$$

2.6. Illustrative solved examples.

Ex. 1. *If the velocity distribution is* $\mathbf{q} = \mathbf{i}\, Ax^2y + \mathbf{j}\, By^2zt + \mathbf{k}Czt^2$, *where A, B, C, are constants, then find the the acceleration and velocity components.*

[Agra 2005; Garhwal 2001; Kanpur 2001; Meerut 2009, 2010, 2011]

Sol. The acceleration $\boldsymbol{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ is given by

$$\boldsymbol{a} = \frac{\partial \mathbf{q}}{\partial t} + u\frac{\partial \mathbf{q}}{\partial x} + v\frac{\partial \mathbf{q}}{\partial y} + w\frac{\partial \mathbf{q}}{\partial z} \qquad ...(1)$$

Also $\qquad \mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = \mathbf{i}Ax^2y + \mathbf{j}By^2zt + \mathbf{k}Czt^2 \qquad ...(2)$

Hence, $\qquad u = Ax^2y, \qquad v = By^2zt, \qquad w = Czt^2 \qquad ...(3)$

Using (2) and (3), (1) reduces to

$$\boldsymbol{a} = By^2z\mathbf{j} + 2Czt\mathbf{k} + Ax^2y \times (2Axy\mathbf{i}) + By^2zt(Ax^2\mathbf{i} + 2Byzt\mathbf{j}) + Czt^2(By^2t\mathbf{j} + Ct^2\mathbf{k})$$
$$= A(2Ax^3y^2 + Bx^2y^2zt)\,\mathbf{i} + B\,(y^2z + 2By^3\,z^2t^2 + Cy^2zt^3)\mathbf{j} + C(2zt + Czt^4)\,\mathbf{k}$$

The components of the acceleration (a_x, a_y, a_z) are given by

$a_x = A(2Ax^3y^2 + Bx^2y^2zt), \qquad a_y = B\,(y^2z + 2By^3z^2t^2 + Cy^2zt^3), \qquad a_z = C(2zt + Czt^4)$

Ex. 2. *The velocity components of a flow in cylindrical polar coordinates are* $(r^2z\cos\theta, rz\sin\theta, z^2t)$. *Determine the components of the acceleration of a fluid particle.*

Sol. Let v_r, v_θ, v_z be the components of velocity in cylindrical polar coordinates (r, θ, z). Then, we have

$$v_r = r^2z\cos\theta, \qquad v_\theta = rz\sin\theta, \qquad v_z = z^2t \qquad ...(1)$$

Let a_r, a_θ and a_z be the components of acceleration. Then using (1), we have

$$a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}$$
$$= 0 + (r^2z\cos\theta)\,(2rz\cos\theta) + \{(rz\sin\theta)/r\}(-2rz\sin\theta) + (z^2t)(r^2\cos\theta) - (rz\sin\theta)^2/r$$
$$= rz^2(2r^2\cos^2\theta - 3\sin^2\theta + rt\cos\theta)$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}$$
$$= (r^2z\cos\theta)(z\sin\theta) + \{(rz\sin\theta)/r\}(rz\cos\theta) + (z^2t)(r\sin\theta) + (1/r)\,(r^2z\cos\theta)(rz\sin\theta)$$
$$= z^2r\sin\theta\,(3r\cos\theta + t)$$

$$a_z = \frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}$$
$$= z^2 + (r^2z\cos\theta)\times 0 + \{(rz\sin\theta)/r\}\times 0 + (z^2t)\,(2zt) = z^2(1 + 2t^2z).$$

Ex. 3. *Determine the acceleration at the point* (2,1, 3) at $t = 0.5$ sec, *if* $u = yz + t$, $v = xz - t$ and $w = xy$.

Sol. Velocity field **q** at the point (x, y, z) is given by

$$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = (yz + t)\mathbf{i} + (xz - t)\mathbf{j} + xy\mathbf{k}. \qquad ...(1)$$

The acceleration $\boldsymbol{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ is given by

$$\boldsymbol{a} = \partial \boldsymbol{q}/\partial t + u(\partial \boldsymbol{q}/\partial t) + v(\partial \boldsymbol{q}/\partial t) + w(\partial \boldsymbol{q}/\partial t)$$

$$= (\mathbf{i} - \mathbf{j}) + (yz + t)\,(z\mathbf{j} + y\mathbf{k}) + (xz - t)\,(z\mathbf{i} + x\mathbf{k}) + xy(y\mathbf{i} + x\mathbf{j})$$

$$= (1 + xz^2 + xy^2 - tz)\mathbf{i} + (-1 + yz^2 + x^2z + zt)\mathbf{j} + (y^2z + x^2z + yt - xt)\mathbf{k}$$

$\therefore$ Acceleration at (2, 1, 3) at $t = 0.5$ is given by $\qquad \boldsymbol{a} = 19.5\,\mathbf{i} + 13.5\,\mathbf{j} + 6.5\,\mathbf{k}$

Hence the components a_x, a_y, a_z of acceleration are given by

$$a_x = 19.5, \qquad a_y = 13.5 \qquad \text{and} \qquad a_z = 6.5$$

EXERCISE 2 (B)

1. Determine the acceleration of a fluid particle from the following flow field :

$$\mathbf{q} = \mathbf{i}\,(Axy^2t) + \mathbf{j}\,(Bx^2yt) + \mathbf{k}\,(Cxyz).$$ **[Meerut 2001, 2012]**

Ans. $a_x = A\,(xy^2 + Axy^4t + 2Bx^3y^2t^2)$, $a_y = B\,(x^2y + 2Ax^2y^3t + Bx^4yt^2)$, $a_z = C\,(Axy^3z + Bx^3yzt + x^2y^2z)$

2. Prove that the acceleration of a fluid element of a fixed quantity can be represented by the material derivative of the velocity vector **q**. **[Kanpur 2001]**

3. Find expressions for the acceleration in cartesian coordinates of an element of fluid in motion.

2.7. Significance of the equation of continuity, (or conservation of mass.)

[Kurukshetra 1999; Meerut 2010; Himachal 2002, 09, 10; Garhwal 2005; Kanpur 2003]

The law of conservation of mass states that fluid mass can be neither created nor destroyed. The equation of continuity aims at expressing the law of conservation of mass in a mathematical form. Thus, in continuous motion, the equation of continuity expresses the fact that the increase in the mass of the fluid within any closed surface drawn in the fluid in any time must be equal to the excess of the mass that flows in over the mass that flows out.

2.8. The equation of continuity (or equation of conservation of mass) by Euler's method.

[Kurukshatra 1999; Himachal 2010; Kanpur 2003, 05, 08; Meerut 2003, 10 Purvanchal 2004, 05]

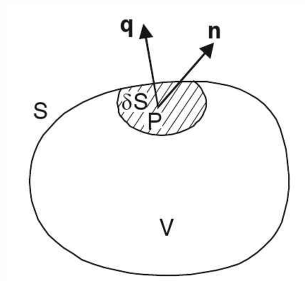
Let S be an arbitrary small closed surface drawn in the compressible fluid enclosing a volume V and let S be taken fixed in space. Let $P\,(x, y, z)$ be any point of S and let $\rho\,(x, y, z, t)$ be the fluid density at P at any time t. Let δS denote element of the surface S enclosing P. Let **n** be the unit outward-drawn normal at δS and let **q** be the fluid velocity at P. Then the normal component of **q** measured outwards from V is **n·q**. Thus,

Rate of mass flow accross $\delta S = \rho\,(\mathbf{n}\cdot\mathbf{q})\,\delta S$

$\therefore$ Total rate of mass flow across S

$$= \int_S \rho\,(\mathbf{n}\cdot\mathbf{q})\,dS = \int_V \nabla\cdot(\rho\,\mathbf{q})\,dV$$

(By Gauss divergence theorem)



$\therefore$ Total rate of mass flow into $V = -\int_V \nabla\cdot(\rho\,\mathbf{q})\,dV \qquad ...(1)$

Again, the mass of the fluid within S at time $t = -\int_V \rho\,dV$

$\therefore$ Total rate of mass increase within $S = \dfrac{\partial}{\partial t}\int_V \rho\, dV = \int_V \dfrac{\partial \rho}{\partial t}\, dV$ ...(2)

Suppose that the region V of the fluid contains neither sources nor sinks (*i.e.* there are no inlets or outlets through which fluid can enter or leave the region). Then by the law of conservation of the fluid mass, the rate of increase of the mass of fluid within V must be equal to the total rate of mass flowing into V. Hence from (1) and (2), we have

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_V \nabla\cdot(\rho\mathbf{q})\, dV \qquad \text{or} \qquad \int_V \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{q})\right] dV = 0$$

which holds for arbitrary small volumes V, if $\dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{q}) = 0.$ ...(3)

Equation (3) is called the *equation of continuity*, or the *conservation of mass* and it holds at all points of fluid free from sources and sinks.

Cor. 1. Since $\nabla\cdot(\rho\mathbf{q}) = \rho\nabla\cdot\mathbf{q} + \nabla\rho\cdot\mathbf{q}$, other forms of (3) are

$$\partial\rho/\partial t + \rho\nabla\cdot\mathbf{q} + \nabla\rho\cdot\mathbf{q} = 0, \qquad ...(4)$$

$$D\rho/Dt + \rho\nabla\cdot\mathbf{q} = 0, \qquad ...(5)$$

and $$D(\log\rho)/Dt + \nabla\cdot\mathbf{q} = 0. \qquad ...(6)$$

Cor. 2. For an incompressible and heterogeneous fluid the density of any fluid particle is invariable with time so that $D\rho/Dt = 0$. Then (5) gives

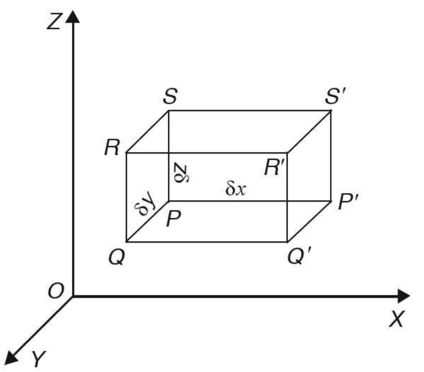
$\nabla\cdot\mathbf{q} = 0$ i.e. div $\mathbf{q} = 0$ or $\partial u/\partial x + \partial v/\partial z + \partial w/\partial z = 0$ if $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$.

Cor. 3. For an incompressible and homogeneous fluid, ρ is constant and hence $\partial\rho/\partial t = 0$. Then (3) gives $\nabla\cdot(\rho\mathbf{q}) = 0$ *i.e.* $\nabla\cdot\mathbf{q} = 0$ or $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$, as ρ is constant.

2.9. The equation of continuity in cartesian coordinates.

[Garhwal 2005; I.A.S. 1999; Kanpur 2011; Meerut 2002; Agra 1997; Bombay 1998' G.N.D.U. Amritsar 2000, 03, 05; Rohilkhand 2005]

Let there be a fluid particle at P (x, y, z). Let $\rho\,(x, y, z, t)$ be the density of the fluid at P at any time t and let u, v, w be the velocity components at P parallel to the rectangular coordinate axes. Construct a small parallelepiped with edges $\delta x, \delta y, \delta z$ of lengths parallel to their respective coordinate axes, having P at one of the angular points as shown in figure. Then, we have



Mass of the fluid that passes in through the face $PQRS$

$= (\rho\delta y\delta z)\, u$ per unit time $= f(x, y, z)$ say ...(1)

$\therefore$ Mass of the fluid that passes out through the opposite face $P'Q'R'S'$

$= f(x+\delta x, y, z)$ per unit time $= f(x,y,z) + \delta x \dfrac{\partial}{\partial x} f(x,y,z) + \ldots$ ...(2)

(expanding by Taylor's theorem)

$\therefore$ The net gain in mass per unit time within the element (rectangular parallelepiped) due to flow through the faces PQRS and $P'Q'R'S'$ by using (1) and (2)

= Mass that enters in through the face $PQRS$ – Mass that leaves through the face $P'Q'R'S'$

$$= f(x,y,z) - \left[f(x,y,z) + \delta x \cdot \frac{\partial}{\partial x} f(x,y,z) + \ldots \right]$$

$$= -\,\delta x \cdot \frac{\partial}{\partial x} f(x,y,z), \text{ to the first order of approximation } = -\,\delta x \cdot \frac{\partial}{\partial x}(\rho u\,\delta y\,\delta z), \text{ by (1)}$$

$$= -\,\delta x\,\delta y\,\delta z\,\frac{\partial(\rho u)}{\partial x} \qquad \ldots(3)$$

Similarly, the net gain in mass per unit time within the element due to flow through the faces $PP'S'S$ and $QQ'RR'$

$$= -\,\delta x\,\delta y\,\delta z\,\frac{\partial(\rho v)}{\partial y} \qquad \ldots(4)$$

and the net gain in mass per unit time within the element due to flow through the faces $PP'Q'Q$ and $SS'R'R$

$$= -\,\delta x\,\delta y\,\delta z\,\frac{\partial(\rho w)}{\partial z} \qquad \ldots(5)$$

$\therefore$ Total rate of mass flow into the elementary parallelepiped

$$= -\,\delta x\,\delta y\,\delta z\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] \qquad \ldots(6)$$

Again, the mass of the fluid within the chosen element at time $t = \rho\,\delta x\,\delta y\,\delta z$

$\therefore$ Total rate of mass increase within the element

$$= \frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z) = \delta x\,\delta y\,\delta z\,\frac{\partial \rho}{\partial t} \qquad \ldots(7)$$

Suppose that the chosen region (bounded by the elementary parallelepiped) of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the mass of the fluid within the element must be equal to the rate of mass flowing into the element. Hence from (6) and (7), we have

$$\delta x\,\delta y\,\delta z\,\frac{\partial \rho}{\partial t} = -\,\delta x\,\delta y\,\delta z\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]$$

or
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad \ldots(8)$$

or
$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y} + \rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z} = 0$$

or
$$\left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]\rho + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

or
$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0, \qquad \text{...(9)}$$

which is the desired equation of continuity in cartesian coordinates and it holds at all point of the fluid free from sources and sinks.

Remark. If the fluid is homogeneous and incompressible, ρ is constant and (9) reduces to

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad \text{...(10)}$$

Further, if the fluid is heterogeneous and incompressible, ρ is a function of x, y, z and t such that $D\rho/Dt = 0$. Hence the corresponding equation of continuity is again given by (10).

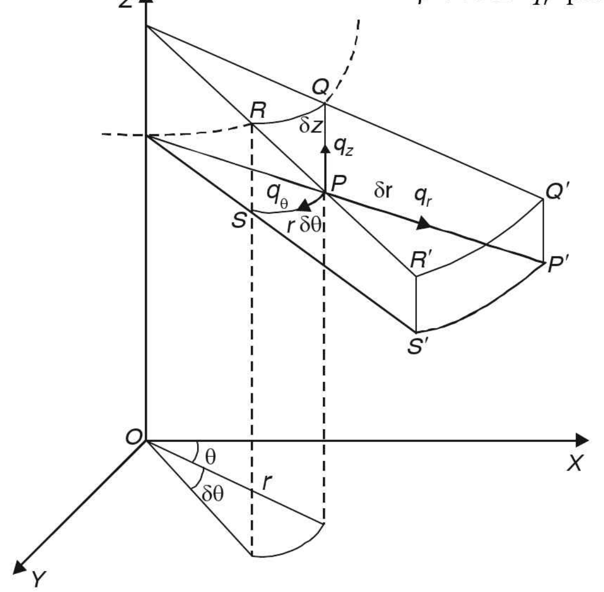
2.10. The equation of continuity in cylindrical coordinates. (Kanpur 2009)

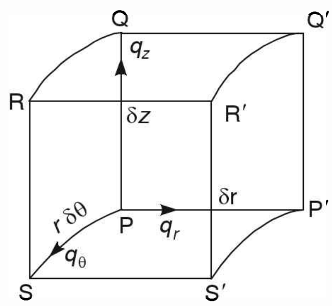
[Agra 2005, Himachal 1998, Meerut 2000, 01, Garhwal 2000, Rajasthan 1998]

Let there be a fluid particle at P whose cylindrical coordinates are (r, θ, z), where $r \geq 0, 0 \leq \theta \leq 2\pi$, $-\infty < z < \infty$. Let $\rho\,(r, \theta, z, t)$ be the density of the fluid at P at any time t. With P as one corner construct a small curvilinear parallelepiped $(PQRS,\ P'Q'R'S')$ with its edges $SS' = \delta r$, arc $SP = r\delta\theta$ and $PQ = \delta z$. Let q_r, q_θ and q_z be the velocity components in the direction of the elements SS', arc SP and PQ respectively. Then, we have

Mass of the fluid that passes in through the face $PSRQ$

$$= \rho \cdot r\delta\theta\,\delta z \cdot q_r \text{ per unit time } = f(r, \theta, z), \text{ say} \qquad \text{...(1)}$$





$\therefore$ Mass of the fluid that passes out through the opposite face $P'S'R'Q'$

$$= f(r + \delta r, \theta, z) \text{ per unit time } = f(r, \theta, z) + \delta r \frac{\partial}{\partial r} f(r, \theta, z) + \ldots \qquad \text{...(2)}$$

(expanding by Taylor's theorem)

$\therefore$ The net gain in mass per unit time within the chosen elementary parallelepiped $(PQRS, P'Q'R'S')$ due to flow through the faces $PSRQ$ and $P'S'R'Q'$ by using (1) and (2)

= Mass that enters in through the face $PQRS$ – Mass that leaves through the face $P'Q'R'S'$

$$= f(r, \theta, z) - \left[f(r, \theta, z) + \delta r \cdot \frac{\partial}{\partial r} f(r, \theta, z) + \ldots\right]$$

$$= -\delta r \cdot \frac{\partial}{\partial r} f(r, \theta, z), \text{ tothe first order of approximation } = -\delta r \cdot \frac{\partial}{\partial r}(\rho\, r\, \delta\theta\, \delta z\, q_r), \text{ by (1)}$$

$$= -\delta r\, \delta\theta\, \delta z \frac{\partial(\rho r q_r)}{\partial r} \qquad \text{...(3)}$$

Similarly, the net gain in mass per unit time within the element due to flow through the faces

$SRR'S'$ and $QPP'Q'$ $$= -\delta r\, \delta\theta\, \delta z \frac{\partial}{\partial \theta}(\rho\, q_\theta) \qquad \text{...(4)}$$

and the net gain in mass per unit time within the element due to flow through the faces $PSS'P'$ and

$QRR'Q'$ $$= -\delta r\, \delta\theta\, \delta z \frac{\partial}{\partial z}(\rho r\, q_z) = -r\delta r\, \delta\theta\, \delta z \frac{\partial(\rho q_z)}{\partial z} \qquad \text{...(5)}$$

$\therefore$ Total rate of mass flow into the chosen element

$$= -\delta r\, \delta\theta\, \delta z \left[\frac{\partial}{\partial r}(\rho r q_r) + \frac{\partial}{\partial \theta}(\rho q_\theta) + r\frac{\partial}{\partial z}(\rho q_z)\right] \qquad \text{...(6)}$$

Again, the mass of the fluid within the element at time $t = \rho r\, \delta r\, \delta\theta\, \delta z$

$\therefore$ Total rate of mass increase within the element $= \dfrac{\partial}{\partial t}(\rho r\, \delta r\, \delta\theta\, \delta z) = r\delta r\, \delta\theta\, \delta z \dfrac{\partial \rho}{\partial t}$ ...(7)

Suppose that the chosen region of the element of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the mass of the fluid within the element must be equal to the rate of mass flowing into the element. Hence from (6) and (7), we have

$$r\,\delta r\, \delta\theta\, \delta z \frac{\partial \rho}{\partial t} = -\delta r\, \delta\theta\, \delta z \left[\frac{\partial}{\partial r}(\rho r q_r) + \frac{\partial}{\partial \theta}(\rho q_\theta) + r\frac{\partial}{\partial z}(\rho q_z)\right]$$

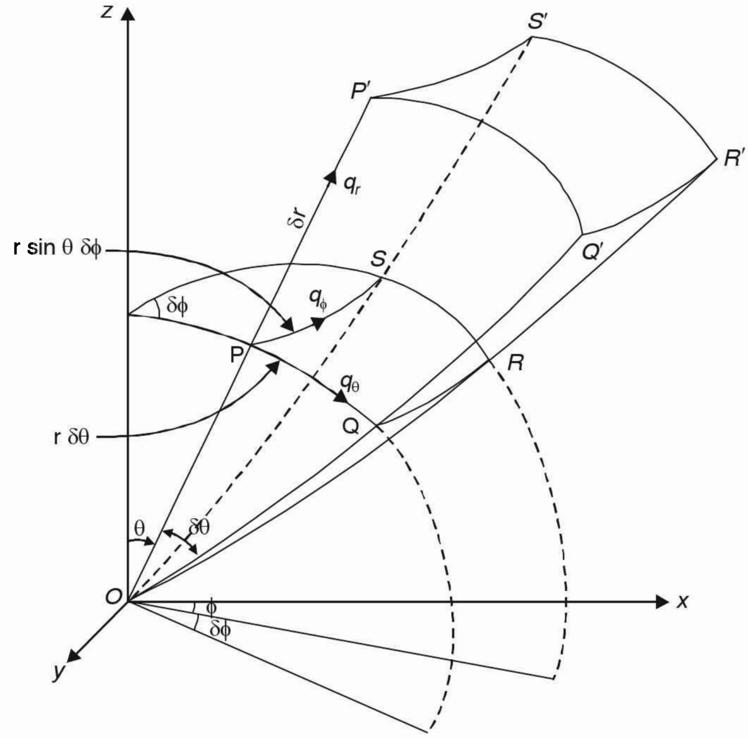
or $$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{1}{\partial z}(\rho q_z) = 0, \qquad \text{...(8)}$$

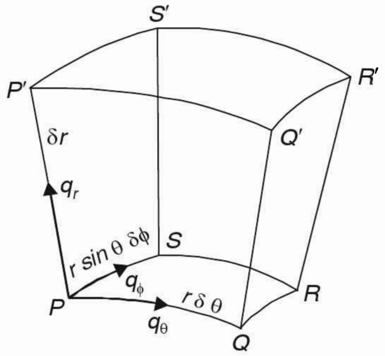
which is the desired equation of continuity in cylindrical coordinates and it holds at all points of the fluid free from sources and sinks.

2.11. The equation of continuity in spherical polar coordinates.

[Meerut 2008; Garhwal 1995, 96; Rajasthan 1997; Rohilkhand 2000]

Let there be a fluid particle at P whose spherical polar coordinates are (r, θ, ϕ), where $r \geq 0, 0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \pi$. Let $\rho(r, \theta, \phi, t)$ be the density of the fluid at P at any time t. With P as one corner construct a small curvilinear parallelopiped $(PQRS, P'Q'R'S')$ with its edges $PP' = \delta r$, arc $PQ = r\delta\theta$, arc $PS = r\sin\theta\delta\phi$. Let q_r, q_θ and q_ϕ be the velocity components in the direction of the elements PP', arc PQ and arc PS respectively. Then, we have





Mass of the fluid that passes in through the face *PQRS*

$$= \rho \cdot r\delta\theta\,\delta r \sin\theta\,\delta\phi \cdot q_r \text{ per unit time } = f(r, \theta, \phi), \text{ say} \qquad ...(1)$$

$\therefore$ Mass of the fluid that passes out through the opposite face *P'Q'R'S'*

$$= f(r + \delta r, \theta, \phi) = f(r, \theta, \phi) + \delta r \frac{\partial}{\partial r}(r, \theta, \phi) + \ldots \qquad ...(2)$$

(expanding by Taylor's theorem)

$\therefore$ The net gain in mass per unit time within the chosen elementary parallelopiped $(PQRS, P'Q'R'S')$ due to flow through the faces *PQRS* and $P'Q'R'S'$ by using (1) and (2)

= Mass that enters in through the face *PQRS* – Mass that leaves through the face $P'Q'R'S'$

$$= f(r, \theta, \phi) - \left[f(r, \theta, \phi) + \delta r \cdot \frac{\partial}{\partial r}(r, \theta, \phi) + \ldots \right]$$

$$= -\delta r \cdot \frac{\partial}{\partial r} f(r, \theta, \phi), \text{ to the first order of approximation}$$

$$= -\delta r \cdot \frac{\partial}{\partial r}(\rho r^2 \sin\theta\, q_r \delta\theta\, \delta\phi), \text{ by (1)} \qquad ...(3)$$

Similarly the net gain in mass per unit time within the element due to flow through the faces $PSS'P'$ and $QRR'Q'$

$$= -r\delta\theta \frac{\partial}{r\delta\theta}(\rho \cdot \delta r \cdot r \sin\theta\, \delta\phi \cdot q_\theta) \qquad ...(4)$$

and the net gain in mass per unit time within the element due to flow through the faces $PQQ'P'$ and $SRR'S'$

$$= -r \sin\theta\, \delta\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}(\rho \cdot \delta r\, r\, \delta\theta \cdot q_\theta) \qquad ...(5)$$

$\therefore$ Total rate of mass flow into the elementary parallelepiped

$$= -\delta r\,\delta\theta\,\delta z\left[\sin\theta\frac{\partial}{\partial r}(\rho r^2 q_r) + r\frac{\partial}{\partial\theta}(\rho\sin\theta q_\theta) + r\frac{\partial}{\partial\phi}(\rho q_\phi)\right] \qquad \text{...(6)}$$

Again, the mass of the fluid within the chosen element at time $t = -\rho\delta r\cdot r\,\delta\theta\cdot r\sin\theta\,\delta\phi$

$\therefore$ Total rate of mass increase within the element

$$= \frac{\partial}{\partial t}(\rho r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi) = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi\frac{\partial\rho}{\partial t} \qquad \text{...(7)}$$

Suppose that the chosen region of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the fluid within the element must be equal to the rate of mass flowing into the element. Hence from (6) and (7), we have

$$-\delta r\,\delta\theta\,\delta z\left[\sin\theta\frac{\partial}{\partial r}(\rho r^2 q_r) + r\frac{\partial}{\partial\theta}(\rho\sin\theta q_\theta) + r\frac{\partial}{\partial\phi}(\rho q_\phi)\right] = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi\frac{\partial\rho}{\partial t}$$

or
$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\sin\theta q_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho q_\phi) = 0, \qquad \text{...(8)}$$

which is the desired equation of continuity in spherical polar coordinates and it holds at all points of the fluid free from sources and sinks.

2.11A. Generalised orthogonal curvilinear coordinates

Let the rectangular cartesian coordinates (x, y, z) of any point P in space be expressed in terms of three independent, single-valued and continuously differentiable scalar point functions u_1, u_2, u_3 as follows :

$$\left.\begin{aligned} x &= x(u_1, u_2, u_3) \\ y &= y(u_1, u_2, u_3) \\ z &= z(u_1, u_2, u_3) \end{aligned}\right\} \qquad \text{...(1)}$$

Suppose that the Jacobian of x, y, z with respect to u_1, u_2, u_3 does not vanish, that is,

$\dfrac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} \neq 0$. Then the transformation (1) can be inverted, *i.e.*, u_1, u_2, u_3 can be expressed in terms of x, y, z giving

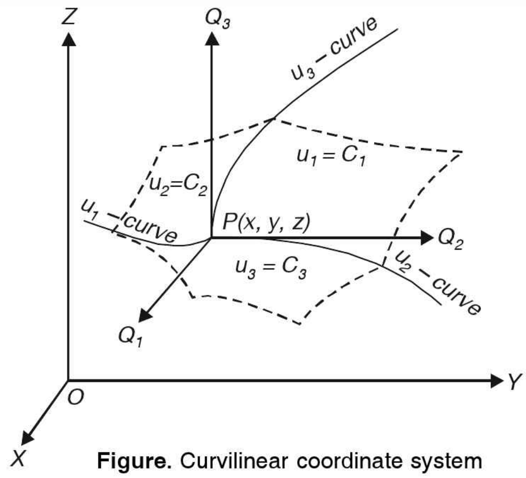


Figure. Curvilinear coordinate system

$$u_1 = u_1(x, y, z), \qquad u_2 = u_2(x, y, z), \qquad u_3 = u_3(x, y, z). \qquad \text{...(2)}$$

Thus to each point $P(x, y, z)$ we can assign a unique set of new coordinates (u_1, u_2, u_3) called the *curvilinear coordinates* of P. In this sense the equations (1) or (2) may be interpreted as defining a *transformation of coordinates*.

The surfaces $u_1(x, y, z) = C_1$, $u_2(x, y, z) = C_2$, $u_3(x, y, z) = C_3$, where C_1, C_2, C_3 are constants, are called *coordinate surfaces* and each pair of these surfaces intersect in curves called *coordinate curves or lines*. The surfaces $u_2 = C_2$ and $u_3 = C_3$ intersect in a curve along which the coordinate 'u_1' alone varies and hence it is called u_1-curve or line. Similarly, we have u_2- line and u_3- line. The coordinate axes are determined by the tangents PQ_1, PQ_2 and PQ_3 to the coordinate curves $u_1 = C_1$, $u_2 = C_2$, $u_3 = C_3$. Note carefully that the directions of these coordinate axes depend on the chosen point P of space and consequently the unit vectors associated with them are not necessarily constant.

If at every point $P(x, y, z)$, the coordinate axes are mutually perpendicular, then u_1, u_2, u_3 are called *orthogonal curvilinear coordinates* of P.

The line element ds in cartesian coordinates in given by

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2. \qquad ...(3)$$

Now, from (1), we have

$$dx = \frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2 + \frac{\partial x}{\partial u_3} du_3, \qquad dy = \frac{\partial y}{\partial u_1} du_1 + \frac{\partial y}{\partial u_2} du_2 + \frac{\partial y}{\partial u_3} du_3$$

and
$$dz = \frac{\partial z}{\partial u_1} du_1 + \frac{\partial z}{\partial u_2} du_2 + \frac{\partial z}{\partial u_3} du_3 \cdot$$

Substituting these values of dx, dy and dz in (3) and using the fact that by orthogonal property coefficents of $du_1 du_2$, $du_2\, du_3$ and $du_3 du_1$ must vanish in the result so obtained, we have

$$(ds)^2 = h_1^2\, (du_1)^2 + h_2^2\, (du_2)^2 + h_3^2\, (du_3)^2, \qquad ...(4)$$

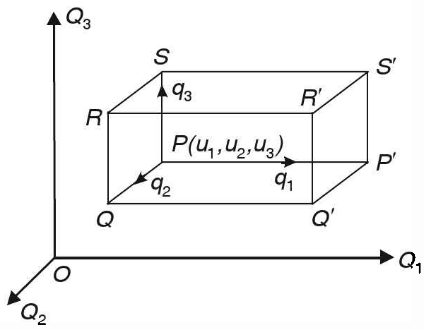
where
$$\left.\begin{aligned} h_1 &= (\partial x / \partial u_1)^2 + (\partial y / \partial u_1)^2 + (\partial z / \partial u_1)^2 \\ h_2 &= (\partial x / \partial u_2)^2 + (\partial y / \partial u_2)^2 + (\partial z / \partial u_2)^2 \\ h_3 &= (\partial x / \partial u_3)^2 + (\partial y / \partial u_3)^2 + (\partial z / \partial u_3)^2, \end{aligned}\right\} \qquad ...(5)$$
and

h_1, h_2, h_3 being known as *scale factors*.

2.11B. Equation of continuity in generalised orthogonal curvilinear coordinates

(Kanpur 2007, 10)

Let there be a fluid particle at P whose orthogonal curvilinear coordinates are (u_1, u_2, u_3). Let $\rho\ (u_1, u_2, u_3, t)$ be the density of fluid at P at any time t and let q_1, q_2, q_3 be the velocity components at P along PP', PQ and PS respectively. Consider an infinitesimal parallelepiped $PQRS$, $P'Q'R'S'$ with one vertex at P as shown in the figure. Then we know that the lengths of edges of parallelopiped are $PP' = h_1 \delta u_1$, $PQ = h_2\, \delta u_2$ and $PS = h_3\, \delta u_3$. Areas of the faces are $h_2 h_3\, \delta u_2 \delta u_3$, $h_3 h_1 \delta u_3\, \delta u_1$ and $h_1\, h_2\, \delta u_1 \delta u_2$ and volume of the parallelepiped is $h_1\, h_2\, h_3\, \delta u_1\, \delta u_2\, \delta u_3$.



Then, mass of the fluid that passes in through the face $PQRS$

$$= \rho(h_2 \delta u_2\, h_3 \delta u_3) q_1 \text{ per unit time} = f(u_1, u_2, u_3), \text{ say.} \qquad ...(1)$$

$\therefore$ Mass of the fluid that passes out through the opposite face $P'Q'R'S'$

$$= f(u_1 + \delta u_1, u_2, u_3) \text{ per unit time} = f(u_1, u_2, u_3) + \delta u_1 \frac{\partial}{\partial u_1} f(u_1, u_2, u_3) + \ldots \qquad ...(2)$$

(expanding by Taylor's theorem)

$\therefore$ The net gain in mass per unit time within the elementary parallelepiped due to flow through the faces $PQRS$ and $P'Q'R'S'$

= Mass that enters in through the face $PQRS$ – Mass that leaves through the face $P'Q'R'S'$

$= f(u_1, u_2, u_3) - [f(u_1, u_2, u_3) + \delta u_1 \dfrac{\partial}{\partial u_1} f(u_1, u_2, u_3) + \ldots]$, by (1) and (2)

$= -\delta u_1 \dfrac{\partial}{\partial u_1} f(u_1, u_2, u_3)$, to the first order of approximation

$= -\delta u_1 \dfrac{\partial}{\partial u_1} (p q_1 h_2 h_3 \delta u_2 \delta u_3)$, using (1)

$$= -\delta u_1 \delta u_2 \delta u_3 \frac{\partial}{\partial u_1} (\rho q_1 h_2 h_3). \qquad ...(3)$$

Similarly, the net gain in mass per unit time within the element due to flow through the faces

$PP'S'S$ and $QQ'R'R$ $$= -\delta u_1 \delta u_2 \delta u_3 \frac{\partial}{\partial u_2}(\rho q_2 h_1 h_3) \qquad ...(4)$$

and the net gain in mass per unit time within the element due to flow through the faces $PP'Q'Q$

and $SS'R'R$ $$= -\delta u_1 \delta u_2 \delta u_3 \frac{\partial}{\partial u_3}(\rho q_3 h_1 h_2). \qquad ..(5)$$

From (3), (4) and (5), total rate of mass flow into the elementary parallelopiped

$$= -\delta u_1 \delta u_2 \delta u_3 \left[\frac{\partial}{\partial u_1}(\rho q_1 h_2 h_3) + \frac{\partial}{\partial u_2}(\rho q_2 h_1 h_3) + \frac{\partial}{\partial u_3}(\rho q_3 h_1 h_2)\right]. \qquad ...(6)$$

Again the mass of the fluid within the chosen element at time $t = \rho h_1 h_2 h_3 \delta u_1 \delta u_2 \delta u_3$.

$\therefore$ Total rate of mass increase within the element

$$= \frac{\partial}{\partial t}(\rho h_1 h_2 h_3 \delta u_1 \delta u_2 \delta u_3) = h_1 h_2 h_3 \delta u_1 \delta u_2 \delta u_3 \frac{\partial \rho}{\partial t}. \qquad ...(7)$$

Now, by the law of conservation of fluid mass, the rate of increase of mass of the fluid within the element must be equal to the rate of flowing into the element. Hence, from (6) and (7), we have

$$h_1 h_2 h_3 \delta u_1 \delta u_2 \delta u_3 \frac{\partial \rho}{\partial t} = -\delta u_1 \delta u_2 \delta u_3 \left[\frac{\partial}{\partial u_1}(\rho q_1 h_2 h_3) + \frac{\partial}{\partial u_2}(\rho q_2 h_1 h_3) + \frac{\partial}{\partial u_3}(\rho q_3 h_1 h_2)\right]$$

or $$\frac{\partial \rho}{\partial t} + \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(\rho q_1 h_2 h_3) + \frac{\partial}{\partial u_2}(\rho q_2 h_1 h_3) + \frac{\partial}{\partial u_3}(\rho q_3 h_1 h_2)\right] = 0 \qquad ...(8)$$

This is the required equation of continuity in orthogonal curvilinear coordinates (u_1, u_2, u_3).

Deductions. *(i)* Rectangular Cartesian Coordinates (x, y, z)

Then, $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2$

$\Rightarrow$ $h_1 = h_2 = h_3 = 1$, $u_1 = x$, $u_2 = y$ and $u_3 = z$.

Also, here $q_1 = u$, $q_2 = v$ and $q_3 = w$

In this case the equation of continuity (8) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0.$$

Deduction *(ii)*. Cylinderical coordinates (r, θ, z)

Cylindrical coordinates (r, θ, z) are defined by means of equations

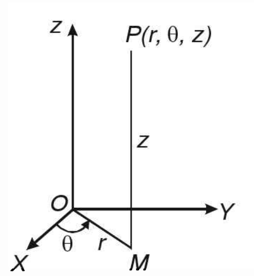
$x = r\cos\theta$, $y = r\sin\theta$, $z = z$,

where $r \geq 0$, $0 \leq \theta \leq 2\pi$, $-\infty < z < \infty$.

Here $(ds)^2 = (dr)^2 + (rd\theta)^2 + (dz)^2 = (h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2$

$\Rightarrow$ $h_1 = 1$, $h_2 = r$, $h_3 = 1$, $u_1 = r$, $u_2 = \theta$, $u_3 = z$.

Also, here $q_1 = q_r$, $q_2 = q_\theta$ and $q_3 = q_z$



Cylindrical Coordinates

In this case the equation of continuity (8) becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial r}(\rho q_r r) + \frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z r)\right] = 0$$

or $$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho q_r r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0.$$

Deduction *(iii)*. Spherical coordinates (r, θ, ϕ).

Spherical coordinates (r, θ, ϕ) are defined by means of equations

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r\cos\theta,$$

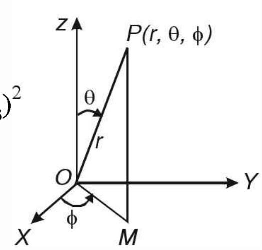
where $\quad r \geq 0, \qquad 0 \leq \theta \leq \pi, \qquad 0 \leq \phi \leq 2\pi.$

Here $(ds)^2 = (dr)^2 + (rd\theta)^2 + (r \sin\theta d\phi)^2 = (h_1 du_1)^2 + (h_2 du_2)^2 + (h_3\, du_3)^2$

$\Rightarrow h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta, \quad u_1 = r, \quad u_2 = \theta, \quad u_3 = \phi.$

Also, here $\quad q_1 = q_r, \qquad q_2 = q_\theta \qquad$ and $\qquad q_3 = q_\phi$

In this case the equation of continuity (8) becomes



Spherical Coordinates

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(\rho r^2 \sin\theta\, q_r) + \frac{\partial}{\partial\theta}(\rho r\sin\theta\, q_\theta) + \frac{\partial}{\partial\phi}(\rho r q_\phi)\right] = 0$$

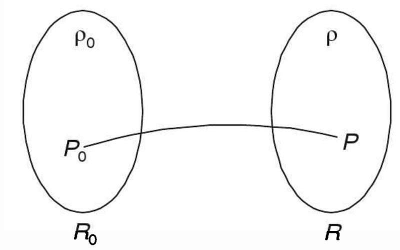
or
$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\sin\theta\, q_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho q_\phi) = 0 \cdot$$

2.12A. The Equation of continuity by the Lagrangian method.

[G.N.D.U - Amritsar 2000, Meerut 2005, 07, Rohilkhand 2005, Kanpur 2003, 04]

Let R_0 be the region occupied by portion of a fluid at the time $t = 0$, and R the region occupied by the same fluid at any time t.

Let (a, b, c) be the initial co-ordinates of a fluid particle P_0 enclosed in this element and ρ_o be its density.



Then mass of the fluid element at $t = 0$ is $\rho_o\, \delta a \delta b \delta c$.

Let P be the subsequent position of P_0 at time t and let ρ be the density of the fluid there.

Then mass of the fluid element at $t = t$ is $\rho \delta x \delta y \delta z$.

From the law of conservation of mass, the mass contained inside a given volume of fluid remains unchanged throughout the motion. Thus, the total mass inside R_0 must be equal to the total mass inside R.

$$\therefore \qquad \iiint_{R_0} \rho_0\, \delta a\, \delta b\, \delta c = \iiint_R \rho\, \delta x\, \delta y\, \delta z \qquad ...(1)$$

From the advanced calculus, we have

$$\delta x\, \delta y\, \delta z = \mathbf{J}\, \delta a\, \delta b\, \delta c \qquad ...(2)$$

where
$$\text{Jacobian } J = \frac{\partial(x,y,z)}{\partial(a,b,c)} = \begin{vmatrix} \partial x/\partial a & \partial x/\partial b & \partial x/\partial c \\ \partial y/\partial a & \partial y/\partial b & \partial y/\partial c \\ \partial z/\partial a & \partial z/\partial b & \partial z/\partial c \end{vmatrix} \qquad ...(3)$$

Using (2), (1) may be re-written as

$$\iiint_{R_0} \rho_0\, \delta a\, \delta b\, \delta c = \iiint_{R_o} \rho\, \mathbf{J}\, \delta a\, \delta b\, \delta c \quad \text{or} \quad \iiint_{R_0} (\rho_0 - \rho\, \mathbf{J})\, \delta a\, \delta b\, \delta c = 0 \qquad ...(4)$$

which holds for all regions R_0 if $\qquad \rho_0 - \rho\mathbf{J} = 0, \qquad ...(5)$

which is the equation of continuity in Lagrangian form.

2.12B. Equivalence between Eulerian and Lagrangian forms of equations of continuity.

[Meerut 2007, Kurukshetra 1997]

Refer figure of Art. 2.12A. Let R_0 be the region occupied by portion of a fluid at the time $t = 0$, and R the region occupied by the same fluid at any time t. Let (a, b, c) be the initial coordinates of a fluid particle P_0 enclosed in this element and ρ_0 be its density. Then mass of the fluid at $t = 0$ is $\rho_0\, \delta a \delta b \delta c$. Let P be the subsequent position of P_0 at time t and let ρ be the density of the fluid there. Then mass of the fluid element at $t = t$ is $\rho \delta x \delta y \delta z$.

The velocity components in the two systems are connected by the equations

$$u = dx/dt, \qquad v = dy/dt, \qquad w = dz/dt \qquad ...(1)$$

Also, $\quad x = x\,(a, b, c, t), \qquad y = y\,(a, b, c, t), \qquad z = z\,(a, b, c, t) \qquad ...(2)$

$$\therefore \quad \frac{\partial u}{\partial a} = \frac{\partial}{\partial a}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{\partial x}{\partial a}\right) \quad \text{so that} \quad \frac{d}{dt}\left(\frac{\partial x}{\partial a}\right) = \frac{\partial u}{\partial a} \quad \text{etc.} \qquad ...(3)$$

The equation of continuity in the Lagrangian form is

$$\rho \mathbf{J} = \rho_0, \qquad ...(4)$$

where
$$J = \frac{\partial(x,y,z)}{\partial(a,b,c)} = \begin{vmatrix} \partial x/\partial a & \partial x/\partial b & \partial x/\partial c \\ \partial y/\partial a & \partial y/\partial b & \partial y/\partial c \\ \partial z/\partial a & \partial z/\partial b & \partial z/\partial c \end{vmatrix} \qquad ...(5)$$

Also, the equation of continuity in the Eulerian form is

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \qquad ...(6)$$

Differentiating both sides of (5) *w.r.t.* '*t*' and using (3), we get

$$\frac{dJ}{dt} = \begin{vmatrix} \partial u/\partial a & \partial u/\partial b & \partial u/\partial c \\ \partial y/\partial a & \partial y/\partial b & \partial y/\partial c \\ \partial z/\partial a & \partial z/\partial b & \partial z/\partial c \end{vmatrix} + \begin{vmatrix} \partial x/\partial a & \partial x/\partial b & \partial x/\partial c \\ \partial v/\partial a & \partial v/\partial b & \partial v/\partial c \\ \partial z/\partial a & \partial z/\partial b & \partial z/\partial c \end{vmatrix} + \begin{vmatrix} \partial x/\partial a & \partial x/\partial b & \partial x/\partial c \\ \partial y/\partial a & \partial y/\partial b & \partial y/\partial c \\ \partial w/\partial a & \partial w/\partial b & \partial w/\partial c \end{vmatrix}$$

or
$$dJ/dt = J_1 + J_2 + J_3, \qquad ...(7)$$

Since $\dfrac{\partial u}{\partial a} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial a} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial a} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial a}$ etc, J_1 can be re-written as (after interchanging its rows and columns) Thus, we have

$$J_1 = \begin{vmatrix} \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial a} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial a} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial a} & \dfrac{\partial y}{\partial a} & \dfrac{\partial z}{\partial a} \\ \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial b} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial b} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial b} & \dfrac{\partial y}{\partial b} & \dfrac{\partial z}{\partial b} \\ \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial c} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial c} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial c} & \dfrac{\partial y}{\partial c} & \dfrac{\partial z}{\partial c} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial a} & \dfrac{\partial y}{\partial a} & \dfrac{\partial z}{\partial a} \\ \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial b} & \dfrac{\partial y}{\partial b} & \dfrac{\partial z}{\partial b} \\ \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial c} & \dfrac{\partial y}{\partial c} & \dfrac{\partial z}{\partial c} \end{vmatrix} + \begin{vmatrix} \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial a} & \dfrac{\partial y}{\partial a} & \dfrac{\partial z}{\partial a} \\ \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial b} & \dfrac{\partial y}{\partial b} & \dfrac{\partial z}{\partial b} \\ \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial c} & \dfrac{\partial y}{\partial c} & \dfrac{\partial z}{\partial c} \end{vmatrix} + \begin{vmatrix} \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial a} & \dfrac{\partial y}{\partial a} & \dfrac{\partial z}{\partial a} \\ \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial b} & \dfrac{\partial y}{\partial b} & \dfrac{\partial z}{\partial b} \\ \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial c} & \dfrac{\partial y}{\partial c} & \dfrac{\partial z}{\partial c} \end{vmatrix}$$

$$= \frac{\partial u}{\partial x}\begin{vmatrix} \dfrac{\partial x}{\partial a} & \dfrac{\partial y}{\partial a} & \dfrac{\partial z}{\partial a} \\ \dfrac{\partial x}{\partial b} & \dfrac{\partial y}{\partial b} & \dfrac{\partial z}{\partial b} \\ \dfrac{\partial x}{\partial c} & \dfrac{\partial y}{\partial c} & \dfrac{\partial z}{\partial c} \end{vmatrix},$$

[the last two determinants vanish because they possess two identical columns]

$\therefore$ $$J_1 = \frac{\partial u}{\partial x} J, \text{ using (5)}$$

Similarly, we have $$J_2 = \frac{\partial v}{\partial y} J \quad \text{and} \quad J_3 = \frac{\partial w}{\partial z} J$$

$\therefore$ (7) becomes $$\frac{dJ}{dt} = J\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \qquad \text{...(8)}$$

Derivation of Eulerian form from Lagragin form :

From (4), $$\frac{d\rho}{dt}(\rho J) = \frac{d}{dt}(\rho_0) = 0 \quad \text{or} \quad \frac{d\rho}{dt} J + \rho \frac{dJ}{dt} = 0$$

$\therefore$ $$\frac{d\rho}{dt} J + \rho J\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0, \text{ using (8)}$$

or $$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0,$$

which is (6) *i.e.* Eulerian form of equation of continuity.

Derivation of Lagrangian form from Eulerian form :

From (6), $$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$\therefore$ $$\frac{d\rho}{dt} + \rho\left(\frac{1}{\mathbf{J}} \frac{d\mathbf{J}}{dt}\right) = 0 \text{ using (8)}$$

or $$J \frac{d\rho}{dt} + \rho \frac{dJ}{dt} = 0 \quad \text{or} \quad \frac{d}{dt}(\rho J) = 0 \qquad \text{...(9)}$$

Integrating (9), $$\rho J = \rho_0,$$

which is (4) *i.e.* Lagrangian equation of continuity.

2.13. Some symmetrical forms of the equation of continuity.

The equation of continuity takes a simplified form in cases when the motion of the fluid possesses certain symmetrical properties as shown below :

(*i*) **Cylindrical Symmetry.** Let there be a fluid particle at P whose cylindrical coordinates are (r, θ, z). Due to cylindrical symmetry, let q_r (r, t) be the velocity at P perpendicular to the axis OZ and let $\rho(r, t)$ be the density of the fluid at P. Consider an element of the fluid consisting of two cylinders of radii r and $r + \delta r$ with OZ as axis, bounded by planes at unit distance apart. Then, we have

Rate of flow across the inner surface $= \rho q_r (2\pi r) = f(r, t)$, say ...(1)

Rate of flow across the outer surface $= f(r + \delta r, t)$ ...(2)

Rate of change of mass within the element $= \dfrac{\partial}{\partial t}(\rho \cdot 2\pi r \cdot \delta r)$ ...(3)

Suppose the element of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the mass within the element must be equal to the rate of mass flowing into the element. Hence from (1), (2) and (3), we have

$$\frac{\partial}{\partial t}(\rho \cdot 2\pi r \cdot \delta t) = f(r,t) - f(r+\delta r, t)$$

or $$2\pi r \delta r \frac{\partial \rho}{\partial t} = f(r,t) - \left[f(r,t) + \delta r \frac{\partial}{\partial r} f(r,t) + \dots \right], \text{ expanding by Taylor's theorem}$$

or $$2\pi r \delta r \frac{\partial \rho}{\partial t} = -\delta r \frac{\partial}{\partial r} f(r,t), \text{ to first order of approximation}$$

or $$2\pi r \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial r}(2\pi r \rho q_r), \text{ by (1)}$$

or $$2\pi r \frac{\partial \rho}{\partial t} = -2\pi \frac{\partial}{\partial r}(r \rho q_r) \qquad \text{or} \qquad \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r \rho q_r) = 0 \qquad \text{...(4)}$$

If ρ is constant, $\partial \rho / \partial t = 0$ and (4) reduces to

$$\frac{\partial}{\partial r}(r \rho q_r) = 0 \qquad \text{...(5)}$$

Integrating (5) *w.r.t.* 'r', we have

$$r \rho q_r = \rho g(t) \qquad \text{or} \qquad r\, q_r = g(t) \qquad \text{...(6)}$$

If the flow is steady, $g(t)$ reduces to an absolute constant. Thus, for a steady flow

$r\, q_r = C$, where C is a constant. ..(7)

Note. The relation (4) may be also be derived as a special case of equation(8) of Art. 2.10 by using the cylindrical symmetry (*i.e.* $\partial/\partial\theta = 0$ and $\partial/\partial z = 0$)

(*ii*) **Spherical Symmetry.** Let there be a fluid particle at P whose spherical polar coordinates are (r, θ, ϕ). Due to spherical symmetry, let $q_r\,(r, t)$ be the velocity at P in the direction of OP and let $\rho(r, t)$ be the density of the fluid at P. Consider an element of the fluid consisting of two concentric spheres of radii r and $r + \delta r$ with O as centre. Then, we have

Rate of flow across the inner surface $= \rho q_r \cdot 4\pi r^2 = f(r,t)$. say ...(1)

Rate of flow across the outer surface $= f(r + \delta r, t)$ ...(2)

Rate of change of mass within the element $= \dfrac{\partial}{\partial t}(\rho \cdot 4\pi r^2 \cdot \delta r)$ ...(3)

Then as in part (*i*) above, we have

$$\frac{\partial}{\partial t}(4\pi r^2 \rho\, \delta r) = f(r,t) - f(r+\delta r, t)$$

or $$4\pi r^2 \delta r \frac{\partial \rho}{\partial t} = f(r,t) - \left[f(r,t) + \delta r \frac{\partial}{\partial r} f(r,t) + \dots \right], \text{ expanding by Taylor's theorem}$$

or $$4\pi r^2 \delta r \frac{\partial \rho}{\partial t} = -\delta r \frac{\partial}{\partial r} f(r,t), \text{ to first order of approximation}$$

or $$4\pi r^2 \delta r \frac{\partial \rho}{\partial t} = -\delta r \frac{\partial}{\partial r}(\rho q_r \cdot 4\pi r^2), \text{ by (1)}$$

or $$4\pi r^2 \frac{\partial \rho}{\partial t} = -4\pi \frac{\partial}{\partial r}(r^2 \rho q_r) \qquad \text{or} \qquad \frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho q_r) = 0 \qquad ...(4)$$

If ρ is constant, $\partial \rho / \partial t = 0$ and (4) reduces to

$$\frac{\partial}{\partial t}(r^2 \rho q_r) = 0 \qquad ...(5)$$

Integrating (5) *w.r.t.* 'r', we have

$$r^2 \rho q_r = \rho g(t) \qquad \text{or} \qquad r^2 q_r = g(t) \qquad ...(6)$$

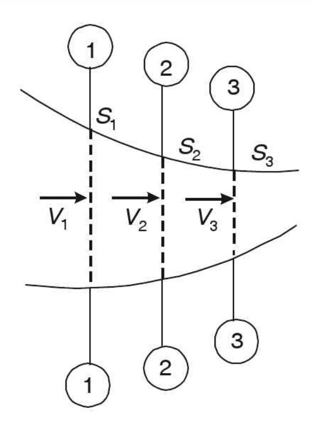
If the flow is steady, $g(t)$ reduces to an absolute constant. Thus, for a steady flow

$r^2 q_r = C$, where C is a constant. ...(7)

Note. The relation (4) may be derived as a special case of equation (8) of Art. 2.11 by using the spherical symmetry $(i.e., \partial/\partial\theta = 0, \partial/\partial\phi = 0)$.

2.14. Equation of continuity of a liquid flow through a channel or a pipe.

Let an incompressible liquid continuously flow through a channel or a pipe whose cross-sectional area may or may not be fixed. Then the quantity of liquid passing per second is the same at all sections.



Suppose some liquid is flowing through a tapering pipe as shown in figure. Let S_1, S_2, S_3 be areas of the pipe at sections 1-1, 2-2, 3-3 respectively. Further, let V_1, V_2 and V_3 be velocities of the liquid at sections 1-1, 2-2, 3-3 respectively. Let Q_1, Q_2, Q_3 be the total quantity of liquid flowing across the sections 1-1, 2-2, 3-3 respectively. Then

$$Q_1 = S_1 V_1, \qquad Q_2 = S_2 V_2, \qquad Q_3 = S_3 V_3 \qquad ...(1)$$

From the law of conservation of mass, the total quantity of liquid flowing across the sections 1-1, 2-2, 3-3 must be the same. Hence

$$Q_1 = Q_2 = Q_3 = \dots \text{ and so on.}$$

Thus, $S_1 V_1 = S_2 V_2 = S_3 V_3 = \dots$ is the equation continuity.

2.15. Working rule of writing the equation of continuity.

Let P be any fluid particle and let ρ be density at P. With P as one corner construct a parallelepiped whose edges are $\lambda\delta\alpha, \mu\delta\beta, \nu\delta\gamma,$ in the chosen coordinate system. Let

Lengths of elements	:	$\lambda\delta\alpha,$	$\mu\delta\beta,$	$\nu\delta\gamma,$
Components of velocity	:	u	v	w

Now calculate the rate of the excess of the flow-in over flow-out along the first length by taking the negative derivative with respect to the first length of the product (density × velocity in the first direction × product of remaining lengths) and finally multiplying this by first length itself. We thus obtain

$$-\lambda\delta\alpha \frac{\partial}{\lambda\partial\alpha}(\rho u \mu\,\delta\beta\, \nu\,\delta\gamma).$$

Similarly calculate the rates of the excess of the flow in over the flow-out along the remaining two lengths and obtain

$$-\mu\,\delta\beta \frac{\partial}{\mu\partial\beta}(\rho v \lambda\delta\alpha\, \nu\delta\gamma) \qquad \text{and} \qquad -\nu\delta\gamma \frac{\partial}{\nu\partial\gamma}(\rho\, w \lambda\delta\alpha\, \mu\,\delta\beta).$$

Now, the total mass of fluid in the element

= density × product of the three edges of the element = $\rho\lambda\,\delta\alpha\,\mu\delta\,\beta v\,\delta\gamma$.

Hence the rate of increase in mass of the element $= \dfrac{\partial}{\partial t}(\rho\,\lambda\delta\alpha\,\mu\delta\beta\; v\delta\gamma)$

For the equation of continuity, we have

Rate of increase in mass of the element

= Total rate of the excess of the flow-in over the flow out along the three lengths of the element

i.e. $\dfrac{\partial}{\partial t}(\rho\,\lambda\delta\alpha\,\mu\delta\beta\; v\delta\gamma) = -\,\lambda\delta\alpha\dfrac{\partial}{\lambda\partial\alpha}(\rho u\,\mu\delta\beta\, v\delta\gamma)\; - \mu\delta\beta\dfrac{\partial}{\mu\partial\beta}(\rho v\,\lambda\delta\alpha\;\; v\delta\gamma) - v\delta\gamma\dfrac{\partial}{v\partial\gamma}(\rho w\lambda\,\delta\alpha\,\mu\delta\beta)$,

which on simplification yields the desired equation of the continuity.

2.16. Illustrative solved examples.

Ex. 1. *The particles of a fluid move symmetrically in space with regard to a fixed centre; prove that the equation of continuity is*

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial r} + \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 u) = 0,$$ *where u is the velocity at distance r.*

[Meerut 2011; Rohilkhand 2005; Himachal 2003; Kanpur 2004]

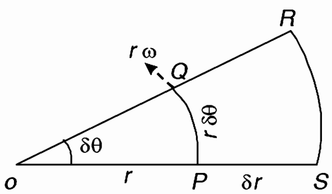
Sol. Here we have spherical symmetry. Proceed as in case (*ii*) Art. 2.13 upto equation (4) and obtain (noting that $q_r = u$ in the present problem).

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho u) = 0 \qquad \text{or} \qquad \frac{\partial\rho}{\partial t} + \frac{1}{r^2}\left\{r^2 u\frac{\partial\rho}{\partial r} + \rho\frac{\partial}{\partial r}(r^2\rho u)\right\} = 0$$

or
$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial r} + \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2\rho u) = 0.$$

Ex. 2. *A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis; show that the equation of continuity is* $\partial\rho/\partial t + \partial(\rho\omega)/\partial\theta = 0$, *where* ω is the angular velocity of a particle whose azimuthal angle is θ at time t

(Meerut 2009; Ranchi 2010)



Sol. Here the motion is confined in a plane. Consider a fluid particle P, whose polar coordinates are (r, θ). Let P describe a circle of radius r. With P as one corner, consider an element $PQRS$ such that $PS = \delta r$ and arc $PQ = r\delta\theta$. Here there is no motion of the fluid along PS. The rate of the excess of the flow-in over the flow-out along PQ

$$= -\,r\delta\theta\frac{\partial}{r\partial\theta}(\rho\cdot r\omega\cdot\delta r).$$

Again, the total mass of the fluid within the element = $\rho\cdot\delta r\cdot r\delta\theta$.

The rate of increase in mass of the element $= \dfrac{\partial}{\partial t}(\rho r\;\delta r\;\delta\theta)$

Hence the equation of continuity is given by

$$\frac{\partial}{\partial t}(\rho r\;\delta r\;\delta\theta) = -\,r\delta\theta\frac{\partial}{r\partial\theta}(\rho r\omega\,\delta r) \qquad \text{or} \qquad r\,\delta r\,\delta\theta\frac{\partial\rho}{\partial r} = -\,r\delta r\,\delta\theta\frac{\partial}{\partial\theta}(\rho\omega)$$

or
$$\partial\rho/\partial t + \partial(\rho\omega)/\partial\theta = 0,$$

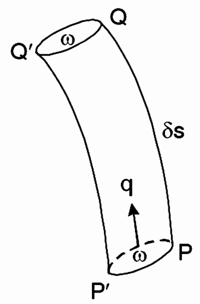
Ex. 3. *If* ω *is the area of cross-section of a stream filament prove that the equation of continuity is*

$$\frac{\partial}{\partial t}(\rho\omega)+\frac{\partial}{\partial s}(\rho\omega q)=0,$$

where δs *in an element of arc of the filament in the direction of flow and q is the speed.*

[Garhwal 1993, 95]

Sol. Let $PP'Q'Q$ be stream filament whose area of cross-section is ω and arc $PQ=\delta s$. The rate of the excess of the flow-in over the flow-out along PQ



$$=-\delta s\frac{\partial}{\partial s}(\rho\omega q)$$

Again, the total mass of the fluid within the stream filament is $\rho\omega\,\delta s$.

$\therefore$ the rate of increase in mass of the stream filament $=\dfrac{\partial}{\partial t}(\rho\omega\,\delta s)$.

Hence the equation of continuity in given by

$$\frac{\partial}{\partial t}(\rho\omega\,\delta s)=-\delta s\frac{\partial}{\partial s}(\rho q\omega)\qquad\text{or}\qquad\frac{\partial}{\partial t}(\rho\omega)+\frac{\partial}{\partial s}(\rho q\omega)=0.$$

Ex. 4. (*i*) *A pulse travelling along a fine straight uniform tube filled with gas causes the density at time t and distance x from the origin where the velocity is* u_0 *to become* $\rho_0\phi(vt-x)$. *Prove that the velocity u (at time t and distance x from the origin) is given by*

$$v+\frac{(u_0-v)\,\phi(vt)}{\phi(vt-x)}$$

(*ii*) *A gas is moving in a uniform straight tube. Prove that if the density be f(at –x) at a point where t is the time and x is the distance of the point from an end of the tube, its velocity is*

$$\frac{af(at-x)+(v-a)f(at)}{f(at-x)},$$

where v is the velocity at that end of the tube and a is a constant.

Sol. (*i*) Let ρ be the density and u the velocity at a distance x. Then we are given that

$$\rho=\rho_0\phi(vt-x)\qquad\text{...(1)}$$

Again the equation of continuity is

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0\qquad\text{or}\qquad\frac{\partial\rho}{\partial t}+\rho\frac{\partial u}{\partial x}+u\frac{\partial\rho}{\partial x}=0\qquad\text{...(2)}$$

From (1), $\dfrac{\partial\rho}{\partial t}=\rho_0 v\phi'(vt-x)$, and $\dfrac{\partial\rho}{\partial x}=-\rho_0\phi'(vt-x)$ ...(3)

Using (1) and (3), (2) reduces to

$$\rho_0 v\phi'(vt-x)+\rho_0\phi(vt-x)\frac{\partial u}{\partial x}-u\rho_0\phi'(vt-x)=0\quad\text{or}\quad(v-u)\phi'(vt-x)+\phi(vt-x)\frac{\partial u}{\partial x}=0$$

or

$$\frac{du}{v-u}+\frac{\phi'(vt-x)}{\phi(vt-x)}dx=0$$

Integrating, $\quad -\log (v - x) - \log \phi (vt - x) = -\log C$, C being an arbitrary constant

or
$$(v - u)\, \phi\, (vt - x) = C \qquad ...(4)$$

Given, $u = u_0$ when x = 0 so that $(v - u_0)\, \phi\, (vt) = C$. With this value of C, (4) reduces to

$$(v - u)\, \phi\, (vt - x) = (v - u_0)\, \phi\, (vt) \qquad \text{or} \qquad u = v + \frac{(u_0 - v)\, \phi(vt)}{\phi(vt - x)}$$

(*ii*) Do just like (*i*) yourself.

Ex. 5. *A mass of fluid is in motion so that the lines of motion lie on the surface of co-axial cylinders. Show that the equation of continuity is*

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u) + \frac{\partial}{\partial z}(\rho v) = 0,$$

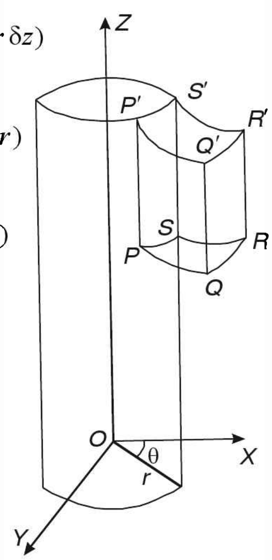
where u,v are the velocity perpendicular and parallel to z.

[Agra 2003; Rohilkhand 2002, Kanpur 2000, 08; Meerut 1999, 2002, 2012]

Sol. Consider a fluid particle P, whose cylindrical coordinates are (r, θ, z). With P as one corner construct an element (curvilinear parallelepiped $PQRS$, $P'Q'R'S'$) with edges $PQ = \delta r$ $PS = r\delta\theta$ and $PP' = \delta z$.

Let ρ be the density of the fluid at P.

Since the lines of motion lie on the surface of co-axial cylinder, there is no motion along PQ. Hence the rate of the excess of the flow-in over flow-out along PQ vanishes. Again, we have



Rate of excess of flow-in over flow-out along $PS = -r\delta\theta \dfrac{\partial}{r\partial\theta}(\rho u\, \delta r\, \delta z)$

Rate of excess of flow-in over flow-out along $PP' = -\delta z \dfrac{\partial}{\partial z}(\rho v r\, \delta\theta\, \delta r)$

Again, the rate of increase in mass of the element $= \dfrac{\partial}{\partial t}(\rho r\, \delta\theta\, \delta r\, \delta z)$

Hence the equation of continuity is given by

$$\frac{\partial}{\partial r}(\rho r\, \delta\theta\, \delta r\, \delta z) = -\delta\theta \frac{\partial}{\partial \theta}(\rho u\, \delta r\, \delta z) - \delta z \frac{\partial}{\partial z}(\rho v r\, \delta\theta\, \delta r)$$

or
$$r\delta\theta\, \delta r\, \delta z \frac{\partial \rho}{\partial t} + \delta r\, \delta\theta\, \delta z \frac{\partial}{\partial \theta}(\rho u) + r\delta r\, \delta\theta\, \delta z \frac{\partial}{\partial z}(\rho v) = 0$$

or
$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u) + \frac{\partial}{\partial z}(\rho v) = 0.$$

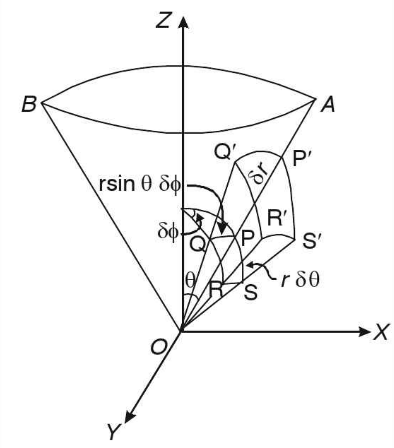
Ex. 6. *If the lines of motion are curves on the surfaces of cones having their vertices at the origin and the axis of z for common surface, prove that the equation of continuity is*

$$\frac{\partial \rho}{\partial r} + \frac{\partial}{\partial r}(\rho u) + \frac{2\rho u}{r} + \frac{\operatorname{cosec}\theta}{\mathrm{r}}\frac{\partial}{\partial \theta}(\rho w) = 0,$$

where u and w are the velocity components in the directions in which r and ϕ increase.

[Agra 2001; Garhwal 2000; Meerut 2001, 02, 03, 04, G.N.D.U. Amritsar 1998; Rohilkhand 2004]

Sol. Let O, the vertex of cones, be the origin and let OZ, their common axis, be the axis of z. Let OAB be a cone of semi-vertical angle θ. Consider a fluid particle P whose spherical polar coordinates are (r, θ, ϕ).. With P as one corner construct an element (curvilinear parallelepiped $PQRS$, $P'Q'R'S'$) with edges $PP' = \delta r$, $PS = r\delta\theta$ and $PQ = r\sin\theta\,\delta\phi$.



Since the lines of motion are curves on the surfaces of cones, there would be no motion perpendicular to the surface of the cone *i.e.*, the velocity in the θ-direction (in the direction of PS) is zero. Further, given that u and w are velocities along PP' and PQ respectively. Since velocity along PS is zero, the excess of flow-in over flow-out along PS vanishes. Again, we have

Rate of excess of flow-in over flow-out along PP'

$$= -\delta r \frac{\partial}{\partial r}(\rho u \cdot r\delta\theta \cdot r\sin\theta\,\delta\phi) = -\sin\theta\,\delta r\,\delta\theta\,\delta\phi \frac{\partial}{\partial r}(r^2 u\rho)$$

Rate of excess of flow-in over flow-out along PQ

$$= -r\sin\delta\phi \frac{\partial}{r\sin\theta\partial\phi}(\rho w \cdot \delta r \cdot r\delta\theta) = -r\delta r\,\delta\theta\,\delta\phi \frac{\partial}{\partial\phi}(\rho w)$$

Also, the rate of increase in mass of the element

$$= \frac{\partial}{\partial t}(\rho\delta r \cdot r\delta\theta \cdot r\sin\theta\,\delta\phi) = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi \frac{\partial\rho}{\partial t}$$

Hence the equation of continuity is given by

$$r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi \frac{\partial\rho}{\partial t} = -\delta r\,\delta\theta\,\delta\phi\left[\sin\theta\frac{\partial}{\partial r}(r^2 u\rho) + r\frac{\partial}{\partial\phi}(\rho w)\right]$$

or $$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho u) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho w) = 0 \quad \text{or} \quad \frac{\partial\rho}{\partial t} + \frac{1}{r^2}\left[r^2\frac{\partial(\rho u)}{\partial r} + 2r\rho u\right] + \frac{1}{r\sin\theta}\frac{\partial(\rho w)}{\partial\phi} = 0$$

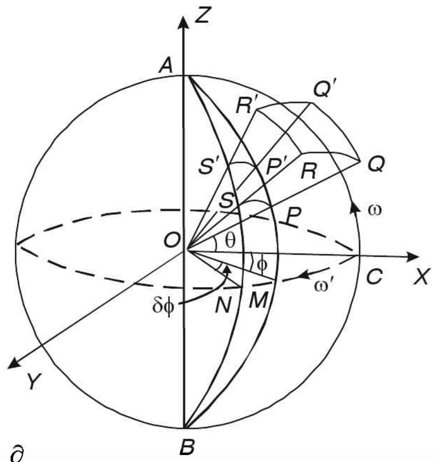
or $$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial r} + \frac{2\rho u}{r} + \frac{\operatorname{cosec}\theta}{r}\frac{\partial(\rho w)}{\partial\phi} = 0.$$

Ex. 7. *If every particle moves on the surface of a sphere prove that the equation of continuity is*

$$\frac{\partial\rho}{\partial t} + \cos\theta + \frac{\partial}{\partial\theta}(\rho\omega\cos\theta) + \frac{\partial}{\partial\phi}(\rho\omega'\cos\theta) = 0,$$

ρ *being the density,* θ, ϕ *the latitude and longitude of any element,* ω *and* ω' *the angular velocities of the element in latitude and longitude respectively.* **(I.A.S. 1991)**

Sol. Consider a fluid particle P on the semi-circle APB making an angle ϕ with semi-circle ACB. Suppose that OP makes an angle θ with OC. With P as one corner construct and elementary parallelepiped $PQRS$, $P'Q'R'S$ on the surface of edges $PQ = \delta r$, $PP' = r\delta\theta$ an $PS = r\cos\theta\,\delta\phi$.



Let ρ be the density of the fluid at P.

Since every particle moves on the surface of the sphere, there will be no velocity normal to the surface of the sphere (*i.e.* radial direction or along PQ). Again the velocities along PP' and PS are $r\omega$ and $r\cos\theta\omega'$ because ω and ω' are the angular velocities in latitude and longitude respectively.

Since velocity along PQ is zero, the rate of excess of flow-in over flow-out along PQ vanishes. Further, we have

Rate of excess of flow-in over flow-out along PP'

$$= -r\delta\theta \frac{\partial}{r\partial\theta}(\rho \cdot r\omega \cdot \delta r \cdot r\cos\theta\,\delta\phi) = -r^2\delta r\,\delta\theta\,\delta\phi \frac{\partial}{\partial\theta}(\rho\omega\cos\theta)$$

Rate of excess of flow-in over flow-out along PS

$$= -r\cos\theta\,\delta\phi \frac{\partial}{r\cos\theta\,\partial\phi}(\rho \cdot r\cos\theta\,\omega' \cdot \delta r \cdot r\delta\theta) = -r^2\delta r\,\delta\theta\,\delta\phi \frac{\partial}{\partial\phi}(\rho\omega'\cos\theta)$$

Again, the rate of increase in mass of the element

$$= \frac{\partial}{\partial t}(\rho \cdot \delta r \cdot r\delta\theta \cdot r\cos\theta\,\delta\phi) = r^2\cos\theta\,\delta r\,\delta\theta\,\delta\phi \frac{\partial\rho}{\partial t}$$

Hence the equation of continuity is given by

$$r^2\cos\theta\,\delta r\,\delta\theta\,\delta\phi \frac{\partial\rho}{\partial t} = -r^2\delta r\,\delta\theta\,\delta\phi\left[\frac{\partial}{\partial\theta}(\rho\omega\cos\theta) + \frac{\partial}{\partial\phi}(\rho\omega'\cos\theta)\right]$$

or
$$\frac{\partial\rho}{\partial t}\cos\theta + \frac{\partial}{\partial\theta}(\rho\,\omega\cos\theta) + \frac{\partial}{\partial\phi}(\rho\omega'\cos\theta) = 0.$$

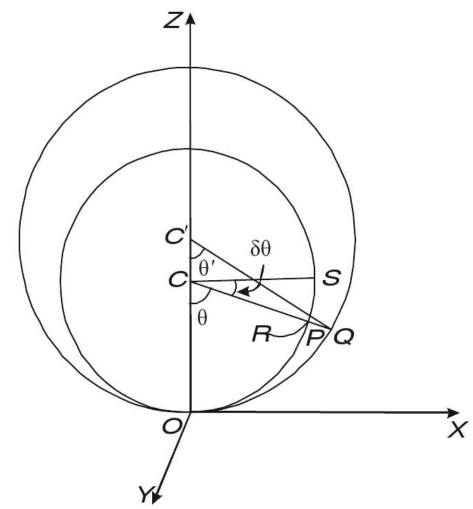
Ex. 8. *If the lines of motion are curves on the surfaces of spheres, all touching the plane of xy at the origin O, the equation of continuity is*

$$r\sin\theta\frac{\partial\rho}{\delta t} + \frac{\partial}{\partial\phi}(\rho v) + \sin\theta\frac{\partial}{\partial\theta}(\rho u) + \rho u\,(1 + 2\cos\theta) = 0,$$

where r is the radius CP of one of the spheres, θ the angle PCO, u the velocity in the plane PCO, v the perpendicular velocity and ϕ *the inclination of the plane PCO to a fixed plane through the axis of z.*

[Agra 1995; Garhwal 2001; I.A.S. 1999, Rohilkhand 2000, U.P.P.C.S. 2002; Rajasthan 1998]

Sol. Let C and C' be the centres of the two consecutive spheres of radii r and $r + \delta r$ respectively touching the plane of xy at the origin O as shown in the figure. Clearly, $CC' = OC' - OC = (r + \delta r) - r = \delta r$. Consider a fluid particle P on the inner sphere. Produce CP so as to meet the outer sphere at Q. Let S be a consecutive point on the circle in the given plane PCO so that $\angle SCO = \theta + \delta\theta$. Hence $\angle SCP = \angle SCO - \angle PCO = (\theta + \delta\theta) - \theta = \delta\theta$. Let PR be an elementary arc in a plane perpendicular to the plane PCO.



In $\Delta CC'Q$, we have form Trigonometry,

$$C'Q^2 = CC'^2 + CQ^2 - 2CC' \cdot CQ \cos QCC'$$

or $\quad (r+\delta r)^2 = \delta r^2 + (r+PQ)^2 - 2\delta r\,(r+PQ)\cos(\pi-\theta)$

or $\quad 2r\delta r = 2r\,PQ + PQ^2 + 2r\,\delta r\cos\theta + 2\delta r\,PQ\cos\theta$

Since PQ and δr are small quantities, to first order of approximation we have

$$2r\delta r\,(1-\cos\theta) = 2r\,PQ$$

or
$$PQ = (1-\cos\theta)\delta r.$$

With P as one corner consider an elementary parallelepiped with edges $PQ = (1-\cos\theta)\delta r$, $PS = r\delta\theta$ and $PR = r\sin\theta\,\delta\phi$, where ϕ is the angle that the plane PCO makes with a fixed plane through the z axis (say, the plane XOZ). The value of PR can be calculated by rotating the plane PCO about Z-axis through an angle $\delta\phi$.

Since the lines of motion are curves on the surfaces of spheres touching the plane of xy, there would be no motion along PQ, *i.e.,* the velocity along PQ is zero. Further, given that u and v are the velocity components along the edges PS and PR in the direction of θ and ϕ increasing.

Since velocity along PQ is zero, the rate of excess of flow in over flow-out along PQ vanishes. Further, we have

Rate of excess of flow-in over flow-out along PS

$$= -r\delta\theta\frac{\partial}{r\partial\theta}\{\rho u\cdot(1-\cos\theta)\,\delta r\cdot r\sin\theta\,\delta\phi\}$$

$$= -r\,\delta r\,\delta\theta\,\delta\phi\left[\sin\theta\,(1-\cos\theta)\frac{\partial}{\partial\theta}(\rho u) + \rho u\cdot\left\{\cos\theta(1-\cos\theta)+\sin^2\theta\right\}\right]$$

$$= -r\,\delta r\,\delta\theta\,\delta\phi\,[\sin\theta\,(1-\cos\theta)\frac{\partial}{\partial\theta}(\rho u) + \rho u\,\{\cos\theta(1-\cos\theta)+(1-\cos^2\theta)\}]$$

$$= -r\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\,[\sin\theta\frac{\partial}{\partial\theta}(\rho u) + \rho u\,(1+2\cos\theta)]$$

Next, rate of excess of flow-in over flow-out along PR

$$= -r\sin\theta\,\delta\phi\frac{\partial}{r\sin\theta\,\partial\phi}\{\rho v\,(1-\cos\theta)\,\delta r\cdot r\delta\theta\} = -r\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\frac{\partial}{\partial\phi}(\rho v)$$

Also, the rate of increase in mass of the element

$$= \frac{\partial}{\partial t}\{\rho\cdot(1-\cos\theta)\,\delta r\cdot r\,\delta\theta\cdot r\sin\theta\,\delta\phi\} = r^2\sin\theta\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\,\frac{\partial\rho}{\partial t}$$

Hence the equation of continuity is given by

$$r^2\sin\theta\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\,\frac{\partial\rho}{\delta t} = -r\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\,[\sin\theta\frac{\partial}{\partial\theta}(\rho u) + \rho u\,(1+2\cos\theta)]$$

$$-r\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\frac{\partial}{\partial\phi}(\rho v)$$

or
$$r\sin\theta\frac{\partial\rho}{\partial t}+\sin\theta\frac{\partial}{\partial\theta}(\rho u)+\frac{\partial}{\partial\phi}(\rho v)+\rho u\,(1+2\cos\theta)=0.$$

Ex. 9. *Show that in a two-dimensional incompressible steady flow field the equation of continuity is satisfied with the velocity components in* rectangular coordinates given by

$$u(x,y)=\frac{k(x^2-y^2)}{(x^2+y^2)^2},\qquad v(x,y)=\frac{2kxy}{(x^2+y^2)^2},$$

where k is an arbitrary constant. **[Meerut 1994; Rolhilkhand 2001, 03, 04]**

Sol. The equation of continuity for incompressible steady flow in cartesian coordinates is

$$\partial u/\partial x+\partial v/\partial y+\partial w/\partial z=0 \qquad \text{...(1)}$$

For a two dimensional flow in xy-plane, $w = 0$ so that (1) reduce to

$$\partial u/\partial x+\partial v/\partial y=0 \qquad \text{...(2)}$$

Differentiating the given values of u and v partially *w.r.t* 'x' and 'y' respectively, we get

$$\frac{\partial u}{\partial x}=k(x^2-y^2)\frac{(-2)\times(2x)}{(x^2+y^2)^3}+\frac{k\times 2x}{(x^2+y^2)^2}=-4kx\frac{(x^2-y^2)}{(x^2+y^2)^3}+\frac{2kx}{(x^2+y^2)^2} \qquad \text{...(3)}$$

$$\frac{\partial v}{\partial y}=2kxy\frac{(-2)\times(2y)}{(x^2+y^2)^3}+\frac{2kx}{(x^2+y^2)^2}=-\frac{8kxy^2}{(x^2+y^2)^3}+\frac{2kx}{(x^2+y^2)^2} \qquad \text{...(4)}$$

From (3) and (4), we have

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{-4kx^3+4kxy^2-8kxy^2}{(x^2+y^2)^3}+\frac{4kx}{(x^2+y^2)^2}=\frac{-4kx^3+4kxy^2-8kxy^2+4kx(x^2+y^2)}{(x^2+y^2)^3}=0$$

Hence, the equation of continuity (2) is satisfied.

Ex. 10. *Consider a two dimensional incompressible steady flow field with velocity components in spherical coordinates* (r,θ,ϕ) *given by*

$$v_r=c_1\left(1-\frac{3}{2}\frac{r_0}{r}+\frac{1}{2}\frac{r_0^3}{r^3}\right)\cos\theta,\qquad v_\phi=0,\qquad v_\theta=-c_1\left(1-\frac{3}{4}\frac{r_0}{r}-\frac{1}{4}\frac{r_0^3}{r^3}\right)\sin\theta,\qquad r\ge r_0>0$$

where c_1 and r_0 are arbitrary constants. Is the equation of continuity satisfied.

Sol. The equation of continuity in spherical polar coordinates is given by (using Art. 2.11 with notations : $q_r=v_r$, $q_\theta=v_\theta$, $q_\phi=v_\phi$)

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r)+\frac{1}{r\sin\theta}\frac{1}{\partial\theta}(\rho\sin\theta\,v_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v_\phi)=0$$

For a two-dimensional incompressible steady flow with $v_\phi=0$, we have ρ = constant and $\partial\rho/\partial t=0$. Hence for the present flow, the equation of continuity is given by

$$\frac{\rho}{r^2}\frac{\partial(r^2v_r)}{\partial r}+\frac{\rho}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,v_\theta)=0 \quad\text{or}\quad \frac{1}{r^2}\left[r^2\frac{\partial v_r}{\partial r}+2rv_r\right]+\frac{1}{r\sin\theta}\left[\sin\theta\frac{\partial v_\theta}{\partial\theta}+\cos\theta\cdot v_\theta\right]=0$$

or
$$\frac{\partial v_r}{\partial r}+\frac{2v_r}{r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_\theta}{r}\cot\theta=0 \qquad \text{...(1)}$$

From the given values of v_r and v_θ, we have

$$\frac{\partial v_r}{\partial r} = c_1\left(0 + \frac{3}{2}\frac{r_0}{r^2} - \frac{3r_0^3}{2r^4}\right)\cos\theta \qquad ...(2)$$

and
$$\frac{\partial v_\theta}{\partial \theta} = -c_1\left(1 - \frac{3r_0}{4r} - \frac{r_0^3}{4r^3}\right)\cos\theta \qquad ...(3)$$

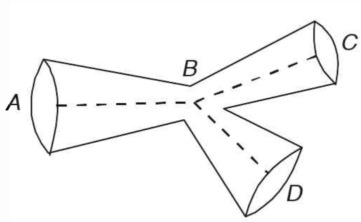
Using (2) and (3), we have

$$\text{L.H.S. of (1)} = c_1\left(\frac{3}{2}\frac{r_0}{r^2} - \frac{3r_0^3}{2r^4}\right)\cos\theta + \frac{2c_1}{r}\left(1 - \frac{3}{2}\frac{r_0}{r} + \frac{1}{2}\frac{r_0^3}{r^3}\right)\cos\theta - \frac{c_1}{r}\left(1 - \frac{3r_0}{4r} - \frac{r_0^3}{4r^3}\right)\cos\theta$$

$$-\frac{c_1}{r}\left(1 - \frac{3}{4}\frac{r_0}{r} - \frac{r_0^3}{4r^3}\right)\cot\theta\sin\theta = 0, \text{ on simplification}$$

Hence, the equation of continuity (1) is satisfied.

Ex. 11. *A pipe branches into two pipes C and D as shown in the adjoining figure. The pipe has diameter of 45 cm at A, 30 cm at B, 20 cm at C and 15 cm at D. Determine the discharge at A, if the velocity at A is 2m/sec. Also determine the velocities at B and D, if the velocity at C is 4 m/sec.*



Sol. Let S_A, S_B, S_C, and S_D be areas of cross sections and let V_A, V_B, V_C, V_D, be velocities at A, B, C and D respectively. Then, we have

$$S_A = \pi\left(\frac{0.45}{2}\right)^2 = 0.159 \text{ square meters}, \qquad S_B = \pi\left(\frac{0.3}{2}\right)^2 = 0.0706 \text{ square meters}$$

$$S_C = \pi\left(\frac{0.2}{2}\right)^2 = 0.0314 \text{ square meters}, \qquad S_D = \pi\left(\frac{0.15}{2}\right)^2 = 0.01767 \text{ square meters}$$

Also, given that $V_A = 2$ m/sec and $V_C = 4$ m/sec

Let Q_A, Q_B, Q_C, and Q_D, be discharges at A, B, C and D respectively. Remembering that Discharge = area of cross-section × velocity, we have

$$Q_A = S_A V_A = 0.159 \times 2 = 0.318 \text{ m}^3/\text{sec}$$

From the equation of continuity (refer Art. 2.14), we have

$$S_A V_A = S_B V_B \quad \text{so that} \quad V_B = \frac{S_A V_A}{S_B} = \frac{Q_A}{S_B} = \frac{0.318}{0.0706} = 4.5 \text{ m/sec.}$$

Again, from the geometry of flow, we have

$$Q_A = Q_C + Q_D \quad \text{or} \quad 0.138 = S_C V_C + S_D V_D$$

or $0.138 = 0.0314 \times 4 + 0.01767 \times V_D$ so that $V_D = 10.6$ m/sec

Ex. 12. *The diameters of a pipe at the sections A and B are 200 mm and 300 mm respectively. If the velocity of water flowing through the pipe at section A is 4m/s, find*

(i) Discharge through the pipe (ii) velocity of water at section B.

Sol. Radii r_1 and r_2 at the section A and B are given by

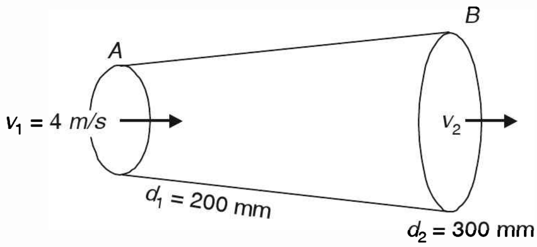
$r_1 = d_1/2 = 100$ mm $= 0.1$ m,

$r_2 = d_2/2 = 150$ mm $= 0.15$ m

S_1 = area of section $A = \pi r_1^2$

$= \pi \times (0.1)^2 = 0.0314 m^2$

v_1 = velocity at section A = 4m/s (given)



S_2 = area of section $B = \pi r_2^2 = \pi \times (0.15)^2 = 0.0707 m^2$

(*i*) **To determine discharge Q through the pipe.** We have

$$Q = S_1 v_1 = 0.0314 \times 4 = 0.1256 \text{ m}^3/\text{s}.$$

(*ii*) **To determine velocity v_2 of water at section B :** Here the continuity equation is

$$S_1 v_1 = S_2 v_2 \quad \Rightarrow \quad v_2 = \frac{S_1 v_1}{S_2} = \frac{0.0314 \times 4}{0.0707} = 1.77 \; m/s$$

Ex.13. *A pipe A 450 mm in diameter branches into two pipe B and C of diameters 300 mm and 200 mm respectively. If the average velocity in 450 mm diameter pipe is 3 m/s, find (i) Discharge through 450 mm diameter pipe (ii) Velocity in 200 mm diameter pipe if the average velocity in 300 mm pipe is 2.5 m/s.*

Sol. S_1 = area of section $A = \pi(d_1/2)^2 = (\pi/4) \times (0.45)^2 = 0.159 \text{ m}^2$

S_2 = area of section $B = \pi(d_2/2)^2 = (\pi/4) \times (0.3)^2 = 0.0707 \text{ m}^2$

S_3 = area of section $C = \pi(d_3/2)^2 = (\pi/4) \times (0.2)^2 = 0.0314 \text{ m}^2$

(*i*) **To find dischange Q_1 through A :**

$Q_1 = S_1 v_1 = 0.159 \times 3 = 0.477$ m³/s.

(*ii*) **To find velocity v_3 in pipe C.**

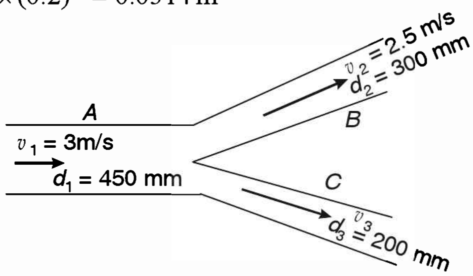
By continuity equation,we have

$S_1 v_1 = S_2 v_2 + S_3 v_3$ so that

$v_3 = (S_1 v_1 - S_2 v_2)/S_3$. ...(1)

But by part (*i*), $S_1 v_1 = 0.477$ m³/s.

Also $S_2 v_2 = 0.0707 \times 2.5 = 0.1767$ m³/s.



Hence (1) reduces to $$v_3 = \frac{0.4770 - 0.1767}{0.0314} = 9.55 \, m/s.$$

Ex. 14. *In a three dimensional incomperessible flow, the velocity components in y and z directions are $v = ax^3 - by^2 + cz^2, w = bx^3 - cy^2 + az^2 x$. Determine the missing component of velocity distribution such that continuity equation is satisfied.*

Sol. Given $v = ax^3 - by^2 + cz^2$ and $w = bx^3 - cy^2 + az^2 x$. ...(1)

The continuity equation for an incompressible fluid flow is

$$(\partial u/\partial x) + (\partial v/\partial y) + (\partial w/\partial z) = 0$$

or $\partial u/\partial x - 2by + 2azx = 0$ or $\partial u/\partial x = 2by - 2azx$.

Integrating (2) *w.r.t.* 'x', $u = 2byx - 2az \times (x^2/2) + f(y, z)$, ...(2)

where $f(y, z)$ is an arbitrary function which is independent of x.

Ex. 15. *Water flows through a pipe of length l which tapers from the entrance radius r_1 to the exist radius r_2 If the entrance velocity is V_1 and the relation between r_1 and r_2 is given by $r_2 = r_1 \pm ml$, where m is the slope, prove that the exist velocity V_2 is*

$$V_2 = V_1\left[1 - \frac{\pm 2m(l/r_1) + m^2(l/r_1)^2}{1 \pm 2m(l/r_1) + m^2(l/r_1)^2}\right].$$

Sol. If S_1 and S_2 be the areas of cross-sections of the pipe at the entrance and exist, then $S_1 = \pi r_1^2$ and $S_2 = \pi r_2^2$. From the equation of continuity, we have

$$S_1 V_1 = S_2\, V_2 \qquad \text{or} \qquad \pi r_1^2 V_1 = \pi r_2^2 V_2$$

Thus, $V_2 = \dfrac{r_1^2 V_1}{r_2^2} = \dfrac{r_1^2 V_2}{(r_1 \pm ml)^2}$, as given $r_2 = r_1 \pm \text{ml}$

$$= \frac{V_1}{\{1 \pm m(l/r_1)\}^2} = \frac{V_1}{1 \pm 2m(l/r_1) + m^2(l/r_1)^2} = V_1\left[1 - \frac{\pm 2m(l/r_1) + m^2(l/r_1)^2}{1 \pm 2m(l/r_1) + m^2(l/r_1)^2}\right]$$

Ex. 16. *Determine the constants l, m and n in order that the velocity* $\mathbf{q} = \{(x + lr)\mathbf{i} + (y + mr)\mathbf{j} + (z + nr)\mathbf{k}\}/\{r(x + r)\}$, *where* $r = (x^2 + y^2 + z^2)^{1/2}$ *may satisfy the equation of continuity* for a liquid. **[Bhopal 2000; Meerut 1996]**

Sol. Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, then we have

$$u = \frac{x + lr}{r(x + r)}, \qquad v = \frac{y + mr}{r(x + r)}, \qquad w = \frac{z + nr}{r(x + r)}, \qquad \ldots(1)$$

Also, given $r = (x^2 + y^2 + z^2)^{1/2}$ so that $r^2 = x^2 + y^2 + z^2$. ...(2)

From (2), differentiating partially *w.r.t.* 'x', we have

$$2r(\partial r/\partial x) = 2x \qquad \text{so that} \qquad \partial r/\partial x = x/r. \qquad \ldots(3)$$

Similarly, from (2), $\partial r/\partial y = y/r$ and $\partial r/\partial z = z/r$. ...(4)

From (1), $\dfrac{\partial u}{\partial x} = \dfrac{1}{r(x + r)}\dfrac{\partial}{\partial x}(x + lr) + (x + lr)\dfrac{\partial}{\partial x}\left\{\dfrac{1}{r(x + r)}\right\}$

$$= \frac{1}{r(x + r)}\left(1 + l\frac{\partial r}{\partial x}\right) + (x + lr)\left[-\frac{1}{r^2}\frac{\partial r}{\partial x} - \frac{1}{r(x + r)^2}\left(1 + \frac{\partial r}{\partial x}\right)\right]$$

$$= \frac{1}{r(x + r)}\left(1 + l\frac{x}{r}\right) + (x + lr)\left[-\frac{1}{r^2}\cdot\frac{x}{r} - \frac{1}{r(x + r)^2}\left(1 + \frac{x}{r}\right)\right], \text{ by (3)}$$

$$= \frac{r + xl}{r^2(x + r)} - \frac{(x + lr)}{r^2}\left(\frac{x}{r} + \frac{1}{x + r}\right) \qquad \ldots(5)$$

Also, $\dfrac{\partial v}{\partial y} = \dfrac{1}{r(x + r)}\dfrac{\partial}{\partial y}(y + mr) + (y + mr)\dfrac{\partial}{\partial y}\left\{\dfrac{1}{r(x + r)}\right\}$

$$= \frac{1}{r(x + r)}\left(1 + m\frac{\partial r}{\partial y}\right) + (y + mr)\left\{-\frac{1}{r^2}\frac{\partial r}{\partial y} - \frac{1}{r(x + r)^2}\left(0 + \frac{\partial r}{\partial y}\right)\right\}$$

$$= \frac{1}{r(x + r)}\left(1 + m\frac{y}{r}\right) + (y + mr)\left\{-\frac{1}{r^2}\frac{y}{r} - \frac{1}{r(x + r)^2}\frac{y}{r}\right\}, \text{ by (4)}$$

$$= \frac{r + my}{r^2(x + r)} - \frac{(y + mr)}{r^2}\left[\frac{y}{r} + \frac{y}{(x + r)^2}\right]. \qquad \ldots(6)$$

Similarly, $$\frac{\partial w}{\partial z} = \frac{r+nz}{r^2(x+r)} - \left(\frac{z+nr)}{r^2}\right)\left[\frac{z}{r} + \frac{z}{(x+r)^2}\right]. \qquad ...(7)$$

For the given velocity to satisfy the equation of continuity, we must have

$$(\partial u/\partial x) + (\partial v/\partial y) + (\partial w/\partial z) = 0 \qquad ...(8)$$

or $$\frac{r+xl}{r^2(x+r)} - \frac{(x+lr)}{r^2}\left(\frac{x}{r} + \frac{1}{x+r}\right) + \frac{r+my}{r^2(x+r)} - \frac{(y+mr)}{r^2}\left[\frac{y}{r} + \frac{y}{(x+r)^2}\right]$$

$$+\frac{r+nz}{r^2(x+r)} - \left(\frac{z+nr)}{r^2}\right)\left[\frac{z}{r} + \frac{z}{(x+r)^2}\right] = 0, \text{ by (5), (6) and (7)}$$

Multiplying both sides by r^3 $(x + r)^2$, we have

$r\,(r + xl)\,(x + r) - (x + lr)\,(x + r)\,[x(x + r) + r] + r(r + my)\,(x + r)$

$\qquad - (y + mr)\,[y(x + r)^2 + yr] + r(r + nz)\,(x + r) - (z + nr)\,[z(x + r)^2 + zr] = 0$

or $\qquad r^2\{r(1- l) + x(1 - l) - my - nz\} = 0$, on simplification.

This is satisfied by all values of x, y, z, if and only if $\quad l = 1, \quad m = 0 \quad$ and $\quad n = 0$.

Ex.17. *From the law of conservation of mass, show that whether the flow field represented by $u = -3x + y^2 - 1/x$ and $v = x^2 + 3y + y \log x$ is a possible velocity field for two-dimenation incomprenible fluid flow.*

Sol. Here $$\partial u/\partial x + \partial v/\partial y = -3 + 2/x^2 + 3 + \log x \neq 0,$$

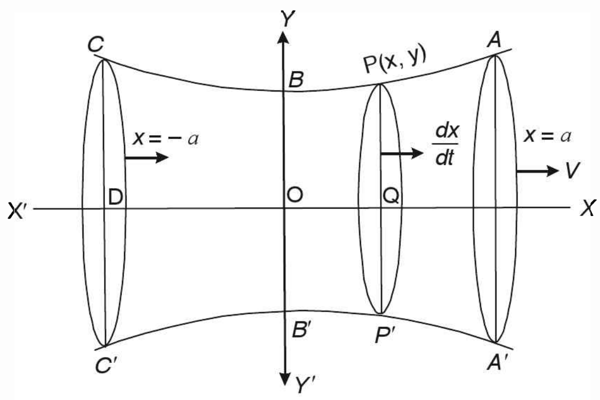
showing that the equation of continuity is not satsfied Hence the given flow does not represent a possible two-dimenation fluid flow

Ex. 18. *Liquid flows through a pipe whose surface is the surface of revolution of the curve $y = a + (kx^2/a)$ about the x - axis $(-a \le x \le a)$. If the liquid enters at the end $x = -a$ of the pipe with velicity V, show that the time taken by a liquid particle to traverse the entire length of the pipe from $x = -a$ to $x = a$ is* $\{2a/V(1 + k)^2\}$ $\{1 + (2k/3) + (k^2/5)\}$. *Assume that k is so small that flow remains appreciably one dimensional thoughtout.* **[I.A.S. 1999]**

Sol. Re-writing the given curve, we have

$$y - a = kx^2/a \qquad \text{or} \qquad (x - 0)^2 = (a/k)\,(y - a), \qquad ...(1)$$

which is a parabola ABC whose vertex is B (0, a). When the given curve (1) *revolved about x-axis,* we get surface of revolution. Figure shows a portion of the above mentioned surface bounded by circular ends $CC'(x = -a)$ and $AA'(x = a)$.



Let $P(x, y)$ be any point on (1). Then from (1), we have

$$PQ = y = a + kx^2/a. \qquad ...(2)$$

Also $C(-a, CD)$ lies on (1), Hence, we have

$$CD - a = k(-a)^2/a \qquad \text{so that} \qquad CD = a(1+k) \qquad ...(3)$$

Velocity at section CC' is given to be V. Again, velocity of arbitrary section PQ is dx/dt. If S_1 and S_2 be areas of sections at C and P respectively, then

$$S_1 = \pi\, CD^2 = \pi a^2(1+k^2) \qquad \text{and} \qquad S_2 = \pi y^2 = \pi(a + kx^2/a)^2.$$

Since the motion is regarded as one-dimensional, by equation of continuity (expressing equal rates of volumetric flow across the cross-sections at CC' and PP'), we have

$$\pi[a(1+k)\}^2 V = \pi(a + kx^2/a)^2 (dx/dt)$$

or

$$dt = \frac{1}{a^2 V(1+k)^2}\left(a + \frac{kx^2}{a}\right)^2 dx. \qquad ...(4)$$

Let the required time of travelling from $x = -a$ to $x = a$ be T. Then integrating w.r.t 't' between $t = 0$ to $t = T$ and integrating w.r.t 'x' between corresponding limits $x = -a$ and $x = a$, (4) gives

$$\int_0^T dt = \frac{1}{a^2 V(1+k)^2}\int_{-a}^{a}\left(a + \frac{kx^2}{a}\right)^2 dx.$$

or

$$T = \frac{1}{V(1+k)^2}\int_{-a}^{a}\left(1 + \frac{kx^2}{a^2}\right)^2 dx = \frac{2}{V(1+k)^2}\int_0^a\left(1 + \frac{kx^2}{a^2}\right)^2 dx$$

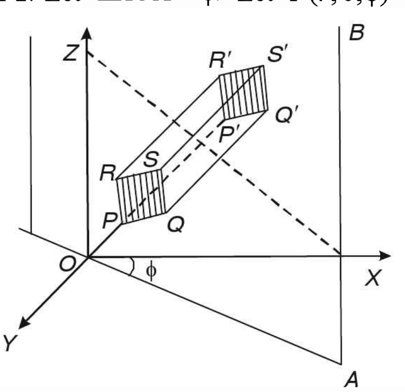
[Since the integrand in an even function]

$$= \frac{2}{V(1+k)^2}\int_0^a\left(1 + \frac{2kx^2}{a^2} + \frac{k^2x^4}{a^4}\right)^2 dx = \frac{2}{V(1+k)^2}\left[x + \frac{2kx^3}{3a^2} + \frac{k^2x^5}{5a^4}\right]_0^a$$

$$= \{2a/V(1+k)^2\}\,\{1 + (2k/3) + (k^2/5)\}$$

Ex. 19. *Each particle of a mass of liquid moves in a plane through the axis of z; find the equation of continuity.*

Sol. Let $ZOAB$ be a plane passing of through the axis of z. Let $\angle XOA = \phi$. Let $P(r, \theta, \phi)$ be the position of fluid particle of a mass of fluid moving on the plane $ZOAB$. We contruct a parallelopiped with edges PQ, PR and PP' such that $PQ = \delta r$, $PR = r\delta\theta$ and $PP' = r\sin\theta\delta\phi$. Clearly, the edges PQ and PR lie on the plane $ZOAB$ while PP' is perpendicular to the plane. Since the fluid particle move only on the plane $ZOAB$, there would be no motion along PP'.



Let u and v be velocity components of the fluid along PQ and PR respectively, We now use working rule of Art 2.15 for writting the equation continuity

The rate of the excess of flow-in over the flow-out along PQ

$$= -\delta r\frac{\partial}{\partial r}(\rho u r\,\delta\theta\, r\sin\theta\,\delta\phi) = -\sin\theta\,\delta r\,\delta\theta\,\delta\phi\frac{\partial}{\partial r}(\rho u r^2)$$

Again, the rate of the excess of flow-in over the flow-out along PR

$$= -r\delta\theta \frac{\partial}{r\,\partial\theta}(\rho v \delta r\, r\sin\theta\,\delta\phi) = -r\,\delta r\,\delta\theta\,\delta\phi \frac{\partial}{\partial\theta}(\rho\, v\sin\theta)$$

Also, the rate of increase in mass of the element

$$= \frac{\partial}{\partial t}(\rho\;\delta r\cdot r\delta\theta\cdot r\sin\theta\,\delta\phi) = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi\frac{\partial\rho}{\partial t}$$

Hence the equation of continuity is given by

$$r^2\,\sin\theta\,\delta r\,\delta\theta\,\delta\phi\frac{\partial\rho}{\partial t} = -\sin\theta\;\delta r\,\delta\theta\,\delta\phi\frac{\partial}{\partial r}(\rho u\, r^2) - r\;\delta r\,\delta\theta\,\delta\phi\frac{\partial}{\partial\theta}(\rho\; v\sin\theta)$$

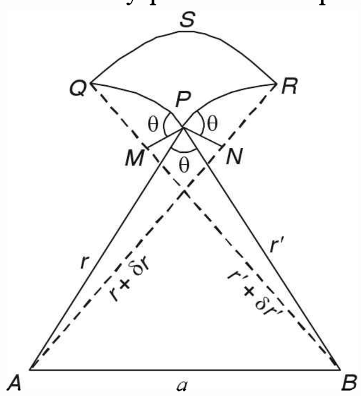
or
$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\; v\sin\theta) = 0$$

Ex. 20. *In the motion of a homogeneous liquid in two dimensions the velocity at any point is given by* v, v' *along the directions which pass through the fixed points distant 'a' from one another. Show that the equation of continuity is*

$$\frac{\partial v}{\partial r} + \frac{\partial v'}{\partial r'} + \frac{r^2 + r'^2 - a^2}{2r\,r'}\left(\frac{\partial v}{\partial r'} + \frac{\partial v'}{\partial r}\right) + \frac{v}{r} + \frac{v'}{r'} = 0,$$

where r *and* r' *are distances of any point of the liquid from the fixed points.* **[Osmania 2005]**

[In this example (r, r') are known as the *di-polar co-ordinates* of any point of the liquid]



Sol. Let A and B be two given fixed points such that $AB = a$. Let r and r' be the distances of any point P of the liquid from A and B respectively. With A as centre draw two circular arcs PQ and RS with r and $r+\delta r$ as radii. Similarly, with B as centre draw two circular arcs PR and QS with r' and $r'+\delta r'$ as radii. Then, we have $AR = r+\delta r$ and $BQ = r'+\delta r'$.

Since arcs PQ and PR are very small hence we can assume that arcs PQ and PR are approximately equal to straight lines PQ and PR respectively.

Let $\angle APB = \theta$. Draw PN and PM perpendicular to AR and BQ. Then,

$AN = r$, $\quad BM = r'$ $\quad NR = \delta r$, $\quad MQ = \delta r'$, $\quad \angle NPR = \theta$ $\quad$ and $\quad \angle MPQ = \theta$

From right-angled ΔPRN, $\quad \sin\theta = NR/PR \quad \Rightarrow \quad PR = (\delta r)/\sin\theta$

Similarly, from ΔPMQ, $\quad \sin\theta = QM/PQ \quad \Rightarrow \quad PQ = (\delta r')/\sin\theta$

Since v and v' are velocities along AP and BP respectively so velocity along normal to PR is $v' + v\cos\theta$ and velocity along normal to PQ is $v + v'\cos\theta$.

Since the liquid is homogeneous, so $\quad \rho$ = constant.

The rate of mass of the liquid flowing through PQ.

$$= \rho\times PQ\times(\text{velocity perpendicular to } PQ) = \rho(\delta r'/\sin\theta)\,(v + v'\cos\theta)$$

$\therefore$ the rate of the excess of the flow-in over the flow-out along PQ

$$= -\delta r \frac{\partial}{\partial r}\left\{\rho \frac{\delta r'}{\sin\theta}(v + v'\cos\theta)\right\} = -\rho \delta r\, \delta r' \frac{\partial}{\partial r}\left(\frac{v + v'\cos\theta}{\sin\theta}\right)$$

Similarly, the rate of the excess of the flow-in over the flow-out along *PR*

$$= -\delta r' \frac{\partial}{\partial r}\left\{\rho \frac{\delta r}{\sin\theta}(v' + v\cos\theta)\right\} = -\rho \delta r\, \delta r' \frac{\partial}{\partial r'}\left(\frac{v' + v\cos\theta}{\sin\theta}\right)$$

Now, area $PQSR = PQ \cdot PR \sin QPR = \dfrac{\delta r'}{\sin\theta} \times \dfrac{\delta r}{\sin\theta} \times \sin(\pi - \theta)$

$\therefore$ the rate of increase of mass of the liquid in *PQSR*

$$= \frac{\partial}{\partial t}\left(\rho \frac{\delta r'}{\sin\theta} \times \frac{\delta r}{\sin\theta} \times \sin\theta\right) = \rho \frac{\delta r\, \delta r'}{\sin\theta} \frac{\partial \rho}{\partial t} = 0, \text{ as } \rho = \text{constant}.$$

Hence the equation of continuity is given by

$$0 = -\rho \delta r\, \delta r' \frac{\partial}{\partial r}\left(\frac{v + v'\cos\theta}{\sin\theta}\right) - \rho \delta r\, \delta r' \frac{\partial}{\partial r'}\left(\frac{v' + v\cos\theta}{\sin\theta}\right)$$

or
$$\frac{\partial}{\partial r}\left(\frac{v + v'\cos\theta}{\sin\theta}\right) + \frac{\partial}{\partial r'}\left(\frac{v' + v\cos\theta}{\sin\theta}\right) = 0$$

or
$$\frac{1}{\sin\theta}\left(\frac{\partial v}{\partial r} + \frac{\partial v'}{\partial r}\cos\theta + v'\frac{\partial \cos\theta}{\partial r}\right) - \frac{v + v'\cos\theta}{\sin^2\theta}\cos\theta \frac{\partial\theta}{\partial r}$$

$$+ \frac{1}{\sin\theta}\left(\frac{\partial v'}{\partial r'} + \frac{\partial v}{\partial r'}\cos\theta + v\frac{\partial \cos\theta}{\partial r'}\right) - \frac{v' + v\cos\theta}{\sin^2\theta}\cos\theta \frac{\partial\theta}{\partial r'} = 0$$

$$\frac{\partial v}{\partial r} + \frac{\partial v'}{\partial r}\cos\theta + v'\frac{\partial \cos\theta}{\partial r} - \frac{v + v'\cos\theta}{\sin^2\theta}\cos\theta\left(-\frac{\partial \cos\theta}{\partial r}\right)$$

$$+ \frac{\partial v'}{\partial r'} + \frac{\partial v}{\partial r'}\cos\theta + v\frac{\partial \cos\theta}{\partial r'} - \frac{v' + v\cos\theta}{\sin^2\theta}\cos\theta\left(-\frac{\partial \cos\theta}{\partial r'}\right) = 0$$

or
$$\frac{\partial v}{\partial r} + \frac{\partial v'}{\partial r'} + \left(\frac{\partial v'}{\partial r} + \frac{\partial v}{\partial r'}\right)\cos\theta + v'\frac{\partial \cos\theta}{\partial r} + v\frac{\partial \cos\theta}{\partial r'}$$

$$+ \frac{\cos\theta}{\sin^2\theta}\left\{(v + v'\cos\theta)\frac{\partial \cos\theta}{\partial r} + (v' + v\cos\theta)\frac{\partial \cos\theta}{\partial r'}\right\} = 0 \quad \text{...(1)}$$

Using cosine formula of trigonometry in ΔABP, we have

$$\cos\theta = \frac{r^2 + r'^2 - a^2}{2rr'} = \frac{r}{2r'} + \frac{r'}{2r} - \frac{a^2}{2rr'} \quad \text{...(2)}$$

$$\therefore \quad \frac{\partial(\cos\theta)}{\partial r} = \frac{1}{2r'} - \frac{r'}{2r^2} + \frac{a^2}{2r^2 r'} = \frac{1}{r'} - \frac{1}{r}\left(\frac{r}{2r'} + \frac{r'}{2r} - \frac{a^2}{2rr'}\right) = \frac{1}{r'} - \frac{\cos\theta}{r}$$

Similarly, from (2), we have $\dfrac{\partial \cos\theta}{\partial r'} = \dfrac{1}{r} - \dfrac{\cos\theta}{r'}$

Substituting the above values of $\partial(\cos\theta)/\partial r$ and $\partial(\cos\theta)/\partial r'$ in (1), we have

$$\frac{\partial v}{\partial r} + \frac{\partial v'}{\partial r'} + \left(\frac{\partial v'}{\partial r} + \frac{\partial v}{\partial r'}\right)\cos\theta + v'\left(\frac{1}{r'} - \frac{\cos\theta}{r}\right) + v\left(\frac{1}{r} - \frac{\cos\theta}{r'}\right)$$

$$+\frac{\cos\theta}{\sin^2\theta}\left\{(v+v'\cos\theta)\left(\frac{1}{r'}-\frac{\cos\theta}{r}\right)+(v'+v\cos\theta)\left(\frac{1}{r}-\frac{\cos\theta}{r'}\right)\right\}=0$$

or
$$\frac{\partial v}{\partial r}+\frac{\partial v'}{\partial r'}+\left(\frac{\partial v'}{\partial r}+\frac{\partial v}{\partial r'}\right)\cos\theta+\frac{v'}{r'}+\frac{v}{r}-\left(\frac{v'}{r}+\frac{v}{r'}\right)\cos\theta$$
$$+\frac{\cos\theta}{\sin^2\theta}\left\{\frac{v}{r'}(1-\cos^2\theta)+\frac{v'}{r}(1-\cos^2\theta)\right\}=0$$

or
$$\frac{\partial v}{\partial r}+\frac{\partial v'}{\partial r'}+\left(\frac{\partial v'}{\partial r}+\frac{\partial v}{\partial r'}\right)\cos\theta+\frac{v'}{r'}+\frac{v}{r}-\left(\frac{v'}{r}+\frac{v}{r'}\right)\cos\theta+\left(\frac{v}{r'}+\frac{v'}{r}\right)\cos\theta=0$$

or
$$\frac{\partial v}{\partial r}+\frac{\partial v'}{\partial r'}+\frac{r^2+r'^2-a^2}{2rr'}\left(\frac{\partial v'}{\partial r}+\frac{\partial v}{\partial r'}\right)+\frac{v}{r}+\frac{v'}{r'}=0,\ \text{using (2)}$$

EXERCISE 2(C)

1. Determine the equation of continuity by vector apprach for incompressible fluid. Interpret it physically. **[Meerut 2003]**

2. A mass of fluid is in motion so that the lines of motion lie on the surface of co-axial cylinders. Show that the equation of continuity is

$\dfrac{\partial\rho}{\partial t}+\dfrac{1}{r}\dfrac{\partial}{\partial\theta}(\rho q_r)+\dfrac{\partial}{\partial z}(\rho q_z)=0,$ where q_r and q_z are velocities perpendicular and parallel to z-axis.

[**Hint.** Do as in Ex. 5 of Art 2.16 by taking $u = q_r$, $v = q_\theta$] **[Meerut 1999, 2001, 02]**

3. Water is flowing through a pipe 10 cm diameter with an average velocity of 10 m/sec. What is the rate of discharge of the water? Also determine the velocity at the other end of the pipe, if the diameter of the pipe is gradually changes to 20 cm.

[**Ans.** Discharge = 0.7854 m^3/sec; velocity = 2.5 m/sec.]

4. Homogeneous liquid moves so that the path the any particle P lies in the plane POX, where OX is fixed axis. Prove that if $OP = r$ and the $\angle XOP=\theta$, the equation of continuity may be written as
$$\frac{\partial}{\partial r}(u\,r^2)-\frac{\partial}{\partial\mu}(v\,r\,sin\,\theta)=0,$$
where u, v are the component velocities along and perpendicular to OP in the plane POX and $\mu=\cos\theta$.

[**Hint:** $\mu=\cos\theta$ so that $d\mu=\sin\theta\, d\theta$. Also ρ = constant. Proceed as Ex. 19 of Art. 2.16 by taking OX in place of OZ]

5. Does the three-dimensional incompressible flow given by

$$u(x,y,z)=\frac{kx}{(x^2+y^2+z^2)^{3/2}},\quad v(x,y,z)=\frac{ky}{(x^2+y^2+z^2)^{3/2}},\quad w(x,y,z)=\frac{kz}{(x^2+y^2+z^2)^{3/2}}$$

satisfy the equation of continuity? K is an arbitrary constant. Thus show the above motion is kinematically possible for an incompressible fluid. **[Purvanchal 2005]**

6. Does the two-dimensional incompressible flow given by

$$v_r(r,\theta)=c_1\left(\frac{1}{r^2}-1\right)cos\,\theta,\qquad v_\theta(r,\theta)=c_1\left(\frac{1}{r^2}+1\right)\sin\theta\qquad (r>0)$$

where c_1 is an arbitrary non-zero constant, satisfy the equation of continuity?

7. Does the two-dimensional incompressible flow given by

$$v_r = c_1 / r^2 + c_2 \cos\theta, \qquad v_\theta = -c_2 \sin\theta, \qquad v_\phi = 0$$

where c_1 and c_2 are arbitrary constants and $r > 0$, satisfy the equation of continuity. **[Ans.** No.**]**

2.17. Boundary conditions (kinematical).

When fluid is in contact with a rigid solid surface (or with another unmixed fluid), the following boundary condition must be satisfied in order to maintain contact:

The fluid and the surface with which contact is preserved must have the same velocity normal to the surface.

Let **n** denote a normal unit vector drawn at the point P of the surface of contact and let **q** denote the fluid velocity at P. When the rigid surface of contact is at rest, we must have **q.n** = 0 at each point of the surface. This expresses the condition that the normal velocities are both zero and hence the fluid velocity is tangential to the surface at its each point as shown in Fig. (*i*).

Next, let the rigid surface be in motion and let **u** be its velocity at P (refer Fig (*ii*)]. Then we must have

$$\mathbf{q} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} \qquad \text{or} \qquad (\mathbf{q} - \mathbf{u}) \cdot \mathbf{n} = 0,$$

which expresses the fact that there must be no normal velocity at P between boundary and fluid, that is, the velocity of the fluid relative to the boundary is tangential to the boundary at its each point.

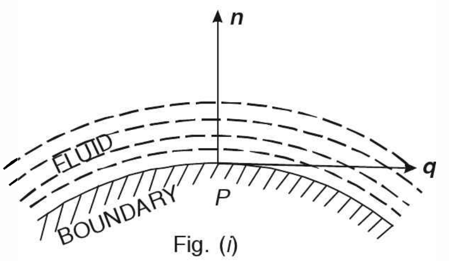


Fig. (*i*)

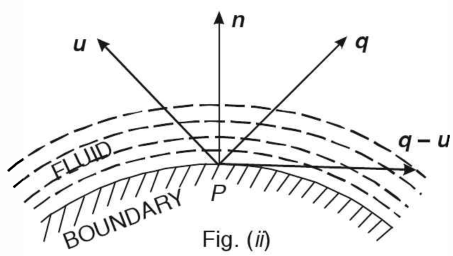


Fig. (*ii*)

Remark. For inviscid fluid the above condition must be satisfied at the boundary. However, for viscous fluid (in which there is no slip), the fluid and the surface with which contact is maintained must also have the same tangential velocity at P.

Boundary conditions (physical). The above mentioned kinematical boundary conditions must hold independently of any particular physical hypothesis. In the case of a non-viscous fluid in contact with rigid boundaries (fixed or moving), the following additional condition must be satisfied:

The pressure of the fluid must act normal to the boundary.

Again, let S denote the surface of separation of two fluids (which do not mix). Then the following additional condition must be satisfied :

The pressure must be continuous at the boundary as we pass from one side of S to the other.

2.18. Conditions at a boundary surface.

[Garhwal 1996, Kanpur 2002, 03, Meerut 1997, Rajasthan 2000, Rohilkhand 2001, 04, Purvanchal 2004]

We propose to derive the differential equation satisfied by a boundary surface of a fluid. Thus, we discuss the following problem :

To find the condition that the surface $F(\mathbf{r}, t) = 0$ *or* $F(x, y, z, t) = 0$ *may be a boundary surface*. For figure, refer figure (*ii*) of Art. 2.17.

Let P be a point on the moving boundary surface $\qquad F(\mathbf{r}, t) = 0. \qquad$ ...(1)

where the fluid velocity is **q** and the velocity of the surface is **u**.

Now in order to preserve contact, the fluid and the surface with which contact is to be maintained must have the same velocity normal to the surface. Thus, we have

$$\mathbf{q}\cdot\mathbf{n} = \mathbf{u}\cdot\mathbf{n} \qquad \text{or} \qquad (\mathbf{q}-\mathbf{u})\cdot\mathbf{n} = 0, \qquad \text{...(2)}$$

where **n** in the unit normal vector drawn at P on the boundary surface (1). We know that the direction ratios of **n** are $\partial F/\partial x,\ \partial F/\partial y,\ \partial F/\partial z$. Again,

$$\nabla F = (\partial F/\partial x)\,\mathbf{i} + (\partial F/\partial y)\mathbf{j} + (\partial F/\partial z)\mathbf{k}, \qquad \text{...(3)}$$

which shows that **n** and ∇F are parallel vectors and hence we may write $\mathbf{n} = k\nabla F$. With this value of **n**, (2) reduces to

$$(\mathbf{q}-\mathbf{u})\cdot k\nabla F = 0 \qquad \text{so that} \qquad \mathbf{q}\cdot\nabla F = \mathbf{u}\cdot\nabla F \qquad \text{...(4)}$$

Let P (**r**, t) move to a point $Q(\mathbf{r}+\delta\mathbf{r}, t+\delta t)$ in time δt. Then Q must satisfy the equation of the boundary surface (1), at time $t+\delta t$, namely

$$F(\mathbf{r}+\delta\mathbf{r}, t+\delta t) = 0$$

Expanding by Taylor's theorem, the above equation gives

$$F(\mathbf{r},t) + \delta\mathbf{r}\cdot\nabla\mathbf{F} + \delta t\left(\frac{\partial F}{\partial t}\right) = 0 \qquad \text{or} \qquad \frac{\partial F}{\partial t} + \frac{\delta\mathbf{r}}{\delta t}\cdot\nabla\mathbf{F} = 0, \ \text{using(1)} \qquad \text{...(5)}$$

Proceeding to the limits as $\delta\mathbf{r}\to 0,\ \delta t\to 0$ and noting that

$$\lim_{\delta t\to 0}\frac{\delta\mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \mathbf{u}, \qquad \text{(5) gives} \qquad \partial F/\partial t + \mathbf{u}\cdot\nabla\mathbf{F} = 0 \qquad \text{...(6)}$$

or

$$\partial F/\partial t + \mathbf{q}\cdot\nabla\mathbf{F} = 0, \ \text{using (4)} \qquad \text{...(7)}$$

which is the required condition for $F(\boldsymbol{r}, t)$ to be a boundary surface.

Remark 1. Let **q** = u**i** + v**j** + w**k**. Then (7) may be re-written as

$$\frac{\partial F}{\partial t} + (u\mathbf{i}+v\mathbf{j}+w\mathbf{k})\cdot\left(\frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}\right) = 0$$

or

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0 \qquad \text{or} \qquad \frac{DF}{Dt} = 0, \qquad \text{...(8)}$$

where

$$D = \partial/\partial t + u(\partial/\partial x) + v(\partial/\partial y) + w(\partial/\partial z)$$

(8) presents the required condition in cartesian coordinates for F (x, y, z, t) = 0 to be a boundary surface. **[Agra 2006, Meerut 1997]**

Remark 2. The normal velocity of the boundary

$$= \mathbf{u}\cdot\mathbf{n} = \mathbf{u}\cdot\frac{\nabla\mathbf{F}}{|\nabla F|} = \frac{-(\partial F/\partial t)}{|(\partial F/\partial x)\mathbf{i} + (\partial F/\partial y)\mathbf{j} + (\partial F/\partial z)\mathbf{k}|}, \text{by (3) and (6)}$$

$$= \frac{-(\partial F/\partial t)}{\sqrt{\{(\partial F/\partial x)^2 + (\partial F/\partial y)^2 + (\partial F/\partial z)^2\}}} \qquad \text{...(9)}$$

$$= \frac{u(\partial F/\partial x) + v(\partial F/\partial y) + w(\partial F/\partial z)}{\sqrt{\{(\partial F/\partial x)^2 + (\partial F/\partial y)^2 + (\partial F/\partial z)^2\}}}, \ \text{using (8)} \qquad \text{...(10)}$$

Remark 3. When the boundary surface is at rest, then $\partial F/\partial t = 0$ and hence the condition (8) reduces to

$$u(\partial F/\partial x) + v(\partial F/\partial y) + w(\partial F/\partial z) = 0 \qquad \text{...(11)}$$

2.19. Illustrative solved examples.

Ex. 1. *Show that the surface* $\dfrac{x^2}{a^2k^2t^4} + kt^2\left(\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}\right) = 1$ *is a possible form of boundary surface of a liquid at time t.* **[I.A.S. 1992; Punjab 2002; Rohilkhand 2001]**

Sol. The given surface

$$F(x, y, z, t) = \frac{x^2}{a^2k^2t^4} + kt^2\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) - 1 = 0 \qquad \text{...(1)}$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$\partial F/\partial t + u\,(\partial F/\partial x) + v\,(\partial F/\partial y) + w\,(\partial F/\partial z) = 0 \qquad \text{...(2)}$$

and the same values of u, v , w satisfy the equation of continuity

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad \text{...(3)}$$

From (1), $\dfrac{\partial F}{\partial t} = -\dfrac{4x^2}{a^2k^2t^5} + 2kt\left(\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}\right),\quad \dfrac{\partial F}{\partial x} = \dfrac{2x}{a^2k^2t^4},\quad \dfrac{\partial F}{\partial y} = \dfrac{2kt^2y}{b^2},\quad \dfrac{\partial F}{\partial z} = \dfrac{2kt^2z}{c^2}$

With these values, (2) reduces to

$$-\frac{4x^2}{a^2k^2t^5} + 2kt\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + \frac{2xu}{a^2k^2t^4} + \frac{2kt^2yv}{b^2} + \frac{2kt^2zw}{c^2} = 0,$$

or

$$\frac{2x}{a^2k^2t^4}\left(u - \frac{2x}{t}\right) + \frac{2kyt}{b^2}(y + vt) + \frac{2ktz}{c^2}(z + wt) = 0,$$

which is identically satisfied if we take

$$u = 2x/t, \qquad v = -\,y/t, \qquad w = -\,z/t \qquad \text{...(4)}$$

From (4),

$$\frac{\partial u}{\partial x} = \frac{2}{t}, \qquad \frac{\partial v}{\partial y} = -\frac{1}{t}, \qquad \frac{\partial w}{\partial z} = -\frac{1}{t} \qquad \text{...(5)}$$

Using (5), we find that (3) is also satisfied by the above values of u, v and w. Hence (1) is a possible boundary surface with velocity components given by (4).

Ex. 2. (*i*) *Determine the restrictions on* f_1, f_2, f_3 *if* $(x^2/a^2) f_1(t) + (y^2/b^2) f_2(t) + (z^2/c^2) f_3(t) = 1$ *is a possible boundary surface of a liquid.*

[Agra 2005; I.A.S.1995; Kanpur 2011; Meerut 2000]

(*ii*) Show that $(x^2/a^2) f(t) + (y^2/b^2)\,\phi(t) + (z^2/c^2)\,\psi(t) = 1$ *is a possible from of the boundary surface if* $f(t)\,\phi(t)\,\psi(t) = 1$.

Sol. (*i*) The given surface

$$F(x, y, z, t) = (x^2/a^2) f_1(t) + (y^2/b^2) f_2(t) + (z^2/c^2) f_3(t) - 1 = 0 \qquad \text{...(1)}$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$(\partial F/\partial t) + u\,(\partial F/\partial x) + v\,(\partial F/\partial y) + w\,(\partial F/\partial z) = 0 \qquad \text{....(2)}$$

and the same values of u, v w satisfy the equation of continuity

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad \text{...(3)}$$

Using dashes for differentiation with respect to t, (1) gives

$$\frac{\partial F}{\partial t} = \frac{x^2}{a^2} f_1'(t) + \frac{y^2}{b^2} f_2'(t) + \frac{z^2}{c^2} f_3'(t), \qquad \frac{\partial F}{\partial x} = \frac{2x}{a^2} f_1(t), \qquad \frac{\partial F}{\partial y} = \frac{2y}{b^2} f_2(t), \qquad \frac{\partial F}{\partial z} = \frac{2z}{c^2} f_3(t)$$

With these values, (2) reduces to

$$\frac{x^2 f_1'}{a^2} + \frac{y^2 f_2'}{b^2} + \frac{z^2 f_3'}{c^2} + \frac{2xf_1u}{a^2} + \frac{2yf_2v}{b^2} + \frac{2zf_3w}{c^2} = 0$$

or
$$\frac{2xf_1}{a^2}\left(u + \frac{xf_1'}{2f_1}\right) + \frac{2yf_2}{b^2}\left(v + \frac{yf_2'}{2f_2}\right) + \frac{2zf_3}{c^2}\left(w + \frac{zf_3'}{2f_3}\right) = 0$$

which is identically satisfied if we take

$$u = -\frac{xf_1'}{2f_1}, \qquad v = -\frac{yf_2'}{2f_2}, \qquad w = -\frac{zf_3'}{2f_3} \qquad \text{...(4)}$$

From (4), $$\frac{\partial u}{\partial x} = -\frac{f_1'}{2f_1}; \qquad \frac{\partial v}{\partial y} = -\frac{f_2'}{2f_2}; \qquad \frac{\partial w}{\partial z} = -\frac{f_3'}{2f_3} \qquad \text{...(5)}$$

Now the required restriction will be obtained if the above velocity components satisfy (3). Hence, we get

$$-\frac{f_1'}{2f_1} - \frac{f_2'}{2f_2} - \frac{f_3'}{2f_3} = 0 \qquad \text{or} \qquad \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \frac{f_3'}{f_3} = 0$$

Integrating, $\qquad \log f_1 + \log f_2 + \log f_3 = \log c$

or $\qquad \log (f_1 f_2 f_3) = \log c \qquad$ or $\qquad f_1 f_2 f_3 = c$, where c is an arbitrary constant.

(*ii*) Proceed as in the above example. There is no loss of generality if c is taken as unity.

Ex. 3. *Show that $(x^2/a^2)\tan^2 t + (y^2/b^2)\cot^2 t = 1$ is a possible form for the bounding surface of a liquid, and find an expression for the normal velocity.*

[Garhwal 2005; I.A.S. 1997; Kanpur 1999, 2004, 08; Rajasthan 2004; Meerut 2003, 05; Rohilkhand 2002; 05; Purvanchel 2004]

Sol. For the present two dimensional motion ($\partial F/\partial z = 0$ and $\partial w/\partial z = 0$), the surface

$$F(x, y, t) = (x^2/a^2)\tan^2 t + (y^2/b^2)\cot^2 t - 1 = 0 \qquad \text{...(1)}$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$\partial F/\partial t + u(\partial F/\partial x) + v(\partial F/\partial y) = 0 \qquad \text{...(2)}$$

and the same values of u and v satisfy the equation of continuity

$$\partial u/\partial x + \partial v/\partial y = 0 \qquad \text{...(3)}$$

From (1), $$\frac{\partial F}{\partial t} = \frac{x^2}{a^2}\cdot 2\tan t \sec^2 t - \frac{y^2}{b^2}\cdot 2\cot t \,\mathrm{cosec}^2 t, \quad \frac{\partial F}{\partial x} = \frac{2x}{a^2}\tan^2 t, \quad \frac{\partial F}{\partial y} = \frac{2y}{b^2}\cot^2 t$$

With these values, (2) reduces to

$$\frac{x\tan t}{a^2}(x\sec^2 t + u\tan t) + \frac{y\cot t}{b^2}(-y\,\mathrm{cosec}^2 t + v\cot t) = 0,$$

which is identically satisfied if we take

$x\sec^2 t + u\tan t = 0 \qquad$ and $\qquad -y\,\mathrm{cosec}^2 t + v\cot t = 0$

i.e. $u = -\dfrac{x}{\sin t \cos t}$ and $v = \dfrac{y}{\sin t \cos t}$ ...(4)

From (4), $\dfrac{\partial u}{\partial x} = -\dfrac{1}{\sin t \cos t}$ and $\dfrac{\partial v}{\partial y} = \dfrac{1}{\sin t \cos t}$ ...(5)

Using (5), we find that (3) is also satisfied by the above values of u, v. Hence (1) is a possible bounding surface with velocity components given by (4).

Using remark 2 of Art. 2.18 (with $\partial F/\partial z = 0$ here), the normal velocity

$$= \frac{u(\partial F/\partial x) + v(\partial F/\partial y)}{\sqrt{\{(\partial F/\partial x)^2 + (\partial F/\partial y)^2\}}} = \frac{-\dfrac{x}{\sin t \cos t}\cdot\dfrac{2x\tan^2 t}{a^2} + \dfrac{y}{\sin t \cos t}\cdot\dfrac{2y\cot^2 t}{b^2}}{\left\{\left(\dfrac{2x\tan^2 t}{a^2}\right)^2 + \left(\dfrac{2y\cot^2 t}{b^2}\right)^2\right\}^{1/2}}$$

$$= \frac{a^2y^2 \cot t \operatorname{cosec}^2 t - b^2x^2 \tan t \sec^2 t}{\sqrt{(x^2b^4 \tan^4 t + y^2a^4 \cot^4 t)}}$$

Ex. 4. (*a*) *Show that the ellipsoid* $x^2/\left(a^2k^2t^{2n}\right) + kt^n\{(y/b)^2 + (z/c)^2\} = 1$ *is a possible form of the boundary surface of a liquid. Derive also velocity components.*

(Kanpur 2009; 2010; Meerut 2007)

(*b*) Show that the variable ellipsoid $x^2/\left(a^2k^2t^4\right) + kt^2\{(y/b)^2 + (z/c)^2\} = 1$ is a possible form for the boundary surface at any time *t*. **(Kanpur 2007)**

Sol. (a) The given surface

$$F(x, y, z, t) = x^2/\left(a^2k^2t^{2n}\right) + kt^n\{(y/b)^2 + (z/c)^2\} - 1 = 0 \qquad \text{... (1)}$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condtion

$$\partial F/\partial t + u(\partial F/\partial x) + v(\partial F/\partial y) + w(\partial F/\partial z) = 0 \qquad \text{...(2)}$$

and the same values of u, v w stisfy the equation of continuity

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0. \qquad \text{...(3)}$$

From (1), $\dfrac{\partial F}{\partial t} = -\dfrac{x^2}{a^2k^2}\cdot\dfrac{2n}{t^{2n+1}} + nkt^{n-1}\left(\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}\right)$,

$\dfrac{\partial F}{\partial t} = \dfrac{2x}{a^2k^2t^{2n}}$, $\dfrac{\partial F}{\partial y} = \dfrac{2kt^n y}{b^2}$ and $\dfrac{\partial F}{\partial z} = \dfrac{2kt^n z}{c^2}$.

With these values, (2) reducces to

$$-\frac{x^2}{a^2k^2}\frac{2n}{t^{2n+1}} + nkt^{n-1}\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + \frac{2xu}{a^2k^2t^{2n}} + \frac{2kt^n vy}{b^2} + \frac{2kt^n zw}{c^2} = 0$$

or $$\left(u - \frac{nx}{t}\right)\frac{2x}{a^2k^2t^{2n}} + \left(v + \frac{ny}{2t}\right)\frac{2kyt^n}{b^2} + \left(w + \frac{nz}{2t}\right)\frac{2kzt^n}{c^2} = 0,$$

which is identically satisfied if we take

$u - (nx/t) = 0$, $v + (ny/2t) = 0$ and $w + (nz/2t) = 0$

or $u = nx/t$, $v = -ny/2t$ and $w = -nz/2t$. ...(4)

From (4), $\partial u/\partial x = n/t$, $\partial v/\partial y = -n/2t$ and $\partial w/\partial z = -n/2t$ ...(5)

Using (5), we find that (3) is also satisfied by the above values of u, v and w. Hence (1) is a possible boundary surface with velocity components given by (4)

(b) Proceed as in part (a) by taking $n = 2$

Ex. 5. *Show that the ellipsoid*

$$\frac{x^2}{a^2 e^{-t}\cos(t+\pi/4)} + \frac{y^2}{b^2 e^{t}\sin(t+\pi/4)} + \frac{z^2}{c^2 \sec 2t} = 1$$

is a possible form of boundary surface of a liquid for any time t and determine the velocity **q** *of any particle on this boundary. Also prove that the equation of continuity is satisfied.*

Sol. The given surface

$$F(x, y, z, t) = (x^2/a^2)e^t \sec(t+\pi/4) + (y^2/b^2)e^{-t}\operatorname{cosec}(t+\pi/4) + (z^2/c^2)\cos 2t - 1 = 0 \quad \ldots(1)$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$\partial F/\partial t + u(\partial F/\partial x) + v(\partial F/\partial y) + w(\partial F/\partial z) = 0 \quad \ldots(2)$$

and the same values of u, v, w satisfy the equation of continuity

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0. \quad \ldots(3)$$

From (1), $$\frac{\partial F}{\partial t} = \frac{x^2}{a^2}e^t \sec\left(t+\frac{\pi}{4}\right) + \frac{x^2}{a^2}e^t \sec\left(t+\frac{\pi}{4}\right)\tan\left(t+\frac{\pi}{4}\right) - \frac{y^2}{b^2}e^{-1}\operatorname{cosec}\left(t+\frac{\pi}{4}\right)$$
$$-\frac{y^2}{b^2}e^{-t}\operatorname{cosec}\left(t+\frac{\pi}{4}\right)\cot\left(t+\frac{\pi}{4}\right) - \frac{2z^2}{c^2}\sin 2t$$

$$\frac{\partial F}{\partial x} = \frac{2x}{a^2}e^t \sec\left(t+\frac{\pi}{4}\right), \qquad \frac{\partial F}{\partial y} = \frac{2y}{b^2}e^{-t}\operatorname{cosec}\left(t+\frac{\pi}{4}\right), \qquad \frac{\partial F}{\partial z} = \frac{2z}{c^2}\cos 2t$$

With these values, (2) reduces to

$$\frac{xe^t}{a^2}\sec\left(t+\frac{\pi}{4}\right)\left[2u + x\left\{1+\tan\left(t+\frac{\pi}{4}\right)\right\}\right] + \frac{ye^{-t}}{b^2}\operatorname{cosec}\left(t+\frac{\pi}{4}\right)\left[2v - y\left\{1+\cot\left(t+\frac{\pi}{4}\right)\right\}\right]$$
$$+(2z/c^2)\times(w\cos 2t - z\sin 2t) = 0,$$

which is identically satisfied if we take

$$2u + x\{1+\tan(t+\pi/4)\} = 0, \qquad 2v - y\{1+\cot(t+\pi/4)\} = 0, \qquad w\cos 2t - z\sin 2t = 0$$

or $$u = -(x/2)\times\{1+\tan(t+\pi/4)\}, \qquad v = -(y/2)\times\{1+\cot(t+\pi/4)\}, \qquad w = z\tan 2t$$

Using these values of u, v, w on the boundary for all t, we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{1}{2}\left\{1+\tan\left(t+\frac{\pi}{4}\right)\right\} - \frac{1}{2}\left\{1+\cot\left(t+\frac{\pi}{4}\right)\right\} + \tan 2t$$

$$= \frac{1-\tan^2(t+\pi/4)}{2\tan(t+\pi/4)} + \tan 2t = \cot\left(2t+\frac{\pi}{2}\right) + \tan 2t = -\tan 2t + \tan 2t = 0,$$

showing that the equation of continuity is satisfied.

EXERCISE 2 (D)

1. Show that $(x^2/a^2)f(t) + (y^2/b^2)\phi(t) = 1$, where $f(t)\phi(t) = 1$ is a possible form of the boundary surface of a liquid. **[Kanpur 2006]**

2. Show that $(x^2/a^2)f(t) + (y^2/b^2)f(t) = 1$ is a possible from of the boundary surface of a liquid.

3. Show that $(x^2/a^2)\cos^2 t + (y^2/b^2)\sec^2 t = 1$ is a possible form for the boundary surface. **[I.A.S. 2007]**

4. Show that $(x^2/a^2)f(t) + y^2/b^2 + (z^2/c^2)f(t) = 1$ is a possible form of the boundary surface of a liquid.

5. A sphere of radius r moves with a steady velocity components (U, V, W) through an initially stationary fluid. If t be measured from the instant the sphere was at the origin, prove that

$$(u - U)(x - Ut) + (v - V)(y - Vt) + (w - W)(z - Wt) = 0$$

where (u, v, w) are components of velocity on the sphere at any point.

6. The parabolic profile $y = kx^{1/2}$ moves in the negative x-direction with a velocity U through a fluid which was initially stationary. If u and v are the instantaneous velocity components of a fluid particle on boundary, show that $v/(u + U) = k^2/2y$.

2.20. Streamline or line of flow. [I.A.S. 1995; Kurkshetra 1998; U. P. P. C. S. 2000, Agra 2004, 2009 Kanpur 2000, 04, Meerut 2001, 02, 05, 12; G N. D. U. Amritsar 1999]

A streamline is a curve drawn in the fluid so that its tangent at each point is the direction of motion (*i.e.* fluid velocity) at that point.

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a point P on a straight line and let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ be the fluid velocity at P. Then $\mathbf{q}$ is parallel to $d\mathbf{r}$ at P on the steamline. Thus, the equation of streamlines is given by

$$\mathbf{q} \times d\mathbf{r} = \mathbf{0} \qquad \text{...(1)}$$

i.e., $$(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \times (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) = \mathbf{0}$$

or $$(vdz - wdy)\,\mathbf{i} + (wdx - udz)\,\mathbf{j} + (udy - vdx)\,\mathbf{k} = \mathbf{0}$$

whence $$vdz - wdy = 0, \qquad wdx - udz = 0, \qquad udy - vdx = 0$$

so that $$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \qquad \text{...(2)}$$

The equations (2) have a double infinite set of solutions. Through each point of the flow field where $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ do not all vanish, there passes one and only one streamline at a given instant. This fact can be verified by employing the well known existence theorem for the system of equations (2). If the velocity vanishes at a given point, various singularities occur there. Such a point is known as a *critical point* or *stagnation point.*

2.21. Path line or path of a particle. [Meerut 2012; Kanpur 2000, 02]

A path line is the curve or trajectory along which a particular fluid particle travels during its motion.

The differential equation of a path line is $$d\mathbf{r}/dt = \mathbf{q} \qquad \text{...(1)}$$

so that $$dx/dt = u, \qquad dy/dt = v \qquad \text{and} \qquad dz/dt = w \qquad \text{...(2)}$$

where $$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \qquad \text{and} \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + w\mathbf{k}.$$

Remark. Let a fluid particle of fixed identity be at (x_0, y_0, z_0) when $t = t_0$, then the path line is determined from equations

$$\left.\begin{aligned} dx/dt &= u(x, y, z, t) \\ dy/dt &= v(x, y, z, t) \\ dz/dt &= w(x, y, z, t) \end{aligned}\right\} \qquad \text{...(3)}$$

with initial conditions $$x(t_0) = x_0, \qquad y(t_0) = y_0, \qquad z(t_0) = z_0 \qquad \text{...(4)}$$

2.22. Streak lines or filament lines. **[Kanpur 2000; Meerut 2005, 12]**

A streak line is a line on which lie all those fluid particles that at some earlier instant passed through a certain point of space. Thus, a streak line presents the instantaneous pictures of the position of all fluid particles, which have passed through a given point at some previous time. When a dye is injected into a moving fluid at some fixed point, the visible lines produced in the fluid are streak lines which have passed through the injected point.

The equation of the streak line at time t can be derived by Lagragian method, Suppose that a fluid particle (x_0, y_0, z_0) passes a fixed point (x_1, y_1, z_1) in the course of time. Then by using the Lagrangian method of description, we have

$$f_1 (x_0, y_0, z_0, t) = x_1, \qquad f_2(x_0, y_0, z_0, t) = y_1, \qquad f_3 (x_0, y_0, z_0, t) = z_1 \qquad \ldots(1)$$

Solving (1) for x_0, y_0, z_0, we have

$$x_0 = g_1 (x_1, y_1, z_1, t), \qquad y_0 = g_2 (x_1, y_1, z_1, t), \qquad z_0 = g_3 (x_1, y_1, z_1, t) \qquad \ldots(2)$$

Now a streak line is the locus of the positions (x, y, z) of the particles which have passed through the fixed point (x_1, y_1, z_1). Hence the equation of the streak line at time t is given by

$$x = h_1 (x_0, y_0, z_0, t), \qquad y = h_2 (x_0, y_0, z_0, t), \qquad z = h_3 (x_0, y_0, z_0, t) \qquad \ldots(3)$$

Substituting the values of x_0, y_0, z_0 in (3), the desired equation of streak line passing through (x_1, y_1, z_1) at time t is given by

$$x = h_1 (g_1, g_2, g_3, t), \qquad y = h_2 (g_1, g_2, g_3, t), \qquad z = h_3 (g_1, g_2, g_3, t) \qquad \ldots(4)$$

2.23. Difference between the streamlines and path lines. **[Agra 2005]**

It is important to note that streamlines are not, in general, the same as the path lines. Streamlines show how each particle is moving at a given instant of time while the path lines present the motion of the particles at each instant. Except in the case of steady motion, u, v, w are always functions of the time and hence the streamlines go on changing with the time, and the actual path of any fluid particle will not in general coincide with a streamline. To understand this, take three consecutive points P, Q, R on a streamline at time t. Then a particle moving through P at this instant will move along PQ but as soon as it arrives at Q at time $t + \delta t$, QR is no longer the direction of the velocity at Q and the particle will therefore cease to move along QR and move instead in the direction of the new velocity at Q. However, in the case of steady motion the streamlines remain unchanged as time progresses and hence they are also the path lines.

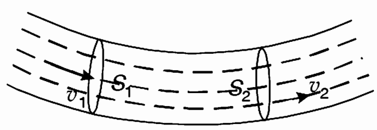
2.24. Stream tube (or tube of flow) and stream filament.

If we draw the streamlines from each point of a closed curve in the fluid, we obtain a tube called the *stream tube*.

A stream tube of infinitesimal cross-section is known as a *stream filament*.

Remark 1. Since there is no movement of fluid across a streamline, no fluid can enter or leave the steam tube except at the ends. So in the case of the steady motion, a stream tube behaves like an actual solid tube through which the fluid is flowing. Due to steady flow, the walls of the tube are fixed in space and hence the motion through the stream tube would remain unchanged on replacing the walls of the tube by a rigid boundary.

Remark 2. Consider a steam filament of liquid in steady motion. Let the cross-sectional area of the filament be so small that the velocity is the same at each point of this area, which may be taken perpendicular to the direction of the velocity. Let v_1, v_2 be the speeds of the flow at places where the cross-sectional areas are S_1, S_2. Let the liquid be incompressible. From the law of conservation of mass, the total quantity of liquid flowing across each section of the filament must be the same. Thus, we have



$$v_1 S_1 = v_2 S_2,$$

from which we arrive at the following theorem :

Theorem : *The product of the speed and cross sectional area is constant along a stream filament of a liquid in steady motion.*

It follows from the above theorem that a stream filament is widest at places where the speed is least and is narrowest at places where the speed is greatest. Furthermore, the stream filament cannot terminate at a point within the liquid unless the velocity is infinite there, which is never possible. Leaving this exceptional case, it follows that, in general, stream filaments are either closed or terminate at the boundary of a liquid. The same results are true for stream lines, because the cross-section of the filament may be considered as small as we please.

2.25. Illustrative solved examples.

Ex. 1. *Obtain the streamlines of a flow* $u = x,\ v = -y$.

OR *If the velocity* $\mathbf{q}$ *is given* $\mathbf{q} = x\mathbf{i} - y\mathbf{j}$, ***determine the equations of the streamlines.***

[Meerut 2012]

Sol. For two-dimensional flow ($w = 0$), we have

$$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = x\mathbf{i} - y\mathbf{j}$$

so that $\qquad u = x, \qquad v = -y, \qquad w = 0$

Streamlines are given by $\qquad (dx)/u = (dy)/v = (dz)/w$

i.e. $(dx)/x = (dy)/(-y) = (dz)/0 \qquad$ so that $\qquad (dx)/x + (dy)/y = 0$ and $dz = 0$

Integrating, $\quad \log x + \log y = \log c_1 \quad$ or $\quad xy = c_1 \quad$ and $\quad z = c_2$

The required straight lines are given by the curves of intersection of $xy = c_1$ and $z = c_2$, c_1 and c_2 being arbitrary constants.

Ex. 2. *The velocity components in a three-dimensional flow field for an incompressible fluid are* $(2x, -y, -z)$. *Is it a possible field? Determine the equations of the streamline passing through the point* $(1, 1, 1)$. Sketch the streamlines.

Sol. Here $\qquad u = 2x, \qquad v = -y, \qquad w = -z$

Streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad i.e. \qquad \frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{-z} \qquad \text{...(1)}$$

Taking the first two members of (1), we have

$$\frac{dx}{2x} = \frac{dy}{-y} \qquad \text{or} \qquad \frac{dx}{x} + 2 \cdot \frac{dy}{y} = 0$$

Integrating, $\quad \log x + 2 \log y = \log c_1 \quad$ or $\quad xy^2 = c_1. \qquad$ ...(2)

Again, taking the first and third members of (1) and proceeding as above, we get

$$xz^2 = c_2. \qquad \text{...(3)}$$

Here c_1 and $c_2$are arbitrary constants. The streamlines are given by the curves of intersection of (2) and (3). The required streamline passes through (1, 1, 1) so that $c_1 = 1$ and $c_2 = 1$. Thus, the desired stream line is given by the intersection of $\quad xy^2 = 1 \quad$ and $\quad xz^2 = 1$.

We also have $\qquad \partial u/\partial x = 2, \qquad \partial v/\partial y = -1, \qquad \partial w/\partial z = -1$

so that $\qquad \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0,$

showing that the equation of continuity is satisfied for the given flow field for an incompressible fluid. Hence the given velocity components correspond to a possible field.

Ex. 3. *The velocity field at a point in fluid is given as* $\mathbf{q} = (x/t, y, 0)$. *Obtain path lines and streak lines.* **[Agra 2008; Meerut 2002; 04]**

Sol. Here $\qquad \mathbf{q} = (u, v, w) = (x/t, y, 0)$

so that $\qquad u = x/t, \qquad v = y, \qquad w = 0 \qquad$ ...(1)

The equations of path lines are

$$dx/dt = u, \qquad dy/dt = v, \qquad dz/dt = w$$

i.e. $\qquad dx/dt = x/t, \qquad dy/dt = y, \qquad dz/dt = 0 \qquad ...(2)$

Suppose that (x_0, y_0, z_0) are coordinates of the chosen fluid particle at time $t = t_0$. Then

$$x = x_0, \qquad y = y_0, \qquad z = z_0 \qquad \text{when } t = t_0 \qquad ...(3)$$

From (2), $\qquad (1/x)dx = (1/t)dt \qquad$ giving $\qquad \log x = \log t + \log c_1$

i.e. $\qquad x = tc_1$, c_1 being an arbitrary constant $\qquad ...(4)$

Using initial conditions (3), (4) gives

$$x_0 = t_0 c_1 \qquad \text{so that} \qquad c_1 = x_0/t_0$$

$\therefore$ From (4), $\qquad x = (x_0\, t)/t_0 \qquad ...(5)$

Similarly, integrating $\quad dy/dt = y \quad$ *i.e.* $\quad (1/y)\, dy = dt$, we get

$$\log y - \log c_2 = t \qquad i.e. \qquad y = c_2 e^t \qquad ...(6)$$

Using (3), (6) gives $\qquad y_0 = c_2\, e^{t_0} \qquad$ *i.e.* $\qquad c_2 = y_0\, e^{-t_0}$

$\therefore$ From (6), $\qquad y = y_0\, e^{-t_0} \qquad ...(7)$

Finally, integrating $\qquad dz/dt = 0, \qquad$ we get $\qquad z = c_3 \qquad ...(8)$

Using (3), $\qquad z_0 = c_3 \qquad$ so that $\qquad z = z_0 \qquad ...(9)$

Hence the required path lines are given by

$$x = (x_0 t)/t_0 \qquad y = y_0 e^{t-t_0}, \qquad z = z_0. \qquad ...(10)$$

Let the fluid particle (x_0, y_0, z_0) passes a fixed point (x_1, y_1, z_1) at time $t = s$ where $t_0 \le s \le t$. Then (10) gives

$$x_1 = (x_0 s)/t_0, \qquad y_1 = y_0 e^{s-t_0}, \qquad z_1 = z_0$$

so that $\qquad x_0 = (x_1 t_0)/s, \qquad y_0 = y_1 e^{t_0 - s}, \qquad z_0 = z_1 \qquad ...(11)$

wherein s is the parameter. Substituting equations (11) into (10), we obtain the equation of streak line passing through (x_1, y_1, z_1) at times t as

$$x = (x_1 t)/s, \qquad y = y_1 e^{t-s}, \qquad z = z_1. \qquad ...(12)$$

Remark. It is easily seen from the above example that for a steady flow, streak lines are identical to path lines, and hence they coincide with streamlines.

Ex. 4. *Find the streamlines and paths of the particles when*

$$u = x/(1 + t), \qquad v = y/(1 + t), \qquad w = z/(1 + t). \qquad \textbf{[I.A.S. 1994]}$$

Sol. Streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad i.e. \qquad \frac{dx}{x/(1+t)} = \frac{dy}{y/(1+t)} = \frac{dz}{z/(1+t)}$$

i.e. $\qquad (dx)/x = (dy)/y = (dz)/z \qquad ...(1)$

Taking the first two members of (1), we get $\qquad x/y = c_1 \qquad ...(2)$

Taking the last two members of (1), we get $\qquad y/z = c_2 \qquad ...(3)$

The desired streamlines are given by the intersection of (2) and (3).

The paths of the particle are given by

$$dx/dt = u, \qquad dy/dt = v, \qquad dz/dt = w$$

i.e. $\qquad dx/dt = x/(1 + t), \qquad dy/dt = y/(1 + t), \qquad dz/dt = z/(1 + t)$

giving $$\frac{dx}{x} = \frac{dt}{1+t}, \qquad \frac{dy}{y} = \frac{dt}{1+t}, \qquad \frac{dz}{z} = \frac{dt}{1+t}$$

Integrating, $\quad x = c_3\,(1 + t), \qquad y = c_4\,(1 + t), \qquad z = c_5\,(1 + t)$
which give the desired paths of the particles, c_3, c_4 and c_5 being arbitrary constants.

Ex. 5. *Consider the velocity field given by* $\boldsymbol{q} = (1 + At)\boldsymbol{i} + x\boldsymbol{j}$. *Find the equation of the streamline at* $t = t_0$ *passing through the point* (x_0, y_0). *Also obtain the equation of the path line of a fluid element which comes to* (x_0, y_0) *at* $t = t_0$. *Show that, if* $A = 0$ *(i.e. steady flow), the streamline and path line coincide.*

Sol. Since $\boldsymbol{q} = u\mathbf{i} + v\mathbf{j}$, $u = 1 + At$ and $v = x$. Hence the streamline at $t = t_0$ is given by

$$\frac{dx}{u} = \frac{dy}{v} \qquad \text{or} \qquad \frac{dx}{1+At_0} = \frac{dy}{x} \qquad \text{or} \qquad xdx = (1 + At_0)dy$$

Integrating, $\quad (1 + At_0)y = x^2/2 + c$, c being an arbitrary constant. ...(*i*)

But $y = y_0$, when $x = x_0$, so we get $\quad (1 + At_0)y_0 = x_0^2/2 + c$ ...(*ii*)

Subtracting (*ii*) from (*i*) to eliminate c, we get $\quad (1 + At_0)\,(y - y_0) = (x^2 - x_0^2)/2,$ ...(1)

which is the required streamline.

We now determine required path line. Consider a fluid element passing through (x_0, y_0) at $t = t_0$. Then its coordinates (x, y) at any instant t (which define the path line) may be written as

$$x = x\,(x_0, y_0, t), \qquad y = y\,(x_0, y_0, t) \qquad \text{...(2)}$$

Now the path line is given by

$$dx/dt = u = 1 + At \qquad \text{...(3)}$$

and
$$dy/dt = v = x \qquad \text{...(4)}$$

Integrating (3) and using the condition $x = x_0$ at $t = t_0$, we get

$$x - x_0 = (t - t_0) + A\,(t^2 - t_0^2)/2 \qquad \text{...(5)}$$

Using (5), (4) may be re-written as

$$dy/dt = x_0 + (t - t_0) + A\,(t^2 - t_0^2)/2 \qquad \text{...(6)}$$

Integrating (6) and using the condition $y = y_0$ at $t = t_0$, we get

$$y - y_0 = (t - t_0)\left[x_0 + \frac{1}{2}(t - t_0)\right] + \frac{1}{6}A(t^3 - t_0^3) - \frac{1}{2}At_0^2(t - t_0) \qquad \text{...(7)}$$

Equations (5) and (7) together give the equation of the path line in parametric form with t as parameter. On elimination of t between (5) and (7), we will get equation of path-line in cartesian coordinates x, y. The resulting equation so obtained will be different from the equation (1) of the streamline.

When $A = 0$, the equation of the streamline (1) gives

$$y - y_0 = (x^2 - x_0^2)/2 \qquad \text{...(8)}$$

and parametric equation of path line given by (5) and (7) reduce to

$$x - x_0 = t - t_0, \qquad y - y_0 = (t - t_0)\left[x_0 + (1/2) \times (t - t_0)\right] \qquad \text{...(9)}$$

Eliminating t from (9), the equation of path line is

$$y - y_0 = (x^2 - x_0^2)/2.$$

Thus, the streamline coindes with the path line.

Ex. 6. *Prove that if the speed is everywhere the same, the streamlines are straight lines.*

Sol. Let u, v, w be the constant speed components of the speed of the fluid particle. Then the equation of the streamlines are given by $\quad (dx)/u = (dy)/v = (dz)/w \quad$ ...(1)

Taking first and second and then first and third fractions in (1), we get

$$vdx - udy = 0 \qquad \text{and} \qquad wdx - udz = 0.$$

Integrating, $\quad vx - uy = c_1 \quad$ and $\quad wx - uz = c_2, \quad$ ...(2)

where c_1 and c_2 are arbitray constants of integration.

The required streamlines are given by the straight lines of intersection of two planes given by (2).

Ex. 7. *Find the equation of the streamlines for the flow* $\mathbf{q} = -\mathbf{i}\,(3y^2) - \mathbf{j}\,(6x)$ at the point (1, 1).

Sol. Here $\mathbf{q} = u\mathbf{i} + v\mathbf{j} = -\mathbf{i}\,(3y^2) - j\,(6x) \quad \Rightarrow \quad u = -3y^2, \qquad v = -6x, \quad$ ...(1)

The equations of streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} \qquad \text{or} \qquad \frac{dx}{-3y^2} = \frac{dy}{-6x} \qquad \text{or} \qquad 6xdx - 3y^2\,dy = 0.$$

Integrating, $\quad 3x^2 - y^3 = c$, c being an arbitrary constant. $\quad$ ...(2)

At the point (1, 1), (2) gives $\quad 3 - 1 = c \quad$ or $\quad c = 2.$

Hence, from (2), the required equation of the streamline is $\quad 3x^2 - y^3 = 2.$

Ex. 8. *The velocity components in a two-dimensonal flow field for an incompressible fluid are given by* $u = e^x \cosh y$ *and* $v = -e^x \sinh y$. *Determine the equation of the streamlines for this flow.* $\quad k \neq 0 \quad$ **[Agra 2003]**

Sol. The equation of the streamlines are given by

$$\frac{dx}{u} = \frac{du}{v} \qquad \text{or} \qquad \frac{dx}{e^x \cosh y} = \frac{dy}{-e^x \sinh y} \qquad \text{or} \qquad \coth y\, dy = -dx.$$

Integrating, $\quad \log \sinh y - \log c = -x \quad$ or $\quad \sinh y = ce^{-x},$

where c is a constant of integration.

Ex. 9. *For an incompressible homogeneous fluid at the point (x, y, z) the velocity distribution is given by* $u = -(c^2y/r^2)$, $v = c^2 x/r^2$, $w = 0$, *where r denotes the distance from the z-axis. Show that it is a possible motion and determine the surface which is orthogonal to streamlines.*

Sol. Since r is distance of point (x, y, z) from the z-axis, we have $r = (x^2 + y^2)^{1/2}$. Hence given velocity distribution becomes

$$u = -\{c^2 y/(x^2 + y^2)\}, \qquad v = (c^2x)/(x^2 + y^2), \qquad w = 0 \qquad \text{...(1)}$$

From (1), $\quad \partial u/\partial x = -\{2c^2yx/(x^2+y^2)^2\}, \qquad \partial v/\partial y = (2c^2xy)/(x^2+y^2)^2, \qquad \partial w/\partial z = 0$

$\therefore \quad \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$, showing that the equation of continutiy is satisfied and so the motion specified by (1) is possble.

The surfaces which are orhogonal to streamlines $(dx)/u = (dy)/v = (dz)/w$ are given by*

$$udx + vdy + wdz = 0 \qquad \text{or} \qquad -\{c^2 y/(x^2 + y^2)\}\, dx + \{c^2 x/(x^2 + y^2)\}dy = 0$$

or $\qquad -y\,dx + x\,dy = 0 \qquad$ or $\qquad (1/y)dy = (1/x)\,dx,$

Integrating, $\quad \log y = \log k + \log x \quad$ or $\quad y = kx, \quad$ k being an arbitrary constant.

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* Refer chapter 3 in part II of author's Ordinary and Partial Differential Equations published by S. Chand & Co., New Delhi

Ex.10. *Determine the streamlines and the path lines of the particle when the components of the velocity field are given by* $u = x / (1 + t)$, $v = y/(2 + t)$ *and* $w = z/(3 + t)$. *Also state the condition for which the steamlines are identical with path lines.* **[I.A.S. 2000]**

Sol. Streamlines are given by $dx/u = dy/v = dz/w$

or $$(1 + t)(1/x)dx = (2 + t)(1/y)dy = (3 + t)(1/z)dz. \qquad ...(1)$$

Taking the first two members of (1), we have

$$(1/x)dx + (t/x)dx = (2/y)dy + (t/y)dy$$

or $$(1/x)dx - (2/y)dy = t\{(1/y)dy - (1/x)dx\}.$$

Integrating, $\log x - 2\log y = t(\log y - \log x) + \log c_1$, c_1 being an arbitrary constant

or $\log (x/y^2) = \log \{c\,(y/x)^t\}$ so that $(y/x)^t = x/c_1y^2$. ...(2)

Similarly, taking the last two members of (1), we have

or $\log (y^2/z^3) = \log\{c_2\,(y/z)^t\}$ or $(y/z)^t = y^2/c_2z^3$. ...(3)

The desired stremlines at a given instant $t = t_0$ are given by the intersection of the surfaces (2) and (3) by substituting t_0 for t.

Again, the path lines are given by

$$dx/dt = u, \qquad dy/dt = v \qquad dz/dt = w$$

or $$dx/dt = x/(1 + t), \qquad dy/dt = y/(2 + t), \qquad dz/dt = z/(3 + t),$$

giving $$dx/x = dt/(1 + t), \qquad dy/y = dt/(2 + t), \qquad dz/z = dt/(3 + t).$$

Integraing, $x = c_3(1 + t)$, $y = c_4(2 + t)$, $z = c_5(3 + t)$, c_3, c_4, c_5 being arbitrary constants which gives the desired paths of the given particle in terms of the parameter t.

Condition under which the stremlines and path linear are indentical.

In the case of steady motion the steamlines remain unchanged as time progresses and hence they are identical with the path lines.

Ex. 11. *In the steady motion of homogenous liquid if the surfaces* $f_1 = a_1$, $f_2 = a_2$, *define the streamlines prove that the most general values of the velocity components* u, v, w *are*

$$F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(y, z)}, \quad F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(z, x)}, \quad F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(x, y)}.$$ **(Meerut 2008)**

Sol. The motion being steady, the streamlines will be independent of time. It follows that the functions f_1 and f_2 will be functions of x, y, z. We have

$$f_1 = a_1 \quad \Rightarrow \quad df_1 = 0 \quad \Rightarrow \quad \frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial z}dz = 0 \qquad ...(1)$$

$$f_1 = a_2 \quad \Rightarrow \quad df_2 = 0 \quad \Rightarrow \quad \frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_2}{\partial z}dz = 0. \qquad ...(2)$$

From (1) and (2), by cross multiplication, we have

$$\frac{dx}{\dfrac{\partial f_1}{\partial y}\dfrac{\partial f_2}{\partial z} - \dfrac{\partial f_1}{\partial z}\dfrac{\partial f_2}{\partial y}} = \frac{dy}{\dfrac{\partial f_1}{\partial z}\dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial x}\dfrac{\partial f_2}{\partial z}} = \frac{dz}{\dfrac{\partial f_1}{\partial x}\dfrac{\partial f_2}{\partial y} - \dfrac{\partial f_1}{\partial y}\dfrac{\partial f_2}{\partial x}}$$

or $$\frac{dx}{J_1} = \frac{dy}{J_2} = \frac{dz}{J_3}, \qquad ...(2)$$

where $$J_1 = \frac{\partial(f_1, f_2)}{\partial(y, z)}, \quad J_2 = \frac{\partial(f_1, f_2)}{\partial(z, x)} \quad \text{and} \quad J_3 = \frac{\partial(f_1, f_2)}{\partial(x, y)}. \qquad ...(3)$$

We know that the equation of streamlines are given by

$$(dx)/u = (dy)/v = (dz)/w.$$

Comparing (2) and (3), $u = FJ_1$, $v = FJ_2$, $w = FJ_3$, ...(5)

where F is an arbitrary function. We now proceed to find F.

For the given liquid motion to be possible, the equation of continuity must be satisfied, *i.e*

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$$

or $$\frac{\partial}{\partial x}(FJ_1) + \frac{\partial}{\partial y}(FJ_2) + \frac{\partial}{\partial z}(FJ_3) = 0 \qquad ...(6)$$

or $$F\left(\frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} + \frac{\partial J_3}{\partial z}\right) + \left(J_1\frac{\partial F}{\partial x} + J_2\frac{\partial F}{\partial y} + J_3\frac{\partial F}{\partial z}\right) = 0. \qquad ...(7)$$

By the property of Jocabains, $\partial J_1/\partial x + \partial J_2/\partial y + \partial J_3/\partial z = 0.$ ...(8)

Using (8), (7) becomes $J_1(\partial F/\partial x) + J_2(\partial F/\partial y) + J_3(\partial F/\partial z) = 0$

or $$\frac{\partial F}{\partial x}\frac{\partial(f_1, f_2)}{\partial(y,z)} + \frac{\partial F}{\partial y}\frac{\partial(f_1, f_2)}{\partial(z,x)} + \frac{\partial F}{\partial z}\frac{\partial(f_1, f_2)}{\partial(x,y)} = 0, \text{using (3)}$$

or $$\frac{\partial F}{\partial x}\left(\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial z} - \frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial y}\right) + \frac{\partial F}{\partial y}\left(\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial z}\right) + \frac{\partial F}{\partial z}\left(\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}\right) = 0$$

or $$\begin{vmatrix} \partial F/\partial x & \partial F/\partial y & \partial F/\partial z \\ \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial x \end{vmatrix} = 0 \quad \text{or} \quad \frac{\partial(F_1, f_1, f_2)}{\partial(x,y,z)} = 0,$$

showing that F, f_1 and f_2 are not independent and hence F is a function of f_1, f_2 only. Therefore, $F = F(f_1, f_2)$.

Hence, from (2) and (5), the values of the velocity components u, v, w respectively are given by FJ_1, FJ_2, FJ_3, that is,

$$F_1(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(y,z)}, \qquad F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(z,x)}, \qquad F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(x,y)}.$$

EXERCISE 2 (E)

1. Determine the streamlines and streaklines for the flow whose velocity field is given by

$u = -x + t + 2$, $v = y - t + 2$. **[Meerut 2005]**

2. Find the streamlines and path lines of the two-dimensional velocity field $u = x/(1 + t)$, $v = y$, $w = 0$. **[Agra 2002, 2004]**

[**Ans.** $z = c_1, y = c_2\, x^{1+t}$; $x = a_1\,(1 + t), y = a_2 e^t, z = a_3$]

3. Distinguish between path lines and streamlines.

4. Find the streamline and path of the paticle when u = $(2xt)/(1 + t^2)$, $v = (2yt)/(1+t^2)$, $w = (2zt)/(1 + t^2)$. **[Purvanchal 2007]**

2.26. The velocity potential or velocity function.

[Meerut 2005, 09; Rohilkhand 2004, 05]

Suppose that the fluid velocity at time t is $\mathbf{q} = (u, v, w)$. Further suppose that at the considered instant t, there exists a scalar function $\phi\ (x, y, z, t)$, uniform throughout the entire field of flow and such that

$$-d\phi = u\,dx + v\,dy + w\,dz \qquad \text{...(1)}$$

i.e.

$$-\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = u\,dx + v\,dy + w\,dz \qquad \text{...(2)}$$

Then the expression on the R.H.S. of (1) is an exact differential and we have

$$u = -\partial\phi/\partial x, \qquad v = -\partial\phi/\partial y, \qquad w = -\partial\phi/\partial z \qquad \text{...(3)}$$

$$\therefore \qquad \mathbf{q} = -\nabla\phi = -\text{grad}\,\phi. \qquad \text{...(4)}$$

ϕ is called the *velocity potential*. The negative sign in (4) is a convention. It ensures that the flow takes place from the higher to lower potentials.

The necessary and sufficient condition for (4) to hold is

$$\nabla \times \mathbf{q} = 0, \qquad i.e. \qquad \text{curl}\ \mathbf{q} = \mathbf{0} \qquad \text{...(5)}$$

or

$$\mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \mathbf{0} \qquad \text{...(6)}$$

Remark 1. The surfaces $\qquad \phi\ (x, y, z, t) = \text{const.} \qquad$ ...(7)

are called the *equipotenials*. The streamlines

$$dx/u = dy/v = dz/w \qquad \text{...(8)}$$

are cut at right angles by the surfaces given by the differential equation

$$udx + vdy + wdz = 0 \qquad \text{...(9)}$$

and the condition for the existence of such orthogonal surfaces is the condition that (9) may posses a solution of the form (7) at the considered instant t, the analytical condition being

$$u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \qquad \text{...(10)}$$

When the velocity potential exists, (3) holds. Then

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = -\frac{\partial^2\phi}{\partial y\partial z} + \frac{\partial^2\phi}{\partial z\partial y} = 0, \qquad i.e., \qquad \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \qquad \text{...(11)}$$

Similarly, $\qquad \partial u/\partial z = \partial w/\partial x \qquad$ and $\qquad \partial v/\partial x = \partial u/\partial y \qquad$ ...(12)

Using (11) and (12), we find that the condition (10) is satisfied. Hence surfaces exist which cut the streamlines orthogonallly. We also conclude that at all points of field of flow the equipotentials are cut orthogonally by the streamlines.

Remark 2. When (5) holds, the flow is known as the *potential kind*. It is also known as *irrototional*. For such flow the field of $\mathbf{q}$ is *conservative*.

Remark 3. The equation of continuity of an incompressible fluid is

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad \text{...(13)}$$

Suppose that the fluid move irrotationally. Then the velocity potential ϕ exists such that

$$u = -\partial\phi/\partial x, \qquad v = -\partial\phi/\partial y, \qquad w = -\partial\phi/\partial z \qquad \text{...(14)}$$

Using (14), (13) reduces to

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = 0, \qquad ...(15)$$

showing that ϕ is a harmonic function satisfying the Laplace euation $\nabla^2\phi = 0$, where

$$\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2. \qquad ...(16)$$

2.27. The Vorticity Vector. [Kanpur 2004, Garhwal 2005]

Let $\mathbf{q} = ui + vi + wk$ be the fluid velocity such that curl $\mathbf{q} \neq \mathbf{0}$. Then the vector

$$\mathbf{\Omega} = \text{curl}\,\mathbf{q} \qquad ...(1)$$

is called the *vorticity vector*.

Let $\mathbf{\Omega}_x, \mathbf{\Omega}_y, \mathbf{\Omega}_z$ be the components of $\mathbf{\Omega}$ in cartesian coordinates. Then (1) reduces to

$$\mathbf{\Omega}_x\mathbf{i} + \mathbf{\Omega}_y\mathbf{j} + \mathbf{\Omega}_z\mathbf{k} = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

so that $\quad \mathbf{\Omega}_x = \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}, \qquad \mathbf{\Omega}_y = \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}, \qquad \mathbf{\Omega}_z = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}.$

Note. Some authors use ξ, η, ζ, for $\mathbf{\Omega}_x, \mathbf{\Omega}_y, \mathbf{\Omega}_z$ and define $\mathbf{\Omega} = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k} = (1/2) \times \text{curl}\,\mathbf{q}$. Thus, we have

$$\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

Remark 1. In the two-dimensional cartesian coordinates, the vorticity is given by

$$\mathbf{\Omega}_z = \partial v/\partial x - (\partial u/\partial y)$$

Remark 2. In the two-dimensional polar coordinates the vorticity is given by

$$\Omega_z = \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

Remark 3. The vorticity components in cylindrical polar coordinates (r, θ, z) are given by

$$\mathbf{\Omega}_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}, \qquad \mathbf{\Omega}_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \qquad \mathbf{\Omega}_z = \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

Remark 4. The vortcity components in spherical polar coordinates (r, θ, ϕ) are given by

$$\mathbf{\Omega}_r = \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_\phi}{r}\cot\theta, \quad \mathbf{\Omega}_\theta = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}, \quad \mathbf{\Omega}_\phi = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

2.28. Vortex line [Agra 2004, 2009; Garhwal 2005]

A vortex line is a curve drawn in the fluid such that the tangent to it at every point is in the direction of the vorticity vector $\mathbf{\Omega}$.

Let $\mathbf{\Omega} = \mathbf{\Omega}_x\mathbf{i} + \mathbf{\Omega}_y\mathbf{j} + \mathbf{\Omega}_z\mathbf{k}$ and let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a point P on a vortex line. Then $\mathbf{\Omega}$ is parallel to $d\mathbf{r}$ at P on the vortex line. Hence the equation of vortex lines is given by

$$\mathbf{\Omega} \times d\mathbf{r} = \mathbf{0},$$

i.e.

$$(\Omega_x\mathbf{i} + \Omega_y\mathbf{j} + \Omega_z\mathbf{k}) \times (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = \mathbf{0}$$

or $$(\Omega_y dz - \Omega_z dy)\mathbf{i} + (\Omega_z dx - \Omega_x dz)\mathbf{j} + (\Omega_x dy - \Omega_y dx)\mathbf{k} = \mathbf{0}$$

whence $$\Omega_y dz - \Omega_z dy = 0, \qquad \Omega_z dx - \Omega_x dz = 0, \qquad \Omega_x dy - \Omega_y dx = 0$$

so that $$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \qquad ...(1)$$

(1) gives the disired equations of vortex lines.

2.29. Vortex tube and vortex filament.

It we draw the vortex lines from each point of a closed curve in the fluid. we obtain a tube callled the *vortex tube*.

A vortex tube of infinitesimal cross-section is known as vortex *filament* or simply a *vortex*.

Remark. It will be shown that vortex lines and tubes cannot originate or terminate at internal points in a fluid. They can only form closed curves or terminate on boundaries. [For proof, refer Art. 11.2 of chapter 11].

2.30. Rotational and irrotationl motion.

[Agra 2011; Garhwal 2005; I.A.S. 2000; G.N.D.U. Amritser 2003; Meerut 2002, 09, 10]

The motion of a fluid is said to be *irrotational* when the vorticity vector Ω of every fluid particle is zero. When the vorticity vector is different from zero, the motion is said to be *rotational*.

Since $\mathbf{\Omega} = \text{curl}\,\mathbf{q}$ and $$\mathbf{\Omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k},$$

we conclude that the motion is irrotation if $\text{curl}\,\mathbf{q} = \mathbf{0}$

or $$\partial w/\partial y = \partial v/\partial z, \qquad \partial u/\partial z = \partial w/\partial x, \qquad \partial v/\partial x = \partial u/\partial y,$$

When the motion is irrotational i.e. when $\text{curl}\,\mathbf{q} = \mathbf{0}$, then $\mathbf{q}$ must be of the form $(-\text{grad}\,\phi)$ for some scalar point function ϕ (say) because curl grad $\phi = 0$. Thus velocity potential exists whenever the fluid motion is irrotational. Again notice that when velocity potential exists, the motion is irrotational because $\mathbf{q} = -\,\text{grad}\,\phi \Rightarrow \text{curl}\,\mathbf{q} = -\,\text{curl grad}\,\phi = \mathbf{0}$.

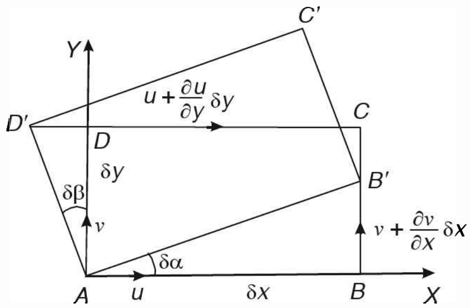
Thus, the fluid motion is irrotational if and only if the velocity potential exists. **(Meerut 2009, 10)**

Rotational motion is also said to be *vortex motion*. Again by definition it follows that there are no vortex lines in an irrotational fluid motion.

2.31. The angular velocity vector. **[Kanpur 2003]**

Consider a rectangular element in two-dimensional flow such that $AB = \delta x$ and $AD = \delta y$ as shown in the figure. Upon rotating about A during a small interval δt, let the element assume the shape indicated by $A'B'C'D'$ in figure, B' and D' approximately lying on BC and CD produced.

Let u, v be the components of velocity at A. Then the components of velocity along BC and DC are respectively $v + (\partial v/\partial x)\delta x$ and $u + (\partial u/\partial y)\delta y$.



$\therefore$ velocity of B relative to A along $BC = \dfrac{\partial v}{\partial x}\delta x$

and velocity of D relative to A along $DC = \dfrac{\partial u}{\partial y}\delta y$.

$$\therefore \; BB' = \frac{\partial v}{\partial x}\delta x \delta t \quad \text{and} \quad DD' = -\frac{\partial u}{\partial y}\delta y \delta t$$

Hence, the angular velocity of AB about z-axis *i.e.* perendicular to the plane through A

$$= \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t} = \lim_{\delta t \to 0} \frac{\tan\delta\alpha}{\delta t} \qquad [\because\ \delta\alpha \text{ is small} \Rightarrow \delta\alpha = \tan\delta\alpha]$$

$$= \lim_{\delta t \to 0} \frac{BB'/\delta x}{\delta t} = \lim_{\delta t \to 0} \frac{\frac{\partial v}{\partial x}\delta x \delta t}{\delta x \delta t} = \frac{\partial v}{\partial x}.$$

Again, the angular velocity of AD about z-axis

$$= \lim_{\delta t \to 0} \frac{\delta\beta}{\delta t} = \lim_{\delta t \to 0} \frac{\tan\delta\beta}{\delta t} = \lim_{\delta t \to 0} \frac{DD'/\delta y}{\delta t} = -\lim_{\delta t \to 0} \frac{\frac{\partial u}{\partial y}\delta y\,\delta t}{\delta y\,\delta t} = -\frac{\partial u}{\partial y}.$$

Let $\overline{\omega}_z$ denote the averge of the angular velocities of AB and AD. Then, we have

$$\overline{\omega}_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \qquad \text{...(1)}$$

The average angular velocity components $\overline{\omega}_x$, $\overline{\omega}_y$ and $\overline{\omega}_z$ in the case of three-dimensional flows may be obetained in a similar manner as follows:

$$\overline{\omega}_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad \overline{\omega}_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad \overline{\omega}_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \text{...(2)}$$

Hence the angular velocity vector $\boldsymbol{\omega}$ of a fluid element is given by

$$\boldsymbol{\omega} = \mathbf{i}\,\overline{\omega}_x + \mathbf{j}\,\overline{\omega}_y + \mathbf{k}\,\overline{\omega}_z$$

or

$$\boldsymbol{\omega} = \frac{1}{2}\left[\mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]$$

Thus, $\boldsymbol{\omega} = (1/2) \times \text{curl } \mathbf{q}$ or $2\boldsymbol{\omega} = \text{curl } \mathbf{q}$ ...(3)

But the vorticity vector Ω is given by $\boldsymbol{\Omega} = \text{curl } \mathbf{q}$ ...(4)

From (3) and (4), we have $\boldsymbol{\Omega} = 2\boldsymbol{\omega}$

Remark 1. $\boldsymbol{\omega}$ is also called the *rotation*. The condition for the two dimensional flow to be irrotational is that the rotation w_z is everywhere zero *i.e.* $\partial v/\partial x = \partial u/\partial y$.

Again, the condition for irrotationality in three-dimendional flow is that,

$\Omega_x = \Omega_y = \Omega_z = 0$ everwhere in the flow, *i.e.*

$$\partial w/\partial y = \partial v/\partial z \qquad \partial u/\partial z = \partial w/\partial x \qquad \partial v/\partial x = \partial u/\partial y$$

Remark 2. A flow, in which the fluid particle also rotate (*i.e.* possess some angular velocity) about their own axes,while flowing, is said to be a *rotational flow*. Again a flow, in which the fluid particles do not rotate about their own axes, and retain their original orientations, is said to be an *irrotational flow*.

2.32. Illustrative solved examples.

Ex. 1. *Give examples of irrotational and rotational flows.* **[Agra 2011, Garhwal 2005]**

Sol. Consider parallel flow with uniform velocity. For example, let there be a fluid motion with the following velocity components:

$$u = kx, \quad k \neq 0 \qquad v = 0, \qquad w = 0 \qquad \text{...(1)}$$

Then, $\partial w/\partial y = \partial v/\partial z$ $\qquad \partial u/\partial z = \partial w/\partial x$ $\qquad \partial v/\partial x = \partial u/\partial y$

Hence the flow is irrotational.

Next, consider a two-dimensional shear flow with the following velocity components;

$$u = ky, \qquad v = 0, \qquad w = 0, \qquad (k \neq 0) \qquad ...(2)$$

Then $$\Omega_z = \partial v/\partial x - \partial u/\partial y = -k \neq 0$$

Hence the rotation Ω_z is non-zero and so the flow is rotational.

Ex. 2. *Determine the vorticity components when velocity distribution is given by* $\mathbf{q} = \mathbf{i}\,(Ax^2yt) + \mathbf{j}\,(By^2zt) + \mathbf{k}\,(Czt^2)$ *where A, B, and C are constants.*

Sol. Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. Hence, here we have

$$u = Ax^2yt, \qquad v = By^2zt, \qquad w = Czt^2 \qquad ...(1)$$

The vorticity components $\mathbf{\Omega}_x, \mathbf{\Omega}_y, \mathbf{\Omega}_z$ are given by

$$\mathbf{\Omega}_x = \partial w/\partial y - (\partial v/\partial z) = 0 - By^2t = -By^2t, \qquad \mathbf{\Omega}_y = \partial u/\partial z - (\partial w/\partial x) = 0 - 0 = 0$$

and $$\mathbf{\Omega}_z = \partial v/\partial x - (\partial u/\partial y) = 0 - Ax^2t = -Ax^2t.$$

Ex. 3. (*a*) *Test whether the motion specified by* $\mathbf{q} = \dfrac{k^2(x\mathbf{j} - y\mathbf{i})}{x^2 + y^2}$ $(k = \text{const})$, *is a possible motion for an incompressible fluid. If so, determine the equation of the streamlines. Also test whether the motion is of the potential kind and if so determine the velocity potential.*

[Kanpur 2006; I.A.S. 1996, Rohilkhand 2003, 04]

(*b*) *Determine the velocity potential for the motion specified by* $\mathbf{q} = \dfrac{k^2(x\mathbf{j} - y\mathbf{i})}{x^2 + y^2}$, $(k = \text{const})$.

[Agra 2007]

Sol. (*a*) Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. Then here

$$u = -\frac{k^2 y}{x^2 + y^2}, \qquad v = \frac{k^2 x}{x^2 + y^2}, \qquad w = 0 \qquad ...(1)$$

The equation of continuity for an incompressible fluid is

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad ...(2)$$

Form (1), $$\frac{\partial u}{\partial x} = \frac{2k^2xy}{(x^2+y^2)^2}, \qquad \frac{\partial v}{\partial y} = -\frac{2k^2xy}{(x^2+y^2)^2}, \qquad \frac{\partial w}{\partial z} = 0$$

Hence (2) is satisfied and so the motion specified by given $\mathbf{q}$ is possible.

The equation of the streamlines are $\qquad dx/u = dy/v = dz/w$

i.e. $$\frac{dx}{-k^2y/(x^2+y^2)} = \frac{dy}{k^2x/(x^2+y^2)} = \frac{dz}{0} \qquad ...(3)$$

Taking the last fraction, $\qquad dz = 0 \qquad$ so that $\qquad z = c_1. \qquad ...(4)$

Taking the first two fractions in (3) and simplifying, we get

$$dx/(-y) = dy/x \qquad \text{or} \qquad 2xdx + 2ydy = 0$$

Integrating, $\qquad x^2 + y^2 = c_2$, c_2 being an arbitrary constant $\qquad ...(5)$

(4) and (5) together give the streamlines. Clearly, the streamlines are circles whose centres are on the z-axis, their planes being perpendicular to this axis.

Again $\quad \operatorname{curl}\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\dfrac{k^2 y}{x^2+y^2} & \dfrac{k^2 x}{x^2+y^2} & 0 \end{vmatrix} = k^2\left\{\dfrac{y^2-x^2}{(x^2+y^2)^2} + \dfrac{x^2-y^2}{(x^2+y^2)^2}\right\}\mathbf{k} = \mathbf{0}.$

Hence the flow is of the potential kind and we can find velocity potential $\phi\ (x, y, z)$ such that $\mathbf{q} = -\nabla\phi$. Thus, we have

$$\frac{\partial\phi}{\partial x} = -u = \frac{k^2 y}{x^2+y^2} \qquad ...(6)$$

$$\frac{\partial\phi}{\partial y} = -v = -\frac{k^2 x}{x^2+y^2} \qquad ...(7)$$

$$\partial\phi/\partial z = -w = 0 \qquad ...(8)$$

Equation (8) shows that the velocity potential ϕ is function of x and y only so that $\phi = \phi(x, y)$.

Integrating (6), $\phi\ (x, y) = k^2 \tan^{-1}(x/y) + f(y)$, where $f(y)$ is an arbitrary function, ...(9)

From (9), $$\frac{\partial\phi}{\partial y} = f'(y) - \frac{k^2 x}{x^2+y^2} \qquad ...(10)$$

Comparing (7) and (10), we have $\quad f'(y) = 0 \quad$ so that

$$f(y) = \text{constant}.$$

Since the constant can be omitted while writting velocity potential, the required velocity potential can be taken as [refer equation (9)]

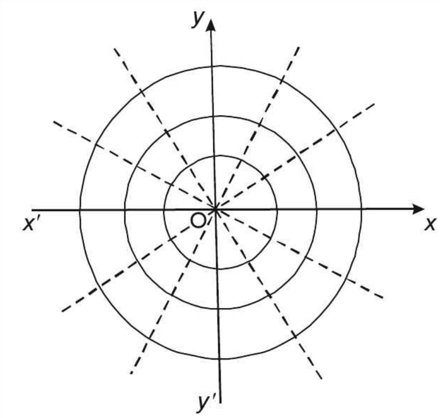
$$\phi(x, y) = k^2 \tan^{-1}(x/y) \qquad ...(11)$$

The equipotentials are given by

$$k^2 \tan^{-1}(x/y) = \text{constant} = k^2 \tan^{-1} c$$

or $\quad x = cy$, c being a constant

which are planes through the z-axis. They are intersected by the streamlines as shown in the figure. Dotted lines represent equipotentials and ordinary lines represent streamlines.



(*b*) Proceed as in part (*a*) upto equation (11). Then the required velocity potential is given by (11).

Ex. 4. *The velocity in the flow field is given by*

$$\mathbf{q} = \mathbf{i}\,(Az - By) + \mathbf{j}\,(Bx - Cz) + \mathbf{k}\,(Cy - Ax)$$

where A, B, C are non-zero constants. Determine the equation of the votex lines.

Sol. Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, Then we have

$$u = Az - By, \qquad v = Bx - Cz, \qquad w = Cy - Ax \qquad ...(1)$$

Let $\Omega_x, \Omega_y, \Omega_z$ be vorticity components. Then

$$\boldsymbol{\Omega}_x = \partial w/\partial y - (\partial v/\partial z) = C + C = 2C, \qquad \boldsymbol{\Omega}_y = \partial u/\partial z - (\partial w/\partial x) = A + A = 2A,$$

and $$\boldsymbol{\Omega}_z = \partial v/\partial x - (\partial u/\partial y) = B + B = 2B.$$

The equation of the vortex lines are $dx/\Omega_x = dy/\Omega_y = dz/\Omega_z$

i.e. $dx/(2C) = dy/(2A) = dz/(2B)$ ...(2)

Taking the first two members in (2) and integrating, we get

$Ax - Cy = C_1$, C_1 being an arbitrary constant ...(3)

Next, taking the last two membes in (2) and integrating, we get

$By - Az = C_2$, C_2 being an arbitrary constants ...(4)

The required vortex lines are the straight lines of the intersection of (3) and (4).

Ex. 5. *At a point in an incomressible fluid having spherical polar co-ordinates* (r, θ, ϕ), *the velocity components are* $[2Mr^{-3}\cos\theta, Mr^{-3}\sin\theta, 0]$, *where M is a constant. Show the velocity is of the potential kind. Find the velocity potential and the equations of the stream lines.*

Sol. Here $q_r = 2Mr^{-3}\cos\theta,$ $q_\theta = Mr^{-3}\sin\theta,$ $q_\phi = 0$.

Then, we have $\mathbf{q} = 2Mr^{-3}\cos\theta\,\mathbf{e_r} + Mr^{-3}\sin\theta\,\mathbf{e_\theta} + 0\,\mathbf{e_\phi}$ and hence

$$\text{curl}\,\mathbf{q} = \frac{1}{r^2\sin^2\theta}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e_\theta} & r\sin\theta\,\mathbf{e_\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ q_r & q_\theta & q_\phi \end{vmatrix} = \frac{1}{r^2\sin^2\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e_\theta} & r\sin\theta\;\mathbf{e_\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 2\mathbf{M}r^{-3}\cos\theta & \mathbf{M}r^{-3}\sin\theta & 0 \end{vmatrix}$$

= 0, on simplification

Hence the flow is of the potential kind.

Let $F(r, \theta, \phi)$ be the required velocity potential. We have used F for velocity potential to avoid confusion. Then by definition

$$-\frac{\partial F}{\partial r} = q_r = 2Mr^{-3}\cos\theta, \qquad -\frac{1}{r}\frac{\partial F}{\partial\theta} = q_\theta = Mr^{-3}\sin\theta, \qquad \text{and} \qquad \frac{1}{r\sin\theta}\frac{\partial F}{\partial\phi} = q_\phi = 0$$

$$\therefore \qquad dF = \frac{\partial F}{\partial r}dr + \frac{\partial F}{\partial\theta}d\theta + \frac{\partial F}{\partial\phi}d\phi.$$

or $$dF = -(2Mr^{-3}\cos\theta)\,dr - (Mr^{-2}\sin\theta)\,d\theta + 0\cdot d\phi = d(Mr^{-2}\cos\theta)$$

Integrating, $F = Mr^{-2}\cos\theta$. omiting constant of integration, for it has no significance in F)

Finally, the streamlines are given by

$$\frac{dr}{q_r} = \frac{rd\theta}{q_\theta} = \frac{r\sin\theta d\phi}{q_\phi} \qquad i.e., \qquad \frac{dr}{2Mr^{-3}\cos\theta} = \frac{rd\theta}{Mr^{-3}\sin\theta} = \frac{r\sin\theta d\phi}{0}$$

given $d\phi = 0$ and $2\cot\theta\,d\theta = (1/r)\,dr$.

Integrating, the equation of the streamlines are given by

$\phi = C_1$ and $r = C_2\sin^2\theta$, C_1 and C_2 being arbitrary constants.

The equation ϕ = constant shows that the required streamlines lie in a plane which pass through the axis of symmetry $\theta = 0$.

Ex. 6. (*a*) *Show that* $u = -\dfrac{2xyz}{(x^2+y^2)^2}$, $v = \dfrac{(x^2-y^2)z}{(x^2+y^2)^2}$, $w = \dfrac{y}{x^2+y^2}$ *are the velocity components of a possible liquid motion. Is this motion irrotational.*

[Garhwal 2004; Agra 2004; Kerala 2001; I.A.S. 2000, 2002 Meerut 2002, 04]

(*b*) *Show that a fluid of constant density can have a velocity* **q** *given by*

$$\mathbf{q} = \left[-\frac{2xyz}{(x^2+y^2)^2}, \frac{(x^2-y^2)z}{(x^2+y^2)^2}, \frac{y}{x^2+y^2} \right]$$

Find the vorticity vector. **[Kanpur 2007; I.A.S. 1988, 98, 2000]**

Sol. Part (a). Here, we have

$$\frac{\partial u}{\partial x} = -2yz\frac{1\cdot(x^2+y^2)^2 - x\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4} = -2yz\frac{x^2+y^2-4x^2}{(x^2+y^2)^3} = -2yz\frac{y^2-3x^2}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = z\frac{-2y(x^2+y^2)^2 - 2(x^2+y^2)\cdot 2y(x^2-y^2)}{(x^2+y^2)^4} = -2yz\frac{x^2+y^2+2(x^2-y^2)}{(x^2+y^2)^3} = -2yz\frac{3x^2-y^2}{(x^2+y^2)^3}$$

and $$\partial w/\partial z = 0$$

Hence the equation of continuity $$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$$

is satisfied and so the liquid motion is possible.

Furthermore, we have

$$\mathbf{\Omega}_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

$$\mathbf{\Omega}_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = -\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} = 0$$

and $$\mathbf{\Omega}_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{2xz(3y^2-x^2)}{(x^2+y^2)^3} - \frac{2xz(3y^2-x^2)}{(x^2+y^2)^3} = 0$$

$\therefore$ $\partial w/\partial y = \partial v/\partial z,$ $\quad \partial u/\partial z = \partial w/\partial x,$ $\quad \partial v/\partial x = \partial u/\partial y$

and hence the motion is irrotational.

Part (b). Let **q** = (u, v, w). Then we have the same values of u, v, w as in part (*a*). By definition, the vorticity vector $\mathbf{\Omega}$ is given by

$$\mathbf{\Omega} = \mathbf{\Omega}_x\mathbf{i} + \mathbf{\Omega}_y\mathbf{j} + \mathbf{\Omega}_z\mathbf{k} = \mathbf{0}, \text{ using part (a)}$$

Ex. 7. *Show that* $\phi = (x-t)(y-t)$ *represents the velocity potential of an incompressible two dimensional fluid. Show that the streamlines at time 't' are the curves*

$$(x-t)^2 - (y-t)^2 = \text{constant},$$

and that the paths of the fluid particles have the equations

$$\log(x-y) = (1/2) \times \{(x+y) - a(x-y)^{-1}\} + b, \text{ where } a, b \text{ are constants.}$$

Sol. Given $$\phi = (x-t)(y-t) \qquad \ldots (1)$$

From (1), we have

$u = -\partial\phi/\partial x = -(y-t),\quad v = -\partial\phi/\partial y = -(x-t)$ and so $\quad \partial u/\partial x = 0 = \partial v/\partial y$

Thus the equation of continuity $\partial u/\partial x + \partial v/\partial y = 0$ is satisfied. Hence ϕ given by (1) represents the velocity potential of an incompressible two-dimensional flow.

Again, the equaiton of streamlines are given by

$$\frac{dx}{u} = \frac{dv}{v} \qquad \text{or} \qquad \frac{dx}{-(y-t)} = \frac{dy}{-(x-t)}$$

or $$(x-t)\,dx-(y-t)\,dy=0$$

Integrating, $$(x-t)^2-(y-t)^2=\text{constant}$$

Finally, the paths of particles are given by

$$u=dx/dt=-(y-t) \quad \text{and} \quad v=dy/dt=-(x-t)$$

$\therefore$ $$dx/dt=t-y \qquad \text{...(2)}$$

and $$dy/dt=t-x \qquad \text{...(3)}$$

From (2) and (3), $$dx/dt+dy/dt=2t-(x+y) \qquad \text{...(4)}$$

Let $x+y=z$ so that $dx/dt+dy/dt=dz/dt$ ...(5)

Then (4) gives $dz/dt=2t-z$ or $dz/dt+z=2t$ ...(6)

which is a linear differential equation.

Its integrating factor $= e^{\int dt}=e^t$. Here solution of (6) is

$$ze^t=\int 2te^t dt+c_1=2t\cdot e^t-\int (2)e^t dt+c_1=2te^t-2e^t+c_1$$

$\therefore$ $z=2t-2+c_1e^{-t}$ or $x+y=2t-2+c_1e^{-t}$, by (5) ...(7)

Again from (2) and (3), $dx/dt-(dy/dt)=x-y$

or $$\frac{dx-dy}{dt}=x-y \quad \text{or} \quad \frac{dx-dy}{x-y}=dt$$

Integrating, $\log(x-y)-\log c_2=t$ or $x-y=c_2e^t$. ...(8)

Using (7) and (8), we have

$\therefore$ $$x+y-a(x-y)^{-1}=2t-2+c_1e^{-t}-\frac{a}{c_2}e^{-t}=2t-2, \quad \text{taking} \quad c_1=\frac{a}{c_2}$$

$\therefore$ $$(1/2)\times\{(x+y)-a(x-y)^{-1}\}=t-1 \qquad \text{...(9)}$$

But from (8), $e^t=(x-y)/c_2$ so that $t=\log(x-y)-\log c_2$

$\therefore$ $$t-1=\log(x-y)-(\log c_2+1)$$

$\therefore$ $t-1=\log(x-y)-b,$ taking $b=-(\log c_2+1)$ ...(10)

Using (10), (9) reduces to the required equations

$$(1/2)\times\{(x+y)-a(x-y)^{-1}\}=\log(x-y)+b.$$

Ex. 8(*a*). *If the velocity of an incompressible fluid at the point* (x, y, z) *is given by*

$$\left(\frac{3xz}{r^5},\ \frac{3yz}{r^5},\ \frac{3z^2-r^2}{r^5}\right)$$

prove that the liquid motion is possible and that the velcity potential is $(\cos\theta)/r^2$. *Also determine the streamlines.*

Sol. Here $$u=\frac{3xz}{r^5}, \quad v=\frac{3yz}{r^5}, \quad w=\frac{3z^2-r^2}{r^5}=\frac{3z^2}{r^5}-\frac{1}{r^3} \qquad \text{...(1)}$$

where $$r^2=x^2+y^2+z^2 \qquad \text{...(2)}$$

From (2), $\partial r/\partial x=x/r,$ $\partial r/\partial y=y/r,$ $\partial r/\partial z=z/r$ ...(3)

From (1), (2) and (3), we have

$$\frac{\partial u}{\partial x}=3z\left[\frac{1}{r^5}+(-5x)r^{-6}\frac{\partial r}{\partial x}\right]=\frac{3z}{r^5}-\frac{15x^2z}{r^7}$$

$$\frac{\partial v}{\partial y} = 3z\left[\frac{1}{r^5} + (-5y)r^{-6}\frac{\partial r}{\partial y}\right] = \frac{3z}{r^5} - \frac{15y^2 z}{r^7}$$

$$\frac{\partial w}{\partial z} = \frac{6z}{r^5} - 15z^2\, r^{-6}\frac{\partial r}{\partial z} + 3r^{-4}\frac{\partial r}{\partial z} = \frac{6z}{r^5} - \frac{15z^2}{r^6}\cdot\frac{z}{r} + \frac{3}{r^4}\cdot\frac{z}{r} = \frac{9z}{r^5} - \frac{15z^3}{r^7}$$

$$\therefore \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{15z}{r^5} - \frac{15z}{r^7}(x^2 + y^2 + z^2) = \frac{15z}{r^5} - \frac{15z}{r^7} \times r^2 = 0.$$

Since the equation of continuity is satisfied by the given values of u, v and w, the motion is possible. Let ϕ be the required velocity potential. Then

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = -(udx + vdy + wdz), \text{ by definition of } \phi$$

$$= -\left[\frac{3xz}{r^5}dx + \frac{3yz}{r^5}dy + \frac{3z^2 - r^2}{r^5}dz\right] = \frac{r^2 dz - 3z\,(xdx + ydy + zdz)}{r^5}$$

Thus, $$d\phi = \frac{r^3 dz - 3r^2 z dr}{(r^3)^2} = d\left(\frac{z}{r^3}\right), \text{ using (2)}$$

Intergrating , $$\phi = z / r^3$$

[Omitting constant of integration, for it has no significance in ϕ]

In spherical polar coordinates (r, θ, ϕ), we know that $z = r \cos\theta$. Hence the required potential is given by $$\phi = (r\cos\theta)/r^3 = (\cos\theta)/r^2$$

We now obtain the streamlines. The equations of streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad i.e., \qquad \frac{dx}{3xz/r^5} = \frac{dy}{3yz/r^5} = \frac{dz}{(3z^2 - r^2)/r^5}$$

or $$\frac{dx}{3xz} = \frac{dy}{3yz} = \frac{dz}{3z^2 - r^2} \qquad \text{...(4)}$$

Taking the first two members of (4) and simplifying, we get

$dx/x = dy/y$ or $dx/x - dy/y = 0$

Integrating, $\log x - \log y = \log c_1$ *i.e.* $x/y = c_1$, c_1 being a constant ...(5)

$$\text{Now, each member in (4)} = \frac{xdx + ydy + zdz}{3x^2 z + 3y^2 z + 3z^3 - r^2 z} = \frac{xdx + ydy + zdz}{3z(x^2 + y^2 + z^2) - r^2 z}$$

$$= \frac{xdx + ydy + zdz}{3z(x^2 + y^2 + z^2) - z(x^2 + y^2 + z^2)} = \frac{xdx + ydy + zdz}{2z(x^2 + y^2 + z^2)}, \text{ by (2)} \qquad \text{...(6)}$$

Taking the first member in (4) and (6), we get

$$\frac{dx}{3xz} = \frac{xdx + ydy + zdz}{2z(x^2 + y^2 + z^2)} \qquad \text{or} \qquad \frac{2}{3}\frac{dx}{x} = \frac{1}{2}\frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2}$$

Integrating, $(2/3) \times \log x = (1/2) \times \log (x^2 + y^2 + z^2) + \log c_2$

or $x^{2/3} = c_2\,(x^2 + y^2 + z^2)^{1/2}$, c_2 being an arbitrary constant ...(7)

The required streamlines are the curves of intersction of (5) and (7).

Ex. 8(*b*). If velocity distributon of an incompressible fluid at point (x, y, z) is gives by $\{3xz/r^5, 3yz/r^5 (kz^2 - r^2)/r^5\}$, determine the parameter k such that it is a possible motion. Hence find its velocity potential. **[I.A.S. 2001]**

Sol. Here $$u = \frac{3xz}{r^5}, \qquad v = \frac{3yz}{r^5}, \qquad w = \frac{kz^2 - r^2}{r^5} = \frac{kz^2}{r^5} - \frac{1}{r^3}, \qquad \text{...(1)}$$

where $$r^2 = x^2 + y^2 + z^2 \qquad \text{...(2)}$$

From (2), $\partial r/\partial x = x/r, \quad \partial r/\partial y = y/r \quad$ and $\quad \partial r/\partial z = z/r$ ...(3)

Now proceed as in solved Ex. 8(*a*) and obtain

$$\frac{\partial u}{\partial x} = \frac{3z}{r^5} - \frac{15x^2 z}{r^7}, \qquad \frac{\partial v}{\partial y} = \frac{3z}{r^5} - \frac{15y^2 z}{r^7} \qquad \text{...(4)}$$

and $$\frac{\partial w}{\partial z} = \frac{2kz}{r^5} - 5kz^2 r^{-6}\frac{\partial r}{\partial z} + 3r^{-4}\frac{\partial r}{\partial z} = \frac{2kz}{r^5} - \frac{5kz^2}{r^6}\cdot\frac{z}{r} + \frac{3}{r^4}\cdot\frac{z}{r} = \frac{(2k+3)z}{r^5} - \frac{15z^3}{r^7} \qquad \text{...(5)}$$

Since (1) gives a possible liquid motion, the equation of continuity must be satisfied and so

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$$

or $$\frac{(2k+9)z}{r^5} - \frac{15z}{r^7}(x^2 + y^2 + z^2) = 0 \quad \text{or} \quad \frac{(2k+9)z}{r^5} - \frac{15z}{r^7}\cdot r^2 = 0, \text{ using (2),(4) and (5)}$$

or $\quad (2k - 6)\, z/r^5 = 0 \quad$ so that $\quad 2k - 6 = 0 \quad$ giving $\quad k = 3.$

Substituting the above value of k in (1), we have

$$u = (3xz)/r^5, \qquad v = (3yz)/r^5, \qquad w = (3z^2 - r^2)/r^5. \qquad \text{...(6)}$$

Using (6) and proceeding as in Ex. 8(*a*), the required velocity potential ϕ is given by $\phi = z/r^3$.

Ex. 9. (*a*) *Show that if the velocity potential of an irroatioanl fluid motion is equal to*

$$A\,(x^2 + y^2 + z^2)^{-3/2}\, z \tan^{-1}(y/x)$$

the lines of flow will be on the series of the surfaces $x^2 + y^2 + z^2 = c^{2/3}\,(x^2 + y^2)^{2/3}$.

[Agra 2004, 06; Kanpur 2002, 11; Meerut 2004]

(*b*) *If the velocity potential of a fluid is* $\phi = (z/r^3)\tan^{-1}(y/x)$ *where* $r^2 = x^2 + y^2 + z^2$, then *show that the streamlines lie on the surfaces* $x^2 + y^2 + z^2 = c\,(x^2 + y^2)^{2/3}$, *c being an arbitrary constant.* **[I.A.S. 2008]**

Sol. (*a*) The veloctiy potential ϕ is given by

$$\phi(x, y, z) = A(x^2 + y^2 + z^2)^{-3/2} z \tan^{-1}(y/x) = Ar^{-3} z \tan^{-1}(y/x) \qquad \text{...(1)}$$

where $$r^2 = x^2 + y^2 + z^2 \qquad \text{...(2)}$$

so that $\quad \partial r/\partial x = x/r, \quad \partial r/\partial y = y/r, \quad \partial z/\partial r = z/r$ ...(3)

$$\therefore \quad u = -\frac{\partial \phi}{\partial x} = 3Azxr^{-5}\tan^{-1}\frac{y}{x} + \frac{Azyr^{-3}}{x^2 + y^2}$$

$$v = -\frac{\partial \phi}{\partial y} = 3Azyr^{-5}\tan^{-1}\frac{y}{x} - \frac{Azxr^{-3}}{x^2 + y^2}$$

$$w = -\frac{\partial \phi}{\partial z} = 3Az^2 r^{-5}\tan^{-1}\frac{y}{x} - Ar^{-3}\tan^{-1}\frac{y}{x}$$

The equation of lines of flow are given by $\quad dx/u = dy/v = dz/w$

i.e. $$\frac{dx}{3Azxr^{-5}\tan^{-1}\frac{y}{x} + \frac{Azyr^{-3}}{x^2 + y^2}} = \frac{dy}{3Azyr^{-5}\tan^{-1}\frac{y}{x} - \frac{Azxr^{-3}}{x^2 + y^2}} = \frac{dz}{A(3z^2 r^{-5} - r^{-3})\tan^{-1}\frac{y}{x}} \qquad \text{...(4)}$$

Each member of (4) is $= \dfrac{xdx + ydy + zdz}{(3x^2 + 3y^2 + 3z^2)r^{-2} - 1} = \dfrac{xdx + ydy}{(3x^2 + 3y^2)/r^2}$ (on simplification)

or
$$\frac{xdx + ydy + zdz}{2} = \frac{r^2(xdx + ydy)}{3(x^2 + y^2)}$$

or
$$\frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2} = \frac{2}{3} \cdot \frac{2xdx + 2ydy}{x^2 + y^2} \qquad ...(5)$$

Integrating (5), $\log (x^2 + y^2 + z^2) = (2/3) \times \log (x^2 + y^2) + (2/3) \times \log c$

or $x^2 + y^2 + z^2 = c^{2/3} (x^2 + y^2)^{2/3}$, c being an arbitrary constant ...(6)

(6) gives the required series of the surfaces on which the desired lines of flow will lie.

(b). Proceed like part (*a*) by taking $A = 1$. Thus obtain (5). Integrating (5), $\log (x^2 + y^2 + z^2) = (2/3) \times \log (x^2 + y^2) + \log c$ giving $x^2 + y^2 + z^2 = c\,(x^2 + y^2)^{2/3}$, c being an arbitrary constant.

Ex. 10. *Given $u = -Wy$, $v = Wx$, $w = 0$, show that the surfaces intersecting the streamlines orthogonally exist and are the planes through z-axis, although the velocity potential does not exist. Discuss the nature of flow.*

Sol. Given $u = -Wy, \qquad v = Wx, \qquad w = 0$ ...(1)

$\therefore \qquad \partial u/\partial x = 0, \qquad \partial v/\partial y = 0, \qquad \partial w/\partial z = 0$

so that
$$\partial u/\partial x + \partial y/\partial y + \partial w/\partial z = 0. \qquad ...(2)$$

(2) shows that the equation of continuity is satisfied and so the motion specified by (1) is possible. The equations of the streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dw}{w} \qquad i.e., \qquad \frac{dx}{-Wy} = \frac{dy}{Wx} = \frac{dz}{0}$$

giving $\qquad xdx + ydy = 0 \qquad$ and $\qquad dz = 0$

Integrating, $x^2 + y^2 = c_1$ and $z = c_2$, c_1 and c_2 being arbitrary constants ...(3)

Hence the streamlines are circles given by the intersection of surfaces (3).

The surfaces which cut the stream lines orthogonally are

$$udx + vdy + wdz = 0$$

i.e. $\qquad -Wydx + Wxdy = 0 \qquad$ or $\qquad dx/x - dy/y = 0$

Integrating, $x/y = c$ or $x = cy$, c being an arbitrary constant, ...(4)

which repressents a plane through z-axis and cuts the stream lines (3) orthogonally

Now $\qquad udx + vdy + wdz = -Wydx + Wxdy$ ...(5)

Here $\qquad \dfrac{\partial}{\partial y}(-Wy) = -W \qquad$ and $\qquad \dfrac{\partial}{\partial x}(Wx) = W.$ ..(6)

Hence $udx + vdy + wdz$ is not a perfect differential and so the velocity potential does not exist. Again, we have

$$\text{curl } \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -Wy & Wx & 0 \end{vmatrix} = 2W\mathbf{k}.$$

Since curl $\mathbf{q} \neq \mathbf{0}$, the motion is rotational. Notice that a rigid body rotating about z-axis with constant vector angular velocity $2W\mathbf{k}$ will produce the above type of motion.

Ex. 11. *Prove that the liquid motion is possible when velocity at (x, y, z) is given by*

$$u = (3x^2 - r^2)/r^5, \qquad v = 3xy/r^5, \qquad w = 3xz/r^5,$$

where $r^2 = x^2 + y^2 + z^2$, *and the streamlines lines are the intersection of the surfaces* $(x^2 + y^2 + z^2)^3 = c\,(y^2 + z^2)^2$ *by the planes passing through OX. State if the motion is irrotational giving reasons for your answer.* **[Kanpur 2011; Agra 2008]**

Sol. Given $u = (3x^2 - r^2)/r^5, \quad v = 3xy/r^5, \quad w = 3xz/r^5$ ...(1)

For the motion to be possible, we must show that the equation of continuity

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad ...(2)$$

must be satisfied.

From (1), $$\frac{\partial u}{\partial x} = \frac{[6x - 2r(\partial r/\partial x)]r^5 - 5r^4(\partial r/\partial x)(3x^2 - r^2)}{r^{10}} \qquad ...(3)$$

But $$r^2 = x^2 + y^2 + z^2 \qquad ...(4)$$

From (4), $\partial r/\partial x = x/r, \quad \partial r/\partial y = y/r$ and $\partial r/\partial z = z/r$ ...(5)

Using (5), (3) gives

$$\frac{\partial u}{\partial x} = \frac{(6x - 2x)r^5 - 5r^3x\,(3x^2 - r^2)}{r^{10}} = \frac{3x(3r^2 - 5x^2)}{r^7}$$

Similarly, $$\frac{\partial v}{\partial y} = \frac{3x(r^2 - 5y^2)}{r^7} \quad \text{and} \quad \frac{\partial w}{\partial z} = \frac{3x(r^2 - 5z^2)}{r^7}.$$

$\therefore$ L.H.S. of (2) $= \dfrac{3x[5r^2 - 5(x^2 + y^2 + z^2)]}{r^7} = 0$, using (4)

(2) is satisfied. So the liquid motion is possible. The equation of streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{or} \quad \frac{dx}{3x^2 - r^2} = \frac{dy}{3xy} = \frac{dz}{3xz} \qquad ...(6)$$

Taking the last two members (6), we get

$dy/y = dz/z$ giving $y = az$, a being an arbitrary constant ...(7)

which is a plane passsing through OX.

Now each member of (6) $= \dfrac{xdx + ydy + zdz}{x(3r^2 - r^2)} = \dfrac{ydy + zdz}{3x(y^2 + z^2)}$

Thus, $$\frac{3(2xdx + 2ydy + 2zdz)}{x^2 + y^2 + z^2} = \frac{2(2ydy + 2zdz)}{y^2 + z^2}$$

Integrating, $3 \log (x^2 + y^2 + z^2) = 2 \log (y^2 + z^2) + \log c$

or $(x^2 + y^2 + z^2)^3 = c(y^2 + z^2)^2$, c being an arbitrary constant ...(8)

The required streamlines are given by the intersection of surfaces (8) by the planes (7) passing through OX.

Finally, to show that the motion is irrotational, we should verify the conditions:

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0, \qquad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \qquad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad ...(9)$$

From (11), we have

$$\frac{\partial u}{\partial y} = -\frac{3y(5x^2 - r^2)}{r^7}, \qquad \frac{\partial u}{\partial z} = -\frac{3z(5x^2 - r^2)}{r^7}, \qquad \frac{\partial v}{\partial x} = \frac{3y(r^2 - 5x^2)}{r^7},$$

$$\frac{\partial v}{\partial z} = -\frac{15xyz}{r^7}, \qquad \frac{\partial w}{\partial x} = \frac{3z(r^2 - 5x^2)}{r^7}, \qquad \frac{\partial w}{\partial y} = -\frac{15xyz}{r^7}.$$

With these values, conditions (9) are all satisfied. Hence the motion is irrotational.

Ex. 12. *Show that in the motion of a fluid in two dimensions if the coordinates (x, y) of an element at any time be expressed in terms of the initial coordinates (a, b) and the time, the motion is irrotational, if* $\dfrac{\partial(\dot{x}, x)}{\partial(a,b)} = \dfrac{\partial(\dot{y}, y)}{\partial(a,b)} = 0.$ [Here $\dot{x} = \dfrac{dx}{dt}$ and $\dot{y} = \dfrac{dy}{dt}$].

[Agra 2010; G.N.D.U. Amritser 2003, 05; Kanpur 1999, 2007; Meerut 2003]

Sol. Let u and v be the velocity components parallel to x-and y-axes respectively so that $\dot{x} = dx/dt = u$, $\dot{y} = dy/dt = v$. Now, we have

$$\left.\begin{aligned} \frac{\partial u}{\partial a} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a}, \qquad & \frac{\partial u}{\partial b} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial b} \\ \frac{\partial v}{\partial a} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial a}, \qquad & \frac{\partial v}{\partial b} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial b} \end{aligned}\right\} \qquad ...(1)$$

$$\therefore \quad \frac{\partial(\dot{x}, x)}{\partial(a,b)} + \frac{\partial(\dot{y}, y)}{\partial(a,b)} = \frac{\partial(u, x)}{\partial(a,b)} + \frac{\partial(v, y)}{\partial(a,b)}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial a} & \dfrac{\partial u}{\partial b} \\ \dfrac{\partial x}{\partial a} & \dfrac{\partial x}{\partial b} \end{vmatrix} + \begin{vmatrix} \dfrac{\partial v}{\partial a} & \dfrac{\partial v}{\partial b} \\ \dfrac{\partial y}{\partial a} & \dfrac{\partial y}{\partial b} \end{vmatrix} = \frac{\partial u}{\partial a}\frac{\partial x}{\partial b} - \frac{\partial u}{\partial b}\frac{\partial x}{\partial a} + \frac{\partial v}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial v}{\partial b}\frac{\partial y}{\partial a}$$

$$= \frac{\partial x}{\partial b}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a}\right) - \frac{\partial x}{\partial a}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial b}\right) + \frac{\partial y}{\partial b}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial a}\right) - \frac{\partial y}{\partial a}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial b}\right),$$

[Using (1)]

$$= \frac{\partial u}{\partial y}\left(\frac{\partial x}{\partial b}\frac{\partial y}{\partial a} - \frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\right) + \frac{\partial v}{\partial a}\left(\frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial y}{\partial a}\frac{\partial x}{\partial b}\right)$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\left(\frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial y}{\partial a}\frac{\partial x}{\partial b}\right) = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\begin{vmatrix} \partial x/\partial a & \partial x/\partial b \\ \partial y/\partial a & \partial y/\partial b \end{vmatrix}$$

$$\therefore \qquad \frac{\partial(\dot{x}, x)}{\partial(a,b)} + \frac{\partial(\dot{y}, y)}{\partial(a,b)} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial(x, y)}{\partial(a,b)} \qquad ...(2)$$

Lagrangian equation of continuity in two dimensional case is given by

$$\rho_0 = \rho J \qquad \text{or} \qquad \rho\frac{\partial(x, y)}{\partial(a,b)} = \rho_0 \qquad ...(3)$$

From (3), we find that $\partial(x, y)/\partial(a,b) \neq 0$, so (2) shows that

$$\frac{\partial(\dot{x}, x)}{\partial(a,b)} + \frac{\partial(\dot{y}, y)}{\partial(a,b)} = 0 \quad \text{if and only if} \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0,$$ *i.e.* the motion is irrotational.

Ex. 13. *Show that all necessary conditions can be satisfied by a velocity potential of the form* $\phi = \alpha x^2 + \beta y^2 + \gamma z^2$, *and a bounding surface of the form* $F = ax^4 + by^4 + cz^4 - \chi(t) = 0$, *where* $\chi(t)$ *is a given function of the time and* $\alpha, \beta, \gamma, a, b, c$ *are suitable functions of the time.*

[Kanpur 2003; Himachel 1994; Gerhwal 1998; I.A.S. 1998; Kurkshetra 2000]

Sol. The given expressions for velocity potential and bounding surface are respectively

$$\phi(x, y, z) = \alpha x^2 + \beta y^2 + \gamma z^2 \qquad \text{...(1)}$$

and
$$F(x, y, z, t) = ax^4 + by^4 + cz^4 - \chi(t) \qquad \text{...(2)}$$

The following conditions must be satisfied :

(*i*) ϕ satisfies the Laplace's equation, namely,

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = 0 \qquad \text{...(3)}$$

(*ii*) F satisfies the condition for boundary surface, namely,

$$\partial F/\partial t + u\,(\partial F/\partial x) + v\,(\partial F/\partial y) + w\,(\partial F/\partial z) = 0 \qquad \text{...(4)}$$

From (1), $\quad \partial^2\phi/\partial x^2 = 2\alpha, \quad \partial^2\phi/\partial y^2 = 2\beta, \quad$ and $\quad \partial^2\phi/\partial z^2 = 2\gamma$

Hence (2) will be satisfied if

$$2\alpha + 2\beta + 2\gamma = 0 \qquad \text{or} \qquad \alpha + \beta + \gamma = 0 \qquad \text{...(5)}$$

for which α, β, γ must be some suitable functions of time.

Now from (2), we have (by using dots for differentiation with respect to time)

$$\partial F/\partial t = x^4\dot{a} + y^4\dot{b} + z^4\dot{c} - \dot{\chi}, \quad \partial F/\partial x = 4ax^3, \quad \partial F/\partial y = 4by^3, \quad \partial F/\partial z = 4cz^3 \qquad \text{...(6)}$$

Remember that if ϕ is the velocity potential function, then u, v, w are given by

$$u = -\partial\phi/\partial x, \qquad v = -\partial\phi/\partial y, \qquad \text{and} \qquad w = -\partial\phi/\partial z \qquad \text{...(7)}$$

Using (1) and (7), we have

$$u = -2\alpha x, \qquad v = -2\beta y, \qquad \text{and} \qquad w = -2\gamma z \qquad \text{...(8)}$$

Using (6) and (8), (4) reduces to

$$x^4(\dot{a} - 8a\alpha) + y^4(\dot{b} - 8b\beta) + z^4(\dot{c} - 8c\gamma) - \dot{\chi} = 0 \qquad \text{...(9)}$$

Since all the points on the surface (2) must also simultaneously satisfy (9), we have

$$\frac{\dot{a} - 8a\alpha}{a} = \frac{\dot{b} - 8b\beta}{b} = \frac{\dot{c} - 8c\gamma}{c} = \frac{\dot{\chi}}{\chi} \qquad \text{...(10)}$$

Taking first and the fourth members of (10), we get

$$\dot{a}/a = 8\alpha + \dot{\chi}/\chi$$

Integrating,
$$\log a = 8\int \alpha\, dt + \log \chi \qquad \text{...(11)}$$

Simlarly,
$$\log b = 8\int \beta\, dt + \log \chi \qquad \text{...(12)}$$

and
$$\log c = 8\int \gamma\, dt + \log \chi \qquad \text{...(13)}$$

In view of (5), α, β and γ are known. Hence equations (11) to (13) determine a, b and c as functions of t.

Thus, velocity potential ϕ given by (1) and the bounding surface $F = 0$ given by (2) satisfy the necessary conditions if a, b, c, α, β and γ are some suitable functions of time.

Ex. 14. *Show that the velocity poremtial* $\phi = (a/2)\times(x^2+y^2-2z^2)$ *satisfies the Laplace equation. Also determine the streamlines.* **[Nagpur 2003, I.A.S. 2002]**

Sol. We know that the velocity **q** of the fluid is given by

$$\mathbf{q} = -\nabla\phi = -\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\left\{\frac{a}{2}(x^2+y^2-2z^2)\right\}$$

or
$$\mathbf{q} = -(a/2)\times(2x\mathbf{i}+2y\mathbf{j}-4z\mathbf{k}). \qquad \text{...(1)}$$

But
$$\mathbf{q} = u\mathbf{i}+v\mathbf{j}+w\mathbf{k}. \qquad \text{...(2)}$$

Comparing (1) and (2), $\quad u = -ax, \quad v = -ay, \quad w = 2az.$

The equaitons of streamlines are given by $\quad dx/u = dy/v = dz/w$

$$\frac{dx}{-ax} = \frac{dy}{-ay} = \frac{dz}{2az} \quad \text{or} \quad \frac{2dx}{x} = \frac{2dy}{y} = \frac{dz}{-z}. \qquad \text{...(3)}$$

Taking the first two fractions of (3), $\quad (1/x)dx = (1/y)\,dy.$

Integrating, $\quad \log x = \log y + \log c_1 \quad$ or $\quad x = c_1 y. \qquad$ ...(4)

Taking the last two fractions of (3), $\quad (2/y)dy + (1/z)dz = 0$

Integrating, $\quad 2\log y + \log z = \log c_2 \quad$ or $\quad y^2 z = c_2. \qquad$ ...(5)

(4) and (5) together give the equations of streamlines, c_1 and c_2 being arbitrary constants of integration.

Now, given that $\quad \phi = (a/2)\times(x^2+y^2-2z^2). \qquad$ ...(6)

From (6), $\quad \partial\phi/\partial x = ax, \quad \partial\phi/\partial y = ay \quad$ and $\quad \partial\phi/\partial z = -2az$

$\Rightarrow \quad \partial^2\phi/\partial x^2 = a, \quad \partial^2\phi/\partial y^2 = a \quad$ and $\quad \partial^2\phi/\partial z^2 = -2a$

$\therefore \quad \partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2+\partial^2\phi/\partial z^2 = a+a-2a \quad$ or $\quad \nabla^2\phi = 0,$

showing that ϕ satisfies the Laplace equation.

Ex. 15. *Show that* $\phi = x f(r)$ *is a possible form for the velocity potential of an incompressible liquid motion. Given that the liquid speed* $q \to 0$ *as* $r \to \infty$, *deduce that the surfaces of constant speed are* $(r^2+3x^2)r^{-8}$ = *constant.*

Sol. Given $\quad \phi = xf(r). \qquad$ ...(1)

$\therefore \quad \mathbf{q} = -\nabla\phi = -\nabla[xf(r)] = -[f(r)\nabla x + x\nabla f(r)]. \qquad$ ...(2)

Now, $\quad r^2 = x^2+y^2+z^2 \quad \Rightarrow \quad 2r(\partial r/\partial x) = 2x \quad \Rightarrow \quad \partial r/\partial x = x/r. \qquad$ ...(3)

Similarly, $\quad \partial r/\partial y = y/r \quad$ and $\quad \partial r/\partial z = z/r. \qquad$ ...(4)

Also, $\quad \nabla x = [\mathbf{i}(\partial/\partial x)+\mathbf{j}(\partial/\partial y)+\mathbf{k}(\partial/\partial z)]x = \mathbf{i}$

and $\quad \nabla f(r) = [\mathbf{i}(\partial/\partial x)+\mathbf{j}(\partial/\partial y)+\mathbf{k}(\partial/\partial z)]f(r)$

$$= \mathbf{i}\, f'(r)\,(\partial r/\partial x) + \mathbf{j}\, f'(r)\,(\partial r/\partial y) + \mathbf{k}\, f'(r)\,(\partial r/\partial z)$$

$$= \mathbf{i}\, f'(r)\,(x/r) + \mathbf{j}\, f'(r)\,(y/r) + \mathbf{k}\, f'(r)\,(z/r), \text{ by (3) and (4)}$$

$$= (1/r) f'(r)\,(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) = (1/r) f'(r)\mathbf{r}.$$

$\therefore$ (2) $\Rightarrow$ $$\mathbf{q} = -f(r)\mathbf{i} - (x/r) f'(r)\mathbf{r}. \qquad \text{...(5)}$$

For a possible motion of an incompressible fluid, we have

$$\nabla\cdot\mathbf{q} = 0 \qquad \text{or} \qquad \nabla\cdot(-\nabla\phi) = 0 \qquad \text{or} \qquad \nabla^2\phi = 0$$

or $$(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial/\partial z^2)[x\, f(r)] = 0, \text{ using (1)} \qquad \text{...(6)}$$

Now, $$\frac{\partial^2}{\partial x^2}[x\, f(r)] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\{x\, f(r)\}\right] = \frac{\partial}{\partial x}\left[f(r) + x\frac{\partial f(r)}{\partial x}\right]$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} = 2\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2}$$

Also $$\frac{\partial^2}{\partial y^2}[x\, f(r)] = x\frac{\partial^2 f}{\partial y^2} \qquad \text{and} \qquad \frac{\partial^2}{\partial z^2}[x\, f(r)] = x\frac{\partial^2 f}{\partial z^2}$$

$\therefore$ (6) becomes $$2\frac{\partial f}{\partial x} + x\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) = 0. \qquad \text{...(7)}$$

Now, $$\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} = f'\frac{x}{r}, \text{ using (3).} \qquad \text{...(8)}$$

and $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(f'\frac{x}{r}\right) = \frac{f'}{r} + x\frac{\partial}{\partial x}\left(\frac{f'}{r}\right)$$

$$= \frac{f'}{r} + x\frac{d}{dr}\left(\frac{f'}{r}\right)\cdot\frac{\partial r}{\partial x} = \frac{f'}{r} + x\cdot\frac{rf'' - f'}{r^2}\cdot\frac{x}{r}.$$

$\therefore$ $$\frac{\partial^2 f}{\partial x^2} = \frac{f'}{r} + \frac{x^2}{r^2}f'' - \frac{x^2}{r^3}f' \qquad \text{...(9)}$$

Similarly, $$\frac{\partial^2 f}{\partial y^2} = \frac{f'}{r} + \frac{y^2}{r^2}f'' - \frac{y^2}{r^3}f' \qquad \text{...(10)}$$

and $$\frac{\partial^2 f}{\partial z^2} = \frac{f'}{r} + \frac{z^2}{r^2}f'' - \frac{z^2}{r^3}f' \qquad \text{...(11)}$$

Adding (9), (10) and (11), we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{3f'}{r} + \frac{x^2 + y^2 + z^2}{r^2}f'' - \frac{x^2 + y^2 + z^2}{r^3}f'$$

$$= \frac{3f'}{r} + f'' - \frac{f'}{r}, \text{ as } x^2 + y^2 + z^2 = r^2.$$

$\therefore$ $$\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 = 2f'/r + f'' \qquad \text{...(12)}$$

Using (8) and (12), (7) reduces to

$$\frac{2f'x}{r} + x\left(\frac{2f'}{r} + f''\right) = 0 \qquad \text{or} \qquad f'' + \frac{4f'}{r} = 0$$

or $$f''/f' + 4/r = 0.$$

Integrating $\quad \log f' + 4\log r = \log c_1 \quad$ so that $\quad f' = c_1 r^{-4}, \qquad$ ...(13)

Integrating (13), $\quad f = -(c_1/3) \times r^{-3} + c_2$, c_2 being an arbitrary constant $\qquad$ ...(14)

Substituting the values of f' and f from (13) and (14) in (5), we get

$$\mathbf{q} = -\{(c_1/3r^2) - c_2\}\mathbf{i} - (c_1 x/r^5)\,\mathbf{r} \qquad \text{...(15)}$$

Given that $\mathbf{q} \to 0$ as $r \to \infty$, hence (15) shows that $c_2 = 0$.

$\therefore$ from (15), $$\mathbf{q} = \frac{c_1}{3r^3}\left(\mathbf{i} - \frac{3x\mathbf{r}}{r^2}\right) \qquad \text{...(16)}$$

Now, $$q^2 = \mathbf{q}\cdot\mathbf{q} = \frac{c_1^2}{9r^6}\left(\mathbf{i} - \frac{3x\mathbf{r}}{r^2}\right)\cdot\left(\mathbf{i} - \frac{3x\mathbf{r}}{r^2}\right) = \frac{c_1^2}{9r^6}\left[\mathbf{i}\cdot\mathbf{i} - \frac{6x}{r^2}\mathbf{r}\cdot\mathbf{i} + \frac{9x^2}{r^4}\mathbf{r}\cdot\mathbf{r}\right]$$

$$= \frac{c_1^2}{9r^6}\left(1 - \frac{6x^2}{r^2} + \frac{9x^2r^2}{r^4}\right), \qquad \text{as} \quad \mathbf{r}\cdot\mathbf{r} = r^2 \quad \text{and} \quad \mathbf{r}\cdot\mathbf{i} = x$$

$$= \frac{c_1^2}{9r^6}\left(1 + \frac{3x^2}{r^2}\right) = \frac{c_1^2}{9r^8}(r^2 + 3x^2).$$

Hence the required surfaces of constant speed are

$q^2 = \text{constant}$ or $(c_1^2/9r^8)\,(r^2 + 3x^2) = \text{constant}$ or $(r^2 + 3x^2)\,r^{-8} = \text{constant}$.

Ex. 16. *What is the irrotational velocity field associated with the velocity potential* $\phi = 3x^2 - 3x + 3y^2 + 16t^2 + 12zt$. *Does the flow field satisfy the incompressible continuity equation?*

Sol. The velocity field is given by

$$u = -\frac{\partial\phi}{\partial x} = -\frac{\partial}{\partial x}(3x^2 - 3x + 3y^2 + 16t^2 + 12zt) = -6x + 3 \qquad \text{...(1)}$$

and $$v = -\frac{\partial\phi}{\partial y} = -\frac{\partial}{\partial y}(3x^2 - 3x + 3y^2 + 16t^2 + 12zt) = -6y. \qquad \text{...(2)}$$

Here $\quad \partial u/\partial x = -6 \quad$ and $\quad \partial v/\partial x = -6. \qquad$ ...(3)

The continuity equation for an incompressible fluid is

$$(\partial u/\partial x) + (\partial v/\partial y) = 0. \qquad \text{...(4)}$$

Using (3) in (4) we find $-6 - 6 = 0$, which is absurd. Hence the velocity field given by (1) and (2) does not satisfy the continuity equation (4).

Ex. 17. *The velocity potential function* ϕ *is given by* $\phi = -(xy^3/3) - x^2 + (x^3y/3) + y^2$. *Determine the velocity components in x and y directions and show that* ϕ *represents a possible case of flow.*

Sol. Here $u = -\partial\phi/\partial x = (y^3/3) + 2x - x^2 y,$ $\qquad v = -\partial\phi/\partial y = xy^2 - (x^3/3) - 2y.$

$\therefore \qquad \partial u/\partial x = 2 - 2xy \qquad$ and $\qquad \partial v/\partial y = 2xy - 2.$

Hence $\partial u/\partial x + \partial v/\partial y = 0,$ showing that the continuity equation is satisfied so ϕ represents a possible case of flow.

Ex. 18. *Prove that the velocity potentials* $\phi_1 = x^2 - y^2$ and $\phi_2 = r^{1/2}\cos(\theta/2)$ *are solutions of the Laplace equation and the velocity potential* $\phi_3 = (x^2 - y^2) + r^{1/2}\cos(\theta/2)$ *satisfies* $\nabla^2\phi_3 = 0.$

Sol. The Laplace's equation in cartesian and cylindrical polar coordinates are given by

$$\nabla^2\phi_1 = \frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} = 0 \qquad \text{and} \qquad \nabla^2\phi_2 = \frac{\partial^2\phi_2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi_2}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi_2}{\partial r} = 0,$$

Here $\quad \partial^2\phi_1/\partial x^2 = 2 \quad$ and $\quad \partial^2\phi_1/\partial y^2 = -2. \quad$ So $\nabla^2\phi_1 = 2 - 2 = 0. \qquad$ ...(1)

Next, $$\frac{\partial\phi_2}{\partial r} = \frac{1}{2}r^{-1/2}\cos\frac{\theta}{2}, \qquad \frac{\partial^2\phi_2}{\partial r^2} = -\frac{1}{4}r^{-3/2}\cos\frac{\theta}{2}, \qquad \frac{\partial^2\phi_2}{\partial\theta^2} = -\frac{r^{1/2}}{4}\cos\frac{\theta}{2}$$

$$\therefore \qquad \nabla^2\phi_2 = -\frac{1}{4r^{3/2}}\cos\frac{\theta}{2} - \frac{1}{4r^{3/2}}\cos\frac{\theta}{2} + \frac{1}{2r^{3/2}}\cos\frac{\theta}{2} = 0. \qquad \text{...(2)}$$

(1) and (2) show that ϕ_1 and ϕ_2 satisfy Laplace's equation.

Now, $$\phi_3 = (x^2 - y^2) + r^{1/2}\cos(\theta/2) = \phi_1 + \phi_2$$

$$\Rightarrow \qquad \nabla^2\phi_3 = \nabla^2(\phi_1 + \phi_2) = \nabla^2\phi_1 + \nabla^2\phi_2 = 0 + 0 = 0, \text{ by (1) and (2).}$$

Hence ϕ_3 satisfies $\nabla^2\phi_3 = 0.$

Ex. 19. *Find the vorticity of the fluid motion for the given velocity components :* (*i*) $u = A(x + y),\ v = -A(x + y),$ (*ii*) $u = 2Axz,\ v = A(c^2 + x^2 - z^2),$ (*iii*) $u = Ay^2 + By + c,\ v = 0,$ *Here A, B, C as constants.*

Sol. The vorticity vector $\boldsymbol{\Omega}$ is given by

$$\boldsymbol{\Omega} = (\partial w/\partial y - \partial v/\partial z)\mathbf{i} + (\partial u/\partial z - \partial w/\partial x)\mathbf{j} + (\partial v/\partial x - \partial u/\partial y)\mathbf{k} \qquad \text{...(1)}$$

(*i*) Using (1), $\qquad \boldsymbol{\Omega} = (0)\mathbf{i} + (0)\mathbf{j} + (-A - A)\mathbf{k} = -2A\mathbf{k}.$

(*ii*) Using (1), $\qquad \boldsymbol{\Omega} = (0 - 2Az)\mathbf{i} + (2Ax - 0)\mathbf{j} + (2Ax - 0)\mathbf{k} = 2A(-z\mathbf{i} + x\mathbf{j} + x\mathbf{k}).$

(*iii*) Using (1), $\qquad \boldsymbol{\Omega} = (0)\mathbf{i} + (0)\mathbf{j} + [0 - (2Ay + B)]\mathbf{k} = -(2Ay + B)\mathbf{k}.$

Ex. 20. *Find the vorticity in the spherical coordinates for the velocity components* $v_r = (1 - A/r^3)\cos\theta,\ v_\theta = -(1 + A/2r^3)\sin\theta,\ v_\phi = 0.$ *Here A is a constant. Find the nature of the fluid motion.*

Sol. Refer remark 4 of Art. 2.27. Let $\boldsymbol{\Omega}(\Omega_r, \Omega_\theta, \Omega_\phi)$ be the vorticity vector in the spherical polar coordinates (r, θ, ϕ). Then, we have

$$\Omega_r = \frac{1}{r}\frac{\partial v_\phi}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} + \frac{v_\phi}{r}\cot\theta = 0, \qquad \Omega_\theta = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r} = 0,$$

and $\Omega_\phi = \dfrac{\partial v_\theta}{\partial r} + \dfrac{v_\phi}{r} - \dfrac{1}{r}\dfrac{\partial v_r}{\partial \theta}$

$$= -\frac{\partial}{\partial r}\left(1+\frac{A}{2r^3}\right)\sin\theta - \frac{1}{r}\left(1+\frac{A}{2r^3}\right)\sin\theta - \frac{1}{r}\frac{\partial}{\partial \theta}\left[\left(1-\frac{A}{r^3}\right)cos\theta\right]$$

$$= \frac{3A\sin\theta}{2r^4} - \frac{1}{r}\left(1+\frac{A}{2r^3}\right)\sin\theta + \frac{1}{r}\left(1-\frac{A}{r^3}\right)\sin\theta = \left(\frac{3A}{2r^4} - \frac{1}{r} - \frac{A}{2r^4} + \frac{1}{r} - \frac{A}{r^4}\right)\sin\theta = 0.$$

Since $\Omega_r = \Omega_\theta = \Omega_\phi = 0$, the motion is irrotational.

Ex. 21. *If the fluid be in motion with a velocity potential* $\phi = z\log r$, *and if the density at a point fixed in space be independent of the time, show that the surfaces of equal density are of the forms* $r^2\{\log r - (1/2)\} - z^2 = f(\theta, \rho)$, *where* ρ *is the density at* (z, r, θ)

Sol. The surfaces of equal density are given by $\qquad \rho(z, r, \theta) = \text{constant}$

or $\qquad \dfrac{D\rho}{Dt} = 0 \qquad$ or $\qquad \dfrac{\partial \rho}{\partial t} + (\mathbf{q}\cdot\nabla)\rho = 0$

or $\qquad \dfrac{\partial \phi}{\partial r}\dfrac{\partial \rho}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial \phi}{\partial \theta}\dfrac{\partial \rho}{\partial \theta} + \dfrac{\partial \phi}{\partial z}\dfrac{\partial \rho}{\partial z} = 0 \qquad$ and $\qquad \mathbf{q} = -\nabla\phi. \qquad$ ...(1)

Also, given $\qquad \phi = z\log r. \qquad$ ...(2)

Then, using (2), (1) reduces to $\qquad z(\partial\rho/\partial r) + r\log r\,(\partial\rho/\partial z) = 0. \qquad$ ...(3)

(3) is of the form of *Lagrange's equation $Pp + Qq = R$ and so here Lagrange's subsidiary equations are

$$\frac{dr}{r} = \frac{dz}{r\log r} = \frac{d\rho}{0}. \qquad \text{...(4)}$$

Third fraction of (4) gives $\qquad d\rho = 0 \qquad$ so that $\qquad \rho = c_1, \qquad$ ...(5)

where c_1 is an arbitary constant.

Taking the first and the second fractions in (4), we have $\qquad 2r\log r\,dr - 2z\,dz = 0.$

Integrating, $\int 2r\log r\,dr - z^2 = c_2$ or $(\log r)\,r^2 - \int \frac{1}{r}\cdot r^2\,dr - z^2 = c_2$, integrating by parts

or $\qquad r^2\log r - (r^2/2) - z^2 = c_2$, c_2 being an arbitrary constant. $\qquad$ ...(6)

From (5) and (6), the solution of (3) is given by

$$r^2\{\log r - (1/2)\} - z^2 = f(\theta, \rho), \ f \text{ being an arbitrary function}$$

which are the surfaces of equal density.

EXERCISE 2(F)

1. Show that the following velocity field is a possible case of irrotational flow of an incompressible flow $u = yzt$, $v = zxt$, $w = xyt$.

2. Show that the equation of an incompressible fluid moving irrotationally is given by $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = 0$, where ϕ is the velocity potential.

3. λ denoting a variable parameter, and f a given function, find the condition that

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* Refer chapter 2 of part III in author's "Ordinary and partial differential equations" published by S. Chand & Co., New Delhi.

$f(x, y, \lambda) = 0$ should be a possible system of streamlines for steady irrotational motion in two dimensions.

4. Find the vorticity in polar coordinates for the following velocity components:

(*i*) $v_r = r\sin\theta, v_\theta = 2r\cos\theta$ (*ii*) $v_r = (A/r)\cos\theta, v_\theta = 0$ (*iii*) $v_r = A/r,, v_\theta = 0$

(*iv*) $v_r = \left(1 - A/r^2\right)\cos\theta,$ $v_\theta = -\left(1 + A/r^2\right)\sin\theta - (B/r).$

5. If $u = \dfrac{ax - by}{x^2 + y^2}$, $v = \dfrac{ay + bx}{x^2 + y^2}$, $w = 0$, investigate the nature of the motion of the liquid.

[**Ans.** Irrotational]

6. Establish the relation $\Omega = 2\,\omega$ connecting the angular velocity ω and the vorticity vector Ω.

(Meerut 2000, 2010)

7. Show that the equation of continuity reduces to Laplace's equation when the liquid is incompressible and irrotational.

[**Hint.** Since the motion is irrotational, there exists velocity potential ϕ such that $\mathbf{q} = -\nabla\phi$. Further $\partial\rho/\partial t = 0$ as the liquid is incompressible. Hence the equation of continuity

$\partial\rho/dt + \rho\nabla\cdot\mathbf{q} = 0$ reduces to $0 + \rho\nabla\cdot(-\nabla\phi) = 0$ or $\nabla^2\phi = 0$

i.e., $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = 0,$ which is Laplace's equation as required]

8. Prove that a surface of the form $ax^4 + by^4 + cz^4 - \chi(t) = 0$ is a possible form of a boundary surface of a homogeneous liquid at time t, the velocity potential of the liquid being

$$\phi = (\beta - \gamma)x^2 + (\gamma - \alpha)y^2 + (\alpha - \beta)z^2$$

where $\chi, \alpha, \beta, \gamma$ are given functions of time and a, b, c are suitable functions of time.

OBJECTIVE QUESTIONS ON CHAPTER 2

Multiple choice questions

Choose the correct alternative from the following questions

1. If the motion is irrotational, we have

(*i*) $\mathbf{w} = (1/2)\times curl\,\mathbf{q} = \mathbf{0}$ (*ii*) $\mathbf{w} = curl\,\mathbf{q} = \mathbf{0}$

(*iii*) $\mathbf{w} = div\,\mathbf{q} = \mathbf{0}$ (*iv*) None of these. **[Agra 2012; Kanpur 2003]**

2. The condition that the surface $F(x, y, z, t) = 0$ may be bounding surface is

(*i*) $DF/Dt = 1$ (*ii*) $DF/Dt = 0$

(*iii*) $DF/Dt = 2$ (*iv*) None of these **[Kanpur 2002, 2003]**

3. With usual notations

(*i*) $\mathbf{q} = -\nabla\,\phi$ (*ii*) $\mathbf{q} = \nabla\,\phi$

(*iii*) $|\,\mathbf{q}\,| = \nabla^2\,\phi$ (*iv*) None of these **[Kanpur 2003]**

4. Differential equations of the path lines are

(*i*) $dx/u = dy/v = dz/w$ (*ii*) $dx/dt = u, dy/dt = v$, $dz/dt = w$

(*iii*) $dx/\xi = dy/\eta = dz/\zeta$ (*iv*) None of these **[Kanpur 2002]**

5. Velocity potential ϕ satisfies the following equation

(*i*) Bernoulli (*ii*) Cauchy (*iii*) Laplace (*iv*) None of these

6. In usual notations, $\rho\dfrac{\partial(x, y, z)}{\partial(a, b, c)} = \rho_0$, is the equation of continuity in

(*i*) Cartesian coordinates (*ii*) Euler's form (*iii*) Lagrange's form (*iv*) None of these

7. Equation of continuity by Euler's method is

(*i*) $\partial\rho/\partial t+\rho\,\nabla\cdot\mathbf{q}=0$ (*ii*) $\partial\rho/\partial t-\rho\,\nabla\cdot\mathbf{q}=0$

(*iii*) $\partial\rho/\partial t+\nabla\cdot(\rho\mathbf{q})=0$ (*iv*) None of these. **[Kanpur 2001]**

8. The condition of continuity in cartesian coordinates is

(*i*) $\dfrac{\partial\rho}{\partial t}+\dfrac{\partial(\rho u)}{\partial x}+\dfrac{\partial(\rho v)}{\partial y}+\dfrac{\partial(\rho w)}{\partial z}=0$ (*ii*) $\dfrac{\partial(\rho u)}{\partial x}+\dfrac{\partial(\rho v)}{\partial y}+\dfrac{\partial(\rho w)}{\partial z}=0$

(*iii*) $\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}+\dfrac{\partial\rho}{\partial t}=0$ (*iv*) $\dfrac{\partial(\rho u)}{\partial x}+\dfrac{\partial(\rho v)}{\partial y}+\dfrac{\partial(\rho w)}{\partial z}-\dfrac{\partial\rho}{\partial t}=0$ **[Kanpur 2001]**

9. The curl of the velocity of any particle of a rigid body is equal to

(*i*) Twice the angular velocity (*ii*) The angular velocity

(*iii*) half the angular velocity (*iv*) Thrice the angular velocity. **[Kanpur 2001]**

10. If the surface is a rigid surface, then the condition is

(*i*) $u\dfrac{\partial F}{\partial x}+v\dfrac{\partial F}{\partial y}+w\dfrac{\partial F}{\partial z}=0$ (*ii*) $u\dfrac{\partial F}{\partial x}+v\dfrac{\partial F}{\partial y}-w\dfrac{\partial F}{\partial z}=0$

(*iii*) $u\dfrac{\partial F}{\partial x}-v\dfrac{\partial F}{\partial y}+w\dfrac{\partial F}{\partial z}=0$ (*iv*) $-u\dfrac{\partial F}{\partial x}+v\dfrac{\partial F}{\partial y}+w\dfrac{\partial F}{\partial z}=0$ **[Kanpur 2001]**

11. The elementary mass in spherical polar co-ordinates is

(*i*) $\rho r\sin^2\theta\,dr\,d\theta\,d\phi$ (*ii*) $\rho r\sin^2\theta\,dr\,d\theta\,d\phi$

(*iii*) $\rho r^2\sin\theta\,dr\,d\theta\,d\phi$ (*iv*) None of these. **[Agra 2004]**

12. If $\mathbf{q}$ is velocity, then rotation is

(*i*) $\nabla\times\mathbf{q}$ (*ii*) $(\nabla\times\mathbf{q})/2$ (*iii*) $\nabla\cdot\mathbf{q}$ (*iv*) None of these. **[Agra 2004]**

13. For incompressible fluid, we have

(*i*) $div\,\mathbf{q}=0$ (*ii*) $d\mathbf{q}/dt=\mathbf{0}$ (*iii*) $D\mathbf{q}/Dt=\mathbf{0}$ (*iv*) None of these **[Agra 2004, 2010]**

14. *D/Dt* is known as

(*i*) Static differential operator (*ii*) Partial differential operator

(*iii*) Total differential operator (*iv*) Differentiation following the motion

15. Fluid motion may be studied by two different methods

(*i*) Lagrangian and Newtonian methods (*ii*) Lagrangian and Eulerian methods

(*iii*) Newtonian and Eulerian methods (*iv*) None of these. **[Agra 2002, 2010]**

16. The differential equations of vortex lines are given by

(*i*) $dx/x=dy/y=dz/z$ (*ii*) $dx/u=dy/v=dz/w$

(*iii*) $dx/\xi=dy/\eta=dz/\zeta$ (*iv*) $dx/dt=u,\ dy/dt=v,\ dz/dt=w.$ **[Agra 2002, 11]**

17. The equation of streamline is

(*i*) $\mathbf{q}\times d\mathbf{r}=\mathbf{0}$ (*ii*) $\mathbf{q}\cdot d\mathbf{r}=0$ (*iii*) $\mathbf{r}\cdot d\mathbf{q}=0$ (*iv*) None of them **[Kanpur 2004]**

18. The motion is along the curve s only and q is velocity at P. Then acceleration is

(*i*) $\partial q/\partial t+q(\partial q/\partial s)$ (*ii*) $q(\partial q/\partial t)$ (*iii*) $\partial q/\partial t$ (*iv*) None of them **[Kanpur 2005]**

19. Let $\mathbf{F}$ be a uniform vector function throughout a region R. Consider the conditions:

(*i*) $\mathbf{F}$ is conservative in R (*ii*) There exists a uniform differential scalar function ϕ such that $\mathbf{F}=\nabla\phi$. Choose the correct statement:

(*a*) Condition (*i*) is necessary for (*ii*) (*b*) Condition (*ii*) is necessary for (*i*)
(*c*) Neither (*i*) is necessary for (*ii*) nor (*ii*) is necessary for (*i*)
(*d*) (*i*) and (*ii*) are equivalent to each other **[Agra 2005]**

20. For incompressible flow, we have
(*a*) div $\mathbf{q} = 0$ (*b*) $Dp/Dt = 0$ (*c*) $d\rho/dt = 0$ (*d*) None of these **[Agra 2005]**

21. For an irrotational flow
(*a*) div $\mathbf{q} = 0$ (*b*) div $\mathbf{q} \neq 0$ (*c*) curl $\mathbf{q} = \mathbf{0}$ (*d*) curl $\mathbf{q} \neq \mathbf{0}$ **[Agra 2007]**

22. For a rotational flow
(*a*) div $\mathbf{q} = 0$ (*b*) div $\mathbf{q} \neq 0$ (c) curl $\mathbf{q} = \mathbf{0}$ (*d*) curl $\mathbf{q} \neq \mathbf{0}$ **[Agra 2008]**

23. Equation of continuity (vector form) by Euler's method for an incompressible and heterogeneous fluid is
(*a*) $\partial \rho/\partial t + \nabla . (\rho \mathbf{q}) = 0$ (*b*) $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$
(*c*) $D\rho/Dt + \rho(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z) = 0$ (*d*) None of these. **[Agra 2008]**

24. Differential equation of path lines are
(*a*) $dx/dt = u,\ dy/dt = v,\ dz/dt = w$ (*b*) $(dx)/u = (dy)/v = (dz)/w$
(*c*) $du/dt = x,\ dv/\mathrm{d}t = y,\ dw/dt = z$ (*c*) $(du)/x = (dv)/y = (dw)/\mathrm{z}$ **[Agra 2008, 09]**

True or False. *Write 'T' for true and 'F' for false statement*:

25. Equation of continuity by Euler's method is $\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{q}) = 0$ **[Agra 2011; Kanpur 2001]**

26. In usual notations, acceleration of a fluid particle is $\partial\mathbf{q}/\partial t$.

27. The velocity potential exists only for irrotational motion. **[Agra 2004]**

Fill in the blank. *Fill in the blanks correctly*

28. Streamlines and path lines become the same when the motion is ....

29. $\dfrac{\partial\rho}{\partial t} + \dfrac{1}{r}\dfrac{\partial}{\partial r}(\rho r q_r) + \dfrac{1}{r}\dfrac{\partial}{\partial\theta}(\rho\, q_\theta) + \dfrac{\partial}{\partial z}(\rho\, q_z) = 0$ is the equation of continuity in ....

Answers/Hints to objective type questions

1. (*i*). See Art. 2.30 and Art. 2.31
2. (*ii*). See Eq.(8) Art. 2.18
3. (*i*). See Eq. (4), Art. 2.26
4. (*iii*). See Art. 2.21,
5. (*iii*). See remark 3, Art. 2.26
6. (*iii*). See Eq. (5), Art. 2.12 A
7. (*iii*). See Eq. (8), Art. 2.7
8. (*i*). See Eq. (8), Art. 2.9
9. (*i*). See Eq. (3), Art. 2.31
10. (*i*). See Eq. (11), Art. 2.18
11. (*iii*). See Art. 2.11,
12. (*i*).
13. (*i*). See Cor. 2, Art. 2.8
14. (*iv*). See Art. 2.4
15. (*iii*). See Art. 2.1,
16. (*iii*). See Art. 2.28. Here $\Omega_x = \xi, \Omega_y = \eta, \Omega_z = \zeta$
17. (*i*). See Art. 2.20
18. (*i*)
19. (*d*)
20. (*c*). See Cor. 2, Art. 2.8
22. (*c*). Refer Art. 2.30
23. (*d*). Refer Art. 2.30
23. (*a*). Refer Art. 2.8
24. (*b*). Refer Art. 2.20
25. *T*. See Eq. (3), Art. 2.8
26. *F*. See Art. 2.5*A*
27. *T*. See Art. 2.30
28. Steady. See Art. 2.23,
29. Cylindrical coordinates (r, θ, z). See Eq. (8), Art 2.10

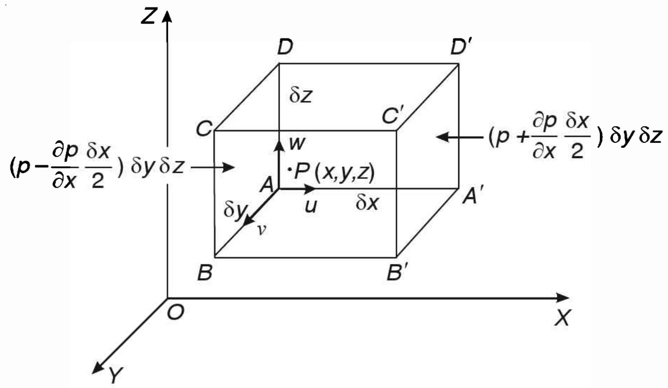
CHAPTER

3 Equations of Motion of Inviscid Fluids

3.1. Euler's equations of motion.

[Meerut 2012; Kanpur 1999; 02, 04; Agra 2005; Garhwal 2001, 05; G.N.D.U. Amritsar 1999; Rohilkhand 2001; Rajasthan 1997, 98; U.P.P.C.S. 1998, Purvanchel 2004; Kurukshetra 1997]

Let p be the pressure and ρ be density at a point P (x, y, z) in an inviscid (perfect) fluid. Consider an elementary parallelepiped with edges of lengths $\delta x, \delta y, \delta z$ parallel to their respective coordinate axes having P at its centre as shown in figure. Let (u, v, w) be the components of velocity and (X, Y, Z) be the components of external force per unit mass at time t at P. Then if $p = f(x, y, z)$, we have



Force on the plane through P parallel to $ABCD = p\,\delta y\,\delta z$.

$\therefore$ Force on the face $ABCD = f\left(x - \frac{1}{2}\delta x, y, z\right)\delta y\,\delta z$

$$= \left\{f - \frac{1}{2}\delta x \frac{\partial f}{\partial x} + \ldots\right\}\delta y\,\delta z, \text{ expanding by Taylor's theorem}$$

and force on the face $A'B'C'D' = f\left(x + \frac{1}{2}\delta x, y, z\right)\delta y\,\delta z = \left\{f + \frac{1}{2}\delta x \frac{\partial f}{\partial x} + \ldots\right\}\delta y\,\delta z$

$\therefore$ The net force in x-direction due to forces on $ABCD$ and $A'B'C'D'$

$$= \left\{f - \frac{1}{2}\delta x \frac{\partial f}{\partial x} + \ldots\right\}\delta y\,\delta z - \left\{f + \frac{1}{2}\delta x \frac{\partial f}{\partial z} + \ldots\right\}\delta y\,\delta z$$

$= -(\partial f/\partial x)\,\delta x\,\delta y\,\delta z$, to first order of approximation

$= -(\partial p/\partial x)\,\delta x\,\delta y\,\delta z$, as $p = f(x, y, z)$

The mass of the element is $\rho\,\delta x\,\delta y\,\delta z$. Hence the external force on the element in x-direction is $X\rho\,\delta x\,\delta y\,\delta z$. Also we know that Du/Dt is the total acceleration of the element in x-direction.

By Newton's second law of motion, the equation of motion in x-direction is

Mass × (acceleration in x-direction) = Sum of the components of external forces in x-direction.

i.e. $$\rho\,\delta x\,\delta y\,\delta z\,\frac{Du}{Dt} = X\rho\,\delta x\,\delta y\,\delta z - \frac{\partial p}{\partial x}\delta x\,\delta y\,\delta z$$

or $$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}. \qquad \ldots(1)$$

Similarly, the equations of motion in y and z-directions are, respectively

$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \ldots(2)$$

and $$\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \ldots(3)$$

Re-writing (1), (2) and (3), the so called *Euler's dynamical equations* of motion in cartesian coordinates are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \ldots(4)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \ldots(5)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \ldots(6)$$

Alternativ form (Vector method). **[Delhi 1997, Punjab 2003, Kurnkshetra 2000]**

Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ and $F = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$. Then, since

$$\mathbf{i}(\partial p/\partial x) + \mathbf{j}(\partial p/\partial y) + \mathbf{k}(\partial p/\partial z) = \nabla p,$$

(1), (2) and (3) may be combined to yield $$\frac{D\mathbf{q}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p, \qquad \ldots(7)$$

which is called the *Euler's equation of motion*.

But $$\frac{D\mathbf{q}}{Dt} = \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} \qquad \ldots(8)$$

Using (8), (7) may be re-written as

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad \ldots(9)$$

Again, $$\nabla(\mathbf{q}\cdot\mathbf{q}) = 2[\mathbf{q}\times \text{curl}\,\mathbf{q} + (\mathbf{q}\cdot\nabla)\,\mathbf{q}]$$

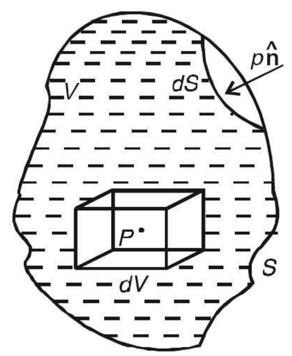
so that $$(\mathbf{q}\cdot\nabla)\,\mathbf{q} = (1/2)\times\nabla\mathbf{q}^2 - \mathbf{q}\times\text{curl}\,\mathbf{q} \qquad \ldots(10)$$

Using (10), (9) takes the form

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{q}^2\right) - \mathbf{q}\times\text{curl}\,\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p$$

or $$\frac{\partial \mathbf{q}}{\partial t} - \mathbf{q}\times\text{curl}\,\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p - \frac{1}{2}\nabla q^2. \qquad \ldots(11)$$

3.1A. The equation of motion of an inviscid fluid (Vector method)



Consider any arbitrary closed surface S drawn in the region occupied by the incompressible fluid and moving with it, so that it contains the same fluid particles at every instant.

By Newton's second law of motion,

the total force acting on this mass of fluid

= the rate of change in linear momentum ...(1)

The mass of fluid under consideration is subjected to the following two forces : (*i*) The normal pressure thrusts on the boundary.

(*ii*) The external force $\mathbf{F}$ (say) per unit mass.

Let ρ be the density of the fluid particle P within the closed surface and let dV be the volume enclosing P. The mass of element ρdV will always remain constant. Let $\mathbf{q}$ be the velocity of fluid particle P, then the momentum $\mathbf{M}$ of the volume V is given by

$$\mathbf{M} = \int_V \mathbf{q}\,\rho\, dV, \qquad \ldots(2)$$

where the integral has been taken over the entire volume V.

The time rate of change of linear momentum is given by differentiating (2) w.r.t. 't' as

$$\frac{D\mathbf{M}}{Dt} = \frac{D}{Dt}\int_V \mathbf{q}\,\rho\, dV = \int_V \frac{D\mathbf{q}}{Dt}\rho\, dV + \int_V \mathbf{q}\,\frac{D}{Dt}(\rho\, dV)$$

or
$$\frac{D\mathbf{M}}{Dt} = \int_V \frac{D\mathbf{q}}{Dt}\rho\, dV, \qquad \ldots(3)$$

noting that the second integral vanishes because the mass ρdV remains constant for all time.

Here D/Dt is the well known material derivative (or differentiation following the motion of the fluid) and is given by
$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla. \qquad \ldots(4)$$

If $\mathbf{F}$ be the external force per unit mass acting on fluid particle P, then the total force on the volume V is given by
$$\int_V \mathbf{F}\,\rho\, dV. \qquad \ldots(5)$$

Finally, if p be the normal pressure thrust at a point of the surface element dS, the total force on the surface S

$= \int_S p\,(-\hat{\mathbf{n}})\, dS$, (negative sign is taken because surface force acts inwards and $\hat{\mathbf{n}}$ is unit vector along the outward normal)

$= -\int_V \nabla p\, dV$, by Gauss theorem

$\therefore$ The total force acting on the volume $V = \int_V \mathbf{F}\,\rho\, dV - \int_V \nabla p\, dV = \int_V (\mathbf{F}\rho - \nabla p)\, dV.$ ...(6)

By Newton's second law as stated in (1), we have

$$\int_V (\mathbf{F}\rho - \nabla p)\, dV = \int_V \frac{D\mathbf{q}}{Dt}\rho\, dV \qquad \text{or} \qquad \int_V \left(\rho\frac{D\mathbf{q}}{Dt} - \rho\mathbf{F} + \nabla p\right) dV = 0. \qquad \ldots(7)$$

Since the volume V enclosed by surface S is arbitrary, (7) holds if

$$\rho \frac{D\mathbf{q}}{Dt} - \rho \mathbf{F} + \nabla p = \mathbf{0} \qquad \text{or} \qquad \frac{D\mathbf{q}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p, \qquad \ldots(8)$$

which is known as *Euler's equation of motion*. It is also known as the *equation of motion by flux method*.

Deduction of Lamb's hydrodynamical equations.

Using (4), (8) may be rewritten as

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\,\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p. \qquad \ldots(9)$$

But $\nabla\,(\mathbf{q}\cdot\mathbf{q}) = 2\,[\mathbf{q}\times \text{curl}\,\mathbf{q} + (\mathbf{q}\cdot\nabla)\,\mathbf{q}]$

so that $(\mathbf{q}\cdot\nabla)\,\mathbf{q} = (1/2)\times\nabla\,(\mathbf{q}\cdot\mathbf{q}) - \mathbf{q}\times\text{curl}\,\mathbf{q}$

or $(\mathbf{q}\cdot\nabla)\,\mathbf{q} = \nabla\,(\mathbf{q}^2/2) + \text{curl}\,\mathbf{q}\times\mathbf{q}.$ ...(10)

Using (10), (9) reduces to

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\,(\mathbf{q}^2/2) + (\text{curl}\,\mathbf{q})\times\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p. \qquad \ldots(11)$$

Now, the vorticity vector Ω is given by $\Omega = \text{curl}\,\mathbf{q}$. ...(12)

Using (12), (11) may be rewritten as

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\,(\mathbf{q}^2/2) + \Omega\times\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p, \qquad \ldots(13)$$

which is known as *Lamb's hydrodynamical equation*. The main advantage of it lies in the fact that it is invariant under a change of co-ordinate system.

3.1B. Conservative field of force.

In a conservative field of force, the work done by the force $\boldsymbol{F}$ of the field in taking a unit mass from one point to the other is independent of the path of motion.

Thus, if $\mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$, then a scalar point function $V(x, y, z)$ exists such that

$$Xdx + Ydy + Zdz = -\,dV \qquad \text{or} \qquad \mathbf{F} = -\nabla V$$

so that $X = -\partial V/\partial x, \quad Y = -\partial V/\partial y, \quad Z = -\partial V/\partial z.$

V is said to be **force potential** and it measures the potential energy of the field.

3.2A. Euler's equations of motion in cylindrical coordinates.

Euler's equation of motion is $\dfrac{D\mathbf{q}}{Dt} = \mathbf{F} - \dfrac{1}{\rho}\nabla p$ ...(1)

Let (q_r, q_θ, q_z) be the velocity components and (F_r, F_θ, F_z) be the components of external force in r, θ, z directions. Then we know that

$$\frac{D\boldsymbol{q}}{Dt} = \left(\frac{Dq_r}{Dt} - \frac{q_\theta^2}{r}, \frac{Dq_\theta}{Dt} + \frac{q_r q_\theta}{r}, \frac{Dq_z}{Dt}\right), \qquad \mathbf{F} = (F_r, F_\theta, F_z), \qquad \nabla p = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{\partial p}{\partial z}\right).$$

Substituting in (1), and equating the coefficients of **i**, **j**, **k**, we obtain Euler's equations of motion in cylindrical coordinates as:

$$\left.\begin{aligned} \frac{Dq_r}{Dt}-\frac{q_\theta^2}{r}&=F_r-\frac{1}{\rho}\frac{\partial p}{\partial r}\\ \frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}&=F_\theta-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}\\ \frac{Dq_z}{Dt}&=F_z-\frac{1}{\rho}\frac{\partial p}{\partial z}\end{aligned}\right\}, \qquad \ldots(2)$$

where $$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial t}+\frac{q_\theta}{r}\frac{\partial}{\partial\theta}+q_z\frac{\partial}{\partial z}. \qquad \ldots(3)$$

3.2B. Euler's equations of motion in spherical coordinates. [Garhwal 2005]

Euler's equation of motion is $$\frac{D\mathbf{q}}{Dt}=\mathbf{F}-\frac{1}{\rho}\nabla p \qquad \ldots(1)$$

Let (q_r, q_θ, q_ϕ) be the velocity components and (F_r, F_θ, F_ϕ) be the components of external force in r, θ, ϕ directions. Then we know that

$$\frac{D\mathbf{q}}{Dt}=\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2+q_\phi^2}{r}, \frac{Dq_\theta}{Dt}-\frac{q_\phi^2\cot\theta}{r}+\frac{q_r q_\theta}{r}, \frac{Dq_\phi}{Dt}+\frac{q_\theta q_\phi\cot\theta}{r}\right)$$

$\mathbf{F}=(\mathrm{F}_r, \mathrm{F}_\theta, \mathrm{F}_\phi)$, and $\nabla p=\left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial r}, \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\right)$.

Substituting in (1) and equating the coefficients of **i**, **j**, **k** we obtain Euler's equations of motion in spherical polar coordinates as :

$$\left.\begin{aligned} \frac{Dq_r}{Dt}-\frac{q_\theta^2+q_\phi^2}{r}&=F_r-\frac{1}{\rho}\frac{\partial p}{\partial r}\\ \frac{Dq_\theta}{Dt}-\frac{q_\phi^2\cot\theta}{r}+\frac{q_r q_\theta}{r}&=F_\theta-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}\\ \frac{Dq_\phi}{Dt}+\frac{q_\theta q_\phi\cot\theta}{r}&=F_\theta-\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \phi}\end{aligned}\right\}, \qquad \ldots(2)$$

where $$\frac{D}{Dt}=\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial r}+\frac{q_\theta}{r}\frac{\partial}{\partial\theta}+\frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \qquad \ldots(3)$$

3.2C. An important theorem.

If the motion of an ideal fluid, for which density is a function of pressure p only, is steady and the external forces are conservative, then there exists a family of surfaces which contain the streamlines and vortex lines.

Proof. Euler's equation in vector form is given by (Refer equation (11), Art. 3.1)

$$\frac{\partial \mathbf{q}}{\partial t}-\mathbf{q}\times\operatorname{curl}\mathbf{q}=\mathbf{F}-\frac{1}{\rho}\nabla p-\frac{1}{2}\nabla q^2 \qquad \ldots(i)$$

For steady flow, $\partial\mathbf{q}/\partial t=\mathbf{0}$.

Since the external forces are conservative, there exists force potential V such that $\mathbf{F}=-\nabla V$. Further, density being a function of pressure p only, there must be a function P such that $\nabla P=(1/\rho)\nabla p$. Using these facts, (i) reduces to

$$\nabla(V + P + q^2/2) = \mathbf{q} \times \text{curl}\, \mathbf{q} \qquad \ldots(ii)$$

Let $\qquad \Omega = \text{curl}\, \mathbf{q} = \text{vorticity vector.}$

Then $$\nabla(V + P + q^2/2) = \mathbf{q} \times \Omega \qquad \ldots(iii)$$

Let $$\mathbf{n} = \nabla(V + P + q^2/2) \qquad \ldots(iv)$$

Then (*iii*) reduces to $$\mathbf{n} = \mathbf{q} \times \Omega \qquad \ldots(v)$$

From (*v*), we get $$\mathbf{n} \cdot \mathbf{q} = (\mathbf{q} \times \Omega) \cdot \mathbf{q} = (\mathbf{q} \times \mathbf{q}) \cdot \Omega = 0$$

and $$\mathbf{n} \cdot \Omega = (\mathbf{q} \times \Omega) \cdot \Omega = \mathbf{q} \cdot (\Omega \times \Omega) = 0.$$

These results show that $\mathbf{n}$ is perpendicular to both $\mathbf{q}$ and Ω.

Since ∇f is perpendicular everywhere to the surface f = constant, (*iv*) shows that $\mathbf{n}$ is perpendicular to the family of surfaces

$$V + P + q^2/2 = C. \qquad \ldots(vi)$$

Thus $\mathbf{q}$ and Ω are both tangential to the surfaces (*vi*). Hence (*vi*) contains the streamlines and vortex lines.

Another Form : *Prove that for steady motion of an inviscid isotropic fluid* $p = f(\rho)$, $\int \frac{dp}{\rho} + \frac{1}{2} q^2 + \Omega = const.$ *over a surface containing the streamlines and vortex lines. Comment on the nature of this constant.*

3.3. Working rule for solving problems.

(*i*) Read and remember all equations of motion given in Art. 3.1, 3.1A, 3.2A and 3.2B. Use an appropriate one in the given problem.

(*ii*) Read and remember all equations of continuity given in Art. 2.8 to 2.14 of chapter 2. Use an appropriate one in the given problem.

(*iii*) Physical relations connecting p and ρ may be used. If the given fluid is at constant temperature, then use $p = k\rho$, where k is a constant. When the change is adiabatic, the relation $p = k\rho^{\gamma}$ is used.

(*iv*) Given initial and boundary conditions are used.

3.4. Illustrative solved examples.

Ex. 1. *A sphere of radius R, whose centre is at rest, vibrates radically in an infinite incompressible fluid of density* ρ, *which is at rest at infinity. If the pressure at infinity is* Π, *show that the pressure at the surface of the sphere at time t is*

$$\Pi + \frac{1}{2}\rho\left\{\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right\}.$$ **[Kanpur 2008; Meernt 2007; Bombay 2000; I.A.S. 1996]**

If $R = a\,(2 + \cos nt)$, show that, to prevent cavitation in the fluid, Π must not be less than $3\rho a^2 n^2$.

Sol. Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the centre. Hence the free surface would be spherical. Thus the fluid velocity v' will be radial and hence v' will be function of r' (the radial distance from the centre of the sphere which is taken as origin), and time t only. Let p be pressure at a distance r'. Let P be the pressure on the surface of the sphere of radius R and V be the velocity there. Then the equation of continuity is

$$r'^2 v' = R^2 V = F(t) \qquad \ldots(1)$$

From (1), $$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \qquad \ldots(2)$$

Again equation of motion is $$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)} \qquad \ldots(3)$$

Integrating with respect to r', (3) reduces to

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C, \ C \text{ being an arbitrary constant}$$

When $r' = \infty$, then $v' = 0$ and $p = \Pi$ so that $C = \Pi/\rho$. Then, we get

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\Pi - p}{\rho} \qquad \text{or} \qquad p = \Pi + \frac{1}{2}\rho\left[2\frac{F'(t)}{r'} - v'^2\right]. \qquad \ldots(4)$$

But $p = P$ and $v' = V$ when $r' = \mathrm{R}$. Hence (4) gives

$$\mathrm{P} = \Pi + \frac{1}{2}\rho\left[\frac{2}{\mathrm{R}}\{F'(t)\}_{r'=R} - \mathrm{V}^2\right] \qquad \ldots(5)$$

Also $V = dR/dt$. Hence using (1), we have

$$\{F'(t)\}_{r'=R} = \frac{d}{dt}(R^2 V) = \frac{d}{dt}\left(R^2\frac{d\mathrm{R}}{dt}\right) = \frac{d}{dt}\left(\frac{R}{2}\cdot\frac{dR^2}{dt}\right)$$

$$= \frac{R}{2}\frac{d^2R^2}{dt^2} + \frac{1}{2}\frac{dR^2}{dt}\frac{dR}{dt} = \frac{R}{2}\frac{d^2R^2}{dt^2} + R\left(\frac{dR}{dt}\right)^2$$

Using the above values of V and $\{F'(t)\}_{r'=R}$, (5) reduces to

$$P = \Pi + \frac{1}{2}\rho\left[\frac{2}{R}\left\{\frac{R}{2}\frac{d^2R^2}{dt^2} + R\left(\frac{dR}{dt}\right)^2\right\} - \left(\frac{dR}{dt}\right)^2\right]$$

or $$P = \Pi + \frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right]. \qquad \ldots(6)$$

Second Part: From $r'^2 v' =$ constant, we conclude that v' is maximum when r' is minimum *i.e.* $r' = R$. So pressure is minimum on $r' = R$ by using Bernoulli's theorem [Refer Chapter 4].

Given $$R = a\,(2 + \cos nt) \qquad \ldots(7)$$

$\therefore$ $$dR/dt = -\,an \sin nt$$

and $$dR^2/dt = 2a^2\,(2 + \cos nt)\,(-\,n \sin nt)$$

$\therefore$ $$d^2R^2/dt^2 = -\,2a^2n^2\,(2 + \cos nt)\cos nt + 2a^2n^2 \sin^2 nt$$

With the above values, (6) reduces to

$$P = \Pi + (3/2) \times \rho a^2 n^2 \sin^2 nt - a^2 n^2 \rho\,(2\cos nt + \cos^2 nt) \qquad \ldots(8)$$

From (7), R varies from $3a$ to a. Thus the sphere has the greatest radius $3a$ when $nt = 0$ or $2m\pi$. Clearly as the sphere shrinks from $R = 3a$, there is a possibility of a cavitation there because pressure would be minimum there. Hence the minimum value of pressure p' (say) on the surface of the sphere is given by replacing $t = 0$ or $nt = 2m\pi$ in (8). We thus obtain

$$P' = \Pi - 3\rho a^2 n^2. \qquad \ldots(9)$$

To prevent cavitation in the fluid. P' given by (9) must be positive *i.e.* Π must not be less than $3\rho a^2 n^2$.

Ex. 2. *An infinite mass of homogeneous incompressible fluid is at rest subject to a uniform pressure* Π *and contains a spherical cavity of radius a, filled with a gas at pressure* $m\Pi$; *prove that if the inertia of the gas be neglected, and Boyle's law be supposed to hold throughout the ensuing motion, the radius of the sphere will oscillate between the values a and na, where n is determined by the equation* $1 + 3m \log n - n^3 = 0$. **(Kanpur 2010)**

If m be nearly equal to 1, *the time of an oscillation will be* $2\pi\sqrt{(a^2\rho/3\pi)}$, ρ *being the density of the fluid.* **[Kanpur 2008; Agra 1998; I.A.S. 1994; Meernt 1999]**

Sol. As in Ex. 1, let at any time t, v' be the velocity at a distance t' and p' the pressure there. Also let v be the velocity at a distance r and p the pressure there. Then the equation of continuity is

$$r'^2 v' = F(t) = r^2 v \qquad \ldots(1)$$

From (1),

$$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \qquad \ldots(2)$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p'}{\partial r'}$$

i.e.

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p'}{\partial r'}, \text{ using (2)} \qquad \ldots(3)$$

Integrating with respect to r', (3) gives

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = C - \frac{p'}{\rho}, \; C \text{ being an arbitrary constant}$$

When $r' = \infty$, then $v' = 0$, $p' = \Pi$ so that $C = \Pi/\rho$. Hence, the above equation yields

$$\therefore \qquad -\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\Pi - p'}{\rho} \qquad \ldots(4)$$

Since gas inside cavity obeys Boyle's law, we get

$$(4/3)\times \pi a^3 m\Pi = (4/3)\times \pi r^3 p \qquad \text{so that} \qquad p = (a^3 m\Pi)/r^3$$

When $r' = r$ then $v' = v$, $p' = p = (a^3 m\Pi)/r^3$. So (4) gives

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = \frac{\Pi}{\rho} - \frac{a^3 m\Pi}{\rho}\cdot\frac{1}{r^3} \qquad \ldots(5)$$

From (1), $\quad F'(t) = 2r\dfrac{dr}{dt}\cdot v + r^2\cdot\dfrac{dv}{dt} = 2rv^2 + r^2\dfrac{dv}{dr}\dfrac{dr}{dt}, \quad$ as $\quad v = \dfrac{dr}{dt}$

or

$$F'(t) = 2rv^2 + r^2 v\,(dv/dr)$$

Hence (5) reduces to

$$-\frac{1}{r}\left(2rv^2 + r^2 v\frac{dv}{dr}\right) + \frac{1}{2}v^2 = \frac{\Pi}{\rho} - \frac{a^3 m\Pi}{\rho}\cdot\frac{1}{r^3}$$

or
$$rv\frac{dv}{dr} + \frac{3}{2}v^2 = -\frac{\Pi}{\rho} + \frac{a^3 m\Pi}{\rho r^3} \qquad \ldots(6)$$

Multiplying both sides of (6) by $2r^2 dr$, we get

$$2r^3 v dv + 3r^2 v^2 dr = \left(-\frac{2\Pi r^2}{\rho} + \frac{2a^3 m\Pi}{\rho r}\right)dr \quad \text{or} \quad d(r^3 v^2) = \left(-\frac{2\Pi r^2}{\rho} + \frac{2a^3 m\Pi}{\rho r}\right)dr$$

Integrating, $r^3 v^2 = -\dfrac{2\Pi}{3\rho}r^3 + \dfrac{2a^3 m\Pi}{\rho}\log r + C'$, C' being an arbitrary constant $\qquad \ldots(7)$

Initially, when $r = a$, then $v = 0$. Hence (7) $\Rightarrow$ $C' = \dfrac{2\Pi a^3}{3\rho} - \dfrac{2a^3 m\Pi}{\rho}\log a$

$\therefore$ From (7),
$$r^3 v^3 = \frac{2\Pi}{3\rho}(a^3 - r^3) + \frac{2a^3 m\Pi}{\rho}\log\left(\frac{r}{a}\right) \qquad \ldots(8)$$

Since the radius of the sphere oscillates between a and na, we have $v = 0$, when $r = a$ and $r = na$. Putting $v = 0$ and $r = na$ in (8), we have

$$0 = \frac{2\Pi}{3\rho}\left\{a^3 - n^3 a^3 + 3ma^3\log\left(\frac{na}{a}\right)\right\}$$

so that $\qquad 1 + 3m\log n - n^3 = 0, \qquad$ as $\qquad a \neq 0$

Second Part. Let m be nearly equal to 1. Then, we take $r = a + x$ where x is small. Again, $v = dr/dt = dx/dt = \dot{x}$. Hence, taking $m = 1$, (8) reduces to

$$(a + x)^3\dot{x}^2 = \frac{2\Pi}{3\rho}\left\{a^3 - (a+x)^3\right\} + \frac{2a^3\Pi}{\rho}\log\left(\frac{a+x}{a}\right)$$

or
$$a^3\left(1+\frac{x}{a}\right)^3\dot{x}^2 = \frac{2\Pi a^3}{3\rho}\left\{1 - \left(1+\frac{x}{a}\right)^3\right\} + \frac{2a^3\Pi}{\rho}\log\left(1+\frac{x}{a}\right)$$

or
$$\left(1+\frac{x}{a}\right)^3\dot{x}^2 = \frac{2\Pi}{3\rho}\left\{1 - \left(1 + \frac{3x}{a} + \frac{3x^2}{a^2} + \ldots\right)\right\} + \frac{2a^3\Pi}{\rho}\left\{\frac{x}{a} - \frac{1}{2}\frac{x^2}{a^2} + \ldots\right\}$$

or
$$\dot{x}^2 = \frac{2\Pi}{3\rho}\left(1+\frac{x}{a}\right)^{-3}\left[-\frac{9}{2}\frac{x^2}{a^2} + \ldots\right] = \frac{2\Pi}{3\rho}\left(1 - \frac{3x}{a} + \frac{6x^2}{a^2} - \ldots\right)\left[-\frac{9}{2}\frac{x^2}{a^2} + \ldots\right]$$

or
$$\dot{x}^2 = -[3\Pi x^2/\rho a^2], \text{ neglecting higher powers of } x$$

Differentiating the above relation with respect to t, we get

$$2\dot{x}\ddot{x} = -\frac{3\Pi}{\rho a}\cdot 2x\dot{x} \qquad \text{or} \qquad \ddot{x} = -\frac{3\Pi}{\rho a^2}x,$$

which represents the standard equation of simple harmonic motion and hence the required time of oscillation (*i.e.* periodic time) is given by

$$2\pi/\sqrt{(3\Pi/\rho a^2)} \qquad i.e. \qquad 2\Pi\sqrt{(\rho a^2/3\Pi)}.$$

Ex. 3. *A mass of gravitating fluid is at rest under its own attraction only, the free surface being a sphere of radius b and the inner surface a rigid concentric shell of radius a. Show that if the shell suddenly disappears, the initial pressure at any point of the fluid at distance r from the centre is* $\frac{2}{3}\pi\gamma\rho^2(b-a)(r-a)\left(\frac{a+b}{r}+1\right)$. **[Bombay 1998]**

Sol. As in Ex. 1, let at time t, v' be the velocity at a distance r' from the centre and let the radius of the inner spherical cavity be r. Let p be the pressure at a distance r'. Then the equation of continuity is

$$r'^2 v' = F(t) \qquad \ldots(1)$$

From (1),

$$\frac{dv'}{dt} = \frac{F'(t)}{r'^2} \qquad \ldots(2)$$

$$\text{Attraction at distance } r' = \frac{(4/3)\times\pi\gamma\rho(r'^3 - r^3)}{r'^2},$$

where γ is the usual constant of gravitation.

Hence the equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{4}{3}\pi\gamma\rho\left(r' - \frac{r^3}{r'^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

i.e.

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{4}{3}\pi\gamma\rho\left(r' - \frac{r^3}{r'^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)}$$

Integrating with respect to r', we obtain

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{4}{3}\pi\gamma\rho\left(\frac{r'^2}{2} + \frac{r^3}{r'}\right) - \frac{p}{\rho} + C, \ C \text{ being an arbitrary constant} \qquad \ldots(3)$$

Initially, when $t = 0$, then $v' = 0$, $r = a$ and p = P (say). Hence (3) yields

$$-\frac{F'(0)}{a} = -\frac{4}{3}\pi\gamma\rho\left(\frac{r'^2}{2} + \frac{a^3}{r'}\right) - \frac{P}{\rho} + C \qquad \ldots(4)$$

But, $P = 0$ when $r' = a$ and $r' = b$. So (4) gives

$$-\frac{F'(0)}{a} = -\frac{4}{3}\pi\gamma\rho\left(\frac{a^2}{2} + a^2\right) + C \qquad \ldots(5)$$

and

$$-\frac{F'(0)}{b} = -\frac{4}{3}\pi\gamma\rho\left(\frac{b^2}{2} + \frac{a^3}{b}\right) + C \qquad \ldots(6)$$

Subtracting (6) from (5), we have

$$F'(0)\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{4}{3}\pi\gamma\rho\left\{\frac{b^2 - a^2}{2} + a^2\left(\frac{a}{b} - 1\right)\right\}$$

$$\therefore \qquad F'(0)\frac{a-b}{ab} = \frac{4}{3}\pi\gamma\rho\left\{\frac{(b-a)(b+a)}{2} + \frac{a^2(a-b)}{b}\right\}$$

or $$F'(0) = -(2/3) \times \pi\gamma\rho ab(a+b) + (4/3) \times \pi\gamma\rho a^3 \qquad \ldots(7)$$

Multiplying (5) by a and (6) by b and then subtracting, we get

$$0 = \frac{4}{3}\pi\gamma\rho\left(\frac{b^3}{2} - \frac{a^3}{2}\right) + C\,(a-b)$$

or $$C\,(b-a) = (2/3)\times\pi\gamma\rho(b-a)\,(b^2+a^2+ba)$$

or $$C = (2/3)\times\pi\gamma\rho(a^2+b^2+ab) \qquad \ldots(8)$$

Putting the values of $F'(0)$ and C in (4), we get

$$-\frac{1}{r'}\left\{-\frac{2}{3}\pi\gamma\rho ab(a+b) + \frac{4}{3}\pi\gamma\rho a^3\right\} = -\frac{4}{3}\pi\gamma\rho\left(\frac{r'^2}{2} + \frac{a^3}{r'}\right) - \frac{P}{\rho} + \frac{2}{3}\pi\gamma\rho(a^2+b^2+ab)$$

$$\therefore \qquad P = \frac{2}{3}\pi\gamma\rho^2\left\{a^2+b^2+ab - \frac{ab\,(a+b)}{r'} - r'^2\right\}$$

or $$P = \frac{2}{3}\pi\gamma\rho^2(r'-a)\,(b-r')\left(\frac{a+b}{r'}+1\right). \qquad \ldots(9)$$

For the required result, replace r' by r in (9).

Ex. 4. *Liquid is contained between two parallel planes, the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated ; prove that if Π be the pressure at the outer surface, the initial pressure at any point on the liquid distant r from the centre is*

$$\Pi\frac{\log r - \log b}{\log a - \log b}.$$ **[Kanpur 2000; Meernt 2000; Agra 1995; I.A.S. 2006]**

Sol. Here the motion of the liquid will take place in such a manner so that each element of the liquid moves towards the axis of the cylinder $|\ z\ | = b$. Hence the free surface would be cylindrical. Thus the liquid velocity v' will be radial and v' will be function of r' (the radial distance from the centre of the cylinder $|\ z\ | = b$ which is taken as origin) and time t only. Let p be the pressure at a distance r'. Then the equation of continuity is

$$r'v' = F\,(t) \qquad \ldots(1)$$

From (1), $$\partial v'/\partial t' = F'(t)/r' \qquad \ldots(2)$$

The equation of motion is $$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{F'(t)}{r} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)}$$

Integrating, $$F'(t)\log r' + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C, \ C \text{ being an arbitrary constant} \qquad \ldots(3)$$

Initially when $t = 0$, $v' = 0$, $p = P$. So (3) $\Rightarrow$ $F'(0)\log r' = -(P/\rho) + C \qquad \ldots(4)$

Again, $P = \Pi$ when $r' = a$ and $P = 0$ when $r' = b$. So (3) yields

$\therefore \qquad F'(0)\log a = -(\Pi/\rho) + C \qquad$ and $\qquad F'(0)\log b = C \qquad \ldots(5)$

Solving (5) for $F'(0)$ and C, we have

$$C = -\log b \frac{\Pi}{\rho \log(a/b)}, \qquad F'(0) = -\frac{\Pi}{\rho\log(a/b)}.$$

Putting these values in (4), we get

$$\frac{P}{\rho} = \frac{\Pi}{\rho\log(a/b)}\log r' - \frac{\Pi}{\rho\log(a/b)}\log b$$

or
$$P = \Pi\frac{\log r' - \log b}{\log(a/b)} = \Pi\frac{\log r' - \log b}{\log a - \log b}. \qquad \ldots(6)$$

For the required result, replace r' by r in (6).

Ex. 5. *A mass of liquid of density* ρ *whose external surface is a long circular cylinder of radius a which is subject to a constant pressure* Π, *surrounds a coaxial long circular cylinder of radius b. The internal cylinder is suddenly destroyed; show that if v is the velocity at the internal surface, when the radius is r, then*

$$v^2 = \frac{2\Pi(b^2 - r^2)}{\rho r^2 \log\{(r^2 + a^2 - b^2)/r^2\}}.$$ **[Garhwal 2000, Meernt 2006, Kanpur 2011]**

Sol. When the inner cylinder is suddenly destroyed, the motion of the liquid will take place along the radii of the normal sections of the cylinder. Hence the velocity will be function of r' (the radial distance from the centre of the cylinder $|z| = a$ which is taken as origin) and time t only. Let p be the pressure at a distance r'. Then the equation of continuity is

$$r'v' = F(t) \qquad \ldots(1)$$

From (1), $$\partial v'/\partial t = F'(t)/r' \qquad \ldots(2)$$

The equation of motion is $$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{F'(t)}{r} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)}$$

Integrating, $$F'(t)\log r' + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C, \text{ being an arbitrary constant} \qquad \ldots(3)$$

Let r and R be the radii of the internal and external surfaces of the cylinder and let v and V be the velocities there at any time t. Hence, we have

When $\qquad r' = r, \qquad v' = v, \qquad p = 0 \qquad \ldots(4)$

and when $\qquad r' = R, \qquad v' = V, \qquad p = \Pi \qquad \ldots(5)$

Using (4) and (5), (2) reduces to

$$F'(t)\log r + v^2/2 = C \qquad \ldots(6)$$

and $$F'(t)\log R + V^2/2 = -\Pi/\rho + C \qquad \ldots(7)$$

Subtracting (7) from (6), we have

$$F'(t)\,(\log r - \log R) + (v^2 - V^2)/2 = \Pi/\rho \qquad \ldots(8)$$

From (1), $$rv = RV = F(t) \qquad \ldots(9)$$

But $v = dr/dt$ and $V = dR/dt$. So (9) becomes

$$2rdr = 2RdR = 2F(t)dt \qquad \ldots(10)$$

Also $$R^2 - r^2 = a^2 - b^2 \qquad \ldots(11)$$

From (9), $$F'(t) = \frac{d}{dt}(rv) = \frac{d}{dr}(rv)\cdot\frac{dr}{dt} = v\frac{d}{dr}(rv), \text{ as } v = \frac{dr}{dt}$$

Putting the values of $F'(t)$ and V, (8) gives

$$v\frac{d}{dr}(rv)\cdot\log\frac{r}{R} + \frac{1}{2}\left(v^2 - \frac{r^2v^2}{R^2}\right) = \frac{\Pi}{\rho} \quad \text{or} \quad rv\frac{d}{dr}(rv)\cdot\log\frac{r}{R} + \frac{1}{2}rv^2\left(1 - \frac{r^2}{R^2}\right) = \frac{\Pi r}{\rho}$$

or $$\frac{1}{2}\frac{d}{dr}\left\{(rv)^2\right\}\cdot\log\frac{r}{R} + \frac{1}{2}r^2v^2\left(\frac{1}{r} - \frac{r}{R^2}\right) = \frac{\Pi r}{\rho} \quad \text{or} \quad \frac{d}{dr}\left(\frac{1}{2}r^2v^2\log\frac{r}{R}\right) = \frac{\Pi r}{\rho}, \qquad \ldots(12)$$

where we have used (10) *i.e.* $RdR = rdr$.

Integrating (12), $$\frac{1}{2}r^2v^2\log\frac{r}{R} = \frac{\Pi r^2}{2\rho} + C', \quad C' \text{ being an arbitrary contant} \qquad \ldots(13)$$

But $v = 0$ when $r = b$. So $C' = -\pi b^2/2\rho$.

$\therefore$ From (13), $$r^2v^2\log\frac{r}{R} = \frac{\Pi}{\rho}(r^2 - b^2)$$

or $$r^2v^2\log\left(\frac{r}{R}\right)^2 = \frac{2\Pi}{\rho}(r^2 - b^2)$$

or $$v^2 = \frac{2\Pi(r^2 - b^2)}{\rho r^2\log(r^2/R^2)} = \frac{2\Pi(r^2 - b^2)}{\rho r^2\log(R^2/r^2)^{-1}} = -\frac{2\Pi(r^2 - b^2)}{\rho r^2\log(R^2/r^2)} = \frac{2\Pi(b^2 - r^2)}{\rho r^2\log(R^2/r^2)}$$

Thus, $$v^2 = \frac{2\Pi(b^2 - r^2)}{\rho r^2\log\left\{(r^2 + a^2 - b^2)/r^2\right\}}, \text{ using (11)}$$

Ex. 6. *A centre of force attracting inversely as the square of the distance is at the centre of a spherical cavity within an infinite mass of incompressible fluid, the pressure on which at an infinite distance is* Π *and is such that the work done by this pressure on a unit area through a unit of length is one-half the work done by the attractive force on a unit volume of the fluid from infinity to the initial boundary of the cavity; prove that the time filling up the cavity will be* $\pi a(\rho/\pi)^{1/2}\left\{2 - (3/2)^{3/2}\right\}$, *a being the initial radius of the cavity, and* ρ *the density of the fluid.*

[Meernt 1999]

Sol. At any time t, let v' be the velocity at a distance r' and p be the pressure there. Let r be the radius of the cavity at that time and v be the velocity there. Then equation of continuity is

$$r'^2v' = \mathrm{F}(t) = r^2v \qquad \ldots(1)$$

From (1), $$\partial v'/\partial t = F'(t)/r'^2 \qquad \ldots(2)$$

The equation of motion is $$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)}$$

Integrating, $$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=\frac{\mu}{r'}-\frac{p}{\rho}+C,$$ C beings an arbitrary constant

But $v'=0$ and $p=\Pi$ when $r'=\infty$. So $C=\Pi/\rho$. Hence the above equation yields

$$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=\frac{\mu}{r'}+\frac{\Pi-p}{\rho} \qquad \ldots(3)$$

Also $v'=v$ and $p=0$ when $r'=r$. So from (3), we get

$$-\frac{F'(t)}{r}+\frac{1}{2}v^2=\frac{\mu}{r}+\frac{\Pi}{\rho} \qquad \ldots(4)$$

From (1), $$F'(t)=\frac{d}{dt}(r^2v)=2r\frac{dr}{dt}\cdot v+r^2\frac{dv}{dt}=2rv\frac{dr}{dt}+r^2\frac{dv}{dr}\frac{dr}{dt}$$

$$=2rv^2+r^2v\frac{dv}{dr}, \qquad \text{as} \qquad v=\frac{dr}{dt}$$

Using the above value of F$'(t)$, (4) gives

$$-\frac{1}{r}\left\{2rv^2+r^2v\frac{dv}{dr}\right\}+\frac{1}{2}v^2=\frac{\mu}{r}+\frac{\Pi}{\rho} \qquad \text{or} \qquad 2rvdv+3v^2dr=-2\left(\frac{\mu}{r}+\frac{\Pi}{\rho}\right)dr$$

or $$2r^3vdv+3v^2r^2dr=-2r^2\left(\frac{\mu}{r}+\frac{\Pi}{\rho}\right)dr \qquad \text{or} \qquad d(r^3v^2)=-2\left(\mu r+\frac{\Pi}{\rho}r^2\right)dr$$

Integrating, $$r^3v^2=-\left(\mu r^2+\frac{2\Pi}{3\rho}r^3\right)+C',$$ C' being an arbitrary constant ...(5)

Initially, when $r=a$, $v=0$. So $C'=\mu a^2+(2\Pi/3\rho)a^3$.

$\therefore$ From (5), $$r^3v^2=\mu(a^2-r^2)+\frac{2\Pi}{3\rho}(a^3-r^3). \qquad \ldots(6)$$

Since the work done by Π is half the work done by the attractive force, we have

$$\Pi\times 1\times 1=\frac{1}{2}\int_\infty^a\left(-\frac{\mu}{r^2}\right)\rho\,dr \qquad \text{so that} \qquad \mu=\frac{2\Pi a}{\rho}.$$

Putting this value of μ in (6), we get

$$r^3v^3=\frac{2\Pi a}{\rho}(a^2-\rho^2)+\frac{2\Pi}{3\rho}(a^3-r^3)$$

or $$r^3v^2=\frac{2\Pi}{3\rho}\{3a(a^2-r^2)+a^3-r^3\} \qquad \text{or} \qquad v^2=\frac{2\Pi}{3\rho}\frac{\{3a(a^2-r^2)+a^3-r^3\}}{r^3}$$

or $$\frac{dr}{dt}=-\left(\frac{2\Pi}{3\rho}\right)^{1/2}\frac{\{3a(a^2-r^2)+a^3-r^3\}^{1/2}}{r^{3/2}} \qquad \ldots(7)$$

wherein negative sign is taken as r decreases when t increases.

Let T be the time of filling the cavity. Then we have, $r=a$ when $t=0$ and $r=0$ when $t=T$. Hence (7) gives on integration

$$\int_0^T dt = -\left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_a^0 \frac{r^{3/2} dr}{\{3a(a^2 - r^2) + a^3 - r^3\}^{1/2}}$$

$$\therefore \qquad T = \left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^a \frac{r^{3/2} dr}{(r + 2a)\sqrt{(a - r)}} \qquad \ldots(8)$$

Put $r = a\sin^2\theta$ so that $dr = 2a\sin\theta\cos\theta$. Then (8) reduces to

$$T = \left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^{\pi/2} \frac{a^{3/2}\sin^3\theta \cdot 2a\sin\theta\cos\theta\, d\theta}{a(2 + \sin^2\theta)\cdot a^{1/2}\cos\theta} = 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^{\pi/2} \frac{\sin^4\theta\, d\theta}{2 + \sin^2\theta}$$

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^{\pi/2} \left(\sin^2\theta - 2 + \frac{4}{2 + \sin^2\theta}\right) d\theta$$

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left[\frac{\pi}{4} - \pi + 4\int_0^{\pi/2} \frac{d\theta}{2 + \sin^2\theta}\right] = 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left[-\frac{3\pi}{4} + 4\int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{2\sec^2\theta + \tan^2\theta}\right]$$

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left[-\frac{3\pi}{4} + 4\int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{2 + 3\tan^2\theta}\right] = 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left[-\frac{3\pi}{4} + \frac{4}{3}\int_0^{\infty} \frac{dt}{(2/3) + t^2}\right]$$

[Putting $\tan\theta = t$ and $\sec^2\theta\, d\theta = dt$]

$$= 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left\{-\frac{3\Pi}{4} + \frac{4}{3}\cdot\sqrt{\frac{3}{2}}\left[\tan^{-1}\left(t\sqrt{\frac{3}{2}}\right)\right]_0^{\infty}\right\} = 2a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \left[-\frac{3\pi}{4} + \frac{4}{3}\left(\frac{3}{2}\right)^{1/2}\cdot\frac{1}{2}\pi\right]$$

$$= a\pi\left(\frac{\rho}{\Pi}\right)^{1/2} \left\{-\frac{3}{2}\times\left(\frac{3}{2}\right)^{1/2} + \frac{4}{3}\times\frac{3}{2}\right\} = \pi a\left(\frac{\rho}{\Pi}\right)^{1/2}\left[2 - \left(\frac{3}{2}\right)^{3/2}\right].$$

Ex. 7. *A spherical hollow of radius a initially exists in an infinite fluid, subject to constant pressure at infinity. Show that the pressure at distance* r' *from the centre when the radius of the cavity is r is to the pressure at infinity as* $3r^2r'^4 + (a^3 - 4r^3)r'^3 - (a^3 - r^3)r^3 : 3r^2r'^4$

[Garhwal 2000]

Sol. Let v' be the velocity at a distance r' at any time t and p be the pressure there. Again, let v be the velocity of the inner surface of radius r. Then the equation of continuity is

$$r'^2 v' = F(t) = r^2 v \qquad \ldots(1)$$

From (1), $\qquad \partial v'/\partial t = F'(t)/r'^2 \qquad \ldots(2)$

The equation of motion is $\qquad \dfrac{\partial v'}{\partial t} + v'\dfrac{\partial v'}{\partial r'} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$

or $\qquad \dfrac{F'(t)}{r'^2} + \dfrac{\partial}{\partial r'}\left(\dfrac{1}{2}v'^2\right) = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$, using (2)

Integrating, $$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C, C \text{ being an arbitrary constant} \qquad \ldots(3)$$

Let Π be the pressue at infinity. Thus $v' = 0$ and $p = \Pi$ when $r' = \infty$. So (3) gives $C = \Pi/\rho$. Then (3) reduces to

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\Pi - p}{\rho} \qquad \ldots(4)$$

But p = 0 and $v' = v$ when $r' = r$. Then (4) gives

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = \frac{\Pi}{\rho} \qquad \ldots(5)$$

From (1), $$F'(t) = \frac{d}{dt}(r^2 v) = 2r\frac{dr}{dt}v + r^2\frac{dv}{dt} = 2rv\frac{dr}{dt} + r^2\frac{dv}{dr}\frac{dr}{dt}$$

$$= 2rv^2 + r^2 v\frac{dv}{dr} \qquad \left[\because\ v = \frac{dr}{dt}\right]$$

Using the above value of $F'(t)$, (5) gives

$$-\frac{1}{r}\left\{2rv^2 + r^2 v\frac{dv}{dr}\right\} + \frac{1}{2}v^2 = \frac{\Pi}{\rho} \quad \text{or} \quad -rv\frac{dv}{dr} - \frac{3}{2}v^2 = \frac{\Pi}{\rho} \qquad \ldots(6)$$

Multiplying both sides by $(-\,2r^2\,dr)$, (6) gives

$$2r^3 v\,dv + 3r^2 v^2 dr = -\frac{2\Pi}{\rho}r^2 dr \quad \text{or} \quad d(r^3 v^2) = -\frac{2\Pi}{\rho}r^2 dr$$

Integrating, $$r^3 v^2 = -\frac{2\Pi r^3}{3\rho} + C', \ C' \text{ being an arbitrary constant.} \qquad \ldots(7)$$

But when $r = a$, $v = 0$. Hence $C' = (2\Pi a^3)/(3\rho)$

$\therefore$ From (7), $$r^3 v^2 = \frac{2\Pi}{3\rho}(a^3 - r^3) \qquad \ldots(8)$$

Putting the value of v from (8) in (5), we get

$$F'(t) = r\left(\frac{1}{2}v^2 - \frac{\Pi}{\rho}\right) = r\left[\frac{\Pi}{3\rho}\frac{a^3 - r^3}{r^3} - \frac{\Pi}{\rho}\right]$$

or $$F'(t) = \frac{\Pi}{3\rho}\frac{a^3 - 4r^3}{r^2}. \qquad \ldots(9)$$

From (1), $$v' = (r^2 v)/r'^2 \qquad \ldots(10)$$

Using (9) and (10), (4) reduces to

$$\frac{\Pi - p}{\rho} = -\frac{1}{r'}\cdot\frac{\Pi}{3\rho}\cdot\frac{a^3 - 4r^3}{r^2} + \frac{1}{2}\frac{v^2 r^4}{r'^4} = -\frac{\Pi}{3\rho}\cdot\frac{a^3 - 4r^3}{r^2 r'} + \frac{\Pi}{3\rho}\cdot\frac{r(a^3 - r^3)}{r'^4}, \text{ using (8)}$$

$$\therefore \qquad \frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{\Pi}{3\rho}\cdot\frac{a^3 - 4r^3}{r^2 r'} - \frac{\Pi}{3\rho}\cdot\frac{r(a^3 - r^3)}{r'^4}$$

or $$\frac{p}{\Pi} = \frac{3r^2 r'^4 + (a^3 - 4r^3) r'^3 - (a^3 - r^3) r^3}{3r^2 r'^4},$$

which gives the required ratio of two pressures under consideration

Ex. 8. *A solid sphere of radius a is surrounded by a mass of liquid whose volume is* $(4\pi c^3)/3$ *and its centre is a centre of attractive force varying directly as the square of the distance. If the solid sphere be suddenly annihilated, show that the velocity of the inner surface, when its radius is x, is given by*

$$\dot{x}^2 x^3 [(x^3 + c^3)^{1/3} - x] = \left(\frac{2\Pi}{3\rho} + \frac{2\mu c^3}{9}\right)(a^3 - x^3)(c^3 + x^3)^{1/3},$$

where ρ *is the density,* Π *the external pressure,* μ *the absolute force and* $\dot{x} = dx/dt$.

[Agra 2000; Himanchal 1999]

Sol. Let v' be the velocity at a distance r' at any time t and p be the pressure there. Let r and R be the radii and v and V the velocities of the inner and outer surfaces at time t. Then the equation of continuity is

$$r'^2 v' = F(t) = r^2 v = R^2 V \qquad \ldots(1)$$

From (1), $$\partial v'/\partial t = F'(t)/r'^2 \qquad \ldots(2)$$

The equation of motion is

$$\frac{\partial v'}{dt} + v'\frac{\partial v'}{\partial r'} = -\mu r'^2 - \frac{1}{\rho}\frac{\partial p}{\partial r'},$$ where here $\mu r'^2$ is the attractive force

or $$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\mu r'^2 - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

Integrating, $$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{\mu r'^3}{3} - \frac{p}{\rho} + C,$$ C being an arbitrary constant ...(3)

Now, when $r' = r$, $v' = v$ and $p = 0$

and when $r' = R$, $v' = \mathrm{V}$ and $p = \Pi$

$\therefore$ (3) yields $$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = -\frac{\mu r^3}{3} + \mathrm{C} \qquad \ldots(4)$$

and $$-\frac{F'(t)}{R} + \frac{1}{2}V^2 = -\frac{\mu R^3}{3} - \frac{\Pi}{\rho} + C \qquad \ldots(5)$$

Subtracting (4) from (5), we have

$$F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) - \frac{1}{2}(v^2 - V^2) = \frac{\mu}{3}(r^3 - R^3) - \frac{\Pi}{\rho}$$

But $(4/3)\times\pi \mathrm{R}^3 - (4/3)\times\pi r^3 = (4/3)\times\pi c^3$ so that $r^3 - R^3 = -c^3$.

$\therefore$ $$F'(t)\left(\frac{1}{r} - \frac{1}{\mathrm{R}}\right) - \frac{1}{2}(v^2 - V^2) = -\frac{\mu c^3}{3} - \frac{\Pi}{\rho} \qquad \ldots(6)$$

From (1), $F'(t) = \dfrac{d}{dt}(r^2 v) = \dfrac{d}{dr}(r^2 v)\cdot\dfrac{dr}{dt}$ or $F'(t) = v\dfrac{d}{dr}(r^2 v)$ ...(7)

Again from (1), we get $V = (r^2 v)/R^2$ ...(8)

Using (7) and (8), (6) gives

$$v\frac{d}{dr}(r^2 v)\cdot\left(\frac{1}{r}-\frac{1}{R}\right)-\frac{1}{2}\left(v^2-\frac{r^4 v^2}{R^4}\right)=-\frac{\mu\rho c^3+3\Pi}{3\rho}$$

Multiplying both sides by r^2, we get

$$2r^2 v\frac{d}{dr}(r^2 v)\cdot\left(\frac{1}{r}-\frac{1}{R}\right)-v^2 r^4\left(\frac{1}{r^2}-\frac{r^2}{R^4}\right)=-\frac{\mu\rho c^3+3\Pi}{3\rho}r^2$$

or
$$\frac{d}{dr}(vr^2)^2\cdot\left(\frac{1}{r}-\frac{1}{R}\right)-(vr^2)^2\cdot\left(\frac{1}{r^2}-\frac{r^2}{R^4}\right)=-\frac{\mu\rho c^3+3\Pi}{3\rho}r^2 \qquad \text{...(9)}$$

From (1), $r^2 v = R^2 V$ or $r^2\dfrac{dr}{dt}=R^2\dfrac{dR}{dt}$

i.e. $r^2 dr = R^2 dR$ ...(10)

Integrating (9) and using (10), we have

$$r^4 v^2\left(\frac{1}{r}-\frac{1}{R}\right)=-\frac{2(\mu c^3\rho+3\Pi)}{3\rho}\int r^2 dr+C'=-\frac{2(\mu c^3\rho+3\Pi)}{9\rho}r^3+C'$$

When $r = a$, $v = 0$ so that $C'=\dfrac{2(\mu c^3\rho+3\Pi)}{9\rho}a^3$

$$\therefore \qquad r^4 v^2\left(\frac{1}{r}-\frac{1}{R}\right)=\frac{2(\mu c^3\rho+3\Pi)}{9\rho}(a^3-r^3)$$

i.e.
$$r^4 v^2\left[\frac{1}{r}-\frac{1}{(r^3+c^3)^{1/3}}\right]=\left(\frac{2\mu c^3}{9}+\frac{2\Pi}{3\rho}\right)(a^3-r^3)$$

Now, for the inner surface, $r = x$, $v=\dot{x}$. Hence, the above relation reduces to

$$\dot{x}^2 x^3[(x^3+c^3)^{1/3}-x]=\left(\frac{2\mu c^3}{9}+\frac{2\Pi}{3\rho}\right)(a^3-x^3)(x^3+c^3)^{1/3}.$$

Ex. 9. *A sphere is at rest in an infinite mass of homogenous liquid of density* ρ, *the pressure at infinity being P. If the radius R of the sphere varies in such a way that* $R = a + b\cos nt$, *where* $b > a$, *show that pressure at the surface of the sphere at any time is*

$$P+\frac{bn^2\rho}{4}(b-4a\cos nt-5b\cos 2nt).$$ **[Agra 2003, Himachal 2000]**

Sol. Let v' be the velocity at a distance r' at any time t and p' be the pressure there. Again, let v be the velocity on the surface of sphere of radius R, where

$$R = a + b\cos nt \qquad \text{...(1)}$$

Then the equation of continuity is $r'^2 v' = F(t) = R^2 v$ ...(2)

From (2), $\partial v'/\partial t = F'(t)/r'^2$ ...(3)

The equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p'}{\partial r'} \qquad \text{or} \qquad \frac{F'(t)}{r'} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p'}{\partial r'}, \text{ using (3)}$$

Integrating, $$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{p'}{\rho} + C, \; C \text{ being an arbitrary constant}$$

Given: when $r' = \infty$, $v = 0$, $p' = P$. So $C = P/\rho$. So the above equation gives

$$\therefore \qquad -\frac{F'(t)}{r} + \frac{1}{2}v'^2 = \frac{P - p'}{\rho} \qquad \ldots(4)$$

Let $p' = p$ when $r' = R$. Also, $v' = v$ when $r' = R$. Then, (4) yields

$$\therefore \qquad -\frac{F'(t)}{R} + \frac{1}{2}v^2 = \frac{P - p}{\rho} \qquad \text{or} \qquad p = P + \rho\left[\frac{F'(t)}{R} - \frac{1}{2}v^2\right] \qquad \ldots(5)$$

From (2), $$F'(t) = \frac{d}{dt}(vR^2) = 2R\frac{dR}{dt}\cdot v + R^2\frac{dv}{dt}$$

$$= 2R\left(\frac{dR}{dt}\right)^2 + R^2\frac{d^2R}{dt^2} \qquad \left[\because\; v = \frac{dR}{dt}\right]$$

Using the above value of $F'(t)$ and noting $v = dR/dt$, we have

$$\frac{F'(t)}{R} - \frac{1}{2}v^2 = 2\left(\frac{dR}{dt}\right)^2 + R\frac{d^2R}{dt^2} - \frac{1}{2}\left(\frac{dR}{dt}\right)^2 = \frac{3}{2}\left(\frac{dR}{dt}\right)^2 + R\frac{d^2R}{dt^2}$$

$= (3/2) \times (-\,bn \sin nt)^2 + (a + b \cos nt)\,(-\,bn^2 \cos nt)$, using (1)

$= (bn^2/2) \times (3b \sin^2 nt - 2b \cos^2 nt - 2a \cos nt)$

$= (bn^2/4) \times [3b\,(1 - \cos 2nt) - 2b\,(1 + \cos 2nt) - 4a \cos nt]$

$= (bn^2/4) \times (b - 4a \cos nt - 5b \cos 2nt)$

Hence (5) reduces to $$p = P + \frac{bn^2\rho}{4}(b - 4a\cos nt - 5b\cos 2nt).$$

Ex. 10. *For an inviscid, incompressible, steady flow with negligible body forces, velocity components in spherical polar coordinates are given by*

$$u_r = V(1 - R^3/r^3)\cos\theta, \qquad u_\theta = -V(1 + R^3/2r^3)\sin\theta, \qquad u_\phi = 0.$$

Show that it is a possible solution of momentum equations (i.e. equations of motion). R and V are constants.

Sol. Here equations of motion in spherical polar coordinates are (refer Art. 14.12)

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots(1)$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{u_r u_\theta}{r} - \frac{u_\phi^2\cot\theta}{r} = F_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} \qquad \ldots(2)$$

$$\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\phi u_r}{r} + \frac{u_\phi u_\theta\cot\theta}{r} = F_\phi - \frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \phi} \qquad \ldots(3)$$

For steady flow $(\partial/\partial t = 0)$ with negligible body forces $(F_r = F_\theta = F_\phi = 0)$, the above equations reduces to

$$u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots(4)$$

$$u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \qquad \ldots(5)$$

$$0 = -\frac{1}{\rho r \sin\phi}\frac{\partial p}{\partial \phi} \qquad \ldots(6)$$

Equation (6) shows that p is function of r and θ only.

Given : $\quad u_r = V\left(1 - \frac{R^3}{r^3}\right)\cos\theta, \qquad u_\theta = -V\left(1 + \frac{R^3}{2r^3}\right)\sin\theta \qquad \ldots(7)$

From (7), $\quad \frac{\partial u_r}{\partial r} = \frac{3VR^3}{r^4}\cos\theta, \qquad \frac{\partial u_r}{\partial \theta} = -V\left(1 - \frac{R^3}{r^3}\right)\sin\theta \qquad \ldots(8)$

Using (7) and (8), (4) reduces to

$$V\left(1-\frac{R^3}{r^3}\right)\cos\theta\cdot\frac{3VR^3}{r^4}\cos\theta - \frac{V}{r}\left(1+\frac{R^3}{2r^3}\right)\sin\theta\cdot\left[-V\left(1-\frac{R^3}{r^3}\right)\sin\theta\right] - \frac{1}{r}\left[V^2\left(1+\frac{R^3}{2r^3}\right)^2\sin^2\theta\right] = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or $$\frac{3V^2R^3}{r^4}\left(1-\frac{R^3}{r^3}\right)\cos^2\theta - \frac{3V^2R^3}{2r^4}\left(1+\frac{R^3}{2r^3}\right)\sin^2\theta = -\frac{1}{\rho}\frac{\partial \rho}{\partial r} \qquad \ldots (9)$$

From (7), $\quad \frac{\partial u_\theta}{\partial r} = \frac{3VR^3\sin\theta}{2r^4}, \qquad$ and $\qquad \frac{\partial u_\theta}{\partial \theta} = -V\left(1+\frac{R^3}{2r^3}\right)\cos\theta \qquad \ldots (10)$

Using (7) and (10), (5) reduces to

$$V\left(1-\frac{R^3}{r^3}\right)\cos\theta\cdot\frac{3VR^3}{2r^4}\sin\theta + \frac{1}{r}\left[-V\left(1+\frac{R^3}{2r^3}\right)\sin\theta\right]\left[-V\left(1+\frac{R^3}{2r^3}\right)\cos\theta\right]$$

$$+\frac{1}{r}\left[V\left(1-\frac{R^3}{r^3}\right)\cos\theta\right]\left[-V\left(1+\frac{R^3}{2r^3}\right)\sin\theta\right] = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

or $$\frac{3V^2R^3}{2r^3}\left(1-\frac{R^3}{r^3}\right)\sin\theta\cos\theta + \frac{3V^2R^3}{2r^3}\left(1+\frac{R^3}{2r^3}\right)\sin\theta\cos\theta = -\frac{1}{\rho}\frac{\partial p}{\partial \theta}$$

Differentiating (9) with respect to θ, we get

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\,\partial r} = \frac{3V^2R^3}{r^4}\left(1-\frac{R^3}{r^3}\right)\cdot 2\cos\theta\,(-\sin\theta) - \frac{3V^2R^3}{2r^4}\left(1+\frac{R^3}{2r^4}\right)\times 2\sin\theta\cos\theta$$

or $$-\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\,\partial r} = \left(-\frac{9V^2R^3}{r^4} + \frac{9V^2R^6}{2r^7}\right)\sin\theta\,\cos\theta \qquad \ldots (12)$$

Next, differentiating (11) with respect to r, we get

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\,\partial\theta} = \frac{3V^2R^3}{2}\left(-\frac{3}{r^4}+\frac{6R^3}{r^7}\right)\sin\theta\cos\theta + \frac{3V^2R^3}{2}\left(-\frac{3}{r^4}-\frac{6R^3}{2r^7}\right)\times\sin\theta\cos\theta$$

or
$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\,\partial\theta} = \left(-\frac{9V^2R^3}{r^4}+\frac{9V^2R^6}{2r^7}\right)\sin\theta\cos\theta \qquad \text{... (13)}$$

Since (12) and (13) are identical, the equations of motion (*i.e.*, monemtum equations) are satisfied.

Ex. 11. *The velocity components* $u_r(r,\theta) = -V\left(1-\frac{a^2}{r^2}\right)\cos\theta,\ u_\theta(r,\theta) = u\left(1+\frac{a^2}{r^2}\right)\sin\theta$ *satisfy the equations of motion for a two-dimensional inviscid incompressible flow. Find the pressure associated with this velocity field. u and a are constants.*

Sol. The equations of motion for inviscid incompressible fluid in cylindrical polar coordinates are given by (Refer Art. 14.11)

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r} = F_r-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{... (1)}$$

$$\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r u_\theta}{r} = F_\theta-\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \qquad \text{... (2)}$$

$$\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta}+u_z\frac{\partial u_z}{\partial z} = F_z-\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \text{... (3)}$$

For steady $(\partial/\partial t = 0)$ and two dimensional flow $(\partial/\partial z \equiv 0, u_z = 0)$ with negligible body forces $(F_r = F_\theta = F_z = 0)$, the above equations (1) to (3) reduces to

$$u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{... (4)}$$

$$u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \qquad \text{... (5)}$$

and $0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}$, which implies that p is function of r and θ only.

Given $$u_r = -U\left(1-\frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = U\left(1+\frac{a^2}{r^2}\right)\sin\theta \qquad \text{... (6)}$$

Using (6), (4) reduces to

$$\left[-U\left(1-\frac{a^2}{r^2}\right)\cos\theta\right]\left(-\frac{2a^2U}{r^3}\right)\cos\theta+\frac{1}{r}U\left(1+\frac{a^2}{r^2}\right)\sin\theta\left[-U\left(1-\frac{a^2}{r^2}\right)\times(-\sin\theta)\right]$$
$$-\frac{1}{r}\cdot U^2\left(1+\frac{a^2}{r^2}\right)^2\sin^2\theta = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or
$$\frac{2U^2a^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\cos^2\theta+\frac{U^2}{r}\sin^2\theta\left[\left(1-\frac{a^4}{r^4}\right)-\left(1+\frac{a^2}{r^2}\right)^2\right] = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or
$$\frac{2U^2a^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\cos^2\theta-\frac{2U^2a^2}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin^2\theta=-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots (7)$$

Again using (6), (5) reduces to

$$-U\left(1-\frac{a^2}{r^2}\right)\cos\theta\times U\left(-\frac{2a^2}{r^3}\right)\sin\theta+\frac{U}{r}\left(1+\frac{a^2}{r^2}\right)\sin\theta\times U\left(1+\frac{a^2}{r^2}\right)\cos\theta$$

$$-\frac{1}{r}U\left(1-\frac{a^2}{r^2}\right)\cos\theta\times U\left(1+\frac{a^2}{r^2}\right)\sin\theta=-\frac{1}{r\rho}\frac{\partial p}{\partial \theta}$$

or
$$\frac{2a^2U^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\sin\theta\cos\theta+\frac{2U^2a^2}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin\theta\cos\theta=-\frac{1}{r\rho}\frac{\partial p}{\partial \theta}$$

or
$$\frac{2a^2U^2}{r^2}\left(1-\frac{a^2}{r^2}\right)\sin\theta\cos\theta+\frac{2a^2U^2}{r^2}\left(1+\frac{a^2}{r^2}\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial p}{\partial \theta}$$

or
$$\frac{4U^2a^2}{r^2}\sin\theta\,\cos\theta=-\frac{1}{\rho}\frac{\partial p}{\partial \theta} \qquad \ldots (8)$$

Differentiating (7) with respect to θ, we have

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\partial r}=-\frac{4U^2a^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\sin\theta\,\cos\theta-\frac{4U^2a^2}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin\theta\,\cos\theta$$

or
$$\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\,\partial r}=\frac{8U^2a^2}{r^3}\sin\theta\,\cos\theta \qquad \ldots (9)$$

Differentiating (8) with respect to r, we have

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial r\,\partial\theta}=\frac{8U^2a^2}{r^3}\sin\theta\,\cos\theta \qquad \ldots (10)$$

Since (9) and (10) are identical, it follows that the given velocity components satisfy the equations of motion.

Since p is function of r and θ, we have

$$dp=(\partial p/\partial r)dr+(\partial p/\partial\theta)d\theta$$

Substituting the values of $\partial p/\partial r$ and $\partial p/\partial\theta$ given by (7) and (8) respectively in the above equation, we obtain

$$\therefore\quad dp=2\rho U^2a^2\left\{\left(\frac{1}{r^3}+\frac{a^2}{r^5}\right)\sin^2\theta-\left(\frac{1}{r^3}-\frac{a^2}{r^5}\right)\cos^2\theta\right\}dr-\frac{4\rho U^2a^2}{r^2}\sin\theta\,\cos\theta\,d\theta \qquad \ldots (11)$$

Let $dp=Mdr+Nd\theta$. Then, by comparison, we have

$$M=2\rho U^2a^2\left\{(1/r^3+a^2/r^5)\sin^2\theta-(1/r^3-a^2/r^5)\cos^2\theta\right\}$$

and $\quad N=-(4\rho U^2a^2/r^2)\times\sin\theta\,\cos\theta$

$$\therefore\quad \frac{\partial M}{\partial\theta}=\frac{\partial}{\partial\theta}\left[2\rho\,U^2a^2\left\{\left(\frac{1}{r^3}+\frac{a^2}{r^5}\right)\sin^2\theta-\left(\frac{1}{r^3}-\frac{a^2}{r^5}\right)\cos^2\theta\right\}\right]$$

$$=2\rho U^2a^2\left\{(1/r^3+a^2/r^5)\times 2\sin\theta\,\cos\theta+(1/r^3-a^2/r^5)\times 2\sin\theta\,\cos\theta\right\}$$

$$=(8/r^3)\times\rho U^2a^2\sin\theta\,\cos\theta$$

and $$\frac{\partial N}{\partial r} = \frac{\partial}{\partial r}\left(-\frac{4\rho U^2 a^2}{r^2}\sin\theta\cos\theta\right) = \frac{8\rho U^2 a^2 \sin\theta\cos\theta}{r^3}$$

Thus, $$\partial M/\partial\theta = \partial N/\partial r.$$

Hence (11) must be exact and so its solution by the usual rule of an exact equation is

$$p = 2\rho U^2 a^2\left[\left(-\frac{1}{2r^2}-\frac{a^2}{4r^4}\right)\sin^2\theta - \left(-\frac{1}{2r^2}+\frac{a^2}{4r^4}\right)\cos^2\theta\right]+C$$

or $$p = 2\rho U^2 a^2\left(\frac{\cos 2\theta}{2r^2}-\frac{a^2}{4r^4}\right)+C,\ C \text{ being an arbitrary constant}$$

Ex. 12(*a*). *A steady inviscid incompressible fluid flow has a velocity field $u = fx$, $v = -fy$, $w = 0$, where f is a constant. Derive an expression for the pressure field $p(x, y, z)$ if the pressure $p(0, 0, 0) = p_0$ and $\boldsymbol{F} = -g\,\boldsymbol{i}\,z$.* **[I.A.S. 2006]**

Sol. Given $u = fx$, $v = -fy$, $w = 0$, f being a constant ... (1)

Also, given that $p = p_0$, when $x = 0$, $y = 0$, $z = 0$ ... (2)

Again, $\boldsymbol{F} = -g\boldsymbol{i}z \Rightarrow X = 0$, $Y = 0$ and $Z = -gz$ ... (3)

Equations of motion for steady motion $(\partial/\partial t) = 0$ of an incompressible fluid flow (see Art 3.1) are given by

$$u(\partial u/\partial x)+v(\partial u/\partial y)+w(\partial u/\partial z) = X-(1/\rho)\times(\partial p/\partial x) \quad \text{... (4)}$$

$$u(\partial v/\partial x)+v(\partial v/\partial y)+w(\partial v/\partial z) = Y-(1/\rho)\times(\partial p/\partial y) \quad \text{... (5)}$$

$$u(\partial w/\partial x)+v(\partial w/\partial y)+w(\partial w/\partial z) = Z-(1/\rho)\times(\partial p/\partial z) \quad \text{... (6)}$$

Using (1) and (3), (4), (5) and (6) reduce to

$$f^2x = -(1/\rho)\times(\partial p/\partial x), \qquad -f^2y = -(1/\rho)\times(\partial p/\partial y), \qquad 0 = -gz-(1/\rho)\times(\partial p/\partial z)$$

$$\Rightarrow \qquad \partial p/\partial x = -f^2\rho x, \qquad \partial p/\partial y = f^2\rho y \quad \text{and} \quad \partial p/\partial z = -\rho g z \quad \text{... (7)}$$

Now, $$dp = (\partial p/\partial x)\,dx+(\partial p/\partial y)\,dy+(\partial p/\partial z)\,dz$$

$\therefore$ $$dp = -(f^2\rho x)\,dx+(f^2\rho y)dy-(\rho g z)\,dz, \quad \text{using (7)}$$

Integrating, $p = -(f^2\rho x^2)/2+(f^2\rho y^2)/2-(\rho g z^2)/2+C$, C being a constant ... (8)

Putting $x = y = z = 0$ and $p = p_0$ (see condition (2)), in (8), we get $C = p_0$

Thus, the required expression for the pressure field is given by

$$p(x,y,z) = p_0-\rho\,(f^2x^2-f^2y^2+g\,z^2)/2$$

Ex. 12(b). *For a steady motion of inviscid incompressible fluid of uniform density under conservative forces, show that the vorticity $\boldsymbol{w}$ and velocity $\boldsymbol{q}$ satisfies*

$$(\boldsymbol{q}\cdot\nabla)\boldsymbol{w} = (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{q}.$$ **[I.A.S. 1989]**

Sol. Vector equation of motion for invicid incompressible fluid is (refer Art. 3.1A)

$$\partial\boldsymbol{q}/\partial t+\nabla(\boldsymbol{q}^2/2)-\boldsymbol{q}\times curl\,\boldsymbol{q} = \boldsymbol{F}-(1/\rho)\,\nabla p \quad \text{... (1)}$$

Since the motion is steady, $\partial\boldsymbol{q}/\partial t = \boldsymbol{0}$ ... (2)

Since ρ is uniform, $\qquad (1/\rho)\,\nabla p = \nabla(p/\rho)$ ... (3)

Since $\mathbf{F}$ is conservative, $\quad \mathbf{F} = -\nabla\Omega$, where Ω is some scalar function. ... (4)

Again, by definition, vorticity vector = $\boldsymbol{w}$ = curl $\boldsymbol{q}$.

Using (2), (3), (4) and (5) in (1), we obtain

$$\nabla(\boldsymbol{q}^2/2) - \boldsymbol{q}\times\boldsymbol{w} = -\nabla\Omega - \nabla(p/\rho) \qquad \text{or} \qquad \boldsymbol{q}\times\boldsymbol{w} = \nabla(\boldsymbol{q}^2/2 + \Omega + p/\rho)$$

Taking the curl of both sides of the above equation and using the vector identity curl gral $\phi = \mathbf{0}$, we have

$$\text{curl }(\boldsymbol{q}\times\boldsymbol{w}) = \mathbf{0} \qquad \text{or} \qquad (\nabla\cdot\boldsymbol{w})\,\boldsymbol{q} - (\boldsymbol{q}\cdot\nabla)\,\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{q} - (\nabla\cdot\boldsymbol{q})\boldsymbol{w} = \mathbf{0}$$

or $$-(\boldsymbol{q}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{q} = \mathbf{0} \qquad \text{or} \qquad (\boldsymbol{q}\cdot\nabla)\boldsymbol{w} = (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{q}.$$

where we have used the following two results

$$\nabla\cdot\boldsymbol{w} = \nabla\cdot\nabla\times\boldsymbol{q} = 0 \qquad \text{and} \qquad \nabla\cdot\boldsymbol{q} = 0 \ \text{(continuity equation).}$$

Ex. 13. *Show that if the velocity field*

$$u(x,y) = \frac{B(x^2-y^2)}{(x^2+y^2)^2}, \qquad v(x,y) = \frac{2Bxy}{(x^2+y^2)^2}, \qquad w(x,y) = 0$$

satisfies the equations of motion for inviscid incompressible flow, then determine the pressure associated with this velocity field, B being a constant.

[Kanpur 2002, 03, 05; Rohilkhand 2000, 05]

Sol. The equations of motion for steady inviscid incompressible flow are given by

$$u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z) = -(1/\rho)\,(\partial p/\partial x), \qquad \text{... (1)}$$

$$u(\partial v/\partial x) + v(\partial v/\partial y) + w(\partial v/\partial z) = -(1/\rho)\,(\partial p/\partial y) \qquad \text{... (2)}$$

and $$u(\partial w/\partial x) + v(\partial w/\partial y) + w(\partial w/\partial z) = -(1/\rho)\,(\partial p/\partial z). \qquad \text{... (3)}$$

From the given values of u, v and w, we have

$$\frac{\partial u}{\partial x} = B\frac{2x(x^2+y^2)^2 - 4x(x^2-y^2)\,(x^2+y^2)}{(x^2+y^2)^4} = \frac{2Bx(3y^2-x^2)}{(x^2+y^2)^3},$$

$$\frac{\partial u}{\partial y} = B\frac{-2y(x^2+y^2)^2 - 4y(x^2-y^2)\,(x^2+y^2)}{(x^2+y^2)^4} = -\frac{2By(3x^2-y^2)}{(x^2+y^2)^3}, \qquad \frac{\partial u}{\partial z} = 0$$

$$\frac{\partial v}{\partial x} = 2B\frac{y(x^2+y^2)^2 - 4x^2y(x^2+y^2)}{(x^2+y^2)^4} = \frac{2By(y^2-3x^2)}{(x^2+y^2)^3},$$

$$\frac{\partial v}{\partial y} = 2B\frac{x(x^2+y^2)^2 - 4xy^2(x^2+y^2)}{(x^2+y^2)^4} = \frac{2Bx(x^2-3y^2)}{(x^2+y^2)^3}, \qquad \frac{\partial v}{\partial z} = 0,$$

$$\partial w/\partial x = 0, \qquad \partial w/\partial y = 0 \qquad \text{and} \qquad \partial w/\partial z = 0.$$

Substituting the given values of u, v and w and also using the above relations, (1), (2) and (3) reduce to

$$\frac{B(x^2-y^2)}{(x^2+y^2)^2}\cdot\frac{2Bx(3y^2-x^2)}{(x^2+y^2)^3} - \frac{2Bxy}{(x^2+y^2)^2}\cdot\frac{2By(3x^2-y^2)}{(x^2+y^2)^3} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{B(x^2-y^2)}{(x^2+y^2)^2}\cdot\frac{2By(y^2-3x^2)}{(x^2+y^2)^3}+\frac{2Bxy}{(x^2+y^2)^2}\cdot\frac{2Bx(x^2-3y^2)}{(x^2+y^2)^3}=-\frac{1}{\rho}\frac{\partial p}{\partial y}.$$

and $$0=-(1/\rho)\,(\partial p/\partial z)$$

Simplifying the above equations, we have

$$\frac{2B^2x}{(x^2+y^2)^5}[(x^2-y^2)(3y^2-x^2)-2y^2(3x^2-y^2)]=-\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{2B^2y}{(x^2+y^2)^5}[(x^2-y^2)(y^2-3x^2)+2x^2(x^2-3y^2)]=-\frac{1}{\rho}\frac{\partial p}{\partial y}$$

and $$0=\partial p/\partial z$$

Again simplifying the above equations, we have

or $$\frac{2B^2x}{(x^2+y^2)^5}(-x^4-2x^2y^2-y^4)=-\frac{1}{\rho}\frac{\partial p}{\partial x}\quad i.e.,\quad \frac{2B^2x\rho}{(x^2+y^2)^3}=\frac{\partial p}{\partial x}\qquad ...(1)$$

$$\frac{2B^2y}{(x^2+y^2)^5}(-x^4-2x^2y^2-y^4)=-\frac{1}{\rho}\frac{\partial p}{\partial z}\quad i.e.,\quad \frac{2B^2y\rho}{(x^2+y^2)^3}=\frac{\partial p}{\partial y}\qquad ...(2)$$

and $$0=\partial p/dz.\qquad ...(3)$$

Relation (3) shows that the pressure p is independent of z, *i.e.*, $p = p\ (x, y)$. Hence, we have

$$dp=(\partial p/\partial x)\,dx+(\partial p/\partial y)\,dy$$

or $$dp=\frac{2B^2x\rho}{(x^2+y^2)^3}dx+\frac{2B^2y\rho}{(x^2+y^2)^3}dy=B^2\rho(x^2+y^2)^{-3}(2xdx+2ydy)$$

or $$dp=B^2\rho(x^2+y^2)^{-3}d(x^2+y^2).$$

Integrating, $$p=C-(1/2)\times B^2\rho(x^2+y^2)^{-2}=C-\{B^2\rho/2(x^2+y^2)^2\},$$

where C is a constant of integratioon. It gives the required pressure distribution.

Ex. 14. *The particle velocity for a fluid motion referred to rectangular axes is given by the components* $u = A\cos(\pi x/2a)\cos(\pi z/2a)$, $v = 0$, $w = A\sin(\pi x/2a)\sin(\pi z/2a)$, *where A is a constant. Show that this is a possible motion of an incompressible fluid under no body forces in an infinite fixed rigid tube,* $-a \le x \le a$, $0 \le z \le 2a$. *Also, find the pressure associated with this velocity field.* **[I.A.S. 1994; Meerut 2003]**

Sol. Given $u = A\cos(\pi x/2a)\cos(\pi z/2a)$, $v = 0$, $w = A\sin(\pi x/2a)\sin(\pi z/2a)$. ...(1)

$$\left.\begin{aligned}&\text{From (1),}\quad \partial u/\partial x=-(A\pi/2a)\sin(\pi x/2a)\cos(\pi z/2a),\qquad \partial v/\partial y=0\\&\text{and}\quad \partial w/\partial z=(A\pi/2a)\sin(\pi x/2a)\cos(\pi z/2a).\end{aligned}\right\}\ ...(2)$$

$$\therefore\qquad \partial u/\partial x+\partial v/\partial y+\partial w/\partial z=0,$$

showing that the given velocity components represent a physically possible flow.

The equations of motion for steady inviscid incompressible flow under no body forces are

$$u(\partial u/\partial x)+v(\partial u/\partial y)+w(\partial u/\partial z)=-(1/\rho)\,(\partial p/\partial x),\qquad ...(3)$$

$$u(\partial v/\partial x)+v(\partial v/\partial y)+w(\partial v/\partial z)=-(1/\rho)\,(\partial p/\partial y)\qquad ...(4)$$

and $$u(\partial w/\partial x)+v(\partial w/\partial y)+w(\partial w/\partial z)=-(1/\rho)\,(\partial p/\partial z).\qquad ...(5)$$

$$\left.\begin{aligned}&\text{From (1)}\ \partial u/\partial y=0,\qquad \partial u/\partial z=-(A\pi/2a)\cos(\pi x/2a)\sin(\pi z/2a)\\&\partial v/\partial x=\partial v/\partial z=0,\qquad \partial w/\partial x=(A\pi/2a)\cos(\pi x/2a)\sin(\pi z/2a)\\&\text{and}\qquad \partial w/\partial y=0.\end{aligned}\right\}\ ...(6)$$

Using (1), (2) and (6), the equations of motion (3), (4) and (5) become

$$-A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\cdot\frac{A\pi}{2a}\sin\frac{\pi x}{2a}\cos\frac{\pi z}{2a}-A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\cdot\frac{A\pi}{2a}\cos\frac{\pi x}{2a}\sin\frac{\pi z}{2a}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$0 = -(1/\rho)\,(\partial p/\partial y)$$

$$A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\cdot\frac{A\pi}{2a}\cos\frac{\pi x}{2a}\sin\frac{\pi z}{2a}+A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\cdot\frac{A\pi}{2a}\sin\frac{\pi x}{2a}\cos\frac{\pi z}{2a}=-\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Simplifying the above equations, we have

$$\partial p/\partial x = (\pi\rho A^2/2a)\cos(\pi x/2a)\sin(\pi x/2a), \qquad \text{...(7)}$$

$$\partial p/\partial y = 0 \qquad \text{...(8)}$$

and
$$\partial p/\partial z = -(\pi\rho A^2/2a)\cos(\pi z/2a)\sin(\pi z/2a). \qquad \text{...(9)}$$

Equation (8) shows that the pressure p is independent of y so that $p = p(x, z)$. Then

$$dp = (\partial p/\partial x)dx + (\partial p/\partial z)dz$$

or $dp = (\pi\rho A^2/2a)\,[\cos(\pi x/2a)\sin(\pi x/2a)\,dx - \cos(\pi z/2a)\sin(\pi z/2a)dz]$, using (7) and (9)

Integrating, $\quad p = (\pi\rho A^2/2a)\,[(a/\pi)\sin^2(\pi x/2a) - (a/\pi)\sin^2(\pi z/2a)] + C$

or $\quad p = (\rho A^2/2)\,[\sin^2(\pi x/2a) - \sin^2(\pi z/2a)] + C$, C being a constant of integration. ...(10)

(10) gives the required pressure associated with the velocity field given by (1).

Ex. 15. *Prove that if* $\lambda = (\partial u/\partial t) - v(\partial v/\partial x - \partial u/\partial y) + w(\partial u/\partial z - \partial w/\partial x)$ *and* μ, ν *are two similar expressions, then* $\lambda\, dx + \mu\, dy + \nu\, dz$ *is a perfect differential, if the external forces are conservative and the density is constant.* **[Agra 2006]**

Sol. Let (X, Y, Z) be the components of external forces. Since the external forces are conservative, there exists force potential $V(x, y, z)$ such that

$$X = -\partial V/\partial x, \qquad Y = -\partial V/\partial y \qquad \text{and} \qquad Z = -\partial V/\partial z. \qquad \text{...(1)}$$

Euler's dynamical equations of motion are

$$Du/Dt = X - (1/\rho)\,(\partial p/\partial x), \qquad \text{...(2)}$$

$$Dv/Dt = Y - (1/\rho)\,(\partial p/\partial y) \qquad \text{...(3)}$$

and
$$Dw/Dt = Z - (1/\rho)\,(\partial p/\partial z), \qquad \text{...(4)}$$

where $p(x, y, z)$ is the pressure at any point (x, y, z).

Using (1), (2), (3) and (4) can be rewritten as

$$Du/Dt = -\partial V/\partial x - (1/\rho)\,(\partial p/\partial x), \qquad \text{...(5)}$$

$$Dv/Dt = -\partial V/\partial y - (1/\rho)\,(\partial p/\partial y) \qquad \text{...(6)}$$

and
$$Dw/Dt = -\partial V/\partial z - (1/\rho)\,(\partial p/\partial z). \qquad \text{...(7)}$$

Multiplying (5), (6) and (7) by dx, dy and dz and then adding, we have

$$\frac{Du}{Dt}dx + \frac{dv}{Dt}dy + \frac{dw}{Dt}dz = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) - \frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right)$$

or
$$\frac{Du}{Dt}dx + \frac{Dv}{Dt}dy + \frac{Dw}{Dt}dz = -dV - \frac{1}{\rho}dp. \qquad \text{...(8)}$$

Re-writing the given value of λ, we have

$$\lambda = \frac{\partial u}{\partial t} - v\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - w\frac{\partial w}{\partial x}$$

$$= \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - \left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}\right)$$

$$= \frac{Du}{Dt} - \frac{1}{2}\frac{\partial}{\partial x}(u^2 + v^2 + w^2) = \frac{Du}{Dt} - \frac{1}{2}\frac{\partial q^2}{\partial x}. \qquad ...(9)$$

$$\left[\because \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \quad \text{and} \quad q^2 = u^2 + v^2 + w^2\right]$$

Similarly, $$\mu = \frac{Dv}{Dt} - \frac{1}{2}\frac{\partial q^2}{\partial y} \quad \text{and} \quad \nu = \frac{Dw}{Dt} - \frac{1}{2}\frac{\partial q^2}{\partial z}. \qquad ...(10)$$

$\therefore$ Using (9) and (10), we have,

$$\lambda\, dx + \mu\, dy + \nu\, dz = \frac{Du}{Dt}dx + \frac{Dv}{Dt}dy + \frac{Dw}{Dt}dz - \frac{1}{2}\left[\frac{\partial q^2}{\partial x}dx + \frac{\partial q^2}{\partial y}dy + \frac{\partial q^2}{\partial z}dz\right]$$

$$= -dV - (1/\rho)\, dp - (1/2) \times dq^2 = -d\,[V + (p/\rho) + (1/2) \times q^2],$$

which is a perfect differential which is what we wished to prove.

Ex. 16. *A sphere whose radius at time t is $b + a\cos nt$, is surrounded by liquid extending to infinity under no forces. Prove that the pressure at distance r from the centre is less than the pressure Π at infinity by*

$$\rho\frac{n^2 a}{r}(b + a\cos nt)\left\{a(1 - 3\sin^2 nt) + b\cos nt + \frac{a^3 \sin^2 nt}{2r^3}(b + a\cos nt)^3\right\}.$$

Prove also that least pressure at the surface of the sphere during the motion is $\Pi - n^2\rho a(a+b)$.

Sol. Let v' be the velocity of the fluid at a distance r' from the origin at any time t and p be the pressure there. Let $r' = b + a\cos nt$ and let r be the radius of any concentric sphere and v be the velocity there. Then the equation continuity is

$$r'^2 v' = F(t) = r^2 v. \qquad ...(1)$$

From (1), $$\partial v/\partial t = F(t)/r^2 \qquad ...(2)$$

The equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \text{or} \quad \frac{F'(t)}{r^2} + \frac{\partial}{\partial r}\left(\frac{1}{2}v^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \text{ using (2)}$$

Integrating it with respect to r, we have

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C, \quad C \text{ being an arbitrary constant} \qquad ...(3)$$

When $r = \infty$, $v = 0$, $p = \Pi$. So (3) gives $C = \Pi/\rho$. Hence (3) reduces to

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = \frac{\Pi - p}{\rho}. \qquad ...(4)$$

Now, $\quad r' = b + a\cos nt \quad \Rightarrow \quad v' = dr'/dt = -an\sin nt.$

Then, $\quad (1) \quad \Rightarrow \quad F(t) = r'^2 v' = (b + a\cos nt)^2(-an\sin nt)$

or $$F(t) = -an\,(b + a\cos nt)^2 \sin nt. \qquad ...(5)$$

Defferentiating (5) with respect to 't', we have

$$F'(t) = 2a^2n^2\,(b + a\cos nt)\sin^2 nt - an^2(b + a\cos nt)^2 \cos nt$$

or $$F'(t) = an^2(b + a\cos nt)\,[2a\sin^2 nt - (b + a\cos nt)\cos nt] \qquad ...(6)$$

Now, (4) $\Rightarrow \Pi - \rho = -(\rho/r)\,F'(t) + (1/2) \times \rho v^2$. ...(7)

or $$\Pi - \rho = -(\rho/r)F'(t) + (\rho/2)\,\{F(t)/r^2\}^2, \text{ using (1)}$$

Using (5) and (6), the above equation becomes

$$\Pi - \rho = -(\rho/r) \times an^2(b + a\cos nt)\,[2a\sin^2 nt - (b + a\cos nt)\cos nt] + (\rho/2r^4) \times a^2n^2(b + a\cos nt)^4 \sin^2 nt$$

$$= (\rho an^2/r) \times (b + a\cos nt)\,\{a(1 - 3\sin^2 nt) + b\cos nt + (a/2r^3) \times \sin^2 nt\,(b + a\cos nt)^3\}$$

Second part : At surface $r = r' = b + a\cos nt$, $v = v' = = dr'/dt = -an\sin t$.

Also, using (6), (4) reduces to

$$\frac{\Pi - p}{\rho} = -\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{1}{b + a\cos nt}\cdot an^2(b + a\cos nt)\,[2a\sin^2 nt - (b + a\cos nt)\cos nt] + (1/2) \times a^2n^2\sin^2 nt$$

$$= n^2a\,[a(1 - 3\sin^2 nt) + b\cos nt + (1/2) \times a\sin^2 nt]. \qquad ...(8)$$

For the maximum or minimum of p, we must have

$$\frac{d}{dt}\left(\frac{\Pi - p}{\rho}\right) = 0$$

i.e. $$n^2a\,[-6an\sin nt\cos nt - bn\sin nt + na\sin nt\cos nt] = 0,$$

giving $\sin nt = 0$ or $\cos nt = -(b/5a)$ *i.e.* $nt = 0$ or $nt = \cos^{-1}(-b/5a)$.

Now, $$\frac{d^2}{dt^2}\left(\frac{\Pi - p}{\rho}\right) = \frac{d}{dt}[n^2a\,\{-3an\sin 2nt - bn\sin nt + (1/2) \times an\sin 2nt\}$$

$$= n^2a\,[-6an^2\cos 2nt - bn^2\cos nt + an^2\cos 2nt]$$

$$= n^2a\,[-6an^2 - bn^2 + an^2], \text{ when } nt = 0$$

$\therefore\ \dfrac{d^2}{dt^2}\left(\dfrac{\Pi - p}{\rho}\right)$ is negative when $nt = 0$ $\Rightarrow$ $\dfrac{d^2p}{dt^2}$ is positive when $nt = 0$.

Putting $nt = 0$ in (8), the least pressure p is given by $(\Pi - p)/\rho = n^2a(a + b)$

and hence the required least pressure $= p = \Pi - \rho n^2a(a + b)$.

Ex. 17. *A sphere of radius a is alone in an unbounded liquid which is at rest at a great distance from the sphere and is subject to no external force. The sphere is forced to vibrate radially keeping its spherical shape, the radius r at any time being given by $r = a + b\cos nt$. Show that if Π is the pressure in the liquid at a great distance from the sphere, the least pressure (assumed positive) at the surface of the sphere during the motion is $\Pi - n^2\rho b(a + b)$.*

Hint. Refer Ex. 16.

Ex. 18. *A volume $(4/3) \times \pi c^3$ of gravitating liquid, of density ρ is initially in the form of a spherical shell of infinitely great radius. If the liquid shell contract under the influence of its own attraction, there being no external or internal pressure, show that when the radius of the inner spherical surface is r, its velocity will be given by*

$$V^2 = (4\pi\gamma\rho R/15r^3)\,(2R^4 + 2R^3r + 2R^2r^2 - 3Rr^3 - 3r^4),$$

where γ is the constant of gravitation, and $R^3 = r^3 + c^3$.

Sol. Let r be the radius of the inner surface and r' be the distance of the point from the centre of the spherical shell, where at time t, p is the pressure and v' is the velocity.

Here attraction F at the point of which distance from the centre is r' is given by

$$F = \frac{(4/3) \times \pi\gamma\rho(r'^3 - r^3)}{r'^2} = \frac{4}{3}\pi\gamma\rho\left(r' - \frac{r^3}{r'^2}\right). \qquad ...(1)$$

The equation of continuity is $\qquad r'^2 v' = F(t) = r^2 v. \qquad ...(2)$

From (2), $\qquad \partial v'/\partial t = F'(t)/r'^2 \qquad ...(3)$

The equation of motion is $\qquad \dfrac{\partial v'}{\partial t} + v'\dfrac{\partial v'}{\partial r'} = -F - \dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$

or
$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{4}{3}\pi\gamma\rho\left(r' - \frac{r^3}{r'^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ by (1) and (3)}$$

Integrating this with respect to r', we have

$$-\frac{F'(t)}{r'^2} + \frac{1}{2}v'^2 = -\frac{4}{3}\pi\gamma\rho\left(\frac{r'^2}{2} + \frac{r^3}{r'}\right) - \frac{p}{\rho} + C, \text{ } C \text{ being an arbitrary constant} \qquad ...(4)$$

Initially, at the inner surface, $r' = r$, $v' = v$ and $p = 0$.

$$\therefore (4) \quad \Rightarrow \quad -\frac{F'(t)}{r} + \frac{1}{2}v^2 = -\frac{4}{3}\pi\gamma\rho \times \frac{3}{2}r^2 + C. \qquad ...(5)$$

Also, initially, at the outer surface, $\quad r' = R, \quad v' = v_1$ (say) $\quad$ and $\quad p = 0$.

$$\therefore (4) \quad \Rightarrow \quad -\frac{F'(t)}{R} + \frac{1}{2}v_1^2 = -\frac{4}{3}\pi\gamma\rho\left(\frac{R^2}{2} + \frac{r^3}{R}\right) + C. \qquad ...(6)$$

Subtracting (6) from (5), we have

$$-F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{1}{2}(v^2 - v_1^2) = -\frac{4}{3}\pi\gamma\rho\left(\frac{3r^2}{2} - \frac{R^2}{2} - \frac{r^3}{R}\right). \qquad ...(7)$$

From equation of continuity, we have $\qquad r^2 v = R^2 v_1 = F(t). \qquad ...(8)$

From (8), $\qquad v = F(t)/r^2 \qquad$ and $\qquad v_1 = F(t)/R^2$. Then (7) becomes

$$-F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{1}{2}\left[\frac{\{F(t)\}^2}{r^4} - \frac{\{F(t)\}^2}{R^4}\right] = -\frac{4}{3}\pi\gamma\rho\left(\frac{3r^2}{2} - \frac{R^2}{2} - \frac{r^3}{R}\right). \qquad ...(9)$$

Now, (8) $\quad \Rightarrow \quad r^2(dr/dt) = R^2(dR/dt) = F(t) \quad \Rightarrow \quad r^2 dr = R^2 dR = F(t)\, dt \qquad ...(10)$

Multiplying each term of (9) by $2r^2\, dr$ or $2R^2 dR$ or $2F(t)\, dt$ (all being equal by virtue of (10)), we have

$$-\left(\frac{1}{r} - \frac{1}{R}\right) \times 2F(t)\, F'(t)\, dt + \frac{\{F(t)\}^2}{2}\left[\frac{2r^2 dr}{r^4} - \frac{2R^2 dR}{R^4}\right] = -\frac{4}{3}\pi\gamma\rho[3r^4 dr - R^4 dR - \frac{r^3}{R} \times 2R^2\, dR]$$

or
$$\left(\frac{1}{r} - \frac{1}{R}\right) d\{F(t)\}^2 + \{F(t)\}^2\, d\left(\frac{1}{r} - \frac{1}{R}\right) = \frac{4}{3}\pi\gamma\rho[3r^4\, dr - R^4\, dR - 2R(R^3 - c^3)\, dR]$$

[$\because$ Given that $R^3 - r^3 = c^3$]

Integrating, $$\left(\frac{1}{r}-\frac{1}{R}\right)\{F(t)\}^2=\frac{4}{3}\pi\gamma\rho\left[\frac{3r^5}{5}-\frac{R^5}{5}-\frac{2R^5}{5}+c^3R^2\right],$$

where we have chosen the constant of integration to be zero

or $$\left(\frac{1}{r}-\frac{1}{R}\right)\{F(t)\}^2=\frac{4}{15}\pi\gamma\rho\{3(r^5-R^5)+5c^3R^2\}. \qquad ...(11)$$

Given that when $r = r$, $v = V$. So from (8), $r^2V = F(t)$.

$\therefore$ (11) gives $$\left(\frac{1}{r}-\frac{1}{R}\right)r^4V^2=\frac{4}{15}\pi\gamma\rho\{3(r^5-R^5)+5R^2(R^3-r^3)], \text{ as } R^3=r^3+c^3$$

or $$V^2=\frac{4}{15}\frac{\pi\gamma\rho R}{r^3}\left\{\frac{3(r^5-R^5)+5R^2(R^3-r^3)}{R-r}\right\}$$

or $$V^2=\frac{4\pi\gamma\rho R}{15r^3}\{-3(r^4+r^3R+r^2R^2+rR^3+R^4)+5R^2(R^2+Rr+r^2)\}$$

$\therefore$ $$V^2=(4\pi\gamma\rho R/15r^3)\,(2R^4+2R^3r+2R^2r^2-3Rr^3-3r^4).$$

Ex. 19. *A homogeneous liquid is contained between two concentric spherical surfaces, the radius of the inner being a and that of the outer indefinitely great. The fluid is attracted to the centre of these surfaces by a force $\phi(r)$ and a constant pressure Π is exerted at the outer surface. Suppose $\int\phi(r)\,dr=\psi(r)$ and $\psi(r)$ vanishes when r is infinite, show that if the inner surface is removed, the pressure at the distance r is suddenly diminished by*

$$\Pi(a/r)-(a\rho/r)\psi(a).$$

Find $\phi(r)$ so that the pressure immediately after the inner surface is removed may be the same as it would be if no attractive force existed. Also with this value of $\phi(r)$, find the velocity of the inner boundary of the fluid at any period of the motion.

Sol. Let v' be the velocity and p the pressure at a distance r' from the centre at any time t. Then the equation of continuity is $$r'^2v'=F(t). \qquad ...(1)$$

From (1), $$\partial v'/\partial t=F'(t)/r'^2. \qquad ...(2)$$

The equation of motion is

$$\partial v'/\partial t+v'(\partial v'/\partial r')=-\phi(r')-(1/\rho)(\partial p/\partial r'),$$

or $$\frac{F'(t)}{r'^2}+\frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right)=-\phi(r')-\frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ by (2)}$$

Integrating it with respect to r', we have

$$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=-\int\phi(r')\,dr'-\frac{p}{\rho}+C, \; C \text{ being an arbitrary constant}$$

or $$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=-\psi(r')-\frac{p}{\rho}+C. \qquad ...(3)$$

($\because$ given that $\int\phi(r')\,dr'=\psi(r')$]

When $r'=\infty$, $v'=0$, $p=\Pi$ and $\psi(\infty)=0$. So (3) $\Rightarrow$ $C=\Pi/\rho$.

$\therefore$ (3) becomes $$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=-\psi(r')+\frac{\Pi-p}{\rho}. \qquad ...(4)$$

Given, initially when $t = 0$, $v' = 0$ and $p = p_0$ (say). Then (4) gives

$$-\frac{F'(0)}{r'}=-\psi(r')+\frac{\Pi-p_0}{\rho}. \qquad ...(5)$$

Again, when $r' = a$, $p_0 = 0$. So (5) reduces to

$$-(1/a)\, F'(0) = -\psi(a) + (\Pi/\rho). \qquad ...(6)$$

Dividing (5) by (6) and re-writting, we have

$$\Pi - p_0 - \rho\psi(r') = (1/r')\Pi a - (1/r')a\, \rho\psi(a). \qquad ...(7)$$

Initially the liquid was at rest. Then hydrostatic pressure is given by

$$dp' = -\rho\, \phi(r')\, dr' \qquad \text{so that} \qquad p' = C' - \rho\psi(r'). \qquad ...(8)$$

$$[\because \int \phi(r')\, dr' = \psi(r')]$$

But, when $r' = \infty$, $\psi(\infty) = 0$ and $p' = \Pi$. So (8) gives $C' = \Pi$ and hence (8) reduces to

$$p' = \Pi - \rho\psi(r'). \qquad ...(9)$$

Then decrease in pressure

$= p' - p_0 = \Pi - \rho\psi(r') - [\Pi - \rho\psi(r') + (1/r')\Pi a + (1/r')\, a\rho\psi(a)]$, using (7) and (9)

$\therefore$ The required decrease in pressure at distance $r = \Pi(a/r) - (a\rho/r)\psi(a)$.

Second Part : In presence of attractive forces, (7) gives

$$p_0 = \Pi - \rho\psi(r') - (1/r')\Pi a + (1/r')a\rho\psi(a). \qquad ...(10)$$

In absence of attractive forces, the terms containing ψ are zero and hence the corresponding pressure p_0' is given by (10) as

$$p_0' = \Pi - (1/r')\Pi a. \qquad ...(11)$$

But, by the condition of the problem, $p_0 = p_0'$. Hence, using (10) and (11), we get

$$\therefore \qquad \Pi - \rho\psi(r') - (1/r')\Pi a + (1/r')a\rho\psi(a) = \Pi - (1/r')\Pi a.$$

$$\therefore \qquad \psi(r') = (a/r')\psi(a), \qquad ...(12)$$

Given $\quad \int \phi(r')\, dr' = \psi(r') \quad$ so that $\quad \phi(r') = \dfrac{d}{dr'}\psi(r')$.

$$\therefore \qquad \phi(r') = \frac{d}{dr'}\left(\frac{a}{r'}\psi(a)\right) = -\frac{a\psi(a)}{r'^2}.$$

Substituting the above value of $\psi(r')$ in (4), we have

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{a\psi(a)}{r'} + \frac{\Pi - p}{\rho}. \qquad ...(13)$$

When $r' = r$, $v' = v$, $p = 0$. Then (13) becomes

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = -\frac{a\psi(a)}{r} + \frac{\Pi}{\rho}. \qquad ...(14)$$

The equation of continuity is $\qquad F(t) = r^2 v$.

$$\therefore \qquad F'(t) = \frac{d}{dt}(r^2 v) = \frac{d}{dr}(r^2 v)\cdot\frac{dr}{dt} = v\frac{d}{dr}(r^2 v), \qquad as \qquad v = \frac{dr}{dt}$$

$$= v\left[2rv + r^2\frac{dv}{dr}\right] = 2rv^2 + r^2 v\frac{dv}{dr}.$$

$\therefore$ Subtituting the above value of $F'(\mathrm{t})$ in (14), we have

$$-\frac{1}{r}\left(2rv^2 + r^2 v\frac{dv}{dr}\right) + \frac{1}{2}v^2 = -\frac{a\psi(a)}{r} + \frac{\Pi}{\rho}$$

or
$$rv\frac{dv}{dr} + 2v^2 - \frac{1}{2}v^2 = \frac{a\psi(a)}{r} - \frac{\Pi}{\rho}$$

or
$$2r^3 v\, dv + 3v^2 r^2 dr = [2ra\psi(a) - (2\Pi/\rho)r^2]\, dr$$

or $$d(r^3v^2) = [2ra\psi(a) - (2\Pi/\rho)r^2]\, dr.$$

Integrating, $r^3v^2 = a\psi(a)r^2 - (2\Pi/3\rho)r^3 + C'$, C' being an arbitrary constant ... (15)

When $r = a$, $v = 0$. So (15) gives $C' = (2\Pi/3\rho)a^3 - \psi(a)a^3$.

Putting this value of C' in (15), the required velocity is given by

$$r^3v^2 = a\psi(a)r^2 + (2\Pi/3\rho)\,(a^3 - r^3).$$

Ex. 20. *A mass of uniform liquid is in the form of a thick spherical shell bounded by concentric spheres of radii a and b (a < b). The cavity is filled with gas the pressure of which varies according to Boyle's law and is initially equal to atmospheric pressure* Π *and the mass of which may be neglected. The outer surface of the shell is exposed to atmospheric pressure. Prove that if the system is symmetrically disturbed, so that particle moves along a line joining it to the centre, the time of small oscillation is*

$$2\pi a\left\{\rho\frac{b-a}{3\Pi b}\right\}^{1/2}, \text{ where } \rho \text{ is the density of the fluid.}$$

Sol. Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the centre. Hence the free surface would be spherical. Thus the fluid velocity v will be radial and v will be function of x (the radial distance from the centre of the spherical shell which is taken as origin) and t only. Let p be pressure at a distance x. Then the equation of continuity is

$$x^2v = F(t) \quad \text{so that} \quad \partial v/\partial t = F'(t)/x^2 \qquad ...(1)$$

The equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{or} \quad \frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \text{ using (1).}$$

Integrating it w.r.t. 'x', we get $$-\frac{F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C. \qquad ...(2)$$

Let r and R be internal and external radii of the shell at any time t. Since the given shell contains gas, it follows that there will be pressure on the inner surface. Let $p = p_1$ when $x = r$.

Since the total mass of the liquid is constant, we have

$$\left(\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3\right)\rho = \left(\frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3\right)\rho \quad \text{or} \quad R^3 - r^3 = b^3 - a^3. \qquad ...(3)$$

Since the initial pressure of the gas is equal to atmospheric pressure Π, Boyle's law gives

$$(4/3) \times \pi r^3 \times p_1 = (4/3) \times \pi a^3 \times \Pi \quad \text{so that} \quad p_1 = (a^3\Pi)/r^3. \qquad ...(4)$$

Since the outer surface is exposed to atmospheric pressure Π, we have

when $x = R$, $v = dR/dt = U$, (say) and $p = \Pi$. So (2) gives

$$-\frac{F'(t)}{R} + \frac{1}{2}U^2 = -\frac{\Pi}{\rho} + C. \qquad ...(5)$$

Again, when $x = r$, $v = dr/dt = u$(say), $p = p_1 = (a^3\Pi)/r^3$, by (4)

So by (2), $$-\frac{F'(t)}{r} + \frac{1}{2}u^2 = -\frac{a^3\Pi}{\rho r^3} + C, \ C \text{ being an arbitrary constant.} \qquad ...(6)$$

Subtracting (6) from (5), we have

$$\left(\frac{1}{r} - \frac{1}{R}\right)F'(t) + \frac{1}{2}(U^2 - u^2) = \frac{\Pi}{\rho}\left(\frac{a^3}{r^3} - 1\right). \qquad ...(7)$$

Since we are to consider small oscillation, so U^2 and u^2 are small quantities and hence we neglect them. Then (7) reduces to

$$F'(t) = \frac{\Pi}{\rho} \times \frac{a^3 - r^3}{r^2} \times \frac{R}{R - r}. \qquad \text{...(8)}$$

By continuity equation (1), $\qquad F(t) = r^2 u.$

$$\therefore \qquad F'(t) = 2r\frac{dr}{dt}u + r^2\frac{du}{dt} = 2ru^2 + r^2\frac{d^2r}{dt^2}, \qquad \text{as} \qquad u = \frac{dr}{dt}$$

or $\qquad F'(t) = r^2(d^2r/dt^2)$, neglecting u^2 as before

Then (8) becomes
$$r^2\frac{d^2r}{dt^2} = \frac{\Pi}{\rho} \times \frac{a^3 - r^3}{r^2} \times \frac{R}{R - r}. \qquad \text{...(9)}$$

Since the displacement is small. We choose small quantities x and x' such that

$$r = a + x \qquad \text{and} \qquad R = b + x'. \qquad \text{...(10)}$$

Then (10) $\Rightarrow$ $\quad \dfrac{d^2r}{dt^2} = \dfrac{d^2}{dt^2}(a + x) \quad$ or $\quad \dfrac{d^2r}{dt^2} = \dfrac{d^2x}{dt^2} = \ddot{x}$. (say)

$\therefore$ (9) becomes
$$(a + x)^2\,\ddot{x} = \frac{\Pi}{\rho} \times \frac{a^3 - (a + x)^3}{(a + x)^2} \times \frac{b + x'}{b + x' - (a + x)}$$

or $\quad \ddot{x} = \dfrac{\Pi}{\rho} \times \dfrac{a^3 - a^3(1 + x/a)^3}{(a + x)^4} \times \dfrac{b + x'}{b - a + x' - x} \quad$ or $\quad \ddot{x} = \dfrac{\Pi}{\rho} \times \dfrac{1 - (1 + x/a)^3}{(1 + x/a)^4} \times \dfrac{b + x'}{x' - x + b - a}$

or $\quad \ddot{x} = \dfrac{\Pi}{a\rho} \times \dfrac{1 - (1 + 3x/a)}{(1 + 4x/a)} \times \dfrac{b + x'}{x' - x + b - a}$, to first order of approximation $\qquad$ ...(11)

Using (10), (3) reduces to

$(b + x')^3 - (a + x)^3 = b^3 - a^3 \qquad$ or $\qquad b^3(1 + x'/b)^3 - a^3(1 + x/a)^3 = b^3 - a^3$

or $\qquad b^3(1 + 3x'/b) - a^3(1 + 3x/a) = b^3 - a^3$, to first order of approximation

or $\qquad 3x'b^2 - 3a^2x = 0 \qquad$ or $\qquad x' = a^2x/b^2. \qquad$ ...(12)

Using (12), (11) reduces to

$$\ddot{x} = -\frac{\Pi}{a\rho}\,\frac{(3x/a)(b + a^2x/b^2)}{(1 + 4x/a)\,[(a^2x/b^2) - x + b - a]} = -\frac{\Pi}{a\rho}\,\frac{(3xb/a)}{(a^2x/b^2) - x + b - a + (4xb/a) - 4x}$$

[To first order approximation]

$$= -\frac{\Pi}{a\rho}\,\frac{(3xb/a)}{x(a^2/b^2 - 5 + 4b/a) + b - a} = -\frac{3b\Pi x}{a^2\rho(b - a)}\left[1 + \frac{(a^2/b^2) - 5 + (4b/a)}{b - a}x\right]^{-1}$$

$$= -\frac{3b\Pi x}{a^2\rho(b - a)}\left[1 - \frac{(a^2/b) - 5 + (4b/a)}{b - a}x + \ldots\right]$$

$$\therefore \qquad \ddot{x} = -\frac{3b\Pi}{a^2\rho(b - a)}x, \text{ to first order of approximation}$$

which represents simple harmonic motion of time period

$$\frac{2\pi}{[3b\Pi/a^2\rho(b - a)]^{1/2}} \qquad \text{or} \qquad 2\pi a\left\{\frac{\rho(b - a)}{3\Pi b}\right\}^{1/2}.$$

EXERCISE 3 (A)

1. Obtain Euler's equation of motion in cartesian form. **[Kanpur 2002; 2004]**

2. Prove that the equation of motion of a homogeneous inviscid liquid moving under conservative forces may be written in the form $\dfrac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \times \text{curl}\, \mathbf{q} = -\,\text{grad}\left(\dfrac{p}{\rho} + \dfrac{1}{2}q^2 + \Omega\right)$

[**Hint.** From Art. 3.1, we have $\dfrac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \times \text{curl}\, \mathbf{q} = \mathbf{F} - \dfrac{1}{\rho}\nabla p - \dfrac{1}{2}\nabla q^2$ ...(1)

Since the forces form a conservative system, there exists a force potential Ω such that $\mathbf{F} = -\nabla\Omega$. Moreover, the fluid being homogeneous, we may write $(1/\rho)\,\nabla p = \nabla(p/\rho)$.

Hence (1) reduces to

$$\frac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \times \text{curl}\, \mathbf{q} = -\nabla\Omega - \nabla\left(\frac{p}{\rho}\right) - \nabla\left(\frac{1}{2}q^2\right) \quad \text{or} \quad \frac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \times \text{curl}\, \mathbf{q} = -\,\text{grad}\left(\frac{p}{\rho} + \frac{1}{2}q^2 + \Omega\right)$$

3. A mass of fluid of density ρ and volume $(4\pi c^3)/3$ is in the form of a speherical shell. There is a constant pressure p on the external surface, and zero pressure on the internal surface. Initially the fluid is at rest and the external radius $2nc$. Show that when the external radius becomes nc the velocity U of the external surface is given by

$$U^2 = \frac{14p}{3\rho}\frac{(n^3-1)^{1/3}}{n-(n^3-1)^{1/3}}$$

Ex. 4. *The particle velocity for a fluid motion referred to rectangular axes is given by* $(A\cos(\pi x/2a)\cos(\pi z/2a), 0, \; A\sin(\pi x/2a)\sin(\pi z/2a))$, *where A, a are constants. Show that this is a possible motion of an inconpressible fluid under no body forces in an infinite fixed rigid tube* $-a \le x \le a, \;\; 0 \le z \le 2a$. *Also find the pressure associated with this velocity field.*

Sol. Let u, v, w be the components of velocity referred to rectangular axes OX, OY, OZ. Then we have $u = A\cos(\pi x/2a)\cos(\pi z/2a)$, $v = 0$, $w = A\sin(\pi x/2a)\sin(\pi z/2a)$.

Now do as in solved example 14 of Art 3.4

Ex. 5. *A sphere is at rest in an infinite mass of homogeneous liquid of density* ρ. *The pressure at infinity being* $\overline{w}$. *Show that, if the radius R of the sphere varies in any manner, the pressure at the surface of the sphere at any time is* $\overline{w} + (\rho/2)\{d^2(R)^2/dt^2 + (dR/dt)^2\}$.

Sol. Refer solved Ex. 1 of Art 3.4 by taking $\Pi = \overline{w}$. **[I.A.S. 1996]**

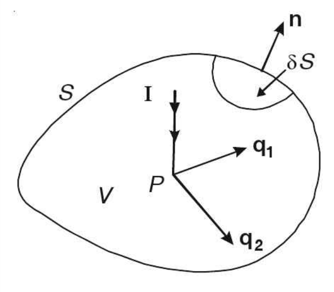
3.5. Impulsive action.

Let sudden velocity changes be produced at the boundaries of an incompressible fluid or that impulsive forces be made to act to its interior. Then it is known that the impulsive pressure at any point is the same in every direction. Moreover the dstrubances produced in both cases are propagated instantaneously throughout the fluid.

3.6. Equation of motion under Impulsive forces (Vector form).

[Meerut 2007; Kanpur 2000, 03, 05, 09; Rohilkhand 2000, 05]

Let S be an arbitrary small closed surface drawn in the incompressible fluid enclosing a volume V. Let $\mathbf{I}$ be the impulsive body force per unit mass. Let this impulse change the velocity at



P ($\mathbf{r}$, t) of V instantaneously from $\mathbf{q}_1$ to $\mathbf{q}_2$ and let it produce impulsive pressure on the boundary S. Let $\tilde{\omega}$ denote the inpulsive pressure on the element δS of S. Let $\mathbf{n}$ be the unit outward drawn normal at δS. Let ρ be density of the fluid.

We now apply Newton's second law for inpulsive motion to the fluid enclosed by S, namely,

Total impulse applied = Change of momentum

$$\therefore \qquad \int_V \mathbf{I}\rho dV - \int_S \mathbf{n}\tilde{\omega} dS = \int_V \rho(\mathbf{q}_2 - \mathbf{q}_1) dV \qquad ...(1)$$

But $$\int_S \mathbf{n}\,\tilde{\omega}\, dS = \int_V \nabla\tilde{\omega}\, dV \qquad \text{(by Gauss divergence theorem)}$$

$$\therefore \text{ From (1),} \qquad \int_V [\mathbf{I}\rho - \nabla\tilde{\omega} - \rho(\mathbf{q}_2 - \mathbf{q}_1)] dV = 0 \qquad ...(2)$$

Since V is an arbitrary small volume, (2) gives

$$\mathbf{I}\rho - \nabla\tilde{\omega} - \rho(\mathbf{q}_2 - \mathbf{q}_1) = 0 \qquad \text{or} \qquad \mathbf{q}_2 - \mathbf{q}_1 = \mathbf{I} - (1/\rho)\nabla\tilde{\omega} \qquad ...(3)$$

Cor. 1. Let $\mathbf{I} = \mathbf{0}$ (*i.e.* external impulsive body forces are absent) whereas impulsive pressures be present. Then (3) reduces to

$$\mathbf{q}_2 - \mathbf{q}_1 = -(1/\rho)\nabla\tilde{\omega} \qquad ...(4)$$

From (4), $$\nabla\cdot(\mathbf{q}_2 - \mathbf{q}_1) = \nabla\cdot[-(1/\rho)\nabla\tilde{\omega}]$$

or $$\nabla\cdot\mathbf{q}_2 - \nabla\cdot\mathbf{q}_1 = -(1/\rho)\nabla^2\tilde{\omega}, \qquad ...(5)$$

For the incompressible fluid, the equation of continuity gives

$$\nabla\cdot\mathbf{q}_2 = \nabla\cdot\mathbf{q}_1 = 0 \qquad ...(6)$$

Making use of (6), (5) reduces to

$$\nabla^2\tilde{\omega} = 0. \qquad \text{(Laplace's equation)} \qquad ...(7)$$

Cor. 2. Let $\mathbf{q}_1 = \mathbf{0}$ and $\mathbf{I} = \mathbf{0}$ so that the motion is started from rest by the application of impulsive pressure at the boundaries but without use of external impulsive body forces. Then, writing $\mathbf{q}_2 = \mathbf{q}$, (3) reduces to

$$\mathbf{q} = -\nabla(\tilde{\omega}/\rho), \qquad ...(8)$$

showing that there exists a velocity potential $\phi = \tilde{\omega}/\rho$ and that the motion is irrotational.

Cor. 3. Let $\mathbf{I} = \mathbf{0}$ *i.e.* let there be no extraneous impulses. Further, let ϕ_1 and ϕ_2 denote the velocity potential just before and just after the impulsive action. Then

$$\mathbf{q}_1 = -\nabla\phi_1 \qquad \text{and} \qquad \mathbf{q}_2 = -\nabla\phi_2 \qquad ...(9)$$

Then (3) reduces to

$$-\nabla\phi_2 + \nabla\phi_1 = -(1/\rho)\nabla\tilde{\omega} \qquad \text{or} \qquad \nabla\tilde{\omega} = \rho\nabla(\phi_2 - \phi_1)$$

Integrating, when ρ is constant $\qquad \tilde{\omega} = \rho(\phi_2 - \phi_1) + C.$

The constant C may be omitted by regarding as an extra pressure and constant throughout the fluid.

$$\therefore \qquad \tilde{\omega} = \rho\phi_2 - \rho\phi_1. \qquad ...(10)$$

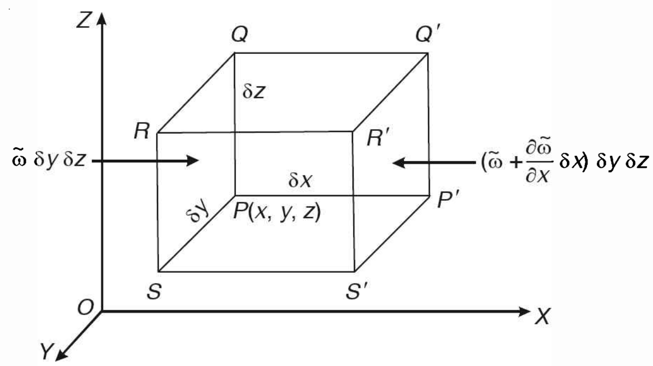
Cor. 4. Physical meaning of velocity potential.

Take $\phi_1 = 0$ and $\rho = 1$ in cor. 3. Then we find that any actual motion, for which a single valued velocity potential exists, could be produced instantaneously from rest by applying appropriate impulses. We then also note that the velocity potential is the impulsive pressure at any point.

It is also easily seen that when a state of rotational motion exists in a fluid, the motion could neither be created nor destroyed by impulsive pressures.

3.7. Equations of motion under Impulsive Force (Cartesian form). [Kanpur 2002, 05]

Let there by a fluid particle at $P(x, y, z)$ and let ρ be the density of the incompressible fluid. Let u_1, v_1, w_1 and u_2, v_2, w_2 be the velocity components at the point P just before and just after the impulsive action. Let I_x, I_y, I_z be the components of the external impulsive forces per unit mass of the fluid. Construct a small parallelepiped with edges of lengths δx, δy, δz parallel to their respective co-ordinate axes, having P at one of the angular points as shown in figure. Let $\tilde{\omega}$ denote the impulsive pressure at P. Then, we have



Force on the face $PQRS = \tilde{\omega}\,\delta y\,\delta z = f(x, y, z)$ say ...(1)

$\therefore$ Force on the face $P'Q'R'S' = f(x + \delta x, y, z) = f(x, y, z) + \delta x \dfrac{\partial}{\partial x} f(x, y, z) + \ldots$ ...(2)

(expanding by Taylor's theorem)

$\therefore$ The net force on the opposite faces $PQRS$ and $P'Q'R'S'$

$$= f(x, y, z) - [f(x, y, z) + \delta x \frac{\partial}{\partial x} f(x, y, z) + \ldots]$$

$$= -\,\delta x \frac{\partial}{\partial x} f(x, y, z), \text{ to the first order of approximation}$$

$$= -\,\delta x \frac{\partial}{\partial x} (\tilde{\omega}\,\delta y\,\delta z), \text{ using (1)}$$

$$= -\,\delta x\,\delta y\,\delta z \frac{\partial \tilde{\omega}}{\partial x}, \text{ which will act along the } x\text{-axis.} \qquad \text{...(3)}$$

Again, the impulse on the elementary parallelepiped along the x-axis due to external impulsive body force I_x $= \rho\,\delta x\,\delta y\,\delta z\,I_x$. ...(4)

Finally, the change in momentum along x-axis $= \rho\,\delta x\,\delta y\,\delta z\,(u_2 - u_1)$ ...(5)

We now apply Newton's second law for impulsive motion to the fluid enclosed by the parallelopiped, namely,

Total impulse applied along x-axis = Change of momentum along x-axis

$$\therefore \qquad -\delta x\delta y\delta z\,\frac{\partial\tilde{\omega}}{\partial x}+\rho\delta x\delta y\delta z\,I_x=\rho\delta x\delta y\delta z\,(u_2-u_1)$$

or
$$\rho(u_2-u_1)=\rho I_x-(\partial\tilde{\omega}/\partial x) \qquad ...(6)$$

Similarly
$$\rho(v_2-v_1)=\rho I_y-(\partial\tilde{\omega}/\partial y) \qquad ...(7)$$

and
$$\rho(w_2-w_1)=\rho I_z-(\partial\tilde{\omega}/\partial z). \qquad ...(8)$$

Equations (6), (7) and (8) are the required equations of motion of an incompressible fluid under impulsive forces.

3.8. Illustrative solved examples.

Ex. 1. *A sphere of radius a is surrounded by infinite liquid of density* ρ, *the pressure at infinity being* Π. *The sphere is suddenly annihilated. Show that the pressure at a distance r from the centre immediately falls to* $\Pi(1-a/r)$. **[Purvanchel 2004, I.A.S. 1996]**

Show further that if the liquid is brought to rest by impinging on a concentric sphere of radius a/2, the impulsive pressure sustained by the surface of this sphere is $(7\Pi\rho^2/6)^{1/2}$.

Sol. Let v' be the velocity at a distance r' from the centre of the sphere at any time t and p the pressure there. Then the equation of continuity is

$$r'^2v'=F(t) \qquad ...(1)$$

From (1),
$$\partial v'/\partial t=F'(t)/r'^2 \qquad ...(2)$$

The equation of motion is

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial P}{\partial r'} \qquad \text{or} \qquad \frac{F'(t)}{r'^2}+\frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right)=-\frac{1}{\rho}\frac{\partial P}{\partial r'}, \text{ using (2)}$$

Integrating,
$$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=-\frac{p}{r'}+C, \; C \text{ being an arbitrary constant.}$$

When $r'=\infty$, then $p=\Pi$ and $v'=0$ so that $C=\Pi/\rho$.

$$\therefore \qquad -\frac{F'(t)}{r'}+\frac{1}{2}v'^2=\frac{\Pi-p}{\rho} \qquad ...(3)$$

When the sphere is suddenly annihilated, we have

$t=0, \qquad r'=a, \qquad v'=0 \qquad$ and $\qquad p=0$

$\therefore$ From (3),
$$-\frac{F'(0)}{a}=\frac{\Pi}{\rho} \qquad \text{so that} \qquad F'(0)=-\frac{a\Pi}{\rho}$$

Hence immediately after the annihilation of the sphere (with $t=0$, $v'=0$), (3) reduces to

$$\frac{a\Pi}{\rho r'}+0=\frac{\Pi-p}{\rho} \qquad \text{or} \qquad p=\Pi\left(1-\frac{a}{r'}\right) \qquad ...(4)$$

Thus at the time of annihilation, when $r'=r$, the pressure is given by

$$p=\Pi(1-a/r'). \qquad ...(5)$$

Second Part. If $\bar{\omega}$ be the impulsive pressure at distance r', then we have

$$d\tilde{\omega} = -\rho v' dr' \qquad ...(6)$$

Let r be the radius of the inner surface and v the velocity there. Then by the equation of continuity, we have

$$F(t) = r^2 v = r'^2 v' \qquad \text{so that} \qquad v' = (r^2 v)/r'^2 \qquad ...(7)$$

$\therefore$ (6) gives $\qquad d\omega' = \rho v (r^2 / r'^2) dr'$

Integrating with respect to r', we get $\qquad \tilde{\omega} = \rho v (r^2 / r') + C' \qquad ...(8)$

When $\quad r' = \infty, \quad \tilde{\omega} = 0 \quad$ so that $\quad C' = 0.$

$$\therefore \qquad \tilde{\omega} = \rho v (r^2 / r'), \qquad ...(9)$$

which gives the impulsive pressure $\tilde{\omega}$ at a distance r'. Since $r = a/2$, (9) reduces to

$$\tilde{\omega} = \frac{1}{4} \rho v a^2 \cdot \frac{1}{r'} \qquad ...(10)$$

We now determine velocity v at the inner surface of the sphere. Setting $r' = r$, $v' = v$ and $p = 0$ in (3), we get

$$-\frac{F'(t)}{r} + \frac{1}{2} v^2 = \frac{\Pi}{\rho} \qquad ...(11)$$

From (7), $\qquad F'(t) = \frac{d}{dt}(r^2 v) = 2r \frac{dr}{dt} v + r^2 \frac{dv}{dt} = 2r \frac{dr}{dt} v + r^2 \frac{dv}{dt} \frac{dr}{dr}$

Thus, $\qquad F'(t) = 2rv^2 + r^2 v \frac{dv}{dr}, \qquad$ as $\qquad v = \frac{dr}{dt}$

$\therefore$ (11) gives $\qquad -\frac{1}{r}\left(2rv^2 + r^2 v \frac{dv}{dr}\right) + \frac{1}{2} v^2 = \frac{\Pi}{\rho}$

Multiplying both sides by $(-2r^2\, dr)$, we get

$$2r^3 v\, dv + 3r^2 v^2\, dr = -\frac{2\Pi r^2}{\rho} dr \qquad \text{or} \qquad d(r^3 v^2) = -\frac{2\Pi r^2}{\rho} dr$$

Integrating, $\qquad r^3 v^2 = -\frac{2\Pi r^3}{3\rho} + C''$, C'' being an arbitrary constant

When $\quad r = a, \quad v = 0 \quad$ so that $\quad C'' = -\frac{2\Pi a^3}{3\rho}.$

$$\therefore \qquad r^3 v^2 = \frac{2\Pi}{3\rho}(a^3 - r^3) \qquad ...(12)$$

The velocity v on the surface of the sphere of radius $a/2$ (which would be the inner surface on which the liquid impinges) is given by (12) by replacing r by $a/2$

$$\therefore \qquad v^2 = \frac{2\Pi}{3\rho} \times \frac{a^3 - a^3/8}{a^3/8} = \frac{14}{3} \times \frac{\Pi}{\rho}$$

Putting this value of v in (10), the impulsive pressure at a distance r' is given by

$$\tilde{\omega} = \frac{\rho}{4}\left(\frac{14}{3} \times \frac{\Pi}{\rho}\right)^{1/2} \frac{a^2}{r'} \qquad \text{...(13)}$$

Hence the desired impulsive pressure on the surface of the sphere of radius $a/2$ is given by setting $r' = a/2$ in (13).

$$\therefore \qquad \tilde{\omega} = \frac{\rho}{4}\left(\frac{14}{3} \times \frac{\Pi}{\rho}\right)^{1/2} \times \frac{a^2}{(a/2)} = \left(\frac{7\Pi\rho a^2}{6}\right)^{1/2}$$

Ex. 2. *A portion of homogeneous fluid is contained between two concentric spheres of radii A and a, and is attracted towards their centre by a force varying inversely as the square of the distance. The inner spherical surface is suddenly annihilated and when the radii of the inner and outer surfaces of the fluid are r and R the fluid impinges on a solid ball concentric with these surfaces, prove that the impulsive pressure at any point of the ball for different values of R and r varies as*

$$\left\{(a^2 - r^2 - A^2 + R^2)\,(1/r - 1/R)\right\}^{1/2}$$ **[Agra 1996; Kanpur 1998]**

Sol. Let v' be the velocity at a distance r' from the centre of the sphere at any time t and p the pressure there. Then the equation of continuity is

$$r'^2 v' = F(t) \qquad \text{...(1)}$$

From (1), $$\partial v'/\partial t = F'(t)/r'^2 \qquad \text{...(2)}$$

Taking μ/r'^2 as the force towards the centre of the sphere, the equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \text{or} \qquad \frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)}$$

Integrating, $$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\mu}{r'} - \frac{p}{\rho} + C, \; C \text{ being an arbitrary constant} \qquad \text{...(3)}$$

Let r and R be the internal and external radii of the fluid at any time t and v and V be the velocities there. Thus, we have

When $r' = R$, $v' = V$, $p = 0$ and also when $r' = 0$, $v' = v$, $p = 0$

$\therefore$ (3) yields $$-\frac{F'(t)}{R} + \frac{1}{2}V^2 = C + \frac{\mu}{R} \qquad \text{...(4)}$$

and $$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = C + \frac{\mu}{r} \qquad \text{...(5)}$$

Subtracting (4) from (5), we have

$$-F'(t)\left[\frac{1}{r} - \frac{1}{R}\right] + \frac{1}{2}(v^2 - V^2) = \mu\left(\frac{1}{r} - \frac{1}{R}\right) \qquad \text{...(6)}$$

From the equation of continuity (1), we have

$$r^2 v = R^2 V = F(t) \qquad \text{...(7)}$$

From (7), $$r^2 \frac{dr}{dt} = R^2 \frac{dR}{dt} = F(t)$$

$\therefore$ $$r^2\, dr = R^2\, dR = F(t)\, dt \qquad ...(8)$$

Using (7), (6) reduces to

$$-F'(t)\left[\frac{1}{r}-\frac{1}{R}\right]+\frac{1}{2}\{F(t)\}^2\left[\frac{1}{r^4}-\frac{1}{R^4}\right]=\mu\left[\frac{1}{r}-\frac{1}{R}\right]$$

Multiplying both sides by $2F(t)\, dt$, we get

$$-2F(t)F'(t)\left(\frac{1}{r}-\frac{1}{R}\right)dt+\frac{1}{2}\{F(t)\}^2\left[\frac{2F(t)}{r^4}-\frac{2F(t)}{R^4}\right]dt=\mu\left[\frac{2F(t)}{r}-\frac{2F(t)}{R}\right]dt$$

or $$-2F(t)F'(t)\left(\frac{1}{r}-\frac{1}{R}\right)dt+\frac{1}{2}\{F(t)\}^2\left[\frac{2dr}{r^2}-\frac{2dR}{R^2}\right]=\mu(2rdr-2RdR), \text{ using (8)}$$

Integrating, $$-\{F(t)\}^2\left[\frac{1}{r}-\frac{1}{R}\right]=\mu(r^2-R^2)+C', \text{ being arbitrary constant} \qquad ...(9)$$

Since velocity is zero when $r = a$ and $R = A$, it follows that $F(t) = 0$. Then (9) reduces to

$$0=\mu(a^2-A^2)+C' \qquad i.e. \qquad C'=-\mu(a^2-A^2)$$

$\therefore$ (9) becommes $$-\{F(t)\}^2\left[\frac{1}{r}-\frac{1}{R}\right]=\mu(r^2-R^2-a^2+A^2) \qquad ...(10)$$

If $\tilde{\omega}$ be the impulsive pressure at a distance r', then we have

$$d\tilde{\omega}=-\rho v'\,dr'=-\rho\frac{F(t)}{r'^2}dr', \text{ using (1)}$$

Integrating, $$\tilde{\omega}=\frac{\rho F(t)}{r'^2}+C'', \quad C'' \text{ being an arbitrary constant}$$

But when, $r'=R$, $\tilde{\omega}=0$ so that $C''=[\rho F(t)]/R$. So the above equation gives

$\therefore$ $$\tilde{\omega}=\rho F(t)\,(1/r'-1/R)$$

Hence the impulsive pressure at any point of the ball where $r'=r$ is given by

$$\tilde{\omega}=\rho F(t)\,(1/r-1/R) \qquad ...(11)$$

From (10), $$F(t)=\left\{\frac{\mu(a^2-r^2-A^2+R^2)}{(1/r-1/R)}\right\}^{1/2}$$

$\therefore$ $$\tilde{\omega}=\rho\sqrt{\mu}\left\{(a^2-r^2-A^2+R^2)\,(1/r-1/R)\right\}^{1/2},$$

showing that the required impulsive pressure varies as $\left\{(a^2-r^2-A^2+R^2)(1/r-1/R)\right\}^{1/2}$

EXERCISE 3(B)

1. If a bomb shell explodes at a great depth beneath the surface of the sea, prove that the impulsive pressure at any point varies inversely as the distance from the centre of the shell.

2. Prove that if $\tilde{\omega}$ be the impulsive pressure, ϕ, ϕ' the velocity potentials immediately before and after an impulse acts, V the potential of the impulses, then $\tilde{\omega}+\rho V+\rho(\phi'-\phi)=\text{const.}$

3. Find the equations of motion of a perfect fluid under extraneous impulses and impulsive pressure. Deduce that any actual irrotational motion of a liquid can be produced instantaneously from rest by a set of impulses properly applied.

4. Prove that in the absence of extraneous impulses the impulsive pressure at any point of a liquid satisfies Laplace's equation.

5. Obtain general equation of motion for impulsive action. **[Kanpur 2002, 03, 05]**

6. Prove that $\tilde{\omega} = \rho\phi$. **[Kanpur 2001]**

7. Derive equation of motion under impulsive forces and prove that the impulse satisfies the Laplace's equation. **[Meerut 2007]**

3.9. The energy equation.

[Kanpue 2007; Agra 2005; Banglore 2006; Patna 2003, 06; Garhwal 2005]

Statement : *The rate of change of total energy (kinetic, potential and intrinsic) of any portion of a compressible inviscid fluid as it moves about is equal to the rate at which work is being done by the pressure on the boundary. The potential due to the extraneous forces is supposed to be independent of time.*

Proof. Consider any arbitrary closed surface S drawn in the region occupied by the inviscid fluid and let V be the volume of the fluid within S. Let ρ be the density of the fluid particle P within S and dV be the volume element surrounding P. Let $\mathbf{q}$ $(\mathbf{r}, t)$ be the velocity of P. Then, the Euler's equation of motion is

$$d\mathbf{q}/dt = -(1/\rho)\nabla p + \mathbf{F}. \qquad \text{... (1)}$$

Let the external forces be conservative so that there exists a force potential Ω which is independent of time. Thus $\mathbf{F} = -\nabla\Omega$ and $\partial\Omega/\partial t = 0$.

Using the above results and then multiplying both sides of (1) scalarly by $\mathbf{q}$, we get

$$\rho\left(\mathbf{q}\cdot\frac{d\mathbf{q}}{dt}\right) = -\mathbf{q}\cdot\nabla p - \rho[\mathbf{q}\cdot\nabla\Omega] \quad \text{or} \quad \rho\left[\frac{d}{dt}\left(\frac{1}{2}q^2\right) + (\mathbf{q}\cdot\nabla)\Omega\right] = -\mathbf{q}\cdot\nabla p \qquad \text{... (2)}$$

But $$\frac{d\Omega}{dt} = \frac{\partial\Omega}{\partial t} + (\mathbf{q}\cdot\nabla)\Omega = (\mathbf{q}\cdot\nabla)\Omega, \qquad \text{since} \quad \frac{\partial\Omega}{\partial t} = 0$$

Hence, equation (2) becomes $$\rho\frac{d}{dt}\left(\frac{1}{2}q^2 + \Omega\right) = -\mathbf{q}\cdot\nabla p \qquad \text{... (3)}$$

Since the elementary mass remains invariant throughout the motion, so $d(\rho V)/dt = 0$ ...(4)

Integrating both sides of (3) over V, we have

$$\int_V \frac{d}{dt}\left(\frac{1}{2}q^2\right)\rho dV + \int_V \frac{d}{dt}(\rho\Omega)\, dV = -\int_V (\mathbf{q}.\nabla p)\, dV$$

or $$\int_V \left\{\frac{d}{dt}\left(\frac{1}{2}q^2\right)\rho dV + \frac{1}{2}q^2\frac{d}{dt}(\rho dV)\right\} + \int_V \frac{d}{dt}(\rho\Omega dV) = -\int_V (\mathbf{q}.\nabla p) dV$$

[Noting that , (4) $\Rightarrow (q^2/2)\times\{d(\rho dV)/dt\} = 0$]

Thus, $$\frac{d}{dt}\int_V \left(\frac{1}{2}\rho q^2\right) dV + \frac{d}{dt}\int_V (\rho\Omega)\, dV = -\int_V (\mathbf{q}.\nabla p)\, dV \qquad \text{... (5)}$$

Let T, W and $\mathbf{I}$ denote the kinetic, potential and intrinsic (internal) energies respectively. Then, by definitions

---

* Here, we write d/dt for D/Dt so that $d/dt = D/Dt = \partial/\partial t + \mathbf{q}\cdot\nabla$ (Refer note of Art. 2.4)

$$T = \int_V \frac{1}{2}\rho q^2\, dV, \qquad W = \int_V \rho\Omega\, dV, \qquad \mathbf{I} = \int_V \rho E\, dV, \qquad ...(6)$$

where E is the intrinsic energy per unit mass,

Since $\nabla \cdot (p\mathbf{q}) = p\nabla\cdot\mathbf{q} + \mathbf{q}\cdot\nabla p$, we have $\mathbf{q}\cdot\nabla p = \nabla\cdot(p\mathbf{q}) - p\nabla\cdot\mathbf{q}$

$$\therefore \text{ R.H.S. of (4)} = -\int_V \nabla\cdot(p\mathbf{q})\, dV + \int_V p\nabla\cdot\mathbf{q}dV = \int_S p\mathbf{q}\cdot\mathbf{n}dS + \int_S p\nabla\cdot\mathbf{q}dV, \qquad ...(7)$$

[By Gauss divergence theorem]

where $\mathbf{n}$ is unit inward normal and dS is the element of the fluid surface S. We now prove that

$$\int_V p\nabla\cdot\mathbf{q}dV = -\frac{d\mathbf{I}}{dt} \qquad ...(8)$$

Now, E is defined as the work done by the unit mass of the fluid against external pressure p (assuming that there exists a relation between pressure and density) from its actual state to some standard state in which p_0 and ρ_0 are the values of pressure and density respectively.

$$\therefore \quad E = \int_V^{V_0} p dV, \qquad \text{where } V\rho = 1, \qquad i.e., \qquad V = 1/\rho$$

or

$$E = \int_\rho^{\rho_0} p\, d\left(\frac{1}{\rho}\right) = -\int_\rho^{\rho_0} \frac{p}{\rho^2}\, d\rho = \int_{\rho_0}^{\rho} \frac{p}{\rho^2}\, d\rho \qquad ...(9)$$

From (9), $\quad \dfrac{dE}{d\rho} = \dfrac{p}{\rho^2} \quad$ and so $\quad \dfrac{dE}{dt} = \dfrac{dE}{d\rho}\dfrac{d\rho}{dt} = \dfrac{p}{\rho^2}\dfrac{d\rho}{dt}$

Multiplying both sides by ρdV and then integrating over a volume V, we have

$$\int_V \frac{dE}{dt}\rho dV = \int_V \frac{p}{\rho}\frac{d\rho}{dt} dV \qquad ...(10)$$

But $\quad \dfrac{d}{dt}(E\rho dV) = \dfrac{dE}{dt}\rho dV + E\dfrac{d}{dt}(\rho dV)$

$$\therefore \quad \frac{d}{dt}(E\rho dV) = \frac{dE}{dt}\rho dV, \text{ using (4)} \qquad ...(11)$$

Also from the equation of continuity, $\quad d\rho/dt = -\rho\nabla\cdot\mathbf{q} \qquad ...(12)$

Using (11) and (12), (10) reduces to

$$\frac{d}{dt}\int_V E\rho dV = -\int_V p\nabla\cdot\mathbf{q}dV \qquad \text{or} \qquad \frac{d\mathbf{I}}{dt} = -\int_V p\nabla\cdot\mathbf{q}dV, \text{ by (6)}$$

which proves (8).

Again the rate of work done by the fluid pressure on an element δS of S is $p\,\delta S\ \mathbf{n}\cdot\mathbf{q}$.

Hence the net rate at which work is being done by the fluid pressure is

$$\int_S p\mathbf{q}\cdot\mathbf{n}\, dS = R, \text{ (say)} \qquad ...(13)$$

Using (8) and (13), (7) reduces to

$$\text{R.H.S. of (4)} = R - d\mathbf{I}/dt \qquad ...(14)$$

Hence using (6) and (14), (4) reduces to $\quad \dfrac{d}{dt}(T + W + I) = R, \qquad ...(15)$

which is the desired energy equation. It is also known as "the *Volume integral form of Bernoulli's equation*".

Re-writing (15), $$\frac{d}{dt}(T+W)=R-\frac{d\mathbf{I}}{dt}=\int_S p\mathbf{q}\cdot\mathbf{n}\,dS+\int_V p\nabla\cdot\mathbf{q}\,dV \qquad \ldots(15)'$$

[on putting values of R and dI/dt]

Corollary. Energy equation for incompressible fluids.

Since $I = 0$ for incompressible fluids, (15) reduces to

$$\frac{d}{dt}(T+W)=R. \qquad \ldots(16)$$

Remark. Many problems solved so far in this chapter may also be solved by using the energy equation. This principle is used to shorten the solution. In what follows, we will give two methods to solve many problems.

The energy equation is stated as follows : *The rate of increase of energy in the system is equal to the rate at which work is done on the system.*

3.10. Illustrative solved examples.

Ex. 1. *An infinite mass of fluid is acted on by a force* $\mu/r^{3/2}$ *per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere* $r = c$ *in it, show that the cavity will be filled up after an interval of time* $(2/5\mu)^{1/2}c^{5/4}$.

[Kanpur 1999, 2009; Meerut 2005; I.A.S. 2003]

Sol. Method I At any time t, let v' be the velocity at distance r' from the centre. Again, let r be the radius of the cavity and v its velocity. Then the equation of continuity yields

$$r'^2v'=r^2v \qquad \ldots(1)$$

When the radius of the cavity is r, then

$$\text{Kinetic energy } =\int_r^\infty \frac{1}{2}(4\pi r'^2\rho dr')\cdot v'^2 \qquad \left[\because \text{Kinetic energy}=\frac{1}{2}\times\text{mass}\times(\text{velocity})^2\right]$$

$$=2\pi\rho r^4v^2\int_r^\infty\frac{dr'}{r'^2},\ \text{using (1)}$$

$$=2\pi\rho r^3v^2.$$

The initial kinetic energy is zero.

Let V be the work function (or force potential) due to external forces. Then, we have

$$-\frac{\partial V}{\partial r'}=\frac{\mu}{r'^{3/2}} \qquad \text{so that} \qquad V=\frac{2\mu}{r'^{1/2}}$$

$\therefore$ the work done $=\int_r^c V\,dm$, dm being the elementary mass

$$=\int_r^c\left(\frac{2\mu}{r'^{1/2}}\right)\cdot 4\pi r'^2dr'\rho=8\pi\mu\rho\int_r^c r'^{3/2}dr'=\frac{16}{5}\pi\rho\mu(c^{5/2}-r^{5/2})$$

We now use energy equation, namely, Increase in kinetic energy $=$ work done

This $\Rightarrow$ $$2\pi\rho r^3v^2-0=(16/5)\times\pi\rho\mu(c^{5/2}-r^{52})$$

$$\therefore \qquad v=\frac{dr}{dt}=-\left(\frac{8\mu}{5}\right)^{1/2}\frac{(c^{5/2}-r^{5/2})^{1/2}}{r^{3/2}} \qquad \ldots(2)$$

wherein negative sign is taken because r decreases as t increases.

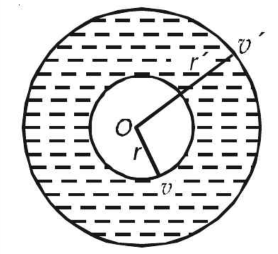
Let T be the time of filling up the cavity. Then (2) gives

$$\int_0^T dt = -\left(\frac{5}{8\mu}\right)^{1/2} \int_c^0 \frac{r^{3/2}\,dr}{\sqrt{(c^{5/2}-r^{5/2})}} \qquad \text{or} \qquad T = \left(\frac{5}{8\mu}\right)^{1/2} \int_0^c \frac{r^{3/2}\,dr}{\sqrt{(c^{5/2}-r^{5/2})}}$$

Put $\quad r^{5/2} = c^{5/2}\sin^2\theta \quad$ so that $\quad (5/2)\times r^{3/2}\,dr = 2c^{5/2}\sin\theta\cos\theta\,d\theta.$

$$\therefore \qquad T = \left(\frac{5}{8\mu}\right)^{1/2} \int_0^{\pi/2} \frac{4}{5} c^{5/4} \sin\theta\,d\theta = \left(\frac{2}{5\mu}\right)^{1/2} c^{5/4}.$$

Second Method. Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the centre. Hence the free surface would be spherical. Thus the fluid velocity v' will be radial and hence v' will be function of r' (the radial distance from the centre of the sphere which is taken as origin) and time t. Also, let v be the velocity at a distance r.



Then the equation of continuity is

$$r'^2 v' = F(t) = r^2 v. \qquad \text{...(1)}$$

From (1), $$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2}. \qquad \text{...(2)}$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{\mu}{r'^{3/2}} - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{\mu}{r'^{3/2}} - \frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)} \qquad \text{...(3)}$$

Integrating (3) with respect to r', we have

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{2\mu}{r'^{1/2}} - \frac{p}{\rho} + C, \; C \text{ being an arbitrary constant} \qquad \text{...(4)}$$

When $r' = \infty$, $v' = 0$, $p = 0$. So from (4), $C = 0$. Then (4) becomes

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{2\mu}{r'^{1/2}} - \frac{p}{\rho}. \qquad \text{...(5)}$$

Now when $r' = r$, $v' = v$ and $p = 0$. So (5) reduces to

$$-\frac{F'(t)}{r} + \frac{1}{2}v'^2 = \frac{2\mu}{r^{1/2}}. \qquad \text{...(6)}$$

Now, $\quad$ (1) $\quad\Rightarrow\quad F(t) = r^2 v \quad\Rightarrow\quad F'(t) = 2rv\,(dr/dt) + r^2(dv/dt)$

or $$F'(t) = 2rv\frac{dr}{dt} + r^2\frac{dv}{dr}\frac{dr}{dt} = 2rv^2 + r^2 v\frac{dv}{dr}, \qquad \text{as} \qquad \frac{dr}{dt} = v.$$

Hence (6) gives

$$-\frac{1}{r}\left[2rv^2 + r^2 v\frac{dv}{dr}\right] + \frac{v^2}{2} = \frac{2\mu}{r^{1/2}} \qquad \text{or} \qquad rv\frac{dv}{dr} + \frac{3}{2}v^2 = -\frac{2\mu}{r^{1/2}}.$$

Multiplying both sides by $2r^2$, the above equation can be written as

$$2r^3 v\,dv + 3r^2 v^2\,dr = -4\mu r^{3/2}\,dr \qquad \text{or} \qquad d\,(r^3 v^2) = -4\mu r^{3/2}\,dr.$$

Integrating, $r^3v^2 = -(8\mu/5)r^{5/2} + D$, D being an arbitrary constant ...(7)

When $r = c, v = 0$. So (7) gives $D = (8\mu/5)c^{5/2}$. Hence (7) reduces to

$$r^3v^2 = (8\mu/5) \times (c^{5/2} - r^{5/2})$$

or $$v = \frac{dr}{dt} = -\left(\frac{8\mu}{5}\right)^{1/2}\left(\frac{c^{5/2} - r^{5/2}}{r^3}\right)^{1/2},$$

taking negative sign for dr/dt since velocity increases as r decreases.

Let T be the time of filling up the cavity, then

$$T = -\left(\frac{5}{8\mu}\right)^{1/2}\int_c^0 \frac{r^{3/2}\,dr}{(c^{5/2} - r^{5/2})^{1/2}}. \qquad \text{...(8)}$$

Let $r^{5/2} = c^{5/2}\sin^2\theta$ so that $(5/2) \times r^{3/2}dr = c^{5/2}\sin\theta\cos\theta\, d\theta$.

$$\therefore \quad T = \frac{4}{5}\left(\frac{5}{8\mu}\right)^{1/2}\int_0^{\pi/2}\frac{c^{5/2}\sin\theta\cos\theta}{c^{5/4}\cos\theta}d\theta = \frac{4c^{5/4}}{5}\left(\frac{5}{8\mu}\right)^{1/2}\int_0^{\pi/2}\sin\theta\, d\theta$$

or $$T = (2/5\mu)^{1/2} \times c^{5/4}.$$

Ex. 4. *An infinite fluid in which a spherical hollow of radius a is initially at rest under the action of no forces. If a constant pressure* Π *is applied at infinity, show that the time of filling up the cavity is*

$$a\left(\frac{\pi\rho}{6\Pi}\right)^{1/2}\frac{\Gamma(5/6)}{\Gamma(4/3)}$$ **[Agra 2004, 05]**

and show that it is equivalent to $2^{5/6}\pi^2\, a(\rho/\Pi)^2\,\{\Gamma(1/3)\}^{-3}$ **[Meerut 2008, 11; Kanpur 2002]**

Sol. At any time t, let v' be the velocity at distance r' from the centre. Again, let r be the radius of the cavity and v its velocity. Then the equation of continuity yields

$$r'^2v' = r^2v \qquad \text{...(1)}$$

When the radius of the cavity is r, then

$$\text{Kinetic energy } = \int_r^\infty \frac{1}{2}(4\pi r'^2\,dr'\rho)\cdot v'^2 = 2\pi\rho r^4v^2\int_r^\infty \frac{dr'}{r'^2}, \text{ using (1)}$$

$$= 2\pi\rho r^3v^2.$$

The initial kinetic energy is zero.

Again, the work done by the outer pressure $= \int_a^r 4\pi r^2\Pi(-dr) = \frac{4}{3}\rho\Pi(a^3 - r^3)$.

Then, by the energy equation, we get $2\pi\rho r^3v^2 - 0 = (4/3)\times\pi\Pi(a^3 - r^3)$

$$\therefore \quad v = \frac{dr}{dt} = -\left(\frac{2\Pi}{3\rho}\right)^{1/2}\frac{a^3 - r^3}{r^3} \qquad \text{...(2)}$$

where negative sign is taken because r decreases as t increases.

Let T be the time of filling up the cavity. Then (2) gives

$$\int_0^T dt = -\int_a^0\left(\frac{3\rho}{2\Pi}\right)^{1/2}\frac{r^{3/2}dr}{\sqrt{(a^3 - r^3)}} \quad \text{or} \quad T = \left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_0^a\frac{r^{3/2}dr}{\sqrt{(a^3 - r^3)}}$$

Put $r^3 = a^3\sin^2\theta$ *i.e.* $r = a\sin^{2/3}\theta$ and $dr = (2a/3)\times(\sin\theta)^{-1/3}\cos\theta\, d\theta$

$$\therefore \quad T = \left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^{\pi/2} \frac{a^{3/2}\sin\theta}{a^{3/2}\cos\theta}\cdot\frac{2a}{3}(\sin\theta)^{-1/3}\cos\theta d\theta \;= \frac{2a}{3}\left(\frac{3\rho}{2\Pi}\right)^{1/2} \int_0^{\pi/2} \sin^{2/3}\theta d\theta$$

$$= \frac{2a}{3}\left(\frac{3\rho}{2\Pi}\right)^{1/2} \times \frac{\Gamma(5/6)\Gamma(1/2)}{2\Gamma(4/3)} \;= a\left(\frac{\pi\rho}{6\Pi}\right)^{1/2} \frac{\Gamma(5/6)}{\Gamma(4/3)}, \qquad \text{as } \Gamma(1/2) = \sqrt{\pi} \qquad \ldots(3)$$

which is the required first part of the result.

From advanced Integral Calculus, we know that

$$\Gamma(n)\Gamma(n+1/2) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}} \qquad \text{(Duplication Formula)} \qquad \ldots(4)$$

and

$$\Gamma(n)\Gamma(1-n) = \pi/\sin n\pi \qquad \ldots(5)$$

Replacing n by 1/3 in (4), we get $\quad \Gamma(1/3)\Gamma(5/6) = \sqrt{\pi}\,2^{1/3}\,\Gamma(2/3)$

$$\therefore \quad \{\Gamma(1/3)\}^2\Gamma(5/6) = \sqrt{\pi}\,2^{1/3}\,\Gamma(1/3)\Gamma(2/3) = \sqrt{\pi}\,2^{1/3}\Gamma(1/3)\Gamma(1-1/3)$$

$$= \sqrt{\pi}\,2^{1/3} \times \{\pi/\sin(\pi/3)\}, \text{ by (5)}$$

Thus, $\quad \{\Gamma(1/3)\}^2\,\Gamma(5/6) = \sqrt{\pi} \times 2^{1/3} \times (2\pi/\sqrt{3})$

$$\therefore \quad \Gamma(5/6) = \sqrt{\pi} \times 2^{1/3} \times (2\pi/\sqrt{3}) \times \{\Gamma(1/3)\}^{-2} \qquad \ldots(6)$$

Also $\quad \Gamma(4/3) = \Gamma(1+1/3) = (1/3) \times \Gamma(1/3) \qquad \ldots(7)$

Using (6) and (7), (3) reduces to

$$T = a\left(\frac{\rho}{\Pi}\right)^{1/2} \cdot \frac{\sqrt{\pi}}{\sqrt{6}} \times \frac{\sqrt{\pi}\times 2^{1/3} \times (2\pi/\sqrt{3}) \times [\Gamma(1/3)]^{-2}}{(1/3)\times\Gamma(1/3)}$$

or

$$T = 2^{5/6}\pi^2 a(\rho/\Pi)^{1/2}\,[\Gamma(1/3)]^{-3}.$$

Second Method. Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the centre. Hence the free surface would be spherical. Thus the fluid velocity v' will be radial and v' will be function of r' (the radial distance from the centre of the spherical shell which is taken as origin). and time t only. Let p be pressure at a distance r'. Then from the continuity equation, we have

$$r'^2\, v' = F(t) = r^2\, v. \qquad \ldots(1)$$

From (1), $\quad \dfrac{\partial v'}{\partial t} = \dfrac{F'(t)}{r'^2}. \qquad \ldots(2)$

The equation of motion is $\quad \dfrac{\partial v'}{\partial t} + v'\dfrac{\partial v'}{\partial r'} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}.$

or

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)} \qquad \ldots(3)$$

Integrating (3) with respect to r', we have

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = C - \frac{p}{\rho}, \text{ where } C \text{ is an arbitrary constant.} \qquad \ldots(4)$$

Initially, when $\quad r' = \infty, \quad v' = 0 \quad$ and $\quad p = \Pi \quad$ so (4) $\Rightarrow C = \Pi/\rho$.

$\therefore$ (4) becomes $$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=\frac{\Pi-p}{\rho}. \qquad ...(5)$$

Let v be the velocity and r be the radius of spherical cavity at any time t so that $v' = v$, $r' = r$ and $p = 0$ (being hollow part of cavity). Then (5) reduces to

$$-\frac{F'(t)}{r}+\frac{1}{2}v^2=\frac{\Pi}{\rho}. \qquad ...(6)$$

Now, from (1), $$F(t) = r^2 v. \qquad ...(7)$$

Differentiating (7) with respect to t, we have

$$F'(t) = r^2\frac{dv}{dt}+2rv\frac{dr}{dt}=r^2\frac{dv}{dr}\frac{dr}{dt}+2rv\frac{dr}{dt}$$

or $$F'(t) = r^2v\,(dv/dr) + 2rv^2, \qquad \text{as} \qquad dr/dt = v.$$

Substituting the above value of $F'(t)$ in (6), we have

$$-\frac{1}{r}\left[r^2v\frac{dv}{dr}+2rv^2\right]+\frac{1}{2}v^2=\frac{\Pi}{\rho} \qquad \text{or} \qquad rv\frac{dv}{dr}+\frac{3v^2}{2}=-\frac{\Pi}{\rho}.$$

Multiplying both sides by $2r^2dr$, we have

$$2r^3\,v\,dv + 3r^2v^2\,dr = -\,(2\Pi/\rho)\,r^2dr.$$

Integrating, $r^3\,v^2 = -\,(2\Pi/3\rho) \times r^3 + D$, where D is an arbitrary constant. ...(8)

Initially, when radius of cavity $r = 0$, then $v = 0$. Hence (8) gives $D = 2\Pi/3\rho$ and so (8) reduces to

$$r^3v^2=\frac{2\Pi}{3\rho}(a^3-r^3) \qquad \text{or} \qquad v=\frac{dr}{dt}=-\left(\frac{2\Pi}{3\rho}\right)^{1/2}\left(\frac{a^3-r^3}{r^3}\right)^{1/3},$$

taking negative sign since v increases as r decreases.

Let T be the required time of filling up the cavity, then

$$T=-\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_a^0\left(\frac{r^3}{a^3-r^3}\right)^{1/2}dr. \qquad ...(9)$$

Putting $r = a\sin^{2/3}\theta$ so that $dr = (2a/3) \times \sin^{-1/3}\theta\cos\theta\, d\theta$, (9) gives

$$T=-\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_{\pi/2}^{0}\frac{a^{3/2}\sin\theta}{a^{3/2}\cos\theta}\cdot\frac{2a}{3}\sin^{-1/3}\theta\cos\theta\,d\theta$$

$$=\frac{2a}{3}\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_0^{\pi/2}\sin^{2/3}\theta\,d\theta=\frac{2a}{3}\left(\frac{3\rho}{2\Pi}\right)^{1/2}\frac{\Gamma(5/6)\Gamma(1/2)}{2\Gamma(4/3)}$$

$$=\frac{a}{3}\left(\frac{3\rho}{2\Pi}\right)^{1/2}\frac{\sqrt{\pi}\,\Gamma(5/6)}{(1/3)\Gamma(1/3)}, \qquad \text{as} \qquad \Gamma\left(\frac{4}{3}\right)=\Gamma\left(1+\frac{1}{3}\right)=\frac{1}{3}\Gamma\left(\frac{1}{3}\right)$$

$\therefore$ $$T=a\left(\frac{3\rho}{2\Pi}\right)^{1/2}\frac{\sqrt{\pi}\,\Gamma(5/6)}{\Gamma(1/3)}. \qquad ...(10)$$

From integral calculus, we know that

$$\Gamma(n)\,\Gamma(n+1/2) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}} \qquad ...(11)$$

and
$$\Gamma(n)\,\Gamma(1-n) = \pi/\sin n\pi. \qquad ...(12)$$

Putting $n = 1/3$ in (11),
$$\Gamma(1/3)\,\Gamma(5/6) = \sqrt{\pi}\;2^{1/3}\,\Gamma(2/3). \qquad ...(13)$$

Multiplying both sides of (13) by $\Gamma(1/3)$, we get

$$[\Gamma(1/3)]^2\,\Gamma(5/6) = \sqrt{\pi}\;2^{1/3}\,\Gamma(1/3)\,\Gamma(2/3)$$

or
$$[\Gamma(1/3)]^2\,\Gamma(5/6) = \sqrt{\pi}\;2^{1/3}\,\Gamma(1/3)\,\Gamma(1-1/3)$$

or
$$[\Gamma(1/3)]^2\,\Gamma(5/6) = \sqrt{\pi}\;2^{1/3}.\{\pi/\sin(\pi/3)\}, \text{ using (12)}$$

$$\therefore \qquad \Gamma(5/6) = \frac{\sqrt{\pi}\,2^{1/3}\,\pi}{(\sqrt{3}/2)\,[\Gamma(1/3)]^2}.$$

Substituting the above value of $\Gamma(5/6)$ in (10), we have

$$T = a\left(\frac{3\rho}{2\Pi}\right)^{1/2} \times \frac{\sqrt{\pi}}{\Gamma(1/3)} \times \frac{\sqrt{\pi}\,2^{1/3}\,\pi}{\left(\sqrt{3}/2\right)[\Gamma(1/3)]^2}$$

or
$$T = \pi^2\,a(\rho/\Pi)^{1/2}\;2^{5/6}\{\Gamma(1/3)\}^{-3}.$$

Ex. 3. *A mass of fluid of density* ρ *and volume* $(4/3)\times\pi c^3$ *is in the form of a spherical shell. A constant pressure* Π *is exerted on the external surface of the shell. There is no pressure on the internal surface and no other forces act on the liquid. Initially the liquid is at rest and the internal radius of the shell is* 2c. *Prove that the velocity of the internal surface when its radius is c, is*

$$\left(\frac{14\Pi}{3\rho}\,\frac{2^{1/3}}{2^{1/3}-1}\right)^{1/2}$$

[Kanpur 1997]

Sol. At any time t, let v' be the velocity at distance r' from the centre. Let r and R be the radii and v and V the velocities of the internal and external surfaces of the shell. Then the equation of continuity yields
$$r'^2 v' = r^2 v \qquad ...(1)$$

Again from conservation of mass, we have

$$\frac{4}{3}\pi R^3\rho - \frac{4}{3}\pi r^3\rho = \text{constant} = \frac{4}{3}\pi c^3\rho$$

$$\therefore \qquad R^3 - r^3 = c^3 \qquad \text{so that} \qquad R^3 = r^3 + c^3. \qquad ...(2)$$

Now, the initial kinetic energy is zero. Again the final kinetic energy

$$= \int_r^R \frac{1}{2}(4\pi r'^2 dr')v'^2 = 2\pi\rho r^4 v^2 \int_r^R \frac{dr'}{r'^2}, \text{ using (1)}$$

$$= 2\pi\rho r^4 v^2\,(1/r - 1/R) = 2\pi\rho r^4 v^2\left[\frac{1}{r} - \frac{1}{(r^3+c^3)^{1/3}}\right], \text{ using (2)}$$

Again, the work done by the external pressure Π in decreasing the shell from radius r to $2c$

$$= \int_{2c}^{r} 4\pi r^2 \Pi(-dr) = \frac{4\pi\Pi}{3}(8c^3 - r^3).$$

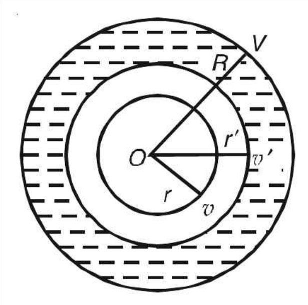
Then the energy equation yields

$$2\pi\rho r^4 v^2 \left[\frac{1}{r} - \frac{1}{(r^3+c^3)^{1/3}}\right] = \frac{4}{3}\pi\Pi(8c^3 - r^3) \qquad \ldots(3)$$

The required value of velocity is given by setting $r = c$ in (3). Thus, we get

$$\rho c^4 v^2 \left(\frac{1}{c} - \frac{1}{c \times 2^{1/3}}\right) = \frac{2}{3} \times \Pi \times (7c^3) \qquad \text{or} \qquad v = \left(\frac{14\pi}{3\rho}\,\frac{2^{1/3}}{2^{1/3}-1}\right)^{1/2} \text{ as required.}$$

Second Method. Let r and R be the internal and external radii of the shell, r' be any radius, where the velocity is v' and the pressure p at any time t.



Volume of liquid $= \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi c^3$ (given)

and so $\qquad R^3 = r^3 + c^3. \qquad \ldots(1)$

The equation of continuity is $r'^2 v' = F(t) = r^2 v = R^2 V \ldots(2)$

From (2), $$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2}. \qquad \ldots(3)$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \text{or} \qquad \frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ by (3)}$$

Integrating it with respect to r', we have

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C, \; C \text{ being an arbitrary constant} \qquad \ldots(4)$$

Initially, when $r' = R$, $v' = V$, $p = \Pi$ and when $r' = r$, $v' = v$, $p = 0$ (as there is no pressure on internal surface). Then (4) reduces to

$$-\frac{F'(t)}{R} + \frac{1}{2}V^2 = -\frac{\Pi}{\rho} + C \qquad \ldots(5)$$

and $$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = C. \qquad \ldots(6)$$

Subtracting (5) from (6), we have

$$-F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{1}{2}(v^2 - V^2) = \frac{\Pi}{\rho} \qquad \text{or} \qquad -F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{1}{2}\left[\frac{\{F(t)\}^2}{r^4} - \frac{\{F(t)\}^2}{R^4}\right] = \frac{\Pi}{\rho}$$

or $$-F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{\{F(t)\}^2}{2}\left[\frac{1}{r^4} - \frac{1}{R^4}\right] = \frac{\Pi}{\rho}. \qquad \ldots(7)$$

From (2), $\qquad r^2 v = R^2 V = F(t)$

$\Rightarrow \quad r^2(dr/dt) = R^2(dR/dt) = F(t) \quad \Rightarrow \quad r^2 dr = R^2 dR = F(t)\, dt. \qquad$ ..(8)

Multiplying both sides of (7) by $2r^2dr$, we get

$$-2F'(t)\, r^2\left(\frac{1}{r}-\frac{1}{R}\right)dr+\frac{\{F(t)\}^2}{2}\left[\frac{2r^2dr}{r^4}-\frac{2r^2dr}{R^4}\right]=\frac{2\Pi r^2 dr}{\rho}. \qquad ...(9)$$

Using relations (8), (9) may be rewritten as

$$-2F'(t)\, F(t)\left(\frac{1}{r}-\frac{1}{R}\right)dt+\frac{\{F(t)\}^2}{2}\left[\frac{2r^2dr}{r^4}-\frac{2R^2dR}{R^4}\right]=\frac{2\Pi r^2 dr}{\rho}$$

or
$$\left(\frac{1}{r}-\frac{1}{R}\right)d\{F(t)\}^2+\{F(t)\}^2 d\left(\frac{1}{r}-\frac{1}{R}\right)=-\frac{2\Pi r^2 dr}{\rho}. \qquad ...(10)$$

Integrating, $\{F(t)\}^2\left(\dfrac{1}{r}-\dfrac{1}{R}\right)=-\dfrac{2\Pi r^3}{3\rho}+D$, D being an arbitrary constant

Initially, when $r = 2c$, $v = 0$ [so that $F(t) = 0$ by (2)]. Hence (10) gives

$$0 = -(2\Pi/3\rho)c^3 + D \qquad \text{or} \qquad D = (2\Pi/3\rho)c^3.$$

$\therefore$ (10) reduces to $\{F(t)\}^2\left(\dfrac{1}{r}-\dfrac{1}{R}\right) = \dfrac{2\Pi}{3\rho}(8c^3 - r^3)$

or
$$r^4v^2\left(\frac{1}{r}-\frac{1}{R}\right) = \frac{2\Pi}{3\rho}(8c^3 - r^3), \text{ using (2).}$$

or
$$r^4v^2\left\{\frac{1}{r}-\frac{1}{(r^3+c^3)^{1/3}}\right\} = \frac{2\Pi}{3\rho}(8c^3 - r^3), \text{ using (1)}$$

or
$$v^2=\frac{2\Pi}{3\rho}\,\frac{8c^3-r^3}{r^4\{1/r-1/(r^3+c^3)^{1/3}\}}. \qquad ...(11)$$

giving velocity v at the inner surface of the cavity. Hence the velocity of the internal surface (where $r = c$) is given by

$$v^2=\frac{2\Pi}{3\rho}\,\frac{7c^3}{c^4\{1/c-1/(c\times 2^{1/3})\}} \qquad \text{or} \qquad v=\left[\frac{14\Pi}{3\rho}\,\frac{2^{1/3}}{2^{1/3}-1}\right]^{1/2}.$$

Ex. 4. *A mass of liquid surrounds a solid sphere of radius a, and its outer surface, which is a concentric sphere of radius b, is subjected to a given constant pressure* Π, *no other force being in action on the liquid. The solid, sphere, suddenly shrinks into a concentric sphere, determine the subsequent motion and the impulsive action on the sphere.*

[Allahabad 2000; Kerala 2004]

Sol. At time t, let v' be the velocity at distance r' from the centre. Again, let R, r be the radii of the external and internal boundaries at time t, and V, v their velocities. Then the equation of continuity yields. $r'^2v' = r^2v$ ...(1)

Again from conservation of mass, we have

$$\frac{4}{3}\pi R^3\rho-\frac{4}{3}\pi r^3\rho=\frac{4}{3}\pi b^3\rho-\frac{4}{3}\pi a^3\rho$$

so that $$R^3 - r^3 = b^3 - a^3 = c^3, \text{ (say)} \qquad ...(2)$$

$\therefore$ $$R = (r^3 + c^3)^{1/3}.$$

Now the initial kinetic energy is zero. Again the final kinetic energy

$$= \int_r^R \frac{1}{2}(4\pi r'^2\, dr'\, \rho)\, v'^2 = 2\pi\rho \int_r^R r'^2 v'^2\, dr' = 2\pi\rho \int_r^R \frac{r^4 v^2}{r'^2}\, dr', \text{ using (1)}$$

$$= 2\pi r^4 \rho v^2\, (1/r - 1/R)$$

and the work done by the outer pressure

$$= \int_b^R 4\pi R^2\, \Pi(-dR) = \frac{4}{3}\pi\Pi(b^3 - R^3) = \frac{4}{3}\pi\,\Pi(a^3 - r^3), \quad \text{using (2)}$$

Therefore, using the energy equation, we have

$$2\pi r^4 \rho v^2\, (1/r - 1/R) = (4/3) \times \pi\Pi(a^3 - r^3)$$

$$\therefore \quad v = \left(\frac{2\Pi}{3\rho}\right)^{1/2} \frac{(a^3 - r^3)^{1/2}}{r^2\,(1/r - 1/R)^{1/2}} \quad \text{or} \quad v = \left(\frac{2\Pi}{3\rho}\right)^{1/2} \frac{(a^3 - r^3)^{1/2}}{r^2\left\{1/r - 1/(r^3 + c^3)^{1/3}\right\}} \qquad ...(3)$$

Expression for impulsive action on the sphere. Let r be the radius of the solid sphere and $\tilde{\omega}$ the impulsive pressure at distance r' from its centre. Then we have

$$d\tilde{\omega} = -\rho v'\, dr' = -\rho \frac{r^2 v\, dr'}{r'^2}, \quad \text{using (1)}$$

Integrating, $$\tilde{\omega} = \frac{\rho r^2 v}{r'} + C, \ C \text{ being an arbitrary constant}$$

Given that $\tilde{\omega} = 0$ when $r' = R$. Hence $C = -(\rho r^2 v)/R$. So $\qquad \tilde{\omega} = \rho r^2 v\, (1/r - 1/R).$

Thus the impulsive pressure when $r' = r$ is given by $\qquad \tilde{\omega} = \rho r^2 v (1/r - 1/R).$

Hence the whole impulsive pressure on the sphere $= 4\pi r^2 \tilde{\omega} = 4\pi r^3 v (R - r)/R,$

and the whole momentum destroyed

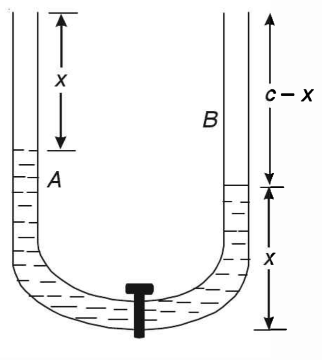
$$= \int_r^R (4\pi r'^2 dr'\, \rho)\, v' = 4\pi\rho \int_r^R v' r'^2 dr' = 4\pi\rho \int_r^R r^2 v\, dr, \text{ using (1)}$$

$$= 4\pi\rho\, r^2 v (R - r).$$

Ex. 5. *Two equal closed cylinders, of height c, with their bases in the same horizontal plane, are filled, one with water and the other with air of such a density as to support a column h of water, h being lees than c. If a communication be opened between them at their bases, the height x, to which the water rises, is given by the equation.* $cx - x^2 + ch \log \{(c - x)/c\} = 0.$

[Meerut 1997; Rajasthan 2000]

Sol. Let A (shown on L.H.S.) and B (shown on R.H.S.) be two cylinders containing water and air respectively. Let α be the cross-section of each cylinder. The water and air are at rest before and after the communication is set up between the two cylinders. Hence the initial and final kinetic energies are zero.



Now, initial potential energy V_1 due to water height in A is given by
$$V_1 = \int_0^c g\rho\alpha x'\, dx' = \frac{1}{2} g\rho\alpha c^2.$$

After communication is set up, a height x of water rises in B and hence a height $(c - x)$ of water is left behind in A. Therefore, the final potential energy V_2 due to water in A and B is given by
$$V_2 = \int_0^{c-x} g\rho\alpha x'\, dx' + \int_0^x g\rho\alpha x'\, dx'$$
$$= \frac{1}{2} g\rho\alpha[(c-x)^2 + x^2] = \frac{1}{2} g\rho\alpha(c^2 - 2cx + 2x^2).$$

$\therefore$ The work done against gravity
$$= V_1 - V_2 = \text{loss in potential energy} = (1/2)\times g\rho\alpha(2cx - 2x^2) = g\rho\alpha x(c-x).$$

Again, work is also done against the compressions of air in B. Let p be the pressure of the air when the water stands to a height x'. Assume that temperature remains constant so that Boyle's law is applicable. Thus, we have
$$g\rho h\alpha c = p\alpha(c - x') \qquad \text{so that} \qquad p = (g\rho hc)/(c - x')$$

Thus the total work done by this pressure $= -\int_0^x \frac{g\rho hc\alpha}{c - x'} dx' = g\rho hc\alpha \log \frac{c-x}{c}.$

Now by energy equation, we have

Increase in K.E. = total work done so that total work done = 0

$\therefore \; g\rho\alpha x(c-x) + g\rho hc\alpha \log\{(c-x)/c\} = 0$ or $cx - x^2 + ch \log\{(c-x)/c\} = 0.$

Ex. 6. *Show that the rate per unit of time at which work is done by the internal pressures between the parts of a compressible fluid obeying Boyle's law is*
$$\iiint p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dx\, dy\, dz,$$
where p is the pressure and u, v, w the velocity components at any point and the integration extends through the volume of the fluid.

Sol. Let W be the work done in compressing the fluid, p is the pressure and dV an elementary volume. Then, we have $W = \int p(-dV) = -\int p\, dV.$

Hence the rate per unit time of work done is given by
$$\frac{DW}{Dt} = -\iiint \frac{Dp}{Dt} dV. \qquad \text{...(1)}$$

The equation of continuity is
$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0. \qquad \text{...(2)}$$

Since the compressible fluid obeys Boyle's law, hence we have
$$p = k\rho \qquad \text{so that} \qquad \rho = p/k. \qquad \text{...(3)}$$

Using (3), (2) becomes
$$\frac{D}{Dt}\left(\frac{p}{k}\right) + \frac{p}{k}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \qquad \text{so that} \qquad \frac{Dp}{Dt} = -p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right).$$

Hence (1) gives $$\frac{DW}{Dt} = \iiint p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)dV,$$

which gives the required rate per unit of time at which work is done.

Ex. 7. *A mass of perfect incompressible fluid of density* ρ *is bounded by concentric spherical surfaces. The outer surface is contained by a flexible envelope which exerts continuously uniform pressure* Π *and contracts from radius* R_1 *to radius* R_2. *The hollow is filled with a gas obeying Boyle's law, its radius contracts from* c_1 *to* c_2 *and the pressure of gas is initially,* p_1. *Initially the whole mass is at rest. Prove that, neglecting the mass of the gas, the velocity* v *of the inner surface when the configuration* (R_2, c_2) *is reached is given by*

$$\frac{1}{2}v^2 = \frac{c_1^3}{c_2^3}\left\{\frac{1}{3}\left(1-\frac{c_2^3}{c_1^3}\right)\frac{\Pi}{\rho} - \frac{p_1}{\rho}\log\frac{c_1}{c_2}\right\}\Big/\left(1-\frac{c_2}{R_2}\right) \qquad \textbf{[I.A.S. 2005]}$$

Sol. Let p_2 be the pressure of the gas when the internal radius is c_2. Then, by Boyle's law,

$$(4/3)\times \pi c_1^3 p_1 = (4/3)\times \pi c_2^3 p_2 \quad \text{so that} \quad p_2 = (c_1^3/c_2^3)p_1. \qquad \ldots(1)$$

Equation of continuity is $$r'^2 v' = F(t) = c_2^2 v. \qquad \ldots(2)$$

From (2), $$v' = c_2^2 v/r'^2. \qquad \ldots(3)$$

Now, the initial kinetic energy (K.E) = 0.

and $$\text{final K.E.} = \int_{c_2}^{R_2}\frac{1}{2}(4\pi r'^2 dr'\rho)v'^2 = 2\pi\rho c_2^4 v^2\int_{c_2}^{R_2}\frac{dr'}{r'^2}, \quad \text{by (3)}$$

$$= 2\pi\rho c_2^4 v^2\left(\frac{1}{c_2}-\frac{1}{R_2}\right) = 2\pi\rho c_2^3 v^2\left(1-\frac{c_2}{R_2}\right). \qquad \ldots(4)$$

Now, work done W by the external pressure Π and the internal pressure p_2 is given by

$$W = \int_{R_1}^{R_2}4\pi R_2^2\,\Pi(-dR_2)+\int_{c_1}^{c_2}4\Pi c_2^2 p_2 dc_2 = -\frac{4}{3}\pi\Pi\left[\frac{R_2^3}{3}\right]_{R_1}^{R_2} + 4\pi\int_{c_1}^{c_2}c_2^2.\frac{c_1^3}{c_2^3}p_1 dc_2, \text{ using (1)}$$

$$= (4/3)\times\pi\Pi(R_1^3-R_2^3)+4\pi p_1 c_1^3\left[\log c_2\right]_{c_1}^{c_2}$$

$$\therefore \quad W = (4/3)\times\pi\Pi(R_1^3-R_2^3)+4\pi p_1 c_1^3\log(c_2/c_1). \qquad \ldots(5)$$

Since the mass of the fluid remains constant, we have

$$(4/3)\times\pi(R_2^3-c_2^3) = (4/3)\times\pi(R_1^3-c_1^3) \quad \text{or} \quad R_1^3-R_2^3 = c_1^3-c_2^3. \qquad \ldots(6)$$

Using (6), (5) reduces to

$$\text{The work done} = W = (4/3)\,\pi\,\Pi(c_1^3-c_2^3)+4\pi p_1 c_1^3\log(c_2/c_1). \qquad \ldots(7)$$

Now, from the energy equation, Increase in K.E. = total work done

or $$2\pi\rho c_2^3 v^2(1-c_2/R_2) = (4/3)\times\pi\Pi(c_1^3-c_2^3)+4\pi p_1 c_1^3\log(c_2/c_1)$$

or $$\frac{1}{2}v^2 c_2^3\left(1-\frac{c_2}{R_2}\right) = c_1^3\left[\frac{1}{3}\left(1-\frac{c_2^3}{c_1^3}\right)\frac{\Pi}{\rho}-\frac{p_1}{\rho}\log\frac{c_1}{c_2}\right]$$

or $$\frac{1}{2}v^2 = \frac{c_1^3}{c_2^3}\left\{\frac{1}{3}\left(1-\frac{c_2^3}{c_1^3}\right)\frac{\Pi}{\rho}-\frac{p_1}{\rho}\log\frac{c_1}{c_2}\right\}\Big/\left(1-\frac{c_2}{R_2}\right).$$

Ex. 8. *A given quantity of liquid moves, under no forces, in a smooth conical tube having a small vertical angle and the distances of its nearer and farther extremities from the vertex at the time t are r and r′, show that*

$$2r\frac{d^2r}{dt^2}+\left(\frac{dr}{dt}\right)^2\left[3-\frac{r}{r'}-\frac{r^2}{r'^2}-\frac{r^3}{r'^3}\right]=0.$$ **[Purvanchal 2005, Agra 2005]**

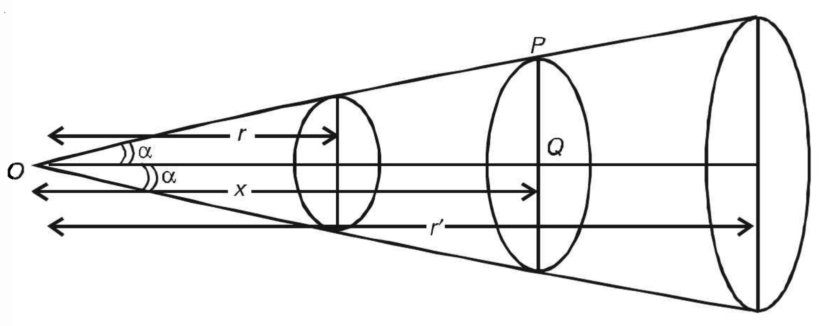
Show that it follows also by taking vis-viva of the mass of the liquid as constant; and that the velocity V of the inner surface is given by the equations

$V^2 = Cr'/(r'-r)r^3$, $\quad r'^3 - r^3 = c^3$, $\quad$ *C, c being constants.*

Sol. At any time t, let p' be the pressure at a distance x from the vertex and v' be the velocity there. Let α be the semi-vertical angle of the conical tube. Then the equation of continuity is given by

$$v'(PQ)^2 = f(t) \quad \text{or} \quad v'(x\tan\alpha)^2 = f(t)$$

or $\quad v'x^2 = F(t),\quad$ where $\quad F(t) = \cot^2\alpha\, f(t).$ ...(1)



The equation of motion is $\quad \dfrac{\partial v'}{\partial t}+v'\dfrac{\partial v'}{\partial x}=-\dfrac{1}{\rho}\dfrac{\partial p'}{\partial x}.$ ...(2)

From (1), $\quad \partial v'/\partial t = (1/x^2)\, F'(t).$ ...(3)

Using (3), $\quad$ (2) $\Rightarrow$ $\quad \dfrac{F'(t)}{x^2}+\dfrac{\partial}{\partial x}\left(\dfrac{1}{2}v'^2\right)=-\dfrac{1}{\rho}\dfrac{\partial p'}{\partial x}.$ ...(4)

Integrating (4) with respect to x, we have

$$-\frac{F'(t)}{x}+\frac{1}{2}v'^2=C-\frac{p'}{\rho},\ C \text{ being an arbitrary constant} \qquad ...(5)$$

Let v and v' be the velocities when $x = r$ and $x = r'$ respectively and p be the pressure there. Then (5) gives

$$-\frac{F'(t)}{r}+\frac{1}{2}v^2=C-\frac{p}{\rho} \qquad ...(6)$$

and $$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=C-\frac{p}{\rho}. \qquad ...(7)$$

Subtracting (6) from (7), $\quad -F'(t)(1/r-1/r')+(1/2)\times(v^2-v'^2)=0.$ ...(8)

From the equation of continuity, $\quad r^2v = r'^2v' = F(t),$ ...(9)

where $\quad v = dr/dt \quad$ and $\quad v' = dr'/dt.$ ...(10)

From (9), $v' = r^2v/r'^2$. Then (8) becomes

$$-F'(t)\left(\frac{1}{r}-\frac{1}{r'}\right)+\frac{1}{2}\left(v^2-\frac{r^4v^2}{r'^4}\right)=0 \quad \text{or} \quad -F'(t)\left(\frac{r'-r}{rr'}\right)+\frac{1}{2}v^2\left(\frac{r'^4-r^4}{r'^4}\right)$$

or $$F'(t)\left(\frac{r'-r}{rr'}\right)-\frac{v^2}{2}\frac{(r'-r)(r'^3+r'^2r+r'r^2+r^3)}{r'^4}=0$$

or $$\frac{2F'(t)}{r}-v^2\left(\frac{r'^3+r'^2r+r'r^2+r^3}{r'^3}\right)=0. \qquad ...(11)$$

From (9), $$F'(t)=\frac{d}{dt}(r^2v)=\frac{d}{dt}\left(r^2\frac{dr}{dt}\right)=2r\left(\frac{dr}{dt}\right)^2+r^2\frac{d^2r}{dt^2}$$

$\therefore$ (11) $\Rightarrow$ $$\frac{2}{r}\left[2r\left(\frac{dr}{dt}\right)^2+r^2\frac{d^2r}{dt^2}\right]-\left(\frac{dr}{dt}\right)^2\left(1+\frac{r}{r'}+\frac{r^2}{r'^2}+\frac{r^3}{r'^3}\right)=0$$

or $$2r\frac{d^2r}{dt^2}+\left(\frac{dr}{dt}\right)^2\left(3-\frac{r}{r'}-\frac{r^2}{r'^2}-\frac{r^3}{r'^3}\right)=0.$$

Second part.

The vis-viva = 2 K.E. = $\int_r^{r'}\pi(x\tan\alpha)^2\,dx\,\rho\,v'^2=\pi\rho\tan^2\alpha\{F(t)\}^2\int_r^{r'}\frac{dx}{x^2}$, by (1)

$$=\pi\rho\tan^2\alpha\,\{F(t)\}^2\left[-\frac{1}{x}\right]_r^{r'}=\pi\rho\tan^2\alpha\{F(t)\}^2\left(\frac{1}{r}-\frac{1}{r'}\right). \qquad ...(12)$$

By the principal of conservation of vis-viva, we have

$$\pi\rho\tan^2\alpha\,\{F(t)\}^2\left(\frac{1}{r}-\frac{1}{r'}\right)=\text{constant} \quad \text{or} \quad \{F(t)\}^2\left(\frac{1}{r}-\frac{1}{r'}\right)=\text{constant}=C$$

or $\qquad (r^2v)^2\,(1/r-1/r')=C$ using (9)

or $\qquad v^2=Cr'/(r'-r)r^3 \qquad$ or $\qquad V^2=Cr'/(r'-r)r^3, \qquad$ taking $\quad V=v$

Since the mass is constant, volume will also be constant.

Hence $\qquad (1/3)\times\pi(r'\tan\alpha)^2\,r'-(1/3)\times\pi\,(r\tan\alpha)^2\,r=\text{constant}$

so that $\qquad r'^3-r^3=\text{constant}=c^3$, say.

Ex. 9. *A spherical mass of liquid of radius b has a concentric spherical cavity of radius a, which contains gas at pressure p whose mass may be neglected; at every point of the external boundary of the liquid an impulsive pressure* ω *per unit area is applied. Assuming that the gas obeys Boyls' law, show that when the liquid first comes to rest, the radius of internal spherical surface will be a* $\exp.\{\{-\omega^2b/(2p\rho a^2(b-a)\}\}]$, *where* $\exp x$ *stands for* e^x.

Sol. Let v' be the velocity at a distance r' from the centre of the spherical cavity at any time t. Then the equation of continuity is

$$r'^2v'=F(t)=b^2V, \qquad ...(1)$$

where we have assumed that $v'=V$ when $r'=b$.

Let ω' be the impulsive pressure at a distance r', then

$$d\omega'=-\rho v'\,dr'=-\rho(b^2V/r'^2)\,dr', \text{ by (1)}$$

Integrating, $\qquad \omega'=(\rho b^2V/r')+C$, C being an arbitrary constant. $\qquad$ ...(2)

Given that, $\qquad$ when $r'=b$, $\quad \omega'=\omega \quad$ and $\quad$ when $r'=a$, $\quad \omega'=0$.

$\therefore$ (2) $\Rightarrow$ $\qquad \omega=(\rho b^2V/b)+C \qquad$ and $\qquad 0=(\rho b^2V/a)+C$.

Subtracting, these give $\qquad \omega=(\rho bV/a)\,(a-b)$. $\qquad$ ...(3)

The initial kinetic energy $= \int_a^b \frac{1}{2}(4\pi r'^2 \rho dr')v'^2 = 2\pi\rho \int_a^b r'^2 \cdot \frac{b^4 V^2}{r'^4} dr'$, by (1)

$$= 2\pi\rho b^4 V^2 \int_a^b \frac{dr'}{r'^2} = 2\pi\rho b^4 V^2 \left[-\frac{1}{r'}\right]_a^b$$

$$= 2\pi\rho b^4 V^2 \left[-\frac{1}{b} + \frac{1}{a}\right] = \frac{2\pi\rho b^3 V^2}{a}(b - a). \qquad \text{...(4)}$$

Again, Final kinetic energy = 0. ...(5)

During the compression let r be the radius of the internal cavity and p_1 the pressure of the gas there. Since the gas obey Boyle's law, we have

$$(4/3) \times \pi r^3 \times p_1 = (4/3) \times \pi a^3 \times p \quad \Rightarrow \quad p_1 = a^3 p / r^3. \qquad \text{...(6)}$$

Now, the work done by internal pressure, *i.e.*, work done in compression of the gas from a sphere of radius a to a sphere of radius r

$$= \int_a^r 4\pi r^2 \, p_1 dr = \int_a^r 4\pi r^2 (a^3 p / r^3)\, dr, \text{ by (6)}$$

$$= 4\pi a^3 p \int_a^r \frac{dr}{r} = 4\pi a^3 p \log \frac{r}{a}.$$

Now, by energy equation, increase in K.E. = work done

or Final K.E. – initial K.E. = work done

or $$0 - \frac{2\pi\rho b^3 V^2}{a}(b - a) = 4\pi a^3 p \log \frac{r}{a} \quad \text{or} \quad \log \frac{r}{a} = -\frac{2\pi\rho b^3 (b - a)}{4\pi a^4 p} \cdot V^2$$

or $$\log \frac{r}{a} = -\frac{2\pi\rho b^3 (b - a)}{4\pi a^4 p} \cdot \frac{\omega^2 a^2}{\rho^2 b^2 (a - b)^2}$$

or $$\log \frac{r}{a} = -\left\{\frac{\omega^2 b}{2p\rho^2 a^2 (b - a)}\right\} \quad \text{or} \quad r = a \exp\left\{-\frac{\omega^2 b}{2p\rho a^2 (b - a)}\right\}$$

EXERCISE 3(C)

1. An infinite mass of liquid acted upon by no forces is at rest and a spherical portion of radius c is suddenly annihilated; the pressure Π at an infinite distance being supposed to remain constant, prove that the pressure at a distance r from the centre of the space in instantaneously diminished in the ratio $(r - c)/r$ and that the cavity will be filled up in the time

$$\sqrt{\left(\frac{\pi\rho c^2}{6\Pi}\right)} \frac{\Gamma(5/6)}{\Gamma(4/3)}.$$

2. A spherical globule of gas of radius a and at pressure P extends in an infinite mass of liquid of density ρ in which the pressure at infinity is zero. The gas is initially at rest and its pressure and volume are governed by the equation $pv^{4/3}$ = const. Prove that the gas doubles its radius in time $(28a/15)\sqrt{2\rho P}$. **[I.A.S. 1999]**

3. An infinite fluid in which there is a spherical hollow of radius a is initially at rest under the action of no forces. If a constant pressure Π is applied at infinity, find the rate at which the radius of the cavity diminishes.

3.11. Lagrange's hydrodynamical equations. [Ranchi 2010; Kanpur 2003, 04]

Let a, b, c be the initial co-ordinates of a particle and x, y, z the co-ordinates of the same particle at time t. Then (refer Lagrangian method in Art. 2.1 in chapter 2) we know that a, b, c, t are the independent variables. We wish to obtain x, y, z in terms of a, b, c and t and hence discuss completely the motion.

Now at time t the component accelerations of the fluid element $\delta x \delta y \delta z$ are $\partial^2 x/\partial t^2$, $\partial^2 y/\partial t^2$, $\partial^2 z/\partial t^2$. Let V be the force potential for the external forces. Then we have (noting that $X = -\partial V/\partial x$, $Y = -\partial V/\partial y$, and $Z = -\partial V/\partial z$) as in Art. 3.1

$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad ...(1)$$

$$\frac{\partial^2 y}{\partial t^2} = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \quad ...(2)$$

$$\frac{\partial^2 z}{\partial t^2} = -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \quad ...(3)$$

We now try to get equations containing only differentiations with respect to a, b, c and t. To this end, we multiply (1), (2) and (3) by $\partial x/\partial a$, $\partial y/\partial a$ and $\partial z/\partial a$ then add. Thus, we get*

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} = -\frac{\partial V}{\partial a} - \frac{1}{\rho}\frac{\partial p}{\partial a} \quad ...(4)$$

Similarly,
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b} = -\frac{\partial V}{\partial b} - \frac{1}{\rho}\frac{\partial p}{\partial b} \quad ...(5)$$

and
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial c} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial c} = -\frac{\partial V}{\partial c} - \frac{1}{\rho}\frac{\partial p}{\partial c} \quad ...(6)$$

These equations, together with the equation of continuity

$$\rho \frac{\partial (x, y, z)}{\partial (a, b, c)} = \rho_0, \quad ...(7)$$

are known as *Lagrange's Hydrodynamical Equations*.

3.12. Cauchy's integrals.

Let a, b, c be the initial co-ordinates of a particle and x, y, z the coordinates of the same particle at time t. Then (refer Lagrangian method in Art. 2.1 in chapter 2) we know that a, b, c, t are the independent variables.

Now at time t the component accelerations of the fluid element $\delta x \delta y \delta z$ are $\partial^2 x/\partial t^2$, $\partial^2 y/\partial t^2$, $\partial^2 z/\partial t^2$. Let V be the force potential for the external forces. Then we have (noting that $X = -\partial V/\partial x$, $Y = -\partial V/\partial y$ and $Z = -\partial V/\partial z$) as in Art. 3.1

$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad ...(1)$$

$$\frac{\partial^2 y}{\partial t^2} = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \quad ...(2)$$

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* Use results: $\dfrac{\partial V}{\partial a} = \dfrac{\partial V}{\partial x}\dfrac{\partial x}{\partial a} + \dfrac{\partial V}{\partial y}\dfrac{\partial y}{\partial a} + \dfrac{\partial V}{\partial z}\dfrac{\partial z}{\partial a}$ etc.; $\dfrac{\partial p}{\partial a} = \dfrac{\partial p}{\partial x}\dfrac{\partial x}{\partial a} + \dfrac{\partial p}{\partial y}\dfrac{\partial y}{\partial a} + \dfrac{\partial p}{\partial z}\dfrac{\partial z}{\partial a}$ etc.

$$\frac{\partial^2 z}{\partial t^2} = -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad ...(3)$$

Taking ρ as a function of p, we take $\qquad Q = V + \int \frac{dp}{\rho} \qquad ...(4)$

Then from (4), we have

$$-\frac{\partial Q}{\partial a} = -\frac{\partial V}{\partial a} - \frac{1}{\rho}\frac{\partial p}{\partial a} \qquad ...(5)$$

$$-\frac{\partial Q}{\partial b} = -\frac{\partial V}{\partial b} - \frac{1}{\rho}\frac{\partial p}{\partial b} \qquad ...(6)$$

$$-\frac{\partial Q}{\partial c} = -\frac{\partial V}{\partial c} - \frac{1}{\rho}\frac{\partial p}{\partial c} \qquad ...(7)$$

Multiplying (1), (2), (3) by $\partial x/\partial a,\ \partial y/\partial a,\ \partial z/\partial a$ respectively, and adding, we have

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} = -\frac{\partial V}{\partial a} - \frac{1}{\rho}\frac{\partial p}{\partial a}$$

or
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} = -\frac{\partial Q}{\partial a}, \quad \text{using (5)} \qquad ...(8)$$

Similarly
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b} = -\frac{\partial Q}{\partial b} \qquad ...(9)$$

and
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial c} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial c} = -\frac{\partial Q}{\partial c} \qquad ...(10)$$

Writing u, v, w for $\partial x/\partial t,\ \partial y/\partial t,\ \partial z/\partial t$, (8), (9) and (10) may be re-written as

$$\frac{\partial u}{\partial t}\frac{\partial x}{\partial a} + \frac{\partial v}{\partial t}\frac{\partial y}{\partial a} + \frac{\partial w}{\partial t}\frac{\partial z}{\partial a} = -\frac{\partial Q}{\partial a} \qquad ...(11)$$

$$\frac{\partial u}{\partial t}\frac{\partial x}{\partial b} + \frac{\partial v}{\partial t}\frac{\partial y}{\partial b} + \frac{\partial w}{\partial t}\frac{\partial z}{\partial b} = -\frac{\partial Q}{\partial b} \qquad ...(12)$$

$$\frac{\partial u}{\partial t}\frac{\partial x}{\partial c} + \frac{\partial v}{\partial t}\frac{\partial y}{\partial c} + \frac{\partial w}{\partial t}\frac{\partial z}{\partial c} = -\frac{\partial Q}{\partial c} \qquad ...(13)$$

Differentiating (12) and (13) partially w.r.t. to c and b respectively, subtracting and noting that $\dfrac{\partial}{\partial c}\left(\dfrac{\partial Q}{\partial b}\right) = \dfrac{\partial}{\partial b}\left(\dfrac{\partial Q}{\partial c}\right)$ etc., we get

$$\left(\frac{\partial^2 u}{\partial b \partial t}\frac{\partial x}{\partial c} - \frac{\partial^2 u}{\partial c \partial t}\frac{\partial x}{\partial b}\right) + \left(\frac{\partial^2 v}{\partial b \partial t}\frac{\partial y}{\partial c} - \frac{\partial^2 v}{\partial c \partial t}\frac{\partial y}{\partial b}\right) + \left(\frac{\partial^2 w}{\partial b \partial t}\frac{\partial z}{\partial c} - \frac{\partial^2 w}{\partial c \partial t}\frac{\partial z}{\partial b}\right) = 0$$

or
$$\left\{\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c} - \frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right) - \frac{\partial u}{\partial b}\frac{\partial^2 x}{\partial t \partial c} + \frac{\partial u}{\partial c}\frac{\partial^2 x}{\partial t \partial b}\right\} + \text{two similar terms} = 0$$

or
$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c} - \frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right) + \text{two similar terms} = 0, \quad \text{as} \quad \frac{\partial^2 x}{\partial c \partial t} = \frac{\partial u}{\partial c}, \quad \frac{\partial^2 x}{\partial t \partial b} = \frac{\partial u}{\partial b} \quad \text{and} \quad u = \frac{\partial x}{\partial t}$$

or
$$\frac{\partial}{\partial t}\left\{\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c}-\frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right)+\left(\frac{\partial v}{\partial b}\frac{\partial y}{\partial c}-\frac{\partial v}{\partial c}\frac{\partial y}{\partial b}\right)+\left(\frac{\partial w}{\partial b}\frac{\partial z}{\partial c}-\frac{\partial w}{\partial c}\frac{\partial z}{\partial b}\right)\right\}=0$$

Integrating the above equation with respect to t and taking u_0, v_0, w_0 as initial values, we get

$$\frac{\partial u}{\partial b}\frac{\partial x}{\partial c}-\frac{\partial u}{\partial c}\frac{\partial x}{\partial b}+\frac{\partial v}{\partial b}\frac{\partial y}{\partial c}-\frac{\partial v}{\partial c}\frac{\partial y}{\partial b}+\frac{\partial w}{\partial b}\frac{\partial z}{\partial c}-\frac{\partial w}{\partial c}\frac{\partial z}{\partial b}=\frac{\partial w_0}{\partial b}-\frac{\partial v_0}{\partial c}, \qquad ...(14)$$

where we have used the following results :

Initially : $x = a, y = b$ and $z = c$ so that $\partial x/\partial a = 1$, $\partial x/\partial b = 0$, $\partial x/\partial c = 0$ etc.

Now,
$$\frac{\partial u}{\partial a}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial a}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial a}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial a} \text{ etc.} \qquad ...(15)$$

Making use of relations of the type (15), (14) may be re-wirtten as

$$\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\frac{\partial(y,z)}{\partial(b,c)}+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\frac{\partial(z,x)}{\partial(b,c)}+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\frac{\partial(x,y)}{\partial(b,c)}=\frac{\partial w_0}{\partial b}-\frac{\partial v_0}{\partial c} \qquad ...(14)'$$

Let ξ, η, ζ be the vorticity components. Then, we have [refer Art. 2.27 of chapter 2]

$$\xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}, \qquad \eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}, \qquad \zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

Then the above equation (14) becomes

$$\xi\frac{\partial(y,z)}{\partial(b,c)}+\eta\frac{\partial(z,x)}{\partial(b,c)}+\zeta\frac{\partial(x,y)}{\partial(b,c)}=\xi_0 \qquad ...(16)$$

Similarly,
$$\xi\frac{\partial(y,z)}{\partial(c,a)}+\eta\frac{\partial(z,x)}{\partial(c,a)}+\zeta\frac{\partial(x,y)}{\partial(c,a)}=\eta_0 \qquad ...(17)$$

and
$$\xi\frac{\partial(y,z)}{\partial(a,b)}+\eta\frac{\partial(z,x)}{\partial(a,b)}+\zeta\frac{\partial(x,y)}{\partial(a,b)}=\zeta_0, \qquad ...(18)$$

where ξ_0, η_0, ζ_0 are the initial vorticity components.

The equation of continuity in Lagrangian system is

$$\rho\frac{\partial(x,y,z)}{\partial(a,b,c)}=\rho_0. \qquad ...(19)$$

Multiplying (16), (17), (18) by $\partial x/\partial a$, $\partial x/\partial b$, $\partial x/\partial c$ respectively, adding and using (19), we obtain

$$\frac{\xi}{\rho}=\frac{\xi_0}{\rho_0}\frac{\partial x}{\partial a}+\frac{\eta_0}{\rho_0}\frac{\partial x}{\partial b}+\frac{\zeta_0}{\rho_0}\frac{\partial x}{\partial c} \qquad ...(20)$$

Similarly,
$$\frac{\eta}{\rho}=\frac{\xi_0}{\rho_0}\frac{\partial y}{\partial a}+\frac{\eta_0}{\rho_0}\frac{\partial y}{\partial b}+\frac{\zeta_0}{\rho_0}\frac{\partial y}{\partial c} \qquad ...(21)$$

and
$$\frac{\zeta}{\rho}=\frac{\xi_0}{\rho_0}\frac{\partial z}{\partial a}+\frac{\eta_0}{\rho_0}\frac{\partial z}{\partial b}+\frac{\zeta_0}{\rho_0}\frac{\partial z}{\partial c}. \qquad ...(22)$$

These are known as *Cauchy's integrals.*

We now state and prove the following theorem.

Statement : *The motion of a inviscid fluid under conservative forces, if once irrotational, is* *always irrotational.*

OR

When the external forces are conservative and are derived from a single valued potential and pressure is a function of density only, then if once the motion of a non-viscous fluid is irrotational, it remains irrotational even afterwards.

Proof. Let the motion be initially irrotational so that $\xi_0 = \eta_0 = \zeta_0 = 0$. Then (20), (21) and (22) show that $\xi = \eta = \zeta = 0$ are always zero. Thus if once the motion is irrotational, it remains irrotational even afterwards.

3.13. Helmholtz equations or Helmholtz vorticity equations

[G.N.D.U. Amritsar 1999, Kanpur 2000]

The Euler's equations of motion are :

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad ...(1a)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y} \quad ...(1b)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=Z-\frac{1}{\rho}\frac{\partial p}{\partial z} \quad ...(1c)$$

Let V be the potential function of the external forces and let ρ be a function of p. Then $1(a)$ may be re-written as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or
$$\frac{\partial u}{\partial t}+\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}\right)+v\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+w\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad ...(2)$$

Let $\Omega = \xi\mathbf{i}+\eta\mathbf{j}+\zeta\mathbf{k}$ be the vorticity vector so that (ξ, η, ζ) are the vorticity components or the components of spin. These are given by

$$\xi=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right), \qquad \eta=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right), \qquad \zeta=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \quad ...(3)$$

Let
$$q^2 = u^2+v^2+w^2$$

Then
$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}=\frac{1}{2}\frac{\partial q^2}{\partial x} \quad ...(4)$$

Using (3) and (4), (2) reduces to

$$\frac{\partial u}{\partial t}-2v\zeta+2w\eta=-\frac{\partial}{\partial x}\left(V+\frac{1}{2}q^2+\int\frac{dp}{\rho}\right) \quad ...(5)$$

Let
$$Q=V+\frac{1}{2}q^2+\int\frac{dp}{\rho} \quad ...(6)$$

Then (5) reduces to

$$\frac{\partial u}{\partial t}-2v\zeta+2w\eta=-\frac{\partial Q}{\partial x} \quad ...(7)$$

Similarly, 1(b) and 1(c) may be re-written as :

$$\frac{\partial v}{\partial t} - 2w\xi + 2u\zeta = -\frac{\partial Q}{\partial y} \qquad \text{...(8)}$$

and
$$\frac{\partial w}{\partial t} - 2u\eta + 2v\xi = -\frac{\partial Q}{\partial z} \qquad \text{...(9)}$$

Differentiating (8) and (9) partially w.r.t. 'z' and 'y' and using the fact $\partial^2 Q/\partial z\partial y = \partial Q/\partial y\partial z$, we obtain

$$\frac{\partial^2 v}{\partial z\partial t} - 2w\frac{\partial \xi}{\partial z} - 2\xi\frac{\partial w}{\partial z} + 2u\frac{\partial \zeta}{\partial z} + 2\zeta\frac{\partial u}{\partial z} = \frac{\partial^2 w}{\partial y\partial t} - 2u\frac{\partial \eta}{\partial y} - 2\eta\frac{\partial u}{\partial y} + 2v\frac{\partial \xi}{\partial y} + 2\xi\frac{\partial v}{\partial y}$$

or
$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - 2u\left(\frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right) + 2v\frac{\partial \xi}{\partial y} + 2w\frac{\partial \xi}{\partial z} + 2\xi\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - 2\eta\frac{\partial u}{\partial y} - 2\zeta\frac{\partial u}{\partial z} = 0 \qquad \text{...(10)}$$

From (3), it easily follows that
$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0 \qquad \text{...(11)}$$

Using (11), (10) reduces to

$$2\frac{\partial \xi}{\partial t} + 2u\frac{\partial \xi}{\partial x} + 2v\frac{\partial \xi}{\partial y} + 2w\frac{\partial \xi}{\partial z} + 2\xi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - 2\xi\frac{\partial u}{\partial x} - 2\eta\frac{\partial u}{\partial z} - 2\zeta\frac{\partial u}{\partial z} = 0$$

or
$$\frac{D\xi}{Dt} + \xi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \left(\xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z}\right) \qquad \text{...(12)}$$

Now the equation of continuity is

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \quad \text{so that} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{D\rho}{Dt}. \qquad \text{...(13)}$$

Using (13), (12) becomes

$$\frac{D\xi}{Dt} - \frac{\xi}{\rho}\frac{D\rho}{Dt} = \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} \quad \text{or} \quad \frac{1}{\rho}\frac{D\xi}{Dt} - \frac{\xi}{\rho^2}\frac{D\rho}{Dt} = \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial u}{\partial y} + \frac{\zeta}{\rho}\frac{\partial u}{\partial z}$$

or
$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial u}{\partial y} + \frac{\zeta}{\rho}\frac{\partial u}{\partial z} \qquad \text{...(14a)}$$

Similarly, we have

$$\frac{D}{Dt}\left(\frac{\eta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial v}{\partial x} + \frac{\eta}{\rho}\frac{\partial v}{\partial y} + \frac{\zeta}{\rho}\frac{\partial v}{\partial z} \qquad \text{...(14b)}$$

$$\frac{D}{Dt}\left(\frac{\zeta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial w}{\partial x} + \frac{\eta}{\rho}\frac{\partial w}{\partial y} + \frac{\zeta}{\rho}\frac{\partial w}{\partial z} \qquad \text{...(14c)}$$

We now re-write (14*a*), (14*b*), (14*c*) in another form. Using (3), we observe that

$$\frac{\eta}{\rho}\frac{\partial u}{\partial y} + \frac{\zeta}{\rho}\frac{\partial u}{\partial z} = \frac{\eta}{\rho}\left[\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\partial v}{\partial x}\right] + \frac{\zeta}{\rho}\left[\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \frac{\partial w}{\partial x}\right]$$

$$= \frac{\eta}{\rho}\cdot(-2\zeta) + \frac{\eta}{\rho}\frac{\partial v}{\partial x} + \frac{\zeta}{\rho}\cdot(2\eta) + \frac{\zeta}{\rho}\frac{\partial w}{\partial x} = \frac{\eta}{\rho}\frac{\partial v}{\partial x} + \frac{\zeta}{\rho}\frac{\partial w}{\partial x} \qquad \text{...(15)}$$

Using (15), (14*a*) reduces to

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right)=\frac{\xi}{\rho}\frac{\partial u}{\partial x}+\frac{\eta}{\rho}\frac{\partial v}{\partial x}+\frac{\zeta}{\rho}\frac{\partial w}{\partial x} \qquad ...(16a)$$

Similarly, (14*b*) and (14*c*) reduce to

$$\frac{D}{Dt}\left(\frac{\eta}{\rho}\right)=\frac{\xi}{\rho}\frac{\partial u}{\partial y}+\frac{\eta}{\rho}\frac{\partial v}{\partial y}+\frac{\zeta}{\rho}\frac{\partial w}{\partial y} \qquad ...(16b)$$

and
$$\frac{D}{Dt}\left(\frac{\zeta}{\rho}\right)=\frac{\xi}{\rho}\frac{\partial u}{\partial z}+\frac{\eta}{\rho}\frac{\partial v}{\partial z}+\frac{\zeta}{\rho}\frac{\partial w}{\partial z}. \qquad ...(16c)$$

Equations (16*a*), (16*b*), (16*c*) are known as *Helmholtz's equations*. Let at any instant $t, \xi=\eta=\zeta=0$. Then the above equations reduce to

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right)=\frac{D}{Dt}\left(\frac{\eta}{\rho}\right)=\frac{D}{Dt}\left(\frac{\zeta}{\rho}\right)=0 \qquad ...(17)$$

or
$$\frac{D\xi}{Dt}=\frac{D\eta}{Dt}=\frac{D\zeta}{Dt}=0, \quad \text{if } \rho = \text{constant} \qquad ...(18)$$

Equation (18) shows that ξ, η, ζ must be constant. Since these are zero at time $t = 0$, it follows that $\xi=\eta=\zeta=0$ at all time afterwards.

Thus those elements of fluid which at any instant have no rotation remain during the motion without rotation.

We discuss the general case by taking $\rho \neq \text{constant}$. Let $\partial u/\partial x, \partial v/\partial x,$ etc. be all finite and let L denote their superior limit. Then ξ/ρ, η/ρ, ζ/ρ cannot increase faster than if they satisfied the equations

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right)=\frac{D}{Dt}\left(\frac{\eta}{\rho}\right)=\frac{D}{Dt}\left(\frac{\zeta}{\rho}\right)=L(\xi+\eta+\zeta)/\rho \qquad ...(19)$$

Let
$$\xi+\eta+\zeta=\rho W \qquad ...(20)$$

Then (19) reduces to
$$\frac{D}{Dt}\left(\frac{\xi}{\rho}+\frac{\eta}{\rho}+\frac{\zeta}{\rho}\right)=\frac{DW}{Dt}=3LW,$$

so that if W is not zero, by dividing by W and integrating, we have

$$W = Ce^{3Lt}, \ C \text{ being an arbitrary constant} \qquad ...(21)$$

But $\xi=\eta=\zeta=0$ at $t = 0$. So $W = 0$ at $t = 0$. Hence (21) shows that $C = 0$ and so W is always zero. But W is the sum of three quantities ξ, η, ζ which evidently cannot be negative. It follows that $\xi=\eta=\zeta=0$. Moreover as ξ, η, ζ remain zero when they satisfy (19), still more will they do so when they satisfy (16*a*) to (16*c*)

Thus, in general, if the motion is irrotational at any instant, it must be so for all time. In other words, if once, the velocity potential exists, it exists for all time. T*his is known as the principle of permanence of irrotational motion.*

An illustrative solved example

Prove that in the steady motion of an incompressible liquid, under the action of conserrative forces, we have $\xi(\partial u/\partial x) + \eta(\partial u/\partial y) + \zeta(\partial u/\partial z) = 0$ *and two similar equation in* v *and* w.

Sol. Helmholtz equations are given by

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial u}{\partial y} + \frac{\zeta}{\rho}\frac{\partial u}{\partial z} \qquad ...(1a)$$

$$\frac{D}{Dt}\left(\frac{\eta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial v}{\partial x} + \frac{\eta}{\rho}\frac{\partial v}{\partial y} + \frac{\zeta}{\rho}\frac{\partial v}{\partial z} \qquad ...(1b)$$

and
$$\frac{D}{Dt}\left(\frac{\zeta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial w}{\partial x} + \frac{\eta}{\rho}\frac{\partial w}{\partial y} + \frac{\zeta}{\rho}\frac{\partial w}{\partial z} \qquad ...(1c)$$

For the steady incompressible liquid, $\quad \dfrac{D}{Dt}\left(\dfrac{\xi}{\rho}\right) = \dfrac{D}{Dt}\left(\dfrac{\eta}{\rho}\right) = \dfrac{D}{Dt}\left(\dfrac{\zeta}{\rho}\right) = 0$

$\therefore \quad (1a) \Rightarrow \quad \xi(\partial u/\partial x) + \eta(\partial u/\partial y) + \zeta(\partial u/\partial z) = 0$

$(1b) \Rightarrow \quad \xi(\partial v/\partial x) + \eta(\partial v/\partial y) + \zeta(\partial v/\partial z) = 0$

and $\quad (1c) \Rightarrow \quad \xi(\partial w/\partial x) + \eta(\partial w/\partial y) + \zeta(\partial w/\partial z) = 0$

OBJECTIVE QUESTIONS ON CHAPTER 3

Multiple choice questions

Choose the correct alternative from the following questions

1. The equation for impulsive action is

(*i*) $\mathbf{q}_2 - \mathbf{q}_1 = \mathbf{I} + (1/\rho)\nabla\tilde{\omega}$ (*ii*) $\mathbf{q}_2 + \mathbf{q}_1 = \mathbf{I} - (1/\rho)\nabla\tilde{\omega}$

(*iii*) $\mathbf{q}_2 - \mathbf{q}_1 = \mathbf{I} - (1/\rho)\nabla\tilde{\omega}$ (*iv*) $\mathbf{q}_1 - \mathbf{q}_2 = \mathbf{I} + \rho\nabla\tilde{\omega}$ **[Kanpur 2001]**

2. The motion of a inviscid fluid under conservative forces, if once irrorational, is always

(*i*) rotational (*ii*) irrotational (*iii*) laminar (*iv*) None of these

3. If $\tilde{\omega}$ denotes the impulsive pressure and external impulsive body forces are absent, then

(*i*) $\nabla^2\tilde{\omega} = 0$ (*ii*) $\nabla\tilde{\omega} = 0$ (*iii*) $\nabla^2\tilde{\omega} \neq 0$ (*iv*) None of these

4. Euler's equation of motion in x-direction is

(*i*) $Du/Dt = X - (1/\rho)\times(\partial p/\partial x)$ (*ii*) $Du/Dt = X + (1/\rho)\times(\partial p/\partial x)$

(*iii*) $\partial u/\partial t = X - (1/\rho)\times(\partial p/\partial x)$ (*iv*) $\partial u/\partial t = X + (1/\rho)\times(\partial p/\partial x)$

Answers/Hints to objective type questions

1. (*iii*). See Eq. (3), Art. 3.6 **2.** (*ii*). Refer Art. 3.1.2

3. (*i*). See Cor. 1, Art. 3.6 **4.** (*i*). See Eq. (1), Art. 3.1

CHAPTER

4 One-Dimensional Inviscid Incompressible Flow (Bernoulli's Equation and its Applications)

4.1. Integration of Euler's equations of motion. Bernoulli's equation. Pressure equation.

[I.A.S. 2005; Kanpur 2002, 04, 05, 09; Meerut 2000, 02, 08]

When a velocity potential exists (so that the motion is irrotational) and the external forces are derivable from a potential function, the equations of motion can always be integrated. Let ϕ be the velocity potential and V be the force potential. Then, by definition, we get

$$u = -\partial\phi/\partial x, \qquad v = -\partial\phi/\partial y, \qquad w = -\partial\phi/\partial z, \qquad \text{...(1)}$$

$$X = -\partial V/\partial x, \qquad Y = -\partial V/\partial y, \qquad Z = -\partial V/\partial y, \qquad \text{...(2)}$$

and
$$\partial u/\partial y = \partial v/\partial x, \qquad \partial v/\partial z = \partial w/\partial y, \qquad \partial w/\partial x = \partial u/\partial z. \qquad \text{...(3)}$$

Then well known Euler's dynamical equation are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Using (1) (2) and (3), these can be re-written as

$$\left.\begin{aligned}
-\frac{\partial^2\phi}{\partial t\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} &= -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}\\
-\frac{\partial^2\phi}{\partial t\partial y} + u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y} &= -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y}\\
-\frac{\partial^2\phi}{\partial t\partial z} + u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z} &= -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z}
\end{aligned}\right\} \qquad \text{...(4)}$$

Re-writing equations (4), we get

$$-\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2+w^2)=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{...(5)}$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial y}(u^2+v^2+w^2)=-\frac{\partial V}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{...(6)}$$

$$-\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial z}(u^2+v^2+w^2)=-\frac{\partial V}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \text{...(7)}$$

Now
$$d\left(\frac{\partial \phi}{\partial t}\right)=\frac{\partial}{dx}\left(\frac{\partial \phi}{\partial t}\right)dx+\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right)dy+\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)dz \qquad \text{...(8)}$$

$$dV=(\partial V/\partial x)\,dx+(\partial V/\partial y)\,dy+(\partial V/\partial z)\,dz \qquad \text{...(9)}$$

$$dp=(\partial p/\partial x)\,dx+(\partial p/\partial y)\,dy+(\partial p/\partial z)\,dz \qquad \text{...(10)}$$

$$d(u^2+v^2+w^2)=\frac{\partial}{\partial x}(u^2+v^2+w^2)\,dx+\frac{\partial}{\partial y}(u^2+v^2+w^2)\,dy+\frac{\partial}{\partial z}(u^2+v^2+w^2)\,dz \qquad \text{...(11)}$$

Multiplying (5), (6) and (7) by dx, dy and dz respectively, then adding and using (8), (9), (10) and (11), we have

$$-d\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}d(u^2+v^2+w^2)=-dV-\frac{1}{\rho}dp$$

or
$$-d\left(\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}dq^2+dV+\frac{1}{\rho}dp=0 \qquad \text{...(12)}$$

where $q^2=u^2+v^2+w^2$ = (velocity of fluid particle)^2

If ρ is a function of p, integration of (12) gives

$$-\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2+V+\int\frac{dp}{\rho}=F(t), \qquad \text{...(13)}$$

where F(t) is an arbitrary function of t arising from integration in which t is regarded as constant. (13) is *Bernoulli's equation* in its most general form. Equation (13) is also known as *pressure equation.*

Special Case I. Let the fluid be homogeneous and inelastic (so that ρ = constant *i.e.*, fluid is incompressible). Then Bernoulli's equation for unsteady and irrotational motion is given by

$$-\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2+V+\frac{p}{\rho}=F(t) \qquad \text{...(14)}$$

Special Case II. If the motion be steady $\partial\phi/\partial t=0,$ the Bernoulli's equation for steady irrotational motion of an incompressible fluid is given by

$q^2/2+V+p/\rho=C$, where C is an absolute constant. **(Kanpur 2010)** ...(15)

4.2. Bernoulli's theorem. (Steady motion with no velocity potential and conservative field of force).

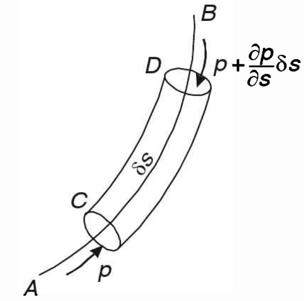
[Agra 2009; Meerut 2009, 2010; Kanpur 2004; Purvanchel 2005; G.N.D.U. Amritsar 2002, 05]

When the motion is steady and the velocity potential does not exist, we have

$$\frac{1}{2}q^2+V+\int\frac{dp}{\rho}=C,$$

where V is the force potential from which the external forces are derivable. **[Meerut 2011]**

Proof. Consider a streamline AB in the fluid. Let δs be an element of this stream line and CD be a small cylinder of cross-sectional area α and δs as axis. If q be the velocity and S be the component of external force per unit mass in direction of the streamline, then by Newton's second law of motion, we have



$$\rho\alpha\,\delta s.\frac{Dq}{Dt}=\rho\alpha\,\delta s\cdot S+p\alpha-\left(p+\frac{\partial p}{\partial s}\delta s\right)\alpha$$

or $$\frac{Dq}{Dt}=S-\frac{1}{\rho}\frac{\partial p}{\partial s}$$

or $$\frac{\partial q}{\partial t}+q\frac{\partial q}{\partial s}=S-\frac{1}{\rho}\frac{\partial p}{\partial s} \qquad \text{...(1)}$$

If the motion be steady $\partial q/\partial t=0$, and if the external forces have a potential function V such that $S=-\partial V/\partial s$, (1) reduces to

$$\frac{1}{2}\frac{\partial q^2}{\partial s}+\frac{\partial V}{\partial s}+\frac{1}{\rho}\frac{\partial p}{\partial s}=0 \qquad \text{...(2)}$$

If ρ is a function of p, integration of (2) along the streamline AB yields

$$\frac{1}{2}q^2+V+\int\frac{dp}{\rho}=C, \qquad \text{...(3)}$$

where C is constant whose value depends on the particular chosen streamline.

Special Case I. If the fluid be homogeneous and incompressible, ρ = constant and hence (3) reduces to

$$q^2/2+V+p/\rho=C. \qquad \textbf{(Kanpur 2008)} \qquad \text{...(4)}$$

Special Case II. Let S be a graviational force per unit mass. Let δh be the vertical distance between C and D. Then we have

$$S=-g\frac{\partial h}{\partial s}=-\frac{\partial}{\partial s}(gh), \qquad \text{as} \qquad V=gh$$

Hence, if the fluid be incompressible, (3) reduces to

$$q^2/2+gh+p/\rho=C. \qquad \text{...(5)}$$

4.3. Illustrative solved examples.

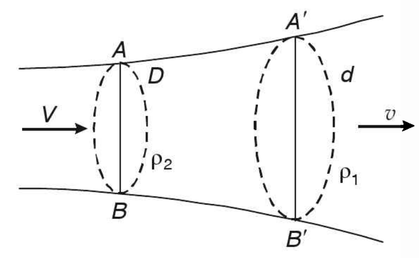
Ex. 1. *A stream is rushing from a boiler through a conical pipe, the diameter of the ends of which are D and d; if V and v be the corresponding velocities of the stream and if the motion be supposed to be that of the divergence from the vertex of the cone, prove that*

$$v/V=(D^2/d^2)e^{(v^2-V^2)/2k} \qquad \textbf{[I.A.S. 1993, 98]}$$

where k is the pressure divided by the density and supposed constant.

Sol. Let AB and $A'B'$ be the ends of the conical pipe such that $A'B'$ = d and AB = D. Let ρ_1 and ρ_2 be densities of the stream at $A'B'$ and AB. *By principle of conservation of mass, the mass of the streem that entres the end AB and leaves at the end A'B'* must be the same. Hence the equation of continuity is

$$\pi(d/2)^2 v\rho_1=\pi(D/2)^2 V\rho_2$$



so that $$\frac{v}{V} = \frac{D^2}{d^2} \times \frac{\rho_2}{\rho_1} \qquad ...(1)$$

By Bernoulli's theorem (in absence of external forces like gravity), we have

$$\int \frac{dp}{\rho} + \frac{1}{2} q^2 = C \qquad ...(2)$$

Given that $p/\rho = k$ so that $dp = k\, d\rho$. ...(3)

$\therefore$ (2) reduces to $k \int \frac{d\rho}{\rho} + \frac{1}{2} q^2 = C$, using (3)

Integrating, $k \log \rho + q^2/2 = C$, C being an arbitrary constant ...(4)

When $q = v$, $\rho = \rho_1$ and when $q = V$, $\rho = \rho_2$. Hence, (4) yields

$$k \log \rho_1 + v^2/2 = C \qquad \text{and} \qquad k \log \rho_2 + V^{2/} 2 = C$$

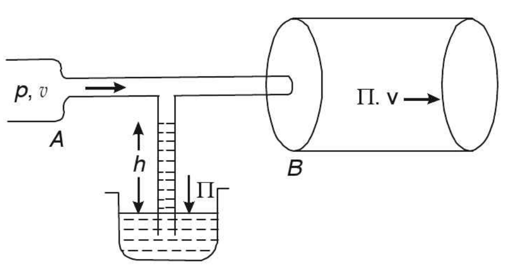
Subtracting, $k(\log \rho_2 - \log \rho_1) + (V^2 - v^2)/2 = 0$

or $\log (\rho_2/\rho_1) = (v^2 - V^2) 2k$ or $\rho_2/\rho_1 = e^{(v^2 - V^2)2k}$. ...(5)

Using (5), (1) reduces to $v/V = (D^2/d^2) \times e^{(v^2 - V^2)/2k}$.

Ex. 2. *A stream in a horizontal pipe, after passing a contraction in the pipe at which its sectional area is A is delivered at atmospheric pressure at a place, where the sectional area is B. Show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservoir at a depth* $(s^2/2g) \times (1/A^2 - 1/B^2)$ *below the pipe, s being the delivery per second.* **[I.A.S. 1997]**

Sol. Let v be the velocity in the tube of smaller section A and p the pressure at that section. Further, let V and Π be the corresponding quantities at the bigger section B of the figure. Then, by Bernoulli's Theorem (in absence of external forces like gravity) for incompressible fluid, namely



$$p/\rho + q^2/2 = \text{constant},$$

we obtain $p/\rho + v^2/2 = \Pi/\rho + V^2/2$

so that $$(\Pi - p)/\rho = (v^2 - V^2)/2 \qquad ..(1)$$

Let h be the height through which water is sucked up. Then

$$(\alpha h)\rho g = \alpha\Pi - \alpha p, \quad \alpha \text{ being area of cross-section of the tube}$$

or $$g\rho h = \text{difference of pressure} = \Pi - p. \qquad ...(2)$$

The equation of continuity is $Av = BV = s$ (delivery per second)

so that $v = s/A$ and $V = s/B$ ...(3)

Using (2) and (3), (1) reduces to

$$\frac{1}{\rho} \times g\rho h = \frac{1}{2}\left(\frac{s^2}{A^2} - \frac{s^2}{B^2}\right) \qquad \text{or} \qquad h = \frac{s^2}{2g}\left(\frac{1}{A^2} - \frac{1}{B^2}\right).$$

Ex. 3. *A mass of homogeneous liquid is moving so that the velocity at any point is proportional to the time and that the pressure is given by*

$$p/\rho = \mu\, x\, y\, z - (t^2/2) \times (y^2 z^2 + z^2 x^2 + x^2 y^2).$$

Prove that this motion may have been generated from rest by natural forces independent of the time and show that, if the direction of motion at every point coincides with the direction of the acting force, each particle of the liquid describes a curve which is the intersection of two hyperbolic cylinders.

Sol. Given that velocity q is proportional to time. So $\quad q = \lambda t \quad$ ...(1)

Also, given $\quad p/\rho = \mu\, x\, y\, z - (t^2/2) \times (y^2 z^2 + z^2 x^2 + x^2 y^2) \quad$ ...(2)

Suppose that the motion is produced by finite natural forces (conservative forces) which are derivable from the potential function V. Then by Bernoulli's equation, we get

$$\frac{p}{\rho} - \frac{\partial \phi}{\partial t} + \frac{1}{2} q^2 + V = F(t)$$

or

$$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - \frac{1}{2} \lambda^2 t^2 - V + F(t), \text{ using (1)} \qquad ...(3)$$

Since (2) and (3) must be identical, equating the coefficients of t^2 on R.H.S. of (2) and (3), we get

$$\lambda^2 = y^2 z^2 + z^2 x^2 + x^2 y^2 \qquad ...(4)$$

Using (4), (1) reduces to $\quad q^2 = t^2 (y^2 z^2 + z^2 x^2 + x^2 y^2) \quad$ ...(5)

But $\quad q^2 = (\partial \phi / \partial x)^2 + (\partial \phi / \partial y)^2 + (\partial \phi / \partial z)^2 \quad$ ...(6)

Comparing (5) and (6), an appropriate value of ϕ is given by

$$\phi = t\, xyz \qquad ...(7)$$

Using (7) and (4), (3) reduces to

$$p/\rho = xyz - (t^2/2) \times (y^2 z^2 + z^2 x^2 + x^2 y^2) - V + F(t) \qquad ...(8)$$

Comparing (2) and (8), we find $\quad F(t) = 0 \quad$ and $\quad xyz - V = \mu\, xyz$

Thus, $\quad V = xyz\,(1 - \mu) \quad$ ...(9)

If u, v, w are the components of velocities and X, Y, Z are the components of forces, then

$$u = -(\partial \phi / \partial x) = -tyz, \qquad v = -(\partial \phi / \partial y) = -txz, \qquad w = -(\partial \phi / \partial z) = -txy$$

and $X = -(\partial V / \partial x) = (\mu - 1) yz$, $Y = -(\partial V / \partial y) = (\mu - 1) xz$, $Z = -(\partial V / \partial z) = (\mu - 1) xy \quad$ ... (9)′

Given that the direction of motion coincides with that of the acting forces. Hence, we have

$$u/X = v/Y = w/Z$$

Again, the equations of the path

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad \text{reduce to} \qquad \frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$$

i.e.,

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}, \text{ using (9)}' \qquad ...(10)$$

Taking the first two members of (10), we get

$$x\,dx - y\,dy = 0 \qquad \text{so that} \qquad x^2 - y^2 = C_1 \qquad ...(11)$$

Taking the last two members of (10), we get

$$ydy - zdz = 0 \qquad \text{so that} \qquad y^2 - z^2 = c_2 \qquad ...(12)$$

Thus each particle of the fluid will be on the curve which is the intersection of two hyperbolic cylinders $x^2 - y^2 = C_1$ and $y^2 - z^2 = C_2$, C_1 and C_2 being arbitrary constants

Ex. 4. A *quantity of liquid occupies a length 2l of a straight tube of uniform small bore under the action of a force to a point in the tube varying as a distance from that point. Determine the pressure at any point.*

OR

A quantity of liquid occupies a length 2l of a straight tube of uniform bore under the action of force which is equal to μx *to a point O in the tube, where x is the distance from O. Find the motion and show that if z be the distance of the nearer free surface from O, pressure at any point is given by* $\quad p/\rho = -(\mu/2) \times (x^2 - z^2) + \mu(x - z)(z + l)$.

Sol. Let p be the pressure and u the velocity at a distance x from the fixed point O; and let z be the distance of the nearer free surface from O. Then the equation of continuity is

$$\partial u / \partial x = 0 \qquad ...(1)$$

Let μx be the external force at a distance x which acts towards O. Then equation of motion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{reduces to} \qquad \frac{\partial u}{\partial t} = -\mu x - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(2)$$

Integrating (2) w.r.t. 'x', we get

$$x\frac{\partial u}{\partial t} = -\frac{1}{2}\mu x^2 - \frac{p}{\rho} + C, \ C \text{ being an arbitrary constant} \qquad ...(3)$$

But $p = 0$ when $x = z$ and $x = z + 2l$. So (3) gives

$$z\frac{\partial u}{\partial t} = -\frac{1}{2}\mu z^2 + C \qquad ...(4)$$

$$(z + 2l)\frac{\partial u}{\partial t} = -\frac{1}{2}\mu(z + 2l)^2 + C \qquad ...(5)$$

Subtracting (4) from (5), we get

$$2l\frac{\partial u}{\partial t} = -\frac{1}{2}\mu[(z + 2l)^2 - z^2] \qquad \text{or} \qquad \frac{\partial u}{\partial t} = -\mu(z + l) \qquad ...(6)$$

or $$d^2 z / dt^2 = -\mu(z + l) \qquad [\because u = dz/dt] \qquad ...(7)$$

Putting $\quad z + l = y \quad$ so that $\quad z = y - l, \quad$ (7) gives

$$d^2 y / dt^2 + \mu y = 0$$

whose solution is $\quad y = A\cos(t\sqrt{\mu} + B)$, A and B being arbitrary constants.

Since $y = z + 1$, it yields $$z = A\cos(t\sqrt{\mu} + B) - l \qquad ...(8)$$

in which A and B may be determined from the knowledge of initial position and velocity.

We now determine pressure. From (4), we get

$$C = z\frac{\partial u}{\partial t} + \frac{1}{2}\mu z^2$$

Putting this value of C in (3), we get

$$\frac{p}{\rho}=-\frac{1}{2}\mu(x^2-z^2)-(x-z)\frac{\partial u}{\partial t} \quad \text{or} \quad \frac{p}{\rho}=-\frac{1}{2}\mu(x^2-z^2)+\mu(x-z)(z+l), \quad \text{using (6)}$$

which gives the pressure at any point.

Ex. 5. *A horizontal pipe gradually reduces in diameter from* 24 *in. to* 12 *in. Determine the total longitudinal thrust exerted on the pipe if the pressure at the larger end is* 50 *lbf/in*^{2} *and the velocity of the water is* 8 *ft./sec.*

Sol. Let S_1 and S_2 be the cross-sections of the larger and the smaller ends. Let q_1 and q_2 be the velocities and p_1 and p_2 be the pressures at the larger and the smaller ends of the pipe. Given

$$S_1=\pi(12)^2\text{in}^2. \qquad \text{and} \qquad S_2=\pi(6)^2\text{in}^2.,$$

Also, $q_1 = 8 \times 12 = 96$ in/sec. and $p_1 = 50$ lbf/in^2.

The equation of continuity $S_1q_1 = S_2q_2$ gives $\pi(12)^2 q_1=\pi(6)^2 q_2$

so that
$$q_2=4q_1 \qquad \ldots(1)$$

By Bernouill's equation, we have

$$\frac{p_1}{\rho}+\frac{1}{2}q_1^2=\frac{p_2}{\rho}+\frac{1}{2}q_2^2 \qquad \text{or} \qquad p_1-p_2=\frac{1}{2}\rho(q_2^2-q_1^2)$$

$$p_1-p_2=(1/2)\times\rho(16q_1^2-q_1^2), \text{ using (1)}$$

or
$$p_2=p_1-(15/2)\times\rho q_1^2 \qquad \ldots(2)$$

$\therefore$ The required longitudinal thrust on the pipe

$$=p_1S_1-p_2S_2=\pi(12)^2p_1-\pi(6)^2p_2=36\pi(4p_1-p_2)$$

$$=36\pi[4p_1-p_1+(15/2)\times\rho q_1^2], \text{ using (2)}$$

$$=36\pi(3p_1+\frac{15}{2}\rho q_1^2)=36\pi\left(150+\frac{15}{2}\times\frac{62.4}{12^3}\times 96^2\right)=95040\pi$$

$$(\because\ \rho=62.4 \text{ lb/ft}^3=(62.4)/12^3 \text{ lb/in}^3.)$$

Ex. 6. *An elastic fluid, the weight of which is neglected, obeying Boyle's law is in motion in a uniform straight tube; show that on the hypothesis of parallel sections the velocity at any time t at a distance r from a fixed point in the tube is defined by the equation*

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left(2v\frac{\partial v}{\partial t}+v^2\frac{\partial v}{\partial r}\right)=k\frac{\partial^2 v}{\partial r^2}.$$

[Kanpur 2006; Rohilkhand 2005]

Sol. The equation of continuity is
$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial r}(\rho v)=0. \qquad \ldots(1)$$

The equation of motion is
$$\frac{\partial v}{\partial r}+v\frac{\partial v}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}. \qquad \ldots(2)$$

Since the given elastic fluid obeys Boyle's law, we have

$$p=k\rho \qquad \text{so that} \qquad \frac{\partial p}{\partial r}=k\frac{\partial\rho}{\partial r}. \qquad \ldots(3)$$

Using (3), (2) becomes $$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}=-\frac{k}{\rho}\frac{\partial \rho}{\partial r}. \qquad ..(4)$$

Differentiating (4) partially with respect to t, we have

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial t}\left(v\frac{\partial v}{\partial r}\right)=-\frac{\partial}{\partial t}\left(\frac{k}{\rho}\frac{\partial \rho}{\partial r}\right). \qquad ...(5)$$

Now, $$\frac{\partial}{\partial t}\left(v\frac{\partial v}{\partial r}\right)=\frac{\partial}{\partial t}\left[\frac{\partial}{\partial r}\left(\frac{1}{2}v^2\right)\right]=\frac{\partial}{\partial r}\left[\frac{\partial}{\partial t}\left(\frac{1}{2}v^2\right)\right]=\frac{\partial}{\partial r}\left(v\frac{\partial v}{\partial t}\right)$$

and $$\frac{\partial}{\partial t}\left(\frac{k}{\rho}\frac{\partial \rho}{\partial r}\right)=\frac{\partial}{\partial t}\frac{\partial}{\partial r}(k\log\rho)=\frac{\partial}{\partial r}\left[\frac{\partial}{\partial t}(k\log\rho)\right]=\frac{\partial}{\partial r}\left[\frac{k}{\rho}\frac{\partial \rho}{\partial t}\right].$$

Hence (5) reduces to

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left(v\frac{\partial v}{\partial t}\right)=-\frac{\partial}{\partial r}\left(\frac{k}{\rho}\frac{\partial \rho}{\partial t}\right) \quad \text{or} \quad \frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left(v\frac{\partial v}{\partial t}+\frac{k}{\rho}\frac{\partial \rho}{\partial t}\right)=0$$

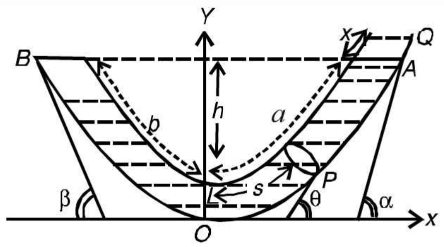
or $$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left[v\frac{\partial v}{\partial t}+\frac{k}{\rho}\left\{-\frac{\partial(\rho v)}{\partial r}\right\}\right]=0, \text{ using (1)}$$

or $$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left\{v\frac{\partial v}{\partial t}-\frac{k}{\rho}\left(\rho\frac{\partial v}{\partial r}+v\frac{\partial \rho}{\partial r}\right)\right\}=0 \quad \text{or} \quad \frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left(v\frac{\partial v}{\partial t}-k\frac{\partial v}{\partial r}-v\frac{k}{\rho}\frac{\partial \rho}{\partial r}\right)=0$$

or $$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left\{v\frac{\partial v}{\partial t}-k\frac{\partial v}{\partial r}+v\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}\right)\right\}, \text{ using (4)}$$

or $$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left(2v\frac{\partial v}{\partial t}+v^2\frac{\partial v}{\partial r}-k\frac{\partial v}{\partial r}\right)=0 \quad \text{or} \quad \frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left(2v\frac{\partial v}{\partial t}+v^2\frac{\partial v}{\partial r}\right)=k\frac{\partial^2 v}{\partial r^2}.$$

Ex. 7. *Water oscillates in a bent uniform tube in a vertical plane. If O be the lowest point of the tube, AB the equilibrium level of the water, α, β the inclinations of the tube to the horizontal at A, B and $OA = a$, $OB = b$, the period of oscillation is given by* $2\pi\{(a+b)/g(\sin\alpha+\sin\beta)\}^{1/2}$ **[Ranchi 2010; Garhwal 2000; Kanpur 2001]**



Sol. Let O be the lowest point of the tube, AB the equilibrium level of water, h the height of AB above O, α, β, the inclinations of the tube to the horizontal at A and B and θ its inclination at P at distance s from O. Let a, b, denote the lengths OA, OB and suppose that at time t the water is displaced a small distance x along the tube from its equilibrium position.

If q is the velocity, the equation of continuity is $\partial q/\partial s = 0$. ...(1)

Again, the equation of motion is $$\frac{\partial q}{\partial t}+q\frac{\partial q}{\partial s}=-g\sin\theta-\frac{1}{\rho}\frac{\partial p}{\partial s}. \qquad ...(2)$$

Using (1), (2) becomes $$\frac{\partial q}{\partial t}=-g\sin\theta-\frac{1}{\rho}\frac{\partial p}{\partial s}$$

or $$\frac{\partial q}{\partial t}=-g\frac{\partial y}{\partial s}-\frac{1}{\rho}\frac{\partial p}{\partial s}, \quad as \quad \sin\theta=\frac{\partial y}{\partial s} \qquad ...(3)$$

Integrating (3) with respect to s, we have

$$s\frac{\partial q}{\partial t} = -gy - \frac{p}{\rho} + f(t), \quad \text{where } f(t) \text{ is an arbitrary function of } t. \quad ...(4)$$

Let Π be the atmospheric pressure. Then, the conditions at the ends of the tube are : When $y = h + x \sin\alpha$, $s = a + x$, $p = \Pi$ and when $y = h - x\sin\beta$, $s = -(b - x)$, $p = \Pi$.

Hence, (4) yields

$$(a + x)\frac{\partial q}{\partial t} = -g(h + x\sin\alpha) - \frac{\Pi}{\rho} + f(t) \quad ...(5)$$

and
$$-(b - x)\frac{\partial q}{\partial t} = -g(h - x\sin\beta) - \frac{\Pi}{\rho} + f(t). \quad ...(6)$$

Subtracting (6) from (5), we have $\quad (a + b)\dfrac{\partial q}{\partial t} = -gx(\sin\alpha + \sin\beta)$

or $\quad \dfrac{d^2x}{dt^2} = -\dfrac{g}{a+b}(\sin\alpha + \sin\beta)x, \quad$ as $\quad q = \dfrac{dx}{dt} \quad$ and $\quad \dfrac{\partial q}{\partial t} = \dfrac{d^2x}{dt^2}$

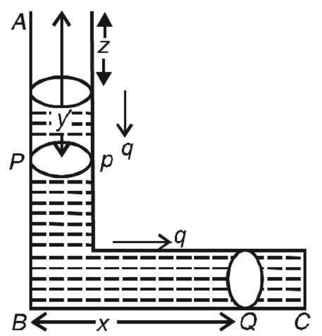
or $\quad \ddot{x} = -\mu x, \quad$ where $\quad \mu = g(\sin\alpha + \sin\beta)/(a + b)$.

This represents a simple harmonic motion. If T be its time period, then we have

$$T = 2\pi/\sqrt{\mu} = 2\mu/[g(\sin\alpha + \sin\beta)/(a+b)]^{1/2} = 2\pi\{(a+b)/g(\sin\alpha + \sin\beta)\}^{1/2}$$

Ex. 8. *A straight tube of small bore, ABC, is bent so as to make the angle ABC a right angle and AB equal to BC. The end C is closed and the tube is placed with end A upwards and AB vertical and is filled with liquid. If the end C be opened, prove that the pressure at any point of the vertical tube is instantaneously diminished one-half and find the instantaneous change of pressure at any point of the horizontal tube, the pressure of the atmosphere being neglected.*

[Aligarh 2005; Bangalore 2003, Nagpur 1999]



Sol. Let ABC be the given tube in which AB is vertical and BC is horizontal. Let $AB = BC = a$.

Let at time t the liquid fall through a depth z and at that instant let q be the downward velocity and p be the pressure at a point P at depth y in the vertical tube. Since the only external force acting downwards is g, the equation of motion is

$$\frac{\partial q}{\partial t} + q\frac{\partial q}{\partial y} = g - \frac{1}{\rho}\frac{\partial p}{\partial y}. \quad ...(1)$$

Since the motion is one-dimensional, the equation of continuity is

$$\partial q/\partial y = 0. \quad ...(2)$$

Using (2), (1) becomes $\quad \partial q/\partial t = g - (1/\rho)(\partial p/\partial y). \quad ...(3)$

Integrating (3) with respect to y, we have

$$(\partial q/\partial t)\, y = gy - (p/\rho) + C, \quad C \text{ being an arbitrary constant.} \quad ...(4)$$

Initially, when $y = z$, $p = 0$. So $\quad (4) \Rightarrow \quad C = z(\partial q/\partial t - g)$.

So (4) gives $\quad \dfrac{\partial q}{\partial t}y = gy - \dfrac{p}{\rho} + z\dfrac{\partial q}{\partial t} - gz$

so that
$$\frac{p}{\rho} = \left(g - \frac{\partial q}{\partial t}\right)(y - z). \quad ...(5)$$

At B, where $y = a$, let $p = p_1$ so that from (5), we have

$$p_1 = \rho\left(g - \frac{\partial q}{\partial t}\right)(a - z). \qquad \text{...(6)}$$

The cross-section of the tube being the same everywhere, let a point Q at a distance x from the point B have the velocity q and let p' be the pressure there.

The equation of continuity is $\qquad \partial q / \partial x = 0. \qquad$ ...(7)

and the equation of motion is $\qquad \dfrac{\partial q}{\partial t} + q\dfrac{\partial q}{\partial x} = -\dfrac{1}{\rho}\dfrac{\partial p'}{\partial x}$

or $\qquad \partial q / \partial t = -(1/\rho)(\partial p'/\partial x)$, using (7). $\qquad$ ...(8)

Integrating (8) with respect to x, $\qquad x\,(\partial q/\partial t) = -(p'/\rho) + C'$. $\qquad$ ...(9)

At C, when $\quad x = a, \quad p' = 0 \quad$ so $\quad (9) \Rightarrow \quad C' = a(\partial q / \partial t)$.

Hence (9) gives $\qquad p' = \rho(-\partial q / \partial t)(x - a)$. $\qquad$ ...(10)

At B, where $x = 0$, $p' = p_1$ so that from (10), we have

$$p_1 = \rho a\,(\partial q / \partial t). \qquad \text{...(11)}$$

Now, $\qquad$ (6) and (11) $\qquad \Rightarrow \qquad \rho\left(g - \dfrac{\partial q}{\partial t}\right)(a - z) = \rho a \dfrac{\partial q}{\partial t}$

or $\qquad (\partial q / \partial t)(2a - z) = g(a - z)$. $\qquad$ ...(12)

Initially, when $z = 0$, (12) gives $\qquad \left(\dfrac{\partial q}{\partial t}\right)_0 = \dfrac{1}{2}g$. $\qquad$ ...(13)

If the initial pressure at P be p_0, then putting $z = 0$ in (5), we get

$$\frac{p_0}{\rho} = \left\{g - \left(\frac{\partial q}{\partial t}\right)_0\right\} y = \left(g - \frac{1}{2}g\right) y = \frac{1}{2}gy.$$

Thus, $\qquad p_0 = (1/2) \times g\rho y$. $\qquad$ ...(14)

When end C is closed, the hydrostatic pressure p_H at P is $\qquad p_H = g\rho y$. $\qquad$ ...(15)

Now, $\qquad$ (14) and (15) $\qquad \Rightarrow \qquad p_0 = (1/2) \times p_H$,

showing that the pressure at any point of the vertical tube is instantaneously diminished by half.

If p'_0 be the initial pressure at Q, then from (10), we get

$$p'_0 = -\rho(x - a)\left(\frac{\partial q}{\partial t}\right)_0 = -\rho \times \frac{1}{2}(x - a)g, \text{ using (13)}$$

When end C is closed, let p_2 be the initial pressure at Q.

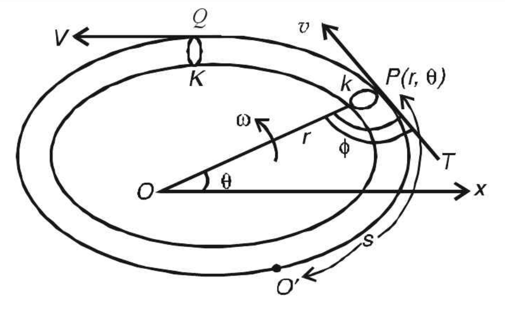
$\therefore \qquad p_2 =$ the initial pressure at $B = g\rho a$.

Hence the change in pressure

$$= p_2 - p'_0 = g\rho a + (1/2) \times \rho(x - a)g = (1/2) \times g\rho(a + x).$$

Ex. 9. *A fine tube whose section k is a function of its length s, in the form of a closed plane curve of area A, filled with ice, is moved in any manner. When the component angular velocity of the tube about a normal to its plane is* Ω, *the ice melts without change of volume. Prove that the velocity of the fluid relatively to the tube at a point where the section is K at any subsequent time when* ω *is the anguler velocity is*

$2A(\Omega - \omega)\Big/\left\{K\displaystyle\int \frac{ds}{k}\right\}$, *the integral being taken once round tube.*



Sol. With O as pole (origin) and OX as initial line, let polar coordinates of an arbitrary point P on the tube be (r, θ). Let O' be taken as a fixed point on the tube and the length of the arc $O'P$ of the tube be s. Let the tube rotate about a normal through O and let at any subsequent time, after the ice melts, the component angular velocity be Ω. Let the cross-section of the tube be k at P and let v be velocity there. Let Q be the point where the cross-section of the tube is K and velocity V. Let p be pressure at P.

The equation of continuity is $\quad vk = VK. \quad$ ...(1)

Relative to the tube, the acceleration of the fluid particle along the tangent at P is

$$(\partial V/\partial t) + V(\partial V/\partial s).$$

Let ϕ be the angle between radius vector OP and tangent PT. Then we know that

$$\sin\phi = r\,(d\theta/ds) \qquad \text{and} \qquad \cos\phi = dr/ds. \qquad ...(2)$$

Now, acceleration of the point P of the tube along the tangent at P

$$= r\dot{\omega}\,\sin\phi - r\omega^2\cos\phi = r\dot{\omega}\frac{rd\theta}{ds} - r\omega^2\frac{dr}{ds}, \qquad \text{where} \qquad \dot{\omega} = \frac{d\omega}{dt}$$

Hence the equation of motion is given by $\quad \dfrac{\partial V}{\partial t} + V\dfrac{\partial V}{\partial s} + r^2\dot{\omega}\dfrac{d\theta}{ds} - r\omega^2\dfrac{dr}{ds} = -\dfrac{1}{\rho}\dfrac{dp}{ds}.$

Integrating both sides of the above equation w.r.t. 's' once round the tube, we have

$$\int\frac{\partial V}{\partial t}ds + \int V\frac{\partial V}{\partial s}ds + \int r^2\dot{\omega}\frac{d\theta}{ds}ds - \int r\omega^2\frac{dr}{ds}ds = -\frac{1}{\rho}\int\frac{dp}{ds}ds \qquad ...(3)$$

But $\int V\dfrac{\partial V}{\partial s}ds = \int V\,dV = [V^2/2]_Q^Q = 0$, Similarly, $\int r\omega^2\dfrac{dr}{ds}ds = \omega^2\left[r^2/2\right]_Q^Q = 0$ and $\int (dp/ds)ds = [p]_Q^Q = 0$. Hence, (3) reduces to

$$\int\frac{\partial V}{\partial t}ds + \int r^2\dot{\omega}\,d\theta = 0 \qquad \text{or} \qquad \int\frac{\partial V}{\partial t}ds = -\dot{\omega}\int r^2 d\theta$$

or $$\int\frac{\partial V}{\partial t}ds = -2A\dot{\omega}, \qquad \text{as} \qquad A = \int\frac{1}{2}r^2\,d\theta \qquad ...(4)$$

Integrating both sides of (4) w.r.t. 't', we get

$$\iint\frac{\partial V}{\partial t}ds\,dt = -2A\int_\Omega^\omega\frac{d\omega}{dt}dt \qquad \text{or} \qquad \int v\,ds = 2A(\Omega - \omega)$$

or $$\int\frac{VK}{k}ds = 2A(\Omega - \omega), \qquad \text{since from (1),} \qquad v = \frac{VK}{k}$$

or $$VK\int\frac{ds}{k} = 2A(\Omega - \omega) \qquad \text{or} \qquad V = 2A(\Omega-\omega)\Big/\left\{K\int\frac{ds}{k}\right\},$$

Ex. 10. *The water is flowing through a tapering pipe having diameters* 300 mm *and* 150 mm *at sections AA′ and BB′ respectively. The discharge through the pipe is* 40 litres/s. *The section AA′ is* 10 m *above datum and section BB′ is* 6 m *above datum. Find the intensity of pressure at section BB′ if that at section AA′ is* 400 KN/m^2.

Sol. At section AA', we have $\quad$ diameter $= d_1 = 300$ mm $= 0.3$ m

$\therefore$ Area of cross-section $= S_1 = (\pi/4) \times (0{\cdot}3)^2 = 0{\cdot}0707$ m^2

Pressure = p_1 = 400 kN/m^2

Height of upper end above the datum = h_1 = 10 m

At section BB', we have

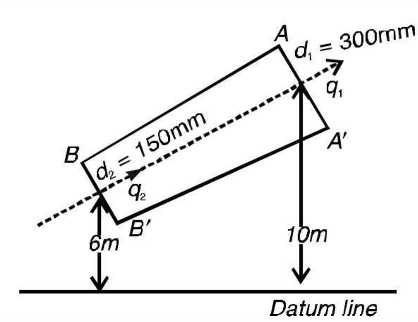
diameter = d_2 = 150 mm = 0.15 m

$\therefore$ Area of crosssection = $S_2 = (\pi/4) \times (0.15)^2$
$= 0.01767$ m^2.

Height of the lower end above the datum = h_2 = 6 m.

Let pressure at section BB' be p_2.

Rate of flow (*i.e.* discharge) = Q



$$= 40 \text{ liters/s} = \frac{40 \times 10^3}{10^6} = 0.04 \text{ m}^3/\text{s}.$$

Let velocity of flow at sections AA' and BB' be q_1 and q_2 respectively. Then, we have

$$q_1 = \frac{Q}{S_1} = \frac{0.04}{0.0707} = 0.566 \text{ m/s} \qquad \text{and} \qquad q_2 = \frac{Q}{S_2} = \frac{0.04}{0.01767} = 2.264 \text{ m/s}.$$

Applying Bernaulli's equation at sections AA' and BB', we get

$$\frac{1}{2}q_1^2 + gh_1 + \frac{p_1}{\rho} = \frac{1}{2}q_2^2 + gh_2 + \frac{p_2}{\rho} \qquad \text{or} \qquad \frac{p_2}{\rho} = \frac{p_1}{\rho} + \frac{1}{2}(q_1^2 - q_2^2) + g(h_1 - h_2)$$

or,
$$\frac{p_2}{w} = \frac{p_1}{w} + \frac{1}{2g}(q_1^2 - q_2^2) + (h_1 - h_2), \qquad \text{as} \qquad \rho g = w$$

$$\Rightarrow \quad \frac{p_2}{w} = \frac{400}{9.81} + \frac{1}{2 \times 9.81}\left[(0.566)^2 - (2.264)^2\right] + (10 - 6) = 40.77 - 0.245 + 4 = 44.525 \text{ m}$$

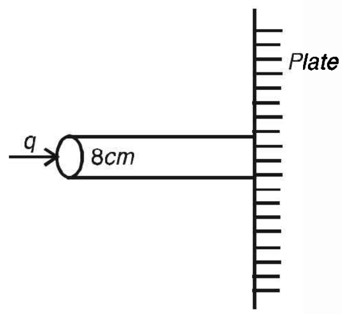
$$\Rightarrow \quad p_2 = 44.525 \times w = 44.525 \times 9.81 = 436.8 \text{ kN/m}^2 \quad \text{as} \quad w = 9.81 \text{kN/m}^3$$

Ex. 11. *A jet of water* 8 *cm in diameter impinges on a plate held normal to its axis. For a velocity of* 4 *m/s, what force will keep the plate in equilibrium?*

Sol. Diameter of jet = d = 8 cm = 0.08 m.

$\therefore$ Area of cross section of the jet

$$= S = (\pi/4) \times d^2 = (\pi/4) \times (0.08)^2 m^2.$$



q = velocity of jet = 4 m/s

w = weight per unit cubic meter of water = 10^3 kg/m^3

F = Force acting on the jet.

Now, force on the plate = charge in momentum

$$\Rightarrow \qquad F = \frac{w(Sq)q}{g} = \frac{wSq^2}{g} = \frac{1000 \times (\pi/4) \times (0.08)^2 \times 4^2}{9.81} = 32.8 kg.$$

Ex. 12. *Air, obeying Boyle's law, is in motion in a uniform tube of a small section prove that if* ρ *be the density and v the velocity at a distance x from a fixed point at time t, then*

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\{\rho(v^2 + k)\}, \qquad \text{where} \qquad k = \frac{p}{\rho}.$$ **[Garhwal 2005. Kanpur 2004, Meerut 2003, 2011]**

Sol. Given that $p/\rho = k$ that is, $p = \rho k$ ... (1)

Since the motion is one-densional, the equations of continuity and motion (refer equation (1) with $S = 0$ in Art. 4.2) are respectively

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \qquad \text{...(2)}$$

and $$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(3)$$

From (1), we have $$\partial p/\partial x = k(\partial \rho/\partial x) \qquad ...(4)$$

Using (4), (3) reduces to $$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{k}{\rho}\frac{\partial \rho}{\partial x} \qquad ...(5)$$

Differentiating (2) partialy w.r.t. 't', we get

$$\frac{\partial^2 \rho}{\partial t^2}+\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial x}(\rho v)\right\}=0 \qquad \text{or} \qquad \frac{\partial^2 \rho}{\partial t^2}+\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial t}(\rho v)\right\}=0$$

or $$\frac{\partial^2 \rho}{\partial t^2}+\frac{\partial}{\partial x}\left\{\rho\frac{\partial v}{\partial t}+v\frac{\partial \rho}{\partial t}\right\}=0 \qquad \text{or} \qquad \frac{\partial^2 \rho}{\partial t^2}+\frac{\partial}{\partial x}\left\{\rho\left(-v\frac{\partial v}{\partial x}-\frac{k}{\rho}\frac{\partial \rho}{\partial x}\right)-v\frac{\partial}{\partial x}(\rho v)\right\}=0$$

[on using (2) and (5)]

or $$\frac{\partial^2 \rho}{\partial t^2}+\frac{\partial}{\partial x}\left\{\rho v\frac{\partial v}{\partial x}+k\frac{\partial \rho}{\partial x}+v\frac{\partial}{\partial x}(\rho v)\right\}=\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}(\rho v\cdot v)+k\frac{\partial \rho}{\partial x}\right\}=\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}(\rho v^2+k\rho)\right\}$$

or $$\frac{\partial^2 \rho}{\partial t^2}=\frac{\partial^2}{\partial x^2}\{\rho(v^2+k)\}.$$

Ex. 13. *If the body force* $\boldsymbol{F}$ *form a conservative system, density* ρ *is a function of* p *only and the flow is steady, prove that* $\Omega+P+\mathbf{q}^2/2$ *is constant along every streamline and vortex line, where* $\boldsymbol{F}=-\nabla\Omega$, $P=\int\left(\frac{1}{\rho}\right)dp$ and $\boldsymbol{q}$ is velocity. **[I.A.S. 1989]**

Sol. Vector equation of motion for invicid incompressible fluid is

$$\partial \boldsymbol{q}/\partial t+\nabla(\boldsymbol{q}^2/2)-\boldsymbol{q}\times \operatorname{curl}\boldsymbol{q}=\boldsymbol{F}-(1/\rho)\nabla p \qquad ...(1)$$

Since the flow in steady, $\partial \boldsymbol{q}/\partial t=\mathbf{0}$. ...(2)

Since ρ in function of p only, $(1/\rho)\nabla p=\nabla(p/\rho)$. ...(3)

Also given that $\boldsymbol{F}=-\nabla\Omega$. ...(4)

By definition, vorticity vector = $\boldsymbol{w}$ = curl $\boldsymbol{q}$. ...(5)

Using (2), (3), (4) and (5), (1) reduces to

$$\nabla(\boldsymbol{q}^2/2)-\boldsymbol{q}\times\boldsymbol{w}=-\nabla\Omega-\nabla(p/\rho) \qquad \text{or} \qquad \nabla\left(\frac{1}{2}\boldsymbol{q}^2+\Omega+\int\frac{dp}{\rho}\right)=\boldsymbol{q}\times\boldsymbol{w}$$

or $$\nabla\left(\Omega+P+\frac{1}{2}\boldsymbol{q}^2\right)=\boldsymbol{q}\times\boldsymbol{w}, \qquad \text{as} \qquad \int\left(\frac{1}{\rho}\right)dp=P \text{ (given)} \qquad ...(6)$$

Taking scalar product of (6) with $d\,\boldsymbol{r}$, *a* time independent variation in the position vector $\boldsymbol{r}$ of the fluid particle, we get

$$d(\boldsymbol{\Omega}+P+\boldsymbol{q}^2/2)=d\boldsymbol{r}\cdot(\boldsymbol{q}\times\boldsymbol{w}). \qquad ...(7)$$

Two cases arise:

Case I. Let $\boldsymbol{q}\times\boldsymbol{w}=\mathbf{0}$. Then have

either (*i*) when $\boldsymbol{q}$ and $\boldsymbol{w}$ are parallel, *i.e.*, when the streamlines and vortex lines coincide. For such motion, $\boldsymbol{q}$ is known as *Beltrami vector.*

or (*ii*) when $\boldsymbol{w}=\mathbf{0}$, *i.e.* the motion is irrotational.

In both cases, (7) gives $$d(\Omega + P + \boldsymbol{q}^2/2) = 0 \qquad ...(8)$$
at all times throughout the entire flow field.

Integrating (8), $$\Omega + P + \boldsymbol{q}^2/2 = \text{constant} \qquad ...(9)$$
throughout the entire field of flow. The constatnt in (9) will remain unchanged throughout the entire field because the differential $\boldsymbol{dr}$ in (7) is an arbitrary small variation of position vector $\boldsymbol{r}$ in the field.

Case II. When $\mathbf{q} \times \mathbf{w} \neq \mathbf{0}$. Since $\boldsymbol{q} \times \boldsymbol{w}$ is perpendicular to the vectors $\boldsymbol{q}$ and $\boldsymbol{w}$, it follows that if $\boldsymbol{dr} \neq \mathbf{0}$, then $\boldsymbol{dr} \cdot (\boldsymbol{q} \times \boldsymbol{w}) = 0$ whenever $\boldsymbol{dr}$ lies in the plane of $\boldsymbol{q}$ and $\boldsymbol{w}$. Therefore, if we take the variation $\boldsymbol{dr}$ in the surface containing both the streamlines and vortex lines, then (7) shows that $$d[\Omega + P + \boldsymbol{q}^2/2] = 0 \qquad \text{over such a surface}$$

and hence $$\Omega + P + \boldsymbol{q}^2/2 = \text{constant} \qquad ...(10)$$
over a surfce containing the streamlines and vortex lines. It may be observed that the constant in (10) is the same everywhere on any one such surface, but that its value varies from one surface to another. It may be noted that (10) holds for steady rotational as well as irrotational motious.

Ex. 14. *Prove that in a steady motion of a liquid,* $H = p/\rho + q^2/2 + V = \text{constant}$ *along a streamline. If this constant has the same value everywhere in the liquid, then prove that the motion must be either irrotational or the vertex lines must coincide with the streamlines.*

In two dimensional motion of a liquid with constant vorticity ζ, *prove that*
$$\nabla(H - 2\zeta\psi) = 0.$$
Show that if the motion be steady, the pressure is given by $p/\rho + q^2 + V - 2\zeta\psi = \text{const.}$, *where* ∇ *is the Laplace's operator.* **[Agra 2000; I.A.S. 1992; Rohilkhand 1999]**

Sol. For the first part, refer Art. 4.2 Thus, we have
$$\int \frac{dp}{\rho} + \frac{1}{2}q^2 + V = \text{const. along a streamline} \qquad ...(1)$$
If the fluid be homogeneous so that $\rho = \text{const}$, then (1) becomes
$$H = p/\rho + q^2/2 + V = \text{constant along a streamline} \qquad ...(2)$$
Second part. We know that for steady motion, Euler's equations of motion for homogeneous liquid moving under conservative forces (so that $X = -\partial V/\partial x$, $y = -\partial V/\partial y$, $Z = -\partial V/\partial z$, where V is force potential and $\boldsymbol{F} = X\,\boldsymbol{i} + Y\,\boldsymbol{j} + Z\,\boldsymbol{k}$ and $\boldsymbol{q} = u\,\boldsymbol{i} + v\,\boldsymbol{j} + w\,\boldsymbol{k}$) (refer Art. of 3.1 chapter3).
$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial y}\right)u = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(3A)$$
$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)v = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad ...(3B)$$
$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)w = -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad ...(3C)$$
Let ξ, η, ζ be the vorticity components, then
$$2\xi = \partial w/\partial y - \partial v/\partial z, \qquad 2\eta = \partial u/\partial z - \partial w/\partial x, \qquad 2\zeta = \partial v/\partial x - \partial u/\partial y, \qquad ...(4)$$

Re-writing (3*A*), we have

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}+v\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+w\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial V}{\partial x}=0$$

or $$\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2+w^2)+v(-2\zeta)+w(2\eta)+\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial V}{\partial x}=0, \text{ using (4)}$$

or $$\frac{\partial}{\partial x}\left(\frac{p}{\rho}+\frac{q^2}{2}+V\right)=2(v\zeta-w\eta), \qquad \text{where} \qquad q^2=u^2+v^2+w^2$$

or $$\partial H/\partial x=2(v\zeta-w\eta), \text{ using (2)} \qquad \text{...(5A)}$$

Similarly (3B) and (3C) reduce to

$$\partial H/\partial y=2(w\xi-u\zeta) \qquad \text{...(5B)}$$

$$\partial H/\partial z=2(u\eta-v\xi) \qquad \text{...(5C)}$$

Multiplying (5A), (5B), (5C) by u, v, w respectively and then adding, we get

$$u(\partial H/\partial x)+v(\partial H/\partial y)+w(\partial H/\partial z)=0 \qquad \text{...(6)}$$

Multiplying (5A), (5B), (5C) by ξ, η, ζ and the adding, we get

$$\xi(\partial H/\partial x)+\eta(\partial H/\partial y)+\zeta(\partial H/\partial z)=0 \qquad \text{...(7)}$$

From (6) and (7), it follows that the surface H = const. contains the streamlines (whose direction cosines are proportional to u, v, w) and vertex lines (whose direction cosines are proportional to ξ, η, ζ).

If H has the same value everywhere in the liquid, we have

$$\partial H/\partial x=0, \qquad \partial H/\partial y=0 \qquad \text{and} \qquad \partial H/\partial z=0$$

$$\Rightarrow \qquad v\zeta-w\eta=0, \qquad w\xi-u\zeta=0, \qquad u\eta-v\xi=0, \text{ by (7A) (7B) and (7C)}$$

$$\Rightarrow \qquad \text{either} \qquad \xi=\eta=\zeta=0 \qquad \text{or} \qquad u/\xi=v/\eta=w/\zeta$$

Now, $\xi=\eta=\zeta=0$ $\Rightarrow$ the motion is irrotational.

and $u/\xi=v/\eta=w/\zeta$ $\Rightarrow$ streamlines given by $(dx)/u=(dy)/v=(dz)/w$

coincide with vortex lines $(dx)/\xi=(dy)/\eta=(dz)/\zeta$

Third part. Consider two dimensional motion such that $\zeta=$ constnat.

Then, $w=0$ and $2\zeta=\partial v/\partial x-\partial u/\partial y$. Also, if ψ be stream function, then we have (refer Art. 5.4, chapter 5)

$$2\zeta=\partial^2\psi/\partial x^2+\partial^2\psi/\partial y^2=\nabla\psi, \qquad \text{...(8)}$$

where $\nabla\ (\equiv \partial^2/\partial x^2+\partial^2/\partial y^2)$ is Laplace's operater ...(8)'

As before, Euler's equations motion (refer Art. 3.1, chapter 3), for two dimensional notion (so that $w=0$), we have

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{...(9A)}$$

and $$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial V}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{...(9B)}$$

Re-writing (9A), $$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+v\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or $$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2)+v\times(-2\zeta)=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or $$\frac{\partial u}{\partial t}-2v\zeta=-\left[\frac{\partial}{\partial x}\left(\frac{q^2}{2}\right)+\frac{\partial V}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}\right],\qquad \text{where}\qquad q^2=u^2+v^2$$

or $$\partial u/\partial t-2v\zeta=-(\partial H/\partial x). \qquad \text{...(10 A)}$$

Similarly, (9B) gives $$\partial v/\partial t+2u\,\zeta=-(\partial H/\partial y) \qquad \text{...(10B)}$$

Differentiating (10A) and (10B) w.r.t.'x' and 'y' respectively and then adding, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-2\zeta\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=-\left(\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}\right) \qquad \text{...(11)}$$

For incompressible fluid in two dimensions, equations of continuity is

$$\partial u/\partial x+\partial v/\partial y=0 \qquad \text{...(12)}$$

Also, for a two dimensinonal, if ψ be velocity potential, then we have

$$u=-\partial\psi/\partial y,\qquad v=\partial\psi/\partial x \qquad \text{so that} \qquad \partial u/\partial y=-\partial^2\psi/\partial y^2,\qquad \partial v/\partial x=\partial^2\psi/\partial x^2$$

$$\therefore \qquad \partial v/\partial x-\partial u/\partial y=\partial^2\psi/\partial x^2+\partial^2\psi/\partial^2 y^2=\nabla\psi, \qquad \text{...(13)}$$

where ∇ is given by (8)'.

Using (12), (13) and (8)', (11) reduces to

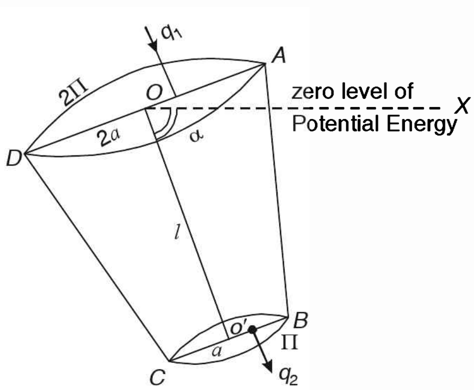
$$-2\zeta\nabla\psi=-\nabla H \qquad \text{or} \qquad \nabla(H-2\zeta\psi)=0 \qquad \text{...(14)}$$

Fourth Part. Let the motion be steady so that $\partial u/\partial t=\partial v/\partial t=0$ and so again (11) reduces to (15),

Integrating (14), $$H-2\zeta\psi=\text{const.}$$

or $$P/\rho+q^2/2+V-2\zeta\psi=\text{const., using (2)}$$

Ex. 15. *A long pipe is of length l and has slowly tapering cross-section. It is inclined at angle α to the horizontal and water flows steadily through it from the upper to the lower end. The section at the upper end has twice the radius of the lower end. At the lower end, the water is delivered at atmospheric pressure. If the pressure at the upper end is twice atmospheric, find the velocity of delivery.*



Sol. Let $ABCD$ be the given pipe of length $l(=OO')$. Let OX be horizontal line which is taken as the zero level of potential energy. Let radii of the ends AD and BC of the given pipe be $2a$ and a respecively Let Π be the atmopsheric pressure. Then, the pressures at upper and lower ends are 2Π and Π respectively. Let q_1 and q_2 be velocities at the entry end AD and exist end BC respectively.

Now, Bernoulli's equation for steady motion is

$$p/\rho+q^2/2+V=C, \qquad \text{...(1)}$$

where V is the force potential and C is a absolute constant.

At end *AD*, $V = 0$ and at end *BC*, $V = -gl \sin \alpha$, where $-gl \sin\alpha$ is the potential ernergy per unit mass of the gravitational force at the end *BC*. Hence, using (1) at the ends *AD* and *BC*,

$$\frac{2\Pi}{\rho} + \frac{1}{2}q_1^2 + 0 = C \qquad \text{and} \qquad \frac{\Pi}{\rho} + \frac{1}{2}q_2^2 - ql\sin\alpha = C \qquad ...(2)$$

Since the fluid is incompressible, equation of continuity of water flowing through the given pipe is (refer Art. 2.14) given by

$$q_1 \times (4\pi a^2) = q_2 \times (\pi a^2) \qquad \text{or} \qquad q_1 = q_2/4 \qquad ...(3)$$

From (2), $\dfrac{2\Pi}{\rho} + \dfrac{q_1^2}{2} = \dfrac{\Pi}{\rho} + \dfrac{q_2^2}{2} - gl\sin\alpha$ or $\dfrac{\Pi}{\rho} + gl\sin\alpha = \dfrac{q_2^2}{2} - \dfrac{q_2^2}{32}$, using (3)

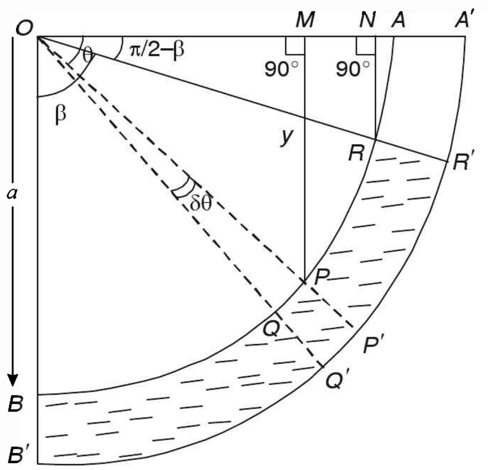
or $\dfrac{15}{32}q_2^2 = \dfrac{\Pi}{\rho} + gl\sin\alpha$ or $q_2 = \left\{\dfrac{32}{15}\left(\dfrac{\Pi}{\rho} + gl\sin\alpha\right)\right\}^{1/2}$,

which yields the desired velocity of delivery at exist *BC*.

Ex. 16. *AB is a tube of small uniform, bore forming a quadrant arc of a circle of radius a and centre O, OA being horizontal and OB vertical with B below O. The tube is full of liquid of density* ρ*, the end B being closed. If B is suddeenly opened, show that intially du/dt* $= 2g/\pi$*, where u = u(t) is the velocity and that the pressure at a point whose angular distance from A is* θ *immediately drops to* $\rho g a(\sin\theta - 2\theta/\pi)$ *above atmospheric pressure. Prove further that when the liquid contained in the tube subtends an angle* β *at the centre,*

$$d^2\beta/dt^2 = -(2g/a\beta)\times\sin^2(\beta/2).$$

Sol. As shown in the figure, *AB* is a tube of small uniform bore $(= AA' = BB' = PP' = QQ')$. Let $\angle AOB = 90°$. Let *P* be any point of the tube such that $\angle AOP = \theta$. Let *Q* be another point on the tube such that $\angle AOQ = \theta + \delta\theta$. Also, let arc $AP = s$ and arc $AQ = s + \delta s$, where *s* is measured from *A*.



Let *u*(*t*) be the velocity of the liquid along arc *AB* so that $\partial u/\partial s = 0$. Then equation of motion of the element *PP'Q'Q* is given by (refer equation (1) of Art. 4.2)

$$\frac{du}{dt} = g\cos\theta - \frac{1}{\rho}\frac{\partial p}{\partial s} \qquad ...(1)$$

But, $\dfrac{du}{dt} = \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial s}\left(\dfrac{1}{2}u^2\right)$ and $\cos\theta = \dfrac{dy}{ds}$,

where *y* is the depth of *P* blow *OA*. Therefore, (1) reduces to

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial s}\left(\frac{u^2}{2}\right) = g\frac{dy}{ds} - \frac{1}{\rho}\frac{\partial p}{\partial s} \qquad ...(2)$$

Integrating (2) w.r.t. '*s*' while treating *t* as constant, we get

$$s(\partial u/\partial t) + u^2/2 = gy - (p/\rho) + f(t), \qquad ...(3)$$

where *f* (*t*) is the arbitary function of *t*. Note that while integrating w.r.t. '*s*' we have assumed that at any time *t*, $\partial u/\partial t$ is the same at all points of the liquid.

Let Π be the atmospheric pressure.

Now, initially at A, when $t = 0$, $s = 0$, $u = 0$, $p = \Pi$ and $y = 0$.

$\therefore$ (3) reduces to $\quad 0 = -(\Pi/\rho) + f(0) \quad$ or $\quad f(0) = \Pi/\rho$ ...(4)

Again, at B, when $t = 0$, $s = \text{arc } AB = (\pi a)/2$, $u = 0$, $p = \Pi$, $y = a$

$\therefore$ (3) reduces to
$$\frac{\pi a}{2} \times \left(\frac{\partial u}{\partial t}\right)_{t=0} = ga - \frac{\Pi}{\rho} + f(0) \qquad \text{...(5)}$$

Subtracting (4) from (5), we have
$$\frac{\pi a}{2} \times \left(\frac{\partial u}{\partial t}\right)_{t=0} = ga \qquad \text{or} \qquad \left(\frac{\partial u}{\partial t}\right)_{t=0} = \frac{2g}{\pi} \qquad \text{...(6)}$$

We now apply (3) at P. Initialy at P, $t = 0$, $u = 0$, $y = OM = a \sin\theta$, s = arc $AP = a\theta$. Also, let $p = p_0$. Then (3) reduces
$$a\theta \times (\partial u/\partial t)_{t=0} = ga \sin\theta - (p_0/\rho) + f(0)$$

or
$$a\theta \times (2g/\pi) = ga\sin\theta - (p_0/\rho) + \Pi/\rho, \text{ by (4) and (6)}$$

or
$$(p_0 - \Pi)/\rho = ga\sin\theta - (2ga\theta)/\pi \qquad \text{or} \qquad p_0 - \Pi = \rho ga(\sin\theta - 2\theta/\pi),$$

showing that if B is suddenly opened, the pressure at P immediately drops to $\rho ga(\sin\theta - 2\theta/\pi)$.

Let us now consider the situation when the liquid contained in the tube AB subtends an angle β at the centre. In this case liquid is shown in part $RBB'R'$ of the tube such that $\angle ROB = \beta$. Then $\angle NOR = \pi/2 - \beta$. Let RN be perpendicular to OA. Thus $RN = OR\sin(\pi/2 - \beta)$ $= a\cos\beta$. Also, arc $AR = a(\pi/2 - \beta)$. For this situation, $P = \Pi$ at the surface RR'. So (3) gives
$$a\left(\frac{\pi}{2} - \beta\right)\frac{\partial u}{\partial t} + \frac{u^2}{2} = ga\cos\beta - \frac{\Pi}{\rho} + f(t) \qquad \text{...(7)}$$

Again, using (3) at B, where $s = (a\pi)/2$, $y = a$, $p = \Pi$, we get

$\therefore$
$$\frac{a\pi}{2} \times \frac{\partial u}{\partial t} + \frac{u^2}{2} = ga - \frac{\Pi}{\rho} + f(t) \qquad \text{...(8)}$$

Subtracting (7) from (8), we have
$$a\beta\frac{\partial u}{\partial t} = ga(1 - \cos\beta) \qquad \text{or} \qquad \frac{\partial u}{\partial t} = \frac{2g}{\beta}\sin^2\frac{\beta}{2} \qquad \text{...(9)}$$

Since arc $AR = a(\pi/2 - \beta)$, we have $\quad u = \dfrac{\partial}{\partial t}\left(\dfrac{a\pi}{2} - a\beta\right)$

or
$$u = -a\frac{d\beta}{dt} \qquad \text{so that} \qquad \frac{\partial u}{\partial t} = -a\frac{d^2\beta}{dt^2} \qquad \text{...(10)}$$

From (9) and (10), $\quad -a\dfrac{d^2\beta}{dt^2} = \dfrac{2g}{\beta}\sin^2\dfrac{\beta}{2} \quad$ or $\quad \dfrac{d^2\beta}{dt^2} = -\dfrac{2g}{a\beta}\sin^2\dfrac{\beta}{2}$

EXERCISE 4 (A)

1. Liquid of density ρ is flowing along a horizontal pipe of variable cross-section, and the pipe is connected with a differential pressure gauge at two points A and B. Show that if $p_1 - p_2$ is the pressure indicated by the gauge, the mass m of liquid through the pipe per second is given by

$m = \sigma_1\sigma_2\sqrt{\dfrac{2\sigma(p_1 - p_2)}{\sigma_1^2 - \sigma_2^2}}$, where σ_1, σ_2 are the cross-sections at A, B respectively.

2. The diameter of a pipe changes from 20 cm at a section 5 metres above datum, to 5 cm at a section 3 metres above datum. The pressure of water at first section is 5 kg/cm^2. If the velocity of flow at the first section is 1 m/sec, determine the intensity of pressure at the second section. [**Ans.** 3.9 kg/cm^2]

3. A pipe 300 metres long has a slope of 1 in 100 and tapers from 1 metre diameter at the high end to 0.5 metre at the low end. Quantity of water flowing is 5,400 litres per minute. If the pressure at the high end is 0.7 kg/cm^2, find the pressure at the low end. [**Ans.** 0.999 kg/cm^2]

4. Find out Bernoulli's equation for unsteady irratational. **[Kanpur 2005; Meerut 2002]**

5. Obtain the well known equation $-(\partial\phi/\partial t) + p/\rho + (q^2/2) + V = C$ **[Kanpur 2001]**

6. Obtain Bernoulli's equation for steady motion. **[Kanpur 2002, 03]**

7. Define pressure equation. **[Kanpur 2000]**

8. Define pressure equation in its most general form by integrating Euler's equation of motion. **[I.A.S. 2005; Meerut 2000]**

9. A vertical tube AB of small section has two apertures close to its base B in which horizontal tubes are fitted and the apertures are closed by valves; a given height a of the tube AB is filled with water and the valves are then opened. The areal section of each horizontal tube being half that of the vertical tube and the length of each greater than AB, prove that the motion is of the simple harmonic type until the vertical tube is emptied which will take place after time $(\pi/2)\sqrt{(a/g)}$.

10. In the case of a steady motion of an ineladtic fluid under no forces the velocities parallel to the axes at the point (x, y, z) are proportional to $y + z$, $z + x$, $x + y$; prove that the surfaces of equal pressure are oblate spheroids, the eccentricity of the generating ellipse being $\sqrt{2/3}$.

11. Derive Bernoulli's equation for unsteady motion of an incompressible fluid and hence derive expression for steady motion.

12. (*a*) State the condition under which Euler's equation of motion can be integrated. Show that

$$-\frac{\partial\phi}{\partial t} + \frac{1}{2}q^2 + V + \int\frac{dp}{\rho} = F(t),$$

where the symbols have this usual meaning. **[I.A.S. 2005]**

[**Hint:** Proceed as in Art. 4.1 upto equation (13)]

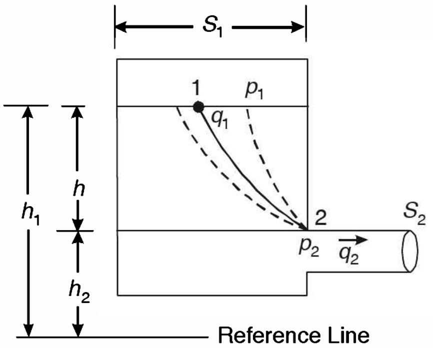
(*b*) When the velocity potential exists and the forces are conservative, show that the Euler's dynamical equations can always be integrated in the form $\displaystyle\int\frac{dp}{\rho} + \frac{1}{2}q^2 - \frac{\partial\phi}{\partial t} + V = f(t)$, where the symbols have their usual meaning. **[Kanpur 2008]**

4.4. Applications of Bernoulli's equation and theorem. [Meerut 2008]

Bernoulli's equation is of fundamental importance in fluid dynamics, especially in hydraulics. It is employed to handle some complicated situations of fluid flow problems in a simple manner. We now discuss some practical applications of the Bernoulli's equation. In each case the fluid will be assumed inviscid and incompressible

4.4A. Flow from a tank through a small orifice. Torrieclli's theorem.

Consider a tank containing a liquid. Let the tank be sealed except for a small orifice near its base. We wish to determine the velocity of efflux from the tank when the orifice is opened. Let S_1 and S_2 be the areas of cross-section of the tank and the orifice respectively.



Now the water will move out steadily in the form of a smooth jet. Let the line connecting point 1 on the liquid surface with the point 2 in the jet represents a streamline of the flow. Then the Bernoulli's theorem [refer equation (5) of Art 4.2] yields

$$\frac{1}{2}q_1^2 + gh_1 + \frac{p_1}{\rho} = \frac{1}{2}q_2^2 + gh_2 + \frac{p_2}{\rho} \qquad ...(1)$$

But from figure $\qquad h_1 - h_2 = h \qquad ...(2)$

Now, from the equation of continuity, we have

$$q_1 S_1 = q_2 S_2 \qquad \text{or} \qquad q_1 = (S_2/S_1) \times q_2 \qquad ...(3)$$

Using (2) and (3), (1) reduces to

$$\frac{1}{2}q_2^2 - \frac{1}{2}\frac{S_2^2}{S_1^2}q_2^2 = g(h_1 - h_2) + \frac{1}{\rho}(p_1 - p_2) \quad \text{or} \quad (q_2^2/2)\times(1 - S_2^2/S_1^2) = gh + (p_1 - p_2)/\rho$$

or
$$q_2 = \sqrt{\frac{2}{(1 - S_2^2/S_1^2)}\left(\frac{p_1 - p_2}{\rho} + gh\right)} \qquad ...(4)$$

which gives the desired velocity of efflux from the tank through the orifice.

We now discuss two special cases of (4) :

Case I. Suppose the tank is vented to the atmosphere or has an open surface, so that $p_1 = p_2$. Further, let $S_2 << S_1$. Then (4) reduces to

$$q_2 = \sqrt{2gh}. \qquad ...(5)$$

Hence the velocity of efflux from the vented tank is equal to that of a rigid body falling freely from a height h.

The above result is known as **Torricelli's theorem.**

Case II. Let $S_2 << S_1$ and $(p_1 - p_2)/\rho >> gh$. Then (4) reduces to

$$q_2 = \sqrt{(2/\rho)\times(p_1 - p_2)}.$$

Illustrative Examples.

Ex. 1. *Calculate the velocity of the water jet in above problem if* $p_2 = 14.7$ *lb/in*^{2}., $p_1 = 30$ *lb/in*^{2}., $S_2/S_1 = 0.01$ and $h = 10$*ft*., $\rho = 1.94$ *lb/ft*^{3}.

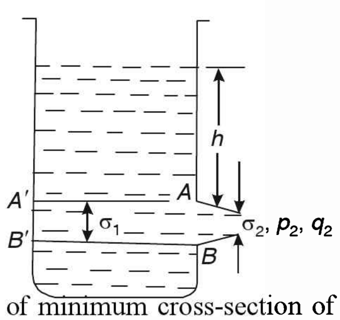
Sol. From (4), we have

$$q_2 = \sqrt{\frac{2}{1-(0.01)^2}\left[\frac{(30-14.7)\times 144}{1.94} + (32.2\times 10)\right]} = \sqrt{2917.8} = 54.01 ft/\text{sec}.$$

If $p_1 = p_2$, the discharge velocity is given by $\qquad q_2 = \sqrt{2\times(32.2)\times(10)} = 25.38 ft/\text{sec}.$

Ex. 2. *Fluid is coming out from a small hole of cross-section* σ_1 *in a tank. If the minimum cross-section of the stream coming out of the hole is* σ_2, *then show that* $\sigma_2/\sigma_1 = 1/2$.

Sol. Let AB be the hole and $A'B'$ be its image on the opposite wall of the tank. Let h be the height of the fluid level in the tank above the orifice. Again, let p_1 be the pressure at AB when the hole is closed. Since the velocity of the fluid coming out from minimum cross-section is at right angles to the hole, the direction of velocity will be horizontal there. If the velocity at the minimum cross-section is q_2 and pressure is p_2 there, then the principle of the conservation of momentum yields



$$\sigma_1(p_1 - p_2) = \sigma_2 \rho q_2^2 \qquad \text{or} \qquad p_1 - p_2 = (\sigma_2/\sigma_1)\rho q_2^2 \qquad \text{...(1)}$$

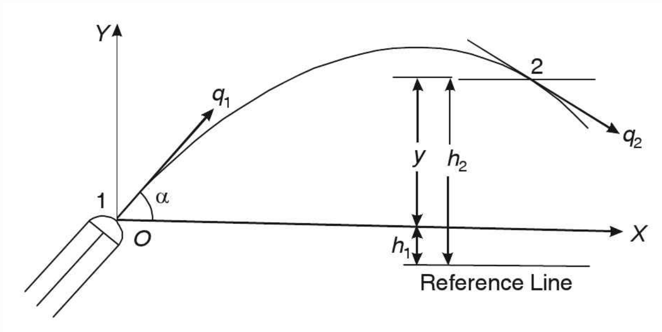
Consider a streamline connecting a point of $A'B'$ and a point of minimum cross-section of the jet. Then the Bernoulli's equation for the above streamline gives

$$\frac{p_1}{\rho} = \frac{p_2}{\rho} + \frac{1}{2}q_2^2 \qquad \text{or} \qquad p_1 - p_2 = \frac{1}{2}\rho q_2^2 \qquad \text{...(2)}$$

Comparing (1) and (2), we find that $\qquad \sigma_2/\sigma_1 = 1/2.$

4.4B. Trajectory of a free jet.

Consider a liquid jet which is coming out from a small hole of area of cross-section S with velocity q_1 and making an angle α with the horizon. Since the entire jet is in the atmosphere, $p_1 = p_2$. Then the Bernoulli's equation between the jet exit 1 and an arbitrary point 2 on the stream line yields



$$gh_1 + q_1^2/2 = gh_2 + q_2^2/2 \qquad \text{or} \qquad q_2^2 = q_1^2 - 2gy, \qquad \text{...(1)}$$

where $y = h_2 - h_1$. Let $Q = q_1 S$ so that $q_1 = Q/S$. Then (1) reduces to

$$q_2^2 = (Q/S)^2 - 2gy \qquad \text{...(2)}$$

To determine the trajectory of the jet, consider the equations of motion for the jet along the horizontal line (x-axis) and the vertical line (y-axis).

$$dx/dt = q_1 \cos\alpha \qquad \text{...(3)}$$

and
$$dy/dt = q_1 \sin\alpha - gt, \qquad \text{...(4)}$$

where (x, y) are coordinates of point 2 with point 1 as the origin. Integrating (3) and (4), we get

$$x = q_1 \cos\alpha \cdot t + c_1,\ c_1 \text{ being an arbitrary constant} \qquad \text{...(5)}$$

and
$$y = q_1 \sin\alpha \cdot t - (1/2) \times gt^2 + c_2,\ c_2 \text{ being an arbitrary constant} \qquad \text{...(6)}$$

Initially at point 1, $t = 0$, $x = 0$, $y = 0$ so that $c_1 = 0$ and $c_2 = 0$. Then (5) and (6) give

$$x = q_1 \cos\alpha \cdot t \qquad \text{...(7)}$$

$$y = q_1 \sin\alpha \cdot t - (1/2) \times gt^2 \qquad ...(8)$$

Eliminating t from (7) and (8), the equation of the jet (or streamline) is given by

$$y = x\tan\alpha - (1/2)\times(g/q_1^2)\sec^2\alpha \cdot x^2 \qquad ...(9)$$

or
$$y = x\tan\alpha - (1/2)\times g(S/Q)^2\sec^2\alpha \cdot x^2 \qquad ...(10)$$

Putting this value of y in (2), the velocity of the liquid at any point of the jet is given by

$$y = (Q/S)^2 - 2gx\tan\alpha + (S/Q)^2 \times g^2x^2\sec^2\alpha \qquad ...(11)$$

Illustrative Solved Examples

Ex. 1. *Calculate the horizontal distance required for a jet striking the ground which is* 3 *feet below the horizontal line of the nozzle. The jet is inclined at an angle of* 60° *with the horizontal at a velocity of* 20 ft./sec. *What is the velocity of the jet just before reaching the ground.*

Sol. Refer equation (9) in above article. Here $\alpha = 60°, q_1 = 20$ ft./sec., $y = -3$ ft. Then we get

$$-3 = x\tan 60° - (1/2)\times(g/20^2)\sec^2 60° \times x^2 \quad \text{or} \quad x^2 - 10.74x - 18.63 = 0, \text{ on simplification}$$

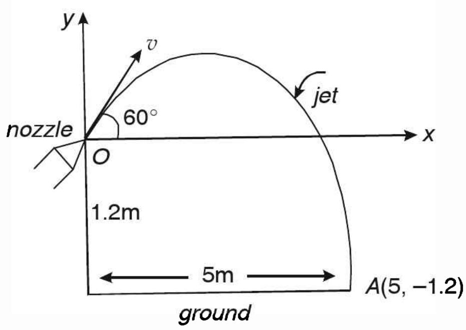
or
$$x = \{10.74 \pm \sqrt{(10.74)^2 + (4\times 18.62)}\}/2$$

Since x cannot be – ve, reject – ve sign in the above value of x. Then (1) gives $x = 12.13$ ft, which is the required horizontal distance between the nozzle and jet striking the ground.

The velocity q_2 of the jet just before reaching the ground can be obtained by using (1) of above article. Thus,
$$q_2 = \sqrt{(20)^2 - 2g(-3)} = \sqrt{593} = 24.24\text{ft/sec}.$$

Ex 2. *A nozzle is situated at a distance of* 1.2 m *above the ground level and is inclined at* 60° *to the horizontal. The diameter of the nozzel is* 40 mm *and the jet of water from the nozzle strikes the ground at a horizontal distance of* 5m.*Find the flow rate.*

Sol. Here the co-ordinates of A, which is on the centre line of the jet of water and is situated on the ground with respect to O as origin are (5, –1.2). Let v be the velocity of the jet. Then the equation of the jet is given by



$$y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha}$$

or
$$-1.2 = 5\tan 60° - \frac{9.81\times 5^2}{2v^2\cos^2 60°}$$

or
$$-1.2 = 5\times 1.732 - (122.62\times 4)/v^2$$
or
$$498.48/v^2 = 8.66 + 1.2$$

or
$$v^2 = 49.74 \quad \text{or} \quad v = 7.05\text{ m/s}.$$

Area of cross-section of nozzle = $(\pi/4)\times(\text{diameter})^2 = (\pi/4)\times(0.04)^2\text{m}^2 = 0.001256\text{m}^2$.

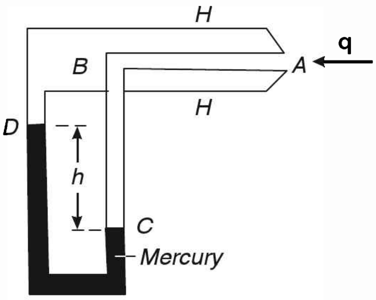
Hence, flow rate = $Q = Sv = 0.001256\times 7.05 = 0.00885\text{m}^3/\text{s}$.

Ex. 2. *Calculate the horizontal distance required for the jet striking the ground which is* 2 *ft. above the horizontal line of the nozzle. The jet is inclined at a* $30°$ *angle with the horizontal at a velocity of* 25 *ft/sec.*

Hint: Do just like Ex. 1.

4.4C. Pitot tube. **[Garhwal [1994, 96]**

A Pitot tube is an instrument to measure the velocity of flow at the required point in a pipe or a stream. Suppose we wish to determine the velocity q of a stream of water. The inner tube BA is kept so as to face the direction of the flow as shown in figure. The outer tube of the Pitot tube has holes such as H. If p is the pressure in the stream where the fluid velocity is q then p is also the pressure on the inside and outside of the hole and therefore p is also the pressure at the meniscus D of the mercury in the U-tube (manometer). Let the steam enter the tube AB and let it be brought to rest at meniscus C. C is called a stagnation point. Let p_0 be pressure at C. Applying the Bernoulli's equation to the steamline passing through A and C, we have



$$\frac{p}{\rho}+\frac{1}{2}q^2=\frac{p}{\rho} \qquad \text{or} \qquad q=\sqrt{\left\{\frac{2(p_0-p)}{\rho}\right\}}, \qquad \text{...(1)}$$

where ρ is the density of the water.

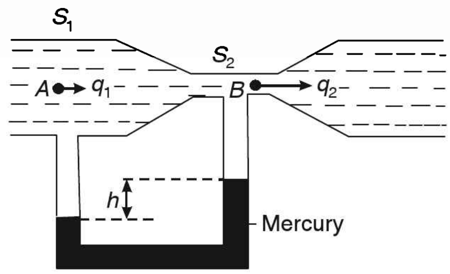
Let h be the difference in level of the mercury in the U-tube and let σ be the density of the mercury. Then we have $\qquad p_0-p=\sigma g h \qquad$ ...(2)

Using (2), (1) reduces to $\qquad q=(2\sigma g h/\rho)^{1/2} \qquad$ ...(3)

which determines the fluid velocity at a point in the flow region.

4.4D. Venturi meter (or tube).

A venturi meter is an instrument to measure the fluid velocity in pipes. The flow rate of a fluid in conduit and the discharge of a fluid flowing in a pipe may also be measured. The venturi meter is made up of a constant cross-section S_1 tapering to a section of smaller cross-section S_2 (also known as throat) and then gradually expanding to the original cross-section. A U-tube serving as a mercury manometer is attached to connect the broad and narrow sections at A and B.



Let q_1, q_2 be the fluid velocities at A, B and p_1, p_2 the pressures. Then by the equation of continuity, we have

$$q_1S_1=q_2S_2 \qquad \text{or} \qquad q_2=(q_1S_1)/S_2 \qquad \text{...(1)}$$

Applying the Bernoulli's equation to the central streamline passing through A and B, we get

$$p_1/\rho+q_1^2/2=p_2/\rho+q_2^2/2, \qquad \text{...(2)}$$

where ρ is the density of the fluid. Eliminating q_2 from (1) and (2) we have

$$q_1=\left\{\frac{2(p_1-p_2)S_2^2}{\rho(S_1^2-S_2^2)}\right\}^{1/2} \qquad \text{...(3)}$$

Let h be the difference in levels of the mercury in the U-tube and let σ be the density of the density of the mercury. Then we have $\qquad p_1-p_2=\sigma g h \qquad$ ...(4)

Using (4), (3) reduces to $$q_1 = \left\{\frac{2\sigma g h S_2^2}{\rho(S_1^2 - S_2^2)}\right\}^{1/2} \qquad \ldots(5)$$

Let Q be flow rate of the fluid flowing through the broad section at A. Then

$$q_1 = \rho q_1 S_1 = \rho S_1 \left\{\frac{2\sigma g h S_2^2}{\rho(S_1^2 - S_2^2)}\right\}^{1/2} \qquad \ldots(6)$$

Remarks. Let the venturi meter be kept inclined at a certain angle to the horizon. With reference to a fixed horizontal line, let vertical heights of A and B be h_1 and h_2 ($h_2 > h_1$) and let $h_2 - h_1 = \Delta h$. Then equation (2) modifies in the following form:

$$p_1/\rho + q_1^2/2 + g h_1 = p_2/\rho + q_2^2/2 + g h_2 \qquad \ldots(7)$$

Eliminating q_1 from (1) and (7), we get

$$q_2 = \left\{\frac{2[(p_1 - p_2)/\rho - g(h_2 - h_1)]}{1 - (S_2/S_1)^2}\right\}^{1/2}$$

and hence the flow rate at either sections is given by

$$Q = S_2 q_2 = S_2 \left\{\frac{2[(\sigma g h)/\rho - g\Delta h]}{1 - (S_2/S_1)^2}\right\}^{1/2} \qquad \ldots(8)$$

Let C be the *coefficient of venturi meter* (or the *coefficient of discharge*). Let Q be the discharge through the venturi meter. Then we know that

$$Q = C S_2 q_2 = C S_2 \left\{\frac{2[(\sigma g h/\rho) - g\Delta h]}{1 - (S_2/S_1)^2}\right\}^{1/2} \qquad \ldots(9)$$

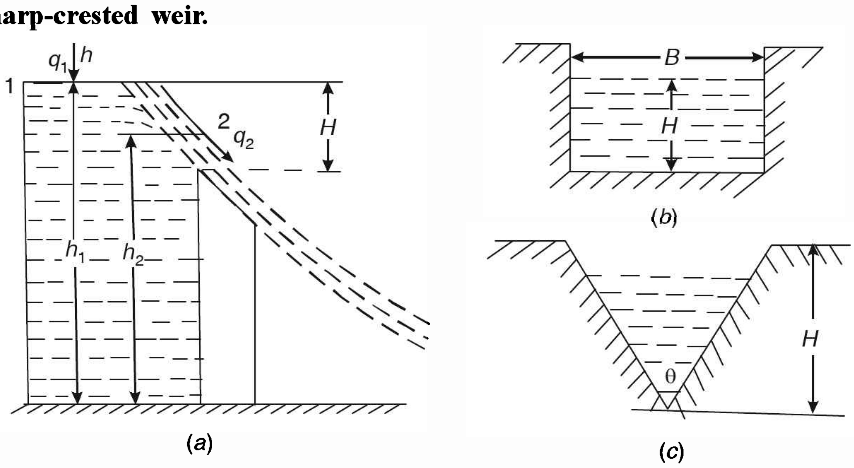
If $\Delta h = 0$ (*i.e.* the venturi meter is horizontal), then (9) reduces to

$$Q = \frac{C S_1 S_2}{\sqrt{S_1^2 - S_2^2}} \sqrt{\frac{\sigma}{\rho}} \sqrt{2gh}. \qquad \ldots(10)$$

4.4 E. Weir.

A structure, used to measure the flow rate of fluid with a free surface (as in an open channel or river), is known as a weir. There are two types of weirs depending on the common physical principle. One type of weir, known as sharp-crested weir, is made up of a sharp edged plate mounted normal to the direction of the flow so as to span the fluid stream. The opening is generally either a rectangle or a triangle (called V-notch also). The other type of weir, known as broad-crested weir, is made up of a obstacle with broad edge. We now discuss these in details.

(*i*) **Sharp-crested weir.**



Consider a streamline lying entirely in the free surface and joining points 1 and 2 as shown in Fig. (*a*). Then the Bernoulli's equation for this streamlines yields

$$\frac{p_1}{\rho}+\frac{1}{2}q_1^2+gh_1=\frac{p_2}{\rho}+\frac{1}{2}q_2^2+gh_2 \qquad ...(1)$$

Since the entire streamline lies in free surface, we have $p_1 = p_2$. Let $h_1 - h_2 = h$ Then, (1) reduces to

$$q_2=\sqrt{q_1^2+2gh}, \qquad ...(2)$$

which gives the velocity at the weir plane. Again the flow rate over the weir is given by

$$Q=\int_o^{\mathrm{H}} q_2 dS_2, \qquad ...(3)$$

where dS_2 is the cross-sectional element parallel to the width lying in the plane of the weir.

For rectangular weir [see Fig. (*b*)] dS_2 = Bdh and hence

$$Q=\int_o^{\mathrm{H}} q_2 B\,dh=\frac{B}{3g}[(2gH+q_1^2)^{3/2}-q_1^3] \qquad ...(4)$$

When the body of water being measured is very large in comparison with the weir opening, q_1 may be neglected. Than, we have

$$Q=(2B/3)\times\sqrt{2g}\,H^{3/2} \qquad ...(5)$$

If C_{dw} is a discharge coefficient, for the actual case, we have

$$Q_{\text{actual}}=\frac{2BC_{dw}}{2}\sqrt{2g}\,H^{3/2} \qquad ...(6)$$

In a similar manner, for the triangular weir the flow rate is given by

$$Q_{\text{actual}}=\frac{8}{15}C_{dw}\sqrt{2g}\,H^{5/2}\tan(\theta/2) \qquad ...(7)$$

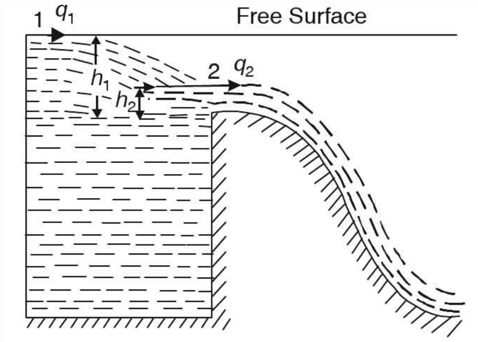
(*ii*) **Broad-crested Wear.**

We again proceed as in (*i*) above. Let there be uniform parallel flow at point 2 and let $q_1 << q_2$. Then Bernoulli's equation for the streamline joining points 1 and 2 gives

$$gh_1=q_2^2/2+gh_2$$

$\therefore$ The flow rate over a weir of width B is given by

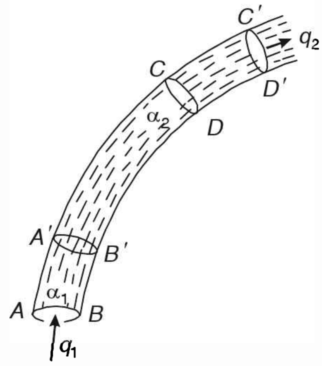
$$Q=Bh_2\sqrt{2g(h_1-h_2)} \qquad ...(8)$$



Then, we have $$Q_{\text{actual}}=C_{dw}B\,h_2\sqrt{2g(h_1-h_2)} \qquad ...(9)$$

4.5. Euler's momentum theorem.

Consider steady motion of a non-viscous liquid contained between, AB and CD of the filament at a given time t. The surrounding fluid will produce a force on the walls and ends of the filament. By Newton's second law of motion, the net force will be equal to the rate of change of momentum of the fluid in the filament $ABCD$ at time t. At time $t+\delta t$, let the new position of the fluid be $A'B'C'D'$. Then notice that the momentum of the given fluid has increased by the momentum of the fluid between CD and $C'D'$ and has decreased by the momentum of the fluid between AB and $A'B'$.



$\therefore$ Gain of momentum at $CD = (\rho\alpha_2 q_2 \delta t) q_2$

and loss of momentum at $AB = (\rho\alpha_1 q_1 \delta t) q_1$,

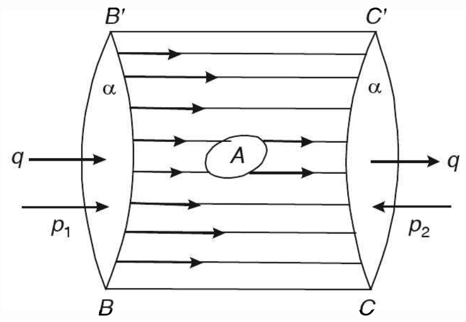
where q_1 and q_2 are the velocities at AB and CD respectively.

Hence the net gain $= \rho\delta t(\alpha_2 q_2^2 - \alpha_1 q_1^2)$ or the net rate of gain $= \rho(\alpha_2 q_2^2 - \alpha_1 q_1^2)$.

This gives the resultant force due to pressure of the surrounding liquid on the walls and ends of the filament. This result is known as *Euler's momentum theorem*.

4.6. D'Alembert's paradox. [Meerut 2000, Kanpur 2001]

Consider a long straight channel of uniform cross section in which a liquid is flowing with a uniform speed q. Let the ends of the tube be bounded by equal cross-sectional area α. If an obstacle A is placed in the middle of the channel, the flow in the immediate neighbourhood of A will be disturbed whereas the flow at a great distance either up-stream or down-stream will remain undisturbed. Suppose F is the force required to hold the obstacle to rest, in the direction of uniform flow.



Let BB' and CC' be two sections at a great distance from A and let the fluid between these sections be split up into stream filaments. Since the outer filaments are bounded by the walls of the channel, the thrust components are normal to the direction of flow. Moreover, the obstacle A acts on those filaments which are in contact with it by a force $-F$.

By Euler's momentum theorem the resultant of all the thrusts on the fluid is $\rho\alpha q^2 - \rho\alpha q^2$.

Let p_1 and p_2 be the pressures on BB' and CC' respectively. Then Bernoulli's theorem gives

$$\frac{p_1}{\rho} + \frac{1}{2}q^2 = C = \frac{p_2}{\rho} + \frac{1}{2}q^2 \qquad \text{so that} \qquad p_1 = p_2$$

Now, the thrust due to pressure p_1 and p_2 is $p_1\alpha - p_2\alpha$.

Thus the equation of motion becomes

$$p_1\alpha - p_2\alpha - F = \rho\alpha q^2 - p\alpha q^2 \qquad \text{so that} \qquad F = 0, \qquad \text{as } p_1 = p_2$$

Let the diameter of the channel increase indefinitely. Then the above problem reduces to that of a obstacle immersed in an infinite uniform stream. As before, again the resultant force exerted by the liquid on the obstacle is zero.

Now let us superimpose a velocity u in the opposite direction on the entire system (the body A and the liquid). Then the body A can be thought as moving with uniform velocity u and the liquid at great distance is reduced to rest.

Thus a body moving with uniform velocity through an infinite liquid, otherwise at rest, will experience no resistance at all. This result is known as *D'Alembert's paradox*.

EXERCISE 4 (B)

1. A venturi meter has its axis vertical, the inlet and throat diameter ratio being 2.5. The throat is 12 in. above the inlet and the coefficient of discharge is 0.97. Determine the pressure difference between the inlet and throat when the velocity of water at the inlet is 6 ft/sec.

2. A jet of water 1 in. in diameter strikes a flat plate at an angle 30° to the normal of the plate with a velocity of 30 ft/sec. Determine the velocity of the plate, moving parallel to itself, if the normal force exerted by the jet is 2.5 Ib.

3. A rectangular plate 4 in. wide and 10 in. long hangs vertically from hinges at its top edge. A jet of water 1 in. in diameter with a velocity of 30 ft/sec strikes the plate at its centre.

Determine the weight of the plate if the plate stays in equilibrium after deflecting 20° from its original position.

4. A jet of water 2 in. in diameter discharges 1.0 ft^3/sec. Calculate the force required to move a flat plate towards the jet with velocity of 25 ft/sec. The jet is perpendicular to the plate.

5. Calculate the force exerted by a jet of water 3/4 in. in diameter which strikes a flat plate at an angle of 30° to the normal of the plate with a velocity of 30 ft/sec if (*a*) the plate is stationary, (*b*) the plate is moving in the direction of the jet with a velocity of 10 ft/sec.

[**Ans.** (*a*) 4.63 Ibf. (*b*) 3.09 Ibf.]

6. State and prove Bernoulli's theorem for steady inviscid flow in a conservative field of force and discuss the nature of the constant.

7. Fluid enters a contracting pipe with velocity V_1 through area A_1 and leaves with velocity V_2 through area A_2 after having been turned through an angle α. Determine the force required to hold the pipe in equilibrium against the pressure of the fluid.

8. A water venturi meter has a throat diameter of 3 in. and a pipe diameter of 6 in. Calculate the velocity at the throat if the deflection of mercury manometer which connects the pipe and the throat is 9.5 in. The coefficient of discharge is 0.96.

9. Define stream tube. Using this obtain the Bernoulli's equation for a steady flow.

10. The venturi meter has an entrance of 6 inch diameter and a throat of 3 inch diameter whose centre is 18 inch above the centre of the entrance. Find the velocity of water at the throat when $p_1 - p_2 = 5$ Ib/in^2. The coefficient of discharge C_d is 0.97. Find the value of the constant for the venturimeter. [**Ans.** 25.4 ft/sec]

11. Calculate the horizontal distance required for the jet striking ground which is 0.5 m above the horizontal line of the nozzle. The jet is inclined at a 30° angle with the horizontal at a velocity of 10 m/s.

12. If the water jet is discharged from a nozzle (inclined at a 60° angle with the horizontal) at 5 m/s, calculate the horizontal distance required for the jet striking the ground which is 1 m below the horizontal line of the nozzle. What is the velocity of the jet just before reaching the ground ?

13. A horizontal straight pipe gradually reduces in diameter from 0.5 m to 0.25 m. Determine the total longitudinal thrust exerted on the pipe if the pressure at the larger end is 0.4 MN/m^2 and the velocity of the water is 2 m/s.

14. A jet of water 0.05 m in diameter strikes a plate at an angle 30° to the normal of the plate with a velocity of 10 m/s. Determine the velocity of the plate, moving parallel to itself, if the normal force exerted by the jet is 10 N.

15. Calculate the force exerted by a jet of water 10 mm in diameter which strikes a flat plate at an angle of 30° to the normal of the plate with a velocity of 10 m/s if (*a*) the plate is stationary, (*b*) the plate is moving in the direction of the jet with a velocity of 2 m/s.

16. A venturi meter has its axis vertical, the inlet and throat diameter ratio being 2.5. The throat is 0.3 m above the inlet and the coefficient of discharge is 0.97. Determine the pressure difference between the inlet and throat when the velocity of water at the inlet is 2 m/s.

17. A water venturi meter has a throat diameter of 0.1 m and a pipe diameter of 0.2 m. Calculate the velocity at the throat if the deflection of the mercury manometer which connects the pipe and the throat is 0.15 m. The coefficient of discharge is 0.96.

18. State and prove D'Alembert's Paradox. **[Meerut 1999, 2000]**

19. Briefly explain the application of Bernoulli's theorem. **[Kanpur 2000]**

Hint. Refer Art. 4.4 and Art 4.4A

OBJECTIVE QUESTIONS ON CHAPTER 4

Multiple choice questions

Choose the correct alternative from the following questions

1. If the motion is steady, velocity potential does not exist and V be the potential function from which the external forces are derivable, then Bernoulli's theorem is

(*i*) $-\dfrac{\partial\phi}{\partial t}+\dfrac{1}{2}q^2+V+\displaystyle\int\dfrac{dp}{\rho}=C$ (*ii*) $\displaystyle\int\dfrac{dp}{\rho}+\dfrac{1}{2}q^2+V=C$

(*iii*) $p/\rho+q^2/2+V=C$ (*iv*) None of these

2. The Bernoulli's equation for unsteady and irrotational motion is given by

(*i*) $-\partial\phi/\partial t+q^2/2+V+p/\rho=F(t)$ (*ii*) $-\partial\phi/\partial t+q^2/2+V=F(t)$

(*iii*) $-\partial\phi/\partial t-q^2/2+V-p/\rho=F(t)$ (*iv*) $q^2/2+V+p/\rho=F(t)$

3. A stream in a horizontal pipe, after passing a contraction in the pipe at which its sectional area is A is delivered at atmospheric pressure at a place, where the sectional area is B. If a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservoir at a depth h below the pipe, s being the delivery per second, where h is given by

(*i*) $(s^2/2g)\times(1/A^2+1/B^2)$ (*ii*) $(s^2/2g)\times(1/A^2-1/B^2)$

(*iii*) $(2g/s^2)\times(1/A^2-1/B^2)$ (*iv*) $(2g/s^2)\times(1/A^2+1/B^2)$

4. The horizontal distance required for a jet striking the ground which is 3 feet below the horizontal line of the nozzle (given that the jet is inclined at a an angle 60° with the horizontal at a velocity of 20 ft/sec.) is

(*i*) 12.20 feet (*ii*) 12.15 feet (*iii*) 12.13 feet (*iv*) None of these

5. A body moving with uniform velocity through an infinite liquid otherwise at rest, will experience no resistance at all. This result is known as

(*i*) Euler's paradox (*ii*) Lagrange's paradox

(*iii*) D'Alembert's paradox (*iv*) none of these

6. If the fluid be homogeneous and incompressible, then in usual, symbols, the Bernoulli's theorem bcomes

(*i*) $q^2/2+V+p=C$ (*ii*) $q^2+V+p/\rho=C$

(*iii*) $q^2/2+V+p/\rho^2=C$ (*iv*) $q^2/2+V+p/\rho=C$

7. The most general form of Bernoulli's equation for motion of fluid is

(*a*) $\dfrac{1}{2}q^2+\Omega+\displaystyle\int\dfrac{dp}{\rho}-\dfrac{\partial\phi}{\partial t}=f(t)$ (*b*) $\dfrac{1}{2}q^2+\Omega-\displaystyle\int\dfrac{dp}{\rho}-\dfrac{\partial\phi}{\partial t}=f(t)$

(*c*) $\dfrac{1}{2}q^2+\Omega-\displaystyle\int\dfrac{dp}{\rho}+\dfrac{\partial\phi}{\partial t}=f(t)$ (*d*) $\dfrac{1}{2}q^2+\Omega+\displaystyle\int\dfrac{dp}{\rho}+\dfrac{\partial\phi}{\partial t}=f(t)$ **[Agra 2005]**

8. The equation $q^2/2+\Omega+p/\rho=$ constant is known as (*a*) Navier equation

(*b*) Bernonilli equation (*c*) Euler equation (*d*) Stokes equation **[Agra 2007]**

9. The Bernoulli's equation for steady motion with the *velocity* potential and conservative field of force is.......... (Fill up the gap) **[Agra 2008]**

Answers/Hints to objective type questions

1. (*ii*). See Eq. (3), Art. 4.2
2. (*i*). See Eq. (14), Art. 4.1
3. (*ii*). See Ex. 2, Art. 4.3
4. (*iii*). See Ex. 1, Art. 4.4B
5. (*iii*). Refer Art. 4.6
6. (*iv*). See Eq. (4), Art. 4.2
7. (*a*). See Art. 4.1
8. (*b*). See Eq. (15), Art. 4.1

9. $q^2/2 + V + \int (1/\rho)dp = C$. Refer Eq. (3), Art. 4.2.

CHAPTER

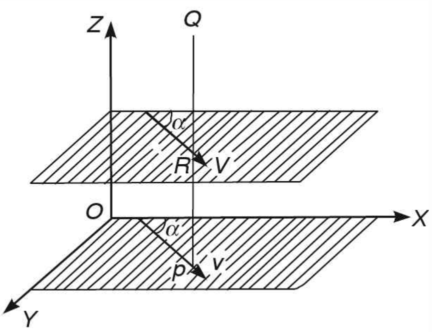
5

Motion in Two-Dimensions Sources and Sinks

5.1. Motion in two-dimensions.

Let a fluid move in such a way that at any given instant the flow pattern in a certain plane (say XOY) is the same as that in all other parallel planes within the fluid. Then the fluid is said to have two-dimensional motion. If (x, y, z) are coordinates of any point in the fluid, then all physical quantities (velocity, density, pressure etc.) associated with the fluid are independent of z. Thus u, are functions of x, y and t and $w = 0$ for such a motion.

To make the concept of two-dimensional motion more clear, suppose the plane under consideration be xy-plane. Let P be an arbitrary point on that plane. Draw a straight line PQ parallel to OZ (or perpendicular to the xy-plane). Then all points on the line PQ are said to correspond to P. Draw a plane (in the fluid) parallel to the xy-plane and meeting PQ in R. Then, if the velocity at P is V in the xy-plane in a direction making an angle α with OX, the velocity at R is also V in magnitude and parallel in direction to the velocity at P as shown in the figure. It follows that the velocity at corresponding points is a function of x, y and the time t, but not of z.



In order to maintain physical reality, we assume that the fluid in two-dimensional motion is confined between two planes parallel to the plane of motion and at a unit distance apart. The reference plane of motion is taken parallel to and midway between the assumed fixed planes. Thus while studying the flow of a fluid past a cylinder in a two-dimensional motion in planes perpendicular to the axis of the cylinder, it is useful to restrict attention to a unit length of cylinder confined between the said planes in place of worrying over the cylinder of infinite length.

Suppose we are dealing with a two-dimensional motion in xy plane. Then by flow across a curve in this plane, we mean the flow across unit length of a cylinder whose trace on the plane xy is the curve under consideration, the generators of the cylinder being parallel to the z-axis. By a point in a flow, we mean a line through that point parallel to z-axis.

5.2. Stream function or current function.

[Agra 2005; Rohilkhand 2002, 03; Meerut 1999, 2010; Kanpur 2010, 09]

Let u and be the components of velocity in two-dimensional motion. Then the differential equation of lines of flow or streamline is

$$dx/u = dy/ \qquad \text{or} \qquad dx - udy = 0 \qquad \text{...(1)}$$

and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}=0 \qquad \text{or} \qquad \frac{\partial}{\partial y}=\frac{\partial(-u)}{\partial x} \qquad ...(2)$$

(2) shows that *L.H.S.* of (1) must be an exact differential, $d\psi$ (say). Thus, we have

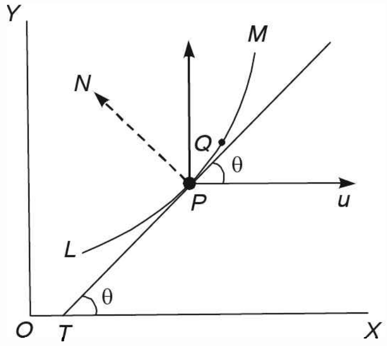
$$dx-udy=d\psi=(d\psi/\partial x)dx+(\partial\psi/\partial y)dy \qquad ...(3)$$

so that $\quad u=-\partial\psi/\partial y \quad$ and $\quad =\partial\psi/\partial x \qquad ...(4)$

This function ψ is known as the *stream function*. Then using (1) and (3), the streamlines are given by $d\psi=0$ *i.e.*, by the equation $\psi=c$, where c is an arbitrary constant. Thus the stream function is constant along a streamline. Clearly the current function exists by virtue of the equation of continuity and incompressibility of the fluid. Hence the current function exists in all types of two-dimensional motion whether rotational or irrotational.

5.3. Physical significance of stream function. **[Kanpur 2005; Rohilkhand 2002, 03]**

Let LM be any curve in the x-y plane and let ψ_1 and ψ_2 be the stream functions at L and M respectively. Let P be an arbitrary point on LM such that arc $LP=s$ and let Q be a neighbouring point on LM such that arc $LQ=s+\delta s$. Let θ be the angle between tangent at P and the x-axis. If u and be the velocity-components at P, then



velocity at P along inward drawn normal PN

$$= \cos\theta-u\sin\theta \qquad ...(1)$$

When ψ is the stream function, then we have

$$u=-\partial\psi/\partial y \qquad \text{and} \qquad =\partial\psi/\partial x \qquad ...(2)$$

Also from Calculus, $\quad \cos\theta=dx/ds \quad$ and $\quad \sin\theta=dy/ds \qquad ...(3)$

Using (1), we get $\quad$ flux across PQ from right to left $=(\ \cos\theta-u\sin\theta)\delta s$

$\therefore$ Total flux across curve LM from right to left

$$=\int_{LM}(\ \cos\theta-u\sin\theta)\,ds=\int_{LM}\left(\frac{\partial\psi}{\partial x}\frac{dx}{ds}+\frac{\partial\psi}{\partial y}\frac{dy}{ds}\right)ds, \text{ using (2) and (3)}$$

$$=\int_{LM}\left(\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy\right)=\int_{\psi_1}^{\psi_2}d\psi=\psi_2-\psi_1$$

Thus a *property of the current function is that the difference of its values at two points represents the flow across any line joining the points.*

Remark 1. The current function ψ at any point can also be defined as the flux (*i.e.* rate of flow of fluid) across a curve LP where L is some fixed point in the plane.

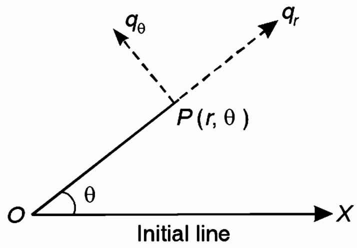
Remark 2. Since the velocity normal to δs will contribute to the flux across δs whereas the velocity along tangent to δs will not contribute towards flux across δs, we have

flux across $\delta s = \delta s\times$ normal velocity

or $\quad (\psi+\delta\psi)-\psi=\delta s\ \times$ velocity from right to left acrss δs

or $\quad$ Velocity from right to left across $\delta s=\partial\psi/\partial s \qquad ...(4)$

Remark 3. Velocity components in terms of ψ in plane-polar coordinates (r, θ) can be obtained by using the method outlined in remark 2 above. Let q_r and q_θ be velocity components in the directions of r and θ increasing respectively. Then



q_r = velocity from right to left across $r\,\delta\theta$

$$= \lim_{\delta\theta \to 0} \frac{\delta\psi}{r\delta\theta} = \frac{1}{r}\frac{\partial\psi}{\partial\theta},$$

and q_θ = velocity from right to left across δr

$$= \lim_{\delta r \to 0} \frac{\delta\psi}{\delta r} = \frac{\partial\psi}{\partial r}.$$

Thus, $$q_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad q_\theta = \frac{\partial\psi}{\partial r}. \qquad \ldots(5)$$

5.4. Spin components in terms of ψ.

We know that the velocity components u and are functions of x, y and t and $w = 0$ in two-dimensional flow. Hence the spin components (ξ, η, ζ) are given by

$$2\xi = \frac{\partial w}{\partial y} - \frac{\partial}{\partial z} = 0, \qquad 2\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

and $$2\zeta = \frac{\partial}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial y}\right) = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}$$

Let the motion be irrotational so that $\zeta = 0$ also. Then we obtain

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0 \qquad \text{or} \qquad \nabla^2\psi = 0$$

showing that ψ satisfies Laplace's equation.

5.5. Some aspects of elementary theory of functions of a complex variables.

Suppose that $z = x + iy$ and that $w = f(z) = \phi(x, y) + i\psi(x, y)$,

where x, y, ϕ, ψ are all real and $i = \sqrt{-1}.$ Also, suppose that ϕ and ψ and their first derivatives are everywhere continuous within a given region. If at any point of the region specified by z the derivative $dw/dz\,(= f'(z))$ is unique, then w is said to be *analytic* or *regular* at that point. If the derivative is unique throughout the region, then w is said to be analytic or regular throughout the region. It can be shown that the necessary and sufficient conditions for w to be analytic at z are

$$\partial\phi/\partial x = \partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y = -\,\partial\psi/\partial x,$$

which are known as the *Cauchy-Riemann equations*. The functions ϕ, ψ are known as *conjugate functions*.

5.6. Irrotational motion in two-dimensions. [Meerut 2007; Purvanchal 2004, 05]

Let there be an irrotational motion so that the velocity potential ϕ exists such that

$$u = -\,\partial\phi/\partial x \qquad \text{and} \qquad = -\,\partial\phi/\partial y \qquad \ldots.(1)$$

In two-dimensional flow the stream function ψ always exists such that

$$u = -\,\partial\psi/\partial y \qquad \text{and} \qquad = \partial\psi/\partial x \qquad \ldots(2)$$

From (1) and (2), we have

$$\partial\phi/\partial x = \partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y = -\,\partial\psi/\partial x \qquad ...(3)$$

which are well known *Cauchy-Riemann's equations*. Hence $\phi + i\psi$ is an analytic function of $z = x + i\,y$. Moreover ϕ and ψ are known as *conjugate functions.*

On multiplying and re-writing, (3) gives

$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial y}=0\,, \qquad ...(4)$$

showing that the families of curves given by ϕ = constant and ψ = constant intersect orthogonally.

Thus the curves of equi-velocity potential and the stream lines intersect orthogonally.

Differentiating the equations given in (3) with respect to x and y respectively, we get

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial^2\psi}{\partial x\partial y} \qquad \text{and} \qquad \frac{\partial^2\phi}{\partial y^2}=-\frac{\partial^2\psi}{\partial x\partial y}. \qquad ...(5)$$

Since $\dfrac{\partial^2\psi}{\partial x\partial y}=\dfrac{\partial^2\psi}{\partial y\partial x}$, adding (5) gives

$$\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2=0. \qquad ...(6)$$

Again, differentiating the equations given in (3) with respect to y and x respectively, we get

$$\frac{\partial^2\phi}{\partial y\partial x}=\frac{\partial^2\psi}{\partial y^2} \qquad \text{and} \qquad \frac{\partial^2\phi}{\partial x\partial y}=-\frac{\partial^2\psi}{\partial x^2}$$

Since $\partial^2\phi/\partial x\partial y=\partial^2\phi/\partial y\,\partial x$, subtracting these, we get $\partial^2\psi/\partial x^2+\partial^2\psi/\partial y^2=0$ ...(7)

Equations (6) and (7) show that *ϕ and ψ satisfy Laplace's equation when a two-dimensional irrotational motion is considered.* **[Meerut 2010]**

5.7. Complex potential. [Meerut 2011]

[G.N.D.U. Amritsar 2003; Rohilkhand 2001; Kanpur 2001, 05; Agra 2005]

Let $w=\phi+i\psi$ be taken as a function of $x + iy$ *i.e., z*. Thus, suppose that $w = f(z)$ *i.e.*

$$\phi+i\psi=f(x+iy) \qquad ...(1)$$

Differentiating (1) w.r.t x and y respectively, we get

$$\partial\phi/\partial x+i(\partial\psi/\partial x)=f'(x+iy) \qquad ...(2)$$

and $$\partial\phi/\partial y+i(\partial\psi/\partial y)=if'\,(x+iy)$$

or $$\partial\phi/\partial y+i(\partial\psi/\partial y)=i\{\partial\phi/\partial x+i(\partial\psi/\partial x)\},\ \text{by (2)}$$

Equating real and imaginary parts, we get

$$\partial\phi/\partial x=\partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y=-\,\partial\psi/\partial x \qquad ...(3)$$

which are *Cauchy-Riemann equations*. Then w is an analytic function of z and w is known as the *complex potential.*

Conversely, if w is an analytic function of z, then its real part is the velocity potential and imaginary part is the stream function of an irrotational two-dimensional motion.

Remarks. If $\phi+i\psi=f(x+iy)$, then $i\phi-\psi=if(x+iy)$

Thus, $\psi-i\phi=-if(x+iy)=g(x+iy)$, say

Hence proceeding as before, we get (3). Hence another irrotational motion is also possible in which lines of equi - velocity potential are given by ψ = constant and the streamlines by ϕ = constant.

5.7A. Cauchy-Riemann equations in polar form. [Kanpur 2003]

Let $$\phi + i\psi = f(z) = f(re^{i\theta}) \qquad \ldots(1)$$

Differentiating (1) w.r.t. r and θ, we get

$$\frac{\partial \phi}{\partial r} + i\frac{\partial \psi}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta} \qquad \ldots(2)$$

and $$\frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta} = f'(re^{i\theta}) \cdot rie^{i\theta} \qquad \ldots(3)$$

From (2) and (3), we easily obtain $$\frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta} = ir\left(\frac{\partial \phi}{\partial r} + i\frac{\partial \psi}{\partial r}\right)$$

Equating real and imadginary parts,we get

$$\frac{\partial \phi}{\partial \theta} = -r\frac{\partial \psi}{\partial r} \qquad \text{and} \qquad \frac{\partial \psi}{\partial \theta} = r\frac{\partial \phi}{\partial r}$$

Thus, $$\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{and} \qquad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}, \qquad \ldots(4)$$

which are *Cauchy-Riemann equations in polar form.*

5.8. Magnitude of velocity. [G.N.D.U. Amritsar 2002; Kanpur 1997]

Let $w = f(z)$ be the complex potential. Then

$$w = \phi + i\psi \qquad \text{and} \qquad z = x + iy \qquad \ldots(1)$$

Also $$\partial\phi/\partial x = \partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y = -\,\partial\psi/\partial x \qquad \ldots(2)$$

For two-diensional irrotational motion, we have (see Art. 5.1.)

$$u = -\partial\phi/\partial x \qquad \text{and} \qquad = -\partial\phi/\partial y \qquad \ldots(3)$$

From (1), $$\frac{dw}{dz}\cdot\frac{\partial z}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \qquad \text{and} \qquad \frac{\partial z}{\partial x} = 1$$

$\therefore$ $$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}, \text{ using (2)} \qquad \ldots(4)$$

or $$dw/dz = -u + i \quad \text{using (3)} \qquad \ldots(5)$$

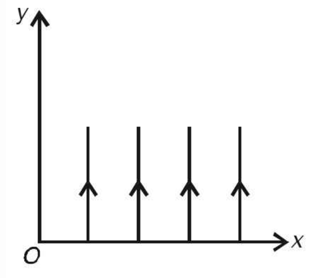
which is called the *complex velocity.*

From (4) and (5), we see that the magnitude of velocity q at any point in a two-dimensional irrotational motion is given by | dw/dz |, where

$$|\,dw/dz\,| = \{(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2\}^{1/2} = (u^2 + {}^2)^{1/2} = q \qquad \ldots(5)$$

Remarks. The points where velocity is zero are known as *stagnation points.*

5.9. Complex potential for some uniform flows



(*i*) Consider $w = ikz$, ...(1)

where k is a real and positive constant

Now, (1) $\Rightarrow$ $dw/dz = -u + i \ = ik \Rightarrow u = 0$ and $= k$, which is clearly a uniform flow parallel to y-axis.

Hence the complex potential for a uniform flow whose magnitude of the stream is V in the positive y-direction is given by $w = iVz$.

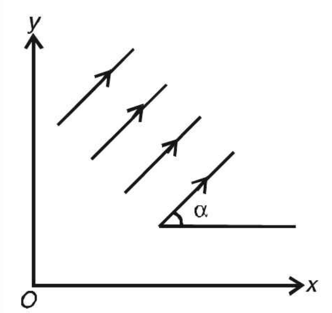
(*ii*) Consider $w = -ke^{-i\alpha}z$, ...(2)

where k and α are real constants.

From (2), $dw/dz = -u + i \;\; = -ke^{-i\alpha}$

$\Rightarrow$ $-u + i \;\; = -k(\cos\alpha - i\sin\alpha)$

$\Rightarrow$ $u = k\cos\alpha$ and $= k\sin\alpha$, which corresponds to a uniform flow inclined at an angle α to the x-axis.



Hence the complex potential for a uniform flow whose magnitude is V and which is inclined is at an angle α to x-axis is given by $w = -Ve^{-i\alpha}z$.

5.10. Illustrative Solved Exaples

Ex. 1. *To show that the curves of constant velocity potential and constant stream functions cut orthogonally at their points of intersection.* **[Meerut 2007; Garhwal 2005]**

OR

To shows that the family of curves $\phi(x, y) = c_1$ *and* $\psi(x, y) = c_2$, c_1, c_2 *being constants, cut orthogonally at their points of intersection.*

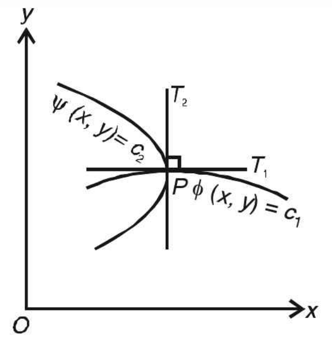
Proof. Let the curves of constant velocity potential and constant stream function be given by

$$\phi(x, y) = c_1 \qquad ...(1)$$

and

$$\psi(x, y) = c_2, \qquad ...(2)$$

where c_1 and c_2 are arbitrary constants. Let m_1 and m_2 be gradients of tangents PT_1 and PT_2 at point of intersection P of (1) and (2). Then, we have



$$m_1 = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} \quad \text{and} \quad m_2 = -\frac{\partial\psi/\partial x}{\partial\psi/\partial y} \qquad ...(3)$$

We know that ϕ and ψ satisfy the Cauchy-Riemann equations, namely,

$$\partial\phi/\partial x = \partial\psi/\partial y \quad \text{and} \quad \partial\phi/\partial y = -\partial\psi/\partial x. \qquad ...(4)$$

Now, from (3),
$$m_1 m_2 = \frac{(\partial\phi/\partial x)(\partial\psi/\partial x)}{(\partial\phi/\partial y)(\partial\psi/\partial y)} = \frac{(\partial\psi/\partial y)(\partial\psi/\partial x)}{-(\partial\psi/\partial x)(\partial\psi/\partial y)}, \text{ by (4)}$$

Hence $m_1 m_2 = -1$, showing that the curves (1) and (2) cut each other orthogonally.

Ex. 2. *If* $\phi = A(x^2 - y^2)$ *represents a possible flow phenomenon, determine the stream function.*

Sol. Here $\phi = A(x^2 - y^2)$ ...(1)

$\therefore$ $\partial\psi/\partial y = \partial\phi/\partial x = 2Ax$, using (1)

Integrating it w.r.t. 'y', $\psi = 2Axy + f(x)$, ...(2)

where $f(x)$ is an arbitrary function of x. (2) gives the required stream function.

Ex. 3. *Determine the stream function* $\psi(x, y, t)$ *for the given velocity field* $u = Ut$, $= x$.

Sol. We know that $u = -(\partial\psi/\partial y)$ and $= \partial\psi/\partial x$.

$\therefore$
$$\partial\psi/\partial y = -Ut\,. \qquad ...(1)$$

and
$$\partial\psi/\partial x = x. \qquad ...(2)$$

Integrating (1), $\psi(x, y, t) = -Uty + f(x, t)$, ...(3)

where $f(x, t)$ is an arbitrary function of x and t.

From (3), $\qquad \partial\psi/\partial x = \partial f/\partial x.$ ...(4)

Then (2) and (4) $\Rightarrow$ $\qquad \partial f/\partial x = x.$ ...(5)

Integrating (5), $\qquad f(x, t) = x^2/2 + F(t),$ ...(6)

where $F(t)$ is an arbitrary function of t.

From (3) and (6), $\qquad \psi\,(x, y, t) = -Uty + x^2/2 + F(t).$

Ex. 4. *The velocity potential function for a two-dimensional flow is* $\phi = x(2y - 1)$. *At a point* $P(4, 5)$ *determine : (i) The velocity and (ii) The value of stream function.*

Sol. Given $\qquad \phi = 2xy - x.$ ...(1)

(*i*) The velocity components u and in x and y directions are given by

$$u = -\partial\phi/\partial x = -2y + 1 \qquad \text{and} \qquad = -\partial\phi/\partial y = -2x. \qquad ...(2)$$

At the point $P(4, 5)$, $\qquad u = -10 + 1 = 9 \qquad$ and $\qquad = -8.$

$\therefore$ Resultant velocity $= V = (u^2 + \;^2)^{1/2} = (81 + 64)^{1/2} = 12{\cdot}04$ units.

(*ii*) Now, $\qquad u = -\partial\psi/\partial y \qquad$ and $\qquad = \partial\psi/\partial x.$ ...(3)

From (2) and (3), $\qquad \partial\psi/\partial x = -2x \qquad$ and $\qquad \partial\psi/\partial y = 2y - 1.$

Now, $\qquad d\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy = -2x\,dx + (2y - 1)dy.$

Integrating, $\qquad \psi = -x^2 + y^2 - y + C$, C being constant of integration.

For $\psi = 0$ at the origin, we have $\qquad 0 = 0 + C \qquad$ or $\qquad C = 0.$

Hence $\qquad \psi = -x^2 + y^2 - y.$

At the point $P(4, 5)$, $\qquad \psi = -4^2 + 5^2 - 5 = 4$ units.

Ex. 5. *The streamlines are represented by (a)* $\psi = x^2 - y^2$ *and (b)* $\psi = x^2 + y^2$. *Then (i) determine the velocity and its direction at* (2, 2) *(ii) sketch the streamlines and show the direction of flow in each case.*

Part (a) Given that $\qquad \psi = x^2 - y^2.$

Now, $\qquad u = \partial\psi/\partial y = -2y \qquad$ and $\qquad = -\partial\psi/\partial x = -2x.$

At (2, 2), $\qquad u = -4 \qquad$ and $\qquad = -4.$

$\therefore$ The resultant velocity $= (u^2 + \;^2)^{1/2} = (16 + 16)^{1/2} = 4\sqrt{2}$ units

and its direction has a slope $= \;/u = 1$ showing that the velocity vector is inclined at 45° to x-axis.

The required streamlines are given by $\psi = c$, where c is a constant, *i.e.* $x^2 - y^2 = c$, which represents a family of hyperbolas. In figure, we have sketched the steamlines for various values of ψ. The direction of arrowhead shows the direction of flow in each case.

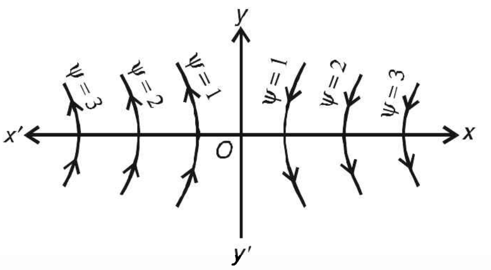


Fig. Pattern of Streamlines for $y = x^2 - y^2$

Part (b) Given that $\qquad \psi = x^2 + y^2$

Now, $\qquad u = \partial\psi/\partial y = 2y, \qquad = -\partial\psi/\partial x = -2x.$

At (2, 2), $\qquad u = 4 \qquad$ and $\qquad = -4.$

$\therefore$ The resultant velocity

$$= (u^2 + \;^2)^{1/2} = (16 + 16)^{1/2} = 4\sqrt{2} \text{ units.}$$

and its direction has a slope $= \;/u = -1$, showing that the velocity vector is inclined at 135° to x-axis.

The required streamlines are given by $\psi = c$, where c is a constant, *i.e.* $x^2 + y^2 = c$, which represents a family of circles. In figure, we have sketched the streamlines for various values of ψ. The direction of arrowhead shows the direction of flow in each case.

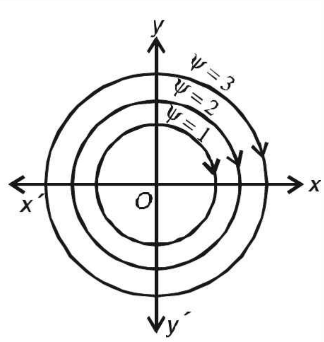


Fig. Pattern of Streamlines for $\psi = x^2 + y^2$

Ex. 6. If $\phi = 3xy$, find x and y components of velocity at (1, 3) and (3, 3). Determine the discharge passing between streamlines passing through these points.

Sol. The velocity components u and in x and y directions are given by

$$u = -\partial\phi/\partial x = -3y \qquad \text{and} \qquad = -\partial\phi/\partial y = -3x. \qquad ...(1)$$

Hence the velocity components at (1, 3) are $u = -9$, $= -3$.

and the velocity components at (3, 3) are $u = -9$, $= -9$.

Now, we have $u = \partial\psi/\partial y$ and $= -\partial\psi/\partial x$. ...(2)

Then, (1) and (2) $\Rightarrow$ $\partial\psi/\partial y = -3y$ and $\partial\psi/\partial x = 3x$. ...(3)

$$d\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy = 3xdx - 3y\, dy.$$

Integrating, $y = (3x^2/2) - (3y^2/2) + C$, where C is constant of integration. ...(4)

Discharge between the streamlines passing through (1, 3) and (3, 3)

$$= \psi(1, 3) - \psi(3, 3) = (3/2) \times (1 - 9) - (3/2) \times (9 - 9) = -12 \text{ units}.$$

Ex. 7. *If the expression for stream function is described by* $\psi = x^3 - 3xy^2$, *determine whether flow is rotational or irrotational. If the flow is irrotational, then indicate the correct value of the velocity potential.* (*a*) $\phi = y^3 - 3x^2y$. (*b*) $\phi = -3x^2y$.

Sol. Now $u = \partial\psi/\partial y = -6xy$, $= -\partial\psi/\partial x = -3(x^2 - y^2)$. ...(1)

Hence, $\partial\ /\partial x = -6x$ and $\partial u/\partial y = -6x$. ...(2)

A two-dimensional flow in xy-plane will be irrotational if the vorticity vector component Ω_z in the z-direction is zero.

Here $\Omega_z = (\partial\ /\partial x) - (\partial u/\partial y) = -6x - (-6x) = 0$, by (2)

Hence the flow is irrotational.

Now, $u = -\partial\phi/\partial x$ and $= -\partial\phi/\partial y$. ...(3)

For an irrotational flow Laplace equation in ϕ must be satisfied, *i.e.* $(\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = 0$.

We now check the validity of each given value of ϕ.

(*a*) Given $\phi = y^3 - 3x^2y \Rightarrow \partial^2\phi/\partial x^2 = -6y$ and $\partial^2\phi/\partial y^2 = 6y$

$\therefore$ $(\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = -6y + 6y = 0$.

(*b*) Given $\phi = -3x^2y \Rightarrow \partial^2\phi/\partial x^2 = -6y$ and $\partial^2\phi/\partial y^2 = 0$

$\therefore$ $(\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = -6y + 0 \neq 0$.

Hence the correct value of ϕ is given by $\phi = y^3 - 3x^2y$.

Ex. 8. *Show that the velocity vector* **q** *is everywhere tangent to lines in the xy-plane along which* $\psi\ (x, y) = $ *const.*

Sol. We have $d\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy$

or $(\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy = 0$ $[\because \psi(x, y) = \text{const.} \Rightarrow d\psi = 0]$

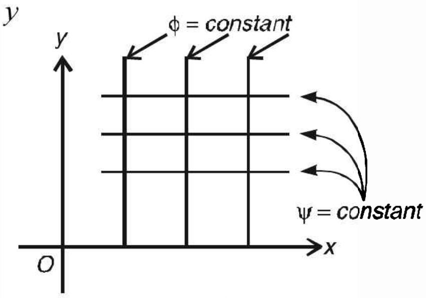
or $dx - udy = 0$, as $u = -\partial\psi/\partial y$ and $= \partial\psi/\partial x$

or $(dx)/u = (dy)/\ $,

showing that the velocity vector $\mathbf{q} = u\mathbf{i} + \ \mathbf{j}$ is tangent to the streamlines $\psi(x, y) = $ const.

Ex. 9. *Find the stream function* ψ *for a given velocity potential* $\phi = cx$, where *c is a constant. Also, draw a set of steamlines and equipotential lines.* **[Rohilkhand 2003]**

Sol. The velocity components u and in x and y directions are given by



$u = -\partial\phi/\partial x = -c$ and $= -\partial\phi/\partial y = 0$. ...(1)

$\therefore$ $u = -\partial\psi/\partial y$ and $= \partial\psi/\partial x$

$\Rightarrow$ $\partial\psi/\partial y = c$ and $\partial\psi/\partial x = 0$. ...(2)

Then, $d\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy = c\, dy$.

Integrating, $\psi = cy + d$, ...(3)

where d is constant of integration.

Now, $\phi = $ constant $\Rightarrow$ $cx = $ constant $\Rightarrow$ $x = $ constant,

showing that the lines of equipotential are parallel to y-axis.

Next, ψ = constant $\Rightarrow$ $cy + d$ = constant $\Rightarrow$ y = constant, showing that the streamlines are parallel to x-axis as shown in the figure.

Ex. 10. *In a two-dimensional incompressible flow, the fluid velocity components are given by* $u = x - 4y$ *and* $= -y - 4x$. *Show that velocity potential exists and determine its form as well as stream function.*

Sol. Given $u = x - 4y$ and $= -y - 4x$. ...(1)

The velocity potential will exist if flow is irrotational. Therefore, the vorticity component Ω_z in the z-direction must be zero.

Here $\Omega_z = (\partial\ /\partial x) - (\partial u/\partial y) = -4 - (-4) = 0$, using (1).

Here the vorticity being zero, the flow is irrotational and so the velocity potential ϕ exists.

Now, we have $d\phi = (\partial\phi/\partial x)dx + (\partial\phi/\partial y)dy = -\ udx - \ \ dy$

or $d\phi = -(x - 4y)dx - (-y - 4x)dy = -xdx + ydy + 4(ydx + xdy)$.

Integration, $\phi = -(x^2/2) + y^2/2 + 4xy + C$, where C is constant of integration. ...(2)

If $\phi = 0$ at the origin, then from (2), we find $C = 0$. Hence (2) reduces to

$$\phi = (y^2 - x^2)/2 + 4xy.$$

Ex. 11. *For a two-dimensional flow the velocity function is given by the expression,* $\phi = x^2 - y^2$. *Then* (*i*) *Determine velocity components in x and y directions* (*ii*) *Show that the velocity components satisfy the conditions of flow continuity and irrotationality* (*iii*) *Determine stream function and flow rate between the streamlines* (2, 0) *and* (2, 2) (*iv*) *Show that the streamlines and potential lines intersect orthogonally at the point* (2, 2).

Sol. (*i*) The velocity components in x and y directions are

$u = -\partial\phi/\partial x = -2x$ and $= -\partial\phi/\partial y = 2y$. ...(1)

(*ii*) Here $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(2y) = -2 + 2 = 0,$$

showing that the velocity components satisfy the flow continuity conditions.

Here $$\text{curl } \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & & w \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2x & 2y & 0 \end{vmatrix}$$

or $$\text{curl } \mathbf{q} = \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2y)\right] \mathbf{i} - \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(-2x)\right] \mathbf{j} + \left[\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(-2x)\right] \mathbf{k}$$

$\Rightarrow$ curl $\mathbf{q} = \mathbf{0}$ $\Rightarrow$ flow is irrotational.

(*iii*) We know that $u = \partial\psi/\partial y$ and $= -\partial\psi/\partial x$. ...(2)

Then (1) and (2) $\Rightarrow$ $\partial\psi/\partial y = -2x$ and $\partial\psi/\partial x = -2y$

$\therefore$ $d\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy = -2(y\,dx + xdy)$.

Integrating, $\psi = -2xy + C$, C being constant of integration.

The required flow between the streamlines through (2, 0) and (2, 2)

$= \psi(2, 0) - \psi(2, 2) = 0 - (-8) = 8$ m^3/s.

Now, we have $\phi = x^2 - y^2$ and $\psi = -2xy + C$. ...(3)

m_1 = The slope of tangent at (x, y) to potential lines $\phi = c_1$

$$= -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} = -\frac{2x}{-2y} = \frac{x}{y}, \quad \text{using (3)}$$

$\therefore$ m_1 = The slope of tangent to $\phi = c_1$ at (2, 2) = 2/2 = 1.

Next, m_2 = the slope of tangent to $\psi = c_2$ at $(x, y) = -\dfrac{\partial\psi/\partial x}{\partial\psi/\partial y} = -\dfrac{-2y}{-2x} = -\dfrac{y}{x}$, by (3)

$\therefore$ m_2 = slope of tangent to streamlines $\psi = c_2$ at $(2, 2) = -(2/2) = -1$

Here $m_1 m_2 = -1$ showing that the streamlines and the potential lines intersect orthogonally

Ex. 12. *Find the lines of flow in the two dimensional fluid motion given by* $\phi + i\psi = -(n/2) \times (x + iy)^2 e^{2int}$. *Prove or verify that the paths of the particles of the fluid (in polar coordinates) may be obtained by eliminating t from the equations.*

$r \cos (nt + \theta) - x_0 = r \sin(nt + \theta) - y_0 = nt\,(x_0 - y_0)$. **[Banaras 2003; I.A.S. 1992]**

Sol. Given $$\phi + i\psi = -(n/2) \times (x + iy)^2 e^{2int}. \qquad \text{...(1)}$$

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

So (1) becomes $$\phi + i\psi = -(n/2) \times (re^{i\theta})^2 e^{2int} = -(n/2) \times r^2 e^{2i(\theta + nt)}$$

or $$\phi + i\psi = -(n/2) \times r^2 [\cos 2(\theta + nt) + i \sin 2(\theta + nt)].$$

Equating the real and imaginary parts on both sides of (2), we get

$$\phi = -(n/2) \times r^2 \cos 2(\theta + nt) \quad \text{and} \quad \psi = -(n/2) \times r^2 \sin 2(\theta + nt). \qquad \text{...(2)}$$

The lines of flow are given by ψ = constant, namely,

$-(n/2) \times r^2 \sin 2(\theta + nt)$ = constant or $r^2 \sin 2(\theta + nt)$ = constant.

We now proceed to find the path of the particles. We have

$$\frac{dr}{dt} = -\frac{\partial\phi}{\partial r} = nr\cos 2(\theta + nt) = nr\cos 2\lambda, \text{ by (2)} \qquad \text{...(3)}$$

and $$r\frac{d\theta}{dt} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -nr\sin 2(\theta + nt) = -nr\sin 2\lambda, \text{ by (2)} \qquad \text{...(4)}$$

where $$nt + \theta = \lambda. \qquad \text{...(5)}$$

Now, (3) $\Rightarrow$ $$nr\cos 2\lambda = \frac{dr}{dt} = \frac{dr}{d\lambda}\frac{d\lambda}{dt} = \frac{dr}{d\lambda}\left(\frac{d\theta}{dt} + n\right), \text{ by (5)}$$

or $$nr\cos 2\lambda = \frac{dr}{d\lambda}(-n\sin 2\lambda + n), \text{ using (4)}$$

or $$(2/r)\,dr - [2\cos 2\lambda/(1 - \sin 2\lambda)]\,d\lambda = 0.$$

Integrating, $2\log r + \log(1 - \sin 2\lambda) = \log C$ or $r^2(1 - \sin 2\lambda) = C$

or $r^2(\sin^2\lambda + \cos^2\lambda - 2\sin\lambda\cos\lambda) = C$ or $[r(\cos\lambda - \sin\lambda)]^2 = C$

or $$r(\cos\lambda - \sin\lambda) = C', \text{ where } C' \;(= \sqrt{C}) \text{ is an arbitrary constant.} \qquad \text{...(6)}$$

Initially, let $\lambda = \theta_0$ and $r = r_0$ when $t = 0$. Then (6) gives

$C' = r_0(\cos\theta_0 - \sin\theta_0) = x_0 - y_0$, where $x_0 = r_0\cos\theta_0$, $y_0 = r_0\sin\theta_0$.

$\therefore$ (6) becomes $$r\cos\lambda - r\sin\lambda = x_0 - y_0 \qquad \text{...(7)}$$

or $$r\cos(\theta + nt) - x_0 = r\sin(\theta - nt) - y_0, \text{ using (5).} \qquad \text{...(8)}$$

Now, from (5), $d\lambda/dt = n + (d\theta/dt)$ or $d\lambda/dt = n - n\sin 2\lambda$, using (4)

or $$\frac{d\lambda}{1 - \sin 2\lambda} = n\,dt \quad \text{or} \quad \frac{d\lambda}{(\cos\lambda - \sin\lambda)^2} = n\,dt$$

$\therefore$ $$\int\frac{\sec^2\lambda\,d\lambda}{(1 - \tan\lambda)^2} = \int n\,dt \quad \text{or} \quad -\int\frac{du}{u^2} = nt + D$$

(Putting $1 - \tan\lambda = u$ so that $-\sec^2\lambda\,d\lambda = du$)

or $\quad \dfrac{1}{u} = nt + D \qquad$ or $\qquad \dfrac{1}{1 - \tan\lambda} = nt + D$

or $$\cos\lambda/(\cos\lambda - \sin\lambda) = nt + D. \qquad ...(9)$$

As before, initially $\lambda = \theta_0$ and $t = 0$. Hence (9) gives

$$D = \frac{\cos\theta_0}{\cos\theta_0 - \sin\theta_0} = \frac{r_0\cos\theta_0}{r_0\cos\theta_0 - r_0\sin\theta_0} = \frac{x_0}{x_0 - y_0}, \text{ as before}$$

Then, (9) becomes $\qquad \dfrac{r\cos\lambda}{r\cos\lambda - r\sin\lambda} = nt + \dfrac{x_0}{x_0 - y_0}$

or $\quad \dfrac{r\cos(\theta + nt)}{x_0 - y_0} = nt + \dfrac{x_0}{x_0 - y_0} \qquad$ or $\qquad r\cos(\theta + nt) = nt(x_0 - y_0) + x_0$

or $$r\cos(nt + \theta) - x_0 = nt(x_0 - y_0). \qquad ...(10)$$

$\therefore$ Then, from (8) and (10), we have

$$r\cos(nt + \theta) - x_0 = r\sin(nt + \theta) - y_0 = nt(x_0 - y_0)$$

Ex. 13. *A single source is placed in an infinite perfectly elastic fluid, which is also a perfect conductor of heat. Show that if the motion be steady, the velocity at a distance r from the source satisfies the equation* $\left(- \dfrac{k}{} \right)\dfrac{\partial}{\partial r} = \dfrac{2k}{r}$ and hence that $\qquad r = \dfrac{1}{\sqrt{}} e^{2/4k}$.

Sol. Since we have an infinite perfectly elastic fluid, there would be hardly any change in temperature, and hence Boyle's law would be obeyed and so $\qquad p = k\rho \qquad ...(1)$

Since the motion is symmetrical about the source, the equation of continuity may be written as

$$\rho r^2 = \text{constant}, \qquad ...(2)$$

where is the velocity at a distance r and ρ is the density of fluid. The pressure equation takes the form

$$\int \frac{dp}{\rho} + \frac{^2}{2} = \text{constant} \qquad \text{or} \qquad k\int \frac{\partial\rho}{\rho} + \frac{^2}{2} = \text{constant, by (1).} \qquad ...(3)$$

Differentiating (2) and (3) w.r.t. 'r', we have

$$r^2 \frac{\partial\rho}{\partial r} + \rho\left[r^2 \frac{\partial}{\partial r} + 2r \right] = 0 \qquad ...(4)$$

and $$\frac{k}{\rho}\frac{\partial\rho}{\partial r} + \frac{\partial}{\partial r} = 0, \qquad \textit{i.e.,} \qquad \frac{\partial\rho}{\partial r} = -\frac{\rho}{k}\frac{\partial}{\partial r}. \qquad ...(5)$$

Substituting the value of $\partial\rho/\partial r$ given by (5) in (4), we get

$$r^2\left(-\frac{\rho}{k}\frac{\partial}{\partial r} \right) + \rho\left(r^2 \frac{\partial}{\partial r} + 2r \right) = 0$$

or $$\frac{r^2}{k}\frac{\partial}{\partial r}(k - {}^2) = -2r \qquad \text{or} \qquad \left(- \frac{k}{} \right)\frac{\partial}{\partial r} = \frac{2k}{r}, \qquad ...(6)$$

which proves the first part of the problem.

Integrating (6), $({}^2/2) - k\log = 2k\log r - 2k\log C$, C being an arbitrary constant.

or $\quad (1/2) \times \log + \log r - \log C = {}^2/4k \qquad$ or $\qquad r\sqrt{} = C e^{2/4k}$

or $$r = (1/\sqrt{})e^{2/4k}, \text{ taking } C = 1.$$

Ex.14. *Prove that the radius of curvature R at any point of a streamline* ψ = *constant is given by*
$$R = (u^2 + \ {}^2)^{3/2} / |\, u^2(\partial \ \ /\partial x) - 2u \ \ (\partial u/\partial x) - \ {}^2(\partial u/\partial y)\,|,$$
where u, are respectively the velocity components of a fluid motion along OX and OY.

Sol. From Differential Calculus, we know that the radius of curvature R at a point (x, y) of streamline $\psi(x, y)$ = constant is given by
$$R = [1 + (dy/dx)^2]^{3/2} / (d^2y/dx^2). \qquad \ldots(1)$$

Given streamline is $$\psi(x, y) = 0. \qquad \ldots(2)$$

Also, we have $$u = -\partial\psi/\partial y \qquad \text{and} \qquad = \partial\psi/\partial x. \qquad \ldots(3)$$

Differentiating (2) w.r.t. x, $$(\partial\psi/\partial x) + (\partial\psi/\partial y)\,(dy/dx) = 0$$

or $$\ - u\,(dy/dx) = 0 \qquad \text{or} \qquad dy/dx = \ \ /u. \qquad \ldots(4)$$

Differentiating (4) w.r.t. x, $$\frac{d^2y}{dx^2} = \frac{\partial}{\partial x}\left(\frac{\ }{u}\right) + \frac{\partial}{\partial y}\left(\frac{\ }{u}\right)\frac{dy}{dx}$$

or $$\frac{d^2y}{dx^2} = \frac{u(\partial \ \ /\partial x) - \ \ (\partial u/\partial x)}{u^2} + \frac{u(\partial \ \ /\partial y) - \ \ (\partial u/\partial y)}{u^2}\cdot\frac{\ }{u}, \text{ uning (4)}$$

or $$\frac{d^2y}{dx^2} = \frac{u[u(\partial \ \ /\partial x) - \ \ (\partial u/\partial x)] + \ \ [u(\partial \ \ /\partial y) - \ \ (\partial u/\partial y)]}{u^3}$$

or $$\frac{d^2y}{dx^2} = \frac{u^2(\partial \ \ /\partial x) - 2u \ \ (\partial u/\partial x) + \ \ (\partial u/\partial y)}{u^3} \qquad \ldots(5)$$

$$\left[\because \ \frac{\partial \ }{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) = -\frac{\partial u}{\partial x}, \text{ by (3)}\right]$$

Putting the values of dy/dx and d^2y/dx^2 from (4) and (5) in (1), we get
$$R = \frac{(1 + \ {}^2/u^2)^{3/2}}{|\{u^2(\partial \ \ /\partial x) - 2u \ \ (\partial u/\partial x) - \ {}^2(\partial u/\partial y)\}/u^3\,|} = \frac{(u^2 + \ {}^2)^{3/2}}{|\,u^2(\partial \ \ /\partial x) - 2u \ \ (\partial u/\partial x) - \ {}^2(\partial u/\partial y)\,|}.$$

Ex.15 *Show that* $u = 2cxy$, $\ = c(a^2 + x^2 - y^2)$ *are the velocity components of a possible fluid motion. Determine the stream function.* **[Rohilkhand 1999]**

Sol. Given $$u = 2cxy, \qquad = c(a^2 + x^2 - y^2) \qquad \ldots(1)$$

Equation of continuity in xy-plane is given by
$$\partial u/\partial x + \partial \ \ /\partial y = 0 \qquad \ldots(2)$$

From (1), $\partial u/\partial x = 2cy$ and $\partial \ \ /\partial y = -2cy$. Putting these values in (2) we get 0 = 0, showing (2) is satisfied by u, given by (1). Hence u and constitute a possible fluid motion.

Let ψ be the required stream function. Then, we have
$$u = -(\partial\psi/\partial y) \qquad \text{or} \qquad \partial\psi/\partial y = -2cxy \qquad \ldots(3)$$

and $$= \partial\psi/\partial x \qquad \text{or} \qquad \partial\psi/\partial x = c(a^2 + x^2 - y^2) \qquad \ldots(4)$$

Integrating (3) partially w.r.t. 'y' $\qquad \psi = -cxy^2 + \phi(x, t)$, ...(5)

where $\phi(x, t)$ is an arbitrary function of x and t.

Differentiating (5) partially w.r.t. 'x', $\qquad \partial\psi/\partial x = -cy^2 + \partial\phi/\partial x$ ...(6)

(4) and (6) $\Rightarrow$ $\quad -cy^2 + \partial\phi/\partial x = c(a^2 + x^2 - y^2)$ $\quad$ or $\quad \partial\phi/\partial x = c(a^2 + x^2)$ ...(7)

Integrating (7) partially w.r.t. 'x', $\qquad \phi(x, t) = c(a^2 x + x^3/3) + \psi(y, t)$,

where $\psi(y, t)$ is an arbitrary function of y and t.

Substituting the above value of $\phi(x, t)$ in (5), we get

$\psi = c(ax^2 + x^3/3 - xy^2) + \psi(y, t)$, which is the required stream function.

Ex. 16. *Show that $u = -\omega y$, $= \omega x$, $w = 0$ represents a possible motion of inviscid fluid. Find the stream function and sketch stream lines. What is the basic difference between this motion and one represented by the potential $\phi = A\log r$, where $r = (x^2 + y^2)^{1/2}$.*

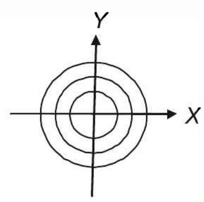
Sol. Given $\qquad u = -\omega y, \qquad = \omega x \qquad$ and $\qquad w = 0$ ...(1)

(1) $\Rightarrow$ $\partial u/\partial x = 0 = \partial \;/\partial y$. Hence the equation of continuity $\partial u/\partial x + \partial \;/\partial y = 0$ is satisfied. Hence these exist a two dimensional motion defined by (1).

Now, $\qquad \partial\psi = (\partial\psi/\partial x)dx + (\partial\psi/\partial y)dy$ ...(2)

But $\qquad \dfrac{\partial\psi}{\partial x} = -\dfrac{\partial\phi}{\partial y} = \;= \omega x \qquad$ and $\qquad \dfrac{\partial\psi}{\partial y} = \dfrac{\partial\phi}{\partial x} = -u = \omega y$

$\therefore$ (3) reduces to $\qquad d\psi = \omega x\,dx + \omega y\,dy = d\{\omega(x^2 + y^2)/2\}$



Integrating, $\psi = \omega(x^2 + y^2)/2 + c$, where c is an arbitrary constant.

The required streamlines are given by ψ = constant = c', say

i.e. $\quad c' = \omega(x^2 + y^2)/2 + c \quad$ or $\quad x^2 + y^2 = 2(c' - c)/w = a^2$, say

Hence the required streamlines are concentric circles with centres at origin as shown in the adjoining figure.

Second part: Given

$$\phi = A\log r = A\log(x^2 + y^2)^{1/2} = (A/2) \times \log(x^2 + y^2) \qquad ...(3)$$

$\therefore \quad u = -\dfrac{\partial\phi}{\partial x} = -\dfrac{Ax}{x^2 + y^2} \qquad$ and $\qquad = -\dfrac{\partial\phi}{\partial y} = -\dfrac{Ay}{x^2 + y^2}$

$\Rightarrow \quad \dfrac{\partial u}{\partial x} = -A\dfrac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = A\dfrac{x^2 - y^2}{(x^2 + y^2)^2}, \qquad \dfrac{\partial}{\partial y} = -A\dfrac{x^2 - y^2 - 2y^2}{(x^2 + y^2)^2} = -\dfrac{x^2 - y^2}{(x^2 + y^2)^2}$

$\therefore$ $\partial u/\partial x + \partial \;/\partial y = 0$ so that the equation of continuity is satisfied.

Hence there exists exists a motion for the given value of ϕ.

Third part. Difference between the two given motions.

For the fluid motion given by (1), we have

$$\text{curl } \boldsymbol{q} = \boldsymbol{i}(\partial w/\partial y - \partial \;/\partial z) + \boldsymbol{j}(\partial u/\partial z - \partial w/\partial x) + \boldsymbol{k}(\partial \;/\partial x - \partial u/\partial y)$$

$$= \boldsymbol{i}(0 - 0) + \boldsymbol{j}(0 - 0) + \boldsymbol{k}(\omega + \omega) \neq \boldsymbol{0},$$

showing that curl $\boldsymbol{q} \neq 0$. Hence velocity potential does not exist for the fluid motion defined by (1) (refer Art. 2.26), whereas velocity potential exist for the second fluid motion.

Ex. 17. *In irrotational motion in two dimensions, prove that*

$(\partial q/\partial x)^2 + (\partial q/\partial y)^2 = q\nabla^2 q$. **(Agra 2012; Kanpur 2002; Meerut 2002,05)**

Sol. Since the motion is irrotational, the velocity potential ϕ exists such that

$$\nabla^2\phi = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0 \qquad ...(1)$$

Again,
$$q^2 = (\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2 \qquad ...(2)$$

Differentiating (2) partially w.r.t. x and y respectively, we get

$$q\frac{\partial q}{\partial x} = \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y} \qquad ...(3)$$

and
$$q\frac{\partial q}{\partial y} = \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2} \qquad ...(4)$$

Differentiating (3) and (4) partially *w.r.t.* x and y respectively, we get

$$q\frac{\partial^2 q}{\partial x^2} + \left(\frac{\partial q}{\partial x}\right)^2 = \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \frac{\partial\phi}{\partial x}\frac{\partial^3\phi}{\partial x^3} + \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 + \frac{\partial\phi}{\partial y}\frac{\partial^3\phi}{\partial x^2\partial y} \qquad ...(5)$$

and
$$q\frac{\partial^2 q}{\partial y^2} + \left(\frac{\partial q}{\partial y}\right)^2 = \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 + \frac{\partial\phi}{\partial x}\frac{\partial^3\phi}{\partial x\partial y^2} + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2 + \frac{\partial\phi}{\partial y}\frac{\partial^3\phi}{\partial y^3} \qquad ...(6)$$

Adding (5) and (6) and simplifying, we get

$$q\nabla^2 q + \left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 = \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2 + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)$$

$$= 2(\partial^2\phi/\partial x^2)^2 + 2(\partial^2\phi/\partial x\partial y)^2 \qquad ...(7)$$

$$\left[\because \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad\Rightarrow\quad \frac{\partial^2\phi}{\partial x^2} = -\frac{\partial^2\phi}{\partial y^2} \quad\Rightarrow\quad \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 = \left(\frac{\partial^2\phi}{\partial y^2}\right)^2\right]$$

Next, squaring and adding (3) and (4), we get

$$q^2\left[\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2\right] = \left(\frac{\partial\phi}{\partial x}\right)^2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right] + \left(\frac{\partial\phi}{\partial y}\right)^2\left[\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2\right]$$

$$+2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)$$

$$= \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] + \left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right], \text{ using (1)}$$

$$= q^2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right], \text{ using (2)}$$

Thus, $$(\partial^2\phi/\partial x^2)^2 + (\partial^2\phi/\partial x\partial y)^2 = (\partial q/\partial x)^2 + (\partial q/\partial y)^2 \qquad ...(8)$$

From (7) and (8), we find

$$q\nabla^2 q + (\partial q/\partial x)^2 + (\partial q/\partial y)^2 = 2[(\partial q/\partial x)^2 + (\partial q/\partial y)^2]$$

or $$q\nabla^2 q = (\partial q/\partial x)^2 + (\partial q/\partial y)^2.$$

Ex. 18. λ *denoting a variable parameter, and f a given function, find the condition that* $f(x, y, \lambda) = 0$ *should be a possible system of stream lines for steady irrotational motion in two dimensions.* **[Kurukshetra 1998]**

Sol. If ψ is the stream function, then streamlines are given by

$$\psi = C \text{ (constant)} \qquad ...(1)$$

Given that $$f(x, y, \lambda) = 0 \qquad ...(2)$$

represents a system of streamlines, λ being parameter. Then for $\lambda = \lambda'$ (say), (2) must give a streamline which corresponds with (1) for $C = C'$. Hence ψ is a function of λ alone. Moreover λ is a function of x and y from (2). Hence, we obtain

$$\frac{\partial\psi}{\partial x} = \frac{d\psi}{d\lambda}\frac{\partial\lambda}{\partial x} \quad \text{and} \quad \frac{\partial\psi}{\partial y} = \frac{d\psi}{d\lambda}\frac{\partial\lambda}{\partial y}$$

Again, $$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{dx}\left(\frac{\partial\psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{d\psi}{d\lambda}\cdot\frac{\partial\lambda}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{d\psi}{d\lambda}\right)\frac{\partial\lambda}{\partial x} + \frac{d\psi}{d\lambda}\frac{\partial}{\partial x}\left(\frac{\partial\lambda}{\partial x}\right)$$

so that $$\frac{\partial^2\psi}{\partial x^2} = \left\{\frac{d}{d\lambda}\left(\frac{d\psi}{d\lambda}\right)\right\}\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial x} + \frac{d\psi}{d\lambda}\frac{\partial^2\lambda}{\partial x^2}$$

Thus, $$\frac{\partial^2\psi}{\partial x^2} = \frac{d^2\psi}{d\lambda^2}\left(\frac{\partial\lambda}{\partial x}\right)^2 + \frac{d\psi}{d\lambda}\frac{\partial^2\lambda}{\partial x^2} \qquad ...(3)$$

Similarly, $$\frac{\partial^2\psi}{\partial y^2} = \frac{d^2\psi}{d\lambda^2}\left(\frac{\partial\lambda}{\partial y}\right)^2 + \frac{d\psi}{d\lambda}\frac{\partial^2\lambda}{\partial y^2} \qquad ...(4)$$

For the irrotational motion, $$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0. \qquad ...(5)$$

Adding (3) and (4) and using (5), we get

$$\frac{d^2\psi}{d\lambda^2}\left[\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2\right] + \frac{d\psi}{d\lambda}\left(\frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2}\right) = 0$$

or $$\left[\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2\right] \Big/ \left[\frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2}\right] = -\frac{d\psi/d\lambda}{d^2\psi/d\lambda^2} \qquad ...(6)$$

Since the R.H.S. of (6) is a function of λ alone, the required condition is that the L.H.S. of (6) should be a function of λ alone.

Ex. 19. *In two-dimensional motion show that, if the streamlines are confocal ellipses*

$$x^2/(a^2+\lambda) + y^2/b^2+\lambda) = 1, \quad \text{then} \quad \psi = A\log(\sqrt{a^2+\lambda} + \sqrt{b^2+\lambda}) + B$$

and the velocity at any point is inversely proportional to the square root of the rectangle under the focal radii of the point. **[Rajasthan 1998]**

Sol. Take $z = C\cos w$ then $$x + iy = C\cos(\phi + i\psi) \qquad ...(1)$$

or $$x + iy = C\ (\cos\phi \cos i\psi - \sin\phi \sin i\psi)$$

or $$x + iy = C\ \cos\phi \cosh\psi - i\, C \sin\phi \sinh\psi \qquad ...(2)$$

Equating real and imaginary parts, (2) gives

$$x = C\ \cos\phi \cosh\psi \qquad \text{and} \qquad y = -C\ \sin\phi \sinh\psi$$

so that $$\cos\phi = \frac{x}{C\ \cosh\psi} \qquad \text{and} \qquad \sin\phi = -\frac{y}{C\ \sinh\psi}$$

Squaring and adding these, we obtain

$$\frac{x^2}{C^2\cosh^2\psi} + \frac{y^2}{C^2\sinh^2\psi} = 1 \qquad ...(3)$$

which give the streamlines in two-dimensions.

Again, given that the streamlines are confocal ellipses

$$x^2/(a^2+\lambda) + y^2/(b^2+\lambda) = 1 \qquad ...(4)$$

Since (3) and (4) must be identical, we have

$$C^2\cosh^2\psi = a^2 + \lambda \qquad \text{and} \qquad C^2\sinh^2\psi = b^2+\lambda$$

$$\therefore \quad C\,(\cosh\psi + \sinh\psi) = \sqrt{a^2+\lambda} + \sqrt{b^2+\lambda} \qquad \text{or} \qquad Ce^{\psi} = \sqrt{a^2+\lambda}+\sqrt{b^2+\lambda}$$

$$[\because\ \cosh\psi = (e^{\psi}+e^{-\psi})/2 \quad \text{and} \quad \sinh\psi = (e^{\psi}-e^{-\psi})/2\ ]$$

or $$\psi = \log(\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda}) - \log C \qquad ...(5)$$

If ϕ, ψ are velocity potential and stream function, so also will be $A\phi$ and $A\psi$ where A is a constant. Hence (5) may be re-written as

$$\psi = A\log(\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda}) + B$$

From (1), $$\frac{dz}{dw} = -C\sin w = -C\sqrt{1-\cos^2 w} = -C(1-z^2/C^2)^{1/2}$$

$$= -\sqrt{C^2-z^2} = -\sqrt{(C+z)\,(C-z)} = -\sqrt{r_1 r_2}$$

where r_1 and r_2 are the focal distances (radii) of any point $P(z)$ from the foci S $(C, 0)$ and S' $(-C, 0)$ of the ellipses.

Thus $$q = |\,dw/dz\,| = 1/\sqrt{r_1 r_2}.$$

Ex. 20. *Show that the velocity potential* $$\phi = \frac{1}{2}\log\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$$ *gives a possible motion. Determine the streamlines and show also that the curves of equal speed are the ovals of Cassini given by rr′ = const.* **[Rajasthan 2000; I.A.S. 1990]**

Sol. Given $$\phi = (1/2)\times\log[(x+a)^2+y^2] - (1/2)\times\log[x-a)^2+y^2]$$

$$\therefore \quad u = -\frac{\partial\phi}{\partial x} = -\frac{x+a}{(x+a)^2+y^2} + \frac{x-a}{(x-a)^2+y^2} \qquad ...(1)$$

and $$= -\frac{\partial\phi}{\partial y} = -\frac{y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} \qquad ...(2)$$

From (1) $$\frac{\partial u}{\partial x} = -\frac{y^2-(x+a)^2}{[(x+a)^2+y^2]^2} + \frac{y^2-(x-a)^2}{[(x-a)^2+y^2]^2} \qquad ...(3)$$

From (2) $$\frac{\partial}{\partial y} = -\frac{(x+a)^2 - y^2}{[(x+a)^2 + y^2]^2} + \frac{(x-a)^2 - y^2}{[(x-a)^2 + y^2]^2} \qquad ...(4)$$

Adding (3) and (4), we see that the equation of continuity $\partial u/\partial x + \partial \ /\partial y = 0$ is satisfied. Hence there exists a motion for the given ϕ.

To determine the streamlines, we use the fact that velocity potential ϕ and the stream function ψ satisfy the Cauchy-Riemann equations, namely,

$$\partial\phi/\partial x = \partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y = -\partial\psi/\partial x \qquad ...(5)$$

From (1) and (5), we have $$\frac{\partial\psi}{\partial y} = \frac{x+a}{(x+a)^2 + y^2} - \frac{x-a}{(x-a)^2 + y^2}$$

Integrating it w.r.t. y, we get

$$\psi = \tan^{-1}\frac{y}{x+a} \tan^{1}\frac{y}{x-a} + f(x), f(x) \text{ being an arbitrary function of } x \qquad ...(6)$$

$$\therefore \qquad \frac{\partial\psi}{\partial x} = -\frac{y}{(x+a)^2 + y^2} + \frac{y}{(x-a)^2 + y^2} + f'(x) \qquad ...(7)$$

Again from (5) and (2), we get

$$\frac{\partial\psi}{\partial x} = -\frac{y}{(x+a)^2 + y^2} + \frac{y}{(x-a)^2 + y^2} \qquad ...(8)$$

Comparing (7) and (8) $f'(x) = 0$ so that $f(x)$ = constant. Omitting the additive constant, (6) gives

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} = \tan^{-1}\frac{[y/(x+a)] - [y/(x-a)]}{1 - [y/(x+a)][y/(x-a)]}$$

$$\therefore \qquad \psi = \tan^{-1}\frac{(-2ay)}{x^2 + y^2 - a^2}$$

Hence the streamlines are given by ψ = const. = $\tan^{-1}$ $(-2a/C)$, that is,

$$x^2 + y^2 - Cy = a^2 \qquad ....(9)$$

which are circles. When $C = 0$, the stream line is the circle passing through $(a, 0)$ and $(-a, 0)$. Again, if C is infinite then stream line $y = 0$ [divide (9) by C and then let $C \to \infty$]

Now, $$w = \phi + i\psi = \frac{1}{2}\log\left[(x+a)^2 + y^2)\right] - \frac{1}{2}\log\left[(x-a)^2 + y^2\right] + i\tan^{-1}\frac{y}{x+a} - i\tan^{-1}\frac{y}{x-a}$$

$$= \log[(x+a) + iy] - \log[(x-a) + iy] = \log(z+a) - \log(z-a), \text{ as } z = x + iy$$

$$\therefore \qquad q = \left|\frac{dw}{dz}\right| = \left|\frac{1}{z+a} - \frac{1}{z-a}\right| = \frac{2a}{|z+a| \cdot |z-a|} = \frac{2a}{rr'},$$

where r, r' are the distances of the point from the points P (x, y) from the points $(a, 0)$ and $(-a, 0)$. The curves of equal speed are given by q = constant or rr' = constant, which are Cassini ovals.

Ex. 21. *A velocity field is given by* $\mathbf{q} = -x\mathbf{i} + (y + t)\mathbf{j}$. *Find the stream function and the streamlines for this field at* $t = 2$. **[Agra 2005; Garhwal 2000; Rohilkhand 2002]**

Sol. We have $$-\partial\psi/\partial y = u = -x \qquad \ldots(1)$$

and $$\partial\psi/\partial x = \quad = y + t \qquad \ldots(2)$$

Integrating (1) and (2), we get $$\psi = xy + f_1(x, t) \qquad \ldots(3)$$

and $$\psi = xy + tx + f_2(x, t) \qquad \ldots(4)$$

Note that f_2 must be a function of t alone, otherwise (4) will not be satisfied; and then $f_1 = tx + f_2$. Thus

$$\psi = xy + tx + f_2(t) \qquad \ldots(5)$$

The function f_2 cannot be obtained from the given data. However since we deal only with differences in ψ values at a given t or with the derivatives $\partial\psi/\partial x$ and $\partial\psi/\partial y$, the determination of f_2 is not necessary. At $t = 2$, (5) becomes

$$\psi = xy + 2x + f_2(2) \qquad \ldots(6)$$

The stream lines (ψ = constant) are given by $\qquad x\,(y + 2) = \text{constant},$

which are rectangular hyperbolas.

Ex. 22. *A two-dimensional flow field is given by* $\psi = xy$. (*a*) Show that the flow is irrotational. (*b*) Find the velocity potential. (*c*) Verify that ψ and ϕ satisfy the Laplace equation. (*d*) Find the streamlines and potential lines. **[Agra 2005, 2011; Garhwal 2005]**

Sol. (*a*) The velocity components are given by $u = -\partial\psi/\partial y = -x, \quad = \partial\psi/\partial x = y$ so that

$$\mathbf{q} = u\mathbf{i} + \ \mathbf{j} \qquad \text{or} \qquad \mathbf{q} = -x\mathbf{i} + y\mathbf{j}$$

and $$\text{curl } \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ -x & y & 0 \end{vmatrix} = \mathbf{0}.$$

Hence the flow is irrotational.

(*b*) We have $$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \qquad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

$\therefore$ $$\phi = \int (\partial\psi/\partial y)dx + f_1(y) = x^2/2 + f_1(y), \qquad \ldots(1)$$

and $$\phi = -\int (\partial\psi/\partial x)\,dy + f_2(x) = -(y^2/2) + f_2(x). \qquad \ldots(2)$$

(1) and (2) show that

$$f_1(y) = -y^2/2 + \text{constant} \qquad \text{and} \qquad f_2(x) = x^2/2 + \text{constant},$$

so that $$\phi = (x^2 - y^2)/2 + \text{constant}$$

(*c*) $\nabla^2\psi = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0 + 0 = 0$ and $\nabla^2\phi = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 1 - 1 = 0$

Hence ψ and ϕ satisfy the Laplace equation.

(*d*) The streamlines (ψ = constant) and the potential lines (ϕ = constant) are given by

$xy = C_1$ and $x^2 - y^2 = C_2$, respectively, where C_1 and C_2 are constants.

EXERCISE 5 A

1. Show that the difference of the values of ψ at two points represents the flux of the fluid across any curve joining the two points. **[Kanpur 2005]**

2. Liquid is streaming steadily and irrotationally in two-dimensions in the region bounded by one branch of a hyperbola and its minor axis, find streamlines.

3. If a homogeneous liquid is acted on by a repulsive force from the origin, the magnitude of which at distance r from the origin is μr per unit mass, show that it is possible for the liquid to move steadily without being constrained by any boundaries, in the space between one branch of the hyperbola $x^2 - y^2 = a^2$ and the asymptotes and find the velocity potential.

4. In a two-dimensional motion, show that a streamline cuts itself at a point of zero velocity and that the two branches are at right angles when the motion is irrotational.

Sketch the streamline which passes through the stagnation point of the motion given by $\psi = u\,(y - a \tan^{-1} y/x)$ and determine the velocity at the points, where this line crosses the axis of y.

5. Prove that in two-dimensions there is always a stream function whether the motion is irrotational or rotational. **[Meerut 1999]**

6. In irrotational motion in two dimensions under conservative forces whose potential V satisfies $\nabla^2 V = 0$, prove that $\nabla^2 (\log \nabla^2 p) = 0$.

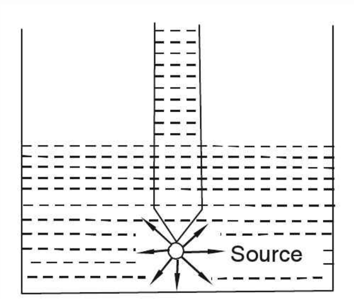
7. Obtain the nature of velocity potential and stream functions if the complex potential is (*i*) $w = Az^2$ (*ii*) $\omega = Az^n$ (*iii*) $z = C \cos w$ (*iv*) $z = C \cosh w$.

8. Verify that the stream function ψ and the velocity potential ϕ for a two-dimensional motion satisfy Laplace's equation.

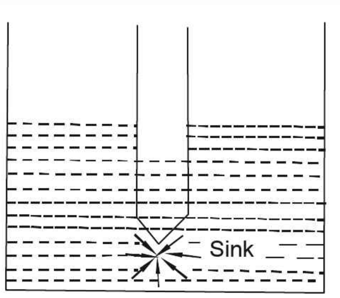
5.11. Sources and sinks.

[Aga 2005; Kanpur 2000, 02; Purvanchal 2004; Meerat 2009; Rohilkhand 2005]

If the motion of a fluid consists of symmetrical radial flow in all directions proceeding from a point, the point is known as a *simple source*. If, however, the flow is such that the fluid is directed radially inwards to a point from all directions in a symmetrical manner, then the point is known as a *simple sink*.



(*i*)



(*ii*)

Obviously a source implies the creation of fluid at a point whereas a sink implies the annihilation of fluid at a point. Sources and sinks are not readily obtained by some dynamical effects of the motion of fluid but may occur due to some external causes. For example, consider a simple source in a tank filled with a fluid. This source may be created by taking a long tube of very small cross-section and injecting fluid through it into the tank as shown in figure (*i*). In such a situation, we find that the fluid is coming out from the tube radially into the tank, Again, a sink can be created by taking a long tube of very small cross-section and sucking fluid through the tube from the tank as shown in figure (*ii*).

Consider a source at the origin. Then the mass m of the fluid coming out from the origin in a unit time is known as the *strength of the source*. Similarly, in a tank at the origin, the amount of fluid going into the sink in a unit time is called the *strength of the sink*.

Remark. Since the velocity is unique at a point, so usually no two streamlines intersect each other. But some flow fields may have singularities, where the velocity vector is not unique. Sources and sinks are examples of singularities of a flow field because infinitely many stream lines meet at such points as indicated in the figures (*i*) and (*ii*).

5.12. Source and sinks in two-dimensions.

[Garhwal 2002; Kanpur 1999; Meerut 2010]

In two-dimensions a source of strength m is such that the flow across any small curve surrounding is $2\pi m$. Sink is regarded as a source of strength $-m$.

Consider a circle of radius r with source at its centre. Then radial velocity q_r is given by

$$q_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{...(1)}$$

or $$q_r = -\frac{\partial \phi}{\partial r}, \qquad \text{as} \qquad \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{...(2)}$$

Then the flow across the circle is $2\pi r q_r$. Hence we have

$$2\pi r q_r = 2\pi m \qquad \text{or} \qquad r q_r = m \qquad \text{...(3)}$$

or $$r\left(-\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) = m, \text{ by (1)}$$

Integrating and omitting constant of integration, we get

$$\psi = -m\theta \qquad \text{...(4)}$$

Using (2) and (3), we obtain as before

$$\phi = -m \log r \qquad \text{...(5)}$$

Equation (4) shows that the streamlines are θ = constant, *i.e.*, straight lines radiating from the source. Again (5) shows that the curves of equi-velocity potential are r = constant, *i.e.*, concentric circles with centre at the source.

5.13. Complex potential due to a source.

[Kurukshetra 2004; Meerut 2001, 2012; Kanpur 2009]

Let there be a source of strength m at origin. Then

$$w = \phi + i\psi = -m\log r - im\theta = -m\,(\log r + i \log e^{i\theta}) = -m \log (re^{i\theta}) = -m \log z.$$

If, however, the source is at z', then the complex potential is given by $w = -m \log (z - z')$

The relation between w and z for sources of strengths $m_1, m_2, m_3, \ldots$ situated at the points $z = z_1, z_2, z_3, \ldots$ is given by

$$w = -m_1 \log (z - z_1) - m_2 \log (z - z_2) - m_3 \log (z - z_3) - \ldots$$

leading to $$\phi = -m_1 \log r_1 - m_2 \log r_2 - m_3 \log r_3 - \ldots$$

and $$\psi = -m_1\theta_1 - m_2\theta_2 - m_3\theta_3 - \ldots$$

where $r_n = |z - z_n|$ and $\theta_n = \arg (z - z_n)$, $n = 1, 2, 3, \ldots$

5.14. Doublet (or dipole) in two dimensions

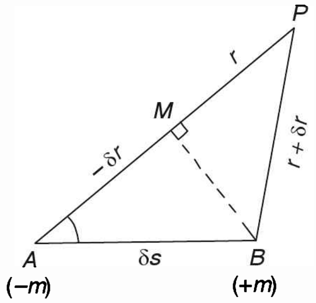
[Agra 2005; Garhwal 2000, 04; Rohilkhand 2000; Kanpur 1999, 2002, 07]

A combination of a source of strength m and a sink of strength $-m$ at a small distance δs apart, where in the limit m is taken infinitely great and δs infinitely small but so that the product $m\delta s$ remains finite and equal to μ, is called a *doublet of strength* μ, and the line δs taken in the sense from $-m$ to $+m$ is taken as the *axis of the doublet*.

Complex potential due to a doublet in two-dimensions

[G.N.D.U. Amritsar 2004, 06; Kanpur 2000, 05, 07; Meerut 2002, 09, 10; Purvanchal 2004, 05]

Let A, B denote the positions of the sink and source and P be any point. Let $AP = r$, $BP = r + \delta r$ and $\angle PAB = \theta$. Let ϕ be the velocity potential due to this doublet.



Then $$\phi = m \log r - m \log (r + \delta r) = -m \log \frac{r+\delta r}{r}$$

or $$\phi = -m \log\left(1 + \frac{\delta r}{r}\right)$$

$\therefore$ $$\phi = -m\frac{\delta r}{r}, \text{ to first order of approximation.} \qquad \ldots(1)$$

Let BM be perpendicular drawn from B on AP. Then,

$$AM = AP - MP = r - (r + \delta r) = -\delta r$$

$\therefore$ $\cos\theta = AM/AB = -\delta r/\delta s$ so that $\delta r = -\delta s \cos\theta$

$\therefore$ From (1), $$\phi = m\delta s \cdot \frac{\cos\theta}{r} = \frac{\mu\cos\theta}{r} \qquad \ldots(2)$$

where $\mu = m\delta s$ = strength of the doublet.

From (2), $$\frac{\partial\phi}{\partial r} = -\frac{\mu\cos\theta}{r^2}$$

or $$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = -\frac{\mu\cos\theta}{r^2}, \quad \text{as} \quad \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

or $$\frac{\partial\psi}{\partial\theta} = -\frac{\mu\cos\theta}{r}$$

Integrating it with respect to θ, we get

$$\psi = -\frac{\mu\sin\theta}{r} + f(r) \qquad \ldots(3)$$

Now, $$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r} \qquad \ldots(4)$$

Using (2) and (3), (4) reduces to

$$\frac{1}{r}\left(-\frac{\mu\sin\theta}{r}\right) = -\left[\frac{\mu\sin\theta}{r^2} + f'(r)\right]$$

or $f'(r) = 0$ so that $f(r)$ = constant Hence omitting the additive constant, (3) reduces to

$$\psi = -\frac{\mu\sin\theta}{r} \qquad \ldots(5)$$

Using (2) and (5), the complex potential due to a doublet is given by

$$w = \phi + i\psi = \frac{\mu}{r}(\cos\theta - i\sin\theta) = \frac{\mu}{r}e^{-i\theta} = \frac{\mu}{re^{i\theta}} = \frac{\mu}{z}.$$

Note 1. Equi-potential curves are given by ϕ = constant, *i.e.*, by

$$(\mu\cos\theta)/r = \text{constant} \quad \text{or} \quad (\cos\theta)/r = C$$

$\therefore$ $$r\cos\theta = Cr^2 \quad \text{or} \quad x = C(x^2 + y^2),$$

which represent circles touching the y-axis at the origin.

Note 2. Streamlines are given by ψ = constant *i.e.*, by

$$(-\mu \sin\theta)/r = \text{constant} \qquad \text{or} \qquad (\sin\theta)/r = C'$$

or $$r\sin\theta = C'r^2 \qquad \text{or} \qquad y = C'(x^2+y^2),$$

which represent circles touching the x-axis at the origin.

Note 3. If the doublet makes an angle θ with x-axis, we have to write $\theta - \alpha$ for θ so that

$$w = \frac{\mu}{re^{i(\theta-\alpha)}} = \frac{\mu e^{i\alpha}}{re^{i\theta}} = \frac{\mu e^{i\alpha}}{z}.$$

If the doublet be at the point $A(x', y')$ where $z' = x' + iy'$ [in place of A being origin (0, 0)]

then we have $$w = \frac{\mu e^{i\alpha}}{z - z'}$$

Note 4. If doublets of strengths $\mu_1, \mu_2, \mu_3, \ldots$ are situated at $z = z_1, z_2, z_3, \ldots$ and their axes making angles $\alpha_1, \alpha_2, \alpha_3, \ldots$ with x-axis, then the complex potential due to the above system is given by

$$w = \frac{\mu_1 e^{i\alpha_1}}{z - z_1} + \frac{\mu_2 e^{i\alpha_2}}{z - z_2} + \frac{\mu_3 e^{i\alpha_3}}{z - z_3} + \ldots$$

5.15. Illustrative solved examples.

Ex. 1. *What arrangement of sources and sinks will give rise to the function* $w = \log(z - a^2/z)$. *Draw a rough sketch of the streamlines. Prove that two of the streamlines subdivide into the circle* $r = a$ *and axis of* y. **[Kanpur 2003, 04; Meerut 2001, 03, 10, 11; [Agra 2005; Garhwal 2004; GNDU Amritsar 2003, 05; Rohilkhand 2000, 03, 05]**

Sol. Given $$w = \log\left(z - \frac{a^2}{z}\right) = \log\left[\frac{(z-a)(z+a)}{z}\right]$$

or $$w = \log(z-a) + \log(z+a) - \log z$$

which shows that there are two sinks of unit strength at the points $z = a$ and $z = -a$ and a source of unit strength at origin. Since $w = \phi + i\psi$ and $z = x + iy$, we obtain

$$\phi + i\psi = \log(x + iy - a) + \log(x + iy + a) - \log(x + iy)$$

$\therefore$ $$\phi + i\psi = \log[(x-a) + iy] + \log[(x+a) + iy] - \log(x + iy)$$

Equating imaginary parts on both sides, we have

$$\psi = \tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x}, \text{ as } \log(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}$$

$$= \tan^{-1}\frac{\dfrac{y}{x-a} + \dfrac{y}{x+a}}{1 - \dfrac{y}{x-a}\cdot\dfrac{y}{x+a}} - \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2xy}{x^2-y^2-a^2} - \tan^{-1}\frac{y}{x}$$

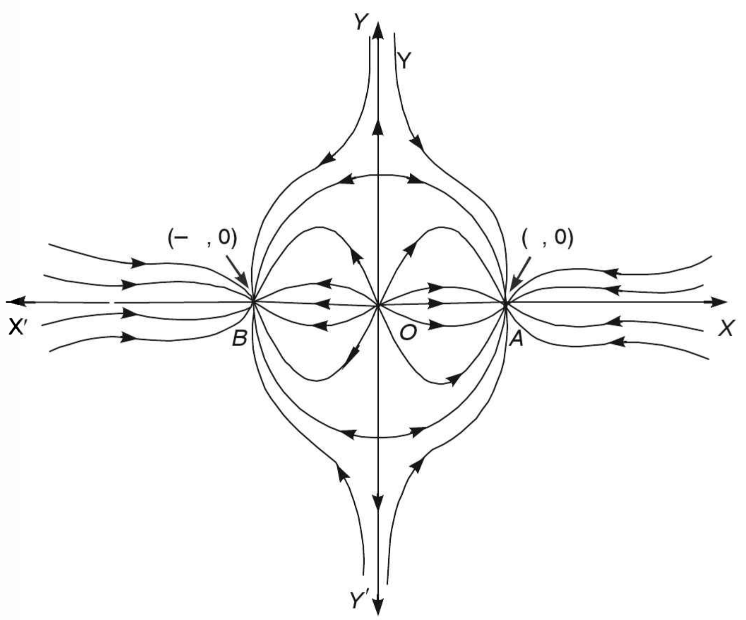
$$= \tan^{-1}\frac{\dfrac{2xy}{x^2-y^2-a^2} - \dfrac{y}{x}}{1 + \dfrac{2xy}{x^2-y^2-a^2}\cdot\dfrac{y}{x}} = \tan^{-1}\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)}.$$

The desired streamlines are given by ψ = constant = $\tan^{-1}(C)$, *i.e.*

$$\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)} = C. \qquad \ldots(1)$$

When $C = 0$, (1) reduces to $y = 0$. Thus x-axis is a streamline. Again, when $C \to \infty$, (1) reduces to $x\,(x^2 + y^2 - a^2) = 0$, *i.e*, $x = 0$ and $x^2 + y^2 = a^2$ or $r = a$, which are streamlines.

Hence the rough sketch of the streamlines is as shown in the following figure. In this figure there is a source of unit strength at origin O and there are two sinks each of unit strength at A $(a, 0)$ and $(-a, 0)$.



Ex. 2. *There is a source of strength m at* (0, 0) *and equal sinks at* (1, 0) *and* (–1, 0). *Discuss two-dimensional motion. Also draw the stream lines.* **[Meerut 2002, 09]**

Sol. Proceed just like Ex. 1. Here, we have

$$w = m \log (z - 1) + m \log (z + 1) - m \log (z - 0)$$

$$\phi + i\psi = m\,[\log (x + iy - 1) + \log (x + iy + 1) - \log (x + iy)]$$

$$\therefore \quad \psi = m\left[\tan^{-1}\frac{y}{x-1} + \tan^{-1}\frac{y}{x+1} - \tan^{-1}\frac{y}{x}\right] \quad \text{or} \quad \frac{\psi}{m} = \tan^{-1}\frac{y(x^2+y^2+1)}{x(x^2+y^2-1)}$$

[As in Ex. 1, here note that $a = 1$]

The desired streamlines are given by ψ/m = constant = $\tan^{-1} C$ *i.e.*

$$\frac{y(x^2+y^2+1)}{x(x^2+y^2-1)} = C. \qquad \text{...(1)}$$

Now give the same discussion and figure as given in Ex. 1 noting that here $a = 1$.

Ex. 3. *Two sources, each of strength m are placed at the points* $(-a, 0)$, $(a, 0)$ *and a sink of strength 2m at the origin. Show that the streamlines are the curves* $(x^2 + y^2)^2 = a^2\,(x^2 - y^2 + \lambda xy)$ *where* λ *is a variable parameter.* **[U.P. P.C.S. 1999; I.A.S. 1999, 2003]**

Show *also that the fluid speed at any point is* $(2ma^2)/(r_1 r_2 r_3)$ *where* r_1, r_2, r_3, *are the distances of the points from the sources and the sink.*

[I.A.S. 1999, 2003; Meerut 2000; Garhwal 2005; Rohilkhand 2002]

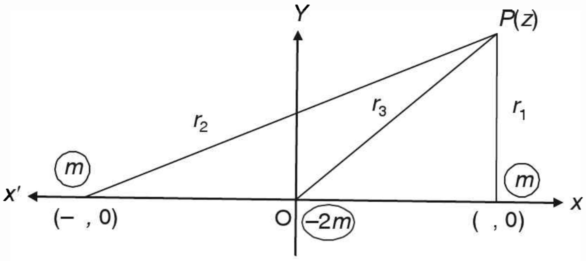
Sol. First Part.

The complex potential w at any point $P(z)$ is given by

$$w = -m \log (z - a) - m \log (z + a) + 2m \log z \qquad \text{...(1)}$$

or $$w = m\,[\log z^2 - \log (z^2 - a^2)]$$

or $$\phi + i\psi = m\,[\log(x^2 - y^2 + 2ixy) - \log(x^2 - y^2 - a^2 + 2ixy)]\,, \text{ as } \quad z = x + iy$$



Equating the imaginary parts, we have

$$\psi = m\left[\tan^{-1}\left\{2xy/(x^2 - y^2)\right\} - \tan^{-1}\left\{2xy/(x^2 - y^2 - a^2)\right\}\right]$$

$\therefore$ $$\psi = m\tan^{-1}\left[\frac{-2a^2xy}{(x^2 + y^2)^2 - a^2(x^2 - y^2)}\right], \text{ on simplification.}$$

The desired streamlines are given by ψ = constant = m $\tan^{-1}$ $(-2/\lambda)$. Then we obtain

$$(-2/\lambda) = (-2a^2xy)/[(x^2 + y^2)^2 - a^2(x^2 - y^2)] \quad \text{or} \quad (x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy).$$

Second Part. From (1), we have

$$\frac{dw}{dz} = -\frac{m}{z-a} - \frac{m}{z+a} + \frac{2m}{z} = -\frac{2a^2m}{z(z-a)\,(z+a)}$$

$\therefore$ $$q = \left|\frac{dw}{dz}\right| = \frac{2a^2m}{|z||z-a||z+a|} = \frac{2a^2m}{r_1r_2r_3}$$

where $\quad r_1 = |z - a|, \quad r_2 = |z + a| \quad$ and $\quad r_3 = |z|.$

Ex. 4. *An area A is bounded by that part of the x-axis for which x > a and by that branch of $x^2 - y^2 = a^2$ which is in the positive quadrant. There is a two-dimensional unit source at (a, 0) which sends out liquid uniformly in all directions. Show by means of the transformation $w = \log (z^2 - a^2)$ that in* steady motion the streamlines of the liquid within the area A are portions *of rectangular hyperbola. Draw the streamlines corresponding to $\psi = 0, \pi/4, \pi/2$. If ρ_1 and ρ_2 are the distances of a point P within the fluid from the points ($\pm$ a, 0), show that the velocity of the fluid at P is measured by $2OP/\rho_1\rho_2$, O being the origin.* **[Grahwal 2001]**

Sol. Given $$w = \log (z^2 - a^2) \qquad ...(1)$$

or $\quad w = \log\,[(x + iy)^2 - a^2] \quad$ or $\quad \phi + i\psi = \log[(x^2 - y^2 - a^2) + 2ixy]$

Equating the imaginary parts, we have

$$\psi = \tan^{-1}\frac{2xy}{x^2 - y^2 - a^2} \qquad ...(2)$$

The streamlines are given by $\quad \psi$ = constant = $\tan^{-1}C$, $\quad$ *i.e.*,

$$(2xy)\,/\,(x^2 - y^2 - a^2) = C \qquad ...(3)$$

When $C = 0$, stream lines (3) reduce to $xy = 0$ *i.e.*, $x = 0$, and $y = 0$. Again, when $C \to \infty$, (3) reduces to $x^2 - y^2 - a^2 = 0$, *i.e.* $x^2 - y^2 = a^2$.

Hence the liquid flows in the area A bounded by $x = 0$, $y = 0$ and $x^2 - y^2 = a^2$ in the positive quadrant.

From (1), $w = \log(z - a) + \log(z + a)$,

which shows that there is a source of unit strength at $(a, 0)$ and an equal source at $(-a, 0)$. Here the source at $(-a, 0)$ is the image of $(a, 0)$ with respect to y-axis.

From (1), $$\frac{dw}{dz} = \frac{2z}{z^2 - a^2} = \frac{2z}{(z-a)(z+a)}$$

$\therefore$ $$q = \left|\frac{dw}{dz}\right| = \frac{2|z|}{|z-a||z+a|} = \frac{2OP}{\rho_1 \rho_2}$$

From (2), the streamline corresponding to $\psi = 0$ is

$$\frac{2xy}{x^2 + y^2 - a^2} = 0 \qquad \text{giving} \qquad x = 0 \qquad \text{and} \qquad y = 0.$$

From (2), the streamline corresponding to $\psi = \pi/4$ is

$$\frac{2xy}{x^2 - y^2 - a^2} = \tan\frac{\pi}{4} = 1 \qquad \text{or} \qquad x^2 - y^2 - a^2 = 2xy.$$

From (2), the streamline corresponding to $\psi = \pi/2$ is

$$\frac{2xy}{x^2 - y^2 - a^2} = \tan\frac{\pi}{2} = \infty \qquad \text{or} \qquad x^2 - y^2 = a^2.$$

Ex. 5. *Find the stream function of the two-dimencial motion due to two equal sources and an equal sink situated midway between them.* **[Kanpur 2008; I.A.S. 1996]**

Sol. Let there be two sources of strength m at the points $z = a$ and $z = -a$ and a sink at of same strength at $z = 0$ (origin). Then complex potential w due to these sources and sink is given by

$$w = -m\log(z - a) - m\log(z + a) + m\log(z - 0)$$

or $$\phi + i\psi = m\log(x + iy) - m\log(x + iy - a) - m\log(x + iy + a)$$

or $$\phi + i\psi = m\log(x + iy) - m\log\{(x - a) + iy\} - m\log\{(x + a) + iy\}$$

or $$\phi + i\psi = m\{(1/2)\times\log(x^2 + y^2) + i\tan^{-1}(y/x)\} - m[(1/2)\times\log\{(x-a)^2 + y^2\}$$
$$+ i\tan^{-1}\{y/(x-a)\}] - m[(1/2)\times\log\{(x+a)^2 + y^2\} + i\tan^{-1}\{y/(x+a)\}]$$

Equating imaginars parts on both sides, we get

$$\psi = m\tan^{-1}(y/x) - m[\tan^{-1}\{y/(x-a)\} + \tan^{-1}\{y/(x+a)\}]$$

or $$\frac{\psi}{m} = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{\{y/(x-a)\} + \{y/(x+a)\}}{1 - \{y/(x-a)\}\{y/(x+a)\}} = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{2xy}{x^2 - y^2 - a^2}$$

or $$\frac{\psi}{m} = \tan^{-1}\frac{(y/x) - \{2xy/(x^2 - y^2 - a^2)\}}{1 + (y/x)\{2xy/(x^2 - y^2 - a^2)\}} \qquad \text{or} \qquad \psi = m\tan^{-1}\frac{y(x^2 + y^2 + a^2)}{x(a^2 - x^2 - y^2)}$$

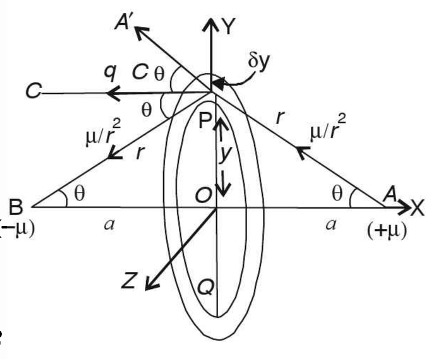
Ex. 6. *An infinite mass of liquid is moving irrotationally and steadily under the influence of a source of strength* μ *and an equal sink at a distance* $2a$ *from it. Prove that the kinetic energy of the liquid which passes in unit time across the plane which bisects at right angles the line joining the source and sink is* $(8\pi\rho\mu^3)/7a^4$, ρ *being the density of the liquid.*

Sol. Let a aource of strength μ and a sink of strength $-\mu$ be situated at A and B such that $AB = 2a$. Let O be the middle point of AB so that $OA = OB = a$. Let OYZ be the plane which bisects AB at right angles. Hence $\angle POA = \angle POB = 90°$. Let $\angle PAB = \angle PBA = \theta$. Let PC be parallel to AB such that $\angle A'PC = \angle BPC = \theta$. Also let $AP = BP = r$. From ΔPAO, we have

$$\cos\theta = a/r \qquad \text{and} \qquad y = a\tan\theta, \qquad \text{where} \qquad OP = y. \qquad \text{Also,} \qquad r = (a^2 + y^2)^{1/2}.$$

Consider an annular strip bounded by circles of radii y and $y+\delta y$. It δS be the area of this strip, then

$$\delta S = 2\pi y\,\delta y. \qquad \ldots(1)$$



At any point inside the circular ring all the fluid particles have the same velocity q in the same direction, namely normal to the plane.

Now, velocity at P due to source $+\mu$ at A $= \mu/r^2$ along AP

and velocity at P due to sink $-\mu$ at $B = \mu/r^2$ along PB

$\therefore$ q = the resultant of the above velocites along PC = $\dfrac{2\mu}{r^2}\cos\theta = \dfrac{2\mu a}{r^3} = \dfrac{2\mu a}{(a^2+y^2)^{3/2}}$.

In unit time the mass δm of the liquid crossing the strip is given by

$$\delta m = \rho(\delta S)q = \rho(2\pi y\delta y)q, \text{ by (1)} \qquad \ldots(2)$$

Hence the required K.E. of the liquid which passes across the plane OYZ in unit time

$$= \int \frac{1}{2}\delta m q^2 = \int_0^\infty \frac{1}{2} q^2 (2\pi\rho\, y\, q\, dy) = \pi\rho\int_0^\infty q^3 y\, dy, \text{ by (2)}$$

$$= \pi\rho\int_0^\infty \left[\frac{2\mu a}{(a^2+y^2)^{3/2}}\right]^3 y\, dy = 8\pi\rho\mu^3 a^3 \int_0^\infty \frac{y\, dy}{(a^2+y^2)^{9/2}}.$$

$$= 8\pi\rho\mu^3 a^3 \int_0^{\pi/2} \frac{a\tan\theta\cdot a\sec^2\theta\, d\theta}{a^9\sec^9\theta}, \text{ putting } y = a\tan\theta \text{ and } dy = a\sec^2\theta\, d\theta$$

$$= \frac{8\pi\rho\mu^3}{a^4}\int_0^{\pi/2}\cos^6\theta\sin\theta\, d\theta = \frac{8\pi\rho\mu^3}{a^4}\left[-\frac{\cos^7\theta}{7}\right]_0^{\pi/2} = \frac{8\pi\rho\mu^3}{7a^4}.$$

Ex. 7. In a two dimensinal liquid motion ϕ and ψ are the velocitly and current functions, show that a second fluid motion exists in which ψ is the velocity potential and $-\phi$ the current function; and prove that if the first motion be due to sources and sinks, the second motion can be built up by replacing a source and an equal sink by a line of doublets uniformly distributed along any curve joining them

Sol. Since ϕ and ψ are the velocity potential and stream function respectively for the two-dimensional motion, we have

$$\partial\phi/\partial x = \partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y = -(\partial\psi/\partial x) \qquad \ldots(1)$$

Again if ψ and $-\phi$ be the velocity potential and strem function respectively for another fluid motion in two-dimensions, then the conditions of the type (1) must be satisfied by ψ and $-\phi$ *i.e.*, we must have

$$\frac{\partial\psi}{\partial x} = \frac{\partial(-\phi)}{\partial y} \qquad \text{and} \qquad \frac{\partial\psi}{\partial y} = -\frac{\partial(-\phi)}{\partial x}$$

i.e., $\qquad \partial\phi/\partial y = -(\partial\psi/\partial x) \qquad$ and $\qquad \partial\phi/\partial x = \partial\psi/\partial y$

which is true by virtue of (1).

If follows that if $w = \phi + i\psi$ exists, then $w' = \psi - i\phi = -i(\phi + i\psi) = -iw$, also exists.

Second part. Consider a source of strength m at $A\,(a, 0)$ and a sink of strength $-m$ at $B\,(-a, 0)$. Then, the complex potential function w due to them is given by

$$w = -m \log (z - a) + m \log (z + a) = m \log \{(z + a)/(z - a)\} \qquad \ldots(2)$$

Join A, B by an arbitrary curve. Then the axis of the doublet on this curve is normal to AB. If w'' be the complex potential due this line of doublets then

$$w'' = \int_A^B \frac{me^{i\pi/2}}{z-t} dt = m e^{i\pi/2} \log \frac{z-a}{z+a} = mi \log \frac{z-a}{z+a} = -iw$$

The required result now follows from the first part.

EXERCISE 5 (B)

1. Find the cartesian equation of the lines of plane flow when fluid is streaming from three equal sources situated at the corners of an equilateral triangle.

2. Let there be a source of strength m at $(a, 0)$ and a sink $-m$ at $(-a, 0)$. Find ϕ, ψ, w and velocity q.

3. Let there be a source of strength m at $(a, 0)$ and a sink $-m$ at $(0, a)$. Find ϕ, ψ, w and velocity q.

4. If there are sources at $(a, 0)$ and $(-a, 0)$ and sinks at $(0, a)$, $(0, -a)$ all of equal strengths, show that the circle through these four points is a streamline. **[I.A.S. 1990]**

5. A source of strength m at A $(a, 0)$ and a sink of strength $-m$ at B $(-a, 0)$ are in the xy plane and in the presence of a uniform stream U-parallel to the x-axis. The stream is directed from the source to the sink. Derive the stream function of the resulting motion.

6. A source and a sink of the same strength are placed at a given distance apart in an infinite fluid which is otherwise at rest. Show that the streamlines are circles and that the fluid speed along any streamline varies inversely as the distance from the line joining the source and sink.

7. Define sources and sinks and explain their utility in hydrodynamics. **[Kanpur 2002]**

8. There is a source at A and an equal sink at B. AB is the direction of a uniform stream. If A is $(a, 0)$, B is $(-a, 0)$ and the ratio of the flow issuing from A in unit time to the speed of the stream is $2\pi b$, show the stream function is $\psi = Vy - Vb \tan^{-1}[2ay/(x^2 + y^2 - a^2)]$ and that the length $2l$, and the breadth $2d$, of the closed wall that forms part of the dividing streamline is given by $l^2 = a^2 + 2ab$, $\quad \tan (d/b) = 2ad/(d^2 - a^2)$ and the locus of the points at which the speed is equal to that of the stream is $x^2 - y^2 = a^2 + ab$.

9. Sources of equal strength are placed at the points $z = nia$ where $n = \ldots, -2, -1, 0, 1, 2, \ldots$. Prove that the complex potential is $w = -m \log \sinh (\pi z/a)$. Hence show that the complex potential for doublets, parallel to x-axis of strength μ at the same points is given by $w = \mu \coth (\pi z/a)$.

If the row of doublets is placed in a uniform stream $-$ U parallel to x-axis, prove that the streamline $\psi = 0$ is

$$\frac{ay}{\pi b^2} = \frac{\sin (2\pi y/a)}{\cosh (2\pi x/a) - \cos (2\pi y/a)}$$

and show that this consists of part of the x-axis and part of an oval curve which is nearly circular (diameter $2b$) if $b << a$.

5.16. Images.

If in a liquid a surface S can be drawn across which there is no flow, then any system of sources, sinks and doublets on opposite sides of this surface is known as the image of the system with regard to the surface. Moreover, if the surface S is treated as a rigid boundary and the liquid removed from one side of it, the motion on the other side will remain unchanged.

As there is no flow across the surface, it must be a streamline. Thus the fluid flows tangentially to the surface and hence the normal velocity of the fluid at any point of the surface is zero.

Images in two dimensions.

If in a liquid a curve C can be drawn across which there is no flow, then any system of sources, sinks and doublets on opposite sides of this curve is known as the image of the system with regard to the curve.

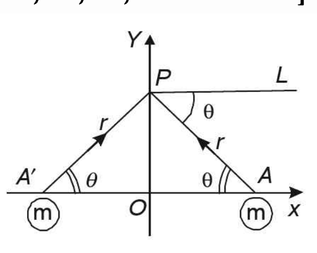
5.17. Advantages of images in fluid dynamics. [Kanpur 2002]

The method of images is used to determine the complex potential due to sources, sinks and doublets in the presence of rigid boundaries. Suppose we wish to determine the flow field outside a rigid boundary due to sources, sinks, doublets lying outside the boundary. To this end we assume the existence of some hypothetical image sources, sinks, doublets within the boundary in such a manner so that the boundary behaves as a streamline or surface. Then the given system of sources, sinks and doublets together with the hypothetical one will be equivalent to the given sources and the rigid boundaries for the region outside the rigid boundary.

5.18. Image of a source with respect to a line.

[Agra 2006; Kanpur 2003, 04, 07, 08; Meerut 2003]

Suppose that image of the source m at A $(a, 0)$ on x-axis is required with respect to OY. Take an equal source at A' $(-a, 0)$. Let P be any point on OY such that $AP = A'P = r$. Then the velocity at P due to source at A is m/r along AP and velocity at P due to source A' is m/r along $A'P$. Let PL be perpendicular to OY. Then, we see that



Resultant velocity at P due to sources at A and A' along PL

$$= (m/r)\cos\theta - (m/r)\cos\theta = 0,$$

showing that there will be no flow across OY. Hence by definition, the image of a simple source with respect to a line in two-dimensions is an equal source equidistant from the line opposite to the source.

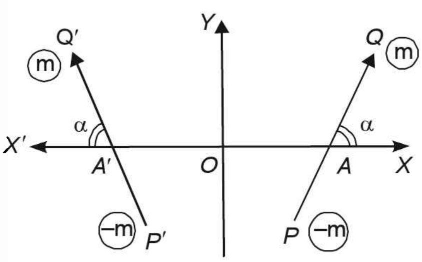
Remark 1. Proceeding as above we can prove that the image of a sink with respect to a line in two-dimensions is an equal sink equidistant from the line opposite to the sink.

Remark 2. The result of Art. 5.18 will still hold good if a line is replaced by a plane.

Corollary. Image of a doublet with respect to a line.

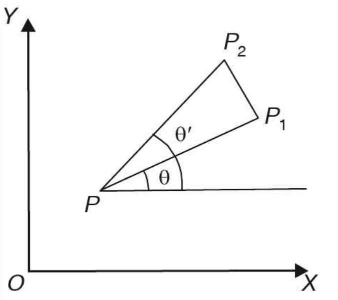
[Kanpur 2002, Rohilkhand 2003]

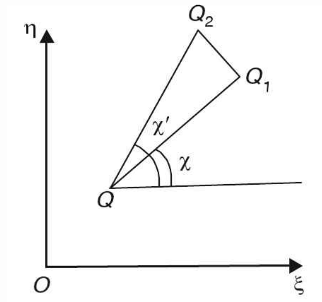
Let PQ be a doublet with its axis inclined at an angle α to OX. Then by using the above result for finding the images of source and sink with respect to OY, we see that the image of the doublet PQ is again an equal doublet $P'Q'$ symmetrically placed as shown in the adjoining figure.



5.19A. Conformal representation (or transformation or mapping.)

Let $f(z)$ be a function of the complex variable $z = x + iy$ and let $f(z)$ be single-valued and differentiable within a closed contour C in the z-plane (*i.e.* xy- plane). Let $\zeta = \xi + i\eta$ be another complex variable in ζ-plane (*i.e* $\xi\eta$-plane) and let there be a relation $\zeta = f(z)$. Then corresponding to each point in the z-plane within or on C, there will be a point ζ in the ζ-plane and points on C or within C will lie on or within a certain contour C' in the ζ-plane. The necessary condition for existence of such a mapping of z-plane into ζ-plane is that $f'(z)$ should never vanish at any point on or within C, or in other words, $d\zeta/dz$ must exist independent of the directions of δz. Thus, let P, P_1, P_2 be neighbouring points z, z_1, z_2 and Q, Q_1, Q_2 the corresponding points ζ, ζ_1, ζ_2. Then, we have





$$\frac{\zeta_1-\zeta}{z_1-z}=\frac{f(z_1)-f(z)}{z_1-z}, \qquad \text{and} \qquad \frac{\zeta_2-\zeta}{z_2-z}=\frac{f(z_2)-f(z)}{z_2-z}$$

In the limit when P_1, P_2 approach P, we have

$$\frac{\zeta_1-\zeta}{z_1-z}=f'(z), \qquad \text{and} \qquad \frac{\zeta_2-\zeta}{z_2-z}=f'(z), \text{ very nearly}$$

$$\therefore \qquad \frac{\zeta_1-\zeta}{z_1-z}=\frac{\zeta_2-\zeta}{z_2-z}=f'(z)=\frac{d\zeta}{dz} \qquad \ldots(1)$$

or
$$\frac{QQ_1e^{i\chi}}{PP_1e^{i\theta}}=\frac{QQ_2e^{i\chi'}}{PP_2e^{i\theta'}} \qquad \text{or} \qquad \frac{QQ_1}{PP_1}e^{i(\chi-\theta)}=\frac{QQ_2}{PP_2}e^{i(\chi'-\theta')}$$

$$\therefore\ \chi-\theta=\chi'-\theta' \qquad \text{or} \qquad \chi'-\chi=\theta'-\theta \qquad i.e. \qquad \angle Q_1QQ_2=\angle P_1PP_2$$

and
$$\frac{QQ_1}{PP_1}=\frac{QQ_2}{PP_2}=|f'(z)|=\left|\frac{d\zeta}{dz}\right|$$

Hence the triangles P_1PP_2 and Q_1QQ_2 are similar. This establishes the similarity of the correspoding infitiesimal elements of the two planes. Such a relation between the two planes is called the *conformal represenation* of either plane on the other.

Again
$$\frac{\Delta\, Q_1QQ_2}{\Delta P_1\, PP_2}=\frac{(1/2)\times QQ_1\times QQ_2\,\sin\angle Q_1OQ_2}{(1/2)\times PP_1\times PP_2\,\sin\angle P_1OP_2}=\frac{QQ_1}{PP_1}\times\frac{QQ_2}{PP_2}=|f'(z)|^2 \qquad \ldots(2)$$

From $\qquad \zeta=f(z) \qquad i.e., \qquad \xi+i\eta=f(x+iy), \qquad$ we have

$$\frac{\delta\xi}{\delta z}=\frac{\delta(\xi+i\eta)}{\delta(x+iy)}=\frac{\frac{\partial\xi}{\partial x}\delta x+\frac{\partial\xi}{\partial y}\delta y+i\left(\frac{\partial\eta}{\partial x}\delta x+\frac{\partial\eta}{\partial y}\delta y\right)}{\delta x+i\delta y}=\frac{\left(\frac{\partial\xi}{\partial x}+i\frac{\partial\eta}{\partial x}\right)\delta x+\left(\frac{\partial\xi}{\partial y}+i\frac{\partial\eta}{\partial y}\right)\delta y}{\delta x+i\delta y}$$

Since $\delta\xi/\delta z$ is independent of $\delta x/\delta y$, we must have

$$\frac{\partial\xi}{\partial y}+i\frac{\partial\eta}{\partial y}=i\left(\frac{\partial\xi}{\partial x}+i\frac{\partial\eta}{\partial x}\right)$$

so that
$$\frac{\partial\xi}{\partial x}=\frac{\partial\eta}{\partial y} \qquad \text{and} \qquad \frac{\partial\xi}{\partial y}=-\frac{\partial\eta}{\partial x} \qquad \ldots(3)$$

Also
$$\frac{d\zeta}{dz}=\frac{\partial\xi}{\partial y}+i\frac{\partial\eta}{\partial y} \qquad \ldots(4)$$

$$\therefore \qquad |f'(z)|^2=\left|\frac{d\xi}{dz}\right|^2=\left(\frac{\partial\xi}{\partial y}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2, \text{ using (1) and (4)}$$

or $$| f'(z)|^2 = \left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 = h^2, \text{ (say), using (3)} \qquad \ldots(5)$$

$\therefore$ From (2), $\quad \Delta Q_1 Q Q_2 / \Delta P_1 P P_2 = h^2 \quad$ so that $\quad d\xi d\eta = h^2 dxdy,$

showing that the corresponding areas in the ζ and z planes are in the ratio $h^2 : 1$.

Let ϕ and ψ be the velocity and current functions of any motion within the contour C' in ζ-plane. Then, within the contour C', we have

$$\phi + i\psi = F_1(\xi + i\eta) \qquad \ldots(6)$$

and C' is given by $$\psi = f_1(\xi, \eta) = \text{const.} \qquad \ldots(7)$$

Substituting the values of ξ, η in terms of x, y, (6) and (7) respectively reduce to

$$\phi + i\psi = F_2(x + iy) \qquad \ldots(8)$$

and $$\psi = f_2(x, y) = \text{constant}, \qquad \ldots(9)$$

where $f_2(x, y)$ is the new value of $f_1(\xi, \eta)$. Thus we fund that ϕ and ψ are the same in the two cases. In other words, $w = \phi + i\psi$ is the same in both the motions so that if q_1 and q_2 be velocities at P and Q respectively, then

$$q_1 = | dw / dz |^2 \qquad \text{and} \qquad q_2^2 = | dw / d\zeta |^2$$

so that $$q_2^2 = q_1^2 \, | dz / d\zeta |^2 = q_1^2 / h^2, \text{ using (5)}$$

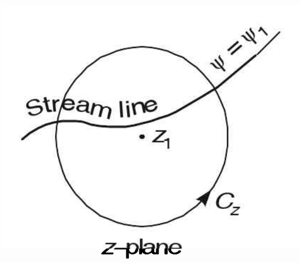
$\therefore$ $$q_2^2 \, d\xi d\eta = (q_1^2 / h^2) \times h^2 \, dx\, dy = q_1^2 dx\, dy,$$

so that $$\frac{1}{2}\int \rho\, q_2^2 \, d\xi d\eta = \frac{1}{2}\int \rho\, q_1^2 dx\, dy,$$

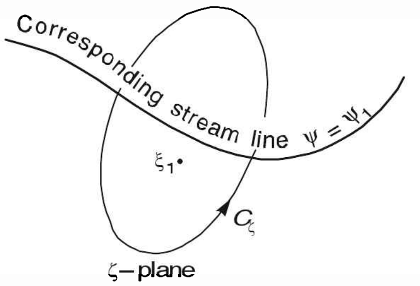
showing that the kinetic energies of the two fields are equal.

5.19B. Two important transformation.

(*i*) **Transformation of a source.**



z-plane



ζ-plane

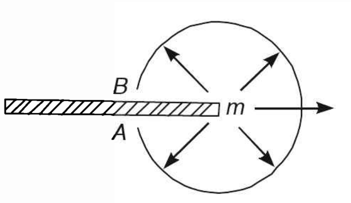
Let there be a source of strength m at z_1 and ζ_1 be the corresponding point in the ζ-plane. Let these be regular points of the transformation. Then a small closed curve C_z may be drawn to enclose z_1 which will transform into a small closed curve C_ζ enclosing ζ_1. Since the value of the stream function is independent of the domain considered, we obtain

$$\int_{C_z} d\psi = \int_{C_\zeta} d\psi, \qquad \ldots(1)$$

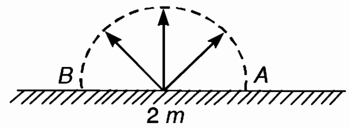
But $$\int_{C_z} d\psi = \int_{C_z} (udy - \; dx) = \text{total flow across the contour } C_z$$

$$= \text{sum of source strength within } C_z$$

Let C_z be chosen sufficiently small so as to isolate the single source of strength m at z_1. It follows from (1) that this will transform into a source at ζ_1 of equal strength. Thus, a source will always transform into a source but the strengths will be equal only if the position of the source is a regular point of the transformation *i.e.* when it is possible to draw a closed contour surrounding the point. However care has to be taken at a zero, infinity or branch point of the function that ζ is of z or that z is of ζ. For example, in the case $\zeta = z^{1/2}$, since a semi circle with centre $\zeta = 0$ transforms into a circle with centre $z = 0$ (as arg $\zeta = (1/2) \times \arg z$). Hence if there be a source of strength m at $z = 0$ the corresponding source at $\zeta = 0$ must be of strength $2m$ as the mass efflux is unchanged by the transformation. (see the following figure).

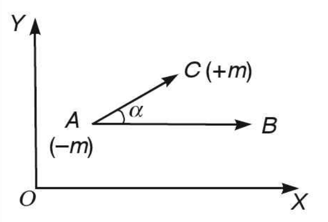


Fluid produced = m x 2π
z - plane

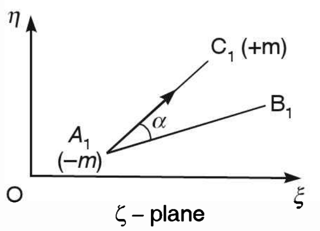


fluid produced = 2 m x π
ζ - plane

(*ii*) **Transformation of a Doublet.**



Z–plane



ζ – plane

Let there be doublet of strength μ at A. Then by above case (*i*), it follows that there will be doublet of strength μ' at the corresponding point A_1 in the ζ-plane.

Also, $\qquad A_1C_1 = h\,AC, \qquad \mu = m \cdot AC, \qquad \mu' = m \cdot A_1C_1$

$\therefore \qquad \dfrac{\mu'}{\mu} = \dfrac{A_1C_1}{AC} = h, \qquad$ *i.e.*, $\qquad \mu' = \mu h$

If the axis AC of the doublet at A makes an angle α with a given direction, the axis A_1C_1 of doublet at A_1 will make angle α with the corresponding direction in ζ-plane.

5.19C. Same theorems concerning conformal transformation of line distribution

[Bangalore 2005, Kurukshetra 1997]

Theorem 1. *Under conformal transformation a uniform line source maps into another uniform line source of the same strength.* **[Bangalore 2005; K.U. Kurukshetra 1997]**

Proof. Let there be a uniform line source of strength m per unit length through the point $z = z_0$ and suppose the conformal transformation $\zeta = f(z)$ is made from the z-plane to the ζ-plane so that the point $z = z_0$ maps into the point $\zeta = \zeta_0$. Let C_{z_0} be a closed curve in the z-plane containing the point $z = z_0$ and C_{z_0} maps into C_{ζ_0} is the ζ-plane. Then $\zeta = \zeta_0$ lies within C_{ζ_0}. The complex potential w is clearly the same for both the systems and has the forms

$$\left.\begin{aligned} w &= \phi + i\psi, \text{ for the } z\text{-plane} \\ &= \phi' + i\psi', \text{ for the } \zeta\text{-plane.} \end{aligned}\right\} \qquad \ldots(1)$$

From (1), $\phi = \phi'$ and $\psi = \psi'$. Since ψ is the same at corresponding points of C_{z_0} and C_{ς_0} we have,

$$\oint_{C_{z_0}} d\psi = \oint_{C_{\varsigma_0}} d\psi' \qquad \ldots(2)$$

But in the z-plane, $w = -m \log(z - z_0)$ and $dw = -m dz/(z - z_0)$. Then, using Cauchy-Residue theorem, we have

$$\therefore \qquad \oint_{C_{z_0}} dw = -m \oint_{C_{z_0}} \frac{dz}{z - z_0} = -m \times (2\pi i), \qquad \ldots(3)$$

since the integrand has a residue of 1 at $z = z_0$. Also, $w = \phi + i\psi \;\Rightarrow\; dw = d\phi + i\,d\psi$. So, (3) reduces to

$$\oint_{C_{z_0}} (d\phi + i d\psi) = -m \times (2\pi i) \qquad \Rightarrow \qquad \oint_{C_{z_0}} d\psi = -2\pi m \qquad \ldots(4)$$

The numerical value of this is clearly the volume of fluid crossing unit thickness of C_{z_0} per unit time. Thus, (2) and (4) show that the same volume crosses unit thickness of C_{ζ_0} per unit time which implies an equal line source of strength m per unit length at $\zeta = \zeta_0$.

Theorem II. *Under conformal transformation a uniform line vortex maps into another uniform line vertex of the same strength.* **[Nagpur 2003, 05]**

Proof. Let there be a uniform line vortex of strength k per unit length through $z = z_0$. Also assume that the conformed transformation $\zeta = f(z)$ is made from the z-plane to the ζ-plane so that the point $z = z_0$ maps into $\zeta = \zeta_0$. Let C_{z_0} be a closed curve containing $z = z_0$ and let C_{ζ_0} be its map in the ζ-plane. Then C_{ζ_0} contains $\zeta = \zeta_0$. The complex potential w is clearly the same for both the systems and has the forms

$$\left.\begin{aligned} w &= \phi + i\psi, \text{ for the z-plane} \\ &= \phi' + i\psi', \text{ for the } \varsigma\text{-plane.} \end{aligned}\right\} \qquad \ldots(1)$$

From (1), $\phi = \phi'$ and $\psi = \psi'$. Since ψ is the same at corresponding points of C_{z_0} and C_{ζ_0}, we have

$$\oint_{C_{z_0}} d\psi = \oint_{C_{\zeta_0}} d\psi' \qquad \ldots(2)$$

Now, in the z-plane, $w = \dfrac{ik}{2\pi} \log(z - z_o)$ so that $dw = \dfrac{ik}{2\pi(z - z_o)} dz$

[Using result (14) of Art. 11.4 of chapter 11]

$$\therefore \qquad \oint_{C_{z_0}} dw = \frac{ik}{2\pi} \oint_{C_{z_0}} \frac{dz}{z - z_0} = \frac{ik}{2\pi} \times 2\pi i = -k, \qquad \ldots(3)$$

since the integrand has a residue of 1 at $z = z_0$.

Also, $w = \phi + i\psi \;\Rightarrow\; dw = d\phi + i\,d\psi$. Hence, (3) reduces to

$$\oint_{C_{z_0}} (d\phi + i\,d\psi) = -k \qquad \Rightarrow \qquad -\oint_{C_{z_0}} d\phi = k \qquad \ldots(4)$$

The integral on the L.H.S. of (4) is the circulation round C_{z_0}. Equations (2) and (4) show that the circulation round C_{ζ_0} is also k. Since C_{z_0} and C_{ζ_0} are arbitrary, it follows that the line source through $z = z_0$ of strength k per unit length maps into an equal line source thorugh $\zeta = \zeta_0$.

Theorem III. *Under conformal transformation a uniform line doublet maps into another uniform line doublet of different strength.* **[K.U.Kurukshetra 2003]**

Proof. Let there be a uniform doublet of strength μ per unit length through P where $z = z_0$. Also assume that under conformal transformation $\zeta = f(z)$, P maps into Q where $\zeta = \zeta_0$.

Let the doublet be replaced by equivalent line sources of strengths $-m$, $+m$ per unit length through P, P' where $\overline{PP'} = \delta z$, $\mu = m|\delta z|$ and $\overline{PP'}$ is in the direction of the axis of the line doublet. Suppose P' maps into Q'. Then by the theorem I, the line sources of strengths $-m$, $+m$ per unit length through P, P' map into ones of strengths $-m$, $+m$ per unit length through Q, Q'. If $\overline{QQ'} = \delta\zeta$, then $\delta\zeta = f'(z)\delta z$, so that $|\delta\zeta| = |f'(z)|\cdot|\delta z|$ and $\arg\,\delta\zeta = \arg f'(z) + \arg \delta z$, showing that the two line sources through Q, Q' give a line doublet at Q of strength μ' where $\mu' = m|\delta\zeta| = \mu|f'(z)|$. The inclination of the axis of the line doublet to the real axis is increased by $\arg f'(z)$.

5.19D. Summery of important results regarding applications of conformal transformations in fluid dynamics [Kanpur 2000]

(*i*) In a conformal transformation a source is transformed into an equal source, a sink into an equal sink and a doublet into an equal doublet.

(*ii*) The complex potential $w = \phi + i\psi$ is invariant under a conformal transformation.

(*iii*) Let $\zeta = f(z)$ be the conformal transformal transformation. Then

total K.E. of fluid in z-plane (per unit depth) = Total K.E. of the liquid in ζ-plane (per unit depth)

(*iv*) Under a conformal transformation, a stream line in z-plane is transtormed into, a stream line in ζ-plane.

(*v*) While using conformal transformation $\zeta = z^n$, n is found by dividing $\pi/2$ by half the angle contained between the rigid boundaries.

5.20. Illustrative solved examples.

Ex. 1. *Between the fixed boundaries $\theta = \pi/6$ and $\theta = -\pi/6$ there is a two-dimensional liquid motion due to a source at the point $(r = c, \theta = \alpha)$ and a sink at the origin absorbing water at the same rate as the source produces. Find the stream function and show that one of the stream lines is a part of the curve $r^3 \sin 3\alpha = c^3 \sin 3\theta$.*

[Kanpur 2000, 07; Meerut 2002 Garhwal 2003, 04; Rohilkhand 2003, 04]

Sol. Consider the following conformal transformation from z-plane (xy-plane) to ζ-plane ($\xi\eta$-plane) :

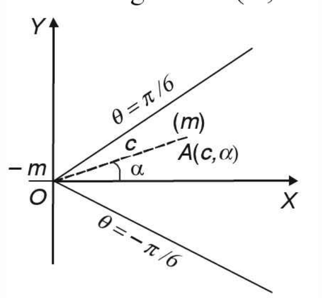
$$*\zeta = z^3 \qquad \text{where} \qquad z = re^{i\theta} \qquad \text{and} \qquad \zeta = Re^{i\Theta}$$

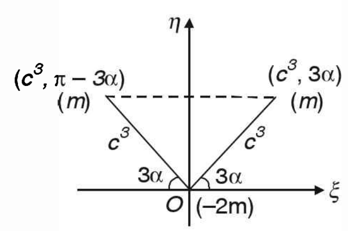
$$\text{This} \quad \Rightarrow \quad Re^{i\Theta} = r^3 e^{3i\theta} \quad \Rightarrow \quad R = r^3 \quad \text{and} \quad \Theta = 3\theta.$$

Hence the boundaries $\theta = \pm\pi/6$ in z-plane transform to $\Theta = \pm\pi/2$, *i.e.*, imaginary axis in ζ-plane. The point (c, α) in z-plane transforms to $(c^3, 3\alpha)$ in ζ-plane. Hence the image system with respect to imaginary axis $(\Theta = \pm\pi/2)$ in ζ-plane consists of

(*i*) a source of strength m at $(c^3, 3\alpha)$, (*ii*) a sink of strength $-m$ at $(0, 0)$,

(*iii*) a source of strength m at $(c^3, \pi - 3\alpha)$ (*iv*) a sink of strength $-m$ at $(0, 0)$





* Refer result (*v*) of Art. 5.19D. Take $\zeta = z^n$. Here, half the angle contained by boundaries = $\pi/6$. Therefore, $n = (\pi/2)/(\pi/6) = 3$ and hence we take $\zeta = z^3$.

$\therefore \quad w = -\log(\zeta - c^3 e^{3i\alpha}) - m\log\{\zeta - c^3 e^{i(\pi - 3\alpha)}\} + 2m\log(\zeta - 0)$

$= -\log(z^3 - c^3 e^{3i\alpha}) - m\log(z^3 + c^3 e^{-3i\alpha}) + 2m\log z^3$

$[\because \ \zeta = z^3 \text{ and } e^{i\pi} = \cos\pi + i\sin\pi = -1]$

$= -m\log[(z^3 - c^3 e^{3i\alpha})(z^3 + c^3 e^{-3i\alpha})] + 6m\log z = -m\log(z^6 - c^6 - 2ic^3 z^3 \sin 3\alpha) + 6m\log z$

$= -m\log\left(\dfrac{z^6 - c^6 - 2ic^3 z^3 \sin 3\alpha}{z^6}\right) = -m\log(1 - c^6 z^{-6} - 2ic^3 z^{-3}\sin 3\alpha)$

$= -m\log(1 - c^6 r^{-6} e^{-6i\theta} - 2ic^3 r^{-3} e^{-3i\theta}\sin 3\alpha)$

$\therefore \quad w = -m\log[1 - c^6 r^{-6}\cos 6\theta - 2c^3 r^{-3}\sin 3\alpha \sin 3\theta + i(c^6 r^{-6}\sin 6\theta - 2c^3 r^{-3}\sin 3\alpha\cos 3\theta)]$

Writing $w = \phi + i\psi$ and equating imaginary parts, we get*

$$\psi = -m\tan^{-1}\frac{c^6 r^{-6}\sin 6\theta - 2c^3 r^{-3}\sin 3\alpha\cos 3\theta}{1 - c^6 r^{-6}\cos 6\theta - 2c^3 r^{-3}\sin 3\theta\sin 3\theta}, \qquad \ldots(1)$$

which is the required stream function. The stream lines are given by ψ = constant. The stream line corresponding to $\psi = 0$ is given by [putting $\psi = 0$ in (1) and noting that $\tan^{-1} = 0$]

$$c^6 r^{-6}\sin 6\theta - 2c^3 r^{-3}\sin 3\alpha\cos 3\theta = 0 \quad \text{or} \quad c^3\sin 6\theta = 2r^3\sin 3\alpha\cos 3\theta$$

or $\quad 2c^3\sin 3\theta\cos 3\theta = 2r^3\sin 3\alpha\cos 3\theta \quad$ or $\quad c^3\sin 3\theta = r^3\sin 3\alpha.$

Ex.2. *Between two fixed boundaries $\theta = \pi/4$ and $\theta = -\pi/4$, there is two-dimensional liquid motion due to a source of strength m at the point ($r = a$, $\theta = 0$) and an equal sink at the point ($r = b$, $\theta = 0$). Show that the stream function is*

$$-m\tan^{-1}\frac{r^4(a^4 - b^4)\sin 4\theta}{r^8 - r^4(a^4 + b^4)\cos 4\theta + a^4 b^4}$$

[I.A.S. 1998; Garhwal 1996; Meerut 2006; Rohilkhand 2000; U.P.P.C.S 2000; Kanpur 1999]

and show that the velocity at (r, θ) is

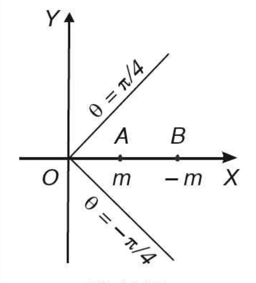
$$\frac{4m(a^4 - b^4)r^3}{(r^8 - 2a^4 r^4\cos 4\theta + a^8)^{1/2}(r^8 - 2b^4 r^4\cos 4\theta + b^8)^{1/2}}$$

[I.A.S. 1991, 94]

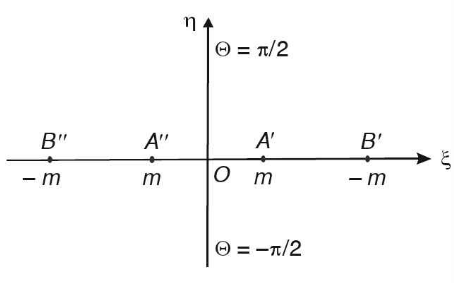
Sol. Consider the following conformal transformation from z-plane (*xy*-plane) to ζ-plane ($\xi\eta$-plane).

$\zeta = z^2$, where $z = re^{i\theta}$ and $\zeta = Re^{i\Theta}$

This $\Rightarrow$ $Re^{i\Theta} = r^2 e^{2i\theta}$ $\Rightarrow$ $R = r^2$ and $\Theta = 2\theta$.



z - PLANE



ζ - PLANE

*$\log(x + iy) = (1/2) \times \log(x^2 + y^2) + i\tan^{-1}(y/x)$, $\quad \log(x - iy) = (1/2) \times \log(x^2 + y^2) - i\tan^{-1}(y/x)$

Hence the boundaries $\theta = \pm\pi/4$ in z-plane transform to $\Theta = \pm\pi/2$ *i.e.*, imaginary axis of ζ-plane. The points A $(a, 0)$ and B $(b, 0)$ in z-plane transform to A' $(a^2, 0)$ and B' $(b^2, 0)$ respectively in ζ-plane. Then, the image system with respect to imaginary axis $(\Theta = \pm\pi/2)$ in ζ-plane consists of

(*i*) a source of strength m at A' $(a^2, 0)$ (*ii*) a sink of strength $-m$ at B' $(b^2, 0)$

(*iii*) a source of strength m at A'' $(-a^2, 0)$ (*iv*) a sink of strength $-m$ at B'' $(-b^2, 0)$

$$\therefore \quad w = -m\log(\zeta - a^2) + m\log(\zeta - b^2) - m\log(\zeta + a^2) + m\log(\zeta + b^2)$$

$$= -m\log(\zeta^2 - a^4) + m\log(\zeta^2 - b^4) = -m\log(z^4 - a^4) + m\log(z^4 - b^4), \text{ as } \zeta = z^2 \qquad \ldots(2)$$

$$= -m\log(r^4e^{4i\theta} - a^4) + m\log(r^4e^{4i\theta} - b^4), \text{ as } z = re^{i\theta}$$

$$\therefore \quad w = -m\log(r^4\cos 4\theta - a^4 + ir^4\sin 4\theta) + m\log(r^4\cos 4\theta - b^4 + ir^4\sin 4\theta) \qquad \ldots(3)$$

Writing $w = \phi + i\psi$ in (3) and equating imaginary parts, we get

$$\psi = -m\left[\tan^{-1}\frac{r^4\sin 4\theta}{r^4\cos 4\theta - a^4} - \tan^{-1}\frac{r^4\sin 4\theta}{r^4\cos 4\theta + b^4}\right]$$

or $\psi = -m\tan^{-1}\dfrac{r^4(a^4 - b^4)\sin 4\theta}{r^8 - r^4(a^4 + b^4)\cos 4\theta - a^4b^4}$, on using $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x - y}{1 + xy}$

From (2), $\dfrac{dw}{dz} = -m\cdot\dfrac{4z^3}{z^4 - a^4} + m\dfrac{4z^3}{z^4 - b^4} = \dfrac{-4mz^3(a^4 - b^4)}{(z^4 - a^4)(z^4 - b^4)}$

$$= \frac{-4mr^3e^{3i\theta}(a^4 - b^4)}{(r^4e^{4i\theta} - a^4)(r^4e^{4i\theta} - b^4)} = \frac{-4mr^3(\cos 3\theta + i\sin 3\theta)(a^4 - b^4)}{(r^4\cos 4\theta - a^4 + 4ir^4\sin 4\theta)(r^4\cos 4\theta - b^4 + 4ir^4\sin^4\theta)}$$

Hence the required velocity $q = |dw/dz|$ is given by

$$q = \frac{4mr^3(a^4 - b^4)}{(r^8 - 2a^4r^4\cos 4\theta + a^8)^{1/2}(r^8 - 2b^4r^4\cos 4\theta + b^8)^{1/2}}$$

Ex. 3. *Use the method of images to prove that if there be a source m at the point z_0 in a fluid bounded by the lines $\theta = 0$ and $\theta = \pi/3$, the solution is*

$$\phi + i\psi = -m\log\{(z^3 - z_0^3)(z^3 - z_0'^3)\} \quad \text{where} \quad z_0 = x_0 + iy_0 \quad \text{and} \quad z_0' = x_0 - iy_0.$$

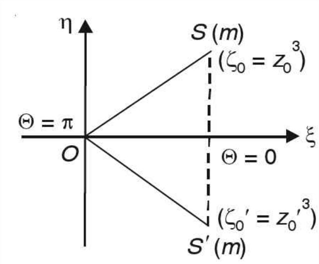
[Agra 2000; Garhwal 2005; Kanpur 2002, 04; I.A.S. 1997]

Sol. Consider the following conformal transformation from z-plane (xy-plane) to ζ-plane ($\xi\eta$-plane) :

$$\zeta = z^3 \quad \text{where} \quad z = re^{i\theta} \quad \text{and} \quad \zeta = Re^{i\Theta} \qquad \ldots(1)$$

This $\Rightarrow$ $Re^{i\Theta} = r^3e^{3i\theta}$ $\Rightarrow$ $R = r^3$ and $\Theta = 3\theta$.

Hence the boundaries $\theta = 0$ and $\theta = \pi/3$ in z-plane transform to $\Theta = 0$ and $\Theta = \pi$ *i.e.*, real axis in ζ-plane. The point z_0 in z-plane transforms to point ζ_0 in z-plane such that $\zeta_0 = z_0^3$. Hence the image system with respect to real axis in ζ-plane consists of



(*i*) a source m at $\zeta_0 = z_0^3$ (*ii*) a source m at $\zeta_0' = z_0'^3$

Hence, $\quad w = -m\log(\zeta - \zeta_0) - m\log(\zeta - \zeta_0')$

or $\quad w = -m\log(z^3 - z_0^3) - m\log(z^3 - z_0'^3)$

or $\quad \phi + i\psi = -m\log\{(z^3 - z_0^3)(z^3 - z_0'^3)\}$.

Ex. 4. *If fluid fills the region of space on the positive side of the x-axis, which is a rigid boundary and if there be a source m at the point* (0, *a*) *and an equal sink at* (0, *b*) *and if the pressure on the negative side be the same as the pressure at infunity, show that the resultant pressure on the boundary is* $\pi \rho m^2 (a-b)^2 / 2ab(a+b)$, *where* ρ *is the density of the fluid.*

[U.P.P.C.S. 1995; I.A.S. 1995, 2008]

Sol. Here the image system with respect to x-axis in z-plane consists of

(*i*) a source m at (0, a) *i.e.*, at $z = ai$ (*ii*) a sink $-m$ at (0, b) *i.e.*, at $z = bi$

(*iii*) a source m at (0, $-a$) *i.e., at* $z = -ai$ (*iv*) a sink $-m$ at (0, $-b$) *i.e.*, at $z = -bi$

Clearly this image system does away with the boundary $y = 0$ (*i.e.*, x-axis). Thus, the complex potential of this entire system is given by

$$\therefore \qquad w = -m \log (z - ai) + m \log (z - bi) - m \log (z + ai) + \log (z + bi)$$

or

$$w = -m \log (z^2 + a^2) + m \log (z^2 + b^2)$$

$$\therefore \qquad \text{velocity} = \left|\frac{dw}{dz}\right| = \left|-\frac{2zm}{z^2+a^2} + \frac{2zm}{z^2+b^2}\right|$$

The velocity q at a point on the boundary (*i.e.*, $y = 0$) is given by (setting $z = x + iy = x$ as $y = 0$)

$$q = \left|-\frac{2xm}{x^2+a^2} + \frac{2xm}{x^2+b^2}\right| = \frac{2xm(a^2-b^2)}{(x^2+a^2)(x^2+b^2)} \qquad \text{... (1)}$$

Let p_0 be the pressure at infinity. Then by Bernoulli's theorem, the pressure p at any piont is given by

$$\frac{1}{2}q^2 + \frac{p}{\rho} = \frac{1}{2} \times 0^2 + \frac{p_0}{\rho} \qquad \text{or} \qquad \frac{p_0 - p}{\rho} = \frac{1}{2}q^2. \qquad \text{... (2)}$$

$\therefore$ The resultant pressure on the boundary

$$= \int_0^\infty (p_0 - p)\, dx = \frac{1}{2}\rho \int_0^\infty q^2 dx = 2\rho m^2 \int_0^\infty \frac{x^2(a^2-b^2)^2}{(x^2+a^2)^2(x^2+b^2)^2} dx, \text{ by (1) and (2)}$$

$$= 2\rho m^2 \int_0^\infty \left[-\frac{a^2+b^2}{a^2-b^2}\left(\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right) - \frac{a^2}{(x^2+a^2)^2} - \frac{b^2}{(x^2+b^2)^2}\right] dx$$

(on resolving into partial fractions)

$$= 2\rho m^2 \left\{\frac{a^2+b^2}{b^2-a^2}\left(\frac{\pi}{2a} - \frac{\pi}{2b}\right) - \frac{\pi}{4a} - \frac{\pi}{4b}\right\}, \text{ on simplification}$$

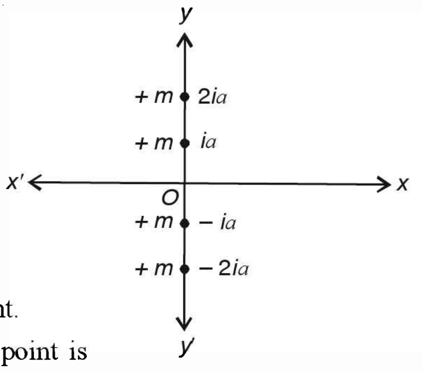
$$= \frac{\pi \rho m^2}{2ab}\left[\frac{2(a^2+b^2) - (a+b)^2}{(a+b)}\right] = \frac{\pi \rho m^2 (a-b)^2}{2ab(a+b)}.$$

Ex. 5. *Parallel line sources (perpendicular to xy-plane) of equal strength m are parallel at the points* $z = nia$ *where* $n = \ldots, -2, -1, 0, 1, 2, \ldots$. *Prove that the complex potential is* $w = -m \log \sinh (\pi z / a)$. *Hence, show that the complex potential for two dimensional doublets (lines doublets), with their axes parallel to the x-axis, of strength* μ *at the same points is given by* $w = \mu \coth (\pi z / a)$.

Sol. The complex potential due to sources of strength m situated at the points $z = 0, ia, -ia, 2ia, -2ia, \ldots$ is given by

$$w = -m \log (z - 0) - m \log (z - ia) - m \log (z + ia) - m \log (z - 2ia) - m \log (z + 2ia) - \ldots$$

$$= -m \log z - m \log \{(z - ia)(z + ia)\} - m \log \{z - 2ia)(z + 2ia)\} - \ldots..$$



$$= -m \log z - m \log (z^2 + a^2) - m \log (z^2 + 2^2 a^2) - \ldots$$

$$= -m \log [z (z^2 + a^2) (z^2 + 2^2 a^2) (z^2 + 3^2 a^2) \ldots]$$

$$= -m \log \left\{ \frac{\pi}{a} z \left(1 + \frac{z^2}{a^2}\right)\left(1 + \frac{z^2}{2^2 a^2}\right)\left(1 + \frac{z^2}{3^2 a^2}\right) \ldots \right\}$$

$$-m \log \left[\left(\frac{a}{\pi}\right) a^2 (2^2 a^2)(3^2 a^2) \ldots \right]$$

$\therefore$ $$w = -m \sinh (\pi z / a) + \text{constant}.$$

The complex potential w_1 for the doublets at the same point is

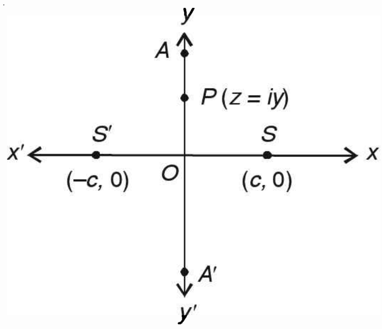
$$w_1 = -\frac{\partial w}{\partial z} = \frac{m\pi}{a} \coth\left(\frac{\pi z}{a}\right) = \mu \coth\left(\frac{\pi z}{a}\right), \quad \text{where} \quad \mu = \frac{m\pi}{a}.$$

Ex. 6. *In the case of the motion of liquid in a part of a plane bounded by a straight line due to a source in the plane, prove that if* $m\rho$ *is the mass of fluid (of density* ρ*) generated at the source per unit of time the pressure on the length 2l of the boundary immediately oppostite to the source is less than that on an equal length at a great distance by*

$$\frac{1}{\rho} \frac{m^2 \rho}{\pi^2} \left[\frac{1}{c} \tan^{-1} \frac{l}{c} - \frac{l}{l^2 + c^2} \right],$$ *where c is the distance of source to the boundary.*

[Rohilkhand 2005; Indore 1998; Meerut 1996; Kanpur 2000]

Sol. Let y-axis be the bounding line and let the given source of strength (μ, say) be situated at S where $OS = c$. Now, by the definition of strength μ of the source, we have $2\pi\mu\rho = m\rho$ so that $\mu = m/2\pi$. Now, the image system consists of



(*i*) a source of strength $m/2\pi$ at $S\,(c, 0)$

(*ii*) a source of strength $m/2\pi$ at $S'\,(-c, 0)$

Here S' is image of S such that $OS = OS' = c$.

The complex potential w is given by

$$w = -(m/2\pi) \log (z - c) - (m/2\pi) \log (z + c) = -(m/2\pi) \log (z^2 - c^2).$$

The velocity is given by $$\left|\frac{dw}{dz}\right| = \left| -\frac{m}{2\pi} \cdot \frac{2z}{z^2 - c^2} \right| = \frac{m}{\pi} \left| \frac{z}{z^2 - c^2} \right|.$$

Hence velocity q at any point P (where $z = iy$) is given by

$$q = \frac{m}{\pi} \left| \frac{iy}{-y^2 - c^2} \right| = \frac{my}{\pi (y^2 + c^2)}. \qquad \ldots(1)$$

Bernoulli's equation for steady motionis given by

$$p/\rho + q^2/2 = \text{constant} = c, \text{ say}. \qquad \ldots (2)$$

Let p_0 be the pressure on y-axis at great distance from O so that $p = p_0$ and $q = 0$ when $y = \infty$. Then (2) reduces to $p_0/\rho = c$ and hence (2) becomes

$$\frac{p}{\rho} + \frac{q^2}{2} = \frac{p_0}{\rho} \quad \text{or} \quad \frac{p_0 - p}{\rho} = \frac{1}{2} q^2$$

or $$p_0 - p = \frac{\rho}{2}q^2 = \frac{\rho}{2}\cdot\frac{m^2y^2}{\pi^2(y^2+c^2)^2}, \text{ using (1).} \qquad \ldots(3)$$

Let $AA' = 2l$, where $OA = OA' = l$. Then pressure on the length AA' of the boundary (*i.e.* y-axis) $= \int_{-l}^{l}(p_0 - p)\,dy = \dfrac{m^2\rho}{2\pi^2}\int_{-l}^{l}\dfrac{y^2dy}{(y^2+c^2)^2} = \dfrac{m^2\rho}{\pi^2}\int_0^{l}\dfrac{y^2dy}{(y^2+c^2)^2}$

[$\because$ The integrand is an even function of y]

$$= \frac{m^2\rho}{\pi^2}\int_0^{\alpha}\frac{c^2\tan^2\theta\cdot c\sec^2\theta\,d\theta}{c^4\sec^4\theta}$$

[Putting $y = c\tan\theta$ so that $dy = c\sec^2\theta\,d\theta$. Here let $\theta = \alpha$ when $y = l$ so that $l = c\tan\alpha$]

$$= \frac{m^2\rho}{\pi^2 c}\int_0^{\alpha}\sin^2\theta\,d\theta = \frac{m^2\rho}{2\pi^2 c}\int_0^{\alpha}(1-\cos 2\theta)\,d\theta = \frac{m^2\rho}{2\pi^2 c}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\alpha} = \frac{m^2\rho}{2\pi^2 c}[\theta - \sin\theta\cos\theta]_0^{\alpha}$$

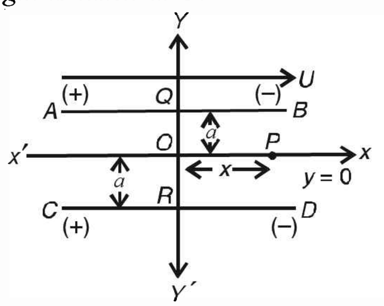
$$= \frac{m^2\rho}{2\pi^2 c}[\alpha - \sin\alpha\,\cos\alpha] = \frac{m^2\rho}{2\pi^2 c}\left[\tan^{-1}\frac{l}{c} - \frac{lc}{l^2+c^2}\right] = \frac{m^2\rho}{2\pi^2}\left[\frac{1}{c}\tan^{-1}\frac{l}{c} - \frac{l}{l^2+c^2}\right].$$

$$\left[\because\ \tan\alpha = l/c \quad\Rightarrow\quad \sin\alpha = l/(l^2+c^2)^{1/2} \quad\text{and}\quad \cos\alpha = c/(l^2+c^2)^{1/2}\right]$$

Ex. 7. *The space on one side of an infinite plane wall $y = 0$, is filled with inviscid, incompressible fluid, moving at infinity with velocity U in the direction of the axis of X. The motion of the fluid is wholly two dimensional, in the (x, y) plane. A doublet of strength μ is at a distance a from the wall and points in the negative direction of the axis of X. Show that if μ is less than $4a^2U$, the pressure of the fluid on the wall is a maximum at points distant $a\sqrt{3}$ from O, the foot of the perpendicular from the doublet on the wall, and is minimum at O.*

If μ is equal to $4a^2U$, find the point where the velocity of the fluid is zero, and show that the streamlines include the circle $x^2 + (y-a)^2 = 4a^2$, where the origin is taken at O.

Sol. We know that the complex potential for a doublet of strength μ at $z = z_0$ inclined at an angle α to the x-axis is given by $\mu\, e^{i\alpha}/(z - z_0)$.



Here the given doublet AB points in the negative direction of x-axis and so the given doublet makes an angle π with OX. Now the given doublet is situated at Q where $z = ia$. Hence the image of the given doublet AB will be an equal doublet of strength μ, similarly oriented, and will be situated at R where $z = -ia$ (note that here $OR = OQ = a$).

Let w be the complex potential for the system, consisting of original doublet AB, image doublet CD and the stream U.

$$\therefore \qquad w = \frac{\mu e^{i\pi}}{z - ia} + \frac{\mu e^{i\pi}}{z + ia} - Uz = -\frac{2\mu z}{z^2 + a^2} - Uz. \qquad \ldots(1)$$

From (1), $$\frac{dw}{dz} = -2\mu\frac{1\times(z^2+a^2) - z\times 2z}{(z^2+a^2)^2} - U = -\frac{2\mu(a^2-z^2)}{(z^2+a^2)^2} - U. \qquad \ldots(2)$$

Let P be any point on the wall $X'X$. Then, at P, $z = x$. Hence the velocity q on the wall is given by

$$q = \left|\frac{dw}{dz}\right| = \left|-\frac{2\mu(a^2-x^2)}{(x^2+a^2)^2} - U\right| = \left|\frac{2\mu(a^2-x^2)}{(x^2+a^2)^2} + U\right|. \qquad \ldots(3)$$

By Bernouilli's theorem, we have

$$p/\rho + q^2/2 = \text{constant} = C, \text{ say.} \qquad \text{...(4)}$$

Now, when $z = \infty$, $p = \Pi$ and $q = U$. Then (4) becomes

$$\Pi/\rho + U^2/2 = C. \qquad \text{...(5)}$$

Subtracting (4) from (5), $(\Pi - p)/\rho + (U^2 - q^2)/2 = 0$

or
$$\Pi - p = (\rho/2) \times (q^2 - U^2)$$

or
$$\Pi - p = \frac{\rho}{2}\left[\left\{\frac{2\mu(a^2 - x^2)}{(a^2 + x^2)^2} + U\right\}^2 - U^2\right], \text{ using (3)}$$

or
$$\Pi - p = 2\mu\rho\left[\frac{\mu(a^2 - x^2)^2}{(a^2 + x^2)^4} + U\frac{a^2 - x^2}{(a^2 + x^2)^2}\right].$$

or
$$p = \Pi - 2\mu^2\rho\frac{(a^2 - x^2)^2}{(a^2 + x^2)^4} - 2\mu\rho U\frac{a^2 - x^2}{(a^2 + x^2)^2}. \qquad \text{...(6)}$$

$$\therefore \quad \frac{dp}{dx} = -2\mu^2\rho\left[\frac{4x(a^2 - x^2)}{(a^2 + x^2)^4} + \frac{8x(a^2 - x^2)^2}{(a^2 + x^2)^5}\right] - 2\mu\rho U\left[\frac{2x}{(a^2 + x^2)^2} + \frac{4x(a^2 - x^2)}{(a^2 + x^2)^3}\right]$$

or
$$\frac{dp}{dx} = -\frac{4\mu x\rho\{2\mu(a^2 - x^2) + U(a^2 + x^2)^2\}\,(3a - x^2)}{(a^2 + x^2)^5} \qquad \text{...(7)}$$

The pressure will be maximum or minimum if $dp/dx = 0$ *i.e.*, if

$$x\,(3a^2 - x^2) = 0 \quad \text{or} \quad 2\mu(a^2 - x^2) + U(a^2 + x^2)^2 = 0, \text{ by (7)}$$

If $x(3a^2 - x^2) = 0$, then we have $x = 0$, $x = a\sqrt{3}$ and $x = -a\sqrt{3}$.

Now on the wall XX' ($y = 0$) at $x = a\sqrt{3}$, by (2), we have

$$\frac{dw}{dz} = -U - \frac{2\mu(a^2 - 3a^2)}{(a^2 + 3a^2)^2} = -U + \frac{\mu}{4a^2} = \frac{\mu - 4a^2U}{4a^2}. \qquad \text{...(8)}$$

If $\mu < 4a^2 U$, the value of d^2p/dx^2 at $x = a\sqrt{3}$ is negative and hence the pressure of the fluid at the wall is a maximum when $x = a\sqrt{3}$.

Again, d^2p/dx^2 at $x = 0$ is positive and hence the pressure of the fluid at the wall is a minimum when $x = 0$.

Now, if $\mu = 4a^2U$, then $dw/dz = 0$ from (8). So (2) reduces to

$$-8a^2\,U(a^2 - z^2)/\,(a^2 + z^2)^2 - U = 0$$

or $z^4 - 6a^2z^2 + 9a^4 = 0$, so that $(z^2 - 3a^2)^2 = 0$ or $z = \pm\, a\sqrt{3}$.

Hence the stragnation points are given by $(a\sqrt{3}, 0)$ and $(-a\sqrt{3}, 0)$.

Writing $\mu = 4a^2U$, $w = \phi + i\psi$ and $z = x + iy$, (1) may be re written as

$$\phi + i\psi = -\frac{8a^2U(x + iy)}{(x^2 + a^2 - y^2) + 2ixy} - U(x + iy)$$

or
$$\phi + i\psi = -\frac{8a^2U(x + iy)\,[(x^2 + a^2 - y^2) - 2ixy]}{[(x^2 + a^2 - y^2) + 2ixy]\,[(x^2 + a^2 - y^2) - 2ixy]} - U(x + iy). \qquad \text{...(9)}$$

Equating the imaginary parts on both sides of (9), we have

$$\psi = -\left[\frac{8a^2U\{y(x^2 + a^2 - y^2) - 2x^2y\}}{(x^2 + a^2 - y^2)^2 + 4x^2y^2} + Uy\right]. \qquad \text{...(10)}$$

The streamlines are given by ψ = constant. Taking constant = 0, the streamlines given by $\psi = 0$ are

$$\frac{8a^2Uy[(x^2+a^2-y^2)-2x^2]}{(x^2+a^2-y^2)^2+4x^2y^2}+Uy=0$$

or $$8a^2[a^2-(x^2+y^2)]+[(x^2-y^2)+a^2]^2+4x^2y^2=0$$

or $$8a^4-8a^2(x^2+y^2)+(x^2-y^2)^2+2a^2(x^2-y^2)+a^4+4x^2y^2=0$$

or $$9a^4-8a^2(x^2+y^2)+(x^2+y^2)^2+2a^2(x^2-y^2)=0$$

or $$9a^4+(x^2+y^2)^2-6a^2x^2-6a^2y^2-4a^2y^2=0 \qquad \text{or} \qquad [(x^2+y^2)-3a^2]^2-4a^2y^2=0$$

or $$(x^2+y^2-3a^2-2ay)(x^2+y^2-3a^2+2ay)=0,$$

which includes the circle $x^2+y^2-3a^2-2ay=0$ or $x^2+(y-a)^2=4a^2$.

EXERCISE 5 (C)

1. Obtain the image of a simple source with respect to a plane (or a straight line). **[Kanpur 2003]**

Hint. Proceeed as in Art 5.18 by replacing line by plane to get the same result.

2. A two-dimensional source of strength m is situated at the point $(a, 0)$, the axis of y being a fixed boundary. Find the points in the boundary at which the fluid velocity is a maximum. Show that the resultant thrust on the part of the axis of y which lies between $y = \pm b$ is

$$2p_0b-2m^2\rho\left[\frac{1}{a}\tan^{-1}\frac{b}{a}-\frac{b}{a^2+b^2}\right],\ \text{where } p_0 \text{ is the pressure at infinity.}$$

Particular Case. When $b = a$, the thrust $= 2p_0a-(m^2\rho/a)\times(\pi/2-1)$

3. OX and OY are fixed rigid boundaries and there is a source at (a, b). Find the stream lines and show that the dividing line is $xy\,(x^2-y^2-a^2+b^2)=0$.

4. The irrotational motion in two-dimensions of a fluid bounded by the lines $y = \pm b$ is due to a doublet of strength μ at the origin, the axis of the doublet being in the positive direction of the axis of x. Prove that the motion is given by

$$\phi+i\psi=\frac{\pi\mu}{2b}\coth\left\{\frac{\pi(x+iy)}{2b}\right\}$$

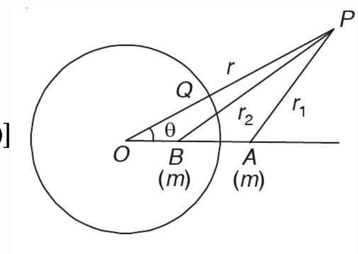
Show also that the points where the fluid is moving parallel to the axis of y lie on the curve $\coth\,(\pi x/b) = \sec\,(\pi y/b)$.

5. In liquid bounded by the axes of x and y in the first quadrant there is a source of strength m at distance a from the origin on the bisector of the angle XOY. Prove that the complex potential is $-m\log(x^4+a^4)$.

5.21. Image of a source with regard to a circle. [Meerut 2000, 06; Rajasthan 2000, 2010; Agra 2005; Garhwal 1999; G.N.D.U. Amritsar 2000, Kanpur 2002, 2010]

Let us determine the image of a source of strength m at a point A with respect to the circle with O as centre. Let $OA = f$ and let B be inverse point of A with respect to the circle. If a be the radius of the circle, then $OA \cdot PB = a^2$ so that $OB = a^2/f$. Let $P(z)$ be an arbitrary point in the plane of the circle.

Let there be a source of strength m at B. If w be the complex potential due to sources at A and B, then we get



$$w = -m\log(z-f)-m\log(z-a^2/f)$$

$$= -m[\log(r\cos\theta-r+ir\sin\theta)+\log(r\cos\theta-a^2/f+ir\sin\theta)]$$

$$[\because\ z=re^{i\theta}=r(\cos\theta+i\sin\theta)=r\cos\theta+ir\sin\theta]$$

Writing $w = \phi + i\psi$ and equating real parts, we get

$$\phi = -(m/2) \times \left[\log\left\{(r\cos\theta - f)^2 + (r\sin\theta)^2\right\} + \log\left\{(r\cos\theta - a^2/f)^2 + (r\sin\theta)^2\right\}\right]$$

$$= -\frac{m}{2}\left[\log(r^2 + f^2 - 2fr\cos\theta) + \log\left(r^2 + \frac{a^4}{f^2} - \frac{2ra^2}{f}\cos\theta\right)\right]$$

$$\therefore \qquad \frac{\partial\phi}{\partial r} = -\frac{m}{2}\left[\frac{2(r - f\cos\theta)}{r^2 + f^2 - 2fr\cos\theta} + \frac{2\{r - (a^2/f)\cos\theta\}}{r^2 + a^4/f^2 - 2r(a^2/f)\cos\theta}\right]$$

Hence normal velocity at any point Q on the circle

$$= -\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = m\left[\frac{a - f\cos\theta}{a^2 + f^2 - 2fa\cos\theta} + \frac{(a/f)(f - a\cos\theta)}{(a^2/f^2)(f^2 + a^2 - 2af\cos\theta)}\right]$$

$$= m\left[\frac{a - f\cos\theta + f^2/a - f\cos\theta}{a^2 + f^2 - 2fa\cos\theta} + \right] = \frac{m}{a}.$$

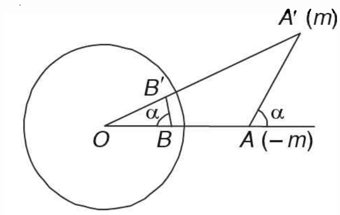
Now, if we place a source of strength $-m$ at O, the normal velocity due to it at Q will be $-(m/a)$ and hence the normal velocity of the system will reduce to zero.

Hence *the image system for a source outside a circle consists of an equal source at the inverse point and an equal sink at the centre of the circle.* **[Kanpur 2002, 08]**

5.22. Image of a doublet with regard to a circle.

[Kanpur 2003, 06; Kurukshetra 2000; Rajasthan 1998]

Let us determine the image of a doublet AA' with its axis making an angle α with OA, outside the circle, there being a sink $-m$ at A and a source m at A′. Join OA and OA'. Let B and B' be the inverse points of A and A' with regard to the circle with O as centre.



Then $$OA \cdot OB = OA' \cdot OB' = a^2, \qquad \ldots(1)$$

where a is the radius of the circle.

Now the image of source m at A' consists of a source m at B' and and a sink $-m$ at O. Similarly, the image of sink $-m$ at A consists of a sink at B and a source m at O. Compounding these, we see that source m and sink $-m$ at O cancel each other and hence the image of the given doublet AA' is another doublet BB'.

Let the strength of the given doublet AA' be μ.

Then $$\mu = \lim_{A \to A'} (m \cdot AA') \qquad \ldots(2)$$

From (1) $$OA/OA' = OB'/OB, \qquad \ldots(3)$$

showing that triangles OAA' and $OB'B$ are similar. From these similar triangles, we have

$$\frac{BB'}{AA'} = \frac{OB'}{OA} = \frac{OB'}{OA} \cdot \frac{OA'}{OA'} = \frac{a^2}{OA \cdot OA'} \qquad \ldots(4)$$

$$\therefore \quad \mu' = \text{strength of doublet } B'B = \lim_{B' \to B} (m \cdot B'B) = \lim_{A \to A'} \frac{a^2}{OA \cdot OA'} \cdot (m \cdot AA'), \text{ by (4)}$$

$$= \mu a^2 / f^2, \text{ using (2) } \textit{and taking } OA = OA' = f$$

Thus *the image of a two-dimensional doublet at A with regard to a circle is another doublet at the inverse point B, the axes of the doublets making supplementary angles with the radius OBA.*

5.23A. The Milne-Thomson circle theorem or simply the circle theorem.

Statement : *Let f(z) be the complex potential for a flow having no rigid boundaries and such that there are no singularities of flow within the circle | z | = a. Then, on introducing the solid circular cylinder | z | = a into the flow, the new complex potential is given by* $w = f(z) + \bar{f}(a^2/z)$ *for* $|z| \geq a$.

[Rohilkhand 2002, 03, 05; Kanpur 2000, 09; Garhwal 2003, 05; Meerut 1998]

Proof. Let C be the cross-section of the circular cylinder | z | = a. Then on C, $z\bar{z} = a^2$ or $\bar{z} = a^2/z$. Hence for points on the circle, we have

$$w = f(z) + \bar{f}(a^2/z) = f(z) + \bar{f}(\bar{z}) \qquad \text{or} \qquad \phi + i\psi = f(z) + \bar{f}(\bar{z}) \qquad \ldots(1)$$

Since the quantity on R.H.S. of (1) is purely real, equating imaginary parts (1) gives $\psi = 0$ on C. Hence C is a streamline in the new flow.

By hypothesis all the singularites of $f(z)$ (at which sources, sinks, doublets or vortices may be present) lie outside the circle | z | = a and so the singularities of $f(a^2/z)$ lie inside the circle | z | = a. Hence the singularities of $\bar{f}(a^2/z)$ also lie inside the circle | z | = a. Thus we find that the additional term $\bar{f}(a^2/z)$ introduces no new singularities into the flow outside the circle | z | = a.

Hence | z | = a is a possible boundary for the new flow and $w = f(z) + \bar{f}(a^2/z)$ is the appropriate complex potential for the new flow.

Remark 1. In the above proof of circle theorem we have used the following important results :

Let $u(t)$ and $v(t)$ be real functions of a real variable t. Let $f(t) = u(t) + iv(t)$ so that $f(t)$ is a complex function of the real variable t. Then conjugate of $f(t)$ is denoted and defined as

$$\bar{f}(t) = u(t) - iv(t).$$

On replacing real variable t by the complex variable z (= $x + iy$), $f(z)$ and $\bar{f}(z)$ are defined as follows :

$$f(z) = u(z) + iv(z), \qquad \bar{f}(z) = u(z) - iv(z)$$

Again, $$f(\bar{z}) = u(\bar{z}) + iv(\bar{z}), \qquad \bar{f}(\bar{z}) = u(\bar{z}) - iv(\bar{z})$$

On comparing the forms of $f(z)$ and $f(\bar{z})$, we find that, since $z = x + iy$, $\bar{z} = x - iy$, the value of $\bar{f}(\bar{z})$ is obtained from $f(z)$ by replacing i throughout by $-i$. It then follows that $\bar{f}(\bar{z})$ is merely the complex conjugate of $f(z)$ and accordingly, we write $\bar{f}(\bar{z}) = \overline{f(z)}$.

Remark 2. When a circular cylinder is present in the field of sources, sinks, doublets or vortices, the above theorem provides an easy method for determining the image system. Furthermore the theorem can also be used to determine modified flows when a long circular cylinder is introduced into a given two-dimensional flow. Consider the following application of "Circle theorem".

523B. To determine image system for a source outside a circle (or a circular cylinder) of radius a with help of the circle theorem.

Refer figure of Art. 5.21. Let $OA = f$. Suppose there is a source of strength m at A where $z = f$, outside the circle of radius a whose centre is at O. When the source is alone in the fluid the complex potential at a point P (z) is given by

$$f(z) = -m \log(z - f) \qquad \text{Then} \qquad \bar{f}(z) = -m \log(z - f)$$

$$\therefore \qquad \bar{f}(a^2/z) = -m \log(a^2/z - f)$$

When the circle of section $|z| = a$ is introduced, then the complex potential in the region $|z| \geq a$ is given by

$$
\begin{aligned}
w &= f(z) + \bar{f}(a^2/z) = -m \log (z - f) - m \log (a^2/z - f) \\
&= -m \log (z - f) - m \log \left(\frac{a^2 - zf}{z}\right) \\
&= -m \log (z - f) - m \log (a^2 - zf) + m \log z \\
&= -m \log (z - f) - m \log [(-f)(z - a^2/f)] + m \log z \\
&= -m \log (z - f) - m \log (z - a^2/f) + m \log z - m \log (-f)
\end{aligned}
$$

$$\therefore \qquad w = -\log (z - f) - m \log (z - a^2/f) + m \log z + \text{constant}, \qquad \ldots(1)$$

the constant (real or complex, $-m \log (-f)$) being immaterial from the view point of analysing the flow. (1) shows that w is the complex potential of

(*i*) a source m at A, $z = f$ (*ii*) a source m at B, $z = a^2/f$ (*iii*) a sink $-m$ at the origin

Since $OA \cdot OB = a^2$, A and B are the inverse points with respect to the circle $|z| = a$ and so B is inside the circle.

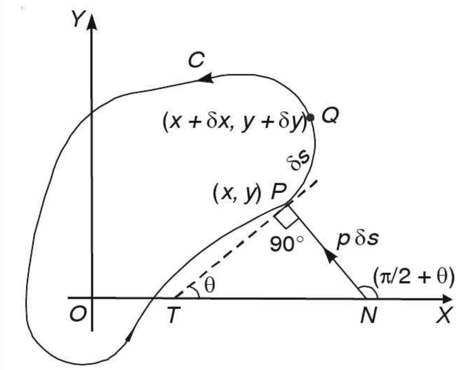
Thus *the image system for a source outside a circle consists of an equal source at the inverse point and an equal sink at the centre of the circle.*

5.24. The Theorem of Blasius. [Agra 2005, 08, 09, 11; I.A.S. 1985; Kanpur 2000; Meerut 2001, 02, 04, 08, 09, 10, 11, 12; G.N.D.U. Amritsar 2003, 05; Rohilkhand 2003]

In a steady two-dimensional irrotational motion of an incompressible fluid under no external forced given by the complex potential $w = f(z)$, if the pressure thrusts on the fixed cylinder of any shape are represented by a force (X, Y) and a couple of moment M about the origin of co-ordinates, then

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz, \qquad M = \text{Real part of } \left\{-\frac{1}{2} i\rho \int_C z \left(\frac{dw}{dz}\right)^2 dz\right\},$$

where ρ is the fluid density and integrals are taken round the contour C of the cylinder.



Proof. Figure shows the section C of the cylinder in plane XOY. Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two neighbouring points on C such that arc $PQ = \delta s$. If θ be the angle which the tangent PT at P on the contour C makes with x-axis, then

$$\cos\theta = dx/ds, \qquad \sin\theta = dy/ds, \quad \ldots(1)$$

and the normal at P makes an angle $(\theta + \pi/2)$ with the x-axis. Now, if p denotes the pressure at p, the force on unit length of the section δs is $p\delta s$ normal to C. Then using (1), we have

$$X = \int_C p\cos(\theta + \pi/2) ds = -\int_C p\sin\theta\, ds = -\int_C p\, dy, \text{ using (1)} \qquad \ldots(2)$$

$$Y = \int_C p\sin(\theta + \pi/2) ds = \int_C p\cos\theta\, ds = \int_C p\, dx, \text{ using (1)} \qquad \ldots(3)$$

$$M = \int_C [x \cdot p\sin(\theta + \pi/2) ds - y \cdot p\cos(\theta + \pi/2) ds] = \int_C p(x\cos\theta\, ds + y\sin\theta\, ds)$$

or

$$M = \int_C p(x\,dx + y\,dy), \text{ using (1)} \qquad \ldots(4)$$

Now Bernoulli's equation in this context is

$$\frac{1}{2}q^2 + \frac{p}{\rho} = B \qquad \text{so that} \qquad p = \rho B - \frac{1}{2}\rho q^2, \qquad \ldots(5)$$

where q is the fluid velocity, ρ the density. Since ρ is constant for an incompressible fluid, take $\rho B = A$ (a contant). Again $q^2 = u^2 + v^2$ where u and v are the velocity components. Then (5) reduces to

$$p = A - (\rho/2) \times (u^2 + v^2) \qquad \ldots(6)$$

Also, $\qquad dw/dz = -u + iv \qquad$ or $\qquad -dw/dz = u - iv \qquad \ldots(7)$

Using (6), (2), (3) and (4) reduce to

$$X = -\int_C \left[A - \frac{1}{2}\rho(u^2 + v^2)\right] dy = \frac{1}{2}\rho\int_C (u^2 + v^2)\, dy \qquad \ldots(8)$$

$$Y = \int_C \left[\mathrm{A} - \frac{1}{2}\rho(u^2 + v^2)\right] dx = -\frac{1}{2}\rho\int_C (u^2 + v^2)\, dx, \qquad \ldots(9)$$

and $$M = \int_C \left[A - \frac{1}{2}\rho(u^2 + v^2)\right](x\,dx + y\,dy) = -\frac{1}{2}\rho\int_C (u^2 + v^2)\,(xdx + ydy) \qquad \ldots(10)$$

While simplifying (8), (9) and (10), we have to use the following results

$$\int_C dy = \int_C dx = \int_C x\, dx = \int_C x\, dy = 0$$

which hold good because C is a closed contour.

Now the contour of the cylinder is a streamline. Hence we have $\qquad dx/u = dy/v.$

Now, $\dfrac{dx}{u} = \dfrac{dy}{v} = \dfrac{dx + idy}{u + iv} = \dfrac{dx - idy}{u - iv}$ or $\dfrac{dx - idy}{dx + idy} = \dfrac{u - iv}{u + iv} = \dfrac{(u - iv)^2}{(u + iv)\,(u - iv)} = \dfrac{(u - iv)^2}{u^2 + v^2}$

$$\therefore \qquad (u - iv)^2\,(dx + idy) = (u^2 + v^2)\,(dx - idy) \qquad \ldots(11)$$

From (8) and (9), we have

$$X - iY = \frac{1}{2}\rho\int_C (u^2 + v^2)\,(dy + idx) = \frac{1}{2}\rho i\int_C (u^2 + v^2)\left(dx + \frac{1}{i}dy\right)$$

$$= \frac{1}{2}\rho i\int_C (u^2 + v^2)(dx - idy) = \frac{1}{2}\rho i\int_C (u - iv)^2 (dx + idy), \text{ by (11)}$$

$$= \frac{1}{2}\rho i\int_C \left(\frac{dw}{dz}\right)^2 dz, \text{ using (7) and the fact } z = x + iy \Rightarrow dz = dx + idy;$$

Re-writing (10), we have

$$M = \text{Reat part of } -\frac{1}{2}\rho\int_C (x + iy)\,(dx - idy)\,(u^2 + v^2)$$

$$= \text{Real part of } -\frac{1}{2}\rho\int_C (x + iy)\,(u - iv)^2\,(dx + idy), \text{ using (11)}$$

$$= \text{Real part of } \left\{-\frac{1}{2}\rho\int_C z\left(\frac{dw}{dz}\right)^2 dz\right\}, \text{ using (7)}$$

Remark 1. The above integrals are to be taken over the contour of the cylinder. If however, we take a large contour surrounding the cylinder such that between this contour and the cylinder there is no singularity of the integrand, then we can take the integrals round such large contours. The singularities of the integrand occur at sources, sinks, doublets etc.

Remark 2. In what follows, we shall often use the following important definitions and results of functions of complex variables.

A point at which a function $f(z)$ ceases to be analytic is known as a *singular point* or *singularity* of the function. If in the neighbourhood of the point $z = a$, $f(z)$ can be expanded in positive and negative powers of $(z - a)$, say

$$f(z) = \dots + A_2 (z-a)^2 + A_1 (z-a) + A_0 + \frac{B_1}{z-a} + \frac{B_2}{(z-a)^2} + \dots$$

then the point $z = a$ is a *singular point* of $f(z)$. If only a finite number of terms contain negative powers of $z - a$, the point $z = a$ is called a *pole*. In this case the coefficient of $1/(z - a)$ is called the *residue* of the function at $z = a$.

Cauchy's Residue theorem. If $f(z)$ is analytic, except at a finite number of poles within a closed contour C and continuous on the boundary C, then

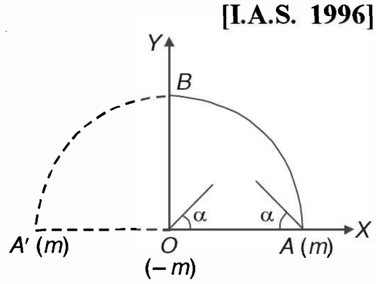
$$\int_C f(z)dz = 2\pi i \times [\text{sum of the residues of } f(z) \text{ at its poles within } C]$$

5.25. Illustrative solved examples.

Ex. 1(*a*). *In the region bounded by a fixed quadrantal arc and its radii, deduce the motion due to a source and an equal sink situated at the ends of one of the bounding radii. Show that the streamline leaving either end at an angle* α *with the radius is* $r^2 \sin(\alpha + \theta) = a^2 \sin(\alpha - \theta)$.

[Kanpur 2011; P.C.S. (U.P.) 2000; Rohilkhand 2003; I.A.S. 1986; Meerut 2002, 07]

(*b*) *In a region bounded by a fixed quadrant arc and its radii, deduce the motion due to a source and an equal sink situated at the ends of one of the bounding radii. Show that the streamline leaving either end at an angle* $\pi/6$ *with radius is* $r^2 \sin(\pi/6 + \theta) = a^2 \sin(\pi/6 - \theta)$, where a is radius of the quadrant. **[I.A.S. 1996]**



Sol. (*a*). Let AOB be the circular quadrant of radius a with OA and OB as bounding radii. Consider a source of strength m at A and a sink of strength $-m$ at O. Then the image system consists of (*i*) a source m at A $(a, 0)$

(*ii*) a source m at A' $(-a, 0)$

(*iii*) a sink $-m$ at O $(0, 0)$.

Hence the complex potential w for the motion of the fluid at any point $P(z + x + iy = re^{i\theta})$ is given by

$$w = -m \log(z-a) - m\log(z+a) + m \log z = -m \log \frac{z^2 - a^2}{z} = -m \log(z - a^2 z^{-1})$$

or $$w = -m \log(re^{i\theta} - a^2 r^{-1} e^{-i\theta}), \qquad \text{as} \qquad z = re^{i\theta}$$

or $$w = -m \log[r(\cos\theta + i\sin\theta) - a^2 r^{-1}(\cos\theta - i\sin\theta)]$$

or $$\phi + i\psi = -m \log[(r - a^2/r)\cos\theta + i(r + a^2/r)\sin\theta]$$

Equating imaginary parts, we obtain

$$\psi = -m \tan^{-1} \frac{(r + a^2/r)\sin\theta}{(r - a^2/r)\cos\theta} = -m \tan^{-1}\left\{\frac{r^2 + a^2}{r^2 - a^2}\tan\theta\right\}$$

The streamline leaving the end A and O at an angle α is given by

$$\psi = -m(\pi - \alpha) \qquad i.e., \qquad -m\tan^{-1}\left\{\frac{r^2+a^2}{r^2-a^2}\tan\theta\right\} = -m(\pi-\alpha)$$

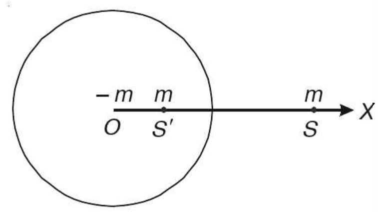
or
$$\frac{(r^2+a^2)\sin\theta}{(r^2-a^2)\cos\theta} = \tan(\pi-\alpha) = -\tan\alpha = -\frac{\sin\alpha}{\cos\alpha}$$

or
$$(r^2+a^2)\sin\theta\cos\alpha = -(r^2-a^2)\cos\theta\sin\alpha \qquad \text{or} \qquad r^2\sin(\alpha+\theta) = a^2\sin(\alpha-\theta).$$

***(b)* Hint.** Proceed as in part (*a*) by taking $\alpha = \pi/6$.

Ex. 2. *In the case of the two-dimensional fluid motion produced by a source of strength m placed at a point S outside a rigid circular disc of radius a whose centre is O, show that the velocity of slip of the fluid in contact with the disc is greatest at the points where the lines joining S to the ends of the diameter at right angles to OS meet the circle, prove that its magnitude at these points is* $(2m \times OS)/(OS^2 - a^2)$

Sol. Let S' be the inverse point of S with respect to the circular disc, with O as its centre. Let $OS = c$. Then $OS \times OS' = a^2$ so that $OS' = a^2/c$.



The equivlent image system consists of

(*i*) a source of strength m at $S\,(c, 0)$,

(*ii*) a source of strength m at $S'\,(a^2/c, 0)$,

(*iii*) a sink of strength $-m$ at $O\,(0, 0)$.

Let OS be taken as x-axis. Then the complex potential for the motion of the fluid at any point $z(= x+iy = re^{i\theta})$ is given by

$$w = -m\log(z-c) - m\log(z-a^2/c) + m\log z$$

$$\therefore \qquad \frac{dw}{dz} = -\frac{m}{z-c} - \frac{m}{z-a^2/c} + \frac{m}{z}$$

Let $q\ (=|dw/dz|)$ be the velocity at any point z. Then

$$q = m\left|\frac{1}{z-c} + \frac{1}{z-a^2/c} - \frac{1}{z}\right| = m\left|\frac{(z-a)(z+a)}{z(z-c)(z-a^2/c)}\right|$$

Hence the velocity at any point $z = ae^{i\theta}$ on the boundary of the circular disc is given by

$$q = m\left|\frac{(ae^{i\theta}-a)(ae^{i\theta}+a)}{ae^{i\theta}(ae^{i\theta}-c)(ae^{i\theta}-a^2/c)}\right| = m\left|\frac{c(e^{i\theta}-1)(e^{i\theta}+1)}{e^{i\theta}(ae^{i\theta}-c)(ce^{i\theta}-a)}\right|$$

or
$$q = mc\left|\frac{(1-e^{-i\theta})(1+e^{i\theta})}{(ae^{i\theta}-c)(ce^{i\theta}-a)}\right| = \frac{2mc\sin\theta}{a^2+c^2-2ac\cos\theta} \qquad \ldots(1)$$

For maximum q, $dq/d\theta = 0$. Hence (1) gives

$$2mc\frac{(a^2+c^2-2ac\cos\theta)\cos\theta - \sin\theta\,(2ac\sin\theta)}{(a^2+c^2-2ac\cos\theta)^2} = 0$$

or
$$(a^2+c^2)\cos\theta - 2ac = 0 \qquad \text{or} \qquad \cos\theta = (2ac)/(a^2+c^2) \qquad \ldots(2)$$

Since $\theta = 0$ gives the minimum velocity [q becomes zero at $\theta = 0$ by (1)], the value of θ given by (2) must correspond to the maximum value of velocity q. Moreover (2) gives the same angles which the diameter through the point where the line joining S to the end of the diameter at right angle to OS cuts the circle, will make with OS.

From (2), $\sin\theta = \sqrt{1-\cos^2\theta} = (c^2 - a^2)/(c^2 + a^2)$ ...(3)

Using (1), (2) and (3), the maximum value of q is given by

$$q = \frac{2mc \cdot \left(\dfrac{c^2 - a^2}{c^2 + a^2}\right)}{a^2 + c^2 - \dfrac{4a^2c^2}{a^2 + c^2}} = \frac{2mc(c^2 - a^2)}{(a^2 + c^2)^2 - 4a^2c^2} \quad \text{or} \quad q = \frac{2mc}{c^2 - a^2} = \frac{2m \cdot OS}{OS^2 - a^2}$$

Since the boundary of the circular disc is a streamline, the velocity on the boundary is the velocity of the slip.

Ex. 3. *A source S and a sink T of equal strengths m are situated within the space bounded by a circle whose centre is O. If S and T are at equal distances from O on opposite sides of it and on the same diameter AOB, show that the velocity of the liquid at any point P is*

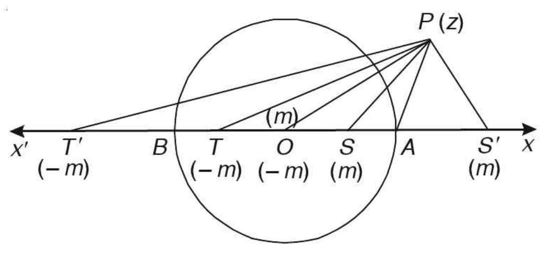
$$2m\frac{OS^2 + OA^2}{OS}\frac{PA \cdot PB}{PS \cdot PS' \cdot PT \cdot PT'},$$

where S′ and T′ are the inverses of S and T with respect to the circle. **[Rohilkhand 2001]**

Sol. Let $OS = OT = c$. Then, we have $OA = a$, $OS \cdot OS' = a^2$ and $OT \cdot OT' = a^2$ so that

$OS' = a^2/c$ and $OT' = a^2/c$. ...(1)

Now the image system of source m at S consists of a source m at S' and a sink $-m$ at O. Again the image system of sink $-m$ at T consists of a sink $-m$ at T' and a source m at O. Compounding these, we find that source m and sink $-m$ at O cancel each other. Hence the equivalent image system finally consists of



(*i*) a source of strength m at $S\,(c, 0)$

(*ii*) a source of strength m at $S'\,(a^2/c, 0)$

(*iii*) a sink of strength $-m$ at $T\,(-c, 0)$

(*iv*) a sink of strength $-m$ at $T'\,(-a^2/c, 0)$

Taking OS as the x-axis, the complex potential at any point $z\ (= x + iy)$ is given by

$$w = -m\log(z - c) - m\log\left(z - \frac{a^2}{c}\right) + m\log(z + c) + m\log\left(z + \frac{a^2}{c}\right)$$

$$\therefore \qquad \frac{dw}{dz} = -\frac{m}{z - c} - \frac{m}{z - a^2/c} + \frac{m}{z + c} + \frac{m}{z + a^2/c}$$

The velocity $q\ (= |dw/dz|)$ at any point is given by

$$q = m\left|-\frac{2c}{z^2 - c^2} - \frac{(2a^2/c)}{z^2 - (a^4/c^2)}\right| = 2m\left|\frac{c(z^2 - a^2) + (a^2/c)(z^2 - a^2)}{(z^2 - c^2)(z^2 - a^4/c^2)}\right|$$

$$= 2m\frac{c^2 + a^2}{c}\left|\frac{z^2 - a^2}{(z^2 - c^2)\left(z^2 - a^4/c^2\right)}\right| = 2m\frac{c^2 + a^2}{c}\,\frac{|z - a||z + a|}{|z - c||z + c||z - \frac{a^2}{c}||z + \frac{a^2}{c}|}$$

$$= 2m\frac{OS^2 + OA^2}{OS} \cdot \frac{PA \cdot PB}{PS \cdot PS' \cdot PT \cdot PT'}$$

Ex. 4. *In the part of an infinite plane bounded by a circular quadrant AB and the production of the radii OA, OB, there is a two-dimensional motion due to the production of the liquid at A and its absorption at B, at the uniform rate m. Find the velocity potential of the motion and show that the fluid which issues from A in the direction making an angle* μ *with OA follows the path whose polar equation is*

$$r = a\sqrt{\sin 2\theta}\,[\cot\mu + \sqrt{(\cot^2\mu + \operatorname{cosec}^2 2\theta)}]^{1/2},$$

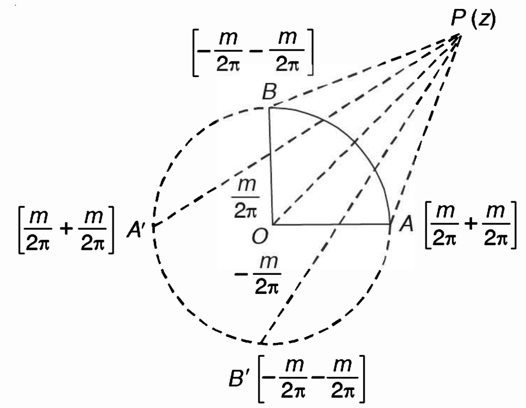
the positive sign being taken for all square roots.

Sol. The image system of source $m/2\pi$ at A with respect to the circular boundary consists of a source $m/2\pi$ at A (since A is the inverse point of itself) and a sink $-m/2\pi$ at O, the centre of the circle. Next, the image of system of the above mentioned image system with respect to the line OA and OB consists of

(*i*) a source of strength $m/2\pi + m/2\pi$ *i.e.* m/π at A $(a, 0)$

(*ii*) a source of strength $m/2\pi + m/2\pi$ *i.e.* m/π at A' $(-a, 0)$

(*iii*) a sink of strength $-\dfrac{m}{2\pi}$ at O $(0, 0)$



Again there is a sink of strength $-\ m/2\pi$ at B. The image system of this sink with respect to the circular boundary consists of a sink $-m/2\pi$ at B (since B is the inverse point of itself) and a source $m/2\pi$ and O. Again the image of the system of the above mentioned image system with respect to lines OA and OB as before consists of

(*i*) a sink of strength $-(m/2\pi) - (m/2\pi)$ *i.e.* $-(m/\pi)$ at B $(0, a)$

(*ii*) a sink of strength $-(m/2\pi) - (m/2\pi)$ *i.e.* $-(m/\pi)$ at B' $(0, -a)$

(*iii*) a source of strength $m/2\pi$ at O $(0, 0)$

Compounding these we find that source $m/2\pi$ and sink $-\ m/2\pi$ at O cancel each other. Taking OA as the x-axis, the complex potential at any point $P\,(z = x + iy = re^{i\theta})$ is given by

$$w = -\frac{m}{\pi}\log(z-a) - \frac{m}{\pi}\log(z+a) + \frac{m}{\pi}\log(z-ai) + \frac{m}{\pi}\log(z+ai)$$

$$\therefore \qquad \phi + i\psi = -\frac{m}{\pi}\log(z^2 - a^2) + \frac{m}{\pi}\log(z^2 + a^2) \qquad \ldots(1)$$

Equating real parts, (1) gives

$$\phi = -\frac{m}{\pi}\log\left|z^2 - a^2\right| + \frac{m}{\pi}\log\left|z^2 - i^2a^2\right| = -\frac{m}{\pi}\{|z-a|\cdot|z+a|\} + \frac{m}{\pi}\log\{|z-ia||z+ia|\}$$

or $$\phi = -\frac{m}{\pi}\log(AP\cdot A'P) + \frac{m}{\pi}\log(BP\cdot B'P) = \frac{m}{\pi}\log\frac{BP\cdot B'P}{AP\cdot A'P}$$

Putting $z = e^{i\theta}$ in (1) and equating imaginary parts, we get

$$\psi = -\frac{m}{\pi}\tan^{-1}\frac{r^2\sin 2\theta}{r^2\cos 2\theta - a^2} + \frac{m}{\pi}\tan^{-1}\frac{r^2\sin 2\theta}{r^2\cos 2\theta + a^2}$$

$$= -\frac{m}{\pi}\tan^{-1}\frac{\dfrac{r^2\sin 2\theta}{r^2\cos 2\theta - a^2} - \dfrac{r^2\sin 2\theta}{r^2\cos 2\theta + a^2}}{1 + \dfrac{r^4\sin^2 2\theta}{r^4\cos^2 2\theta - a^4}} = -\frac{m}{\pi}\tan^{-1}\frac{2a^2r^2\sin 2\theta}{r^4 - a^4}$$

The required streamline that leaves A at an inclination μ is given by $\psi = -(m/\pi)\mu$, *i.e.*,

$$-\frac{m}{\pi}\mu = -\frac{m}{\pi}\tan^{-1}\frac{2a^2r^2\sin 2\theta}{r^4 - a^4} \quad \text{or} \quad r^4 - 2a^2r^2\sin 2\theta\cot\mu - a^4 = 0$$

$$\therefore \quad r^2 = [2a^2\sin 2\theta\cot\mu + \sqrt{(4a^4\sin^2 2\theta\cot^2\mu + 4a^4)}]\Big/2$$

wherein negative sign has been omitted because r^2 is non-negative quantity. Thus, we have

$$r = a\sqrt{\sin 2\theta}\,[\cot\mu + \sqrt{(\cot^2\mu + \mathrm{cosec}^2 2\theta)}]^{1/2}.$$

Ex. 5. *Prove that in the two-dimensional liquid motion due to any number of sources at points on a circle, the circle is a streamline provided that there is no boundary and that the algebraic sum of the strengths of sources is zero. Show that the same is true if the region of flow is bounded by a circle which cuts orthogonally the circle in question.*

[Kanpur 1997, 2000; Rohilkhand 2000]

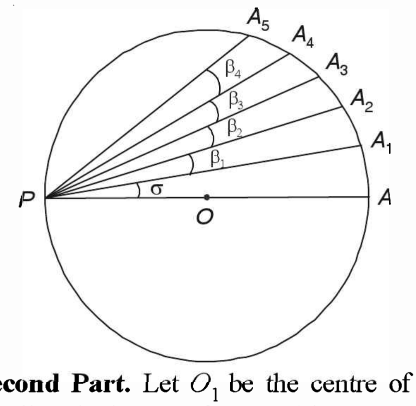
Sol. Let $A_1, A_2, A_3,\ldots$ be the positions of the sources of strengths $m_1, m_2, m_3,\ldots$ respectivly. Let P be any point on the circle and let the diameter through P be taken as the initial line.

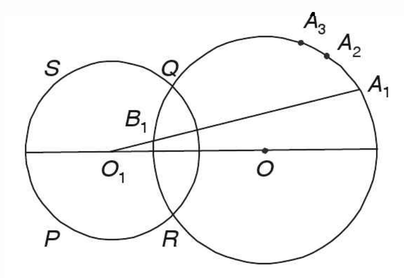
Let $\angle A_1PA = \delta$, $\angle A_2PA_1 = \beta_1$, $\angle A_3PA_2 = \beta_2$ and so on. Then the stream function ψ of the system is given by

$$\psi = -m_1\delta - m_2(\delta + \beta_1) - m_3(\delta + \beta_1 + \beta_2) - \ldots$$

$$= -\delta(m_1 + m_2 + m_3 + \ldots) - [m_2\beta_1 + m_3(\beta_1 + \beta_2) + \ldots] = -\delta\ (m_1 + m_2 + m_3 + \ldots) - \text{constant},$$

since $\beta_1, \beta_2, \beta_3, \ldots$ do not depend on the position of P. If we take $m_1 + m_2 + m_3 + \ldots = 0$, then ψ = constant is a streamline *i.e.* the circle is a stream line.





Second Part. Let O_1 be the centre of a circle which cuts the above circle (with centre O) orthogonally. The image of m_1 at A is m_1 at B_1, the inverse point of A and a sink $-m_1$ at O_1. If the barriers are omitted, we see that the system reduces to a source 2 $(m_1 + m_2 + \ldots)$ on the boundary of the given circle and a sink $-(m_1 + m_2 + \ldots)$ at O_1. Since $m_1 + m_2 + \ldots = 0$, the result follows.

Ex. 6. *A line source is in the presence of an infinite plane on which is placed a semi-circular cylindrical boss, the direction of the source is parallel to the axis of the boss, the source is at a disdtance c from the plane and the axis of the boss, whose radius is a. Show that the radius to the point on the boss at which the velocity is a maximum makes an angle* θ *with the radius to the source, where*

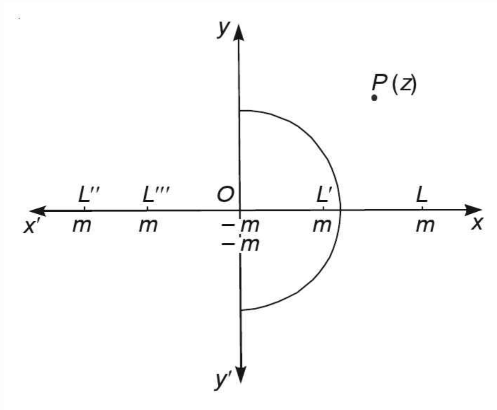
$$\theta = \cos^{-1}\frac{a^2+c^2}{\sqrt{\{2(a^4+c^4)\}}}$$ **[Agra 1999, 2000]**

OR If the axis of y and the circle $x^2 + y^2 = a^2$ are fixed boundaries and there is a two-dimensional source at the point (c, 0) where c > a, show that the radius drawn from the origin to the point on the circle, where the velocity is a maximum, makes with the axis of x an angle

$$\cos^{-1}\frac{a^2+c^2}{\sqrt{\{2(a^4+c^4)\}}}.$$

When c = 2a, show that the required angle is $\cos^{-1}(5/\sqrt{34})$.

Sol. Let there be a source of strength m at L $(c, 0)$. Let L' be the inverse point of L with respect to the circular boundary so that $OL \times OL' = a^2$ *i.e.* $OL' = a^2/c$. The image of source m at L in the circular boundary (cylindrical boundary) is a source m at L' and a sink $-m$ at O.



For the above system the equivalent image system with respect to the y-axis (*i.e.* the line $x = 0$) consists of

(*i*) a source m at L $(c, 0)$ and L'' $(-c, 0)$

(*ii*) a source m at L' $(a^2/c, 0)$ and $L'''(-a^2/c, 0)$

(*iii*) a sink $-m-m$ *i.e.* $-2m$ at O $(0, 0)$

Thus, if P $(z = x + iy = re^{i\theta})$ is any point in the fluid, the complex potential at P due to the above system is given by

$$w = -m\log(z-c) - m\log(z+c) - m\log(z-a^2/c) - m\log(z+a^2/c) + 2m\log z$$

or $$w = 2m\log z - m\log(z^2-c^2) - m\log(z^2-a^4/c^2)$$

$$\therefore \quad \frac{dw}{dz} = \frac{2m}{z} - \frac{2mz}{z^2-c^2} - \frac{2mz}{z^2-a^4/c^2} \quad \text{or} \quad \frac{dw}{dz} = -\frac{2m(z^4-a^4)}{z(z^2-c^2)(z^2-a^4/c^2)}$$

The velocity $q\ (=|dw/dz|)$ at any point $P(z = ae)^{i\theta}$ on the circular boundary is given by

$$q = \frac{2m|a^4e^{4i\theta}-1|}{|ae^{i\theta}(a^2e^{2i\theta}-c^2)(a^2e^{2i\theta}-a^4/c^2)|} \quad \text{or} \quad q = \frac{4mac^2\sin 2\theta}{a^4+c^4-2a^2c^2\cos 2\theta}$$

or $$(4mac^2/q) = (a^4+c^4-2a^2c^2\cos 2\theta)/\sin 2\theta \qquad \ldots(1)$$

Let $f = 4mac^{2\prime}/q$. When q is maximum, then f will be minimum. From (1), we have

$$f = (a^4+c^4)\operatorname{cosec} 2\theta - 2a^2c^2\cot 2\theta \qquad \ldots(2)$$

$$\therefore \qquad df/d\theta = -2(a^4 + c^4)\operatorname{cosec} 2\theta \cot 2\theta + 4a^2c^2 \operatorname{cosec}^2 2\theta \qquad \ldots(3)$$

$$d^2 f/d\theta^2 = 4(a^4 + c^4)\operatorname{cosec} 2\theta (\operatorname{cosec}^2 2\theta + \cot^2 2\theta) - 8a^2c^2 \operatorname{cosec}^2 2\theta \cot 2\theta$$

$$= 4\operatorname{cosec} 2\theta [(a^2 \operatorname{cosec} 2\theta - c^2 \cot 2\theta)^2 + a^4 \cot^2 2\theta + c^4 \operatorname{cosec}^2 2\theta]$$

Since $\theta \le \pi/2$, clearly $d^2 f/d\theta^2$ is positive and hence f will be minimum and consequently q will be maximum. From (3), setting $df/d\theta = 0$, we get

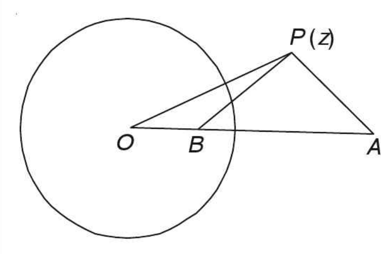
$$(a^4 + c^4)\operatorname{cosec} 2\theta \cot 2\theta = 4a^2c^2 \operatorname{cosec}^2 2\theta \qquad \text{or} \qquad \cos 2\theta = 2a^2c^2/(a^4 + c^4)$$

$$\therefore \qquad 2\cos^2\theta - 1 = 2a^2c^2/(a^4 + c^4), \qquad \text{or} \qquad \cos^2\theta = (a^2 + c^2)^2/2(a^4 + c^4)$$

so that
$$\cos\theta = (a^2 + c^2)/\sqrt{\{2(a^4 + c^4)\}}.$$

Ex. 7. *A source of fluid situated in space of two dimensions, is of such strength that* $2\pi\rho\mu$ *represents the mass of fluid of density* ρ *emitted pet unit of time. Show that the force necessary to hold a circular disc at rest in the plane of source is* $2\pi\rho\mu^2 a^2/r(r^2 - a^2)$, *where a is the radius of the disc and r the distance of the source from its centre. In what direction is the disc urged by the pressure ?* **[Kanpur 2005, 06; Meerut 2005, 11; Rohilkhand 2002]**

Sol. Since the mass of fluid emitted is $2\pi\rho\mu$ per unit of time, by definition the strength of the given source is μ. Let this source be situated at A such that $OA = r$ and let B be the inverse point of A. Then, $OA \cdot OB = a^2$ so that $OB = a^2/r$. Here the equivalent image system consists of (taking OA as x-axis and using Art. 5.21)



(*i*) a source of strength μ at A $(r, 0)$

(*ii*) a source of strength μ at B $(a^2/r, 0)$

(*iii*) a sink of strength μ at O $(0, 0)$

Hence the complex potentioal at any point P $(z = x + iy)$ is given by

$$w = -\mu \log(z - r) - \mu \log(z - a^2/r) + \mu \log z$$

$$\therefore \qquad \frac{dw}{dz} = -\frac{\mu}{z - r} - \frac{\mu}{z - a^2/r} + \frac{\mu}{z} \qquad \ldots(1)$$

If the pressure thrusts on the given circular disc are represented by (X, Y), then by Blasius' theorem, we have

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz \qquad \ldots(2)$$

where C is the boundary of the disc. Again, by Cauchy's residue theorem, we have

$$\int_C \left(\frac{dw}{dz}\right)^2 dz = 2\pi i \times [\text{sum of the residues}], \qquad \ldots(3)$$

wherein the indicated sum of the residues is calcualated at poles of $(dw/dz)^2$ lying within the circular boundary. Using (3), (2) reduces to

$$X - iY = -\pi\rho \times [\text{sum of the residues}] \qquad \ldots(4)$$

We proceed to find the residues of $(dw/dz)^2$. From (1), we have

$$\left(\frac{dw}{dz}\right)^2 = \mu^2 \left[\frac{1}{(z - r)^2} + \frac{1}{(z - a^2/r)^2} + \frac{1}{z^2} - \frac{2}{z(z - r)} - \frac{2}{z(z - a^2/r)} + \frac{2}{(z - r)(z - a^2/r)}\right]$$

$$= \mu^2 \left[\frac{1}{(z-r)^2} + \frac{1}{(z-a^2/r)^2} + \frac{1}{z^2} - \frac{2}{z(z-r)} + \frac{2}{rz} - \frac{2}{(a^2/r)(z-a^2/r)} \right.$$

$$\left. + \frac{2}{(a^2/r)z} + \frac{2}{(r-a^2/r)(z-r)} + \frac{2}{(a^2/r-r)(z-a^2/r)} \right] \qquad \ldots(5)$$

[Resolving R.H.S. into partial fractions]

From (5), we find that the poles inside the circular contour C are $z = 0$ and $z = a^2/r$.

$\therefore$ The required sum of the residues

= the sum of the coffecients of z^{-1} and $(z - a^2/r)^{-1}$ in R.H.S. of (5)

$$= \frac{2\mu^2}{r} + \frac{2\mu^2}{a^2/r} - \frac{2\mu^2}{a^2/r} + \frac{2\mu^2}{a^2/r - r} = \frac{2\mu^2 a^2}{r(a^2 - r^2)} \qquad \ldots(6)$$

Using (6) in (4) and then equating real and imaginary parts, we have

$$X = 2\pi\rho\mu^2 a^2 / r(r^2 - a^2) \qquad \text{and} \qquad Y = 0.$$

Thus the disc is attracted towards the source along OA. Hence the disc will be urged to move along OA.

Ex. 8. *Within a circular boundary of radius a there is a two-dimensional liquid motion due to source producing liquid at the rate m, at a distance f from the centre, and an equal sink at the centre. Find the velocity potential and show that the resultant pressure on the boundary is* $\rho m^2 f^3 / 2a^2(a^2 - f^2)$, *where* ρ *is the density. Deduce as a limit velocity potential due to a doublet at the centre.* **[Rohilkhand 2000; Agra 2004; Kanpur 1997; Meerut 1999, 2005]**

Sol. Since the rate of production of liquid is m, by definition the strength of the given source is $m/2\pi$. Let this sources be situated at B such that $OB = f$ (refer figure of Ex. 7). Let A be the inverse point of B. Then $OA \cdot OB = a^2$ so that $OA = a^2/f$.

Taking OA as x-axis, the equivalent image system consists of

(*i*) a source of strength $m/2\pi$ at B $(f, 0)$ (*ii*) a source of length $m/2\pi$ at A $(a^2/f, 0)$

(*iii*) a sink of strength $-\ m/2\pi$ at O (0, 0)

Hence the complex potential w at any point P $(z = x + iy)$ is

$$w = -(m/2\pi)\log(z-f) - (m/2\pi)\log(z - a^2/f) + (m/2\pi)\log z \qquad \ldots(1)$$

$$\therefore \qquad \frac{dw}{dz} = -\frac{m}{2\pi}\left[\frac{1}{z-f} + \frac{1}{z-a^2/f} - \frac{1}{z}\right]$$

$$\therefore \quad \left(\frac{dw}{dz}\right)^2 = \frac{m^2}{4\pi^2}\left[\frac{1}{(z-f)^2} + \frac{1}{(z-a^2/f)^2} + \frac{1}{z^2} + \frac{2}{(z-f)(z-a^2/f)} - \frac{2}{z(z-a^2/f)} - \frac{2}{z(z-f)}\right]$$

$$= \frac{m^2}{4\pi^2}\left[\frac{1}{(z-f)^2} + \frac{1}{(z-a^2/f)^2} + \frac{1}{z^2} + \frac{2}{(f-a^2/f)(z-f)} + \frac{2}{(a^2/f - f)(z-a^2/f)} \right.$$

$$\left. + \frac{2}{za^2/f} - \frac{2}{(a^2/f)(z-a^2/f)} - \frac{2}{f(z-f)} + \frac{2}{fz}\right] \qquad \ldots(5)$$

[Resolving R.H.S. into partial fraction]

If the pressure thrusts on the given circular disc are represented by (X, Y), then by Blasius' theorem, we have

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz \qquad \ldots(6)$$

where C is the boundary of the disc. Again, by Cauchy's residue theorem, we have

$$\int_C \left(\frac{dw}{dz}\right)^2 dz = 2\pi i \times [\text{sum of the residues}] \qquad \ldots(7)$$

Using (7), (6) reduces to $\qquad X - iY = \pi\rho \times [\text{sum of the residues}] \qquad \ldots(8)$

From (5), we find that poles inside the circular contour C are at $z = 0$ and $z = f$.

$\therefore$ The required sum of the residues

= the sum of the coefficients of z^{-1} and $(z - f)^{-1}$ in R.H.S. of (5)

$$= \frac{m^2}{4\pi^2}\left[\frac{2}{(f - a^2/f)} + \frac{2}{a^2/f} - \frac{2}{f} + \frac{2}{f}\right] = \frac{m^2 f^3}{2a^2\pi^2(f^2 - a^2)}$$

Using this in (8) and equating real and imaginary parts, we have

$$X = \rho m^2 f^3 / 2a^2(a^2 - f^2) \qquad \text{and} \qquad Y = 0, \qquad \ldots(9)$$

giving the required pressure.

We now obtain the velocity potential. Note that real part of $\log z = \log |z|$. Writing $w = \phi + i\psi$ (1) and equating real parts, we have

$$\phi = -\frac{m}{2\pi}[\log|z - f| + \log|z + f| - \log|z|]$$

$$= -\frac{m}{2\pi}[\log PB + \log PA - \log PO] = -\frac{m}{2\pi}\log\frac{PB \cdot PA}{PO}.$$

Second part. In order to obtain the doublet at the centre, make $f \to 0$, $(m/2\pi)f \to \mu$ so that $a^2/f \to \infty$. Then, (1) reduces to

$$w = -\frac{m}{2\pi}\left[\log\left(1 - \frac{f}{z}\right) + \log\left(1 - \frac{fz}{a^2}\right)\right],$$

where we have rejected the constant term. Using the expansions of log $(1 \pm x)$ and rejecting powers of f higher than the first, we get

$$w = \frac{m}{2\pi}\left(\frac{f}{z} + \frac{fz}{a^2}\right) \qquad \text{or} \qquad \phi + i\psi = \frac{\mu}{z} + \frac{\mu z}{a^2} \qquad \left[\because \frac{mf}{2\pi} = \mu\right]$$

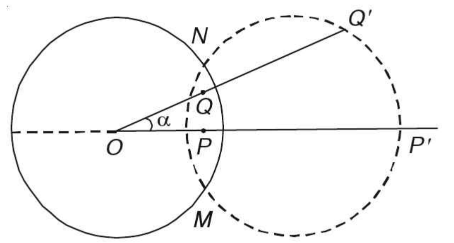
or $$\phi + i\psi = \frac{\mu}{r}e^{-i\theta} + \frac{\mu r}{a^2}e^{i\theta}, \qquad \text{as} \qquad z = re^{i\theta}$$

Equating real parts, the velocity potential due to doublet at centre is given by

$$\phi = \mu(r^{-1} + a^{-2}r)\cos\theta$$

Ex. 9. *Find the velocity potential when there is a source and an equal sink within a circular cavity and show that one of the stream lines is an arc of the circle which passes through the source and the sink and cuts orthogonally the boundary of the cavity.*

Sol. Let the source m and the sink $-m$ be situated at the P and Q within the circular cavity with centre at O (0, 0). Let P' and Q' be the inverse points of P and Q respectively. Now, to do away with the circular cavity, we proceed to get an equivalent system. The image system of source m at P consists of a source m at P' and a sink $-m$ at O. Similarly, the image system of sink $-m$ at Q consists of a sink $-m$ at Q' and a source m at O. The source and sink at O cancel each other and then the resulting equivalent system consists of



(*i*) a source of strength m at P $(z = c)$

(*ii*) a source of strength m at P' $(z = a^2/c)$

(*iii*) a sink of strength $-m$ at $Q\,(z = be^{i\alpha})$

(*iv*) a sink of strength $-m$ at $Q'[z = (a^2/b)e^{i\alpha}]$

wherein $OP = c$, $OQ = b$, a = radius of circle with centre O, $OP{\cdot}OP' = a^2$ so that $OP' = a^2/c$, $OQ{\cdot}OQ' = a^2$ so that $OQ' = a^2/b$ and $\angle QOP = \alpha$.

Hence the complex potential at any point $P\,(z)$ is

$$w = -m\log(z-c) - m\log(z - a^2/c) + m\log(z - be^{i\alpha}) + m\log[z - (a^2/b)e^{i\alpha}]$$

$$\phi + i\psi = m\log\frac{(z - be^{i\alpha})[z-(a^2/b)e^{i\alpha}]}{(z-c)(z-a^2/c)} \qquad \ldots(1)$$

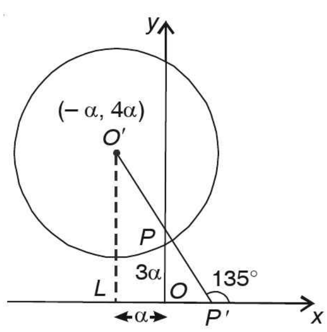
The desired velocity potential ϕ and stream function ψ may be obtained by equating real and imaginary parts in (1).

Since $OP{\cdot}OP' = OQ \cdot OQ' = a^2$, the points P, Q, P', Q' are concyclic. Let the circle passing through P, Q, P', Q' cut given circle (with centre O) at N and M. Since $OP \cdot OP' = a^2 = ON^2$, ON must be tangent to the circle through N. Hence the two circles cut orthogonally. Again the circle $PQMN$ passes through P and Q (*i.e.* source and sink) and so it is a streamline.

Ex. 10. *With a rigid boundary in the form of the circle* $(x+\alpha)^2 + (y-4\alpha)^2 = 8\alpha^2$, *there is a liquid motion due to a doublet of strength* μ *at the point* $(0, 3\alpha)$ *with its axis along the axis of y. Show that the velocity potential is*

$$\mu\left\{\frac{4(x-3\alpha)}{(x-3\alpha)^2+y^2} + \frac{y-3\alpha}{x^2+(y-3\alpha)^2}\right\}. \qquad \textbf{[Meerut 2008]}$$

Sol. The given circle has centre $O'(-\alpha, 4\alpha)$ and radius $= \sqrt{(8\alpha^2)} = 2\sqrt{2}\alpha$. Let the given doublet be at $P\,(0, 3\alpha)$.



$$\text{Gradient of } O'P = \frac{3\alpha - 4\alpha}{0-(-\alpha)} = -1 = \tan\frac{3\pi}{4}.$$

Hence $O'P$ makes an angle $\pi/4$ with OY. Let P' be the image of P. Then the axis of doublet at P' will make an angle of 45°with PP' and hence it will be parallel to x-axis.

We now show that P' lies on x-axis. We have

$$O'P \cdot O'P' = 8\alpha^2 \quad \text{or} \quad O'P\,(O'P + PP') = 8\alpha^2 \qquad \ldots(1)$$

$$\text{But} \quad O'P = \sqrt{[(-\alpha-0)^2 + (4\alpha-3\alpha)^2]} = \alpha\sqrt{2} \qquad \ldots(2)$$

$\therefore$ From (1), $\alpha\sqrt{2}(\alpha\sqrt{2} + PP') = 8\alpha^2$ or $PP' = 3\alpha\sqrt{2} = 3\alpha\sec 45° = OP\sec 45°$

This shows that P' lies on the x-axis and that co-ordinates of P' are $(3\alpha, 0)$. Let μ be the strength of P.

Then, the strength of doublet at $P' = \mu \cdot \dfrac{(\text{radius})^2}{(O'P)^2} = \mu \dfrac{8\alpha^2}{2\alpha^2} = 4\mu \cdot$

Thus the equivalent image system consists of doublets at P and P'. Hence the complex potential of motion at point z $(= x + iy)$ is given by

$$w = \frac{\mu e^{\pi i/2}}{z - 3i\alpha} + \frac{4\mu e^{0.i}}{z - 3\alpha}, \qquad \textit{where} \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

or $$\phi + i\psi = \mu\left[\frac{4}{x + iy - 3\alpha} + \frac{i}{x + iy - 3i\alpha}\right] = \mu\left[4\frac{(x - 3\alpha) - iy}{(x - 3\alpha)^2 + y^2} + \frac{i\{x - i(y - 3\alpha)\}}{x^2 + (y - 3\alpha)^2}\right] \qquad \ldots (3)$$

Equating real parts, (3) gives $$\phi = \mu\left[\frac{4(x - 3\alpha)}{(x - 3\alpha)^2 + y^2} + \frac{y - 3\alpha}{x^2 + (y - 3\alpha)^2}\right].$$

Ex. 11. *Find image of a line source in a circular cylinder.*

Sol. Let there be a uniform line source of strength m per unit length through the point $z = c$, where $z > a$. Then the complex potential at a point z is given by

$$f(z) = -m \log (z - c)$$

Then $$\bar{f}(z) = -m \log (z - c)$$

and so $$\bar{f}(a^2/z) = -m \log \{(a^2/z) - c\}$$

Let a circular cylinder of section $|z| = a$ be introduced. Then the new complex potential by Milne-Thomson's circle theorem is given by

$$w = f(z) + \bar{f}(a^2/z) \qquad \text{for} \qquad |z| \ge a$$

i.e. $$w = -m \log (z - c) - m \log \{(a^2/z) - c\}$$

or $$w = -m \log (z - c) - m \log \{z - (a^2/c)\} + m \log z + \text{constant}, \qquad \ldots(1)$$

the constant (real or complex) being immaterial for the discussion of the flow. The point $z = a^2/c$ is the inverse point of the point $z = c$ with regard to the circle $|z| = a$. Hence (1) shows that the image of a line source in a right circular cylinder is an equal line source through the inverse point in the circular section in the plane of flow together with an equal line sink through the centre of the section.

Ex. 12. *Determine image of a line doublet parallel to the axis of a right circular cylinder.*

Sol. Let there be a uniform line doublet of strength μ per unit length through the point $z = c > a$. Furthermore let the axis of the line doublet be inclined at an angle α to x-axis. Then the complex potential at a point z is given by

$$f(z) = (\mu e^{i\alpha})/(z - c)$$

Then $$\bar{f}(z) = (\mu e^{-i\alpha})/(z - c)$$

and so $$\bar{f}(a^2/z) = \frac{\mu e^{-i\alpha}}{(a^2/z) - c}.$$

Let a circular cylinder of section $|z| = a$ be introduced. Then the new complex potential by Milne-Thomson's circle theorem is given by $$w = \frac{\mu e^{-i\alpha}}{z - c} + \frac{\mu e^{-i\alpha}}{(a^2/z) - c}.$$

Ex. 13. A source and sink of equal strength are placed at the points ($\pm a/2$, 0) within a fixed circular boundary $x^2 + y^2 = a^2$. Show that the streamlines are given by
$(r^2 - a^2 / 4)(r^2 - 4a^2) - 4a^2 y^2 = ky(r^2 - a^2)$. **[Bhopal 1999, 2000; I.A.S. 1984, 86]**

Sol. Corresponding to a source of strength m, say at $(a/2, 0)$ and an equal sink of strength $-m$ at $(-a/2, 0)$, the complex potential $f(z)$ in absence of the given circular boundary $x^2 + y^2 = a^2$, is given by

$$f(z) = -m \log (z - a/2) + m \log (z + a/2) \qquad \ldots(1)$$

$$(1) \quad \Rightarrow \quad f(a^2/z) = -m \log (a^2/z - a/2) + m \log (a^2/z + a/2)$$

When the circular boundary $x^2 + y^2 = a^2$ is inserted, the complex potential w at any interior point of the boundary is given by

$$w = f(z) + f(a^2/z), \qquad \text{that is,}$$

$$w = m \log (z + a/2) - m \log (z - a/2) + m \log \{(z + 2a)/2z\} - m \log \{(2a - z)/2z\}$$

or $\quad w = m \log (z + a/2) - m \log (z - a/2) + m \{\log (2a + z) - \log 2z\} - m\{\log (2a - z) - \log 2)\}$

or $\quad \phi + i\psi = m \log (x + a/2 + iy) - m \log (x - a/2 + iy) + m \log (2a + x + iy) - m \log (2a - x - iy)$

Using results $\quad \log (x + iy) = (1/2) \times \log (x^2 + y^2) + i \tan^{-1} (y/x) \quad$ and $\log (x - iy) = (1/2) \times \log (x^2 + y^2) - i \tan^{-1} (y/x)$ on R.H.S. of the above equation and then equation imaginary parts on both sides, we obtain

$$\frac{\psi}{m} = \tan^{-1} \frac{y}{x + a/2} - \tan^{-1} \frac{y}{x - a/2} + \tan^{-1} \frac{y}{2a + x} - \left(-\tan^{-1} \frac{y}{2a - x}\right)$$

$$= \tan^{-1} \frac{\dfrac{y}{x + a/2} - \dfrac{y}{x - a/2}}{1 + \dfrac{y^2}{x^2 - a^2/4}} + \tan^{-1} \frac{\dfrac{y}{2a + x} + \dfrac{y}{2a - x}}{1 - \dfrac{y^2}{4a^2 - x^2}}$$

$$= \tan^{-1}\left(\frac{4ay}{4a^2 - r^2}\right) - \tan^{-1}\left(\frac{ay}{r^2 - a^2/4}\right), \quad \text{where} \quad r^2 = x^2 + y^2$$

$$= \tan^{-1} \frac{\dfrac{4ay}{4a^2 - r^2} - \dfrac{ay}{r^2 - a^2/4}}{1 + \dfrac{4a^2 y^2}{(4a^2 - r^2)(r^2 - a^2/4)}} = \tan^{-1} \frac{5ay(r^2 - a^2)}{(4a^2 - r^2)(r^2 - a^2/4) + 4a^2 y^2}$$

The required streamlines in the desired form can be obtained by choosing ψ = constant $= m \tan^{-1} (-5a/k)$. Thus, the required streamlines are given by

$$-\frac{5a}{k} = \frac{5ay(r^2 - a^2)}{(4a^2 - r^2)(r^2 - a^2/4) + 4a^2 y^2} \quad \text{or} \quad \frac{1}{k} = \frac{y(r^2 - a^2)}{(r^2 - a^2/4)(r^2 - 4a^2) - 4a^2 y^2}$$

or $$(r^2 - a^2/4)(r^2 - 4a^2) - 4a^2 y^2 = ky(r^2 - a^2).$$

Ex. 14. *Verify that $w = ik \log \{(z - ia)/(z + ia)\}$ is the complex potential of a steady flow of liquid about a circular cylinder the plane $y = 0$ being a rigid boundary. Find the force exerted by the liquid on unit length of the cylinder.* **[Bhopal 1993; Rohilkhand 1998]**

Sol. We have

$$w = \phi + i\psi = ik \log |(z - ia)/(z + ia)| \qquad \ldots(1)$$

Hence,

$$\psi = k \log |(z - ia)/(z + ia)|$$

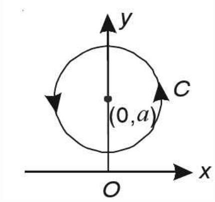
and so the streamlines are given by ψ = constant = $k\lambda$, say *i.e.*,

$$|(z - ia)/(z + ia)| = \lambda, \qquad \text{or} \qquad |z - ia| = \lambda |z + ia|, \qquad \ldots(2)$$

which are non-intersecting co-axial circles having $z = \pm\, ia$ as the limiting points. In particular, for $\lambda = 1$, (2) represents the straight line $|z - ia| = |z + ia|$, *i.e.*, $|x + i(y - a)| = |x + i(y + a)|$, *i.e.*, $x^2 + (y - a)^2 = x^2 + (y + a)^2$, *i.e.*, $y = 0$, showing that $y = 0$ is a rigid boundary.

From (1), $$w = ik\,\{\log(z - ia) - \log(z + ia)\}$$

so that $$\frac{dw}{dz} = ik\left(\frac{1}{z - ia} - \frac{1}{z + ia}\right) = \frac{2ka}{z^2 + a^2} \qquad \ldots(3)$$



The adjoining figure shows that circular section C of the cylinder and the rigid plane.

If the pressure thrusts on the given circular disc are represented by (X, Y) then by Blasius theorem, we have

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz = 2k^2a^2\,\rho i \int_C \frac{dz}{(z^2 + i^2)^2} \qquad \ldots(4)$$

Again, by Cauchy's residue theorem, we have

$$\int_C \frac{dz}{(z^2 + a^2)^2} = 2\pi i \times \text{(sum of the resides)}$$

$\therefore$ (4) becomes, $$X - iY = -\,4k^2a^2\rho\pi \times \text{(sum of the residues)} \qquad \ldots(5)$$

where the indicated sum of the residues is calculated at poles of $1/(z^2 + a^2)^2$ lying within the circular boundary C.

We now proceed to find the residues of $1/(z^2 + a^2)^2$. The only poles of $1/(z^2 + a^2)^2$ are at $z = \pm\, ia$. But only $z = ia$ lies within the boundadry C as shown in the figure. Hence we shall find residue at $z = ia$.

Now, $$\frac{1}{z^2 + a^2} = \frac{1}{(z - ia)(z + ia)} = \frac{1}{2ia}\left(\frac{1}{z - ia} - \frac{1}{z + ia}\right)$$

$\therefore$ $$\frac{1}{(z^2 + a^2)^2} = -\frac{1}{4a^2}\left\{\frac{1}{(z - ia)^2} + \frac{1}{(z + ia)^2} - \frac{2}{(z - ia)(z + ia)}\right\}$$

$$= -\frac{1}{4a^2}\left\{\frac{1}{(z - ia)^2} + \frac{1}{(z + ia)^2} - \frac{1}{2ia}\left(\frac{1}{z - ia} - \frac{1}{z + ia}\right)\right\}$$

Hence, Reissue of $1/(z^2 + a^2)^2$ at $z = ia$ is $1/(8ia^3)$.

$\therefore$ (5) becomes $$X - iY = -\,(4k^2a^2\rho\pi) \times (1/8ia^3) = \{(\pi\rho k^2)/2a\}i$$

$\Rightarrow$ $X = 0$ and $Y = -(\pi\rho k^2/2a)$,

showing that the liquid exerts a downward force on the cylinder of numerical value $(\pi\rho k^2/2a)$ per unit length.

Ex. 15. *In the two-dimensial motion of an infinite liquid there is a rigid boundary consisting of that part of the circle $x^2 + y^2 = a^2$ which lies in the first and fourth quadrants and the parts of y-axis which lie outside the circle. A simple source of strength m is placed at the point $(f, 0)$ where $f > a$. Prove that the speed of the fluid at the point $(a\cos\theta, a\sin\theta)$ of the semi-circular boundary is $(4\,a\,m\,f^2 \sin 2\theta)/(a^4 + f^4 - 2a^2f^2\cos 2\theta)$. Find at what point of the boundary the pressure is least ?*

Sol. Refer solution of Ex. 6. Here $f = c$. Then equation (1) gives the required value of speed of the fluid.

Second part. By Bernoulli's equation. $p + (\rho q^2)/2$ = constant. So it follows that p is least when q is maximum. Hence as explained in solution of Ex. 6 at a point $P(a \cos\theta, a \sin\theta)$, where θ is given by $\cos\theta = (a^2 + f^2)/[2(a^4 + f^4)]^{1/2}$, the pressure is least. At every other point, p is greater than that at P.

Ex. 16. *A circular cylinder is placed in a uniform stream, find the forces acting on the cylinder.*

Sol. For undisturbed motion, the complex potential is given by $\qquad w = (u - iv)z.$

Then by circle theorem, complex potential for the disturbed motion is

$$w = (u - iv)z + (u + iv)(a^2/z).$$

so that
$$dw/dz = (u - iv) - (u + iv)(a^2/z^2). \qquad \ldots(1)$$

If the pressure thrusts on the given cylinder are represented by a force (X, Y) and a couple of moment M about the origin of co-ordinates, then by Blasius's throrem, we have

$$X - iY = \frac{1}{2} i\rho \int_C (dw/dz)^2 dz \qquad \ldots(2)$$

and
$$N = \text{Real part of } -\frac{1}{2}\rho \int_C z\,(dw/dz)^2 dz \qquad \ldots(3)$$

where ρ is the fluid density and integrals are taken round the contour C of the cylinder.

From (1) and (2), we have

$$X - iY = (1/2) \times i\rho \int_C \left\{(u - iv) - (u + iv)(a^2/z^2)\right\}^2 dz = 0 \Rightarrow X = 0 \quad \text{and} \quad Y = 0$$

From (1) and (3), we have

$$N = \text{Real part of } -\frac{1}{2}\rho \int_C \left[(u - iv) - (u + iv)(a^2/z^2)\right]^2 z\,dz$$

$$= \text{Real part of } -\frac{1}{2}\rho \int_C \left[(u - iv)^2 - 2(u^2 + v^2)(a^2/z^2) + \cdots\right] z\,dz$$

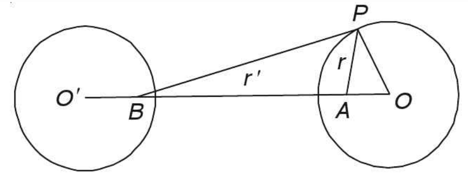
$$= \text{Real part of} - (1/2) \times \rho\{-2(u^2 + v^2)a^2\} 2\pi i = 0$$

Thus, we find that no force or couple acts on the cylinder.

Ex. 17. *Prove that for liquid circulating irrotationally in part of the fluid between two non-intersecting circles the curves of constant veloctity are Cassini's Ovals.*

[U.P.P.C.S 1997; Rohilkhand 1994; I.A.S. 1993]

Sol. Let O and O' be the centres of the two non-intersecting circles. Let $A(a, 0)$ and $B(-a, 0)$ be the inverse points with respect to both the circles. Let P be any point on one of the given circles such that $PA = r$ and $PB = r'$.



Since A and B are inverse points of the circle with centre O, so by defintion, we have

$$OA \cdot OB = OP^2$$

Now, from similar triangles OPA and OPB, we have

$$PA/PB = OP/OB = \text{constant} \quad \Rightarrow \quad r/r' = \text{constant}.$$

Hence the equations of the two circles may taken as $r/r' = c_1$ and $r/r' = c_2$, where c_1 and c_2 are constants. Since these circles are two streamlines, it follows that the stream function ψ is of the form $f(r/r')$ and it being a harmonic, we take $\psi = k \log (r/r')$ because $\log r$ is the only function of r which is plane harmonic. Here k is a constant.

Now, if θ is the conjugate harmonic of r, $\phi + i\psi$ or $\psi - i\phi$ must be an analytic function of z, so that

$$\phi = -k(\theta - \theta').$$

$$\therefore \quad w = \psi - i\phi = k\log(r/r') + ik(\theta - \theta') = k[\log r - \log r' + i\theta - i\theta']$$

$$= k[(\log r + i\theta) - (\log r' + i\theta')] = k[\log(re^{i\theta}) - \log(r'e^{i\theta'})]$$

or $\quad w = k[\log(z-a) - \log(z+a)], \quad$ as $\quad re^{i\theta} = z - a \quad$ and $\quad r'e^{i\theta'} = z + a$

$$q = \left|\frac{dw}{dz}\right| = \left|k\left[\frac{1}{z-a} - \frac{1}{z+a}\right]\right| = \frac{2ak}{|z-a||z+a|} = \frac{2ak}{rr'}.$$

Hence the curves of equal velocity are given by q = constant or $(2ak)/rr'$ = constant or rr' = constant, which are Cassini's ovals.

EXERCISE 5 (D)

1. Show that the image system of a source outside a circle consists of an equal source at the inverse point and an equal sink at the centre of the circle. **[Meerut 2000]**

2. Show that the force per unit length extered on a circular cylinder, radius a, due a source of strength m, at a distance c from the the axis is $(2\pi\rho m^2 a^2)/c(c^2 - a^2)^2$ **[Kanpur 2005]**

Hint. Refer Ex. 7 of Art. 5.25. Here $\mu = m$, $r = a$.

3. The boundary of a semi-infinite liquid consists of an infinite plane surmounted by the cylinder boss of semi-circular cross-section of radius a and the liquid contains a line source everywhere at a distance c from the plane and the axis of the boss, where $c = a \tan\lambda$. Show that the velocity at points on the boss is a maximum along the generators lying in the axial planes, making an angle an angle θ with the axial plane containing the line source, given by $\tan\theta = \pm\cos 2\lambda$.

4. A source is situated at the point (c, c) on the region bounded by the x-axis and the circle $x^2 + y^2 = a^2$, the source being outside the circle. Show that the fluid velocity vanishes at the points $(\pm a, 0)$ and that it will vanish at any other point on the circle provided that $2c < (2+\sqrt{2})a$.

5. If a circle be cut in half by the y-axis, forming rigid boundary and a source of strength m be on the x-axis at a distance a, equal to half the radius, from the centre, prove that the stream lines are given by

$$(16a^4 + r^4)\cos 2\theta - 17a^2r^2 = (16a^4 - r^4)\sin 2\theta \cot(\psi/m).$$

Show that stream line $\psi = m\pi/2$ leves the source in a direction perpendicular to OX and entres the sink at and angle $\pi/4$ with OX.

6. Within a rigid circular boundary of radius a there is a source of strength m at a point P distance b from the centre; at Q, R the extremities of the diameter through P are equal sinks. Find the velocity potential and stream function of two dimensional fluid motion.

7. A simple source, of strength m, is fixed at the origin O in a uniform stream of incompessible fluid moving with velocity U. Show that the velocity potential ϕ at any point P of the stream is

$(m/r) - U\, r\cos\theta$, where $OP = r$ and θ is the angle OP makes with the direction $\boldsymbol{i}$. Find the differential equation of the stream lines and show that they lie on the surfaces

$$Ur^2 \sin^2\theta - 2m\cos\theta = \text{constant.}$$

OBJECTIVE QUESTIONS ON CHAPTER 5

Choose the correct alternative from the following questions

1. The image of source $+ m$ with respect to a circle is a source $+ m$ at the inverse point and
(*i*) a source $+ m$ at the centre (*ii*) a source $+ m$ at the same point
(*iii*) a sink $- m$ at the centre (*iv*) None of these. **[Kanpur 2003]**

2. The relation between ϕ and ψ is
(*i*) $\partial\phi/\partial x = \partial\psi/\partial y$ and $\partial\phi/\partial y = \partial\psi/\partial x$ (*ii*) $\partial\phi/\partial x = \partial\psi/\partial y$ and $\partial\phi/\partial y = -\partial\psi/\partial x$
(*iii*) $\partial\phi/\partial x = -\partial\psi/\partial y$ and $\partial\phi/\partial y = \partial\psi/\partial x$ (*iv*) None of these **[Kanpur 2002]**

3. With usual notations complex potential of a doublet is
(*i*) $\mu\, e^{i\alpha}/(z-a)$ (*ii*) $\mu\, e^{-i\alpha}/(z-a)$
(*iii*) $\mu\, e^{i\alpha}/(z+a)$ (*iv*) None of these. **[Kanpur 2002]**

4. If w be the complex potential, then the magnitude of the velocity of the fluid is given by
(*i*) $|dw/d\phi|$ (*ii*) $|dw/d\psi|$ (*iii*) $|dw/dz|$ (*iv*) None of these

5. The complex potential due to a source m at $z = z'$ is
(*i*) $-m\log(z-z')$ (*ii*) $m\log(z-z')$ (*iii*) $-m\log(z+z')$ (*iv*) $m\log(z+z')$

6. How many sinks are there if the complex potential is given by $w = \log\{z-(a^2/z)\}$?
(*i*) 1 (*ii*) 2 (*iii*) 3 (*iv*) None of these

7. The family of curves given by ϕ = constant and ψ = constant intersect at
(*i*) 30° (*ii*) 45° (*iii*) 60° (*iv*) 90°

8. The velocity vector $\mathbf{q}$ is everywhere tangent to the lines in xy-plane along which
(*i*) $\phi(x, y)$ = const. (*ii*) $\psi(x, y)$ = const. (*iii*) w = const. (*iv*) None of these

9. A two-dimensional flow field is given by $\psi = xy$. Then flow is
(*i*) rotational (*ii*) irrotational (*iii*) laminar (*iv*) None of these

10. The stream function is contant along a particular stream line flow
(*i*) false statememnt (*ii*) true statement (*iii*) both of above (*iv*) None of these

11. In a conformal transformation, a source is transformed into
(*a*) an equal source (*b*) an equal sink (*c*) an equal doublet (*d*) None of these **[Agra 2005]**

12. Cauchy–Riemann equations in polar form are
(*a*) $\dfrac{\partial\phi}{\partial r} = r\dfrac{\partial\psi}{\partial\theta}$, $\dfrac{\partial\phi}{\partial\theta} = -\dfrac{1}{r}\dfrac{\partial\psi}{\partial r}$ (*b*) $\dfrac{\partial\phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}$, $\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} = -\dfrac{\partial\psi}{\partial r}$
(*c*) $\dfrac{\partial\phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}$, $r\dfrac{\partial\phi}{\partial\theta} = -\dfrac{\partial\psi}{\partial\theta}$ (*c*) $\dfrac{\partial\phi}{\partial r} = -r\dfrac{\partial\psi}{\partial\theta}$, $\dfrac{\partial\phi}{\partial\theta} = \dfrac{1}{r}\dfrac{\partial\psi}{\partial r}$ **[Agra 2008]**

Answer/Hints to objective type questions

1. (*iii*). Refer result of Art. 5.21 **2.** (*ii*). See Eq. (3), Art 5.6
3. (*i*). See note 3, Art. 5.14 **4.** (*iii*). See Art. 5.8
5. (*i*). See Art. 5.13 **6.** (*ii*). See Ex. 1, Art. 5.15
7. (*iv*). See Art. 5.6 **8.** (*ii*). See Ex. 8, Art. 5.10
9. (*ii*). See Ex. 22, Art. 5.10 **10.** (*i*) Refer Art. 5.2
11. (*a*). See Art. 5.19B **12.** (*b*). See Art. 5.7A

Miscellaneous Problems on Chapter 5

1. Show that at the points of fields of flow the equipotential surfaces cut streamlines orthogonally. **(Agra 2009)**

Hint : Use Ex. 1, page 5.6

2. Explain velocity potential and stream function and derive expressions for velocity components in terms of ϕ and ψ. Also, prove that ϕ and ψ satisfy Laplace's equation. **(Meerut 2010)**

Hint : Refer Art. 2.26 on page 2.56, Art. 5.2 on page 5.1 and Art. 5.6 on page 5.3.

3. Find complex potential of a two-dimensional source. (Meerut 2012)

Hint : Refer Art. 5.13, page 5.20

4. Write 'T' for true and 'F' for false statements :

The stream function ψ exists only in irrotation motions. **(Agra 2004, 11)**

Ans. 'F'. Refer Art.5.2

CHAPTER 6

General Theory of Irrotational Motion

6.1. Connectivity. Definition.

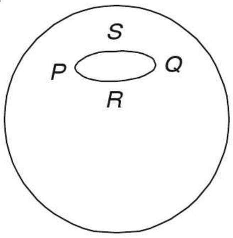
A region of space is said to be connected if a continuous curve joining any two points of the region lies entirely in the given region.

Thus the region interior to a sphere, or the region between two coaxial infinitely long cylinders are connected.

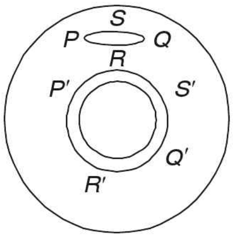
Reducible and irreducible circuits.

Definition. *A closed circuit, all of whose points lie in the region, is said to be reducible if it can be shrunk to a point of the region without passing outside of the region.*

The circuit *PRQS* in figures (*i*) and (*ii*) are reducible; the circuit $P'Q'R'S'$ in figure (*ii*) is irreducible, for it cannot be made smaller than the circumference of the inner cylinder.



(*i*)



(*ii*)

Simply connected region.

Definition. *A region in which every circuit is reducible is known as simply connected.*

Thus the region interior to a sphere, the region exterior to a sphere, the region between two concentric spheres, unbounded space etc. are simply connected regions. The region between the concentric cylinders in figure (*ii*) above is not simply connected, for it contains irreducible circuits. This region can be made simply connected by inserting one boundary or barrier which may not be crossed, such as *AB* containing a generating line of each cylinder as shown in figure (*iii*).

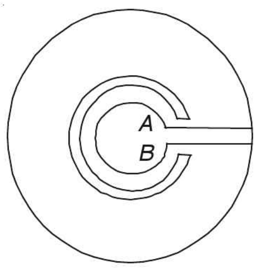


Fig. (*iii*)

With the insertion of this barrier each circuit in the modified region becomes reducible and hence the modified region is simply connected.

Doubly connected and *n*-ply connected regions.

Definition. *A region is said to be doubly connected, if it can be made simply connected by the insertion of one barrier. Similarly, a region is said to be n-ply connected, if can be made simply connected by the insertion of* $n - 1$ *barriers.*

Thus the region between two coaxial infinitely long cylinders, the region exterior to an infinitely long cylinder, the region interior (or exterior) to an anchor ring etc. are doubly connected regions.

Reconcilable or irreconcilable paths and circuits.

Definition. *The paths joining two points P and Q of the region are said to be reconcilable, if either can be continuously deformed into the other without ever passing out of the region.*

Thus the paths PRQ and PSQ in figures (*i*) and (*ii*) are reconcilable and the paths $P'\,R'\,Q'$ and $P'\,S'\,Q'$ in figure (*ii*) are irreconcilable.

Note that two reconcilable paths taken together form a reducible circuit.

Two closed circuits are said to be reconcilable, if either can be continuously deformed into the other without ever passing out of the region.

Clearly reconcilable circuits are not always reducible.

6.2. Flow and circulation. [Himachal 2003]

If A and P be any two points in a fluid, then the value of the integral

$$\int_A^P (udx + vdy + wdz),$$

taken along any path from A to P, is called the *flow* along that path from A to P.

When a velocity potential ϕ exists, the flow from A to P is

$$= -\int_A^P \left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right) = -\int_A^P d\phi = \phi_A - \phi_P.$$

The flow round a closed curve is known as the *circulation round* the curve. Let C be closed curve and Γ be the circulation. Also, let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. Then,

$$\Gamma = \int_C (udx + vdy + wdz) = \int_C \mathbf{q} \cdot d\mathbf{r},$$

where the line integral is taken round C in a counter clockwise direction and $\mathbf{q}$ is the velocity vector.

Remark. When a single-valued velocity potential exists the circulation round any closed curve is clearly zero. Again, in what follows, we shall prove that if the velocity potential is many-valued there are curves for which the circulation is zero, though it is not zero for all such paths.

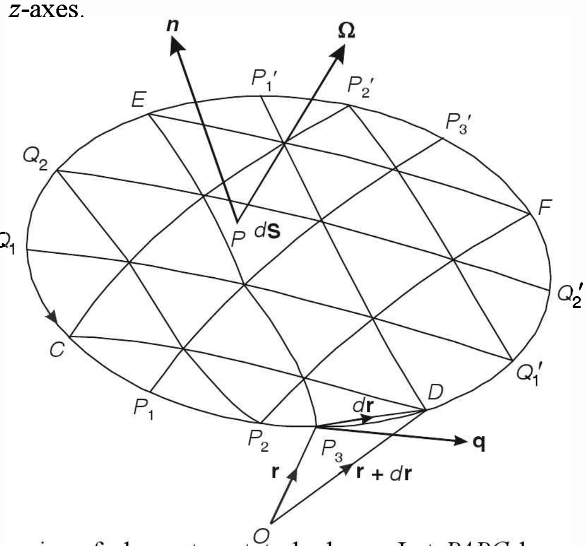
6.3. Stokes's theorem. [Kanpur 2000, 01; Meerut 2000, 02, 09]

Let $\mathbf{q}$ *be the velocity vector,* Ω *the vorticity vector and S be a surface bounded by a closed curve C. Then*

$$\int_C \mathbf{q} \cdot d\mathbf{r} = \int_S \text{Curl}\,\mathbf{q} \cdot d\mathbf{S} \qquad i.e. \qquad \Gamma = \int_S \Omega \cdot \mathbf{n}\, dS,$$

where Γ *is the circulation round C and the unit normal vector* $\mathbf{n}$ *at any point of S is drawn in the sense in which a right-handed screw would move when rotated in the sense of description of C.*

Proof. As shown in figure we observe that the given surface S can be divided up into a network of infinitesimally small triangles ΔS. Let lines be drawn from the vertices of such triangles parallel to the x, y, and z-axes.



Then we obtain a series of elementary tetrahedrons. Let $PABC$ be one of these tetrahedrons, with edges, PA, PB and PC equal to dx, dy and dz, respectively, as shown in the following figure. Let D, E, F be the mid-points of the AB, BC and CA, respectively. Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

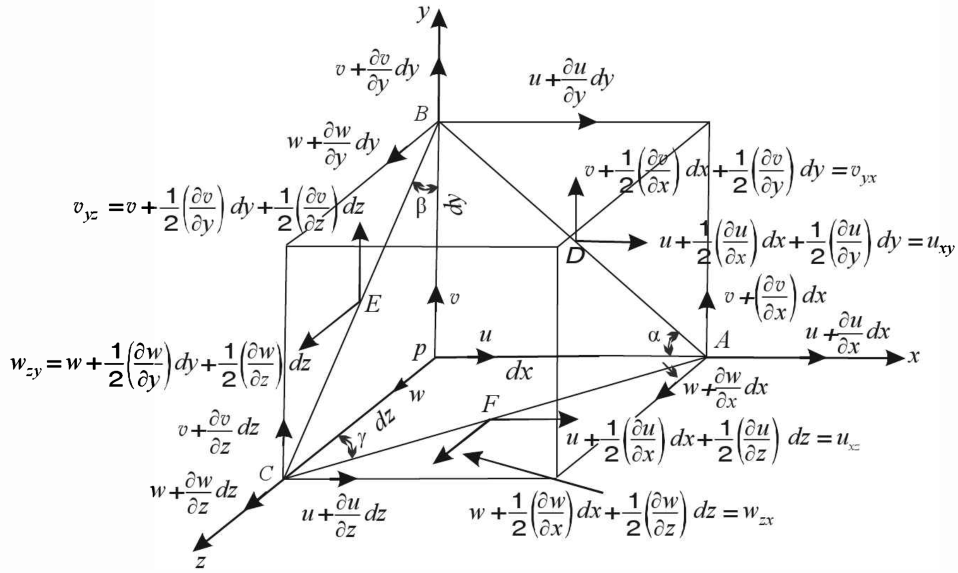
From right-angled triangle PAB, we have

$$\cos\alpha = dx/AB \qquad \text{and} \qquad \sin\alpha = dy/AB$$

so that
$$AB = dx/\cos\alpha = dy/\sin\alpha \qquad \text{...(1)}$$

Similarly,
$$BC = dy/\cos\beta = dz/\sin\beta \qquad \text{...(2)}$$

and
$$CA = dz/\cos\gamma = dx/\sin\gamma \qquad \text{...(3)}$$



Let co-ordinates of P be (x, y, z). Also, we have

$$\text{Velocity at } P \text{ parallel to } x\text{-axis} = u = f(x, y, z), \text{ say} \qquad \text{..(4)}$$

Then velocity at A parallel to x-axis $= f(x + dx, y, z) = f(x, y, z) + dx\,(\partial f/\partial x)$

[To first order of approximation by using Taylor's theorem]

$$= u + dx(\partial u/\partial x), \text{ using (4)}$$

Proceeding likewise, the velocity components at D, E and F (which contribute to the desired circulation can be calculated from those at the vertices P, A, B and B and are given by

$$u_{xy} = u + \frac{1}{2}\frac{\partial u}{\partial x}dx + \frac{1}{2}\frac{\partial u}{\partial y}dy \qquad \text{...(5)}$$

$$u_{xz} = u + \frac{1}{2}\frac{\partial u}{\partial x}dx + \frac{1}{2}\frac{\partial u}{\partial z}dz \qquad \text{...(6)}$$

$$v_{yx} = v + \frac{1}{2}\frac{\partial v}{\partial y}dy + \frac{1}{2}\frac{\partial v}{\partial x}dx \qquad \text{...(7)}$$

$$v_{yz} = v + \frac{1}{2}\frac{\partial v}{\partial y}dy + \frac{1}{2}\frac{\partial v}{\partial z}dz \qquad \text{...(8)}$$

$$w_{zx} = w + \frac{1}{2}\frac{\partial w}{\partial z}dz + \frac{1}{2}\frac{\partial w}{\partial x}dx \qquad \text{...(9)}$$

$$w_{zy} = w + \frac{1}{2}\frac{\partial w}{\partial z}dz + \frac{1}{2}\frac{\partial w}{\partial y}dy \qquad \text{...(10)}$$

Let the circulation be taken as positive if it rotates according to the right-handed screw rule with normal outward. Then the circulation along the sides of the triangle ABC is given by

$$d\Gamma = -u_{xy}\cos\alpha\,(AB) + v_{yx}\sin\alpha\,(AB) - v_{yz}\cos\beta(BC) + w_{zy}\sin\beta(BC) - w_{zx}\cos\gamma\,(CA) + u_{xz}\sin\gamma\,(CA)$$

$$= -u_{xy}\,dx + v_{yx}\,dy - v_{yz}\,dy + w_{zy}\,dz - w_{zx}\,dz + u_{xz}\,dx, \text{ using (1), (2) and (3)}$$

$$= (u_{xz} - u_{xy})\,dx + (v_{yx} - v_{yz})\,dy + (w_{zy} - w_{zx})\,dz$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial z}dz - \frac{\partial u}{\partial y}dy\right)dx + \frac{1}{2}\left(\frac{\partial v}{\partial x}dx - \frac{\partial v}{\partial z}dz\right)dy + \frac{1}{2}\left(\frac{\partial w}{\partial y}dy - \frac{\partial w}{\partial x}dx\right)dz, \text{ using (5) to (10)}$$

$$= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dy\,dz + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz\,dx + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy \qquad ...(11)$$

But, $$\text{Curl}\,\mathbf{q} = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

and $$d\mathbf{S} = \mathbf{i}\left(\frac{1}{2}dy\,dz\right) + \mathbf{j}\left(\frac{1}{2}dz\,dx\right) + \mathbf{k}\left(\frac{1}{2}dx\,dy\right)$$

$\therefore$ $$\text{Curl}\,\mathbf{q}\cdot d\mathbf{S} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dy\,dz + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz\,dx + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy \qquad ...(12)$$

Using (12), (11) reduces to $$d\Gamma = \text{Curl}\,\mathbf{q}\cdot d\mathbf{S} \qquad ...(13)$$

Proceeding likewise we can obtain the circulation around all other elementary triangles, dS, of the entire surface S. We observe that the circulation along the elementary sides common to two triangles cancels and hence the remaining circulation will be that round the closed contour C. Thus, circulation round C is equal to the sum of the circulation in all elementary triangles, *i.e.*,

$$\Gamma = \int_S \text{Curl}\,\mathbf{q}\cdot d\mathbf{S} \qquad ...(14)$$

But $$\Gamma = \int_C \mathbf{q}\cdot d\mathbf{r} \quad \text{and} \quad \mathbf{\Omega} = \text{Curl}\,\mathbf{q}$$

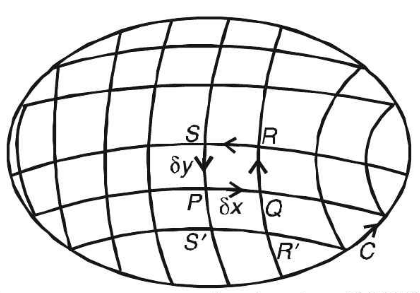
$\therefore$ $$\int_C \mathbf{q}\cdot d\mathbf{r} = \int_S \mathbf{\Omega}\cdot d\,\mathbf{S} = \int_S \mathbf{\Omega}\cdot\mathbf{n}\,d\,S \qquad ...(15)$$

6.3A. Stakes' theorem (Alternative form with proof)

The circulation Γ round any closed curve C drawn in a fluid is equal to the surface integral of the normal component of spin (i.e. vorticity vector Ω) taken over any surface S, provided the surface lies wholly in the fluid, that is,

$\int_C \mathbf{q}\cdot d\mathbf{r} = \int_S \text{curl}\,\mathbf{q}\,dS$ *so that* $\Gamma = \int_S \Omega\cdot\mathbf{n}\,dS$,

where $\boldsymbol{n}$ is the unit normal vector at any point of S.

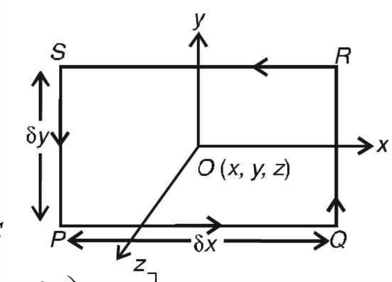


Proof. Let the given surface S be divided into small meshes by drawing a net work of lines across it as shown in the adjoining figure. Then the circulation round the edge of any finite surface is equal to the sum of the circulations, taken all in the same sense, round the boundaries of the infinitely small meshes into which the surface has been divided.

Suppose an elementary mesh be in the form of an elementary rectangular lamina $PQRS$ whose sides are δx, δy. Let the positive direction of circulation for $PQRS$ be taken from the axis of X to that of Y. Let $\boldsymbol{q}$ (u, v, w) be the velocity at the centre of intertia O (x, y, z) of the rectangle.

Now, the circulation due to the two sides QR and SP

$$= \left(\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial v}{\partial x}dx\right)dy - \left(\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial x}dx\right)dy = \frac{\partial v}{\partial x}dx\,dy.$$



Similarly, the circulation due to sides RS and $PQ = -\,(\partial u/\partial y)\,dx\,dy$.

$\therefore$ The circulation $\Gamma_{PQRS} = (\partial v/dx - \partial u/dy)\,dx\,dy$. ...(1)

It follows that the circulation round the boundary C of S

$$= \iint_S \left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dy\,dz + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz\,dx + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy\right]. \qquad ...(2)$$

While computing the L.H.S. of (1), we find that no contribution to

$$\sum_{\text{Rectangles}} \mathbf{q}\,.\,d\mathbf{r} \qquad ...(3)$$

will be made by boundary lines (such as PQ between two adjoining rectangles $PQRS$ and $PQR'S'$) because each of such a line will give equal and opposite contribution to the two rectangles adjoining it. It follows that the result of the sum (3) will be simply $\int_C \mathbf{q}\cdot d\mathbf{r}$ taken over the boundary curve C. Hence , we have

$$\int_C \mathbf{q}\cdot d\mathbf{r} = \iint_S \left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dy\,dz + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz\,dx + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy\right]$$

or $$\int_C \mathbf{q}\cdot d\mathbf{r} = \int_S \text{curl}\,\mathbf{q}\,d\text{S},$$

where the surface integral is taken over given surface S bounded by closed curve C and the line integral is taken once round the curve.

6.4. Kelvin's circulation theorem. **[Garhwal 2001, 03, 05; Rohilkhand 2001 Agra 2006, 08; Himanchal 2007; Garhwal 2005; Kanpur 1999; Meerut 2005, 09, 10, 12]**

When the external forces are conservative and derivable from a single valued potential function and the density is a function of pressure only, the circulation in any closed circuit moving with the fluid is constant for all time.

Proof. Let C be a closed circuit moving with the fluid so that C always consists of the same fluid particles. Let $\mathbf{q}$ be the fluid velocity at any point P of the circuit and let $\mathbf{r}$ be its position vector. Then the circulation along the closed circuit C is given by

$$\Gamma = \int_C \mathbf{q}\cdot d\mathbf{r} \qquad \text{or} \qquad \frac{D\Gamma}{Dt} = \frac{D}{Dt}\int_C \mathbf{q}\cdot d\mathbf{r} \qquad ...(1)$$

Since the above integration is performed at constant time, reversing the order of integration and differentiation is justified. Then (1) may be re-written as

$$\frac{D\Gamma}{Dt} = \int_C \frac{D}{Dt}(\mathbf{q}\cdot d\mathbf{r}) \qquad ...(2)$$

But $$\frac{D}{Dt}(\mathbf{q}\cdot d\mathbf{r}) = \frac{D\mathbf{q}}{Dt}\cdot d\mathbf{r} + \mathbf{q}\cdot\frac{D}{Dt}d\mathbf{r} = \frac{D\mathbf{q}}{Dt}\cdot d\mathbf{r} + \mathbf{q}\cdot d\mathbf{q} \qquad ...(3)$$

The Euler's equation of motion is

$$\frac{D\mathbf{q}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad ...(4)$$

Let the external forces be conservative and derivable from a single valued potential function

V. Then $\mathbf{F} = -\nabla \mathbf{V}$ and hence (4) becomes

$$\frac{D\mathbf{q}}{Dt} = -\nabla \mathbf{V} - \frac{1}{\rho}\nabla p \qquad \text{...(5)}$$

$$\therefore \qquad \frac{D\mathbf{q}}{Dt}\cdot d\mathbf{r} = -\nabla \mathbf{V}\cdot d\mathbf{r} - \frac{1}{\rho}\nabla p \cdot d\mathbf{r} = -d\,\mathbf{V} - \frac{dp}{\rho} \qquad \text{...(6)}$$

Also
$$\mathbf{q}\cdot d\mathbf{q} = \frac{1}{2}d\,(\mathbf{q}\cdot\mathbf{q}) = \frac{1}{2}dq^2, \qquad \text{...(7)}$$

where q denotes the magnitude of the velocity vector $\boldsymbol{q}$.

Using (6) and (7), (3) reduces to
$$\frac{D}{Dt}(\mathbf{q}\cdot d\mathbf{r}) = -d\mathbf{V} - \frac{1}{\rho}dp + \frac{1}{2}dq^2 \qquad \text{...(8)}$$

Using (8) and assuming that ρ is a single-valued function of p, (2) reduces to

$$\frac{D\Gamma}{Dt} = \left[\frac{1}{2}q^2 - \mathbf{V} - \int_C \frac{dp}{\rho}\right]_C \qquad \text{...(9)}$$

where $[\ ]_C$ denotes change in the quantity enclosed within brackets on moving once round C. Since q, V and p are single-valued functions of $\mathbf{r}$, so R.H.S. of (9) vanishes. Equation (9) gives the rate of change of flow along any closed circuit moving with the fluid. Thus, it follows that the circulation in any closed circuit moving with the fluid is constant for all time.

6.5. Permanence of irrotational motion.

[G.N.D.U. Amritser 2000, 04; Meerut 1999, 2002; Garhwal 2001, 02; Kurkshetra 1998; Rohilkhand 2002]

When the external forces are conservative and derivable from a single valued potential, and density is a function of pressure only, then the motion of an inviscid fluid, if once irrotational, remains irrotational even afterwards.

Proof. *From Stokes' theorem, the circulation is given by*

$$\Gamma = \int_C \mathbf{q}\cdot d\mathbf{r} = \int_S \text{Curl}\,\mathbf{q}\cdot d\,\mathbf{S}. \qquad \text{...(1)}$$

At any time t, let the motion be irrotational so that curl $\mathbf{q} = \mathbf{0}$. Then (1) shows that $\Gamma = 0$ at that instant. Hence it follows from Kelvin's circulation theorem that $\Gamma = 0$ for all time. Hence at any subsequent time, (1) shows that

$$\int_S \text{Curl}\,\mathbf{q}\cdot d\mathbf{S} = 0 \qquad \text{...(2)}$$

Since S is arbitrary, (2) shows that Curl $\mathbf{q} = \mathbf{0}$ at all subsequent time *i.e.* the motion remains irrotational even afterwards.

6.6. Green's theorem.

(Agra 2012)

If ϕ, ϕ' are both single-valued and continuously differentiable scalar point functions such that $\nabla\phi$ and $\nabla\phi'$ are also continuously differentiable, then

$$\int_V (\nabla\phi\cdot\nabla\phi')\,d\,\mathbf{V} = -\int_S \phi\frac{\partial\phi'}{\partial n}d\,\mathbf{S} - \int_V \phi\,\nabla^2\phi'\mathbf{dV}$$
$$= -\int_S \phi'\frac{\partial\phi}{\partial n}d\,\mathbf{S} - \int_V \phi'\nabla^2\phi\,\mathbf{dV},$$

where S is closed surface bounding any simply-connected region, δn is an element of inward normal at a point on S, and V is the volume enclosed by S.

Proof. From vector calculus, we have
$$\nabla\cdot(\phi\mathbf{a}) = \mathbf{a}\cdot(\nabla\phi) + \phi(\nabla\cdot\mathbf{a}), \qquad \text{...(1)}$$
where ϕ is a scalar point function and $\mathbf{a}$ is a vector point function.

Replacing **a** by $\nabla\phi'$ in (1), we get $\qquad \nabla\cdot(\phi\nabla\phi') = (\nabla\phi')\cdot(\nabla\phi) + \phi(\nabla\cdot\nabla\phi')$ ...(2)

Integrating both sides of (2) over volume V, we get

$$\int_V \nabla\cdot(\phi\nabla\phi')\,dV = \int_V (\nabla\phi')\cdot(\nabla\phi)\,dV + \int_V \phi(\nabla\cdot\nabla\phi')\,dV \qquad \text{...(3)}$$

By Gauss divergence theorem, we have $\qquad \int_V \nabla\cdot(\phi\nabla\phi')\,dV = -\int_S \boldsymbol{n}\cdot(\phi\nabla\phi')\,dS,$

where $\boldsymbol{n}$ is the unit vector drawn to the surface S.

or $\quad \int_V \nabla\cdot(\phi\nabla\phi')\,dV = -\int_S \phi(\boldsymbol{n}\cdot\nabla\phi')\,dS \quad$ or $\quad \int_V \nabla\cdot(\phi\nabla\phi')\,dV = -\int_S \phi\frac{\partial\phi'}{\partial n}\,dS$

...(4)

Again, $\qquad \nabla\cdot\nabla\phi' = \nabla^2\phi' \qquad$ and $\qquad \nabla\phi'\cdot\nabla\phi = \nabla\phi\cdot\nabla\phi'$ ...(5)

Using (4) and (5), (3) reduces to

$$-\int_S \phi\frac{\partial\phi'}{\partial n}\,dS = \int_V (\nabla\phi\cdot\nabla\phi')\,dV + \int_V \phi\nabla^2\phi'\,dV$$

or
$$\int_V (\nabla\phi\cdot\nabla\phi')\,dV = -\int_S \phi\frac{\partial\phi'}{\partial n}\,dS - \int_V \phi\nabla^2\phi'\,dV \qquad \text{...(6)}$$

Interchanging ϕ and ϕ' in (6), we have

$$\int_V (\nabla\phi'\cdot\nabla\phi)\,dV = -\int_S \phi'\frac{\partial\phi}{\partial n}\,dS - \int_V \phi'\nabla^2\phi\,dV$$

or
$$\int_V (\nabla\phi\cdot\nabla\phi')\,dV = -\int_S \phi'\frac{\partial\phi}{\partial n}\,dS - \int_V \phi'\nabla^2\phi\,dV \qquad \text{...(7)}$$

(6) and (7) together prove the Green's theorem

6.7. Deductions from Green's theorem.

Deduction I. Let ϕ, ϕ' be the velocity potentials of two liquid motions taking place within S. Then $\nabla^2\phi = 0 = \nabla^2\phi'$ and hence Green's theorem yields

$$\int_S \phi\frac{\partial\phi'}{\partial n}\,dS = \int_S \phi'\frac{\partial\phi}{\partial n}\,dS \qquad \text{or} \qquad \int_S \rho\phi\left(-\frac{\partial\phi'}{\partial n}\right)dS = \int_S \rho\phi'\left(-\frac{\partial\phi}{\partial n}\right)dS \qquad \text{...(1)}$$

But $-\partial\phi/\partial n$ is the normal velocity inwards and $\rho\phi$ is the impulsive pressure at any point on the surface which will produce velocity potential ϕ from rest. Hence (1) shows that if there be two possible motions inside S by means of two different impulsive pressures on the boundary, then the work done by the first in acting through the displacement produced by the second must be equal to the work done by the second in acting through the displacement produced by the first.

Deduction II. Let ϕ' = constant (= k, say). Then $\nabla^2\phi' = 0 = \partial\phi'/\partial n$ everywhere. If ϕ be the velocity potential of a liquid motion within **S**, then by Green's theorem, we have

$$\int_S k\frac{\partial\phi}{\partial n}\,dS = 0 \qquad \text{or} \qquad \int_S \frac{\partial\phi}{\partial n}\,dS = 0. \qquad \text{...(2)}$$

Since $\partial\phi/\partial n$ is the normal velocity outwards, $(\partial\phi/\partial n)$ dS represents the flow across dS per unit time. Then (2) shows that the total flow across S is zero, *i.e.*, the quantity of a liquid inside S remains constant.

Deduction III. Let $\phi = \phi'$ and let ϕ be the velocity potential of a liquid motion within S. Then $\nabla^2\phi = 0$ and hence Green's theorem gives

$$\int_V (\nabla\phi\cdot\nabla\phi)\,d\mathrm{V} = -\int_S \phi\frac{\partial\phi}{\partial n}d\mathrm{S} \quad \text{or} \quad \int_V \left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]d\mathrm{V} = -\int_S \phi\frac{\partial\phi}{\partial n}d\mathrm{S} \quad \text{...(3)}$$

Let q be the velocity and ρ the density of the liquid, then (3) reduces to

$$\frac{1}{2}\rho\int_V q^2\,d\mathrm{V} = -\frac{1}{2}\rho\int_S \frac{\partial\phi}{\partial n}d\mathrm{S} \quad \text{...(4)}$$

Clearly the L.H.S. of (4) represents the kinetic energy T of the liquid within S. Hence (4) reduces to

$$T = -\frac{1}{2}\rho\int_S \frac{\partial\phi}{\partial n}d\mathrm{S}. \quad \text{...(5)}$$

Now $\rho\phi$ is the impulsive pressure that would set up the motion instantaneously from rest, and $-\partial\phi/\partial n$ is the inward normal velocity at the surface. Hence (5) shows that the kinetic energy set up by impulses, in a system starting from rest, is the sum of the products of each impulse and half the velocity of its point of application. From (5), we also find that the kinetic energy of a given mass of liquid moving irrotationally in simply-connected region depends only on the motion of its boundaries.

Suppose on the boundary $\partial\phi/\partial n = 0$. Then (4) reduces to

$$\int_V q^2\,d\mathrm{V} = 0 \quad \text{...(6)}$$

Since q^2 is positive, (6) implies that $q = 0$ everywhere. Hence the liquid is at rest. Thus a cyclic irrotational motion is impossible in a liquid bounded by fixed rigid boundary.

6.8. Kinetic energy of infinite liquid. [Meerut 2007]

Consider an infinite mass of liquid moving irrotationally, at rest at infinity, and bounded internally by a solid surface S and externally by a large surface S'. Let ϕ be the single-valued velocity potential. Then from deduction III of Art. 6.7, the kinetic energy T of the liquid contained in the region bounded by S and S' is given by

$$T = -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}d\mathrm{S} - \frac{1}{2}\rho\int_{S'} \phi\frac{\partial\phi}{\partial n}d\mathrm{S}' \quad \text{...(1)}$$

Since there is no flow into the region across S, the equation of continuity takes the form

$$\int_S \frac{\partial\phi}{\partial n}d\mathrm{S} + \int_{S'} \frac{\partial\phi}{\partial n}d\mathrm{S}' = 0 \quad \text{...(2)}$$

Multiplying (2) by $C/2$, a constant, and subtracting from (1), we get

$$T = -\frac{1}{2}\rho\int_S (\phi - C)\frac{\partial\phi}{\partial n}d\mathrm{S} - \frac{1}{2}\rho\int_{S'} (\phi - C)\frac{\partial\phi}{\partial n}d\mathrm{S}' = 0 \quad \text{...(3)}$$

Since for the solid boundary S, $\int_S \frac{\partial\phi}{\partial n}d\mathrm{S} = 0$, it follows from (2) that $\int_{S'} \rho\frac{\partial\phi}{\partial n}d\mathrm{S}' = 0$, *i.e.*, $\int_{S'} \frac{\partial\phi}{\partial n}d\mathrm{S}'$ is independent of S'. Let $\phi \to C$ at infinity and let the surface S' be enlarged indefinitely in all directions. Then the second integral in (3) vanishes and hence the required kinetic energy of infinite liquid is given by

$$T = -\frac{1}{2}\rho\int_S (\phi - C)\frac{\partial\phi}{\partial n}d\mathrm{S}$$

i.e., $$T = -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}d\mathrm{S} \quad \text{as} \quad \int_S \frac{\partial\phi}{\partial n}d\mathrm{S} = 0 \quad \text{...(4)}$$

Remark. For the motion of liquid to exist, T must not vanish. Hence all internal boundaries must not be at rest.

6.9A. Acyclic and cyclic motions.

The motion in which the velocity potential is single-valued is called *acyclic* whereas the motion in which the velocity potential is not a single-valued is called *cyclic*.

6.9B. Some uniqueness theorems related to acyclic irrotational motion.

In what follows, we shall use the following equivalence of the expressions for the kinetic energy

$$T = \frac{1}{2}\rho\int_V q^2 dV = -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}dS \qquad ...(1)$$

where the symbols have their usual meaning.

Theorem I. *There cannot be two different forms of acyclic irrational motions of a confined mass of incompressible inviscid liquid, when the boundaries have prescribed velocities.*

Proof. If possible, let ϕ_1, ϕ_2 be the velocity potentials of two different motions subject to the condition

$$\partial\phi_1/\partial n = \partial\phi_2/\partial n \text{ at each point of } S \qquad ...(2)$$

Also $$\nabla^2\phi_1 = \nabla^2\phi_2 = 0 \qquad ...(3)$$

Let $\phi = \phi_1 - \phi_2$, then $\nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0.$

Hence ϕ is a solution of Laplace's equation and so it represents irrotational motion in which

$$\frac{\partial\phi}{\partial n} = \frac{\partial}{\partial n}(\phi_1 - \phi_2) = \frac{\partial\phi_1}{\partial n} - \frac{\partial\phi_2}{\partial n} = 0, \text{ by (2)}.$$

Hence $q = 0$ by (1). But $q^2 = (\nabla\phi)^2$. Thus, we have

$(\nabla\phi)^2 = 0$ so that ϕ = constant or $\phi_1 - \phi_2$ = constant.

Since the constant is of no significance, it follows that the two motions are the same.

Theorem II. *There cannot be two different forms of irrotational motion for a given confined mass of incompressible inviscid liquid whose boundaries are subject to the given impulses.*

Proof. If possible, let ϕ_1, ϕ_2 be the velocity potentials of two motions subject to the conditions

$$\rho\phi_1 = \rho\phi_2 \text{ at each point } S \qquad ...(4)$$

Also, $$\nabla^2\phi_1 = \nabla^2\phi_2 = 0 \qquad ...(5)$$

Let $\phi = \phi_1 - \phi_2$, then $\nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0.$

Hence ϕ is a solution of Laplace's equation and so it represents irrotational motion in which

$$\rho\phi = \rho(\phi_1 - \phi_2) = 0, \text{ by (4)}$$

Hence $q = 0$ by (1). But $q^2 = (\nabla\phi)^2$. Thus, we have

$(\nabla\phi)^2 = 0$ so that ϕ = constant *i.e.* $\phi_1 - \phi_2$ = constant,

showing that the two motions are the same.

Theorem III. *Acyclic irrotational motion is impossible in a liquid bounded entirely by fixed rigid walls.*

Proof. Since at every point of the rigid boundary walls $\partial\phi/\partial n = 0$, it follows by (1) that

$$\int_V q^2 dV = 0.$$

Since q^2 cannot be negative, $q = 0$ everywhere and hence the motion will be impossible.

Theorem IV. *Acyclic irrotational motion of a liquid bounded by rigid walls will instantly cease if the boundaries are brought to rest.*

Proof. This is an immediate corollary to Theorem III.

Theorem V. *Acyclic irrotational motion is impossible in a liquid which is at rest at infinity and is bounded internally by fixed rigid walls.*

Proof. Since the liquid is at rest at infinity and there is no flow over the internal boundaries, the kinetic energy is still given by (1). Hence here $\partial\phi/\partial n = 0$ at each point of *S*. Hence as shown in theorem III, the motion is impossible.

Theorem VI. *The acyclic irrotational motion of a liquid at rest at infinity and bounded internally by rigid walls will instantly cease if the boundaries are brought to rest.*

Proof. This is an immediate corollary to Theorem V.

Theorem VII. *The acyclic irrotational motion of a liquid at rest at infinity, due to the prescribed motion of an immersed solid, is uniquely determined by the motion of the solid.*

Proof. If possible, let ϕ_1, ϕ_2 be the velocity potentials of two different motions.Then

$$\nabla^2\phi_1 = \nabla^2\phi_2 = 0 \qquad ...(6)$$

Also given $\quad \partial\phi_1/\partial n = \partial\phi_2/\partial n,$ at each point of surface ...(7)

and $\quad q_1 = q_2 \quad$ at infinity ...(8)

Let $\phi = \phi_1 - \phi_2$ and $q = q_1 - q_2$ Then, we have $\nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0$ by (6). Hence ϕ must be the velocity potential of a possible motion. Furthermore,

$$\partial\phi/\partial n = \partial\phi_1/\partial n - \partial\phi_2/\partial n = 0, \quad \text{at each point of surface} \qquad ...(9)$$

and $\quad q = q_1 - q_2 = 0 \quad$ at infinity. ... (10)

From (1) and (7), we have $q = 0$. But $q^2 = (\nabla\phi)^2$. Thus, we have

$(\nabla\phi)^2 = 0 \quad$ so that $\quad \phi = \text{constant} \quad$ or $\quad \phi_1 - \phi_2 = \text{constant}$

Since the constant is of no significance, it follows that the two motions are the same.

Theorem VIII. *If the liquid is in motion at infinity with uniform velocity, the acyclic irrotational motion, due to the prescribed motion of an immersed solid, is uniquely determined by the motion of the solid.*

Proof. Let us superimpose on the whole system of solid and liquid a velocity equal in magnitude and opposite in direction to the velocity at infinity. The relative kinematical conditions remain unchanged and the liquid is reduced to rest at infinity. The resulting motion is then determined by theorem VII and we return to the given motion by reimposing the velocity at infinity.“

6.10. Kelvin's minimum energy theorem. [Meerut 2003, 05, 08]

The irrotational motion of a liquid occupying a simply connected region has less kinetic energy than any other motion consistent with the same normal velocity of the boundary.

Proof. Let T_1 be the kinetic energy, $\mathbf{q}_1$ the fluid velocity of the actual irrotational motion with a velocity potential ϕ Then $\quad \mathbf{q}_1 = -\nabla\phi \quad$ ...(1)

Let T_2 be the kinetic energy, $\mathbf{q}_2$ the fluid velocity of any other possible state of motion consistent with the same normal velocity of the boundary *S*.

Continuity equations for the above two motions give

$$\nabla\cdot\mathbf{q}_1 = 0 \quad \text{and} \quad \nabla\cdot\mathbf{q}_2 = 0 \qquad ...(2)$$

Let **n** denote the unit normal at a point of *S*. Then using the fact that the boundary has the same normal velocity in both motions, we have

$$\mathbf{n}\cdot\mathbf{q}_1 = \mathbf{n}\cdot\mathbf{q}_2 \qquad \text{...(3)}$$

Now, $\quad T_1 = \frac{1}{2}\rho\int_V q_1^2 dV = \frac{1}{2}\rho\int_V \mathbf{q}_1^2 dV \quad$ and $\quad T_2 = \frac{1}{2}\rho\int_V q_2^2 dV = \frac{1}{2}\rho\int_V \mathbf{q}_2^2 dV$

$$\therefore \quad T_2 - T_1 = \frac{1}{2}\rho\int_V (\mathbf{q}_2^2 - \mathbf{q}_1^2)dV = \frac{1}{2}\rho\int_V \left\{2\mathbf{q}_1\cdot(\mathbf{q}_2-\mathbf{q}_1)+(\mathbf{q}_2-\mathbf{q}_1)^2\right\}dV$$

$$= \rho\int_V \mathbf{q}_1\cdot(\mathbf{q}_2-\mathbf{q}_1)\,d\mathrm{V} + \frac{1}{2}\rho\int_V (\mathbf{q}_2-\mathbf{q}_1)^2 d\mathrm{V}$$

$$= -\rho\int_V (\nabla\phi)\cdot(\mathbf{q}_2-\mathbf{q}_1)\,d\mathrm{V} + \frac{1}{2}\rho\int_V (\mathbf{q}_2-\mathbf{q}_1)^2 d\mathrm{V}\text{, using (1)} \qquad \text{...(4)}$$

But $\quad \nabla\cdot[\phi(\mathbf{q}_2-\mathbf{q}_1)] = \phi[\nabla\cdot(\mathbf{q}_2-\mathbf{q}_1)] + (\nabla\phi).(\mathbf{q}_2-\mathbf{q}_1) = (\nabla\phi)\cdot(\mathbf{q}_2-\mathbf{q}_1)$, using (2)

$\therefore \quad \int_V (\nabla\phi).(\mathbf{q}_2-\mathbf{q}_1)dV = \int_V \nabla\cdot[\phi(\mathbf{q}_2-\mathbf{q}_1)]dV = \int_S \phi\mathbf{n}\cdot(\mathbf{q}_2-\mathbf{q}_1)\,dS$, by divergrnce Theorem

Thus, $\quad \int_V (\nabla\phi)\cdot(\mathbf{q}_2-\mathbf{q}_1)dV = 0$, Using (3) ...(5)

Making use of (5), (4) reduces to

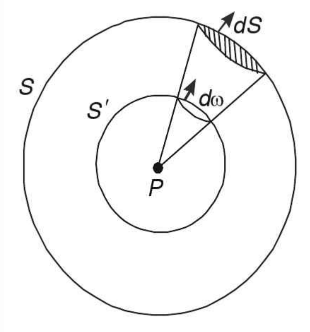
$$T_2 - T_1 = \frac{1}{2}\rho\int_V (\mathbf{q}_2-\mathbf{q}_1)^2 dV \qquad \text{...(6)}$$

Since R.H.S. of (6) is non-negative, we have $T_2 - T_1 \ge 0$, *i.e.*, $T_1 \le T_2$. Hence the result.

6.11. Mean potential over spherical surface.

The mean value of ϕ over any spherical surface, throughout whose interior $\nabla^2\phi = 0$, is equal to the value of ϕ at the centre of the sphere.

Proof. Describe a sphere S of radius r with P as its centre. Let ϕ_P and $\bar{\phi}$ denote the value of ϕ at P and the mean value of ϕ over S. Describe another concentric sphere S' of radius unity. Then we know that a cone with vertex P which intercepts dS from S also intercepts $d\omega$ (the solid angle) from S'. We then have



$$dS/d\omega = r^2/1^2 \quad \text{so that} \quad dS = r^2 d\omega \qquad \text{...(1)}$$

Now, $\quad \bar{\phi} = \frac{1}{4\pi r^2}\int_S \phi dS = \frac{1}{4\pi r^2}\int_{S'} \phi\, r^2 d\omega = \frac{1}{4\pi}\int_{S'} \phi\, d\omega$

$$\therefore \quad \frac{\partial\bar{\phi}}{\partial r} = \frac{1}{4\pi}\int_{S'}\frac{\partial\phi}{\partial r}d\omega = \frac{1}{4\pi r^2}\int_S \frac{\partial\phi}{\partial r}dS \qquad \text{...(2)}$$

Let $\mathbf{n}$ denote unit normal at any point of S. Then, we have

$$\int_S \frac{\partial\phi}{\partial r}dS = \int_S \frac{\partial\phi}{\partial n}dS = \int_V \boldsymbol{n}\cdot\nabla\phi\, dS = \int_V \nabla^2\phi dV\text{, by Divergence theorem}$$

$$= 0, \quad \text{as} \quad \nabla^2\phi = 0 \text{ (given)}$$

Hence (2) reduces to $\partial\bar{\phi}/\partial r = 0$ so that $\bar{\phi}$ is independent of r. It follows that $\bar{\phi}$ has the same value over all concentric spheres with P as centre. Hence, by shrinking S to a point P, we have $\bar{\phi} = \phi_P$,. Hence the result.

Cor. 1. *ϕ cannot be a maximum or minimum in the interior of any region throughout which $\nabla^2\phi = 0$.*

Proof. If possible, let ϕ_P be a maximum value of ϕ at a point P. Describe a sphere S of radius r with P as its centre such that r is very small. Let $\bar{\phi}$ be the mean value of ϕ over S. Then in our case $\bar{\phi} < \phi_P$, which contradicts the above theorem. Similarly, we can show that ϕ cannot be a minimum.

Cor. 2. *In irrotational motion the maximum values of the speed must occur at the boundary.*

OR

In irrotational motion the velocity cannot be a maximum in the interior of the fluid.

Proof. Let P be a point interior to the fluid. Take P as origin and the axis of x in the direction of motion at P. Let Q be a point near to P and let q and q' be the speeds at P and Q respectively. Then we have

$$q^2 = \left(\frac{\partial \phi}{\partial x}\right)_P^2, \qquad q'^2 = \left(\frac{\partial \phi}{\partial x}\right)_Q^2 + \left(\frac{\partial \phi}{\partial y}\right)_Q^2 + \left(\frac{\partial \phi}{\partial z}\right)_Q^2$$

Now $$\nabla^2\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}(\nabla^2 \phi) = 0,$$

showing that $\partial\phi/\partial x$ satisfies Laplace's equation and hence cannot be a maximum or minimum at P. It follows that there exist points such as Q very near to P such that

$$\left(\frac{\partial \phi}{\partial x}\right)_Q^2 > \left(\frac{\partial \phi}{\partial x}\right)_P^2 \qquad \text{so that} \qquad q'^2 > q^2.$$

Thus q cannot be a maximum in the interior of the fluid, and its maximum value, if any, must occur only on the boundary.

Remark. q^2 may be minimum in the interior of the fluid, for $q = 0$ at a *stagnation point*.

Cor. 3. I*n steady irrotational motion the hydrodynamical pressure has its minimum values on the boundary.*

Proof. By Bernoulli's theorem in absence of external forces, we have

$$p/\rho + q^2/2 = \text{constant}$$

Thus p is least when q^2 is greatest, and this cannot occur inside the fluid by corollary 2. Hence the minimum value of p, if any, must occur on the boundary.

Remark. The maximum value of p occurs at the stagnation points.

6.12. Mean value of velocity potential in a region with internal boundaries.

If Σ is the sold boundary of a large spherical surface of radius R, containing fluid in motion and also enclosing one or more closed surfaces, then the mean value of ϕ on Σ is of the from.

$$\bar{\phi} = (M/R) + C$$

where M, C are constants, provided that the fluid extends to infinity and is at rest there.

Proof. Let the volume of fluid crossing each of the internal surfaces contained within Σ per unit time is a finite quantity $4\pi M$. Then the equation of continuity gives

$$\int_\Sigma \left(-\frac{\partial \phi}{\partial R}\right) dS = 4\pi M.$$

Let dS subtends a solid angle $d\omega$ at the centre of Σ. Then $dS = R^2 d\omega$ and hence

$$\frac{1}{4\pi}\int_\Sigma \frac{\partial \phi}{\partial R} d\omega = -\frac{M}{R^2} \qquad \text{or} \qquad \frac{1}{4\pi}\frac{\partial}{\partial R}\int_\Sigma \phi\, d\omega = -\frac{M}{R^2}$$

Integrating, $$\frac{1}{4\pi}\int_{\Sigma}\phi\, d\omega = \frac{M}{R}+C,\ C \text{ being an arbitrary constant}$$

or $$\frac{1}{4\pi R^2}\int_{\Sigma}\phi\, dS = \frac{M}{R}+C, \quad \text{as} \quad dS = R^2\, d\omega \qquad ...(1)$$

or $$\bar{\phi} = M/R + C, \qquad ...(2)$$

where $\bar{\phi}$ is the mean value of ϕ on S and C is independent of R.

We now prove that C is an absolute constant. For this we must prove that C is independent of the coordinates of the centre of Σ. To prove this, let the sphere be displaced a distance δx in any direction, keeping R constant. Then (1) and (2) give

$$\frac{\partial\bar{\phi}}{\partial x}\delta x = \frac{1}{4\pi r^2}\int_{\Sigma}\frac{\partial\phi}{\partial x}\delta x\, dS = \frac{\partial C}{\partial x}\delta x \qquad ...(3)$$

Since the liquid is at rest at infinity, $\partial\phi/\partial x = 0$ on Σ when $R \to \infty$. Hence for large R, (3) shows that $\partial C/\partial x = 0$. Thus we see that C is an absolute constant and the required result follows from (2).

6.13. Illustrative solved examples.

Ex. 1. (*i*) *A velocity field is given by* $\mathbf{q} = (-\mathbf{i}\,y + \mathbf{j}\,x)/(x^2 + y^2)$. *Determine whether the flow is irrotational. Calculate the circulation round* (*a*) *a square with its corners at* (1, 0), (2, 0), (2, 1), (1, 1); (*b*) *a unit circle with centre at the origin.*

[Agra 2008; Rohilkhand 2003; 04; Meerut 2002 04, 05, 07, 08; Kanpur 2002]

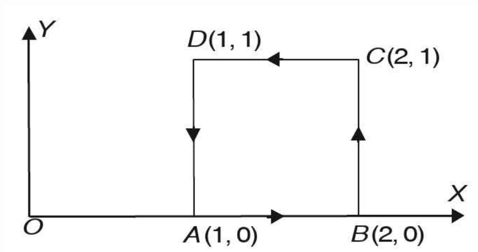
(*ii*) *Find the circulation about the square enclosed by the lines* $x = \pm 2$, $y = \pm 2$ *for the flow* $u = x + y$, $v = x^2 - y$.

Sol. Part (*i*) We have,

$$\text{Curl}\,\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y/(x^2+y^2) & x/(x^2+y^2) & 0 \end{vmatrix} = \mathbf{k}\left\{\frac{(y^2-x^2)+(x^2-y^2)}{(x^2+y^2)^2}\right\} = \mathbf{0}$$

Hence curl $\mathbf{q} = \mathbf{0}$ every where except at the origin. Thus the flow is irrotational. It has a singularity at the origin where the velocity becomes infinite.

(*a*) Draw a square in the cartesian plane as follows : $A(1, 0)$, $B(2, 0)$, $C(2, 1)$, $D(1,1)$. Then circulation around the square $ABCD$ is given by



$$\Gamma = \int \mathbf{q}\cdot d\mathbf{r}$$

$$= \int_A^B \mathbf{q}\cdot d\mathbf{r} + \int_B^C \mathbf{q}\cdot d\mathbf{r} + \int_C^D \mathbf{q}\cdot d\mathbf{r} + \int_D^A \mathbf{q}\cdot d\mathbf{r}. \qquad ...(1)$$

Now, $$\int_A^B \mathbf{q}\cdot d\mathbf{r} = \int_{x=1}^{x=2}[(-\mathbf{i}\,y+\mathbf{j}\,x)/(x^2+y^2)]_{y=0}\cdot(dx\,\mathbf{i}) = \int_1^2 (x^{-1}\mathbf{j})\cdot(dx\,\mathbf{i})$$

[$\because$ along AB (*i.e.* x-axis), $y = 0$ so $dy = 0$ and hence $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} = dx\,\mathbf{i}$]

$= 0$, as $\mathbf{j}\cdot\mathbf{i} = 0$ ...(2)

Next, $$\int_C^D \mathbf{q}\cdot d\mathbf{r} = \int_{x=2}^{x=1}[(-\mathbf{i}y+\mathbf{j}x)/(x^2+y^2)]_{y=1}\cdot(dx\,\mathbf{i}) = \int_1^2 \frac{dx}{x^2+1} = \tan^{-1}2 - \tan^{-1}1 \qquad ...(3)$$

Also, $\int_B^C \mathbf{q}\cdot d\mathbf{r} = \int_{y=0}^{y=1} [(-\mathbf{i}\, y + \mathbf{j}\, x)/(x^2+y^2)]_{x=2}\cdot(dy\, \mathbf{j})$ [$\because$ along BC (*i.e.* parallel to y-axis), $x = 2$ so $dx = 0$ and hence $d\mathbf{r} = dx\, \mathbf{i} + dy\, \mathbf{j} = dy\, \mathbf{j}$]

$$= \int_0^1 \frac{2dy}{y^2+4} = 2\cdot\frac{1}{2}\left[\tan^{-1}\frac{y}{2}\right]_0^1 = \tan^{-1}\frac{1}{2} \qquad ...(4)$$

and $$\int_D^A \mathbf{q}\cdot d\mathbf{r} = \int_{y=1}^{y=0} [(-\mathbf{i}\, y + \mathbf{j}\, x)/(x^2+y^2)]_{x=1}\cdot(dy\mathbf{j}) = \int_1^0 \frac{dy}{y^2+1} = -\tan^{-1}1 \qquad ...(5)$$

Using (2), (3), (4) and (5), (1) becomes

$$\Gamma = \tan^{-1}(1/2) + \tan^{-1}2 - \tan^{-1}1 - \tan^{-1}1 = \cot^{-1}2 + \tan^{-1}2 - \pi/4 - \pi/4$$

$$= \pi/2 - \pi/4 - \pi/4 = 0, \text{ as } \tan^{-1}2 + \cot^{-1}2 = \pi/2$$

Since curl $\mathbf{q} = \mathbf{0}$ everywhere inside the square path, we could have got the same result directly from Stokes's theorem.

(*b*) To obtain circulation around the unit circle with its centre at the origin, we use polar coordinates for convenience. Writing $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\mathbf{q} = \frac{-r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j}}{r^2} = -\frac{\sin\theta}{r}\mathbf{i} + \frac{\cos\theta}{r}\mathbf{j} \quad \text{and} \quad \mathbf{q} = u\mathbf{i} + v\mathbf{j}$$

so that $$u = -\frac{\sin\theta}{r} \quad \text{and} \quad v = \frac{\cos\theta}{r}$$

$$\therefore \quad q_r = u\cos\theta + v\sin\theta = 0 \quad \text{and} \quad q_\theta = -u\sin\theta + v\cos\theta = 1/r.$$

$$\therefore \quad \Gamma = \int \mathbf{q}\cdot d\mathbf{r} = \int_0^{2\pi}\{(1/r)\times r\}d\theta = 2\pi.$$

Note. The answer indicates the following facts :

1. Unlike part (a), Γ is not equal to zero, because the circle encloses the origin where a singularity exists (*i.e.*, the continuity conditions for Stokes's theorem do not hold there).

2. The circulation is independent of the radius of the circle ; in fact, it can be shown that $\Gamma = 2\pi$ for every curve enclosing the origin.

Part (*ii*) Proceed like part (*a*) of part (*i*).

Ex. 2. *Show that if* $\phi = -(ax^2 + by^2 + cz^2)/2$, $V = -(lx^2 + my^2 + cn^2)/2$ *where a, b, c; l, m, n are functions of time and a + b + c = 0, irrotational motion is possible with a free surface of equi-pressure if*

$$(l + a^2 + \dot{a})e^{2\int adt}, \quad (m + b^2 + \dot{b})e^{2\int bdt}, \quad (n + c^2 + \dot{c})e^{2\int cdt} \text{ are constants.}$$

Sol. Given $$\phi = -(ax^2 + by^2 + cz^2)/2 \qquad ...(1)$$

$$V = (lx^2 + my^2 + nz^2)/2 \qquad ...(2)$$

and $$a + y + z = 0 \qquad ...(3)$$

From (1), $$\partial\phi/\partial x = -ax, \quad \partial\phi/\partial y = -by, \quad \partial\phi/\partial z = -cz \qquad ...(4)$$

and $$\partial^2\phi/\partial x^2 = -a, \quad \partial^2\phi/\partial y^2 = -b, \quad \partial^2\phi/\partial z^2 = -c. \qquad ...(5)$$

$$\therefore \quad \nabla^2\phi = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = -(a+b+c) = 0, \text{ by (3)}$$

Thus the Laplace equation $\nabla^2\phi = 0$ is satisfied and hence ϕ is velocity potential for a possible irrotational motion.

Bernoulli's equation for non-steady irrotational motion under conservative external forces with potential V is given by

$$\frac{p}{\rho} - \frac{\partial\phi}{\partial t} + \frac{1}{2}q^2 + V = F(t) \text{ where } F(t) \text{ is an arbitrary function of } t. \quad ...(6)$$

Let dot denote differentiation *w.r.t.* 't'. Then (1) gives

$$\frac{\partial\phi}{\partial t} = -\frac{1}{2}\Sigma\dot{a}x^2 \quad ....(7)$$

Also $$q^2 = \Sigma(\partial\phi/\partial x)^2 = \Sigma a^2x^2 \quad ...(8)$$

In order that a free surface of a equal pressure may exist, we have p = const. so that $Dp/Dt = 0$.

$$\therefore \quad \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = 0 \quad ...(9)$$

Using (2), (7) and (8), (6) gives

$$\frac{p}{\rho} = -\frac{1}{2}\Sigma\dot{a}x^2 - \frac{1}{2}\Sigma a^2x^2 - \frac{1}{2}\Sigma lx^2 + F(t)$$

$$\therefore \quad \frac{\partial p}{\partial t} = -\frac{1}{2}\rho[\Sigma\ddot{a}x^2 + 2\Sigma a\dot{a}x^2 + \Sigma\dot{l}x^2] + \rho F'(t)$$

and $$u\frac{\partial p}{\partial x} = \left(-\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial p}{\partial x}\right) = -\rho a x^2(\dot{a} + a^2 + l).$$

Similarly, we have

$$v\frac{\partial p}{\partial y} = -\rho b y^2(\dot{b} + b^2 + m), \qquad w\frac{\partial p}{\partial z} = -\rho c z^2(\dot{c} + c^2 + n).$$

Substituting these in (9), we obtain

$$-\frac{1}{\rho}\Sigma(\ddot{a} + 2a\dot{a} + \dot{l})x^2 - \rho\Sigma a(\dot{a} + a^2 + l)x^2 + \rho F'(t) = 0,$$

which is an identity in t and so the coefficients of x^2, y^2, z^2 and the constant must vanish separately. Thus, we obtain

$$(1/2)\times(\ddot{a} + 2a\dot{a} + \dot{l}) + a(\dot{a} + a^2 + l) = 0 \quad ...(10)$$

$$(1/2)\times(\ddot{b} + 2b\dot{b} + \dot{m}) + b(\dot{b} + b^2 + m) = 0 \quad ...(11)$$

$$(1/2)\times(\ddot{c} + 2c\dot{c} + \dot{n}) + c(\dot{c} + c^2 + n) = 0 \quad ...(12)$$

$$F'(t) = 0 \quad ...(13)$$

Now (13) gives $F(t)$ = const. (= C, say). Thus $F(t)$ is an absolute constant.

Re writing (10), we have

$$\frac{\ddot{a} + 2\dot{a}a + \dot{l}}{\dot{a} + a^2 + l}dt + 2a\,dt = 0$$

Integrating, $$\log(\dot{a} + a^2 + l) - \log c' = -2\int a\,dt$$

or $$(\dot{a} + a^2 + l)/c' = e^{-\int 2a\,dt} \quad \text{or} \quad (\dot{a} + a^2 + l)e^{2\int a\,dt} = c'$$

Similarly, (11) and (12) also yields two similar expressions.

Ex. 3. *A space is bounded by an ideal fixed surface S drawn in a homogeneous incompressible fluid satisfying the conditions for the continued existence of a velocity potential* ϕ *under conservative forces. Prove that the rate per unit time at which energy flows across S into the space bounded by S is*

$$-\rho\iint \frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial n}\,dS,$$

where ρ *is the density and* δn *an element of the mormal to* δS *drawn into the space considered.*

Sol. The kinetic energy T is given by
$$T = -\frac{1}{2}\rho\iint \phi\frac{\partial \phi}{\partial n}\,dS$$

$$\therefore \qquad \frac{dT}{dt} = -\frac{1}{2}\rho\iint\left[\frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial n}dS + \phi\frac{\partial^2 \phi}{\partial n \partial t}\,dS\right] \qquad \text{...(1)}$$

But
$$\iint \phi\frac{\partial \phi'}{\partial n}dS = \iint \phi'\frac{\partial \phi}{\partial n}dS \qquad \text{...(2)}$$

Taking $\phi' = \partial\phi/\partial t$, (2) reduces to
$$\iint \phi\frac{\partial^2 \phi}{\partial n\partial t}dS = \iint \frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial n}dS \qquad \text{...(3)}$$

Using (3), (1) reduces to
$$\frac{dT}{dt} = -\rho\iint \frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial n}dS.$$

Ex. 4. *Prove that if the velocity potential at any instant be* λxyz, *the velocity at any point* $(x+\xi, y+\eta, z+\zeta)$ *relative to the fluid at the point (x, y, z) where* ξ, η, ζ *are small, is normal to the quadratic* $x\eta\zeta + y\zeta\xi + z\xi\eta$ = *contant, with centre at (x, y, z).* **[Meerut 2006]**

Sol. Let $\mathbf{q} = (u, v, w)$ and $\mathbf{q}' = (u', v', w')$ be the fluid velocities at $P(x, y, z)$ and $P'(x+\xi, y+\eta, z+\zeta)$ respectivley.

Given velocity potential = $\lambda xyz = \phi$, say Hence $\quad u = -\partial\phi/\partial x = -\lambda yz$

Similarly, $\quad v = -\lambda xz \quad$ and $\quad w = -\lambda xy.$

Again
$$u' = u + \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} = -u - \lambda(\eta z + \zeta y)$$

Similarly, $\quad v' = v - \lambda(\zeta x + \xi z) \quad$ and $\quad w' = w - \lambda(\xi y + \eta x).$

$\therefore$ velocity of P' relative to P *i.e.* $(u'-u, v'-v, w'-w)$ is given by

$$\{-\lambda(\eta z + \zeta y), -\lambda(\zeta x + \xi z), -\lambda(\xi y + \eta x)\} \qquad \text{...(1)}$$

Let
$$F \equiv x\eta\zeta + y\zeta\xi + z\xi\eta = \text{const.} \qquad \text{...(2)}$$

Then direction ratios of the normal at P' are $\quad \partial F/\partial\xi, \quad \partial F/\partial\eta, \quad \partial F/\partial\zeta$

i.e. $\quad y\zeta + z\eta, \quad x\zeta + z\xi, \quad x\eta + y\xi. \qquad$ ...(3)

From (1) and (3), it follows that the velocity at P' relative to that at P is normal to the quadratic (2).

Ex. 5. *Deduce from the principle that the kinetic energy set up is a minimum that, if a mass of incompressible liquid be given at rest, completely filling a closed vessel of any shape and if any motion of the liquid be produced suddenly by giving arbitrarily prescribed normal velocities at all points of its bounding surface subject to the condition of constant volume, the motion produced is irrotational.*

Sol. Let T be the kinetic energy and let u, v, w be the components of velocity at any point.

Then, $$T = \frac{1}{2}\iint (u^2 + v^2 + w^2)\,dxdydz$$

Since T is minimum, $\delta T = 0$ and so we get

$$\iiint (u\,\delta u + v\,\delta v + w\,\delta w)\,dxdydz = 0 \qquad ...(1)$$

Since normal velocity $lu + mv + nw$ is prescribed on the bounding surface, we have

$$l\,\delta u + m\,\delta v + n\,\delta w = 0 \qquad \text{on the boundary } S. \qquad ...(2)$$

Equation of continuity $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$ gives

$$\frac{\partial}{\partial x}\delta u + \frac{\partial}{\partial y}\delta v + \frac{\partial}{\partial z}\delta w = 0, \qquad \text{which holds at each point within the fluid.}$$

Since ϕ is finite, we have everywhere

$$\iiint \phi\left(\frac{\partial}{\partial x}\delta u + \frac{\partial}{\partial y}\delta v + \frac{\partial}{\partial z}\delta w\right)dxdydz = 0$$

or $$\iiint\left[\frac{\partial}{\partial x}(\phi\delta u) + \frac{\partial}{\partial y}(\phi\delta v) + \frac{\partial}{\partial z}(\phi\delta w)\right]dxdydz - \iiint\left[\delta u\frac{\partial\phi}{\partial x} + \delta v\frac{\partial\phi}{\partial y} + \delta w\frac{\partial\phi}{\partial z}\right]dxdydz = 0$$

or $$-\iint \phi\,(l\delta u + m\delta v + n\delta w)\,dS - \iiint\left[\delta u\frac{\partial\phi}{\partial x} + \delta v\frac{\partial\phi}{\partial y} + \delta w\frac{\partial\phi}{\partial z}\right]dxdydz = 0 \qquad ..(3)$$

Making use of (2), we have everywhere

$$\iiint\left[\delta u\frac{\partial\phi}{\partial x} + \delta v\frac{\partial\phi}{\partial y} + \delta w\frac{\partial\phi}{\partial z}\right]dxdydz = 0 \qquad ...(4)$$

Adding (1) and (4), we have

$$\iiint\left[\left(u + \frac{\partial\phi}{\partial x}\right)\delta u + \left(v + \frac{\partial\phi}{\partial y}\right)\delta v + \left(w + \frac{\partial\phi}{\partial z}\right)\delta w\right]dxdydz = 0$$

$$\therefore \qquad u + \partial\phi/\partial x = 0, \qquad v + \partial\phi/\partial y = 0, \qquad w + \partial\phi/\partial z = 0,$$

showing that u, v and w can be obtained from ϕ and hence the motion is irrotational.

Ex. 6. *Show that the theorem that under certain conditions, the motion of a frictionless fluid, if once irrotational, will always be so, is true also when each particle is acted on by a resistance varying as the velocity.*

Sol. Let P' be a point very near to P such that $PP' = \delta s$.

Let flow along PP' be Q. Then, we have $\qquad Q = u\delta x + v\delta y + w\delta z$

$$\therefore \qquad \frac{DQ}{Dt} = \frac{D}{Dt}(u\delta x + v\delta y + w\delta z) \qquad ...(1)$$

But $$\frac{D}{Dt}(u\delta x) = \delta x\frac{Du}{Dt} + u\frac{D\delta x}{Dt} = \delta x\frac{Du}{Dt} + u\delta u. \qquad ...(2)$$

Let the components of the resistance be $(-\,ku,\ -\,kv,\ -\,kw)$. Then the equation of motion along x-axis is

$$\frac{Du}{Dt} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} - ku \qquad ...(3)$$

Using (3), (2) reduces to

$$\frac{D}{Dt}(u\delta x) = \delta x\left[-\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} - ku\right] + u\delta u \qquad ...(4)$$

Similarly,
$$\frac{D}{Dt}(v\delta y) = \delta y\left[-\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} - kv\right] + v\delta v \qquad ...(5)$$

and
$$\frac{D}{Dt}(w\delta z) = \delta z\left[-\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} - kw\right] + w\delta w \qquad ...(6)$$

Using (4), (5) and (6), (1) reduces to

$$\frac{DQ}{Dt} = -\delta V - \frac{1}{\rho}\delta p + \frac{1}{2}\delta(u^2 + v^2 + w^2) - k(u\delta x + v\delta y + w\delta z)$$

Let Γ be the circulation along the closed curve *APA*. Then

$$\Gamma = \int_A^A (u\delta x + v\delta y + w\delta z)$$

$$\therefore \qquad \frac{D\Gamma}{Dt} = \left[-V - \int\frac{dp}{\rho} + \frac{1}{2}q^2\right]_A^A - k\int_A^A (u\delta x + v\delta y + w\delta z)$$

$= -\, k\,\Gamma$ [when p is a function of ρ and V is a single valued function]

$\therefore \qquad \Gamma = \Gamma_0 e^{-kt}$, where Γ_0 is independent of t.

Let initially the motion be irrotational so that $\xi = \eta = \zeta = 0$. Thus $\Gamma = 0$ when $t = 0$. Hence $\Gamma_0 = 0$ and so $\Gamma = 0$ is always true. Therefore by Stokes's theorem, we always have

$$\iint (l\xi + m\eta + n\zeta)\, d\mathbf{S} = 0, \qquad \text{so that} \qquad \xi = \eta = \zeta = 0 \qquad \text{is always true.}$$

Thus if the fluid motion is once irrotational, it will be always so.

Ex. 7. *Obtain Cauchy's integral using circulation theorem.*

[**Note :** For the alternative method of getting Cauchy's integrals, refer Art. 3.12]

Sol. Let a, b, c be the initial co-ordinates of a particle and x, y, z the co-ordinates of the same particle at time t. Let C_0 be the initial position of the closed curve C in yz-plane.

From circulation theorem, we have

$$\int_{C_0} (v_0 db + w_0 dc) = \int_C (udx + vdy + wdz),$$

where u, v, w are velocity components at any time t and u_0, v_0, w_0 are initial velocity components.

Let ξ, η, ζ be the vorticity components at any time t and ξ_0, η_0, ζ_0 be the initial vorticity components.

Let l, m, n be direction cosines of normal to surface S which was initially in shape S_0. Then, by Stokes' theorem, we have

$$\iint_{S_0}\left(\frac{\partial w_0}{\partial b} - \frac{\partial v_0}{\partial c}\right)dS = \iint_S \left\{l\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + m\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + n\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right\}dS$$

or $$\iint_{S_0} \xi_0 \, db\, dc = \iint_S (l\xi + m\eta + n\zeta)\, dS$$

or $\displaystyle\iint_{S_0} \xi_0 \, db\, dc = \iint_S \left\{\xi \frac{\partial(y,z)}{\partial(b,c)} + \eta \frac{\partial(z,x)}{\partial(b,c)} + \zeta \frac{\partial(x,y)}{\partial(b,c)}\right\} dbdc$, as $ldS = dydz = \dfrac{\partial(x,y)}{\partial(b,c)} db\, dc$ etc.

$$\Rightarrow \qquad \xi_0 = \xi \frac{\partial(y,z)}{\partial(b,c)} + \eta \frac{\partial(z,x)}{\partial(b,c)} + \zeta \frac{\partial(x,y)}{\partial(b,c)}. \qquad ...(1)$$

Similarly, $$\eta_0 = \xi \frac{\partial(y,z)}{\partial(c,a)} + \eta \frac{\partial(z,x)}{\partial(c,a)} + \zeta \frac{\partial(x,y)}{\partial(c,a)}. \qquad ...(2)$$

and $$\zeta_0 = \xi \frac{\partial(y,z)}{\partial(a,b)} + \eta \frac{\partial(z,x)}{\partial(a,b)} + \zeta \frac{\partial(x,y)}{\partial(a,b)}. \qquad ...(3)$$

Multiplying (1), (2) and (3) by $\partial x/\partial a$, $\partial x/\partial b$ and $\partial x/\partial c$ respectively and then adding, we have

$$\xi_0 \frac{\partial x}{\partial a} + \eta_0 \frac{\partial x}{\partial b} + \zeta_0 \frac{\partial x}{\partial c} = \xi \left\{\frac{\partial x}{\partial a}\frac{\partial(y,z)}{\partial(b,c)} + \frac{\partial x}{\partial b}\frac{\partial(y,z)}{\partial(c,a)} + \frac{\partial x}{\partial c}\frac{\partial(y,z)}{\partial(a,b)}\right\}$$

or $$\xi_0 \frac{\partial x}{\partial a} + \eta_0 \frac{\partial x}{\partial b} + \zeta_0 \frac{\partial x}{\partial c} = \xi \frac{\partial(x,y,z)}{\partial(a,b,c)}, \text{ on simplification.} \qquad ...(4)$$

In Lagrangian coordinates, the equation of continuity is

$$\rho \frac{\partial(x,y,z)}{\partial(a,b,c)} = \rho_0 \qquad \text{so that} \qquad \frac{\partial(x,y,z)}{\partial(a,b,c)} = \frac{\rho_0}{\rho}. \qquad ...(5)$$

Using (5), (4) may be re-written as

$$\frac{\xi}{\rho} = \frac{\xi_0}{\rho_0}\frac{\partial x}{\partial a} + \frac{\eta_0}{\rho_0}\frac{\partial x}{\partial b} + \frac{\zeta_0}{\rho_0}\frac{\partial x}{\partial c}. \qquad ...(6)$$

Proceeding likewise, we also obtain

$$\frac{\eta}{\rho} = \frac{\xi_0}{\rho_0}\frac{\partial y}{\partial a} + \frac{\eta_0}{\rho_0}\frac{\partial y}{\partial b} + \frac{\zeta_0}{\rho_0}\frac{\partial y}{\partial c} \qquad ...(7)$$

and $$\frac{\zeta}{\rho} = \frac{\xi_0}{\rho_0}\frac{\partial z}{\partial a} + \frac{\eta_0}{\rho_0}\frac{\partial z}{\partial b} + \frac{\zeta_0}{\rho_0}\frac{\partial z}{\partial c}. \qquad ...(8)$$

Relations (6), (7) and (8) are known as *Cauchy's integrals*.

Ex. 8. *A rigid envelope is filled with homogeneous frictionless liquid; show that it is not possible, by any movements applied to the envelope, to set its contents into motion which will persist after the envelope has come to rest.*

Sol. Liquid motion is produced by movements on the boundary. Equations of motion are given by [Refer Art. 3.7]

$$u' - u = -(1/\rho)\times(\partial\tilde{w}/\partial x), \qquad ...(1)$$

$$v' - v = -(1/\rho)\times(\partial\tilde{w}/\partial y) \qquad ...(2)$$

and $$w' - w = -(1/\rho)\times(\partial\tilde{w}/\partial z), \qquad ...(3)$$

where u, v, w and u', v', w' are the velocity components at the point $P(x, y, z)$ just before and just after the impulsive action and $\tilde{w}$ is the impulsive pressure at P. Here, we have $u = v = w = 0$. Hence, (1), (2), (3) reduce to

$$u' = -(1/\rho)\times(\partial\tilde{w}/\partial x), \qquad v' = -(1/\rho)\times(\partial\tilde{w}/\partial y), \qquad \text{and} \qquad w' = -(1/\rho)\times(\partial\tilde{w}/\partial z).$$

$$\therefore \qquad u'\,dx + v'\,dy + w'dz = -\frac{1}{\rho}\left[\frac{\partial \widetilde{w}}{\partial x}dx + \frac{\partial \widetilde{w}}{\partial y}dy + \frac{\partial \widetilde{w}}{\partial z}dz\right] = -\frac{1}{\rho}d\widetilde{w} = -\,d\phi, \text{ say}$$

When the density ρ is constant and therefore the motion produced is irrotational. Since the pressure at any point is single valued, ϕ is single valued, *i.e.,* the motion is acyclic, then

$$\iiint q^2\,dx\,dy\,dz = -\iint \phi\frac{\partial \phi}{\partial n}dS\,. \qquad \text{...(4)}$$

If $\partial\phi/\partial n = 0$ on the boundary, then (4) shows that q is zero everywhere, that is, the liquid comes to rest.

Ex. 9. *Prove that in a cylic irrotational motion of a homogeneous fluid the total momentum of the fluid contained within the sphere of any radius is equivalent to a single vector through the centre of the sphere.*

Sol. Let S and V denote surface and volume of the given sphere whose centre is O. Let ρ be the density and $\mathbf{q}$ the velocity of the fluid. Let $\mathbf{M}$ be the momentum of the fluid contained within the sphere. Then, we have

$$\mathbf{M} = \int_V \rho\mathbf{q}\,dV. \qquad \text{...(1)}$$

Let $\mathbf{N}$ be the moment of momentum $\mathbf{M}$ about O. Then, we have

$$\mathbf{N} = \int_V \mathbf{r}\times(\rho\mathbf{q}\,dV) = \rho\int_V (\mathbf{r}\times\mathbf{q})\,dV\,. \qquad \text{...(2)}$$

Since the motion is irrotational, velocity potential ϕ exists such $\mathbf{q} = -\nabla\phi$. Hence (2) becomes

$$\mathbf{N} = -\rho\int_V \mathbf{r}\times\nabla\phi\,dV. \qquad \text{...(3)}$$

Now, $\qquad \nabla\times(\phi\mathbf{r}) = \nabla\phi\times\mathbf{r} + \phi(\nabla\times\mathbf{r}),\quad$ [Refer Art. 1.6 for vector identities] ...(4)

Also, $\qquad \nabla\times\mathbf{r} = \nabla\times(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$

$$\therefore \qquad \nabla\times\mathbf{r} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\mathbf{k} = \mathbf{0}$$

Hence $\quad (4) \Rightarrow \nabla\times(\phi\mathbf{r}) = \nabla\phi\times\mathbf{r} = -\mathbf{r}\times\nabla\phi \quad \Rightarrow \quad \mathbf{r}\times\nabla\phi = -\nabla\times(\phi\mathbf{r}).$ ...(5)

Using (5), (3) $\Rightarrow$ $\mathbf{N} = \rho\int_V \nabla\times(\phi\mathbf{r})\,dV$ or $\mathbf{N} = \rho\iint_S \hat{\mathbf{n}}\times(\phi\mathbf{r})\,dS = \rho\iint_S \phi(\hat{\mathbf{n}}\times\mathbf{r})\,dS,$ ...(6)

where $\hat{\mathbf{n}}$ is the inward drawn normal unit vector. Again $\hat{\mathbf{n}}$ and $\mathbf{r}$ are parallel vectors on the surface of the sphere and so $\hat{\mathbf{n}}\times\mathbf{r} = \mathbf{0}$. Then, by (6), $\mathbf{N} = \mathbf{0}$. Hence the moment of momentum $\mathbf{N}$ about O is zero and therefore $\mathbf{N}$ must pass through the centre O of the sphere.

Ex. 10. *If p denotes the pressure, V the potential of the external forces and q the velocity of* a homogeneous liquid moving irrotationally, show that *$\nabla^2 q^2$ is positive and $\nabla^2 p$ is negative provided $\nabla^2 V = 0$. Hence prove that the velocity cannot have a maximum value and the pressure cannot have a minimum value at a point in the interior of the liquid.*

Sol. Since the motion is irrotational, the velocity potential ϕ exists such that

$$\boldsymbol{q} = -\nabla\phi = -[(\partial\phi/\partial x)\mathbf{i} + (\partial\phi/\partial y)\mathbf{j} + (\partial\phi/\partial z)\mathbf{k}].$$

$$\therefore \qquad q^2 = (\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2 + (\partial\phi/\partial z)^2. \qquad \text{...(1)}$$

From vector calculus (Refer vector identities in Art. 1.6, we have

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\therefore \qquad \nabla\cdot\{\nabla(\phi\psi)\} = \nabla\cdot(\phi\,\nabla\psi) + \nabla\cdot(\psi\nabla\phi)$$

or $$\nabla^2(\phi\psi) = [\nabla\phi\cdot\nabla\psi + \phi\,\nabla\cdot\nabla\psi] + [\nabla\psi\cdot\nabla\phi + \psi\,\nabla\cdot\nabla\phi]$$

or $$\nabla^2(\phi\psi) = 2\nabla\phi\cdot\nabla\psi + \psi\,\nabla^2\phi + \phi\nabla^2\psi. \qquad \text{...(2)}$$

Replacing ϕ and ψ by $\partial\phi/\partial x$ in result (2), we have

$$\nabla^2\left(\frac{\partial\phi}{\partial x}\right)^2 = 2\left(\nabla\frac{\partial\phi}{\partial x}\cdot\nabla\frac{\partial\phi}{\partial x}\right) + 2\frac{\partial\phi}{\partial x}\nabla^2\left(\frac{\partial\phi}{\partial x}\right). \qquad ...(3)$$

But $$\nabla^2\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}\nabla^2\phi = 0. \qquad ...(4)$$

Also, $$\nabla\frac{\partial\phi}{\partial x} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\frac{\partial\phi}{\partial x} = \mathbf{i}\frac{\partial^2\phi}{\partial x^2} + \mathbf{j}\frac{\partial^2\phi}{\partial y\partial x} + \mathbf{k}\frac{\partial^2\phi}{\partial z\partial x}.$$

$\therefore$ $$\nabla\frac{\partial\phi}{\partial x}\cdot\nabla\frac{\partial\phi}{\partial x} = \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial y\partial x}\right)^2 + \left(\frac{\partial^2\phi}{\partial z\partial x}\right)^2. \qquad ...(5)$$

Using (4) amd (5), (3) reduces to

$$\nabla^2\left(\frac{\partial\phi}{\partial x}\right)^2 = 2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial y\partial x}\right)^2 + \left(\frac{\partial^2\phi}{\partial z\partial x}\right)^2\right] > 0. \qquad ...(6)$$

Similarly, $\nabla^2(\partial\phi/\partial y)^2 > 0$ and $\nabla^2(\partial\phi/\partial z)^2 > 0$. ...(7)

Now, (6) and (7) $\Rightarrow$ $\nabla^2[(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2 + (\partial\phi/\partial z)^2] > 0$. ...(8)

Then (1) and (9) $\Rightarrow$ $\nabla^2 q^2 > 0$, as required. ...(9)

From Bernoulli's equation, we have

$$(p/\rho) - (\partial\phi/\partial t) + q^2/2 + V = f(t).$$

$\therefore$ $$\nabla^2\left(\frac{p}{\rho}\right) - \frac{\partial}{\partial t}(\nabla^2\phi) + \frac{1}{2}\nabla^2 q^2 + \nabla^2 V = \nabla^2 f(t).$$

Now, $\nabla^2\phi = 0$ and $\nabla^2 f(t) = 0$. Also given that $\nabla^2 V = 0$.

$\therefore$ From (10), $\nabla^2 p = -\frac{1}{2}\rho\,\nabla^2 q^2 < 0$ as $\nabla^2 q^2 > 0$, using (9)

Second part. From Gauss' theorem, we have

$$\iint_S (l'U + m'V + n'W)dS = -\iiint_V \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right)dxdydz, \qquad ...(1)$$

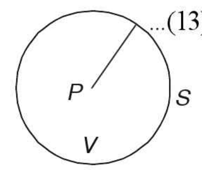
where l', m', n' are the direction cosines of the inward normal to an element δS of surface S and V is the volume enclosed by S.

Let $U = \partial\phi/\partial x$, $V = \partial\phi/\partial y$, $W = \partial\phi/\partial z$. Then (1) reduces to

$$\iint_S \left(l'\frac{\partial\phi}{\partial x} + m'\frac{\partial\phi}{\partial y} + n'\frac{\partial\phi}{\partial z}\right)dS = -\iiint_V \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)dx\,dy\,dz$$

or $\displaystyle\iint_S \frac{\partial\phi}{\partial n}dS = -\iiint_V \nabla^2\phi\,dxdydz$, where δn is an element of inward normal. ...(12)

If $\phi = q^2$, (12) $\Rightarrow$ $\displaystyle\iint_S \frac{\partial q^2}{\partial n}dS = -\iiint_V \nabla^2 q^2\,dxdydz$, ...(13)



We now apply the above result (13) to the case of a liquid contained in a small sphere (See figure) Then, since $\nabla^2 q^2 > 0$, (13) shows that

$$\iint_S \frac{\partial q^2}{\partial n}dS < 0. \qquad ...(14)$$

Since $\delta n = -\delta r$, $\quad$ (14) $\Rightarrow$ $\quad$ $\iint_S \frac{\partial q^2}{\partial r} dS > 0.$ ...(15)

If q^2 is maximun at P, then $\partial q^2/\partial r$ is negative on the surface of a small sphere surrounding P,

that is, $\quad$ $\iint_S \frac{\partial q^2}{\partial r} dS < 0,$ ...(16)

which contradicts the results (15) and hence q^2 cannot have a maximum within the liquid. Thus q^2 can be maximum only on the boundary.

Similarly, putting $\phi = p$ in (12), we have

$$\iint_S \frac{\partial p}{\partial n} dS = -\iiint_V \nabla^2 p \, dx\, dy\, dz. \qquad ...(17)$$

Since $\nabla^2 p < 0$, $\quad$ (17) $\Rightarrow$ $\quad$ $\iint_S \frac{\partial p}{\partial n} dS > 0.$ ...(18)

Now proceeding as before, we find that p cannot have a minimum at a point within the liquid. Hence the pressure can be minimum only on the boundary.

Ex. 11. *Prove that irrotational acyclic motion of a liquid contained in a boundary cannot be created or destroyed by application of impulses.*

Sol. The equations of motion under inpulsive forces are

$$u' - u = I_x - (1/\rho)(\partial \tilde{w}/\partial x) \qquad ...(1)$$

$$v' - v = I_y - (1/\rho)(\partial \tilde{w}/\partial y) \qquad ...(2)$$

and $$w' - w = I_z - (1/\rho)(\partial \tilde{w}/\partial z), \qquad ...(3)$$

where u, v, w and u', v', w' are the velocity components at any point $P(x, y, z)$ just before and just after the impulsive action, I_x, I_y, I_z are the components of external impulsive forces per unit mass of the fluid and $\tilde{w}$ is the impulsive pressure at P.

Multiplying (1), (2) and (3) *by*, *dx*, *dy* and *dz* respectively and then adding, we have

$$(u'dx + v'dy + w'dz) - (udx + vdy + wdz)$$

$$= (I_x\, dx + I_y dy + I_z dz) - (1/\rho)\left[(\partial \tilde{w}/\partial x)dx + (\partial \tilde{w}/\partial y)dy + (\partial \tilde{w}/\partial z)dz\right]$$

$$= -dV - (1/\rho)\, d\tilde{w}, \text{ if } V \text{ is potential of the external impulses}$$

$$= -d(V + \tilde{w}/\rho), \text{ if } \rho \text{ is constant}$$

$\therefore$ $$(u'dx + v'dy + w'dz) - (udx + vdy + wdz) = -d(V + \tilde{w}/\rho). \qquad ...(4)$$

Now R.H.S. of (4) is an exact differential and hence L.H.S. of (4) must be an exact differential. Hence if $udx + vdy + wdz$ is not an exact differential, then $u'dx + v'dy + w'dz$ will also be not an exact differential, that is, when the motion is not irrotational we cannot make it irrotational by application of impulses.

Let $$udx + vdy + wdz = -d\phi, \qquad ...(5)$$

then, clearly $$u'dx + v'dy + w'dz = -d\phi', \text{ say} \qquad ...(6)$$

Then (4) becomes $$-d\phi' + d\phi = -d(V + \tilde{w}/\rho).$$

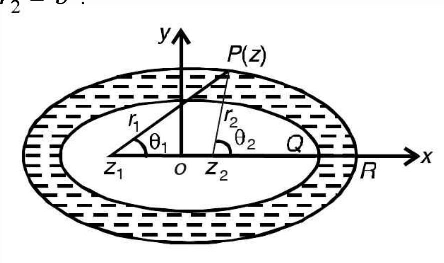
Integrating, $\quad -\phi' + \phi = -(V + \tilde{w}/\rho) - C \quad$ or $\quad \phi' - \phi = V + \tilde{w}/\rho + C,$ ...(7)

where C is an arbitrary constant.

Thus, if V and $\tilde{w}$ be single valued, then from (7) we see that $\phi' - \phi$ is also single-valued. It follows that if ϕ be single valued, ϕ' must be single valued and if ϕ be many valued, ϕ' must also be many valued so that $\phi' - \phi$ is single-valued. Hence the required result follows.

Ex. 12. *Liquid of density ρ is flowing in two dimensions between the oval curves $r_1r_2 = a^2$ and $r_1r_2 = b^2$ where r_1, r_2 are the distances measured from two fixed points. If the motion is irrotational and quantity m per unit time crosses any line joining the bounding curves, then prove that the kinetic energy is $(\pi\rho m^2)/\log(b/a)$.*

Sol. Here we have two-dimensional irrotational motion in a region bounded by the given curves $\qquad r_1r_2 = a^2 \qquad$ and $\qquad r_1r_2 = b^2$.



Let the complex potential w be

$$w = iE \log \{(z - z_1)(z - z_2)\}. \qquad ...(1)$$

But $\quad z - z_1 = r_1 e^{i\theta_1} \quad$ and $\quad z - z_2 = r_2 e^{i\theta_2}$

$$\therefore \quad (z - z_1)(z - z_2) = (r_1r_2)\, e^{i(\theta_1+\theta_2)} \qquad ...(2)$$

Using (2), (1) reduces to

$$\phi + i\psi = iE\,[\log (r_1r_2) + i(\theta_1 + \theta_2)].$$

$$\therefore \quad \phi = -E(\theta_1 + \theta_2) \quad \text{and} \quad \psi = E \log r_1r_2. \qquad ...(3)$$

Let the barrier be taken at $\theta_1 = \theta_2 = 0$. On the positive side $\theta_1 = \theta_2 = 0$ and hence $\phi = 0$. Again, on the negative side $\theta_1 = \theta_2 = 2\pi$ and so $\phi = 4\pi E$.

If k is the circulation, we have $\qquad k = 4\pi E \qquad$ so that $\qquad E = k/4\pi$.

$$\therefore \text{ From (3),} \quad \phi = -(k/4\pi)(\theta_1 + \theta_2) \quad \text{and} \quad \psi = (k/4\pi)\log r_1r_2. \qquad ...(4)$$

Now, $\qquad m = \psi_B - \psi_A = (k/4\pi)\log b^2 - (k/4\pi)\log a^2$, by (4)

or $\qquad m = (k/4\pi)\log (b^2/a^2) = (k/4\pi)\log (b/a)^2 = (k/2\pi)\log (b/a)$

and so $\qquad k = (2\pi m)/\log (b/a). \qquad ...(5)$

Let T be required the kinetic energy. Then, we have

$$T = -\frac{1}{2}\rho k \iint \phi \frac{\partial \phi}{\partial n} dS = -\frac{1}{2}\rho k m, \qquad \text{as} \qquad m = -\iint \phi \frac{\partial \phi}{\partial n} dS$$

$$= \frac{1}{2}\rho\, m \times \frac{2\pi m}{\log (b/a)} = \frac{\pi \rho m^2}{\log (b/a)}, \text{ using (5).}$$

Ex. 13. *Incompressible fluid of density ρ is contained between two co-axial circular cylinders, of radii a and b (a < b), and between two rigid planes perpendicular to the axis at a distance l apart. The cylinders are at rest and the fluid is circulating in irrotational motion, its velocity V at the surface of the inner cylinder. Prove that the kinetic energy is $\pi\rho l a^2 V^2 \log (b/a)$.*

Sol. For the case of irrotational two-dimensional fluid motion, we have

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0. \qquad ...(1)$$

Here ψ is function of r only. So $\qquad \partial^2\psi/\partial\theta^2 = 0$.

Then (1) becomes $\quad \dfrac{d^2\psi}{dr^2} + \dfrac{1}{r}\dfrac{d\psi}{dr} = 0 \quad$ or $\quad r\dfrac{d^2\psi}{dr^2} + \dfrac{d\psi}{dr} = 0 \quad$ or $\quad d\{r(d\psi/dr)\} = 0$.

Integrating, $\qquad r(d\psi/dr) = C \qquad$ so that $\qquad d\psi/dr = C/r. \qquad ...(2)$

But given that $d\psi/dr = V$ when $r = a$. So (2) gives $V = C/a$ and hence $\qquad C = Va$.

Thus, $\qquad$ transverse velocity $= d\psi/dr = (Va)/r$.

Again, the radial velocity is zero as $\partial\psi/\partial\theta = 0$, ψ being a function of r only.

Hence, $\qquad q =$ resultant velocity $= (Va)/r$.

Let T be the required kinetic energy of the fluid. Then, we have

$$T = \int_a^b \frac{1}{2}(2\pi r\, dr\, l\rho)q^2 = \int_a^b \frac{1}{2}(2\pi r\, l\rho)\frac{V^2a^2}{r^2}dr = \pi\rho\, la^2V^2 \int_a^b \frac{1}{r}dr = \pi\, \rho l a^2 V^2 \log (b/a).$$

Ex. 14. *In a two-dimensional flow the velocity components are $u = Cy$, $v = 0$ (where C is a constant). Find the circulation about the circle $x^2 + y^2 - 2ay = 0$ situated in the flow.*

Sol. We know that $$\Gamma = \Omega A, \qquad \text{...(1)}$$

where A = area of circular boundary of radius $a = \pi a^2$

and Ω = vorticity $= (\partial v/\partial x) - (\partial u/\partial y) = 0 - C = -C$

$\therefore$ $\Gamma = (-C) \times \pi a^2 = -C\pi a^2$ square meters per second.

EXERCISES

1. Show that the kinetic energy of a volume V of liquid of constant density ρ that is moving irrotationally with velocity potential ϕ is $-\dfrac{1}{2}\rho \int_S \phi \dfrac{\partial \phi}{\partial n} dS$, where S denotes the surface of V and n the normal into the liquid. **[Meerut 2007]**

[**Hint.** Refer deduction **III** of Art 6.7]

2. Prove that the circulation in any closed path moving with the fluid is constatnt for all time, provided that the fluid is barotropic and the external forces are conservative. Deduce the theorem of the permanence of irrorational motion.

3. Prove that acyclic irrotatinal motion produces in an infinite liquid bounded intermally and extermally by given velocities on the boundary may not cease when the boundaries are brought to rest.

4. State and prove uniqueness, theorem. **[Kanpur 2001, G.N.D.U. 1998]**

[**Solution : Statment of uniqueness theorem.** *There cannot be two diffeent forms of irrotational motion for a given confined mass of liquid when boundaries have prescribed velocities or are subject to given impulses.*

Proof. Give proofs of theorem I and theorem II given in Art. 6.9 B.

5. State and prove Kelvin's circulation theorem. Also prove that the irrotational motion is permanent. **[Agra 2008; Meerut 1999; 2002]**

[**Hint.** Refer Art. 6.4 and Art 6.5]

6. State and prove Kelvin's theorem of constancy of circulation. **[Meerut 2001, 02]**

[**Hint.** Refer Art 6.4]

7. State and prove Stokes' theorem for circulation. **[Agra 2005]**

OBJECTIVE QUESTIONS ON CHAPTER 6

Multiple choice questions

Choose the correct alternative from the following questions

1. Let C be a closed curve and Γ be the circulation, then

(*i*) $\Gamma = \int_C \mathbf{q} \cdot d\mathbf{r}$ (*ii*) $\Gamma = \int_C \mathbf{q} \times d\mathbf{r}$ (*iii*) $\Gamma = \int_C |\mathbf{q}|\, dr$ (*iv*) None of these

2. In usual notations, Stoke's theorem is

(*i*) $\int_C \mathbf{q} \cdot d\mathbf{r} = \int_S \text{curl}\, \mathbf{q} \times d\mathbf{S}$ (*ii*) $\int_C \mathbf{q} \cdot d\mathbf{r} = \int_S \text{curl}\, \mathbf{q} \cdot dS$

(*iii*) $\int_C \mathbf{q} \times d\mathbf{r} = \int_S \text{curl}\, \mathbf{q} \cdot dS$ (*iv*) $\int_C \mathbf{q} \times d\mathbf{r} = \int_C \text{curl}\, \mathbf{q} \times dS$

3. The motion in which the velocity potential is single–valued is called

(*i*) Laminar (*ii*) Turbulent (*iii*) Cyclic (*iv*) Acyclic **(Agra 2011)**

4. A velocity field is given by $\mathbf{q} = (-\mathbf{i}\,y + \mathbf{j}\,x)/(x^2 + y^2)$. Then circulation round a unit circle with centre at the origin is

(*i*) π (*ii*) 2π (*iii*) 3π (*iv*) 4π

5. In the usual notations, relation $\Gamma = \int_S \Omega \cdot \mathbf{n}\, dS$ holds for

(*i*) Gauss theorem (*ii*) Kelvin' theorem (*iii*) Stokes' theorem (*iv*) Green's theorem

6. "The irrotational motion of a liquid occuping a simply connected region has less kinetic energy than any other motion consistent with the same normal velocity of the boundary" is the statement of

(*i*) Green's theorem (*ii*) Kelvin's minimum energy theorem
(*iii*) Theorem of Blasius (*iv*) Stoke's theorem. **[Agra 2002]**

7. The result, namely, "when the external forces are conservative and derivable from a single valued potential function and the density is a function of pressure only, then the circulation in any closed circuit moving with the fluid is constant for all time" is due to

(*i*) Stokes (*ii*) Kelvin (*iii*) Green (*iv*) Lagrange

8. Let $\boldsymbol{q}$ be the velocity vector, Ω the vorticity vector and S be a surface bounded by a closed curve S, then $\int_C \boldsymbol{q} \cdot \boldsymbol{dr} = \int_S curl\ \boldsymbol{q} \cdot \boldsymbol{dS}$ is the statement of

(*i*) Kelvin's Circulation theorem (*ii*) Stokes' theorem
(*iii*) Green's theorem (*iv*) Gauss theorem **[Agra 2005]**

Answers/Hint to objective type questions

1. (*i*). See Art. 6.2 **2.** (*ii*). See Art. 6.3
3. (*iv*). See Art. 6.9 *A* **4.** (*ii*). See Ex. 1, Art. 6.13
5. (*iii*). See Art. 6.3, **6.** (*ii*). See Art. 6.10
7. (*ii*). See Art. 6.4 **8.** (*ii*). See Art. 6.3

Miscellaneous problems on chapter 6

1. Prove that if the motion is irrotational, then circulation is zero **(Agra 2008, 2009)**

Sol. Let C be a closed curve. Then, by definition 6.2, the circulation Γ is given by

$$\Gamma = \int_C (udx + vdy + wdz) \qquad \text{... (1)}$$

Since the given motion is irrotational, there exists velocity potential ϕ such that

$$u = -(\partial\phi/\partial x), \qquad v = -(\partial\phi/\partial y) \qquad \text{and} \qquad w = -(\partial\phi/\partial z) \qquad \text{... (2)}$$

Using (2), (1) reduces to $\quad \Gamma = -\int_C \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = -\int_A^A d\phi = -[\phi]_A^A = 0,$

where we have used the fact that for a closed path initial and final points of C are the same. Also, let the initial and final points of closed curve C be denoted by A.

2. State and prove Green's theorem. Hence show that the kinetic energy of a given mass of liquid moving irrotationally in a simply connected surface depends only on the motion of boundaries.

Hint. Refer Art. 6.6 and deduction III of Art. 6.7 **(Agra 2012)**

CHAPTER

7 Motion of Cylinders

7.1. General motion of a cylinder [Kanpur 2004; G.N.D.U. 2003, 04]

The present chapter is devoted to study two dimensional irrotational motion produced by the motion of a cylinder in an infinite mass of liquid at rest at infinity, or when a cylinder is inserted in a steady stream. For the sake of simplicity we shall suppose the cylinder to be of unit length, and the liquid and the cylinder to be confined between two smooth parallel planes at right angles to the axis of the cylinder.

We know that the velocity potential ϕ and the stream function ψ are connected with the help of Cauchy-Riemann equations, namely,

$$\partial\phi/\partial x = \partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y = -\partial\psi/\partial x \qquad ...(1)$$

In view of (1), the complex potential

$$w(z) = \phi(x, y) + i\psi(x, y) \qquad ...(2)$$

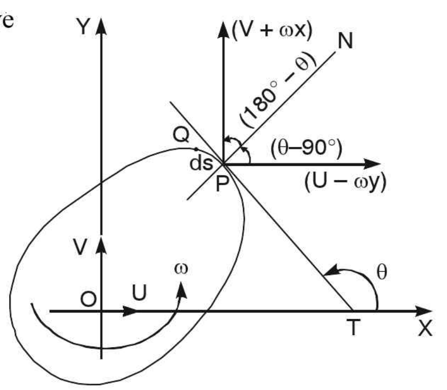
can be determined by finding only ψ. The stream function ψ must satisfy the Laplace's equation $\nabla^2\psi = 0$ at all points of the liquid and must also satisfy the following boundary conditions :

(*i*) Since the liquid is at rest at infinity, we must have

$$\partial\psi/\partial x = 0 \qquad \text{and} \qquad \partial\psi/\partial y = 0 \qquad \text{at infinity}$$

(*ii*) At any fixed boundary the normal velocity must be zero, or the boundary must coincide with a streamline ψ = const.

(*iii*) At the boundary of the moving cylinder, the normal component of the velocity of the liquid must be equal to the normal component of the velocity of the cylinder.



We now express the condition (*iii*) by a formula for ψ as follows :

Let a point O of the cross-section of any cylinder be taken as origin. Let U and V be the velocities parallel to the axes of X and Y at O and let the cylinder turn with angular velocity ω. If $P(x, y)$ be any point on the surface of the cylinder, then the velocity components of P are $U - \omega y$ and $V + \omega x$. If θ is the inclination of the tangent at P with OX, then from Differential Calculus,

$$\cos\theta = dx/ds \qquad \text{and} \qquad \sin\theta = dy/ds, \qquad ...(3)$$

where arc $PQ = ds$, P and Q being two neighbouring points.

$\therefore$ the outward normal velocity at P $= (U - \omega y)\cos(\theta - 90°) + (V + \omega x)\cos(180° - \theta)$

$$= (U - \omega y)\sin\theta - (V + \omega x)\cos\theta$$

$$= (U - \omega y)(dy/ds) - (V + \omega x)(dx/ds) \qquad ...(4)$$

Also the velocity of the liquid in the direction of the outward normal is $-\partial\psi/\partial s$.

On equating the above two expressions for the normal component of velocities in accordance with condition (*iii*), we have

$$-\frac{\partial \psi}{\partial s} = (U - \omega y)\frac{dy}{ds} - (V + \omega x)\frac{dx}{ds} \qquad ...(5)$$

On integrating (5) along the arc, we get

$$\psi = Vx - Uy + (1/2) \times \omega(x^2 + y^2) + C, \text{ where } C \text{ is an arbitrary constant.} \qquad ...(6)$$

Note that (6) holds for rotational or irrotational motion. Thus (6) is the condition for the most general type of motion of a cylinder of arbitrary cross-section.

If the cylinder is rotating about a fixed axis with angular velocity (so that $U = V = 0$), then (6) reduces to

$$\psi = (1/2) \times \omega\,(x^2 + y^2) + C \qquad ...(7)$$

or

$$\psi = (1/2) \times \omega\, z\bar{z} + C \qquad ...(8)$$

Suppose the equation of the cross-section of the boundary of the cylinder be of the form

$$z\bar{z} = f(z) + f(\bar{z})$$

Then the complex potential satisfying the boundary condition (8) is

$$w = i\omega\, f(z) \qquad ...(9)$$

Next, let the cylinder move along the x-axis with velocity U without rotation (so that $V = 0$ and $\omega = 0$). Then (6) reduces to

$$\psi = -Uy + C \qquad ...(10)$$

Similarly, if the cylinder moves along the y-axis with velocity V without rotation, then (6) reduces to

$$\psi = Vx + C \qquad ...(11)$$

7.2. Kinetic energy.

In any type of a cylinder moving in liquid at rest at infinity, the kinetic energy is given by, as in Art. 6.7 of chapter 6,

$$T = -\frac{1}{2}\rho \iint_S \phi \frac{\partial \phi}{\partial n} dS, \qquad ...(1)$$

where dS is the elementary surface and integration is taken round a closed surface S.

Suppose the liquid is confined between two smooth planes at unit distance apart. Let ds be the elementary arc of the cylinder so that $dS = 1 \cdot ds = ds$. Then (1) reduces to

$$T = -\frac{1}{2}\rho \int \phi \frac{\partial \phi}{\partial n} ds, \qquad ...(2)$$

where the integration is now round the perimeter of the cross-section of the cylinder. But $-\partial\phi/\partial n$ is the normal velocity outwards, and $\partial\psi/\partial s$ is the normal velocity inwards, so that $\partial\phi/\partial n = \partial\psi/\partial s$. Then (2) reduces to

$$T = -\frac{1}{2}\rho \int \phi\, d\psi \qquad ...(3)$$

PART I : MOTION OF A CIRCULAR CYLINDER

7.3. Motion of a circular cylinder.

[Meerut 2000, 02; Garhwal 2002; G.N.D.U. Amritsar 2003, 05, 06; Rohilkhand 2000]

To determine the motion of a circular cylinder moving in an infinite mass of the liquid at rest at infinity, with velocity U in the direction of x-axis. **[Kanpur 2008, 09]**

To find the velocity potential ϕ that will satisfy the given boundary conditions, we have the following considerations :

(*i*) ϕ satisfies the Laplace's equation $\nabla^2\phi = 0$ at every point of the liquid. In polar coordinates in two dimensions $\nabla^2\phi = 0$ takes the form

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \qquad ...(1)$$

We know that (1) has solutions of the forms

$$r^n \cos n\theta \qquad \text{and} \qquad r^n \sin n\theta,$$

where n is any integer, positive or negative. Hence the sum of any number of terms of the form

$$A_n r^n \cos n\theta \qquad \text{or} \qquad B_n r^n \sin n\theta,$$

is also a solution of (1),

(*ii*) Normal velocity at any point of the cylinder
= velocity of the liquid at that point in that direction

i.e.
$$-\frac{\partial\phi}{\partial r} = U\cos\theta, \qquad \text{when } r = a \qquad ...(2)$$

(*iii*) Since the liquid is at rest at infinity, velocity must be zero there.

Thus,
$$-\frac{\partial\phi}{\partial r} = 0 \qquad \text{and} \qquad -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = 0 \qquad \text{at} \qquad r = \infty \qquad ...(3)$$

The above considerations suggest that we must assume the following suitable form of ϕ.

$$\phi = Ar\cos\theta + (B/r)\cos\theta \qquad ...(4)$$

From (4),
$$-\frac{\partial\phi}{\partial r} = -\left(A - \frac{B}{r^2}\right)\cos\theta \qquad ...(5)$$

Putting $r = a$ in (5) and using (2), we get

$$U\cos\theta = -\left(A - \frac{B}{a^2}\right)\cos\theta \qquad \text{so that} \qquad A - \frac{B}{a^2} = -U \qquad ...(6)$$

Putting $r = \infty$ in (5) and using (3), we get $0 = -A\cos\theta$ so that $A = 0$

Then (6) gives $B = Ua^2$. Hence (4) reduces to

$$\phi = (Ua^2/r)\cos\theta \qquad ...(7)$$

It may be noted that (7) also satisfies the second condition given by (3). Hence (7) gives the required velocity potential.

But
$$\frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} \qquad ...(8)$$

$\therefore$
$$\partial\psi/\partial r = (Ua^2/r^2)\sin\theta, \text{ using (7) and (8)}$$

Integrating,
$$\psi = -(Ua^2/r)\sin\theta \qquad ...(9)$$

which gives the stream function of the motion. The complex potential $w(=\phi + i\psi)$ is given by

$$w = \frac{Ua^2}{r}(\cos\theta - i\sin\theta) = \frac{Ua^2}{r}e^{-i\theta} = \frac{Ua^2}{z} \qquad ...(10)$$

where
$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Note 1. From (7) and (9), we find that the velocity potential and stream function are the same as for a two-dimensional doublet of strength Ua^2 on the axis of the cylinder in an infinite mass of liquid.

Note 2. The streamlines are given by ψ = const. *i.e.* by

$$-\frac{Ua^2}{r}\sin\theta = -\frac{Ua^2}{c} \qquad \text{or} \qquad cr\sin\theta = r^2$$

i.e. $\quad x^2 + y^2 - cy = 0 \quad$ or $\quad x^2 + (y - c/2)^2 = (c/2)^2$,

which are circles all touching x-axis at origin and having the centre $(0, c/2)$.

7.4. Liquid streaming past a fixed circular cylinder.

[Kanpur 2010; Meerut 2001, 04, Agra 2007]

Let the cylinder be at rest and let the liquid flow past the cylinder with velocity U in the negative direction of x-axis. This motion may be deduced from that of Art. 7.3 by imposing a velocity $-U$ parallel to the x-axis on both the cylinder and the liquid. The cylinder is then reduced to rest and we must add to the velocity potential a term Ux (*i.e.* $Ur\cos\theta$) to account for the additional velocity; consequently a term $Ur\sin\theta$ must be added to ψ. Thus, we have

$$\phi = U\left(r + a^2/r\right)\cos\theta, \qquad \psi = U(r - a^2/r)\sin\theta \qquad \text{...(1)}$$

and $$w = \phi + i\psi = U(r\cos\theta + ir\sin\theta) + \frac{Ua^2}{r}(\cos\theta - i\sin\theta) = Ure^{i\theta} + \frac{Ua^2}{re^{i\theta}} = Uz + \frac{Ua^2}{z} \qquad \text{...(2)}$$

[Using the fact that $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$]

Note 1. Here the equation $(r - a^2/r)\sin\theta$ = const. represents the streamlines relative to the cylinder, and this is true whether the cylinder be moving or be at rest.

Note 2. The velocity distribution at any point $z = ae^{i\theta}$ on the cylinder is given by

$$q = |dw/dz| \text{ when } z = ae^{i\theta}$$

$$= \left|U - (Ua^2/z^2)\right| \text{ when } z = ae^{i\theta}, \text{ by (2)}$$

$$= \left|U - \frac{Ua^2}{a^2e^{2i\theta}}\right| = \left|U - Ue^{-2i\theta}\right| = |U|\,\left|1 - e^{-2i\theta}\right| = |U|\,|1 - \cos 2\theta + i\sin 2\theta|$$

Thus, $$q = |U|\left\{(1-\cos 2\theta)^2 + \sin^2 2\theta\right\}^{1/2} = |U|(2 - 2\cos 2\theta)^{1/2} = 2|U|\sin\theta \qquad \text{...(3)}$$

The maximum value of q occurs where $\sin\theta = 1$ *i.e.* $\theta = \pi/2$. Thus, we have

$$q_{\max} = 2|U| = \text{twice the velocity of free stream}$$

Stagnation points (or critical points) occur where $q = 0$ *i.e.* $\sin\theta = 0$ *i.e.* $\theta = 0$ or $\theta = \pi$.

Note 3. We now determine pressure on the boundary of the cylinder. Let Π be the pressure at infinity and U be free stream velocity at infinity. Then Bernoulli's equation gives

$$p + \frac{1}{2}\rho q^2 = \text{const.} = \Pi + \frac{1}{2}\rho U^2$$

or $$p - \Pi = (1/2)\times\rho U^2(1 - 4\sin^2\theta), \text{ by (3)} \qquad \text{...(4)}$$

We know that a liquid is not able to sustain a negative pressure. Moreover it maintains contact with boundary so long as the pressure remains positive everywhere. When the pressure p given by (4) becomes negative, the theory breaks down and cavitation occurs. Hence for p to be positive, we must have

$$\Pi + \frac{1}{2}\rho U^2(1 - 4\sin^2\theta) > 0 \qquad \text{at} \qquad \theta = \frac{\pi}{2}$$

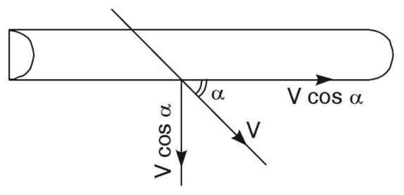
i.e. $\qquad \Pi - \frac{3}{2}\rho U^2 > 0 \qquad$ or $\qquad U^2 < \frac{2\Pi}{3\rho}$

Thus, if $U > \sqrt{2\Pi/3\rho}$, cavitation occurs.

7.5. Illustrative solved examples.

Ex. 1. *A stream of water of great depth is flowing with uniform velocity V over a plane level bottom.An infinite cylinder of which the cross-section is a semi-circle of radius r, lies on its flat side with its generating lines making an angle* α *with the undisturbed streamlines. Prove that the resultant fluid pressure per unit length on the curved surface is* $2a\Pi - (5/3)\times \rho a V^2 \sin\alpha$, *where* Π *is the fluid pressure at a great distance from the cylinder.* **[Meerut 2003]**

Sol. The components of the velocity V are $V\cos\alpha$ along the generators and $V\sin\alpha$ perpendicular to the generators of the cylinder. Since $V\cos\alpha$ is parallel to the generators, there is no pressure on account of it. The component $V\sin\alpha$ will give rise to the velocity potential given by



$$\phi = V\sin\alpha\,(r + a^2/r)\cos\theta \qquad ...(1)$$

$$\therefore \qquad \partial\phi/\partial r = V\sin\alpha\,(1 - a^2/r^2)\cos\theta \qquad ...(2)$$

and $$\partial\phi/\partial\theta = -V\sin\alpha\,(r + a^2/r)\sin\theta \qquad ...(3)$$

Let q be the velocity at any point (r, θ). Then

$$q^2 = \left(-\frac{\partial\phi}{\partial r}\right)^2 + \left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 = V^2\sin^2\alpha\left(1 + \frac{a^4}{r^4} - \frac{2a^2}{r^2}\cos 2\theta\right), \text{ using (2) and (3)}$$

The pressure p at any point (r, θ) is given by $\qquad p/\rho = C - q^2/2$

$$\therefore \qquad \frac{p}{\rho} = C - \frac{1}{2}V^2\sin^2\alpha\left(1 + \frac{a^4}{r^4} - \frac{2a^2}{r^2}\cos 2\theta\right) \qquad ...(4)$$

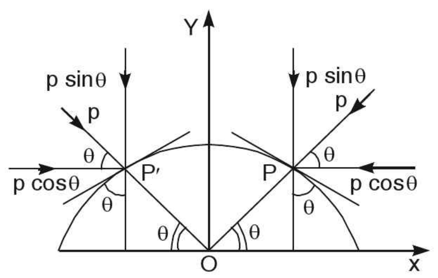
Given $p = \Pi$ when $r = \infty$. So $C = \Pi/\rho + (V^2/2)\times\sin^2\alpha$ and hence (4) reduces to

$$\frac{p}{\rho} = \frac{\Pi}{\rho} - \frac{1}{2}V^2\sin^2\alpha\left(\frac{a^4}{r^4} - \frac{2a^2}{r^2}\cos 2\theta\right)$$

Taking $r = a$, the pressure at any point (a, θ) on the circular cross-section of the cylinder is given by

$$p = \Pi - \rho V^2(1 - 2\cos 2\theta)\sin^2\alpha \qquad \text{or} \qquad p = \Pi - (1/2)\times\rho V^2(3 - 4\cos^2\theta)\sin^2\alpha$$

As shown in figure, the components $p\cos\theta$ on the semi-circular cylinder neutralize each other while the components $p\sin\theta$ add up. Hence the resultant fluid pressure per unit length on the semi-circular cylinder



$$= 2\int_0^{\pi/2} p\sin\theta\cdot a\,d\theta$$

$$= 2a\int_0^{\pi/2} \{\Pi - (1/2) \times \rho V^2 (3 - 4\cos^2\theta)\sin^2\alpha\} \sin\theta \, d\theta$$

$= 2a\Pi - (5/3) \times \rho a V^2 \sin^2\alpha$, on integration and simplification.

Ex. 2. *A circular cylinder of radius a is moving with velocity U along the x-axis; show that the motion produced by the cylinder in a mass of fluid at rest is given by the complex function*

$$w = \phi + i\psi = \frac{a^2 U}{z - Ut}, \qquad \text{where } z = x + iy$$

Find the magnitude and direction of velocity in the fluid and deduce that for a marked particle of the fluid, whose polar coordinates are (r, θ) *referred to the centre of the cylinder as origin,*

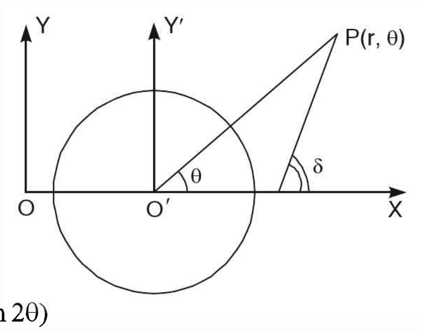
$$\frac{1}{r}\frac{dr}{dt} + i\frac{d\theta}{dt} = \frac{U}{r^2}\left(\frac{a^2}{r^2}e^{i\theta} - e^{-i\theta}\right) \quad \text{and} \quad \left(r - \frac{a^2}{r}\right)\sin\theta = b.$$

Hence prove that the path of such a particle is the elastic curve given by, $\rho(y - b/2) = a^2/4$, *where* ρ *is the radius of curvature of the path.* **[Guru Nanak Dev Unvi. 1998; I.A.S. 1986]**

Sol. Let O' be the centre of the circular cross-section of the cylinder at any time t. Then coordinates of O' are $(Ut, 0)$. Now, the complex potential of the fluid motion, referred to O' as origin, is Ua^2/z. Hence, when referred to the fixed origin O, the complex potential is given by

$$w = \phi + i\psi = Ua^2/(z - Ut), \qquad \text{...(1)}$$

which proves the first part of the problem.



From (1), $$-\frac{dw}{dz} = \frac{Ua^2}{(z - Ut)^2}$$

or $$-(-u + iv) = \frac{Ua^2}{r^2 e^{2i\theta}}, \qquad \text{where} \quad z - Ut = re^{i\theta}$$

or $$u - iv = (Ua^2/r^2)(\cos 2\theta - i\sin 2\theta)$$

$$\therefore \quad u = (Ua^2/r^2)\cos 2\theta \quad \text{and} \quad v = (Ua^2/r^2)\sin 2\theta$$

and q = the magnitude of velocity $= \sqrt{(u^2 + v^2)} = Ua^2/r^2$

with δ = direction of velocity $= \tan^{-1}(v/u) = 2\theta$.

Now consider fixed axes OX, OY at the instant when the centre of the cylinder is at O'. To determine the path of any particle referred to the cylinder, we reduce the cylinder to rest. Hence, relative to the cylinder, the complex potential of the motion at any point $P(r, \theta)$ is

$$w = Uz + (Ua^2/z) = Ure^{i\theta} + (Ua^2/r)e^{-i\theta}$$

$$\therefore \quad \phi = U(r + a^2/r)\cos\theta \quad \text{and} \quad \psi = U(r - a^2/r)\sin\theta \qquad \text{...(2)}$$

Then at the instant under consideration, the particle is moving along the streamline ψ = const. *i.e.*,

$$U(r - a^2/r)\sin\theta = Ub \quad \text{or} \quad (r - a^2/r)\sin\theta = b \qquad \text{...(3)}$$

Again, $$\frac{dr}{dt} = -\frac{\partial\phi}{\partial r} = -U\cos\theta + \frac{Ua^2}{r^2}\cos\theta, \text{ by (2)}$$

and $$r\frac{d\theta}{dt} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = U\sin\theta + \frac{Ua^2}{r^2}\sin\theta, \text{ by (2)}$$

Then, $$\frac{1}{r}\frac{dr}{dt} + i\frac{d\theta}{dt} = -\frac{U\cos\theta}{r} + \frac{Ua^2\cos\theta}{r^3} + i\left(\frac{U\sin\theta}{r} + \frac{Ua^2\sin\theta}{r^3}\right)$$

$$= \frac{U}{r}\left[\frac{a^2}{r^2}(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)\right] = \frac{U}{r}\left(\frac{a^2}{r^2}e^{i\theta} - e^{-i\theta}\right) \qquad ...(4)$$

From (3) and (4), the required second result follows.

Let (x, y) be the co-ordinates of P referred to O as origin and let (r, θ) be the polar coordinates of P referred to O' as origin. Then

$$x = Ut + r\cos\theta \qquad \text{and} \qquad y = r\cos\theta. \qquad ...(5)$$

Let ρ be the radius of curvature of the path. Then

$$\frac{1}{\rho} = \frac{d\delta}{ds} = \frac{d\delta}{dy}\frac{dy}{ds} = \frac{d\delta}{d\theta}\frac{d\theta}{dy}\frac{dy}{ds} = 2\sin 2\theta\frac{d\theta}{dy} \qquad ...(6)$$

since $\delta = 2\theta$ and $dy/ds = \sin\delta = \sin 2\theta$.

From (3) and (5), we have $$y - (a^2/y)\times\sin^2\theta = b \qquad ...(7)$$

Differentiating (7) w.r.t. 'y', we get

$$1 + \frac{a^2}{y^2}\sin^2\theta - \frac{2a^2}{y}\sin\theta\cos\theta\frac{d\theta}{dy} = 0 \qquad i.e. \qquad \sin 2\theta\frac{d\theta}{dy} = \frac{y}{a^2}\left(1 + \frac{a^2}{y^2}\sin^2\theta\right)$$

or $$\sin 2\theta\frac{d\theta}{dy} = \frac{y}{a^2}\left(1 + \frac{y-b}{y}\right), \text{ by (7)} \qquad ...(8)$$

Using (8), (6) reduces to

$$\frac{1}{\rho} = \frac{2y}{a^2}\left(2 - \frac{b}{y}\right) \qquad \text{or} \qquad \rho\left(y - \frac{b}{2}\right) = \frac{a^2}{4}.$$

Ex. 3. *A circular cylinder is placed in a uniform stream, find the forces acting on the cylinder.* **[Agra 2006; Meerut 2007; Garhwal 2005]**

Sol. We know that the complex potential for the undisturbed motion is given by $w = (u - iv)z$. Using Milne-Thomson's circle theorem, the complex potential for the present problem is

$$w = (u - iv)z + (u + iv)\,(a^2/z)$$

$$\therefore \qquad dw/dz = u - iv - (u + iv)(a^2/z^2)$$

If the pressure thrusts on the contour of the fixed circular cylinder be represented by a force (X, Y) and a couple of moment N about the origin of co-ordinates, then by Blasius' theorem,

$$X - iY = \frac{1}{2}i\rho\int_C\left(\frac{dw}{dz}\right)^2 dz = \frac{1}{2}i\rho\int_C\left\{(u - iv) - (u + iv)(a^2/z^2)\right\}^2 dz = 0$$

so that $X = 0$ and $Y = 0$.

Also $$N = \text{Real part of } -\frac{1}{2}\rho\int_C z\left(\frac{dw}{dz}\right)^2 dz = \text{real part of } -\frac{1}{2}\rho\int_C z\left\{u - iv - (u + iv)\frac{a^2}{z^2}\right\}^2 dz$$

Hence, N = real part of $-(\rho/2)\left\{-2(u^2+v^2)a^2\right\}2\pi i = 0$, on simplification

$\therefore$ $X = Y = N = 0$, showing that neither a force nor a couple acts on the cylinder.

Ex. 4. *In the case of two dimensional motion of a liquid streaming past a fixed circular disc, the velocity at infinity is u in a fixed direction, where u is a variable. Show that the maximum value of the velocity at any point of the fluid is 2u. Prove that the force necessary to hold the disc is* $2m\dot{u}$, *where m is the mass of the liquid displaced by the disc and* $\dot{u} = du/dt$.

Sol. The velocity potential for the liquid streaming past a fixed circular disc is given by

$$\phi = u(r + a^2/r)\cos\theta, \qquad \text{...(1)}$$

where a is the radius of the disc. This gives

$$\frac{\partial\phi}{\partial r} = u\left(1 - \frac{a^2}{r^2}\right)\cos\theta \qquad \text{and} \qquad \frac{\partial\phi}{\partial\theta} = -u\left(r + \frac{a^2}{r}\right)\sin\theta$$

$$\therefore \qquad q^2 = \left(-\frac{\partial\phi}{\partial r}\right)^2 + \left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 = u^2\left(1 - \frac{a^2}{r^2}\right)^2\cos^2\theta + u^2\left(1 + \frac{a^2}{r^2}\right)^2\sin\theta$$

or

$$q^2 = u^2\left(1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right), \qquad \text{...(2)}$$

which is maximum with respect to θ when $\cos 2\theta = -1$ *i.e.* $2\theta = \pi$ and then

$$q^2 = u^2\left(1 + \frac{2a^2}{r^2} + \frac{a^4}{r^4}\right) = u^2\left(1 + \frac{a^2}{r^2}\right)^2 \qquad \text{so that} \qquad q = u\left(1 + \frac{a^2}{r^2}\right).$$

Now q is further maximum with respect to r when r is minimum, *i.e.*, when $r = a$. Hence the required maximum value of q is given by $\quad q = u\,(1 + a^2/a^2) = 2u.$

By Bernoulli's equation, the pressure p is given by

$$\frac{p}{\rho} = \frac{\partial\phi}{\partial t} - \frac{1}{2}q^2 + F(t) \qquad \text{...(3)}$$

Using (1) and (2), (3) reduces to

$$\frac{p}{\rho} = F(t) - \frac{1}{2}u^2\left(1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right) + \dot{u}\left(r + \frac{a^2}{r}\right)\cos\theta$$

Putting $r = a$, the pressure on the boundary of the disc is given by

$$p/\rho = F(t) - 2u^2\sin^2\theta + \dot{u}\cdot 2a\cos\theta \qquad \text{...(4)}$$

Then the resultant pressure on the disc

$$= \int_0^{2\pi}(-p\cos\theta)\,a\,d\theta = -\rho a\int_0^{2\pi}[F(t) - 2u^2\sin^2\theta + 2\dot{u}a\cos\theta]\,d\theta, \text{ using (4)}$$

$$= -2\rho a^2\dot{u}\int_0^{2\pi}\cos^2\theta\,d\theta = -2\pi a^2\rho\dot{u} = -2m\dot{u}, \text{ since } m = \pi a^2\rho \text{ (given)}$$

Hence the desired force necessary to hold the disc is $2m\dot{u}$

Ex. 5. *If a long circular cylinder of radius a moves in a straight line at right angles to its length in liquid at rest at infinity, show that when a particle of liquid in the plane of symmetry, initially at distance b in advance of the axis of the cylinder has moved through a distance c, then the cylinder has moved through a distance* $\quad c + (b^2 - a^2)/\{b + a\coth(c/a)\}$

Sol. Refer figure of Ex. 2. Let the line of motion and a line perpendicular to it be taken as the co-ordinate axes. Let the cylinder move with velocity U so that at any time t, $OO' = Ut$. Here O' is the position of the centre of the cylinder at time t. The complex potential at any point z is

$$w = \frac{Ua^2}{z - Ut} = \frac{Ua^2}{x + iy - Ut}, \quad \text{where } z = x + iy \qquad \text{...(1)}$$

$$\therefore \quad \frac{dw}{dz} = -\frac{Ua^2}{(z - Ut)^2} \quad \text{or} \quad -u + iv = -\frac{Ua^2}{(z - Ut)^2} \qquad \text{...(2)}$$

Along the real axis $y = 0$, velocity is $\dot{x}$. Therefore, (2) reduces to

$$\dot{x} = Ua^2/(x - Ut)^2 \qquad \text{...(3)}$$

Putting $x - Ut = \eta$ so that $\dot{x} = (d\eta/dt) + U$ ...(4)

From (3) and (4), $\dot{x} = Ua^2/\eta^2$ ...(5)

From (4) and (5), $d\eta/dt = U(a^2 - \eta^2)/\eta^2$ ...(6)

or $$Udt = \frac{\eta^2}{a^2 - \eta^2} d\eta$$

Integrating, $Ut + A = -\int \left[1 + \dfrac{a^2}{\eta^2 - a^2}\right] d\eta$, where A is an arbitrary constant.

or $$Ut + A = -\eta - \frac{a}{2}\log\frac{\eta - a}{\eta + a} \quad \text{or} \quad Ut + A = -x + Ut - \frac{a}{2}\log\frac{x - Ut - a}{x - Ut + a}$$

or $$x + A = -\frac{a}{2}\log\frac{x - Ut - a}{x - Ut + a} \qquad \text{...(7)}$$

Given $x = b$ when $t = 0$. Then (7) gives

$$b + A = -\frac{a}{2}\log\frac{b - a}{b + a} \qquad \text{...(8)}$$

Subtracting (8) from (7), we have

$$x - b = -\frac{a}{2}\log\frac{x - Ut - a}{x - Ut + a} + \frac{a}{2}\log\frac{b - a}{b + a} \qquad \text{...(9)}$$

Let $t = t$ when $x = b + c$. Then (9) gives

$$c = -\frac{a}{2}\log\frac{b + c - Ut - a}{b + c - Ut + a} + \frac{a}{2}\log\frac{b - a}{b + a}$$

or $$-\frac{2c}{a} = \log\frac{b + a}{b - a}\left[\frac{(b - a) + (c - Ut)}{(b + a) + (c - Ut)}\right] \quad \text{or} \quad e^{-(2c/a)} = \frac{(b^2 - a^2) + (b + a)(c - Ut)}{(b^2 - a^2) + (b - a)(c - Ut)}$$

or $$\frac{1 + e^{-(2c/a)}}{1 - e^{-(2c/a)}} = \frac{(b^2 - a^2) + b(c - Ut)}{-a(c - Ut)} \quad \text{or} \quad \frac{e^{c/a} + e^{-c/a}}{e^{c/a} - e^{-c/a}} = -\frac{1}{a}\left[b + \frac{b^2 - a^2}{c - Ut}\right]$$

or $$-a\coth(c/a) = b + \frac{b^2 - a^2}{c - Ut} \quad \text{or} \quad (b^2 - a^2)/(c - Ut) = -[b + a\coth(c/a)]$$

or $$c - Ut = -(b^2 - a^2)/[b + a\coth(c/a)] \quad \text{or} \quad Ut = c + \frac{b^2 - a^2}{b + a\coth(c/a)}.$$

Hence, the required distance $= c + (b^2 - a^2)/\{b + a\coth(c/a)\}$

Ex. 6. *Show that when a circular cylinder moves uniformly in a given straight line in an* *infinite liquid, the path of any point in the fluid is given by the equations*

$$\frac{dz}{dt} = \frac{Va^2}{(\bar{z} - Vt)^2} \qquad \text{and} \qquad \frac{d\bar{z}}{dt} = \frac{Va^2}{(z - Vt)^2},$$

where V is the velocity of the cylinder, radius a and $z = x + iy$, $\bar{z} = x - iy$, where x, y are the co-ordinates measured from the starting point of the axis, along and perpendicular to its direction of motion. **[Garhwal 2001]**

Sol. Suppose the line in which the centre of the cross-section of the cylinder moves is taken as x-axis. Hence, if (x, y) be the co-ordinates of centre after a time t, then $x = Vt$ and $y = 0$. Then complex potential of the motion is given by

$$w = K/(z - Vt), \; K \text{ being a constant} \qquad \text{...(1)}$$

$$\therefore \qquad \phi + i\psi = K/(re^{i\theta}) \quad \text{where } z - Vt = re^{i\theta}$$

or $\quad \phi + i\psi = (K/r)(\cos\theta - i\sin\theta) \quad$ so that $\quad \phi = (K/r)\cos\theta$

$$\therefore \qquad \partial\phi/\partial r = (-K/r^2)\cos\theta \qquad \text{...(2)}$$

Now the following boundary condition must be satisfied :

The normal velocity of the cylinder at its surface

= Velocity of the fluid at that point along the normal to the cylinder

i.e., $\quad V\cos\theta = (-\partial\theta/\partial r)_{r=a} \quad$ or $\quad V\cos\theta = (K/a^2)\cos\theta$, by (2)

so that $K = Va^2$, since $\cos\theta \neq 0$. Hence (1) becomes

$$w = \frac{Va^2}{z - Vt} \qquad \text{so that} \qquad \frac{dw}{dz} = -\frac{Va^2}{(z - Vt)^2}$$

or $\quad -u + iv = -\dfrac{Va^2}{(z - Vt)^2} \quad$ or $\quad u - iv = \dfrac{Va^2}{(z - Vt)^2}$

or $\quad \dfrac{dx}{dt} - i\dfrac{dy}{dt} = \dfrac{Va^2}{(z - Vt)^2}, \quad$ as $\quad u = \dot{x} \quad$ and $\quad v = \dot{y}$

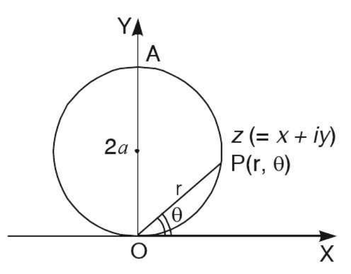
or $\quad \dfrac{d}{dt}(x - iy) = \dfrac{Va^2}{(z - Vt)^2}, \quad$ *i.e.*, $\quad \dfrac{d\bar{z}}{dt} = \dfrac{Va^2}{(z - Vt)^2}$

Replacing i by $-i$ in the above equation, we get $\qquad dz/dt = Va^2/(\bar{z} - Vt)^2$

Ex. 7. *A circular cylinder of radius a and infinite length lies on a plane in an infinite depth of liquid. The velocity of the liquid at a great distance from the cylinder is U perpendicular to the generators, and the motion is irrotational and two-dimensional. Verify that the stream function is the imaginary part of $w = \pi aU\coth(\pi a/z)$, where z is a complex variable zero on the line of contact and real on the plane. Prove that the pressure at the two ends of the diameter of the cylinder normal to the plane differs by $(1/32) \times \pi^4 \rho U^2$.* **[Kanpur 1999; Rohilkhand 2000]**

Sol. Let $P(r, \theta)$ be a point on the cross-section of the circular cylinder. Then, we have

$$w = \pi aU\coth(\pi a/z) \qquad \text{...(1)}$$



or $\quad \phi + i\psi = \pi a U \coth\left(\dfrac{\pi a}{r\theta^{i\theta}}\right)$, *as* $z = x + iy = re^{i\theta}$

$$= \pi a U \coth\left(\frac{\pi a}{r^2}\cdot re^{-i\theta}\right) = \pi a U \coth\left[\frac{\pi a}{r^2}(x - iy)\right]$$

$$[\because\ re^{-i\theta} = r(\cos\theta - i\sin\theta) = x - iy]$$

$$= \pi a U \frac{\cosh\left[\dfrac{\pi a}{r^2}(x-iy)\right]}{\sinh\left[\dfrac{\pi a}{r^2}(x-iy)\right]} = \pi a U \frac{\cosh\left[\dfrac{\pi a}{r^2}(x-iy)\right]\sinh\left[\dfrac{\pi a}{r^2}(x+iy)\right]}{\sinh\left[\dfrac{\pi a}{r^2}(x-iy)\right]\sinh\left[\dfrac{\pi a}{r^2}(x+iy)\right]}$$

$$= \pi a U \frac{\sinh\dfrac{2\pi a x}{r^2} + i\sin\dfrac{2\pi a y}{r^2}}{\cosh\dfrac{2\pi a x}{r^2} - \cos\dfrac{2\pi a y}{r^2}}$$

Equating imaginary parts, we get $\qquad \psi = \pi a U \dfrac{\sin\dfrac{2\pi a y}{r^2}}{\cosh\dfrac{2\pi a x}{r^2} - \cos\dfrac{2\pi a y}{r^2}}$

Streamlines $\psi = 0$ take the form

$$\sin\left(\frac{2\pi a y}{r^2}\right) = 0 \qquad \text{so that} \qquad y = 0 \qquad \text{and} \qquad \frac{2\pi a y}{r^2} = \pi.$$

Thus $y = 0$ and $r = 2a \sin\theta$ are the streamlines, *i.e.*, x-axis and the circle OPA are the streamlines.

From (1), $\quad \dfrac{dw}{dz} = -\dfrac{\pi^2 a^2 U}{z^2}\operatorname{cosech}^2\dfrac{\pi a}{z} \quad$ or $\quad -u + iv = \dfrac{\pi^2 a^2 U}{z^2 \sinh^2(\pi a/z)}, \qquad$ ...(2)

showing that velocity on a point on x-axis at infinite distance is U.

Now the pressure p is given by $\qquad p/\rho + q^2/2 = C. \qquad$ ...(3)

Let Π be the pressure at infinity. Thus $p = \Pi$ when $q = U$. Then (3) gives $C = \Pi/\rho + U^2/2$ and hence (3) reduce to

$$\frac{p}{\rho} + \frac{1}{2}q^2 = \frac{\Pi}{\rho} + \frac{1}{2}U^2 \qquad \text{or} \qquad p = \Pi + \frac{1}{2}\rho(U^2 - q^2) \qquad \text{...(4)}$$

Let p_0 be the pressure at origin O, where the velocity is zero. Then (4) gives

$$p_0 = \Pi + (\rho U^2)/2 \qquad \text{...(5)}$$

Let p_1 be pressure at A and q_1 be velocity there. Then from (2), we get (note that velocity $q = |dw/dz|$)

$$q_1 = \frac{\pi^2 a^2 U}{r^2}\left|\frac{1}{\sinh^2(\pi a/z)}\right|, \text{ where } \quad x = 0, \quad y = 2a, \quad r = 2a \qquad \text{...(6)}$$

Now, $\qquad \left|\sinh^2\dfrac{\pi a}{z}\right| = \left|\sinh\dfrac{\pi a}{z}\right|^2 = \left|\sinh\dfrac{\pi a\bar{z}}{z\bar{z}}\right|^2 = \left|\sinh\dfrac{\pi a(x-iy)}{|z|^2}\right|^2$

$$= \left| \sinh\left(\frac{\pi a x}{r^2} - i\frac{\pi a y}{r^2}\right) \right|^2, \text{ as } |z| = r$$

Thus, $$\left|\sinh^2 \frac{\pi a}{z}\right| = \sinh^2 \frac{\pi a x}{r^2}\cos^2\frac{\pi a y}{r^2} + \cosh^2\frac{\pi a x}{r^2}\sin^2\frac{\pi a y}{r^2} \qquad ...(7)$$

Using (7), (6) reduces $$q_1 = \frac{\pi^2 a^2 U}{(2a^2)}\left|\frac{1}{0+1}\right| = \frac{\pi^2 a^2 U}{4a^2} \qquad ...(8)$$

Again from (4), we have $$p_1 = \Pi + (\rho/2)\times(U^2 - q_1^2) \qquad ...(9)$$

(5) and (9) $\Rightarrow$ $p_0 - p_1 = (\rho q_1^2)/2 = (\pi^4 \rho U^2)/32$, using (8)

which proves the required result.

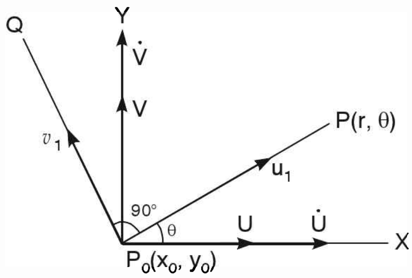
Ex. 8. *An infinite circular cylinder of radius a is in motion in homogeneous fluid which extends to infinity and is at rest there. Show that at any moment, the presure at any point of the fluid at a distance r from the axis of the cylinder exceeds the hydrostatic by*

$$\rho\left[(a^2/r)\times f_1 + (a^2/r^2)\times\left\{(1 - a^2/2r^2)u_1^2 - (1 + a^2/2r^2)v_1^2\right\}\right],$$

where f_1 is the component of the acceleration of the centre of the cylinder in the direction of r, u_1 and v_1 are the components of velocity in and perpendicular to the direction.

Sol. As shown in the adjoining figure let (x_0, y_0) be the coordinates of the centre P_0 of the cylinder at any time t. Let U and V the velocity components of the centre of the cylinder along OX and OY. Then, in usual symbols, we have

$$\dot{x}_0 = dx_0/dt = U \quad \text{and} \quad \dot{y}_0 = dy_0/dt = V. \quad \text{Also } z_0 = x_0 + iy_0 \qquad ...(1)$$



Differentiating w.r.t. 't', (1) gives $\dot{z}_0 = \dot{x}_0 + i\,\dot{y}_0$

or $$\dot{z}_0 = U + iV \qquad ...(2)$$

The complex potential is given by

$$\phi + i\psi = w = \{(U + iV)a^2\}/(z - z_0), \qquad ...(3)$$

where $$z - z_0 = re^{i\theta}, \quad \text{as} \quad |z - z_0| = r \qquad ...(4)$$

$\therefore$ Magnitude of velocity = $$q = \left|\frac{dw}{dz}\right| = \left|\frac{(U+iV)a^2}{(z-z_0)^2}\right| = \frac{a^2}{r^2}(U^2 + V^2)^{1/2}, \text{ using (2)} \qquad ...(5)$$

Differentiating w.r.t. 't', (3) yieds

$$\frac{\partial\phi}{\partial t} + i\frac{\partial\psi}{\partial t} = \frac{a^2}{z - z_0}(\dot{U} + i\dot{V}) - (U + iV)a^2 \times \frac{1}{(z - z_0)^2}(-\dot{z}_0)$$

or $$\frac{\partial\phi}{\partial t} + i\frac{\partial\psi}{\partial t} = \frac{a^2(\dot{U} + i\dot{V})}{r}e^{-i\theta} + \frac{a^2}{r^2}(U + iV)^2 e^{-2i\theta}, \text{ using (2) and (3)}$$

or $$\frac{\partial\phi}{\partial t} + i\frac{\partial\psi}{\partial t} = \frac{a^2}{r}(\dot{U} + i\dot{V})\,(\cos\theta - i\sin\theta) + \frac{a^2}{r^2}(U^2 - V^2 + 2iUV)\,(\cos 2\theta - \sin 2\theta)$$

Equating real parts on both sides of the above equation, we get

$$\partial\phi/\partial t = (a^2/r)\times(\dot{U}\cos\theta + \dot{V}\sin\theta) + (a^2/r^2)\times\left\{(U^2 - V^2)\cos 2\theta + 2UV\sin 2\theta\right\}$$

Bernoulli's equation for unsteady and irrotational motion of incompressible fluid in absence of external forces is (refer special case I of Art. 4.1, Chapter 4)

$$-(\partial\phi/\partial t)+q^2/2+p/\rho=F(t)$$

or
$$-(a^2/r)\times(\dot{U}\cos\theta+\dot{V}\sin\theta)-(a^2/r^2)\times\{(U^2-V^2)\cos 2\theta+2UV\sin 2\theta\}$$
$$+(a^4/2r^4)\times(U^2+V^2)+p/\rho=F(t) \qquad \text{...(6)}$$

Let p_0 be the hydrostatic pressure so that $p=p_0$ when $r\to\infty$. Also, $q=0$ when $r\to\infty$ so that $U=V=0$ when $r\to\infty$. Setting $r\to\infty$ and using the above values, (6) yields $F(t)=p_0/\rho$. Hence, (6) reduces to

$$\frac{p-p_0}{\rho}=\frac{a^2}{r}(\dot{U}\cos\theta+\dot{V}\cos\theta)+\frac{a^2}{r^2}\left\{(U^2-V^2)\cos 2\theta+2UV\sin 2\theta\right\}-\frac{a^4}{2r^4}(U^2+V^2) \quad \text{...(7)}$$

Given f_1 = component of acceleration along $OP=\dot{U}\cos\theta+\dot{V}\sin\theta$

u_1 = component of velocity along $OP=U\cos\theta+V\sin\theta$

v_1 = component of velocity along $OQ=U\cos(\pi/2+\theta)+V\sin(\pi/2+\theta)=-U\sin\theta+V\cos\theta$

$$\therefore \qquad u_1^2+v_1^2=(U\cos\theta+V\sin\theta)^2+(-U\sin\theta+V\cos\theta)^2=U^2+V^2$$

and
$$u_1^2-v_1^2=(U\cos\theta+V\sin\theta)^2-(-U\sin\theta+V\cos\theta)^2=(U^2-V^2)\cos 2\theta+2UV\sin 2\theta$$

Using above results, (7) may be re-written as

$$(p-p_0)/\rho=(a^2/r)\times f_1+(a^2/r^2)\times(u_1^2-v_1^2)+(a^4/2r^4)\times(u_1^2+v_1^2)$$

$$\Rightarrow \qquad p-p_0=\rho\left[(a^2/r)\times f_1+(a^2/r^2)\times\left\{(1-a^2/2r^2)u_1^2-(1+a^2)/2r^2)v_1^2\right\}\right],$$

which gives the required amount of the pressure by which the pressure at any point P of the fluid exceeds the hydrostatic pressure p_0.

Ex. 9. *The centre of a circular cylinder of radius a is moving with uniform velucity U along the x-axis through fluid which is at rest at infinity. Show that the co-ordinates (X, Y) of a fluid particle satisfy the equation*

$$dX/dt=(Ua^2/r)\times\cos 2\theta,\quad dY/dt=(Ua^2/r^2)\times\sin 2\theta, \quad \textit{where} \quad X-Ut=r\cos\theta, \quad Y=r\sin\theta.$$

Hence deduce that $Y(1-a^2/r^2)=Y_\infty=a,$ *constant and*

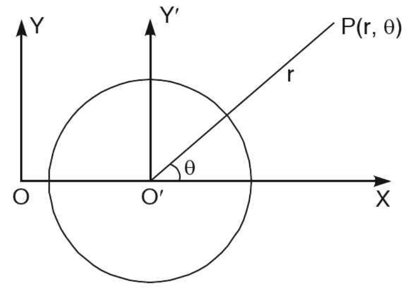
$$X=-\int_\theta^{\pi/2}\frac{a^2\cos 2\theta\, d\theta}{(Y_\infty^2+4a^2\sin^2\theta)^{1/2}},$$

if the particle is initially at $X=0$, $Y=0$.

Sol. Let O' be the centre of the circular cross-section of the given cylinder at any time t. Then the co-ordinates of O' are $(Ut, 0)$. Hence, the complex potential of the fluid motion, referred to O' as origin, is $(Ua^2)/z$. Therefore, when referred to the fixed origin O, the complex potential is given by

$$w=(Ua^2)/(z-Ut) \qquad \text{...(1)}$$

Also, $\qquad z-Ut=r^{i\theta} \qquad \text{...(2)}$



From (1), $-\dfrac{dw}{dz} = \dfrac{Ua^2}{(z-Ut)^2}$ or $-(-u+iv) = \dfrac{Ua^2}{r^2 e^{2i\theta}}$, using (2)

$\therefore\ u - iv = (Ua^2/r^2)\times e^{-2i\theta}$ or $dX/dt - i(dY/dt) = (Ua^2/r^2)\times(\cos 2\theta - i\sin 2\theta)$

Equating real imaginary parts, we have

$$dX/dt = (Ua^2/r^2)\times\cos 2\theta \quad \text{and} \quad dY/dt = (Ua^2/r^2)\times \sin 2\theta \qquad \text{...(3)}$$

Again, (2) $\Rightarrow$ $X + iY - Ut = r(\cos\theta + i\sin\theta)$

$$\Rightarrow \quad X - Ut = r\cos\theta \quad \text{and} \quad Y = r\sin\theta \qquad \text{...(4)}$$

Deduction. We consider the fixed axes, OX, OY at the instant when the centre of the cylinder is at O'. Relative to the cylinder, the complex potential of the motion at any point $P(r, \theta)$ is given by

$$w = Uz + (Ua^2)/z \quad \text{or} \quad \phi + i\psi = Ure^{i\theta} + (Ua^2)/(re^{i\theta})$$

or $$\phi + i\psi = Ur(\cos\theta + i\sin\theta) + (Ua^2/r)\times(\cos\theta - i\sin\theta)$$

Equating imaginary parts, we get $\psi = U(r - a^2/r)\sin\theta$

Then at the instant under consideration, the particle is moving along the streamline

$$\psi = \text{constant}, \quad \text{that is,} \quad U(r - a^2/r)\sin\theta = \text{constant}$$

or $$(1 - a^2/r^2)r\sin\theta = \text{constant} \quad \text{or} \quad (1 - a^2/r^2)Y = \text{constant} = Y_\infty, \text{ say,} \qquad \text{...(5)}$$

Using (4), (5) $\Rightarrow$ $\left\{1 - (a^2\sin^2\theta)/Y^2\right\}Y = Y_\infty$ or $Y^2 - a^2\sin^2\theta = YY_\infty$ ...(6)

or $$Y^2 - YY_\infty - a^2\sin^2\theta = 0 \quad \Rightarrow \quad Y = \left\{Y_\infty \pm (Y_\infty^2 + 4a^2\sin^2\theta)^{1/2}\right\}\big/2$$

Taking positive square root, we get $2Y - Y_\infty = (Y_\infty^2 + 4a^2\sin^2\theta)^{1/2}$ ...(7)

Now, from (5), $1 - (a^2/r^2) = Y_\infty/Y$ ...(8)

$$\Rightarrow \quad d\{1 - (a^2/r^2)\} = d(Y_\infty/Y) \quad \Rightarrow \quad (2a^2/r^3)\,dr = -(Y_\infty/Y^2)dY$$

$$\Rightarrow \quad dr = -(Y_\infty r^3/2a^2Y^2)\,dY \qquad \text{...(9)}$$

Now, (4) $\Rightarrow$ $dY = d(r\sin\theta)$ $\Rightarrow$ $dY = \sin\theta\, dr + r\cos\theta\, d\theta$

or $$dY = -\sin\theta\times(Y_\infty r^3/2a^2Y^2)\,dY + r\cos\theta\, d\theta, \text{ using (9)}$$

$$\left(1 + \frac{Y_\infty r^3\sin\theta}{2a^2Y^2}\right)dY = r\cos\theta\, d\theta \quad \text{or} \quad \left(1 + \frac{Y_\infty r^3\sin\theta}{2a^2r^2\sin^2\theta}\right)dY = r\cos\theta\, d\theta$$

or $$\left(1 + \frac{Y_\infty r\sin\theta}{2a^2\sin^2\theta}\right)dY = r\cos\theta\, d\theta \quad \text{or} \quad \left(1 + \frac{YY_\infty}{2a^2\sin^2\theta}\right)dY = r\cos\theta\, d\theta$$

or $$dY = \frac{2a^2 r\sin\theta\cos\theta\, d\theta}{2a^2\sin^2\theta + YY_\infty} \quad \text{or} \quad dY = \frac{a^2\sin 2\theta\cdot(r\sin\theta)\,d\theta}{2a^2\sin^2\theta + YY_\infty} \qquad \text{...(10)}$$

From (3), $dY/dX = \tan 2\theta$ so that $dY = \tan 2\theta\, dX$ ...(11)

(10) and (11) $\Rightarrow$ $$\tan 2\theta\, dX = \frac{a^2 Y \sin 2\theta\, d\theta}{2a^2 \sin^2\theta + YY_\infty}, \text{ using (4)} \qquad ...(12)$$

Now, (6) $\Rightarrow$ $Y^2 - YY_\infty = a^2 \sin^2\theta$ $\Rightarrow$ $2Y - 2Y_\infty = (2a^2/Y)\times \sin^2\theta$

or $2Y - Y_\infty = Y_\infty + (2a^2/Y)\times\sin^2\theta$ or $2Y - Y_\infty + (2a^2\sin^2\theta + YY_\infty)/4$

$$\Rightarrow \qquad 2a^2\sin^2\theta + YY_\infty = Y(2Y - Y_\infty) \qquad ...(13)$$

$$\Rightarrow \qquad 2a^2\sin^2\theta + YY_\infty = Y(Y_\infty^2 + 4a^2\sin^2\theta)^{1/2}, \text{ by (7)} \qquad ...(14)$$

Now, (13) and (14) $\Rightarrow$ $$2a^2\sin^2\theta + YY_\infty = Y(Y_\infty^2 + 4a^2\sin^2\theta)^{1/2} \qquad ...(15)$$

Form (12) and (15), $\tan 2\theta\, dX = \dfrac{a^2 \sin 2\theta\, d\theta}{(Y_\infty^2 + 4a^2\sin^2\theta)^{1/2}}$ or $dX = \dfrac{a^2\cos 2\theta\, d\theta}{(Y_\infty^2 + 4a^2\sin^2\theta)^{1/2}}$

Integrating, $$\int_{X=0}^{X=X} dX = \int_{\theta'}^{\theta} \frac{a^2\cos 2\theta\, d\theta}{(Y_\infty^2 + 4a^2\sin^2\theta)^{1/2}}, \qquad ...(16)$$

where θ' is the value of θ when $X = 0$

Now, $X = 0$ $\Rightarrow$ $r'\cos\theta' = 0$ $\Rightarrow$ $\theta' = \pi/2$

$\therefore$ (16) $\Rightarrow$ $$X = \int_{\pi/2}^{\theta} \frac{a^2\cos 2\theta\, d\theta}{(Y_\infty^2 + 4a^2\sin^2\theta)^{1/2}} = -\int_{\theta}^{\pi/2} \frac{a^2\cos 2\theta\, d\theta}{(Y_\infty^2 + 4a^2\sin^2\theta)^{1/2}}$$

EXERCISE 7 (A)

1. Prove that when a cylinder moves in a finite liquid; $\psi = Vx - Uy + (\omega/2)\times(x^2 + y^2) + C$, where the symbols have their usual meaning. **[Kanpur 2004]**

Hint : Give entire matter of Art. 7.1 upto equaiton (6).

2. Find the velocity potential ϕ and steam function ψ in the case of a liquid streaming with velocity U past a fixed cylinder of radius a.

Hint : Refer Art. 7.4 **[Kanpur 2001]**

3. (*a*) Show that in the case of the cylinder moving forward in a fluid otherwise at rest, the speed of the fluid varies inversely as the square of the distance from the centre.

(*b*) A stationary infinite right circular cylinder of radius a is placed in a uniform stream, its axis being perpendicular to the direction of flow. Find the resultant velocity potential

Hint : Refer Art. 7.4 **[Agra 2007]**

4. A circular cylinder is moving with velocity U parallel to OX in an infinite fluid. Prove that the motion is the same as it would be if the cylinder were removed and a doublet placed at the centre of the cylinder with its axis pointing in the direction of the motion of the liquid.

5. Prove, or verify, that the velocity function $\phi = U(r + a^2/r)\cos\theta$ represents a streaming past a fixed circular cylinder.

The pressure at infinity being given, calculate the resultant fluid action per unit length on half the cylinder lying on one side through the axis and parallel to the streamline.

6. A circular cylinder consists of two halves. There is a uniform flow of inviscid fluid with circulation around the cylinder in such a way that the dividing plane is perpendicular to the free stream direction. If the weight of the cylinder is neglected, determine the force required to hold two halves together.

7. (*a*) Prove that if a right circular cylinder of infinite length is introduced into a given two-dimensional flow, its axis being aligned with the co-ordinate axis perpendicular to the plane of flow, it is possible to derive a general expression for the new velocity potential outside the cylinder.

(*b*) Use the above result to discuss uniform flow past a stationary cylinder.

7.6. To find complex potential due to circulation about a circular cylinder.

[Kanpur 2001; Meerut 2002]

Let k be the constant circulation about the cylinder. Then the suitable form of ϕ may be obtained by equating to k the circulation round a circle of radius r. Thus, we have

$$\left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)(2\pi r) = k \qquad i.e. \qquad \frac{\partial\phi}{\partial\theta} = -\frac{k}{2\pi}$$

so that $$\phi = -(k\theta/2\pi)$$

Since ϕ and ψ are conjugate functions, we have $\psi = (k/2\pi)\log r$.

Thus, the complex potential due to the circulation about a circular cylinder is given by

$$w = \phi + i\psi = -\frac{k\theta}{2\pi} + \frac{ik}{2k}\log r = \frac{ik}{2\pi}(\log r + i\theta) = \frac{ik}{2\pi}(\log r + \log e^{i\theta}) = \frac{ik}{2\pi}\log(re^{i\theta})$$

Thus $\qquad w = (ik/2\pi)\log z, \qquad$ as $\qquad z = re^{i\theta}$.

7.7. Streaming and circulation about a fixed circular cylinder.

We know that the complex potential w_1 due to the circulation of strength k about the cylinder is given by

$$w_1 = (ik/2\pi)\log z. \qquad \text{...(1)}$$

Again, the complex potential w_2 for streaming past a fixed circular cylinder of radius a, with velocity U, in the negative direction of x-axis is given by

$$w_2 = Uz + (Ua^2/z). \qquad \text{...(2)}$$

Hence the complex potential $w = (= \phi + i\psi)$ due to the combined effects at any point $z = re^{i\theta}$

is given by $$w = w_1 + w_2 = U\left(z + \frac{a^2}{z}\right) + \frac{ik}{2\pi}\log z \qquad \text{...(3)}$$

or $$w = U(re^{i\theta} + a^2 e^{-\theta}) + (ik/2\pi) \times \log\,(re^{i\theta}), \; as \;\; z = re^{i\theta}$$

or $$\phi + i\psi = U\left[r\cos\theta + ir\sin\theta + \frac{a^2}{r}(\cos\theta - i\sin\theta)\right] + \frac{ik}{2\pi}(\log r + i\theta)$$

so that $$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{k\theta}{2\pi} \qquad \text{and} \qquad \psi = U\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{k}{2\pi}\log r.$$

Since the velocity will be only tangential at the boundary of the cylinder, $(-\partial\phi/\partial r) = 0$ and hence the magnitude of the velocity q is given by

$$q = \left|-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right|_{r=a} = \left|2U\sin\theta + \frac{k}{2\pi a}\right| \qquad \text{...(4)}$$

If there were no circulation ($k = 0$) there would be points of zero velocity on the cylinder at $\theta = 0$ and $\theta = \pi$, the former being the point at which the on-coming stream divides. However, in the presence of circulation, the stagnation (or critical) points are given by $q = 0$, *i.e.*

$$\sin\theta = -k/4\pi Ua \qquad \text{...(5)}$$

and such points exist when $$|k| < 4\pi Ua. \qquad \text{...(6)}$$

The lines of flow are then as shown in figure (i) N, N' being the stagnation points. When, $|k| = 4\pi aU$, the stagnation points N and N' coincide at the bottom point M of the cylinder as shown in figure (*ii*). When $|k| > 4\pi aU$, there are no stagnation points on the cylinder but there is such a point below the cylinder on the axis of y as shown in the figure (*iii*).

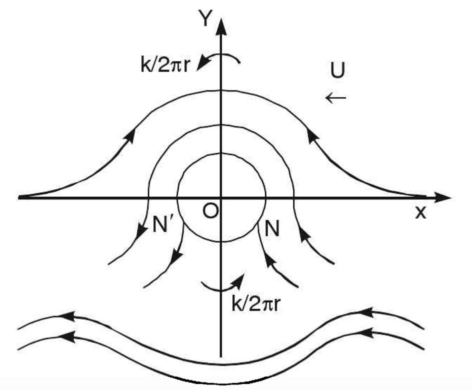


Fig. (*i*) $|k| < 4\pi aU$

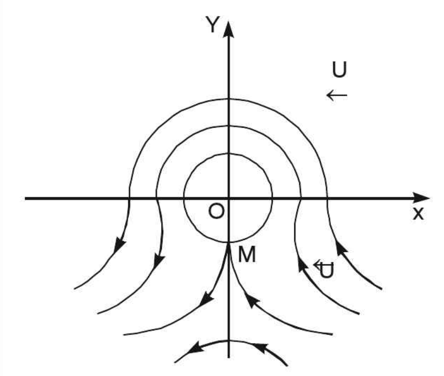


Fig. (*ii*) $|k| = 4\pi aU$

Remarks. From the above discussion, it follows that any point on the circumference might be made a critical point by a suitable choice of the ratio *k*/*U*. This fact is employed in the theory of aerofoils.

We now determine the pressure at points of the cylinder. The pressure p is given by Bernoulli's theorem $\quad p/\rho = F(t) - q^2/2 \qquad \ldots(7)$

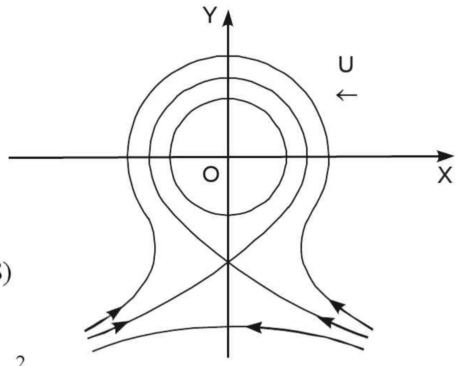


Fig. (*iii*) $|k| > 4\pi aU$

Let Π be the pressure at infinity. Then $p = \Pi$ and $q = U$, so that

$$\frac{\Pi}{\rho} = F(t) - \frac{1}{2}U^2 \quad \text{or} \quad F(t) = \frac{\Pi}{\rho} + \frac{1}{2}U^2 \qquad \ldots(8)$$

Using (8), (7) reduces to $\quad \dfrac{p}{\rho} = \dfrac{\Pi}{\rho} + \dfrac{1}{2}U^2 - \dfrac{1}{2}q^2$

or
$$p = \Pi + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(2U\sin\theta + \frac{k}{2\pi a}\right)^2 \qquad \ldots(9)$$

If X, Y be the components of thrust on the cylinder, we have

$$X = -\int_0^{2\pi} p\cos\theta \cdot (a d\theta) \quad \text{and} \quad Y = -\int_0^{2\pi} p\sin\theta \cdot (a d\theta) \qquad \ldots(10)$$

Using (9), (10) reduces to (after simplification)

$$X = 0, \quad \text{and} \quad Y = \rho kU, \qquad \ldots(11)$$

showing that the cylinder experiences an upward lift. This effect may be attributed to circulation phenomenon.

7.8. Illustrative aolved examples.

Ex. 1. *The space between two fixed co-axial circular cylinders of radii a and b, and between two planes perpendicular to the axis and distance c apart is occupied by the liquid of density* ρ. *Show that the velocity potential of a motion whose kinetic energy shall equal to a given quantity T is given by* $A\theta$, *where A is given by* $\pi\rho A^2 c\log(b/a) = T$. **[Kanpur 2009]**

Sol. As the liquid moves only in the space between the two co-axial circular cylinders $r = a$, $r = b$, the motion must be purely due to a circulation

$\therefore$ $$\phi = A\theta, \; A \text{ being a constant.} \qquad ...(1)$$

Now $$q^2 = \left(-\frac{\partial \phi}{\partial r}\right)^2 + \left(-\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)^2 \quad \text{or} \quad q^2 = \frac{A^2}{r^2}, \text{ by (1)} \qquad ...(2)$$

Then $$T = \frac{1}{2}\int_a^b (2\pi r dr \rho c) q^2 = \rho A^2 c\pi \int_a^b \frac{1}{r} dr = \pi\rho A^2 c \log(b/a)$$

Ex. 2. *A circular cylinder is fixed across a stream of velocity U with a circulation k round the cylinder. Show that the maximum velocity in the liquid is* $2U + (k/2\pi a)$, *where a is the radius of the cylinder.* **[Meerut 2007]**

Sol. *The velocity potential* ϕ *for the motion is*

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{k\theta}{2\pi}, \qquad ...(1)$$

where r is measured from the centre of the cross-section of the cylinder. Then the velocity q is given by

$$q^2 = \left(-\frac{\partial \phi}{\partial r}\right)^2 + \left(-\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)^2 = U^2\left(1 - \frac{a^2}{r^2}\right)^2 \cos^2\theta + \left\{U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{k}{2\pi r}\right\}^2$$

$$= U^2\left(1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right) + \frac{Uk}{\pi r}\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{k^2}{4\pi^2 r^2},$$

which is maximum with respect to r when r is minimum, *i.e.*, when $r = a$. Then

$$q^2 = U^2(2 - 2\cos\theta) + \frac{2Uk}{\pi a}\sin\theta + \frac{k^2}{4\pi^2 a^2} = 4U^2\sin^2\theta + \frac{2Uk}{\pi a}\sin\theta + \frac{k^2}{4\pi^2 a^2}$$

Thus, $$q^2 = (2U\sin\theta + k/2\pi a)^2 \qquad ...(2)$$

Now q is further maximum with respect to θ when $\sin\theta = 1$ *i.e.* $\theta = \pi/2$. Thus, from (2) the desired maximum velocity is given by

$$q^2 = (2U + k/2\pi a)^2 \quad \text{i.e.} \quad q = 2U + k/2\pi a.$$

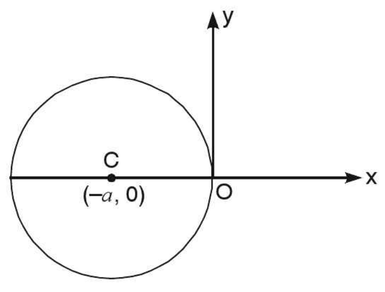
Ex. 3. *The circle* $(x + a)^2 + y^2 = a^2$ *is placed in an on coming wind of velcity U and there is a circulation* $2\pi k$. *Find the complex potential and show that the moment about the orgin is* $2\pi k \rho a U$. **[Meerut 2008; Garhwal 2003, 04]**

Sol. The given circle is $$(x + a^2) + y^2 = a^2 \qquad ...(1)$$

The complex potential for the motion of the wind is given by

$$w = Uz + (Ua^2)/z + \{(2\pi i k)/(2\pi)\}\log(z - a)$$

or $$w = Uz + (Ua^2)/z + ik\log(z - a) \qquad ...(2)$$



By Blasius theroem, we know that in a steady two-dimensional irrotational motion of an incompressible fluid under no external forces given by the complex potential $w = f(z)$, if the pressure thrusts on the fixed cylinder (with cross section given by (1)) are represented by a force (X, Y) and a couple of moment M about the origin O, then

$$M = \text{Real part of } \left\{ -\frac{1}{2}\rho \int_C z \left(\frac{dw}{dz}\right)^2 dz \right\} \qquad ...(3)$$

Again, by Cauchy's residue theroem, we have

$$\int_C z(dw/dz)^2 dz = 2\pi i \times \{\text{sum of the residues}\}, \qquad ...(4)$$

where the indicated sum of the residues is calculated at poles of $z(dz/dz)^2$ lying within the circular boundary C given by (1). Using (4), (3) reduce to

$$M = -(\pi\rho) \times \text{Real part of } \{i \times \text{sum of the residues})\} \qquad ...(5)$$

(2) $\Rightarrow$ $\qquad \dfrac{dw}{dz} = U - \dfrac{Ua^2}{z^2} + \dfrac{ik}{z-a} \qquad$ or $\qquad \left(\dfrac{dw}{dz}\right)^2 = \left\{ U\left(1 - \dfrac{a^2}{z^2}\right) + \dfrac{ik}{z-a} \right\}^2$

or
$$\left(\frac{dw}{dz}\right)^2 = U^2\left(1 - \frac{a^2}{z^2}\right)^2 - \frac{k^2}{(z-a)^2} + \frac{2ikU}{z-a}\left(1 - \frac{a^2}{z^2}\right)$$

or
$$z\left(\frac{dw}{dz}\right)^2 = zU^2\left(1 - \frac{a^2}{z^2}\right)^2 - \frac{zk^2}{z^2(1-a/z)^2} + \frac{2ikUz}{z(1-a/z)}\left(1 - \frac{a^2}{z^2}\right)$$

or
$$z\left(\frac{dw}{dz}\right)^2 = zU^2\left(1 - \frac{a^2}{z^2}\right)^2 - \frac{k^2}{z}\left(1 - \frac{a}{z}\right)^{-2} + 2ikU\left(1 - \frac{a}{z}\right)^{-1}\left(1 - \frac{a^2}{z^2}\right)$$

or
$$z\left(\frac{dw}{dz}\right)^2 = zU^2\left(1 - \frac{2a^2}{z^2} + \frac{a^4}{z^4}\right) - \frac{k^2}{z}\left(1 + \frac{2a}{z} + \dots\right) + 2ikU\left(1 + \frac{a}{z} + \dots\right)\left(1 - \frac{a^2}{z^2}\right) \qquad ...(6)$$

From (6), we find that the only pole of $z(dw/dz)^2$ lying within the circular boundary (1) is at $z = 0$.

$\therefore$ The required sum of the residues
= the sum of the coefficients of z^{-1} in R.H.S. of (6) $= -2U^2a^2 - k^2 + 2ikUa$

$\therefore$ (5) $\quad\Rightarrow\quad M = -(\pi\rho) \times \text{Real part of } \{i(-2U^2a^2 - k^2 + 2ikUa)\} = 2\pi k\rho aU$

Ex. 4. *A mass of liquid whose outer boundary is an infinitely long cylinder of radius b is in a state of cyclic irrotational motion is under the action of a uniform pressure P over the external surface. Prove that there must be a concentric cylindrical hollow whose radius a is given by* $8\pi^3a^2b^2P = Mk^2$, *where M is the mass of unit length of the liquid and k the circulation.*

Sol. The complex potential is given by

$$w = (ik/2\pi)\log(z - z_0) = (ik/2\pi)\log z = (ik/2\pi)\log(re^{i\theta}), \qquad \text{as } z_0 = 0$$

or
$$\phi + i\psi = (ik/2\pi)(\log r + \log e^{i\theta}) = (ik/2\pi)(\log r + i\theta)$$

Equating real parts, we obtain $\qquad \phi = -(k\theta/2\pi)$

$\Rightarrow \qquad u = -(\partial\phi/\partial r) = 0 \qquad$ and $\qquad v = -(1/r) \times (\partial\phi/\partial\theta) = k/(2\pi r)$

$\therefore \qquad q = (u^2 + v^2)^{1/2} = v = k/(2\pi r)$

Bernoulli's equation for steady motion is given by

$p/\rho + q^2/2 = C$; C being a constant $\qquad$ or $\qquad p/\rho + k^2/8\pi^2r^2 = C \qquad ...(1)$

Using the given bundary conditions $p = P,\ q = k/(2\pi b)$ when $r = b$, (1) gives

$$P/\rho + k^2/(8\pi^2 b^2) = C \qquad \ldots(2)$$

From (1) and (2), $\quad \dfrac{p}{\rho} + \dfrac{k^2}{8\pi^2 r^2} = \dfrac{P}{\rho} + \dfrac{k^2}{8\pi^2 b^2} \quad \Rightarrow \quad p = P + \dfrac{k^2\rho}{8\pi^2}\left(\dfrac{1}{b^2} - \dfrac{1}{r^2}\right) \qquad \ldots(3)$

From (3), it follows that when r is very small, then $p < 0$. Therefore, there must exist a cavity, $r = a$ (given). Since pressure vanishes on the surface of the cavity, we have $p = 0$ when $r = 0$. Hence (3) reduces to

$$0 = P + \frac{k^2\rho}{8\pi^2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = P - \frac{k^2\rho}{8\pi^2}\left(\frac{b^2 - a^2}{a^2 b^2}\right) \quad \text{or} \quad 8\pi^3 a^2 b^2 P = k^2 \pi\rho (b^2 - a^2)$$

or $\quad 8\pi^3 a^2 b^2 P = Mk^2$, as M = Mass of unit length of the liquid $= \pi\rho\,(b^2 - a^2)$

Ex. 5. *Verify that the stream functions for uniform streaming parallel to the axis past a solid, bounded by those parts of the circles,* $(x+1)^2 + y^2 = 2$, $(x-1)^2 + y^2 = 2$ *which are external to each other are given by*

$$\psi = y\left\{1 + \frac{1}{x^2 + y^2} - \frac{2}{(x+1)^2 + y^2} - \frac{2}{(x-1)^2 + y^2}\right\}$$

and

$$\psi = -x + \frac{x}{x^2 + y^2} + \frac{2(x+1)}{(x+1)^2 + y^2} + \frac{2(x-1)}{(x-1)^2 + y^2}$$

and, when the stream is inclined at an angle α *to the line of centres, find the equation to the streamline that divides on the solid.*

Ex. 6. Show that the function $w = U(z + a^2/z) + (ik/2\pi) \times \log z$ gives the solution of the problem of flow of a stream of velocity U past a fixed circular cylinder of radius a there being a circulation k round the cylinder.

A cylinder of radius a is placed in a stream of velocity V and pressure p_0 at infinity. Show that the resultant thrust (per unit thickness) on a quadrant of the cylinder between $\theta = 0$ and $\theta = \pi/2$ where θ points up stream is given by $X = a\{-p_0 + (\rho V^2/6)\}$ and $Y = a\{-p_0 + (5\rho V^2/6)\}$, where ρ is the density of the liquid.

Ex. 7. Show that the formula $w = ik\log\{(z-c)/(z+c)\};\ z = x + iy$ in which k is real, gives the irrotational motion of fluid, circulating about two fixed circles, the circulations being $2\pi k$ for one and $-\,2\pi k$ for the other.

Determine the motion obtained by applying the transformation $z' = a^2/(z-c),\ z' = x' + iy'$, where a is real, obtaining the boundaries of the region in which it takes place.

EXERCISE 7 (B)

1. State and prove the theorem of Blasius. Hence discuss the flow past an infinite circular cylinder in a uniform stream with circulation.

2. Liquid is streaming past a fixed circular cylinder with velocity U and circulation k about the cylinder. Find the velocity potential and determine the points of no velocity.

3. Verify that $w = ik \log\{(z-ia)/(z+ia)\}$ is the complex potential of a steady flow of liquid about a circular cylinder, the plane $y = 0$ being rigid boundary. Find the force exerted by the liquid on unit length of the cylinder.

4. Show that the complex potential for the streaming motion past a circular cylinder of radius a, round which there is a circulation k, is

$$w = V\left(z + \frac{a^2}{z}\right) + \frac{ik}{2\pi}\log\frac{z}{a}.$$

Discuss the general form of the stream lines and the position of the stagnation points.

The boundary of an obstacle in the z-plane is mapped on the circle $|\zeta| = a$ in the ζ-plane by the transformation $\qquad \zeta = z + a_1/z + a_2/z^2 + \cdots$

Prove that the resultant force per unit length of the obstacle has a magnitude $k\rho V$.

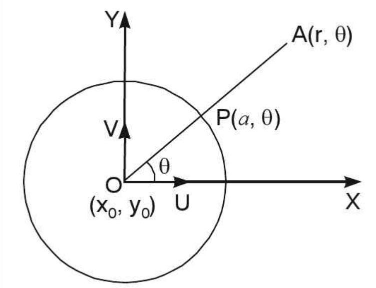
5. Show that the complex potential $w = U[z + (a^2/z)] + ik\log z$ represents a possible flow past a circular cylinder. Sketch the stream lines, find the stagnation points and calculate the force on the cylinder.

[**Hint :** Refer Art. 7.7. To obtain the desired form $ik \log z$ in place of $(ik/2\pi)\log z$, the reader should note that some authors define $2\pi k$ as strength of circulation in place of k.]

7.9. Equations of motion of a circular cylinder.

[Garhwal 2005; Kurukshetra 2000, 01]

A circular cylinder is moving in a liquid at rest at infinity. To calculate the forces acting on the cylinder owing to the pressure of the fluid.



Let U, V be the components of velocity of the cylinder when the centre of the cross-section O is (x_0, y_0). Then,

$$U = \dot{x}_0 \quad \text{and} \quad V = \dot{y}_0 \qquad \ldots(1)$$

Let $\quad z_0 = x_0 + iy_0 \quad$ and $\quad z - z_0 = re^{i\theta}, \qquad \ldots(2)$

where r denotes the distance from the axis of the cylinder.

On the surface of the cylinder $r = a$, we must have

Normal velocity of the liquid = Normal velocity of the cylinder

i.e. $\quad$ at $\quad r = a, \quad -\partial\phi/\partial r = U\cos\theta + V\sin\theta \qquad \ldots(3)$

Since liquid is at rest at infinity, so $\quad -\partial\phi/\partial r = 0 \quad$ at $\quad r = \infty \qquad \ldots(4)$

Keeping (3) and (4) in mind, we take

$$\phi = \left(Ar + \frac{B}{r}\right)\cos\theta + \left(Cr + \frac{D}{r}\right)\sin\theta \qquad \ldots(5)$$

$$\therefore \qquad \frac{\partial\phi}{\partial r} = \left(A - \frac{B}{r^2}\right)\cos\theta + \left(C - \frac{D}{r^2}\right)\sin\theta \qquad \ldots(6)$$

Using (6), (3) reduces to

$$-\left(A-\frac{B}{a^2}\right)\cos\theta-\left(C-\frac{D}{a^2}\right)\sin\theta=U\cos\theta+V\sin\theta \qquad ...(7)$$

so that $\qquad B/a^2-A=U \qquad$ and $\qquad D/a^2-C=V \qquad ...(8)$

Again, using (6), (4) reduces to $\qquad -A\cos\theta-C\sin\theta=0$

so that $\qquad A=0 \qquad$ and $\qquad C=0 \qquad ...(9)$

From (8) and (9), $\qquad B=a^2U, \qquad$ and $\qquad D=a^2V \qquad ...(10)$

Using (9) and (10), (5) reduces to

$$\phi=(a^2/r)(U\cos\theta+V\sin\theta) \qquad ...(11)$$

But
$$\frac{\partial\psi}{\partial r}=-\frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

$\therefore$
$$\frac{\partial\psi}{\partial r}=-\frac{a^2}{r^2}(-U\sin\theta+V\cos\theta)$$

Integrating,
$$\psi=(a^2/r)(-U\sin\theta+V\cos\theta) \qquad ...(12)$$

Hence the complex potential is given by

$$w=\phi+i\psi=\frac{a^2}{r}[U(\cos\theta-i\sin\theta)+iV(\cos\theta-i\sin\theta)]=\frac{a^2e^{-i\theta}}{r}(U+iV)$$

Thus,
$$w=\frac{a^2(U+iV)}{z-z_0},\ \text{by (2)} \qquad ...(13)$$

$$\therefore \quad \frac{\partial w}{\partial t}=\frac{a^2(\dot U+i\dot V)}{z-z_0}-\frac{a^2(U+iV)}{(z-z_0)^2}(-\dot z_o)=\frac{a^2(\dot U+i\dot V)}{z-z_0}-\frac{a^2(U+iV)^2}{(z-z_0)^2} \qquad \left[\because\ \dot z_0=\dot x_0+i\dot y_0 = \mathrm{U}+i\mathrm{V}\right]$$

or
$$\frac{\partial\phi}{\partial t}+i\frac{\partial\psi}{\partial t}=\frac{a^2}{r}(\dot U+i\dot V)(\cos\theta+i\sin\theta)+\frac{a^2}{r^2}(U+iV)^2(\cos2\theta-i\sin2\theta),\ \text{by (2)}$$

$$\therefore \quad \frac{\partial\phi}{\partial t}=\frac{a^2}{r}(\dot U\cos\theta+\dot V\sin\theta)+\frac{a^2}{r^2}[(U^2-V^2)\cos2\theta+2UV\sin2\theta] \qquad ...(14)$$

The velocity q is given by help of (13). Thus we have

$$q^2=\left|\frac{dw}{dz}\right|^2=\left|-a^2\frac{U+iV}{(z-z_0)^2}\right|^2=\frac{a^4(U^2+V^2)}{r^4},\ as\ z-z_0=re^{i\theta} \qquad ...(15)$$

Omititing the external forces, the pressure at any point is gives by Bernoulli's equation,

$$\frac{p}{\rho}=F(t)+\frac{\partial\phi}{\partial t}-\frac{1}{2}q^2 \qquad ...(16)$$

Using (14) and (15), (16) reduces to

$$\frac{p}{\rho}=F(t)+\frac{a^2}{r}(\dot U\cos\theta+\dot V\sin\theta)+\frac{a^2}{r^2}[(U^2-V^2)\cos2\theta+2UV\sin2\theta]-\frac{1}{2}\frac{a^4}{r^4}(U^2+V^2) \qquad ...(17)$$

Let p_1 be the pressure at (a,θ) on the boundary of the cylinder. Then p_1 is given by putting $r=a$ in (17). Thus, we have

$$p_1=\rho F(t)+\rho a(\dot U\cos\theta+\dot V\sin\theta)+\rho[(U^2-V^2)\cos2\theta+2UV\sin2\theta]-(U^2+V^2)/2 \qquad ...(18)$$

Let X and Y be the components of forces on the cylinder due to fluid thrusts. Then, we have

$$X = -\int_0^{2\pi} p_1 \cos\theta \cdot a d\theta \qquad \text{and} \qquad Y = -\int_0^{2\pi} p_1 \sin\theta \cdot a d\theta \qquad ...(19)$$

Using (18), (19) gives $\quad X = -\rho a^2 \int_0^{2\pi} \dot{U} \cos^2\theta \; d\theta$, on simplifiaction

Thus, $$X = -\pi a^2 \rho \dot{U} = -M'\dot{U}, \qquad ...(20)$$

where $M' = \pi a^2 \rho$ = the mass of the liquid displaced by the cylinder of unit length.

Similarly, $$Y = -\pi a^2 \rho \dot{V} = -M'\dot{V} \qquad ...(21)$$

Corollary *To show that the effect of the presence of the liquid is to reduce the extraneous forces in the ratio* $\sigma - \rho;\ \sigma + \rho$ *where* σ, ρ *are the densities of the cylinder and liquid respectively.*

Let M the mass of the cylinder per unit length and X', Y' be the components of the extraneous (external) forces on the cylinder if there were no liquid.

Let f_x be the acceleration of the extraneous forces in x-direction. Then, due to presence of liquid the resultant force in x-direction

$$= \pi a^2 \sigma f_x - \pi a^2 \rho f_x = \frac{\sigma - \rho}{\sigma}(\pi a^2 \sigma f_x) = \frac{\sigma-\rho}{\sigma} X'.$$

Now, the equation of motion in x-direction is of the form

$$M\dot{U} = -M'\dot{U} + \frac{\sigma-\rho}{\sigma}X' \qquad \text{or} \qquad (M + M')\dot{U} = \frac{\sigma-\rho}{\sigma}X'$$

or $$M\dot{U} = \frac{M}{M+M'} \cdot \frac{\sigma-\rho}{\sigma}X' = \frac{\pi a^2 \sigma}{\pi a^2 \sigma + \pi a^2 \rho} \cdot \frac{\sigma-\rho}{\sigma}X'$$

$\therefore$ $$M\dot{U} = \frac{\sigma-\rho}{\sigma+\rho}X' \qquad ...(22)$$

Similarly, $$M\dot{V} = \frac{\sigma-\rho}{\sigma+\rho}Y' \qquad ...(23)$$

Hence the effect of the presence of the liquid is to reduce the external forces in the ratio $\sigma - \rho : \sigma + \rho$.

7.10. Equations of motion of a circular cylinder with circulation.

Here we use the notation of Art 7.9. Let k be the circulation about the cylinder. Then using Art 7.6 and Art. 7.9, the complex potential due to the combined effects at any point $z = re^{i\theta}$ is

$$w = \frac{\mathrm{U} + i\mathrm{V}}{z - z_0} + \frac{ik}{2\pi}\log(z - z_0) \qquad ...(1)$$

or $$\phi + i\psi = \frac{a^2(\mathrm{U} + i\mathrm{V})}{r}(\cos\theta - i\sin\theta) + \frac{ik}{2\pi}(\log r + i\theta), \qquad \text{as} \qquad z - z_0 = re^{i\theta}$$

so that $$\phi = \frac{a^2}{r}(U\cos\theta + V\sin\theta) - \frac{k\theta}{2\pi}$$

$\therefore$ $$\frac{\partial\phi}{\partial r} = -\frac{a^2}{r^2}(U\cos\theta + V\sin\theta) \qquad \text{and} \qquad \frac{\partial\phi}{\partial\theta} = \frac{a^2}{r}(-U\sin\theta + V\cos\theta) - \frac{k}{2\pi}$$

$$\therefore \quad q^2 = \left(-\frac{\partial \phi}{\partial r}\right)^2 + \left(-\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)^2$$

$$= \frac{a^4}{r^4}(U\cos\theta + V\sin\theta)^2 + \left[\frac{a^2}{r^2}(U\sin\theta - V\cos\theta)^2 + \frac{k}{2\pi r}\right]^2 \qquad ...(2)$$

Again, from (1), we have [as in Art. 7.9]

$$\frac{\partial w}{\partial t} = \frac{a^2(\dot{U}+i\dot{V})}{z-z_0} - \frac{a^2(U+iV)}{(z-z_0)^2}(-\dot{z}_0) + \frac{ik}{2\pi(z-z_0)}(-\dot{z}_0)$$

$$= \frac{a^2(\dot{U}+i\dot{V})}{z-z_0} - \frac{a^2(U+iV)^2}{(z-z_0)^2} - \frac{ik}{2\pi}\frac{U+iV}{z-z_0}$$

$$\therefore \quad \frac{\partial \phi}{\partial t} + i\frac{\partial \psi}{\partial t} = \frac{a^2}{r}(U+iV)(\cos\theta - i\sin\theta)$$

$$+ \frac{a^2}{r^2}(U+iV)^2(\cos 2\theta - i\sin 2\theta) - \frac{ik}{2\pi r}(U+iV)(\cos\theta - i\sin\theta)$$

Equating real parts on both sides, we get

$$\therefore \quad \frac{\partial \phi}{\partial t} = \frac{a^2}{r}(\dot{U}\cos\theta + \dot{V}\sin\theta) + \frac{a^2}{r^2}[(U^2-V^2)\cos 2\theta + 2UV\sin 2\theta] + \frac{k}{2\pi a}(V\cos\theta - U\sin\theta) \quad ...(3)$$

Let X and Y be the components of forces on the cylinder due to fluid thrusts. Then as explained in Art. 7.9, we have

$$X = -M'\dot{U} - k\rho V \qquad \text{and} \qquad Y = -M'\dot{V} + k\rho U \qquad ...(4)$$

Hence if, as before, M denotes the mass of unit length of the cylinder and there are no extraneous forces, the equations of motion take of form

$$(M+M')\dot{U} = -k\rho V \qquad ...(5)$$

and

$$(M+M')\dot{V} = k\rho U \qquad ...(6)$$

From (5) and (6), $\quad U\dot{U} + V\dot{V} = 0 \quad$ so that $\quad U^2 + V^2 = \text{const.} \qquad ...(7)$

and $\quad \dfrac{U\dot{V} - \dot{U}V}{U^2+V^2} = \dfrac{k\rho}{M+M'} \quad$ or $\quad \dfrac{d}{dt}\left[\tan^{-1}\dfrac{V}{U}\right] = \dfrac{k\rho}{M+M'}, \qquad ...(8)$

which gives $\alpha = \tan^{-1}(V/U)$ as the inclination of the direction of motion to the axis of x.

Equations (5) and (6) show that the cylinder is acted on by a force $k\rho$ × (velocity) at right angles to the path. It can be verified that this force is independent of the cross-section of the cylinder.

Equations (7) and (8) show that the cylinder describes a circle of radius $(M+M')(U^2+V^2)^{1/2}/k\rho$ with constant velocity $(U^2 + V^2)^{1/2}$ in the sense of the cyclic motion.

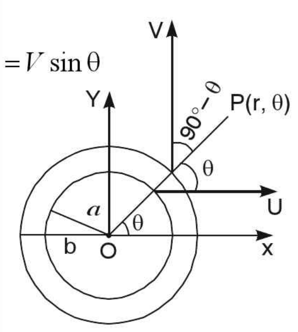
Remark. When U and V are constants and there is no circulation, the cylinder will not experience any resistance to its motion. This is referred to as *D'Alembert's Paradox*.

7.11. Two coaxial cylinders (problem of initial motion).

To determine the velocity potential and stream function at any point of a liquid contained between two coaxial cylinders of radii a and b (a < b) when the cylinders are moved suddenly parallel to themselves in directions at right angles with velocities U and V respectively.

Let ϕ be the velocity potential and ψ the current function at any point (r, θ) in the liquid. Here the boundary conditions for the velocity potential ϕ are:

(*i*) when $r = a$, $-\partial\phi/\partial r = U\cos\theta$ (*ii*) when $r = b$, $-\partial\phi/\partial r = V\sin\theta$



ϕ must satisfy the Laplace's equation $\nabla^2\phi = 0$ at every point of the liquid. In polar co-ordinates of two dimensions $\nabla^2\phi = 0$ takes the form

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \qquad ...(1)$$

Since (1) has solutions of the form $r^n \cos n\theta$, $r^n \sin n\theta$, where n is any integer (positive or negative), hence the sum of any number of terms of the form $A_n r^n \cos n\theta$, $B_n r^n \sin n\theta$ is also a solution of (1).

The above considerations suggest that we must assume the following suitable form of ϕ

$$\phi = \left(Ar + \frac{B}{r}\right)\cos\theta + \left(Cr + \frac{D}{r}\right)\sin\theta \qquad ...(2)$$

so that

$$\frac{\partial\phi}{\partial r} = \left(A - \frac{B}{r^2}\right)\cos\theta + \left(C - \frac{D}{r^2}\right)\sin\theta \qquad ...(3)$$

Using the boundary conditions (*i*) and (*ii*), (3) gives

$$-U\cos\theta = \left(A - \frac{B}{a^2}\right)\cos\theta + \left(C - \frac{D}{a^2}\right)\sin\theta \qquad ...(4)$$

and

$$-V\sin\theta = \left(A - \frac{B}{b^2}\right)\cos\theta + \left(C - \frac{D}{b^2}\right)\sin\theta \qquad ...(5)$$

Since (4) and (5) must hold for all values of θ, we have

$$A - \frac{B}{a^2} = -U, \qquad C - \frac{D}{a^2} = 0, \qquad A - \frac{B}{b^2} = 0, \qquad C - \frac{D}{b^2} = -V$$

$$\therefore \quad A = \frac{Ua^2}{b^2 - a^2}, \qquad B = \frac{Ua^2b^2}{b^2 - a^2}, \qquad C = -\frac{Vb^2}{b^2 - a^2}, \qquad D = -\frac{Va^2b^2}{b^2 - a^2}.$$

Hence,

$$\phi = \frac{a^2U}{b^2 - a^2}\left(r + \frac{b^2}{r}\right)\cos\theta - \frac{b^2V}{b^2 - a^2}\left(r + \frac{a^2}{r}\right)\sin\theta \qquad ...(6)$$

Since $\partial\phi/\partial r = (1/r) \times (\partial\psi/\partial\theta)$, (6) gives

$$\frac{\partial\psi}{\partial\theta} = \frac{a^2U}{b^2 - a^2}\left(r - \frac{b^2}{r}\right)\cos\theta - \frac{b^2V}{b^2 - a^2}\left(r - \frac{a^2}{r}\right)\sin\theta$$

$$\therefore \quad \psi = \frac{a^2U}{b^2 - a^2}\left(r - \frac{b^2}{r}\right)\sin\theta + \frac{b^2V}{b^2 - a^2}\left(r - \frac{a^2}{r}\right)\cos\theta \qquad ...(7)$$

Remark. Equations (6) and (7) only represent the motion at the instant when the cylinders are coaxial. Thus they give the initial motion.

7.12. Illustrative solved examples.

Ex. 1. *The space between two infinitely long coaxial cylinders of radii a and b (b > a) respectively is filled with homogeneous liquid of density* ρ.

The inner cylinder is suddenly moved with velocity U perpendicular to the axis, the outer one being kept fixed. Show that the resultant impulsive pressure on a length l of the inner cylinder is

$$\pi\rho a^2 l\frac{b^2+a^2}{b^2-a^2}U.$$ **[Agra 2008; Meerut 2005; I.A.S. 2004; Kanpur 2008, 10]**

Sol. Refer figure of Art. 7.11. For this problem $V = 0$ and hence you need not show V in that figure. The boundary conditions for the velocity potential ϕ are :

(*i*) when $r = a$, $-\partial\phi/\partial r = U\cos\theta$ (*ii*) when $r = b$, $-\partial\phi/\partial r = 0$

Since ϕ must satisfy the Laplace's equation $\nabla^2\phi = 0$ *i.e.*, $\dfrac{\partial^2\phi}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\phi}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}=0,$

we assume the suitable form of ϕ as

$$\phi = (Ar + B/r)\cos\theta + (Cr + D/r)\sin\theta \qquad ...(1)$$

so that
$$\frac{\partial\phi}{\partial r} = \left(A-\frac{B}{r^2}\right)\cos\theta + \left(C-\frac{D}{r^2}\right)\sin\theta \qquad ...(2)$$

Applying the boundary conditions (*i*) and (*ii*), (2) gives

$$-U\cos\theta = \left(A-\frac{B}{a^2}\right)\cos\theta + \left(C-\frac{D}{a^2}\right)\sin\theta \qquad ...(3)$$

and
$$0 = \left(A-\frac{B}{b^2}\right)\cos\theta + \left(C-\frac{D}{b^2}\right)\sin\theta \qquad ...(4)$$

Since (3) and (4) must hold for all values of θ, we get

$$A-\frac{B}{a^2} = -U, \qquad C-\frac{D}{a^2}=0, \qquad A-\frac{B}{b^2}=0, \qquad C-\frac{D}{b^2}=0$$

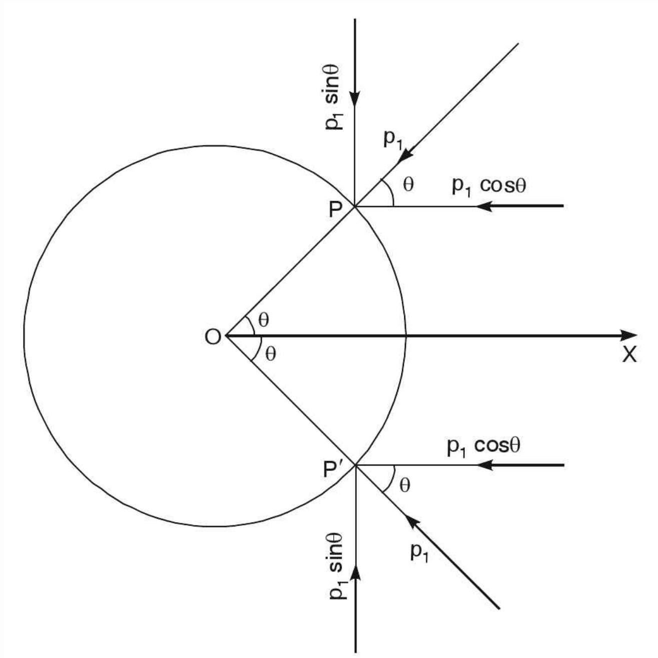
$$\therefore \qquad A=\frac{Ua^2}{b^2-a^2}, \qquad B=\frac{Ua^2b^2}{b^2-a^2}, \qquad C=0, \qquad D=0.$$

$\therefore$ from (1),
$$\phi = \frac{Ua^2}{b^2-a^2}\left(r+\frac{b^2}{r}\right)\cos\theta.$$

Impulsive pressure at any point is $\rho\phi$. Hence impulsive pressure p_1 at any point P (a, θ) of the inner cylinder is given by

$$p_1 = \rho\phi \qquad \text{at} \qquad r = a$$

$$\therefore \qquad p_1 = \rho\frac{Ua^2}{b^2-a^2}\left(a+\frac{b^2}{a}\right)\cos\theta \qquad \text{or} \qquad p_1 = \rho Ua\frac{b^2+a^2}{b^2-a^2}\cos\theta$$



Hence the total impulsive pressure on the cylinder of length l

$$= \int_0^{2\pi} (p_1 \cos\theta)(a d\theta)\, l = l\rho U a^2 \frac{b^2+a^2}{b^2-a^2} \int_0^{2\pi} \cos^2 \theta d\theta = \pi\rho a^2 l U \frac{b^2+a^2}{b^2-a^2}$$

Ex. 2. *The space between two infinitely long coaxial cylinders of radii a and b respectively is filled with homogeneous liquid of density ρ and is suddenly moved with velocity V perpendicular to the axis, the outer one being kept fixed. Show that the resultant impulsive pressure on unit length of the inner cylinder is* $\pi\rho a^2 \dfrac{b^2+a^2}{b^2-a^2} V.$

Sol. Proceed as in Ex. 1. So the resultant impulsive per unit length is given by putting $l = 1$.

Ex. 3. *An infinite cylinder of radius a and density σ is surrounded by a fixed concentric cylinder of radius b and the intervening space is filled with liquid of density ρ. Prove that the impulse per unit length necessary to start the inner cylinder with velocity V is*

$$\frac{\pi a^2}{b^2-a^2}\left\{(\sigma+\rho) b^2 - (\sigma-\rho) a^2\right\} V. \qquad \textbf{[I.A.S. 1989]}$$

Sol. As in Ex. 2. above, the total impulsive pressure on the cylinder of unit length

$$= \pi\rho a^2 V^2 \frac{b^2+a^2}{b^2-a^2}$$

Moreover, the impulse needed to move the inner cylinder with a velocity V

$$= \text{(mass of the cylinder)} \times V = \pi a^2 \sigma V.$$

Hence the total impulse per unit length to move inner cylinder with velocity V

$$= \pi a^2 \sigma V + \pi\rho a^2 V \frac{b^2+a^2}{b^2-a^2} = \frac{\pi a^2}{b^2-a^2}\left\{(\sigma+\rho) b^2 - (\sigma-\rho) a^2\right\} V$$

EXERCISE 7 (C)

1. Liquid is contained between two infinite cylinders of radii a, b ($b > a$) with the same axis. A two-dimensional motion is suddenly generated from rest by moving the inner and outer cylinders with velocities U, V respectively, the directions of the velocities are perpendicular to each other and to the axis of the cylinders. Prove that initially the kinetic energy of the liquid contained in unit length of the cylinders is $\dfrac{\pi\rho}{2}\dfrac{b^2+a^2}{b^2-a^2}(a^2U^2+b^2V^2)$.

2. An infinite cylinder of radius a is surrounded by a fixed coaxial cylinder of radius b, the intervening space being filled with a liquid, prove that the potential at a point (r, θ) of the liquid at the instant, when the inner cylinder is made to move with velocity U, is given by

$\phi = \dfrac{Ua^2}{b^2-a^2}\left(r+\dfrac{b^2}{r}\right)\cos\theta$, θ being measured from the direction of V.

3. Obtain the velocity potential at any point of a liquid contained between two coaxial cylinders of radii a and $2a$ when the cylinders are moved suddenly parallel to themselves in directions at right angles with velocities v and $2v$ respectively.

4. The space between two infinitely long coaxial cylinders of radii a and $2a$ is filled with homogeneous liquid of density ρ and the inner cylinder is suddenly moved with velocity u perpendicular to the axis, the outer being kept fixed. Show that the resultant impulsive pressure on unit length of the inner cylinder is $(5/3) \times \pi\rho a^2 U$.

5. Obtain velocity potential for motion between two co-axial cylinders. **[Kanpur 2003]**

Hint. Refer Art 7.11

PART II MOTION OF ELLIPTIC CYLINDERS

7.13. Elliptic co-ordinates.

Let $z = c\cosh\zeta$, where $z = x + iy$ and $\zeta = \xi + i\eta$.

Then $x + iy = c\cosh(\xi + i\eta) = c(\cosh\xi\cos\eta + i\sinh\xi\sin\eta)$

so that $x = c\cosh\xi\cos\eta \qquad y = c\sinh\xi\sin\eta \qquad ...(1)$

Eliminating η from (1), we have for all values of η from 0 to 2π

$$\frac{x^2}{c^2\cosh^2\xi}+\frac{y^2}{c^2\sinh^2\xi}=1 \qquad ...(2)$$

Eliminating ξ from (1), we have for all values of ξ from 0 to ∞

$$\frac{x^2}{c^2\cosh^2\eta}-\frac{y^2}{c^2\sinh^2\eta}=1 \qquad ...(3)$$

From (2) and (3) we find that ξ = const. and η = const. represent confocal ellipses and hyperbolas respectively and distance between the foci in each case is $2c$ (verify yourself). If a and b are the semi-axes of the ellipse (2) when $\xi = \alpha$, we have on comparison

$a = c\cosh\alpha$, and $b = c\sinh\alpha$ so that $a^2 - b^2 = c^2$

$$\left.\begin{aligned} &\text{Also} && a+b = c(\cosh\alpha+\sinh\alpha) = ce^{\alpha} \\ &\text{and} && a-b = c(\cosh\alpha-\sinh\alpha) = ce^{-\alpha} \\ &\text{so that} && e^{2\alpha} = (a+b)/(a-b) \quad \text{or} \quad \alpha = (1/2)\times\log[(a+b)/(a-b)] \end{aligned}\right\} \qquad ...(4)$$

The parameters ξ, η are known as the *elliptic co-ordinates*.

The equations $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0$ and $\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0$ can be transformed respectively into

$$\partial^2\phi/\partial\xi^2 + \partial^2\phi/\partial\eta^2 = 0 \qquad \text{and} \qquad \partial^2\psi/\partial\xi^2 + \partial^2\psi/\partial\eta^2 = 0 \qquad \text{...(5)}$$

with the help of (1). It is known that equations in (5) have solutions of the type

$$\left.\begin{matrix}\cosh\\ \sinh\\ \exp\end{matrix}\right\}(n\xi) \times \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}(n\eta). \qquad \text{...(6)}$$

When the liquid extends to infinity (where the liquid is at rest), solution containing $e^{-n\xi}$ must be used. Again, for conical ellipse the form $(A\cosh n\xi + B\sinh n\xi)\,{}^{\cos}_{\sin}\,(n\eta)$ must be used.

7.14. Motion of an elliptic cylinder.

(I) *To determine the velocity potential and stream function when an elliptic cylinder moves in an infinite liquid with velocity U parallel to the axial plane through the major axis of a cross-section.* **[Kanpur 2009]**

For any cylinder moving with velocities U and V parallel to axes and rotating with an angler velocity ω, we know that

$$\psi = Vx - Uy + (1/2)\times\omega\,(x^2 + y^2) + \text{const.}$$

Here $V = 0$, $\omega = 0$. Hence the stream function is given by

$$\psi = -Uy + A \qquad \text{...(1)}$$

Let the cross-section be the ellipse $x^2/a^2 + y^2/b^2 = 1$. This is the same as $\xi = \alpha$, if $a = c\cosh\alpha$, $b = c\sinh\alpha$ and $c^2 = a^2 - b^2$. The elliptic co-ordinates are

$$x = c\cosh\alpha\cos\eta, \qquad y = c\sinh\alpha\sin\eta \qquad \text{...(2)}$$

Using (2), (1) becomes $\qquad \psi = -Uc\sinh\alpha\sin\eta + A \qquad$ ...(3)

Since ψ contains $\sin\eta$ and the liquid is at rest at infinity, ψ must be of the form $e^{-\xi}\sin\eta$. We, therefore, assume that

$$\phi + i\psi = Be^{-(\xi + i\eta)} \qquad \text{...(4)}$$

so that

$$\psi = -Be^{-\xi}\sin\eta \qquad \text{...(5)}$$

Then at boundary $\xi = \alpha$, we must have

$$-Be^{-\alpha}\sin\eta = -Uc\sinh\alpha\sin\eta + A \text{ for all values of } \eta.$$

This gives $\qquad A = 0 \qquad$ and $\qquad B = Uce^{\alpha}\sinh\alpha.$

$$\therefore \qquad \psi = -Uce^{\alpha-\xi}\sinh\alpha\sin\eta, \qquad \text{...(6)}$$

is a stream function which will make the boundary of the ellipse a stream line, when the cylinder moves with velocity U.

But $\quad c\sinh\alpha = b \quad$ and $\quad e^{\alpha} = \sqrt{[(a+b)/(a-b)]}$. by Art. 7.13 $\qquad$ ...(7)

Using (7), (6) may be re-written as

$$\psi = -Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}\sin\eta \qquad \text{...(8)}$$

From (4), $\phi = Be^{-\xi}\cos\eta$. Hence, as before

$$\phi = Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}\cos\eta \qquad \text{...(9)}$$

$$\therefore \qquad w = \phi + i\psi = Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-(\xi+i\eta)} \qquad \text{...(10)}$$

(II) *To determine the velocity function and the stream function when an elliptic cylinder moves in an infinite liquid with velocity V parallel to the axial plane through the minor axis of a cross-section.*

Proceeding as above, here we have

$$\phi = Va\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}\cos\eta, \qquad \text{...(11)}$$

$$\psi = Va\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}\sin\eta \qquad \text{...(12)}$$

and
$$w = \phi + i\psi = iVa\left(\frac{a+b}{a-b}\right)^{1/2} e^{-(\xi+i\eta)} \qquad \text{...(13)}$$

(III) *To determine the complex potential when an elliptic cylinder moves in an infinite liquid with a velocity in a direction making an angle θ with the major axis of the cross section of the cylinder.*

The components of v along the axes of co-ordinates are

$$U = v\cos\theta \qquad \text{and} \qquad V = v\sin\theta$$

Let w_1 and w_2 be the complex potentials corresponding to the motion of the cylinder with velocities U and V respectively. Then from (10) and (13), we have

$$w_1 = Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-(\xi+i\eta)} = bv\cos\theta\left(\frac{a+b}{a-b}\right)^{1/2} e^{-(\xi+i\eta)}$$

and
$$w_2 = iVa\left(\frac{a+b}{a-b}\right)^{1/2} e^{-(\xi+i\eta)} = iav\sin\theta\left(\frac{a+b}{a-b}\right)^{1/2} e^{-(\xi+i\eta)}$$

Hence the complex potential due to velocity v is given by

$w = w_1 + w_2$

$$= v\left(\frac{a+b}{a-b}\right)^{1/2} e^{-(\xi+i\eta)}(b\cos\theta + ia\sin\theta) = v\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\zeta}(c\sinh\alpha\cos\theta + ic\cosh\alpha\sin\theta)$$

$$[\because\ \zeta = \xi + i\eta,\quad b = c\sinh\alpha,\quad a = c\cosh\alpha]$$

$$= cv\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\zeta}\sinh(\alpha + i\theta) = v\,(a+b)e^{-\zeta}\sinh(\alpha + i\theta) \qquad [\because\ c^2 = a^2 - b^2]$$

7.15. Liquid streaming past a fixed elliptic cylinder.

To determine ϕ and ψ for a liquid streaming past a fixed elliptic cylinder with velocity U parallel to major axis of the section. **[Kanpur 1997; 2005]**

Superimpose a velocity U on the cylinder and liquid both in the sense opposite to the velocity of the liquid. This brings the liquid at rest and the cylinder in motion with velocity U.

Hence, some suitable term must be added to each of the expressions for ϕ and ψ obtained in Art 7.14 (I). When the stream flows from positive x-axis to negative x-axis, we have

$$-\partial\phi/\partial x = -\partial\psi/\partial y = -U \qquad \text{... (1)}$$

Accordingly, we must add a term Ux to ϕ and Uy to ψ as obtained in Art. 7.14 (I). Thus, we have (noting results of Art 7.13)

$$\phi = Ux + Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}\cos\eta = Uc\cosh\xi\cos\eta + Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}\cos\eta$$

$$= U\sqrt{(a^2-b^2)}\cosh\xi\cos\eta + Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}\cos\eta\text{, as } c^2 = a^2-b^2 \qquad \text{...(2)}$$

$$\psi = Uy - Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}\sin\eta \qquad \text{...(3)}$$

$$= U\sqrt{(a^2-b^2)}\sinh\xi\sin\eta - Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}\sin\eta \qquad \text{...(4)}$$

Then the complex potential is given by

$$w = \phi + i\psi = U(x+iy) + Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}(\cos\eta - i\sin\eta),\text{ by (2) and (3)}$$

$$= Uz + Ube^{\alpha}e^{-\xi}e^{-i\eta} = Uz + Ube^{\alpha}e^{-\zeta}, \qquad \text{as} \qquad \zeta = \xi + i\eta$$

Thus, $$w = Uz + Ube^{\alpha-\zeta} \qquad \text{...(5)}$$

Another from for ϕ, ψ and w.

Re-writing (2), we have

$$\phi = U\left(\frac{a+b}{a-b}\right)^{1/2}\cos\eta[be^{-\xi} + (a-b)\cosh\xi]$$

$$= U\left(\frac{a+b}{a-b}\right)^{1/2}\cos\eta[b(\cosh\xi - \sinh\xi) + (a-b)\cosh\xi] = U\left(\frac{a+b}{a-b}\right)^{1/2}\cos\eta\,(a\cosh\xi - b\sinh\xi)$$

$$= U\left(\frac{a+b}{a-b}\right)^{1/2}\cos\eta\,(c\cosh\alpha\cosh\xi - c\sinh\alpha\sinh\xi) = Uc\left(\frac{a+b}{a-b}\right)^{1/2}\cos\eta\cosh(\xi-\alpha)$$

Thus, $$\phi = U(a+b)\cos\eta\cosh(\xi-\alpha), \qquad \text{as} \qquad c = \sqrt{(a^2-b^2)} \qquad \text{...(6)}$$

or $$\phi = Uce^{\alpha}\cos\eta\cosh(\xi-\alpha), \qquad \text{as} \qquad e^{\alpha} = \left(\frac{a+b}{a-b}\right)^{1/2} \qquad \text{...(7)}$$

Similarly, (4) gives $$\psi = U(a+b)\sin\eta\sinh(\xi-\alpha) \qquad \text{...(8)}$$

or $$\psi = Uce^{\alpha}\sin\eta\sinh(\xi-\alpha) \qquad \text{...(9)}$$

Using (6) and (8), the complex potential is given by

$$w = \phi + i\psi = U(a+b)[\cos\eta\cosh(\xi-\alpha) + i\sin\eta\sinh(\xi-\alpha)]$$

$$= U(a+b)\cosh\{(\xi-\alpha)+i\eta\} = U(a+b)\cosh(\zeta-\alpha)\text{, as } \zeta = \xi + i\eta \qquad \text{...(10)}$$

7.16. Rotating elliptic cylinder.

To determine ϕ and ψ and complex potential when an elliptic cylinder is rotating with angular velocity ω in an infinite mass of the liquid at infinity.

[Kanpur 2001, 06, 10; Rohilkhand 2001; Garhwal 1997, 98; Kurukshetra 1997]

For any cylinder moving with velocity U and V parallel to axes and rotating with an angular velocity ω, we know that

$$\psi = Vx - Uy + (1/2) \times \omega\,(x^2 + y^2) + \text{const.} \qquad ...(1)$$

Let the cross section be the ellipse $x^2/a^2 + y^2/b^2 = 1$. This is the same as $\xi = \alpha$, if $a = c\cosh\alpha$, $b = c\sinh\alpha$ and $c^2 = a^2 - b^2$. The elliptic co-ordinates are

$$x = c\cosh\alpha\cos\eta, \qquad y = c\sinh\alpha\sin\eta \qquad ...(2)$$

Here $U = V = 0$. So using (2), (1) gives

$$\psi = (1/2) \times \omega c^2(\cosh^2\alpha\cos^2\eta + \sinh^2\alpha\sin^2\eta) + A$$

or
$$\psi = (1/4) \times \omega c^2(\cosh 2\alpha + \cos 2\eta) + A \qquad ...(3)$$

Since ψ contains $\cos 2\eta$ and the liquid is at rest at infinity, ψ must be taken as

$$\psi = Be^{-2\xi}\cos 2\eta \qquad ....(4)$$

and hence
$$\phi = Be^{-2\xi}\sin 2\eta \qquad ...(5)$$

Then at boundary $\xi = \alpha$, we must have

$$Be^{-2\alpha}\cos 2\eta = (1/4) \times \omega c^2(\cosh 2\alpha + \cos 2\eta) + A$$

for all values of η. This gives

$$Be^{-2\alpha} = (1/4) \times \omega c^2 \qquad \text{and} \qquad 0 = (1/4) \times \omega c^2\cosh 2\alpha + A$$

$$\therefore \quad B = (1/4) \times \omega c^2 e^{2\alpha} \qquad \text{and} \qquad A = (-1/4) \times \omega c^2\cosh 2\alpha$$

(4) and (5) reduce to
$$\psi = (1/4) \times \omega c^2 e^{-2\alpha} e^{-2\xi}\cos 2\eta \qquad ...(6)$$

$$= (1/4) \times \omega(a^2 - b^2)\frac{a+b}{a-b}e^{-2\xi}\cos 2\eta, \text{ using Art 7.13}$$

$$= (1/4) \times \omega\,(a+b)^2 e^{-2\xi}\cos 2\eta \qquad ...(7)$$

and
$$\phi = (1/4) \times \omega c^2 e^{2\alpha} e^{-2\xi}\sin 2\eta \qquad ...(8)$$

$$= (1/4) \times \omega(a+b)^2 e^{-2\xi}\sin 2\eta, \text{ as before} \qquad ...(9)$$

$$\therefore \qquad w = \phi + i\psi = (1/4) \times i\omega\,(a+b)^2 e^{-2(\xi + i\eta)}$$

or
$$\omega = (1/4) \times i\omega(a+b)^2 e^{-2\zeta} \qquad ...(10)$$

7.17. Kinetic energy (K.E.) for elliptic cylinder.

In what follows, we use formula (3) explained in Art. 7.2.

(I) *Kinetic energy T when the elliptic cylinder moves with velocity U parallel to x-axis.*

For an elliptic cylinder, $\xi = \alpha$, moving with velocity U parallel to x-axis, we have [from

Art. 7.14 (I)]. $\phi = Ub\left(\dfrac{a+b}{a-b}\right)^{1/2} e^{-\alpha}\cos\eta = Ub\cos\eta,$ as $e^{\alpha} = \left(\dfrac{a+b}{a-b}\right)^{1/2}$

and $\psi = -Ub\left(\dfrac{a+b}{a-b}\right)^{1/2} e^{-\alpha}\sin\eta = -Ub\sin\eta,$

where η varies from 0 to 2π for one complete round of the perimeter of the cross-section of the ellipse.

$$\therefore \qquad T = -\frac{1}{2}\rho\int\phi d\psi = \frac{1}{2}\rho U^2 b^2\int_0^{2\pi}\cos^2\eta d\eta = \frac{1}{2}\rho U^2 b^2\pi.$$

(II) *Kinetic energy T when the elliptic cylinder moves with velocity V parallel to y-axis.*

As in (I), $T = (1/2)\times\rho V^2 a^2\pi$

(III) *Kinetic energy T when the elliptic cylinder moves with velocity v in a direction making an angle* θ *with the major axis.*

As in (II), $T = (1/2)\times\pi\rho v^2(b^2\cos^2\theta + a^2\sin^2\theta)$

(IV) *Kinetic energy when the elliptic cylinder rotates in an infinite mass of liquid at rest at infinity.* **[Kanpur 2000, 04, 07]**

For an elliptic cylinder, $\xi = \alpha$, rotation with angular velocity ω, we have [Refer Art 7.16]

$$\phi = (1/4)\times\omega\,(a+b)^2 e^{-2\alpha}\sin 2\eta \qquad \text{and} \qquad \psi = (1/4)\times\omega\,(a+b)^2 e^{-2\alpha}\cos 2\eta,$$

where η varies from 0 to 2π for one complete round of the perimeter of the cross-section of the ellipse.

$$\therefore \quad T = -\frac{1}{2}\rho\int\phi d\psi = \frac{1}{16}\rho\omega^2(a+b)^4 e^{-4\alpha}\int_0^{2\pi}\sin^2 2\eta\, d\eta = \frac{1}{16}\pi\rho\omega^2(a+b)^4 e^{-4\alpha}$$

$$= \frac{1}{16}\pi\rho\omega^2(a^2+b^2)^2, \qquad \text{as} \qquad e^{\alpha} = \left(\frac{a+b}{a-b}\right)^{1/2}$$

7.18. Motion of a liquid in rotating elliptic cylinders.

Let the elliptic cylinder containing liquid rotate with angular velocity ω. The stream function ψ must satisfy the Laplace's equation $\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0$ and on the boundary it satisfies the condition

$$\psi = (1/2)\times\omega\,(x^2+y^2) + A \qquad \text{...(1)}$$

We, therefore, assume that

$$\psi = B\,(x^2 - y^2) \qquad \text{...(2)}$$

On the boundary of the cylinder, we must have

$$B\,(x^2-y^2) = (1/2)\times\omega\,(x^2+y^2) + A \qquad \text{or} \qquad (B-\omega/2)x^2 - (B+\omega/2)y^2 = A$$

or

$$\frac{x^2}{A/(B-\omega/2)} + \frac{y^2}{A/(-B-\omega/2)} = 1 \qquad \text{...(3)}$$

Let the boundary of the cylinder be $x^2/a^2 + y^2/b^2 = 1$. Then comparing it with (3), we have

$$\frac{A/(B-\omega/2)}{a^2} = \frac{A/(-B-\omega/2)}{b^2} \qquad \text{so that} \qquad B = \frac{1}{2}\omega\frac{a^2-b^2}{a^2+b^2}.$$

$$\therefore \qquad \psi = \frac{1}{2}\omega\frac{a^2-b^2}{a^2+b^2}(x^2-y^2) \qquad \text{...(4)}$$

The form (2) for ψ, suggests that we must take

$$\phi = -2Bxy \qquad \text{so that} \qquad \phi = -w\frac{a^2-b^2}{a^2+b^2}xy \qquad ...(5)$$

The velocity q is given by

$$q^2 = (-\partial\phi/\partial x)^2 + (-\partial\phi/\partial y)^2 = \omega^2\left(\frac{a^2-b^2}{a^2+b^2}\right)(x^2+y^2), \text{ using (5)}$$

K.E. of the liquid contained in rotating cylinder is given by

$$T = \frac{1}{2}\rho\iint q^2 dxdy = \frac{1}{2}\rho\omega^2\left(\frac{a^2-b^2}{a^2+b^2}\right)^2 \iint (x^2+y^2)\,dxdy$$

$$= \frac{1}{2}\rho\omega^2\left(\frac{a^2-b^2}{a^2+b^2}\right)^2\left[\iint x^2dxdy + \iint y^2\,dxdy\right] \qquad ...(6)$$

But $\iint x^2dxdy$ = Moment of inertia of the ellipse $x^2/a^2 + y^2/b^2 = 1$ about its minor axis

$$= (1/4)\times Mb^2 = (1/4)\times(\pi ab)\times b^2 = (1/4)\times \pi ab^3 \qquad ...(7)$$

Similarly, $$\iint y^2\,dxdy = (1/4)\times \pi ba^3 \qquad ...(8)$$

Using (7) and (8), (6) reduces to

$$T = \frac{1}{2}\rho\omega^2\left(\frac{a^2-b^2}{a^2+b^2}\right)^2\cdot\frac{1}{4}\pi ab\,(a^2+b^2) = \frac{1}{8}\pi\rho ab\omega^2\frac{(a^2-b^2)^2}{a^2+b^2}.$$

Remark. To determine kinetic energy T if the liquid were rotating with the boundary as one rigid mass with angular velocity ω, we use the formula

$$T = \frac{1}{2}Mk^2\omega^2 = \frac{1}{2}(\pi ab\rho)\frac{a^2+b^2}{4}\omega^2 = \frac{1}{8}\pi ab\rho\omega^2(a^2+b^2).$$

Here M = mass of the elliptic section = $\pi ab\rho$

and k^2 = the square of the radius of gyration = $(a^2+b^2)/4$.

7.19. Circulation about an elliptic cylinder.

In Art. 7.14 (III), the irrotational motion is cyclic, with circulation k round the cylinder, we can take it into account by taking

$$w = \phi + i\psi = \frac{ik}{2\pi}(\xi + i\eta) = \frac{ik\zeta}{2\pi} \qquad ...(1)$$

To verify the above choice of w, we find that the circulation taken round the cylinder

$$= \int\left(-\frac{\partial\phi}{\partial s}\right)ds = \int_0^{2\pi}\left(-\frac{\partial\phi}{\partial\eta}\right)d\eta = \int_0^{2\pi}\frac{k}{2\pi}d\eta = k, \quad \text{as from (1)}, \quad \phi = -\frac{k\eta}{2\pi}$$

7.20. Illustrative solved examples.

Ex. 1. *Liquid of density* ρ *is circulating irrotationally between two confocal ellipses* $\xi = \alpha$, $\xi = \beta$ *where* $x + iy = c \cosh\,(\xi + i\eta)$. *Prove that if* k *is the circulation, the kinetic energy per unit length of the cylinder is* $(1/4)\times\rho k^2[(\beta-\alpha)/\pi]$.

Sol. For irrotational cyclic motion of circulation k round the elliptic cylinder, we have from Art 7.19

$$w = \phi + i\psi = \frac{ik}{2\pi}(\xi + i\eta) \qquad \text{so that} \qquad \phi = -\frac{k\eta}{2\pi} \qquad \text{and} \qquad \psi = \frac{k\xi}{2\pi}$$

$$\therefore \qquad \text{K.E.} = \frac{1}{2}\rho\iint\left\{\left(\frac{\partial\phi}{\partial\xi}\right)^2 + \left(\frac{\partial\phi}{\partial\eta}\right)^2\right\}d\xi\, d\eta = \frac{1}{2}\rho\cdot\frac{k^2}{4\pi^2}\int_{\eta=0}^{2\pi}\int_{\xi=\alpha}^{\beta} d\xi d\eta = \frac{k^2\rho}{4\pi}(\beta - \alpha).$$

Ex. 2. *Prove that when an infinitely long cylinder of density* σ *whose cross-section is an ellipse of semi-axes a, b is immersed in an infinite liquid of density* ρ*, the square of its radius of gyration about its axis is effectively increased by* $\rho(a^2 - b^2)^2/8\sigma ab$.

[Garhwal 2003; Kanpur 2000, 2006; Rohilkhand 2000, 05]

Sol. The kinetic energy T_1 of an elliptic cylinder rotating about its axis with angular velocity ω is given by

$$T_1 = \frac{1}{2}Mk^2\omega^2 = \frac{1}{2}\pi\sigma ab\frac{a^2+b^2}{4}\omega^2$$

When the elliptic cylinder is immersed in an infinite liquid of density ρ, then the kinetic energy of the liquid [as in Art 7.17 (IV)] is given by

$$T_2 = \frac{1}{16}\pi\rho\omega^2(a^2 - b^2)^2$$

$$\therefore \qquad \text{Total K.E.} = T_1 + T_2 = (1/8)\times\pi\sigma ab(a^2+b^2)\omega^2 + (1/16)\times\pi\rho\omega^2(a^2-b^2)^2$$

Let K' be the effective increase in the radius of gyration, then we have

$$\frac{1}{2}\pi\sigma ab\left[\frac{a^2+b^2}{4} + K'^2\right]\omega^2 = \frac{1}{8}\pi\sigma ab(a^2+b^2)\,\omega^2 + \frac{1}{16}\pi\rho\omega^2(a^2-b^2)^2$$

so that

$$K'^2 = \frac{\rho}{8\sigma}\frac{(a^2-b^2)^2}{ab}$$

Ex. 3. *If an elliptic cylinder of semi-axes a, b filled with a liquid, rotates with a uniform velocity about its axes; show that the kinetic energy of liquid contained is less than if it were moving as solid in the ratio* $(a^2 - b)^2 : (a^2 + b^2)^2$. **[Garhwal 2005]**

Sol. Kinetic energy T_1 of the liquid contained in the cylinder [as in Art 7.18] is given by

$$T_1 = \frac{1}{8}\pi\rho ab\omega^2\frac{(a^2-b^2)^2}{a^2+b^2}$$

Kinetic energy T_2 of the liquid when moving as a sold (as in Art 7.18) is given by

$$T_2 = (1/8)\times\pi ab\rho\omega^2(a^2+b^2)$$

$$\therefore \qquad T_1 : T_2 = (a^2 - b)^2 : (a^2 + b^2)^2.$$

Ex. 4. *A thin shell in the form of an infinite long elliptic cylinder, semi axes a and b, is rotating about its axes in an infinite liquid otherwise at rest. It is filled with the same liquid. Prove that the ratio of the kinetic energy of the liquid inside to that of that of the liquid outside is* $2ab : (a^2 + b^2)$. **[Garakhpur 2001, 05; Rohilkhand 2003, 04]**

Sol. K.E. T_1 of the liquid inside the elliptic boundary [as in Art 7.18] is given by

$$T_1 = \frac{1}{8}\pi\rho ab\omega^2\frac{(a^2-b^2)^2}{a^2+b^2}$$

Again the K.E. T_2 of the liquid outside the boundary [as in Art. 7.17 (IV)] is given by

$$T_2 = (1/16) \times \pi\rho w^2 (a^2 - b^2)^2$$

$$\therefore \qquad T_1 : T_2 = 2ab : (a^2 + b^2).$$

Ex. 5. *An infinite elliptic cylinder with semi axes a, b is rotating round its axes with angular velocity* ω *in an infinite liquid of density* ρ *which is at rest at infinity. Show that if the fluid is under the action of no force, the moment of the fluid pressure on the cylinder round the centre is*

$(1/8) \times \pi\rho c^4 \times (d\omega/dt)$ where $c^2 = a^2 - b^2$. **[Garhwal 2000, 02]**

Sol. Using Bernoulli's equation, pressure p at any point is given by

$$p/\rho = C - q^2/2 + \partial\phi/\partial t \qquad \text{...(1)}$$

Now for an elliptic cylinder rotating with an angular velocity ω in an infinite fluid velocity potential ϕ and complex potential w (as in Art 7.16) are given by

$$\phi = (1/4) \times \omega (a+b)^2 e^{-2\xi} \sin 2\eta \qquad \text{...(2)}$$

and
$$w = (1/4) \times i\omega (a+b)^2 e^{-2\zeta}, \qquad \text{...(3)}$$

where
$$z = x + iy = c\cosh\zeta \quad \text{and} \quad \zeta = \xi + i\eta \qquad \text{...(4)}$$

$$\therefore \qquad q^2 = \left|\frac{dw}{dz}\right|^2 = \left|\frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz}\right|^2 = \left|\frac{1}{4} i\omega (a+b)^2 e^{-2\zeta} \times (-2)\frac{1}{c\sinh\zeta}\right|^2$$

$$= \frac{\omega^2 (a+b)^4}{4c^2}\left|\frac{e^{-2\xi}\cdot e^{-2i\eta}}{\sinh(\xi + i\eta)}\right|^2, \quad \text{as} \quad \zeta = \xi + i\eta$$

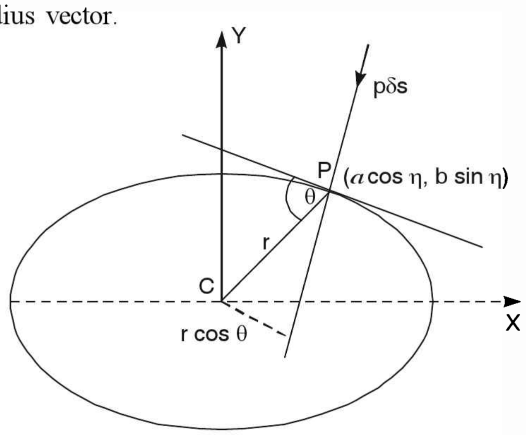
$$= \frac{\omega^2 (a+b)^4 e^{-4\xi}}{4c^2} \times \frac{1}{\sinh^2\xi + \sin^2\eta} \qquad \text{...(5)}$$

and
$$\frac{\partial\phi}{\partial t} = \frac{1}{4}(a+b)^4 e^{-2\xi} \sin 2\eta \frac{d\omega}{dt} \qquad \text{...(6)}$$

Using (1), (5) and (6), the pressure at any point on the boundary of the ellipse $\xi = \alpha$ is given by

$$\frac{p}{\rho} = C - \frac{\omega^2 (a+b)^2 e^{-4\alpha}}{8c^2 (\sinh^2\alpha + \sin^2\eta)} + \frac{1}{4}(a+b)^2 e^{-2\alpha} \sin 2\eta \frac{d\omega}{dt} \qquad \text{...(7)}$$

Now pressure on an elementary arc ds of elliptic boundary at a point P (of eccentric angle η) is $p\, ds$. Let θ be the angle between tangent and radius vector.



Then from calculus, we have

$$\cos\theta = dr/ds \qquad \text{...(8)}$$

Now the moment of the fluid pressure on the element ds about the centre

$= -\,p\,ds\,r\cos\theta = -\,pr\,dr$, using (8)

$= (p/2) \times (a^2 - b^2) \sin 2\eta\, d\eta$

$$\left[\begin{array}{ll} \text{since,} & r^2 = a^2\cos^2\eta + b^2\sin^2\eta \\ \Rightarrow & rdr = -\dfrac{1}{2}(a^2 - b^2)\sin 2\eta\, d\eta \end{array}\right]$$

$\therefore$ The required total moment of the fluid pressure on the elliptic cylinder about the centre is

$$= \frac{a^2 - b^2}{2} \int_0^{2\pi} p \sin 2\eta \, d\eta$$

$$= \frac{a^2 - b^2}{2} \rho \int_0^{2\pi} \left[C - \frac{\omega^2 (a+b)^4}{8c^2} \frac{e^{-4\alpha}}{\sinh^2 \alpha + \sin^2 \eta} + \frac{(a+b)^2}{4} e^{-2\alpha} \sin 2\eta \frac{d\omega}{dt} \right] \sin 2\eta \, d\eta$$

$$= \frac{a^2 - b^2}{2} \rho \int_0^{2\pi} \frac{(a+b)^2}{4} e^{-2\alpha} \sin^2 2\eta \frac{d\omega}{dt} d\eta, \quad \text{since other integrals vanish}$$

$$= \frac{(a^2 - b^2)(a+b)^2 e^{-2\alpha}}{8} \rho \frac{d\omega}{dt} \int_0^{2\pi} \sin^2 2\eta \, d\eta = \frac{c^2 (a+b)^2}{8} \frac{a-b}{a+b} \rho \frac{d\omega}{dt} \int_0^{2\pi} \frac{1 - \cos 4\eta}{2} d\eta$$

$$\left[\because \ c^2 = a^2 - b^2 \text{ and } e^{2\alpha} = (a+b)/(a-b) \right]$$

$$= \frac{c^2 (a^2 - b^2)}{8} \rho \frac{d\omega}{dt} \pi = \frac{1}{8} \pi \rho c^4 \frac{d\omega}{dt}$$

Ex. 6. *Prove that if* $2a$, $2b$ *are axes of the cross-section of an elliptic cylinder placed across a stream in which the velocity at infinity is U parallel to the major axis of the cross-section, the velocity at a point* $(a \cos \eta, b \sin \eta)$ *on the surface is*

$$\frac{U(a+b) \sin \eta}{(b^2 \cos^2 \eta + a^2 \sin^2 \eta)^{1/2}}$$

and that, in consequence of the motion, the resultant thrust per unit length on that half cylinder on which the stream impinges is diminished by

$$\frac{2b^2 \rho U^2}{a-b} \left[1 - \left(\frac{a+b}{a-b} \right)^{1/2} \tan^{-1} \left(\frac{a-b}{a+b} \right)^{1/2} \right], \text{ where } \rho \text{ is the density of the liquid.}$$

Sol. The velocity potential ϕ for the given motion (as in Art 7.15) is given by

$$\phi = U(a+b) \cos \eta \cosh (\xi - \alpha) \qquad \text{...(1)}$$

We know that the velocity q is given by

$$q^2 = h^2 \left\{ (\partial \phi / \partial \xi)^2 + (\partial \phi / \partial \eta)^2 \right\}^{1/2} \qquad \text{...(2)}$$

where $$1/h^2 = (\partial x / \partial \xi)^2 + (\partial x / \partial \eta)^2 \qquad \text{...(3)}$$

and $$x = c \cosh \xi \cos \eta$$

$$\therefore \quad 1/h^2 = c^2 (\sinh^2 \xi \cos^2 \eta + \cosh^2 \xi \sin^2 \eta) = c^2 (\sinh^2 \xi + \sin^2 \eta)$$

$$\therefore \quad q^2 = \frac{U^2 (a+b)^2}{c^2} \frac{\cos^2 \eta \sinh^2 (\xi - \alpha) + \sin^2 \eta \cosh^2 (\xi - \alpha)}{\sinh^2 \xi + \sin^2 \eta} = \frac{U^2 (a+b)^2}{c^2} \frac{\sinh^2 (\xi - \alpha) + \sin^2 \eta}{\sinh^2 \xi + \sin^2 \eta}$$

$$= U^2 \frac{a+b}{a-b} \frac{\sinh^2 (\xi - \alpha) + \sin^2 \eta}{\sinh^2 \xi + \sin^2 \eta}, \qquad \text{as} \quad c^2 = a^2 - b^2 \qquad \text{...(5)}$$

Now the point $(a \cos \eta, b \sin \eta)$ lies on the ellipse $\xi = \alpha$ where

$$a = c \cosh \alpha, \qquad b = c \sinh \alpha, \qquad c^2 = a^2 - b^2 \qquad \text{...(6)}$$

$\therefore$ Putting $\xi = \alpha$ in (5), the velocity at any point $(a\cos\eta, b\sin\eta)$ of the ellipse is given by

$$q^2 = U^2 \frac{a+b}{a-b}\frac{\sin^2\eta}{\sinh^2\alpha + \sin^2\eta} = U^2 \frac{(a+b)^2}{a^2-b^2}\frac{\sin^2\eta}{\sinh^2\alpha+\sin^2\eta}$$

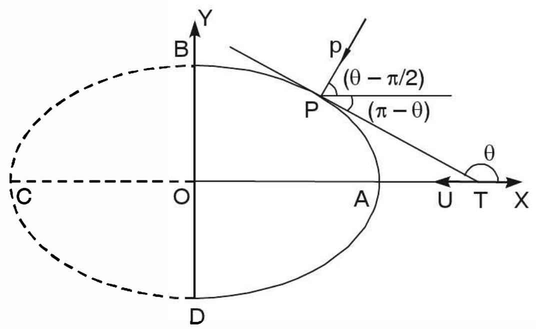
$$= U^2\frac{(a+b)^2}{c^2}\frac{\sin^2\eta}{\sinh^2\alpha+\sin^2\eta} = \frac{U^2(a+b)^2\sin^2\eta}{c^2\sinh^2\alpha + c^2\sin^2\eta} = \frac{U^2(a+b)^2\sin^2\eta}{b^2+(a^2-b^2)\sin^2\eta}, \text{ by (6)}$$

Thus, $$q^2 = \frac{U^2(a+b)^2\sin^2\eta}{b^2\cos^2\eta + a^2\sin^2\eta} \qquad ...(7)$$

Let Π pressure at infinity and $-U$ be the velocity at infinity parallel to the major axis. Then from Bernoulli's equation, we have

$$\frac{p}{\rho} + \frac{1}{2}q^2 = \frac{\Pi}{\rho} + \frac{1}{2}U^2 \qquad \text{so that} \qquad \Pi - p = \frac{1}{2}\rho(q^2 - U^2) \qquad ...(8)$$

Since Π would have been pressure, had there been no motion, the diminution in pressure is $\Pi - p$. Let θ be the angle which the tangent PT to the element δs of the surface makes with major axis. Then the total diminution in the resultant thrust on the half cylinder BAD on which the stream impinges



$$= \int (\Pi - p)\cos(\theta - \pi/2)\,ds = \int (\pi - p)\sin\theta\,ds$$

$$= \int (\Pi - p)\,dy, \text{ as } \sin\theta = dy/ds$$

$$= \int (\Pi - p)\,b\cos\eta\,d\eta, \text{ as } y = b\sin\eta$$

$$= \frac{1}{2}\rho U^2 \int_{-\pi/2}^{\pi/2}\left[\frac{(a+b)^2\sin^2\eta}{b^2\cos^2\eta + a^2\sin^2\eta} - 1\right] b\cos\eta\,d\eta, \text{ using (7) and (8)}$$

$$= \rho b U^2 \int_0^{\pi/2}\left[\frac{(a+b)^2\sin^2\eta}{b^2(1-\sin^2\eta)+a^2\sin^2\eta} - 1\right]\cos\eta\,d\eta = \rho b U^2\int_0^1\left[\frac{(a+b)^2t^2}{b^2(1-t^2)+a^2t^2} - 1\right]dt$$

[Putting $\sin\eta = t$ and $\cos\eta\,d\eta = dt$]

$$= \rho b U^2\int_0^1\left[\frac{2b}{a-b} - \frac{b^2(a+b)}{a-b}\cdot\frac{1}{b^2+(a^2-b^2)t^2}\right]dt = \frac{2\rho U^2 b^2}{a-b}\left[t - \frac{a+b}{2(a^2-b^2)^{1/2}}\tan^{-1}\frac{t\sqrt{(a^2+b^2)}}{b}\right]_0^1$$

$$= \frac{2\rho U^2 b^2}{a-b}\left[1 - \frac{1}{2}\left(\frac{a+b}{a-b}\right)^{1/2}\tan^{-1}\frac{\sqrt{(a^2+b^2)}}{b}\right] = \frac{2\rho U^2 b^2}{a-b}\left[1 - \left(\frac{a+b}{a-b}\right)^{1/2}\tan^{-1}\left(\frac{a-b}{a+b}\right)^{1/2}\right]$$

$$\left[\because \frac{1}{2}\tan^{-1}\frac{\sqrt{(a^2-b^2)}}{b} = \tan^{-1}\left(\frac{a-b}{a+b}\right)^{1/2}\right]$$

Ex. 7. *An elliptic cylinder whose semi-axes are* $c\cosh\alpha$, $c\sinh\alpha$ *is divided in two by a plane through the axis of the cylinder and the major axis of its cross-section. An infinite liquid of density* ρ *streams past the cylinder, its velocity U at infinity being uniform and parallel to the major axis of the cross-section of the cylinder. Show that in consequence of the motion of the liquid the pressure between the two portions of the cylinder is diminished by*

$\rho c U^2 e^{\alpha} \sinh\alpha\,[2\cosh\alpha + e^{\alpha}\sinh\alpha \log\tanh(\alpha/2)]$ *per unit length of the cylinder.*

Sol. This example differs from Ex. 6 in one way only. Here the diminution of the pressure is to be calculated on *ABC* (and not on *BAD*).

So, here the required total diminution of the liquid pressure

$$= \int (\Pi - p)\sin(\theta - \pi/2)\,ds = -\int (\Pi - p)\,dx, \qquad \text{as} \qquad \cos\theta = dx/ds$$

$$= -\int (\Pi - p)(-a\sin\eta\,d\eta), \qquad \text{as} \qquad x = a\cos\eta$$

$$= \frac{1}{2}\rho a U^2 \int_0^{\pi} \left[\frac{(a+b)^2 \sin^2\eta}{a^2\sin^2\eta + b^2\cos^2\eta} - 1\right]\sin\eta\,d\eta$$

$$= \frac{1}{2}\rho a U^2 \times 2\int_0^{\pi/2} \left[\frac{(a+b)^2(1-\cos^2\eta)}{a^2(1-\cos^2\eta) + b^2\cos^2\eta} - 1\right]\sin\eta\,d\eta$$

$$= \rho a U^2 \int_1^0 \left[\frac{(a+b)^2(1-t^2)}{a^2(1-t^2)+b^2t^2} - 1\right](-dt), \text{ putting } \cos\eta = t \text{ and } \sin\eta\,d\eta = -dt$$

$$= \rho a U^2 \int_0^1 \left[\frac{2b}{a-b} - \frac{b^2(a+b)}{a-b}\frac{1}{a^2-(a^2-b^2)t^2}\right]dt$$

$$= \frac{\rho a b U^2}{a-b}\left[2t - \frac{b(a+b)}{2a\sqrt{(a^2-b^2)}}\log\frac{a+\sqrt{(a^2-b^2)}\,t}{a-\sqrt{(a^2-b^2)}\,t}\right]_0^1$$

$$= \frac{\rho a b U^2}{a-b}\left[2 - \frac{b(a+b)}{2a\sqrt{(a^2-b^2)}}\log\frac{a+\sqrt{(a^2-b^2)}}{a-\sqrt{(a^2-b^2)}}\right]$$

$$= \frac{\rho U^2 c^2 \sinh\alpha\cosh\alpha}{ce^{-\alpha}}\left[2 - \frac{c\sinh\alpha}{2c\cosh\alpha}\left(\frac{e^{\alpha}}{e^{-\alpha}}\right)^{1/2}\log\frac{\cosh\alpha+1}{\cosh\alpha-1}\right]$$

$$= \rho c U^2 e^{\alpha}\sinh\alpha[2\cosh\alpha + e^{\alpha}\sin\alpha\log\tanh(\alpha/2)], \qquad \text{as} \quad a = c\cosh\alpha, \quad b = c\sinh\alpha$$

Ex. 8. *If the liquid is contained between the elliptic cylinders* $x^2/a^2 + y^2/b^2 = 1$ *and* $x^2/a^2 + y^2/b^2 = k^2$ *and the whole rotates about OZ with angular velocity* ω, *prove that the velocity potential* ϕ *referred to the axes OX, OY is given by*

$$\phi = -\omega\frac{a^2-b^2}{a^2+b^2}xy$$

and that the surface of equal pressure are the hyperbolic cylinders

$$\frac{x^2}{3a^2+b^2} - \frac{y^2}{a^2+3b^2} = K.$$

Sol. As in Art 7.18, we have
$$\phi = -\omega \frac{a^2 - b^2}{a^2 + b^2} xy \qquad \text{...(1)}$$

When the axes are rotating, the pressure is given (in two-dimensions) by the equation*
$$\int \frac{dp}{\rho} - \frac{\partial \phi}{\partial t} + \frac{1}{2} q^2 + \omega \left(x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} \right) + V = F(t) \qquad \text{...(2)}$$

Here the motion is steady, *i.e.* $\partial \phi / \partial t = 0$ and $V = 0$. Since the fluid is incompressible, (2) reduces to
$$\frac{p}{\rho} + \frac{1}{2} q^2 + \omega \left(x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} \right) = C \qquad \text{...(3)}$$

or
$$\frac{p}{\rho} + \frac{1}{2} \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right] + \omega \left(x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} \right) = C$$

or
$$\frac{p}{\rho} + \left(\frac{a^2 - b^2}{a^2 + b^2} \right)^2 \omega^2 (x^2 + y^2) + \omega^2 \left(\frac{a^2 - b^2}{a^2 + b^2} \right) (-x^2 + y^2) = C$$

Putting ρ = const. in the above equation, the required surfaces of equal pressure are given by
$$\frac{a^2 - b^2}{a^2 + b^2} (x^2 + y^2) + 2 (y^2 - x^2) = \text{const.} \quad \text{or} \quad x^2 \left(\frac{a^2 + 3b^2}{a^2 + b^2} \right) - y^2 \left(\frac{3a^2 + b^2}{a^2 + b^2} \right) = \text{const.}$$

or
$$\frac{x^2}{3a^2 + b^2} - \frac{y^2}{a^2 + 3b^2} = \text{const.} \ (= K, \text{say}).$$

Ex. 9. *Liquid is contained in a rotating elliptic cylinder, show that the stream function of the motion is*
$$\psi = \frac{1}{2} \omega \frac{a^2 - b^2}{a^2 + b^2} (x^2 - y^2).$$

Hence prove that the paths of the particles are similar ellipses described in time $\pi (a^2 + b^2) / \omega ab$.

OR

An elliptic cylinder is filled with liquid which has molecular rotation ω *at every point and whose particles move in planes perpendicular to the axis. Prove that the streamlines are similar ellipses described in periodic time* $\pi (a^2 + b^2) / \omega ab$. **[Kanpur 2006]**

Sol. For the value of ψ, refer Art 7.18. Again, the velocity potential ϕ is given by
$$\phi = -\omega \frac{a^2 - b^2}{a^2 + b^2} xy \qquad \text{...(1)}$$

We now determine the paths of the particles. Let (x, y) be the co-ordinates of a point. Now $(-\partial \phi / \partial x)$ is the velocity of the fluid particle parallel to x-axis. Then, we have
$$-\partial \phi / \partial x = \dot{x} - \omega y \qquad \text{...(2)}$$

Similarly,
$$-\partial \phi / \partial y = \dot{y} + \omega x \qquad \text{...(3)}$$

* Refer Art. 2.41 page 24 in Hydromechanics, part II, by Besant and Ramsey

From (1), $\dfrac{\partial\phi}{\partial x} = -\omega\dfrac{a^2-b^2}{a^2+b^2}y$ and $\dfrac{\partial\phi}{\partial y} = -\omega\dfrac{a^2-b^2}{a^2+b^2}x$ ...(4)

From (2), $\dot{x} = \omega y - \dfrac{\partial\phi}{\partial x} = \omega y\left[1+\dfrac{a^2-b^2}{a^2+b^2}\right]$, by (4)

$\therefore$ $$\dot{x} = \frac{2a^2}{a^2+b^2}\omega y \qquad ...(5)$$

Similarly, (3) and (4) give $$\dot{y} = -\frac{2a^2}{a^2+b^2}\omega x \qquad ...(6)$$

From (5), $$\ddot{x} = \frac{2a^2}{a^2+b^2}\omega\dot{y}$$

or $$\ddot{x} = -\frac{4a^2b^2}{(a^2+b^2)^2}\omega^2 x, \text{ using (6)} \qquad ...(7)$$

Equation (7) represents a simple harmonic motion whose periodic time is given by

$$2\pi\Big/\left\{\frac{2ab\omega}{a^2+b^2}\right\} \qquad i.e., \qquad \frac{\pi(a^2+b^2)}{abw}$$

Then general solution of (7) is $$x = A\cos\left(\frac{2ab\omega}{a^2+b^2}t+B\right) \qquad ...(8)$$

From (8), $$\dot{x} = -A\frac{2ab\omega}{a^2+b^2}\sin\left(\frac{2ab\omega}{a^2+b^2}t+B\right)$$

or $$\frac{2a^2}{a^2+b^2}\omega y = -\frac{2ab\omega A}{a^2+b^2}\sin\left(\frac{2ab\omega}{a^2+b^2}t+B\right), \text{ by (5)}$$

or $$\frac{ay}{b} = -A\sin\left(\frac{2ab\omega}{a^2+b^2}t+B\right) \qquad ...(9)$$

Squaring and adding (8) and (9), we have

$$x^2+\frac{a^2y^2}{b^2} = A^2 \qquad \text{or} \qquad \frac{x^2}{A^2}+\frac{y^2}{(bA/a)^2} = 1,$$

showing that the paths of the particles are similar ellipses.

Ex. 10. *In the two-dimensional irrotational motion of a liquid streaming past a fixed elliptic disc $x^2/a^2 + y^2/b^2 = 1$, the velocity at infinity being parallel to the major axis and equal to U, prove that if* $x+iy = c\cosh(\xi+i\eta)$ $a^2 - b^2 = c^2$, $a = c\cosh\alpha$, $b = c\sinh\alpha$,

the velocity at any point is given by $$q^2 = U^2\frac{a+b}{a-b}\cdot\frac{\sinh^2(\xi-\alpha)+\sin^2\eta}{\sinh^2\xi+\sin^2\eta}$$

and that it has maximum value $[U(a+b)]/a$ at the end of the minor axis.

Sol. As in Art. 7.15, the complex potential is given by

$$w = U(a+b)\cosh(\zeta-\alpha) \qquad ...(1)$$

Now, $$q = \left|\frac{dw}{dz}\right| = \left|\frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz}\right|$$

$$\therefore \qquad q = \frac{U(a+b)}{c}\left|\frac{\sinh(\zeta-\alpha)}{\sinh\zeta}\right|, \text{ using (1) and relation } z = c\cosh\zeta \qquad ...(2)$$

But $|\sinh(\zeta-\alpha)| = |\sinh(\xi-\alpha+i\eta)|$, as $\zeta=\xi+i\eta$

$$= |\sinh(\xi-\alpha)\cos\eta + i\cosh(\xi-\alpha)\sin\eta|$$

$$= \sqrt{[\sinh^2(\xi-\alpha)\cos^2\eta+\cosh^2(\xi-\alpha)\sin^2\eta]} = \sqrt{[\sinh^2(\xi-\alpha)+\sin^2\eta]}$$

Similarly, $$|\sinh\zeta| = \sqrt{(\sinh^2\xi+\sin^2\eta)}$$

$$\therefore \qquad q = \frac{U(a+b)}{\sqrt{(a^2-b^2)}}\left[\frac{\sinh^2(\xi-\alpha)+\sin^2\eta}{\sinh^2\xi+\sin^2\eta}\right]^{1/2}, \text{ using (2) and relation } c=\sqrt{(a^2-b^2)}$$

or $$q^2 = U^2\left(\frac{a+b}{a-b}\right)\frac{\sinh^2(\xi-\alpha)+\sin^2\eta}{\sinh^2\xi+\sin^2\eta} \qquad ...(3)$$

(3) gives the required value of velocity.

To determine the maximum value of q, we re-write (3) as follows:

$$q^2 = U^2\left(\frac{a+b}{a-b}\right)\left[1-\frac{\sinh^2\xi-\sinh^2(\xi-\alpha)}{\sinh^2\xi+\sin^2\eta}\right] \qquad ...(4)$$

But $\sinh\xi > \sinh(\xi-\alpha)$. Hence for a given ξ, (4) shows that q will be maximum when $\sin\eta$ is maximum *i.e.* $\eta=\pi/2$. Then (3) gives

$$q^2 = U^2\left(\frac{a+b}{a-b}\right)\frac{1+\sinh^2(\xi-\alpha)}{1+\sinh^2\xi} = U^2\left(\frac{a+b}{a-b}\right)\frac{\cosh^2(\xi-\alpha)}{\cosh^2\xi} \qquad ...(5)$$

$$= U^2\left(\frac{a+b}{a-b}\right)\left[\frac{\cosh\xi\cosh\alpha-\sinh\xi\sin\alpha}{\cosh\xi}\right]^2 = U^2\left(\frac{a+b}{a-b}\right)(\cosh\alpha-\tanh\xi\sin\alpha)^2, \qquad ...(6)$$

showing that q will be maximum when $\tanh\xi$ is minimum *i.e.* ξ is minimum. Since we have an elliptic cylinder surrounded by liquid, the minimum value of ξ is α. Hence putting $\xi=\alpha$ in (5), the required maximum value of q is given by

$$(q_{\max})^2 = U^2\left(\frac{a+b}{a-b}\right)\frac{1}{\cosh^2\alpha} = U^2\left(\frac{a+b}{a-b}\right)\cdot\frac{c^2}{a^2}, \text{ as } a = c\cosh\alpha$$

$$= U^2\left(\frac{a+b}{a-b}\right)\cdot\frac{a^2-b^2}{a^2}, \text{ as } c^2 = a^2-b^2$$

Thus, $$q_{\max} = [U(a+b)]/a,$$

which occurs at $\xi=\alpha$, $\eta=\pi/2$, *i.e.*, at the end of the major axis.

Ex. 11. *An elliptic cylinder, the semi-axes of whose cross-sections are a and b, is moving with velocity U parallel to the major axis of the cross-section, through an infinite liquid of density* ρ *which is at rest at infinity, the pressure there being* Π. *Prove that in order that the pressure may everywhere be positive* $\rho U^2 < (2a^2\Pi)/(2ab+b^2)$. **[Kanpur 2011]**

Sol. In order to get rid of ϕ in the Bernoulli's equation for pressure, the motion is reduced to a steady motion by superimposing a velocity U, in the direction opposite to the motion of the cylinder, both on the cylinder and liquid. Note that this arrangement will not affect the pressure at any point. Moreover, the liquid now streams past a fixed elliptic cylinder with velocity U and hence the maximum value of velocity is [refer Ex.10] given by

$$q_{\max.} = [\mathrm{U}(a+b)]/a \qquad \text{...(1)}$$

For a steady motion, Bernoulli's equation is

$$P/\rho + q^2/2 = C \qquad \text{...(2)}$$

But at infinity, $p = \Pi$ and $q = -U$. Hence (2) gives $C = \Pi/\rho + U^2/2$ With this value of C, (2) reduces to $\quad p = \Pi + (1/2) \times \rho(U^2 - q^2) \qquad$ ...(3)

From (3), it follows that the pressure will be maximum everywhere,

if $\quad \Pi + \dfrac{1}{2}\rho(U^2 - q^2) > 0$ for each q $\quad$ or $\quad \Pi + \dfrac{1}{2}\rho U^2 > \dfrac{1}{2}\rho q_{\max}^2$.

or $\quad \Pi + \dfrac{1}{2}\rho U^2 > \dfrac{1}{2}\rho \dfrac{U^2(a+b)^2}{a^2}$ $\quad$ or $\quad \Pi > \dfrac{1}{2}\rho U^2 \left[\dfrac{(a+b)^2}{a^2} - 1\right]$

or $\quad \Pi > \dfrac{1}{2}\rho U^2 \cdot \dfrac{2ab + b^2}{a^2}$ $\quad$ *i.e.* $\quad \rho U^2 < \dfrac{2a^2\Pi}{2ab + b^2}$

Ex. 12. *Show that the motion of a liquid streaming past the elliptic disc $x^2/a^2 + y^2/b^2 = 1$, the velocity at infinity being parallel to the x-axis and equal to U, can be represented by the relation* $\phi + i\psi = [U\{az - b\sqrt{(z^2 - c^2)}\}]/(a-b)$

Sol. As in Art. 7.15, we have $\quad w = \phi + i\psi = \mathrm{U}z + \mathrm{U}be^{\alpha - \zeta}$,

$$\therefore \quad \phi + i\psi = \mathrm{U}z + \mathrm{U}b\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\zeta}, \quad \text{as} \quad e^{\alpha} = \left(\frac{a+b}{a-b}\right)^{1/2}$$

$$= U\left(\frac{a+b}{a-b}\right)^{1/2} \cdot \left[z\left(\frac{a-b}{a+b}\right)^{1/2} + be^{-\zeta}\right] = U\left(\frac{a+b}{a-b}\right)^{1/2}\left[z\frac{a-b}{\sqrt{a^2-b^2}} + b(\cosh\zeta - \sinh\zeta)\right]$$

$$= U\left(\frac{a+b}{a-b}\right)^{1/2}\left[z\frac{a-b}{\sqrt{a^2-b^2}} + b\left\{\cosh\zeta - \sqrt{(\cosh^2\zeta - 1)}\right\}\right]$$

$$= U\left(\frac{a+b}{a-b}\right)^{1/2}\left[z\frac{a-b}{c} + b\left\{\frac{z}{c} - \left(\frac{z^2}{c^2} - 1\right)^{1/2}\right\}\right], \quad \text{as } c^2 = a^2 - b^2 \text{ and } z = c\cosh\zeta$$

$$= \frac{U}{c}\left(\frac{a+b}{a-b}\right)^{1/2}[az - b\sqrt{(z^2 - c^2)}] = \frac{U}{\sqrt{(a^2 - b^2)}} \cdot \left(\frac{a+b}{a-b}\right)^{1/2}[az - b\sqrt{(z^2-c^2)}]$$

$$= [U\{az - b\sqrt{(z^2 - c^2)}\}/(a-b).$$

Ex. 13. *Show that with proper choice of units the motion of an infinite liquid produced by the motion of an elliptic cylinder parallel to one of its principal axes is given by the complex function* $w = e^{-\zeta}$ where $z = 2\cosh\zeta$ *Deduce the formulas* **[Kanpur 2010]**

$$x=\phi\left(1+\frac{1}{\phi^2+\psi^2}\right) \quad \text{and} \quad y=\psi\left(1-\frac{1}{\phi^2+\psi^2}\right).$$

Sol. For an elliptic cylinder moving parallel to major axis with a velocity U, we have as in Art. 7.14 (I)

$$\phi=Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}\cos\eta \quad \text{and} \quad \psi=-Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}\sin\eta$$

$$\therefore \qquad w=\phi+i\psi=Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\xi}(\cos\eta-i\sin\eta)$$

or
$$w=Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-(\xi+i\eta)}=Ub\left(\frac{a+b}{a-b}\right)^{1/2} e^{-\zeta} \qquad \text{...(1)}$$

Again $\qquad x+iy=c\cosh(\xi+i\eta) \qquad$ or $\qquad z=c\cosh\zeta \qquad$ ...(2)

Taking $Ub\left(\frac{a+b}{a-b}\right)^{1/2}=1$ and $c=2$, (1) and (2) give

$$w=e^{-\zeta} \quad \text{with} \quad z=2\cosh\zeta \qquad \text{...(3)}$$

From (3), $\qquad x+iy=(e^{\zeta}+e^{-\zeta})=w^{-1}+w,\ as\ w=e^{-\zeta}$

$$=\frac{1}{\phi+i\psi}+\phi+i\psi, \text{ as } w=\phi+i\psi$$

Thus, $\qquad x+iy=\dfrac{\phi-i\psi}{\phi^2+\psi^2}+\phi+i\psi$

Equation real and imaginary parts, we have

$$x=\phi\left(1+\frac{1}{\phi^2+\psi^2}\right) \quad \text{and} \quad y=\psi\left(1-\frac{1}{\phi^2+\psi^2}\right).$$

Ex. 14. *An elliptic cylinder, semi-axes a and b, is held with its length perpendicular to, and its major axis making an angle θ with the direction of a stream of velocity V. Prove that the magnitude of the couple per unit length on the cylinder due to the fluid pressure is $\pi\rho(a^2-b^2)V^2\sin\theta\cos\theta$ and determine its sense.* **[Kanpur 1999]**

Sol. As in Art. 7.15, the complex potential is given by

$$w=V(a+b)\cosh(\zeta-\zeta_0), \qquad \text{...(1)}$$

where $\qquad \zeta=\xi+i\eta \qquad$ and $\qquad \zeta_0=\alpha+i\theta \qquad$ ...(2)

Now, $\dfrac{dw}{dz}=\dfrac{dw}{d\zeta}\cdot\dfrac{d\zeta}{dz}=\dfrac{V(a+b)\sinh(\zeta-\zeta_0)}{c\sinh\zeta}$, as $z=c\cosh\zeta$

$$=\frac{V(a+b)}{c}\,\frac{\sinh\zeta\cosh\zeta_0-\cosh\zeta\sinh\zeta_0}{\sinh\zeta}=\frac{V(a+b)}{c}\left[\cosh\zeta_0-\frac{\cosh\zeta\sinh\zeta_0}{\sqrt{(\cosh^2\zeta-1)}}\right]$$

$$=\frac{V(a+b)}{c}\left[\cosh\zeta_0-\frac{z}{\sqrt{(z^2-c^2)}}\sin\zeta_0\right], \text{ as } z=c\cosh\zeta$$

$$= \frac{V(a+b)}{c}\left[\cosh\zeta_0 - \left(1-\frac{c^2}{z^2}\right)^{1/2}\sin\zeta_0\right] = \frac{V(a+b)}{c}\left[\cosh\zeta_0 - \left(1+\frac{c^2}{2z^2}\right)\sin\zeta_0\right],$$

for large values of z.

$$= \frac{V(a+b)}{c}\left[(\cosh\zeta_0 - \sinh\zeta_0) - \frac{c^2}{2z^2}\sinh\zeta_0\right] = \frac{V(a+b)}{c}\left[e^{-\zeta_0} - \frac{c^2}{2z^2}\sinh\zeta_0\right]$$

$$\therefore \qquad z\left(\frac{dw}{dz}\right)^2 = \frac{V^2(a+b)^2}{c^2}\cdot z\left(e^{-\zeta_0} - \frac{c^2}{2z^2}\sinh\zeta_0\right)^2,$$

which has a pole at origin. Its residue at origin, Res (0), is given by

Res (0) = Coeff. of $(1/z)$ in the expansion of the above function

$$= -\frac{V^2(a+b)^2}{c^2}\cdot c^2\sinh\zeta_0 e^{-\zeta_0} = -V^2(a+b)^2\sinh\zeta_0 e^{-\zeta_0}. \qquad \text{...(3)}$$

Let M be the couple about the origin. Then by Blasius's theorem, we have

$$M = \text{Real part of } -\frac{1}{2}\rho\int_C z\left(\frac{dw}{dz}\right)^2 dz, \qquad \text{...(4)}$$

where the integral is taken round the contour C of the elliptic cylinder. By Cauchy's Residue Theorem, (4) gives

$$\text{M} = \text{Real part of } (-\rho/2)\times\{2\pi i\times \text{ sum of residues of } z\times(dw/dz)^2 \text{ inside the contour C]}$$

$$= \text{real part of } -(\rho/2)\times 2\pi i \text{ Res. (0)}. = \text{Real part of } \pi\rho i V^2(a+b)^2\sinh\zeta_0 e^{-\zeta_0}, \text{ by (3)}$$

$$= \text{Real part of } \pi\rho i V^2(a+b)^2\sinh(\alpha+i\theta)e^{-(\alpha+i\theta)}$$

$$= \text{Real part of } \pi\rho i V^2(a+b)^2(\sinh\alpha\cos\theta + i\cosh\alpha\sin\theta)\times e^{-\alpha}(\cos\theta - i\sin\theta)$$

$$= -\pi\rho V^2(a+b)^2 e^{-\alpha}(\cosh\alpha - \sinh\alpha)\sin\theta\cos\theta = -\pi\rho V^2(a+b)^2 e^{-2\alpha}\sin\theta\cos\theta$$

$$= -\pi\rho V^2(a^2-b^2)\sin\theta\cos\theta, \text{ as } e^{\alpha} = \sqrt{(a+b)/(a-b)}$$

The negative sign shows that the couple tends to set the cylinder broad-sides to the stream.

Ex. 15. *The space between two confocal cylinders* (a_0, b_0) *and* (a_1, b_1) *and two planes perpendicular to their axes is filled with liquid. If both the cylinders be made to rotate with the same angular velocity* ω, *prove that the kinetic energy of the motion set up is*

$$\{M\omega^2c^4(b_1a_0 - b_0a_1)\}/\{8(a_1a_0 - b_1b_0)(a_1b_1 - a_0b_0)\},$$

M being the mass of the liquid and $2c$ *the distance between the foci.*

Sol. We know that, if the cross section of the elliptic cylinder be $x^2/a^2 + y^2/b^2 = 1$, then it is same as $\xi = \alpha$, if $a = c\cosh\alpha$ and $b = c\sinh\alpha$ (Refer Art. 7.13)

Let the ellipitc cylinder (a_0, b_0) be given by $\xi = \alpha_0$ and the elliptic cylinder (a, b) given by $\xi = \alpha_1$. Then we must have

$$a_0 = c\cosh\alpha_0, \qquad b_0 = c\sinh\alpha_0, \qquad a_1 = c\cosh\alpha_1, \qquad b_1 = c\sinh\alpha_1 \qquad \text{...(1)}$$

Stream function ψ when an elliptic cylinder $\psi = \alpha$ is rotating with angular velocity ω is given by (Refer Art. 7.16)

$$\psi = (1/4)\times\omega c^2(\cosh 2\xi + \cos 2\eta) + C, \qquad ...(2)$$

where C is a const. Since ψ satisfies Laplace's equation and the expression for ψ given by (2) involves $\cos 2\eta$, the suitable form of ψ for the present problem is given by

$$\psi = (A\cosh 2\xi + B\sinh 2\xi)\cos 2\eta \qquad ...(3)$$

Clearly, at $\xi = \alpha_0$ and $\xi = \alpha_1$, expresions for ψ given by (2) and (3) must be the same. Thus, we have

$$(1/4)\times\omega c^2(\cosh 2\alpha_0 + \cos 2\eta) + C = (A\cosh 2\alpha_0 + B\sinh 2\alpha_0)\cos 2\eta \qquad ...(4)$$

and
$$(1/4)\times\omega c^2(\cosh 2\alpha_1 + \cos 2\eta) + C = (A\cosh 2\alpha_1 + B\sinh 2\alpha_1)\cos 2\eta \qquad ...(5)$$

Equating coefficients of $\cos 2\eta$ from both sides of (4) and (5), we have

$$A\cosh 2\alpha_0 + B\sinh 2\alpha_0 = (\omega c^2)/4 \qquad ...(6)$$

and
$$A\cosh 2\alpha_1 + B\sinh 2\alpha_1 = (\omega c^2)/4 \qquad ...(7)$$

Multiplying (6) by $\sinh 2\alpha_1$ and (7) by $\sinh 2\alpha_0$ and then subtracting the resulting equations, we have

$$A(\sinh 2\alpha_1 \cosh 2\alpha_0 - \sinh 2\alpha_0 \cosh 2\alpha_1) = (\omega c^2/4)\times(\sinh 2\alpha_1 - \sinh 2\alpha_0)$$

or
$$A\sinh(2\alpha_1 - 2\alpha_0) = (\omega c^2/4)\times 2\times\cosh(\alpha_1+\alpha_0)\sinh(\alpha_1-\alpha_0)$$

[$\because$ $\sinh A \cosh B - \cosh A \sinh B = \sinh(A - B)$,
and $\sinh A - \sinh B = 2\cosh\{(A+B)/2\}\sinh\{(A-B)/2\}$]

Thus, we have

$$A = \frac{(\omega c^2/2)\times\cosh(\alpha_1+\alpha_0)\sinh(\alpha_1-\alpha_0)}{\sinh 2(\alpha_1-\alpha_0)} = \frac{(\omega c^2/2)\times\cosh(\alpha_1+\alpha_0)\sinh(\alpha_1-\alpha_0)}{2\sinh(\alpha_1-\alpha_0)\cosh(\alpha_1-\alpha_0)}$$

or
$$A = \frac{\omega c^2\cosh(\alpha_1+\alpha_0)}{4\cosh(\alpha_1-\alpha_0)} \qquad ...(8)$$

Similarly, eliminating A from (6) and (7), we have

$$B = \frac{\omega c^2\sinh(\alpha_1+\alpha_0)}{4\cosh(\alpha_1-\alpha_0)} \qquad ...(9)$$

Substituting the values of A and B given by (8) and (9) in (3), we get

$$\psi = \frac{\omega c^2\cos 2\eta}{4}\cdot\frac{\cos 2\xi - \cosh(\alpha_1+\alpha_0) + \sin 2\xi \sinh(\alpha_1+\alpha_0)}{\cosh(\alpha_1-\alpha_0)}$$

or
$$\psi = \frac{\omega c^2}{4}\frac{\cosh(2\xi-\alpha_1-\alpha_0)}{\cosh(\alpha_1-\alpha_0)}\cos 2\eta \qquad ...(10)$$

Remembering that $\partial\phi/\partial\xi = \partial\psi/\partial\eta$, the velocity potential ϕ is given by

$$\phi = -\frac{\omega c^2}{4}\frac{\sinh(2\xi-\alpha_1-\alpha_0)}{\cosh(\alpha_1-\alpha_0)}\sin 2\eta \qquad ...(11)$$

Now, $\quad M$ = mass of the liquid lying in the space between the given ellipses

$\therefore \qquad M = \pi\rho a_1 b_1 - \pi\rho a_0 b_0 = \pi\rho(a_1 b_1 - a_0 b_0)$ ...(12)

The required kinetic energy T of the motion is given by

$$T = -\frac{1}{2}\rho\int \phi\, d\psi = -\frac{1}{2}\rho\int_{\xi=\alpha_0} \phi\, d\psi - \left\{-\frac{1}{2}\int_{\xi=\alpha_1} \phi\, d\psi\right\}$$

$$= \frac{\omega^2 c^4 \rho}{16}\frac{\sinh(\alpha_1 - \alpha_0)}{\cosh(\alpha_1 - \alpha_0)}\int_0^{2\pi} \sin^2 2\eta\, d\eta + \frac{\omega^2 c^4 \rho}{16}\frac{\sinh(\alpha_1 - \alpha_0)}{\cosh(\alpha_1 - \alpha_0)}\int_0^{2\pi} \sin^2 2\eta\, d\eta$$

$$= 2\times\frac{\omega^2 c^4 \rho}{16}\frac{\sinh(\alpha_1 - \alpha_0)}{\cosh(\alpha_1 - \alpha_0)}\int_0^{2\pi} \frac{1-\cos 4\eta}{2}\, d\eta$$

$$= \frac{\omega^2 c^4 \rho}{8}\frac{\sinh\alpha_1 \cosh\alpha_0 - \cosh\alpha_1 \sinh\alpha_0}{\cosh\alpha_1 \cosh\alpha_0 - \sinh\alpha_1 \sinh\alpha_0}\times\frac{1}{2}\left[\eta - \frac{\sin 4\eta}{4}\right]_0^{2\pi}$$

$$= \frac{\omega^2 c^4 \rho}{8}\times\frac{(c\sinh\alpha_1)(c\cosh\alpha_0) - (c\cosh\alpha_1)(c\sinh\alpha_0)}{(c\cosh\alpha_1)(c\cosh\alpha_0) - (c\sinh\alpha_1)(c\sinh\alpha_0)}\times\pi$$

$$= \frac{\omega^2 c^4}{8}\times\frac{M}{\pi(\alpha_1 b_1 - a_0 b_0)}\times\frac{b_1 a_0 - a_1 b_0}{a_1 \alpha_0 - b_1 b_0}\times\pi, \text{ by (1) and (12)}$$

Thus, $\qquad T = \{M\omega^2 c^4 (b_1 a_0 - b_1 a_1)\}/\{8(a_1 b_1 - a_0 b_0)(a_1 a_0 - b_1 b_0)\}$

Ex. 15. *An elliptic cylinder of semi-axes a, b is filled with incompressible fluid and rotates about its axis with angular velocity ω. Prove that the velocity components (u, v) Parallel to OX, OY (the axes of the ellipse) are given by* $u = \{\omega(a^2 - b^2)y\}/(a^2 + b^2)$ *and* $v = \{\omega(a^2 - b^2)x\}/(a^2 + b^2)$

Show that the co-ordinates X, Y (relative to axis through O fixed in space) of a given particle at time t can be written as

$$X = \lambda\left[(a+b)\cos\left\{\frac{(a-b)^2\omega t}{a^2+b^2}\right\} + (a-b)\cos\left\{\frac{(a+b)^2\omega t}{a^2+b^2}\right\}\right]$$

and

$$Y = \lambda\left[(a+b)\sin\left\{\frac{(a-b)^2\omega t}{a^2+b^2}\right\} + (a-b)\sin\left\{\frac{(a+b)^2\omega t}{a^2+b^2}\right\}\right],$$

where λ is a constant depending on the particle and t = 0, when the particle crosses the axis OX.

Sol. For the given elliptic cylinder, we have (See Art. 7.18)

$$\phi = -\omega\left(\frac{a^2 - b^2}{a^2 + b^2}\right)xy \qquad ...(1)$$

(1) $\Rightarrow \qquad u = -\frac{\partial\phi}{\partial x} = \omega\left(\frac{a^2 - b^2}{a^2 + b^2}\right)y, \qquad v = -\frac{\partial\phi}{\partial y} = \omega\left(\frac{a^2 - b^2}{a^2 + b^2}\right)x,$ ...(2)

giving the required values of u and v. From Art. 7.1, we have

$u = \dot{x} - \omega y \qquad$ and $\qquad v = \dot{y} + \omega x, \qquad$ where $\qquad \dot{x} = dx/dt, \qquad \dot{y} = dy/dt,$

$\Rightarrow \qquad \dot{x} = u + \omega y = \omega\left(\frac{a^2 - b^2}{a^2 + b^2}\right)y + \omega y, \qquad \dot{y} = v - \omega x = \omega\left(\frac{a^2 - b^2}{a^2 + b^2}\right)x - \omega x,$ by (2)

Thus, $$\dot{x} = (2\omega a^2 y)/(a^2+b^2) \qquad ...(3)$$

and $$\dot{y} = -(2\omega b^2 y)/(a^2+b^2) \qquad ...(4)$$

Differentating (3) w.r.t. 't' we have

$$\ddot{x} = \frac{2\omega a^2}{a^2+b^2}\dot{y} \qquad \text{or} \qquad \ddot{x} = -\left(\frac{2\omega a^2}{a^2+b^2}\right)\left(\frac{2\omega b^2}{a^2+b^2}\right)x = -\left(\frac{2\omega ab}{a^2+b^2}\right)^2 x$$

Thus, $$\ddot{x} = \mu x, \qquad ...(5)$$

where $$\mu = \{(2\omega ab)/(a^2+b^2)\}^2 \qquad ...(6)$$

(5) represents a simple harmonic motion and hence

$$x = C_1 \cos(t\sqrt{\mu} + C_2),\ C_1 \text{ and } C_2 \text{ being arbtrary constants} \qquad ...(7)$$

From (7), $$\dot{x} = -C_1\sqrt{\mu}\sin(t\sqrt{\mu}+C_2) \qquad ...(8)$$

Re-writing (3), $$\dot{x} = \frac{a}{b}\left(\frac{2\omega ab}{a^2+b^2}\right)y = \frac{a}{b}\sqrt{\mu}\,y, \text{ using (6)} \qquad ...(9)$$

Equating the two values of $\dot{x}$ given by (8) and (9), we get

$$(a/b)\times\sqrt{\mu}\,y = -C_1\sqrt{\mu}\sin(t\sqrt{\mu}+C_2) \qquad \text{so that} \qquad y = -(b/a)\times C_1\sin(t\sqrt{\mu}+C_2) \qquad ...(10)$$

When $t = 0$, $x = \lambda'$, (say) and $y = 0$ (as the particle crosses the axis OX at a distance x, from O at time $t = 0$). Thus, using initial conditions $x = \lambda'$, $y = 0$ when $t = 0$, (7) and (10) give

$$\lambda' = C_1\cos C_2 \qquad \text{and} \qquad 0 = -(b/a)\,C_1 \sin C_2 \qquad ...(11)$$

For non-zero value of x, we must have $C_1 \neq 0$. Hence (11) $\Rightarrow$ $\sin C_2 = 0 \Rightarrow C_2 = 0$. Then again, from (11), $C_1 = \lambda'$. Hence (7) and (9) reduce to

$$x = \lambda'\cos t\sqrt{\mu} \qquad \text{and} \qquad y = -(b\lambda'/a)\sin t\sqrt{\mu}; \qquad ...(12)$$

which are the co-ordinates with respect to rotating axes. We now proceed to determine the co-ordinates X, Y, (relative to axis through O fixed in space).

We have $$Z = ze^{i\omega t} \qquad \text{and} \qquad X + iY = (x+iy)\,(\cos\omega t + i\sin\omega t)$$

$$\Rightarrow \qquad X = x\cos\omega t - y\sin\omega t \qquad ...(13)$$

and $$Y = y\cos\omega t + x\sin\omega t \qquad ...(14)$$

Substituting the values of x and y from (12) in (13), we get

$$X = \lambda'\cos t\sqrt{\mu}\cos\omega t + (b\lambda'/a)\times\sin t\sqrt{\mu}\sin\omega t$$

or $$X = \frac{\lambda'}{2a}\left\{2a\cos\left(\frac{2\omega abt}{a^2+b^2}\right)\cos\omega t + 2b\sin\left(\frac{2\omega abt}{a^2+b^2}\right)\sin\omega t\right\}$$

or $$X = \lambda\left[a\left\{\cos\left(\frac{2\omega abt}{a^2+b^2}+\omega t\right)+\cos\left(\omega t - \frac{2\omega abt}{a^2+b^2}\right)\right\}\right.$$

$$\left.+\,b\left\{\cos\left(\omega t - \frac{2\omega abt}{a^2+b^2}\right) - \cos\left(\omega t + \frac{2\omega abt}{a^2+b^2}\right)\right\}\right], \text{ where } \lambda = \frac{\lambda'}{2a}$$

[$\because$ $2\cos A\cos B = \cos(A+B) + \cos(A-B)$, $2\sin A\sin B = \cos(A-B) - \cos(A+B)$]

or
$$X = \lambda\left[(a+b)\cos\left\{\frac{(a-b)^2\omega t}{a^2+b^2}\right\} + (a-b)\cos\left\{\frac{(a+b)^2\omega t}{a^2+b^2}\right\}\right]$$

Similarly (14) yields the required values of Y, namely
$$Y = \lambda\left[(a+b)\sin\left\{\frac{(a-b)^2\omega t}{a^2+b^2}\right\} + (a-b)\sin\left\{\frac{(a+b)^2\omega t}{a^2+b^2}\right\}\right]$$

Ex. 17. *Show that the curvature of a streamline in steady motion is* $\dfrac{1}{q^2}\dfrac{\partial}{\partial n}\left(\dfrac{p}{\rho}+V\right)$, *where p, ρ, q are the pressure, density and velocity of the liquid, V the potential of the external forces, and δn is an element of the principal normal to the streamlines and hence obtain the velocity potential of the two- dimensional irrotational motion for which the stremlines are confocal ellipses.*

Sol. If R be the radius of curvature, then the normal acceleration of the fluid element is q^2/R. Hence the normal equation in steady motion is given by
$$\frac{q^2}{R} = -\frac{\partial V}{\partial n} - \frac{1}{\rho}\frac{\partial p}{\partial n} \qquad \text{so that} \qquad -\frac{1}{R} = \frac{1}{q^2}\frac{\partial}{\partial n}\left(V+\frac{p}{\rho}\right), \qquad \ldots(1)$$
which give the required value of the curvature of a streamline.

Second Part. Bernoulli's equation for steady motion is
$$(p/\rho) + V + (q^2/2) = \text{constant}$$
$$\Rightarrow \quad \frac{p}{\rho}+V = \text{constant} - \frac{q^2}{2} \quad \Rightarrow \quad \frac{\partial}{\partial n}\left(\frac{p}{\rho}+V\right) = -q\frac{\partial q}{\partial n}. \qquad \ldots(2)$$

Then, (1) becomes $\quad -\dfrac{1}{R} = \dfrac{1}{q^2}\left(-q\dfrac{\partial}{\partial n}\right) \quad$ or $\quad \dfrac{q}{R} = \dfrac{\partial q}{\partial n}.$

or
$$\frac{1}{R}\frac{\partial \phi}{\partial s} = \frac{\partial}{\partial n}\left(\frac{\partial \phi}{\partial s}\right). \qquad \ldots(3)$$

If (x, y) and (ξ, η) be cartesian and elliptic coordinates of any point P, then by Art. 7.13
$$x = c\cosh\xi\cos\eta \qquad \text{and} \qquad y = c\sinh\xi\sin\eta.$$

These give ξ = const. as ellipses and η = const. as orthogonal hyperbolas.

Element of arc on ξ = const. is $\quad ds_1 = (d\eta)/h$

and element of arc on η = const. is $\quad ds_2 = (d\xi)/h,$

where $\quad 1/h^2 = c^2(\cosh^2\xi - \cos^2\eta).$

For the present problem, streamlines are ellipses and velocity potential are hyperbolas.

$\therefore \quad ds = ds_1 = (d\eta)/h \quad$ and $\quad ds = ds_2 = (d\xi)/h.$

Hence (1) reduces to $\quad \dfrac{1}{R}\dfrac{\partial \phi}{\partial s_1} = \dfrac{\partial}{\partial s_2}\left(\dfrac{\partial \phi}{\partial s_1}\right).$

Now, for s_1, ξ = const. and for s_2, η = const. So we obtain

$$\frac{1}{R}\frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial s_1}=\frac{\partial}{\partial\xi}\left(\frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial s_1}\right)\frac{d\xi}{\partial s_2} \qquad ...(4)$$

Now, $\dfrac{\partial\eta}{\partial s_1}=\dfrac{1}{c\sqrt{\cos^2\xi-\cos^2\eta}}$ and $\dfrac{\partial\xi}{\partial s_2}=\dfrac{1}{c\sqrt{\cos^2\xi-\cos^2\eta}}$.

Hence (4) reduces to $\dfrac{1}{R}\dfrac{\partial\phi}{\partial\eta}=\dfrac{\partial}{\partial\xi}\left(\dfrac{\partial\phi}{\partial\eta}\dfrac{\partial\eta}{\partial s_1}\right)$. ...(5)

If a and b are the semi-axes of the ellipse $x^2/(c\cosh\xi)^2+y^2/(c\sinh\xi)^2=1$

then we have $R=(a^2-e^2x^2)^{3/2}/ab,$ ...(6)

where $a=c\cosh\xi,\quad b=c\sinh\xi$ and $e^2=1-(b/a)^2=1-\tanh^2\xi=\text{sech}^2\xi$.

$$\therefore\ (6)\Rightarrow\qquad R=\frac{(c^2\cosh^2\xi-\text{sech}^2\xi\cdot c^2\cosh^2\xi\cos^2\eta)^{3/2}}{c^2\cosh\xi\sinh\xi}$$

or $$R=c(\cosh^2\xi-\cos^2\eta)^{3/2}/(\cosh\xi\sinh\xi).$$

Then (5) reduces

$$\frac{\cosh\xi\sinh\xi}{c(\cosh^2\xi-\cos^2\eta)^{3/2}}\frac{\partial\phi}{\partial\eta}=\frac{\partial}{\partial\xi}\left[\frac{\partial\phi}{\partial\eta}\frac{1}{c(\cosh^2\xi-\cos^2\eta)^{1/2}}\right]$$

$$=\frac{\partial^2\phi}{\partial\xi\partial\eta}\frac{1}{c\sqrt{\cosh^2\xi-\cos^2\eta}}+\frac{\partial\phi}{\partial\eta}\frac{\cosh\xi\sinh\xi}{c(\cosh^2\xi-\cos^2\eta)^{3/2}}$$

$\Rightarrow\quad \partial^2\phi/\partial\xi\partial\eta=0 \quad\Rightarrow\quad \partial\phi/\partial\eta=-A$ or $f'(\eta)$ [Here A is a constant]

$\Rightarrow\quad \phi=-A\eta$ or $\phi=f(\eta)$.

When $\phi=-A\eta,\ \psi=A\xi$ so that $\psi-i\phi=A(\xi+i\eta)$, showing that $\phi=-A\eta$ gives the correct velocity potential. Now, we have

$$q=-\frac{\partial\phi}{\partial s_1}=-\frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial s_1}=-\frac{\partial\phi}{\partial\eta}\frac{1}{c(\cosh^2\xi-\cos^2\eta)^{1/2}}=\frac{A}{c(\cosh^2\xi-\cos^2\eta)}$$

which varies as $1/(r_1r_2)^{1/2}$, where r_1, r_2 are the focal distnces of the point which is a standard well known result.

EXERCISE 7 (D)

1. An infinite flat plate of breadth $2l$ is rotating. Prove that the couple (per unit thickness) necessary to maintain the rotation is $(\pi\rho l^4/8)\,(d\omega/dt)$, where ω is the angular velocity and ρ is the density of the fluid.

2. Determine the character of the two-dimensional fluid motion inside the ellipse (a, b) for which the stream function $\psi=k(x^2/a^2+y^2/b^2)$ and find the pressure at each point in the corss-section when there is no field of force.

3. An elliptic cylinder is placed in a steady stream which at infinity makes an angle α with the major axis of the cylinder. Show that on the ellipse the pressure is greatest at the points where the stream divides and least at the points where the fluid is moving parallel to the stream as it meets the ellipse.

4. If the ellipse $a(x^2 - y^2) + 2bxy - (1/2) \times \omega (x^2 + y^2) + c = 1$

is full of liquid and is rotated round the origin with angular velocity ω, prove that the stream function ψ is given by $\psi = a(x^2 - y^2)^2 + 2bxy$.

5. With the usual notation show that for liquid streaming past an elliptic cylinder in a direction parallel to the minor axis, the stream function is $\psi = -Vce^{\alpha} \sinh(\xi - \alpha)\cos\eta$.

Hence show that for stream of velocity U in a direction making an angle θ with OX the stream function is $\psi = Uce^{\alpha} \sinh(\xi - \alpha)\cos(\eta - \theta)$.

6. Obtain an expression for the stream function of the two dimensional motion produced in an infinite liquid by the motion through it of an elliptic cylinder, which has a velocity of translation V in a direction making an angle θ with the major axis. Show that the kinetic energy of the liquid between two planes perpendicular to the generators of the cylinder at unit distance apart is $(1/2) \times \Pi \rho V^2 (b^2 \cos^2\theta + a^2 \sin^2\theta)$.

PART III : MOTION OF A PARABOLIC CYLINDER

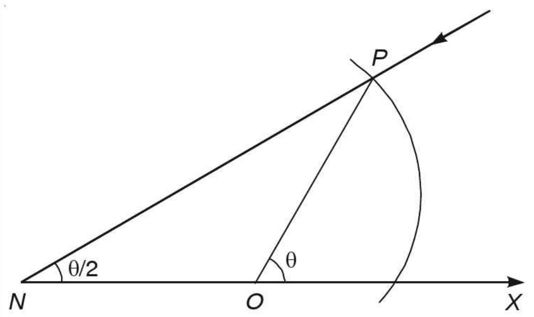
7.21. Motion of a parabolic cylinder.

Let a parabolic cylinder move along the axis of its section with velocity U. Suppose the equation of the parabolic section be

$$r = 2a/(1 + \cos\theta), \qquad \text{...(1)}$$

which is shown in figure with O as pole and OX as the initial line. From (1), we have

$$r\cos^2(\theta/2) = a \qquad \text{or} \qquad r^{1/2}\cos(\theta/2) = a^{1/2} \qquad \text{...(2)}$$



For any cylinder moving with velocities U and V parallel to axes and rotating with an angular velocity ω, we know that

$$\psi = Vx - Uy + (1/2) \times \omega (x^2 + y^2) + \text{const.}$$

Here $V = 0$, $\omega = 0$. Hence the stream function is given by $\psi = Uy + A$

or $\psi = -Ur\sin\theta + A = -2Ur\sin(\theta/2)\cos(\theta/2)$

or
$$\psi = -2Ua^{1/2}r^{1/2}\sin(\theta/2) + A, \text{ using (2)} \qquad \text{...(3)}$$

Since $r^{1/2}\sin(\theta/2)$ is plane harmonic, ψ given by (3) will satisfy Laplace's equation $\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0$ and also the condition on the boundary. Hence omitting the constant A, the stream function for the present discussion is given by

$$\psi = -2U a^{1/2} r^{1/2} \sin(\theta/2) \qquad \text{...(4)}$$

and hence
$$\phi = -2U a^{1/2} r^{1/2} \cos(\theta/2). \qquad \text{...(5)}$$

Since both $\partial\phi/\partial r$ and $(1/r) \times (\partial\phi/\partial\theta)$ vanish at infinity, the liquid is obviously at rest there.

7.22. Liquid streaming past a fixed parabolic cylinder.

To determine ϕ and ψ for a liquid streaming past a fixed parabolic cylinder with velocity U parallel to the axis of its section.

Superimpose a velocity U on the cylinder and liquid both in the sense opposite to the velocity of the liquid. This brings the liquid at rest and the cylinder in motion with velocity U.

Hence some suitable term must be added to each of the expressions for ϕ and ψ obtained in Art. 7.21. When stream flows in the direction opposite to that of the initial line (*i.e.*, *x*-axis), we have

$$-\partial\phi/\partial x = -\partial\psi/\partial y = -U.$$

Accordingly, we have to add a term Ux to ϕ and Uy to ψ. Thus using Art 7.21, we have

$$\phi = Ux - 2Ua^{1/2}r^{1/2}\cos(\theta/2) \quad \text{or} \quad \phi = Ur\cos\theta - 2Ua^{1/2}r^{1/2}\cos(\theta/2)$$

and $$\psi = Uy - 2Ua^{1/2}r^{1/2}\sin(\theta/2) \quad \text{or} \quad \psi = Ur\sin\theta - 2Ua^{1/2}r^{1/2}\sin(\theta/2)$$

To determine the resultant thrust on the cylinder per unit length.

Pressure at any point, by Bernoulli's theorem for steady motion, is given by

$$p/\rho = C - q^2/2 \qquad \text{...(3)}$$

Suppose that the pressure at infinity is zero so that $p = 0$ when $q = U$. Then (3) gives $C = U^2/2$ and hence, we obtain

$$p = (1/2)\times\rho(U^2 - q^2) \qquad \text{...(4)}$$

From (1), $$\frac{\partial\phi}{\partial r} = U\cos\theta - U\frac{a^{1/2}}{r^{1/2}}\cos(\theta/2) \qquad \text{...(5)}$$

and $$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta + U\frac{a^{1/2}}{r^{1/2}}\sin(\theta/2) \qquad \text{...(6)}$$

On the cylinder $r^{1/2}\cos(\theta/2) = a^{1/2}$. So (5) and (6) reduce to

$$\partial\theta/\partial r = U\left(\cos^2(\theta/2) - \sin^2(\theta/2)\right) - U\cos^2(\theta/2) = -U\sin^2(\theta/2) \qquad \text{...(7)}$$

and $$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -2U\sin(\theta/2)\cos(\theta/2) + U\sin(\theta/2)\cos(\theta/2) = -U\sin(\theta/2)\cos(\theta/2) \qquad \text{...(8)}$$

Hence at any point on the cylinder, velocity is given by

$$q^2 = \left(-\frac{\partial\phi}{\partial r}\right)^2 + \left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 = U^2\sin^2(\theta/2), \text{ by (7) and (8)}$$

Putting the above value of q^2 in (4), the pressure at any point P (r, θ) on the boundary of the section is given by

$$p = \frac{1}{2}\rho\left(U^2 - U^2\sin^2(\theta/2)\right) = \frac{1}{2}\rho U^2\cos^2(\theta/2) \qquad \text{...(9)}$$

From figure, we see that the normal at P makes an angle $\theta/2$ with *x*-axis. Due to symmetry, the resultant thrust on the cylinder will be along *x*-axis.

Now, from Differential Calculus, we have

$$(ds)^2 = (dr)^2 + (rd\theta)^2 = a^2\sec^6(\theta/2)(d\theta)^2, \quad \text{as} \quad r = a\sec^2(\theta/2)$$

$$\therefore \qquad ds = a\sec^3(\theta/2)\,d\theta \qquad \text{...(10)}$$

$$\therefore \text{ Resultant thrust } = -\int p\cos\frac{1}{2}\theta\,ds = \frac{1}{2}\rho U^2 a\int_{-\pi}^{\pi}\cos^3\frac{1}{2}\theta\sec^3\frac{1}{2}\theta\,d\theta, \text{ using (9) and (11)}$$

$$= -\pi\rho a U^2.$$

EXERCISE 7 (E)

1. Homogeneous liquid streams past the infinite parabolic cylinder $r\cos^2(\theta/2) = a$, the velocity at infinity being U in the positive direction of the x-axis.

Prove that $\phi = -Ur\cos\theta + 2Ua^{1/2}r^{1/2}\cos(\theta/2)$ and that the resultant thrust on the cylinder per unit length is $\Pi\rho aU^2$, the pressure at infinity is supposed to be zero.

[**Hint.** Proceed like Art. 7.22 with U in the just opposite direction.]

2. If $x+iy = (\xi+i\eta)^2$, prove that the streaming motion with velocity U parallel to the axis past the parabola $\xi = \xi_0$ is given by $\qquad \psi = 2U(\xi - \xi_0)\eta$.

3. Prove that $w^2 = z$, gives the motion in the space bounded by two confocal and co-axial parabolic sections.

OBJECTIVE QUESTIONS ON CHAPTER 7

Multiple choice questions

Choose the correct alternative from the following questions

1. When a circular cylinder is in motion with velocity U along x-axis, we have

(*i*) $w = Ua^2/r$ (*ii*) $w = Ua^2/z$

(*iii*) $w = Uz/a^2$ (*iv*) None of these. **[Kanpur 2003]**

2. When a circular cylinder of radius a moves in an infinite mass of liquid, with velocity U, the values of ϕ and ψ respectively are

(*i*) $(Ua^2\cos\theta)/r$ and $-(Ua^2\sin\theta)/r$ (*ii*) $(Ua^2\sin\theta)/r$ and $-(Ua^2\cos\theta)/r$

(*iii*) $-(Ua^2\cos\theta)/r$ and $-(Ua^2\sin\theta)/r$ (*iv*) None of these. **[Kanpur 2002]**

3. For circulation about a circular cylinder the complex potential is given by

(*i*) $(ik/2\pi)\log z$ (*ii*) $(2\pi/ik)\log z$

(*iii*) $(2k/i\pi)\log z$ (*iv*) $(2i/\pi k)\log z$ **[Kanpur 2001]**

4. For liquid streaming past a fixed circular cylinder the complex potential is

(*i*) $2Uz + Ua^2/z$ (*ii*) $Uz + a^2/z$ (*iii*) $Uz + Ua^2/z$ (*iv*) $Uz + z/Ua^2$

5. For circulation about a circular cylinder, velocity potential is

(*i*) $k\theta/2\pi$ (*ii*) $2\pi/k\theta$ (*iii*) $-(k\theta/2\pi)$ (*iv*) $-(2\pi/k\theta)$

6. The space between two fixed co-axial circular cylinders of radii a and b, and between two planes perpendicular to the axis and distance c apart is occupied by the liquid of density ρ. Show that the velocity potential of a motion whose kinetic energy shall equal to a given quantity T is given by $A\theta$, where T is equal to

(*i*) $\pi\rho A^2c^2\log(b/a)$ (*ii*) $\pi\rho Ac^2\log(b/a)$ (*iii*) $\pi\rho A^2c\log(b/a)$ (*iv*) None of these

7. When an elliptic cylinder moves in an infinite liquid with velocity U parallel to the axial plane through the major axis of the cross-section, then with usual notations, the stream function ψ is given by (*i*) $Uce^{\alpha-\xi}\sinh\alpha\sin\eta$ (*ii*) $Uce^{\alpha-\xi}\cosh\alpha\cos\eta$

(*iii*) $-Uce^{\alpha-\xi}\sinh\alpha\sin\eta$ (*iv*) $-Uce^{\alpha-\xi}\cosh\alpha\cos\eta$

8. When an elliptic cylinder is rotating with angular velocity ω in an infinite mass of the liquid at infinity, then complex potential is

(*i*) $4i\omega(a+b)^2 e^{-2\zeta}$ (*ii*) $2\omega(a+b)^2 e^{-\zeta}$ (*iii*) $\{i\omega(a+b)^2/4\}e^{-2\zeta}$ (*iv*) None of these

9. Liquid of density ρ is circulating irrotationally between two confocal ellipses $\zeta=\alpha$, $\xi=\beta$, where $x+iy=c\cosh(\xi+i\eta)$. If k is the circulation, then the kinetic energy per unit length of the cylinder is

(*i*) $\rho k^2(\beta-\alpha)/8\pi$ (*ii*) $\rho k^2(\beta-\alpha)/4\pi$ (*iii*) $\rho k^2(\beta-\alpha)/2\pi$ (*iv*) None of these

10. When a circular cylinder is placed in a uniform stream, the force acting on the cylinder is

(*i*) zero (*ii*) Infinite (*iii*) finite but not zero (*iv*) None of these

11. When a circular cylinder is placed in a uniform stream, the couple acting on the cylinder is

(*i*) Infinite (*ii*) finite but not zero (*iii*) zero (*iv*) None of these

12. A circular cylinder is placed in a uniform stream, then the force or couple acting on the cylinder is

(*i*) zero (*ii*) non-zero (*iii*) infinite (*iv*) oscillating **[Agra 2006, 09, 12]**

Answers/Hints to objective type questions

1. (*ii*). See Eq. (10), Art. 7.3
2. (*i*). See Eqs. (7), (8), Art. 7.3
3. (*i*). See Art. 7.6.
4. (*iii*). See Eq. (2), Art. 7.4
5. (*iii*). See Art. 7.6.
6. (*iii*). See Ex. 1, Art. 7.8
7. (*iii*). See Eq. (6), Art 7.14
8. (*iii*). See Eq. (10), Art. 7.16
9. (*i*). See Ex. 1, Art. 7.20
10. (*i*). See Ex. 3, Art. 7.5
11. (*iii*). See Ex. 3, Art. 7.5
12. (*i*). See Ex. 3, Art. 7.5

CHAPTER

8 The use of conformal representation. Aerofoils

8.1. Introduction

For definition and implication of conformal representation, refer Art. 5.19A, chepter 5. By choosing an appropriate formulae of transformation, fluid motion with a complicated boundary can be deduced from that with a simpler boundary. Accordingly, an extensive use is made of several sets of conformal transformations. Thus a problem which cannot be solved in one physical configuration (say, z-plane) may be solved into another configuration (say ζ-plane) by applying a suitable conformal tansformation. The problem may thus be regarded not as that of finding a direct solution, but of finding a suitable conformal transoformation which admits of an immediate solution. Though not always applicable, it is the most reliable technique to arrive at exact solutions.

8.2. Kutta-Joukowski's theorem. [Agra 2006, 09, 10, 12; Meerut 1997]

When a cylinder of any shape is placed in a uniform stream of speed U, the resultant thrust on the cylinder is a lift of magnitude $k\rho U$ *per unit length and at right angles to the stream, where k is the circulation around the cylinder.*

Proof. Let there be a fixed cylinder of some form in the finite region of the plane, its cross section containing the origin. The disturbance of the stream caused by the cylinder can be represented at a great distance in the form

$$w = Az + B/z^2 + \dots \qquad \dots(1)$$

where $A, B\dots$ depend on U and k. Let the direction of the stream make an angle α with x-axis. Then complex potential w_2 due to uniform stream velocity U is given by

$$w_2 = Ue^{-i\alpha}z \qquad \dots(2)$$

Again the complex potential w_3 due to circulation k is given by

$$w_3 = (ik/2\pi) \times \log z \qquad \dots(3)$$

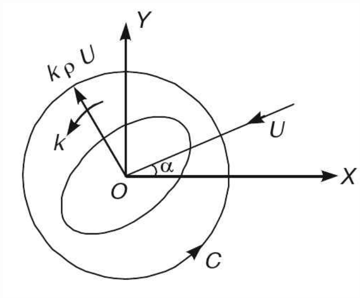
Using (1), (2) and (3), the complex potential at a great distance from the origin is given by

$$w = w_1 + w_2 + w_3 \qquad i.e., \qquad w = Ue^{-i\alpha}z + \frac{ik}{2\pi}\log z + \frac{A}{z} + \frac{B}{z^2} + \dots \qquad \dots(4)$$

so that

$$\frac{dw}{dz} = Ue^{-i\alpha} + \frac{ik}{2\pi z} - \frac{A}{z^2} - \frac{2B}{z^3} - \dots \qquad \dots(5)$$

Hence, by Blasius theorem, the force (X, Y) exerted on the cylinder is given by



$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz = (1/2) \times i\rho \cdot 2\pi i \times \text{ (sum of}$$

the residues of $(dw/dz)^2$ within C) ...(6)

(By Cauchy-Residue Theorem)

From (5), we see that $(dw/dz)^2$ has the pole at $z = 0$ inside the boundary and hence the required sum of the residues

$$= \text{Residue (at } z = 0) = (ikUe^{-i\alpha})/\pi.$$

Hence (6) reduces to $\quad X - iY = -ik\rho Ue^{-i\alpha} = -ik\rho U(\cos\alpha - i\sin\alpha)$

so that $\quad X = -k\rho U \sin\alpha \quad$ and $\quad Y = k\rho U\cos\alpha \qquad$ ...(7)

Thus, $\quad$ the resultant lift $= \sqrt{(X^2 + Y^2)} = \rho k U.$

Hence the maximum lift is $k\rho$ U which always acts at right angles to the stream.

8.3. A Joukowski transformation. Joukowski hypothesis (condition), Joukowski aerofoil.

Consider a Joukowski transformation $\qquad \zeta = z + a^2/z. \qquad$ ...(1)

Let A, B be the points $z = a$, $z = -a$. These will map into points A', B' given by $\zeta = 2a$, $\zeta = -2a$.

so that $\qquad \zeta - 2a = z + \dfrac{a^2}{z} - 2a = \dfrac{(z-a)^2}{z}$

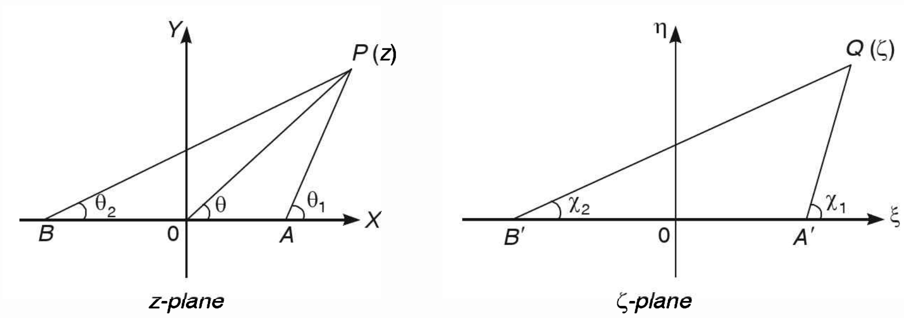


Fig. (*i*)

Then, $\qquad A'Q\ e^{i\chi_1} = \dfrac{(APe^{i\theta_1})^2}{OPe^{i\theta}}$

Hence, $\quad A'Q = AP^2/OP \quad$ and $\quad \chi_1 = 2\theta_1 - \theta \qquad$ ...(2)

Similarly, $\qquad \zeta + 2a = z + \dfrac{a^2}{z} + 2a = \dfrac{(z+a)^2}{z}$

so that $\quad B'Q = BP^2/OP \quad$ and $\quad \chi_2 = 2\theta_2 - \theta \qquad$ ...(3)

$\therefore \qquad \angle A'QB' = \chi_1 - \chi_2 = 2(\theta_1 - \theta_2) = 2\angle APB$

and $\quad A'Q + B'Q = \dfrac{AP^2 + BP^2}{OP} = \dfrac{2OP^2 + 2OA^2}{OP}$, since OP is the median of triangle APB.

Again when | z | is very large, $\zeta = z$ approximately so that distant parts of the planes are the same.

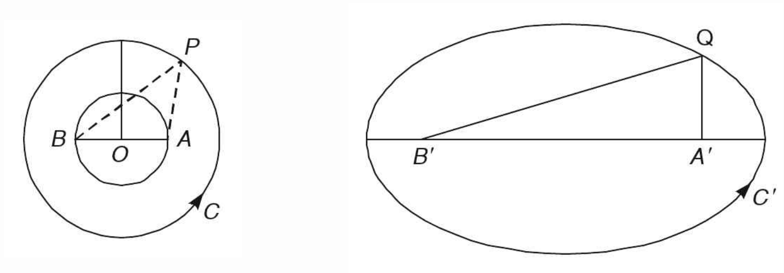


Fig. (*ii*)

Consider a circle C with centre O at origin in the z-plane. If P be a point on this circle C, then $A'Q + B'Q$ = constant because OP and OA are both constants. It follows that Q will describe an ellipse C' with A' and B' as foci. Similarly if P lies on a bigger circle, Q will describe a bigger ellipse which shows that points exterior to circle C will be mapped into points exterior to C'. Hence Joukowski transformation (1) transforms a circle in the z-plane with centres at the origin into confocal ellipses in the ζ-plane. Now consider the circle with AB as diameter. If P be a point on this circle then $\angle APB = \pi/2$ and hence $\angle A'QB' = \pi$, so that Q lies on the line $A'B'$. Hence the circle with AB as diameter is mapped into the straight line $A'B'$ of length $4a$.

Let us now consider a circle in the z-plane touching at B the circle on AB as diameter shown the figure given below.

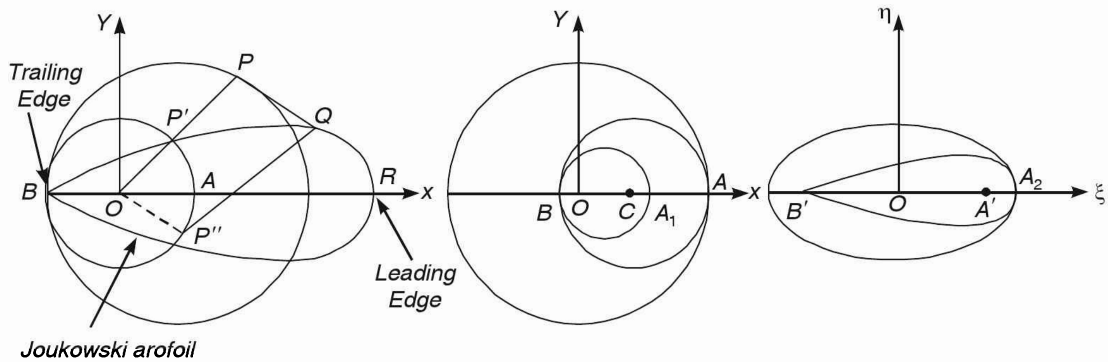


Fig. (*iii*)

Let P be a point z on this circle, P' its inverse point with respect to the circle AB and P'', reflection of P' on x-axis, then P'' is the point a^2/z. If we draw the parallelogram with OP and OP'' as adjacent sides, OQ is the diagonal of the parallelogram and Q will be the point $\zeta = z + a^2/z$. The locus of the point Q is a fish-shaped contour which touches the line BA on both sides. Such contour is known as *Joukowski aerofoil* (on account of its resemblance with the section of an aeroplane wing), the point B is known as the *trailing edge* and R as *leading edge*.

The circle with centre O and diameter AB (= $2a$) is mapped into the line $B'A'$ of length $4a$. Circle with centre O and passing through A_1 ($OA_1 > OA$) is transformed into the ellipse passing through A_2. Circle with centre C and touching the other two circles at B and A_1 is mapped into symmetrical aerofoil with trailing edge at B' and blunt nose at A_2.

From (1), we get $$d\zeta/dz = 1 - (a^2/z^2)$$

$\therefore \quad d\zeta/dz = 0 \quad \Rightarrow \quad z = \pm\, a.$

Now $z = a$ is the point A which transforms to $\zeta = 2a$ and this point A' falls inside the aerofoil, whereas $z = -a$ is the point B which transforms to the point B', $\zeta = -2a$. Thus at the trailing edge $B, d\zeta/dz = 0$ or $dz/d\zeta = \infty$.

If q be the velocity at B for the circle and q' the velocity at B' for the aerofoil, then

$$q' = \left|\frac{dw}{d\zeta}\right| = \left|\frac{dw}{dz}\right| \times \left|\frac{dz}{d\zeta}\right| = q\left|\frac{dz}{d\zeta}\right|$$

Since $dz/d\zeta = \infty$ at the trailing edge, $q' = \infty$ unless $q = 0$. Hence in order to avoid infinite velocity at the trailing edge of the aerofoil, the velocity at B is taken to be zero *i.e.*, B is taken at the stagnation point of the flow in the z-plane. This is known as *Joukowski's hypothesis.*

8.4. The aerofoil. Definition.

The aerofoil has a profile of fish type. It is employed in construction of modern aeroplanes. Such an aerofoil has a blunt leading edge and a sharp trailing edge. The theory of the flow round an aerofoil is based on the following assumptions:

(*i*) The air behaves as an incompressible fluid.

(*ii*) The aerofoil is a cylinder whose cross-section is a curve of fish type.

(*iii*) The flow is two-dimensional irrotational cyclic motion.

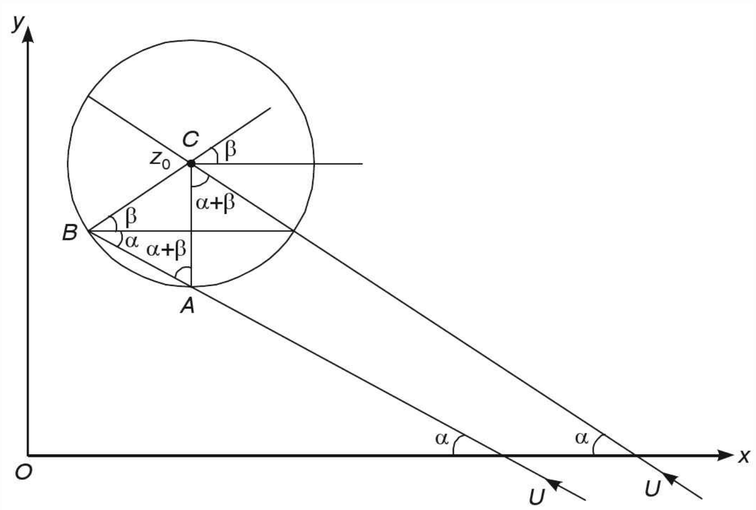
These assumptions are simply approximations to the real position of flow. The profiles obtained by application of conformal transformation of a circle by Joukowski transformation produce good wing shapes. Further-more, the lift is obtained by using the given flow with respect to a circular cylinder.

8.5. Flow round a circle.

Let U be the velocity of the stream at infinity, its direction making an angle α with the negative direction of the x-axis, k the circulation round the circle whose centre is at z_0 and radius b. Then we have

$$w = Ue^{i\alpha}(z - z_0) + \frac{Ub^2 e^{-i\alpha}}{z - z_0} + \frac{ik}{2\pi}\log(z - z_0)$$

$$\frac{dw}{dz} = Ue^{i\alpha} - \frac{Ub^2 e^{-i\alpha}}{(z - z_0)^2} + \frac{ik}{2\pi(z - z_0)} \qquad \text{...(1)}$$



For a stagnation point, $dw/dz = 0$. Taking stagnation point at $z = z_0 + be^{i(\pi+\beta)} = z_0 - be^{i\beta}$, (1) reduces to

$$U[e^{i\alpha} - e^{-i(\alpha+2\beta)}] - \frac{ik}{2\pi b}e^{-i\beta} = 0$$

or $\quad 2\pi bU[e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}] = ik \quad$ or $\quad 2\pi bU \cdot 2i\sin(\alpha+\beta) = ik$

$\therefore \quad k = 4\pi bU\sin(\alpha+\beta), \quad$ provided $\quad k < 4\pi bU \quad$ ...(2)

The stagnation points are given by $z - z_0 = be^{i(\pi+\beta)}$. Hence taking $z - z_0 = t$ and using (1), the stagnation points are given by

$$Ue^{i\alpha} - \frac{1}{t^2}U^2b^2e^{-i\alpha} + \frac{ik}{2\pi t} = 0 \quad \text{or} \quad t^2 + \frac{ike^{-i\alpha}}{2\pi U}t - b^2e^{-2i\alpha} = 0$$

$$\therefore \quad t = -\frac{ike^{-i\alpha}}{4\pi U} \pm \frac{1}{2}\sqrt{4b^2\,e^{-2i\alpha} - \frac{k^2e^{-2i\alpha}}{4\pi^2U^2}} = be^{-i\alpha}[-i\sin(\alpha+\beta) \pm \cos(\alpha+\beta)], \text{ using (2)}$$

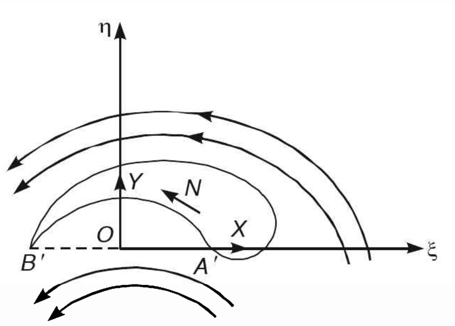
Hence, $t = be^{-i(2\alpha+\beta)}$ or $-be^{i\beta}$ so that $t = be^{i[2\pi-(2\alpha+\beta)]}$ or $t = be^{i(\pi+\beta)}$

Thus $z - z_0 = be^{i(\pi+\beta)}$ gives the point B and $z - z_0 = be^{i(2\pi-(2\alpha+\beta))}$ gives the point A as shown in the figure.

8.6. Flow round an aerofoil and lift.

We now transform the circle (see Art. 8.5) into an aerofoil by the transformation

$$\zeta = z + a^2/z \qquad \text{...(1)}$$



Then the stagnation point B transforms into the trailing edge B' and the other stagnation point A into stagnation point $A' \cdot B'$ lies on the negative ξ-axis such that $OB' = -2a$. Let U be the velocity of the stream at infinity, its direction making an angle α with ξ-axis. Streamlines divide at A' normally and at B' tangentially. If we take the origins of both the z and ζ planes to coincide, then B and B' both will lie on the left of the origin O. Then by theorem of Blasius,

$$X - iY = \frac{1}{2}i\rho\int\left(\frac{dw}{d\zeta}\right)^2 d\zeta \qquad \text{...(2)}$$

and $$N = \text{real part of}\left[-\frac{1}{2}\rho\int\left(\frac{dw}{d\zeta}\right)^2 \zeta\, d\zeta\right], \qquad \text{...(3)}$$

where the symbols have their usual meanings.

From (1), $$\frac{dw}{d\zeta} = \frac{dw}{dz}\cdot\frac{dz}{d\zeta} = \frac{dw}{dz}\Big/\frac{d\zeta}{\partial z} = \frac{dw/dz}{1 - a^2/z^2}$$

$$\therefore \quad \left(\frac{dw}{d\zeta}\right)^2 = \frac{dw}{d\zeta}\cdot\frac{dw}{d\zeta} = \frac{dw}{dz}\cdot\frac{dz}{\partial\zeta}\cdot\frac{dw/dz}{1-a^2/z^2} \quad \text{or} \quad \left(\frac{dw}{d\zeta}\right)^2 d\zeta = \frac{(dw/dz)^2}{(1-a^2/z^2)}dz \qquad \text{...(4)}$$

But in z-plane, we have

$$w = Ue^{i\alpha}z + \frac{Ub^2e^{-i\alpha}}{z - z_0} + \frac{ik}{2\pi}\log(z - z_0) \quad \Rightarrow \quad \frac{dw}{dz} = Ue^{i\alpha} - \frac{Ub^2e^{-i\alpha}}{(z-z_0)^2} + \frac{ik}{2\pi}\cdot\frac{1}{z - z_0}.$$

$$\therefore \quad \frac{1}{1 - a^2/z^2}\left(\frac{dw}{dz}\right)^2 = \frac{1}{1 - a^2/z^2}\left[Ue^{i\alpha} - \frac{Ub^2e^{-i\alpha}}{(z-z_0)^2} + \frac{ik}{2\pi(z - z_0)}\right]^2$$

$$= \left(1 + \frac{a^2}{z^2} + \ldots\right)\left[Ue^{i\alpha} - \frac{Ub^2e^{-i\alpha}}{z^2}\left(1 + \frac{2z_0}{z} + \ldots\right) + \frac{ik}{2\pi z}\left(1 + \frac{z_0}{z} + \ldots\right)\right]^2 \quad \ldots(5)$$

From (2) and (4), we have

$$X - iY = \frac{1}{2}i\rho\int\frac{(dw/dz)^2}{1 - a^2/z^2}dz = \frac{1}{2}i\rho\cdot 2\pi i\left[\text{sum of the residues of } \frac{(dw/dz)^2}{1 - a^2/z^2}\right]$$

[By Cauchy-Residue theorem]

$$= -\pi\rho\cdot\text{coeff. of } \frac{1}{z} \text{ in R.H.S. of (5)} = -\pi\rho\cdot 2Ue^{i\alpha}\frac{ik}{2\pi} = -i\rho kUe^{i\alpha}$$

$$X = \rho kU\sin\alpha \qquad \text{and} \qquad Y = -\rho kU\cos\alpha \quad \ldots(6)$$

These are components of lift of magnitude ρkU at right angles to the stream. Since $k = 4\pi bU\sin(\alpha + \beta)$, lift is $4\pi b\rho U^2\sin(\alpha + \beta)$.

[$\because k = 4\pi b\, U\sin(\alpha + \beta)$,

Further, $$\left(\frac{dw}{d\zeta}\right)^2\zeta d\zeta = \frac{(z + a^2/z)}{1 - a^2/z}\left(\frac{dw}{dz}\right)^2 dz.$$

Now, $$\frac{z + a^2/z}{1 - a^2/z^2}\left(\frac{dw}{dz}\right)^2$$

$$= \left(z + \frac{a^2}{z}\right)\left(1 + \frac{a^2}{z^2} + \ldots\right)\left[Ue^{i\alpha} - \frac{Ub^2e^{-i\alpha}}{z^2}\left(1 + \frac{2z_0}{z} + \ldots\right) + \frac{ik}{2\pi z}\left(1 + \frac{z_0}{z} + \ldots\right)\right]^2$$

$$= \left(z + \frac{2a^2}{z} + \ldots\right)\left[U^2e^{2i\alpha} - \frac{2U^2b^2}{z^2} - \frac{k^2}{4\pi^2z^2}\left(1 + \frac{2z_0}{z}\right) + 2Ue^{i\alpha}\cdot\frac{ik}{2\pi z}\left(1 + \frac{z_0}{z}\right) + \ldots\right]$$

$$\therefore \qquad \text{Coeff. of } \frac{1}{z} = -2U^2b^2 - \frac{k^2}{4\pi^2} + \frac{iUe^{i\alpha}kz_0}{\pi} + 2a^2U^2e^{2i\alpha}$$

$$\therefore \text{ As before,} \qquad N = \text{real part of } -\frac{1}{2}\rho\left[-2b^2U^2 - \frac{k^2}{4\pi^2} + \frac{iUe^{i\alpha}kz_0}{\pi} + 2a^2U^2e^{2i\alpha}\right]\cdot 2\pi i$$

Taking $z_0 = ce^{i\gamma}$, we have

$$N = \pi\rho\left[\frac{Ukc}{\pi}\cos(\alpha + \gamma) + 2a^2U^2\sin 2\alpha\right] = \pi\rho\left[\frac{Uc}{\pi}\cdot 4\pi bU\sin(\alpha + \beta)\cos(\alpha + \gamma) + 2a^2U^2\sin 2\alpha\right]$$

[$\because k = 4\pi b\, U\sin(\alpha + \beta)$, using result (2)of Art. 8.5]

$$= 2\pi\rho U^2[2bc\sin(\alpha + \beta)\cos(\alpha + \gamma) + a^2\sin 2\alpha]$$

X, Y, N are the components of forces and their moment about the origin O. Hence the moment of the forces about the trailing end B is $N + aY$, where

$$Y = \rho kU \cos\alpha = 4\pi\rho bU^2 \cos\alpha \sin(\alpha+\beta).$$

Transferring the origin to B, we have for the centre $\quad z = z_0 - be^{i\beta}, \quad$ where $z = 0$.

Hence $z_0 = be^{i\beta}$ and so $c = b$ and $\gamma = \beta$. Then, we have

$$N = \pi\rho\,[(Uk/\pi)\times b\cos(\alpha+\beta) + 2a^2U^2 \sin 2\alpha]$$

$$= \pi\rho[(Ub/\pi)\times 4b\pi \sin(\alpha+\beta)\cos(\alpha+\beta) + 2a^2U^2\sin 2\alpha], \text{ as } k = 4\pi bU\sin(\alpha+\beta)$$

$$= 2\pi\rho U^2[2b^2\sin(\alpha+\beta)\cos(\alpha+\beta) + a^2\sin 2\alpha] = 2\pi\rho U^2[b^2\sin 2(\alpha+\beta) + a^2\sin 2\alpha]$$

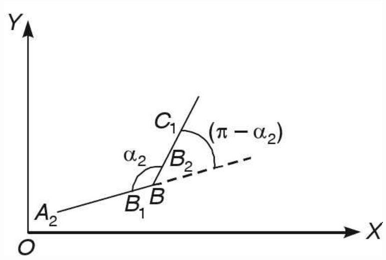
Thus, the moment about $B = 2\pi\rho U^2[b^2\sin 2(\alpha+\beta) + a^2\sin 2\alpha + 2ab\cos\alpha\sin(\alpha+\beta)]$

8.7. Schwarz-Christoffel transformations. **[Rohilkhand 2004; Meerut 1997]**

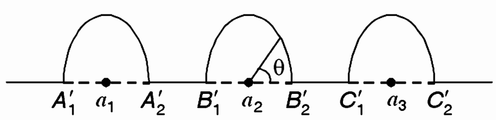
Any closed polygon with n vertices in the z (= x + iy) plane can be transformed into the real axis in the $\zeta\,(=\xi+i\eta)$ *plane, the interior points of the polygon corresponding to points on one side of the real axis* $\eta = 0$, *by the transformation*

$$\frac{dz}{d\zeta} = k(\zeta-a_1)^{\frac{\alpha_1}{\pi}-1}(\zeta-a_2)^{\frac{\alpha_2}{\pi}-1}\ldots(\xi-a_n)^{\frac{\alpha_n}{\pi}-1},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are the interior angles of the simple closed polygon and* $a_1, a_2, \ldots a_n$ *are the points on the real axis* $\eta = 0$, *that correspond to the angular points of the polygon in the z-plane and k is a constant which may be complex.*



z - plane



ζ - plane

Proof. Let us suppose that the points $a_1, a_2, \ldots a_n$ are so arranged that $a_1 < a_2 < a_3 \ldots < a_n$. Using the property of the angles of a closed polygon, we have

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = (n-2)\pi \qquad \ldots(1)$$

From the transformation equation

$$\frac{dz}{d\zeta} = k(\zeta-a_1)^{\frac{\alpha_1}{\pi}-1}(\zeta-a_2)^{\frac{\alpha_2}{\pi}-1}\ldots(\xi-a_r)^{\frac{a_r}{\pi}-1}\ldots(\zeta-a_n)^{\frac{\alpha_n}{\pi}-1}, \qquad \ldots(2)$$

we notice that at $\zeta = a_r$, $dz/d\zeta = 0$ if $\alpha_r < \pi$ and $dz/d\zeta = \infty$ if $\alpha_r < \pi$, for $r = 1, 2, \ldots$, n. Hence we avoid these points by surrounding them by means of small semicircles in the upper half of the ζ-plane. Suppose that ζ moves along $\eta = 0$ between the points $A_2', B_1', B_2', C_1' \ldots$ and goes round the semicircles with centres $a_2, a_3, \ldots$. Let $A_1, A_2, B_1, B_2, \ldots$ be the points in the z-plane corresponding to the points $A_1', A_2', B_1', B_2', \ldots$ in the ζ-plane.

Let $\qquad k = \lambda e^{i\beta}$ where $\lambda > 0 \qquad$ and $\qquad \lambda, \beta$ are real.

Taking arguments of both sides of (2), we get

$$\arg(dz) - \arg(d\zeta) = \beta + \sum_{r=1}^{n} \left(\frac{\alpha_r}{\pi} - 1 \right) \arg(\zeta - a_r).$$

As ζ moves from A_2' to B_1', $\arg(d\zeta)$ remains zero, $\arg(\zeta - a_1) = 0$ because $(\zeta - a_1)$ is real and positive but each of $\arg(\zeta - a_r) = \pi$ for r = 2, 3, ..., n because each of $(\zeta - a_r)$ is real and negative. Hence, we have

$$\therefore \qquad \arg(dz) = \beta + 0 + \sum_{r=2}^{n} \left(\frac{\alpha_r}{\pi} - 1 \right) \pi \;= \beta + (\alpha_2 + \alpha_3 + \ldots + \alpha_n) - (n-1)\pi$$

$$= \beta - \alpha_1 + (n-2)\pi - (n-1)\pi \quad \text{using (1)}$$

$$= \beta_1 - \alpha_1 - \pi \qquad \ldots(3)$$

showing that arg (dz) is constant as ζ moves from A_2' to B_1' and hence A_2B_1 is a straight line segment.

As ζ moves over from B_2' to C_1', we obtain

$$\arg(dz) = \beta + 0 + 0 + \sum_{r=3}^{n} \left(\frac{\alpha_r}{\pi} - 1 \right) = \beta - \alpha_1 - \alpha_2, \qquad \ldots(4)$$

which is constant and so $B_2\, C_1$ is a straight line segment. Using (3) and (4), we have

arg (dz) on $B_2\, C_1$ – arg (dz) on $A_2\, B_1$ $= (\beta - \alpha_1 - \alpha_2) - (\beta - \alpha_1 - \pi) = \pi - \alpha_2$,

showing that the direction of motion of z has turned through $\pi - \alpha_2$ in the positive sense. Now on the semicircle $B_1'\, B_2'$, we have $\quad \zeta - a_2 = re^{i\theta} \quad$ so that $\quad d\zeta = ire^{i\theta}\, d\theta$.

Taking r << 1, we have approximately

$$\frac{dz}{ire^{i\theta}\, d\theta} = \lambda e^{i\beta} (a_2 - a_1)^{\frac{\alpha_1}{\pi} - 1} (re^{i\theta})^{\frac{\alpha_2}{\pi} - 1} (a_2 - a_3)^{\frac{\alpha_3}{\pi} - 1} \ldots$$

so that $$dz / d\theta = i r^{a_2/\pi} e^{i(\beta + \alpha_2 \theta/\pi)} f,$$

where f is independent of r and θ. Integrating it, we obtain

$$z = z_1 + (\pi / \alpha_2) \times r^{\alpha_2/\pi} e^{i(\beta + \alpha_2\theta/\pi)} f, \qquad \ldots(5)$$

where z_1 is a constant. Since α_2 is positive, we find that $z \to z_1$ as $r \to 0$ so that z_1 is the point B where the lines $A_2\, B_1$ and $B_2\, C_1$ meet. Thus the transformation (2) makes z describe a polygon whose vertices correspond to the points a_1, a_2, a_3, ... and whose interior angles are $\alpha_1, \alpha_2, \ldots$

Now, from (5), $$\arg(z - z_1) = \beta + (\alpha_2\theta / \pi) + \arg f$$

As ζ describes the semi circle, θ decreases from π to 0, arg $(z - z_1)$ decreases by α_2 and therefore z describes a circular arc, centre B, situated inside the polygon when it is a simple polygon. Thus points in the upper half of the ζ-plane correspond to points within the polygon.

Finally, if the vertex of the polygon in the z-plane corresponds to the point at infinity on $\eta = 0$, say $a_1 \to -\infty$, we can write by suitable choice of k

$$\frac{dz}{d\zeta} = \lambda e^{i\beta}(-a_1)^{-\frac{\alpha_1}{\pi}+1}(\zeta - a_1)^{\frac{\alpha_1}{\pi}-1}(\zeta - a_2)^{\frac{\alpha_2}{\pi}-1}\ldots$$

When $a_1 \to -\infty$, $\left(\dfrac{\zeta - a_1}{-a_1}\right)^{\frac{\alpha_1}{\pi}-1} \to 1$, and so the transformation reduces to

$$\frac{dz}{d\zeta} = \lambda e^{i\beta}(\zeta - a_2)^{\frac{\alpha_2}{\pi}-1}(\zeta - a_3)^{\frac{\alpha_3}{\pi}-1}\ldots$$

showing that the factor corresponding to a_1 and α_1 disappear.

Working Rule. Suppose we wish to transform a polygon of z-plane into ξ-axis of the ζ-plane. Then angles $\alpha_1, \alpha_2, \ldots, \alpha_n$ will be given. Select any three of $a_1, a_2, \ldots, a_n$. Then others will depend on the dimensions of the given polygon. In case any of $a_1, a_2, \ldots, a_n$ is infinite, then the factor corresponding to a_1 and α_1 will not appear in the expression for $dz/d\zeta$.

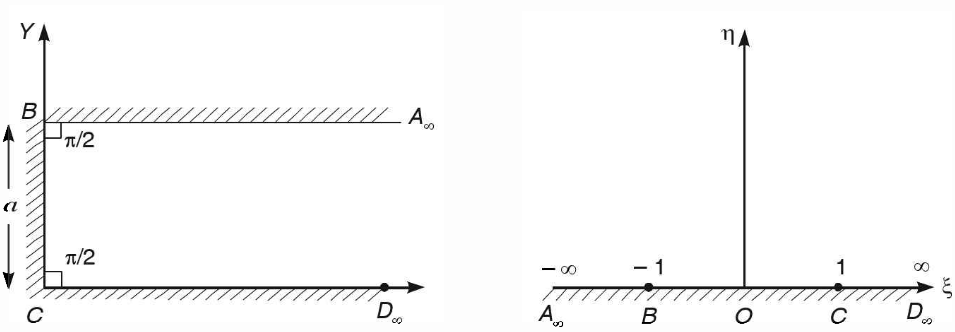
8.8. Transformation of semi-infinite strip.

Consider a semi-infinite strip $A_\infty BCD_\infty$ of breadth a in the z-plane, regarded as a rectangle with two vertices at infinity. We wish to transform this strip into ξ-axis of ζ-plane by using Schwarz-Christoffel transformation

$$\frac{dz}{d\zeta} = k(\zeta - a_1)^{\frac{\alpha_1}{\pi}-1}(\zeta - a_2)^{\frac{\alpha_2}{\pi}-1}(\zeta - a_3)^{\frac{\alpha_3}{\pi}-1}(\zeta - a_4)^{\frac{\alpha_4}{\pi}-1} \quad \ldots(1)$$

Let us map A_∞, B, C on the points $\zeta = -\infty$, $\zeta = -1$, $\zeta = 1$ on ξ-axis of ζ-plane. It is convenient to cut this rectangular strip at A_∞ and open the boundary out into ξ-axis of ζ-plane such that A_∞ maps on to $\zeta = -\infty$. Then, clearly D_∞ will be mapped on to $\zeta = \infty$.

$\therefore$ Here, $a_1 = -\infty$, $a_2 = -1$, $a_3 = 1$, $a_4 = \infty$, $\alpha_2 = \alpha_3 = \pi/2$.



z - plane *ζ - plane*

Since $a_1 = -\infty$ and $a_4 = \infty$, the brackets corresponding to a_1 and a_4 will not occur in (1). Thus, (1) becomes

$$\frac{dz}{d\zeta} = k(\zeta + 1)^{-\frac{1}{2}}(\zeta - 1)^{-\frac{1}{2}} \quad \text{or} \quad dz = \frac{k\,d\zeta}{\sqrt{(\zeta^2 - 1)}}$$

Integrating, $z = k\cosh^{-1}\zeta + A$, A being an arbitary constant ...(2)

At C, $\zeta = 1$, $z = 0$, so $0 = k\cosh^{-1} 1 + A$ ...(3)

But, $\cosh^{-1} x = \log[x + (x^2 - 1)^{1/2}]$ ...(4)

$\therefore$ $\cosh^{-1} = \log 1 = 0$, using (4). Hence (3) gives $A = 0$.

$\therefore$ From (2), we get $z = k \cosh^{-1}(\zeta)$ ...(5)

Now, at B, $\zeta = -1$, $z = ia$, so from (5)

$$ia = k \cosh^{-1}(-1) = k \log[-1 + \sqrt{\{(-1)^2 - 1\}}], \text{ using (4)}$$

or $$ia = k\log(-1) = k\log e^{i\pi} = k\,i\pi$$

$\therefore$ $k = a/\pi$ and so from (5), we have

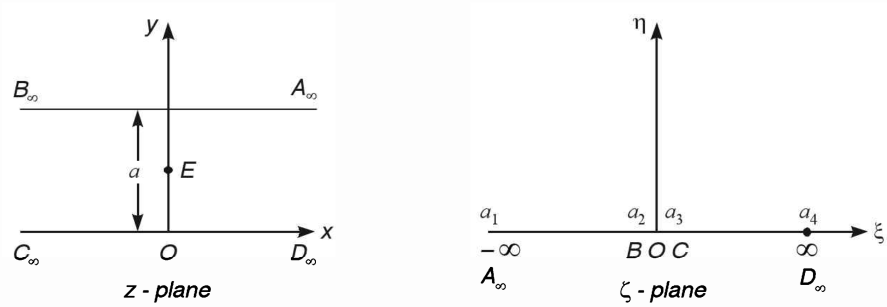
$$z = (a/\pi)\cosh^{-1}\zeta \quad \text{or} \quad \zeta = \cosh(\pi z/a),$$

which is the required transformation.

8.9. Transformation of an infinite strip.

We wish to transform infinite strip $A_\infty B_\infty C_\infty D_\infty$ in z-plane bounded by the straight lines $y = 0$ and $y = \infty$ into the upper half of the ζ-plane by using Schwarz-Christoffel transformation

$$\frac{dz}{d\zeta} = k(\zeta - a_1)^{\frac{\alpha_1}{\pi}-1} (\zeta - a_2)^{\frac{\alpha_2}{\pi}-1} (\zeta - a_3)^{\frac{\alpha_3}{\pi}-1} (\zeta - a_4)^{\frac{\alpha_4}{\pi}-1} \quad ...(1)$$



Let us suppose that A_∞, B_∞, C_∞ be mapped on $A_\infty\,(\zeta = -\infty)$, $B\,(\zeta = 0)$, $C\,(\zeta = 0)$ respectively. Then, clearly D_∞ will be mapped on $D_\infty\,(\zeta = \infty)$. Hence, here

$$a_1 = -\infty, \qquad a_2 = a_3 = 0, \qquad a_4 = \infty, \qquad \alpha_2 = \alpha_3 = 0.$$

Note that the brackets corresponding to a_1 and a_4 will not occur in the transformation (1). Again, since a_2 and a_3 are coincident, we take $k(\zeta - a_2)^{0-1}$ in place of $k(\zeta - a_2)^{0-1}(\zeta - a_3)^{0-1}$ while writing transformation (1). Accordingly (1) reduces to

$$\frac{dz}{d\zeta} = k(\zeta - 0)^{0-1} \quad \text{or} \quad dz = k\frac{d\zeta}{\zeta}$$

Integrating, $z = k\log\zeta + A$, A being an arbitrary constant ...(2)

Next, when $z = \infty + ai$ at A_∞, $\zeta = -\infty$ at A_∞,

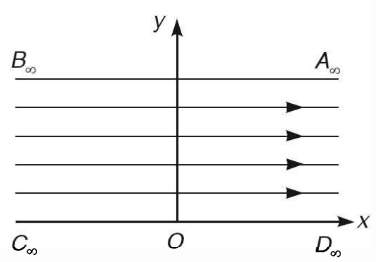
$\therefore$ From (3), $\infty + ai = k\log(-\infty) = k\log[(-1)\,\infty] = k[\log(-1) + \infty]$

or $\infty + ai = k\log(-1) + \infty$ so that $ai = k \log(-1)$

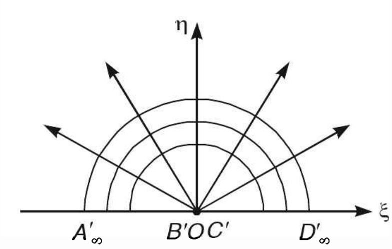
or $ai = k\log e^{i\pi} = ki\pi$, giving $a = k\pi$ or $k = a/\pi$

$\therefore$ From (3), $z = (a/\pi)\log\zeta$, which is the required transformation,

Remark 1. Note that the point $z = ai$ corresponds to $\zeta = -1$ and the point $z = 0$ corresponds to $\zeta = 1$. The lines x = constant transform into circles $|\zeta|$ = constant and the lines y = constant transform into lines $\arg\zeta$ = constant radiating from the origin in the ζ-plane as shown in the following figure.



z - plane



ζ - plane

Remark 2. If we take the origin at the point E, midway between the walls, we have by writing $z + i\ (a/2)$ for z,

$$z + \frac{ia}{2} = \frac{a}{\pi}\log\zeta \qquad \text{or} \qquad z = \frac{a}{\pi}\log\zeta - \frac{ia}{2} \qquad \text{so that} \qquad \zeta = ie^{\pi z/a}.$$

(*i*) Transformation for semi-infinite region $x = 0$, $y = \pm\ (a/2)$ is $\zeta = i\sinh(\pi z/a)$

(*ii*) Transformation for infinite region $x = 0$, $y = 0$, $x = a$ is $\zeta = -\cos(\pi z/a)$

(*iii*) Transformation for semi-infinite region $y = 0$, $x = \pm\ (a/2)$ is $\zeta = \sin(\pi z/a)$

(*iv*) Transformation for infinite region $y = \pm\ (a/2)$ is $\zeta = ie^{\pi z/a}$

(*v*) Transformation for infinite region $x = 0$, $x = a$ is $\zeta = e^{i\pi z/a}$

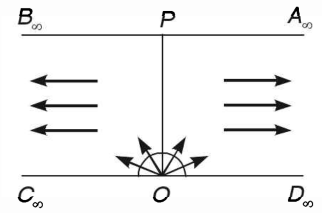
(*vi*) Transformation for infinite region $x = \pm\ (a/2)$ is $\zeta = ie^{\pi z/a}$

(*vii*) Transformation for infinite region $y = 0$, $y = a$ is $\zeta = e^{\pi z/a}$

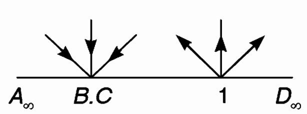
(*viii*) Transformation for semi-infinite region $x = 0$, $y = 0$, $y = a$, is $\zeta = \cosh(\pi z/a)$

8.10. Flow into channel through a narrow slit in a wall.

Let one side of the channel be real axis and slit the origin in the z-plane. The channel is of breadth a. The infinite strip $A_\infty\ B_\infty\ C_\infty\ D_\infty$ in the z-plane is transformed into the upper half of the real axis in ζ-plane by transformation $\zeta = e^{\pi z/a}$ where the origin O of the z-plane goes to $\zeta = 1$ and $B_\infty\ C_\infty$, coincide at the origin (B, C) of the ζ-plane.



z - plane



ζ - plane

If the strength of the source be m, then flow at O upwards in the z-plane is πm. At infinite distance from O there will be parallel flow and there will be sinks of strength $m/4$ at

A_∞, B_∞, C_∞, D_∞. In the ζ-plane there will be a source of strength m at $\zeta = 1$ and a sink of strength $m/2$ at the origin (B, C), hence

$$w = -m\log(\zeta-1)+(m/2)\log\zeta = -m\log(\zeta^{1/2}-\zeta^{1/2}) = -m\log(e^{\pi z/2a}-e^{-\pi z/2a})$$

$$= -m\log\left(2\sinh\frac{\pi z}{2a}\right) = -m\log\sinh\frac{\pi z}{2a}, \text{ omitting the constant term.}$$

$$\therefore \qquad \frac{dw}{dz} = -\frac{m\pi}{2a}\coth\frac{\pi z}{2a}, \text{ which is zero at } z = ai.$$

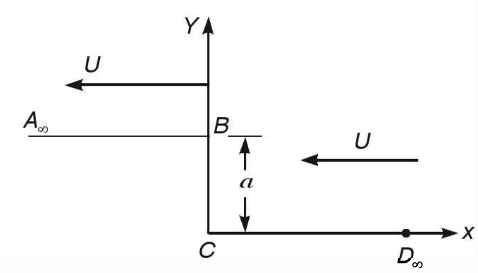
Thus the velocity at P, exactly opposite O, in the other wall, is zero. Hence the pressure is maximum since $p/\rho+q^2/2$ = const. Thus at other parts of this wall, the pressure will be lesser and lesser, the effect will be that the wall will be drawn towards the other wall *i.e.* the channel will collapse. The velocity is infinity at $\pi m/2a$.

8.11A. Flow past a step in a deep stream.

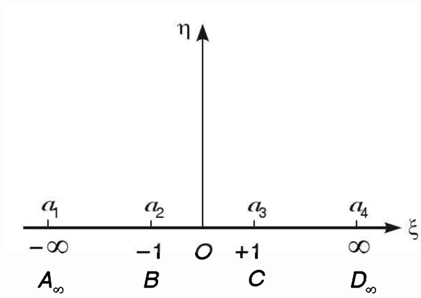
We wish to transform step $A_\infty BC\,D_\infty$ in z-plane bounded by the straight lines $y = a$, $x = 0$ and $y = 0$ into $\xi-axis$ of ζ-plane by using Schwarz-Christoffel transformation :

$$\frac{dz}{d\zeta} = k(\zeta-a_1)^{\frac{\alpha_1}{\pi}-1}(\zeta-a_2)^{\frac{\alpha_2}{\pi}-1}(\zeta-a_3)^{\frac{\alpha_3}{\pi}-1}(\zeta-a_4)^{\frac{\alpha_4}{\pi}-1} \qquad ...(1)$$

Let us suppose that A_∞, B, C be mapped on $A_\infty(\zeta = -\infty)$, $B(\zeta=-1)$, $C(\zeta=1)$ respectively. Then clearly D_∞ will be mapped on $D_\infty(\zeta=\infty)$.



z - plane



ζ - plane

$\therefore$ Here, $\qquad a_1 = -\infty, \qquad a_2 = -1, \qquad a_3 = 1, \qquad a_4 = \infty,$

$\qquad \alpha_2 = 3\pi/2 \qquad$ and $\qquad \alpha_3 = \pi/2$

Note that brackets corresponding to a_1, a_4 will not occur in the transformation (1). Hence

(1) reduces to $$\frac{dz}{d\zeta} = k(\zeta+1)^{3/2-1}(\zeta-1)^{1/2-1} = k\frac{(\zeta+1)^{1/2}}{(\zeta-1)^{1/2}} \qquad ...(2)$$

or $$\frac{dz}{d\zeta} = k\frac{(\zeta+1)}{\sqrt{(\zeta^2-1)}} = k\left[\frac{\zeta}{\sqrt{(\zeta^2-1)}}+\frac{1}{\sqrt{(\zeta^2-1)}}\right]$$ or $$dz = k\left[\frac{\zeta}{\sqrt{(\zeta^2-1)}}+\frac{1}{\sqrt{(\zeta^2-1)}}\right]d\zeta$$

Integrating, $\zeta = k[\sqrt{(\zeta^2-1)}+\cosh^{-1}\zeta]+A$, A being an arbitrary constant $\qquad$ ...(3)

Also, we have $\cosh^{-1}\zeta = \log\,[\zeta + (\zeta^2 - 1)^{1/2}]$ ...(4)

But when $z = 0$, $\zeta = 1$, so from (3) $0 = k\cosh^{-1} 1 + A$

Hence, $0 = k \log 1 + A$, using (4). Thus, $A = 0$ and so (3) reduces to

$$z = k\,[(z^2 - 1)^{1/2} + \cosh^{-1}\zeta] \qquad ...(5)$$

Again, when $z = ai$, $\zeta = -1$, so (5) gives

$ai = k\cosh^{-1}(-1)$ or $ai = k\log(-1)$, using (4)

or $ai = k\log e^{i\pi} = ki\pi$ so that $k = a/\pi$

$\therefore$ From (5), we have $z = (a/\pi)\,[(\zeta^2 - 1)^{1/2} + \cosh^{-1}\zeta]$ ...(6)

Let V be the velocity of the stream in ζ-plane. Then the complex potential w is given by

$$w = V\zeta \qquad ...(7)$$

Now, $\dfrac{dw}{dz} = \dfrac{dw}{d\zeta}\cdot\dfrac{d\zeta}{dz}$ or $\dfrac{dw}{dz} = \dfrac{V}{k}\dfrac{\sqrt{(\zeta-1)}}{\sqrt{(\zeta+1)}}$, using (2) and (7)

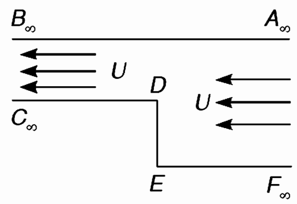
$\therefore$ $\lim\limits_{\zeta\to\infty} dw/dz = V/k$

At infinity, $\dfrac{dw}{dz} = U$, $\zeta = \infty$ So $U = \dfrac{V}{k} = \dfrac{V\pi}{a}$, as $k = \dfrac{a}{\pi}$

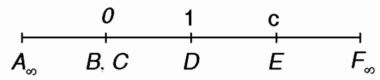
Thus, $V = aU/\pi$. Hence, from (7) $w = (aU/\pi)\zeta$.

8.11B. Flow past a step in a channel.

At the end $A_\infty F_\infty$, breadth of the channel is a, and at the other end, b. If U be the velocity at A_∞, velocity at B_∞ is Ua/b. In the ζ-plane, B_∞, C_∞ coincide with origin, D goes to $\zeta = 1$ and E to $\zeta = c$. Then, $dz/d\zeta = k\zeta^{-1}(\zeta - 1)^{1/2}(\zeta - c)^{-1/2}$



z - plane



ζ - plane

At A_∞ the flow is from a source of output Ua and at B_∞ the flow is due to a sink of intake Ua, hence in the ζ-plane we have a sink at the origin of strength Ua/π.

$\therefore$ $w = \dfrac{Ua}{\pi}\log\zeta$ so that $\dfrac{dw}{dz} = \dfrac{Ua}{\pi\zeta}$

Thus $$\frac{dw}{dz} = \frac{Ua}{\pi k}\,\frac{(\zeta - c)^{1/2}}{(\zeta - 1)^{1/2}} \qquad ...(1)$$

At A_∞, $\zeta = \infty$, $dw/dz = U$ so $U = Ua/\pi k$ so that $k = a/\pi$

At B_∞, $\zeta = 0$, $dw/dz = (Ua)/b$

$\therefore \quad \dfrac{Ua}{b} = \dfrac{Ua}{\pi k}\sqrt{c}$ so that $\sqrt{c} = \dfrac{\pi k}{b} = \dfrac{a}{b}$, as $k = \dfrac{a}{\pi}$

$\therefore \quad c = a^2/b^2$

To integrate (1), put $\zeta = (c^2 - t^2)/(1-t^2)$

so that $\dfrac{\zeta - 1}{\zeta - c^2} = \dfrac{1}{t^2}$ and $\dfrac{d\zeta}{\zeta} = \left(\dfrac{2t}{1-t^2} - \dfrac{2t}{c^2 - t^2}\right)dt$

Thus, $\dfrac{dz}{dt} = k\left(\dfrac{2t}{1-t^2} - \dfrac{2t}{c^2-t^2}\right)$, whose integration gives

$$z = \frac{a}{\pi}\left(\log\frac{1+t}{1-t} - \frac{1}{c}\log\frac{c+t}{c-t}\right) + A., \ A \text{ being an arbitrary constant} \qquad ...(2)$$

Since $z = 0$ corresponds to $\zeta = c$, we have $t = 0$ and hence (2) gives $A = 0$. So (2) reduces to

$$z = \frac{a}{\pi}\left(\log\frac{1+t}{1-t} - \frac{1}{c}\log\frac{c+t}{c-t}\right) \qquad ...(3)$$

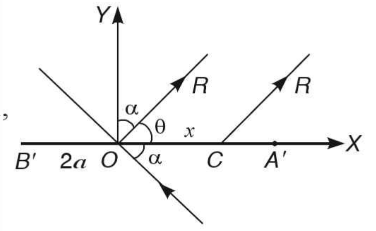
Further $$w = \frac{Ua}{\pi}\log\zeta = \frac{Ua}{\pi}\log\frac{c^2-t^2}{1-t^2} \qquad ...(4)$$

Eliminating t from (3) and (4), we get $w = f(z)$.

8.12. Illustrative solved examples.

Ex. 1. *A flat plate of infinite length and width l is placed in a current of incompressible fluid with its plane at an angle α to the undisturbed streamlines and its edge perpendicular to them. Determine the resulting flow on the circulation theory, assuming the velocity at the trailing edge finite. By considering the pressures and velocities over a large cylinder whose axis is the median line of the plate, show that the forces on the plate are equivalent to a force $\pi\rho U^2 l \sin\alpha$ per unit length perpendicular to the current, acting at a distance $l/4$ from the leading edge.*

Sol. The circle of radius a on BOA as diameter transforms into the line $B'A'$ of length $4a$, by means of transformation.



$$\zeta = z + a^2/z$$

Thus $l = 4a$. Taking the centre of the circle as origin, the complex potential in z-plane is given by

$$w = Uze^{i\alpha} + \frac{Ua^2 e^{-i\alpha}}{z} + \frac{ik}{2\pi}\log z$$

$$\therefore \quad \frac{dw}{dz} = Ue^{i\alpha} - \frac{Ua^2 e^{-i\alpha}}{z^2} + \frac{ik}{2\pi z} \qquad ...(1)$$

Stagnation points corresponding to $z = -a$ are given by $dw/dz = 0$

i.e. $Ue^{i\alpha} - Ue^{-i\alpha} - \dfrac{ik}{2\pi a} = 0$ or $U \times 2i\sin\alpha = \dfrac{ik}{2\pi a}$

$\therefore \quad k = 4\pi a U \sin\alpha = \pi l U \sin\alpha,$ as $l = 4a$

We now use the Blasius theorem and obtain

$$X - iY = \frac{1}{2}i\rho\int\left(\frac{dw}{d\zeta}\right)^2 d\zeta = \frac{1}{2}i\rho\int\frac{(dw/dz)^2}{1-a^2/z^2}dz \qquad \text{[Do as in Art.8.6]}$$

and $\quad N = \text{real part of } -\dfrac{1}{2}\rho\displaystyle\int \left(\frac{dw}{dz}\right)^2 \zeta\, d\zeta = -\frac{1}{2}\rho\int \frac{z+a^2/z}{1-a^2/z^2}\left(\frac{dw}{dz}\right)^2 dz$ $\quad$ [do as in Art. 8.6]

Now $$\frac{1}{1-a^2/z^2}\left(\frac{dw}{dz}\right)^2 = \left(1+\frac{a^2}{z^2}+\ldots\right)\left[Ue^{i\alpha} - \frac{Ue^{-i\alpha}}{z^2} + \frac{ik}{2\pi z}\right]^2, \text{ using (1)}$$

$$\therefore \qquad \text{Coeff. of } \frac{1}{z} = \frac{2Uike^{i\alpha}}{2\pi}$$

Hence $\quad X - iY = \dfrac{1}{2}i\rho\dfrac{2Uike^{i\alpha}}{2\pi}\times 2\pi i = -i\rho kUe^{i\alpha} = -i\rho l\pi U^2 e^{i\alpha}\sin\alpha, \quad \text{as } k = \pi lU\sin\alpha$

Thus, $$X - iY = -i\rho l\pi U^2 \sin\alpha\,(\cos\alpha + i\sin\alpha)$$

$$\therefore \qquad X = \pi\rho lU^2\sin^2\alpha \qquad \text{and} \qquad Y = \pi\rho lU^2 \sin\alpha\cos\alpha$$

and $\qquad$ resultant force $= R = \sqrt{X^2+Y^2} = \pi\rho lU^2\sin\alpha$

acting at angle $\qquad \theta = \tan^{-1}(Y/X) = \pi/2 - \alpha,$

showing that R is perpendicular to the stream.

Now, $$\frac{z+a^2/z}{1-a^2/z^2}\left(\frac{dw}{dz}\right)^2 = \left(z+\frac{2a^2}{z}\right)\left[Ue^{i\alpha} - \frac{Ua^2c^{-i\alpha}}{z^2} + \frac{ik}{2\pi z}\right]^2$$

$$x - iY = -ipl\pi U^2 \sin\alpha\,(\sin\alpha + i\cos\alpha)$$

$$\therefore \qquad \text{Coeff. of } \frac{1}{z} = Ue^{2i\alpha}\cdot 2a^2 - \frac{k^2}{4\pi^2} - 2U^2a^2$$

$$\therefore \qquad N = \text{real part of } -(1/\rho)\times[2\,U^2a^2e^{2i\alpha} - 2a^2U^2]\times 2\pi i$$

$$= 2\pi\rho U^2a^2\sin 2\alpha = (1/8)\times\pi\rho l^2U^2\sin 2\alpha$$

Thus the system reduces to a force R at O, and a couple N about O. Let it be equivalent to a force R at C where $OC = x$. Then $\qquad R\,x\cos\alpha = N$

$$\therefore \qquad \pi\rho lU^2\sin\alpha\cdot x\cos\alpha = (1/8)\times\pi\rho l^2U^2\sin^2\alpha \qquad \text{so that} \qquad x = l/4$$

Ex. 2. *Explain the derivation of Joukowski aerofoil by the transformation* $\zeta = z + \sum_{r=1}^{n}(a_r/z^r)$ *applied to a circle of centre* z_0 *and radius a. Obtain the lift formula* $L = 4\pi\rho aU^2\sin(\alpha+\beta)$ *and show that the momentum about the point* $\zeta = z_0$ *is* $M = 2\pi\rho b^2U^2\sin 2(\alpha+\gamma)$ *where* α *is the angle of attack and b,* β, γ *constants of transformation.*

Sol. Refer Art 8.5 and take a in place of b. Then for the present problem $k = 4\pi aU\sin(\alpha+\beta)$ and the stagnation points are given by $z - z_0 = ae^{i(\pi+\beta)}$.

Let X and Y be the components of lift and N be the moment of the lift force in ζ-plane. Then by Blasius theorem, we have

$$X - iY = \frac{1}{2}i\rho\int (dw/d\zeta)^2\, d\zeta \qquad \ldots(1)$$

and $$N = \text{real part of } -\frac{\rho}{2}\int (dw/d\zeta)^2\, \zeta\, d\zeta \qquad ...(2)$$

Then [refer Art. 8.5], we have

$$\frac{dw}{dz} = Ue^{i\alpha} - \frac{Ua^2 e^{-i\alpha}}{(z-z_0)^2} + \frac{ik}{2\pi(z-z_0)} \qquad ...(3)$$

Given $$\zeta = z + \sum_{r=1}^{n} \frac{a_r}{z^r} = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots \qquad ...(4A)$$

so that $$\frac{d\zeta}{dz} = 1 - \frac{a_1}{z^2} + \dots \qquad \text{or} \qquad d\zeta = \left(1 - \frac{a_1}{z^2} + \dots\right) dz \qquad ...(4B)$$

Then, $$\frac{dw}{d\zeta} = \frac{dw}{dz}\cdot\frac{dz}{d\zeta} = \left(1 - \frac{a_1}{z^2} + \dots\right)^{-1} \frac{dw}{dz}, \text{ using (4B)} \qquad ...(5)$$

Using (4) and (5), we have

$$\left(\frac{dw}{d\zeta}\right)^2 d\zeta = \left(\frac{dw}{dz}\right)^2 \cdot \left(1 - \frac{a_1}{z^2} + \dots\right)^{-2} \cdot \left(1 - \frac{a_1}{z^2} + \dots\right) dz$$

$$= \left(1 - \frac{a_1}{z^2} + \dots\right)^{-1} \left(\frac{dw}{dz}\right)^2 dz = \left(1 + \frac{a_1}{z^2} + \dots\right)\left(\frac{dw}{dz}\right)^2 dz \qquad ...(6)$$

From, (4A) and (6), we have

$$\left(\frac{dw}{d\zeta}\right)^2 \zeta\, d\zeta = \left(1 + \frac{a_1}{z^2} + \dots\right)\left(z + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots\right)\left(\frac{dw}{dz}\right)^2 dz = \left(z + \frac{2a_1}{z_1} + \dots\right)\left(\frac{dw}{dz}\right)^2 dz \qquad ...(7)$$

Using (1), (3) and (6) and proceeding as Art. 8.6, we have

$$X - iY = i\rho k U e^{i\alpha} = i\rho k U (\cos\alpha + i\sin\alpha)$$

so that $$X = \rho k U \sin\alpha \qquad \text{and} \qquad Y = \rho k U \cos\alpha \qquad ...(8)$$

Then the lift force $L\left(= \sqrt{X^2 + Y^2}\right)$ is given by

$$L = k\rho U = 4\pi a \rho U^2 \sin(\alpha + \beta), \text{ putting value of } k$$

Again, using (2), (3) and (7) and proceeding as Art. 8.6, we have

$$N = 2\pi\rho U^2 [a^2 \sin 2(\alpha+\beta) + a_1 \sin 2\alpha]$$

$\therefore$ M = moment about z_0

$= N + xY - yX = N + Ya\cos\beta - Xa\sin\beta = N + a\rho k U \cos(\alpha+\beta)$, by (8)

$= 2\pi\rho U^2 [a^2 \sin 2(\alpha+\beta) + a_1 \sin 2\alpha] + 2\pi\rho a^2 U^2 \sin 2(\alpha+\beta)$

$= 2\pi\rho U^2 [2a^2 \sin 2(\alpha+\beta) + a_1 \sin 2\alpha] = 2\pi\rho U^2 [(2a^2\cos 2\beta + a_1)\sin 2\alpha + 2a^2 \sin 2\beta \cos 2\alpha]$

$= 2\pi\rho U^2 [b^2 \cos 2\gamma \sin 2\alpha + b^2 \sin 2\gamma \cos 2\alpha]$,

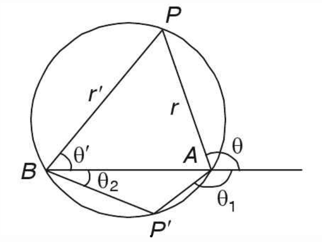
taking $\quad 2a^2 \cos 2\beta \sin + a_1 = b^2 \cos 2\gamma \quad$ and $\quad 2a^2 \sin 2\beta = b^2 \sin 2\gamma$

Thus, $$M = 2\pi\rho U^2 b^2 \sin 2(\alpha + \gamma)$$

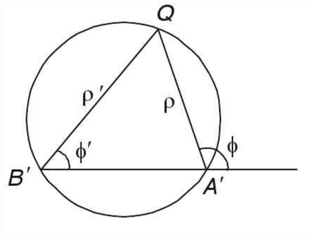
Ex. 3. *Show that the relation* $\zeta = z + 1/z$ *can be used to transform the circumference of a given circle into (i) a circular arc of given angle, (ii) a circular arc whose chord is of any prescribed length not exceeding twice the diameter of the given circle.*

Sol. Given $$\zeta = z + 1/z \qquad \text{...(1)}$$

$$\therefore \qquad \frac{\zeta - 2}{\zeta + 2} = \left(\frac{z-1}{z+1}\right)^2 \qquad \text{...(2)}$$



z - plane



ζ - plane

Then (2) shows that the points A (z = 1), B (z = − 1) are transformed into points $A'(\zeta = 2)$, $B'(\zeta = -2)$. Let a point P be mapped on Q. Then (2) reduces to

$$\frac{\rho e^{i\phi}}{\rho' e^{i\phi'}} = \left(\frac{re^{i\theta}}{r'e^{i\theta'}}\right)^2 = \left(\frac{r}{r'}\right)^2 e^{2i(\theta-\theta')} \qquad \text{so that} \qquad \phi - \phi' = 2(\theta - \theta')$$

For a point P' on the circle, we have

$$\left(\frac{z-1}{z+1}\right)^2 = \left(\frac{r_1 e^{-i\theta_1}}{r_1' e^{-i\theta_2}}\right)^2 = \left(\frac{r_1}{r_1'}\right)^2 e^{-2i(\theta_1-\theta_2)} = \left(\frac{r_1}{r_1'}\right)^2 e^{-2i[\pi-(\theta-\theta')]} = \left(\frac{r_1}{r_1'}\right)^2 e^{2i(\theta-\theta')}$$

For a given circle $AB/\sin(\theta - \theta')$ is constant. Hence when $\theta - \theta'$ is given, AB is known or when AB is given $\theta - \theta'$ is known. When $\theta - \theta'$ is given, we see that for both the points P and P', $\phi - \phi'$ is the same and hence we obtain only one point Q. Hence the circle is transformed into the arc $A'QB'$. When AB is given (say equal to l and less than the diameter of the given circle) then by suitably choosing units, AB is transformed into $A'B'$, where $A'B' = 2l <$ twice the diameter of the given circle, Since $A'B'$ is given, therefore the circle will be transformed into a circular arc whose chord is $A'B'$.

Ex. 4. *The irrotational motion in two dimensions of a fluid bounded by the lines y = 0, y = a is due to a doublet of strength μ at the origin, the axis of the doublet being in the positive direction of x. Prove that the motion is given by $w = (\pi\mu/2a)\coth(\pi z/2a)$.*

Find the streamlines and show that those points where the fluid is moving parallel to the axis of y lie on the curve $\cosh(\pi x/a) = \sec(\pi y/a)$.

Sol. Refer figure of Art 8.9 The necessary transformation for mapping the infinite strip of z-plane bounded by y = 0, y = a into real axis of ζ-plane is

$$\zeta = e^{\pi z/a} \qquad \text{...(1)}$$

From (1) we see that origin (*i.e.* z = 0) in z-plane corresponds to $\zeta = 1$ in ζ-plane. Further the doublet of strength μ at z = 0 is transformed to a doublet strength μ' at $\zeta = 1$ with its axis along positive direction of ξ-axis.

Also $$\mu' = \mu h = \left|\frac{d\zeta}{dz}\right| \mu = \left|\frac{\pi\mu}{a} e^{\pi z/a}\right| = \frac{\pi\mu}{a} \quad \text{when } z = 0$$

The complex potential is given by

$$w = \frac{\mu'}{\zeta - 1} = \mu' = \left[\frac{1}{e^t - 1}\right], \quad \text{where } t = \frac{\pi z}{a}$$

$$= \mu' \left[\frac{1}{e^t - 1} + \frac{1}{2} - \frac{1}{2}\right] = \frac{\mu'}{2}\frac{e^t + 1}{e^t - 1} - \frac{\mu'}{2} = \frac{\mu'}{2}\frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}} - \frac{\mu'}{2}$$

$$= \frac{\mu'}{2}\left(\coth\frac{t}{2} - 1\right) = \frac{\pi\mu}{2a}\coth\frac{\pi z}{2a} - \frac{\pi\mu}{2a}$$

$\therefore$ $$w = \frac{\pi\mu}{2a}\coth\frac{\pi z}{2a}. \qquad (\because \text{ the constant may be omitted})$$

Let $w = \phi + i\psi$ and $z = x + iy$. Then, we obtain

$$\phi + i\psi = \frac{\pi\mu}{2a}\coth\frac{\pi(x+iy)}{2a} = \frac{\pi\mu}{2a}\,\frac{2\cosh\dfrac{\pi(x+iy)}{2a}\sinh\dfrac{\pi(x-iy)}{2a}}{2\sinh\dfrac{\pi(x+iy)}{2a}\sinh\dfrac{\pi(x-iy)}{2a}} = \frac{\pi\mu}{2a}\,\frac{\sinh\dfrac{\pi x}{a} - i\sin\dfrac{\pi y}{a}}{\cosh\dfrac{\pi x}{a} - \cos\dfrac{\pi y}{a}}$$

Hence the streamlines are given by

$$\psi = -\frac{\pi\mu}{2a}\,\frac{\sin(\pi y/a)}{\cosh(\pi x/a) - \cos(\pi y/a)} = \text{const.} \qquad \text{...(2)}$$

Points where the motion is parallel to the y-axis, $u = 0$

$\therefore$ $\partial\psi/\partial y = 0$, as $u = -\partial\psi/\partial y$

or $$-\frac{\pi\mu}{2a}\,\frac{\dfrac{\pi}{a}\cos\left(\dfrac{\pi y}{a}\right)\left[\cosh\dfrac{\pi x}{a} - \cos\dfrac{\pi y}{a}\right] - \dfrac{\pi}{a}\sin^2\dfrac{\pi y}{a}}{[\cosh(\pi x/a) - \cos(\pi y/a)]^2} = 0$$

or $$\left(\cosh\frac{\pi x}{a} - \cos\frac{\pi y}{a}\right)\cos\frac{\pi y}{a} - \sin^2\frac{\pi y}{a} = 0$$

or $$\cosh\frac{\pi x}{a}\cos\frac{\pi y}{a} - 1 = 0 \qquad \text{or} \qquad \cosh\frac{\pi x}{a} = \sec\frac{\pi y}{a}.$$

Ex. 5. *Find the motion in the space bounded by* $x = 0$, $y = 0$, $y = b$ *due to a source at the origin.* **[Rohilkhand 2000, 05]**

Sol. Refer figure of Art. 8.8 We have the transformation :

$$\zeta = \cosh(\pi z/b) \qquad \text{...(1)}$$

Here we have a source $+m$ at $z = 0$ *i.e.* at $\zeta = 1$ (when $z = 0$, from (1), $\zeta = \cosh 0 = 1$).

The image of source $+m$ at $\zeta = 1$ relative to ξ-axis is a source $+m$ at $\zeta = 1$. The complex potential is given by

$$w = -2m\log(\zeta - 1) \qquad \text{...(2)}$$

or $$w = -2m\log\left(\cosh\frac{\pi z}{b} - 1\right) \qquad \text{...(3)}$$

From (2), $\phi + i\psi = -2m\log(\xi + i\eta - 1)$, giving $\psi = -2m\tan^{-1}\{\eta/(\xi - 1)\}$

Hence lines of flow are given by ψ = constant, *i.e.*, $\xi - 1 = c\eta$ ...(4)

From (1), $\xi + i\eta = \cosh\dfrac{\pi(x+iy)}{b}$ or $\xi + i\eta = \cos\dfrac{\pi(ix - y)}{b}$ $[\because \cosh\theta = \cos i\theta]$

or $$\xi + i\eta = \cos\frac{\pi ix}{b}\cos\frac{\pi y}{b} + \sin\frac{\pi ix}{b}\sin\frac{\pi y}{b}$$

or, $$\xi + i\eta = \cosh\frac{\pi x}{b}\cos\frac{\pi y}{b} + i\sinh\frac{\pi x}{b}\sin\frac{\pi y}{b} \qquad \text{...(5)}$$

Equating real and imaginary parts on both sides of (5), we get

$$\xi = \cosh\frac{\pi x}{b}\cos\frac{\pi y}{b} \qquad \text{and} \qquad \eta = \sinh\frac{\pi x}{b}\sin\frac{\pi y}{b}$$

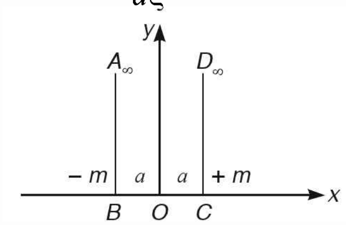
Putting these values in (4), the lines of flow are given by

$$\cosh(\pi x/b)\cos(\pi y/b) - 1 = c\sinh(\pi x/b)\sin(\pi y/b).$$

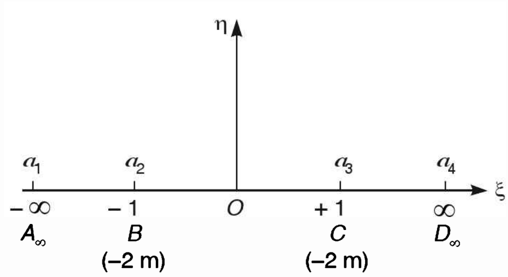
Ex. 6. *Fluid motion is taking place in the part of the plane bounded by the real axis and the lines $x = \pm a$, which is due to a source at the corner and a sink at the other corner of the strip, each of strength m, show that the motion is given by* $\tanh(w/8m) = \tan(\pi z/4a)$ *and that the streamline which leaves the source at angles* $\pi/4$ *to the sides is* $\cos(\pi x/2a) = \sinh(\pi y/2a)$

Sol. We wish to transform the rectangle $A_\infty BCD_\infty$ bounded by the lines $x = -a$, $y = 0$, $x = a$ of z-plane into ξ-axis of ζ-plane by using Schwarz – Christoffel transformation

$$\frac{dz}{d\zeta} = k(\zeta - a_1)^{\frac{\alpha_1}{\pi}-1}(\zeta - a_2)^{\frac{\alpha_2}{\pi}-1}(\zeta - a_3)^{\frac{\alpha_3}{\pi}-1}(\zeta - a_4)^{\frac{\alpha_4}{\pi}-1} \qquad \text{...(1)}$$



z - plane



ζ - plane

Let us suppose that A_∞, B, C be mapped on $A_\infty(\zeta = -\infty)$, $B(\zeta = -1)$, $C(\zeta = 1)$ respectively. Then clearly D_∞ will be mapped on $D_\infty(\zeta = \infty)$. Hence, here $a_1 = -\infty$, $a_2 = -1$, $a_3 = 1$, $a_4 = \infty$, $\alpha_2 = \alpha_3 = \pi/2$. Note that brackets corresponding to a_1, a_4 will not occur in the transformation (1). Hence (1) reduces to

$$\frac{dz}{d\zeta} = k(\zeta+1)^{1/2-1}(\zeta-1)^{1/2-1} \qquad \text{or} \qquad dz = \frac{k\,d\zeta}{\sqrt{(\zeta^2-1)}} \qquad \text{...(2)}$$

Integrating (2), $\quad z = k\cosh^{-1}\zeta + A$, A being an arbitrary constant ...(3)

$$\left.\begin{array}{llll} \text{But} & \text{when} & z = -a, & \zeta = -1 \\ \text{and} & \text{when} & z = a, & \zeta = 1 \end{array}\right\} \qquad \text{...(4)}$$

Using conditions (4), (3) gives

$$-a = k\cosh^{-1}(-1) + A = ki\pi + A \qquad \text{...(5)}$$

and $$a = k\cosh^{-1}(1) + A = A \qquad \text{...(6)}$$

$$[\because \cosh^{-1}(-1) = \log[-1 + \sqrt{\{(-1)^2 - 1^2\}}] = \log(-1) = \log e^{i\pi} = i\pi$$

$$\text{and } \cosh^{-1}1 = \log[1 + \sqrt{(1^2 - 1^2)}] = \log 1 = 0].$$

Solving (5) and (6), $A = a,$ and $k = 2ai/\pi$

$\therefore$ From (3), $$z = (2ai/\pi)\cosh^{-1}\zeta + a \qquad \text{...(7)}$$

From (7), $$\zeta = \cosh\left\{\frac{\pi}{2ai}(z-a)\right\} = \cosh\left\{\frac{\pi i(a-z)}{2a}\right\} = \cos\left\{\frac{\pi(a-z)}{2a}\right\} = \cos\left(\frac{\pi}{2}-\frac{\pi z}{2a}\right) = \frac{\pi z}{2a}$$

The required transformation is $$\zeta = \sin(\pi z/2a) \qquad \text{...(8)}$$

Hence if there be a source of strength m at $x = a$, and an equal sink at $x = -a$, we will have a source of strength $2m$ at $\xi = +1$, and a sink of strength $-2m$ at $\xi = -1$. Therefore, the complex potential is given by

$$w = -2m\log(\zeta-1) + 2m\log(\zeta+1)$$

or $$w = -2m\log\frac{\zeta-1}{\zeta+1} = -2m\log\left[(-1)\frac{1-\zeta}{1+\zeta}\right] = -2m\log\frac{1-\zeta}{1+\zeta} = -2m\log(-1).$$

Omitting the constant, we have

$$w = -2m\log\frac{1-\zeta}{1+\zeta} = -2m\log\frac{1-\sin(\pi z/2a)}{1+\sin(\pi z/2a)}, \text{ by (8)}$$

$\therefore$ $$w = -2m\log\frac{1-\sin t}{1+\sin t}, \quad \text{taking} \quad t = \pi z/2a$$

$$= -2m\log\left(\frac{\cos t/2 - \sin t/2}{\cos t/2 + \sin t/2}\right)^2 = -4m\log\frac{\cos t/2 - \sin t/2}{\cos t/2 + \sin t/2}$$

$\therefore$ $$-\frac{w}{4m} = \log\frac{\cos t/2 - \sin t/2}{\cos t/2 + \sin t/2} = \log\frac{1-\tan t/2}{1+\tan t/2} \qquad \text{...(9)}$$

or $$e^{-w/4m} = \frac{1-\tan t/2}{1+\tan t/2} \quad \text{or} \quad \frac{1-e^{-w/4m}}{1+e^{-w4m}} = \tan t/2 \quad \text{or} \quad \frac{e^{w/8m}-e^{-w/8m}}{e^{w/8m}+e^{-w8m}} = \tan\frac{\pi z}{4a}$$

or $$\tanh(w/8m) = \tan(\pi z/4a) \qquad \text{...(10)}$$

Again, from (9), we have

$$w = -4m\log\tan\left(\frac{\pi}{4}-\frac{\pi z}{4a}\right), \textit{as } t = \frac{\pi z}{2a}$$

$$= -4m\log\tan\frac{\pi}{4a}(a-z) = -4m\log\tan\frac{\pi}{4a}(a-x-iy)$$

$$= -4m\log\frac{2\sin\frac{\pi}{4a}(a-x-iy)\cos\frac{\pi}{4a}(a-x+iy)}{2\cos\frac{\pi}{4a}(a-x-iy)\cos\frac{\pi}{4a}(a-x+iy)} = -4m\log\frac{\sin\frac{\pi}{2a}(a-x)-i\sinh\frac{\pi y}{2a}}{\cos\frac{\pi}{2a}(a-x)+\cosh\frac{\pi y}{2a}}$$

or $$\phi + i\psi = -4m\left[\log\left\{\sin\frac{\pi}{2a}(a-x)-i\sinh\frac{\pi y}{2a}\right\}-\log\left\{\cos\frac{\pi}{2a}(a-x)+\cosh\frac{\pi y}{2a}\right\}\right]$$

$$\therefore \qquad \psi = -4m\tan^{-1}\left\{\frac{\sinh(\pi y/2a)}{\sin\{\pi(a-x)/2a\}}\right\} = 4m\tan^{-1}\frac{\sinh(\pi y/2a)}{\cos(\pi x/2a)}$$

Hence the streamlines are given by ψ = const. *i.e.*, by

$$\sinh(\pi y/2a) = C\cos(\pi x/2a) \qquad \text{...(11)}$$

For the streamlines leaving the source at $\pi/4$, we have

$$(dy/dx)_{x=a,\,y=0} = -1 \qquad \text{...(12)}$$

From (11), on differentiation w.r.t. 'x' we get

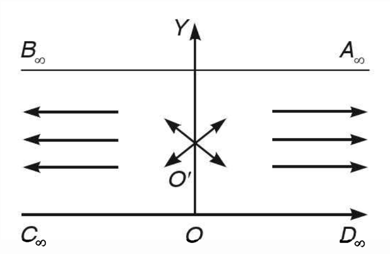
$$\frac{\pi}{2a}\cosh\frac{\pi y}{2a}\cdot\frac{dy}{dx} = -C\cdot\frac{\pi}{2a}\sin\frac{\pi x}{2a} \qquad \text{...(13)}$$

Putting $x = a$, $y = 0$ and using (12), (13) gives $C = 1$.

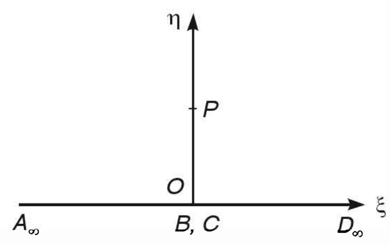
Hence the required streamlines are given by $\sinh(\pi y/2a) = \cos(\pi x/2a)$.

Ex. 7. *A source is placed midway between two planes whose distance from one another is 2a. Find the equation of the streamlines when the motion is in two dimensions and show that those particles which at an infinite distance are distance a/2 from one of the boundaries, issued from the source in a direction making an angle* $\pi/4$ *with it.*

Sol. The transformation $\zeta = ie^{\pi z/2a}$ transforms the strip of breadth $2a$ in the z-plane into the upper half of the plane ζ-plane, the origin O' in the z-plane being midway between the two walls. The points B_∞, C_∞ coincide with (B, C), $\zeta = 0$.



z - plane



ζ - plane

When $z = 0$, $\zeta = i$, *i.e.*, the point P in the ζ-plane.

Thus in the z-plane there is a source m at O' and equal sink at infinite distance, so in the ζ-plane there will be a source m at P and a sink ($-m$) at (B, C) and hence an image source m at the point $\zeta = -i$.

$$\therefore \qquad w = -m\log(\zeta - i) - m\log(\zeta + i) + m\log\zeta = -m\log\frac{\zeta^2 - 1}{\zeta} = -m\log(\zeta + \zeta^{-1})$$

$$= -m\log\left(ie^{\frac{\pi z}{2a}} - ie^{-\frac{\pi z}{2a}}\right) = -m\log i\left(e^{\frac{\pi z}{2a}} - e^{-\frac{\pi z}{2a}}\right) = -m\log\left(e^{\frac{\pi z}{2a}} - e^{-\frac{\pi z}{2a}}\right) - m\log i$$

Omitting the constant, we take

$$w = -m\log\left(e^{\frac{\pi z}{2a}} - e^{-\frac{\pi z}{2a}}\right) \qquad \text{or} \qquad w = -m\log(e^{cz} - e^{-cz}), \qquad \text{...(1)}$$

where $$c = \pi/2a \qquad ...(2)$$

$\therefore \quad w = -m\log[e^{c(x+iy)} - e^{-c(x+iy)}] = -m\log[e^{cx}(\cos cy + i\sin cy) - e^{-cx}(\cos cy - i\sin cy)]$

$= -m\log[\cos cy(e^{cx} - e^{-cx}) + i\sin cy(e^{cx} + e^{-cx})]$

$\therefore\ \phi + i\psi = -m\log[2\cos cy\sinh cx + 2i\sin cy\cosh cx]$, as $w = \phi + i\psi$

and so $$\psi = -m\tan^{-1}\frac{2\sin cy\cosh cx}{2\cos cy\sinh cx} = -m\tanh^{-1}\left(\frac{\tan cy}{\tanh cx}\right)$$

Streamlines are given by ψ = const, *i.e.*, $\tan cy = K\tanh cx$

or $\tan(\pi y/2a) = K\tanh(\pi x/2a)$, Using (2)

When $x = \infty$, $y = a/2$. Hence $K = 1$

$\therefore$ Streamlines become $$\tan(\pi y/2a) = \tanh(\pi x/2a) \qquad ...(3)$$

Differentiating (3) w.r.t 'x' we have

$$\sec^2\frac{\pi y}{2a}\cdot\frac{dy}{dx} = \operatorname{sech}^2\frac{\pi x}{2a} \quad \text{or} \quad \frac{dy}{dx} = \frac{\operatorname{sech}^2(\pi x/2a)}{\sec^2(\pi y/2a)}$$

At the origin (where the source is situated in z-plane) $x = y = 0$ so that $dy/dx = 1$ there. Hence we have $\tan\theta = 1$ or $\theta = \pi/4$, as required.

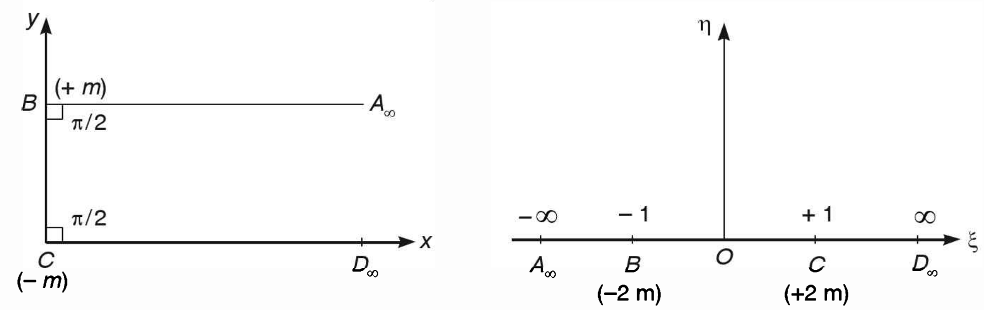
Ex. 8. *A rectangle open at infinity in the x-direction has solid boundaries along $x = 0$, $y = 0$ and $y = a$. Fluid of amount $2\pi m$ flows into and out of the rectangle at the corners $x = 0$, $y = 0$ and $x = 0$ $y = a$ respectively. Prove that the motion of the fluid is given by*

$$w = 4m\log\tanh(\pi z/2a)$$

Also show that half the streamlines lie between $x = 0$ and the streamline which cuts $y = a/2$ at the point $x = (a/\pi)\log(1+\sqrt{2})$.

Sol. As in Art. 8.8, the given semi-infinite strip in z-plane is transformed into ζ-plane by transformation

$$\zeta = \cosh(\pi z/a) \qquad ...(1)$$



Given that there is a source of strength m at $B(0, a)$ and a sink of strength $-m$ at $C(0, 0)$. Since B and C of z-plane are mapped at $B(\zeta = -1)$ and $C(\zeta = 1)$ in ζ-plane, we have a source of strength $+2m$ at $\zeta = -1$ and sink of strength $-2m$ at $\zeta = 1$.

$\therefore$ the complex potential is given by

$$w = -2m\log(\zeta+1) + 2m\log(\zeta-1)$$

$$= 2m\log\frac{\zeta-1}{\zeta+1} = 2m\log\frac{\cosh(\pi z/a)-1}{\cosh(\pi z/a)+1} = 2m\log\left(\tanh\frac{\pi z}{2a}\right)^2$$

Thus, $$w = 4m\log\tanh(\pi z/2a) \qquad ...(2)$$

From (2), we have

$$\phi + i\psi = 4m \log \tanh \frac{\pi(x+iy)}{2a}$$

or $$\phi + i\psi = 4m \log \frac{2\sinh \frac{\pi(x+iy)}{2a} \cosh \frac{\pi(x-iy)}{2a}}{2\cosh \frac{\pi(x+iy)}{2a} \cosh \frac{\pi(x-iy)}{2a}} = 4m \log \frac{\sinh(\pi x/a) - i\sin(\pi y/a)}{\cosh(\pi x/a) - \cos(\pi y/a)}$$

or $$\phi + i\psi = 4m \log[\sinh(\pi x/a) - i\sin(\pi y/a)] - 4m\log[\cosh(\pi x/a) - \cos(\pi y/a)]$$

Equating imaginary parts on both sides, we have

$$\psi = 4m \tan^{-1} \frac{\sin(\pi y/a)}{\sinh(\pi x/a)} \qquad ...(3)$$

At $x = 0$, $\psi = \psi_0 = 4m \tan^{-1} \infty = 4m \times (\pi/2) = 2\pi m$

At $x = \infty$, $\psi = \psi_\infty = 4m \tan^{-1} 0 = 4m \times 0 = 0$

$\therefore$ Total flow $= \psi_0 - \psi_\infty = 2\pi m - 0 = 2\pi m \Rightarrow$ half the total flow $= (1/2) \times 2\pi m = \pi m$

Hence the corresponding streamline is given by

$$\pi m = 4m \tan^{-1} \frac{\sin(\pi y/a)}{\sinh(\pi x/a)} \quad \text{or} \quad \frac{\sin(\pi y/a)}{\sinh(\pi x/a)} = \tan\frac{\pi}{4}$$

$$\therefore \qquad \sin(\pi y/a) = \sinh(\pi x/a) \qquad ...(4)$$

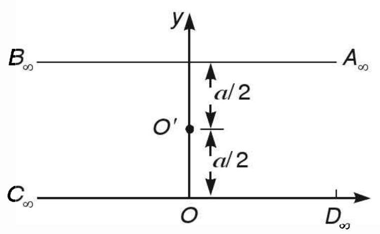
Now, when $y = a/2$, from (4) we have $\sinh(\pi x/a) = \sin \pi/2 = 1$

or $\pi x/a = \sinh^{-1} 1 = \log(1 + \sqrt{(1^2 + 1^2)})]$ so that $x = (a/\pi) \times \log(1 + \sqrt{2})$.

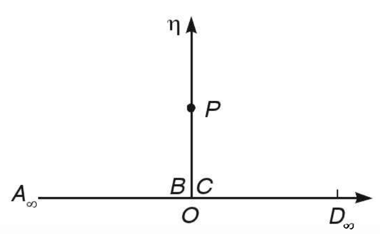
Ex. 9. *Use the transformation* $\zeta = e^{\pi z/a}$ *to find the streamlines of the motion in two dimensions due to a source midway between two infinite parallel boundaries. (Assume the liquid drawn off equally by sinks at the ends of the region). If the pressure tends to zero at the ends of the streams, prove that planes are pressed apart with a force which varies inversely as their distance from each other.*

Sol. We know that the transformation $\zeta = e^{\pi z/a}$ ...(1)

transforms the infinite strip $A_\infty B_\infty C_\infty D_\infty$ in the z-plane with origin at O into the upper half in the ζ-plane with origin at (B, C) where B_∞, C_∞ go. The point $z = ai/2$ goes to $\zeta = e^{i\pi/2} = i$ the point P in ζ-plane. There is a source at O' in the z-plane and equal sinks at infinity, therefore in the ζ-plane there is a source of strength m at P, sink of strength (– m) at (B, C) and an image source at $\zeta = -i$.



z - plane



ζ - plane

The complex potential is given by

$$w = -m\log(\zeta - i) - m\log(\zeta + i) + m\log\zeta$$
$$= -m\log[(\zeta^2+1)/\zeta] = -m\log(\zeta + \zeta^{-1}) = -m\log(e^{\pi z/a} + e^{-\pi z/a}), \text{ using (1)}$$
$$= -m\log[2\cosh(\pi z/a)] = -m\log 2 - m\log\cosh(\pi z/a)$$

$\therefore$ $\quad w = -m\log\cosh(\pi z/a)$, omitting the constant term in w. ... (2)

From (2), $\quad q = \dfrac{dw}{dz} = -\dfrac{m\pi}{a}\tanh\dfrac{\pi z}{a}$ and $\quad q_\infty = \dfrac{m\pi}{a}$

We know that $\quad \dfrac{p}{\rho} + \dfrac{1}{2}q^2 = \text{const.} = \dfrac{1}{2}q_\infty^2,$ as $\quad p_\infty = 0$

$$\Rightarrow \quad \frac{p}{\rho} = \frac{\pi^2 m^2}{2a^2}\left(1 - \tanh^2\frac{\pi z}{a}\right) = \frac{\pi^2 m^2}{2a^2}\,\text{sech}^2\frac{\pi z}{a} \qquad \text{...(3)}$$

Now, any point on the upper boundary is $z = x + ia$ and hence (3) gives

$$\frac{p}{\rho} = \frac{\pi^2 m^2}{2a^2}\,\text{sech}^2\left(\frac{\pi x}{a} + i\pi\right) = \frac{\pi^2 m^2}{2a^2}\,\text{sech}^2\frac{\pi x}{a}$$

If F be the force with which the planes are pressed apart, then we have

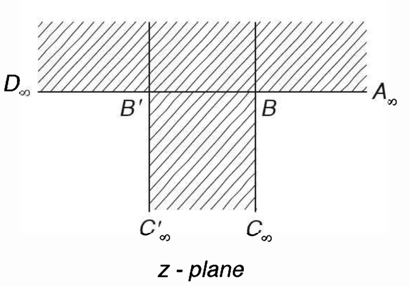
$$F = 2\int_0^\infty p\,dx = \frac{\pi^2 \rho m^2}{a^2}\int_0^\infty \text{sech}^2\frac{\pi x}{a}\,dx = \frac{\pi\rho m^2}{a^2}\cdot\frac{a}{\pi}\left[\tanh\frac{\pi x}{a}\right]_0^\infty = \frac{\pi\rho m^2}{a},$$

showing that $F \propto 1/a$ *i.e.* the force varies inversely as the distance between the planes apart.

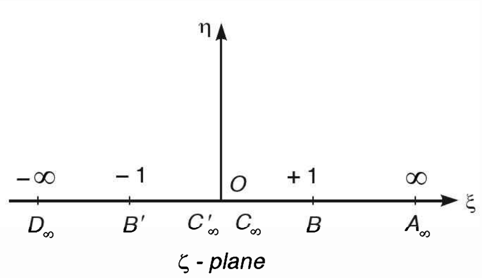
Ex. 10. *Show that the transformation* $z = (a/\pi)\times\{(\zeta^2-1)^{1/2} - \sec^{-1}\zeta\}$, $\zeta = e^{-\pi w/aV}$ *where* $z = x + iy$, $w = \phi + i\psi$, *give the flow of a straight river of breadth a, running with velocity V at right angles to the straight shore of an otherwise unlimited sea of water into which it flows.*

Sol. Given $\quad z = (a/\pi)\times[(\zeta^2-1)^{1/2} - \sec^{-1}\zeta]$

$$\therefore \quad \frac{dz}{d\zeta} = \frac{a}{\pi}\left[\frac{\zeta}{(\zeta^2-1)^{1/2}} - \frac{1}{\zeta(\zeta^2-1)^{1/2}}\right] = \frac{a}{\pi}\frac{(\zeta^2-1)^{1/2}}{\zeta}$$



z - plane



ζ - plane

The points A_∞, B in the z-plane correspond to points A_∞, $B(\zeta = 1)$ in ζ-plane, C_∞, C'_∞ in z-plane coincide at $\zeta = 0$, B' in the z-plane goes to $\zeta = -1$, D_∞ to D_∞. Since $dz/d\zeta$ can be re-written as

$$\frac{dz}{d\zeta} = \frac{a}{\pi}\zeta^{\frac{0}{\pi}-1}(\zeta-1)^{\frac{3\pi}{2\pi}-1}(\zeta+1)^{\frac{3\pi}{2\pi}-1},$$

it follows that the angles at the points $B, C_\infty, C'_\infty, B'$ in the z-plane are $3\pi/2$, 0, 0 $3\pi/2$, hence the bends in the z-plane.

Now, from $\zeta = e^{-\pi w/aV}$ or $w = -(aV/\pi)\log\zeta$, we see that as ζ varies from ∞ to 1 then to 0, *i.e.* along $A_\infty BO$, w varies from $-\infty$ to 0 then to ∞ i.e. ϕ from $-\infty$ to ∞ *i.e.* ϕ varies from ∞ at C_∞, C'_∞ to $\phi = -\infty$ at A_∞ and $\psi = 0$ along $A_\infty\, B\, O'$ *i.e.*, along $A_\infty\, B\, C_\infty \ldots$

Again when ζ varies from 0 to -1 then $-\infty$, w varies from $\infty + i\pi(aV/r)$ to iaV then to $-\infty + iaV$ *i.e.*, from $OB'D_\infty$ in ζ-plane, $\psi = aV$ along $OB'D_\infty$ *i.e.* along $C'_\infty B' D_\infty$ since breadth $BB' = a$, the velocity of the stream is V.

EXERCISE

1. State and prove the theorem of Schwarz and Christoffel in conformal transformation.

2. Write a short note on Joukowski condition.

3. Use the transformation $\zeta = e^{\pi z/a}$ to find the streamline in two dimensions due to a source midway between infinite parallel boundaries.

4. There is a source of strength $+ m$ in the liquid filling the space between two parallel planes distance c apart. Find the complex potential and the paths of the particles.

5. Two dimensional irrotational motion of a fluid in the semi-infinite rectangular region $0 \le x < \infty$, $0 \le y \le \pi$ is due to a doublet of strength μ, at the point $z_0 = \log 2 + i\pi/2$, with axis in the negative x-direction. Show that the complex potential is given by $w = 30\mu/(9 + 16\cosh^2 z)$.

6. Write note on Schwarz-Christoffel transfrmation and its application to hydrdnamics.

7. Prove that the velocity potential and stream function of the two dimensional motion between walls $y = 0, y = \pi$, due to a source of strength m at (x_1, y_1) and an equal sink at (x_0, y_0) are given by

$$\phi + i\psi = m\log\frac{[\exp(x+iy) - \exp(x_0+iy_0)]\,[\exp(x+iy) - \exp(x_0-iy_0)]}{[\exp(x+iy) - \exp(x_1+iy_1)]\,[\exp(x+iy) - \exp(x_1-iy_1)]}$$

[**Hint.** Here the transformation is $\zeta = e^{\pi a/a} = e^z$ $[\because a = \pi]$

$\therefore$ $\zeta = e^{x+iy}$

The source and sink in -plane are at points $\zeta_1 = e^{x_1+iy_1}$, $\zeta_0 = e^{x_0+iy_0}$ and their corresponding images are at points

$\zeta_1' = e^{x_1-iy_1}$, $\zeta_0' = e^{x_0+iy_0}$ so that $w = -m\log\dfrac{(\zeta-\zeta_1)\,(\zeta-\zeta_1')}{(\zeta-\zeta_0)\,(\zeta-\zeta_0')}$

Now proceed to get the required result]

8. Determine the nature of the fluid motion in the space bounded by $y = 0$, $\pi(x^2 + y^2) - 2y = 0$ which is given by $\phi + i\psi = \coth(x+iy)^{-1}$.

OBJECTIVE QUESTION ON CHAPTER 8

1. With help of transformatin $\zeta = z + a^2/z$, The circle transforms into
   (*i*) Parbola (*ii*) aerofoil (*iii*) circle (*iv*) ellipse
2. The transformation $J = z + a^2/z$ is known as
   (*i*) Blasius transformation (*ii*) Kutta transformation
   (*iii*) Joukowski transformation (*v*) None of these

3. The Joukowski transformation is, for $c > 0$

(*a*) $t = z + c/z$ (*b*) $t = z + c^2/2$ (*c*) $t = z - c/z$ (*d*) $t = z + c^2/2z$ **(Agra 2007)**

4. For Kutta-Joukowski condition, k is equal to

(*a*) $4\pi\, aU \sin\alpha$ (*b*) $2\pi\, aU \sin\alpha$ (*c*) $2\pi\, aU \cos\alpha$ (*d*) $2\pi\, aU \sin 2\alpha$ **(Agra 2007, 08)**

Answers/Hints to objective type questions.

1. (*iii*). See Art. 8.6 **2.** (*ii*). See Art. 8.3 **3.** (*a*). See Art. 8.3

CHAPTER

9 Discontinuous Motion

9.1. Free streamlines.

Consider a streamline μ in two-dimensional motion separating the fluid into two regions A and B. Neglecting external forces, we have on streamlines of the two regions

$$\frac{p_A}{\rho_A}+\frac{1}{2}q_A^2=C_A \qquad \text{and} \qquad \frac{p_B}{\rho_B}+\frac{1}{2}q_B^2=C_B, \qquad \text{...(1)}$$

for an inviscid liquid in steady motion, the suffix denoting the region considered. Let P be any point on the streamline μ If we approach P from the region A, we obtain a value p_1 for the pressure and q_1 for the velocity. Similarly if we approach P from the region B, we obtain p_2 for pressure and q_2 for velocity. Hence from (1), we obtain

$$\frac{p_1}{\rho_A}+\frac{1}{2}q_1^2=C_1 \qquad \text{and} \qquad \frac{p_2}{\rho_B}+\frac{1}{2}q_2^2=C_2. \qquad \text{...(2)}$$

Since the pressure must be continuous, so $p_1 = p_2$. Further let $\rho_A=\rho_B$. Then, from (2),

$$\rho_A q_1^2-\rho_B q_2^2=\text{constant}. \qquad \text{...(3)}$$

We consider a motion in which the velocity is discontinuous, for example, let us suppose that the fluid in the region A is at rest so that $q_1 = 0$. Then, from (3), we see that along streamline μ, q_2 = constant. A streamline which separates fluid in motion from fluid at rest is called a *free streamline.*

9.2. Properties of the free streamlines.

Neglecting external forces, free streamlines have the following properties:

(*i*) Along a free streamline the stream function ψ is constant. This is true for all streamlines.

(*ii*) Along a free streamline the speed is a constant called the *skin speed*. Free streamlines are thus *isotachic lines*, or *lines of constant speed.*

(*iii*) Along a free streamline the pressure is constant. Free streamlines are thus *isobaric lines or isobars, i.e.lines of constant pressure*.

(*iv*) Along a fixed boundary which is also a streamline the direction of the velocity is known but not its magnitude.

Remark. To obtain a solution *i.e.* to find a suitable relation between z and w, we introduce the Kirchhoff method.

Now, $$f'(z)=\frac{dw}{dz}=\frac{d\phi+i\,d\psi}{dx+i\,dy}=\frac{(d\phi/dx+i\,d\psi/dx)\,dx+(d\phi/dy+id\psi/dy)\,dy}{dx+i\,dy}$$

And since $\dfrac{d\phi}{dy}+i\dfrac{d\psi}{dy}=i\left(\dfrac{d\phi}{dx}+i\dfrac{d\psi}{dx}\right)$, this ratio is independent of the ratio *dy/dx*.

If ϕ and ψ be the velocity potential and current function of a liquid

$$\frac{dw}{dz} = \frac{d\phi}{dx} + i\frac{d\psi}{dx} = u - iv.$$

Let q be the velocity and let the direction of the velocity be given by θ. Then, we have

$$-\frac{dz}{dw} = \frac{1}{u - iv} = \frac{u + iv}{u^2 + v^2} = \frac{u + iv}{q^2} = \frac{q\cos\theta + i\,q\sin\theta}{q^2} = \frac{e^{i\theta}}{q}.$$

Along a fixed boundary, θ is constant and along a free streamline q is constant.

Let $$\Omega = \log\left(-\frac{dz}{dw}\right) = \log\frac{1}{q} + i\theta.$$

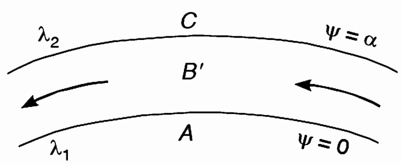
We can conformably represent z-plane on the Ω-plane. The figure in the z-plane is bounded by lines for which either θ = constant or q = constant. The fixed boundaries θ = const. on the z-plane will correspond to lines parallel to the real axis on the Ω-plane and the free streamlines (q = 1) on the z-plane will correspond to portions of the imaginary axis on the Ω-plane. It follows that a rectangular figure bounded by straight lines is represented on the Ω-plane.

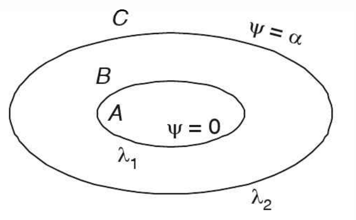
9.3. Discontinuous motion.

In the perfect fluid theory the velocity is supposed to be continuous. However, there exist motions in which the velocity is discontinuous, for example, a layer of oil flowing over a layer of calm water. The region of fluid can be divided by the surfaces of free streamlines and there exists discontinuity in the tangential velocity as we cross the surface. We know that such a surface should be unstable, but in the analysis which follows we shall proceed on the hypothesis of the existence of a steady state. In particular, we shall discuss some cases of discontinuous two-dimensional motion, such as the flow of liquid through an aperture, and the impact of a stream on a plane lamina.

9.4. Flow in jets and currents.

Neglecting external forces, consider the motion, (in two dimensions) between free streamlines λ_1, λ_2. These streamlines separate the plane into three regions A, B, C, the liquid in motion occupying the region B. If there be no liquid in A and C we have a jet. If A and C are occupied by liquids at rest we have a current. A jet or current may be closed or may extend to infinity as shown in the following figures.



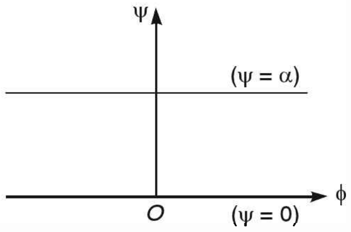


On the free streamlines both ψ and q are constants.

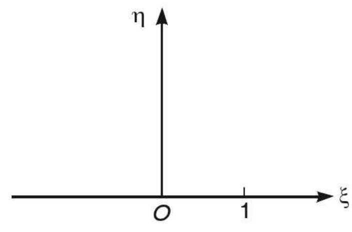
$$\left.\begin{array}{lll} \text{On } \lambda_1, & \text{let } \psi = 0, & q = U \\ \text{on } \lambda_2, & \text{let } \psi = \alpha, & q = V \end{array}\right\} \qquad \text{...(1)}$$

Thus in the region in w-plane there is an infinite strip of breadth α parallel to ϕ-axis. We transform this into the upper half of the ζ-plane so that w = 0 corresponds to $\zeta = 1$, by transformation

$$w = (\alpha/\pi)\log\zeta. \qquad \text{...(2)}$$



w - plane



ζ - plane

Now, from Art. 9.2 $\qquad \Omega = \log(1/q) + i\theta. \qquad$ ...(3)

$$\text{on } \lambda_1, \quad q = U \quad \text{and} \quad \text{on } \lambda_2, \quad q = V.$$

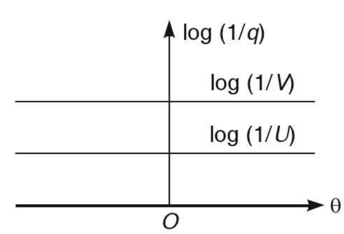
$$\therefore \quad \Omega_1 = \log(1/U) + i\theta \quad \text{and} \quad \Omega_2 = \log(1/V) + i\theta$$

$$\therefore \quad \Omega_1 - \Omega_2 = \log(U/V).$$

Thus in the Ω-plane we have a strip of breadth $\log(U/V) = \beta$ (say) parallel to θ-axis. This can be transformed into the half of the ζ-plane by means of the transformation

$$\zeta = e^{(i\pi/\beta)\Omega} \qquad \text{...(4)}$$

where $\Omega = 0$, corresponds $\zeta = 1$.



Ω - plane

From (4), $\quad \Omega = (\beta/\pi i)\log\zeta \quad$ or $\quad \log\zeta = (\pi i \Omega)/\beta$

$\therefore$ from (2), $$w = \frac{i\alpha}{\beta}\Omega$$

Also $$\Omega = \log(-dz/dw)$$

$$\therefore \quad -\frac{dz}{dw} = e^{\Omega} = e^{\beta w/i\alpha} = e^{-(i\beta w)/\alpha}$$

$$\therefore \quad dz/dw = -e^{-(i\beta w)/\alpha} \qquad \text{...(5)}$$

When $\quad V = U, \quad$ we have $\quad \beta = 0.$

$\therefore \quad dw/dz = \text{const.} \quad$ so that $\quad w = -Uz,$

showing that a jet in which the speed is the same on both boundaries must be a straight line.

When $V \neq U$, $\beta \neq 0$, we have by integration of (5)

$$z - z_0 = -\int e^{-(\beta i w)/\alpha} dw = \frac{\alpha}{i\beta} e^{-(\beta i w)/\alpha} = -\frac{i\alpha}{\beta} e^{-\frac{i\beta}{\alpha}(\phi + i\psi)} = -\frac{i\alpha}{\beta} e^{-(i\beta/\alpha)\phi} e^{(\beta/\alpha)\psi}$$

$$\therefore \quad |z - z_0| = (\alpha/\beta)\, e^{(\beta\psi)/\alpha}.$$

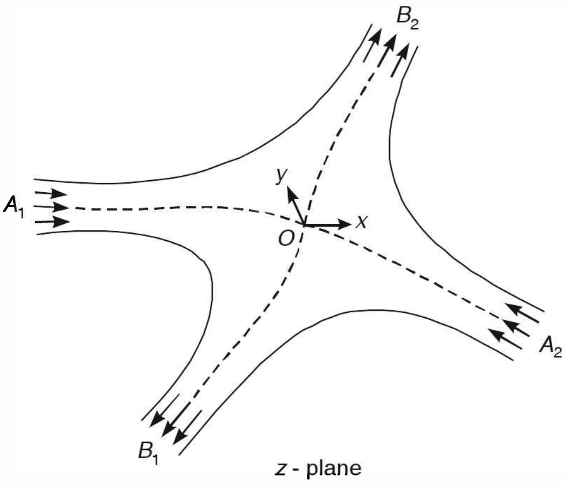
When ψ = const., $|z - z_0|$ is constant *i.e.* z describes a circle whose centre is z_0 and radius $(\alpha/\beta)\, e^{(\beta\psi)/\alpha}$ and if r_1, r_2 be the radii of λ_1, λ_2, we have

$$\frac{r_1}{r_2} = \frac{e^{(\beta\lambda_1)/\alpha}}{e^{(\beta\lambda_2)/\alpha}} = e^{-\frac{\beta}{\alpha}(\lambda_2 - \lambda_1)} = e^{-\beta} = \frac{V}{U},$$

showing that the current speeds on the free streamlines are inversely proportional to the radii. It is evident that currents bounded by free streamlines can exist and these streamlines are either parallel lines or concentric circles. Also note that in the latter case, the motion being irrotational, fluid does not rotate like a rigid ring.

9.5. Motion of two impinging jets.

Consider two uniform streams A_1, A_2 of the same speed U at infinity, meeting and branching off into two other streams, B_1, B_2. Assume that the motion is steady and A_1 and A_2 are well known. Then we wish to determine the streams B_1, B_2. If we imagine the streams or currents A_1, A_2 to advance from infinity, it is physically possible that when they meet a stagnation point O will arise. It follows that when the motion has become steady a stagnation point will continue to exist. Take this stagnation point O as origin and the x-axis as parallel to and in the direction of flow of A_1.



The free streamlines A_1B_1, B_1A_2, A_2B_2, B_2A_1 will be lines of constant speed and so the speed U at infinity of all four streams must be the same. Let h_1, h_2, k_1, k_2 be the breadths at infinity of A_1, A_2, B_1, B_2. To preserve continuity, the inflow and outflow must balance and hence we obtain

$$h_1 + h_2 = k_1 + k_2, \text{ where } h_1 \text{ and } h_2 \text{ are given and } k_1, k_2 \text{ are unknown.} \quad \ldots(1)$$

To determine complex velocity. Writing as usual,

$$\upsilon = qe^{-i\theta} = u - iv, \quad \ldots(2)$$

where q is the speed and θ is the direction of the velocity, we have on the free streamlines.

$$\upsilon = Ue^{-i\theta}, \quad \ldots(3)$$

and hence as we go round the free streamlines starting at A_1 and describing A_1B_1, B_1A_2, A_2B_2, B_2A_1 in turn, θ will vary from 0 to -2π. and therefore $-\theta$ will vary from 0 to 2π.

It follows that the representative point υ drawn on the Argand diagram in the υ-plane will describe a circle and whose radius is U.

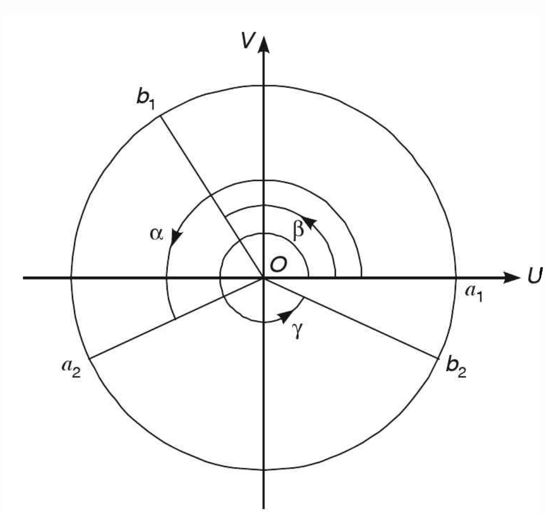
The points A_1, A_2, B_1, B_2 are then represented by

$$a_1 = U, \qquad a_2 = Ue^{i\alpha}, \qquad b_1 = Ue^{i\beta}, \qquad b_2 = Ue^{i\gamma}, \quad \ldots(4)$$

where $-\alpha$, $-\beta$, $-\gamma$ are the asymptotic directions of the streams A_2, B_1, B_2. Here α is given but β and γ are unknown.

The values of the flux at A_1, A_2, B_1, B_2 are respectively h_1U, h_2U, k_1U, k_2U, and hence if we assume $\psi = 0$ on A_1B_2, *i.e.* on the arc a_1b_2, we shall have

$$\left.\begin{aligned} &\psi = h_1U \text{ on the arc } a_1b_1 \\ &\psi = (k_2 - h_2)U \text{ on the arc } b_1a_2, \\ &\psi = k_2U \text{ on the arc } a_2b_2 \end{aligned}\right\} \quad \ldots(5)$$



Expression of the complex potential in terms of υ.

We now determine the complex potential $w = \phi + i\psi$, satisfying conditions (5). Note that ψ is the real part of $-iw$ and so we can apply the formula of Schwarz* giving

$$2\pi i w = \int_0^\beta h_1 U \frac{U e^{i\theta} + \upsilon}{U e^{i\theta} - \upsilon} d\theta + \int_\beta^\alpha (k_2 - h_2) U \frac{U e^{i\theta} + \upsilon}{U e^{i\theta} - \upsilon} d\theta + \int_\alpha^\gamma k_2 U \frac{U e^{i\theta} + \upsilon}{U e^{i\theta} - \upsilon} d\theta \qquad \text{...(6)}$$

Now,
$$\int \frac{U e^{i\theta} + \upsilon}{U e^{i\theta} - \upsilon} d\theta = \int \left(-1 + \frac{2U e^{i\theta}}{U e^{i\theta} - \upsilon} \right) d\theta = -\theta - 2i \log (U e^{i\theta} - \upsilon)$$

$$= -\theta - 2i \log \left\{ U e^{i\theta} \left(1 - \frac{\upsilon}{U e^{i\theta}} \right) \right\} = -\theta - 2i \log (U e^{i\theta}) - 2i \log \left(1 - \frac{\upsilon}{U e^{i\theta}} \right)$$

$$= \theta - 2i \log U - 2i \log \left(1 - \frac{\upsilon}{U e^{i\theta}} \right),$$

where the logarithm is determined so as to vanish when $\upsilon = 0$.

$\therefore$ (6) reduces to

$$-\frac{2\pi i w}{U} = h_1 \left\{ \beta - 2i \log \left(\frac{\upsilon}{b_1} \right) + 2i \log \left(1 - \frac{\upsilon}{U} \right) \right\} + (k_2 - h_2) \left\{ (\alpha - \beta) - 2i \log \left(1 - \frac{\upsilon}{a_2} \right) \right.$$

$$\left. + 2i \log \left(1 - \frac{\upsilon}{a_2} \right) \right\} + k_2 \left\{ (\gamma - \alpha) - 2i \log \left(1 - \frac{\upsilon}{b_2} \right) + 2i \log \left(1 - \frac{\upsilon}{a_2} \right) \right\}$$

$\therefore$ Omitting a constant, we find

$$w = -\frac{U}{\pi} \left\{ h_1 \log \left(1 - \frac{\upsilon}{a_1} \right) + h_2 \log \left(1 - \frac{\upsilon}{a_2} \right) - k_1 \log \left(1 - \frac{\upsilon}{b_1} \right) - k_2 \log \left(1 - \frac{\upsilon}{b_2} \right) \right\}, \qquad \text{...(7)}$$

giving the desired expression for the complex potential in terms of υ.

Relations between the breadths and directions of currents.

Since momentum is conserved in x-and y-directions, we obtain

$$h_1 + h_2 \cos \alpha - k_1 \cos \beta - k_2 \cos \gamma = 0, \qquad \text{...(8)}$$

$$h_2 \sin \alpha - k_1 \sin \beta - k_2 \sin \gamma = 0. \qquad \text{...(9)}$$

Expression for z in terms of υ.

Since $\upsilon = -\dfrac{dw}{dz}$, hence $dz = -\dfrac{1}{\upsilon} dw = -\dfrac{1}{\upsilon} \dfrac{dw}{d\upsilon} d\upsilon$ ...(10)

---

* **Formula of Schwarz.** Given a circle, centre O, radius R, the function $f(z)$, which is analytic within the circle and whose real part takes the value $\phi(\theta)$ on the circumference, is given, save for an imaginary constant, by

$$f(z) = \frac{1}{2} \int_0^{2\pi} \phi(\theta) \frac{R e^{i\theta} + z}{R e^{i\theta} - z} d\theta$$

Now, from (7), we have

$$\frac{1}{\upsilon}\frac{dw}{d\upsilon}=\frac{U}{\pi}\left\{\frac{h_1}{\upsilon(a_1-\upsilon)}-\frac{h_2}{\upsilon(a_2-\upsilon)}-\frac{k_1}{\upsilon(b_1-\upsilon)}-\frac{k_2}{\upsilon(b_2-\upsilon)}\right\}$$

$$=\frac{U}{\pi}\left(\frac{h_1}{a_1}\frac{1}{a_1-\upsilon}+\frac{h_2}{a_2}\frac{1}{a_2-\upsilon}-\frac{k_1}{b_1}\frac{1}{b_1-\upsilon}-\frac{k_2}{b_2}\frac{1}{b_2-\upsilon}\right)+\frac{U}{\pi\upsilon}\left(\frac{h_1}{a_1}+\frac{h_2}{a_2}-\frac{k_1}{b_1}-\frac{k_2}{b_2}\right)$$

$$=\frac{U}{\pi}\left(\frac{h_1}{a_1}\frac{1}{a_1-\upsilon}+\frac{h_2}{a_2}\frac{1}{a_2-\upsilon}-\frac{k_1}{b_1}\frac{1}{b_1-\upsilon}-\frac{k_2}{b_2}\frac{1}{b_2-\upsilon}\right)$$

[Using (4), (8) and (9), noting that the second term vanishes]

$\therefore$ (10) reduces to

$$dz=-\frac{U}{\pi}\left(\frac{h_1}{a_1}\frac{1}{a_1-\upsilon}+\frac{h_2}{a_2}\frac{1}{a_2-\upsilon}-\frac{k_1}{b_1}\frac{1}{b_1-\upsilon}-\frac{k_2}{b_2}\frac{1}{b_2-\upsilon}\right)d\upsilon$$

Integrating and noting that $z = 0$ when $\upsilon = 0$, we get

$$z=-\frac{U}{\pi}\left\{\frac{h_1}{a_1}\log\left(1-\frac{\upsilon}{a_1}\right)+\frac{h_2}{a_2}\log\left(1-\frac{\upsilon}{a_2}\right)-\frac{k_1}{b_1}\log\left(1-\frac{\upsilon}{b_1}\right)-\frac{k_2}{b_2}\log\left(1-\frac{\upsilon}{b_2}\right)\right\}, \quad ...(11)$$

The above result (11) shows that the motion is reversible because the above expression of z remains unchanged on changing the sign of U, a_1, a_2, b_1, b_2 and υ.

The expression of the free streamlines.

On a free streamline, we have $\quad \upsilon = Ue^{-i\theta}$. ...(12)

With this value of υ, (11) gives

$$\pi z=h_1\log(1-e^{-i\theta})+h_2e^{-i\alpha}\log(1-e^{-i\theta-i\alpha})-k_1e^{-i\beta}\log(1-e^{-i\theta-i\beta})-k_2e^{-i\gamma}\log(1-e^{-i\theta-i\gamma}) \quad ...(13)$$

Now, $\quad 1-e^{-ik}=e^{-ik/2}(e^{ik/2}-e^{-ik/2})=2i\sin(k/2)\,e^{-ik/2}$ ...(14)

Using formula (14) in (13), we obtain

$$\pi z=h_1\left(\log 2i+\log\sin\frac{1}{2}\theta-\frac{1}{2}i\theta\right)+h_2e^{-i\alpha}\left\{\log 2i+\log\sin\frac{1}{2}(\theta+\alpha)-\frac{1}{2}i(\theta+\alpha)\right\}$$

$$-k_1e^{-i\beta}\left\{\log 2i+\log\sin\frac{1}{2}(\theta+\beta)-\frac{1}{2}i(\theta+\beta)\right\}-k_2e^{-i\gamma}\left\{\log 2i+\log\sin\frac{1}{2}(\theta+\gamma)-\frac{1}{2}i(\theta+\gamma)\right\} \quad ...(15)$$

But, from (8) and (9), we have

$$h_1+h_2e^{-i\alpha}-k_1e^{-i\beta}-k_2e^{-i\gamma}=0. \quad ...(16)$$

Using (16), (15) reduces to

$$\pi z=\frac{1}{2}i(-h_2\alpha e^{-i\alpha}+k_1\beta e^{-i\beta}-k_2\gamma e^{-i\gamma})+h_1\log\sin\frac{1}{2}\theta+h_2e^{-i\alpha}\log\sin\frac{1}{2}(\theta+\alpha)$$

$$-k_1e^{-i\beta}\log\sin\frac{1}{2}(\theta+\beta)-k_2e^{-i\gamma}\log\sin\frac{1}{2}(\theta+\gamma) \quad ...(17)$$

Equating real and imaginary parts, we can obtain the coordinates (x, y) of a point on the free streamline involving the parameter θ.

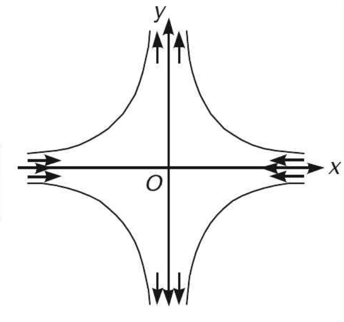
Particular case : Direct impact of two equal jets.

In this case there is symmetry about both the axes, and hence we assume that

$$\alpha = \pi, \qquad \beta = \pi/2, \qquad \gamma = 3\pi/2, \qquad \text{...(18)}$$

$$h_1 = h_2 = k_1 = k_2 = h, \text{ say} \qquad \text{...(19)}$$

$\therefore$ from (7), we obtain



$$w = -\frac{Uh}{\pi}\left[\log\left(1-\frac{\upsilon}{U}\right)+\log\left(1+\frac{\upsilon}{U}\right)-\log\left(1-\frac{\upsilon}{iU}\right)-\log\left(1+\frac{\upsilon}{iU}\right)\right]$$

$$w = -\frac{Uh}{\pi}\log\frac{U^2-\upsilon^2}{U^2+\upsilon^2} \quad \text{or} \quad \exp\left(-\frac{\pi w}{Uh}\right) = \frac{U^2-\upsilon^2}{U^2+\upsilon^2}. \qquad \text{...(20)}$$

Again, (11) reduces to $$z = \frac{h}{\pi}\left[\log\frac{U-\upsilon}{U+\upsilon}+i\log\frac{U+i\upsilon}{U-i\upsilon}\right] \qquad \text{...(21)}$$

The elimination of υ between (20) and (21) gives the relation between w and z.

On the free streamlines, $\upsilon = Ue^{-i\theta}$. So, from (21), we have

$$\frac{\pi z}{h} = \log\frac{1-e^{-i\theta}}{1+e^{-i\theta}}+i\log\frac{1+ie^{-i\theta}}{1-ie^{-i\theta}} \quad \text{or} \quad \frac{\pi z}{h} = \log\frac{1-e^{-i\theta}}{1+e^{-i\theta}}-i\log\frac{1-ie^{-i\theta}}{1+ie^{-i\theta}}$$

or $$\frac{\pi z}{h} = \log\frac{1-e^{-i\theta}}{1+e^{-i\theta}}-i\log\frac{1-e^{-i(\theta-\pi/2)}}{1+e^{-i(\theta-\pi/2)}}, \quad \text{as} \quad i = e^{i\pi/2} \qquad \text{...(22)}$$

Now, $$\log\frac{1-e^{-i\theta}}{1+e^{-i\theta}} = \log\frac{e^{i\theta/2}-e^{-i\theta/2}}{e^{i\theta/2}+e^{-i\theta/2}} = \log\left(i\tan\frac{\theta}{2}\right)$$

Similarly, $$\log\frac{1-e^{-i(\theta-\pi/2)}}{1+e^{-i(\theta-\pi/2)}} = \log\left\{i\tan\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\right\} = \log\left\{-i\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}$$

$\therefore$ (22) reduces $$\frac{\pi z}{h} = \log\left(i\tan\frac{\theta}{2}\right)-i\log\left\{-i\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\} \qquad \text{...(23)}$$

Now, on the streamline in the first quadrant, $-3\pi/2 < \theta < -\pi$.

Hence, if we put $\theta = -\pi-\theta'$, then $0 < \theta' < \pi/2$, and so (23) reduces to

$$\frac{\pi z}{h} = \log\left\{-i\left(\frac{\pi}{2}+\frac{\theta'}{2}\right)\right\}-i\log\left\{-i\tan\left(\frac{3\pi}{4}+\frac{\theta'}{2}\right)\right\} = \log\left(i\cot\frac{\theta'}{2}\right)-i\log\left\{i\cot\left(\frac{\pi}{4}+\frac{\theta'}{2}\right)\right\}$$

$$= \log i - i\log i + \log\cot\frac{\theta'}{2} - i\log\cot\left(\frac{\pi}{4}+\frac{\theta'}{2}\right) = \frac{i\pi}{2}-i\left(\frac{i\pi}{2}\right)+\log\cot\frac{\theta'}{2}+i\log\tan\left(\frac{\pi}{4}+\frac{\theta'}{2}\right). \qquad \text{...(24)}$$

Taking $t = \tan\theta'/2$, (24) reduces to

$$\frac{\pi}{h}(x+iy) = \frac{i\pi}{2}+\frac{\pi}{2}+\log\frac{1}{t}+i\log\frac{1+t}{1-t}$$

Equating real and imaginary parts, we get

$$\frac{\pi x}{h} = \frac{\pi}{2}+\log\frac{1}{t} \quad \text{and} \quad \frac{\pi y}{h} = \frac{\pi}{2}+\log\frac{1+t}{1-t}$$

$$\therefore \quad t = \exp\left\{\frac{\pi}{2}\left(1-\frac{2x}{h}\right)\right\} \quad \text{and} \quad \frac{1+t}{1-t} = \exp\left\{\frac{\pi}{2}\left(1-\frac{2y}{h}\right)\right\}.$$

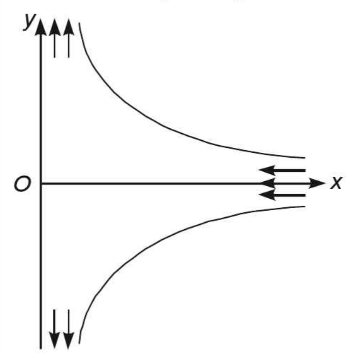
Eliminating t from the above two relations, we obtain

$$\exp\frac{\pi}{2}\left(1-\frac{2y}{h}\right)=\frac{1-\exp\frac{\pi}{2}\left(1-\frac{2x}{h}\right)}{1+\exp\frac{\pi}{2}\left(1-\frac{2x}{h}\right)}=-\tanh\frac{\pi}{4}\left(1-\frac{2x}{h}\right)=\tanh\frac{\pi}{4}\left(\frac{2x}{h}-1\right)$$

$$\therefore \quad \left(\frac{2y}{h}-1\right)\frac{\pi}{2}=\log\coth\frac{\pi}{4}\left(\frac{2x}{h}-1\right) \quad \text{or} \quad y=\frac{h}{2}+\frac{h}{\pi}\log\coth\frac{\pi}{4}\left(\frac{2x}{h}-1\right).$$

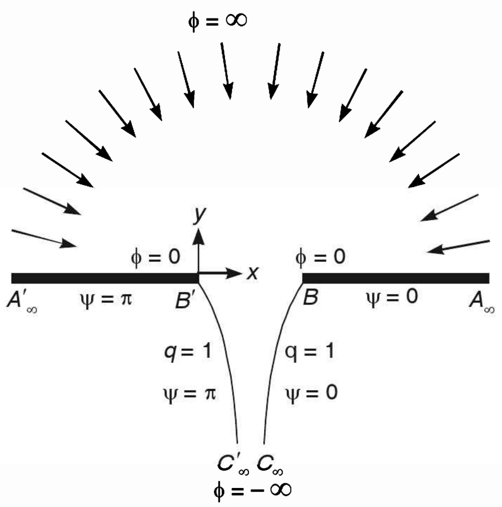
Treating the steamline $x = 0$ as a rigid barrier, we have also solved the problem of the direct impact of a jet on an infinite plane.

We can obtain the thrust on the plane (per unit thickness of liquid) by noting that momentum is advancing through the jet at the rate $\rho h U^2$ perpendicular to the plane and that the momentum of the fluid in contact with the plane is zero in the direction of the normal to the plane. So the required thrust is $\rho h U^2$.



9.6. Jet of liquid through a slit in a plane barrier. Flow through an aperture.

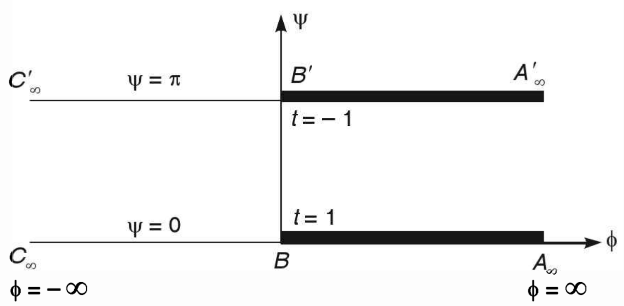
Suppose that the sides of the vessel containing the liquid are infinitely distant from the slit compared to its breadth. In all the figures of this article, fixed boundaries and lines that correspond to them will be shown by thick lines, free lines by thin lines, and the arrows indicate the direction of flow. Consider fluid issuing from a very large vessel through a slit in one of the walls. The fluid will issue as jet bounded by streamlines along which the speed is constant, and at infinity the flow in the jet will be uniform and parallel.



z - plane

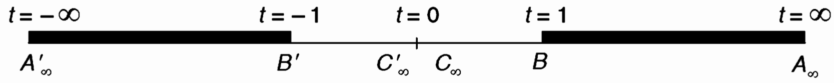
Here $A_\infty B$ and $A'_\infty B'$ are the fixed boundaries and BB' is the slit under consideration. Since $q=-\partial\phi/\partial s$, *i.e.* the velocity is in the direction of ϕ decreasing, we assume that the stream comes from infinity where $\phi=\infty$ and goes to infinity where $\phi=-\infty$. We take $\phi=0$ at B and B'.

If the fixed boundaries $A_\infty B$ and $A'_\infty B'$ are the streamlines $\psi=0$ and $\psi=\pi$, then free streamlines BC_∞ and $B'C'_\infty$ will be the same streamlines $\psi=0$ and $\psi=\pi$.



w - plane

The region on the w-plane which is to correspond to the given region on the z-plane is therefore seen to be a strip of width π extending along and above the axis of ϕ from $\phi = -\infty$ to $\phi = \infty$.

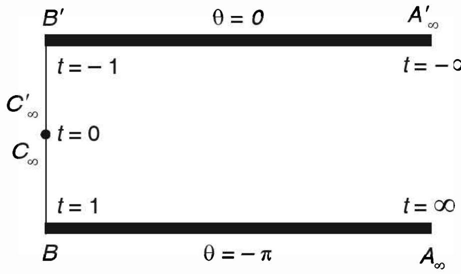


t - plane

To transform the z-plane on to the Ω plane, we take

$$\Omega = \log(1/q) + i\theta. \qquad \text{...(1)}$$

In the z-plane we take the origin at B', then for the velocity along $A'_\infty B'$ we have $\theta = 0$ and along AB, $\theta = -\pi$. Hence in Ω-plane, the lines $A'_\infty B'$, $A_\infty B$ are $\theta = 0$ and $\theta = -\pi$ and the lines corresponding to the free streamlines BC_∞ $B'C'_\infty$ for which $q = 1$ are parts of the imaginary axis.



Ω - plane

We now wish to transform the areas in the w-plane and in the Ω-plane into the upper half of the t-plane so that corresponding corners in the w and Ω planes are represented by the same point on the real t-axis.

Before lettering w and Ω diagrams it will be convenient to choose particular points on the real t-axis to correspond to them, since three such points may be chosen arbitrarily. Thus we may take the edges of the slit B, B' to correspond to $t = 1$, $t = -1$ and A_∞ correspond to $t = \infty$. The w-diagram is then as indicated, where we may take the line BB' to be $\phi = 0$ so that B is the origin in this diagram.

The relation between w and t is given by

$$w = C_1 \log t + C_2. \qquad \text{... (2)}$$

Since $w = 0$, when $t = -1$, so (2) gives $C_2 = 0$. ...(3)

$$\therefore \quad w = C_1 \log t. \qquad \text{...(3)}$$

Again $w = i\pi$ when $t = -1$, so from (3), we have

$$i\pi = C_1 \log(-1) \quad \text{or} \quad i\pi = C_1 i\pi \quad \text{and so} \quad C_1 = 1.$$

$$\therefore \quad w = \log t. \qquad \text{...(4)}$$

The diagram in the Ω-plane has the point B' for origin, and the relation between Ω and t is given by

$$\Omega = C_3 \cosh^{-1} t + C_4 \qquad \text{...(5)}$$

Since $\Omega = -i\pi$ when $t = 1$, so (5) gives

$$-i\pi = C_3 \cosh^{-1}(1) + C_4 \quad \text{or} \quad C_4 = -i\pi, \quad \text{as} \quad \cosh^{-1} 1 = 0.$$

$\therefore$ from (5), $$\Omega = C_3 \cosh^{-1} t - i\pi. \qquad \text{...(6)}$$

Since $\Omega = 0$ when $t = -1$ so (6) gives $0 = C_3 \cosh^{-1}(-1) - i\pi$

But $\cosh^{-1}(-1) = i\pi$ so that $C_3 = 1.$

$\therefore$ From (6),
$$\left.\begin{aligned} \Omega &= \cosh^{-t} t - i\pi \\ t &= -\cosh\Omega \end{aligned}\right\} \qquad \text{...(7)}$$

But $\Omega = \log\zeta$ or $\log(-dz/dw)$, ...(8)

where $$\zeta = -dz/dw. \qquad \text{...(9)}$$

$\therefore$ $$\cos\log\zeta = \log\Omega = -t, \text{ using (7)}$$
$$= -e^w, \text{ using (4)}$$

$\therefore$ $(1/2)\times(e^{\log\zeta}+e^{-\log\zeta}) = -e^w$ or $(\zeta+\zeta^{-1})/2 = -e^w$

or $$\zeta^2 + 2e^w\zeta + 1 = 0$$

$\therefore$ $\zeta = -e^w \pm\sqrt{e^{2w}-1}$ or $-(dz/dw) = -e^w \pm\sqrt{e^{2w}-1}$, using (9) ...(10)

Since ζ or $e^{i\theta}/q$ is infinite when $\psi = 0$ and $\phi = \infty$, the lower sign must be taken in (10).

$\therefore$ we get $$dz/dw = e^w + \sqrt{e^{2w}-1}$$

Integrating, $$z = e^w + \sqrt{e^{2w}-1} - \tan^{-1}\sqrt{e^{2w}-1} - 1, \qquad \text{...(11)}$$
adjusting the constant so that $z = 0$ when $w = 0$.

Equations (1), (4), (7) and (11) constitute the solution of the problem.

To determine the equation of a free streamline.

Along the free stream line $B'C'_\infty$, $-\partial\phi/\partial s = q = 1$ or $\phi = -s$,

where s is measured from the origin B'. Hence on this stream line $\psi = \pi$, we have $s = -\phi$ = – real part of w = – real part of $\log t$, where t is real and lies between – 1 (at B') and o (at C'_∞), also $q = 1$ so from (1)

$i\theta = \Omega$ or $i\theta = \cosh^{-1} t - i\pi$, using (7)

or $$t = -\cos\theta \qquad \text{...(12)}$$

where θ varies from 0 to $-\pi/2$.

On the free stream line $B'C'_\infty$,
$$s = -\log t = -\log(-\cos\theta) = \log(-\sec\theta) \qquad \text{...(13)}$$

From (13), $$ds/d\theta = \tan\theta \qquad \text{...(14)}$$

Now, $$\frac{dx}{d\theta} = \frac{dx}{ds}\cdot\frac{ds}{d\theta} = \cos\theta\tan\theta = \sin\theta.$$

Integrating, $x = -\cos\theta + C_5$, C_5 being an arbitrary constant ...(15)

Since $\theta = 0$ when $x = 0$, so $C_5 = 1$

$\therefore$ $$x = 1 - \cos\theta. \qquad \text{...(16)}$$

Again, $\dfrac{dy}{d\theta} = \dfrac{dy}{ds}\cdot\dfrac{ds}{d\theta} = \sin\theta\tan\theta = \dfrac{1-\cos^2\theta}{\cos\theta}$ or $\dfrac{dy}{d\theta} = \sec\theta - \cos\theta.$

Integrating, $\qquad y = \log(\sec\theta + \tan\theta) - \sin\theta + C_6.$ ...(17)

Since $\theta = 0$, when $y = 0$, so $C_6 = 0$.

$\therefore \qquad y = \log(\sec\theta + \tan\theta) - \sin\theta.$ ...(18)

Thus the parametric form of the equation of the free streamline is

$$x = 1 - \cos\theta, \qquad y = \log(\sec\theta + \tan\theta) - \sin\theta. \qquad ...(19)$$

The ultimate breadth $C'_\infty C_\infty$ of the jet when the free streamlines become parallel is given by

$$q \cdot C'_\infty C_\infty = \psi_{C'_\infty} - \psi_{C_\infty} = \pi - 0 = \pi.$$

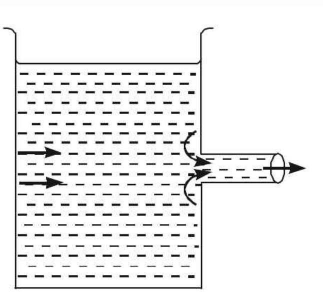
Since $\qquad q = 1, \qquad$ so $\qquad C'_\infty C_\infty = \pi$ ...(20)

which is attained when $\theta = -(\pi/2)$, for which $x = 1$ [using (16)]

$\therefore \qquad$ The breadth of the slit $= BB' = \pi + 2(x)_{C'_\infty} = \pi + 2 \times 1 = \pi + 2.$ ...(21)

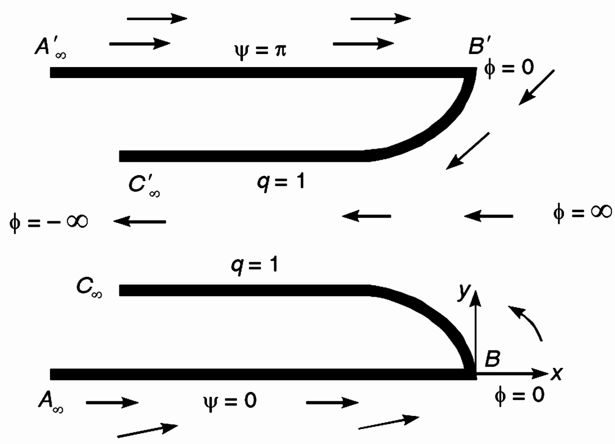
$\therefore$ The coefficient of contraction $= \dfrac{C'_\infty C_\infty}{B'B} = \dfrac{\pi}{\pi + 2} = 0.611$, using (20) and (21)

9.7. Borda's mouthpiece.



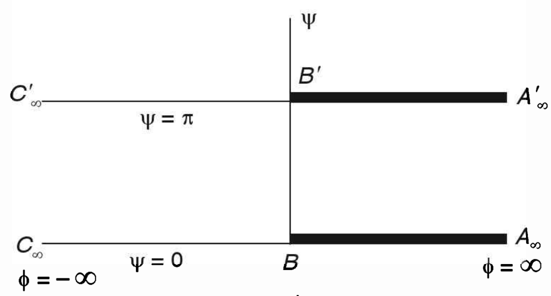
Borda's mouth piece consists of a long straight tube projecting inwards in a large vessel. We shall now consider the efflux of liquid through a pipe projecting into the containing vessel and we shall assume that the sides of the vessel are so far away as not to affect the flow. In all the figures of this article, fixed boundaries and lines that correspond to them will be shown by thick lines, free lines by thin lines, and the arrows indicate the direction of flow.

The z-plane diagram represents a section of the mouth piece whose fixed walls are $A_\infty B$, $A'_\infty B'$. The wall $A_\infty B$ is part of a streamline. The fluid flows along $A_\infty B$, turns at B, and flows out of the tube along BC_∞. If $A_\infty B$ and $A'_\infty B'$ are the streamlines $\psi = 0$ and $\psi = \pi$, then free stream lines BC_∞ and $B'C'_\infty$ will be same streamlines $\psi = 0$ and $\psi = \pi$.



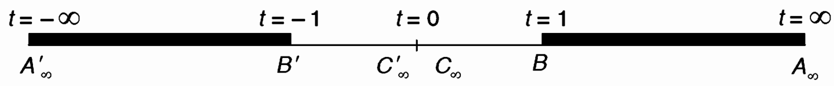
z - plane

Since $q = -\partial\phi/\partial x$, *i.e.* the velocity is in the direction of ϕ decreasing, we assume that the stream comes from infinity where $\phi = \infty$ and goes to infinity where $\phi = -\infty$. We take $\phi = 0$ at B and B'.



w - plane

The region on the w-plane which is to correspond to the given region on the z-plane is therefore seen to be a strip of width π extending along and above the axis of ϕ from $\phi = -\infty$ to $\varphi = \infty$.



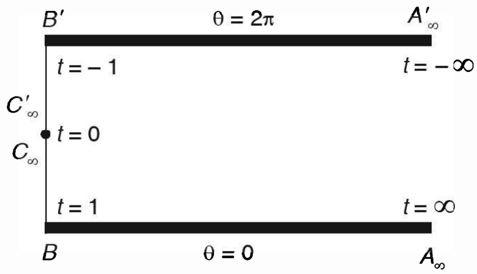
t - plane

To transform the z-plane on to the Ω -plane, we take

$$\Omega = \log(1/q) + i\theta. \qquad \text{...(1)}$$

In the z-plane we take the origin at B, then for the velocity along $A_\infty B$ we have $\theta = 0$ and $A'_\infty B'$, $\theta = 2\pi$. Hence in Ω -plane the lines $A_\infty B$ and $A'_\infty B'$ are $\theta = 0$ and $\theta = 2\pi$ and the lines corresponding to the free streamlines BC_∞, $B'C'_\infty$ for which $q = 1$ are parts of the imaginary axis.

We now wish to transform the areas in the w-plane and in the Ω -plane into the upper half of the t-plane so that corresponding corners in the w and Ω planes are represented by the same point on the real t-axis.



Ω - plane

Before lettering our w and Ω diagrams it will be convenient to choose particular points on the real t - axis to correspond to them, since three such points may be chosen arbitrarily. Thus we may take B, B' to correspond to $t = 1$, $t = -1$ and A_∞ correspond to $t = \infty$. The w-diagram is then as indicated, where we may take the line BB' to be $\phi = 0$ so that B is the origin in this diagram. The relation between w and t is given by

$$w = C_1 \log t + C_2. \qquad \text{...(2)}$$

Since $w = 0$ when $t = 1$, so (2) gives $C_2 = 0$.

$$\therefore \qquad w = C_1 \log t. \qquad \text{...(3)}$$

Again, since $w = i\pi$ when $t = -1$ so from (3), we have

$$i\pi = C_1 \log(-1) = C_1 i\pi \qquad \text{and so} \qquad C_1 = 1$$

$$\therefore \qquad w = \log t \qquad \text{...(4)}$$

The diagram in the Ω -plane has the point B for origin, and the relation between Ω and t is given by

$$\Omega = C_3 \cosh^{-1} t + C_4. \qquad \text{...(5)}$$

Since $\Omega = 0$ when $t = 1$, so (5) gives

$$0 = C_3 \cosh^{-1}(1) + C_4 \qquad \text{so that} \qquad C_4 = 0.$$

$\therefore$ from (5), $$\Omega = C_3 \cosh^{-1} t \qquad ...(6)$$

Again, since $\Omega = 2i\pi$ when $t = -1$, so (6) gives

$$2i\pi = C_3 \cosh^{-1}(-1) = C_3\, i\pi, \qquad \text{giving} \qquad C_3 = 2$$

$\therefore$ from (6), $$\Omega = 2\cosh^{-1} t. \qquad ...(7)$$

Equations (1), (4) and (7) constitute the solution of the problem.

To determine the equations of a free streamline.

Along the free streamline BC_∞

$$-(\partial\phi/\partial s) = q = 1, \qquad \text{so that} \qquad \phi = -s + C_5, \qquad ...(8)$$

where s is measured from the origin B.

At B, $\phi = 0$, $s = 0$ so $C_5 = 0$.

$$\therefore \qquad -s = \phi. \qquad ...(9)$$

But along BC_∞, $\psi = 0$ and since $w = \phi + i\psi$ so along BC_∞, $\phi = w$.

$\therefore$ from (9) $-s = \omega$ or $-s = \log t$, using (4) ...(10)

where t varies from 1 to 0.

Again, since $q = 1$, so from (1), we have

$$i\theta = \Omega \qquad \text{or} \qquad i\theta = 2\cosh^{-1} t, \text{ using (7)}$$

or $$t = \cosh(i\theta/2) = \cos(\theta/2)$$

$\therefore$ from (10), $$s = -\log\cos(\theta/2) = \log\sec(\theta/2). \qquad ...(11)$$

From (11), $$\frac{ds}{d\theta} = \frac{1}{2}\tan\frac{1}{2}\theta \qquad ...(12)$$

Now, $$\frac{dx}{d\theta} = \frac{dx}{ds}\cdot\frac{ds}{d\theta} = \frac{1}{2}\cos\theta\tan\frac{1}{2}\theta = \frac{1}{2}\left(2\cos^2\frac{1}{2}\theta - 1\right)\tan\frac{1}{2}\theta$$

$$dx = \left(\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta - \frac{1}{2}\tan\frac{1}{2}\theta\right)d\theta.$$

Integrating, $x = \sin^2(\theta/2) - \log\sec(\theta/2) + C_6$, C_6 being an arbitrary constant

Since $\theta = 0$ when $x = 0$ so $C_6 = 0$.

$$\therefore \qquad x = \sin^2(\theta/2) - \log\sec(\theta/2). \qquad ...(13)$$

Again, $$\frac{dy}{d\theta} = \frac{dy}{ds}\cdot\frac{ds}{d\theta} = \frac{1}{2}\sin\theta\tan\frac{1}{2}\theta = \sin^2\frac{1}{2}\theta = \frac{1}{2}(1-\cos\theta)$$

$$\therefore \qquad dy = (1/2)\times(1-\cos\theta)\,d\theta$$

Integrating, $$y = (1/2)\times(\theta - \sin\theta) + C_7.$$

Since when $y = 0$, so $C_7 = 0$.

$$\therefore \qquad y = (1/2)\times(\theta - \sin\theta). \qquad ...(14)$$

Thus the equations for the free streamline BC_∞ in parametric form are

$$x = \sin(\theta/2) - \log\sec(\theta/2), \qquad y = (1/2)\times(\theta - \sin\theta). \qquad ...(15)$$

When the two free streamlines BC_∞ and $B'C'_\infty$ ultimately become parallel, the distance $C_\infty C'_\infty$ between them is given by

$$q\cdot C_\infty C'_\infty = \psi_{C'_\infty} - \psi_{C_\infty} = \pi - 0.$$

Since $q = 1$, so $C_\infty C'_\infty = \pi$.

For C_∞, $y' = (y)_{\theta=\pi} = \pi/2$

$\therefore$ $BB' = C_\infty C'_\infty + 2(y)_{\theta=\pi} = \pi + \pi = 2\pi.$

The walls $A_\infty B$ and $A'_\infty B'$ are at a distance 2π apart.

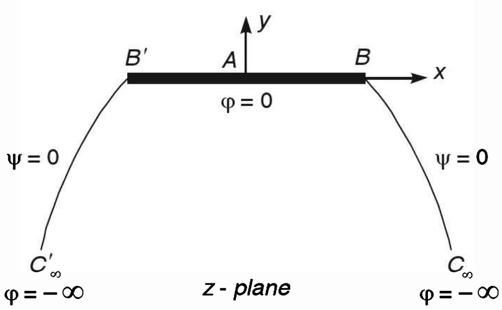
Now, the coefficient of contraction $= \dfrac{C_\infty C'_\infty}{BB'} = \dfrac{\pi}{2\pi} = \dfrac{1}{2},$

which is in agreement with well known Borda's theory.

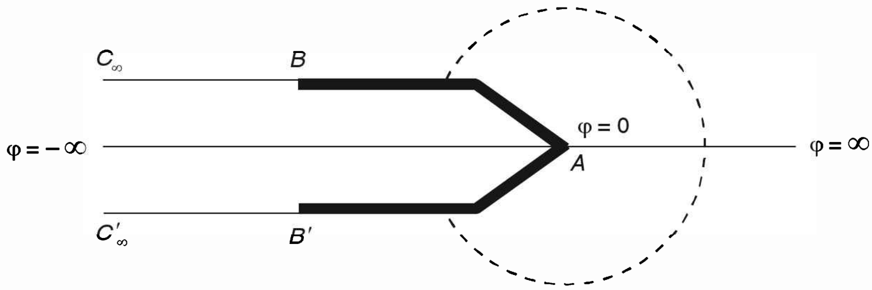
9.8. Impact of a stream on a lamina.

Let the width of the stream to be infinite compared to that of the lamina and the lamina to be fixed at right angles to the stream.

In all the figures of this article, fixed boundaries and lines that correspond to them will be shown by thick lines, free lines by thin lines, and the arrows indicate the direction of flow. The streamline $\psi = 0$ which strikes the lamina at its middle point A divides there into the branches ABC_∞, $AB'C'_\infty$ at A. As stream comes from $\phi = \infty$, we take $\phi = 0$ at A and $\phi = -\infty$ at C_∞ and C'_∞.



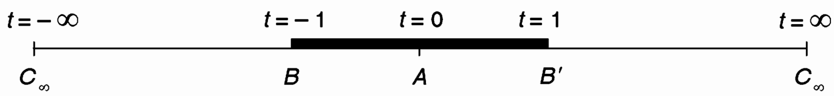
In w-plane, the portions $C_\infty BA$ and $C'_\infty B'A$ coincide with the negative ϕ-axis as ϕ varies from $\phi = 0$ at A to $\phi = -\infty$ at C_∞ or C'_∞. Thus the region on the z-plane occupied by liquid corresponds to the whole w-plane regarded as bounded by the double line from the origin to $\phi = -\infty$, $\psi = 0$.



w - plane

We transform this into the upper half of the t-plane with the following correspondence : C'_∞ goes to $t = \infty$, B' to $t = 1$, A to $t = 0$, B to $t = -1$ and C_∞ to $t = -\infty$. Hence the necessary Schwarz-Christoffel transformation is

$$dw/dt = k(t-1)^{(\pi/\pi - 1)}\,(t-0)^{(2\pi/\pi - 1)}\,(t+1)^{(\pi/\pi - 1)} = kt. \quad \text{...(1)}$$



t - plane

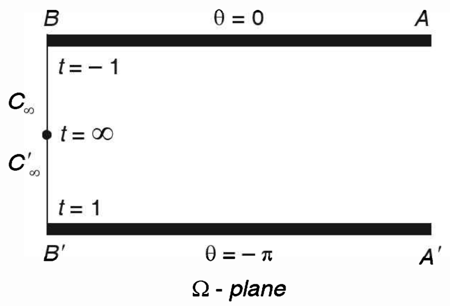
Integrating (1), $w = (1/2) \times kt^2 + C_1$. ...(2)

But $w = 0$ when $t = 0$, so $C_1 = 0$.

Thus, $$w = (1/2) \times kt^2. \quad ...(3)$$

We now transform the z-plane on to the Ω-plane, where

$$\Omega = \log(1/q) + i\theta \quad ...(4)$$



In the z-plane we take the origin at A. Then to get the diagram on the Ω-plane we have $\theta = 0$ along AB, $\theta = -\pi$ along AB' and $q = 1$ along BC_∞ and $B'C'_\infty$.

Hence the region in Ω-plane is a semi-infinite strip of width π bounded by the imaginary axis and $\theta = 0$ $\theta = -\pi$. We transform this strip into the upper half of the t-plane with the following correspondence : A goes to $t = 0$, B to $t = -1$, C'_∞ to $t = \infty$, C_∞ to $t = -\infty$, B' to $t = 1$. Hence the necessary Schwarz-Christoffel transformation is

$$\frac{d\Omega}{dt} = k'(t-0)^{(0-1)}(t-1)^{(\pi/2\pi-1)}(t+1)^{(\pi/2\pi-1)} = \frac{k'}{t(t^2-1)^{1/2}}$$

Integrating, $\Omega = k' \cosh^{-1}(-1/t) + C_2$, C_2 being an arbitrary constant ...(5)

At B, $t = -1$, $\Omega = 0$, so from (5), $0 = k' \cosh^{-1}(1) + C_2$ or $C_2 = 0$

$\therefore$ from (5), $$\Omega = k' \cosh^{-1}(-1/t) \quad ...(6)$$

At B', $t = 1$, $\Omega = -i\pi$, so from (6)

$$-i\pi = k' \cosh^{-1}(-1) \quad \text{or} \quad -i\pi = k' i\pi \quad \text{so} \quad k' = -1$$

$\therefore$ From (6) $\Omega = -\cosh^{-1}(-1/t)$ or $1/t = -\cosh\Omega$. ...(7)

But $$\Omega = \log(-dz/dw) = \log\zeta, \quad ...(8)$$

where $$\zeta = -dz/dw. \quad ...(9)$$

From (8), $\cosh\log\zeta = \cosh\Omega$ or $\cosh\log\zeta = -1/t$, using (7)

or $$\frac{e^{\log\zeta} + e^{-\log\zeta}}{2} = -\frac{1}{t}, \quad \text{or} \quad \frac{\zeta + \zeta^{-1}}{2} = -\frac{1}{t}$$

or $$t = -\frac{2\zeta}{1+\zeta^2}. \quad ...(10)$$

We have to determine the constant k in equation (3), and its value must depend on the width of the lamina.

Along the stream AB, since $\theta = 0$, (4) gives $\Omega = \log(1/q)$ or $1/q = e^{\Omega}$.

$\therefore$ $$1/q = e^{\log\zeta}, \text{ using (8)} \quad ...(11)$$

giving $\zeta = 1/q$ and so (10) gives

$$t = -\frac{2q}{1+q^2} \quad \text{giving} \quad q = \frac{-1+\sqrt{1-t^2}}{t}, \quad ...(12)$$

taking the positive sign in order to make $q = 0$ when $t = 0$, for the velocity must be zero at the point A where the stream line breaks into two branches. Again, along AB, $\psi = 0$ and so $w = \phi + i\psi$.

So using (3), we have $\phi = w = (1/2)\times kt^2$ ...(13)

Since the velocity is wholly along the x-axis, hence $-q = \partial\phi/\partial x = kt\,(dt/dx)$

or $-\dfrac{-1+\sqrt{1-t^2}}{t} = kt\dfrac{dt}{dx}$, using (13) or $dx = \dfrac{kt^2\,dt}{1-\sqrt{1-t^2}}$. ...(14)

If l is the width of the lamina, then

$$l = 2\int dx = 2k\int_0^{-1}\frac{t^2 dt}{1-\sqrt{1-t^2}}, \text{ using (14)}$$

Putting $t = \sin x$ and $dt = \cos x\, dx$, we get

$$l = 2k\int_0^{-\pi/2}\frac{\sin^2 x\cos x\,dx}{1-\cos x} = 2k\int_0^{-\pi/2}(1+\cos x)\cos x\,dx$$

$$= 2k\int_0^{-\pi/2}\left[\cos x+\frac{1}{2}(1+\cos 2x)\right]dx = 2k\left[\sin x+\frac{1}{2}x+\frac{1}{4}\sin 2x\right]_0^{-\pi/2} = 2k\left[-1-\frac{\pi}{4}\right]$$

$\therefore$ $k = -2l/(\pi+4)$ ...(15)

$\therefore$ (3) gives, $w = -lt^2/(\pi+4)$ ...(16)

Relations (10) and (16) contain the solution of the problem.

Determination of the equations of the streamline BC_∞.

Since $q = 1$ along BC_∞, from (4), we have $\Omega = i\theta$. Also from (7), we have

$$\Omega = -\cosh^{-1}(-1/t).$$

$i\theta = -\cosh^{-1}(-1/t)$ or $\cos\theta = -1/t$. ...(17)

Along BC_∞, $\psi = 0$ so $w = \phi + i\psi = \phi$. ...(18)

From (16) and (18), $\phi = -\dfrac{lt^2}{\pi+4} = -\dfrac{l\sec^2\theta}{\pi+4}$, using (17) ...(19)

Again, $\partial\phi/\partial s = -q = -1$.

Integrating, $s = -\phi + C_3$ or $s = \dfrac{l\sec^2\theta}{4+\pi} + C_3$. ...(20)

Measuring s from B where $\theta = 0$, we have $s = 0$ when $\theta = 0$ and so (20) gives $C_3 = -l\,(4+\pi)$.

$\therefore$ From (20), we have $s = \dfrac{l(\sec^2\theta - 1)}{4+\pi}$. ...(21)

Now, $\dfrac{dx}{d\theta} = \dfrac{dx}{ds}\cdot\dfrac{ds}{d\theta} = \cos\theta\cdot\dfrac{2l}{4+\pi}\sec^2\theta\tan\theta$ or $dx = \dfrac{2l}{4+\pi}\sec\theta\tan\theta\,d\theta$

Integrating, $dx = \dfrac{2l}{4+\pi}\sec\theta + C_4$, C_4 being an arbitrary constant ...(22)

Taking the origin at A, we have at B, $\theta = 0$ when $x = l/2$, so (22) gives

$\dfrac{l}{2} = \dfrac{2l}{4+\pi} + C_4$ or $C_4 = \dfrac{\pi l}{2(4+\pi)}$.

$\therefore$ (22) gives $x = \dfrac{2l}{4+\pi}\left(\sec\theta + \dfrac{\pi}{4}\right)$ ...(23)

Again, $\dfrac{dy}{d\theta} = \dfrac{dy}{ds} \cdot \dfrac{ds}{d\theta} = \sin\theta \cdot \dfrac{2l}{4+\pi} \sec^2\theta \tan\theta$ or $dy = \dfrac{2l}{4+\pi} \sec\theta \tan^2\theta \, d\theta.$

Integrating, $$y = \frac{l}{4+\pi}[\sec\theta\tan\theta - \log(\sec\theta + \tan\theta)] + C_5 \qquad ...(2)$$

At B, $y = 0$, $\theta = 0$ so $C_5 = 0$ so (24) gives

$$y = \frac{l}{4+\pi}\{\sec\theta\tan\theta - \log(\sec\theta + \tan\theta)\}. \qquad ...(25)$$

Thus the parametric form of the equations of the streamline $B_\infty C_\infty$ is

$$x = \frac{2l}{4+\pi}\left(\sec\theta + \frac{\pi}{4}\right), \qquad y = \frac{l}{4+\pi}\{\sec\theta\tan\theta - \log(\sec\theta + \tan\theta)\} \qquad ...(26)$$

Determination of the drag or thrust on the lamina.

Consider $$p/\rho = C - q^2/2. \qquad ...(27)$$

On the free stream lines, $q = 1$ and let $p = p_0$.

$$\therefore \qquad p_0/\rho = C - (1/2), \qquad ...(28)$$

which gives the pressure p_0 in the wake or dead water. The drag D on the lamina is given by

$$D = \int_{-l/2}^{l/2} (p - p_0)\,dx = \frac{1}{2}\rho \int_{-l/2}^{l/2} (1 - q^2)\,dx, \text{ using (27) and (28)}$$

$$= \frac{1}{2}\rho \int_{-1}^{1} \left\{1 - \frac{1 + 1 - t^2 - 2\sqrt{1-t^2}}{t^2}\right\} \frac{kt^2\,dt}{1 - \sqrt{1-t^2}}, \text{ using (12) and (14)}$$

$$= \rho k \int_{-1}^{1} \sqrt{1-t^2}\,dt = -\frac{1}{2}\pi\rho k = \frac{\pi l \rho}{x+4}, \text{ using (15)}$$

9.9. Illustrative solved examples.

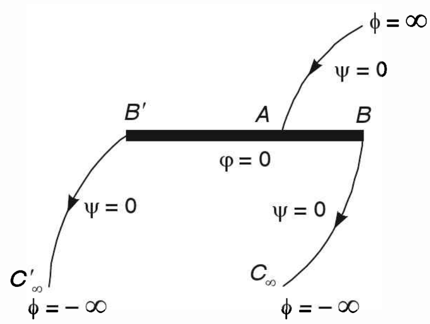
Ex. 1. *When a flat plate of breadth l is set at angle of incidence α to the main stream and a dead water region is formed behind the plate, prove that the drag on the plate is*

$$(\pi\rho l \sin\alpha)/(4 + \pi\sin\alpha).$$

***Sol.** Let the width of the stream to be infinite compared to that of the lamina and the lamina to be fixed at an angle α to the stream.

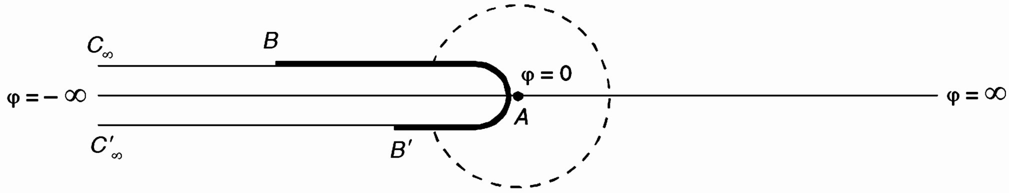
In all the figures of this solution, fixed boundaries and lines that correspond to them will be shown by thick lines, free lines by thin lines and arrows indicate the direction of flow.

The stream line $\psi = 0$ which strikes the lamina at A (which is not the middle point of the lamina) makes an acute angle α with the plate and divides there into the branches ABC_∞ and $AB'C'_\infty$ at A. As stream comes from $\phi = \infty$, we take $\phi = 0$ at A and $\phi = -\infty$ at C_∞ and C'_∞



z - plane

* We proceed in the same way as in Art. 9.8, but the streamline that divides is not in this case the one that strikes the barrier at its middle point.

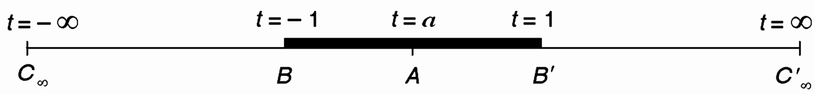


w - plane

In w-plane, the portion $C_\infty BA$ and $C'_\infty B'A$ coincide with the negative ϕ-axis as ϕ varies from $\phi = 0$ at A to $\phi = -\infty$ at C_∞ or C'_∞. Thus the region on the z-plane occupied by liquid corresponds to the whole w-plane regarded as bounded by the double line from the origin to $\phi = -\infty,\ \psi = 0.$

We transform this into the upper half of the t-plane with the following correspondence : C'_∞ goes to $t = \infty$, B' to $t = 1$, A to $t = a$, B to $t = -1$ and C_∞ to $t = -\infty$. Hence the necessary Schwarz-Christoffel transformation is

$$dw/dz = k(t-1)^{(\pi/\pi-1)}\,(t-a)^{(2\pi/\pi-1)}\,(t+1)^{(\pi/\pi-1)} = k\,(t-a) \qquad ...(11)$$



t - plane

Integrating (1), $\quad w = (1/2)\times k\,(t-a)^2 + C_1,\ C_1$ being an arbitrary constant. $\qquad$ ...(2)

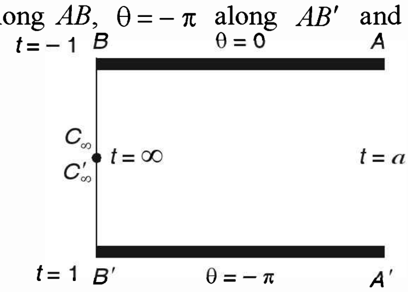
But $\quad w = 0 \quad$ when $\quad t = a, \quad$ so $\quad C_1 = 0.$

$\therefore$ (2) reduces to $\qquad w = (1/2)\times k\,(t-a)^2. \qquad$ ...(3)

We now transform the z-plane on to the Ω-plane, where

$$\Omega = \log(1/q) + i\theta. \qquad ...(4)$$

To get the diagram on the Ω-plane we have $\theta = 0$ along AB, $\theta = -\pi$ along AB' and $q = 1$ along BC_∞ and $B'C'_\infty$



Ω - plane

Hence the region in Ω-plane is a semi-infinite strip of width π bounded by the imaginary axis and $\theta = 0$, $\theta = -\pi$. We transform this strip into the upper half of the t-plane with the following correspondence: A goes to $t = a$, B to $t = -1$, C'_∞ to $t = \infty$, C_∞ to $t = \infty$, B to $t = -1$. Hence the necessary Schwarz-Christofftel transformation is

$$\frac{d\Omega}{dt} = k'(t-a)^{(0-1)}\,(t-1)^{(\pi/2\pi-1)}\,(t+1)^{(\pi/2\pi-1)} = \frac{k'}{(t-a)\sqrt{t^2-1}}$$

Integrating, $\qquad \Omega = k'\cosh\dfrac{at-1}{t-a} + C_2,\ C_2$ being an arbitrary constant $\qquad$ ...(5)

But when $t = -1$, the diagram shows that $\Omega = 0$. So (5) gives

$0 = k'\cosh^{-1}(1) + C_2 \qquad$ or $\qquad C_2 = 0 \qquad [\because\ \cosh^{-1} 1 = 0]$

$\therefore$ from (5) $\qquad \Omega = k'\cosh^{-1}\dfrac{at-1}{t-a}. \qquad$ ...(6)

Again, when $t = 1$, we have $\Omega = -i\pi$, so (6) gives

$$-i\pi = k' \cosh^{-1}(-1) \qquad \text{or} \qquad k' = -1 \qquad [\because \cosh^{-1}(-1) = i\pi]$$

$\therefore$ from (6)
$$\Omega = -\cosh^{-1}\frac{at-1}{t-a} \qquad \text{...(7)}$$

or
$$(at-1)/(t-a) = \cosh\Omega \qquad \text{...(8)}$$

But
$$\Omega = \log(-dz/dw) = \log\zeta, \text{ say} \qquad \text{...(9)}$$

where
$$\zeta = -dz/dw.$$

From (9),
$$\cosh\log\zeta = \cosh\Omega \qquad \text{or} \qquad \frac{e^{\log\zeta} - e^{-\log\zeta}}{2} = \frac{at-1}{t-a}, \text{ by (8)}$$

$\therefore$
$$(at-1)/(t-a) = \cosh\Omega = (\zeta + \zeta^{-1})/2. \qquad \text{...(10)}$$

Since the stream makes an acute angle α with the plate, the final direction of BC_∞ and $B'C'_\infty$ is given by $\theta = -(\pi - \alpha)$ where $t = \infty$, *i.e.*, $\Omega = -i(\pi - \alpha)$. So (7) gives

$$i(\pi - \alpha) = \cosh^{-1} a \qquad \text{so that} \qquad a = -\cos\alpha. \qquad \text{...(11)}$$

Therefore, (10) may be re-written as

$$-\frac{t\cos\alpha + 1}{t + \cos\alpha} = \cosh\Omega = \frac{1}{2}(\zeta + \zeta^{-1}). \qquad \text{...(12)}$$

On the plate from A to B, since $\theta = 0$, so from (4), we have

$$\Omega = \log(1/q) \qquad \text{or} \qquad 1/q = e^{\Omega}$$

$$1/q = e^{\log\zeta}, \text{ using (9)} \qquad \text{or} \qquad \zeta = 1/q. \qquad \text{...(13)}$$

On the plate from B' to A, since $\theta = -\pi$, so from (4), we have

$$\Omega = \log(1/q) - i\pi \qquad \text{or} \qquad 1/q = e^{\Omega + i\pi} = -e^{\Omega}$$

or
$$1/q = -e^{\log\zeta} = -\zeta \qquad \text{so} \qquad \zeta = -1/q. \qquad \text{...(14)}$$

Using (13) and (14), (12) gives
$$\frac{q^2+1}{2q} = \pm\frac{t\cos\alpha + 1}{t + \cos\alpha}, \qquad \text{...(15)}$$

taking the upper or lower sign according as t lies between 1 and $-\cos\alpha$ or between $-\cos\alpha$ and -1.

From (15),
$$q = \pm\frac{t\cos\alpha + 1 - \sin\alpha\sqrt{1-t^2}}{t + \cos\alpha} \qquad \text{...(16)}$$

the signs being adjusted so that q shall not become infinite when $t = -\cos\alpha$.

Also along the plate,
$$\psi = 0 \qquad \text{and} \qquad \phi = w = k(t-a)^2/2,$$

so that
$$q = \mp\frac{\partial\varphi}{\partial x} = \mp k(t-a)\frac{dt}{dx}, \qquad \text{...(17)}$$

taking the upper or lower signs according as we are on AB' or AB, since these are the directions of q. From (16) and (17), we have

$$\pm\frac{t\cos\alpha + 1 - \sin\alpha\sqrt{1-t^2}}{t + \cos\alpha} = \mp k(t-a)\frac{dt}{dx}$$

or $$k(t+\cos\alpha)\frac{dt}{dx} = -\frac{t\cos\alpha + 1 - \sin\alpha\sqrt{1-t^2}}{t+\cos\alpha}, \text{ as form (11)}, \qquad a = -\cos\alpha$$

or $$dx = -k\frac{(t+\cos\alpha)^2}{t\cos\alpha + 1 - \sin\alpha\sqrt{1-t^2}}dt \quad \text{or} \quad dx = \frac{-k(t+\cos\alpha)^2(t\cos\alpha+1+\sin\alpha\sqrt{1-t^2})}{(t\cos\alpha+1)^2 - \sin^2\alpha\,(1-t^2)}dt$$

$$dx = \frac{-k(t+\cos\alpha)^2(t\cos\alpha+1+\sin\alpha\sqrt{1-t^2})}{(t+\cos\alpha)^2}dt \quad \text{or} \quad dx = -k(t\cos\alpha + 1 + \sin\alpha\sqrt{1-t^2})\,dt$$

Integrating, $$x = -k[(t^2/2)\times\cos\alpha + t + \sin\alpha\,\{(t/2)\times(1-t^2)^{1/2} + (1/2)\times\sin^{-1}t\}] + C_3$$

Taking the origin at the middle point of the plate $B'B$ so that $t = 1$, $t = -1$ give equal and **opposite values of** x, we obtain $C_3 = 0$. Hence, we obtain

$$x = -(k/2)\times\{(t^2-1)\cos\alpha + 2t + \sin\alpha\,(t(1-t^2)^{1/2} + \sin^{-1}t)\}, \qquad \text{...(18)}$$

Put $t = -1$ and $x = l/2$ in (18), we get

$$l = k(4+\pi\sin\alpha)/2 \qquad \text{or} \qquad k = 2l/(4+\pi\sin\alpha) \qquad \text{...(19)}$$

$\therefore$ from (3), $$w = \frac{l(t+\cos\alpha)^2}{4\pi + \sin\alpha}, \text{ using (11) and (14)} \qquad \text{...(20)}$$

If in (18) we put $t = -\cos\alpha$, we get for the distance from the middle of the plate to the point where the stream is divided. Thus,

$$x = (k/2)\times\{2\cos\alpha\,(1+\sin^2\alpha) + (\pi/2-\alpha)\sin\alpha\}$$

Thus, $$x = l\{2\cos\alpha\,(1+\sin^2\alpha) + (\pi/2-\alpha)\sin\alpha\}/(4+\pi\sin\alpha), \text{ using (19)} \qquad \text{...(21)}$$

Determination of the drag or thrust on the plate.

Consider $$p/\rho = C - q^2/2. \qquad \text{...(22)}$$

On the free stream lines, $q = 1$ and let $p = p_0$ so that

$$p_0/\rho = C - 1/2, \qquad \text{...(23)}$$

giving pressure p_0 of the 'dead water' behind the plate. The drag D on the plate is given by

$$D = \int_{-l/2}^{l/2}(p-p_0)\,dx = \frac{1}{2}\rho\int_{-l/2}^{l/2}(1-q^2)\,dx, \text{ using (22) and (23)}$$

or $$D = \frac{1}{2}\rho\int_{-l/2}^{l/2}(q^{-1}-q)\,q\,dx \qquad \text{...(24)}$$

But $$q = \pm\{t\cos\alpha + 1 - \sin\alpha\,(1-t^2)^{1/2}\}/(t+\cos\alpha)$$

$\therefore$ $$q^{-1} = \pm\{t\cos\alpha + 1 + \sin\alpha\,(1-t^2)^{1/2}\}/(t+\cos\alpha) \quad \text{and} \quad q\,dx = \mp k(t+\cos\alpha)\,dt.$$

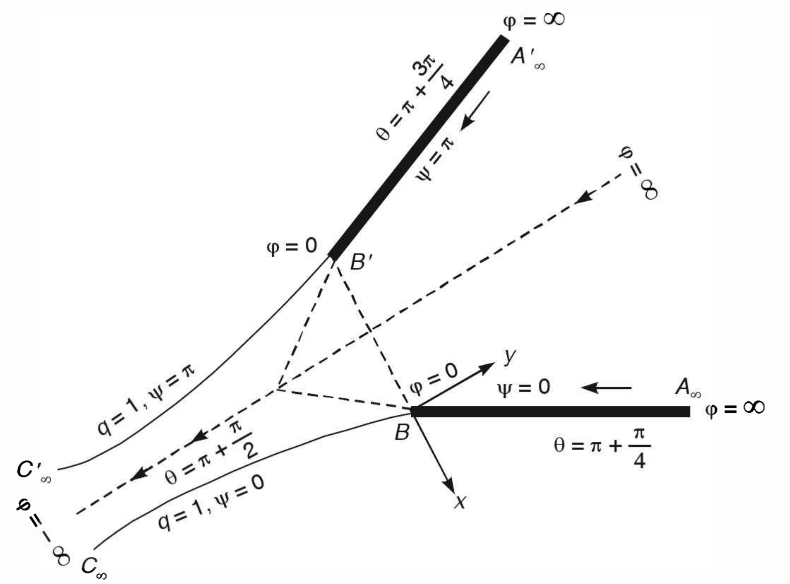
$\therefore$ From (24), $$D = -\rho k\int_1^{-1}\sin\alpha(1-t^2)^{1/2}dt = \frac{1}{2}\pi\rho k\sin\alpha = \frac{\pi\rho l\sin\alpha}{4+\pi\sin\alpha}.$$

Ex. 2. *In the case of unipolar efflux from a large vessel with two plane sides at right angles and an aperture in the corner equally inclined to the two sides, show that the coefficient of contraction is* $\pi/\{\pi + 2\sqrt{2} - \log_e(1+\sqrt{2})\}$ *or* 0.747.

Sol. Suppose that $A_\infty B$ and $A'_\infty B'$ be the sides of a large vessel and BB' be an aperture in the corner equally inclined to the two sides so that $\angle A'_\infty BB' = \angle A'_\infty B'B = 3\pi/4$. Let $A_\infty BC_\infty$ be the streamline $\psi = 0$ and $A'_\infty B'C'_\infty$ the stream line $\psi = \pi$. With suitable choice of units, let $q = 1$ on both BC_∞ and $B'C'_\infty$. In the w-plane we get a strip of breadth π, which is converted into the upper half of the t-plane by the transformation

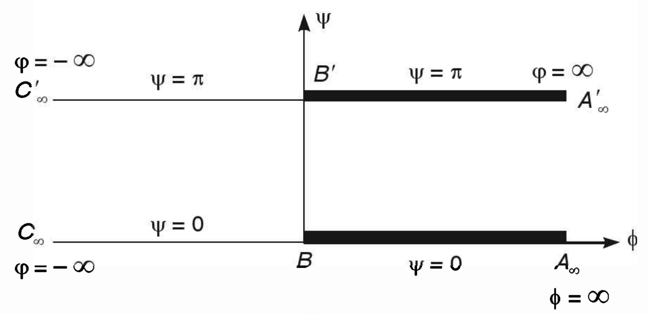
$$w = \log t \qquad \text{...(1)}$$

where B' corresponds to $t = -1$, C'_∞, C_∞ coincide at origin, B to $t = 1$.

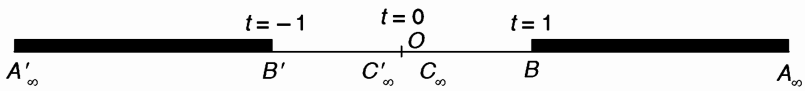


z - plane

In all the figures of this solution, fixed boundaries and lines that correspond to them will be shown by thick lines, free lines by thin lines, and arrows indicate the direction of flow.



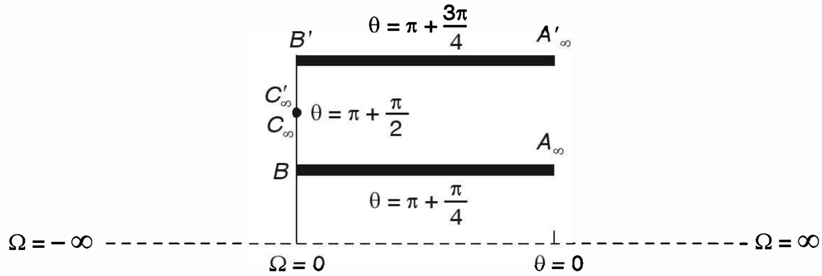
w - plane



t - plane

To find $$\Omega = \log(1/q) + i\theta, \qquad \text{...(2)}$$

we take in the z-plane the origin at B, x-axis along $B'B$. So we have along $A_\infty B$, $\theta = \pi + \pi/4$; along $A'_\infty B'$, $\theta = \pi + 3\pi/4$. Thus the region in Ω-plane is the semi infinite strip bounded by the imaginary axis and the lines $\theta = \pi + \pi/4$, $\theta = \pi + 3\pi/4$. We also note that at $C_\infty\ C'_\infty$, $\theta = \pi + \pi/2$.



Ω - plane

We now transform this strip into the upper half of the t-plane by using

$$\Omega = k\cosh^{-1} t + C_1. \qquad \text{...(3)}$$

At B, $\Omega = i\theta$, by (2) $\quad [\because \log(1/q) = 0]$

$$= i(\pi + \pi/4) \text{ at } t = 1 \qquad \text{...(4)}$$

At B' $\quad \Omega = i(\pi + 3\pi/4)$ at $t = -1$ ...(5)

Using (4) and (5), (3) gives

$i(\pi + \pi/4) = k\cdot\cosh^{-1} 1 + C_1 = C_1$ and $i(\pi + 3\pi/4) = k\cosh^{-1}(-1) + C_1 = ki\pi + C_1$

so that $k = 1/2$ and $C_1 = i(\pi + \pi/4)$.

$\therefore$ from (3),
$$\Omega = \frac{1}{2}\cos^{-1} t + i\left(\pi + \frac{\pi}{4}\right) \qquad \text{...(6)}$$

or
$$t = \cosh\left[2\Omega - i\left(\frac{\pi}{2} + 2\pi\right)\right] \qquad \text{...(7)}$$

Along BC_∞, $q = 1$ so that $-\partial\phi/\partial s = 1$.

$\therefore$ $s = -\phi$ = – real part of w along BC_∞.

But $\psi = 0$ along BC_∞

$\therefore$ $s = -w = -\log t$. ...(8)

Along BC_∞, $\Omega = i\theta$ as $\log(1/q) = 0$. ...(9)

From (7) and (9), $t = \cosh[2i\theta - i(2\pi + \pi/2)] = \cos[2\theta - (2\pi + \pi/2)] = \sin 2\theta$

$\therefore$ $\log t = \log\sin 2\theta$ or $s = -\log\sin 2\theta$, using (8) ...(10)

Now,
$$\frac{dx}{d\theta} = \frac{dx}{ds}\cdot\frac{ds}{d\theta} = \cos\theta\cdot\left(-\frac{2\cos\theta}{\sin 2\theta}\right) = -\frac{\cos 2\theta}{\sin\theta} = -\frac{1 - 2\sin^2\theta}{\sin\theta}$$

Thus, $dx = (2\sin\theta - \operatorname{cosec}\theta)\,d\theta$.

Integrating, $x = -2\cos\theta - \log\tan(\theta/2) + C_2$, C_2 being an arbitrary constant. ...(11)

At B, $x = 0$, $\theta = \pi + \pi/4$, so (11) gives $C_2 = \log\tan(5\pi/8) - \sqrt{2}$.

$\therefore$ from (11), $x = -2\cos\theta - \log\tan(\theta/2) + \log\tan(5\pi/8) - \sqrt{2}$ ...(12)

For C_∞, $\theta = \pi + \pi/4$ and let its x-coordinate be $-x'$, then from (12), we have

$$-x' = -2\cos(3\pi/2) - \log\tan(3\pi/8) + \log\tan(5\pi/4) - \sqrt{2}$$

$$\therefore \quad x' = \sqrt{2} + \log\frac{\tan(3\pi/4)}{\tan(5\pi/8)} = \sqrt{2} + \log\frac{\tan(\pi/2 + \pi/4)}{\tan(\pi/2 + \pi/8)} = \sqrt{2} + \log\frac{\cot(\pi/4)}{\cot(\pi/8)}$$

$$= \sqrt{2} + \log\tan(\pi/8) = \sqrt{2} + \log(\sqrt{2} - 1) = \sqrt{2} - \log(\sqrt{2} + 1)$$

Now, $C_\infty C'_\infty = \psi_{C'_\infty} - \psi_{C_\infty} = \pi$.

But $BB' = 2x' + C_\infty C'_\infty = 2\{\sqrt{2} - \log(\sqrt{2} + 1)\} + \pi$.

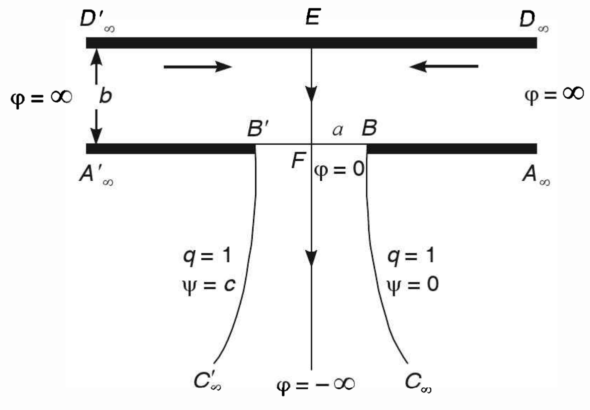
$$\therefore \quad \text{coefficient of contraction} = \frac{C_\infty C'_\infty}{BB'} = \frac{\pi}{\pi + 2\sqrt{2} - 2\log(\sqrt{2} + 1)} = 0.747.$$

Ex. 3. *The fixed boundaries of a liquid moving in two dimensions are given by $y = 0$ from $x = -\infty$ to $x = 0$ and from $x = a$ to $x = \infty$, together with $y = b$ from $x = -\infty$ to $x = \infty$; prove that if c denote the ultimate breadth of the jet escaping through the opening in $y = 0$ from $x = 0$ to $x = c$, c is given by*

$$a = c + \frac{c}{\pi}\left(\frac{2b}{c} + \frac{c}{2b}\right)\log\frac{2b+c}{2b-c}$$

and show that if $a = b$, then ratio of contraction is approximately 4 / 7.

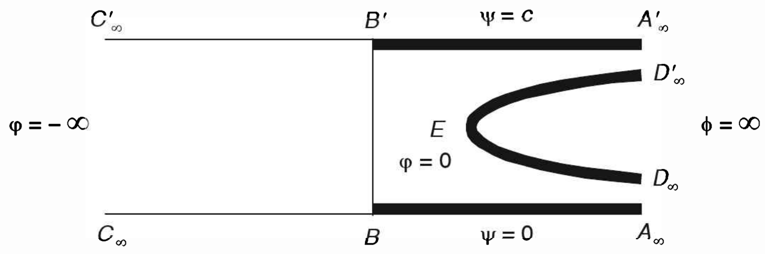
So. Let BC_∞ and $B'C'_\infty$ be free streamlines. Let $A_\infty BC_\infty$ be $\psi = 0$ and $A'_\infty B'C'_\infty$ be $\psi = c$. Then since the flow along EF is symmetrical, $D_\infty EF$ and $D'_\infty EF$ will be $\psi = c/2$.



z - plane

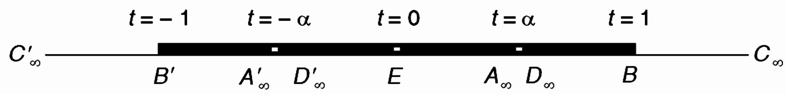
The velocity at E is zero. Let velocity at A_∞ and A'_∞ be U. Then

$bU = (c/2) \times 1$ so that $U = c/2b$. ...(1)



w - plane

In the w-plane the points A_∞, D_∞, A'_∞, D'_∞ correspond to $\phi = \infty$, $C_\infty\ C'_\infty$ to $\phi = -\infty$, B, B' to $\phi = 0$ and the lines $A_\infty BC_\infty$ corresponds to $\psi = 0$, $A'_\infty B'C'_\infty$ to $\psi = c$ and $D_\infty ED'_\infty$ to $\psi = c/2$. Thus in the w-plane there is a strip of breadth c with a cut along $D_\infty ED'_\infty$ midway.



t - plane

We now transform the above mentioned strip of w-plane in the t-plane with the following correspondence :

C_∞ to $t = \infty$, B to $t = 1$, A_∞ or D_∞ to $t = \alpha$ (say),

E to $t = 0$, A'_∞ or D'_∞ to $t = -\alpha$, B' to $t = -1$, C'_∞ to $t = -\infty$.

$\therefore$ By Schwarz-Christoffel's transformation, we get

$$\frac{dw}{dz} = 2k(t-\alpha)^{0-1}(t-0)^{(2\pi/\pi-1)}(t+\alpha)^{0-1} = \frac{2kt}{t^2-\alpha^2}$$

Integrating, $$w = k\log(t^2-\alpha^2)+C_1,\ C_1 \text{ being an arbitrary constant} \qquad ...(2)$$

$\therefore$ $w = 0$ when t = 1, from (2) $$C_1 = -k\log(1-\alpha^2).$$

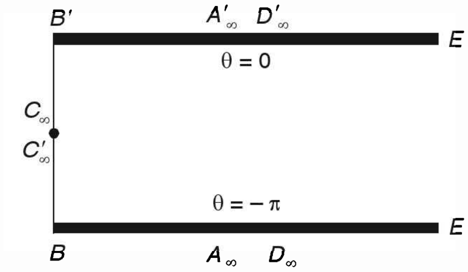
$\therefore$ From (2), $$w = k\log\frac{t^2-\alpha^2}{1-\alpha^2} \qquad ...(3)$$

Again, when $t = -1$, $w = ic$, so form (3), we have

$$ic = -k(2\pi i) \quad \text{so that} \quad k = -c/2\pi.$$

$\therefore$ From (3), $$w = -\frac{c}{2\pi}\log\frac{t^2-\alpha^2}{1-\alpha^2}. \qquad ...(4)$$

Here along ED'_∞ and $A'_\infty B'$, $\theta = 0$; along BA_∞, $D_\infty E$, $\theta = -\pi$; along BC_∞, $B'C'_\infty$, $q = 1$; hence they belong to parts of imaginary axis.



Ω - plane

Using Schwarz-Christofell's transformation, we transform the above mentioned strip in Ω-plane to t-plane with correspondence as before

$$\frac{d\Omega}{dt} = k'(t-1)^{(\pi/2\pi-1)}(t-0)^{0-1}(t+1)^{(\pi/2\pi-1)} = \frac{k'}{t\sqrt{t^2-1}}$$

Integrating, $$\Omega = k'\cosh^{-1}(-1/t)+C_2.\ C_2 \text{ being an arbitrary constant} \qquad ...(5)$$

$\therefore$ $t = -1$ when $\Omega = 0$; so (5) gives $0 = k'\cosh^{-1}(-1)+C_2$ or $C_2 = 0$. So (5) gives

$$\Omega = k'\cosh^{-1}(-1/t) \qquad ...(6)$$

Again $t = 1$ when $\Omega = -i\pi$ so (5) gives $-i\pi = k'\cosh^{-1}(-1)$

or $-i\pi = k'(i\pi)$ so that $k' = -1$. So (6) gives

$$\Omega = -\cosh^{-1}(1/t) \qquad ...(7)$$

or $$1/t = -\cosh\Omega \qquad ...(8)$$

At A'_∞ or D'_∞, $t = -\alpha$, $\theta = 0$, $q = c/2b$.

$\therefore$ $$\Omega = \log(1/q)+i\theta, \qquad ...(9)$$

reduces to $$\Omega = \log(2b/c) \qquad ...(10)$$

and hence from (8), $$-\frac{1}{\alpha} = -\cosh\Omega = -\cosh\log\frac{2b}{c}$$

$$\therefore \quad \frac{1}{\alpha} = \cosh\log\frac{2b}{c} = \frac{1}{2}\left[e^{\log\frac{2b}{c}}+e^{-\log\frac{2b}{c}}\right] = \frac{1}{2}\left(\frac{2b}{c}+\frac{c}{2b}\right) = \frac{4b^2+c^2}{4bc}$$

$\therefore \qquad \alpha = \dfrac{4bc}{4b^2 + c^2} \qquad$ and $\qquad \dfrac{1}{\alpha} = \dfrac{1}{2}\left(\dfrac{2b}{c} + \dfrac{c}{2b}\right) \qquad$ ...(11)

Along the free stream line BC_∞, $q = 1$ so from (9) $\Omega = i\theta$,

But from (8), $\Omega = -\cosh^{-1}(-1/t)$

$\therefore \qquad i\theta = -\cosh^{-1}(-1/t) \qquad$ or $\qquad 1/t = -\cos\theta. \qquad$ ...(12)

Also, we have $\partial\varphi/\partial s = -q = -1.$

$$\therefore \qquad s = -\phi = -w = \frac{c}{2\pi}\log\frac{t^2 - \alpha^2}{1 - \alpha^2}, \text{ using (4)}$$

Then, $s = \dfrac{c}{2\pi}\log\dfrac{\sec^2\theta - \alpha^2}{1 - \alpha^2}$, using (12) ...(13)

Now, $\dfrac{dx}{d\theta} = \dfrac{dx}{ds}\cdot\dfrac{ds}{d\theta} = \cos\theta\dfrac{c}{2\pi}\dfrac{1 - \alpha^2}{\sec^2\theta - \alpha^2} \times \dfrac{2\sec^2\theta\tan\theta}{1 - \alpha^2}$

Thus, $dx = \dfrac{c}{\pi}\dfrac{\sec\theta\tan\theta}{\sec^2\theta - \alpha^2}d\theta$

Integrating, $x = \dfrac{c}{\pi}\displaystyle\int\frac{du}{u^2 - \alpha^2} + C_3$, putting $u = \sec\theta$ and $\sec\theta\tan\theta\, d\theta = du$

$$= \frac{c}{2\pi}\log\frac{u - \alpha}{u + \alpha} + C_3 = -\frac{c}{2\pi}\log\frac{u + \alpha}{u - \alpha} + C_3$$

$\therefore \qquad x = -\dfrac{c}{\pi}\log\dfrac{\sec\theta + \alpha}{\sec\theta - \alpha} + C_3$, C_3 being an arbitrary constant ...(14)

But $x = 0$ when $\theta = 0$; so (14) gives $\qquad C_3 = \dfrac{c}{2\pi}\log\dfrac{1 + \alpha}{1 - \alpha}.$

$\therefore$ from (14), we gave $\qquad x = -\dfrac{c}{2\pi}\log\dfrac{\sec\theta + \alpha}{\sec\theta - \alpha} + \dfrac{c}{2\pi}\log\dfrac{1 + \alpha}{1 - \alpha}. \qquad$ ...(15)

Again, since $x = (a - c)/2$ when $\theta \to -(\pi/2)$, so (15) gives

$$\frac{a - c}{2} = \frac{c}{2\pi}\log\frac{1 + \alpha}{1 - \alpha} \qquad \text{so that} \qquad a = c + \frac{c}{2\pi}\log\frac{1 + \alpha}{1 - \alpha}$$

or $\qquad a = c + \dfrac{c}{2\pi}\left(\dfrac{2b}{c} + \dfrac{c}{2b}\right)\log\dfrac{2b + c}{2b - c}$, using (11) ...(16)

Second Part : Given $\qquad a = b \qquad$ ...(17)

$k = c/a =$ *ratio of contraction* ...(18)

Replacing b by a and then dividing by a, (16) gives

$$1 = \frac{c}{a} + \frac{c}{\pi a}\left(\frac{2a}{c} + \frac{c}{2a}\right)\log\frac{2a + c}{2a - c} \qquad \text{...(19)}$$

Using (18), (19) becomes

$$1 = k + \frac{k}{\pi}\left(\frac{2}{k} + \frac{k}{2}\right)\log\frac{2 + k}{2 - k} = k + \frac{1}{\pi}\left(2 + \frac{k^2}{2}\right)\left\{\log\left(1 + \frac{k}{2}\right) - \log\left(1 - \frac{k}{2}\right)\right\}$$

or $$1 = k + \frac{1}{\pi}\left(2 + \frac{k^2}{2}\right)\left\{\frac{k}{2} - \frac{1}{2}\left(\frac{k}{2}\right)^2 + \ldots + \frac{k}{2} + \frac{1}{2}\left(\frac{k}{2}\right)^2 + \ldots\right\}$$

[expanding log series and neglecting higher powers of k since $k < 1$ and small]

$$\therefore \quad 1 = k + \frac{2k}{\pi} \quad \Rightarrow \quad 1 = k\left(1 + \frac{2}{\pi}\right) \quad \Rightarrow \quad k = \frac{\pi}{\pi + 2} = \frac{4}{7} \text{ (approximately).}$$

Ex. 4. *Liquid moving in the plane (x, y) escapes from an opening between two fixed boundaries given by y = 0, x < 0, and y = h, x > b, the part of the plane for which y is greater that its value on the fixed boundaries being completely filled with liquid which is at rest at infinite distances. Find the equations of the free streamlines, and prove that the ultimate direction of the jet makes with the axis of x an angle* α *is given by the equations*

$$\frac{b}{h} = \frac{1}{2}\tan\alpha + \frac{1}{\pi}\sec\alpha + \frac{1}{\pi}\log\left(\tan\frac{1}{2}\alpha\right).$$

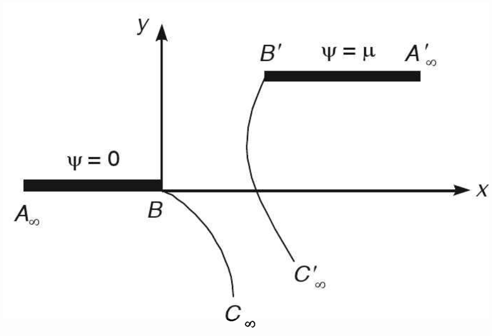
Sol. Suppose that in the z-plane $A_\infty B$, $A'_\infty B'$ be the fixed boundaries and let BC_∞ and $B'C'_\infty$ be the free streamlines $\psi = 0$ and $\psi = k$ respectively.

The region on the w-plane which is to correspond to the given region on the z-plane is therefore seen to be a strip of width μ.

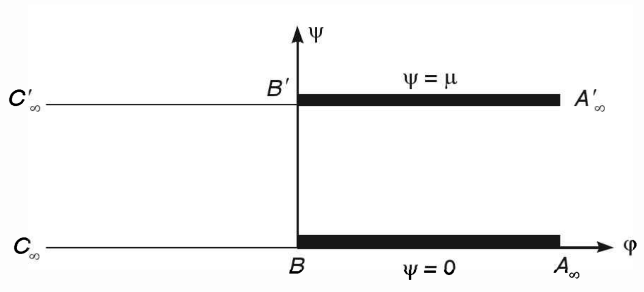
This area of w-plane is transformed into the upper half of the t-plane by relation

$$dw/dt = k/(t - \lambda) \qquad \ldots(1)$$

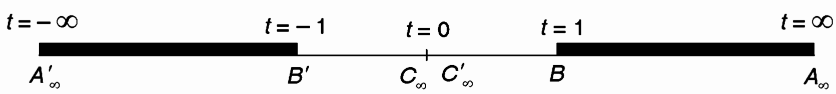
where $t = \lambda$ corresponds to C_∞, C'_∞.



z - plane



w - plane



t - plane

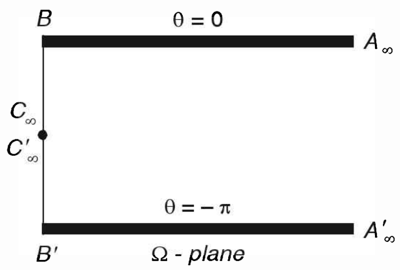
Integrating (1), $\quad w = k\log(t - \lambda) + C_1$, C_1 being an arbitrary constant $\quad \ldots(2)$

Since at $t = 1$, $w = 0$ so (2) gives $\quad C_1 = -k\log(t - \lambda)$.

$\therefore$ from (2), $$w = k\log\frac{t - \lambda}{1 - \lambda}. \qquad \ldots(3)$$

The necessary transformation for Ω-plane to t-plane is given by

$$\Omega = k'\cosh^{-1}t + C_2. \qquad \ldots(4)$$



$\because$ At B', $\Omega = 0$, $t = 1$ and at B, $\Omega = -i\pi$, $t = -1$

$\because$ (4) gives $0 = k' \cosh^{-1}(1) + C_2$ and $-i\pi = k' \cosh^{-1}(1) + C_2$.

These give $k' = -1$ and $C_2 = 0$. So (4) becomes $\Omega = -\cosh^{-1} t$. ...(5)

Let C_∞, C'_∞ correspond to $t = \lambda$., Then $\lambda = \cosh(-i\alpha) = \cos\alpha$. ...(6)

From (5), $t = \cosh \Omega$. ...(7)

If $\zeta = -dz/dw$, ...(8)

then $\Omega = \log \zeta$. ...(9)

From (7) and (9), we have

$$t = \cosh \log \zeta = \frac{1}{2}(e^{\log \zeta} + e^{-\log \zeta}) = \frac{1}{2}(\zeta + \zeta^{-1}) = \frac{1+\zeta^2}{2\zeta}$$

$\therefore$ $\zeta^2 - 2t\zeta + 1 = 0$ giving $\zeta = t + \sqrt{t^2 - 1}$, ...(10)

positive sign being taken as ζ is infinite at $t = \infty$.

From (8) and (10), we have

$$-\frac{dz}{dw} = t + \sqrt{t^2 - 1} \quad \text{or} \quad -dz = (t + \sqrt{t^2-1})\, dw$$

or $\quad -dz = (t + \sqrt{t^2 - 1}).\dfrac{k}{t - \lambda} dt$, using (1) $\quad$ or $\quad -dz = k\left[\dfrac{t}{t-\lambda} + \dfrac{t^2-1}{(t-\lambda)\sqrt{t^2-1}}\right] dt$

Integrating, we have

$$C_3 - z = k\left[t + \lambda \log(t - \lambda) + \sqrt{t^2 - 1} + \lambda \cosh^{-1} t + \sqrt{1-\lambda^2}\sin^{-1}\frac{1-\lambda t}{t-\lambda}\right] \qquad ...(11)$$

Since $z = 0$ when $t = 1$, (11) reduces to

$$C_3 = k[1 + \lambda \log(1-\lambda) + \sqrt{1-\lambda^2} \times (\pi/2)] \qquad ...(12)$$

Since $z = b + ih$ when $t = -1$, (11) reduces to

$$C_3 - (b + ih) = k[-1 + \lambda \log(-1-\lambda) + \cosh^{-1}(-1) + \sqrt{1-\lambda^2}) \times (-\pi/2)]$$

or

$$C_3 - (b+ih) = k[-1 + \lambda\{(-\pi i + \log(1+\lambda)\} - \lambda i\pi - \sqrt{1-\lambda^2} \times (-\pi/2)] \qquad ...(13)$$

Subtracting (13) from (12), we get

$$b + ih = k\left[2 + \lambda \log\frac{1-\lambda}{1+\lambda} + \pi\sqrt{1-\lambda^2} + 2\lambda\pi i\right] \qquad ...(14)$$

Equating real and imaginary parts from both sides of (14), we have

$$b = k\left[2 + \lambda \log\frac{1-\lambda}{1+\lambda} + \pi\sqrt{1-\lambda^2}\right] \qquad ...(15)$$

and
$$h = 2k\lambda\pi. \qquad ...(16)$$

Dividing (15) by (16), we have

$$\frac{b}{h} = \frac{1}{\lambda\pi} + \frac{1}{2\pi}\log\frac{1-\lambda}{1+\lambda} + \frac{\sqrt{1-\lambda^2}}{2\lambda} \quad \text{or} \quad \frac{b}{h} = \frac{1}{\pi\cos\alpha} + \frac{1}{2\pi}\log\frac{1-\cos\alpha}{1+\cos\alpha} + \frac{\sqrt{1-\cos^2\alpha}}{2\cos\alpha}$$

or $\dfrac{b}{h} = \dfrac{1}{2}\tan\alpha + \dfrac{1}{\pi}\sec\alpha + \dfrac{1}{\pi}\log\tan\dfrac{\alpha}{2}$, which is what we wished to prove.

Ex. 5. *The sides of a vessel are two planes which extend to infinity in one direction. The straight lines in the section, made by a plane perpendicular to the sides, are inclined at an angle π/n; and they are symmetrically situated with respect to the line joining those extremities that lie in the first part of the plane of section. Fluid escapes from the orifice, the motion being parallel to the plane of section. Show that the coefficient of contraction is*

$$1\bigg/\left\{1 + \frac{2}{\pi}\int_0^{\pi/2} \sin(\theta/n)\cot\theta\, d\theta\right\}.$$

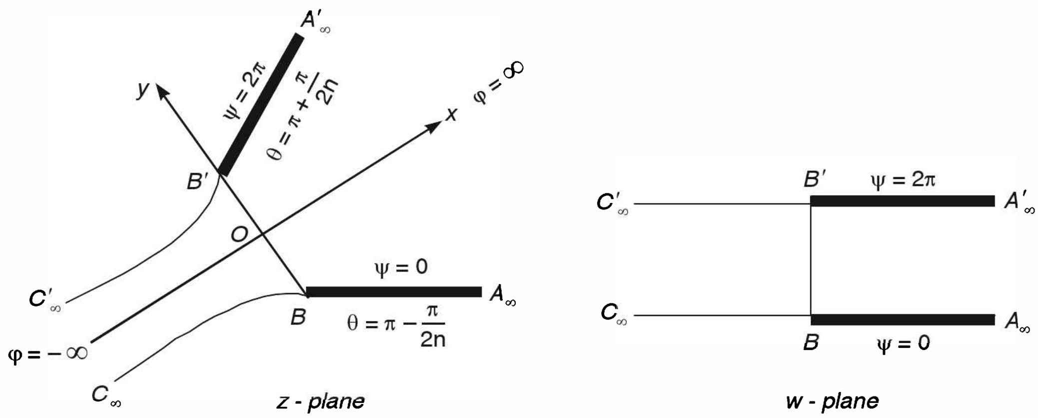
In the case when $n = 2$, show that the coordinates of any point in the free streamline may be expressed as

$$x = 2\tanh^{-1}\sqrt{1+e^{-s/2}} + 2\tanh^{-1}\sqrt{1-e^{-s/2}} - 2[\sqrt{1+e^{-s/2}} + \sqrt{1-e^{-s/2}}],$$

$$y = \pi + 2(\sqrt{1+e^{-s/2}} - \sqrt{1-e^{-s/2}}) - 2\tanh^{-1}\sqrt{1+e^{-s/2}} + 2\tanh^{-1}\sqrt{1-e^{-s/2}},$$

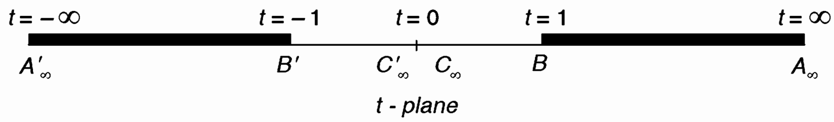
where the middle streamline is the axis of x, the distance along the free streamline from the edge of the nozzle is s, and the scale of measurement is so chosen that the final breadth of the stream is 2π.

Sol. Let $A_\infty B$ and $A'_\infty B'$ denote the sides of a vessel BC_∞ and $B'C'_\infty$ represent the free streamlines of the motion of the fluid. Let $A_\infty BC_\infty$ and $A'_\infty B'C'_\infty$ be the streamlines $\psi = 0$ and $\psi = 2\pi$ respectively.



In order to transform the w-plane into t-plane, we take the relation

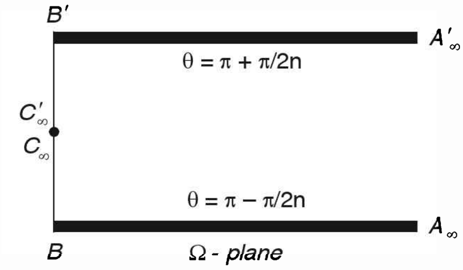
$$w = k\log t + C_1. \qquad ...(1)$$



At B, $w = 0$, $t = 1$ and at B', $w = 2\pi i$, $t = -1$.

$\therefore$ From (1), $0 = C_1$ and $2\pi i = k\log(-1) + C_1$.

$\therefore$ $C_1 = 0$ and $k = 2$ $[\because \log(-1) = \log e^{i\pi} = i\pi]$



$\therefore$ from (1), $$w = 2\log t. \qquad ...(2)$$

In order to transform the Ω -plane into t-plane, we take the relation

$$\Omega = k' \cosh^{-1} t + C_2. \qquad ...(3)$$

Now, $\Omega = i\pi$, when $t = 0$, and $\Omega = i(\pi - \pi/2n)$ when $t = 1$,

Hence, from (3), $k' = 1/n$ and $C_2 = i(\pi - \pi/2n)$

$\therefore$ (3) becomes $$\Omega = \frac{1}{n}\cosh^{-1} t + i\left(\pi - \frac{\pi}{2n}\right). \qquad ...(4)$$

Now, along the streamline BC_∞, $\psi = 0$ and $-(\partial\phi/\partial s) = 1$

$\therefore$ $$\phi = -s + C_3.$$

But $\phi = 0$ when $s = 0$. Hence $C_3 = 0$. So, we have

$$-s = \phi = w = 2\log t. \qquad ...(5)$$

On the free streamline BC_∞, $q = 1$ so from relation

$$\Omega = \log(1/q) + i\theta, \quad \text{we have} \quad \Omega = i\theta$$

or $$\frac{1}{n}\cosh^{-1} t + i\left(\pi - \frac{\pi}{2n}\right) = i\theta, \text{ using (4)} \quad \text{or} \quad \cosh^{-1} t = i\left\{n\theta - \left(n\pi - \frac{\pi}{2}\right)\right\}$$

or $$t = \cos\left(n\theta - n\pi + \frac{\pi}{2}\right) = -\sin(n\theta - n\pi) = -(\sin n\theta \cos n\pi - \cos n\theta \sin n\pi)$$

Thus, $t = (-1)^{n+1}\sin n\theta$, as $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ ...(6)

Now, $$\frac{dy}{d\theta} = \frac{dy}{ds}\cdot\frac{ds}{d\theta} = \frac{dy}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{d\theta}$$

$$= \sin\theta \times \left(\frac{-2}{t}\right) \times (-1)^{n+1} \times n\cos n\theta, \text{ since from (5), } s = -2\log t$$

$$= \frac{2(-1)^{n+1} n \sin\theta \cos n\theta}{(-1)^{n+1}\sin n\theta}$$

$\therefore$ $$dy = -2n\sin\theta\cot n\theta\, d\theta$$

$$\therefore \qquad \int_{-BO}^{-\pi} dy = -2n \int_{\pi-\pi/2n}^{\pi} \sin\theta \cot n\theta \, d\theta$$

or
$$BO - \pi = 2n \int_{-\pi/2n}^{0} \sin u \cot nu \, du, \quad \text{putting } u = \theta - \pi$$

or
$$BO = \pi + 2\int_{\pi/2}^{0} \sin\left(-\frac{\phi}{n}\right) \cot(-\phi)(-d\phi), \quad \text{putting } \phi = -nu \quad \text{and} \quad d\phi = -n du$$

$$= \pi + 2\int_{0}^{\pi/2} \sin\frac{\theta}{n} \cot\theta \, d\theta$$

$$\therefore \qquad BB' = 2 \times BO = 2\pi + 4\int_{0}^{\pi/2} \sin\frac{\theta}{n} \cot\theta \, d\theta$$

Hence the coefficient of contraction is

$$= \frac{C_\infty C'_\infty}{BB'} = \frac{2\pi}{2\pi + 4\int_0^{\pi/2} \sin\frac{\theta}{n} \cot\theta \, d\theta} = \frac{1}{1 + \frac{2}{\pi}\int_0^{\pi/2} \sin\frac{\theta}{n} \cot\theta \, d\theta},$$

Second Part. When $n = 2$, from (5) and (6), we get

$$t = e^{-s/2} \qquad \text{and} \qquad t = -\sin 2\theta. \qquad \text{...(7)}$$

Now
$$dy/ds = \sin\theta = [\sqrt{1+\sin 2\theta} - \sqrt{1-\sin 2\theta}]/2$$

$$= [\sqrt{1-t} - \sqrt{1+t}]/2, \text{ using (7)}$$

$$= [\sqrt{1-e^{-s/2}} - \sqrt{1+e^{-s/2}}]/2, \text{ using (7)}$$

Integrating,

$$y = (1/2) \times [-4\sqrt{1-e^{-s/2}} + 4\tanh^{-1}\sqrt{1-e^{-s/2}} - 4\sqrt{1+e^{-s/2}} - 4\tanh^{-1}\sqrt{1+e^{-s/2}}]$$

or
$$y = \pi + 2\left[\sqrt{1+e^{-s/2}} - \sqrt{1-e^{-s/2}}\right] - 2\tanh^{-1}\sqrt{1-e^{-s/2}} + 2\tanh^{-1}\sqrt{1-e^{-s/2}}, \qquad \text{...(8)}$$

the constant of integration is determined as $y = \pi$ when $s \to \infty$.

Similarly, $dx/ds = \cos\theta$

$$= (1/2) \times [\sqrt{1+\sin 2\theta} + \sqrt{1-\sin 2\theta}] = (1/2) \times [\sqrt{1-t} + \sqrt{1+t}], \text{ using (7)}$$

$$= (1/2) \times [\sqrt{1-e^{-s/2}} + \sqrt{1+e^{-s/2}}]$$

Integrating as before, we get

$$x = 2\tanh^{-1}\sqrt{1-e^{-s/2}} + 2\tanh^{-1}\sqrt{1-e^{-s/2}} - 2[\sqrt{1+e^{-s/2}} + \sqrt{1-e^{-s/2}}] \qquad \text{...(9)}$$

(8) and (9) give the desired values of x and y.

EXERCISE

1. Prove that the formula

$$\frac{dz}{dw} = A\frac{1-au+\sqrt{1-a^2}\sqrt{1-u^2}}{u-a}\cdot\frac{1+au+\sqrt{1-a^2}\sqrt{1-u^2}}{u+a},$$

where $u = e^w$, represents (in two dimensions) the efflux of liquid by a Borda's mouth piece (inward poining tube) from the base of a cylindrical vessel, the vessel and the tube being coaxial, and the aperture of the tube at a distance 'a' from the base.

Prove that the coefficient of contraction is equal to $n-\sqrt{n(n-1)}$, where n is the ratio of the breadth of the vessel to that of the tube.

2. Water escapes, under pressure, from the wall of a vessel by means of a large number of parallel, equal, and equidistant slits. The breadth of each slit is a, and the distance between the centres of consecutive slits in b. Prove that the final breadth c of each issuing jet is given by the equation

$$\frac{a}{c} = 1+\frac{2}{\pi}\left(\frac{b}{c}-\frac{c}{b}\right)\tan^{-1}\frac{c}{b}.$$

Calculate the mean pressure on the wall, having given the velocity v of the issuing jets.

3. Show that the transformations $z = (a/\pi)\times(\sqrt{t^2-1}-\sec^2 t)$; $t = e^{-\pi w/aV}$,

where $z = x + iy$, $w = \phi = i\psi$, give the velocity potential ϕ and the stream function ψ for the flow of a straight river of breadth a running with velocity V at right angles to the straight shore of an otherwise unlimited sheet of water, into which it flows; the motion being treated as two-dimensional. Show that the real axis in the t-plane corresponds to the whole boundary of the liquid.

4. Prove that by proper adjustment of the constants (α, β, γ, δ) the assumption

$$z = \alpha w + \beta e^{\gamma w} + \delta, \qquad (z = x + iy,\ w = \phi + i\psi),$$

may be made to give the solution for the two-dimensional motion of a liquid in a straight pipe of breadth b, and sides $y = \pm b/2$, extending from $x = -\infty$ to $x = 0$, the velocity in the pipe at $x = -\infty$ being V, and the pipe opening into an otherwise unbounded liquid at rest at infinity. Find the values of these constants, assuming that at the point $(0, b/2)$ the value of ϕ is ϕ_0.

5. Exemplify the treatment of problems in discontinuance two-dimensional liquid motion by investigating the case of a stream whose breadth and velocity at infinity are a and V respectively, whose course is disturbed by a symmetrically placed transverse straight barrier of length b. Show that the force necessary to keep the barrier in position is $\rho a V^2(1-\sin\alpha)$, where

$$b/a = 1-\sin\alpha + (1/\pi)\cos\alpha\log(\cot^2\alpha/2).$$

6. Determine the nature of the fluid motion in the space bounded by $y = 0$, $\pi(x^2 + y^2) - 2y = 0$, which is given by $\phi + i\psi = \coth(x+iy)^{-1}$.

7. Explain how to obtain solution of two-dimensional problems in which the liquid is bounded partly by the fixed boundaries and partly by free stream lines.

CHAPTER

10 Irrotational Motion in Three-Dimensions. Motion of A Sphere. Stokes's Stream Function

10.1. Introduction.

We propose to study irrotational motion in three-dimensions with a particular reference to the motion of a sphere. We shall consider certain special forms of solution of the equation

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = 0\,, \qquad \text{...(1)}$$

which, in spherical polar co-ordinates (r, θ, ω)*, reduces to

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\omega^2} = 0. \qquad \text{...(2)}$$

When there is symmetry about a line (say, z-axis), ϕ is independent of ω and hence (2) reduces to

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} = 0 \qquad \text{...(3)}$$

In the case of motion of a sphere the velocity potential is known to have the form $f(r)\cos\theta$. Substituting $\phi = f(r)\cos\theta$ in (3), we have

$$\left(\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}\right)\cos\theta - \frac{f(r)}{r^2}\cos\theta - \frac{\cos\theta}{r^2}f(r) = 0 \qquad \text{or} \qquad \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{2f}{r^2} = 0$$

or

$$r^2(d^2 f/dr^2) + 2r(df/dr) - 2f = 0,$$

which is homogeneous differential equation. As usual, its solution is $f(r) = Ar + B/r^2$. Hence a solution of (3) of the form $f(r)\cos\theta$ may be taken as

$$\phi = (\mathrm{A}r + \mathrm{B}/r^2)\cos\theta \qquad \text{...(4)}$$

10.2. Motion of a sphere through an infinite mass of a liquid at rest at infinity.

[Agra 2007; Kanpur 2009, 11; Rohilkhand 2004, 05; Grahwal 2005; Meerut 2005, 08, 12]

Take the origin at the centre of the sphere and the axis of z in the direction of motion. Let the sphere move with velocity U along the z-axis. To determine the velocity potential ϕ that will satisfy the given boundary conditions, we have the following considerations :

(*i*) ϕ satisfies the Laplace's equation

* In this chapter, we shall write ω in place for ϕ while writing spherical polar coordinates. This is done so because velocity potential is denoted by ϕ.

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \phi}{\partial \theta} = 0, \qquad ...(1)$$

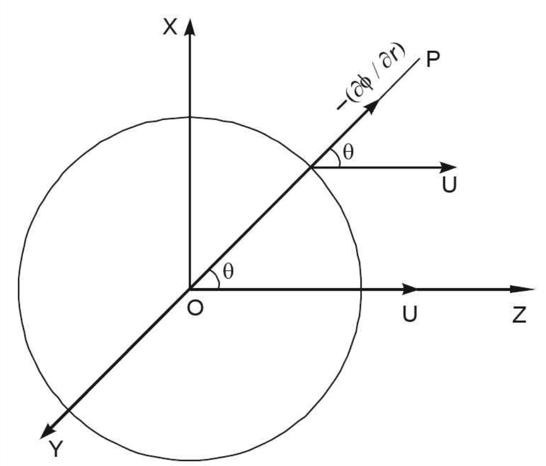
wherein we have used the fact that there is symmetry of flow about z-axis.

(*ii*) Boundary condition at the surface of the sphere $r = a$, namely,

Normal velocity at any point of the sphere = velocity of the liquid at that point in that direction

i.e., $-(\partial\phi/\partial r) = U\cos\theta$, when $r = a$ ...(2)

(*iii*) Since the liquid is at rest at infinity, we must have



$$-(\partial\phi/\partial r) = 0, \quad \text{at} \quad r = \infty \qquad ...(3)$$

The above considerations (*i*) and (*ii*) suggest that ϕ must be of the form $f(r)\cos\theta$ and hence it may be assumed as

$$\phi = (Ar + B/r^2)\cos\theta \qquad ...(4)$$

From (4), $\quad -\partial\phi/\partial r = -(A - 2B/r^3)\cos\theta$ ...(5)

Putting $r = \infty$ in (5) and using (3), we get

$0 = A\cos\theta$ so that $A = 0$. ...(6)

Putting $r = a$ in (5) and using (2) and (6), we get

$U\cos\theta = (2B/a^3)\cos\theta$ so that $B = Ua^3/2$ ...(7)

Thus, $$\phi = \frac{1}{2}Ua^3\frac{\cos\theta}{r^2}, \qquad ...(8)$$

which determines the velocity potential for the flow.

We now determine the equations of lines (streamlines) of flow. The differential equation of the lines of flow at the instant the centre of sphere is passing through the origin is given by

$$\frac{dr}{\partial\phi/\partial r} = \frac{rd\theta}{\partial\phi/r\partial\theta} \quad \text{or} \quad \frac{dr}{(Ua^3/r^3)\times\cos\theta} = \frac{rd\theta}{(Ua^3/2r^3)\times\sin\theta}, \text{ using (8)}$$

or $$(1/r)dr = 2\cot\theta\, d\theta \qquad ...(9)$$

Integrating (9), $\log r = 2\log\sin\theta + \log c$ or $r = c\sin^2\theta$

which is the equation of the lines of flow.

10.3. Liquid streaming past a fixed sphere. [Kanpur 2007, 08]

Let the sphere be at rest and let the liquid flow past the sphere with velocity U in the negative direction of z-axis. This motion may be deduced from that of Art. 10.2 by imposing a velocity $-U$ parallel to the z-axis on both the sphere and the liquid. The sphere is then reduced to rest and we must add to the velocity potential a term $Ur\cos\theta$ to account for the additional velocity. Thus

$$\phi = \frac{1}{2}Ua^3\frac{\cos\theta}{r^2} + Ur\cos\theta = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta. \qquad ...(1)$$

To determine the lines of flow relative to the sphere.

Now the streamlines are given by

$$\frac{dr}{\partial\phi/\partial r} = \frac{rd\theta}{\partial\phi/r\partial\theta} \qquad \text{or} \qquad \frac{dr}{U(1-a^3/r^3)\cos\theta} = \frac{rd\theta}{-U\,(1+a^3/2r^3)\sin\theta}$$

or
$$2\cot\theta d\theta = \frac{2r^2+a^3}{r^3-a^3}\cdot\frac{dr}{r} = \left(\frac{3r^2}{r^3-a^3} - \frac{1}{r}\right)dr$$

Integrating, $\quad -2\log\sin\theta = \log(r^3-a^3) - \log r - \log c$, where c is a constant

i.e.,
$$\sin^2\theta = \frac{cr}{r^3-a^3} \qquad \text{or} \qquad r^2\sin^2\theta\left(1-\frac{a^3}{r^3}\right) = c. \qquad \ldots(2)$$

(2) gives, for either this article or the last article, the lines of flow relative to the sphere.

10.4. Illustrative solved examples.

Ex. 1. *Show that when a sphere of radius a moves with uniform velocity U through a perfect incompressible infinite fluid, the acceleration of a particle of the fluid at* $(r, 0)$ *is* $3U^2(a^3/r^4 - a^6/r^7)$. **[Meerut 2007; Garhwal 2003; Rohilkhand 2001]**

Sol. Superimpose a velocity $-U$ both to the sphere and the liquid. This reduces the sphere to rest and the velocity potential of the flow is gives by [Refer Art. 10.3]

$$\phi = U(r + a^3/2r^2)\cos\theta \qquad \ldots(1)$$

$\therefore$
$$\dot{r} = -\partial\phi/\partial r = -U\,(1-a^3/r^3)\cos\theta \qquad \ldots(2)$$

and
$$r\dot{\theta} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = U\left(1+\frac{a^3}{2r^3}\right)\sin\theta \qquad \ldots(3)$$

Again, from (2), we have

$$\ddot{r} = U\left(1-\frac{a^3}{r^3}\right)\sin\theta\,\dot{\theta} - U\frac{3a^3}{r^4}\dot{r}\cos\theta = U\left(1-\frac{a^3}{r^3}\right)\sin\theta\,\dot{\theta} + \frac{3a^3}{r^4}U^2\left(1-\frac{a^3}{r^3}\right)\cos^2\theta, \text{ by (2)}$$

Clearly for a point (r, θ), the velocity is only along the direction of r and hence the acceleration will also be only along r so that $\theta = 0$.

Thus the required acceleration $= \ddot{r}$ only {at $(r, 0)$}

$$= \frac{3a^3}{r^4}U^2\left(1-\frac{a^3}{r^3}\right), \text{ from (3) with } \theta = \dot{\theta} = 0.$$

$$= 3U^2(a^3/r^4 - a^6/r^7)$$

Ex. 2. *A solid sphere moves through quiescent frictionless liquid whose boundaries are at a distance from it great compare with its radius. Prove that at each instant the motion in the liquid depends only on the position and velocity of the sphere at that instant. Prove that the liquid streams past the sides of the sphere with half the velocity of the sphere.* **[Garhwal 2003]**

Sol. As in Art. 10.2, the velocity potential is given by (taking O, the origin, as the centre of sphere and U the velocity at any time t)

$$\phi = \frac{1}{2}\frac{a^3U}{r^2}\cos\theta, \qquad \ldots(1)$$

which depends on the position and the velocity U of the sphere at instant t under consideration.

Now the velocity with which the liquid streams past the sides of the sphere, *i.e.*, velocity of slip,

$$= \left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)_{r=a} = \left(\frac{1}{r}\frac{a^3U}{r^3}\sin\theta\right)_{r=a} = \frac{1}{2}U\sin\theta$$

= half the velocity of the sphere along the tangent.

Ex. 3. *An infinite ocean of an incompressible perfect liquid of density ρ is streaming past a fixed spherical obstacle of radius a. The velocity is uniform and equal to U except in so far as it is distributed by the sphere and the pressure in the liquid at a great distance from the obstacle is Π. Show that the thrust on that half of the sphere on which the liquid impinges is* $\pi a^2 (\Pi - \rho U^2/16)$. **[Garhwal 2000; Kurukshetra 2000]**

Sol. The velocity potential of the motion of the liquid streaming past the fixed sphere with velocity U in the negative direction of z-axis is given by [refer Art. 10.3]

$$\phi = U(r + a^3/2r^2)\cos\theta \qquad \ldots(1)$$

$$\Rightarrow \quad \left(\frac{\partial \phi}{\partial r}\right)_{r=a} = \left[U\left(1 - \frac{a^3}{r^3}\right)\cos\theta\right]_{r=a} = 0, \quad \left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)_{r=a} = \left[\frac{U}{r}\left(r + \frac{a^3}{2r^2}\right)(-\sin\theta)\right]_{r=a} = -\frac{3}{2}U\sin\theta$$

Let q be the velocity at any point of the boundary of the sphere $r = a$. Then, we have

$$q^2 = \left\{\left(-\frac{\partial \phi}{\partial r}\right)^2 + \left(-\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)^2\right\}_{r=a} = \frac{9}{4}U^2\sin^2\theta \qquad \ldots(2)$$

In steady motion in absence of external forces, the pressure at any point by Bernoulli's equation is given by

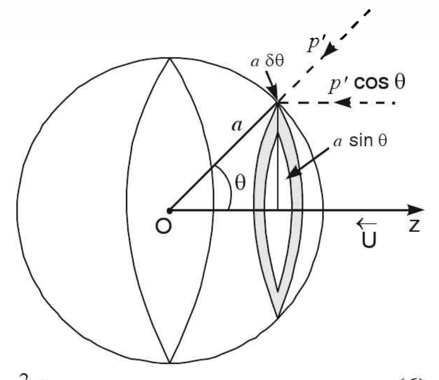
$$p/\rho + q^2/2 = C \qquad \ldots(3)$$

But $p = \Pi$, $q = U$ at infinity. So (3) gives

$$\Pi/\rho + U^2/2 = C \qquad \ldots(4)$$

Subtracting (4) from (3), we obtain

$$p = \Pi + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho q^2 \qquad \ldots(5)$$



Using (2), the pressure p' at any point P on the surface of the sphere $r = a$ is given by

$$p' = \Pi + (\rho U^2/2) - (9\rho U^2/8)\sin^2\theta. \qquad \ldots(6)$$

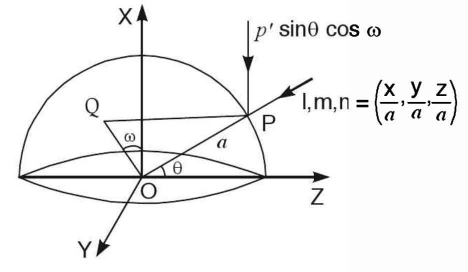
Hence the required thrust on that half of the sphere on which the liquid impinges

$$= \int_0^{\pi/2} (p'\cos\theta)\, 2\pi a \sin\theta \cdot a\, d\theta = 2\pi a^2 \int_0^{\pi/2} \left[\Pi + \frac{1}{2}\rho U^2 - \frac{9}{8}\rho U^2 \sin^2\theta\right]\sin\theta\cos\theta\, d\theta$$

$$= 2\pi a^2 \left[(\pi + \frac{1}{2}\rho U^2)\cdot\frac{1}{2} - \frac{9}{8}\rho U^2 \cdot \frac{1}{4}\right] = \pi a^2 (\Pi - \rho U^2/16).$$

Ex. 4. *A stream of water of great depth is flowing with a uniform velocity V over a plane level bottom. A hemisphere of weight W in water and of radius a, rests with its base on the bottom. Prove that the average pressure between the base of the hemisphere and the bottom is less than the fluid pressure at any point of the bottom at a great distance from the hemisphere if* $V^2 \geq (32W)/(11\pi a^2 \rho)$. **[Kanpur 2011; Kurukshtra1998; Rohilkhand 2000, 05]**

Sol. Let water be flowing past a fixed hemisphere with velocity V along z-axis and (r, θ, ω) be the spherical polar co-ordinates of a point referred to the centre of the hemisphere as the origin. Then as in solved example 3 above, pressure at point P on the surface of the hemisphere is given by



$$p' = \Pi + (\rho V^2/2) - (9\rho V^2/8)\sin^2\theta \qquad ...(1)$$

Relations between (x, y, z) and (r, θ, ω) are given by

$$x = r\sin\theta\cos\omega, \qquad y = r\sin\theta\sin\omega, \qquad z = r\cos\theta \qquad ...(2)$$

Direction cosines of OP are $(x/r, y/r, z/r)$ where $OP = r = a$. Using (2), direction cosines of OP are $(\sin\theta\cos\omega, \sin\theta\sin\omega, \cos\theta)$.

Hence the component of p' along x-axis is $p'\sin\theta\cos\omega$.

Taking $a\ \sin\theta d\omega \cdot a d\theta$ as an element on the surface of the hemisphere, the total thrust on the hemisphere due to water along XO

$$= \int_{\theta=0}^{\pi}\int_{\omega=-\pi/2}^{\omega=\pi/2} (p'\sin\theta\cos\omega)(a\sin\theta d\omega \cdot a d\theta)$$

$$= a^2\int_0^{\pi}\int_{-\pi/2}^{\pi/2}\left[(\Pi + \frac{1}{2}\rho V^2) - \frac{9}{8}\rho V^2\sin^2\theta\right]\sin\theta\cos\omega\, d\theta d\omega, \text{ using (1)}$$

$$= 2a^2\int_0^{\pi}\left[(\Pi + \frac{1}{2}\rho V^2) - \frac{9}{8}\rho V^2\sin^2\theta\right]\sin^2\theta d\theta$$

$$= 2a^2\int_0^{\pi}\left[(\Pi + \frac{1}{2}\rho V^2)\frac{\pi}{4} - \frac{9}{8}\rho V^2\frac{\Gamma(5/2)\,\Gamma(1/2)}{2\Gamma(3)}\right] = \pi a^2\left(\Pi - \frac{11\rho V^2}{32}\right).$$

Since there is a weight W on the base, the total thrust on the base

$$= \pi a^2\left(\Pi - \frac{11\rho V^2}{32}\right) + W$$

$\therefore$ Average pressure on the base $= \dfrac{\text{pressure on base}}{\text{area of the base}} = \Pi - \dfrac{11\rho V^2}{32} + \dfrac{W}{\pi a^2}.$

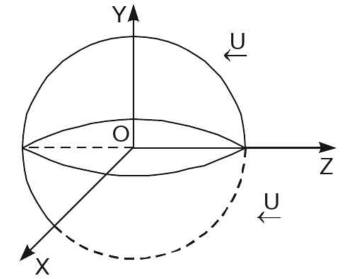
$\therefore$ The average pressure < pressure at great distance,

if $\quad \Pi - \dfrac{11\rho V^2}{32} + \dfrac{W}{\pi a^2} < \Pi, \qquad$ *i.e.*, if $\quad V^2 > \dfrac{32W}{11\rho\pi a^2}.$

Ex. 5. *An infinite homogeneous liquid is flowing steadily past a rigid boundary consisting partly of the horizontal plane y = 0 and partly of a hemispherical boss $x^2 + y^2 + z^2 = a^2$ with irrotational motion which tends, at a great distance from the origin to uniform velocity U parallel to the axis of z. Find the velocity potential and the surface of equal pressure.*

Sol. From Art. 10.3 the velocity potential of the motion of a liquid streaming past a fixed sphere with velocity U in the negative direction of z-axis is given by

$$\phi = U\,(r + a^3/2r^2)\cos\theta \qquad ...(1)$$



Let y-axis be taken in vertical upward direction as shown in figure. Then the motion under consideration is such that velocity perpendicular to the plane $y = 0$ (*i.e.* xz-plane) vanishes. Hence $y = 0$ may be taken as a stream surface. Also the hemisphere above $y = 0$ is also a stream surface. Accordingly, for the hemispherical boss $x^2 + y^2 + z^2 = a^2$ on $y = 0$, the velocity potential is given by (1).

Since U is uniform and the hemisphere is at rest, the motion is steady. Hence by Bernoulli's theorem, the pressure at any point is given by

$$p/\rho + q^2/2 = c. \qquad ...(2)$$

Hence the surfaces of equal pressure are given by putting p = const. in (2). Therefore these are given by q^2 = const. (as ρ is constant), *i.e.*, by

$$\left(-\frac{d\phi}{dr}\right)^2 + \left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 = \text{const.}$$

or $$\left[U(1 - a^3/r^3)\cos\theta\right]^2 + \left[\frac{U}{r}(r + a^3/2r^2)\sin\theta\right]^2 = \text{const., using (1)}$$

or $$(1 - a^3/r^3)^2\cos^2\theta + (1 + a^3/2r^3)^2\sin^2\theta = \text{const, as } U \text{ is a constant}$$

EXERCISE 10 (A)

1. A sphere of radius a is moving with constant velocity U through an infinite liquid at rest at infinity. If p_0 be the pressure at infinity, show that the pressure at any point of the surface of the sphere, the radius to which point makes an angle θ with the direction of motion is given by

$$p = p_0 + \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right)$$

2. For the case of irrotational motion of incompressible fluid in which a sphere of radius a is moving with velocity U, show that the equation to the path of a particle relative to the centre of the moving sphere is $$r^2\sin^2\theta\,(1 - a^3/r^3) = b^2,$$
where b is a constant depending on the particle.

3. Determine the irrotational motion in a liquid due to the motion of a sphere through it.

4. A liquid is moving in a frictionless liquid at rest at infinity. Calculate the velocity potential and equations of lines of flow.

5. For a frictionless liquid streaming past a fixed sphere, obtain the lines of flow relative to the sphere.

6. Determine the velocity potential and stream function, if a sphere is moving in a liquid at rest at infinity. Hence obtain the equations to the lines of flow. **[Meerut 2012; Kanpur 2004; 05]**

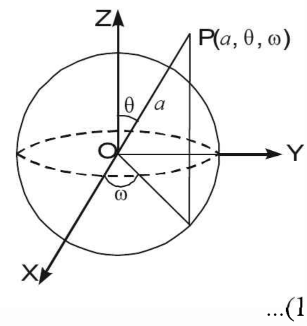
7. Find the velocity potential when a sphere is moving with constant velocity in a liquid which in otherwise at rest **[Agra 2007, 2012]**

[Hint. See equation (8), Art. 10.2]

10.5. Equations of motion of a sphere.

Take the origin at the centre of the sphere and the axis of z in the direction of motion. Let the sphere move with velocity W along the z-axis in an infinite mass of liquid at rest at infinity. Then as in Art. 10.2, velocity potential of the motion is given by

$$\phi = \frac{Wa^3}{2r^2}\cos\theta \qquad \text{so that} \qquad \frac{\partial\phi}{\partial r} = -\frac{Wa^3}{r^3}\cos\theta$$



Let P (a, θ, ω) be the spherical polar co-ordinates of any point on the surface of the sphere. Then elementary surface area dS at P is $a\,d\theta.a\sin\theta\,d\omega$ Again the value of $\phi\,(\partial\phi/\partial r)$ at P is given by

$$\left(\phi\frac{\partial\phi}{\partial r}\right)_{r=a} = \left[\frac{Wa^3}{2r^2}\cos\theta\cdot\left(-\frac{Wa^3}{r^3}\cos\theta\right)\right]_{r=a} = -\frac{W^2a}{2}\cos^2\theta \qquad ...(1)$$

From Art. 6.8 the kinetic energy T_1 of the liquid is given by $\quad T_1 = -\dfrac{1}{2}\rho\displaystyle\int\int \phi \frac{\partial \phi}{\partial n} dS,$

integrated over the surface. Using (1), we obtain

$$T_1 = \frac{1}{2}\rho \int_{\omega=0}^{2\pi} \int_{\theta=0}^{\pi} \left(-\frac{1}{2}W^2 a \cos^2\theta\right) \cdot \left(a^2 \sin\theta \, d\theta \, d\omega\right) = \frac{1}{4} W^2 \rho a^3 \left[\int_0^{\pi} \cos^2\theta \sin\theta d\theta\right] \times \left[\int_0^{2\pi} d\omega\right]$$

Thus, $$T_1 = \frac{1}{3}\pi\rho a^3 W^3 = \frac{1}{4}\cdot\frac{4}{3}\pi\rho a^3 \times W^2 = \frac{1}{4} M' W^2, \qquad \text{...(2)}$$

where $$M' = (4/3) \times \pi \rho a^3 \qquad \text{...(3)}$$

is the mass of the liquid displaced by the sphere, ρ being the density of the liquid. Let σ be the density of the sphere and M be the mass of the sphere so that

$$M = (4/3) \times \pi \sigma a^3 \qquad \text{...(4)}$$

and $$\text{K.E. of the sphere} = T_2 = (1/2) \times M W^2. \qquad \text{...(5)}$$

Let T be the total kinetic energy of the liquid and the sphere. Then, we have

$$T' = \frac{1}{2}(M + \frac{1}{2}M')W^2, \text{ using (2) and (5).}$$

Let Z be the external force parallel to the z-axis (*i.e.* in the direction of motion of the sphere). Then from the principle of energy, we have

Rate of increase of total *K.E.* = rate at which work is being done

i.e., $$\frac{d}{dt}\left[\frac{1}{2}\left(M + \frac{1}{2}M'\right)W^2\right] = ZW \quad \text{or} \quad \left(M + \frac{1}{2}M'\right) W \dot{W} = ZW, \text{where } \dot{W} = \frac{dW}{dt}$$

or $$M\dot{W} = Z - (1/2) \times M'\dot{W} \qquad \text{...(6)}$$

Let Z' be the external force on the sphere when no liquid is present. Then from hydrostatical considerations, there exists a relation between Z and Z' of the form

$$Z = [(\sigma - \rho)/\sigma] Z'. \qquad \text{...(7)}$$

From (6) and (7), we have

$$M\dot{W} + (1/2) \times M'\dot{W} = [(\sigma - \rho)/\sigma] Z' \quad \text{or} \quad (M + M'/2)\dot{W} = [(\sigma - \rho)/\sigma] Z'$$

or $$M\dot{W} = \frac{M}{M + M'/2} \cdot \frac{\sigma - \rho}{\sigma} \cdot Z' = \frac{(4/3) \times \pi\sigma a^3}{(4/3) \times \pi\sigma a^3 + (1/2) \times (4/3) \times \pi\rho a^3} \cdot \frac{\sigma - \rho}{\sigma} \cdot Z', \text{ by (3) and (4)}$$

or $$M\dot{W} = \frac{\sigma - \rho}{\sigma + \rho/2} Z' \qquad \text{...(8)}$$

(8) shows that the whole effect of the presence of the liquid is to reduce the external force in the ratio $\sigma - \rho : \sigma + \rho/2$.

Remark 1. When liquid is absent (so that $M' = 0$), (6) reduces to

$$M\dot{W} = Z. \qquad \text{...(9)}$$

Comparing (9) with (6), we find that the presence of liquid offers resistance of amount (1/2) × $M'\dot{W}$ to the motion of the sphere.

Remark 2. When U, V, W are the components of velocity of the centre of the sphere and X', Y', Z' are the components of the external force on the sphere in absence of liquid, then equations of motion of the sphere are of the form

$$\left.\begin{aligned} M\dot{U} &= \frac{\sigma-\rho}{\sigma+\rho/2}X' \\ M\dot{V} &= \frac{\sigma-\rho}{\sigma+\rho/2}Y' \\ M\dot{W} &= \frac{\sigma-\rho}{\sigma+\rho/2}Z' \end{aligned}\right\} \qquad ...(10)$$

10.6. Sphere projected in a liquid under gravity.

To show that a sphere projected in a liquid under gravity describes a parabola of latus rectum $\dfrac{2\sigma+\rho}{\sigma-\rho}\times\dfrac{W^2}{g}$, *where* σ *and* ρ *are the densities of the sphere and the liquid and* W *is the horizontal velocity.* **[Meerut 2007]**

Proof. The present problem illustrates Art 10.5. Here the external force is gravity only. Since there is no horizontal component of external force, the horizontal velocity is constant. Hence as in Art. 10.5, the vertical motion is the same as if the sphere moved in vacuo and gravity were reduced in the ratio $\sigma-\rho ; \sigma+\rho/2$. Thus if g' is the value of gravity, then we have

$$g' = \frac{\sigma-\rho}{\sigma+\rho/2}g.$$

Hence the centre of the sphere describes parabola of latus rectum

$$= 2\ (\text{Horezontal component of velocity})^2/g' = \frac{2W^2}{\{(\sigma-\rho)/(\sigma+\rho/2)\}g} = \frac{2\sigma+\rho}{\sigma-\rho}\times\frac{W^2}{g}.$$

10.7. Pressure distribution on a sphere.

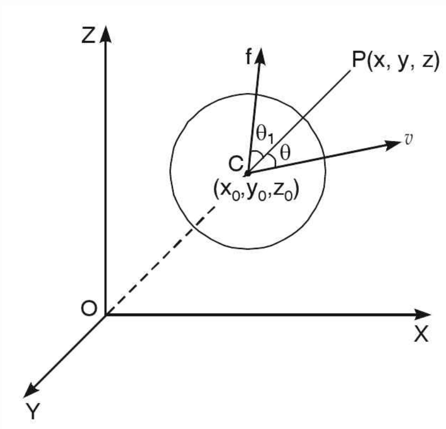
To show that at a point on a sphere moving through an infinite liquid the pressure is given by the formula

$$\frac{p-p_0}{\rho} = \frac{1}{2}af\cos\theta_1 + \frac{1}{8}v^2(9\cos^2\theta - 5),$$

where v *is the velocity,* f *the acceleration of the sphere, and* θ, θ_1 *are the angles between the radii and the direction of* v, f *respectively, and* p_0 *is the pressure at infinity.* **[Garhwal 2003; I.A.S 1987]**

Proof. Let the coordinates of the centre C of the moving sphere referred to fixed axes be (x_0, y_0, z_0) and let

$$\dot{x}_0 = U, \qquad \dot{y}_0 = V, \qquad \dot{z}_0 = W \qquad ...(1)$$



Let (x, y, z) be the co-ordinates of any point P in the liquid.

Let θ, θ_1 be the angles between CP and the directions of v, f respectively.

Let $CP = r$. Then, we have

$$r^2 = (x - x_0)^2 + (y - y_0)^2 + (z - y_0)^2 \qquad ...(2)$$

Let l, m, n be the direction cosines of CP, then

$$l = \frac{x - x_0}{r}, \qquad m = \frac{y - y_0}{r}, \qquad n = \frac{z - z_0}{r} \qquad \text{...(3)}$$

Also $$v^2 = U^2 + V^2 + W^2 \qquad \text{...(4)}$$

$v \cos\theta$ = resolved part of v along $CP = Ul + Vm + Wn$

$$= U\frac{x - x_0}{r} + V\frac{y - y_0}{r} + W\frac{z - z_0}{r} \qquad \text{...(5)}$$

and $f\cos\theta_1$ = resolved part of f along $CP = \dot{U}l + \dot{V}m + \dot{W}n$

$$= \dot{U}\frac{x - x_0}{r} + \dot{V}\frac{y - y_0}{r} + \dot{W}\frac{z - z_0}{r} \qquad \text{...(6)}$$

Then from Art. 10.2 the velocity potential at a fixed point of space (x, y, z) is given by

$$\phi = \frac{a^3}{2r^2}\cdot v\cos\theta \qquad \text{...(7)}'$$

or $$\phi = \frac{a^3}{2r^3}[U(x - x_0) + V(y - y_0) + W(z - z_0)] \qquad \text{...(7)}$$

From (2), $$2r\frac{\partial r}{\partial x} = 2(x - x_0) \quad \text{so that} \quad \frac{\partial r}{\partial x} = \frac{x - x_0}{r} \qquad \text{...(8)}$$

Differentiating (7) partially w.r.t. 'x', we get

$$\frac{\partial\phi}{\partial x} = \frac{1}{2}\frac{a^3U}{r^3} - \frac{3a^3}{2r^4}\frac{\partial r}{\partial x}[U(x - x_0) + V(y - y_0) + W(z - z_0)]$$

$$= \frac{1}{2}\frac{a^3U}{r^3} - \frac{3a^3}{2r^4}(x - x_0)v\cos\theta, \text{ by (5) and (8)}$$

Similarly, differentiating (7) partially w.r.t. y and z, we get

$$\frac{\partial\phi}{\partial y} = \frac{1}{2}\frac{a^3V}{r^3} - \frac{3a^3}{2r^4}(y - y_0)v\cos\theta \quad \text{and} \quad \frac{\partial\phi}{\partial z} = \frac{1}{2}\frac{a^3W}{r^3} - \frac{3a^3}{2r^4}(z - z_0)v\cos\theta.$$

$$\therefore\ q^2 = \left(-\frac{\partial\phi}{\partial x}\right)^2 + \left(-\frac{\partial\phi}{\partial y}\right)^2 + \left(-\frac{\partial\phi}{\partial z}\right)^2 = \frac{1}{4}\frac{a^6}{r^6}(U^2 + V + W^2) - \frac{3a^6}{2r^7}v\cos\theta[U(x - x_0)$$

$$+ V(y - y_0) + W(z - z_0)] + \frac{9a^6}{4r^8}v^2\cos^2\theta[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]$$

$$= \frac{1}{4}\frac{a^6v^2}{r^6} - \frac{3a^6}{2r^6}v^2\cos^2\theta + \frac{9a^6}{4r^6}v^2\cos^2\theta, \text{ by (2) and (5)}$$

Thus, $$q^2 = (a^6v^2/4r^6) \times (1 + 3\cos^2\theta) \qquad \text{...(9)}$$

From (2), $$r\frac{\partial r}{\partial t} = -(x - x_0)\dot{x}_0 - (y - y_0)\dot{y}_0 - (z - z_0)\dot{z}_0$$

$$= -U(x - x_0) - V(y - y_0) - W(z - z_0), \text{ using (1)} \qquad \text{...(10)}$$

Differentiating (7) partially w.r.t. 't', we get

$$\frac{\partial \phi}{\partial t} = \frac{a^3}{2r^3}[\dot{U}(x-x_0)+\dot{V}(y-y_0)+\dot{W}(z-z_0)-(U\dot{x}_0+V\dot{y}_0+W\dot{z}_0)$$

$$-\frac{3a^3}{2r^4}\frac{\partial r}{\partial t}[U(x-x_0)+V(y-y_0)+W(z-z_0)]$$

$$= \frac{a^3}{2r^3}[fr\cos\theta_1-(U^2+V^2+W^2)]+\frac{3a^2}{2r^5}[U(x-x_0)+V(y-y_0)+W(z-z_0)]^2, \text{ by (1), (6) and (10)}$$

$$= \frac{a^3}{2r^3}(fr\cos\theta_1-v^2)+\frac{3a^3}{2r^5}(r^2v^2\cos^2\theta), \text{ by (5)}$$

Thus, $$\partial\phi/\partial t = (a^3/2r^3)\times(fr\cos\theta_1 - v^2+3v^2\cos^2\theta) \qquad ...(11)$$

Let P the potential function due to external forces. Then the pressure at any point in the liquid is given by Bernoulli's equation, namely,

$$\frac{p}{\rho}-\frac{\partial\phi}{\partial t}+\frac{1}{2}q^2+P = F(t) \qquad ...(12)$$

At infinity $r=\infty, p = p_0$ and so $\partial\phi/\partial t = 0$ and $q = 0$ from (11). Hence (12) gives $F(t) = p_0/\rho+P$. So (12) reduces to

$$\frac{p-p_0}{\rho} = \frac{\partial\phi}{\partial t}-\frac{1}{2}q^2 = \frac{a^3}{2r^3}(fr\cos\theta_1-v^2+3v^2\cos^2\theta)-\frac{1}{8}\frac{a^6v^2}{r^6}(1+3\cos^2\theta)$$

$$\therefore \qquad \frac{p-p_0}{\rho} = \frac{a^3 f}{2r^2}\cos\theta_1 - \frac{a^3v^2}{8r^6}(4r^3+a^3)+\frac{3a^3}{8r^6}v^2(4r^3-a^3)\cos^2\theta \qquad ...(13)$$

Putting $r = a$ in (13), pressure at any point on the surface of the sphere is given by

$$\frac{p-p_0}{\rho} = \frac{1}{2}af\cos\theta_1+\frac{1}{8}v^2(9\cos^2\theta-5) \qquad ...(14)$$

Cor. 1. When sphere moves uniformly, *i.e.*, when $f = 0$, pressure at point on the surface of the sphere $r = a$ is given by [putting $f = 0$ in (14)]

$$\frac{p-p_0}{\rho} = \frac{1}{8}v^2(9\cos^2\theta-5) \qquad ...(15)$$

or $$\frac{p-p_0}{\rho} = \frac{1}{8}v^2\left[9\frac{(1+\cos 2\theta)}{2}-5\right] \quad \text{or} \quad \frac{p-p_0}{\rho} = \frac{1}{16}v^2(9\cos 2\theta - 1) \qquad ...(16)$$

Cor. 2. Resultant thrust when there is no acceleration.

In this case pressure p is given by (15), *i.e.*,

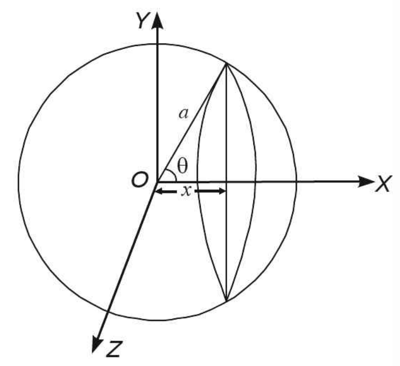
$$p = p_0+(1/8)\times\rho v^2(9\cos^2\theta-1) \qquad ...(17)$$

So the resultant thrust on the sphere

$$= -\int p\cos\theta\, ds = -\int_0^\pi p\cos\theta\cdot d\theta\cdot 2\pi a\sin\theta$$

$$= -2\pi a^2\int_0^\pi\left[p_0+\frac{1}{8}v^2\rho(9\cos^2\theta-5)\right]\sin\theta\cos\theta d\theta = 0,$$

which is in conformity with D' Alembert's paradox.



Cor. 3. Resultant thrust when there is acceleration.

When f is not zero, the resultant thrust due to that part will be

$$= -\int_0^{\pi} \rho \cdot \frac{1}{2} af \cos\theta_1 \cdot 2\pi a \sin\theta_1 \cdot a\, d\theta_1 \;= -\pi a^3 \rho f \int_0^{\pi} \cos^2\theta_1 \sin\theta_1 d\theta_1 = -\frac{2}{3}\pi a^3 f\rho - \frac{1}{2}M' f,$$

where $\quad M' = (4/3)\times \pi a^3 \rho$ = mass of the liquid displaced.

10.8. Illustrative solved examples.

Ex. 1. *Prove that at a point on the sphere moving through an infinite liquid the pressure is given by the formula* $\quad \dfrac{p - p_0}{\rho} = \dfrac{1}{2} af\cos\theta_1 + \dfrac{1}{8} v^2 (9\cos^2\theta - 5)$,

where v is the velocity, f the acceleration of the sphere, and θ, θ_1 are the angles between the radius and the directions of v, f respectively, and p_0 is the hydrostatic pressure.

Sol. Proceed as in Art 10.7 upto equation (12). If p_0 is hydrostatic pressure, *i.e.* when there is no motion so that and $q = 0$, $\partial\phi/\partial t = 0$ and $p = p_0$. Then (12) gives $F(t) = p_0/\rho + P$ as before in Art. 10.7 and hence we get the required formula from equation (14) of Art 10.7.

Ex. 2. *When a sphere of radius a moves in an infinite liquid, show that the pressure at any point exceeds what would be the pressure, if the sphere were at rest by*

$$\frac{a^3 f}{2r^2} - \frac{a^2}{8r^6}(4r^3 + a^3) q^2 + \frac{3a^3}{8r^6}(4r^3 - a^3) q'^2$$

where q is the velocity of the sphere and q' and f are the resolved parts of its velocity and acceleration in the direction of r and density of the liquid is unity. **[Garhwal 2000]**

Sol. Proceed as in Art. 10.7 upto equation (12) by using q_1 for q and f_1 for f in the entire discussion. Thus (12) takes the form

$$p/\rho - (\partial\phi/\partial t) + q_1^2/2 + P = F(t). \qquad \ldots(12)'$$

Let p_0 be the pressure when the sphere is not moving. Then $p = p_0, \partial\phi/\partial t = 0$ and $q_1 = 0$ so (12)$'$ gives $F(t) = p_0/\rho + P$. Then as before in Art. 10.7, we get [refer equation (13)]

$$\frac{p - p_0}{\rho} = \frac{a^3 f_1}{2r^2}\cos\theta_1 - \frac{a^3 v^2}{8r^6}(4r^3 + a^3) + \frac{3a^3}{8r^6} v^2 (4r^2 - a^3)\cos^2\theta.$$

Here in the present problem $\rho = 1$, $v = q$, $v\cos\theta = q'$ and $f_1 \cos\theta_1 = f$. Hence, we obtain

$$p - p_0 = \frac{a^3 f}{2r^2} - \frac{a^3}{8r^6}(4r^3 + a^3) q^2 + \frac{3a^3}{8r^6}(4r^3 - a^3) q'^2,$$ which gives the desired result.

Ex. 3. *Obtain the pressure distribution in the form*

$$\frac{p - \Pi}{\rho} = \frac{1}{2} a\cos\theta \frac{dU}{dt} + \frac{1}{8} U^2 (9\cos^2\theta - 1)$$

for the motion induced by a sphere of radius a moving with velocity U through an infinite fluid otherwise at rest.

Sol. Refer equation (14) in Art. 10.7. Here $\theta_1 = \theta$, $v = U$ and $f = dv/dt$.

Ex. 4. *A sphere of centre O and radius a moves through an infinite liquid of constant density at rest at infinity, O describing a straight line with velocity V(t). If there are no body* *forces, show that the pressure p at points on the surface of the sphere in a plane perpendicular to the straight line at a distance x from O measured positively in the direction of v is given by*

$$p = p_0 - \frac{5}{8}\rho V^2 + \frac{9}{8}\rho V^2 \frac{x^2}{a^2} + \frac{1}{2}\rho x \frac{dV}{dt},$$ where p_0 is the pressure at infinity.

Sol. Refer figure of cor. 2 in Art 10.7. Proceed like Art. 10.7 upto equation (14). Here $\theta_1 = \theta$ so that $\cos\theta = x/a$. Also $v = V$ and $f = dV/dt$. With these changes in (14), we have

$$\frac{p - p_0}{\rho} = \frac{1}{2} a \frac{dV}{dt}\frac{x}{a} + \frac{V^2}{8}\left(9\frac{x^2}{a^2} - 5\right) \quad i.e. \quad p = p_0 - \frac{5}{8}\rho V^2 + \frac{9}{8}\rho V^2 \frac{x^2}{a^2} + \frac{1}{2}\rho x \frac{dV}{dt}.$$

Ex. 5. *A solid sphere is moving through frictionless liquid. Compare the velocities of slip of the liquid past it at different parts of its surface.* **[Garhwal 2005]**

Prove that when the sphere is in motion with uniform velocity v, the pressure at the part of its surface where the radius makes an angle θ with the direction of motion is increased on account of the motion by the amount, $(\rho v^2/16) \times (9\cos 2\theta - 1)$, where ρ is the density of the liquid.

Sol. Proceed like Art 10.7 upto equation (16). Again from (7)′, the velocity of slip at any point (a, θ) on the surface of the sphere

$$= \left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)_{r=a} = \left(\frac{1}{r}\cdot\frac{a^2}{2r^2} v \sin\theta\right)_{r=a} = \frac{1}{2} v \sin\theta.$$

Ex. 6. *Find the pressure at any point of a liquid, of infinite extent and at rest at a great distance, through which a sphere is moving under no external forces with constant velocity U, and show that the mean pressure over the sphere is in defect of the pressure Π at a great distance by $\rho U^2/4$, it being supposed that Π is sufficiently large for the pressure everywhere to be positive, that is that $\Pi > (5\rho U^2)/8$*

Sol. Proceed like Art 10.7. Here $p_0 = \Pi$ and $v = U$. Then, from equation (15) of Art 10.7,

$$\frac{p - \Pi}{\rho} = \frac{1}{8} U^2 (9\cos^2\theta - 5) \quad \text{or} \quad p = \Pi + \frac{1}{8}\rho U^2 (9\cos^2\theta - 5) \qquad ...(A)$$

Now the mean pressure on the sphere

$$= \frac{\int_0^\pi p \cdot d\theta \cdot 2\pi a \sin\theta}{\text{Total surface area of the sphere}} = \frac{2\pi a^2 \int_0^\pi \left[\Pi + \frac{1}{8}\rho U^2 (9\cos^2\theta - 5)\right] \sin\theta\, d\theta}{4\pi a^2}$$

$= \Pi - (\rho U^2/4)$, on simplification.

∴ Required defect = Π – mean pressure $= \rho U^2/4$.

Now from (A) we see that the pressure will be minimum when $\cos\theta$ is minimum, *i.e.*, when $\theta = \pi/2$.

$$\therefore \qquad \text{Minimum pressure} = \Pi - (5\rho U^2)/8 \qquad ....(B)$$

Hence the pressure will be positive everywhere only if the minimum pressure given by (*B*) is positive, *i.e.* when $\Pi - (5\rho U^2/8) > 0$ *i.e.* $\Pi > (5\rho U^2/8)$.

Ex. 7. *A sphere of radius a is made to move in incompressible perfect fluid with non-uniform velocity v along the x-axis. If the pressure at infinity is zero, prove that at a point x in advance of the centre,*

$$p = \frac{1}{2}\rho a^3 \left[\frac{\dot{v}}{x^2} + v^2\left(\frac{2}{x^3} - \frac{a^3}{x^6}\right)\right]$$ **[Garwhal 2001, 02; Kanpur 2010]**

Sol. Refer Art 10.7 upto equation (13). Here velocity and acceleration are both in x-direction so that $\theta_1 = \theta = 0$. Since pressure at infinity is zero, so $p_0 = 0$. Also here $r = x$ and $f = \dot{v}$. Then equation (13) of Art. 10.7 reduces to

$$\frac{p}{\rho} = \frac{a^3\dot{v}}{2x^2} - \frac{a^3v^2}{8x^6}(4x^3 + a^3) + \frac{3a^3}{8x^6}v^2(4x^3 - a^3) = \frac{1}{2}a^3\left(\frac{\dot{v}}{x^2} + \frac{2v^2}{x^3} - \frac{v^2a^3}{x^6}\right)$$

$$\therefore \qquad p = \frac{1}{2}\rho a^3\left\{\frac{\dot{v}}{x^2} + v^2\left(\frac{2}{x^3} - \frac{a^3}{x^6}\right)\right\}$$

Ex. 8. *A sphere of radius a is in motion in fluid, which is at rest at infinity, the pressure there being* Π, *determine the pressure at any point of the fluid, and show that the pressure on the front hemisphere cut off by a plane perpendicular to the direction of motion is the resultant of pressures* $\pi a^2(\Pi - \rho v^2/16)$ *and* $(\pi\rho a^3 f)/3$ *in the directions respectively opposite to those of the velocity v, and the acceleration of the centre of the sphere.*

Sol. Proceed as in Art 10.7 upto (14) after replacing by p_0 by Π. Thus, we have

$$\frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{1}{8}v^2(9\cos^2\theta - 5) + \frac{1}{2}af\cos\theta_1 = \frac{p_1}{\rho} + \frac{p_2}{\rho}, \text{ say}$$

where $$\frac{p_1}{\rho} = \frac{\Pi}{\rho} + \frac{1}{8}v^2(9\cos^2\theta - 5) \qquad \text{...(A)}$$

and $$\frac{p_2}{\rho} = \frac{1}{2}af\cos\theta_1. \qquad \text{...(B)}$$

Using (A), the pressure on the hemisphere along the direction opposite to v

$$= \int_0^{\pi/2}(p_1\cos\theta)\cdot a\,d\theta\cdot 2\pi a\sin\theta = 2\pi a^2\int_0^{2\pi}\left[\Pi + \frac{1}{8}\rho v^2(9\cos^2\theta - 5)\right]\sin\theta\cos\theta\, d\theta$$

$$= \pi a^2\Pi - \frac{1}{16}\pi\rho a^2v^2 = \pi a^2\left(\Pi - \frac{1}{16}\rho v^2\right)$$

Using (B), the pressure on the hemisphere opposite to f

$$= \int_0^{\pi/2}(p_2\cos\theta_1)\cdot a\,d\theta_1\cdot 2\pi a\sin\theta_1 = 2\pi a^2\int_0^{\pi/2}\frac{1}{2}a\rho f\cos\theta_1\cdot\cos\theta_1\sin\theta_1\, d\theta_1 = \frac{1}{3}\pi\rho a^3 f.$$

Therefore the pressure on the hemisphere is resultant of $\pi a^2(\Pi - \rho v^2/16)$ and $(\pi\rho a^3 f)/3$ which are in directions opposite to v and f respectively.

Ex. 9. *A rigid sphere of radius a is moving in a straight line with velocity v and acceleration f through an infinite incompressible liquid; prove that the resultant fluid pressure over the two hemispheres into which the sphere is divided by a diametral plane perpendicular to its direction of motion are* $\Pi\pi a^2 \pm (1/4)\times Mf - (3Mv^2/64a)$, *where* Π *is the pressure at a great distance and M is the mass of the fluid displaced by the sphere.*

Sol. As in equation (14) of Art 10.7 with $\theta = \theta_1$ and $p_0 = \Pi$ here, we have

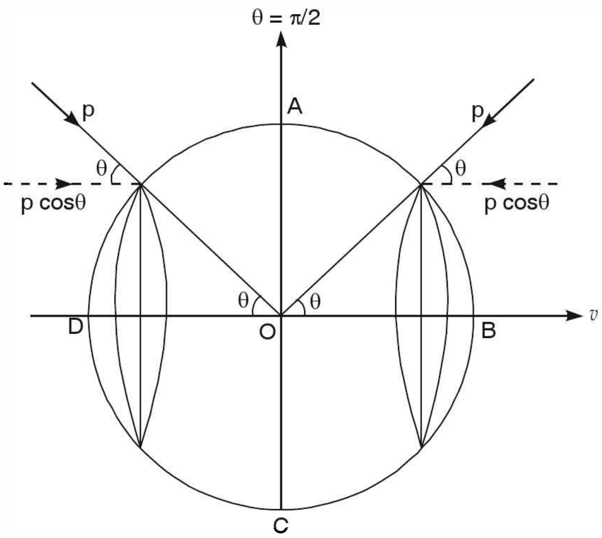
$$p = \Pi + (1/2) \times a\rho f \cos\theta + (1/8) \times \rho v^2 (9\cos^2\theta - 5). \qquad \ldots(A)$$

Then the thrust on the front hemisphere $ABC = \int_0^{\pi/2} (p\cos\theta)\cdot a d\theta \cdot 2\pi a \sin\theta$

$$= 2\pi a^2 \int_0^{\pi/2} \left[\Pi + \frac{1}{2} a\rho f \cos\theta + \frac{1}{8}\rho v^2 (9\cos^2\theta - 5)\right] \sin\theta\cos\theta\, d\theta$$

$$= \Pi\pi a^2 + \frac{1}{3}\pi a^3 \rho f - \frac{1}{16}\pi a^2 \rho v^2 = \Pi\pi a^2 + \frac{1}{4} Mf - \frac{3}{64}\frac{Mv^2}{a},$$

where $\qquad M = (4/3) \times \pi a^3 \rho$ = mass of the fluid displaced by the sphere.



Again, the thrust on the other hemisphere $ADC = \int_{\pi/2}^{\pi} p\cos(\pi - \theta)\cdot a d\theta \cdot 2\pi a \sin\theta$

$$= -2\pi a^2 \int_{\pi/2}^{\pi} \left[\Pi + \frac{1}{2} a\rho f \cos\theta + \frac{1}{8}\rho v^2 (9\cos^2\theta - 5)\right] \sin\theta\cos\theta\, d\theta$$

$$= \Pi\pi a^2 - \frac{1}{3}\pi a^3 \rho f - \frac{1}{16}\pi a^2 \rho v^2 = \Pi\pi a^2 - \frac{1}{4} Mf - \frac{3}{64}\frac{Mv^2}{a}, \text{ as before.}$$

Ex. 10. *A sphere of radius a is moving with constant velocity v through infinite liquid at infinity. If p_0 be the pressure at infinity, prove that the pressure past any point P distant r from the centre O of the sphere and such that OP makes an angle θ with the velocity of the sphere, is given by*

$$p = p_0 - \frac{\rho v^2 a^2}{2r^3}\left[1 + \frac{a^3}{4r^3} - \frac{3x^2}{r^2}\left(1 - \frac{a^3}{4r^3}\right)\right]$$

Show further that if $v > \sqrt{(8p_0/5\rho)}$, a hollow ring is formed in the liquid round the equator of the sphere.

Sol. Proceed as Art in 10.7 upto equation (13). Here $f = 0$, $\cos\theta = x/r$. Hence (13) reduces to

$$\frac{p-p_0}{\rho} = -\frac{a^3 v^2}{8r^6}(4r^3 + a^3) + \frac{3a^3 v^2}{8r^6} \cdot \frac{x^2}{r^2}(4r^3 - a^3)$$

or $$p = p_0 - \frac{\rho v^2 a^3}{2r^3}\left[\frac{1}{4r^3}(4r^3 + a^3) - \frac{3x^2}{4r^5}(4r^3 - a^3)\right] = p_0 - \frac{\rho v^2 a^3}{2r^3}\left[1 + \frac{a^3}{4r^3} - \frac{3x^2}{r^2}\left(1 - \frac{a^3}{4r^3}\right)\right]$$

Second part : From (14) of Art. 10.7, we have

$$p = p_0 + (1/8) \times v^2 \rho (9\cos^2\theta - 5) \qquad \text{...(A)}$$

From (A) we see that the pressure will be minimum when $\cos\theta$ is minimum *i.e.* when $\theta = \pi/2$.

$\therefore$ Minimum pressure $= p_0 - (5/8) \times \rho v^2$ ...(B)

For cavity minimum pressure given by (B) should be negative, that is,

$$p_0 - (5\rho v^2)/8 < 0 \qquad i.e. \qquad v > \sqrt{(8p_0/5\rho)}.$$

Thus at $\theta = \pi/2$ (*i.e.* at equator), when $v > \sqrt{(8p_0/5\rho)}$, pressure will be negative and a hollow ring is formed in the liquid round the equator of the sphere.

EXERCISE 10 (B)

1. A sphere of radius a is moving in an infinite liquid with a variable velocity v in the direction of the axis of x. Show that the pressure at the surface of the sphere is least over the small circle $x = (-2a^2/9v^2)\dot{v}$, the centre of the sphere being the origin.

2. A sphere of radius a is placed in an infinite stream of liquid flowing with uniform velocity V. If the sphere is divided into two parts by a diametral plane perpendicular to the direction of motion of the stream, show that the resultant force between the two parts is less than it would be if the liquid were at rest, the pressure at infinity remaining the same, by an amount $(\pi\rho a^2 v^2)/16$.

3. A sphere of radius a is placed in an incompressible fluid extending to infinity. Each point of the sphere is moving normally outwards with a velocity da/dt, also the fluid at points very distant from the sphere is moving with velocity V in a given direction. Find the velocity potential at any point of the fluid.

Also prove that the resultant pressure on the sphere is the force $(1/2) \times (dM/dt)\, V$ in the direction of the stream, where M is the mass of the fluid displaced by the sphere at the instant considered.

4. A sphere whose radius at time t is $a + b\cos nt$, is held in a stream of liquid of density ρ, whose velocity at a great distance is U. Prove that the resultant thrust on the sphere is

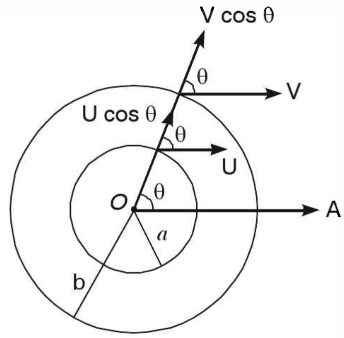
$$2\pi\rho\,(a + b\cos nt)^2\, nb \sin nt.$$

5. Find the velocity potential when a sphere of radius a moves with a speed U along OX in an inviscid, incompressible liquid. Show that the fluid pressure exerts a force $(1/2) \times M'\dot{U}$ opposing the motion where M' = mass of the liquid displaced by the sphere.

10.9. Concentric spheres (problem of initial motion)

A sphere of radius a is surrounded by a concentric sphere of radius b, the space between being filled with liquid at rest. The inner sphere is given a velocity U and outer sphere a velocity V in the same direction. To determine the initial motion of the liquid. **[Meerut 2005]**

Let O be the common centre and ϕ be the velocity potential of the initial motion. Let U and V be in the direction of initial line OA as shown in the figure. Then to determine ϕ we have the following considerations :



(*i*) ϕ satisfies the Laplace's equation

$$\frac{\partial^2 \phi}{\partial r^2}+\frac{2}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r}\frac{\partial^2 \phi}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial \phi}{\partial \theta}=0, \qquad \ldots(1)$$

wherein we have used the fact that there is symmetry about the initial line.

(*ii*) ϕ satisfies the following boundary conditions :

$$-(\partial\phi/\partial r)=U\cos\theta, \qquad \text{when } r=a \qquad \ldots(2)$$

and $$-(\partial\phi/\partial r)=V\cos\theta, \qquad \text{when } r=b. \qquad \ldots(3)$$

The above considerations (*i*) and (*ii*) suggest that ϕ must be of the form $f(r)\cos\theta$ and hence it may be assumed as

$$\phi=(Ar+B/r^2)\cos\theta \qquad \ldots(4)$$

so that $$-\partial\phi/\partial r=-(A-2B/r^3)\cos\theta \qquad \ldots(5)$$

Using boundary conditions (2) and (3), (5) gives

$$U\cos\theta=-(A-2B/a^3)\cos\theta \quad \text{or} \quad -A+2B/a^3=U \qquad \ldots(6)$$

and $$V\cos\theta=-(A-2B/b^3)\cos\theta \quad \text{or} \quad -A+2B/b^3=V. \qquad \ldots(7)$$

Solving (6) and (7) for A and B, we get

$$A=\frac{Ua^2-Vb^3}{b^3-a^3} \quad \text{and} \quad B=\frac{(U-V)\,a^3b^3}{2\,(b^3-a^3)}$$

Therefore, at the instant of starting the motion, the velocity potential is given by

$$\phi=\frac{Ua^3-Vb^3}{a^3-b^3}\,r\cos\theta+\frac{(U-V)a^3b^3}{2\,(a^3-b^3)}\,\frac{\cos\theta}{r^2} \qquad \ldots(8)$$

Cor. *A sphere of radius a is surrounded by a concentric spherical shell of radius b, the space between is filled with liquid. If the sphere be moving with velocity U, to show that*

$$\phi=\frac{Ua^3}{b^3-a^3}\left(r+\frac{b^3}{2r^2}\right)\cos\theta.$$

Also, to discuss the motion so produced.

Since the outer sphere is at rest, $V=0$ and hence (8) reduces to the desired result. Let M be the mass of the inner sphere.

Let I be the impulse necessary to produce the velocity U in the inner sphere. Then by principle of momentum, we have

$$I=MU+\iint \bar{\omega}\cos\theta\, dS, \qquad \ldots(A)$$

where $\bar{\omega}=(\rho\phi)_{r=a}$ is the impulsive pressure of the liquid at point on the sphere. Hence, we have

$$\bar{\omega}=\frac{Ua^3\rho}{b^3-a^3}\left(a+\frac{b^3}{2a^3}\right)\cos\theta.$$

$$\therefore \qquad \iint \varpi \cos\theta\, dS = \int_0^{\pi} \frac{Ua^3\rho}{b^3-a^3}\left(a+\frac{b^3}{2a^2}\right)\cos\theta\cdot\cos\theta\cdot 2\pi a\sin\theta\cdot a d\theta$$

$$= \frac{2\pi\rho U a^3(2a^3+b^3)}{3(b^3-a^3)} = \frac{1}{2}\frac{M'U(2a^3+b^3)}{b^3-a^3}, \qquad \text{...(B)}$$

where $\qquad M'$ = mass of the liquid displace by the sphere $=(4/3)\times \pi a^3\rho$ ...(C)

So from (A) and (B), we have

$$I = MU + \frac{1}{2}\cdot\frac{M'U(2a^3+b^3)}{b^3-a^3} \qquad \text{...(D)}$$

Let $b\to\infty$, then $\qquad \lim_{b\to\infty}\frac{2a^3+b^3}{b^3-a^3} = \lim_{b\to\infty}\frac{2(a^3/b^3)+1}{1-(a^3/b^3)} = 1.$

Thus, if the outer sphere becomes infinitely large (*i.e.* $b\to\infty$), the impulse required to give a sphere in unbounded liquid, a velocity U is

$$I = MU + (1/2)\times M'U = (M + M'/2)U,$$

showing that it effectively increases the mass of the sphere by an amount $M'/2$.

We also notice that the impulse required to impart a velocity U is the same when the sphere is in a mass of liquid at rest at infinity or is surrounded by a fixed spherical envelope of a very large radius.

10.10. Illustrative solved examples.

Ex. 1. *Prove that for liquid contained between the two instantaneously concentric spheres, when the outer (radius a) is moving parallel to the x-axis with a velocity u and the inner (radius b) is moving parallel to the axis of y with velocity v, the velocity potential is*

$$-\frac{1}{a^3-b^3}\left\{a^3ux\left(1+\frac{b^3}{2r^3}\right)-b^3vy\left(1+\frac{a^3}{2r^3}\right)\right\}$$

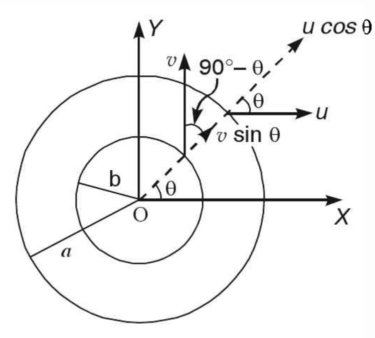
and find the kinetic energy. **[Rohilkhand 2002, 03; Kanpur 1997; Meerut 2003]**

Sol. Proceed like Art. 10.9. Here boundary conditions are

$$-\partial\phi/\partial r = u\cos\theta, \qquad \text{when } r = a \qquad \text{...(1)}$$

and

$$-\partial\phi/\partial r = v\sin\theta, \qquad \text{when } r = b. \qquad \text{...(2)}$$



Moreover ϕ must satisfy the Laplace's equation

$$\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta}=0. \qquad \text{...(3)}$$

The above considerations suggest that ϕ must involve terms containing $\sin\theta$ and $\cos\theta$. So we assume that

$$\phi = (Ar + B/r^2)\cos\theta + (Cr + D/r^2)\sin\theta \qquad \text{...(4)}$$

$$\therefore \qquad -\frac{\partial\phi}{\partial r} = \left(-A+\frac{2B}{r^3}\right)\cos\theta + \left(-C+\frac{2D}{r^3}\right)\sin\theta \qquad \text{...(5)}$$

Using (1) and (2), (5) gives

$$(-A+2B/a^3)\cos\theta+(-C+2D/a^3)\sin\theta=u\cos\theta \qquad ...(6)$$

and

$$(-A+2B/b^3)\cos\theta+(-C+2D/b^3)\sin\theta=v\sin\theta \qquad ...(7)$$

Comparing the coefficients of $\cos\theta$ and $\sin\theta$, (6) and (7) give

$$-A+2B/a^3=u, \qquad -C+2D/a^3=0 \qquad ...(8)$$

$$-A+2B/b^3=0, \qquad -C+2D/b^3=v \qquad ...(9)$$

Solving (8) and (9), we get

$$A=-\frac{ua^3}{a^3-b^3}, \qquad B=-\frac{ua^3b^3}{2(a^3-b^3)}, \qquad C=\frac{ub^3}{a^3-b^3}, \qquad D=\frac{ua^3b^3}{2(a^3-b^3)}.$$

$$\therefore \qquad \phi=-\frac{ua^3}{a^3-b^3}\left(r+\frac{b^3}{2r^2}\right)\cos\theta+\frac{vb^3}{a^3-b^3}\left(r+\frac{a^3}{2r^2}\right)\sin\theta$$

$$=-\frac{1}{a^3-b^3}\left[a^3u\left(1+\frac{b^3}{2r^3}\right)r\cos\theta-b^3v\left(1+\frac{a^3}{2r^3}\right)r\sin\theta\right]$$

$$=-\frac{1}{a^3-b^3}\left[a^3u\left(1+\frac{b^3}{2r^3}\right)x-b^3v\left(1+\frac{a^3}{2r^3}\right)y\right], \quad \begin{matrix} as\ x=r\cos\theta \\ and\ y=r\sin\theta\end{matrix} \qquad ...(10)$$

To determine K.E. The kinetic energy of the liquid is given by

$$T=-\frac{1}{2}\rho\iint\phi\frac{\partial\phi}{\partial n}dS=-\frac{1}{2}\rho\iint\left(\phi\frac{\partial\phi}{\partial r}\right)_{r=a}dS-\frac{1}{2}\rho\iint\left(\phi\frac{\partial\phi}{\partial r}\right)_{r=b}dS$$

[Note that the normal direction is along the direction of r. Also, $-\partial\phi/\partial n$ denotes the outwards normal velocity]

$$=\frac{1}{2}\rho\frac{1}{a^3-b^3}\iint_{r=a}\left[a^3ux\left(1+\frac{b^3}{2a^3}\right)-b^3vy\left(1+\frac{1}{2}\right)\right](u\cos\theta)dS$$

$$-\frac{1}{2}\rho\frac{1}{a^3-b^3}\iint_{r=b}\left[a^3ux\left(1+\frac{1}{2}\right)-b^3vy\left(1+\frac{a^3}{2b^3}\right)\right](v\sin\theta)dS$$

$$\left[\text{Using (10) and}\left(-\frac{\partial\phi}{\partial r}\right)_{r=a}=-u\cos\theta,\ \left(-\frac{\partial\phi}{\partial r}\right)_{r=b}=v\sin\theta\right]$$

$$=\frac{1}{4}\rho\frac{u^3(2a^3+b^3)}{(b^3-a^3)a}\iint_{r=a}x^2dS-\frac{3}{4}\rho\frac{uvb^3}{(a^3-b^3)}\iint_{r=a}xy\,dS$$

$$-\frac{3}{4}\rho\frac{uva^3}{(b^3-a^3)b}\iint_{r=b}xy\,dS+\frac{1}{4}\rho\frac{v^2(a^3+2b^3)}{(a^3-b^3)b}\iint_{r=b}y^2\,dS$$

[Since, when $r=a$, $a\cos\theta=x$ and $a\sin\theta=y$ and when $r=b$, $b\cos\theta=x$ and $b\sin\theta=y$]

$$=\frac{1}{4}\rho\frac{u^2(2a^3+b^3)}{(a^3-b^3)a}\cdot\frac{4}{3}\pi a^4-0-0+\frac{1}{4}\rho\frac{v^2(a^3+2b^3)}{(a^3-b^3)b}\cdot\frac{4}{3}\pi b^4$$

$$\left[\because \iint_{r=a} x^2 dS = \text{M.I. of the hollow sphere of radius } a \text{ about a diameter}\right.$$

$$= \frac{1}{2}\cdot\frac{2Ma^2}{3} = \frac{Ma^2}{3} = \frac{4\pi a^2 . a^2}{3} = \frac{4\pi a^4}{3}. \qquad \text{Similarly, } \iint_{r=b} y^2\, dS = \frac{4\pi b^4}{3}$$

$$\left.\text{Also, } \iint_{r=a} xy\, dS = 0 \quad \text{and} \quad \iint_{r=b} xy\, dS = 0 \text{ (being product of inertia)}\right]$$

$$\therefore \qquad T = \frac{1}{3}\frac{\pi\rho}{a^3 - b^3}[2(u^2 a^6 + v^2 b^6) + a^3 b^3 (u^2 + v^2)]$$

Ex. 2(*a*). *A hollow spherical shell of inner radius a contains a concentric solid uniform sphere of radius b and density* σ *and the space between the two is filled with liquid of density* ρ. *If the shell is suddenly made to move with speed u, prove that a velocity* v *is imparted to the inner sphere, where*

$$v = \frac{3ua^3}{2(\sigma/\rho)(a^3 - b^3) + a^3 + 2b^3}.$$

[Kanpur 2009]

(*b*) *A spherical shell of internal radius a contains a concentric sphere of radius* λa *and density* σ, *the intervening space being filled with liquid of density* ρ *and the whole system is at rest. If a velocity u is communicated to the shell, prove that the initial velocity v communicated to the shell is given by*

$$v = \frac{3u}{2(\sigma/\rho)(1-\lambda^3) + 1 + 2\lambda^3}.$$

[Garhwal 1998]

Sol. **(*a*)** The velocity potential ϕ must satisfy Laplace's equation $\nabla^2\phi = 0$ and it must satisfy the following boundary conditions

$$-\partial\phi/\partial r = u\cos\theta, \quad \text{when} \quad r = a \qquad \text{...(1)}$$

and
$$-\partial\phi/\partial r = v\cos\theta, \quad \text{when} \quad r = b \qquad \text{...(2)}$$

Accordingly, we assume that
$$\phi = (Ar + B/r^2)\cos\theta \qquad \text{...(3)}$$

$$\therefore \qquad -\partial\phi/\partial r = -(A - 2B/r^3)\cos\theta \qquad \text{...(4)}$$

Using (1) and (2), (4) gives

$$(-A + 2B/a^3)\cos\theta = u\cos\theta \quad \text{and} \quad (-A + 2B/b^3)\cos\theta = v\cos\theta.$$

These give
$$A = \frac{b^3 v - a^3 u}{a^3 - b^3} \quad \text{and} \quad B = \frac{a^3 b^3 (v-u)}{2(a^3 - b^3)}$$

$$\therefore \qquad \phi = \frac{1}{a^3 - b^3}\left[(b^3 v - a^3 u) r + \frac{a^3 b^3 (v-u)}{2r^2}\right]\cos\theta \qquad \text{...(5)}$$

The impulsive pressure at any point of the solid sphere $r = b$ is given by

$$\varpi = (\rho\phi)_{r=b} = \frac{\rho b}{a^3 - b^3}\left[b^3 v - a^3 u + \frac{a^3}{2}(v-u)\right]\cos\theta$$

$\therefore$ Resultant impulsive pressure on the inner sphere

$$= \int_0^\pi \varpi\cos\theta \cdot b\,d\theta \cdot 2\pi b\sin\theta = \frac{2\rho b^3 \pi}{a^3 - b^3}\left[b^3 v - a^3 u + \frac{a^3}{2}(v-u)\right]\int_0^\pi \cos^2\theta\sin\theta\, d\theta$$

$$= -\frac{4\rho b^3\pi}{3(a^3-b^3)}\left[b^3 v - a^3 u + \frac{1}{2}a^3(v-u)\right].$$

Since the solid sphere of density σ and radius b moves with velocity v, the equation of motion gives

$$\frac{4}{3}\pi b^3\sigma\, \dot{v} = -\frac{4\rho b^3\pi}{3(a^3-b^3)}\left[b^3 \dot{v} - a^3 \dot{u} + \frac{1}{2}a^3(\dot{v}-\dot{u})\right] \quad \text{or} \quad \dot{v} = \frac{3\dot{u}a^3}{3(\sigma/\rho)(a^3-b^3)+a^3+2b^3}.$$

Part (b). Here $b = a\lambda$. Proceed as in part (*a*).

Ex. 3. *Liquid of density ρ fills the space between a solid sphere of radius a and density σ and a fixed concentric spherical envelope of radius b. Prove that the work done by an impulse which starts the solid sphere with velocity U is*

$$\frac{1}{3}\pi a^3 U^2\left(2\sigma + \frac{2a^3+b^3}{b^3-a^3}\rho\right).$$ **[I.A.S. 1983; Garhwal 1997; Kurukshetra 1999, Kanpur 2008]**

Sol. As in corollary of Art 10.9, the total impulse I is given by

$$I = MU + \iint \overline{\omega}\cos\theta\, dS$$

But $$\iint \overline{\omega}\cos\theta\, dS = \frac{2}{3}\pi\rho U a^3\,\frac{2a^3+b^3}{b^3-a^3},$$ by result (B) of Art. 10.9

and M = mass of inner solid sphere = (4/3) × $\pi a^3\sigma$

$$\therefore \qquad I = \frac{2\pi a^3 U}{3}\left(2\sigma + \frac{2a^3+b^3}{b^3-a^3}\rho\right)$$

Hence the work done by impulse I = I × (mean of the initial and final velocities)

$$= I \times \frac{0+U}{2} = \frac{1}{2}UI = \frac{\pi a^3 U^2}{3}\left(2\sigma + \frac{2a^3+b^3}{b^3-a^3}\rho\right).$$

Ex. 4. *The space between two concentric spherical shells of radii a and b (a > b) is filled with an incompressible fluid of density ρ and the shells suddenly begin to move with velocities U, V, in the same direction. Prove that the resultant impulsive pressure on the inner shell is*

$$\frac{2\pi\rho b^3}{3(a^3-b^3)}\left[3a^3U - (a^3+2b^3)V\right].$$ **[Jiwaji 1998, Meerut 2004]**

Further show that the K.E. of the liquid is

$$\frac{\pi\rho}{3(a^3-b^3)}\left[a^3b^3(V-U)^2 + 2(b^3V - a^3U)^2\right].$$ **[Agra 1995; Kurukshetra 1998]**

Sol. As in Art. 10.9, the velocity potential is given by

$$\phi = \frac{Ua^3 - Vb^3}{b^3-a^3}\, r\cos\theta + \frac{(U-V)a^3b^3}{2(b^3-a^3)}\frac{\cos\theta}{r^2} = \frac{1}{a^3-b^3}\left[r(b^3V - a^3U) + \frac{a^3b^3(V-U)}{2r^2}\right]\cos\theta \quad ...(1)$$

The impulsive pressure at a point on the sphere $r = b$ is given by

$$\varpi = (\rho\phi)_{r=b} = \frac{\rho\cos\theta}{a^3 - b^3}\left[b(b^3V - a^3U) + \frac{1}{2}a^3 b(V-U)\right] \qquad \text{...(2)}$$

The resultant impulsive pressure on the inner shell ($r = b$)

$$= \int_0^\pi \varpi\cos\theta \cdot b\theta \cdot 2\pi b\sin\theta = \frac{2\pi b^3\rho}{a^3-b^3}\left[b^3V - a^3U + \frac{1}{2}a^3(V-U)\right]\int_0^\pi \cos^2\theta\sin\theta\, d\theta, \quad \text{by (2)}$$

$$= \frac{\pi b^3\rho}{a^3-b^3}\left[2(b^3V - a^3U) + a^3(V-U)\right]\cdot\left(-\frac{2}{3}\right) = \frac{2\pi\rho b^3}{3(a^3-b^3)}\left[3a^3U - (a^3+2b^3)V\right].$$

From (1),
$$\frac{\partial\phi}{\partial r} = \frac{1}{a^3-b^3}\left[b^3V - a^3U - \frac{a^3b^3(V-U)}{r^3}\right]\cos\theta$$

and
$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{a^3-b^3}\left[b^3V - a^3U + \frac{a^3b^3(V-U)}{2r^3}\right](-\sin\theta)$$

$$\therefore\ q^2 = (-\partial\phi/\partial r)^2 + (-\partial\phi/r\partial\theta)^2$$

$$= \frac{1}{(a^3-b^3)^2}\left[\left\{b^3V - a^3U - \frac{a^3b^3(V-U)}{r^3}\right\}^2\cos^2\theta + \left\{b^3V - a^3U + \frac{a^3b^3(V-U)}{2r^2}\right\}^2\sin^2\theta\right]$$

$$= \frac{1}{(a^3-b^3)^2}\left[(b^3V - a^3U)^2 + \frac{a^6b^6(V-U)^2}{r^6}\left(\cos^2\theta + \frac{1}{4} + \sin^2\theta\right)\right.$$
$$\left. - \frac{2a^3b^3(b^3V - a^3U)(V-U)}{r^3}\cos^2\theta + \frac{a^3b^3(b^3V - a^3U)(V-U)}{r^3}\sin^2\theta\right]$$

$$= \frac{1}{(a^3-b^3)^2}\left[(b^3V - a^3U)^2 + \frac{a^6b^6(V-U)^2}{4r^6}(1+3\cos^2\theta) + \frac{a^3b^3(b^3V - a^2U)(V-U)}{r^3}(1-3\cos^2\theta)\right]$$

$$\text{K.E.} = \int_0^\pi\int_b^a \left(\frac{1}{2}\rho q^2\right)\cdot 2\pi r\sin\theta\cdot r\,d\theta\,dr = \frac{\pi\rho}{(a^3-b^3)^2}\int_b^a\left[2(b^3V - a^3U)^2 + \frac{a^6b^6}{r^6}(V-U)^2\right]r^3\,dr$$

[on putting value of q^2 and integrating w.r.t. θ]

$$= \frac{\pi\rho}{3(a^3-b^3)}[a^3b^3(V-U)^2 + 2(b^3V - a^3U)^2].$$

Ex. 5. *Incompressible fluid of density* ρ *is contained between two rigid concentric spherical surfaces, the outer one of mass* M_1 *and radius a, the inner one of mass* M_2 *and radius b. A normal blow P is given to the outer surface. Prove that the initial velocities of the two containing surfaces (U for the outer and V for the inner) are given by the equations*

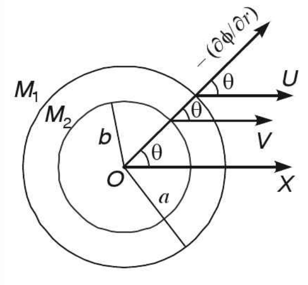
$$\left\{M_1 + \frac{2\pi\rho a^3(2a^3+b^3)}{3(a^3-b^3)}\right\}U - \frac{2\pi\rho a^3b^3}{a^3-b^3}V = P,$$

$$\left\{M_2 + \frac{2\pi\rho b^3(2a^3+b^3)}{3(a^3-b^3)}\right\}V = \frac{2\pi\rho a^3b^3}{a^3-b^3}U. \qquad \textbf{[Meerut 2006]}$$

Sol. As in Art. 10.9, we have

$$\phi = \frac{1}{a^3 - b^3}\left[(Vb^3 - Ua^3)\,r + \frac{(V-U)\,a^3b^3}{2r^2}\right]\cos\theta \qquad \text{...(1)}$$

The normal blow P in the outer surface imparts velocity U to the outer and V to the inner spherical surface. Let ϖ_1, ϖ_2 be the impulsive pressures on an element dS of the boundary surfaces $r = a$ and $r = b$ respectively. Then



$$M_1U = P - \iint \varpi_1 \cos\theta\, dS \qquad \text{on} \qquad r = a \qquad \text{...(2)}$$

and $$M_2V = -\iint \varpi_2 \cos\theta\, dS \qquad \text{on} \qquad r = b \qquad \text{...(3)}$$

On $r = a$, from (1), $$\varpi_1 = (\rho\phi)_{r=a} = \frac{1}{a^3 - b^3}\left[(Vb^3 - Ua^3)\,a + \frac{1}{2}ab^3(V-U)\right]\cos\theta$$

$\therefore$ $$(2) \Rightarrow M_1U = P - \int_0^{\pi} \varpi \cos\theta \cdot a\,d\theta \cdot 2\pi a \sin\theta$$

$$= P - \frac{2\pi a^2}{a^3 - b^3}\left[(Vb^3 - Ua^3)\,a + \frac{1}{2}ab^3(V-U)\right]\int_0^{\pi} \cos^2\theta \sin\theta\, d\theta$$

$$= P - \frac{\pi a^3}{a^3 - b^3}[3Vb^3 - U(2a^3 + b^3)] \times \left(-\frac{2}{3}\right)$$

Thus, $$\left\{M_1 + \frac{2\pi\rho a^3(2a^3 + b^3)}{3(a^3 - b^3)}\right\}U - \frac{2\pi\rho a^3b^3}{a^3 - b^3}V = P. \qquad \text{...(4)}$$

Again, on $r = b$, $$\varpi_2 = (\rho\phi)_{r=b} = \frac{1}{a^3 - b^3}\left[(Vb^3 - Ua^3)\,b + \frac{(V-U)\,a^3b}{2}\right]\cos\theta$$

$\therefore$ $(3) \Rightarrow$ $$M_2V = -\int_0^{\pi} \varpi_2 \cos\theta\, b\,d\theta \cdot 2\pi b \sin\theta$$

$$= -\frac{2\pi b^2\rho}{a^3 - b^3}\left[(Vb^3 - Ua^3)b + \frac{(V-U)\,a^3b}{2}\right]\int_0^{\pi} \cos^2\theta \sin\theta\, d\theta$$

$$= -\frac{2\pi b^3\rho}{a^3 - b^3}\left[Vb^3 - Ua^3 + \frac{1}{2}a^3(V-U)\right]\cdot\left(-\frac{2}{3}\right)$$

Thus, $$\left\{M_2 + \frac{2\pi\rho b^3(2b^3 + a^3)}{3(a^3 - b^3)}\right\}V = \frac{2\pi\rho b^3a^3}{a^3 - b^3}U.$$

EXERCISE 10 (C)

1. Detemine the velocity potential potential of the initial motion when an impulsive force is so applied that the inner sphere starts moving with velocity U and the outer sphere with velocity V in the same direction. Also show that the impulse I is given by $I = (M + M'/2)\,U$, where M is mass of the inner sphere and M' is mass of the liquid displaced. **[Meerut 2005]**

Hint. Refer Art 10.9 and its corollary.

2. The space between two concentric spheres of radii a and b is filled with liquid. The spheres have velocities U and V in the same direction. Determine the kinetic energy of the liquid.

3. A sphere of radius a surrounded by a concentric spherical shell of radius b, and the space between is filled with liquid. If the sphere be moving with velocity V, show that

$$\phi = \frac{Va^3}{b^3 - a^3}\left(r + \frac{b^3}{2r^2}\right)\cos\theta.$$

If the radius of the spherical shell be twice the radius of the sphere and the liquid density be ρ, show that the kinetic energy of liquid motion is $10\pi\rho a^3 V^2/21$.

4. Find the velocity potential for the liquid contained between two instantaneously concentric spheres when the outer (radius $2a$) is moving parallel to the axis of x with velocity u and the inner (radius a) is moving parallel to the axis of y with velocity u.

5. The space between two spherical shells of radii $2a$ and a is filled with an incompressible fluid of density ρ and the shell suddenly begins to move with velocity v, $2v$ in the same direction, prove that the resultant impulsive pressure on the inner shell is $8\pi\rho v a^3/21$.

6. The space between a solid sphere of radius a and a concentric spherical shell of radius $2a$, is filled with homogeneous liquid and the system being at rest, an impulse is applied to the shell causing it to start with a velocity V, show that the sphere starts with velocity $(12\rho V)/(7\sigma + 5\rho)$, where σ, ρ are respectively the densities of the sphere and the liquid.

[Kanpur 2006]

7. The space between two concentric spherical shells of radii a, b ($a < b$) is filled with liquid of density ρ. If the shells are in motion the outer one with velocity U in the x-direction and the inner with velocity V in the y-direction, show that the initial motion of the liquid is given by the velocity potential $\phi = \{a^3 \text{ U } (1 + b^3 r^{-3}/2)x - b^3 V (1 + a^3 r^{-3}/2)y\} / (b^3 - a^3)$, where $r^2 = x^2 + y^2 + z^2$. Evaluate the velocity at any point of the liquid and hence prove that the total momentum communicated has components $[(4/3) \times \pi\rho a^3 U, (4/3) \times \pi\rho b^3 V, 0]$. **[Agra 2007]**

Hint : Refer solved Ex.1, page 10.17.

10.11. Three dimensional sources and sinks.

If the motion of a fluid consists of a symmetrical flow in all directions proceeding from a point, the point is known as a *source in three dimensions*. If the total flow across a small surface surrounding the point is $4\pi m$, m is known as the *strength of the source*.

If ϕ be the velocity potential due to a simple source of strength m in fluid at rest at infinity, the velocity at a distance r from the source is $-\partial\phi/\partial r$ in radial direction only. Hence the flow across a sphere of radius r is $4\pi r^2(-\partial\phi/\partial r)$ and hence we have

$$-4\pi r^2(\partial\phi/\partial r) = 4\pi m \qquad \text{or} \qquad \partial\phi/\partial r = -(m/r^2)$$

Integrating, $\qquad \phi = m/r^2.$

If, however, the flow is such that the fluid is directed radially inwards to a point from all directions in a symmetrical manner, then the point is known as *sink*. In other words, a source of negative strength, or inward radial flow, is called a sink.

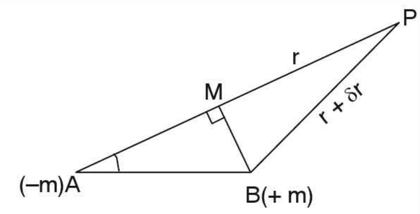
Remark. For more details of sources, sinks, doublets and images, please refer Chapter 5.

10.12. Three dimensional doublet.

A combination of a three dimensional source of strength m and a three dimensional sink of strength $-m$ at a small distance δs apart, where in the limit m is taken infinitely great and δs infinitely small but so that the product $m\delta s$ remains finite and equal to μ is called a *three dimensional doublet* of strength m; and the line δs taken in the sense from $-m$ to $+m$ is taken as the *axis of the doublet*.

10.13. Velocity potential due to a three dimensional doublet. [Kanpur 2007]

Let a doublet be made up of a sink $-m$ at A and a source $+m$ at B. Let P be any point. Let $AP = r$, $BP = r + \delta r$ and $\angle PAB = \theta$. Let ϕ be the velocity potential due to this doublet. Then,



$$\phi = -\frac{m}{r} + \frac{m}{r+\delta r} = -\frac{m}{r} + \frac{m}{r(1+\delta r/r)}$$

$$= -\frac{m}{r} + \frac{m}{r}\left(1+\frac{\delta r}{r}\right)^{-1} = -\frac{m}{r}\left[1-\left(1+\frac{\delta r}{r}\right)^{-1}\right]$$

$$= -\frac{m}{r}\left[1-\left(1-\frac{\delta r}{r}+\ldots\right)\right], \text{ to first order of approximation}$$

$$= \frac{m\delta r}{r^2}$$

$$= \frac{m\delta s\cos\theta}{r^2}, \qquad \text{as } AB = \delta s \quad \text{and} \quad \delta r = AM = \delta s\cos\theta$$

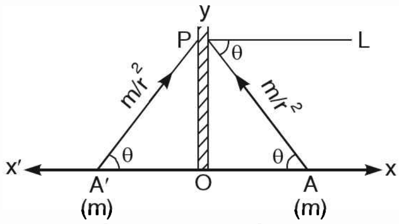
$$= \frac{\mu\cos\theta}{r^2} \qquad [\because \ \mu = m\delta s = \text{strength of the doublet}]$$

$$= \mu\frac{\partial}{\partial s}\left(\frac{1}{r}\right), \text{ as } \cos\theta = \frac{\delta r}{\delta s}.$$

10.14. Image of a three-dimensional source with regard to a plane.

Let there be two sources each of strength m at A and A' on opposite sides of and equidistant from the plane. Since the velocity potential ϕ due to a source of strength m is m/r, we have

$$\therefore \ q_r = \text{radial velocity } = -\frac{\partial\phi}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{m}{r}\right) = \frac{m}{r^2}.$$



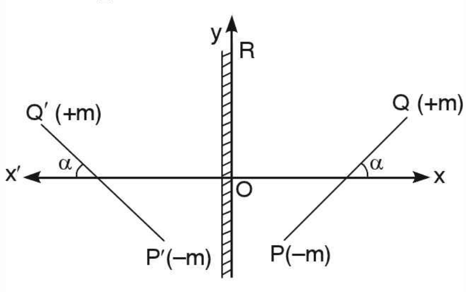
Hence the velocity at P due to source m at A is m/r^2 along AP and velocity at P due to source m at A' is m/r^2 along $A'P$

$\therefore$ the resultant velocity at P (perpendicular to the plane) $= -(m/r^2)\cos\theta + (m/r^2)\cos\theta = 0$.

It follows that there is no flow across the plane OP. *Hence the image of a simple source with regard to plane is an equal source equidistant from the plane.*

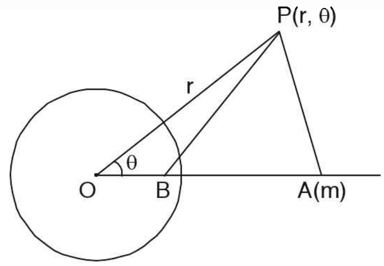
Cor. Image of a three-dimensional doublet with regard to a plane.

Let PQ be a doublet with its axis inclined at angle α to OX. Then by using the above result for finding the images of the source and sink with respect to the plane OR, we see that the image of the doublet PQ is again an equal doublet symmetrically placed as shown in the figure. Here sink at P' and source at Q' are the images of sink P and source Q respectively.



10.15. Image of a three-dimensional source with regard to a sphere. [Rohilkhand 2000]

Let the source of strength m be placed at a distance f ($> a$) from the centre of the sphere of radius a. Suppose that the velocity potential ϕ is made up of two parts, *i.e.*, let $\phi = \phi_1 + \phi_2$, where ϕ_1 is the velocity potential due to the source alone in the absence of the sphere and ϕ_2 is the contribution to the velocity potential due to the presence of the sphere. Let $\mu = \cos\theta$ and $P_n(\mu)$ be the Legendre's polynomial of degree n. Then, if $r < f$, we have



$$\phi_1 = \frac{m}{AP} = \frac{m}{(f^2 + r^2 - 2fr\cos\theta)^{1/2}}$$

$$= \frac{m}{f}\left(1 - \frac{2r}{f}\cos\theta + \frac{r^2}{f^2}\right)^{-1/2} = \frac{m}{f}\left[1 - 2\frac{r}{f}\mu + \left(\frac{r}{f}\right)^2\right]^{-1/2}$$

$$\therefore \qquad \phi_1 = \frac{m}{f}\sum_{n=0}^{\infty}\left(\frac{r}{f}\right)^n P_n(\mu) \qquad \text{...(1)}$$

$$\left[\text{Using the well known formula*}, \quad (1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)\right]$$

Since the motion is symmetrical about OA and ϕ_2 has to satisfy Laplace's equation, we take

$$\phi_2 = \sum_{n=0}^{\infty} A_n \frac{a^n}{r^{n+1}} P_n(\mu). \qquad \text{...(2)}$$

For the proposed image system, the sphere must be a streamline surface and hence the velocity normal to the sphere will be zero. For this, we must have

$$\partial(\phi_1 + \phi_2)/\partial r = 0, \qquad \text{when} \quad r = a.$$

i.e.,
$$\frac{m}{f}\sum_{n=0}^{\infty}\frac{na^{n-1}}{f^n}P_n - \sum_{n=0}^{\infty}(n+1)\frac{A_n}{a^2}P_n = 0.$$

$$\therefore \qquad A_0 = 0 \qquad \text{and} \qquad A_n = m\,n\,a^{n+1}/(n+1)f^{n+1}$$

$$\therefore \qquad \phi_2 = m\sum_{n=1}^{\infty}\frac{n}{n+1}\frac{a^{2n+1}}{r^{n+1}f^{n+1}}P_n = m\sum_{n=1}^{\infty}\frac{n+1-1}{n+1}\frac{a^{2n+1}}{r^{n+1}f^{n+1}}P_n$$

$$= m\sum_{n=1}^{\infty}\frac{a^{2n+1}}{r^{n+1}f^{n+1}}P_n - m\sum_{n=1}^{\infty}\frac{a^{2n+1}}{r^{n+1}f^{n+1}}\frac{P_n}{n+1}$$

Let B be the inverse point of A with respect to the sphere. Then $OA \times OB = a^2$, so that $OB = a^2/f = c$, say. Then the above value of ϕ_2 may be re-written as

$$\phi_2 = \frac{ma}{f}\sum_{n=0}^{\infty}\frac{c^n}{r^{n+1}}P_n - \frac{ma}{f}\sum_{n=0}^{\infty}\frac{c^n}{r^{n+1}}\frac{P_n}{n+1}, \quad \text{on adding and subtracting } \frac{ma}{f}\frac{P_0}{r}$$

$$= \frac{ma}{fr}\left(1 - \frac{2c}{r}\cos\theta + \frac{c^2}{r^2}\right)^{-1/2} - \frac{ma}{f}\sum_{n=0}^{\infty}\frac{c^n}{r^{n+1}}\frac{P_n}{n+1} = \frac{ma}{f(r^2 + c^2 - 2rc\cos\theta)^{1/2}} - \frac{ma}{f}\sum_{n=0}^{\infty}\frac{c^n}{r^{n+1}}\frac{P_n}{n+1}$$

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* Refer chapter 9 in part II of author's "Ordinary and partial differential equations", published by S. Chand & Co., New Delhi.

Thus, $$\phi_2 = \frac{(ma/f)}{BP} - \frac{ma}{f}\sum_{n=0}^{\infty}\frac{c^n}{r^{n+1}}\frac{P_n}{n+1} \qquad ...(3)$$

The first term on R.H.S. of (3) is velocity potential due to a source of strength ma/f at B.

Next consider a source of strength ma/f at any point on OB distant λ from the centre. Then velocity potential due to this source

$$= \frac{ma}{f}(r^2 + \lambda^2 - 2r\lambda\cos\theta)^{-1/2} = \frac{ma}{f}\sum_{n=0}^{\infty}\frac{\lambda^n}{r^{n+1}}P_n.$$

Suppose the line OB is formed of sources of strength ma/cf at every point of it. Then velocity potential due to this line source

$$= \frac{ma}{cf}\int_0^c\left[\sum_{n=0}^{\infty}\frac{\lambda^n}{r^{n+1}}P_n\right]d\lambda = \frac{ma}{cf}\sum_{n=0}^{\infty}\int_0^c\frac{\lambda^n}{r^{n+1}}P_n d\lambda = \frac{ma}{cf}\sum_{n=0}^{\infty}\frac{c^{n+1}P_n}{r^{n+1}(n+1)}$$

This shows that the second term on *R.H.S.* of (3) is the velocity potential due to a continuous line distribution of sinks of strength $-ma/cf$ or $-ma/a^2$ or $-m/a$ (as $a^2/f = c$) per unit length extending from O to B.

Thus, the image system of a source of strength m placed outside with regard to a sphere consists of a source of strength ma/f at an inverse point and a line sink of strength m/a per unit length extending from the centre of the sphere to the inverse point.

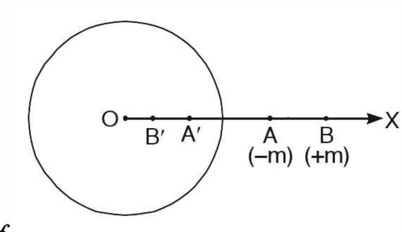
10.16. Image of a doublet in front of a sphere.

Let the axis of the given doublet AB be along OX; the sink $-m$ at A ($OA = f$) and source m at B ($OB = f + \delta f$). Then

μ = the strength of the doublet = $\lim m\,\delta f$, as $m \to \infty$ and $\delta f \to 0$

Let A' and B' be the inverse points of A and B respectively with regard to the sphere.

Then, $OA' = a^2/f$ and $OB' = a^2/(f + \delta f)$.



Now, $$OB' = \frac{a^2}{f}\left(1 + \frac{\delta f}{f}\right)^{-1} = \frac{a^2}{f}\left(1 - \frac{\delta f}{f}\right),$$

(to first order of approximation)

Thus, $$A'B' = OA' - OB' = (a^2/f^2)\delta f.$$

Now, image of sink $-m$ at A is a sink $(-ma/f)$ at A' and a line source of strength (m/a) per unit length distributed from O to A'. Similarly, image of source m at B is a source $[ma/(f + \delta f)]$ at B' and a line sink of strength $(-m/a)$ per unit length distributed from O to B'. Considering the above two line distributions, we find that the line distributions of sinks and sources from O upto B' cancel each other and then we obtain a line distribution of sources of strength m/a from B' to A'. This gives rise to a source of strength $(m/a)\,B'A' = (ma/f^2)\,\delta f = \mu a/f^2$, ultimately at B'.

Now, $$\frac{ma}{f + \delta f} = \frac{ma}{f}\left(1 + \frac{\delta f}{f}\right)^{-1} = \frac{ma}{f}\left(1 - \frac{\delta f}{f}\right) = \frac{ma}{f} - \frac{\mu a}{f^2}$$

This shows that source of strength $ma/(f+\delta f)$ at B is equivalent to a source (ma/f) at B' and a sink $\mu a/f^2$ at B'. But there is already a source $(\mu a/f^2)$ at B'. Thus source and sink at B' cancel each other.

Thus finally we are left with a sink $(-ma/f)$ at A' and a source (ma/f) at B'. When $AB \to 0$ then $A'B' \to 0$ also and hence the source and sink at B' and A' give rise to another doublet of strength μ' (say).

Now, $\mu' = \lim \{(ma/f)\, A'B'\} = \lim \{(ma/f) \times (a^2\delta f/f^2)\} = (a^3/f^3) \lim (m\delta f) = a^3\mu/f^3$.

Clearly, the axis of the image doublet is opposite to that of the given doublet.

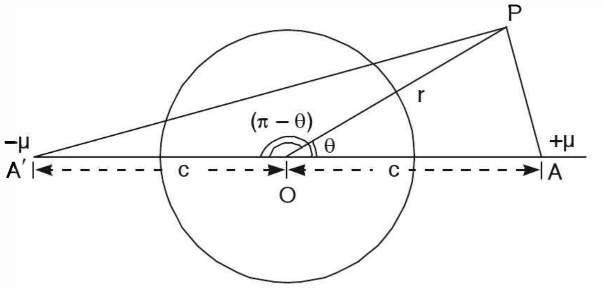
10.16A. Ilustrative solved examples.

Ex. 1. *A source and a sink, each of strength μ, exist in an infinite liquid on opposite sides of, and at equal distances c, from the centre of a rigid sphere of radius a. Show that the velocity potential ϕ may be expressed in the form*

$$\phi = \frac{2\mu}{c}\sum_{n=0}^{\infty}\left[\left(\frac{r}{c}\right)^{2n+1} + \frac{2n+1}{2n+2}\cdot\frac{c}{a}\left(\frac{a^2}{rc}\right)^{2n+2}\right]P_{2n+1}(\cos\theta),$$

θ being the vectorial angle measured from the diameter of the sphere on which the source and sink lie and $r < c$. Also find an expression for ϕ when $r > c$.

Sol. Case 1. When $r < c$. Let a source m and a sink $-m$ be placed at A $(OA = c)$ and $A'(OA' = c)$ respectively. Suppose that the velocity potential ϕ is made up of two parts, $\phi = \phi_1 + \phi_2$, where ϕ_1 is velocity potential due to the source and sink alone in absence of the sphere and ϕ_2 is the contribution to the velocity potential due to the presence of the sphere. Then, we have



$$\phi_1 = \mu/AP - \mu/A'P = \mu(r^2+c^2-2rc\cos\theta)^{-1/2} - \mu(r^2+c^2+2rc\cos\theta)^{-1/2}$$

$$= \frac{\mu}{c}\left[\left(1-\frac{2r}{c}\cos\theta+\frac{r^2}{c^2}\right)^{-1/2} - \left(1+\frac{2r}{c}\cos\theta+\frac{r^2}{c^2}\right)^{-1/2}\right]$$

$$= \frac{\mu}{c}\left[\sum_{n=0}^{\infty}\left(\frac{r}{c}\right)^n P_n(\cos\theta) - \sum_{n=0}^{\infty}\left(-\frac{r}{c}\right)^n P_n(\cos\theta)\right]$$

Thus, $\phi_1 = \dfrac{2\mu}{c}\displaystyle\sum_{n=0}^{\infty}\left(\frac{r}{c}\right)^{2n+1} P_{2n+1}(\cos\theta)$, since all even terms cancel each other ... (1)

Since ϕ_2 must be zero at infinity and satisfies Laplace's equation, we take (noting that the motion is symmetrical about OA)

$$\phi_2 = \sum_{n=0}^{\infty}\frac{A_n}{r^{n+1}}P_n(\cos\theta). \qquad \text{... (2)}$$

Since velocity normal to the sphere must be zero, $\partial(\phi_1+\phi_2)/\partial r = 0$, when $r = a$,

i.e.,
$$\frac{2\mu}{c}\sum_{n=0}^{\infty}\frac{(2n+1)a^{2n}}{c^{2n+1}}P_{2n+1}(\cos\theta) - \sum_{n=0}^{\infty}\frac{A_n(n+1)}{a^{n+2}}P_n(\cos\theta) = 0.$$

Since Legendre's polynomials are all independent, equating the coefficients of P_n's, we have

$$A_{2n} = 0 \text{ for each value of } n,$$

and
$$A_{2n+1}\frac{(2n+2)}{a^{2n+3}} - \frac{2\mu}{c}\frac{(2n+1)}{c^{2n+1}}a^{2n} = 0 \quad \Rightarrow \quad A_{2n+1} = \frac{\mu(2n+1)a^{4n+3}}{c^{2n+2}(n+1)} = \frac{\mu}{a}\frac{2n+1}{n+1}\left(\frac{a^2}{c}\right)^{2n+2}$$

$\therefore$ From (2), $\phi_2 = \sum_{n=0}^{\infty}\frac{A_{2n+1}}{r^{2n+2}}P_{2n+1}(\cos\theta)$ or $\phi_2 = \sum_{n=0}^{\infty}\frac{\mu}{a}\cdot\frac{2n+1}{n+1}\left(\frac{a^2}{rc}\right)^{2n+2}P_{2n+1}(\cos\theta)$

Thus,
$$\phi_2 = \frac{2\mu}{c}\sum_{n=0}^{\infty}\frac{2n+1}{2n+2}\cdot\frac{c}{a}\cdot\left(\frac{a^2}{rc}\right)^{2n+2}P_{2n+1}(\cos\theta) \qquad \ldots(3)$$

Using (1) and (3), we have

$$\therefore \phi = \phi_1 + \phi_2 = \frac{2\mu}{c}\sum_{n=0}^{\infty}\left[\left(\frac{r}{c}\right)^{2n+1} + \frac{2n+1}{2n+2}\times\frac{c}{a}\times\left(\frac{a^2}{rc}\right)^{2n+2}\right]P_{2n+1}(\cos\theta)$$

Case II. When $r > c$. Let the new values of ϕ, ϕ_1 and ϕ_2 be ϕ', ϕ'_1 and ϕ'_2 respectively. Then

$$\phi' = \phi'_1 + \phi'_2.$$

When $r > c$, $c/r < 1$. Hence interchanging c and r in the expression for ϕ_1, we obtain

$$\phi'_1 = \frac{2\mu}{r}\sum_{n=0}^{\infty}\left(\frac{c}{r}\right)^{2n+1}P_{2n+1}(\cos\theta) = 2\mu\sum_{n=0}^{\infty}\frac{c^{2n+1}}{r^{2n+2}}P_{2n+1}(\cos\theta)$$

Similarly, we take
$$\phi'_2 = 2\mu\sum_{n=0}^{\infty}\frac{B_{2n+1}}{r^{2n+2}}P_{2n+1}(\cos\theta), \text{ as before}$$

Thus,
$$\phi' = \phi'_1 + \phi'_2 = 2\mu\sum_{n=0}^{\infty}\left[\frac{c^{2n+1}}{r^{2n+2}} + \frac{B_{2n+1}}{r^{2n+2}}\right]P_{2n+1}(\cos\theta)$$

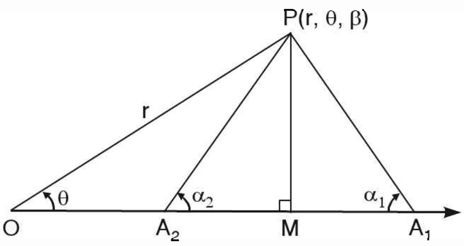
To obtain the constants B_{2n+1}, we use the fact that $\phi = \phi'$ when $r = c$ and obtain

$$B_{2n+1} = \frac{2n+1}{2n+2}\frac{a^{4n+3}}{c^{2n+2}}, \quad \text{and so} \quad \phi' = \frac{2\mu}{r}\sum_{n=0}^{\infty}\left[\left(\frac{c}{r}\right)^{2n+1} + \frac{2n+1}{2n+2}\frac{a}{c}\left(\frac{a^2}{rc}\right)^{2n+1}\right]P_{2n+1}(\cos\theta).$$

Ex. 2. Doublets of strengths μ_1, μ_2 are situated at points A_1, A_2 whose cartesian coordinates are $(0, 0, c_1)$, $(0, 0, c_2)$, their axes being directed towards and away from the origin respectively. Find the condition that there is no transport of fluid over the surface of the sphere $x^2 + y^2 + z^2 = c_1 c_2$.

Sol. With OA_2A_1 as initial line, let spherical polar coordinates* of P be (r, θ, β). Let the axes of the doublet at A_1 and A_2 make angles α_1, α_2 with A_1P, A_2P. Then the velocity potential ϕ at P is

$$\phi = \frac{\mu_2\cos\alpha_2}{A_2P^2} + \frac{\mu_1\cos\alpha_1}{A_1P^2} \qquad \ldots(1)$$



Given $OA_1 = c_1$ and $OA_2 = c_2$. Then from figure,

$$\left.\begin{aligned} A_1P &= (r^2 - 2rc_1\cos\theta + c_1^2)^{1/2}, \\ \text{and} \quad A_2P &= (r^2 - 2rc_2\cos\theta + c_2^2)^{1/2} \end{aligned}\right\} \qquad \ldots(2)$$

$$\cos\alpha_1 = \frac{MA_1}{A_1P} = \frac{OA_1 - OM}{A_1P} = \frac{c_1 - r\cos\theta}{A_1P} \qquad \ldots(3)$$

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* In order to avoid confusion we have taken (r, θ, β) in place of (r, θ, ϕ). Note that ϕ is used to denote velocity potential in this solution.

and $$\cos\alpha_2 = \frac{MA_2}{A_2P} = \frac{OM - OA_2}{A_2P} = \frac{r\cos\theta - c_2}{A_2P}, \qquad \text{...(4)}$$

where M is the foot of the perpendicular drawn from P on OA_1.

Using (3) and (4), (1) reduces to

$$\phi(r,\theta) = \frac{\mu_2(r\cos\theta - c_2)}{A_2P^3} + \frac{\mu_1(c_1 - r\cos\theta)}{A_1P^3} \qquad \text{...(5)}$$

Using (2), (5) reduces to

$$\phi(r,\theta) = \mu_2(r\cos\theta - c_2)(r^2 - 2rc_2\cos\theta + c_2^2)^{-3/2} + \mu_1(c_1 - r\cos\theta)(r^2 - 2rc_1\cos\theta + c_1^2)^{-3/2}$$

$$\therefore\ \partial\phi/\partial r = \mu_2\{\cos\theta\,(r^2 - 2rc_2\cos\theta + c_2^2)^{-3/2}$$

$$-3(r\cos\theta - c_2)(r - c_2\cos\theta)(r^2 - 2rc_2\cos\theta + c_2^2)^{-5/2}\} + \mu_1\{-\cos\theta\,(r^2 - 2rc_1\cos\theta + c_1^2)^{-3/2}$$

$$-3(c_1 - r\cos\theta)(r - c_1\cos\theta)(r^2 - 2rc_1\cos\theta + c_1^2)^{-5/2}\} \qquad \text{...(6)}$$

Since there is no transport of fluid over the sphere $x^2 + y^2 + z^2 = (\sqrt{c_1c_2})^2$, we have

$$\partial\phi/\partial r = 0 \qquad \text{when} \qquad r = \sqrt{c_1c_2}. \qquad \text{...(7)}$$

Hence using (7), (6) reduces to

$$\mu_2[\cos\theta\,\{c_1c_2 - 2c_2(c_1c_2)^{1/2}\cos\theta + c_2^2\}^{-3/2} - 3\{(c_1c_2)^{1/2}\cos\theta - c_2\}\ \{(c_1c_2)^{1/2} - c_2\cos\theta\}$$

$$\times\{c_1c_2 - 2c_2(c_1c_2)^{1/2}\cos\theta + c_2^2\}^{-5/2}] = \mu_1[\cos\theta\,\{c_1c_2 - 2c_1(c_1c_2)^{1/2}\cos\theta + c_1^2\}^{-3/2}$$

$$+3\{c_1 - (c_1c_2)^{1/2}\cos\theta\}\ \{(c_1c_2)^{1/2} - c_1\cos\theta\}\{c_1c_2 - 2c_1(c_1c_2)^{1/2}\cos\theta + c_1^2\}^{-5/2}]$$

or $$\mu_2c_2^{-3/2}[\cos\theta\{c_1 - 2(c_1c_2)^{1/2}\cos\theta + c_2\}^{-3/2} + 3\{c_1^{1/2}\cos\theta - c_2^{1/2}\}\{c_2^{1/2}\cos\theta - c_1^{1/2}\}$$

$$\times\{c_1 - 2(c_1c_2)^{1/2}\cos\theta + c_2\}^{-5/2}] = \mu_1c_1^{-3/2}[\cos\theta\{c_1 - 2(c_1c_2)^{1/2}\cos\theta + c_2\}^{-3/2}$$

$$+ 3\{c_1^{1/2}\cos\theta - c_2^{1/2}\}\{c_2^{1/2}\cos\theta - c_1^{1/2}\}\{c_1 - 2(c_1c_2)^{1/2}\cos\theta + c_2\}^{-5/2}]$$

or $\mu_2c_2^{-3/2} = \mu_1c_1^{-3/2}$ or $\mu_2/\mu_1 = (c_2/c_1)^{3/2}$, which is the required condition.

Ex. 3. *Prove that the velocity potential at a point P due to a uniform finite line source AB of strength m per unit length is of the form* $\phi = m\log f$, *where*

$$f = \frac{r_2 + x_2}{r_1 + x_1} = \frac{r_1 - x_1}{r_2 - x_2} = \frac{a + l}{a - l},$$

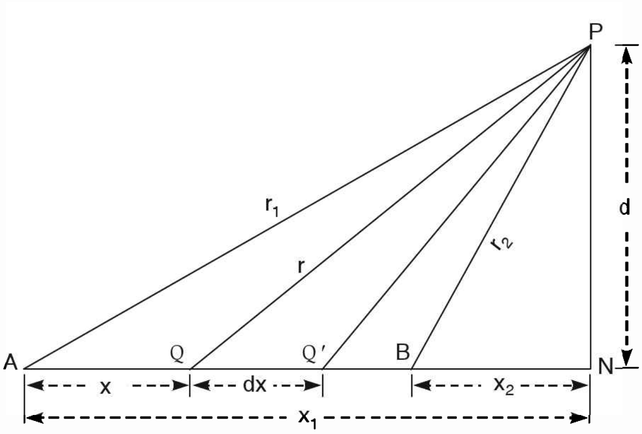
in which AB = 2l, PA = r_1*, PB =* r_2*, NA =* x_1*, NB =* x_2*, N being the foot of the perpendicular from P on the line AB, and 2a the length of the major axis of the spheroid through P having A, B as foci.* **[I.A.S. 1988]**

Sol. The line section QQ' of length δx in AB at a distance x from A can be treated as a point source of strength $m\delta x$ giving a velocity potential at P of amount $(m\delta x)/r$, where $QP = r$.

$\therefore$ The total velocity potential at P due to the entire line distribution is

$$\phi = \int_0^{2l}\frac{m\,dx}{r} = m\int_0^{2l}\frac{dx}{r} \qquad \text{...(1)}$$

From ΔPQN, $r^2 = PN^2 + QN^2 = d^2 + (x_1 - x)^2$, as $QN = AN - AQ = x_1 - x$



$\therefore$ From (1), we have

$$\phi = m\int_0^{2l} \frac{dx}{\{d^2 + (x_1 - x)^2\}^{1/2}} = -m\left[\sinh^{-1}\frac{x_1 - x}{d}\right]_0^{2l} = m\left[\sinh^{-1}\frac{x_1}{d} - \sinh^{-1}\frac{x_1 - 2l}{d}\right]$$

$$= m\left[\sinh^{-1}\frac{x_1}{d} - \sinh^{-1}\frac{x_2}{d}\right], \text{ as } x_1 - 2l = AN - AB = x_2$$

$$= m\left[\log\left\{\frac{x_1}{d} + \left(1 + \frac{x_1^2}{d^2}\right)^{1/2}\right\} - \log\left\{\frac{x_2}{d} + \left(1 + \frac{x_2^2}{d^2}\right)^{1/2}\right\}\right] = m\log\frac{x_1 + \sqrt{(x_1^2 + d^2)}}{x_2 + \sqrt{(x_2^2 + d^2)}}$$

$\therefore$
$$\phi = m\log\frac{x_1 + r_1}{x_2 + r_2} \qquad \text{...(2)}$$

Now, $\qquad r_1^2 - x_1^2 = r_2^2 - x_2^2 = d^2$, so that

$$\frac{r_1 + x_1}{r_2 + x_2} = \frac{r_2 - x_2}{r_1 - x_1} = \frac{r_1 + x_1 + r_2 - x_2}{r_2 + x_2 + r_1 - x_1} = \frac{r_1 + r_2 + 2l}{r_1 + r_2 - 2l} \qquad \text{...(3)}$$

$[\because\ x_1 - x_2 = NA - NB = AB = 2l]$

At P on the spheroid through P having A and B as foci, we know that

$$r_1 + r_2 = 2a \qquad \text{...(4)}$$

Using (4), (3) reduces to $\qquad \dfrac{r_1 + x_1}{r_2 + x_2} = \dfrac{r_2 - x_2}{r_1 - x_1} = \dfrac{a + l}{a - l} = f$, (given)

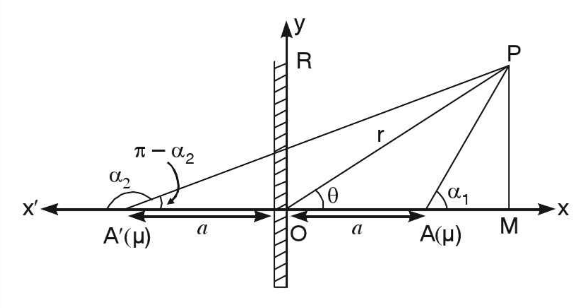
$\therefore$ $(x_1 + r_1)/(x_2 + r_2) = f$ and hence from (2), we obtain $\phi = m\log f$, as required.

Ex. 4. *A three-dimensional doublet of strength* μ *whose axis is in the direction OX distant a from the rigid plane* $x = 0$ *which is the sole boundary of liquid of density* ρ, *infinite in extent. Find the pressure at a point on the boundary distant r from the doublet given that the pressure at infinity is* p_∞. *Show that the pressure on the plane is least at a distance* $\sqrt{5}/2$ *from the doublet.*

[Kolkata 1998]

Sol. Let OR be the rigid boundary. Then we know that the image system of a doublet of strength μ at A $(a, 0, 0)$ consists of a doublet of the same strength μ at A' $(-a, 0, 0)$. Then the velocity potential at P is given by

$$\phi = \frac{\mu\cos\alpha_1}{AP^2} + \frac{\mu\cos\alpha_2}{A'P^2} \qquad ...(1)$$



From figure, we have $$AP^2 = r^2 + a^2 - 2ra\cos\theta \qquad ...(2)$$

and $$A'P^2 = r^2 + a^2 - 2ra\cos(\pi - \theta) = r^2 + a^2 + 2ra\cos\theta \qquad ...(3)$$

Also, $$\cos\alpha_1 = \frac{AM}{AP} = \frac{OM - OA}{AP} = \frac{r\cos\theta - a}{AP} \qquad ...(4)$$

and $$\cos\alpha_2 = -\cos(\pi - \alpha_2) = -\frac{A'M}{A'P} = -\frac{A'O + OM}{A'P} = -\frac{a + r\cos\theta}{A'P} \qquad ...(5)$$

where M is the foot of the perpendicular drawn from P on OX. Using (4) and (5), (1) reduces to

$$\phi = \frac{\mu(r\cos\theta - a)}{AP^3} - \frac{\mu(r\cos\theta + a)}{A'P^3} = \frac{\mu(r\cos\theta - a)}{(r^2 + a^2 - 2ra\cos\theta)^{3/2}} = \frac{\mu(r\cos\theta + a)}{(r^2 + a^2 + 2ra\cos\theta)^{3/2}},$$

[Using (2) and (3)]

$$\therefore \quad \phi = \mu\{(r\cos\theta - a)(r^2 + a^2 - 2ra\cos\theta)^{-3/2} - (r\cos\theta + a)(r^2 + a^2 + 2ra\cos\theta)^{-3/2}\} \qquad ...(6)$$

$$\therefore \quad q_r = -\partial\phi/\partial r = -\mu\{\cos\theta(r^2 + a^2 - 2ra\cos\theta)^{-3/2}$$
$$-3(r - a\cos\theta)(r\cos\theta - a)(r^2 + a^2 - 2ra\cos\theta)^{-5/2} - \cos\theta(r^2 + a^2 + 2ra\cos\theta)^{-3/2}$$
$$+3(r + a\cos\theta)(r\cos\theta - a)(r^2 + a^2 + 2ra\cos\theta)^{-5/2}\}, \text{ using (6)}$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\mu}{r}\{-r\sin\theta(r^2 + a^2 - 2ra\cos\theta)^{-3/2}$$
$$-3ra\sin\theta(r\cos\theta - a)(r^2 + a^2 - 2ra\cos\theta)^{-5/2} + r\cos\theta(r^2 + a^2 + 2ra\cos\theta)^{-3/2}$$
$$-3ra\sin\theta(r\cos\theta + a)(r^2 + a^2 + 2ra\cos\theta)^{-5/2}\}, \text{ using (6)}$$

When $\theta = \pi/2$, we have $$q_r = -6\mu ra(r^2 + a^2)^{-5/2}, \qquad q_\theta = 0.$$

Now, Bernoulli's equation is $$p/\rho + q^2/2 = p_\infty/\rho \qquad ...(7)$$

Here $$q = q_r = -6\mu ra(r^2 + a^2)^{-5/2}$$

$\therefore$ (7) gives $$p/\rho + 18\mu^2 a^2 r^2 (r^2 + a^2)^{-5} = p_\infty/\rho$$

$$\therefore \quad p = p_\infty - 18\mu^2 a^2 r^2 \rho (r^2 + a^2)^{-5} \qquad ...(8)$$

From (8), $dp/dr = -18\mu^2 a^2 \rho\{2r(r^2+a^2)^{-5} - 5r^2(r^2+a^2)^{-6}\cdot 2r\}$

$$= -18\mu^2 a^2 \rho \cdot 2r(r^2+a^2)^{-5}\left\{1 - \frac{5r^2}{r^2+a^2}\right\} = -36\mu^2 a^2 \rho r(r^2+a^2)^{-5}\frac{a^2-4r^2}{r^2+a^2}$$

Thus, $$dp/dr = 36\mu^2 a^2 \rho r(r^2+a^2)^{-6}(4r^2-a^2). \qquad ...(9)$$

For maximum and minimum values of p, we must have

$$dp/dr = 0 \qquad \text{or} \qquad 36\mu^2 a^2 r(r^2+a^2)^{-6}(4r^2-a^2) = 0, \text{ by (9)}.$$

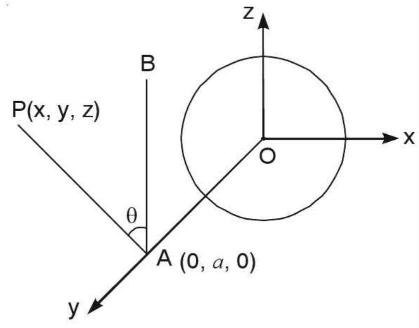
giving $r = 0,\ a/2,\ -a/2$. Now, from (9), we get

$$dp/dr = 144\mu^2 a^2 \rho r(r^2+a^2)^{-6}(r^2-a^2/4)$$

Thus, $$dp/dr = 144\mu^2 a^2 \rho r(r^2+a^2)^{-6}(r-a/2)(r+a/2) \qquad ...(10)$$

$\therefore$ When r is slightly less than $a/2$, dp/dr is negative and when r is slightly greater than $a/2$, dp/dr is positive. Hence p must be minimum at $r = a/2$ on the plane *i.e.*, at a distance $(a^2 + a^2/4)^{1/2}$ *i.e.*, $a\sqrt{5}/2$ from the doublet.

Ex. 5. *A doublet of strength m is placed at the point* (0, *a*, 0) *with its axis parallel to the z-axis. Prove that at points close to the origin the velocity potential of the doublet is approximately* $(mz)/a^3 + (3myz)/a^4$, *neglecting terms of order* r^3/a^5 *and higher powers. Deduce that if a small sphere of radius c be placed with its centre at the origin, the velocity potential is then increased by the terms* $(mc^3/2a^3)\times(z/r^3) + (2mc^5/a^4)\times(yz/r^5)$



Sol. Let a doublet of strength m be placed at A (0, a, 0) with its axis parallel to the z-axis. Let P (x, y, z) be any point in the liquid and let PA make an angle θ with a line AB parallel to z-axis. Since direction cosines of AP and AB are x/AP, $(y-a)/AP$, z/AP and 0, 0, 1 respectively, hence

$$\cos\theta = \frac{x}{AP}\cdot 0 + \frac{y-a}{AP}\cdot 0 + \frac{z}{AP}\cdot 1 = \frac{z}{AP} \qquad ...(1)$$

$\therefore$ The velocity potential due to given doublet at P is given by

$$\phi = \frac{m\cos\theta}{AP^2} = \frac{mz}{AP^3}, \text{ by (1)} \qquad ...(2)$$

From figure, $$OP^2 = r^2 = x^2 + y^2 + z^2 \qquad ...(3)$$

and $$AP^2 = x^2 + (y-a)^2 + z^2 = x^2 + y^2 + z^2 - 2ya + a^2$$

Now, using (3), we get $AP^2 = r^2 - 2ya + a^2$ so that $AP = (r^2 - 2ya + a^2)^{1/2}$

$\therefore$ From (2), we have

$$\phi = \frac{mz}{(r^2-2ya+a^2)^{3/2}} = \frac{mz}{a^3}\left[1 - \left(\frac{2y}{a} - \frac{r^2}{a^2}\right)\right]^{-3/2} = \frac{mz}{a^3}\left[1 + \left(\frac{3}{2}\right)\left(\frac{2y}{a}\right) + \dots\right]$$

[neglecting r^2 and higher powers]

$$= (mz)/a^3 + (3myz)/a^4 \text{ as required.}$$

Second Part. Let a sphere of radius c be placed with its centre at the origin. Let ϕ' be the new value of the velocity potential at any point P. Then, the boundary conditions are :

(*i*) the velocity of the liquid remains unchanged at infinity (*ii*) $\partial\phi'/\partial r = 0$ at $r = c$

Suppose that the increase in the velocity potential be

$$(Az)/r^3 + (Byz)/r^5 \qquad \ldots(4)$$

Then,
$$\phi' = \frac{mz}{a^3} + \frac{3myz}{a^4} + \frac{Az}{r^3} + \frac{Byz}{r^5} \qquad \ldots(5)$$

Let spherical polar coordinates of P be (r, θ, ω). Then, we know that

$$x = r\sin\theta\cos\omega, \qquad y = r\sin\theta\sin\omega, \qquad z = r\cos\theta \qquad \ldots(6)$$

Using (6), (5) may be re-written as

$$\phi' = \frac{mr\cos\theta}{a^3} + \frac{3mr^2\sin\theta\cos\theta\sin\omega}{a^4} + \frac{A\cos\theta}{r^2} + \frac{B\sin\theta\cos\theta\sin\omega}{r^3}$$

$$\therefore \quad \frac{\partial\phi'}{\partial r} = \frac{m\cos\theta}{a^3} + \frac{6mr\sin\theta\cos\theta\sin\omega}{a^4} - \frac{2A\cos\theta}{r^3} - \frac{3B\sin\theta\cos\theta\sin\omega}{r^4} \qquad \ldots(7)$$

Putting $r = c$ in (7) and using boundary condition (*ii*) *i.e.* $\partial\phi'/\partial r = 0$, we have

$$0 = \frac{m\cos\theta}{a^3} + \frac{6mc\sin\theta\cos\theta\sin\omega}{a^4} - \frac{2A\cos\theta}{c^3} - \frac{3B\sin\theta\cos\theta\sin\omega}{c^4}$$

or
$$\left(\frac{m}{a^3} - \frac{2A}{c^3}\right)\cos\theta + \left(\frac{6mc}{a^4} - \frac{3B}{c^4}\right)\sin\theta\cos\theta\sin\omega = 0$$

Equating the coefficients of $\cos\theta$ and $\sin\theta\cos\theta\sin\omega$ on both sides, we have

$$\frac{m}{a^3} - \frac{2A}{c^3} = 0 \quad \text{and} \quad \frac{6mc}{a^4} - \frac{3B}{c^4} = 0 \quad \text{so that} \quad A = \frac{mc^3}{2a^3} \quad \text{and} \quad B = \frac{2mc^5}{a^4}$$

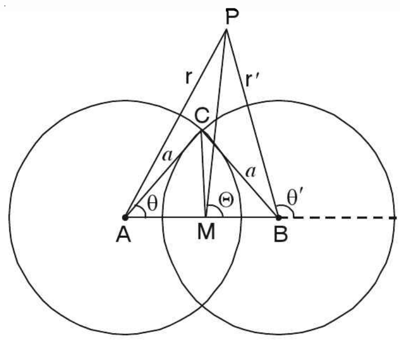
$\therefore$ From (4), the required increase in the velocity potential

$$= \left(\frac{mc^2}{2a^3}\right)\left(\frac{z}{r^3}\right) + \left(\frac{2mc^5}{a^4}\right)\left(\frac{yz}{r^5}\right), \text{ as required.}$$

Ex. 6. *A solid is bounded by the exterior portions of two equal spheres (or radius a) which cut one another orthogonally and is surrounded by infinite mass of liquid. If the solid is set in motion with velocity u in the direction of the line of centres,show the the velocity potential of the resulting motion is*

$$\frac{1}{2}a^3u\left(\frac{\cos\theta}{r^2} + \frac{\cos\theta'}{r'^2} - \frac{\cos\Theta}{2\sqrt{2}R^2}\right),$$

where r, r', R' *are the radii vectors of a point, measured positively from the centres of the two spheres and from the points midway between them and* θ, θ', Θ *are the angles which these radii vectors make with the direction of motion of the solid.*



Sol. Let A and B be the centres of the given circles and let M be the middle point of AB. Let the two circles cut each other at C. Since the given circles cut each other orhogonally, so $\angle ACB = 90°$. Now $AC = BC = a$. Hence, from ΔABC, $\quad AB^2 = AC^2 + BC^2$

$$\Rightarrow \quad AB^2 = 2a^2 \quad \Rightarrow \quad AB = a\sqrt{2}.$$

Hence, $\quad AM = BM = (a/\sqrt{2})/2 = a/\sqrt{2}.$

Furthermore, we have $AB \cdot MA = BA \cdot MB = (a\sqrt{2}) \times (a/\sqrt{2}) = a^2$,

showing that M is the inverse point of B with respect to the circle with centre A and M is also the inverse point of A with respect to the circle with centre B.

In what follows we shall use the fact that the effect to the motion of a sphere of radius a with velocity u is the same as that of a doublet of strength $(ua^3)/2$ placed at the centre of the sphere. Accordingly, we can replace given spheres with doublets of strength $(ua^3)/2$ each placed at their centress A and B.

Now, the image of doublet at B is a doublet at its inverse point M, the strength of this image doublet

$$= \frac{1}{2}ua^3 \times \frac{a^3}{(AB)^3} = \frac{1}{2}ua^3 \times \frac{a^3}{(a\sqrt{2})^3} = \frac{1}{4\sqrt{2}}ua^3, \text{ with sense opposite to that at } B.$$

Similarly, the image of doublet at A is a doublet at its inverse point M, the strength of thin image doublet $= (ua^3)/4\sqrt{2}$, with sense opposite to that A

Thus, the image system for the given boundaries is

(*i*) a doublet of strength $(ua^3)/2$ at A (*ii*) a doublet of strength $(ua^3)/2$ at B (*iii*) a doublet of strength $-\{(ua^3)/\,4\sqrt{2}\}$ at M.

Hence its velocity potential ϕ of the required resulting motion is given by

$$\phi = \frac{1}{2}ua^3\frac{\cos\theta}{r^2} + \frac{1}{2}ua^3\frac{\cos\theta'}{r'^2} - \frac{1}{4\sqrt{2}}ua^3\frac{\cos\Theta}{R^2} \quad \text{or} \quad \phi = \frac{1}{2}a^3u\left(\frac{\cos\theta}{r^2} + \frac{\cos\theta'}{r'^2} - \frac{\cos\Theta}{2\sqrt{2}\,R^2}\right)$$

Ex. 7. *In the case of irrotational motion in two dimensions on the surface of a sphere, show that the velocity potential is of the form* $f\left(\dfrac{x+iy}{r+z}\right) + f\left(\dfrac{x-iy}{r+z}\right)$, *r being the radius of the sphere and x, y, z the coordinates of a point referred to rectangular axes through the centre of the sphere.*

Sol. We know that $\qquad \phi + i\psi = g(e^{i\omega}\tan\theta/2) = g[\tan(\theta/2)(\cos\omega + i\sin\omega)] \qquad \ldots(1)$

Replacing i by $-i$ in (1), $\qquad \phi - i\psi = g[\tan(\theta/2)(\cos\omega - i\sin\omega)] \qquad \ldots(2)$

Adding (1) and (2), $2\phi = g[\tan(\theta/2)(\cos\omega + i\sin\omega)] + g[\tan(\theta/2)(\cos\omega - i\sin\omega)] \quad \ldots(3)$

Also,
$$\frac{x+iy}{r+z} = \frac{r\sin\theta\cos\omega + ir\sin\theta\sin\omega}{r + r\cos\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)(\cos\omega + i\sin\omega)}{2\cos^2(\theta/2)}$$

or, $\qquad g[(x+iy)/(r+z)] = g[\tan(\theta/2)(\cos\omega + i\sin\omega)] \qquad \ldots(4)$

Replacing i by $-i$ in (4), $\qquad g[(x-iy)/(r+z)] = g[\tan(\theta/2)(\cos\omega - i\sin\omega)] \qquad \ldots(5)$

Adding (4) and (5) and using (3), we get

$$2\phi = g\left(\frac{x+iy}{r+z}\right) + g\left(\frac{x-iy}{r+z}\right) \quad \text{or} \quad \phi = f\left(\frac{x+iy}{r+z}\right) + f\left(\frac{x-iy}{r+z}\right),$$

which is the required form of the velocity potential.

10.17A. Butler sphere theorem [Kerala 2001]

If a solid sphere r = a is introduced (avoiding singularities) into a field of an axisymmetrical irrotational flow in an incompressible inviscid fluid with no rigid boundaries, previously of current function $\psi_0 = \psi_0(r,\theta)$, *where* $\psi_0 = O(r^2)$ *at the origin, then the stream function becomes*

$$\psi = \psi_0 - \psi_0^* = \psi_0(r, \theta) - (r/a) \times \psi_0(a^2/r, \theta).$$

Proof. The following four conditions are to be satisfied:

(*i*) The flow given by ψ must be irrotational

(*ii*) When $r = a$, *i.e.*, on the boundary of the sphere, ψ = constant.

(*iii*) ψ_0^* has no singularities outside $r = a$. Here ψ_0^* denotes the value of ψ^* at origin

(*iv*) The velocity due to ψ_0^* must tend to zero as $r \to \infty$, and ψ_0^* must not introduce any net flux over the sphere at infinity.

We now establish the truth of these conditions one by one.

Condition (*i*) Since the motion is irrotational, the velocity potential ϕ exists at any point in the fluid such that the velocity components in the directions or r and θ increasing in terms of ϕ and ψ are given by

$$q_r = -\frac{\partial \phi}{\partial r} = -\frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} \qquad \text{and} \qquad q_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}$$

$$\therefore \quad \frac{\partial}{\partial \theta}\left(\frac{\partial \phi}{\partial r}\right) = \frac{\partial}{\partial \theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \theta}\left(\frac{\partial \phi}{\partial \theta}\right) = -\frac{\partial}{\partial r}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial r}\right)$$

Since $\quad \dfrac{\partial}{\partial \theta}\left(\dfrac{\partial \phi}{\partial r}\right) = \dfrac{\partial}{\partial r}\left(\dfrac{\partial \phi}{\partial \theta}\right), \qquad$ so $\quad \dfrac{1}{r^2}\dfrac{\partial}{\partial \theta}\left(\dfrac{1}{\sin\theta}\dfrac{\partial \psi}{\partial \theta}\right) = -\dfrac{1}{\sin\theta}\dfrac{\partial^2 \psi}{\partial r^2}$

or
$$r^2\frac{\partial^2 \psi}{\partial r^2} + \sin\theta\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \theta}\right) = 0 \qquad \ldots(1)$$

By directly differentiating the functions ψ_0 and ψ_0^* (at origin), we find that (1) is satisfied. Hence the flow given by ψ is irrotational. Thus, we see that the condition (*i*) satisfied.

Condition (*ii*) Clearly when $r = a$, $\psi = 0$. Hence the condition (*ii*) is satisfied.

Condition (*iii*) Since r and a^2/r are the inverse points with respect to the sphere $r = a$, therefore if one point is inside the sphere, the other must be outside the sphere. By hypothesis, all singularities of ψ_0 are outside the sphere, so that all the singularities of ψ_0^* (at origin) must be inside the sphere. Thus we see that the condition (*iii*) is also satisfied.

Condition (*iv*) Since ψ_0 is regular inside the sphere and near the origin, we have $\psi_0 = O(r^2)$ and therefore at infinity ψ_0^* (at origin) $= O(1/r)$. Since $q_r = -(1/r^2\sin\theta)\times(\partial\psi/\partial\theta)$, it follows that velocity at infinity due to ψ_0^* (at origin) is $O(1/r^2)$ which tends to zero as $r \to \infty$. For the flux, we have

$$\int q_r \cdot ds = O(1/r), \text{ which tends to zero as } r \to \infty.$$

So condition (*iv*) is also satisfied. This completes the proof.

10.17B. Image in solid spheres

In Art 10.3, we examined the effect of placing a solid sphere in a uniform stream of incompressible flow. In the present article, we propose to study the perturbations obtained when such a sphere is placed in a more general field of flow. In what follows we shall get results which do not presuppose axial symmetry of flow.

Some useful properties of harmonic functions satisfying the Laplace's equation in spherical polar coordinates (r, θ, ω), *namely*

$$\nabla^2\phi = \frac{1}{r^2 \sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2 \sin^2\theta \frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\phi}{\partial\theta}\right) + \frac{\partial}{\partial\omega}\left(\frac{1}{\sin\theta}\frac{\partial\phi}{\partial\omega}\right)\right\} = 0 \qquad ...(1)$$

Theorem I. *If $r^n S_n(\theta, \omega)$ is a harmonic function, so also is $r^{-(n+1)} S_n(\theta, \omega)$.*

Proof. According to the statement of theorem, $\phi = r^n S_n(\theta, \omega)$ satisfies (1), so

$$\frac{\partial}{\partial r}(nr^{n+1} S_n \sin\theta) + \frac{\partial}{\partial\theta}\left(r^n \sin\theta \frac{\partial S_n}{\partial\theta}\right) + \operatorname{cosec}\theta \frac{\partial}{\partial\omega}\left(r^n \frac{\partial S_n}{\partial\omega}\right) = 0$$

or
$$n(n+1) S_n \sin\theta + \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial S_n}{\partial\theta}\right) + \operatorname{cosec}\theta \frac{\partial^2 S_n}{\partial\omega^2} = 0 \qquad ...(2)$$

Again, taking $\phi = r^{-(n+1)} S_n(\theta, \omega)$, we have

$$r^2 \sin\theta \nabla^2\phi = \frac{\partial}{\partial r}\left\{-(n+1) r^{-n} \sin\theta S_n\right\} + r^{-(n+1)} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial S_n}{\partial\theta}\right) + \operatorname{cosec}\theta\, r^{-(n+1)} \frac{\partial^2 S_n}{\partial\omega^2}$$

$$= r^{-(n+1)}\left\{(n+1) n S_n \sin\theta + \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial S_n}{\partial\theta}\right) + \operatorname{cosec}\theta \frac{\partial^2 S_n}{\partial\omega^2}\right\} = 0, \text{ using (2)}$$

Hence, we arrive at the required result.

Theorem II. If $\phi(r, \theta, \omega)$ has an expansion of the form $\phi(r, \theta, \omega) = \sum_{n=0}^{\infty} \alpha_n r^n S_n(\theta, \omega)$, the series on the right being uniformly convergent with respect to r, then for constant λ,

$$\frac{1}{r^\lambda}\int_0^r R^{\lambda-1} \phi(R, \theta, \omega)\, dR = \sum_{n=0}^{\infty} \frac{\alpha_n r^n S_n(\theta, \omega)}{n+\lambda}$$

Proof. Since $R^{\lambda-1} \phi(R, \theta, \omega) = \sum_{n=0}^{\infty} \alpha_n R^{n+\lambda-1} S_n(\theta, \omega)$, we have

$$\int_0^r R^{\lambda-1} \phi(R, \theta, \omega)\, dR = \sum_{n=0}^{\infty} \frac{\alpha_n r^{n+\lambda} S_n(\theta, \omega)}{n+\lambda},$$

from which the required result follows

Theorem III. Weiss's sphere theorem

Let $\phi(r, \theta, \omega)$ be the velocity potential at a point $P(r, \theta, \omega)$ in an incompressible fluid having irrotational motion, there being no rigid bounderies. Also suppose that ϕ has no singularities within the region $r \le a$. Then when a solid sphere of radius a is introduced into the flow with centre at the origin of coordinates, the new velocity potential in the fluid is

$$\phi(r, \theta, \omega) + \frac{a}{r}\phi(a^2/r, \theta, \omega) - \frac{1}{a}\int_0^{a^2/r} \phi(R, \theta, \omega)\, dR \quad (r > a)$$

Proof. Let ϕ be regular near the origin and let it possess an expansion of the form

$$\phi(r, \theta, \omega) = \sum_{n=0}^{\infty} \alpha_n r^n S_n(\theta, \omega),$$

where each $r^n S_n(\theta, \omega)$ is harmonic and each α_n is known.

When the sphere is introduced into the stream, it will produce a perturbation potential which will be zero at infinity and which must be harmonic. By theorem I, a suitable form for the perturbation potential can be taken as $\sum_{n=0}^{\infty} \beta_n r^{-(n+1)} S_n(\theta, \omega)$ where the coefficients β_n are undertermined. Thus the new velocity potential at P is $\phi_1\,(r, \theta, \omega)$, where

$$\phi_1\,(r, \theta, \omega) = \sum_{n=0}^{\infty} \alpha_n r^n S_n(\theta, \omega) + \sum_{n=0}^{\infty} \beta_n r^{-(n+1)} S_n(\theta, \omega)$$

On $r = a$, we required $\partial\phi_1/\partial r = 0$. Hence, equating to zero the coefficient of each $S_n(\theta, \omega)$, we obtain

$$\alpha_n n a^{n-1} - (n+1)\beta_n a^{-(n+2)} = 0 \qquad \text{or} \qquad \beta_n = \left(1 - \frac{1}{n+1}\right) a^{2n+1} \alpha_n \qquad (n = 0, 1, 2, \ldots)$$

$$\therefore \text{ Perturbation potential } = \frac{a}{r} \sum_{n=0}^{\infty} \alpha_n \left(\frac{a^2}{r}\right)^n S_n - \frac{1}{a} \sum_{n=0}^{\infty} \frac{\alpha_n\,(a^2/r)^{n+1} S_n}{n+1}$$

$$= \frac{a}{r}\phi(a^2/r, \theta, \omega) - \frac{1}{a}\int_0^{a^2/r} \phi(R, \theta, \omega)\, dR,$$

the integral following by taking $\lambda = 1$ in theorem II. This proves the required result.

Example. *Find image of a source in a solid sphere.*

Sol. The velocity potential $\phi\,(r, \theta)$ at any point (r, θ, ω) in a fluid due to simiple source of srength m at $(f, 0, 0)$ is given by

$$\phi\,(r, \theta, \omega) = m\,(r^2 - 2r\,f\cos\theta + f^2)^{-1/2}.$$

Introduce a solid sphere in a region $r \le a$, where $a < f$. Using Weiss's sphere theorem, we obtain a perturbation potential of

$$\frac{ma/f}{\{r^2 - 2r\,(a^2/f)\cos\theta + (a^2/f)^2\}^{1/2}} - \frac{m}{a}\int_0^{a^2/r} \frac{dR}{(R^2 - 2R\,f\cos\theta + f^2)^{1/2}},$$

showing that the image system of a point source of stregth m placed at f (> a) from the centre of the solid consists of a source of strength ma/f at the inverse point in the sphere, together with a continuous line distribution of sinks of strength $-\,m/a$ per unit length extending from the centre to the inverse point.

10.18. Motion symmetrical about an axis, the lines of motion being in planes passing through the axis : Stokes's stream function. [Meerut 1997; Kanpur 1998]

When the motion is the same in every plane through a given line, called the axis, the motion is called *axi-symmetrical*. Such a motion occurs, for example, in uniform flow past a stationary sphere, the sphere moving with uniform velocity in a fluid at rest, the motion of a solid of revolution moving in the direction of the axis of revolution etc. Such motions give rise to some analogies with the two-dimensional case; for example a stream function can be defined for such motions as illustrated below.

The equation of continuity in cylindrical coordinates for the case of incompressible fluid is,

$$\frac{1}{r}\frac{\partial}{\partial r}(rq_r) + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z} = 0. \qquad \ldots(1)$$

When the motion in symmetrical about z-axis, $q_\theta = 0$ and hence (1) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}(rq_r) + \frac{\partial q_z}{\partial z} = 0. \qquad \ldots(2)$$

Now, let x-axis be taken as the axis of symmetry in place of z-axis and let $\overline{\omega}[=\sqrt{(y^2+z^2)}]$ denote the distance from the x-axis. Let u, v denote components of velocity in the directions of x and $\overline{\omega}$ respectively. Then replacing r and z by $\overline{\omega}$ and x respectively and replacing q_r and q_z by v and u respectively in (2), we have

$$\frac{1}{\overline{\omega}}\frac{\partial}{\partial\overline{\omega}}(\overline{\omega}v)+\frac{\partial u}{\partial x}=0 \quad \text{or} \quad \frac{\partial}{\partial\overline{\omega}}(\overline{\omega}v)=\frac{\partial}{\partial x}(-\overline{\omega}u) \qquad \text{...(3)}$$

But (3) is the condition that $\overline{\omega}\,v\,dx-\overline{\omega}\,u\,d\overline{\omega}$,

may be an exact differential, $d\psi$, say,

Thus, $$\overline{\omega}\,v\,dx-\overline{\omega}\,u\,d\overline{\omega}=d\psi \qquad \text{...(4)}$$

$\therefore$ $$\overline{\omega}\,v\,dx-\overline{\omega}\,u\,d\overline{\omega}=(d\psi/\partial x)\,dx+(\partial\psi/\partial\overline{\omega})\,d\overline{\omega}$$

so that $$u=-\frac{1}{\overline{\omega}}\frac{\partial\psi}{\partial\overline{\omega}} \quad \text{and} \quad v=\frac{1}{\overline{\omega}}\frac{\partial\psi}{\partial x} \qquad \text{...(5)}$$

The function ψ defined by (5) is known as *Stokes's stream function.*

The streamlines are given by

$$(dx)/u=(d\overline{\omega})/v \quad \text{or} \quad \overline{\omega}vdx-\overline{\omega}ud\overline{\omega}=0$$

or $$d\psi=0, \text{ using (4)}$$

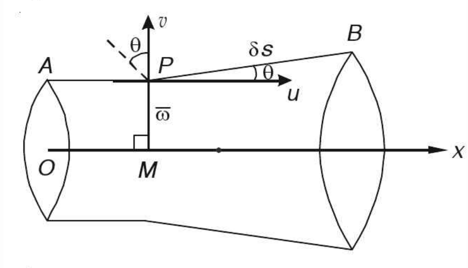
Integrating, ψ = constant, *which represents the streamlines.*

Remark. Stokes's stream function ψ represents the streamlines ψ = const. in an analogous way to the stream function in two-dimensional flow as defined in Art. 5.2. But the existence of Stokes's stream function does not depend upon the existence of velocity potential ϕ, *i.e.*, the Stokes's stream function exists even if the motion is not irrotational which is not true in the case of two-dimensional stream function defined in Art. 5.2.

10.19. A Property of Stokes's stream function.

2π *times the difference of the values of Stokes's stream function at two points in the meridian plane is equal to the flow across the annular surface obtained by the revolution round the axis of curve joining the points.*

Proof. Let AB be an arc of a curve which when rotated about the axis (x-axis) will describe an annular surface. Let P be a point in AB, ds an elementary arc at P. Let θ be the inclination of ds to the axis.



Then, velocity across the element ds

$$=v\cos\theta-u\sin\theta$$

$$=\frac{1}{\overline{\omega}}\frac{\partial\psi}{\partial x}\frac{dx}{ds}+\frac{1}{\overline{\omega}}\frac{\partial\psi}{\partial\overline{\omega}}\frac{d\overline{\omega}}{ds}=\frac{1}{\overline{\omega}}\frac{\partial\psi}{ds} \qquad \text{...(1)}$$

$\therefore$ Flow across the annular surface

$$=\int_A^B (v\cos\theta-u\sin\theta)\cdot 2\pi\overline{\omega}\,ds=2\pi\int_A^B\frac{1}{\overline{\omega}}\frac{\partial\psi}{\partial s}\overline{\omega}\,ds, \text{ using (1)}$$

$$=2\pi\int_A^B d\psi=2\pi(\psi_B-\psi_A),$$

which proves the required result.

10.20. Irrotational motion.

Since the motion is irrotational, the velocity potential ϕ exists such that

$$u = -(\partial\phi/\partial x) \qquad \text{and} \qquad v = -(\partial\phi/\partial \overline{w}) \qquad \ldots(1)$$

Again Stokes's stream function always exists such that

$$u = -\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}, \qquad \text{and} \qquad v = \frac{1}{\varpi}\frac{\partial\psi}{\partial x} \qquad \ldots(2)$$

$$\therefore \quad \frac{\partial\phi}{\partial x} = \frac{1}{\varpi}\frac{d\psi}{d\varpi}, \qquad \text{and} \qquad \frac{\partial\phi}{\partial\varpi} = -\frac{1}{\varpi}\frac{\partial\psi}{dx} \qquad \ldots(3)$$

From (3), $$\frac{\partial}{\partial\varpi}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}\right) \qquad \text{and} \qquad \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial\varpi}\right) = -\frac{\partial}{\partial x}\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}\right)$$

so that $$\frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}\right) = -\frac{\partial}{\partial x}\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial x}\right) \qquad \left[\because \frac{\partial^2\phi}{\partial\varpi\partial x} = \frac{\partial^2\phi}{\partial x\partial\varpi}\right]$$

or $$-\frac{1}{\varpi^2}\frac{\partial\psi}{\partial\varpi} + \frac{1}{\varpi}\frac{\partial^2\psi}{\partial\varpi^2} = -\frac{1}{\varpi}\frac{\partial^2\psi}{\partial x^2} \qquad \text{or} \qquad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial\varpi^2} - \frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi} = 0. \qquad \ldots(4)$$

Again from (3), $$\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial\varpi}\right) = \frac{\partial}{\partial x}\left(\varpi\frac{\partial\phi}{\partial x}\right), \qquad \text{and} \qquad \frac{\partial}{\partial\varpi}\left(\frac{\partial\psi}{\partial x}\right) = -\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial\phi}{\partial\varpi}\right)$$

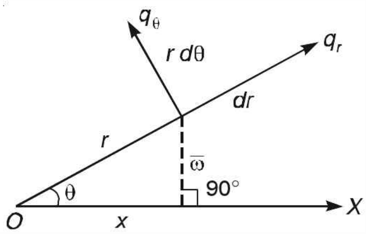
so that $$\frac{\partial}{\partial x}\left(\varpi\frac{\partial\phi}{\partial x}\right) = -\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial\phi}{\partial\varpi}\right), \qquad \text{as} \qquad \frac{\partial^2\psi}{\partial x\partial\varpi} = \frac{\partial^2\psi}{\partial\varpi\partial x}$$

or $$\varpi\frac{\partial^2\phi}{\partial x^2} = -\varpi\frac{\partial^2\phi}{\partial\varpi^2} - 1\cdot\frac{\partial\phi}{\partial\varpi} \qquad \text{or} \qquad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial\varpi^2} + \frac{1}{\varpi}\frac{\partial\phi}{\partial\varpi} = 0. \qquad \ldots(5)$$

Equations (4) and (5) show that ϕ and ψ are not interchangeable in the way that applied to the velocity potential and stream function of two-dimensional irrotational motions.

We now re-write (4) and (5) in polar co-ordinates. Let q_r and q_θ be the velocities in the directions of dr and $rd\theta$. Then, since $\varpi = r\sin\theta$ and noting that the velocity from right to left across ds is $\partial\psi/\varpi\partial s$, we get

$$\left.\begin{aligned} q_r &= -\frac{1}{\varpi}\frac{\partial\psi}{r\partial\theta} = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \\ q_\theta &= \frac{1}{\varpi}\frac{\partial\psi}{\partial r} = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \end{aligned}\right\} \qquad \ldots(6)$$



But in irrotational motion, we know that

$$q_r = -\frac{\partial\phi}{\partial r}, \qquad \text{and} \qquad q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} \qquad \ldots(7)$$

$$\therefore \quad \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r} \qquad \text{and} \qquad \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} \qquad \ldots(8)$$

and so $$\frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = \frac{\partial^2\phi}{\partial\theta\partial r} = -\frac{\partial}{\partial r}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial r}\right)$$

i.e., $$r^2 \frac{\partial^2 \psi}{\partial r^2} + \sin\theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \theta}\right) = 0 \qquad ...(9)$$

Let $\mu = \cos\theta$ so that $\sin\theta \dfrac{\partial}{\partial \mu} = -\dfrac{\partial}{\partial \theta}$. ...(10)

Then (9) reduces to

$$r^2 \frac{\partial^2 \psi}{\partial r^2} + \sin^2\theta \frac{\partial}{\partial \mu}\left(\frac{\partial \psi}{\partial \mu}\right) = 0 \quad \text{or} \quad r^2 \frac{\partial^2 \psi}{\partial r^2} + (1-\mu^2)\frac{\partial^2 \psi}{\partial \mu^2} = 0. \qquad ...(11)$$

Similarly eliminating ψ from (8), we get

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0 \quad i.e., \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{\partial \mu}\left[(1-\mu^2)\frac{\partial \phi}{\partial \mu}\right] = 0, \qquad ...(12)$$

which is Laplace's equation and has solutions of the forms

$$r^n P_n(\mu) \quad \text{and} \quad r^{-n-1} P_n(\mu).$$

Again from (8), we have

$$\frac{\partial \psi}{\partial \mu} = -r^2 \frac{\partial \phi}{\partial r} = -nr^{n+1} P_n \quad \text{or} \quad (n+1) r^{-n} P_n \qquad ...(13)$$

$$\frac{\partial \psi}{\partial r} = (1-\mu^2)\frac{\partial \phi}{\partial \mu} = (1-\mu^2) r^n \frac{\partial P_n}{\partial \mu} \quad \text{or} \quad (1-\mu^2) r^{-n-1} \frac{\partial P_n}{\partial \mu} \qquad ...(14)$$

On integrating, (14) gives us possible solutions for ψ, namely

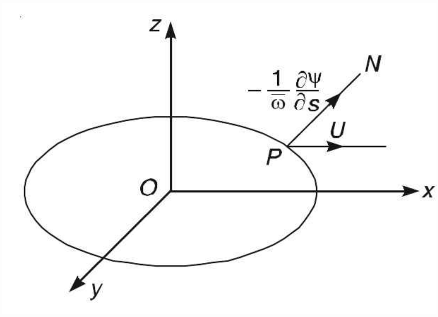
$$\psi = \frac{(1-\mu^2)}{n+1} r^{n+1} \frac{\partial P_n}{\partial \mu} \quad \text{or} \quad \psi = -\frac{(1-\mu^2)}{n}\frac{1}{r^n}\frac{\partial P_n}{\partial \mu} \qquad ...(15)$$

It may be verified that solutions (15) satisfy (11).

10.21. Solids of revolution moving along their axes in an infinite mass of liquid.

Suppose the solid moves along OX with velocity U and let OX be the axis of revolution. Since the motion is symmetrical about OX, Stokes's stream function ψ exists.

Now, the normal velocity of the liquid in contact with the surface at P is $-(1/\varpi)(\partial\psi/\partial s)$.



$\therefore$ On the boundary, we have

$$-\frac{1}{\varpi}\frac{\partial \psi}{\partial s} = \text{velocity of the solid along normal } PN$$

$$\therefore \quad -\frac{1}{\varpi}\frac{\partial \psi}{\partial s} = U\cos\theta = U\frac{\partial \varpi}{\partial s}, \quad \text{as} \quad \cos\theta = \frac{\partial \varpi}{\partial s}$$

or $$d\psi = -U\,\varpi\, d\varpi$$

Integrating, $$\psi = -(1/2) \times U\varpi^2 + \text{const.}$$

or $$\psi = -(1/2) \times Ur^2 \sin^2\theta + \text{const}, \quad \text{as} \quad \varpi = r\sin\theta \qquad ...(1)$$

or $$\psi = -(1/2) \times U(1-\mu^2) + \text{const}, \quad \textit{as} \quad \mu = \cos\theta \qquad ...(2)$$

which is the boundary condition.

Again ψ must satisfy the equation

$$r^2(\partial^2\psi/\partial r^2) + (1-\mu^2)(\partial^2\psi/\partial\mu^2) = 0, \qquad \text{where } \mu = \cos\theta, \qquad ...(3)$$

and it is known that (3) has solutions of the form

$$\frac{1-\mu^2}{n+1} r^{n+1} \frac{\partial P_n}{\partial\mu} \qquad \text{and} \qquad \frac{1-\mu^2}{nr^n} \frac{\partial P_n}{\partial\mu}. \qquad ...(4)$$

As an example, consider the case of a sphere of radius a. Then, with $r = a$ in (2), we must have

$$\psi = -(1/2) \times Ua^2(1-\mu^2) + C \qquad ...(5)$$

Taking $n = 1$ in (4), we have a solution of the form $\psi = A(1-\mu^2)/r$, ...(6)

then at the boundary we must have

$$A(1-\mu^2)/a = -(1/2) \times Ua^2(1-\mu^2) + C \qquad \text{for all values of } \mu.$$

This requires that $C = 0$ and $A = -(1/2) \times Ua^3$. Hence putting these values and noting that $\mu = \cos\theta$, (6) gives

$$\psi = -\frac{1}{2}\frac{Ua^3 \sin^2\theta}{r}. \qquad ...(7)$$

Again we know that

$$(1-\mu^2)\frac{\partial\phi}{\partial\mu} = \frac{\partial\psi}{\partial r} = \frac{1}{2}\frac{Ua^3}{r^2}\sin^2\theta \qquad \text{or} \qquad \frac{\partial\phi}{\partial\mu} = \frac{1}{2}\frac{Ua^3}{r^2}$$

Integrating, $$\phi = \frac{1}{2}\frac{Ua^3}{r^2}\mu = \frac{1}{2}\frac{Ua^3}{r^2}\cos\theta, \qquad ...(8)$$

which is the same as obtained in Art 10.2.

10.22. Values of Stokes's stream function in simple cases.

(*i*) **A simple source on the axis of *x*.** From Art 10.11, we get $\phi = m/r$.

But $$\frac{\partial\psi}{\partial\mu} = -r^2\frac{\partial\phi}{\partial r} = m \qquad \text{[use Art 10.20]}$$

$\therefore$ $\psi = m\mu = m\cos\theta$ or $\psi = (mx)/r$

(*ii*) **A doublet along the axis of *x*.** From Art 10.12, we get

$$\phi = \frac{\mu_1\cos\theta}{r^2} = \frac{\mu\mu_1}{r^2},$$

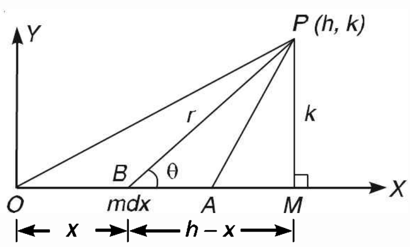
where μ_1 is the strength of the doublet and $\mu = \cos\theta$. Then, by Art. 10.20, we get

$$\frac{\partial\psi}{\partial r} = (1-\mu^2)\frac{\partial\phi}{\partial\mu} = \frac{(1-\mu^2)\mu_1}{r^2}$$

Integrating, $$\psi = -\frac{\mu_1\sin^2\theta}{r} \qquad [\because \mu = \cos\theta]$$

(*iii*) **A uniform line source along the axis of *x*.**

Let m be the strength per unit length. If the source extends form O to A, we have, at any point $P(h, k)$,



$$\psi = \int_{x=0}^{OA} m\cos\theta\, dx = \int_{x=0}^{OA} \frac{m(h-x)\,dx}{\sqrt{\{(h-x)^2 + k^2\}}}$$

$$= -\frac{m}{2}\int_{x=0}^{OA} [(h-x)^2 + k^2]^{-1/2}\{-2(h-x)\,dx\}$$

$$= m\left[-\left\{(h-x)^2+k^2\right\}^{1/2}\right]_0^{OA}$$

$$= m[\sqrt{(h^2+k^2)} - \sqrt{\{(h-OA)^2+k^2\}}] = m[OP - \sqrt{AM^2+PM^2}] = m\,(OP - AP).$$

10.23. Illustrative solved examples.

Ex. 1. *Discuss the motion for which Stokes' stream function is given by*

$$\psi = (V/2) \times (a^4 r^{-2} \cos\theta - r^2) \sin^2\theta,$$

where r is the distance from a fixed point and θ *is the angle this distance makes with the fixed direction.* **[Agra 2002]**

Sol. If a liquid flows with velocity V at infinity in x-direction, then we have

$$-\frac{1}{\overline{\omega}}\frac{\partial\psi}{\partial s} = V \qquad \text{so that} \qquad V = -\frac{1}{2}V\overline{\omega}^2 = -\frac{1}{2}Vr^2\sin^2\theta,$$

since $\overline{\omega} = r\sin\theta$. Hence the given Stokes's stream function

$$\psi = \frac{1}{2}\frac{Va^4}{r^2}\sin^2\theta\cos\theta - \frac{1}{2}Vr^2\sin^2\theta, \qquad \text{...(1)}$$

is the stream function when a solid of revolution is at rest in a liquid moving with velocity V in x-direction so that the term

$$\psi = \frac{1}{2}\frac{Va^4}{r^2}\sin^2\theta\cos\theta \qquad \text{...(2)}$$

gives the stream function when the solid is moving with velocity V in the negative x-direction. But in this case the boundary condition is

$$\psi = (1/2) \times Vr^2\sin^2\theta + \text{constant.} \qquad \text{...(3)}$$

Using (2) and (3), on the boundary, we have

$$\frac{1}{2}\frac{Va^4}{r^2}\sin^2\theta\cos\theta = \frac{1}{2}Vr^2\sin^2\theta$$

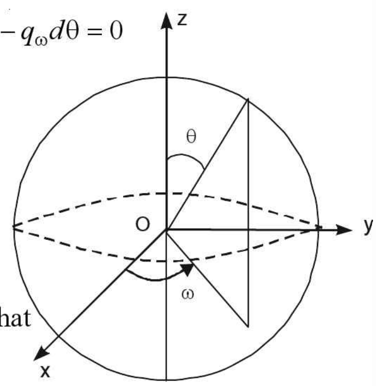
$\therefore$ the boundary is given by $\qquad r^4 = a^4\cos\theta.$

Hence the given stokes's stream function (1) gives the motion of a liquid flowing past a solid $r^4 = a^4\cos\theta$, moving with velocity V in x-direction at infinity.

Ex. 2. *A thin stream of incompressible fluid is contained between two concentric spheres; show that the velocity at any point is equivalent to the components* $-\dfrac{1}{\sin\theta}\dfrac{\partial\psi}{\partial\omega}$, $\dfrac{\partial\psi}{\partial\theta}$ *along the meridian and parallel to latitude respectively. Also if the fluid be homogeneous and the motion irrotational, prove that* $\dfrac{\partial\phi}{\partial\theta} = \dfrac{1}{\sin\theta}\dfrac{\partial\psi}{\partial\omega}$, $-\dfrac{1}{\sin\theta}\dfrac{\partial\phi}{\partial\omega} = \dfrac{\partial\psi}{\partial\theta}$ *and deduce that* $\phi + i\psi = F(e^{i\omega}\tan(\theta/2))$

Sol. ρ being constant and there being no motion along the radius vector, the equation of continuity is (setting $\partial\rho/\partial t = 0$, $q_r = 0$, $q_\phi = q_\omega$ and $\phi = \omega$ in Art. 2.11, chapter 2)

$$\frac{\partial(q_r\sin\theta)}{\partial\theta} + \frac{\partial q_\omega}{\partial\omega} = 0 \qquad \text{...(1)}$$



But (1) is also the condition that $q_\theta \sin\theta \; d\omega - q_\omega d\theta = 0$

may be an exact differential, $d\psi$ (say). Hence, we have

$$q_\theta \sin\theta \; d\omega - q_\omega d\theta = -d\psi = -\left(\frac{\partial \psi}{\partial \omega} d\omega + \frac{\partial \psi}{\partial \theta} d\theta\right)$$

so that $$q_\theta = -\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \omega}, \qquad \text{and} \qquad q_\omega = \frac{\partial \psi}{\partial \theta}. \qquad ...(2)$$

Since motion is irrotational, velocity potential ϕ exists such that

$$q_r = -\frac{1}{a}\frac{\partial \phi}{\partial \theta}, \qquad \text{and} \qquad q_\omega = -\frac{1}{a\sin\theta}\frac{\partial \phi}{\partial \overline{\omega}}.$$

Here we take either $a = 1$ or include a in ϕ so that

$$q_r = -\frac{\partial \phi}{\partial \theta} \qquad \text{and} \qquad q_\omega = -\frac{1}{\sin\theta}\frac{\partial \phi}{\partial \omega} \qquad ...(3)$$

From (2) and (3), $$\frac{\partial \phi}{\partial \theta} = \frac{1}{\sin\theta}\frac{\partial \psi}{\partial \omega}, \qquad \text{and} \qquad \frac{1}{\sin\theta}\frac{\partial \phi}{\partial \omega} = -\frac{\partial \psi}{\partial \theta} \qquad ...(4)$$

or $$\sin\theta \frac{\partial \phi}{\partial \theta} = \frac{\partial \psi}{\partial \omega} \qquad \text{and} \qquad \frac{\partial \phi}{\partial \omega} = -\sin\theta \frac{\partial \psi}{\partial \theta} \qquad ...(5)$$

Let $$\sin\theta \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \mu} \qquad \text{or} \qquad d\mu = \frac{d\theta}{\sin\theta} \qquad ...(6)$$

Integrating, $\mu = \log\tan(\theta/2)$ so that $e^\mu = \tan(\theta/2)$ ...(7)

Using (6), (5) reduces to

$$\frac{\partial \phi}{\partial \mu} = \frac{\partial \psi}{\partial \omega} \qquad \text{and} \qquad \frac{\partial \phi}{\partial \omega} = -\frac{\partial \psi}{\partial \mu}$$

which are the Cauchy-Riemann's equations. Hence we must have

$$\phi + i\psi = f(\mu + i\omega) = F(e^{\mu + i\omega}) = F(e^\mu \cdot e^{i\omega}) \quad \text{or} \quad \phi + i\psi = F(e^{i\omega}\tan(\theta/2)), \text{ using (7).}$$

Ex. 3. *A solid of revolution is moving along its axis in an infinite liquid; show that the kinetic energy of the liquid is* $-\frac{1}{2}\pi\rho \int \frac{\psi}{\overline{\omega}}\frac{\partial \psi}{\partial n} ds$, *where* ψ *is the Stokes's stream function of the motion,* $\overline{\omega}$ *the distance of a point from the axis and the integral is taken once round a meridian curve of the solid. Hence obtain the kinetic energy of infinite liquid due to the motion of a sphere through it with velocity V.*

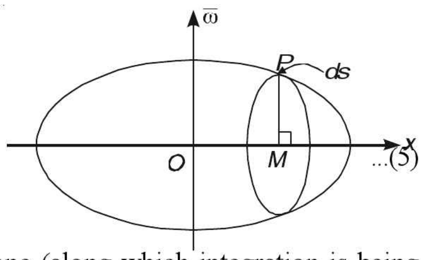
Sol. Kinetic energy T of the liquid is given by

$$T = -\frac{1}{2}\rho \int \phi \frac{\partial \phi}{\partial n} dS \qquad ...(1)$$

Let ds be an elementary arc of the meridian curve of the boundary. Then, for a solid of revolution, we have $dS = 2\pi\overline{\omega}\, ds$ ...(2)

Also $-(\partial\phi/\partial n)$ = outward normal velocity $= -(1/\overline{\omega})(\partial\psi/\partial s)$ ...(3)

and $$-(\partial\phi/\partial s) = (1/\varpi)(\partial\psi/\partial n) \quad ...(4)$$



Using (2) and (3), (1) reduces to

$$T = -\pi\rho\int \phi\, d\psi \quad \text{or} \quad T = -\pi\rho\left\{[\phi\psi]_A^A - \int \psi\, d\phi\right\} \quad ...(5)$$

[on integration by parts]

Since ψ = constant along the curve in the meridian plane (along which integration is being taken), the first term on R.H.S. of (5) vanishes. So (5) reduces to

$$T = \pi\rho\int \psi\, d\phi = \pi\rho\int \psi\,\frac{\partial\phi}{\partial s}ds = -\pi\rho\int \frac{\psi}{\varpi}\frac{\partial\psi}{\partial n}ds = -\frac{1}{2}\pi\rho\int \frac{\psi}{\varpi}\frac{\partial\psi}{\partial n}ds, \text{ using (4)}$$

the integration being taken once round the entire boundary which is double that of the meridian curve.

We now determine K.E. T of the moving sphere. For the motion of a sphere through an infinite liquid with velocity V parallel to the axis of revolution (*i.e.* x-axis), we have [Refer Art. 10.21]

$$\phi = \frac{1}{2}\frac{Va^3}{r^2}\cos\theta, \qquad \text{and} \qquad \psi = -\frac{1}{2}\frac{Va^3}{r}\sin^2\theta \quad ...(5)$$

$$\therefore \qquad T = -\pi\rho\int \phi\, d\psi \qquad \text{on} \qquad r = a$$

$$= -\pi\rho\int_0^\pi \left(\frac{1}{2}\frac{Va^3}{r^2}\cos\theta\right)_{r=a}\left(-\frac{Va^3}{r}\sin\theta\cos d\theta\right)_{r=a}$$

$$= \frac{1}{2}\pi\rho V^2 a^3\int_0^\pi \cos^2\theta\sin\theta\, d\theta = \frac{1}{3}\pi\rho V^2 a^3 = \left(\frac{4}{3}\pi\rho a^3\right)\cdot\frac{1}{4}V^2 = \frac{1}{4}MV^2,$$

where M is the mass of the liquid displaced by the sphere.

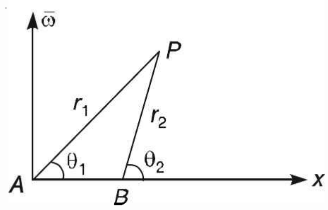
Ex. 4. *If AB be a uniform line source, and A, B equal sinks of such strength that there is no total gain or loss of fluid, show that the stream function at any point is*

$$\psi = K[(r_1 - r_2)^2 - c^2](1/r_1 - 1/r_2),$$

where c is the length of AB, and r_1, r_2 are the distances of the point considered from A and B and K is a constant depending on the strengths of the sources.

Sol. If m be the strength of the source per unit length, then the total strength of the line source is mc. Let $-m'$ be the strength of each sink situated at A and B. Since the total flow is zero, we have

$$-4\pi m' - 4\pi m' + 4\pi(mc) = 0 \qquad \text{so that} \qquad m' = (mc)/2$$



It follows that to neutralize the gain of fluid due to the line source, there must be two sinks of strength – (1/2) × mc at A and B.

Let P be any point and let $\angle PAX = \theta_1$, $\angle PBX = \theta_2$, Then, from figure, we have

$$\cos\theta_1 = \frac{r_1^2 + c^2 - r_2^2}{2r_1 c} \qquad \text{and} \qquad \cos(\pi - \theta_2) = \frac{r_2^2 + c^2 - r_1^2}{2r_2 c} \quad ...(1)$$

If ψ is the Stokes's stream function at P, then, we have by cases (*i*) and (*iii*) of Art. 10.22.

$$\psi = m(r_1 - r_2) - \frac{1}{2} mc\cos\theta_1 - \frac{1}{2} mc\cos\theta_2 = m(r_1 - r_2) - \frac{1}{2} mc\cos\theta_1 + \frac{1}{2} mc\cos(\pi - \theta_2)$$

$$= m(r_1 - r_2) - \frac{1}{2} mc \cdot \frac{r_1^2 + c^2 - r_2^2}{2r_1 c} + \frac{1}{2} mc \cdot \frac{r_2^2 + c^2 - r_1^2}{2r_2 c}, \text{ using (1)}$$

$$= m(r_1 - r_2) - \frac{m}{4}(r_1 - r_2) - \frac{mc^2}{4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{m}{4}\left(\frac{r_2^2}{r_1} - \frac{r_1^2}{r_2}\right)$$

$$= \frac{3m}{4}(r_1 - r_2) - \frac{mc^2}{4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{m}{4}\frac{r_2^3 - r_1^3}{r_1 r_2} = \frac{m}{4}\left[\frac{r_2^3 - r_1^3}{r_1 r_2} + 3(r_1 - r_2)\right] - \frac{mc^2}{4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= \frac{m}{4}\frac{(r_2 - r_1)(r_2^2 + r_1^2 + r_2 r_1) - 3r_1 r_2 (r_2 - r_1)}{r_1 r_2} - \frac{mc^2}{4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

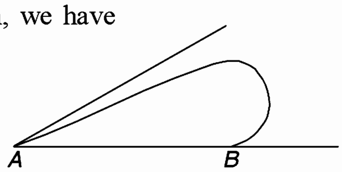
$$= \frac{m}{4}\frac{(r_2 - r_1)(r_1 - r_2)^2}{r_1 r_2} - \frac{mc^2}{4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{m}{4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)(r_1^2 - r_2^2) - \frac{mc^2}{4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= \frac{m}{4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\left[(r_1 - r_2)^2 - c^2\right] = K\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\left[(r_1 - r_2)^2 - c^2\right], \qquad \text{where } K = \frac{m}{4}$$

Ex. 5. *A and B are a simple source and sink of strengths m and* m' *respectively in an infinite liquid. Show that the equation of the streamlines is* $m\cos\theta - m'\cos\theta' = \text{constant}$, *where* θ, θ' *are the angles which AP, BP make with AB, P being any point. Prove also that if* $m > m'$, *the cone defined by the equation* $\cos\theta = 1 - (2m'/m)$ *divides the streamlines issuing from A into two sets, one extending to infinity and the other terminating at B.*

Sol. Let Stokes's stream function at any point P be ψ, Then, we have

$$\psi = m\cos\theta - m'\cos\theta'$$



$\therefore$ The required streamlines are given by ψ = constant,

i.e. $\qquad m\cos\theta - m'\cos\theta' = \text{constant} = c. \qquad \ldots(1)$

For the extreme streamline leaving A (say at angle α) and leaving B, we find that when P is very near to A, $\theta = \alpha, \theta' = \pi$ and when P is very near B, $\theta = 0$ and $\theta' = 0$. Hence for such streamline, (1) gives

$$m\cos\alpha + m' = m - m' \qquad \text{or} \qquad \cos\alpha = 1 - (2m'/m).$$

This generates the cone $\qquad \cos\theta = 1 - (2m'/m).$

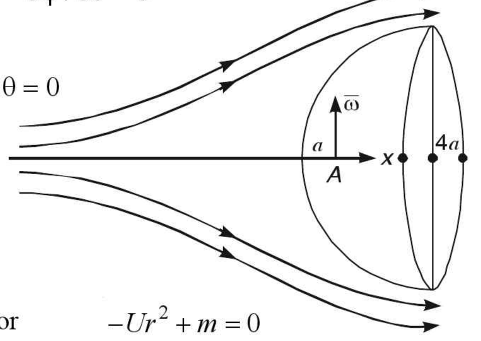
Ex. 6. *Find the Stokes's stream function* ψ *where fluid motion is due to a source of strength m (flux* $4\pi m$*) at a fixed point A and a translation of the fluid of velocity U. Explain how this solution can be used to deduce the motion of fluid past a blunt nosed cylindrical body whose diameter is ultimately* $4a$, *where* $a^2 = m/U$.

Sol. For the present problem, the Stokes's stream function ψ is given by

$$\psi = -(1/2) \times Ur^2 \sin^2\theta + m\cos\theta \qquad \text{...(1)}$$

$\therefore$ At the stagnation point, components of velocity are zero,

i.e. $\quad \partial\psi/\partial r = 0 \quad$ and $\quad \partial\psi/\partial\theta = 0$



From (1), we have

$-Ur\sin^2\theta = 0 \quad$ and $\quad -Ur^2\sin\theta\cos\theta - m\sin\theta = 0$

or $\quad \sin^2\theta = 0 \quad$ and $\quad \sin\theta\,(Ur^2\cos\theta + m) = 0,$

giving $\quad \sin\theta = 0 \quad$ so that $\quad \theta = \pi \qquad$ ...(2)

and $\quad Ur^2\cos\theta + m = 0 \qquad$ ...(3)

Using (2), (3) gives $\quad Ur^2\cos\pi + m = 0 \quad$ or $\quad -Ur^2 + m = 0$

$$\therefore \qquad r^2 = m/U = a^2 \text{ (given)} \qquad \text{...(4)}$$

Hence the streamlines which passes through the stagnation point are given by

$-(1/2)\times Ur^2\sin^2\theta + m\cos\theta = -Ua^2$ or $-(1/2)\times Ur^2\sin^2\theta + Ua^2\cos\theta = -Ua^2$, using (4)

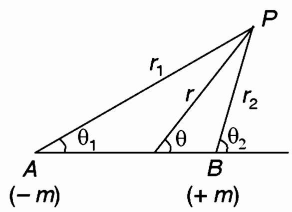
or $\quad (r\sin\theta)^2 = 2Ua^2(1+\cos\theta) \quad$ or $\quad \varpi^2 = 2Ua^2(1+\cos\theta) \qquad [\because\ \varpi = r\sin\theta]$

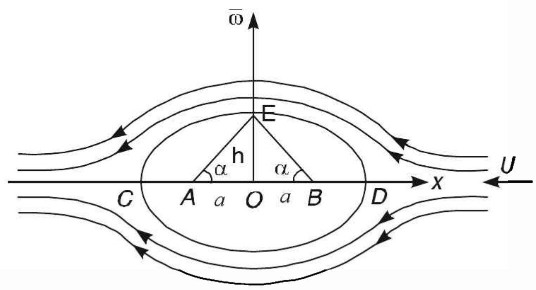
This is the dividing streamline, which when rotated gives a cylindrical surface and this can be taken as the boundary of the solid. We also find that as $\theta \to 0$, $\varpi \to 2a$ and hence the diameter of the cylindrical body is ultimately $4a$.

Ex. 7. *Find the Stokes's stream function* ψ *where fluid motion is due to a source of strength m (flux* $4\pi m$*) at a fixed point A, a sink – m at another fixed point B, a translation of the fluid of velocity U in the direction AB. Explain how this solution can be used to deduce the motion of fluid past a certain solid of revolution. If U =* $(8m)/(9a^2)$*, where AB =* $2a$*, prove that the solid is of axial length* $4a$*, of equatorial radius approximately* $1.6a$*.*

Sol. Combining the source and sink with a uniform stream U in the negative x-direction, the Stokes's stream function may be written as

$$\psi = (1/2)\times Ur^2\sin^2\theta + m(\cos\theta_2 - \cos\theta_1) \qquad \text{...(1)}$$





Let P be any point. When P is on the axis, $\theta = 0$ or $\theta = \pi$, $\theta_1 = \theta_2 = 0$, when P is to the right of B, $\theta_1 = \theta_2 = \pi$ when left of the A but when P is between A and B, $\theta_1 = 0$, $\theta_2 = \pi$.

Hence, $\psi = 0$ for all points on the axis except the portion AB, hence it gives the dividing streamline. Its equation is $\quad (1/2)\times Ur^2\sin^2\theta + m(\cos\theta_2 - \cos\theta_1) = 0$

or $\quad \varpi^2 + b^2(\cos\theta_2 - \cos\theta_1) = 0, \quad$ where $\quad b^2 = 2m/U. \qquad$ ...(2)

Since $\cos\theta_1, \cos\theta_2$ are each numerically less than 1, ϖ^2 cannot exceed $2b^2$ and hence the dividing streamline is closed. By rotating this about the axis, we get a surface of revolution which may be taken to be the boundary of a solid.

If C and D be the stagnation points, then at these points velocity is zero. If $CD = 2l$, so that $OC = OD = l$, then at point D, we have

$$-\frac{m}{(l+a)^2}+\frac{m}{(l-a)^2}=U \quad \text{or} \quad (l^2-a^2)^2 = 2ab^2 l \qquad ...(3)$$

If $OE = h$ = breadth, then

$$h^2+b^2(-\cos\alpha-\cos\alpha)=0 \quad \text{or} \quad h^2 = 2b^2\cos\alpha$$

$$\therefore \quad \frac{h^2}{b^2}=\frac{2a}{\sqrt{(h^2+a^2)}}, \quad \text{since from} \quad \Delta OBE, \cos\alpha = \frac{OB}{BE}=\frac{a}{(h^2+a^2)^{1/2}} \qquad ...(4)$$

If $\quad U = \dfrac{8m}{9a^2}, \quad$ we have $\quad b^2 = \dfrac{2m}{U} = \dfrac{9a^2}{4} \qquad ...(5)$

Form (3), $\quad (l^2 - a^2)^2 = (9a^3 l)/2, \quad$ giving $\quad l = 2a$

$\therefore$ the axial length of the solid $= 2l = 4a$.

From (4), we have $\quad h^2\sqrt{(h^2+a^2)} = 2ab^2 = (9a^3/2)$, using (5)

giving $h = 1.6\ a$ approximately.

Ex. 8. *The resolved attractions of a body symmetrical about the axis of x are $f_1(x\varpi)$ and $f_2(x\varpi)$ respectively perpendicular and parallel to that axis. The equation of a solid of revolution is $\varpi f_1(x\varpi) = a\varpi^2 + b$ where a and b are constants. Prove that if this solid be made to move parallel to its axis in an infinite fluid the streamlines are given by equating the left hand side of this equation to any constant and the velocity function is $-f_2(x\varpi)$ multiplied by a constant.*

Sol. If V be the velocity function, we get

$$-(\partial V/\partial\varpi) = f_1(x\varpi), \quad \text{and} \quad \partial V/\partial x = f_2(x\varpi) \qquad ...(1)$$

The equation of the solid is $\quad \varpi f_1(x\varpi) = a\varpi^2 + b \qquad ...(2)$

On the boundary, the Stokes's stream function is given by

$$\psi = -(1/2)\times U\varpi^2 + \text{constant}, \qquad ...(3)$$

so that if we assume $\psi = A\varpi f_1(x\varpi)$, then we can satisfy the boundary condition, hence we have to show that

$$\psi = A\varpi f_1(x\varpi) \qquad ...(4)$$

is a solution of

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial\varpi^2}-\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}=0 \qquad ...(5)$$

Since V satisfies the Laplace's equation, we have

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial\varpi^2}+\frac{1}{\varpi}\frac{\partial V}{\partial\varpi}=0 \qquad ...(6)$$

Differentiating both sides of (6) w.r.t. ϖ, we get

$$\frac{\partial^3 V}{\partial x^2\partial\varpi}+\frac{\partial^3 V}{\partial\varpi^3}+\frac{1}{\varpi}\frac{\partial^2 V}{\partial\varpi^2}-\frac{1}{\varpi^2}\frac{\partial V}{\partial\varpi}=0 \qquad ...(7)$$

Putting $f_1 = -\partial V/\partial\varpi$ in (7), we have

$$\frac{\partial^2 f_1}{\partial x^2}+\frac{\partial^2 f_1}{\partial\varpi^2}+\frac{1}{\varpi}\frac{\partial f_1}{\partial\varpi}-\frac{f_1}{\varpi^2}=0 \qquad \text{...(8)}$$

Put $$\psi = \varpi f_1 \qquad \text{...(9)}$$

so that $$f_1 = \psi/\varpi \qquad \text{...(10)}$$

From (10), $\dfrac{\partial f_1}{\partial\varpi}=\dfrac{1}{\varpi}\dfrac{\partial\psi}{\partial\varpi}-\dfrac{1}{\varpi^2}\psi$, $\quad\dfrac{\partial^2 f_1}{\partial\varpi^2}=\dfrac{1}{\varpi}\dfrac{\partial^2\psi}{\partial\varpi^2}-\dfrac{2}{\varpi^2}\dfrac{\partial\psi}{\partial\varpi}+\dfrac{2\psi}{\varpi^3}$, $\quad\dfrac{\partial^2 f_1}{\partial x^2}=\dfrac{1}{\varpi}\dfrac{\partial^2\psi}{\partial x^2}$.

Substituting these values in (8) and simplifying, we have

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial\varpi^2}-\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}=0.$$

Hence, we have $$\psi = A\varpi f_1 \qquad \text{...(11)}$$

so that streamlines are given by $$\varpi f_1 = \text{constant}. \qquad \text{...(12)}$$

If ϕ be the velocity potential, then

$$-\frac{\partial\phi}{\partial x}=-\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}=-A\frac{\partial f_1}{\partial\varpi}-\frac{Af_1}{\varpi}, \text{ using (11)}$$

$$=A\frac{\partial^2 V}{\partial x^2}+\frac{A}{\varpi}\frac{\partial V}{\partial\varpi}, \text{ using (1)}$$

$$=-A(\partial^2 V/\partial x^2), \text{ using (6)}$$

$$=A\,(\partial f_2/\partial x) \text{ using (1)}$$

and $$\frac{\partial\phi}{\partial\varpi}=\frac{1}{\varpi}\frac{\partial\psi}{\partial x}=A\frac{\partial f_1}{\partial x}, \text{ using (11)}$$

$$=-A\,(\partial^2 V/dx\partial\varpi) \text{ using (1)}$$

$$=A\,(\partial f_2/\partial\varpi), \text{ using (1)}$$

Hence, $$d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial\varpi}d\varpi=-A\left(\frac{\partial f_2}{\partial x}dx+\frac{\partial f_2}{\partial\varpi}d\varpi\right),$$

Thus, $d\phi = -A\,df_2$ so that $\phi = -Af_2$. or $\phi = A\times\{-f_2(x\varpi)\}$

showing that the required velocity potential ϕ is $-f_2(x\varpi)$ multiplied by a constant.

Ex. 9. *The space bounded by the paraboloids $x^2+y^2=az$, $x^2+y^2=b(z-c)$ (where a, b, c are positive and $b>a$), outside the former and inside the latter, contains liquid at rest. Suddenly the bounding surfaces are made to move with velocities U, V respectively in the direction of the axis of z. Prove that in the motion instantaneously set up the surfaces over which the current function is constant are parabolic of latus rectum $ab(U-V)/(aU-bV)$.*

Sol. The given equations of paraboloids are $x^2+y^2=az$ or $\varpi^2=az$

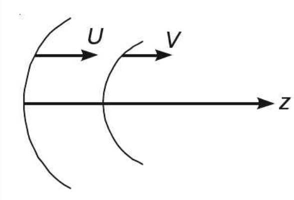
i.e., $$r^2\sin^2\theta = ar\cos\theta \qquad \text{...(1)}$$

and $x^2+y^2=b\,(z-c)$ or $\varpi^2=b(z-c)$

i.e., $$r^2\sin^2\theta = a(r\cos\theta - c) \qquad \text{...(2)}$$

On the boundaries, Stokes's stream function is gives by

$$\psi = -(1/2)\times Ur^2 \sin^2\theta + \text{constant}.$$



On the first parabola, we have

$$\psi = -(1/2)\times aUr\cos\theta + \text{constant}. \qquad ...(3)$$

and on the second parabola, we have

$$\psi = -(1/2)\times bV(r\cos\theta - c) + \text{constant}$$

or $$\psi = -(1/2)\times bVr\cos\theta + \text{constant} \qquad ...(4)$$

Now, ψ satisfies the equation $$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi} = 0$$

whose solutions are $\psi = z$ and $\psi = \varpi^2$

i.e., $\psi = r\cos\theta$ and $\psi = r^2\sin^2\theta$.

Hence, we assume a solution of the form $$\psi = Ar^2\sin^2\theta + Br\cos\theta. \qquad ...(5)$$

On the first paraboloid, by (1), (3) and (5), we have

$$Aar\cos\theta + Br\cos\theta = -(1/2)\times aUr\cos\theta + \text{constant}. \qquad ...(6)$$

On the second paraboloid, by (2), (4) and (5), we have

$$Ab(r\cos\theta - c) + Br\cos\theta = -(1/2)\times bVr\cos\theta + \text{constant}. \qquad ...(7)$$

From (6) and (7), we have

$$Aa + B = -(1/2)\times aU, \quad \text{and} \quad Ab + B = -(1/2)\times bV \qquad ...(8)$$

From (8), $$A = \frac{1}{2}\frac{bV - aU}{a-b} \quad \text{and} \quad B = \frac{1}{2}\frac{ab(U-V)}{a-b} \qquad ...(9)$$

Therefore, ψ = constant reduces to

$$Ar^2\sin^2\theta + Br\cos\theta = \text{constant} \quad i.e., \quad A\varpi^2 + Bz = \text{constant}$$

or $$A(x^2 + y^2) + Bz = \text{constant},$$

which represents a paraboloid of latus rectum $-\dfrac{B}{A}$ *i.e.*, $\dfrac{ab(U-V)}{aU - bV}$, using (9)

EXERCISE 10 (D)

1. A source of strength m is situated in a fluid bounded internally by fixed sphere of radius a, at a distance c from the centre of the sphere. Prove that the velocity at any point on the surface is

$$\frac{2m}{r} - \frac{m}{a}\log\frac{r+c+a}{r+c-a},$$

r being the distance of the point from the source. Find the magnitude of the velocity at any point of the surface.

2. If a fixed sphere of radius a is placed with its centre on the axis of a doublet of strength m and at a distance c from it, show that the velocity of the fluid in contact with the sphere is

$$\{3m(c-a)R/ar^5\}$$

where r is the distance of the point considered from the doublet and R its distance form the axis.

3. Show that the image with regard to a sphere of a doublet whose axis passes through the centre is a doublet at the inverse point.

4. Show that a uniform stream of velocity U can be obtained as the limit $a \to \infty$ of the field due to a source of strength $2\pi a^2 U$ at $(-a, 0, 0)$ and a sink of strength $-2\pi a^2 U$ at $(a, 0, 0)$.

5. Prove the if O, C_1, C_2 are points on the axis of x such that $OC_1 = c_1$, $OC_2 = c_2$ and $C_1C_2 = a^2$, the function
$$\psi = m\left[\frac{r_2 - r}{a} + \frac{a}{c_1}\frac{x - c_1}{r_2} + \frac{x - c_1}{r_1}\right],$$
where r, r_1, r_2 are the distances of any point from O, C_1, C_2 respectively and O is the origin, gives the motion of liquid due to a simple source of strength m at C_1, in the presence of a fixed sphere. (C_1 is outside the sphere, C_2 is its inverse point).

6. Verify that $\psi = \left(Ar^{-2}\cos\theta + Br^2\right)\sin^2\theta$, is a possible form of Stokes's stream function, find the corresponding velocity potential.

7. A source and sink of equal strengths are placed at the points (0, 0, $\pm c$) inside a sphere of radius a with its centre at (0, 0, 0). Find the velocity at points within the sphere.

8. The equation of the meridian section of a surface of revolution is $r = a\sec(\theta/2)$, where $0 \le \theta \le \pi$. The surface is placed in a steady stream of velocity U. Show that the stream function is $U\{(r^2/2)\times\sin^2\theta - a^2(1-\cos\theta)\}$, and find the velocity potential.

9. Find the image of a source with regard to a sphere. O is the centre, P, Q are points outside the sphere on the same radius, Q being nearer the sphere, and P', Q' are their inverse points. Prove that a source of strength m at Q and one of strength $\mu a/OQ$ at Q' produce the small radial flow at every part of the surface of the sphere a line source uniformly distributed along QP of total strength m, together with a line source uniformly along $P'Q'$ of total strength $\mu a/OQ$.

10. Define Stokes's stream function for motion of incompressible fluid symmetrical about an axis, show that the following are possible Stokes's functions and give their interpretation.
$$r - r' \quad \text{and} \quad \cos\theta, \quad \text{where} \quad r = OP, \quad r' = O'P, \quad \angle POO' = \theta;$$
O, O' being any two fixed points on the axis of symmetry

Prove that
$$\psi = m\left[\cos\theta + \frac{a}{c}\cos\theta' + \frac{r' - R}{a}\right]$$
gives the motion due to a simple source S of strength m *placed at a distance* c from the centre of a fixed sphere of radius a, R being measured from the centre of the sphere, (r, θ) from S and (r', θ') from the inverse point of S with respect to the sphere.

11. Show that the image of a radial doublet in a sphere is another radial doublet, and compare their magnitudes; show also that the velocity at any point of the sphere is proportional to ϖr^{-5}, where r is the distance from the doublet, and ϖ the perpendicular on the diameter on which it lies.

12. Find the value of Stokes's stream function in case of a simple source on the axis of x and a uniform line source along the axis.

13. (*a*) Prove that there is always a stream function ψ known as Stokes's stream function in the case of motion of a liquid symmetrical about an axis such that the lines of motion are in planes passing through the axis.

(*b*) Obtain the differential equation satisfied by Stokes's stream function. When the motion is irrotational? Also obtain the solutions of the equation in the forms
$$\psi = \frac{1-\mu^2}{n+1} r^{n+1}\frac{dP_n}{d\mu} \quad \text{and} \quad \psi = -\frac{1-\mu^2}{n}\frac{1}{r^n}\frac{dP_n}{d\mu},$$
where the symbols have their usual meanings.

14. Define Stokes' stream function using the cylindrical coordinates. Show that it in not a harmonic function. **[Agra 2007, 09]**

10.24. Ellipsoidal boundaries. Motion of liquid inside a rotating ellipsoidal shell.

Let $$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \qquad ...(1)$$

be the equation of ellipsoidal shell and $\omega_x, \omega_y, \omega_z$ the components of the angular velocity, referred to axes fixed in space and coincident with the axes of the ellipsoid at the instant under consideration. Then the components of velocity at a point P (x, y, z) of the shell are $z\omega_y - y\omega_z,\ x\omega_z - z\omega_x,\ y\omega_x - x\omega_y$ and the direction cosines of the normal at P are proportional to $x/a^2,\ y/b^2,\ z/c^2$.

Let ϕ be the velocity potential of the liquid motion. Then boundary condition is

$$-\frac{x}{a^2}\frac{\partial\phi}{\partial x} - \frac{y}{b^2}\frac{\partial\phi}{\partial y} - \frac{z}{c^2}\frac{\partial\phi}{\partial z} = \frac{x}{a^2}(z\omega_y - y\omega_z) + \frac{y}{b^2}(x\omega_z - z\omega_x) + \frac{z}{c^2}(y\omega_x - x\omega_y) \qquad ...(2)$$

To satisfy Laplace's equation and the boundary condition (2), we assume that

$$\phi = Ayz + Byz + Cxy \qquad ...(3)$$

Using (3), the equation (1) becomes

$$Ayz(1/b^2 + 1/c^2) + Bzx(1/c^2 + 1/a^2) + Cxy(1/a^2 + 1/b^2)$$
$$= yz\omega_x(1/b^2 - 1/c^2) + zx\omega_y(1/c^2 - 1/a^2) + xy\omega_z(1/a^2 - 1/b^2),$$

$$\therefore \qquad A = -\frac{b^2 - c^2}{b^2 + c^2}\omega_x, \qquad B = -\frac{c^2 - a^2}{c^2 + a^2}\omega_y, \qquad C = -\frac{a^2 - b^2}{a^2 + b^2}\omega_z.$$

Substituting these values in (3), we have

$$\phi = -\frac{b^2 - c^2}{b^2 + c^2}\omega_x yz - \frac{c^2 - a^2}{c^2 + a^2}\omega_y zx - \frac{a^2 - b^2}{a^2 + b^2}\omega_z xy \qquad ...(4)$$

Since (4) depends only on the mutual ratios of a, b, c and not on their absolute magnitudes, the motion considered would remain unchanged in all ellipsoids of the same shape rotating with the same angular velocity.

To find the paths of the particles relative to the ellipsoid.

Let (ξ, η, ζ) be the coordinates of a particle P referred to the axes of the ellipsoid, then the velocity components of P referred to axes fixed in space are

$$\dot{\xi} - \eta\omega_z + \zeta\omega_y, \qquad \dot{\eta} - \zeta\omega_x + \xi\omega_z, \qquad \dot{\zeta} - \xi\omega_y + \eta\omega_x,$$

where dot denotes differentiation with respect to t.

$$\therefore \qquad \dot{\xi} - \eta\omega_z + \zeta\omega_y = -\frac{\partial\phi}{\partial x} = \frac{c^2 - a^2}{c^2 + a^2}\omega_y\zeta + \frac{a^2 - b^2}{a^2 + b^2}\omega_z\eta, \text{ using (4)}$$

or $$\dot{\xi} = \left(\frac{c^2 - a^2}{c^2 + a^2} - 1\right)\omega_y\zeta + \left(\frac{a^2 - b^2}{a^2 + b^2} + 1\right)\omega_z\eta \qquad \text{or} \qquad \dot{\xi} = \frac{2a^2}{a^2 + b^2}\omega_z\eta - \frac{2a^2}{c^2 + a^2}\omega_y\zeta$$

Thus,
Similarly,

$$\left.\begin{aligned} \dot{\xi} &= a^2(\gamma\eta - \beta\zeta) \\ \dot{\eta} &= b^2(\alpha\zeta - \gamma\xi) \\ \dot{\zeta} &= c^2(\beta\xi - \alpha\eta) \end{aligned}\right\} \qquad ...(5)$$

where $\alpha = \dfrac{2\omega_x}{b^2 + c^2}$, $\beta = \dfrac{2\omega_y}{c^2 + a^2}$ and $\gamma = \dfrac{2\omega_z}{a^2 + b^2}$ ...(6)

Multiplying equations in (5) by α / a^2, β / b^2, γ / c^2 respectively and adding, we get

$$\frac{\alpha\dot{\xi}}{a^2} + \frac{\beta\dot{\eta}}{b^2} + \frac{\gamma\dot{\zeta}}{c^2} = \alpha(\gamma\eta - \beta\zeta) + \beta(\alpha\zeta - \gamma\xi) + \gamma(\beta\xi - \alpha\eta) \quad \text{or} \quad \frac{d}{dt}\left(\frac{\alpha\xi}{a^2} + \frac{\beta\eta}{b^2} + \frac{\gamma\zeta}{c^2}\right) = 0$$

Integrating, $\dfrac{\alpha\xi}{a^2} + \dfrac{\beta\eta}{b^2} + \dfrac{\gamma\zeta}{c^2} = c_1$, c_1 being an arbitrary constant ...(7)

Again, multiplying equations in (5) by ξ / a^2, η / b^2, ζ / c^2, respectively adding and then integrating as before, we get

$$\xi^2 / a^2 + \eta^2 / b^2 + \zeta^2 / c^2 = c_2, \; c_2 \text{ being an arbitrary constant} \qquad ...(8)$$

Hence path of the particle lies on the curve of intersection of the plane (7) and the ellipsoid (8) and so the path of the particle is an ellipse.

Now, suppose that equations (5) have solutions of the form

$$\xi = Pe^{ipt}, \qquad \eta = Qe^{ipt}, \qquad \zeta = Re^{ipt} \qquad ...(9)$$

Substituting these values in (5) and re-writing, we get

$$\left.\begin{aligned} (ip/a^2) \times P - \gamma Q + \beta R &= 0 \\ \gamma P + (ip/b^2) \times Q - \alpha R &= 0 \\ -\beta P + \alpha Q + (ip/c^2) \times R &= 0 \end{aligned}\right\} \qquad ...(10)$$

Eliminating P, Q, R from equations (10), we get

$$\begin{vmatrix} ip/a^2 & -\gamma & \beta \\ \gamma & ip/b^2 & -\alpha \\ -\beta & \alpha & ip/c^2 \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, we get

$$p = abc(\alpha^2 / a^2 + \beta^2 / b^2 + \gamma^2 / c^2)^{1/2} \qquad ...(11)$$

Thus, we find that each particle of the liquid describes an ellipse relative to the ellipsoid, like a particle moving under a law of force varying as the distance from a fixed point. Also, periodic time for each particle is $2\pi / p$, where

$$p = 2abc\left\{\left(\frac{\omega_x / a}{b^2 + c^2}\right)^2 + \left(\frac{\omega_y / b}{c^2 + a^2}\right)^2 + \left(\frac{\omega_z / c}{a^2 + b^2}\right)^2\right\}^{1/2}$$

As a particular case, consider a sphere of radius a so that $b = c = a$ and hence

$$p = \left(\omega_x^2 + \omega_y^2 + \omega_z^2\right)^{1/2},$$

showing that the period of revolution of the liquid relative to the spherical shell is the same as the period of revolution of the shell; this means that the liquid is left at rest in space, the shell revolving alone.

10.25. Motion of an ellipsoid in an infinite mass of liquid.

In what follows, we shall use (without proof) the following solutions of Laplace's equation and formulae connected with the ellipsoid (refer theory of attractions in any text book of statics for more details).

Let the equation of the boundary of solid homogeneous ellipsoid of unit density be

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \qquad ...(1)$$

Potential V due to (1) at an external point (x, y, z) is gien by

$$V = \pi abc \int_{\lambda}^{\infty} \left(1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right) \frac{du}{\{(a^2+u)(b^2+u)(c^2+u)\}^{1/2}}, \qquad ...(2)$$

where λ is the positive root of the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1 = 0. \qquad ...(3)$$

We may write this as $$V = \pi(\delta - \alpha x^2 - \beta y^2 - \gamma z^2), \qquad ...(4)$$

where $$\delta = abc\int_{\lambda}^{\infty} \frac{du}{\Delta}, \qquad \alpha = abc\int_{\lambda}^{\infty} \frac{du}{(a^2+u)\Delta},$$

$$\left.\begin{aligned} \beta = abc\int_{\lambda}^{\infty} \frac{du}{(b^2+u)\Delta}, \quad \gamma = abc\int_{\lambda}^{\infty} \frac{du}{(c^2+u)\Delta} \end{aligned}\right\} \qquad ...(5)$$

and $$\Delta = \left\{(a^2+u^2)(b^2+u)(c^2+u)\right\}^{1/2}$$

The potential V_0 at an internal point of (1) is a similar expression with $\lambda = 0$. Using the above notations, we may write

$$V_0 = \pi(\delta_0 - \alpha_0 x^2 - \beta_0 y^2 - \gamma_0 z^2), \qquad ...(6)$$

where $\delta_0, \alpha_0, \beta_0, \gamma_0$ denote what $\delta, \alpha, \beta, \gamma$ become when we put $\lambda = 0$.

Let X, Y, Z be the components of attraction at an external point, then we have

$$X = \frac{\partial V}{\partial x} = -2\pi\alpha x + \frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial x} = -2\pi\alpha x, \qquad [\because \partial V/\partial\lambda = 0 \text{ by (1)}]$$

Similarly, $\quad Y = -2\pi\beta y \quad$ and $\quad Z = -2\pi\gamma z.$

$$\therefore \quad X = -2\pi\alpha x, \quad Y = -2\pi\beta y, \quad Z = -2\pi\gamma z, \qquad ...(7)$$

where α, β, γ are not constants but functions of λ or x, y, z.

We know that V in a solution of Laplace's equation and hence X, Y, Z are also solutions of Laplace's equation.

Let ellipsoid (1) move with velocity U in the direction of the x-axis. The boundary condition is

$$-\frac{x}{a^2}\frac{\partial\phi}{\partial x} - \frac{y}{b^2}\frac{\partial\phi}{\partial y} - \frac{z}{c^2}\frac{\partial\phi}{\partial z} = U\frac{x}{a^2}, \quad \text{over the ellipsoid,} \quad i.e., \quad \text{where } \lambda = 0. \qquad ...(8)$$

To satisfy (8), we assume that $$\phi = AX = -2A\pi\alpha x \qquad ...(9)$$

From (9), $$\frac{\partial\phi}{\partial x} = -2\pi A\left(\alpha + x\frac{\partial\alpha}{\partial\lambda}\frac{\partial\lambda}{\partial x}\right) \qquad ...(10)$$

When $\lambda = 0$, we have $$\partial\alpha/\partial\lambda = -1/a^2 \qquad ...(11)$$

Again, differentiating (3) w.r.t. 'x', we get

$$\frac{2x}{a^2+\lambda}-\frac{\partial\lambda}{\partial x}\left[\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}\right]=0$$

or
$$\left.\begin{aligned}&\frac{\partial\lambda}{\partial x}=\frac{2p^2x}{a^2+\lambda}\\ &\frac{\partial\lambda}{\partial y}=\frac{2p^2y}{b^2+\lambda},\qquad \frac{\partial\lambda}{\partial z}=\frac{2p^2z}{c^2+\lambda}\end{aligned}\right\},\qquad \ldots(12)$$
and similarly,

where
$$\frac{1}{p^2}=\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}\qquad \ldots(13)$$

Hence when $\lambda = 0$, using (11) and (12), (10) reduces to

$$\frac{\partial\phi}{\partial x}=-2\pi A\left(\alpha_0-\frac{2p^2x^2}{a^4}\right)$$

Similarly, $$\frac{\partial\phi}{\partial y}=-2\pi A\left(-\frac{2p^2xy}{a^2b^2}\right),\quad\text{and}\quad \frac{\partial\phi}{\partial z}=-2\pi A\left(-\frac{2p^2xz}{a^2c^2}\right)\qquad \ldots(14)$$

Using (14), (8) reduces to

$$2\pi A\left\{\frac{x}{a^2}\left(\alpha_0-\frac{2p^2x^2}{a^4}\right)-\frac{y}{b^2}\left(\frac{2p^2xy}{a^2b^2}\right)-\frac{z}{c^2}\left(\frac{2p^2xz}{a^2c^2}\right)\right\}=\frac{Ux}{a^2}$$

or
$$2\pi A\left\{\frac{\alpha_0 x}{a^2}-\frac{2p^2x}{a^2}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)\right\}=\frac{Ux}{a^2}\qquad \ldots(15)$$

Also, with $\lambda = 0$, (13) gives $$x^2/a^2+y^2/b^2+z^2/c^2=1/p^2\qquad \ldots(16)$$

Using (16), (15) reduces to $$2\pi A\left\{\frac{\alpha_0 x}{a^2}-\frac{2p^2x}{a^2}\cdot\frac{1}{p^2}\right\}=\frac{Ux}{a^2}$$

giving $$A=U/2\pi(\alpha_0-2)]$$

$\therefore$ From (9), $\phi=U\alpha x/(2-\alpha_0)$, giving the velocity potential of the liquid.

If U, V, W be the components of the velocity of the ellipsoidal (1), then the velocity potential is given by
$$\phi=\frac{U\alpha x}{2-\alpha_0}+\frac{V\beta y}{2-\beta_0}+\frac{W\gamma z}{2-\gamma_0}.$$

10.26. Illustrative solved examples.

Ex. 1. *An ellipsoidal cavity (semi-axes a, b, c) in a solid initially at rest is filled with an incompressible frictionless fluid initially at rest. Prove that if the solid be moved with velocities u, v, w parallel to axes of the cavity, and be rotated with angular velocities p, q, r round the semi-axes, the angular momentum of the fluid round the semi-axis a at any instant is*

$$\frac{4}{15}\pi\rho abc\frac{(b^2-c^2)^2}{b^2+c^2}p.$$ **[Kanpur 2000]**

Sol. Here $\omega_x = p$, $\omega_y = q$, $\omega_z = r$, So from Art 10.24, we get

$$\phi = -\frac{b^2 - c^2}{b^2 + c^2} pyz - \frac{c^2 - a^2}{c^2 + a^2} qzx - \frac{a^2 - b^2}{a^2 + b^2} rxy \qquad ...(1)$$

Let the ellipsoid be reduced to simple rotation about axes by superimposing velocities $-u$, $-v$, $-w$ along the axes. Accordingly, term $(-ux - vy - wz)$ must be added in the expression given by (1). Thus, we obtain

$$\phi = -ux - vy - wz - \frac{b^2 - c^2}{b^2 + c^2} pyz - \frac{c^2 - a^2}{c^2 + a^2} qzx - \frac{a^2 - b^2}{a^2 + b^2} rxy \qquad ...(2)$$

Let U, V, W be the components of velocity at any point P (x, y, z). Then, we have

$$U = -(\partial\phi/\partial x), \qquad V = -(\partial\phi/\partial y) \qquad W = -(\partial\phi/\partial z) \qquad ...(3)$$

Hence the angular momentum of the whole liquid about x-axis

$$= \iiint (yW - zV)\,\rho\, dx\, dy\, dz, \qquad \text{where} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

$$= -\rho \iiint \left(y\frac{\partial\phi}{dz} - z\frac{\partial\phi}{\partial y} \right) dx\, dy\, dy, \text{ by (3)}$$

$$= \rho \iiint \left[y\left\{ w + \frac{b^2 - c^2}{b^2 + c^2} py + \frac{c^2 - a^2}{c^2 + a^2} qx \right\} - z\left\{ v + \frac{b^2 - c^2}{b^2 + c^2} pz + \frac{a^2 - b^2}{a^2 + b^2} rx \right\} \right] dx\, dy\, dz$$

$$= \rho \frac{b^2 - c^2}{b^2 - c^2} p \iiint (y^2 - z^2)\, dx\, dy\, dz, \text{ since other integrals vanish}$$

$$= \frac{b^2 - c^2}{b^2 + c^2} p \iiint \rho[(x^2 + y^2) - (x^2 + z^2)]\, dx\, dy\, dz = \frac{b^2 - c^2}{b^2 + c^2} p \left[M\frac{a^2 + b^2}{5} - M\frac{a^2 + c^2}{5} \right]$$

$$\left\{ \begin{aligned} &\because \iiint \rho(x^2 + y^2)\, dx\, dy\, dz = \text{moment of inertia of the ellipsoid about } z-\text{axis} \\ &\qquad\qquad = (1/5) \times M\,(a^2 + b^2) \\ &\text{Similarly, } \iiint \rho(x^2 + z^2)\, dx\, dy\, dz = (1/5) \times M\,(a^2 + c^2) \end{aligned} \right.$$

$$= \frac{b^2 - c^2}{b^2 + c^2} p \frac{M}{5}(b^2 - c^2) = \frac{4}{15}\pi\rho\, abc \frac{(b^2 - c^2)^2}{b^2 + c^2} p \qquad \left[\because M = \frac{4}{3}\pi\rho abc \right]$$

Ex. 2. *A rigid ellipsoidal envelope, without mass, encloses a perfect incompressible fluid of mass M. The equatin of the ellipsoid is* $x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 = 0$.

An inpulsive couple in the xy-plane causes the envelope to rotate initially with angular velocity ω. *Find the initial velocity potential of the fluid, and prove that the moment of the couple is*

$$\frac{1}{5} M\omega \frac{(a^2 - b^2)^2}{a^2 + b^2}.$$

Sol. Here $\omega_x = 0$, $\omega_y = 0$, $\omega_z = \omega$. Hence as in Art. 10.24, the components of velocity at any point P (x, y, z) are $-y\omega, x\omega, 0$. The direction consines of the normal at P are proportional to x/a^2, y/b^2, z/c^2. Let ϕ be the velocity potential of the liquid motion. Then boundary condition is

$$-\frac{x}{a^2}\frac{\partial\phi}{\partial x} - \frac{y}{b^2}\frac{\partial\phi}{\partial y} - \frac{z}{c^2}\frac{\partial\phi}{\partial z} = \frac{x}{a^2}(-y\omega) + \frac{y}{b^2}(x\omega). \qquad \ldots(1)$$

Also given $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ ...(2)

To satisfy Laplace's equation and (1), take $\phi = cxy$ ...(3)

Then equation (1) becomes

$$-\frac{x}{a^2}cy - \frac{y}{b^2}cx = xy\omega\left(\frac{1}{b^2} - \frac{1}{a^2}\right) \qquad \text{so that} \qquad c = -\omega\frac{a^2 - b^2}{a^2 + b^2}$$

and hence $$\phi = -\omega\frac{a^2 - b^2}{a^2 + b^2}xy, \qquad \text{...(4)}$$

which gives the required initial velocity potential.

Let U, V, W be the components of velocity at any point of the fluid.

Then $U = -(\partial\phi/\partial x)$, $V = -(\partial\phi/\partial y)$ and $W = 0$ ...(5)

The required initial moment of the couple

$$= \text{the total moment of momentum } = \iiint (xV - yU)\,\rho dxdydz, \text{ where } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$

$$= -\rho\iiint\left(x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x}\right)dx\,dy\,dz = \rho\omega\frac{a^2 - b^2}{a^2 + b^2}\iiint (x^2 - y^2)\,dx\,dy\,dz, \text{ using (4) and (5)}$$

$$= \omega\frac{a^2 - b^2}{a^2 + b^2}\iiint \rho[(x^2 + z^2) - (x^2 + y^2)]\,dx\,dy\,dz = \omega\frac{a^2 - b^2}{a^2 + b^2}\left[M\frac{a^2 + c^2}{5} - M\frac{a^2 + b^2}{5}\right], \text{ as in Ex. 1}$$

$$= \frac{\omega M}{5}\cdot\frac{(a^2 - b^2)^2}{a^2 + b^2}.$$

Ex. 3. *Prove that if two rigid surfaces of revolution one of which surrounds the other, are moving alng their common axis with velcities U_1, U_2 and space between them is filled with homogeneous liquid, the momentum of the liquid is $M_2U_2 - M_1U_1$, where M_1, M_2 are the masses of liquid which either surface would contain.*

Sol. Let x-axis be taken as the axis of revolution. Due to symmetry, the moment of moemntum of the liquid along the y-axis and z-axis is zero. Then

The momentum of the liquid along x-axis = $\iiint \rho u\; dxdydz$. ... (1)

If ϕ be the velocity potential at any point P (x, y, z) of the liquid, then $u = -\partial\phi/\partial x$ and hence (1) reduces to

The momentum of the liquid along x-axis = $-\iiint \rho\frac{\partial\phi}{\partial x}dxdydz$, ...(2)

the integration extends over the whole volume of the liquid.

Using Green's theorem, (2) can be re-written as

The momentum of the liquid along x-axis $= \iint x\frac{\partial\phi}{\partial n}dS$, ...(3)

where δn is an element of the outward normal at the element of the bounding surface δS.

Hence the momentum of the liquid along x-axis is

$$= \rho\iint_{\text{inner}} x\frac{\partial\phi}{\partial n}dS_1 + \rho\iint_{\text{outer}} x\frac{\partial\phi}{\partial n}dS_2 = -\rho\iint_{\text{inner}} xl_1U_1dS_1 + \rho\iint_{\text{outer}} xl_2U_2dS_2, \qquad \text{...(4)}$$

where l_1 and l_2 are cosines of the angles which the outer drawn normals at dS_1, dS_2 make with x-axis. But $l_1 dS_1 = dydz$ and $l_2 dS_2 = dydz$. So (4) yields

The momentum of the liquid along x-axis

$$= -\rho \iint_{\text{inner}} xU_1 dy\,dz + \rho \iint_{\text{outer}} xU_2 dy\,dz = -U_1 \iint_{\text{inner}} \rho x\,dy\,dz + U_2 \iint_{\text{outer}} \rho x\,dy\,dz = M_2 U_2 - M_1 U_1,$$

where M_1, M_2 are the masses of the liquids which either surface would contain.

EXERCISE 10 (E)

1. Prove that the period of revolution of the liquid relative to the spherical shell is the same as the period of revolution of the shell.

2. Determine the velocity potential ϕ of the liquid motion if the ellipsoid has velocity components U, V, W parallel to the coordinate axes.

3. Prove that, when an oblate spheroid of eccentricity $\sin\alpha$ moves parallel to its axis of figure with velocity V in infinite fluid, the kinetic energy of the fluid is

$\frac{1}{2} M V^2 \frac{\tan\alpha - \alpha}{\alpha - \sin\alpha\cos\alpha}$, where M' denotes the mass of the displaced fluid.

4. Find the equations of motion of sphere through a liquid extending to infinity in all directions and at rest there, the extraneous force, whose potential is Ω being supposed to act on the sphere and the liquid alike. Prove that the velocity potential at a point P due to a uniform line source OA of strength m per unit length is $m\ (OP + AP + OA)\ /\ (OP + AP - OA)$ the liquid is supposed to extend to infinity in all directions and to be a rest there.

OBJECTIVE QUESTIONS ON CHAPTER 10

Multiple choice questions

Choose the correct alternative from the following questions

1. The equation of lines of flow relative to a sphere is

   (*i*) $\sin^2\theta = (r^3 - a^3)/cr$ (*ii*) $\sin^2\theta = (r^3 + a^3)/cr$

   (*iii*) $\sin^2\theta = cr/(r^3 - a^3)$ (*iv*) $\sin^2\theta = cr/(r^3 + a^3)$ **[Kanpur 2001, Agra 2012]**

2. For motion of a sphere through an infinite mass of liquid at rest at infinity, velocity potential ϕ is given by

   (*i*) $(U a^3 \cos\theta)/2r$ (*ii*) $(U a^3 \cos\theta)/2r^2$ (*iii*) $(U a^3 \cos\theta)/2r^3$ (*iv*) None of these

3. When a sphere of radius a moves with uniform velocity U through a perfect incompressible fluid, the acceleration of a particle of the fluid at $(r, 0)$ is

   (*i*) $3U^2 (a^3/r^4 - a^6/r^7)$ (*ii*) $4U^2 (a^3/r^4 - a^6/r^7)$

   (*iii*) $5U^2 (a^3/r^4 - a^6/r^7)$ (*iv*) None of these

4. A sphere of radius a is surrounded by a concentric spherical shell of radius b, the space between is filled with liquid. If the sphere be moving with velocity U, then velocity potential ϕ is given by

   (*i*) $\frac{U a^3}{b^3 - a^3}(r + b^3/2r^2)\sin\theta$ (*ii*) $\frac{U a^3}{b^3 - a^3}(r + b^3/2r^2)\cos\theta$

   (*iii*) $U a^3 (r + b^3/2r^2)\sin\theta$ (*iv*) $U a^3 (r + b^3/2r^2)\cos\theta$

5. If ϕ be the velocity potential due to a simple three dimensional source, then in usual symbols,

(*i*) $\phi = m/r$ (*ii*) $\phi = -m/r$ (*iii*) $\phi = m/r^2$ (*iv*) $\phi = -m/r^2$

6. In usual symbols, the velocity potential due to a three dimensional doublet is

(*i*) $2\mu \dfrac{\partial}{\partial s}\left(\dfrac{1}{r}\right)$ (*ii*) $3\mu \dfrac{\partial}{\partial s}\left(\dfrac{1}{r}\right)$ (*iii*) $4\mu \dfrac{\partial}{\partial s}\left(\dfrac{1}{r}\right)$ (*iv*) None of these

7. In usual notations, the Stokes' stream function for a simple source on the axis of x is

(*i*) $m\sin\theta$ (*ii*) mx (*iii*) mx/r (*iv*) mx/r^2

8. For a simple source of strength m at the origin, the value of Stokes' stream function at the point $P(r, \theta, \phi)$ in

(*a*) $m\sin\theta$ (*ii*) $m\cos\theta$ (*iii*) $m\sin 2\theta$ (*iv*) $m\cos 2\theta$ **[Agra 2007]**

Answers/Hints to objective type questions

1. (*iii*). See Art, 10.3,
2. (*ii*). See Eq. (8), Art. 10.2
3. (*i*). See Ex. 1, Art. 10.4,
4. (*ii*). See Cor., Art. 10.9
5. (*i*). See Art. 10.11,
6. (*iv*). See Art. 10.13,
7. (*iii*). See part (i) of Art. 10.22
8. (*ii*). See part (i) of Art. 10.22

Miscellaneous Problems

1. Fill up gap : The stokes stream function for a simple source on the axis of x is .....

Ans. $(mx)/r$. Refer part (*i*) of Art. 10.22 (*Agra 2011*)

CHAPTER

11 Vortex Motion (Rectilinear Vortices)

11.1A. Introduction.

It is known that all possible motions of an ideal liquid can be subdivided into two classes; vortex free irrotational or potential flows, whose characteristics can be derived from a velocity potential $\phi(x, y, z, t)$, and vortex or rotational motions for which this is not the case. Rotational motions differ from potential flows in that, as the name applies, all particles of the fluid or at least part of them rotate about an axis which moves with the fluid. Potential flow, on the other hand, is irrotational by definition. So far we paid attention almost entirely to cases involving irrotational motion only. In the present chapter we wish to discuss the theory of rotational or vortex motion.

11.1B. Vorticity, vorticity components (or components of spin).

If $\mathbf{q}$ be the velocity vector of a fluid particle, then the vector quantity, $\mathbf{\Omega}$ (= curl $\mathbf{q}$), is called the *vorticity vector* or simply the *vorticity* and is a measure of the angular velocity of an infinitesimal element. Let $\mathbf{\Omega} = \Omega_x\,\mathbf{i} + \Omega_y\,\mathbf{j} + \Omega_z\,\mathbf{k}$ so that (Ω_x, Ω_y, Ω_z) are the *vorticity components* or the *components of the spin.* Then, if $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$.

$$\Omega_x = \partial w/\partial y - \partial v/\partial z \qquad \Omega_y = \partial u/\partial z - \partial w/\partial x \qquad \Omega_z = \partial v/\partial x - \partial u/\partial y,$$

If Ω_x, Ω_y, Ω_z, are all zero, the motion is *irrotational* and the velocity function ϕ exists and if Ω_x, Ω_y, Ω_z are not all zero, the motion is *rotational.*

In the case of two-dimensional motion, we know that $w = 0$ and u and v are functions of x and y only and hence for two-dimensional case,

$$\Omega_x = 0, \qquad \Omega_y = 0 \qquad \text{and} \qquad \Omega_z = (\partial v/\partial x) - (\partial u/\partial y)$$

It follows that in two-dimensional motion there can be at the most only one component of spin and its axis is perpendicular to the plane of the motion.

11.1C. Vortex line. [Kanpur 1999]

A *vortex line* is a curve drawn in the fluid such that the tangent to it at every point is in the direction of the vorticity vector. The differential equations of the vortex lines are

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}.$$

In two dimensional motion, since axis of rotation at every point is perpendicular to this plane of motion and hence the vortex lines must be all parallel (because all vortex line will be perpendicular to the plane of motion).

11.1D. Vortex tube and vortex filament (or vortex). [Kanpur 2001]

The vertex lines drawn through each point of a closed curve enclose a tubular space in the fluid which is called the *vortex tube*.

A vortex tube of infinitesimal cross section is known as a *vortex filament or simply* a vortex.

11.2. Helmholtz's vorticity theorems. Properties of vortex tube.

(1) *The product of the cross section and vorticity (or angular velocity) at any point on a vortex filament is constant along the filament and for all time when the body forces are conservative and the pressure is a single-valued function of density only*

Let Ω be the vorticity vector and let $\boldsymbol{\omega}$ be the angular velocity vector. Then we have

$$\Omega = \text{Curl}\,\mathbf{q} \qquad \text{and} \qquad \Omega = 2\omega. \qquad \ldots(1)$$

Let δS_1, δS_2 be two sections of a vortex tube and let n_1, n_2 be the unit normals to these sections drawn outwards from the fluid between them. Again, suppose δS be the curved surface of the vortex tube and

ΔS = total surface area of element = $\delta S_1 + \delta S + \delta S_2$

ΔV = total volume which ΔS contains.

Then $$\int_{\Delta S} \mathbf{n}\cdot\Omega\, dS = \int_{\Delta V} \nabla\cdot\Omega\, dV = 0, \text{ by Gauss divergence theorem} \qquad \ldots(2)$$

Since $\nabla\cdot\Omega = \nabla\cdot\text{curl}\,\mathbf{q} = 0$. Hence (2) gives

$$\int_{\delta S_1} \mathbf{n}\cdot\Omega\, dS + \int_{\delta S} \mathbf{n}\cdot\Omega\, dS + \int_{\delta S_2} \mathbf{n}\cdot\Omega\, dS = 0 \qquad \ldots(3)$$

Since Ω is tangential to the curved surface of the vortex tube, $\mathbf{n}\cdot\Omega = 0$ at each point of δS. Hence (3) reduces to

$$\int_{\delta S_1} \mathbf{n}\cdot\Omega\, dS = -\int_{\delta S_2} \mathbf{n}\cdot\Omega\, dS \qquad \text{or} \qquad \int_{\delta S_1} \Omega\cdot d\boldsymbol{S} = -\int_{\delta S_2} \Omega\cdot d\boldsymbol{S} \qquad \ldots(4)$$

or $$\int_{\delta S_1} 2\omega\cdot d\boldsymbol{S} = -\int_{\delta S_2} 2\omega\cdot d\boldsymbol{S} \qquad \ldots(5)$$

Thus, to the first order of approximation, (4) and (5) give

$$\Omega_1\,\delta S_1 = \Omega_2\,\delta S_2 \qquad \text{and} \qquad \omega_1\,\delta S_1 = \omega_2\,\delta S_2. \qquad \ldots(6)$$

Equation (6) shows that $\Omega\,\delta S$ or $2\omega\,\delta S$ is constant over every section δS of the vortex tube. Its value is known as the *strength of the vortex tube.* A vortex tube whose strength is unity is called a *unit vortex tube.*

(2) *Vortex lines and tubes cannot originate or terminate at internal points in a fluid.*

Let S be any closed surface containing a volume V. Then we have

$$\int_S \Omega\cdot d\boldsymbol{S} = \int_S \mathbf{n}\cdot\Omega\, dS = \int_V \nabla\cdot\Omega\, dV = 0,$$

which shows that the total strength of vortex tubes emerging from S must be equal to that entering S. Hence vortex lines and tubes cannot begin or end at any point within the liquid. They must either form closed curves or have their extremities on the boundary of the liquid.

(3) *Vortex lines move along with the liquid (i.e. they are composed of the same elements of the liquid) provided that body forces are conservative and the pressure is a single-valued function of density.*

Let C be a closed circuit of liquid particles and let S be an open surface with C as rim. Then the circulation Γ is constant in the moving circuit C by Kelvin's circulation theorem (refer Art. 6.4). Thus we have

$$\Gamma = \int_C \mathbf{q} \cdot d\mathbf{r} = \int_S \text{curl}\,\mathbf{q} \cdot d\boldsymbol{S} = \int_S \Omega \cdot d\boldsymbol{S} \qquad \text{so that} \qquad \int_S \Omega \cdot d\boldsymbol{S} = \text{constant} \qquad ...(7)$$

since Γ is constant. Thus for a surface S moving with the fluid (7) holds. The L.H.S. of (7) represents the total strength of vortex tubes passing through S. This shows that the vortex tubes move with the fluid. By taking $S \to 0$, it follows that the vortex lines move with the liquid.

11.3. Illustrative solved examples.

Ex. 1. *If* $u = (ax - by)/(x^2 + y^2)$ $v = (ay + bx)/(x^2 + y^2)$ $w = 0$, *investigate the nature of motion of the liquid. Also show that*

(*i*) *the velocity potential is* $-\{(a/2) \times \log(x^2 + y^2) + b\tan^{-1}(y/x)\}$,

(*ii*) *the pressure at any point* (x, y) *is given by* $\qquad \dfrac{p}{\rho} = \text{const.} - \dfrac{1}{2}\dfrac{a^2 + b^2}{x^2 + y^2}$

Sol. Given $\quad u = (ax - by)/(x^2 + y^2) \qquad v = (ay + bx)/(x^2 + y^2), \qquad w = 0 \quad ...(1)$

From (1),
$$\frac{\partial u}{\partial x} = \frac{a(x^2 + y^2) - 2x(ax - by)}{(x^2 + y^2)^2} = \frac{ay^2 - ax^2 + 2bxy}{(x^2 + y^2)^2}$$

and
$$\frac{\partial v}{\partial y} = \frac{a(x^2 + y^2) - 2y(ay + bx)}{(x^2 + y^2)^2} = \frac{ax^2 - ay^2 - 2bxy}{(x^2 + y^2)^2}.$$

We see that $\partial u/\partial x + \partial v/\partial y = 0$ and hence the equation of continuity is satisfied by (1). Therefore (1) represents a possible motion. Moreover (1) represents a two-dimensional motion and hence vorticity components are given by

$$\Omega_x = 0, \qquad \Omega_y = 0, \qquad \Omega_z = (\partial v/\partial x) - (\partial u/\partial y) \qquad ...(2)$$

From (1),
$$\frac{\partial u}{\partial y} = \frac{-b(x^2 + y^2) - 2y(ax - by)}{(x^2 + y^2)^2} = \frac{-bx^2 + by^2 - 2ayx}{(x^2 + y^2)^2}$$

and
$$\frac{\partial v}{\partial y} = \frac{b(x^2 + y^2) - 2x(ay + bx)}{(x^2 + y^2)^2} = \frac{-bx^2 + by^2 - 2axy}{(x^2 + y^2)^2}$$

so from (1), $\Omega_z = 0$. Thus, $\Omega_x = \Omega_y = \Omega_z = 0$, showing that the motion is irrotational.

Part (*i*) We have, $\quad d\phi = (\partial\phi/\partial x)\,dx + (\partial\phi/\partial y)dy = -\,udx\,-\,vdy$

$$= -\left[\frac{ax - by}{x^2 + y^2}dx + \frac{ay + bx}{x^2 + y^2}dy\right] = -\left[\frac{a}{2}\cdot\frac{2xdx + 2ydy}{x^2 + y^2} + b\frac{xdy - ydy}{x^2 + y^2}\right]$$

Thus,
$$d\phi = -\left[\frac{a}{2}d\{\log(x^2 + y^2)\} + b\,d\left(\tan^{-1}\frac{y}{x}\right)\right]$$

Integrating,
$$\phi = -\{(a/2) \times \log(x^2 + y^2) + b\tan^{-1}(y/x)\}.$$

Part (*ii*). Let q be the fluid velocity. Then, pressure is given by

$$p/\rho + q^2/2 = \text{const.} \qquad \text{or} \qquad p/\rho = \text{const.} - (q^2/2) \qquad ...(3)$$

But $$q^2 = u^2 + v^2 = \frac{(ax - by)^2 + (ay + bx)^2}{(x^2 + y^2)^2}, \text{ using (1)}$$

or $$q^2 = \frac{a^2(x^2 + y^2) + b^2(x^2 + y^2)}{(x^2 + y^2)^2} = \frac{a^2 + b^2}{x^2 + y^2} \qquad ...(4)$$

Using (4), (3) gives the required result.

Ex. 2. *Prove that the necessary and sufficient condition that the vortex lines may be at right angles to the stream lines are*

$$u, v, w, = \mu\,(\partial\phi/\partial x, \partial\phi/\partial y, \partial\phi/\partial z), \textit{ where } \mu, \phi \textit{ are functions of } x, y, z, t.$$

[G.N.D.U. Amritsar 1998; I.A.S. 2005; Agra 2001, 03, 05]

OR

Find the necessary and sufficient condition that vortex lines may be at right angles to the streamlines. **[G.N.D.U. Amritsar 2002; Kanpur 1999]**

Sol. Streamlines are given by $$dx/u = dy/v = dz/w \qquad ...(1)$$

and vortex lines are given by $$dx/\Omega_x = dy/\Omega_y = dz/\Omega_z \qquad ...(2)$$

(1) and (2) will be at right angles, if $$u\,\Omega_x + v\,\Omega_y + w\,\Omega_z = 0 \qquad ...(3)$$

But $$\Omega_x = \partial w/\partial y - \partial v/\partial z, \qquad \Omega_y = \partial u/\partial z - \partial w/\partial x \qquad \Omega_z = \partial v/\partial x - \partial u/\partial y \qquad ...(4)$$

Using (4), (3) may be re-written as

$$u\,(\partial w/\partial y - \partial v/\partial z) + v\,(\partial u/\partial z - \partial w/\partial x) + w\,(\partial v/\partial x - \partial u/\partial y) = 0,$$

which is the necessary and sufficient condition in order that $udx + vdy + wdy$ may be a perfect differential. So we may write

$$udx + vdy + wdz = \mu d\phi = \mu\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$\therefore \quad u = \mu(\partial\phi/\partial x), \qquad v = \mu(\partial\phi/\partial y) \qquad \text{and} \qquad w = \mu(\partial\phi/\partial z).$$

Ex. 3. *In an incompressible fluid the vorticity at every point is constant in magnitude and direction; show that the components of velocity u, v, w are solutions of Laplace's equation.*

[Agra 2012; I.A.S. 2004; Meerut 1999; Rajasthan 2000, 05]

Sol. Let Ω_x, Ω_y, Ω_z be the components of vorticity Ω so that $\Omega = (\Omega_x^2 + \Omega_y^2 + \Omega_z^2)^{1/2}$ and direction cosines of its direction are Ω_x/Ω, Ω_y/Ω, Ω_z/Ω.

Ω and its direction being given, it follows that Ω_x, Ω_y, Ω_z are all constants. Moreover,

$$\Omega_x = \partial w/\partial y - \partial v/\partial z, \qquad \Omega_y = \partial u/\partial z - \partial w/\partial x, \qquad \Omega_z = \partial v/\partial x - \partial u/\partial y \qquad ...(1)$$

Differentiating second equation in (1) w.r.t. 'z' and third equation in (1) w.r.t 'y' and then subtracting, we get

$$\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial x \partial z} - \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{or} \qquad \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \qquad ...(2)$$

The equation of continuity is

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad i.e. \qquad \partial v/\partial y + \partial w/\partial z = -(\partial u/\partial x) \qquad ...(3)$$

Using (3), (2) reduces to $\qquad \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0,$

showing that u satisfies Laplace's equation. Similarly we can show that v and w also satisfy Laplace's equation.

Ex. 4. *Assuming that in an infinite unbounded mass of incompressible fluid, the circulation in any closed circuit is independent of time, show that the angular velocity of any element of the fluid moving rotationally varies as the length of the element measured in the direction of the axis of rotation.* **(Guwahati 2000; 03; Roorkee 2000)**

Sol. Let ω be the angular velocity of the element ds and let σ be its area of cross-section. Then circulation in any closed circuit surrounding this element is $2\omega\sigma$. Let the element ds form a part of the vortex filament s so that the circulation is constant all along this element. Furthermore, since circulation is assumed to be independent of time, we get

$$2\omega\sigma = \text{constant}. \qquad \text{...(1)}$$

Again, since the liquid is incompressible $\qquad \sigma\, ds = \text{constant}. \qquad$ ...(2)

Dividing (1) by (2), we get ω / ds = constant, which proves the required result.

Ex. 5. *It* $udx + vdy + wdz = d\theta + \lambda d\chi$, *where* θ, λ, χ *are functions of* x, y, z, t, *prove that the vortex lines at any time are the lines of inter- section of the surfaces* λ = *constant and* χ = *constant* **(Agra 2012)**

Sol. Given $\qquad udx + vdy + wdz = d\theta + \lambda d\chi$

$$\therefore \quad udx + vdy + wdz = \frac{\partial\theta}{\partial x}dx + \frac{\partial\theta}{\partial y}dy + \frac{\partial\theta}{\partial z}dz + \frac{\partial\theta}{\partial t}dt + \lambda\left(\frac{\partial\chi}{\partial x}dx + \frac{\partial\chi}{\partial y}dy + \frac{\partial\chi}{\partial z}dz + \frac{\partial\chi}{\partial t}dt\right)$$

$$\Rightarrow \quad \left.\begin{aligned} u &= \frac{\partial\theta}{dx} + \lambda\frac{\partial\chi}{\partial x}, & v &= \frac{\partial\theta}{dy} + \lambda\frac{\partial\chi}{\partial y} \\ w &= \frac{\partial\theta}{\partial z} + \lambda\frac{\partial\chi}{\partial z}, \quad \text{and} & 0 &= \frac{\partial\theta}{\partial t} + \lambda\frac{\partial\chi}{\partial t} \end{aligned}\right\} \qquad \text{...(1)}$$

Hence the components of spin Ω_x, Ω_y, Ω_z are given by

$$2\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial\theta}{\partial z} + \lambda\frac{\partial\chi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\theta}{\partial y} + \lambda\frac{\partial\chi}{\partial y}\right), \text{ using (1)}$$

$$= \frac{\partial^2\theta}{\partial y\partial z} + \frac{\partial\lambda}{\partial y}\frac{\partial\chi}{\partial z} + \lambda\frac{\partial^2\chi}{\partial y\partial z} - \frac{\partial^2\theta}{\partial z\partial y} - \lambda\frac{\partial^2\chi}{\partial z\partial y} - \frac{\partial\lambda}{\partial z}\frac{\partial\chi}{\partial y} = \frac{\partial\lambda}{\partial y}\frac{\partial\chi}{\partial z} - \frac{\partial\lambda}{\partial z}\frac{\partial\chi}{\partial y}$$

or
$$2\Omega_x = \begin{vmatrix} \partial\lambda/\partial y & \partial\lambda/\partial z \\ \partial\chi/\partial y & \partial\chi/\partial z \end{vmatrix}$$

Similarly, $\qquad 2\Omega_y = \begin{vmatrix} \partial\lambda/\partial z & \partial\lambda/\partial x \\ \partial\chi/\partial z & \partial\chi/\partial x \end{vmatrix} \quad$ and $\quad 2\Omega_z = \begin{vmatrix} \partial\lambda/\partial x & \partial\lambda/\partial y \\ \partial\chi/\partial x & \partial\chi/\partial y \end{vmatrix}$

$$\therefore \quad 2\left(\Omega_x\frac{\partial\lambda}{\partial x} + \Omega_y\frac{\partial\lambda}{\partial y} + \Omega_z\frac{\partial\lambda}{\partial z}\right) = \begin{vmatrix} \partial\lambda/\partial x & \partial\lambda/\partial y & \partial\lambda/\partial z \\ \partial\lambda/\partial x & \partial\lambda/\partial y & \partial\lambda/\partial z \\ \partial\chi/\partial x & \partial\chi/\partial y & \partial\chi/\partial z \end{vmatrix} = 0.$$

$$\therefore \quad \Omega_x(\partial\lambda/\partial x) + \Omega_y(\partial\lambda/\partial y) + \Omega_z(\partial\lambda/\partial z) = 0. \qquad \text{...(2)}$$

Similarly, we have $\qquad \Omega_x(\partial\chi/\partial x) + \Omega_y(\partial\chi/\partial y) + \Omega_z(\partial\chi/\partial z) = 0. \qquad$ ...(3)

Equations (2) and (3) show that the vortex lines at any time are the lines of intersection of the surfaces λ = constant and χ = constant.

EXERCISE 11 (A)

1. Each particle of a mass of liquid is revolving uniformly about a fixed axis with the angular velocity varying as the nth power of the distance from the axis. Show that the motion is irrotational only if $n + 2 = 0$.

If a very small spherical portion of the liquid be suddenly solidified, prove that it will begin to rotate about a diameter with an angular velocity $(n + 2)/2$ of that with which it was revolving about the fixed axis.

11.4. Rectilinear vortices.

Vortex lines being straight and parallel, all vortex tubes are cylindrical, with generators perpendicular to the plane of motion. Such vortices are known as *rectilinear vortices*.

Derivation of velocity potential, stream function, velocity components and complex potential due to a rectilinear vortex filament.

Consider a rectilinear vortex filament with its axis parallel to the axis of z. The motion being similar in all planes parallel to xy-plane, we have no velocity along the axis *i.e.* $w = 0$. Moreover u and v are independent of z *i.e.*

$$\partial u / \partial z = 0 \qquad \text{and} \qquad \partial v / \partial z = 0 \qquad ...(1)$$

If Ω_x, Ω_y, Ω_z be the vorticity components, then

$$\Omega_x = 0, \qquad \Omega_y = 0 \qquad \text{and} \qquad \Omega_z = \partial v/\partial x - \partial u / \partial y \qquad ...(2)$$

Now the equations of lines of flow are

$$dx/u = dy/v \qquad i.e. \qquad vdx - udy = 0. \qquad ...(3)$$

The equation of continuity is $\quad \partial u / \partial x + \partial v / \partial y = 0$

so that
$$\partial v/\partial y = \partial (-u) / \partial x \qquad ...(4)$$

Equation (4) shows that $vdx - udy$ must be perfect differential, $d\psi$ (say). Thus, we have

$$vdx - udy = (\partial\psi/\partial x)\, dx + (\partial\psi / \partial y)\, dy$$

so that
$$u = -(\partial\psi / \partial y) \qquad \text{and} \qquad v = \partial\psi / \partial x. \qquad ...(5)$$

Then the lines of flow are given by $d\psi = 0$ *i.e.* ψ = const. Hence ψ is the stream function.

Using (5), (2) gives
$$\Omega_z = \partial^2\psi / \partial x^2 + \partial^2\psi / \partial y^2 \qquad ...(6)$$

Thus the stream function ψ satisfies

$$\partial^2\psi / \partial x^2 + \partial^2\psi / \partial y^2 = \Omega_z, \text{ on the vortex filament} \qquad ...(7A)$$

$$= 0, \text{ outside the filament} \qquad ...(7B)$$

Let $P\,(r, \theta)$ be any point outside the vortex filament. Since the motion outside the vortex is irrotational, the velocity potential ϕ exists such that

$$\frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}. \qquad ...(8)$$

Moreover, outside vortex filament, ψ satisfies the equation (re-writing (7B) in polar co-ordinates)

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0. \qquad ...(9)$$

There being symmetry about the origin, ψ must be independent of θ and so (9) reduces to

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} = 0 \qquad \text{or} \qquad \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = 0$$

Integrating, $r\,(d\psi/dr) = c$, c being an an arbitrary constant ...(10)

Integrating (10), $\psi = c\log r.$ ...(11)

Since ψ is independent of θ, (8) $\Rightarrow$ $d\psi/dr = -(1/r)\,(\partial\phi/\partial\theta)$ ...(11)′

Now (10) and (11)′ give $\dfrac{c}{r} = -\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}$ so that $\phi = -c\theta.$ ...(12)

If $w(=\phi + i\psi)$ be the complex potential outside the filament, then we have

$$w = -c\theta + ic\log r = ic(\log r + i\theta) = ic\log(re^{i\theta}) = ic\log z.$$

Let k be the circulation in the circuit embracing the vortex. Then, we have

$$\therefore \qquad k = \int_0^{2\pi}\left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) r d\theta = c\int_0^{2\pi} d\theta = 2\pi c, \text{ by (12)}$$

$\therefore$ $c = k/2\pi.$ Hence, we have

$$\phi = -(k/2\pi)\theta, \qquad \psi = (k/2\pi)\log r \qquad \text{and} \qquad w = (ik/2\pi)\log z, \qquad \text{...(13)}$$

Here k is called the *strength of the vortex*.

If there be a rectilinear vortex of strength k at z_0 $(= x_0 + iy_0)$, then

$$w = (ik/2\pi)\log(z - z_0) \qquad \text{...(14)}$$

We now determine velocity components due to a rectilinear vortex of strength k at A_0 (z_0). Let P (x, y) be any point in the fluid. Then, if r_0 be the distance between A_0 (z_0) and P (z),

$$r_0^2 = (x - x_0)^2 + (y - y_0)^2 \qquad \text{and} \qquad \psi = (k/2\pi)\times\log r_0$$

$$\therefore \qquad u = -\frac{\partial\psi}{\partial y} = -\frac{\partial\psi}{\partial r_0}\frac{\partial r_0}{\partial y} = -\frac{k}{2\pi r_0}\frac{y - y_0}{r_0} = -\frac{k}{2\pi}\frac{y - y_0}{r_0^2} \qquad \text{...(15)}$$

and

$$v = \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial r_0}\frac{\partial r_0}{\partial x} = \frac{k}{2\pi r_0}\frac{x - x_0}{r_0} = \frac{k}{2\pi}\frac{x - x_0}{r_0^2} \qquad \text{...(16)}$$

Hence,

$$q = \sqrt{u^2 + v^2} = \frac{k}{2\pi r_0^2}\sqrt{\{(x - x_0)^2 + (y - y_0)^2\}} = \frac{k}{2\pi r_0}, \qquad \text{...(17)}$$

which gives velocity at P (x, y).

Remark 1. Some authors define $K = k/2\pi$ as the strength of the vortex. So, they take

$$\left.\begin{array}{llll} \phi = -K\theta, & \psi = K\log r, & w = iK\log z, & w = iK\log(z - z_0) \\ u = -K\dfrac{y - y_0}{r_0^2}, & v = K\dfrac{x - x_0}{r_0^2}, & & q = \dfrac{K}{r_0} \end{array}\right\} \qquad \text{...(18)}$$

However, we shall not use these results in the present discussion unless otherwise stated.

Remark 2. The case of several rectilinear vortices.

Let there be a number of vortices of strengths $k_1, k_2, k_3, \ldots$ situated at $z_1, z_2, z_3, \ldots$ Then the complex potential is given by

$$w = \frac{ik_1}{2\pi}\log(z - z_1) + \frac{ik_2}{2\pi}\log(z - z_2) + \frac{ik_3}{2\pi}\log(z - z_3) + \ldots = \frac{i}{2\pi}\sum k_n\log(z - z_n) \qquad \text{...(19)}$$

Here vortices of strengths $k_1, k_2, k_3, \ldots$ are situated at (x_1, y_1), (x_2, y_2), (x_3, y_3), ... Hence, using (15) and (16), the velocity components u and v due to these vortices at any point outside there filamentsare given by

$$u = -\frac{1}{2\pi}\sum k_n \frac{y - y_n}{r_n^2} \qquad \text{and} \qquad v = \frac{1}{2\pi}\sum k_n \frac{x - x_n}{r_n^2} \qquad \text{...(20)}$$

where $$r_n^2 = (x - x_n)^2 + (y - y_n)^2, \qquad n = 1, 2, 3, \ldots \qquad \text{...(21)}$$

Result (20) may also be deduced from $u - iv = -\dfrac{dw}{dz} = -\dfrac{i}{2\pi}\sum \dfrac{k_n}{z - z_n}$

Let k_m be the strength of a vortex situated at (x_m, y_m). Then we omit the term containing k_m while finding the velocity of that vortex. Thus the motion of the m th vortex situated at (x_m, y_m) is given by

$$\dot{x}_m = -\frac{1}{2\pi}\sum k_n \frac{y_m - y_n}{r_{mn}^2}, \qquad \text{and} \qquad \dot{y}_m = \frac{1}{2\pi}\sum k_n \frac{x_m - x_n}{r_{mn}^2} \qquad \text{...(22)}$$

where $$m \neq n \qquad \text{and} \qquad r_{mn}^2 = (x_m - x_n)^2 + (y_m - y_n)^2. \qquad \text{...(23)}$$

Using (22), we have $$\sum_m k_m \dot{x}_m = -\frac{1}{2\pi}\sum_m \sum_n k_m k_n \frac{y_m - y_n}{r_{mn}^2} = 0, \qquad \text{...(24)}$$

since m, n can be interchanged and the denominator is positive.

Similarly, $$\sum k_m \dot{y}_m = 0 \qquad \text{...(25)}$$

Since k_m is independent of t, integration of (24) and (25) yield

$$\sum k_m x_m = \text{const.} \qquad \text{and} \qquad \sum k_m y_m = \text{const.} \qquad \text{...(26)}$$

Also, $$\bar{x} = \sum k_m x_m \Big/ \sum k_m \qquad \text{and} \qquad \bar{y} = \sum k_m y_m \Big/ \sum k_m. \qquad \text{...(27)}$$

Using (26), (27) show that $\bar{x}$, $\bar{y}$ are constants. Hence if $k_1, k_2, k_3, \ldots$ be supposed to be the masses situated at $z_1, z_2, z_3, \ldots$, then their centre of gravity is fixed throughout the motion. This point is known as the *centre of vortices*. Thus if there be several vortices, they move in such a manner that their centre is stationary. If $\sum k_m = 0$, the centre of vortices is at infinity or else indeterminate

Remark 3. Single vortex in the field of several vortices.

"*To show that a single rectilinear vortex in an unlimited mass of liquid remains stationary, and when such a vortex is in the presence of other vortices it has tendency to move of itself but its motion through the liquid is entirely due to the vortices caused by the other vortices.*"

Proof. The value of stream function at any point inside of a circular vortex tube is given by

$$2\zeta = \frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} \qquad \text{or} \qquad \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = 2\zeta \qquad \text{or} \qquad \frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = 2\zeta dr$$

Integrating it twice, $\psi = (1/2) \times \zeta r^2 + c_1 \log r + c_2$, c_1, c_2 being arbitrary constants ...(1)

$\therefore$ Velocity at right angles to the radius vector $= d\psi/dr = \zeta r + c/r$ ...(2)

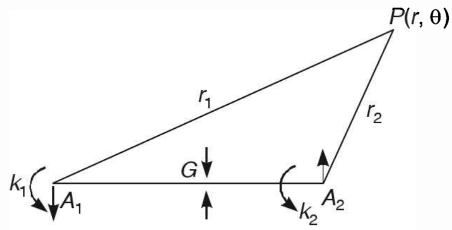
Since the velocity at the origin is finite, c must be zero. Then (2) gives

$$(d\psi/dr)_{r=0} = 0,$$

showing that the velocity at the origin due to a single vortex must vanish. It follows that a vortex filament (vortex) induces no velocity at its centre. Thus, if a vortex is in the presence of other vortices it has no tendency to move of itself but its motion through the liquid will be caused by the other vortices.

11.5. Two vortex filaments.

Case I. When the filaments are in the same sense.



Let us consider two rectilinear vortices of strengths k_1 and k_2 at A_1 ($z = z_1$) and A_2 ($z = z_2$). Then complex potential due to stationary system is

$$w = (ik_1/2\pi)\log(z - z_1) + (ik_2/2\pi)\log(z - z_2) \qquad ...(1)$$

However, vortices situated at A_1 and A_2 would start moving due to the presence of each other. Let u_1, v_1 be the components of the velocity q_1 of A_1 which is due to A_2 alone. Then, we have

$$u_1 - iv_1 = \left[\frac{1}{2\pi}\frac{ik_1}{z - z_1} + \left(-\frac{dw}{dz}\right)\right]_{z = z_1} = -\frac{ik_2}{2\pi}\frac{1}{z_1 - z_2} \qquad ...(2)$$

$$q_1 = |u_1 - iv_1| = \frac{k_2}{2\pi |z_1 - z_2|} = \frac{k_2}{2\pi(A_1 A_2)} \qquad ...(3)$$

Similarly, $$u_2 - iv_2 = -\frac{ik_1}{2\pi}\frac{1}{z_2 - z_1} \qquad ...(4)$$

and $$q_2 = k_1/2\pi(A_1A_2) \qquad ...(5)$$

From (2) and (4), $\quad (u_1 - iv_1)/k_2 = -(u_2 - iv_2)/k_1$

i.e. $\quad k_1(u_1 - iv_1) + k_2(u_2 - iv_2) = 0 \quad$ or $\quad (k_1u_1 + k_2u_2) - i(k_1v_1 + k_2v_2) = 0$

so that $\quad k_1u_1 + k_2u_2 = 0 \quad$ and $\quad k_1v_1 + k_2v_2 = 0. \qquad ...(6)$

Since $k_1 + k_2 \neq 0$, (6) shows that a point G, the centroid of masses k_1, k_2 at z_1 and z_2, moving with velocities (u_1, v_1), (u_2, v_2) is at rest. Hence the line $A_1 A_2$ rotates about G. Since G is C. G. of k_1 and k_2, we have $\quad k_1 \cdot A_1G = k_2 \cdot A_2G$

or $$\frac{A_1G}{k_2} = \frac{A_2G}{k_1} = \frac{A_2G + A_1G}{k_1 + k_2} = \frac{A_1 A_2}{k_1 + k_2}$$

so that $$A_1G = \frac{k_2}{k_1 + k_2}A_1A_2, \quad \text{and} \quad A_2G = \frac{k_1}{k_1 + k_2}A_1A_2 \qquad ...(7)$$

Re-writing (3), we have $$q_1 = \frac{k_2 A_1 A_2}{k_1 + k_2}\cdot\frac{k_1 + k_2}{2\pi(A_1 A_2)^2} = A_1G \cdot \omega, \qquad ...(8)$$

where $$\omega = \frac{k_1 + k_2}{2\pi(A_1 A_2)^2} \quad \text{and} \quad A_1G = \frac{k_2 A_1 A_2}{k_1 + k_2}. \qquad ...(9)$$

Thus angular velocity of A_1 is ω about G. Similarly, we may show that the angular velocity of A_2 is ω about G. Hence the line A_1A_2 revolves about G with uniform angular velocity ω.

Remark. As a particular case, let $k_1 = k_2 = k$ and $A_1A_2 = 2a$. Then, we have

$$q_1 = k/4\pi a \qquad q_2 = k/4\pi a \qquad \text{and} \qquad \omega = k/4\pi a^2$$

and the stream function is given by

$$\psi = (k/2\pi) \times \log r_1 + (k/2\pi) \times \log r_2 = (k/2\pi) \times \log(r_1 r_2)$$

where $r_1 = A_1P$, $r_2 = A_2P$ and P is any point in the fluid. The streamlines are given by

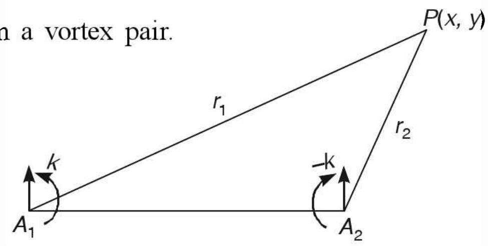
$\psi =$ const, $\quad$ *i.e.* $\quad r_1 r_2 =$ const, $\quad$ which are Cassini's ovals.

Case II. When the filaments are in the opposite sense.

Let k_1 and k_2 be of opposite signs. Then G will not lie in between A_1 and A_2. However, if $k_1 > k_2$, then G will lie on A_1A_2 produced and if $k_2 > k_1$, it will lie on A_2A_1 produced. As before, it can be shown that the line A_1A_2 revolves about G with uniform angular velocity ω.

11.6A. Vortex pair.

Two vortex filaments of strengths k and $-k$ form a vortex pair.



Let us consider two rectilinear vortices of strengths k and $-k$ at A_1 $(z = z_1)$ and A_2 $(z = z_2)$. Then complex potential at any point $P(x, y)$ due to stationary system is

$$w = \frac{ik}{2\pi}\log(z - z_1) - \frac{ik}{2\pi}\log(z - z_2) \qquad \text{...(1)}$$

However, the vortices situated at A_1 and A_2 would start moving due to the presence of each other. Let u_1, v_1 be the components of the velocity q_1 of A_1 which is due to A_2 alone. Then,

$$u_1 - iv_1 = \left[\frac{1}{2\pi}\frac{ik}{z - z_1} + \left(-\frac{dw}{dz}\right)\right]_{z = z_1} = \frac{ik}{2\pi}\frac{1}{z_1 - z_2} \qquad \text{...(2)}$$

$$\therefore \qquad q_1 = |u_1 - iv_1| = \frac{k}{2\pi|z_1 - z_2|} = \frac{k}{2\pi(A_1A_2)} \qquad \text{...(3)}$$

Similarly,
$$u_2 - iv_2 = \frac{ik}{2\pi}\frac{1}{z_2 - z_1} \qquad \text{...(4)}$$

and
$$q_2 = k/2\pi(A_1A_2) \qquad \text{...(5)}$$

Let
$$q_1 = q_2 = q, \text{ (say)} \qquad \text{...(6)}$$

Thus the velocity q_1 of A_1 due to A_2 is q perpendicular to A_1A_2. Similarly, the velocity q_2 of A_2 due to A_1 is q perpendicular to A_2A_1 in the same sense as that of A_1. Hence the vortices situated at A_1 and A_2 move in the same direction perpendicular to A_1A_2 with uniform velocity q. However, the line may move forward or backward according to the directions of rotation.

Let $\quad w = \phi + i\psi, \qquad z = (x, y), \qquad z_1 = (x_1, y_1) \quad$ and $\quad z_2 = (x_2, y_2)$.

$$\therefore \ (1) \Rightarrow \ \phi + i\psi = (ik/2\pi)\log[(x - x_1) + i(y - y_1)] - (ik/2\pi)\log[(x - x_2) + i(y - y_2)]$$

Equating imaginary parts of both sides, we get

$$\psi = (k/4\pi)\log[(x - x_1)^2 + (y - y_1)^2] - (k/4\pi)\log[(x - x_2)^2 + (y - y_2)^2]$$

or
$$\psi = (k/4\pi)\,(\log r_1^2 - \log r_2^2) = (k/4\pi)\log(r_1/r_2)^2 = (k/2\pi)\log(r_1/r_2)$$

where $\qquad r_1 = A_1P \qquad$ and $\qquad r_2 = A_2P$.

The streamlines are given by $\quad \psi = \text{const.} \quad$ *i.e.* $\quad r_1/r_2 = \text{const.}$

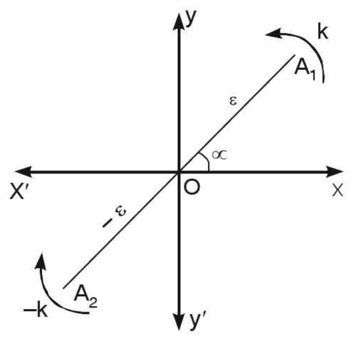
which clearly form a system of coaxial circles having A_1 and A_2 as their limiting points.

11.6B. Vortex doublet or dipole.

A vortex pair consisting of two vortices of strengths k and $-k$ at a distance δs apart, where $\delta s \to 0$ and $k \to \infty$ in such a manner that $\mu = \delta s \times (k/2\pi)$ is finite, is called a *vortex doublet* of *strength* μ. To simplify out working, we take $\delta s = 2\varepsilon$ where $\varepsilon \to 0$ as $\delta s \to 0$.

Thus
$$\mu = 2\varepsilon \times (k/2\pi) = (\varepsilon k)/\pi \qquad \text{...(1)}$$

Consider two vortex filaments of strengths k and $-k$ at $A_1 (z = \varepsilon e^{i\alpha})$ and $A_2 (z = -\varepsilon e^{i\alpha})$ so that $A_1 A_2 = 2\varepsilon$. Then $A_1 A_2$ will be the axis of the doublet, inclined at an angle α to the x-axis. The complex potential is given by



$$w = (ik/2\pi)[\log(z - \varepsilon e^{i\alpha}) - \log(z + \varepsilon e^{i\alpha})]$$

$$= \frac{ik}{2\pi}\left[\log\left(1 - \frac{\varepsilon e^{i\alpha}}{z}\right) - \log\left(1 + \frac{\varepsilon e^{i\alpha}}{z}\right)\right]$$

$$= \frac{ik}{2\pi}\left[-\frac{\varepsilon e^{i\alpha}}{z} - \frac{\varepsilon^2 e^{2i\alpha}}{z^2} - \frac{\varepsilon^3 e^{3i\alpha}}{z^3} - \dots - \left(\frac{\varepsilon e^{i\alpha}}{z} - \frac{\varepsilon^2 e^{2i\alpha}}{z^2} + \frac{\varepsilon^3 e^{3i\alpha}}{z^3} - \dots\right)\right]$$

$$= -\frac{ik}{2\pi}\cdot\frac{2\varepsilon e^{i\alpha}}{z} = -\frac{\varepsilon ik}{\pi}\frac{e^{i\alpha}}{re^{i\theta}}, \text{ as } z = re^{i\theta}$$

[To first order of approx. of small quantity ε]

Thus, $$w = -(\mu i/r)\, e^{i(\alpha-\theta)}, \text{ by (1)} \qquad \dots(2)$$

$\therefore$ $$\phi + i\psi = -(\mu i/r)[\cos(\alpha - \theta) + i\sin(\alpha - \theta)]$$

$\therefore$ $$\phi = (\mu/r)\sin(\alpha - \theta) \quad \text{and} \quad \psi = -(\mu/r)\cos(\alpha - \theta) \qquad \dots(3)$$

As a particular case, let the doublet be at O and the axis along y-axis. Then putting $\alpha = \pi/2$ in (3), we have

$$\phi = (\mu/r)\cos\theta \quad \text{and} \quad \psi = -(\mu/r)\sin\theta \qquad \dots(4)$$

Remark 1. If $\mu = Ua^2$, then (4) gives $\psi = -(Ua^2/r)\sin\theta$,

which is the stream function for a circular cylinder of radius a moving with velocity U along the x-axis. Thus the motion due to a circular cylinder is the same as that due to a suitable vortex doublet placed at the centre with axis perpendicular to the direction of motion.

Remark 2. Some authors define $\nu = 2\pi\mu$ as the strength of the vortex doublet. Then (4) takes the form

$$\phi = (\nu/2\pi r)\cos\theta \quad \text{and} \quad \psi = -(\nu/2\pi r)\sin\theta. \qquad \dots(5)$$

11.7. Motion of any vortex.

When there are any number of vortices in an infinite liquid, we can find the motion of any one of them. It depends not on itself but on others, hence to find the motion we have to subtract from the stream function of the system the term that corresponds to it.

Let there be a number of vortices of strengths $k_1, k_2, k_3, \dots$ situated at $z_1, z_2, z_3, \dots$ respectively, where $z_n = x_n + iy_n$. Then the complex potential of the system at any outside point is given by

$$w = (i/2\pi)\sum k_n \log(z - z_n) \quad \text{or} \quad \phi + i\psi = (i/2\pi)\sum k_n \log[(x - x_n) + i(y - y_n)]$$

or $$\phi + i\psi = \frac{i}{2\pi}\sum k_n\left[\frac{1}{2}\log\{(x - x_n)^2 + (y - y_n)^2\} + i\tan^{-1}\frac{y - y_n}{x - x_n}\right]$$

$\therefore$ $$\psi = \sum\frac{k_n}{4\pi}\log\{(x - x_n)^2 + (y - y_n)^2\}$$

$\therefore$ The stream function ψ' at the vortex (x_m, y_m) is given by

$$\psi' = \sum\frac{k_n}{4\pi}\log\{(x - x_n)^2 + (y - y_n)^2\} - \frac{k_m}{4\pi}\log\{(x - x_m)^2 + (y - y_m)^2\}$$

If χ be the stream function for the motion of the vortex (x_m, y_m), we have

$$-\frac{\partial \chi}{\partial y_m} = \left(-\frac{\partial \psi'}{\partial y}\right)_m \qquad \text{and} \qquad \frac{\partial \chi}{\partial x_m} = \left(\frac{\partial \psi'}{\partial x}\right)_m$$

by equating the components of velocity of the vortex (x_m, y_m).

Suppose there is a single vortex k at (x_1, y_1) in front of a fixed wall taken as $y = 0$.

We have to introduce the image $-k$ at $(x_1, -y_1)$ and the stream function of the system is

$$\psi = (k/4\pi)\log\left\{(x-x_1)^2 + (y-y_1)^2\right\} - (k/4\pi)\log\left\{(x-x_1)^2 + (y+y_1)^2\right\}$$

$$\therefore \qquad \psi' = -(k/4\pi)\log\left\{(x-x_1)^2 + (y+y_1)^2\right\}$$

$$\therefore \quad -\frac{\partial \chi}{\partial y_1} = \left(-\frac{\partial \psi'}{\partial y}\right)_1 = \left\{\frac{k}{4\pi}\frac{2(y+y_1)}{(x-x_1)^2+(y-y_1)^2}\right\}_{x=x_1,\, y=y_1} = \frac{k}{4\pi} \times \frac{4y_1}{4y_1^2} = \frac{k}{4\pi y_1}$$

and

$$\frac{\partial \chi}{\partial x_1} = \left(\frac{\partial \psi'}{\partial x}\right)_1 = \left\{-\frac{k}{4\pi}\,\frac{2(x-x_1)}{(x-x_1)^2+(y+y_1)^2}\right\}_{x=x_1,\, y=y_1} = 0$$

$$\therefore \qquad \chi = -(k/4\pi)\log y_1$$

Hence the path of the vortex is the streamline for the vortex, *i.e.* y_1 = constant.

11.8. Kirchhoff vortex theorem. General system of vortex filament. (Agra 2011)

If (r_1, θ_1), (r_2, θ_2), ..., (r_n, θ_n) *be the polar coordinates at any time t of a system of rectilinear vortices of strengths* $k_1, k_2 \ldots, k_n$ *then*

$$\sum_{p=1}^{n} k_p\, x_p = A, \qquad \sum_{p=1}^{n} k_p\, y_p = B, \qquad \sum_{P=1}^{n} k_p\, r_p^2 = C, \qquad \sum_{P=1}^{n} k_p\, r_p^2\, \dot{\theta}_p = D,$$

where A, B, C, D are constants and $\dot{\theta}_p = d\theta_p/dt$.

Proof. The complex potential w due to n vortex filaments of strengths k_p at the points $z_p = x_p + iy_p = r_p(\cos\theta_p + i\sin\theta_p)$ is given by

$$w = \sum_{P=1}^{n} \frac{ik_p}{2\pi}\log(z - z_p).$$

Hence, the velocity at any point of the fluid, not occupied by any vortex is given by

$$u - iv = -\frac{dw}{dz} = -\sum_{P=1}^{n} \frac{ik_p}{2\pi(z - z_p)}.$$

Since the velocity (u_p, v_p) of vortex k_p is produced by the remaining other vortices (because any particular vortex cannot move solely on its own account), we have

$$u_p - iv_p = \left(-\frac{dw_p}{dz}\right)_{z=z_p} = \left[-\frac{d}{dz}\sum_{q\neq p}\frac{ik_q}{2\pi}\log(z - z_q)\right]_{z=z_p} = -\sum_{q\neq p}\frac{ik_q}{2\pi(z_p - z_q)} \qquad \ldots(1)$$

Multiplying (1) by k_p and summing up from $p = 1$ to $p = n$, we obtain

$$\sum_{p=1}^{n} k_p(u_p - iv_p) = -\sum_{p=1}\sum_{q\neq p}\frac{ik_pk_q}{2\pi(z_p - z_q)} = 0, \qquad \ldots(2)$$

the double summation on R.H.S is zero because the terms cancel in pairs, for example $ik_p k_q/(z_p - z_q)$ cancels $ik_p k_q/(z_q - z_p)$ and there are no terms in k_p^2 etc.

Equating real and imaginary parts, (2) gives

$$\sum k_p u_p = 0 \quad \text{and} \quad \sum k_p v_p = 0$$

or
$$\sum k_p \frac{dx_p}{dt} = 0 \quad \text{and} \quad \sum k_p \frac{dy_p}{dt} = 0. \qquad \ldots(3)$$

Integrating (3), $\quad \sum k_p x_p = A \quad$ and $\quad \sum k_p y_p = B,$

where A and B are constants of integration.

Again, multiplying (1) by $k_p z_p$ and summing from $p = 1$ to $p = n$, we obtain

$$\sum_{p=1}^{n} k_p z_p (u_p - iv_p) = -\sum_{p=1}^{n} \sum_{q \neq p} \frac{ik_p k_q z_p}{2\pi(z_p - z_q)}$$

or
$$\sum_{p=1}^{n} k_p (x_p + iy_p)(u_p - iv_p) = -\frac{i}{2\pi}\sum_{p=1}^{n} \sum_{q \neq p} \frac{k_p k_q z_p}{z_p - z_q}$$

or
$$\sum_{p=1}^{n} k_p \left[(x_p u_p + y_p v_p) - i(x_p v_p - y_p u_p) \right] = -\frac{i}{2\pi}\sum k_p k_q, \qquad \ldots(4)$$

because in the double summation on R.H.S., the sum of pairs of terms such as $k_p k_q z_p/(z_p - z_q)$ and $k_p k_q z_q/(z_q - z_p)$ reduces to $k_p k_q$ and there are no terms in k_p^2.

Equating real and imaginary parts, (4) gives

$$\sum k_p (x_p u_p + y_p v_p) = 0 \qquad \ldots(5)$$

and
$$\sum k_p (x_p v_p - y_p u_p) = \frac{1}{2\pi}\sum k_p k_q = \text{constant} = D, \text{ say} \qquad \ldots(6)$$

Re-writing (5), we have $\quad \sum k_p \left[2x_p \frac{dx_p}{dt} + 2y_p \frac{dy_p}{dt} \right] = 0$

or $\quad \sum k_p \frac{d}{dt}(x_p^2 + y_p^2) = 0 \quad$ or $\quad \sum k_p \frac{dr_p^2}{dt} = 0 \qquad \left[\because\ r_p^2 = x_p^2 + y_p^2\right]$

Integrating, $\quad \sum k_p r_p^2 = C$, where C is constant of integration.

From (6), $$\sum k_p \left(x_p \frac{dy_p}{dt} - y_p \frac{dx_p}{dt} \right) = D \qquad \ldots(7)$$

But, $\quad y_p / x_p = \tan\theta_p \qquad \left[\because\ x_p = r_p \cos\theta_p \text{ and } y_p = r_p \sin\theta_p\right]$

Differentiating both sides of the above equation w.r.t. 't', we get

$$\frac{x_p \dfrac{dy_p}{dt} - y_p \dfrac{dx_p}{dt}}{x_p^2} = \sec^2\theta_p \frac{d\theta_p}{dt}$$

or
$$x_p \frac{dy_p}{dt} - y_p \frac{dx_p}{dt} = x_p^2 \sec^2\theta_p\, \dot{\theta}_p = r_p^2\, \dot{\theta}_p \qquad \left[\because\ x_p = r_p \cos\theta_p\right]$$

$\therefore$ (7) can be re-written as $\quad \sum k_p r_p^2\, \dot{\theta}_p = D.$

11.8A. Illustrative solved examples

Ex. 1. *Verify that the stream function* ψ *and velocity potential* ϕ *of a two-dimensional vortex flow satisfies the Laplace equation.*

Sol. We know that the stream function ψ and velocity potential ϕ of a two-dimensional vortex flow are given by

$$\phi = -(k\theta/2\pi) \quad ...(1)$$

and
$$\psi = (k/2\pi)\log r \quad ...(2)$$

From (1), $\partial\phi/\partial r = 0,\qquad \partial^2\phi/\partial r^2 = 0,\qquad \partial^2\phi/\partial\theta^2 = 0.$

so that
$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0. \quad ..(3)$$

Form (2), $\dfrac{\partial\psi}{\partial r} = \dfrac{k}{2\pi r},\qquad \dfrac{\partial^2\psi}{\partial r^2} = -\dfrac{k}{2\pi r^2},\qquad \dfrac{\partial^2\psi}{\partial\theta^2} = 0$

so that
$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0. \quad ...(4)$$

The equations (3) and (4) show that ϕ and ψ satisfy Laplace equation.

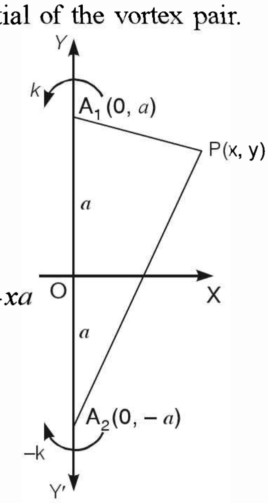
Ex. 2. (*a*) *When an infinite liquid contains two parallel, equal and opposite rectilinear vortices at a distance* $2a$, *prove that the streamlines relative to the vortex are given by the equation,*

$$\log\frac{x^2+(y-a)^2}{x^2+(y+a)^2} + \frac{y}{a} = c,$$

the origin being the middle point of the join, which is taken for axis of y. **[Kanpur 2009]**

(*b*) *Show that for a vortex pair the relative streamlines are given by* $k\{(y/2a) + \log(r_1/r_2)\}$ = *constant, where* $2a$ *is the distance between the vortices and* r_1, r_2 *are the distances of any point from them.*

Sol. Part (a) Let there be two rectilinear vortices of strengths k and $-k$ at A_1 ($z = 0 + ia$) and A_2 ($z = 0 - ia$) respectively. Thus $A_1A_2 = 2a$, origin being the middle point of A_1A_2 and y-axis being taken along A_1A_2 as shown in figure. Here we have a vortex pair and hence (by Art. 11.6) the vortex pair will move with a uniform velocity $k/(2\pi A_1A_2)$ or $k/4\pi a$ perpendicular to the line A_1A_2 (*i.e.* along the x-axis). To determine the streamlines relative to the vortices, we must impose a velocity on the given system equal and opposite to the velocity $k/4\pi a$ of motion of the vortex pair. Accordingly, we add a term $kz/4\pi a$ to the complex potential of the vortex pair. Note that



$$-\frac{d}{dz}\left(\frac{kz}{4\pi a}\right) = -\frac{k}{4\pi a},$$

and hence the term added is justified. So, for the case under consideration, the complex potential is given by

$$w = (ik/2\pi)\log(z-ia) - (ik/2\pi)\log(z+ia) + (kz/4\pi a)$$

or $\phi + i\psi = (ik/2\pi)\log\{x+i(y-a\} - (ik/2\pi)\log\{x+i(y+a)\} + k(x+iy)/4xa$

Equating the imaginary parts, we have

$$\psi = \frac{k}{4\pi}\log[x^2+(y-a)^2] - \frac{k}{4\pi}\log[x^2+(y+a)^2] + \frac{ky}{4\pi a}$$

or $$\psi = \frac{k}{4\pi}\left[\log\frac{x^2+(y-a)^2}{x^2+(y+a)^2}+\frac{y}{a}\right]. \qquad ...(1)$$

Hence the required relative streamlines are given by $\psi = \text{const.}$, that is,

$$\log\frac{x^2+(y-a)^2}{x^2+(y+a)^2}+\frac{y}{a}=c.$$

Part (b). As in part (*a*), do upto (1). Let $r_1 = A_1P$ and $r_2 = A_2P$ so that $r_1^2 = x^2+(y-a)^2$ and $r_2^2 = x^2+(y+a)^2$. Putting these in (1) of part (a), we obtain

$$\psi = \frac{k}{4\pi}\left[\log\frac{r_1^2}{r_2^2}+\frac{y}{a}\right]=\frac{k}{2\pi}\left(\log\frac{r_1}{r_2}+\frac{y}{2a}\right) \qquad ...(2)$$

Hence the relative streamlines are given by $\psi = \text{const.}$, *i.e.* $k\{(y/2a) + \log(r_1/r_2)\} = \text{const.}$

Ex. 3. *An infinite liquid contains two parallel, equal and opposite rectilinear vortex filaments at a distance 2a. Show that the paths of the fluid particles relative to the vortices can be represented by the equation*

$$\log\frac{r^2+a^2-2ar\cos\theta}{r^2+a^2+2ar\cos\theta}+\frac{r\cos\theta}{a}=\text{const.}$$ **[Kanpur 1997]**

Sol. To obtain the desired result, modify solution of example 2(*a*) as follows : Let the vortex pair lie along *x*-axis in place of *y*-axis. Then interchanging *x* and *y*, we obtain

$$\psi = \frac{k}{4\pi}\left[\log\frac{y^2+(x-a)^2}{y^2+(x+a)^2}+\frac{x}{a}\right] \quad \text{or} \quad \psi = \frac{k}{4\pi}\left[\log\frac{x^2+y^2+a^2-2ax}{x^2+y^2+a^2+2ax}+\frac{x}{a}\right]. \qquad ...(1)$$

Let $x = r\cos\theta$, $y = r\sin\theta$. Then, in polar coordinates, (1) takes the form

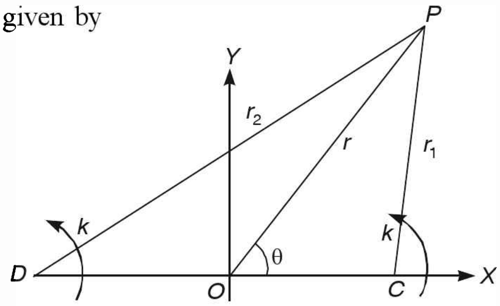
$$\psi = \frac{k}{4\pi}\left[\log\frac{r^2+a^2-2ar\cos\theta}{r^2+a^2+2ar\cos\theta}+\frac{r\cos\theta}{a}\right]. \qquad ...(2)$$

Hence the relative streamlines are given by $\psi = \text{const.}$,

i.e. $$\log\frac{r^2+a^2-2ar\cos\theta}{r^2+a^2+2ar\cos\theta}+\frac{r\cos\theta}{a}=\text{const.}$$

Ex. 4. *If two vortices are of the same strength and the spin is the same in both, show that the relative stream lines are given by* $\log(r^4+a^4-2a^2r^2\cos 2\theta) - (r^2/2a) = \text{const.}$, θ being measured from the join of vortices, the origin being its middle point, $2a$ being the distance between the vortices.

Sol. Let there be two vortices each of strength *k* at *C* (*a*, 0) and *D* (– *a*, 0) and let *O*, the middle point of CD be taken as the origin as shown in figure. Then, by case I of Art. 11.5, line *DC* revolves about *O* with uniform angular velocity ω given by



$$\omega = \frac{k+k}{2\pi(A_1A_2)^2}=\frac{2k}{2\pi(2a)^2}=\frac{k}{4\pi a^2} \qquad ...(1)$$

Let $P(r, \theta)$ be any point in the fluid. Then,

$$CP = r_1 = (r^2+a^2-2ar\cos\theta)^{1/2}$$

and $$DP = r_2 = (r^2+a^2+2ar\cos\theta)^{1/2}$$

Corresponding to the given vortices at C and D, the stream function ψ is given by

$$\psi = \frac{k}{2\pi}\log r_1 + \frac{k}{2\pi}\log r_2 = \frac{k}{2\pi}\log(r_1 r_2) = \frac{k}{2\pi}\log\left[(r^2+a^2-2ar\cos\theta)(r^2+a^2+2ar\cos\theta)\right]^{1/2}$$

$$= \frac{k}{4\pi}\log[(r^2+a^2)^2 - 4a^2r^2\cos^2\theta] = \frac{k}{4\pi}\log[r^4+a^4-2a^2r^2(2\cos^2\theta-1)]$$

Thus, $$\psi = (k/4\pi)\times\log(r^4+a^4-2a^2r^2\cos 2\theta) \qquad ...(2)$$

Since the vortices are moving with angular velocity ω about O, there would be linear velocity due to vortex system. To determine the streamlines relative to the vortices, we must impose velocity $-\omega r$ on the given system so as to reduce to the whole system to rest. If ψ' be the stream function to produce the desired velocity, then we have

$$\frac{\partial\psi'}{\partial r} = -\omega r = -\frac{kr}{4\pi a^2}, \text{ by (1)} \qquad \text{so that} \qquad \psi' = -\frac{kr^2}{8\pi a^2}. \qquad ...(3)$$

Hence for the case under consideration, the stream function $\psi''(=\psi+\psi')$ is given by

$$\psi'' = (k/4\pi)\left[\log(r^4+a^4-2a^2r^2\cos 2\theta) - r^2/2a^2\right]. \qquad ...(4)$$

Hence the required streamlines are given by $\psi = \text{const.}$,

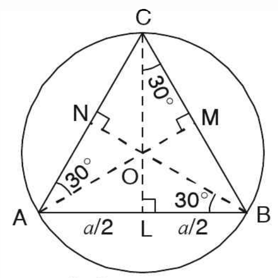
i.e. $$\log(r^4+a^4-2a^2r^2\cos 2\theta) - (r^2/2a^2) = \text{const.}$$

Ex. 5. *Three parallel rectilinear vortices of the same strength k and in the same sense meet any plane perpendicular to them in an equilateral triangle of side a. Show that the vortices all move round the same cylinder with uniform speed in time* $(4\pi^2a^2)/3k$. **(Kanpur 2011)**

Sol. Let r be the radius of the circumcircle of the equilateral triangle ABC. Let O be the circumcentre. From figure, $r = OB = (a/2)\sec 30° = a/\sqrt{3}$. There are three vortices of strength K at A, B, C which are situated at the points $z_m = re^{2m\pi i/3}$, $m = 1, 2, 3$. Then the complex potential of the vortices at B, C, A, is given by

$$w = (ik/2\pi)\times[\log(z-re^{2i\pi/3}) + \log(z-re^{4\pi i/3}) + \log(z-re^{6\pi i/3})] = (ik/2\pi)\times\log(z^3-r^3)$$

Then the velocity induced at $z = re^{6\pi i/3} = r$, by others is given by



$$u_1 - iv_1 = -\frac{d}{dz}\left[\frac{ik}{2\pi}\log(z^3-r^3) - \frac{ik}{2\pi}(z-r)\right] = -\frac{ik}{2\pi}\,\frac{2z+r}{z^2+zr+r^2}$$

Thus, $$q_1 = |u_1 - iv_1| = \frac{k}{2\pi}\left[\frac{2z+r}{z^2+zr+r^2}\right]_{z=r} = \frac{k}{2\pi r}$$

$$\text{The required time} = \frac{\text{circumference of the circumcircle}}{\text{velocity at } z=r} = \frac{2\pi a/\sqrt{3}}{k/2\pi r} = \frac{4\pi^2 ra}{k\sqrt{3}} = \frac{4\pi^2a^2}{3k}, \textit{ as } r = \frac{a}{\sqrt{3}}$$

Ex. 6. *If n rectilinear vortices of the same strength k are symmetrically arranged as generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time* $8\pi^2a^2/(n-1)k$, *and find the velocity of any part of the liquid.*

[Kanpur 2005, 09; Rohilkhand 2000; Nagpur 2003, 05; Himanchal 1999, 2003; Kurukshetra 1999]

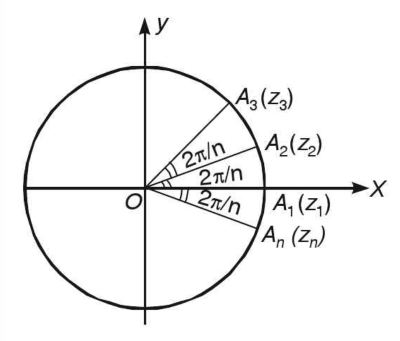
Sol. Let $A_1 A_2 A_3$ be the circle of radius a. Suppose that n rectilinear vortices each of strength k be situated at points $z_m = ae^{2\pi im/n}$, $m = 0, 1, 2, \ldots, n-1$ of the circle. Then the complex potential due to these n vortices is given by

$$w = \frac{ik}{2\pi}\sum_{m=0}^{n-1}\log(z - ae^{2\pi im/n}) = \frac{ik}{2\pi}\log\prod_{m=0}^{n-1}(z - ae^{2\pi im/n}) = \frac{ik}{2\pi}\log(z^n - a^n).$$

[using a well known result of algebra]

Now the fluid velocity q at any point out of all the n vortices is given by

$$q = \left|\frac{dw}{dz}\right| = \left|\frac{ik}{2\pi}\frac{nz^{n-1}}{z^n - a^n}\right| = \frac{kn}{2\pi}\left|\frac{z^{n-1}}{z^n - a^n}\right|$$



Again the velocity induced at A_1 $(z = a)$, by others is given by the complex potential

$$w' = \frac{ik}{2\pi}\log(z^n - a^n) - \frac{ik}{2\pi}\log(z - a) = \frac{ik}{2\pi}\log\frac{z^n - a^n}{z - a}$$

$$\therefore \quad w' = (ik/2\pi)\log(z^{n-1} + z^{n-2}a + \ldots + za^{n-2} + a^{n-1})$$

so that
$$\frac{dw'}{dz} = \frac{ik}{2\pi}\frac{(n-1)z^{n-2} + (n-2)z^{n-2}a + \ldots + a^{n-2}}{z^{n-1} + z^{n-2}a + \ldots + za^{n-2} + a^{n-1}}$$

$$\therefore \quad \left(\frac{dw'}{dz}\right)_{z=a} = \frac{ik}{2\pi}\frac{(n-1) + (n-2) + \ldots + 2 + 1}{na} = \frac{ik(n-1)}{4\pi a}$$

[$\because$ By algebra $(n-1) + (n-2) + \ldots + 2 + 1 = \{(n-1)/2\} \times \{(n-1) + 1] = n(n+1)/2$

or
$$u_1 - iv_1 = \left(-\frac{dw'}{dz}\right)_{z=a} = -\frac{ik(n-1)}{4\pi a},$$

so that $u_1 = 0$ and $v_1 = k(n-1)/4\pi a$. If q_r and q_θ be the radial and transverse velocity components of the velocity at $z = a$, then we have $q_r = 0$ and $q_\theta = k(n-1)/4\pi a$. Due to symmetry of the problem, it follows that each vortex moves with the same transverse velocity $k(n-1)/4\pi a$. Hence the required time T is given by
$$T = \frac{2\pi a}{k(n-1)/4\pi a} = \frac{8\pi^2 a^2}{(n-1)k}.$$

Ex. 6 (*b*). *Prove that if n rectangular vortices of equal strength k are symmetrically arranged as generators of a right cicular cylinder of radius a and infinite length in an incompressible liquid, then the two dimensional motion of the liquid is given by* $w = (ik/2\pi)\log(z^n - a^n)$, *the origin of co-ordinates being the centre of the cross-section of the cylinder. Show that the vortices move round the cylinder with speed* $(n-1)k/4\pi a$. **[G.N.D.U. Amritsar, 1999, 2001]**

Sol. Proceed as in Ex. 6 (*a*)

Ex. 7. *If* $(r_1, \theta_1), (r_2, \theta_2), \ldots,$ *be polar coordinates at time t of a system of rectilinear vortices of strength* k_1, k_2, ... *prove that* $\sum kr^2 = const.$ *and* $\sum kr^2\dot{\theta} = (1/2\pi)\sum k_1 k_2$.

[Agra 2003, 10; Allahabad 2000; Jadavpur 2003, 04; Kanpur 2001; Mumbai 1999; Rohilkhand 2000]

Sol. Refer remark 2 of Art. 11.4. We have

$$\left.\begin{aligned} \dot{x}_m &= -\frac{1}{2\pi}\sum_n k_n \frac{y_m - y_n}{r_{mn}^2} \\ \dot{y}_m &= \frac{1}{2\pi}\sum_n k_n \frac{x_m - x_n}{r_{mn}^2} \end{aligned}\right\}, \quad m \neq n \qquad \text{... (1)}$$

where $$r_{mn}^2 = (x_m - x_n)^2 + (y_m - y_n)^2. \qquad \text{... (2)}$$

$$\therefore \quad \sum_m k_m (x_m \dot{x}_m + y_m \dot{y}_m) = -\frac{1}{2\pi}\sum_m \sum_n k_n k_m \frac{x_m (y_m - y_n) - y_m (x_m - x_n)}{r_{mn}^2}, \text{ using (1)}$$

$$= -\frac{1}{2\pi}\sum_m \sum_n k_m k_n \frac{x_n y_m - x_m y_n}{r_{mn}^2} = 0, \text{ since } m \text{ and } n \text{ can be interchanged.}$$

$$\therefore \quad \sum_m k_m \frac{d}{dt}(x_m^2 + y_m^2) = 0 \qquad \text{or} \qquad \sum_m k_m \frac{dr_m^2}{dt} = 0.$$

$\therefore$ Integrating, $\sum_m k_m r_m^2 = \text{const.}$, *i.e.* $\sum kr^2 = \text{const.}$

Also, $$\sum_m k_m (x_m \dot{y}_m - y_m \dot{x}_m) = \frac{1}{2\pi}\sum_m \sum_n k_m k_n \frac{x_m (x_m - x_n) + y_m (y_m - y_n)}{r_{mn}^2}, \text{ using (1)}$$

$$= \frac{1}{2\pi}\sum_m \sum_n k_m k_n \qquad \left[\begin{array}{l} \text{Since interchanging } m, n, \text{ we get} \\ x_m (x_m - x_n) + y_m (y_m - y_n) + x_n (x_n - x_m) \\ + y_n (y_n - y_m) = (x_m - x_n)^2 + (y_m - y_n)^2 = r_{mn}^2, \\ \text{using(2)} \end{array}\right]$$

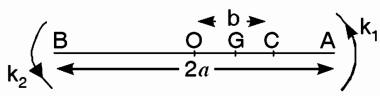
$$\therefore \quad \sum k_m r_m^2 \frac{d\theta_m}{dt} = \frac{1}{2\pi}\sum_m \sum_n k_m k_n, \qquad \text{as} \qquad x\dot{y} - \dot{x}y = r^2\dot{\theta}$$

i.e. $$\sum kr^2 \dot{\theta} = \left(\frac{1}{2\pi}\right)\sum k_1 k_2.$$

Ex. 8. *Two parallel rectilinear vortices of strengths* k_1 *and* k_2 $(k_1 > k_2)$ *are at a distance* $2a$ *apart in an infinite mass of liquid. If the vortices intersect a plane perpendicular to their length at points A and B, show that the point on AB at a distance b from the mid-point on the same side of mid-point as the vortex of strength* k_1*, is always occupied by the same fluid element if .* $(k_1 - k_2)/(k_1 + k_2) = (b^3 - 5a^2 b)/(ab^2 + 3a^3)$.

Sol. In what follows, we shall need results (7) and (9) of Art. 11.8.

If G be the centre of vertex, then $$AG = \frac{k_2}{k_1 + k_2} AB = \frac{2ak_2}{k_1 + k_2}$$



Similarly, $BG = (2ak_1)/(k_1 + k_2)$

Let O be the mid-point of AB. Let $OC = b$.

$$\therefore \quad OG = OA - AG = a - \frac{2ak_2}{k_1 + k_2} = \frac{k_1 - k_2}{k_1 + k_2} a$$

and $$GC = OC - OG = b - \frac{k_1 - k_2}{k_1 + k_2} a = \frac{(k_1 + k_2) b - (k_1 - k_2) a}{k_1 + k_2}$$

Now, the line BA rotates about G with angular velocity

$$= \frac{k_1 + k_2}{2\pi \cdot AB^2} = \frac{k_1 + k_2}{8\pi a^2}, \quad \text{as} \quad AB = 2a.$$

Hence the velocity of the point C of AB

$$= GC \cdot \frac{k_1 + k_2}{8\pi a^2} = \frac{(k_1 + k_2) b - (k_1 - k_2) a}{8\pi a^2} \qquad ...(1)$$

Suppose that O is origin and OA produced is x-axis. Then, we have two rectilinear vortices of strengths k_1 and k_2 at A ($z = z_1 = a$) and B ($z = z_2 = -a$). Hence, the complex potential at any point P ($z = x + iy$) is given by

$$w = (i/2\pi)\left[k_1 \log (z - z_1) + k_2 \log (z - z_2)\right] = (i/2\pi)\left[k_1 \log (z - a) + k_2 \log (z + a)\right]$$

$$\therefore \quad u - iv = -\frac{dw}{dz} = -\frac{i}{2\pi}\left[\frac{k_1}{z - a} + \frac{k_2}{z + a}\right] = -\frac{i}{2\pi}\left[\frac{(k_1 + k_2) z + k_1 a - k_2 a}{(z - a)(z + a)}\right] \qquad ...(3)$$

Using (3) for the point C for which $z = b$, we have

$$u - iv = -\frac{1}{2\pi}\left[\frac{(k_1 + k_2) b + k_1 a - k_2 a}{(b - a)(b + a)}\right],$$

giving $u = 0,$ and $$v = \frac{1}{2\pi} \frac{(k_1 + k_2) b + (k_1 - k_2) a}{b^2 - a^2}$$

$$\therefore \quad \text{Velocity at } C = \sqrt{(u^2 + v^2)} = \frac{1}{2\pi} \frac{(k_1 + k_2) b + (k_1 - k_2) a}{b^2 - a^2} \qquad ...(4)$$

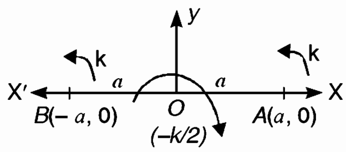
By the given problem, relative velocity of two C's must be zero and so the two different expressions (1) and (4) must be the same, *i.e.*,

$$\frac{(k_1 + k_2) b - (k_1 - k_2) a}{8\pi a^2} = \frac{1}{2\pi} \frac{(k_1 + k_2) b + (k_1 - k_2) a}{b^2 - a^2}$$

or $$(k_1 - k_2)\,[4a^3 + a\,(b^2 - a^2)] = (k_1 + k_2)\,[b\,(b^2 - a^2) - 4a^2 b]$$

$$\therefore \quad (k_1 - k_2)/(k_1 + k_2) = (b^3 - 5a^2 b)/(ab^2 + 3a^3)$$

Ex. 9. *Two point vortices each of strength k are situated at* ($\pm$ *a*, 0) *and a point vortex of strength –k/2 is situated at the origin. Show that the fluid motion is stationary and find the equations of streamlines. Show that the streamline which passes through the stagnation points meet the x-axis at* ($\pm$ *b*, 0) *where,* $3\sqrt{3}\,(b^2 - a^2)^2 = 16a^3 b.$



Sol. The complex potential of the fluid motion is given by

$$w = (ik/2\pi)\log (z - a) + (ik/2\pi)\log (z + a) - (ik/4\pi)\log z$$

$$\therefore \quad w = (ik/2\pi) \times [\log (z^2 - a^2) - (1/2) \times \log z] \qquad ...(1)$$

Now, for the motion of the vortex A (a, 0) the complex potential is given by

$$w' = w - \frac{ik}{2\pi}\log (z - a) = \frac{ik}{2\pi}\left[\log (z^2 - a^2) - \frac{1}{2}\log z - \log (z - a)\right]$$

$\therefore \qquad w' = (ik/2\pi)\left[\log(z+a) - (1/2)\times\log z\right]$ ...(2)

The velocity (u_A, v_A) of the vortex K at A is produced solely by the other vertices as it will not move on its own account. Its velocity is therefore given by

$$u_A - iv_A = \left(-\frac{dw'}{2\pi}\right)_{z=a} = -\frac{ik}{2\pi}\left[\frac{1}{z+a} - \frac{1}{2z}\right]_{z=a}, \text{ using (2)}$$

Thus, $u_A - iv_A = 0$. Therefore, $u_A = v_A = 0$, showing that A is stationary. Similarly O and B are stationary. Hence the fluid motion is stationary.

Determination of streamlines. From (1), we have

$$\phi + i\psi = \frac{ik}{2\pi}\left[\log\left\{(x+iy)^2 - a^2\right\} - \frac{1}{2}\log(x+iy)\right] = \frac{ik}{2\pi}\left[\log(x^2 - y^2 - a^2 + 2ixy) - \frac{1}{2}\log(x+iy)\right]$$

$$\psi = (k/2\pi)\left[(1/2)\times\log\left\{(x^2 - y^2 - a^2)^2 + 4x^2y^2\right\} - (1/4)\times\log(x^2+y^2)\right] \quad ...(3)$$

Hence stream lines are given by $\qquad \psi = \text{constant} = (k/4\pi)\log C$

or $$\frac{k}{4\pi}\log\frac{(x^2 - y^2 - a^2)^2 + 4x^2y^2}{(x^2+y^2)^{1/2}} = \frac{k}{4\pi}\log C, \text{ using (3)}$$

or $$(x^2 - y^2 - a^2)^2 + 4x^2y^2 = C\,(x^2+y^2)^{1/2}$$

or $$(x^2 - y^2)^2 + a^4 - 2a^2(x^2 - y^2) + 4x^2y^2 = C\,(x^2+y^2)^{1/2}$$

or $$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) + a^4 = C\,(x^2+y^2)^{1/2}, \quad ...(4)$$

which are the required streamlines, C being an arbitrary constant.

Determination of stagnation points.

From (1), $$\frac{dw}{dz} = \frac{ik}{2\pi}\left[\frac{2z}{z^2 - a^2} - \frac{1}{2z}\right]. \quad ...(5)$$

Required stagnation points are given by $\qquad dw/dz = 0$

or $$\frac{ik}{2\pi}\left[\frac{2z}{z^2 - a^2} - \frac{1}{2z}\right] = 0, \text{ using (5)}$$

or $\qquad 3z^2 + a^2 = 0 \qquad$ so that $\qquad z = \pm ia/\sqrt{3}$

Hence $(0,\ a/\sqrt{3})$ and $(0,\ -a/\sqrt{3})$ are stagnation points. Since the streamlines (4) pass through $(0,\ \pm a/\sqrt{3})$, we have

$(a^2/3)^2 - 2a^2\times(-a^2/3) + a^4 = (Ca)/\sqrt{3} \qquad$ giving $\qquad C = \left(16\sqrt{3}a^3\right)/9$

Hence the streamlines (4) reduces to

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) + a^4 = \left(6\sqrt{3}a^3/9\right)\times(x^2+y^2)^{1/2} \quad ...(6)$$

Again, streamline (6) also passes through ($\pm\, b$, 0) and hence

$$b^4 - 2a^2b^2 + a^4 = \frac{16\sqrt{3}\,a^3b}{9} \qquad \text{or} \qquad (b^2 - a^2)^2 = \frac{16a^3b}{3\sqrt{3}}$$

or $$3\sqrt{3}\,(b^2 - a^2)^2 = 16a^3b.$$

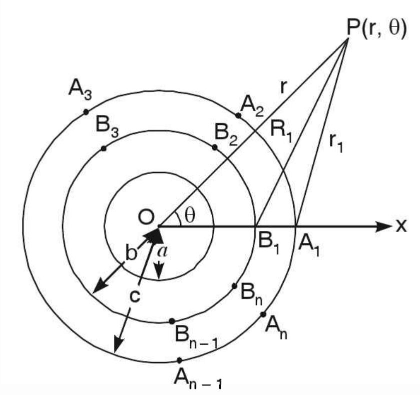
Ex. 10. *A fixed cylinder of radius a is surrounded by incompressible homogeneous fluid extending to infinity. Symmetrically arranged round it as generators an a cylinder of radius c (c > a) co-axial with the given one are n straight parallel vortex filaments each of strength k.*

Show that the filaments will remain on this cylinder throughout the motion and revolve round its axis with angular velocity $\dfrac{k}{4\pi c^2}\dfrac{(n+1)\,c^{2n}+(n-1)\,a^{2n}}{c^{2n}-a^{2n}}$, *where* $a^2 = bc$.

Find also the velocity at any point of the fluid.

Sol. Let the given vortices of the same strength k be situated at $A_1, A_2, \ldots, A_n$ and arranged symmetrically round a circle of radius c ($c > a$). Then, by geometry, we have

$$\angle A_1OA_2 = \angle A_2OA_3 = \ldots = \angle A_{n-1}OA_n = 2\pi/n.$$



Let $A_1P = r_1$, $A_2P = r_2$, $\ldots$, $A_nP = r_n$, $OP = r$ and $\angle POX = \theta$, OA_1X being taken as initial line. In order that the cylinder of radius a may be a streamline we must place vortices each of strength $-k$ at $B_1, B_2, \ldots, B_n$, the inverse points of $A_1, A_2, \ldots, A_n$ with respect to the circle of radius a. Hence, we have

$OA_i \cdot OB_i = a^2$ for $i = 1, 2, 3, \ldots, n$.

Let $OB_i = b$ for $i = 1, 2, 3, \ldots, n$.

So $bc = a^2$ and hence the points $B_1, B_2, \ldots, B_n$ lie an a circle of radius b.

Let $B_iP = R_i$ for $i = 1, 2, 3, \ldots, n$.

From ΔOPA_1, $$r_1^2 = r^2 + c^2 - 2rc\cos\theta \qquad \ldots(1)$$

From ΔOPB_1, $$R_1^2 = r^2 + b^2 - 2rb\cos\theta \qquad \ldots(2)$$

Similarly, $$r_2^2 = r^2 + c^2 - 2rc\cos(\theta + 2\pi/n) \qquad \ldots(1)'$$

$$R_2^2 = r^2 + b^2 - 2rb\cos(\theta + 2\pi/n) \qquad \ldots(2)'$$

and so on

The stream function ψ at $P(r, \theta)$ due to the given vortices and their images is given by

$$\psi = \frac{k}{2\pi}(\log r_1 + \ldots + \log r_n) - \frac{k}{2\pi}(\log R_1 + \ldots + \log R_n) = \frac{k}{2\pi}\log(r_1 r_2 \ldots r_n) - \frac{k}{2\pi}\log(R_1 R_2 \ldots R_n)$$

[Using results (13) of Art. 11.4]

$$= \frac{k}{4\pi}\log(r_1 r_2 \ldots r_n)^2 - \frac{k}{4\pi}\log(R_1 R_2 \ldots R_n)^2 = \frac{k}{4\pi}\log(r_1^2 r_2^2 \ldots r_n^2) - \frac{k}{4\pi}\log(R_1^2 R_2^2 \ldots R_n^2)$$

$$= (k/4\pi)\log\left[(r^2 + c^2 - 2rc\cos\theta)\left\{r^2 + c^2 - 2rc\cos\left(\theta + 2\pi/n\right)\right\}\ldots\right]$$

$$-(k/4\pi)\log\left[(r^2 + b^2 - 2rb\cos\theta)\left\{r^2 + b^2 - 2rb\cos\left(\theta + 2\pi/n\right)\right\}\ldots\right]$$

[Using (1), (2), (1)′, (2)′ etc ...]

$$= \frac{k}{4\pi}[\log(r^{2n} + c^{2n} - 2r^n c^n \cos n\theta) - \log(r^{2n} + b^{2n} - 2r^n b^n \cos n\theta)]$$

$$= \frac{k}{4\pi}\log\frac{r^{2n} + c^{2n} - 2r^n c^n \cos n\theta}{r^{2n} + b^{2n} - 2r^n b^n \cos n\theta}$$

The motion of a particular vortex A_1 is produced soley by the other vortices as it will not move on its own account. To find the motion of the vortex A_1, we take

$$\psi' = \psi - (k/4\pi)\log(r^2 + c^2 - 2rc\cos\theta)$$

or $\quad \psi' = (k/4\pi)[\log(r^{2n} + c^{2n} - 2r^n c^n \cos n\theta) - \log(r^{2n} + b^{2n} - 2r^n b^n \cos n\theta)$

$$- \log(r^2 + c^2 - 2rc\cos\theta)] \qquad ...(3)$$

If χ be the stream function for the motion of A_1, we have

$$-\frac{\partial\chi}{\partial c} = \left(-\frac{\partial\psi'}{\partial r}\right)_{r=c,\,\theta=0} \quad \text{and} \quad \left(\frac{\partial\chi}{c\partial\theta}\right)_{\theta=0} = \left(\frac{\partial\psi'}{r\partial\theta}\right)_{r=c,\,\theta=0}$$

Now $\displaystyle \frac{\partial\psi'}{\partial r} = \frac{k}{4\pi}\left[\frac{2nr^{2n-1} - 2nr^{n-1}c^n\cos n\theta}{r^{2n} + c^{2n} - 2r^n c^n \cos n\theta} - \frac{2nr^{2n-1} - 2nr^{n-1}b^n \cos n\theta}{r^{2n} + b^{2n} - 2r^n b^n \cos n\theta} - \frac{2r - 2c\cos\theta}{r^2 + c^2 - 2rc\cos\theta}\right]$, by (3)

$$\therefore \quad \left(\frac{\partial\psi'}{\partial r}\right)_{r=c} = \frac{k}{4\pi}\left[\frac{2nc^{2n-1}}{2c^{2n}} - \frac{2nc^{n-1}(c^n - b^n\cos n\theta)}{c^{2n} + b^{2n} - 2c^n b^n \cos n\theta} - \frac{1}{c}\right]$$

$$\therefore \quad \left(\frac{\partial\psi'}{\partial r}\right)_{r=c,\,\theta=0} = \frac{k}{4\pi}\left[\frac{n-1}{c} - \frac{2nc^{n-1}(c^n - b^n)}{(c^n - b^n)^2}\right] = \frac{k}{4\pi}\left[\frac{n-1}{c} - \frac{2nc^{n-1}}{c^n - b^n}\right]$$

$$= \frac{k}{4\pi}\left[\frac{n-1}{c} - \frac{2nc^{n-1}}{c^n - (a^{2n}/c^n)}\right], \quad \text{as} \quad bc = a^2 \quad \Rightarrow \quad b^n = a^{2n}/c^n$$

$$= \frac{k}{4\pi}\left[\frac{n-1}{c} - \frac{2nc^{2n-1}}{c^{2n} - a^{2n}}\right] = \frac{k}{4\pi}\frac{(n-1)(c^{2n} - a^{2n}) - 2nc^{2n}}{c(c^{2n} - a^{2n})}$$

$$= -\frac{k}{4\pi c}\frac{(n+1)c^{2n} + (n-1)a^{2n}}{c^{2n} - a^{2n}} = v, \text{ say} \qquad ...(4)$$

Again, from (3), we have $\quad \displaystyle \left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)_{r=c,\,\theta=0} = 0 \qquad ...(5)$

Hence the velocity is tangential and of amount v.

$$\therefore \quad \text{angular velocity } = \frac{v}{c} = \frac{k}{4\pi c^2}\frac{(n+1)c^{2n} + (n-1)a^{2n}}{c^{2n} - a^{2n}}, \text{ using (4)}$$

For velocity v at any point, we have $\quad \displaystyle v^2 = \left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)^2.$

Now, complete the solution yourself.

EXERCISE 11 (B)

1. If a rectilinear vortex moves in two dimensions in fluid bounded by a fixed plane, prove that a streamline can never coincide with a line of constant pressure.

2. When an infinite liquid contains two parallel equal rectilinear vortices at a distance $2a$ apart with the spin in the same sense, show that the relative streamlines are given by

$$\log(r^4 + a^4 - 2a^2r^2\cos 2\theta) - (r^2/2a^2) = \text{const.},$$

θ being measured from the join of vortices, the origin being its middle point. Show that the surfaces of equi-pressure at any instant are given by $r^4 + a^4 - 2a^2r^2\cos 2\theta = \lambda(r^2\cos^2 2\theta + a^2)$.

3. Investigate the motion of two infinitely long parallel straight line vortices of the same strength, in infinite liquid. Prove that the equation of the streamlines of the liquid relative to moving axes, so chosen that the coordinates of the vortices are (± c, 0) is

$$\log\{(x+c)^2+y^2\}\{(x-c)^2+y^2\}(x^2+y^2)/2c^2 = \text{const.}$$

4. Find the stream function due to a single line-vortex of strength k.

When an infinite liquid contains two parallel and equal rectilinear vortices at a distance $2b$ apart, prove that the streamlines relative to the vortices are given by the equation

$$x^2+y^2-2b^2 \log [(x+b)^2+y^2] [(x-b)^2+y^2] = c,$$

the origin being the middle point of the join which is taken for the axis of x.

5. Two parallel line vortices of strengths k_1, k_2 $(k_1+k_2 \neq 0)$, in unlimited liquid across the z-plane at right angles at points A, B respectively, the centre of mass of masses k_1 at A and k_2 at B is G. Show that if the motion of the liquid is due solely to these vortices, G is a fixed point about which A and B move in circles with angular velocity $(k_1+k_2)/AB^2$. Show also that the speed at any point P in the z-plane is $(k_1+k_2)\, CP/(AP.\, BP)$, whence C is the centre of masses k_2 at A and k_1 at B.

[**Hint :** Let given vortices of strengths k_1 and k_2 be situated at A $(z = z_1)$ and B $(z = z_2)$ respectively. Let P (z) be any point and C $(z = z_3)$ be given point. Then, we have (To get the result in desired form, we use result (18) of page 11.7)

$$w = ik_1 \log (z-z_1) + ik_2 \log (z-z_2) \qquad \text{so that} \qquad dw/dz = ik_1/(z-z_1) + ik_2/(z-z_2)$$

or
$$\frac{dw}{dz} = \frac{i(k_1+k_2)\left[z-\dfrac{k_1z_1+k_2z_2}{k_1+k_2}\right]}{(z-z_1)(z-z_2)} = \frac{i(k_1+k_2)(z-z_3)}{(z-z_1)(z-z_2)} \qquad \text{...(1)}$$

Also $\quad |z-z_1| = AP, \qquad |z-z_2| = BP \qquad$ and $\qquad |z-z_3| = CP.$

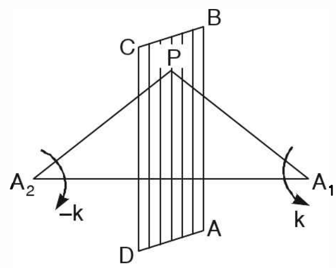
$\therefore$ The required speed $= \left|\dfrac{dw}{dz}\right| = \dfrac{(k_1+k_2)CP}{AP\cdot BP}.$ using (1)

11.9. Image of a vortex filament in a plane. (Agra 2011)

To show that the image of a vortex filament in a plane to which it is parallel is an equal and opposite vortex filament at its optical image in the plane.

Proof. Let the vortex filaments of strength k and $-k$ be situated at $A_1(z = z_1)$ and $A_2(z = z_2)$ respectively. Then, complex potential due to these vortices at any point P(z) is given by

$$w = (ik/2\pi)\log(z-z_1) - (ik/2\pi)\log(z-z_2)$$



$$\therefore \qquad \phi + i\psi = (ik/2\pi)[\log(r_1e^{i\theta_1}) - \log(r_2 e^{i\theta_2})], \qquad \text{...(1)}$$

where $r_1 = |z-z_1|$, $\theta_1 = \arg(z-z_1)$, $r_2 = |z-z_2|$, $\theta_2 = \arg(z-z_2)$.

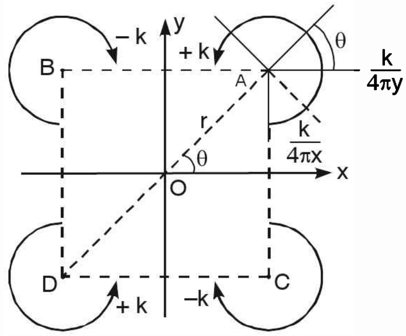
Equating imaginary parts, $\quad \psi = \dfrac{ik}{2\pi}\log\dfrac{r_1}{r_2}. \qquad$ ...(2)

Let $ABCD$ be a plane bisecting A_1A_2 at right angles and let P be any point on it. Then $r_1 = r_2$ on P so that $\psi = 0$ from (2). Thus there would be no flow across the plane AB. Hence the motion would remain unchanged if the plane were made a rigid barrier. This proves the required result.

Remark. Let $A_1A_2 = 2a$. Then the uniform velocity of vortex filament A_1 parallel to the plane AB (induced by B) is given by
$$\left|\frac{dw}{dz}\right| = \left|\frac{ik}{2\pi}\,\frac{1}{z_1-z_2}\right| = \frac{k}{2\pi A_1A_2} = \frac{k}{4\pi a}. \qquad \text{...(3)}$$

Moreover the velocity midway A_1 and A_2 due to both the vortices is $k/\pi a$. Thus the vortex moves parallel to the plane with one-fourth of the velocity of the liquid at the boundary.

11.10. Image of vortex in a quadrant.



The image system of vortex of strength k, at the point A (x, y) in xy-plane with respect to quadrant XOY consists of (*i*) a vortex of strength $-k$ at B $(-x, y)$

(*ii*) a vortex of strength – k at C $(x, -y)$

(*iii*) a vortex of strength k at D $(-x, -y)$

The velocity at A is only on account of its images and hence its components (making use of remark of Art. 11.9) are as indicated in the figure. Thus the radial and transverse components of velocity at A are given by

$$\frac{dr}{dt}=\frac{k\cos\theta}{4\pi y}-\frac{k\sin\theta}{4\pi x}=\frac{k\cos\theta}{4\pi r\sin\theta}-\frac{k\sin\theta}{4\pi r\cos\theta}=\frac{k(\cos^2\theta-\sin^2\theta)}{2\pi r\sin 2\theta}=\frac{k\cos 2\theta}{2\pi r\sin 2\theta} \quad ...(1)$$

$$r\frac{d\theta}{dt}=\frac{k}{4\pi r}-\frac{k\sin\theta}{4\pi y}-\frac{k\cos\theta}{4\pi x}=\frac{k}{4\pi r}-\frac{k\sin\theta}{4\pi r\cos\theta}=-\frac{k\cos\theta}{4\pi r\cos\theta}=-\frac{k}{4\pi r} \quad ...(2)$$

On dividing (1) by (2), $\quad \dfrac{1}{r}\dfrac{dr}{d\theta}=-2\dfrac{\cos 2\theta}{\sin 2\theta} \quad$ or $\quad \dfrac{1}{r}\,dr=-\dfrac{\cos 2\theta}{\sin 2\theta}\,d\theta$

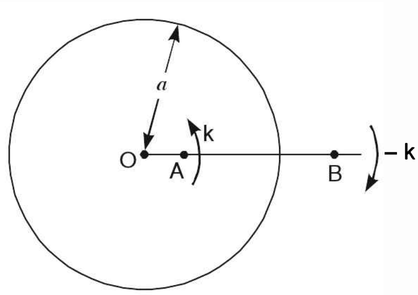
Integrating it, $\quad \log r=-\log\sin 2\theta+\log c \quad$ *i.e.* $\quad r\sin 2\theta=c,$

which is Cote's spiral. Transforming into cartesian, it becomes (using $x=r\cos\theta,\ \ y=r\sin\theta$)

$2r\sin\theta\cos\theta=c \quad$ or $\quad 4r^4\cos^2\theta\sin^2\theta=c^2r^2 \quad$ or $\quad 4(r\cos\theta)^2\,(r\sin\theta)^2=c^2r^2$

i.e. $\quad 4x^2y^2=c^2\,(x^2+y^2) \quad$ or $\quad 1/x^2+1/y^2=4/c^2.$

11.11(*A*). Vortex inside an infinite circular cylinder.



Let the vortex of strength k be situated at A $(OA=f)$ inside the circular cylinder of radius a with axis parallel to the axis of the cylinder. Let a vortex of strength $-k$ be placed at B, where B is the inverse point of A with respect to the circular section of the cylinder so that

$OB\cdot OA=a^2 \quad$ or $\quad OB\cdot f=a^2$

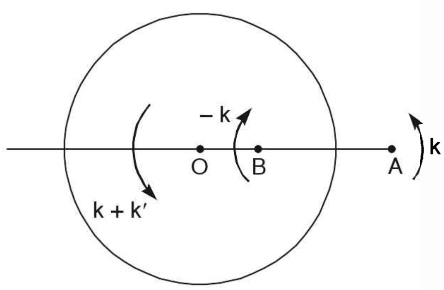
$\Rightarrow \quad OB=a^2/f.$

The circle is one of the co-axial system having A and B as limiting points and so it is a streamline.

The velocity of A $=\dfrac{k}{2\pi\cdot AB}=\dfrac{k}{2\pi(OB-OA)}=\dfrac{k}{2\pi(a^2/f-f)}=\dfrac{kf}{2\pi(a^2-f^2)},$ by Art. 11.6A

which is perpendicular to OA. B also has the above mentioned velocity so that OAB will not remain a straight line at the next instant. But if A describes a circle about O with the above velocity, then at every instant the circle will be a streamline, the positions of B, of course, changing from instant to instant.

11.11(*B*). Vortex outside a circular cylinder.



Let the vortex of strength k be situated at A $(OA=f)$ outside the circular cylinder of radius a with axis parallel to the axis of the cylinder. Let a vortex of strength $-k$ be placed B, where B is the inverse point of A with respect to the circular section of the cylinder so that

$$OB \cdot OA = a^2 \quad \Rightarrow \quad OB \cdot f = a^2 \quad \Rightarrow \quad OB = a^2/f.$$

Then the circle will be an instantaneous streamline due to this vortex pair and A will describe a circle with velocity (refer Art. 11.6 A)

$$= \frac{k}{2\pi \cdot AB} = \frac{k}{2\pi(OB - OA)} = \frac{k}{2\pi(f - a^2/f)} = \frac{kf}{2\pi(f^2 - a^2)}.$$

But the introduction of a vortex of strength $-k$ at B gives a circulation $-k$ about the cylinder and let the circulation about the cylinder be k'. The circulation $-k$ about the cylinder due to the vortex B can be annuled by putting a vortex k at O and therefore to get the final circulation k' about the cylinder, we must put an additional vortex k' at O.

Thus we have a vortex k at A, $-k$ at B, $k + k'$ at O. Hence the velocity of A due to the above system

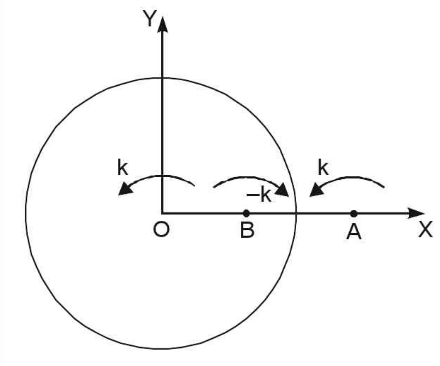
$$= \frac{k+k'}{2\pi \cdot OA} - \frac{k}{2\pi \cdot AB} = \frac{k+k'}{2\pi f} - \frac{k}{2\pi(AB - OB)} = \frac{k+k'}{2\pi f} - \frac{k}{2\pi(f - a^2/f)} = \frac{k+k'}{2\pi f} - \frac{kf}{2\pi(f^2 - a^2)}$$

and A describes a circle with this velocity

11.12(A). Image of a vortex outside a circular cylinder.

To show that the image system of a vortex k outside the circular cylinder consists of a vortex of strength $-k$ at the inverse point and a vortex of strength k at the centre.

[Kurukshetra 2000, 04]



Let us determine the image of a vortex filament of strength k placed at $A(z = c > a)$ with respect to a circular cylinder $|z| = a$ with O as centre. Let B be the inverse point of A with respect to $|z| = a$ so that $OA \times OB = a^2$ and so $OB = a^2/c$.

In absence of $|z| = a$, the complex potential at any point due to vortex at A is given by

$$(ik/2\pi) \times \log(z - c).$$

When the circular cylinder $|z| = a$ is inserted in the fluid, the modified complex potential by Milne-Thomson's circle theorem is given by

$$w = \frac{ik}{2\pi}\log(z-c) - \frac{ik}{2\pi}\log\left(\frac{a^2}{z} - c\right) = \frac{ik}{2\pi}\log(z-c) - \frac{ik}{2\pi}\log\left[-\frac{c}{z}\left(z - \frac{a^2}{c}\right)\right]$$

$$= (ik/2\pi)\left[\log(z-c) - \log(z - a^2/c) + \log z - \log(-c)\right]$$

On adding the constant term $(ik/2\pi)\log(-c)$ to the above value, the complex potential takes the form

$$w = \frac{ik}{2\pi}\log(z-c) - \frac{ik}{2\pi}\log\left(z - \frac{a^2}{c}\right) + \frac{ik}{2\pi}\log z. \qquad \text{...(1)}$$

Putting $w = \phi + i\psi$, $z = ae^{i\theta}$ for any point on $|z| = a$ and equating imaginary parts, (1) gives $\psi = 0$. Thus there would be no flow across the boundary $|z| = a$. Hence the motion would remain unchanged if the cylindrical boundary $|z| = a$ were made a rigid barrier. From (1) the required image system follows.

Remark 1. Complex potential w' induced at A, by a vortex $-k$ at B and a vortex k at O is

$$w' = w - (ik/2\pi)\log(z-c) = -(ik/2\pi)\log(z - a^2/c) + (ik/2\pi)\log z$$

$$\therefore \qquad -\frac{dw'}{dz} = -\frac{ik}{2\pi} \times \frac{1}{z - a^2/c} + \frac{ik}{2\pi} \times \frac{1}{z} = -\frac{ik}{2\pi}\left[\frac{c}{cz - a^2} - \frac{1}{z}\right]$$

$$\therefore \qquad \left| -\frac{dw'}{dz} \right|_{z=c} = \frac{k}{2\pi} \left| \frac{c}{cz - a^2} - \frac{1}{z} \right|_{z=c} = \frac{ik}{2\pi c} \times \frac{a^2}{c^2 - a^2},$$

which gives the velocity of the vortex A with which it moves round the cylinder.

Remark 2. Since the term $ik \log z$ denotes the circulation round the cylinder, the result of the above image system may be restated as under:

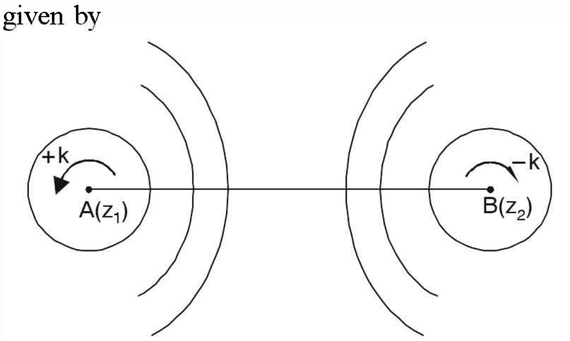
The image system of a vortex k outside the circular cylinder consists of a vortex of strength $-k$ at the inverse point and a circulation of strength k round the cylinder.

Remark 3. *Proceeding as above, we can also show that the image system of a vortex $-k$ outside the circular cylinder consists of a vortex of strength k at the inverse paint and a vortex of strength $-k$ at the centre.*

11.12(B). Image of a vortex inside a circular cylinder.

To show that the image of a vortex inside a circular cylinder would be an equal and opposite vortex at the inverse point.

Let there be a vortex pair consisting of two vortices of strength k at A $(z = z_1)$ and $-k$ at B $(z = z_2)$. Then the complex potential at any point is given by



$$w = \frac{ik}{2\pi}\log(z - z_1) - \frac{ik}{2\pi}\log(z - z_2)$$

or $$\phi + i\psi = \frac{ik}{2\pi}\log(r_1 e^{i\theta_1}) - \frac{ik}{2\pi}\log(r_2 e^{i\theta_2})$$

$$\therefore \qquad \psi = \frac{ik}{2\pi}\log\frac{r_1}{r_2},$$

where $r_1 = |z - z_1|$, $\qquad r_2 = |z - z_2|$.

Hence the streamlines are given by $\psi = \text{const.}$ *i.e.* $r_1/r_2 = c$, which represents a family of co-axial circles with A and B as limiting points.

Moreover the motion is unsteady and hence streamlines go on changing and following the vortices which move through the liquid. However, if a particular circle of the family of co-axial circle be replaced by a similar rigid boundary and held fixed, then it follows that the image of a vortex inside a circular cylinder would be an equal and opposite vortex at the inverse point.

Remark. Let O be the centre of the cylinder. Let $OA = c$. Then, if B is the inverse point of A, $OB = a^2/c$, where a is the radius of the circular cylinder. The vortex at A will move round the circular cylinder with velocity q given by

$$q = \frac{k}{2\pi AB} = \frac{k}{2\pi(OB - c)} = \frac{k}{2\pi(a^2/c - c)} = \frac{kc}{2\pi(a^2 - c^2)}.$$

Let ω be the angular velocity of vortex at A. Then

$$\omega = \frac{q}{OA} = \frac{q}{c} = \frac{k}{2\pi(a^2 - c^2)}.$$

11.13. Illustrative solved examples.

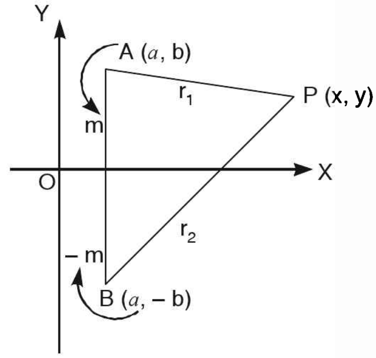
Ex. 1. *An infinitely long line vortex of strength m, parallel to the axis of z, is situated in infinite liquid bounded by a rigid wall in the plane $y = 0$. Prove that, if there be no field of force,* *the surfaces of equal pressure are given by*

$$\{(x-a)^2+(y-b)^2\}\ \{(x-a)^2+(y+b^2)\} = c\ \{(y^2+b^2)-(x-a)^2\},$$

where (a, b) are the coordinates of the vortex, and c is a parametric constant.

[G.N.D. Univ. 1998, 1997]

Sol. The image of the vortex of strength m at $A(a, b)$ is a vortex of strength $-m$ at $B(a, -b)$. Then two vortices at A and B form a vortex pair with line joining them perpendicular to x-axis and $AB = 2b$. Hence these vortices move parallel to x-axis with velocity $m/4\pi b$. The above system of vortices can be brought to rest by superimposing a velocity $-m/4\pi b$ parallel to x-axis.



Hence the components u and v of velocity q at a point $P(x, y)$ are given by

$$u = -\frac{m}{2\pi}\left[\frac{y-b}{r_1^2}-\frac{y+b}{r_2^2}\right]-\frac{m}{4\pi b}, \qquad v = \frac{m}{2\pi}\left[\frac{x-a}{r_1^2}-\frac{x-a}{r_2^2}\right].$$

$\therefore\ q^2 = u^2 + v^2$

$$= \frac{m^2}{4\pi^2}\left[\frac{(x-a)^2+(y-b)^2}{r_1^4}+\frac{(x-a)^2+(y+b)^2}{r_2^4}-\frac{2\{(x-a)^2+y^2-b^2\}}{r_1^2 r_2^2}+\frac{1}{b}\left(\frac{y-b}{r_1^2}-\frac{y+b}{r_2^2}\right)+\frac{1}{4b^2}\right]$$

Thus,
$$q^2 = \frac{m^2}{4\pi^2}\left[\frac{1}{r_1^2}+\frac{1}{r_2^2}-\frac{2\{(x-a)^2+y^2-b^2\}}{r_1^2 r_2^2}+\frac{1}{b}\left(\frac{y-b}{r_1^2}-\frac{y+b}{r_2^2}\right)+\frac{1}{4b^2}\right] \quad ...(1)$$

Since the system of vortices has been reduced to rest, the motion may be regarded as steady and hence in the absence of external field of force the pressure at any point (by Bernouilli's equation) is given by

$$p/q + q^2/2 = \text{const.} \quad ...(2)$$

Hence the surfaces of equal pressure are given by p = const. Using (2), the surfaces of equal pressure are given by q^2 = const., *i.e.*, by

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}-2\frac{\{(x-a)^2+y^2-b^2\}}{r_1^2 r_2^2}+\frac{1}{b}\left(\frac{y-b}{r_1^2}-\frac{y+b}{r_2^2}\right) = \text{const.} = 1/c, \text{ say}$$

or
$$c[r_1^2+r_2^2-2(x-a)^2-2(y^2-b^2)+(y/b)(r_2^2-r_1^2)-(r_1^2+r_2^2)] = r_1^2 r_2^2 \quad ...(3)$$

But $\quad r_1^2 = (x-a)^2+(y-b)^2 \quad$ and $\quad r_2^2 = (x-a)^2+(y+b)^2 \quad ...(4)$

so that
$$r_2^2 - r_1^2 = 4yb. \quad ...(5)$$

Using (4) and (5), (3) becomes

$$c[(y^2+b^2)-(x-a)^2] = \{(x-a)^2+(y-b)^2\}\{(x-a)^2+(y+b)^2\}.$$

Ex. 2. *Prove that a thin cylindrical vortex of strength k, running parallel to a plane boundary at distance a will travel with velocity $(k/4\pi a)$; and show that a stream of fluid will flow past between the travelling vortex and the boundary, of total amount $(k/2\pi)[\log(2a/c)-1/2]$ per unit length along the vortex, where c is the (small) radius of the cross section of the vortex.*

Sol. The image of the cylindrical vortex of strength k at A $(z = ai)$ is a cylindrical vortex of strength $-k$ at B $(z = -ia)$. The complex potential at any point $P(x, y)$ due to this system is

$$w = (ik/2\pi)\log(z-ia) - (ik/2\pi)\log(z+ia) \quad ...(1)$$

The velocity induced at A by the vortex at B

$$= \left| \frac{d}{dz}\left\{ w - \frac{ik}{2\pi}\log(z - ia) \right\} \right|_{z=ia} = \left| \frac{d}{dz}\left\{ -\frac{ik}{2\pi}\log(z+ia) \right\} \right|_{z=ia} = \left| -\frac{ik}{2\pi(z+ia)} \right|_{z=ia} = \frac{k}{4\pi a}.$$

Let $w = \phi + i\psi$. Then, equating imaginary parts, (1) gives

$$\psi = (k/2\pi)\log(r_1/r_2) \qquad \text{... (2)}$$

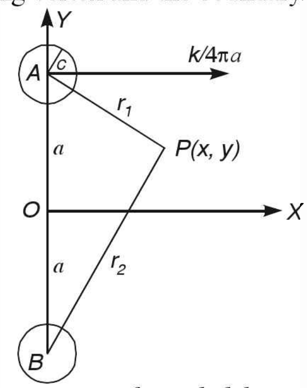
Let OX be the x-axis and O, the middle point of AB be the origin. Since vortex system is moving with uniform velocity $k/4\pi a$ perpendicular to the line AB, the vortex system can be reduced to relative rest, by superimposing a velocity $(-k/4\pi a)$. Let ψ' be the stream function of the superimposed system. Then, we have

$$-\frac{\partial \psi'}{\partial y} = -\frac{k}{4\pi a} \qquad \text{so that} \qquad \psi' = \frac{ky}{4\pi a}. \qquad \text{...(3)}$$

Let ψ'' be the stream function of the relative motion. Then, from (2) and (3), we have

$$\psi'' = \frac{k}{2\pi}\log\frac{r_1}{r_2} + \frac{ky}{4\pi a} = \frac{k}{4\pi}\left[2\log\frac{r_1}{r_2} + \frac{y}{a}\right] = \frac{k}{4\pi}\left[\log\frac{r_1^2}{r_2^2} + \frac{y}{a}\right] = \frac{k}{4\pi}\left[\log\frac{x^2+(y-a)^2}{x^2+(y+a)^2} + \frac{y}{a}\right].$$

Let Q be the total amount of fluid flowing past between the travelling vortex and the boundary. Then $Q = \psi''$ at $(0, 0) - \psi''$ at $(0,\, a - c)$



$$= 0 - \frac{k}{4\pi}\left[\log\frac{c^2}{(2a-c)^2} + \frac{a-c}{a}\right] = \frac{k}{4\pi}\left[\log\left(\frac{2a-c}{c}\right)^2 - \frac{a-c}{a}\right]$$

$$= \frac{k}{2\pi}\left[\log\left(\frac{2a}{c} - 1\right) - \frac{a-c}{2a}\right] = \frac{k}{2\pi}\left[\log\left\{\frac{2a}{c}\left(1 - \frac{c}{2a}\right)\right\} - \frac{a-c}{2a}\right]$$

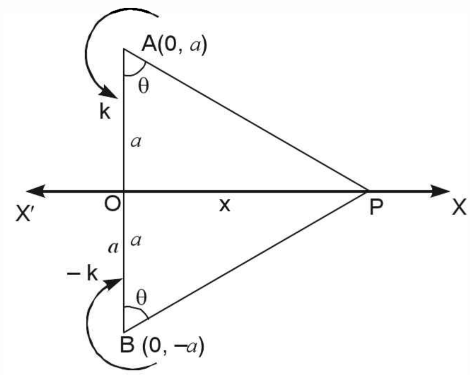
$$= \frac{k}{2\pi}\left[\log\frac{2a}{c} + \log\left(1 - \frac{c}{2a}\right) - \frac{a-c}{2a}\right] = \frac{k}{2\pi}\left[\log\frac{2a}{c} - \frac{1}{2}\right]$$

nearly, as c is small.

Ex. 3. *Find the motion of a straight vortex filament in an infinite region bounded by an infinite plane wall to which the filament is parallel and prove that the pressure defect at any point of the wall due to the filament is proportional to* $\cos^2\theta\cos 2\theta$, *where* θ *is the inclination of the plane through the filament and the point to the plane through the filament perpendicular to the wall.*

Sol. The image of the vortex of strength k at A is a vortex of strength $-k$ at B which is the optical image in the plane. Let middle point of AB be taken as origin and AB be taken along the y-axis as shown in the figure. Let coordinates of A and B be $(0, a)$ and $(0, -a)$ respectively and $AB = 2a$. Then the complex potential of the system is given by

$$w = \frac{ik}{2\pi}\log(z - ia) - \frac{ik}{2\pi}\log(z+ia) = \frac{ik}{2\pi}\log\frac{z-ia}{z+ia} \qquad \text{...(1)}$$



Then we know that the system moves parallel to x-axis with velocity u given by

$$u = k/4\pi a \qquad \text{...(2)}$$

The complex potential of the system at any time t is given by

$$w' = \frac{ik}{2\pi}\log\frac{z - ia - ut}{z + ia - ut} \qquad \text{...(3)}$$

$$\therefore \qquad \frac{\partial w'}{\partial t} = -\frac{iku}{2\pi}\left[\frac{1}{z-ia-ut} - \frac{1}{z+ia-ut}\right] \qquad ...(4)$$

Let P $(x, 0)$ be any point on x-axis. Since $w' = \phi + i\psi$, (4) gives

$$\left(\frac{\partial \phi}{\partial t} + i\frac{\partial \psi}{\partial t}\right)_{t=0} = \frac{uka}{\pi(x^2+a^2)} = \frac{k^2\cos^2\theta}{4\pi^2 a^2}, \qquad \text{as} \qquad u = \frac{k}{4\pi a} \quad \text{and} \quad \tan\theta = \frac{x}{a}$$

$$\therefore \qquad \left(\frac{\partial \phi}{\partial t}\right)_{t=0} = \frac{k^2\cos^2\theta}{4\pi^2 a^2} \qquad ...(5)$$

From (1), $$\frac{dw}{dz} = \frac{ik}{2\pi}\left[\frac{1}{z-ia} - \frac{1}{z+ia}\right] = \frac{-ka}{\pi(x^2+y^2+a^2)} \qquad ...(6)$$

Let q be velocity at P, where $x = a\tan\theta$, $y = 0$. Then $\quad q = \left|-\frac{dw}{dz}\right| = \frac{k}{\pi a}\cos^2\theta$, by (6)

The pressure is given by Bernoulli's theorem

$$p/\rho + q^2/2 - \partial\phi/\partial t = F(t) \qquad ...(7)$$

Let p' be pressure at P at $t = 0$. Then, we get

$$p = p', \qquad q = \frac{k}{\pi a}\cos^2\theta, \qquad \left(\frac{\partial \phi}{\partial t}\right)_{t=0} = \frac{k^2\cos^2\theta}{4\pi^2 a^2}$$

$\therefore$ From (7), $$\frac{p'}{\rho} + \frac{k^2\cos^4\theta}{2\pi^2 a^2} - \frac{k^2\cos^2\theta}{4\pi^2 a^2} = F(0). \qquad ...(8)$$

Let Π be the pressure at infinity. Then, we have $p \to \Pi$ when $P \to \infty$ *i.e.* $\theta = \pi/2$. Then (8) gives $$\Pi/\rho = F(0) \qquad ...(9)$$

From (8) and (9), we have

$$\frac{p'-\Pi}{\rho} = -\frac{k^2\cos^4\theta}{2\pi^2 a^2} + \frac{k^2\cos^2\theta}{4\pi^2 a^2} \qquad \text{or} \qquad \Pi - p' = \frac{\rho k^2}{4\pi^2 a^2}\cos^2\theta\,(1 - 2\cos^2\theta)$$

or $$\Pi - p' = c\cos^2\theta\cos 2\theta, \quad c \text{ being a constant}$$

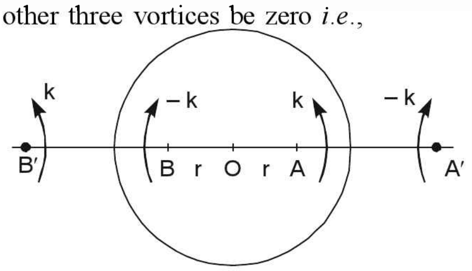
showing that the pressure defect $\Pi - p'$ is proportional to $\cos^2\theta\cos 2\theta$.

Ex. 4. *A vortex pair is situated within a cylinder. Show that it will remain at rest if the distance of either from the centre is given by* $(\sqrt{5}-2)^{1/2}a$, *where a is the radius of the cylinder.* **(Agra 2010)**

Sol. Let a vortex pair be situated at A, B where $OA = OB = r$. Let A' and B' be the inverse points of A and B respectively with regard to the circular cylinder so that $OA' = a^2/r = OB'$.

The vortex will remain at rest if its velocity due to other three vortices be zero *i.e.*,

$$\frac{k}{2\pi}\left[\frac{1}{AA'} - \frac{1}{BA} + \frac{1}{B'A}\right] = 0$$



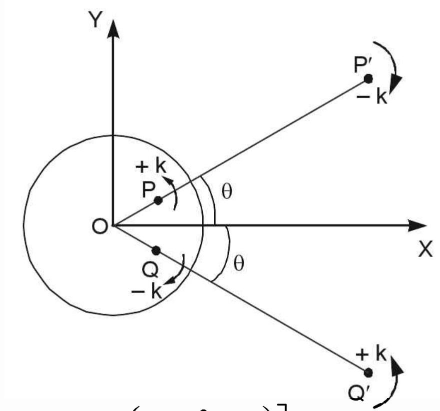
or $$\frac{1}{a^2/r - r} - \frac{1}{2r} + \frac{1}{a^2/r + r} = 0$$

or $$\frac{r}{a^2-r^2} + \frac{r}{a^2+r^2} - \frac{1}{2r} = 0$$

or $\quad r^4 + 4a^2r^2 - a^4 = 0$ $\quad$ or $\quad (r^2/a^2)^2 + 4\,(r^2/a^2) - 1 = 0$

or $\quad r^2/a^2 = \sqrt{5} - 2$ $\quad$ or $\quad r = (\sqrt{5} - 2)^{1/2}$.

Ex. 5. *When a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal distances from its axis, show that the path of each vortex is given by the equation* $(r^2 \sin^2\theta - b^2)(r^2 - a^2)^2 = 4a^2b^2r^2 \sin^2\theta$, θ *being measured from the line through the centre perpendicular to the joint of the vortices.*



Sol. Let k be the strength of the vortex at $P\,(r, \theta)$ and $-k$ at $Q\,(r, -\theta)$. Let P' and Q' be the inverse points of P and Q respectively with regard to the circular cylinder $|z| = a$ so that $OP' = a^2/r = OQ'$. Then the image of vortex k at P is a vortex $-k$ at P' and the image of vortex $-k$ at Q is a vortex k at Q'.

Hence the complex potential of the system of four vortices is given by

$$w = \frac{ik}{2\pi}\left[\log(z - re^{i\theta}) - \log\left(z - \frac{a^2}{r}e^{i\theta}\right) - \log(z - re^{-i\theta}) + \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

or
$$w = (ik/2\pi)\log(z - re^{i\theta}) + w',$$

Since the motion of vortex P is solely due to other vortices, the complex potential of the vortex at P is given by value of w' at $z = re^{i\theta}$.

$$\therefore \quad [w']_{z=re^{i\theta}} = \frac{ik}{2\pi}\left[-\log\left(z - \frac{a^2}{r}e^{i\theta}\right) - \log(z - re^{-i\theta}) + \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]_{z=re^{i\theta}}$$

$$\therefore \quad \phi + i\psi = -\frac{ik}{2\pi}\left[\log\left(re^{i\theta} - \frac{a^2}{r}e^{i\theta}\right) + \log(re^{i\theta} - re^{-i\theta}) - \log\left(re^{i\theta} - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

or
$$\phi + i\psi = -\frac{ik}{2\pi}\left[\log\left(r - \frac{a^2}{r}\right) + i\theta + \log(2ir\sin\theta) - \log\left[\left(r - \frac{a^2}{r}\right)\cos\theta + i\left(r + \frac{a^2}{r}\right)\sin\theta\right]\right]$$

$$\therefore \quad \psi = -\frac{k}{2\pi}\left[\log\left(r - \frac{a^2}{r}\right) + \log(2r\sin\theta) - \frac{1}{2}\log\left\{\left(r - \frac{a^2}{r}\right)^2\cos^2\theta + \left(r + \frac{a^2}{r}\right)^2\sin^2\theta\right\}\right]$$

[Using the formula: $\log(x + iy) = (1/2) \times \log(x^2 + y^2) + i\tan^{-1}(y/x)$]

$$= -\frac{k}{2\pi}\left[\log\left(r - \frac{a^2}{r}\right) + \log(2r\sin\theta) - \frac{1}{2}\log\left\{\left(r^2 + \frac{a^4}{r^2} - 2r\cdot\frac{a^2}{r}\cos 2\theta\right)\right\}\right]$$

Thus,
$$\psi = -\frac{k}{4\pi}\log\frac{(r - a^2/r)^2 \times (2r\sin\theta)^2}{r^2 + a^4/r^2 - 2a^2\cos 2\theta}$$

So the required streamlines are given by ψ = const., *i.e.*, $\dfrac{(r^2 - a^2)^2 r^2 \sin^2\theta}{r^4 + a^4 - 2a^2r^2\cos 2\theta} = b^2$, say

i.e.
$$b^2(r^4 + a^4 - 2a^2r^2\cos 2\theta) = r^2(r^2 - a^2)^2\sin^2\theta$$

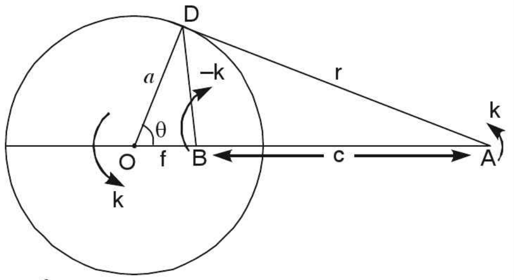
i.e.
$$b^2\{(r^2 - a^2)^2 + 2a^2r^2(1 - \cos 2\theta)\} = r^2(r^2 - a^2)^2\sin^2\theta$$

i.e. $$2a^2b^2r^2(1-\cos 2\theta) = (r^2-a^2)^2(r^2\sin^2\theta - b^2)$$

or $$4a^2b^2r^2\sin^2\theta = (r^2-a^2)^2(r^2\sin^2\theta - b^2).$$

Ex. 6. *A long fixed cylinder of radius a is surrounded by infinite frictionless incompressible liquid, and there is in the liquid a vortex filament of strength k which is parallel to the axis of the cylinder at a distance c (c > a) from this axis. Given that there is no circulation round any circuit enclosing the cylinder but not the filament, show that the speed q of the fluid at the surface of the cylinder is* $\dfrac{k}{2\pi a}\left(1-\dfrac{c^2-a^2}{r^2}\right)$, *r being the distance of the point considered from the filament.*

Sol. Since there is no circulation round the circle, the image system of vortex k at A $(z = c)$ consists of a vortex of strength $-k$ at the inverse point B $(z = a^2/c = f$, say) and a vortex of strength k at the centre $(z = 0)$. Hence the complex potential of the entire system is given by



$$w = (ik/2\pi)[\log(z-c) - \log(z-f) + \log z]$$

$$\therefore \quad \frac{dw}{dz} = \frac{ik}{2\pi}\left[\frac{1}{z-c} - \frac{1}{z-f} + \frac{1}{z}\right]$$

$$= \frac{ik}{2\pi}\frac{z^2-2fz+cf}{z(z-c)(z-f)}$$

$$\therefore \quad q = \left|\frac{dw}{dz}\right| = \frac{k}{2\pi}\frac{|z^2-2fz+a^2|}{OD\times AD\times BD}, \text{ as } a^2/c = f \qquad \ldots(1)$$

Now, from similar triangles ODB and ODA, we get

$$\frac{BD}{OD} = \frac{DA}{OA} \qquad z = ae^{i\theta} \quad i.e. \qquad \frac{BD}{a} = \frac{r}{c} \qquad \ldots(2)$$

Also, $z^2-2fz+a^2 = (z-f)^2+a^2-f^2 = (ae^{i\theta}-f)^2+a^2-f^2$

[$\because$ On the surface of the given cylinder, $z = ae^{i\theta}$]

$$= (a\cos\theta - f + ia\sin\theta)^2 + a^2 - f^2 = 2a[a\cos^2\theta - f\cos\theta + i\sin\theta\,(a\cos\theta - f)]$$

$$\therefore \quad |z^2-2fz+a^2| = 2a\sqrt{[(a\cos^2\theta - f\cos\theta)^2 + \sin^2\theta\,(a\cos\theta - f)^2]} = 2a(f - a\cos\theta)$$

$$= 2a\left[f - a\frac{a^2+c^2-r^2}{2ac}\right] = \frac{a}{c}[2cf-(a^2+c^2-r^2)] = \frac{a(a^2-c^2+r^2)}{c}, \text{ as } \frac{a^2}{c} = f \qquad \ldots(3)$$

Using (2) and (3), (1) gives

$$q = \frac{k}{2\pi}\frac{(a/c)\times(a^2-c^2+r^2)}{ar\times(ar/c)} = \frac{k}{2\pi a}\left(1-\frac{c^2-a^2}{r^2}\right).$$

Ex. 7. *A vortex of strength k is placed at the point* $(f, 0)$ *outside a circular cylinder, centre* $(0, 0)$ *of radius a. By calculating the force exerted on the image system, prove that the cylinder is acted on the a force of magnitude* $2\pi\rho k^2a^2/f(f^2-a^2)$. *In what direction is the cylinder urged by this force.*

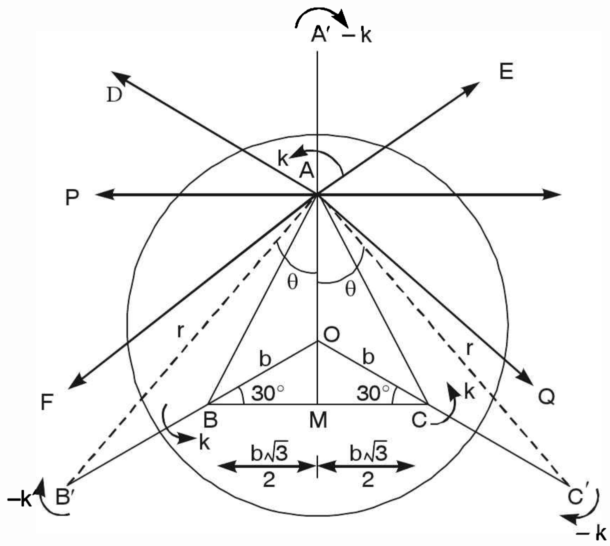
[**Hint.** Replace μ by k and c by f in Ex. 7, Art 5.25 of chapter 5]

Ex. 8. *Three vortex filaments, each of strength k, are symmetrically placed inside a circular cylinder of radius a, and pass through the corners of an equilateral triangle of side* $b\sqrt{3}$. *If there is no circulation in the fluid other than that due to the vortices, show that they will revolve about the axis of the cylinder with angular velocity* $k(a^6+2b^6)/b^2(a^6-b^6)$

Sol. Let vortex filaments, each of strength k be placed at A, B, C. Let A', B', C' be the inverse points of A, B, C respectively with regard to the circular boundary.

Since each side of $\Delta\, ABC$ is $b\sqrt{3}$, hence $OA = OB = OC = b$. Also, $OA \cdot OA' = a^2$

$$\therefore \qquad OA' = a^2 / b = b', \text{ say}$$



Thus, $OA' = OB' = OC' = b' = a^2 / b.$ and $AA' = BB' = CC' = b' - b = a^2 / b - b.$

Now, the image system for the vortices at A, B, C consists of the vortices, each of strength $-k$ at A', B', C'. Let $AB' = AC' = r$ and $\angle OAB' = \angle OAC' = \theta$.

Now, from $\Delta OAB'$, $$\cos\theta = \frac{OA^2 + AB'^2 - OB'^2}{2OA \cdot AB'} = \frac{b^2 + r^2 - b'^2}{2br} \qquad ...(1)$$

and $$AB'^2 = OA^2 + OB'^2 - 2OA \cdot OB' \cos 120°$$

or $$r^2 = b^2 + b'^2 - 2bb' \times (-1/2) = b^2 + b'^2 + b \times (a^2/b) = a^2 + b^2 + b'^2. \qquad ...(2)$$

We know that each vortex pair moves perpendicular to their join (AA', say for example) with velocity k/r, where r is the distance between them.

Now, the vortex at A moves due to the other vortices at B, C, A', B' and C' and describes a circle with certain angular velocity about O. Hence we shall resolve all velocities in a direction perpendicular to OA (*i.e.* along AP, where AP is perpendicular to OA). Let AQ, AF, AD and AE be perpendicular to AB', AC', AB and AC respectively.

(*i*) Velocity due to A' is along AP and its magnitude $= \dfrac{k}{AA'} = \dfrac{k}{(a^2/b) - b} = \dfrac{kb}{a^2 - b^2}$

(*ii*) Velocity due to B' is along AQ and its magnitude $= k / B'A = k / r$

(*iii*) Velocity due to B is along AD and its magnitude $= k / AB = k / b\sqrt{3}$

(*iv*) Velocity due to C is along AE and its magnitude $= k / AC = k / b\sqrt{3}$

(*v*) Velocity due to C' is along AF and its magnitude $= k / C'A = k / r$

As explained before, resolving all velocities in a direction perpendicular to OA *i.e.* along AP, the algebraic sum of all velocities

$$= k\left[\frac{b}{a^2-b^2} - \frac{1}{r}\cos\theta + \frac{1}{b\sqrt{3}}\cos 30° + \frac{1}{b\sqrt{3}}\cos 30° - \frac{1}{r}\cos\theta\right]$$

$$= k\left[\frac{b}{a^2-b^2} + \frac{1}{b} - \frac{2\cos\theta}{r}\right] = k\left[\frac{b}{a^2-b^2} + \frac{1}{b} - \frac{1}{r^2}\left(\frac{b^2+r^2-b'^2}{b}\right)\right], \text{ using (1)}$$

$$= k\left[\frac{b}{a^2-b^2} + \frac{1}{b} - \frac{1}{r^2 b}\left(b^2+r^2-\frac{a^4}{b^2}\right)\right], \text{ as } b' = \frac{a^2}{b}$$

$$= k\left[\frac{b}{a^2-b^2} + \frac{a^4-b^4}{b^3 r^2}\right] = k\left[\frac{b}{a^2-b^2} + \frac{a^4-b^4}{b^3(a^2+b^2+b'^2)}\right], \text{ by (2)}$$

$$= k\left[\frac{b}{a^2-b^2} + \frac{a^4-b^4}{b^3(a^2+b^2+a^4/b^2)}\right], \quad \text{as } b' = \frac{a^2}{b}$$

$$= k\left[\frac{b}{a^2-b^2} + \frac{a^4-b^4}{b(a^4+b^4+a^2b^2)}\right] = k\frac{b^2(a^4+b^4+a^2b^2)+(a^2-b^2)(a^4-b^4)}{b(a^2-b^2)(a^4+b^4+a^2b^2)}$$

$$= k(a^6+2b^6)/b(a^6-b^6)$$

Hence, $b\dot{\theta} = k(a^6+2b^6)/b(a^6-b^6)$ or $\dot{\theta} = k(a^6+2b^6)/b^2(a^6-b^6)$ giving the required angular velocity.

EXERCISE 11 (C)

1. The space enclosed between the planes $x = 0$, $x = a$, $y = 0$ on the positive side of $y = 0$ is filled with uniform incompressible liquid. A rectilinear vortex parallel to the axis of z has coordinates (x', y'). Determine the velocity at any point of the liquid and show that the path of the vortex is given by $\cot^2(\pi x/a) + \coth^2(\pi y/a) = \text{const.}$ **[Kanpur 2003]**

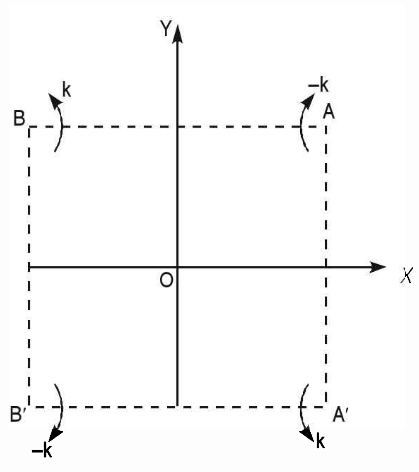
2. Determine the image of a vortex in a liquid outside a circular cylinder.

A rectilinear vortex of strength k is situated in an infinite fluid surrounding a fixed circular cylinder of radius a. The vortex is parallel to and at a distance f from the axis of the cylinder and there is no circulation in any circuit which does not enclose the vortex. Show that the vortex moves about the axis of the cylinder with a constant angular velocity equal to $ka^2(f^2 - a^2)/f^2$.

Find the velocity of the fluid at a point on the cylinder such that the axial plane through the point makes an angle θ with the axial plane through the vortex and proceed to show how the resultant thrust on a unit length of the cylinder may be calculated.

11.14. Four vortices.

Let there be four vortices in an infinite liquid such that the lines joining them form a rectangle $ABB'A'$ at any instant. Let the strength of vortices at A' and B be k and that of A and B' be $-k$. Notice that the centres will always form a rectangle.



We wish to study the motion of a vortex pair moving upwards or forming a parallel plane boundary between planes meeting at right angles. Let the planes which bisect AB and AA' at right angles be taken as fixed boundaries.

If (x, y) be the coordinates of the vortex A relative to the planes of symmetry (*i.e.* due to other three vortices), the velocity components are given by

k

$$u = \frac{k}{2\pi AA'} - \frac{k}{2\pi AB'} \cdot \frac{AA'}{AB'} = \frac{k}{4\pi y} \frac{x^2}{(x^2+y^2)} = \frac{kx^2}{4\pi y r^2}, \text{ as } r^2 = x^2 + y^2 \quad ...(1)$$

and
$$v = -\frac{k}{2\pi AB} + \frac{k}{2\pi AB'} \cdot \frac{AB}{AB'} = -\frac{k}{4\pi x} \frac{y^2}{x^2+y^2} = -\frac{ky^2}{4\pi x r^2} \quad ...(2)$$

The path of the vortex P is given by $dx/dt = u$ and $dy/dt = v$ so that

$$\frac{dx}{u} = \frac{dy}{v} \qquad \text{or} \qquad \frac{dx}{x^3} + \frac{dy}{y^3} = 0, \text{ using (1) and (2)}$$

Integrating, $1/x^2 + 1/y^2 = \text{const.} = 1/a^2$, say or $(x^2 + y^2)\, a^2 = x^2 y^2$

or $4r^2 a^2 = 4r^4 \sin^2\theta \cos^2\theta$, as $x = r\cos\theta$ and $y = r\sin\theta$

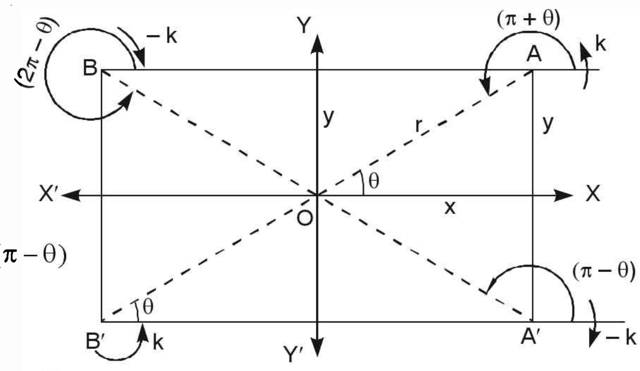
or $r^2 \sin^2 2\theta = 4a^2$ or $r \sin 2\theta = 2a$, which is Cote's spiral.

11.15. Illustrative solved examples.

Ex. 1. *A rectilinear vortex moves parallel to two rigid planes which intersect at right angles, prove that on the line of intersection of the planes the excess of pressure due to the vortex varies inversely as the square of the distance of the vortex from the line of intersection.*

[Kanpur 2011]

Sol. The Image system is as shown in the figure. Velocity at O is zero, because the velocities due to vortices A and B' cancel each other and that due to B and A' cancel each other. The velocity potential of the entire system is given by



$$\phi = -\frac{k}{2\pi}(\pi+\theta) + \frac{k}{2\pi}(2\pi-\theta) - \frac{k}{2\pi}\theta + \frac{k}{2\pi}(\pi-\theta)$$

or $\phi = \text{costant}-(2k/\pi)\theta$

$\therefore \quad \dot{\phi} = -(2k/\pi)\dot{\theta} \qquad ...(1)$

From figure $\tan\theta = y/x$ so that $(\sec^2\theta)\,\dot{\theta} = (x\dot{y} - \dot{x}y)/x^2 \qquad ...(2)$

For the vortex at $A\,(x, y)$, the velocity components are given by

$$\dot{x} = \frac{k}{2\pi \cdot 2y} - \frac{k}{2\pi \cdot 2r}\sin\theta = \frac{k}{4\pi y} - \frac{ky}{4\pi r^2} = \frac{k}{4\pi y} - \frac{ky}{4\pi(x^2+y^2)} = \frac{k}{4\pi} \cdot \frac{x^2}{y(x^2+y^2)}$$

$$[\because x^2 + y^2 = r^2]$$

$$\dot{y} = -\frac{k}{2\pi \cdot 2x} + \frac{k}{2\pi \cdot 2r}\cos\theta = -\frac{k}{4\pi x} + \frac{kx}{4\pi r^2} = -\frac{k}{4\pi x} + \frac{kx}{4\pi(x^2+y^2)} = -\frac{k}{4\pi} \cdot \frac{y^2}{x(x^2+y^2)}$$

$$\therefore \text{ From (2), } \dot{\theta} = \frac{x\dot{y} - \dot{x}y}{x^2 \sec^2\theta} = \frac{x\dot{y} - \dot{x}y}{r^2} = -\frac{k}{4\pi r^2}\left[\frac{xy^2}{y(x^2+y^2)} + \frac{yx^2}{y(x^2+y^2)}\right] = -\frac{k}{4\pi r^2}$$

$\therefore$ From (1), $\dot{\phi} = k^2/2\pi^2 r^2 \qquad ...(3)$

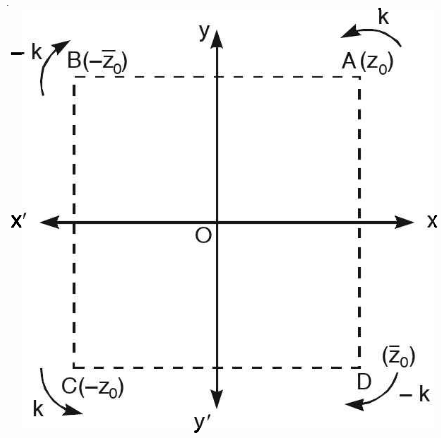
Now, from pressure equation, the excess of pressure at origin is given by

$$p - p_0 = \rho\left[\frac{\partial\phi}{\partial t} - \frac{1}{2}q^2\right] = \frac{k^2\rho}{2\pi^2} \cdot \frac{1}{r^2} \qquad [\because q = 0 \text{ at } O]$$

$\therefore \quad p - p_0$ is proportional to $1/r^2$ which proves the desired result.

Ex. 2. *A rectilinear vortex filament of strength k is in an infinite liquid bounded by two perpendicular infinite plane walls whose line of intersection is parallel to the filament. Show that the filament will trace out a curve (in a plane at right angles to the walls)* $r \sin 2\theta = \text{const.}$, *where r is the distance of the vortex from the line of intersection of the walls, and* θ *the angle between one of the walls and the plane containing the filament and the line of intersection.*

Sol. Let the vortex of strength k be situated at $A\ (z = z_0)$ where $z_0 = x_0 + iy_0$. To reduce OX and OY as streamlines, we place a vortex of strength k at $C\ (z = -z_0)$ and vortices of strength $-k$ at $B(z = -\bar{z}_0)$. and $D(z = \bar{z}_0)$. Let $z\ (= x + iy)$ be any point of fluid. Then complex potential due to all the four vortices at A, B, C, D is given by



$$w = (ik/2\pi)[\log(z - z_0) + \log(z + z_0) - \log(z - \bar{z}_0) - \log(z + \bar{z}_0)] \qquad \ldots(1)$$

Let w' be the complex potential of the vortex at A due to the vortices at B, C, D. To find the motion of vortex at A, we must omit the part due to it. So from (1), we have

$$w' = (ik/2\pi)\,[\log(z + z_0) - \log(z - \bar{z}_0) - \log(z + \bar{z}_0)] = (ik/2\pi)[\log(z + z_0) - \log(z^2 - \bar{z}_0^2)]$$

$$\therefore\ u - iv = \lim_{z \to z_0}\left(-\frac{dw'}{dz}\right) = -\frac{ik}{2\pi}\lim_{z \to z_0}\left[\frac{1}{z + z_0} - \frac{2z}{z^2 - \bar{z}_0^2}\right]$$

$$= -\frac{ik}{2\pi}\left[\frac{1}{2z_0} - \frac{2z_0}{z_0^2 - \bar{z}_0^2}\right] = -\frac{ik}{2\pi}\left[\frac{1}{2(x_0 + iy_0)} - \frac{2(x_0 + iy_0)}{(x_0 + iy_0)^2 - (x_0 - iy_0)^2}\right]$$

$$= -\frac{ik}{4\pi}\left[\frac{x_0 - iy_0}{x_0^2 + y_0^2} - \frac{x_0 + iy_0}{ix_0 y_0}\right] = -\frac{ik}{4\pi}\left[i\left\{\frac{1}{y_0} - \frac{y_0}{x_0^2 + y_0^2}\right\} + \left\{\frac{x_0}{x_0^2 + y_0^2} - \frac{1}{x_0}\right\}\right]$$

Thus,
$$u - iv = \frac{k}{4\pi}\left[\frac{x_0^2}{y_0(x_0^2 + y_0^2)} - i\frac{y_0^2}{x_0(x_0^2 + y_0^2)}\right]$$

$$\therefore\quad u = \frac{k}{4\pi}\frac{x_0^2}{y_0(x_0^2 + y_0^2)}, \qquad \text{and} \qquad v = -\frac{k}{4\pi}\frac{y_0^2}{x_0(x_0^2 + y_0^2)} \qquad \ldots(2)$$

For the path of vortex at z_0, $\quad u = dx_0/dt \quad$ and $\quad v = dy_0/dt \qquad \ldots(3)$

$\therefore$ From (2) and (3), $\dfrac{dx_0}{dy_0} = \dfrac{u}{v} = -\dfrac{x_0^3}{y_0^3} \quad$ or $\quad \dfrac{dx_0}{x_0^3} + \dfrac{dy_0}{y_0^3} = 0$

Integrating, $\quad -(1/2x_0^2) - (1/2y_0^2) = -(C/2) \quad$ or $\quad 1/x_0^2 + 1/y_0^2 = C$

$\therefore \qquad x_0^2 + y_0^2 = C\,x_0^2 y_0^2$, C being an arbitrary constant. $\qquad \ldots(4)$

Let $x_0 = r\cos\theta,\ y_0 = r\sin\theta$, Then (4) reduces to

$$r^2 = C\,r^4 \sin^2\theta\cos^2\theta = (C/4) \times r^4 \sin^2 2\theta$$

$\therefore \qquad r \sin 2\theta = 4/C = \text{constant} = A, \text{ say} \qquad ...(5)$

Again, $\tan\theta = y_0/x_0$, as $x_0 = r\cos\theta$, $y_0 = r\sin\theta$

Differentiating both sides of $\tan\theta = y_0/x_0$ w.r.t. 't', we get

$$\sec^2\theta\,\dot\theta = (x_0\dot y_0 - y_0\dot x_0)/x_0^2 \qquad \text{or} \qquad x_0^2\sec^2\theta\cdot\dot\theta = x_0\dot y_0 - y_0\dot x_0$$

or $$r^2\dot\theta = x_0\,\dot y_0 - y_0\,\dot x_0 = x_0 v - y_0 u, \text{ by (3)} \qquad [\because\ x_0 = r\cos\theta]$$

Thus, $$r^2\dot\theta = -\frac{k}{4\pi}\frac{y_0^2}{x_0^2+y_0^2} - \frac{k}{4\pi}\frac{x_0^2}{x_0^2+y_0^2}, \text{ using (2)}$$

$$\therefore \quad r^2\frac{d\theta}{dt} = -\frac{k}{4\pi} \qquad \text{or} \qquad \frac{A^2}{\sin^2 2\theta}\frac{d\theta}{dt} = -\frac{k}{4\pi}, \text{ using (5)}$$

or $$dt = -(4\pi A^2/k)\,\text{cosec}^2 2\theta\, d\theta$$

Integrating, $t = (2\pi A^2/k)\cot 2\theta$, so that t is proportional to $\cot 2\theta$

11.16. Vortex rows.

When a body moves slowly through a liquid rows of vortices are often generated in its wake. When these vortices are stable, then they can be photographed. In the next two articles we wish to consider infinite systems of parallel rectilinear vortices in two dimensional flow.

11.17A. Infinite number of parallel vortices of the same strength in one row.

To show that the motion due to a set of line vortices of strength k at points $z = \pm\, na$ (n = 0, 1, 2, 3, ...) is given by the relation $w = (ik/2\pi)\log\sin(\pi z/a)$

Also to get velocity components and streamlines.

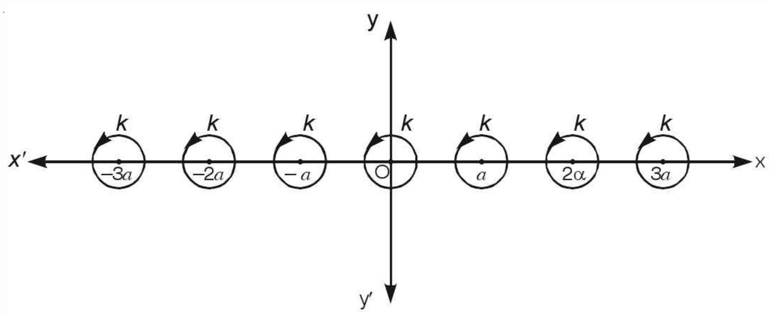
[Agra 2006; Kurukshetra 1998; Kolkata 2001, 05, 06]

Proof. Let there be $(2n + 1)$ vortices of strength k each situated at the points $(0, 0)$, $(\pm\, a, 0)$ $(\pm\, 2a, 0)$, $(\pm\, 3a, 0)$, ..., $(\pm\, na, 0)$. The complex potential of these $(2n + 1)$ vortices at any point z is given by

$$w_{2n+1} = (ik/2\pi)\,[\log z + \log(z-a) + \log(z+a) + \log(z-2a) + \log(z+2a) + \ldots\ldots\ldots + \log(z-na) + \log(z+na)]$$

$$= (ik/2\pi)\,\log[z(z^2-a^2)(z^2-2^2a^2)(z^2-3^2a^2)\ldots(z^2-n^2a^2)]$$

$$= \frac{ik}{2\pi}\log\left[\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\cdots\left(1-\frac{z^2}{n^2a^2}\right)\right] + \frac{ik}{2\pi}\log\left[(-1)^n\frac{a}{\pi}\cdot a^2\cdot 2^2a^2\ldots n^2a^2\right] \qquad ...(1)$$



The second term on R.H.S. of (1) being constant, it may be neglected for the purpose of complex potential. Hence the complex potential given by (1) may be also written as

$$w_{2n+1} = \frac{ik}{2\pi}\log\left[\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\cdots\left(1-\frac{z^2}{n^2a^2}\right)\right] \qquad ...(2)$$

Making $n\to\infty$ in (2), the complex potential w of the entire system of vortices at points $z = \pm\, na$ (n = 0, 1, 2, 3, ...,) is given by

$$w = \frac{ik}{2\pi}\log\left[\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\left(1-\frac{z^2}{3^2a^2}\right)\cdots\cdots\right] \qquad ...(3)$$

But $$\sin\theta = \theta\,(1-\theta^2/\pi^2)\,(1-\theta^2/2^2\pi^2)\,(1-\theta^2/3^2\pi^2)\ldots\ldots \qquad ...(4)$$

Putting $\theta = \pi z/a$ *i.e.* $z/a = \theta/\pi$ *in* (4), we get

$$\sin(\pi z/a) = (\pi z/a)\,(1-z^2/a^2)\,(1-z^2/2^2a^2)\ldots\ldots \qquad ...(5)$$

Using (5) , (3) becomes $$w = (ik/2\pi)\log\sin(\pi z/a) \qquad ...(6)$$

Let u and v be the velocity components at any point of the fluid not occupied by any vortex filament. Then, we have

$$u - iv = -\frac{dw}{dz} = -\frac{ik}{2a}\cot\frac{\pi z}{a}, \text{ using (6)}$$

$$= -\frac{ik}{2a}\cot\frac{\pi(x+iy)}{a} = -\frac{ik}{2a}\,\frac{\cos\frac{\pi}{a}(x+iy)\,\sin\frac{\pi}{a}(x-iy)}{\sin\frac{\pi}{a}(x+iy)\,\sin\frac{\pi}{a}(x-iy)}$$

$$= -\frac{ik}{2a}\,\frac{\sin(2\pi x/a)-\sin(2\pi iy/a)}{\cos(2\pi iy/a)-\cos(2\pi x/a)} = -\frac{ik}{2a}\,\frac{\sin(2\pi x/a)-i\sinh(2\pi y/a)}{\cosh(2\pi y/a)-\cos(2\pi x/a)}$$

Equating real and imaginary parts, we have

$$u = -\frac{k}{2a}\,\frac{\sinh(2\pi y/a)}{\cosh(2\pi y/a)-\cos(2\pi x/a)} \qquad ...(7)$$

$$v = -\frac{k}{2a}\,\frac{\sinh(2\pi x/a)}{\cosh(2\pi y/a)-\cos(2\pi x/a)} \qquad ...(8)$$

Since the motion of the vortex at the origin is due to other vortices only, the velocity q_0 of vortex at the origin is given by

$$q_0 = -\left\{\frac{d}{dz}\left[\frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log z\right]\right\}_{z=0} = -\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a} - \frac{1}{z}\right]_{z=0}$$

$$= -\frac{ik}{2\pi}\lim_{z\to 0}\left[\frac{\pi\cos(\pi z/a)}{a\sin(\pi z/a)} - \frac{1}{z}\right] \qquad \text{Form: } [\infty-\infty]$$

$$= -\frac{ik}{2\pi a}\lim_{z\to 0}\frac{\pi z\cos(\pi z/a) - a\sin(\pi z/a)}{z\sin(\pi z/a)} \qquad \text{Form: } \left[\frac{0}{0}\right]$$

$$= -(ik/2\pi a)\times 0 = 0 \qquad \text{[By } L\text{' Hospital's rule]}$$

[on evaluating the above indeterminate form as usual]

Hence the vortex at origin is at rest. Similarly, it can be shown that the remaining vortices are also at rest. Thus we find that the vortex row induces no velocity on itself.

We now determine streamlines. From (6), we get

$$\phi + i\psi = \frac{ik}{2\pi}\log\sin\left\{\frac{\pi}{a}(x+iy)\right\} \qquad \text{...(9)}$$

$$\therefore \qquad \phi - i\psi = -\frac{ik}{2\pi}\log\sin\left\{\frac{\pi}{a}(x-iy)\right\} \qquad \text{...(10)}$$

Subtracting (10) from (9), $\qquad 2i\psi = \dfrac{ik}{2\pi}\left[\log\sin\left\{\dfrac{\pi}{a}(x+iy)\right\} + \log\sin\left\{\dfrac{\pi}{a}(x-iy)\right\}\right]$

or $\qquad \psi = \dfrac{k}{4\pi}\log\left[\sin\left\{\dfrac{\pi}{a}(x+iy)\right\}\sin\left\{\dfrac{\pi}{a}(x-iy)\right\}\right] = \dfrac{k}{4\pi}\log\left[\dfrac{1}{2}\left(\cos\dfrac{2\pi iy}{a} - \cos\dfrac{2\pi x}{a}\right)\right]$

or $$\psi = \frac{k}{4\pi}\log\left(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right), \qquad \text{...(11)}$$

on omitting the irrelevent constant. The required streamlines are given by ψ = const.,

i.e. $$\cosh(2\pi y/a) - \cos(2\pi x/a) = \text{const.} \qquad (12)$$

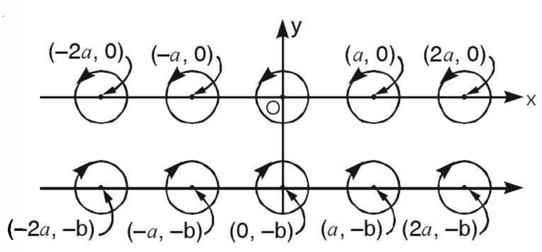
When y is very large, the second therm on L.H.S. of (12) may be omitted. Then the resulting streamlines are given by

$$\cosh(2\pi y/a) = \text{const}, \qquad \text{so that} \qquad y = \text{const.},$$

showing that at a great distance from the row of vortices the streamlines are parallel to the row.

11.17B. Two infinite rows of parallel rectilinear vortices.

Let there be two infinite rows of vortices one above the other at a distance b, the upper one having vortices each of strength k and lower one each of strength $-k$, one vertex of the upper row being exactly above each of the lower row. Taking the upper row as x-axis and y-axis passing through the centre of one of the vortices of strength k each are at the points $(0, 0)$, $(\pm a, 0)$, $(\pm 2a, 0)$... and those of strength $-k$ each are at the points $(0, -b)$, $(\pm a, -b)$, $(\pm 2a, -b)$,.....



The complex potential of the entire system (using Art 11.17A) is given by

$$w = \frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log\sin\frac{\pi}{a}(z+ib) \qquad \text{...(1)}$$

Let u and v be the velocity components at any point of the fluid not occupied by any vortex filament. Then

$$u - iv = -\frac{dw}{dz} = -\frac{ik}{2a}\cot\frac{\pi z}{a} + \frac{ik}{2\pi}\cot\frac{\pi}{a}(z+ib) \qquad \text{...(2)}$$

The velocity of the vortex at the origin is given by

$$u_0 - iv_0 = -\left\{\frac{d}{dz}\left[\frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log z - \frac{ik}{2\pi}\log\sin\frac{\pi}{a}(z+ib)\right]\right\}_{z=0}$$

$$\Rightarrow \quad u_0 - iv_0 = -\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a} - \frac{1}{z} - \frac{\pi}{a}\coth\frac{\pi}{a}(z+ib)\right]_{z=0} = -\frac{ik}{2a}\cot\frac{i\pi b}{a} = \frac{k}{2a}\coth\frac{\pi b}{a},$$

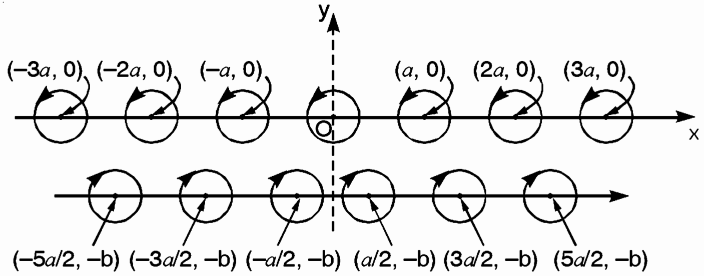
$$\left[\because \lim_{z\to 0}\{(\pi/a)\cot(\pi z/a) - (1/z)\} = 0 \text{. Prove yourself as in Art.11.17A}\right]$$

so that $\qquad u_0 = (k/2a)\coth(\pi b/a), \qquad$ and $\qquad v_0 = 0,$

showing that the vortex system moves parallel to itself with velocity $(k/2a)\coth(\pi b/a)$.

11.18. Karman Vortex Street. [Agra 2009, 12; Rohilkhand 2000]

Let there be two parallel rows of vortices of equal but opposite strength placed in such a way that each vortex in one row is opposite to the point midway between two vortices of the other row. Accordingly, let vortices of strength k each be situated at the points (0, 0), ($\pm a$, 0), ($\pm 2a$, 0), ... and the vortices of strength $-k$ each be situated at the points $(\pm a/2, -b), (\pm 3a/2, -b), \ldots$.



The complex potential of the entire system is given by

$$w = \frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log\sin\frac{\pi}{a}\left(z + \frac{1}{2}a + ib\right) \qquad \ldots(1)$$

If u and v be the velocity components at any point of the fluid not occupied by any vortex filament, then

$$u - iv = -\frac{dw}{dz} = -\frac{ik}{2a}\cot\frac{\pi z}{a} + \frac{ik}{2a}\cot\frac{\pi}{a}\left(z + \frac{1}{2}a + ib\right) \qquad \ldots(2)$$

Since the motion of the vortex at the origin is due to other vortices only, the velocity of vortex at the origin is given by

$$u_0 - iv_0 = -\left[\frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log z - \frac{ik}{2\pi}\log\sin\frac{\pi}{a}\left(z + \frac{1}{2}a + ib\right)\right\}\right]_{z=0}$$

$$= -\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a} - \frac{1}{z} - \frac{\pi}{a}\cot\frac{\pi}{a}\left(z + \frac{1}{2}a + ib\right)\right]_{z=0} = -\frac{ik}{2a}\cot\left(\frac{\pi}{2} + \frac{i\pi b}{a}\right) = -\frac{ik}{2a}\tan\frac{i\pi b}{2}$$

$$\left[\because \lim_{z\to 0}\{(\pi/a)\cot(\pi z/a) - (1/z)\} = 0 \text{. Prove yourself as in Art 11.17A}\right]$$

Thus, $$u_0 - iv_0 = (k/2a)\tanh(\pi b/a)$$

so that $u_0 = (k/2a)\tanh(\pi b/a)$ and $v_0 = 0$,

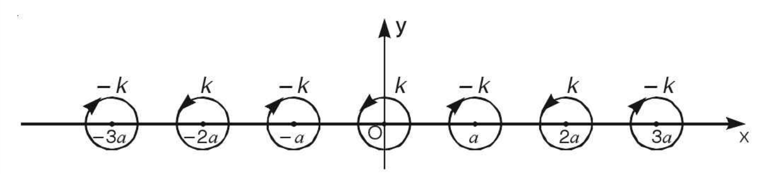
showing that the entire system would move parallel to itself with a uniform velocity $(k/2a)\tanh(\pi b/a)$.

Remark. A Karman vortex street is often realized when a flat plate moves broadside through a liquid.

11.19. Illustrative solved examples.

Ex. 1. *An infinite row of equidistant rectilinear vortices are at a distance a apart. The vortices are of the same numerical strength k but they are alternately of opposite signs. Find the complex function that determines the velocity potential and the stream function. Show also that, if α be the radius of a vortex, the amount of flow between any vortex and the next is* $(k/\pi)\log\cot(\pi\alpha/2a)$. **[Bhopal 2000, Kolkata 2005; Agra 2003]**

Sol. Let the row of vortices be taken along the x-axis. Let there be vortices of strength k each at the points (0, 0), ($\pm 2a$, 0), ($\pm 4a$, 0), ... and those of strength $-$ k each at the points ($\pm a$, 0), ($\pm 3a$, 0), ($\pm 5a$, 0), ...



The complex potential of the entire system is given by

$$w = (ik/2\pi)\,[\{\log z + \log(z-2a) + \log(z+2a) + \log(z-4a) + \log(z+4a) + \ldots\}$$
$$- \{\log(z-a) + \log(z+a) + \log(z-3a) + \log(z+3a) + \ldots\}]$$

$$= \frac{ik}{2\pi}\log\frac{z(z^2-2^2a^2)(z^2-4^2a^2)\ldots}{(z^2-a^2)(z^2-3^2a^2)\ldots} = \frac{ik}{2\pi}\log\frac{\dfrac{z}{2a}\left[1-\left(\dfrac{z}{2a}\right)^2\right]\left[1-\left(\dfrac{z}{4a}\right)^2\right]\ldots}{\left[1-\left(\dfrac{z}{a}\right)^2\right]\left[1-\left(\dfrac{z}{3a}\right)^2\right]\ldots} + \text{a const.}$$

Thus,
$$w = \frac{ik}{2\pi}\log\frac{\sin(\pi z/2a)}{\cos(\pi z/2a)} = \frac{ik}{2\pi}\log\tan\left(\frac{\pi z}{2a}\right), \qquad \text{..(1)}$$

[Using well known expansions of $\sin(\pi z/2a)$ and $\cos(\pi z/2a)$ given in Art. 1.10]
which is the desired potential function that determines the velocity potential and stream function.

From (1), we get

$$\phi + i\psi = \frac{ik}{2\pi}\log\tan\frac{\pi}{2a}(x+iy) \qquad \text{so that} \qquad \phi - i\psi = -\frac{ik}{2\pi}\log\tan\frac{\pi}{2a}(x-iy)$$

$\sin(\pi z/2a)$ and $\cos(\pi z/2a)$]

Subtracting, these give
$$2i\psi = \frac{ik}{2\pi}\left[\log\tan\frac{\pi}{2a}(x+iy) + \log\tan\frac{\pi}{2a}(x-iy)\right]$$

$$\therefore \quad \psi = \frac{k}{4\pi}\log\frac{\sin\dfrac{\pi}{2a}(x+iy)\sin\dfrac{\pi}{2a}(x-iy)}{\cos\dfrac{\pi}{2a}(x+iy)\cos\dfrac{\pi}{2a}(x-iy)} = \frac{k}{4\pi}\log\frac{\cosh(\pi y/a)-\cos(\pi x/a)}{\cosh(\pi y/a)+\cos(\pi x/a)} \qquad \text{...(2)}$$

Since the motion of the vortex at the origin is due to other vortices only, the velocity, q_0 of vortex at the origin is given by

$$q_0 = -\left\{\frac{d}{dz}\left[\frac{ik}{2\pi}\log\tan\frac{\pi z}{2a} - \frac{ik}{2a}\log z\right]\right\}_{z=0} = -\frac{ik}{2\pi}\left[\frac{\sec^2(\pi z/2a)}{\tan(\pi z/2a)}\times\frac{\pi}{2a} - \frac{1}{z}\right]_{z=0} = 0.$$

[On simplifying the indeterminate form with help of L'Hospital's rule]

Hence the vortex at origin is at rest. Similarly, it can be shown that the remaining vortices are also at rest. Thus we find that the vortex row induces no velocity on itself.

We now determine the required flow. For any point on the x-axis, $y = 0$ and hence ψ' at any point on the x-axis is given by [putting $y = 0$ in (2)]

$$\psi' = \frac{k}{4\pi}\log\frac{1-\cos(\pi x/a)}{1+\cos(\pi x/a)} = \frac{k}{4a}\log\frac{2\sin^2(\pi x/2a)}{2\cos^2(\pi x/2a)} = \frac{k}{4\pi}\log\tan\frac{\pi x}{2a} \qquad \text{...(3)}$$

$\therefore$ The required flow between two consecutive vortices

$$= (\psi')_{a-\alpha} - (\psi')_{\alpha} = \frac{k}{2\pi}\left[\log\tan\frac{\pi(a-\alpha)}{2a} - \log\tan\frac{\pi\alpha}{2a}\right]$$

$$= \frac{k}{2\pi}\log\frac{\tan(\pi/2-\pi\alpha/2a)}{\tan(\pi\alpha/2a)} = \frac{k}{2\pi}\log\cot^2\frac{\pi\alpha}{2a} = \frac{k}{\pi}\log\cot\frac{\pi\alpha}{2a}.$$

Ex. 2. *An infinite row of equidistant rectilinear vortices of equal numerical strength k, but alternately of opposite signs, are placed at distances a apart in infinite fluid. Show that the complex potential is* $w = (ik/2\pi)\log\tan(\pi z/2a)$, *the origin of coordinates being at one of the vortices of positive sign and hence show that the row remains at rest in this configuration.*

Show further that if the very small radius of cross section of each vortex filament is εa, *then the amount of flow between two consecutive vortices is approximately* $(k/\pi)\log(2/\pi\varepsilon)$.

Sol. First re-write the solution of Ex. 1. For approximation, replacing α by $a\varepsilon$, we have

$$\text{Required flow } = \frac{k}{\pi}\log\cot\frac{\pi a\varepsilon}{2a} = \frac{k}{\pi}\log\cot\left(\frac{\pi\varepsilon}{2}\right) = \frac{k}{\pi}\log\frac{\cos(\pi\varepsilon/2)}{\sin(\pi\varepsilon/2)}$$

$$= \frac{k}{\pi}\log\left(\frac{1}{\pi\varepsilon/2}\right), \text{ as } \varepsilon \text{ is small}$$

$$= (k/\pi)\times\log(2/\pi\varepsilon), \text{ as required.}$$

Ex. 3. *An infinite street of linear parallel vortices is given as : x = ra, y = b, strength k ; x = ra, y = – b, strength = – k, where r is any positive or negative integer or zero. Prove that if the liquid at infinity is at rest, the street moves as a whole, in the direction of its length with the speed* $(k/2a)\coth(2\pi b/a)$. **(Agra 2006)**

Sol. Here vortices of strength k each are situated in the first row at points

$(0, b)$, $(\pm a, b)$, $(\pm 2a, b)$, ...

and vortices of strength $-k$ each are situated in the second row at points

$(0, -b)$, $(\pm a, -b)$, $(\pm 2a, -b)$, ...

Hence the complex potential w of the above system of vortices is given by

$$w = \frac{ik}{2\pi}\log\sin\frac{\pi(z-ib)}{a} - \frac{ik}{2\pi}\log\sin\frac{\pi(z+ib)}{a} \qquad \text{....(1)}$$

Let w' be the complex potential of the vortex at $z = ib$ due to vortices situated at the remaining points. To find the motion of vortex at $z = ib$, we must omit the part due to it. So from (1), we have

$$w' = \frac{ik}{2\pi}\log\sin\frac{\pi(z-ib)}{a} - \frac{ik}{2\pi}\log\sin\frac{\pi(z+ib)}{a} - \frac{ik}{2\pi}\log(z-ib)$$

Let u, v be the velocity components of the vortex at $z = ib$. Then, we have

$$u - iv = -\left(\frac{dw'}{dz}\right)_{z=ib} = -\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi(z-ib)}{a} - \frac{\pi}{a}\cot\frac{\pi(z+ib)}{a} - \frac{1}{z-ib}\right]_{z=ib} = \frac{k}{2a}\coth\frac{2\pi b}{a},$$

[On simplifying the indeterminate form with help of L'Hospital's rule]

$\therefore \quad u = (k/2a)\coth(2\pi b/a) \quad$ and $\quad v = 0$.

Hence the required velocity $= \sqrt{(u^2+v^2)} = u = (k/2a)\coth(2\pi b/a)$

EXERCISE 11 (D)

1. An infinite row of parallel rectilinear vortices, each of circulation $2\pi k$, intersects the z-plane at right angles at the points $z = na + ib/2$, where $n = 0, \pm 1, \pm 2, \ldots$ Another parallel row in which each vortex has circulation $-2\pi k$, meets the z-plane at the points $z = (n + 1/2)a - ib/2$, the two rows together forming a Karman street. Show that the complex velocity potential in the z-plane is

$$ik\log\frac{\pi}{a}(z-ib) - ik\log\sin\frac{\pi}{a}\left(z - \frac{1}{2}a + \frac{1}{2}ib\right)$$

and prove that the velocity of advance of the rows is $(\pi k/a)\tanh(\pi b/a)$.

The following formula may be assumed: $\sin x = \left(1 - \frac{x^2}{1^2 \times \pi^2}\right)\left(1 - \frac{x^2}{2^2 \times \pi^2}\right)\left(1 - \frac{x^2}{3^2 \times \pi^2}\right)\ldots$

2. Who do you mean by Karman's vortex street? Show that the entire upper row advances with the same velocity. **(Agra 2007)**

11.20. Rectilinear vortex with circular section.

Here we wish to study a case of vortex with finite cross section. Let the section be a circle of radius a, and let ζ be the uniform spin throughout the circle $r = a$, the vortex being rectilinear. Then we know that the stream function satisfies the equations

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 2\zeta, \text{ inside the vortex } r = a \qquad \ldots(1)$$

and
$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0 \text{ outside the vortex } r = a \qquad \ldots(2)$$

Due to symmetry ψ is a function of r alone and hence (1) and (2), in polar, coordinates, take the form

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} = 2\zeta, \text{ when } r < a \qquad \ldots(3)$$

and
$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} = 0, \text{ when } r > a \qquad \ldots(4)$$

Solving (3) and (4), we obtain

$$\psi = A\log r + B + (1/2)\times\zeta r^2, \text{ when } r < a \qquad \ldots(5)$$

and
$$\psi = C\log r + D, \text{ when } r > a \qquad \ldots(6)$$

Since ψ is not infinite when $r = 0$, we take $A = 0$ in (5). Thus (5) reduces to

$$\psi = B + (1/2)\times\zeta r^2, \text{ when } r < a. \qquad \ldots(7)$$

For the motion to be continuous at the surface, ψ and $\partial\psi/\partial r$ must be the same on $r = a$, as calculated from within and from outside. Hence we must have

$$B + (1/2)\times\zeta a^2 = C\log a + D, \qquad \ldots(8)$$

and
$$\zeta a = C/a. \qquad \ldots(9)$$

From (8) and (9), $C = \zeta a^2$ and $B = \zeta a^2 \log a + D - (1/2)\times\zeta a^2$.

Using these values and omitting an additive constant so as to make $\psi = 0$ on $r = a$, (7) and (6) give

$$\psi = -(1/2)\times\zeta(a^2 - r^2), \text{ when } r < a \qquad \ldots(10)$$

and
$$\psi = \zeta a^2 \log(r/a), \text{ when } r > a \qquad \ldots(11)$$

The velocity is wholly transversal both inside and outside the vortex, its values being ζr and $\zeta a^2/r$. Outside the vortex the motion is irrotational and the velocity potential can be obtained by assuming

$$w = i\zeta a^2 \log(z/a) \qquad \ldots(12)$$

Taking $w = \phi + i\psi$ and equating real and imaginary parts, we get

$$\psi = \zeta a^2 \log(r/a) \quad \text{and} \quad \phi = -\zeta a^2\theta. \qquad \ldots(13)$$

Let k denote the circulation or the strength of the vortex. Then, we have

$$k = -\int_0^{2\pi} \frac{\partial\phi}{r\partial\theta}\cdot r d\theta = \zeta a^2 \int_0^{2\pi} d\theta = 2\pi\zeta a^2 \qquad \ldots(14)$$

$\therefore$ From (13), $\quad \psi = (k/2\pi)\log(r/a) \quad$ and $\quad \phi = -(k\theta/2\pi), \quad$ ...(15)

as for a thin filament.

For a point inside the vortex, the velocity is given by $\partial\psi/\partial r = kr/2\pi a^2$ and for a point outside, velocity is $k/2\pi r$ and these are in transverse directions. It follows that the velocity at the centre of the vortex is zero. Thus the vortex has no tendency to move itself. If the vortex is left alone in an infinite liquid, it will simply rotate as a rigid body without advancing through the liquid. Since the motion is steady outside the vortex, the pressure equation is

$$p/\rho + q^2/2 = C$$

or $$\frac{p}{\rho} = C - \frac{k^2}{8\pi^2 r^2}, \quad \text{as} \quad q = \frac{k}{2\pi r} \quad \text{when } r > a \qquad \text{...(16)}$$

At infinity, $p = \Pi$ (say), $r \to \infty$. So (16) gives $C = \Pi/\rho$. With this value of C, (16) reduces to

$$\frac{p}{\rho} = \frac{\Pi}{\rho} - \frac{k^2}{8\pi^2 r^2} \qquad \text{...(17)}$$

Again, inside the vortex the liquid rotates uniformly with angular velocity ζ so that

$$(1/\rho)\, d\rho = \zeta^2 r\, dr.$$

Integrating $\quad p/\rho = (1/2)\times\zeta^2 r^2 + D,\ D$ being an arbitrary constant ...(18)

Let P be the pressure at the centre of the vortex. Then $p = P$ when $r = 0$ so that $D = P/\rho$. Then (18) reduces to

$$\frac{p}{\rho} = \frac{1}{2}\zeta^2 r^2 + \frac{P}{\rho} = \frac{k^2 r^2}{8\pi^2 a^4} + \frac{P}{\rho}, \text{ using (14)} \qquad \text{...(19)}$$

Since the values of p must be same when $r = a$, (17) and (19) give

$$\frac{\Pi}{\rho} - \frac{k^2}{8\pi^2 a^2} = \frac{k^2}{8\pi^2 a^2} + \frac{P}{\rho} \quad \text{so that} \quad P = \Pi - \frac{k^2\rho}{4\pi^2 a^2}$$

With this value of P (19) gives

$$\frac{p}{\rho} = \frac{k^2 r^2}{8\pi^2 a^4} + \frac{\Pi}{\rho} - \frac{k^2}{4\pi^2 a^2}, \text{ when } r < a \qquad \text{...(20)}$$

Equation (20) shows that if $\Pi < (k^2\rho/4\pi^2 a^2)$ there will be a value of $r < a$ for which $p < 0$, implying that a cylindrical hollow must exist inside the vortex.

Hence a cyclic irrotational motion is possible outside a cylindrical hollow, the condition for which is $p = 0$ when $r = a$; that is $\Pi = k^2\rho/8\pi^2 a^2$.

11.21. Rankine's combined vortex.

It consists of a circular vortex with axis vertical in a mass of liquid moving irrotationaly under the action of gravity, the upper surface being exposed to atmospheric pressure. Here the external forces are derivable from the potential gz. The kinematical equations are the same as in Ar. 11.20. Hence the pressures are (measuring z downwards)

$$\frac{p}{\rho} = A - \frac{k^2}{8\pi^2 r^2} + gz, \quad \text{when} \quad r > a \qquad \text{...(1)}$$

and $$\frac{p}{\rho} = B + \frac{k^2 r^2}{8\pi^2 a^4} + gz, \quad \text{when} \quad r < a \qquad \text{...(2)}$$

Since p is the same on $r = a$, we have

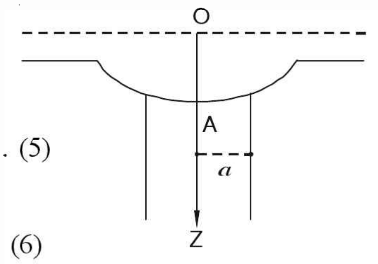
$$A - \frac{k^2}{8\pi^2 a^2} = B + \frac{k^2}{8\pi^2 a^2} \quad \text{so that} \quad A = B + \frac{k^2}{4\pi^2 a^2}.$$

Thus, $$\frac{p}{\rho} = B + \frac{k^2}{4\pi^2 a^2} - \frac{k^2}{8\pi^2 r^2} + gz, \quad \text{for } r > a, \quad \text{by (1)} \qquad \text{...(3)}$$

and $$\frac{p}{\rho} = B + \frac{k^2 r^2}{8\pi^2 r^4} + gz, \quad \text{for } r < a, \quad \text{by (2)} \qquad \text{...(4)}$$

Since when $r \to \infty$, $z \to 0$ and $p \to \Pi$, we get

$$\frac{\Pi}{\rho} = B + \frac{k^2}{4\pi^2 a^4} \quad \text{so that} \quad B = \frac{\Pi}{\rho} - \frac{k^2}{4\pi^2 a^2}$$



$\therefore$ by (3), $$\frac{p}{\rho} = \frac{\Pi}{\rho} - \frac{k^2}{8\pi^2 r^2} + gz, \quad \text{for } r > a \quad \text{... (5)}$$

and by (4), $$\frac{p}{\rho} = \frac{\Pi}{\rho} - \frac{k^2}{4\pi^2 a^2} + \frac{k^2 r^2}{8\pi^2 a^4} + gz, \quad \text{for } r < a \quad \text{... (6)}$$

Since $p = \Pi$ on the free surface, the free surface is given by

$$gz = k^2/8\pi^2 r^2 \quad \text{for } r > a, \text{ using (5)} \qquad \text{...(7)}$$

$$gz = (k^2/4\pi^2 a^2)(1 - r^2/2a^2) \quad \text{for } r < a, \text{ using (6)} \qquad \text{...(8)}$$

The depth of the depression z' at A can be calculated by putting $r = 0$ in (8). Thus

$$gz' = k^2/4\pi a^2 \quad \text{or} \quad z' = k^2/4\pi^2 a^2 g \qquad \text{...(9)}$$

11.22. Rectilinear vortices with elliptic section.

To show that a rectilinear vortex whose cross section is an ellipse and whose spin is constant can maintain its form rotating as if it were a solid cylinder in an infinite liquid.

Let the elliptic cross section $$x^2/a^2 + y^2/b^2 = 1 \qquad \text{...(1)}$$

be given by $\xi = \alpha$, where $x + iy = c\cosh(\xi + i\eta)$

*so that $a = c\cosh\alpha$, $b = c\sinh\alpha$, $a - b = ce^{-\alpha}$, $c^2 = a^2 - b^2$ ...(2)

Let Ω be the constant spin. Then the stream function satisfies the equations

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 2\Omega, \text{ inside the boundary } \xi = \alpha \qquad \text{...(3)}$$

and $$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0, \text{ outside the boundary } \xi = \alpha \qquad \text{...(4)}$$

As in articles 7.16 *and* 7.19, *we know that if a rigid elliptic cylinder of semi-axes a, b rotates with uniform angular velocity* ω *in an infinite mass of liquid the stream function for cyclic irrotational motion with circulation k is given by*

$$\psi = (1/4) \times \omega\,(a+b)^2 e^{-2\xi} \cos 2\eta + k\xi/2\pi \qquad \text{...(5)}$$

But $k = 2 \times$ angular velocity $\times$ area of cross section

* For details refer Art. 7.13.

so that $$k = 2\Omega \pi ab \qquad ...(6)$$

Suppose that inside the boundary $\xi = \alpha$, $$\psi = \Omega (Ax^2 + By^2) \qquad ...(7)$$

From (3) and (7), $$A + B = 1 \qquad ...(8)$$

If l, m be direction cosines of the normal to the section (1), then

$$l = x / pa^2, \qquad m = y / pb^2 \qquad \text{where} \qquad p = (x^2 / a^4 + y^2 / b^4)^{1/2} \qquad ...(9)$$

The boundary condition is that the velocity of the liquid normal to the boundary must be equal to that of the boundary.

Now the velocity components of any point of the elliptic section are $-\omega y$ and ωx. Again from (7), the velocity components of any point inside the boundary are given by

$$u = -\partial\psi / \partial y = -2By \qquad \text{and} \qquad v = \partial\psi / \partial x = 2Ax.$$

So the boundary condition $lu + mv = (-\omega y) l + (\omega x) m$ reduces to

$$\frac{x}{pa^2}(-2By) + \frac{y}{pb^2}(2Ax) = (-\omega y)\frac{x}{pa^2} + (\omega x)\frac{y}{pb^2}$$

i.e. $$Aa^2 - Bb^2 = (\omega / 2) \times (a^2 - b^2). \qquad ...(10)$$

An additional condition of the continuity of the tangential velocity at the boundary requires that the values of $\partial\psi / \partial\xi$ given by (5) and (7) must be equal. Putting $x = c \cosh\alpha \cos\eta$, $y = c \sinh\alpha \sin\eta$ in (7), the condition at the boundary reduces to

$$-(\omega / 2) \times (a+b)^2 e^{-2\alpha} \cos 2\eta + \Omega ab = \Omega c^2 \cosh\alpha \sinh\alpha \, \{A + B + (A - B) \cos 2\eta\}$$

for all values of η from 0 to 2π. Equating coefficients of $\cos 2\eta$, we get

$$-\frac{1}{2}\omega (a+b)^2 e^{-2\alpha} = \Omega c^2 (A - B) \cosh\alpha \sinh\alpha \quad \text{or} \quad -\frac{1}{2}\omega (a+b)^2 \frac{(a-b)^2}{c^2} = \Omega c^2 (A - B)\frac{ab}{c^2}, \quad \text{by (2)}$$

or $$A - B = -\frac{\omega}{2\Omega}\,\frac{a^2 - b^2}{ab} \qquad ...(11)$$

From (8), (10) and (11), we have $A = b/(a + b)$, $\qquad B = a/(a + b) \qquad ...(12)$

and $$\omega = \frac{2ab\Omega}{(a+b)^2} = \frac{2\sqrt{1-e^2}\,\Omega}{(1+\sqrt{1-e^2})^2}, \quad \text{as} \quad b^2 = a^2 (1 - e^2) \qquad ...(13)$$

Equation (13) gives the velocity of rotation of the cylinder as a whole in terms of the spin Ω and eccentricity e of the elliptic section.

To find the paths of the particle. Let (x, y) be coordinates of a particle of the vortex referred to the axes of the cross section. Then, we have

$$\dot{x} - \omega y = u = -\partial\psi / \partial y = -2\Omega By = -\omega y (a+b) / b,$$

and $$\dot{y} + \omega x = v = \partial\psi / \partial x = 2\Omega Ax = \omega x (a+b) / a.$$

Hence $$\dot{x} = -\omega ya / b \qquad \text{and} \qquad \dot{y} = \omega yb / a. \qquad ...(14)$$

From (14), $$\ddot{x} = -\omega \dot{y} a / b = -\omega^2 x. \qquad ...(15)$$

Solution of (15), is $x = aL \cos(\omega t + \varepsilon)$, L and ε being arbitrary constants ...(16)

From (16), $$\dot{x} = -aL\omega \sin(\omega t + \varepsilon) \qquad ...(17)$$

$\therefore$ From (14) and (17), $$-a\omega y/b = -aL\omega \sin(\omega t + \varepsilon)$$

so that $$y = Lb\sin(\omega t + \varepsilon). \qquad \text{...(18)}$$

From (16) and (18), $$x^2/a^2 + y^2/b^2 = L^2, \qquad \text{...(19)}$$

showing that the paths of the particles of the vortex relative to the boundary are similar ellipses, and the period of the relative motion is the same as that of the rotation of the cylinder.

Here, $$\text{periodic time} = \frac{2\pi}{\omega} = \frac{\pi(a+b)^2}{\Omega ab}.$$

Illustrative Example. *An elliptic cylinder is filled with liquid which has molecular rotation* ω *at every point, and whose particles move in planes perpendicular to the axes; prove that the streamlines are similar ellipses described in periodic time* $\pi(a+b)^2/\omega ab$.

Sol. Since there is liquid only inside the cylinder, the stream function ψ satisfies

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 2\omega. \qquad \text{...(1)}$$

Assume that $$\psi = \omega(Ax^2 + By^2) \qquad \text{...(2)}$$

From (1) and (2), $$A + B = 1 \qquad \text{...(3)}$$

From (2) $$u = -\partial\psi/\partial y = -2B\omega y, \qquad v = \partial\psi/\partial x = 2A\omega x \qquad \text{...(4)}$$

If l, m be direction cosines of normals to the $$x^2/a^2 + y^2/b^2 = 1, \qquad \text{...(5)}$$

then, $$l = x/pa^2, \qquad m = y/pb^2 \qquad \text{where} \qquad p = (x^2/a^4 + y^2/b^4)^{1/2} \qquad \text{...(6)}$$

Since the elliptic boundary is at rest, the boundary condition $lu + mv = 0$ reduces to

$$(x/pa^2)(-2B\omega y) + (y/pb^2)(2A\omega x) = 0 \qquad \text{or} \qquad Aa^2 - Bb^2 = 0 \qquad \text{...(7)}$$

From (3) and (7), $$A = b^2/(a^2+b^2) \qquad \text{and} \qquad B = a^2/(a^2+b^2) \qquad \text{...(8)}$$

$\therefore$ (2) gives $$\psi = \frac{\omega}{a^2+b^2}(b^2x^2 + a^2y^2) \qquad \text{...(9)}$$

Let (x, y) be any point in the fluid. Then

$$\dot{x} = -\frac{\partial\psi}{\partial y} = -\frac{2a^2y\omega}{a^2+b^2} \qquad \text{and} \qquad \dot{y} = \frac{\partial\psi}{\partial x} = \frac{2b^2x\omega}{a^2+b^2} \qquad \text{...(10)}$$

From (10), $$\ddot{x} = -\frac{2a^2\omega}{a^2+b^2}\dot{y} = -\frac{4a^2b^2\omega^2}{(a^2+b^2)^2}x \qquad \text{...(11)}$$

Solving (11), $$x = La\cos\left(\frac{2ab\omega}{a^2+b^2}t + \varepsilon\right) \qquad \text{...(12)}$$

From (12) $$\dot{x} = -La\cdot\frac{2ab\omega}{a^2+b^2}\sin\left(\frac{2ab\omega}{a^2+b^2}t + \varepsilon\right) \qquad \text{...(13)}$$

From (10) and (13), $$-\frac{2a^2y\omega}{a^2+b^2} = -\frac{2a^2b\omega L}{a^2+b^2}\sin\left(\frac{2ab\omega}{a^2+b^2}t + \varepsilon\right)$$

so that $$y = Lb\sin\left(\frac{2ab\omega}{a^2+b^2}t + \varepsilon\right) \qquad \text{...(14)}$$

From (12) and (14), $$x^2/a^2 + y^2/b^2 = L^2,$$

showing that the paths of the particles are similar ellipses and periodic time T is given by

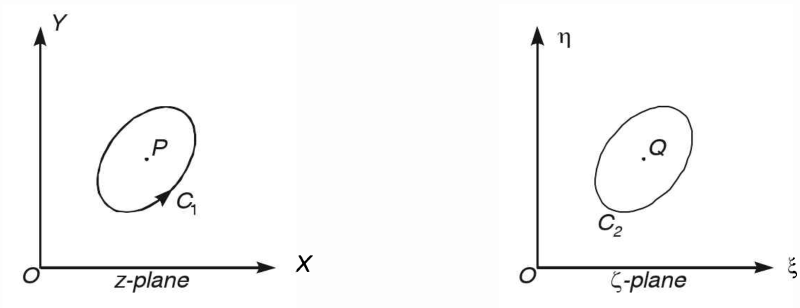
$$T = 2\pi \div \left[\frac{4a^2b^2\omega^2}{(a^2+b^2)^2}\right]^{1/2} = \frac{\pi(a^2+b^2)}{\omega ab}$$

EXERCISE 11 (E)

1. Show that a rectilinear vortex whose cross-section is an ellipse and whose spin is constant can maintain its form rotating as if it were a solid cylinder in an infinite liquid. Prove that the paths of the particles of the vortex relative to the boundary are similar ellipses, and the period of the relative motion is the same as that of the rotation of the cylinder.

11.23. Conformal transformation. Routh's theorem.

Let k be the strength of a vortex at Q in the ζ-plane and let P be the point corresponding to Q in the z-plane, by the conformal mapping $\zeta = f(z)$. Since the complex potential w, stream function ψ and velocity potential ϕ possess equal values at the corresponding points in the two planes, it follows that $-\int d\phi$ taken round corresponding curves in the two planes must be the same, and hence the strengths of vortices at the corresponding points must be equal. Let P be z_1 and Q be ζ_1. Then the complex potential W in ζ-plane at any point ζ is given by



$$W = W_\zeta + (ik/2\pi)\log(\zeta - \zeta_1), \qquad \ldots(1)$$

where W_ζ is the complex potential omitting the term due to vortex ζ_1. Similarly, for the z-plane,

$$W = W_z + (ik/2\pi)\log(z - z_1) \qquad \ldots(2)$$

Equating the values of W given by (1) and (2), we get

$$W_z = W_\zeta + \frac{ik}{2\pi}\log\frac{\zeta - \zeta_1}{z - z_1}$$

The velocity of vortex ζ_1 is obtained from $\qquad u - iv = -\left(\dfrac{dW_\zeta}{d\zeta}\right)_{\zeta=\zeta_1}$

Similarly, the velocity of vortex z_1 can be obtained.
Then for the motion of the vortex z_1, we get

$$W_{z_1} = W_{\zeta_1} + \left[\frac{ik}{2\pi}\log\frac{\zeta - \zeta_1}{z - z_1}\right]_{z = z_1,\, \zeta = \zeta_1}$$

If $\psi_1(\zeta_1, \eta_1)$ and ψ_2 be the stream functions giving the motions of the vortices, then

$$\psi_2 = \psi_1 + \psi, \qquad \text{where} \qquad \ldots(3)$$

$$\frac{\partial \psi}{\partial y_1} = \text{real part of}\left[\frac{\partial}{\partial y}\frac{k}{2\pi}\log\frac{\zeta - \zeta_1}{z - z_1}\right]_{z = z_1} = \text{real part of}\left[\frac{ik}{2\pi}\frac{d}{dz}\log\frac{\zeta - \zeta_1}{z - z_1}\right]_{z = z_1}, \quad \text{as } \frac{\partial}{\partial y} = i\frac{d}{dz}$$

Now expanding $\zeta-\zeta_1$ in terms of $z-z_1$, we get

$$\zeta-\zeta_1=(z-z_1)\left(\frac{d\zeta}{dz}\right)_1+\frac{1}{2}(z-z_1)^2\left(\frac{d^2\zeta}{dz^2}\right)_1+\ldots \quad \text{or} \quad \frac{\zeta-\zeta_1}{z-z_1}=\left(\frac{d\zeta}{dz}\right)_1+\frac{1}{2}(z-z_1)\left(\frac{d^2\zeta}{dz^2}\right)_1+\ldots$$

$$\therefore \text{ Real part of } \frac{ik}{2\pi}\frac{d}{dz}\log\frac{\zeta-\zeta_1}{z-z_1}=\text{real part of } \frac{ik}{2\pi}\frac{d}{dz}\log\left[\left(\frac{d\zeta}{dz}\right)_1+\frac{1}{2}(z-z_1)\left(\frac{d^2\zeta}{dz^2}\right)_1+\ldots\right]$$

$$=\text{real part of } \frac{ik}{2\pi}\frac{\frac{1}{2}\left(\frac{d^2\zeta}{dz^2}\right)_1+\ldots}{\left(\frac{d\zeta}{dz}\right)_1+\frac{1}{2}(z-z_1)\left(\frac{d^2\zeta}{dz^2}\right)_1+\ldots}=\text{real part of } \frac{ik}{4\pi}\frac{\left(\frac{d^2\zeta}{dz^2}\right)_1}{\left(\frac{d\zeta}{dz}\right)_1},$$

[approximately as $z \to z_1$, $\zeta\to\zeta_1$, as $i(d/dz)=\partial/\partial y$]

$$=\frac{k}{4\pi}\frac{\partial}{\partial y}\log\left|\frac{d\zeta}{dz}\right|_1$$

Therefore, $$\psi=\frac{k}{4\pi}\log\left|\frac{d\zeta}{dz}\right|_1$$

Thus (3) reduces to $$\psi_2(x_1,y_1)=\psi_1(\xi_1,\eta_1)+(k/4\pi)\log|d\zeta/dz|_1$$
which is the well known *Routh's theorem*.

11.24. Illustrative solved examples.

Ex. 1. *To find the path of a vortex in the angle between two planes to which it is parallel.* **[Agra 2002, 2010; Kanpur 2002]**

Sol. Let the two planes in z-plane be inclined at an angle π/n. Consider the conformal transformation $$\zeta=c(z/c)^n. \qquad \ldots(1)$$

If $\zeta=\rho e^{i\alpha}$ and $z=re^{i\theta}$, then (1) reduces to $\rho\, e^{i\alpha}=c(re^{i\theta}/c)^n$

so that $$\rho=c(r/c)^n \quad \text{and} \quad \alpha=n\theta \qquad \ldots(2)$$

This transforms the $\xi=$ axis $(\alpha=0, \alpha=\pi)$ in ζ-plane to the straight lines $\theta=0$, $\theta=\pi/n$ in z-plane. Let $P(x_1,y_1)$ in z-plane be mapped on $Q(\xi_1,\eta_1)$ in ζ-plane.

The vortex ζ_1 is in a plane bounded by ξ-axis. The image of vortex of strength k at $Q(\zeta_1)$ with respect to ξ-axis is a vortex $(-k)$ at $Q'(\bar{\zeta}_1)$. The complex potential W at any point ζ in ζ-plane is given by (Taking $\zeta=\xi+i\eta$, $\zeta_i=\xi_1+i\eta_i$ and $\bar{\zeta}_1=\xi_1-i\eta_i$)

$$W=\frac{ik}{2\pi}\log(\zeta-\zeta_1)-\frac{ik}{2\pi}\log(\zeta-\bar{\zeta}_1)=\frac{ik}{2\pi}\log\frac{(\xi-\xi_1)+i(\eta-\eta_1)}{(\xi-\xi_1)+i(\eta+\eta_1)}$$

$$\therefore \quad \psi=\frac{k}{4\pi}\log\frac{(\xi-\xi_1)^2+(\eta-\eta_1)^2}{(\xi-\xi_1)^2+(\eta+\eta_1)^2}=\frac{k}{4\pi}\log\frac{\xi^2+\eta^2+\xi_1^2+\eta_1^2-2(\xi\xi_1+\eta\eta_1)}{\xi^2+\eta^2+\xi_1^2+\eta_1^2-2(\xi\xi_1-\eta\eta_1)}$$

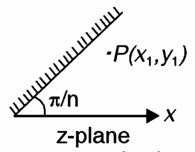
Now, (1) $\Rightarrow$ $$\xi+i\eta=(r^n/c^{n-1})(\cos n\theta+i\sin\theta)$$

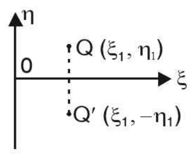
$$\therefore \quad \xi=(r^n/c^{n-1})\cos n\theta \quad \text{and} \quad \eta=(r^n/c^{n-1})\sin n\theta.$$

Hence stream function for the vortex $P(z_1)$ [*i.e.* (x_1,y_1) or (r_1,θ_1)] is given by

Hence $$\psi=\frac{k}{4\pi}\log\frac{r^{2n}+r_1^{2n}-2r^n r_1^n\cos n(\theta-\theta_1)}{r^{2n}+r_1^{2n}-2r^n r_1^n\cos n(\theta+\theta_1)}$$

The paths of particles are given by ψ = const.





Again $\qquad \left|\dfrac{d\zeta}{dz}\right| = \dfrac{d\rho}{dr} = n\left(\dfrac{r}{c}\right)^{n-1}$

So for the path of the vortex z_1, using Routh's theorem, we obtain

$$\psi_2(x_1, y_1) = \psi_1(\xi_1, \eta_1) + (k/4\pi)\log r_1^{n-1}, \text{ omitting the constant}$$

But, by Art. 11.7, $\qquad \psi_1(\xi_1, \eta_1) = -(k/4\pi)\log \eta_1$

$$\therefore \qquad \psi_2(x_1, y_1) = -(k/4\pi)\log(r_1^n \sin n\theta_1) + (k/4\pi)\log r_1^{n-1}$$

$$= -(k/4\pi)\log(r_1 \sin n\theta_1), \text{ omitting the constant.}$$

Hence the path of the vortex z_1 is $\psi_2 = \text{const.}$, *i.e.* $r_1 \sin n\theta_1 = \text{const.}$

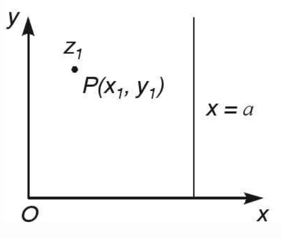
Thus the required path is given by $\qquad r \sin n\theta = \text{constant.}$

Ex. 2. *The space enclosed between the planes $x = 0$, $x = a$, $y = 0$ on the positive side of $y = 0$ is filled with uniform incompressible liquid. A rectilinear vortex parallel to the axis of z has coordinates (x_1, y_1). Determine the velocity at any point of the liquid and show that the path of the vortex is given by* $\cot^2(\pi x/a) + \coth^2(\pi y/a) = \text{constant.}$

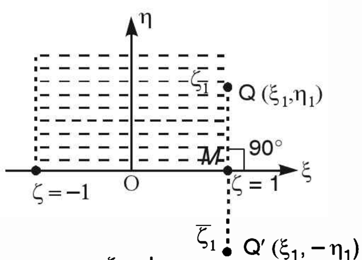
[Gorakhpur 2004, 06; Purvanchal 2004, 06; Kanpur 2003]

Sol. Consider the transformation $\qquad \zeta = -\cos(\pi z/a), \qquad \ldots(1)$

which transforms the semi-infinite strip in z-plane to the upper half of the ζ-plane such that $z = 0$ goes to $\zeta = -1$ and $z = a$ to $\zeta = 1$. Let $P(x_1, y_1)$ in z-plane be mapped on $Q(\xi_1, \eta_1)$ in ζ-plane.



z - plane



ζ - plane

The image of vortex of strength k at Q relative to the boundary ξ-axis is a vortex of strength $-k$ at $Q'(\bar{\zeta}_1)$. The complex potential at any point in ζ- plane (not occupied by the vortex) is given by

$$w = \frac{ik}{2\pi}\left[\log(\zeta - \zeta_1) - \log(\zeta - \bar{\zeta}_1)\right] = \frac{ik}{2\pi}\left[\log\left(-\cos\frac{\pi z}{a} + \cos\frac{\pi z_1}{a}\right) - \log\left(-\cos\frac{\pi z}{a} + \cos\frac{\pi \bar{z}_1}{a}\right)\right]$$

$$\frac{dw}{dz} = \frac{ik}{2\pi}\cdot\frac{\pi}{a}\left[\frac{\sin(\pi z/a)}{-\cos(\pi z/a) + \cos(\pi z_1/a)} - \frac{\sin(\pi z/a)}{-\cos(\pi z/a) + \cos(\pi \bar{z}_1/a)}\right]$$

or
$$\frac{dw}{dz} = \frac{ik}{2a}\,\frac{\sin\dfrac{\pi z}{a}\left(\cos\dfrac{\pi \bar{z}_1}{a} - \cos\dfrac{\pi z_1}{a}\right)}{\left(-\cos\dfrac{\pi z}{a} + \cos\dfrac{\pi z_1}{a}\right)\left(-\cos\dfrac{\pi z}{a} + \cos\dfrac{\pi \bar{z}_1}{a}\right)}$$

Let $\lambda = \pi/2a$ so that $\pi/a = 2\lambda$. Then the above equaton reduces to

$$\therefore \qquad \frac{dw}{dz} = \frac{ik}{2a}\ \frac{\sin 2\lambda z\,(\cos 2\lambda \overline{z}_1 - \cos 2\lambda z_1)}{(-\cos 2\lambda z + \cos 2\lambda z_1)\,(-\cos 2\lambda z + \cos 2\lambda \overline{z}_1)}$$

$$= \frac{ik}{2a}\ \frac{\sin 2\lambda z \times 2 \sin \lambda\,(z_1 + \overline{z}_1) \sin \lambda\,(z_1 - \overline{z}_1)}{2 \sin \lambda (z + z_1)\, \sin \lambda\,(z - z_1) \times 2 \sin \lambda\,(z + \overline{z}_1) \sin \lambda\,(z - \overline{z}_1)}$$

$$= \frac{ik}{4a}\ \frac{\sin 2\lambda z\, \sin 2\lambda x_1\, \sin 2\lambda i y_1}{\sin \lambda\,(z + z_1) \sin \lambda\,(z - z_1) \sin \lambda\,(z + \overline{z}_1) \sin \lambda\,(z - \overline{z}_1)}$$

$$[\because z_1 = x_1 + iy \ \textit{and}\ \overline{z}_1 = x_1 - iy_i]$$

$\therefore$ Velocity at any point

$$= \left|\frac{dw}{dz}\right| = \left| \frac{-k \sin 2\lambda x_1 \sinh 2\lambda y_1 \sin 2\lambda z}{4a \sin \lambda\,(z + z_1) \sin \lambda\,(z - z_1) \sin \lambda\,(z + \overline{z}_1) \sin \lambda\,(z - \overline{z}_1)} \right|$$

$$[\because \sin 2\lambda i y_1 = i \sinh 2\lambda y_1,\ |\,i\,| = 1 \text{ and } i^2 = -1]$$

$$= \frac{k \sin 2\lambda x_1 \sinh 2\lambda y_1}{4a} \left| \frac{\sin 2\lambda z}{\sin \lambda\,(z + z_1) \sin \lambda\,(z - z_1) \sin \lambda\,(z + \overline{z}_1) \sin \lambda\,(z - \overline{z}_1)} \right|$$

Second part : The stream function $\chi_1(\xi_1, \eta_1)$ due to vortex at (ξ_1, η_1) is ζ-plane is

$$\chi_1(\xi_1, \eta_1) = -(k/4\pi) \log \eta_1 \qquad \text{...(2)}$$

From (1), $\xi_1 + i\eta_1 = -\cos\dfrac{\pi}{a}(x_1 + iy_1) = -\left[\cos\dfrac{\pi x_1}{a} \cos\dfrac{i\pi y_1}{a} - \sin\dfrac{\pi x_1}{a} \sin\dfrac{i\pi y_1}{a}\right]$

$$= -\cos\frac{\pi x_1}{a} \cosh\frac{\pi y_1}{a} + i \sin\frac{\pi x_1}{a} \sinh\frac{\pi y_1}{a}$$

$$\therefore \qquad \eta_1 = \sin(\pi x_1 / a) \sinh(\pi y_1 / a)$$

$\therefore$ From (2),
$$\chi_1(\xi_1, \eta_1) = -\frac{k}{4\pi} \log\left(\sin\frac{\pi x_1}{a} \sinh\frac{\pi y_1}{a}\right) \qquad \text{...(3)}$$

From (1),
$$d\zeta / dz = (\pi / a) \sin(\pi z / a)$$

$$\therefore \quad \left|\frac{d\zeta}{dz}\right|_1 = \left|\frac{\pi}{a} \sin\frac{\pi z_1}{a}\right| = \left|\frac{\pi}{a} \sin\frac{\pi}{a}(x_1 + iy_1)\right| = \frac{\pi}{a}\left|\sin\frac{\pi x_1}{a} \cos\frac{i\pi y_1}{a} + \cos\frac{\pi x_1}{a} \sin\frac{i\pi y_1}{a}\right|$$

$$= \frac{\pi}{a}\left|\sin\frac{\pi x_1}{a} \cosh\frac{\pi y_1}{a} + i \cos\frac{\pi x_1}{a} \sinh\frac{\pi y_1}{a}\right| = \frac{\pi}{a}\left(\sin^2\frac{\pi x_1}{a} \cosh^2\frac{\pi y_1}{a} + \cos^2\frac{\pi x_1}{a} \sinh^2\frac{\pi y_1}{a}\right)^{1/2}$$

By Routh's theorem, we have
$$\chi_2(x_1, y_1) = \chi_1(\xi_1, \eta_1) + \frac{k}{4\pi} \log\left|\frac{d\zeta}{dz}\right|_1$$

$$\therefore \quad \chi_2(x_1, y_1) = -\frac{k}{4\pi} \log\left(\sin\frac{\pi x_1}{a} \sinh\frac{\pi y_1}{a}\right)$$

$$+ \frac{k}{4\pi} \log\left[\frac{\pi}{a}\left(\sin^2\frac{\pi x_1}{a} \cosh^2\frac{\pi y_1}{a} + \cos^2\frac{\pi x_1}{a} \sinh^2\frac{\pi y_1}{a}\right)^{1/2}\right]$$

$$= \frac{k}{4\pi} \log\left[\frac{\sin^2(\pi x_1 / a) \cosh^2(\pi y_1 / a) + \cos^2(\pi x_1 / a) \sinh^2(\pi y_1 / a)}{\sin^2(\pi x_1 / a) \sinh^2(\pi y_1 / a)}\right]^{1/2} + \frac{k}{4\pi} \log\frac{\pi}{a}$$

$$\therefore \qquad \chi_2(x_1, y_1) = \frac{k}{4\pi}\left[\log\left(\coth^2\frac{\pi y_1}{a} + \cot^2\frac{\pi x_1}{a}\right)^{1/2} + \log\frac{\pi}{a}\right] \qquad \text{...(4)}$$

The path of the vortex (x_1, y_1) is $\qquad \chi_2(x_1, y_1) = \text{constant}.$

or
$$\frac{k}{4\pi}\left[\frac{1}{2}\log\left(\coth^2\frac{\pi y_1}{a}+\cot^2\frac{\pi x_1}{a}\right)+\log\frac{\pi}{a}\right] = \text{constant},$$

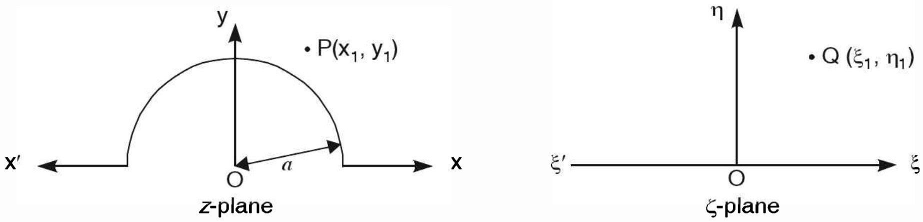
giving
$$\coth^2(\pi y_1/a)+\cot^2(\pi x_1/a) = \text{constant}.$$

Hence the path of vortex at any point is
$$\coth^2(\pi y/a)+\cot^2(\pi x/a) = \text{constant}.$$

Ex. 3. *A vortex is in an infinite liquid occupying the upper half of the z-plane bounded by a circle of radius a, centre O and parts of the x-axis outside the circle. Find the path of the vortex.* **(Agra 2006)**

Sol. Consider the transformation $\qquad \zeta = z + a^2/z, \qquad$ ...(1)

which transforms the portion in the z-plane occupied by the liquid into the upper half of the ζ-plane. Let $P(x, y)$ in z-plane be mapped on $Q(\xi_1, \eta_1)$ in ζ-plane.



The stream function $\chi_1(\xi_1, \eta_1)$ due to vortex at (ξ_1, η_1) in ζ-plane is given by
$$\chi_1(\xi_1, \eta_1) = -(k/4\pi)\log\eta_1 \qquad ...(2)$$

From (1), $\quad \xi_1 + i\eta_1 = x_1 + iy_1 + \dfrac{a^2}{x_1+iy_1} \quad$ or $\quad \xi_1 + i\eta_1 = x_1 + iy_1 + \dfrac{a^2(x_1-iy_1)}{x_1^2+iy_1^2}$

Hence,
$$\eta_1 = y_1 - \frac{a^2 y_1}{x_1^2+y_1^2} = y_1\frac{x_1^2+y_1^2-a^2}{x_1^2+y_1^2}$$

$\therefore$ From (2),
$$\chi_1(\xi_1, \eta_1) = -\frac{k}{4\pi}\log\frac{y_1(x_1^2+y_1^2-a^2)}{x_1^2+y_1^2} \qquad ...(3)$$

From (1),
$$\frac{d\zeta}{dz} = 1-\frac{a^2}{z^2} = \frac{z^2-a^2}{z^2} = -\frac{x^2-y^2-a^2+2ixy}{x^2-y^2+2ixy}$$

$\therefore$
$$\left|\frac{d\zeta}{dz}\right|_1 = \left[\frac{(x_1^2-y_1^2-a^2)^2+4x_1^2y_1^2}{(x_1^2-y_1^2)^2+4x_1^2y_1^2}\right]^{1/2} = \left[\frac{(x_1^2+y_1^2-a^2)^2+4a^2y_1^2}{(x_1^2+y_1^2)^2}\right]^{1/2}$$

By Routh's theorem, we have $\qquad \chi_2(x_1, y_1) = \chi_1(\xi_1, \eta_1) + \dfrac{k}{4\pi}\log\left|\dfrac{d\zeta}{dz}\right|_1$

$$\therefore\ \chi_2(x_1, y_1) = -\frac{k}{4\pi}\log\frac{y_1(x_1^2+y_1^2-a^2)}{x_1^2+y_1^2} + \frac{k}{4\pi}\log\frac{[(x_1^2+y_1^2-a^2)^2+4a^2y_1^2]^{1/2}}{x_1^2+y_1^2}$$
$$= \frac{k}{4\pi}\log\frac{[(x_1^2+y_1^2-a^2)^2+4a^2y_1^2]^{1/2}}{y_1(x_1^2+y_1^2-a^2)}$$

$\therefore$ The path of the vortex is $\qquad \chi_2(x_1, y_1) = \text{constant}.$

or $\quad \dfrac{k}{4\pi}\log\dfrac{[(x_1^2+y_1^2-a^2)^2+4a^2y_1^2]^{1/2}}{y_1(x_1^2+y_1^2-a^2)}=\dfrac{k}{4\pi}\log A^{1/2}$, A being a constant

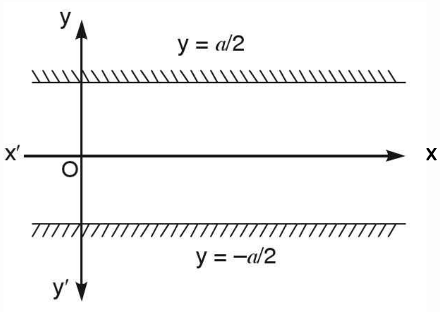
giving $$(x_1^2+y_1^2-a^2)^2+4a^2y_1^2=Ay_1^2(x_1^2+y_1^2-a^2)^2$$

Hence the path of the vortex at any point is

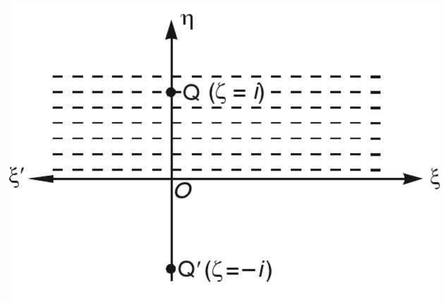
$$(x^2+y^2-a^2)^2+4a^2y^2=Ay^2(x^2+y^2-a^2)^2.$$

Ex. 4. *Determine the motion and the paths of particles due to a vortex in a liquid filling the space between two infinite parallel planes, the vortex being mid-way between them.*

Sol. Let the liquid be confined between planes $y = \pm\, a/2$, the vortex being at O.



z-plane



ζ–plane

Consider the transformation $$\zeta = ie^{\pi z/c}, \qquad ...(1)$$

which transforms the region $y = \pm\, a/2$ in z-plane to the upper half of ζ-plane. The point O $(z = 0)$ in z-plane is mapped on $Q(\zeta = i)$ on ζ-plane. Let k be the strength of the vortex at Q.

The image of vortex of strength $+\,k$ at Q relative to ξ-axis is a vortex of strength $-\,k$ at $Q'(\zeta=-i)$. The complex potential in ζ-plane is given by

$$w=\frac{ik}{2\pi}\log(\zeta-i)-\frac{ik}{2\pi}\log(\zeta+i)=\frac{ik}{2\pi}\log\frac{\zeta-i}{\zeta+i}=\frac{ik}{2\pi}\log\frac{ie^{\pi z/c}-i}{ie^{\pi z/c}+i},\ \text{by (1)}$$

$$=\frac{ik}{2\pi}\log\frac{e^{\pi z/2c}-e^{-\pi z/2c}}{e^{\pi z/2c}+e^{-\pi z/2c}}=\frac{ik}{2\pi}\log\tanh\frac{\pi z}{2c} \qquad ...(2)$$

From (2), $$\phi+i\psi=\frac{ik}{2\pi}\log\tanh\frac{\pi(x+iy)}{2c} \qquad ...(3)$$

$\therefore$ $$\phi-i\psi=-\frac{ik}{2\pi}\log\tanh\frac{\pi(x-iy)}{2c} \qquad ...(4)$$

Subtracting (4) from (3), we have

$$2i\psi=\frac{ik}{2\pi}\log\left[\tanh\frac{\pi(x+iy)}{2c}\tanh\frac{\pi(x-iy)}{2c}\right]$$

$\therefore$ $$\psi=\frac{k}{4\pi}\log\frac{2\sinh\frac{\pi(x+iy)}{2c}\sinh\frac{\pi(x-iy)}{2c}}{2\cosh\frac{\pi(x+iy)}{2c}\cosh\frac{\pi(x-iy)}{2c}}=\frac{k}{4\pi}\log\frac{\cosh\frac{\pi x}{c}-\cosh\frac{\pi yi}{c}}{\cosh\frac{\pi x}{c}+\cosh\frac{\pi yi}{c}}$$

or $$\psi=\frac{k}{4\pi}\log\frac{\cosh\frac{\pi x}{c}-\cos\frac{\pi y}{c}}{\cosh\frac{\pi x}{c}+\cos\frac{\pi y}{c}} \qquad ...(5)$$

Hence the streamlines are given by

$$\psi = \text{constant} = (k/4\pi)\log\lambda,\ \lambda \text{ being a constant}$$

or $$\frac{k}{4\pi}\log\frac{\cosh\frac{\pi x}{c}-\cos\frac{\pi y}{c}}{\cosh\frac{\pi x}{c}+\cos\frac{\pi y}{c}}=\frac{k}{4\pi}\log\lambda,$$ or $$\frac{\cosh\frac{\pi x}{c}-\cos\frac{\pi y}{c}}{\cosh\frac{\pi x}{c}+\cos\frac{\pi y}{c}}=\lambda, \text{ by (5)}$$

Applying componendo and dividendo, we get $$\frac{2\cosh(\pi x/c)}{2\cos(\pi y/c)}=\frac{\lambda+1}{1-\lambda}=\mu, \text{ say}$$

$$\therefore \qquad \cosh(\pi x/c)=\mu\cos(\pi y/c), \text{ giving the paths of the particles.}$$

Now, the motion of the vortex O is given by $(dw'/dz)_{z=0}$, where

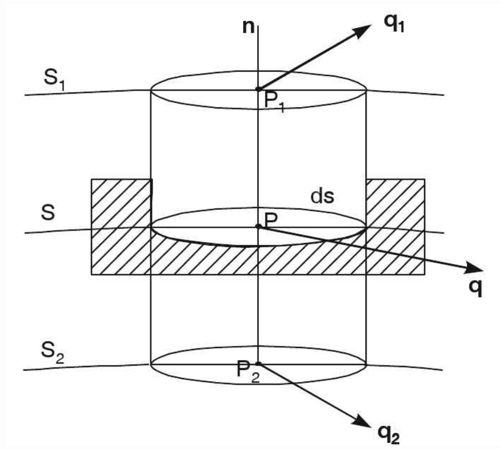
$$w' = w-\frac{ik}{2\pi}\log z=\frac{ik}{2\pi}\left[\log\tanh\frac{\pi z}{2c}-\log z\right], \text{ by (2)}$$

$$\therefore \qquad \frac{dw'}{dz}=\frac{ik}{2\pi}\left[\frac{\pi}{2c}\frac{\operatorname{sech}^2(\pi z/2c)}{\tanh(\pi z/2c)}-\frac{1}{z}\right]=\frac{ik}{2\pi}\left[\frac{\pi}{c}\operatorname{cosech}\frac{\pi z}{c}-\frac{1}{z}\right]$$

$\therefore$ As $z\to 0$, $dw'/dz\to 0$, showing that the vortex at O remains at rest.

11.25. Vortex sheet. [Meerut 1997; GNDU Amritsar 2003, 05]

Let $\mathbf{n}$ be the unit normal vector at the point P of a surface S. Let ε be an infinitesimal positive scalar and let the position vectors of two points P_1 and P_2 be $(\varepsilon\mathbf{n})/2$ and $(-\varepsilon\mathbf{n})/2$ respectively with P as origin of vectors. Hence when P describes the surface S, the points P_1 and P_2 describe surfaces S_1 and S_2 parallel and on the opposite sides of and equidistant from S. Take an infinitesimal area of S, say dS, whose centroid is P. Then the normals to S at the boundary of dS together with the surfaces S_1 and S_2 enclose a cylindrical element of volume $dV=\varepsilon\, dS$.



Now, imagine the above surfaces to be drawn in fluid which is moving irrotationally everywhere except in that part which lies between S_1 and S_2, Let Ω be the vorticity vector at P, then

$$\Omega dV = \Omega\varepsilon\, dS = \omega\, dS,$$

where $$\omega = \Omega\,\varepsilon \qquad \text{...(1)}$$

Suppose that $\varepsilon\to 0$ and $\mathbf{\Omega}\to\infty$ in such a manner that $\boldsymbol{\omega}$ remains unaltered. Then the surface S is called a *vortex sheet* of vorticity $\boldsymbol{\omega}$ per unit area. It is to be noted that the normal component of velocity is continuous across the vortex sheet.

If $\mathbf{q}$, $\mathbf{q}_1$, $\mathbf{q}_2$ be velocities at P, P_1 and P_2 respectively, then we have

$$\mathbf{q_1} = \mathbf{q}+(1/2)\times\varepsilon(\mathbf{n}\cdot\nabla)\mathbf{q} \qquad \text{...(2)}$$

$$\mathbf{q_2} = \mathbf{q}-(1/2)\times\varepsilon(\mathbf{n}\cdot\nabla)\mathbf{q} \qquad \text{...(3)}$$

Adding (2) and (3), $$\mathbf{q}=(1/2)\times(\mathbf{q}_1+\mathbf{q}_2),$$

which is independent of ε. Thus, the velocity of a point P of a vortex sheet is the arithmetic mean of the velocities just above and just below P on the normal at P.

By Gauss divergence theorem, we have

$$\int_V \text{curl}\,\mathbf{q}\,dV = \int_S d\mathbf{S}\times\mathbf{q} \qquad \text{or} \qquad \int_V \mathbf{\Omega}\,dV = \int_S \mathbf{n}\times\mathbf{q}\,dS \qquad \text{...(4)}$$

If we apply Gauss divergence theorem (4) to the elementary cylinder of volume dV in the given figure, we get approximately $\qquad \Omega\,\varepsilon\,dS = \mathbf{n}\times(\mathbf{q}_1-\mathbf{q}_2)\,dS$

Dividing by dS, $\qquad \Omega\varepsilon = \mathbf{n}\times(\mathbf{q}_1-\mathbf{q}_2)$. ...(5)

Let $\varepsilon\to 0$ and $\Omega\to\infty$ as before and using (1), (5) reduces to

$$\boldsymbol{\omega} = \mathbf{n}\times(\mathbf{q}_1-\mathbf{q}_2). \qquad \text{...(6)}$$

It follows that a non-zero value of $\boldsymbol{\omega}$ is associated with a discontinuity of the components of $\mathbf{q}_1$, $\mathbf{q}_2$ perpendicular to $\mathbf{n}$. Hence a surface across which the tangential velocity changes abruptly is a vortex sheet.

From (6), $\qquad \boldsymbol{\omega}\cdot\mathbf{n} = \mathbf{n}\cdot\mathbf{n}\times(\mathbf{q}_2-\mathbf{q}_1)$, $\qquad$ so that $\qquad \boldsymbol{\omega}\cdot\mathbf{n} = 0$,

showing that $\boldsymbol{\omega}$ is perpendicular to $\mathbf{n}$ and is therefore tangential to the vortex sheet *i.e.* a vortex sheet may be thought of as covered by a system of vortex filaments

Remark. A two-dimensional vortex sheet is represented by a line AB in the plane of the motion, such that there is an abrupt change in the tangential velocity, but no change in normal velocity, on crossing the line AB.

OBJECTIVE QUESTIONS ON CHAPTER 11

Multiple choice questions

Choose the correct alternative from the following questions

1. The image of vortex + k outside a circular boundary is
(*i*) a vortex – k at the inverse point and vortex + k at the centre
(*ii*) a vortex + k at the inverse point and a vortex – k at the centre
(*iii*) a vortex + k at the inverse point and the centre
(*iv*) None of these. **[Kanpur 2003]**

2. If k be the circulation in the circuit embracing the vortex, Then
(*i*) $w=(ik\log z)/2\pi$ (*ii*) $w=-(ik\log z)/2\pi$
(*iii*) $w=(k\log z)/2\pi i$ (*iv*) None of these. **[Kanpur 2002]**

3. The differential equations of vortex lines are
(*i*) $dx/u = dy/v = dz/w$ (*ii*) $dx/\Omega_x = dy/\Omega_y = dz/\Omega_z$
(*iii*) $dx/dt = u,\ dy/dt = v,\ dz/dt = w$ (*iv*) None of these.

4. At an internal point in a fluid, vortex lines
(*i*) can originate (*ii*) can terminate (*iii*) cannot originate (*iv*) None of these

5. In usual symbols if $u, v, w = \mu\,(\partial\phi/\partial x, \partial\phi/\partial y, \partial\phi/\partial z)$, then angle θ between the vortex lines and stream lines is
(*i*) 0° (*ii*) 45° (*iii*) 90° (*iv*) None of these

6. If there be a rectilinear vortex of strength k at $z_0\ (= x_0 + i\,y_0)$, then complex potential is given by
(*i*) $(2ik/\pi)\log(z-z_0)$ (*ii*) $(ik/2\pi)\log(z-z_0)$ (*iii*) $k\log(z-z_0)$ (*iv*) None of these

7. In usual notations, complex potential of dipole is
(*i*) $(\mu i/r)\,e^{i(\alpha-\theta)}$ (*ii*) $\mu r\,e^{i(\alpha-\theta)}$ (*iii*) $\mu\,e^{i(\alpha-\theta)}$ (*iv*) None of these **[Agra 2011]**

8. The motion due to a set of line of vortices of strength k at points $z = \pm\, n\, a$ (n = 0, 1, 2, 3, ...) is given by the relation

(*i*) $w = (i\,k/2\pi) \times \log(\pi z/a)$ (*ii*) $w = (i\,k/3\pi) \times \log(\pi z/a)$

(*iii*) $w = (i\,k/4\pi) \times \log(\pi z/a)$ (*iv*) None of these

9. If (r_1, θ_1), (r_2, θ_2), ..., (r_n, θ_n) be the polar co-ordinates at any time t of a system of rectilinear vortices of strengths k_1, k_2, ..., k_n, then

(*i*) $\Sigma k\, r^2$ = const. (*ii*) $\Sigma k\, r^2\, (d\theta/dt) = 0$

(*iii*) $\Sigma k\, r^2 = (1/2\pi) \times \Sigma k_1\, k_2$ (*iv*) None of these. **[Agra 2002, 2012]**

10. The vortex system moving with uniform velocity $(k/2a) \times \tanh(\pi\, b/a)$ is

(*i*) Routh's vortex street (*ii*) Rankine's vortex street

(*iii*) Karman vortex street (*iv*) None of these

11. In an infinite liquid at rest at infinity when the components of spin are known, then there could

(*i*) Infinite types of motion (*ii*) Only one type of motion

(*iii*) Two types of motion (*iv*) None of these

12. If a rectilinear vortex moves in two dimensions in a fluid bounded by a fixed plane, then a streamline can never coincide with a line of

(*i*) Constant velocity (*ii*) Constant density

(*iii*) Constant pressure (*iv*) None of these

Fill in the blank. Fill in the blanks correctly

13. If components of spin are all ...... the motion is called irrotational. **[Agra 2003]**

14. Stream function is ..... outside of the vortex filament. **[Agra 2012]**

Answers/Hints to objective type questions

1. (*i*). See Art. 11.12 (*A*) **2.** (*i*). See Eq. (13), Art. 11.4

3. (*ii*). See Art. 11.1 *C* **4.** (*iii*). See result (2), Art. 11.2

5. (*iii*). See Ex. 2, Art 11.3 **6.** (*ii*). See Eq. (14), Art. 11.4

7. (*iv*). See Eq. (2), Art. 11.6 *B* **8.** (*i*). See Art. 11.17 *A*

9. (*i*). See Art. 11.8 **10.** (*iii*). See Art. 11.18

11. (*i*). **12.** (*iii*).

13. Zero. See Art. 11.1 *B* **14.** Zero. See Eq. (7*B*), Art. 11.4

MISCELLANEOUS PROBLEMS ON CHAPTER 11

1. A vortex of circulation $2\pi k$ is at rest a point $z = na\,(n > 1)$ in the presence of a plane circular boundary $|z| = a$ around which there is a circulation $2\pi\lambda k$. Show that $\lambda(n^2 - 1) = 1$. **(Agra 2007)**

2. State Karman's vortex street and discuss velocities. **(Agra 2012)**

Hint. Refer Art.11.18, page 12.39.

CHAPTER

12 Waves

12.1. Introduction.

The dynamics of wave motion is very important in physical investigations, as wave motion is one of the main modes of transmission of energy. The energy from the sun is transmitted by waves. When some musical instrument is played upon in a room, sound waves spread through the room. If we throw a stone in a pond we find waves in the pond which start from the point of striking of the stone and spread in all directions. Such water waves are also produced by pressure of wind upon the surface of water, by the relative motion of bodies like a ship moving in a sea and by obstructions in the bed of the stream. These are all examples of wave motion and they possess two characteristic properties : firstly, energy is propagated to distant points and secondly, the disturbance travels through the medium without any transference of the medium itself. Irrespective of the nature of the medium which transmits the waves, these two properties persist and enable us to relate them together. Accordingly, each wave motion is governed by certain *differential equation of wave motion. The solution of such equation under suitable boundary conditions explain the given problem in its right perspective.

12.2. General expression of a wave motion.

A wave motion of a liquid acted upon by gravity and having a free surface is a motion in which the elevation of the free surface above some chosen fixed horizontal plane varies. We now show that, in general, a wave motion may be represented by an equation of the form

$$y = f(x - ct) \qquad \text{...(1)}$$

by taking the axis of x to be horizontal and the axis of y to be vertically upwards. Here y represents the displacement of a particle situated at x at time t. Increasing t by T and x by cT, we find

R.H.S. of (1) $= f\{x + cT - c(t + T)\} = f(x - ct) =$ L.H.S. of (1)

Thus (1) shows that the wave profile $y = f(x)$ moves with velocity c in the positive x-direction. Likewise we can show that $y = f(x + ct)$ represents a progressive wave travelling in the negative x-direction with a velocity c.

A *progressive wave* is one in which the surface pattern moves forward. In two-dimensional waves the curve in which the free surface meets the plane of motion is called a *wave profile*.

12.3. Mathematical representation of wave motion.

Taking the x-axis to be horizontal and the y-axis to be vertically upwards, a motion in which the equation of the vertical section of the free surface at time t is of the form

$$y = a \sin (mx - nt), \qquad \text{...(1)}$$

where a, m, n are constants, is known as *simple harmonic progressive wave*. (1) may also be written as

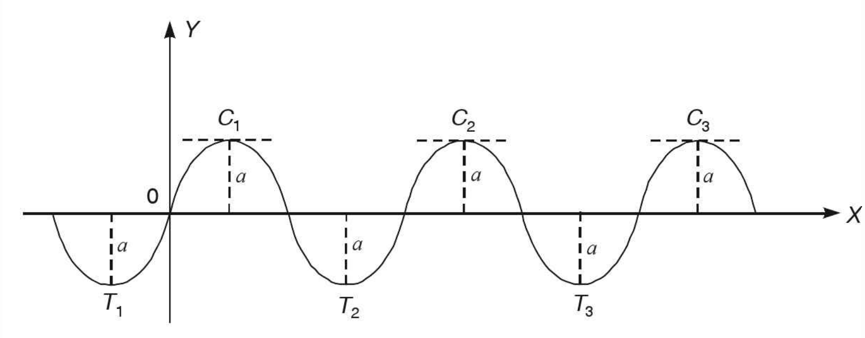
$$y = a \sin m(x - nt/m) \qquad \text{...(2)}$$

* $\nabla^2\phi = (1/c^2)(\partial^2\phi/\partial t^2)$ is the most general equation of wave motion. For its derivation and particular solution, refer part III of author's ''Advanced Differential equations'', published by S. Chand & Co, Delhi.

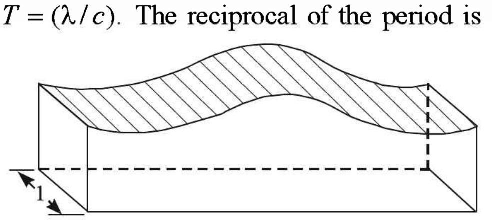
If we increase t by T and x by $(n/m) \times T$, then we find

$$\text{R.H.S. of (2)} = a \sin m\left(x + \frac{n}{m}T - \frac{n}{m}(t+T)\right) = a \sin m\left(x - \frac{n}{m}t\right) = \text{L.H.S. of (1)}.$$

Hence (2) shows that the wave profile $y = a \sin mx$ at time $t = 0$ moves with velocity n/m ($= c$, say) in the positive x-direction. c is called the *velocity of propagation* of the wave. When $a = 0$, the profile of the liquid is $y = 0$, which is the *mean level*. The maximum value of y, namely a, is known as the *amplitude* of the wave.



The points $C_1, C_2, \ldots$, of maximum elevation are known as *crests* and the points $T_1, T_2, \ldots$, of maximum depression are known as *troughs*. The distance between two consecutive crests is known as the *wave length* and is denoted by λ. Thus $\lambda = (2\pi/m)$. Again the nature of the free surface (1) remains unchanged by replacing t by $t + (2\pi/n)$. The time $T = (2\pi/n)$ is known as the *period* of the wave. Since $c = (n/m)$ and $\lambda = (2\pi/m)$, we get $T = (\lambda/c)$. The reciprocal of the period is known as the *frequency*; it denotes the number of oscillations per second, the angle $mx - nt$ is known as the *phase angle*. If the equation of wave motion be $y = a \sin (mx - nt + \varepsilon)$ then ε is called the *phase* of the wave.



Since the wave motion given by (1) is a two-dimensional motion, we shall consider only two-dimensional wave motions in this chapter. For such motions we shall assume that they take place between two vertical planes unit distance apart as shown in the above figure.

12.4. Standing or stationary waves. [Rohilkhand 2001]

Two simple harmonic progressive waves of the same amplitude, wave length and period traveling in opposite directions are given by the surface elevations

$$y_1 = (a/2) \times \sin(mx - nt) \qquad \text{and} \qquad y_2 = (a/2) \times \sin(mx + nt) \qquad \ldots(1)$$

By the principle of superposition, the resulting surface elevation is represented by the equation

$$y = y_1 + y_2, \qquad i.e. \qquad y = a \sin mx \cos nt. \qquad \ldots(2)$$

A motion of this type is known as stationary wave. At a given value of x the surface of water moves up and down. At any instant the equation represents a sine curve of amplitude a cos nt, which therefore varies between 0 and a. Thus a wave of this nature is not propagated. The points of intersection of the curve with the x-axis are given by $\sin mx = 0$, namely

$$x = n\pi/m = \lambda n/2 \qquad \text{where} \qquad n = 0, \pm 1, \pm 2, \ldots$$

These points are called *nodes* and the intermediate points where the amplitude is maximum are called *antinodes*.

Again, if $y_3 = a \sin mx \cos nt$ and $y_4 = a \cos mx \sin nt$ be two stationary waves, the result of superposing these is the elevation

$$y = y_3 \pm y_4, \qquad i.e. \qquad y = a \sin (mx \pm nt).$$

Hence a progressive wave can be regarded as the combination of two systems of stationary waves of the same amplitude, wave length and period, the crests and troughs of one system coinciding with the nodes of the other and their phases differing by a quarter period.

12.5. Types of liquid waves.

Roughly speaking the liquid waves may be divided into the following two classes:

(*i*) **Long waves in shallow water or tidal waves.**

Such waves arise when the depth of the liquid is small compared to the wave length and the disturbance affects the motion of the whole of the liquid. In these waves the vertical acceleration of the liquid is negligible as compared with the horizontal acceleration and the plane of the liquid moves as a whole.

(*ii*) **Surface waves.**

Such waves occur when the wave length of the oscillations is small compared to the depth of the liquid and hence the disturbance does not extend far below the surface. In these waves the vertical acceleration is appreciable and so it cannot be neglected. Wind waves and surface tension waves are examples of surface waves. Such waves occur in deep and unbounded (in horizontal directions) liquids like lakes and oceans.

12.6. Surface waves. **[Garhwal 2005]**

Such waves occur at and near the free surface of an unbounded sheet of liquid where the depth is considerable compared to the wave length. For these waves the vertical acceleration is comparable with the horizontal acceleration, and so we consider forces both in horizontal and vertical directions.

Let the x-axis be taken in the undisturbed surface in the direction of propagation of the waves and the y-axis vertically upwards. Taking the motion to be irrotational, incompressible and two-dimensional, the velocity potential ϕ exists such that throughout the liquid

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0 \qquad \text{...(1)}$$

and at a fixed boundary $\qquad \partial\phi/\partial x = 0. \qquad$ ...(2)

The pressure can be obtained from the Bernoull's equation

$$p/\rho = \partial\phi/\partial t - gy - q^2/2 = F(t) \qquad \text{...(3)}$$

Since the free surface is a surface of equipressure p = const., hence on the free surface

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} = 0, \qquad \text{...(4)}$$

where u and v are the velocity components on the free surface in x and y directions respectively.

Since, $\qquad u = -\partial\phi/\partial x \qquad$ and $\qquad v = -\partial\phi/\partial y, \qquad$ ...(5)

hence at the free surface the relation (4) becomes

$$\frac{\partial p}{\partial t} - \frac{\partial\phi}{\partial x}\frac{\partial p}{\partial x} - \frac{\partial\phi}{\partial y}\frac{\partial p}{\partial y} = 0. \qquad \text{...(6)}$$

Let the motion be so small that the squares of small quantities (*e.g.* velocities) may be omitted. So we omit q^2 in (3). Again, without loss of generality we may include $F(t)$ in ϕ and hence we may take $F(t) = 0$ in (3). Then, (3) reduces

$$p/\rho = \partial\phi/\partial t - g\,y. \qquad \text{...(7)}$$

Substituting the value of p from (7) in (6), we get

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial t} - \frac{\partial\phi}{\partial y}\left(\frac{\partial^2\phi}{\partial y\partial t} - g\right) = 0, \qquad \text{...(8)}$$

or, omitting the second and third terms which are of the same order as q^2, we get

$$\partial^2\phi/\partial t^2 + g(\partial\phi/\partial y) = 0. \qquad ...(9)$$

Condition (1) must be satisfied at the free surface.

If η the elevation of the free surface at time t above the point whose abscissa is x, the equation of the free surface is given by

$$\eta = f(x,t) \qquad \text{or} \qquad \eta - f(x,t) = 0. \qquad ...(10)$$

But we know that if $F(x,\eta,t) = \eta - f(x-t) = 0$ be the boundary surface, then we must have as in Art. 2.18

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial \eta} = 0 \qquad i.e. \qquad -\frac{\partial f}{\partial t} - u\frac{\partial f}{\partial x} + v = 0. \qquad ...(11)$$

Now $\partial f/\partial t$ is $\dot{\eta}$. Again $\partial f/\partial x$ or $\partial\eta/\partial x$ being the tangent of the slope of the free surface is small so that the second term in (11) can be omitted. Then (11) reduces to

$$\dot{\eta} = v = -\partial\phi/\partial y, \qquad ...(12)$$

which holds at free surface.

Thus for the surface waves the velocity potential is a solution of Laplace's equation (1) *which makes* $\partial\phi/\partial x = 0$ *as a fixed boundary and satisfies* (9) *and* (12) *at the free surface of the liquid.*

To illustrate the above theory, we consider the following cases.

Case I. Progressive waves on the surface of a canal. **[Garhwal 2005]**

Consider the propagation of simple harmonic waves of the type

$$\eta = a\sin(mx - nt) \qquad ...(13)$$

at the surface of canal of uniform depth h and having parallel vertical walls. Let the free surface be along the x-axis (*i.e.* $y = 0$) so that equation of the bottom (rigid boundary) is $y = -h$. Then we must find ϕ satisfying (1) and subjected to the following boundary conditions

$$\partial\phi/\partial y = 0, \qquad \text{at} \qquad y = -h \qquad ...(14)$$

$$\partial^2\phi/\partial t^2 + g(\partial\phi/\partial y) = 0, \qquad \text{at} \qquad y = 0 \qquad ...(15)$$

$$v = \partial\eta/\partial t = -\partial\phi/\partial y \qquad \text{at} \qquad y = 0 \qquad ...(16)$$

Using (13), (16) gives $\quad \partial\phi/\partial y = an\cos(mx - nt) \quad$ at $\quad y = 0 \quad ...(17)$

Equation (17) suggests that we should take the solution of (1) of the form

$$\phi = f(y)\cos(mx - nt). \qquad ...(18)$$

Substituting this in (1), we obtain $\quad d^2 f/dy^2 - m^2 f = 0, \quad ...(19)$

whose solution is $\quad f(y) = Ae^{my} + Be^{-my}$, A and B being arbitrary contants.

and hence $\quad \phi = (Ae^{my} + Be^{-my})\cos(mx - nt) \quad ...(20)$

Using (14), (20), gives $\quad Ae^{-mh} = Be^{mh} = D/2$, say,

so that $\quad \phi = D\cosh m(y+h)\cos(mx - nt) \quad ...(21)$

Again, using (15), (21) gives $\quad n^2 = gm \tanh mh. \quad ...(22)$

Let $\quad c = n/m \quad$ and $\quad \lambda = 2\pi/m \quad ...(23)$

denote the velocity of propagation and the wave length respectively. Then (22) reduces to

$$c^2 = (g/m)\tanh mh \qquad ...(24)$$

or $$c^2 = (g\lambda / 2\pi) \tanh\, (2\pi h / \lambda) \qquad \text{...(25)}$$

We now determine the constant D of (21) in terms of the amplitude a of the wave. Using (13) and (21), the boundary condition (16) gives $\quad -na = mD \sin mh$

so that $$\phi = \frac{na}{m} \frac{\cosh m(y+h)}{\sinh mh} \cos(mx - nt) \qquad \text{...(26)}$$

or using (22) $$\phi = \frac{ga}{n} \frac{\cosh m(y+h)}{\cosh mh} \cos(mx - nt) \qquad \text{...(27)}$$

From (26) the velocity components of the particles are

$$u = -\frac{\partial \phi}{\partial x} = na \frac{\cosh m(y+h)}{\sinh mh} \sin(mx - nt) \qquad \text{...(28)}$$

$$v = -\frac{\partial \phi}{\partial y} = -na \frac{\sinh m(y+h)}{\sinh mh} \cos(mx - nt) \qquad \text{...(29)}$$

We now determine the path of the particles. Let (x', y') be the coordinates a particle relative to its mean position (x, y) such that $|z'| = |x' + i\,y'|$ is very small. Neglecting the squares of small quantities, for a wave of small elevation the velocities at $z = x + iy$ and $z + z' = (x + x') + i(y + y')$ will be equal. Hence we may write

$$\frac{dx'}{dt} = u = na \frac{\cosh m(y+h)}{\sinh mh} \sin(mx - nt) \qquad \text{...(30)}$$

and $$\frac{dy'}{dt} = v = -na \frac{\sinh m(y+h)}{\sinh mh} \cos(mx - nt) \qquad \text{...(31)}$$

Integrating w.r.t. 't', (30) and (31) gives

$$x' = a \frac{\cosh m(y+h)}{\sinh mh} \cos(mx - nt), \qquad \text{...(32)}$$

$$y' = a \frac{\sinh m(y+h)}{\sinh mh} \sin(mx - nt) \qquad \text{...(33)}$$

Let $$a' = \frac{a \cosh m(y+h)}{\sinh mh} \quad \text{and} \quad b' = \frac{a \sinh m(y+h)}{\sinh mh} \qquad \text{...(34)}$$

Using (34) and eliminating t from (32) and (33), we get

$$x'^2 / a'^2 + y'^2 / b'^2 = 1, \qquad \text{...(35)}$$

showing that the particle describes the ellipse about its mean position. For a given particle (mx – nt) plays the part of the eccentric angle in the ellipse and increases at a constant rate. The distance between the foci is given by

$$2\sqrt{a'^2 + b'^2} = 2a \operatorname{cosech} mh, \qquad \text{...(36)}$$

which is constant and so it is the same for all such ellipses. Their major axes are horizontal and the lengths of both axes decrease as the depth of the particle increases, the minor axis vanishing at the bottom (where $y = -h$), degenerating ellipse into a straight line where the particles execute to-and fro-motion.

Case II. Progressive waves on a deep canal.

If the depth h of the canal is sufficiently great in comparison with λ for e^{-mh} to be *neglected, then in Case I we must have $B = 0$. Thus, we have, instead of (20)

$$\phi = Ae^{my}\cos(mx - nt), \qquad \text{...(20)}'$$

instead of (22)

$$n^2 = gm \qquad \text{..(22)}'$$

and instead of (25)

$$c^2 = g\lambda/2\pi. \qquad \text{...(25)}'$$

We now determine the constant A of (20)′ in terms of the amplitude a of the wave. Using (13) and (20)′, the boundary condition (16) gives $na = mA$ so that

$$\phi = (na/m)\, e^{my}\cos(mx - nt), \qquad \text{...(26)}'$$

or

$$\phi = (ga/m)\, e^{my}\cos(mx - nt). \qquad \text{...(27)}'$$

The velocity components of the particles are

$$u = -(\partial\phi/\partial x) = nae^{my}\sin(mx - nt) \qquad \text{...(28)}'$$

and

$$v = -(\partial\phi/\partial y) = -nae^{my}\cos(mx - nt). \qquad \text{...(29)}'$$

Following the procedure of case I we obtain in this case for the displacement (x', y') of a particle from its mean position (x, y)

$$x' = ae^{my}\cos(mx - nt) \qquad \text{and} \qquad y' = ae^{my}\sin(mx - nt),$$

and hence the path of the particle is a circle $\quad x'^2 + y'^2 = (ae^{my})^2,$

of radius ae^{my}, which decreases with depth of a particle under consideration. For fluid particles on the surface of the liquid the radius is equal to the amplitude of the wave. The circle is described with uniform angular velocity n, which in this case is equal to $(gm)^{1/2}$ or $(2\pi g/\lambda)^{1/2}$.

Remark 1. Comparing (25) and (25)′ we observe that the velocity of a surface wave in a liquid of finite depth differs from that in deep liquid (theoretically of infinite depth) by a factor $\sqrt{\tan(2\pi h/\lambda)}$, which in turn differs from unity by less than 1 per cent by taking $2\pi h/\lambda = \pi$ or $h = \lambda/2$. Thus, if $h > \lambda/2$, then we can treat theoretically the deep canal having an infinite depth, and in all such cases the wave velocity is given by (25)′ and is independent of the depth.

Remark 2. Complex potential for a progressive wave in a deep canal.

Since $n/m = c$, (26)′ reduces to $\quad \phi = ace^{my}\cos(mx - nt),$

Using $\partial\psi/\partial y = \partial\phi/\partial x$, we have $\quad \partial\psi/\partial y = -acme^{my}\sin(mx - nt),$

Integrating w.r.t, y, we have $\quad \psi = -ace^{my}\sin(mx - nt).$

Hence the complex potential $w (= \phi + i\psi)$ is given by

$$w = ace^{my}[\cos(mx - nt) - i\sin(mx - nt)] = ace^{my}e^{-i(mx - nt)}$$

$$= ace^{my}e^{-imx + int} = ace^{-im(x+iy)+int} = ace^{-imz+int}$$

$\therefore \quad w = ace^{-i(mz - nt)}$, which is the desired complex potential.

Case III. Stationary waves on the surface of a canal.

Consider a stationary wave of the type $\quad \eta = a\sin mx\cos nt \qquad \text{...(13A)}$

at the surface of canal of uniform depth h and having parallel vertical walls. Let the free surface

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* This also follows as follows : Here we have, instead of (14) $\quad \partial\phi/\partial y = 0$ at $y = -\infty \qquad$ ...(14)′

So (20) shows that we must take $B = 0$ to satisfy (14)′.

be along the x-axis (*i.e.* $y = 0$) so that equation of the bottom (rigid boundary) is $y = -h$. Then we must find ϕ satisfying (1) and subjected to the following boundary conditions

$$\partial\phi/\partial y = 0 \quad \text{at} \quad y = -h \qquad ...(14A)$$

$$\partial^2\phi/\partial t^2 + g(\partial\phi/\partial y) = 0 \quad \text{at} \quad y = 0 \qquad ...(15A)$$

$$v = \partial\eta/\partial t = -\partial\phi/\partial y \quad \text{at} \quad y = 0 \qquad ...(16A)$$

Using (13A), (16A) gives $\quad \partial\phi/\partial y = an \sin mx \sin nt \quad$ at $\quad y = 0$ ...(17A)

Equation (17A) suggests that we should take the solution of (1) of the form

$$\phi = f(x) \sin mx \sin nt \qquad ...(18A)$$

Substituting this in (1), we obtain $\quad df^2/dy^2 - m^2 f = 0,$ ...(19A)

whose solution is $\quad f(y) = Ae^{my} + Be^{-my}$, A, B being arbitrary constants.

and hence

$$\phi = (Ae^{my} + Be^{-my}) \sin mx \sin nt \qquad ...(20A)$$

Using (14A), (20A) gives $\quad Ae^{-mh} = Be^{mh} = D/2,$ say,

so that

$$\phi = D \cosh m(y+h) \sin mx \sin nt \qquad ...(21A)$$

Again, using (15A), (21A) gives $\quad n^2 = gm \tanh mh$ ...(22A)

Let $\quad c = n/m \quad$ and $\quad \lambda = 2\pi/m$ ...(23A)

denote the velocity of propagation and the wave length respectively.

Then (22A) reduces to

$$c^2 = (g/m) \tanh mh \qquad ...(24A)$$

or

$$c^2 = (g\lambda/2\pi) \tanh (2\pi h/\lambda) \qquad ...(25A)$$

We now determine the constant D of (21A) in terms of the amplitude a of the wave. Using (13A) and (21A), the boundary condition (16A) gives

$$-na = -mD \sinh mh$$

so that

$$\phi = \frac{na}{m} \frac{\cosh m(y+h)}{\sinh mh} \sin mx \sin nt \qquad ...(26A)$$

or using (22A)

$$\phi = \frac{ga}{n} \frac{\cosh m(y+h)}{\sinh mh} \sin mx \sin nt \qquad ...(27A)$$

From (26A) the velocity components of the particles are

$$u = -\frac{\partial\phi}{\partial x} = -na \frac{\cosh m(y+h)}{\sinh mh} \cos mx \sin nt. \qquad ...(28A)$$

and

$$v = -\frac{\partial\phi}{\partial y} = -na \frac{\sinh m(y+h)}{\sinh mh} \sin mx \sin nt \qquad ...(29A)$$

To determine the path of the particles in stationary waves.

With the same notations and method as in Case I, we have

$$\frac{dx'}{dt} = -\frac{\partial\phi}{\partial x} = -na \frac{\cosh m(y+h)}{\sinh mh} \cos mx \sin nt. \qquad ...(30A)$$

and

$$\frac{dy'}{dt} = -\frac{\partial\phi}{\partial y} = -na \frac{\sinh m(y+h)}{\sinh mh} \sin mx \sin nt \qquad ...(31A)$$

Integrating w.r.t. 't', (30A) and (31A) give

$$x' = a\frac{\cosh m(y+h)}{\sinh mh}\cos mx \cos nt \qquad ...(32A)$$

and
$$y' = a\frac{\sinh m(y+h)}{\sinh mh}\sin mx \sin nt \qquad ...(33A)$$

Hence $\qquad y'/x' = \tan m(y+h)\tan mx, \qquad ...(34A)$

and since this is independent of t the motion of each particle is rectilinear. The direction of motion varies from vertical below the crests and troughs [where $mx = (n+1/2)\pi$], to horizontal below the nodes (where $mx = n\pi$).

Case IV. Stationary waves on a deep canal.

If the depth h of the canal is sufficiently great in comparison with λ for e^{-mh} to be neglected, then in case III we must have B = 0. Thus, we have, instead of (20A)

$$\phi = Ae^{my}\sin mx \sin nt, \qquad ...(20A)'$$

instead of (22A) $\qquad n^2 = gm \qquad ...(22A)'$

and instead (25A) $\qquad c^2 = g\lambda/2\pi \qquad ...(25A)'$

We now determine the constant A of (20A)′ in terms of the amplitude of the wave. Using (13A) and (20A)′, the boundary condition (16A) gives $na = mA$,so that

$$\phi = (na/m)\, e^{my}\sin mx \sin nt, \qquad ...(26A)'$$

or
$$\phi = (ga/n)\, e^{my}\sin mx \sin nt \qquad ...(27A)'$$

The velocity components of the particles are

$$u = -(\partial\phi/\partial x) = -nae^{my}\cos mx \sin nt \qquad ...(28A)'$$

$$v = -(\partial\phi/\partial y) = -nae^{my}\sin mx \sin nt \qquad ...(29A)'$$

Following the procedure of Case III we obtain in this case

$x' = ae^{my}\cos mx \cos nt$ and $y' = ae^{my}\sin mx \cos nt$

Hence $\qquad y'/x' = \tan mx,$

showing that the path of the particle is a straight line. The amplitude of oscillations is ae^{my} which decreases as depth increases. Moreover the particles oscillate in the vertical direction at the antinodes, whereas they oscillate in the horizontal direction at the nodes.

12.7. The energy of progressive waves.

Kinetic energy. Definition. *The kinetic energy possessed by the liquid (per unit thickness), stretching between two vertical planes situated at a distance of one wave length apart and perpendicular to the direction of flow, is known as the kinetic energy of progressive wave.*

Consider a train of progressive waves at the surface of liquid (in a canal) of depth h, given, as in Case I of Art. 12.6, by $\qquad \eta = a\sin(mx - nt) \qquad ...(1)$

and
$$\phi = \frac{ga}{n}\frac{\cosh m(y+h)}{\cosh mh}\cos(mx - nt) \qquad ...(2)$$

Since the motion is irrotational, the kinetic energy is given by

$$T = -\frac{1}{2}\rho\int \phi\frac{\partial\phi}{\partial n}ds, \qquad ...(3)$$

δn being normal drawn into the liquid and integration being performed along the profile of a wave length. For the present problem, upto the first order of small quantities, (3) becomes

$$T = \frac{1}{2}\rho\int_0^\lambda \left[\phi\,\frac{\partial\phi}{\partial y}\right]_{y=0} dx = \frac{1}{2}g\rho a^2\int_0^\lambda \cos^2(mx - nt)\,dx, \text{ using (2)}$$

or $$T = (1/4)\times g\rho a^2\lambda. \qquad \ldots(4)$$

Potential energy. Definition. *The potential energy due to the elevated liquid in a wave length (the energy being calculated relative to the undisturbed state) is known as the potential energy (per unit thickness) of a progressive wave.*

Let us calculate the potential energy of liquid between two vertical planes parallel to the direction of propagation at unit distance apart. Then, for a single wave length, the potential energy is given by

$$V = \frac{1}{2}g\rho\int_0^\lambda \eta^2 dx = \frac{1}{4}g\rho a^2\lambda, \qquad \text{as} \qquad \lambda = 2\pi/m \qquad \ldots(5)$$

$\therefore$ Total energy per wave length $= T + V = (1/2)\times g\rho a^2\lambda.$

Hence it follows that the total energy is half kinetic and half potential.

12.8. The energy of stationary waves.

Consider a train of stationary waves at the surface of liquid (in a canal) of depth h, given, as in Case III of Art 12.6, by

$$\eta = a\sin mx\cos nt. \qquad \ldots(1)$$

and $$\phi = \frac{ga}{n}\frac{\cosh m(y+h)}{\cosh mh}\sin mx\sin nt \qquad \ldots(2)$$

Then, as in Art. 12.7, $\quad T = (1/4)\times g\rho a^2\lambda\sin^2 nt \quad$ and $\quad V = (1/4)\times g\rho a^2\lambda\cos^2 nt$

$\therefore$ Total energy per wave length at any time $= T + V = (1/4)\times g\rho a^2\lambda.$

Again the amounts of kinetic and potential energy change continuously with the time.

12.9. Progressive waves reduced to a case of steady motion.

In any case in which waves propagate in one direction only without change of shape, the problem of determining the velocity of propagation can be simplified as follows : Impose on the whole liquid a velocity equal and opposite to the velocity of propagation of the waves. Then the wave profile having the same relative velocity as before becomes fixed in space and the problem becomes one of steady motion. We now illustrate this technique by means of the following two cases:

Case I. Progressive waves on the surface of a canal.

As in case I of Art. 12.6, let progressive waves move on the surface of a canal of uniform depth h and having parallel vertical walls. Let the progressive wave move towards the positive direction of x-axis with velocity c and without change of form. Impose on the whole liquid a velocity c in the negative direction of x-axis. Then as explained before the problem becomes one of steady motion. Since the problem is two-dimensional, the problem reduces to find appropriate expressions for the velocity potential ϕ and stream function ψ so that the free surface (*i.e.* $y = 0$) and the bottom of the liquid (*i.e.*, $y = -h$) may satisfy the conditions for streamlines.

Consider the complex potential*

$$w = cz + A\cos mz - iB\sin mz$$

or $$\phi + i\psi = c\,(x + iy) + A\cos m\,(x + iy) - iB\sin m\,(x + iy)$$

---

* In absence of wave motion, we would have a uniform flow with velocity c in the negative direction of x-axis. For such a flow the complex potential is of the form $\qquad w = c\,(x + iy) = cz.$

$\therefore \qquad \phi = cx + (A \cosh my + B \sinh my) \cos mx$ ...(1)

and $\qquad \psi = cy - (A \sinh my + B \cosh my) \sin mx$ ...(2)

Since ϕ and ψ given by (1) and (2) satisfy Laplace's equation, they represent a possible motion.

For the bottom of the canal to be a streamline we must have $\psi = \text{const.}$ at $y = -h$, so that (2) gives

$-A \sinh mh + B \cosh mh = 0$ so that $A/\cosh mh = B/\sinh mh = D$, say ...(3)

Using (3), (1) and (2) become

$$\phi = cx + D \cosh m(y+h) \cos mx \qquad ...(4)$$

$$\Psi = cy - D \sinh m(y+h) \sin mx \qquad ...(5)$$

Let the free surface be a simple curve $\eta = a \sin mx$. Then (5) will make this the streamline $\psi = 0$ provided $\qquad ca - D \sinh mh = 0,$ ...(6)

when squares of small quantities are neglected.

Now the pressure is given by Bernoulli's equation

$$p/\rho + gy + (1/2) \times \{(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2\} = \text{const.} \qquad ...(7)$$

At the free surface, $y = a \sin mx$. Using (4), (6) and neglecting a^2, (7) reduces to (at free surface)

$$p/\rho + ga \sin mx + (c^2/2) \times \{1 - 2ma \coth mh \sin mx\} = \text{const.} \qquad ...(8)$$

But p is constant at the free surface. Hence (8) holds if the coefficient of $\sin mx$ vanishes, *i.e.* $\qquad ga - c^2 ma \coth mh = 0 \qquad$ or $\qquad g = mc^2 \coth mh$

or $\qquad c^2 = (g\lambda/2\pi) \tanh(2\pi h/\lambda),$ ...(9)

which is the same as already obtained in case I of Art. 12.6.

Case II. Progressive waves on a deep canal.

For this case (where $h \to \infty$) we consider

$$\phi = cx + Ee^{my} \cos mx \qquad ...(10)$$

and $\qquad \psi = cy - Ee^{my} \sin mx$ ...(11)

with a free surface $\qquad \eta = a \sin mx$ ...(12)

The free surface is the streamline $\psi = 0$, if $ca = D$ and hence

$$\phi = cx + cae^{my} \cos mx \qquad ...(13)$$

and $\qquad \psi = cy - cae^{my} \sin mx$ ...(14)

Now the pressure is given by (7). Since pressure is constant on the free surface, (7) reduce to

$$gy + (1/2) \times [\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2]_{y=0} = \text{cosnt.}$$

or $\qquad ga \sin mx + (1/2) \times c^2 (1 - 2am \sin mx + m^2a^2) = \text{const.}$

Neglecting a^2 and equating to zero the coefficient of $\sin mx$, we obtain

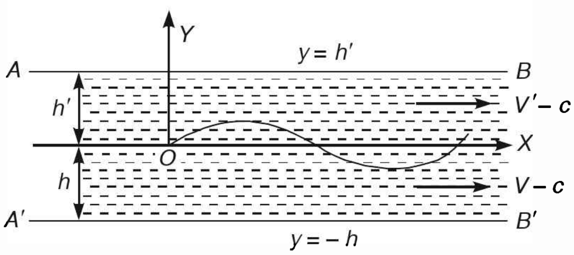
$$c^2 = g/m = g\lambda/2\pi, \qquad ...(15)$$

which is the same as already obtained in case II of Art. 12.6. If a uniform velocity c is superimposed on the motion just considered, we get motion of progressive waves on deep canal.

12.10. Waves at the interface (i.e. common surface) of two liquids. (Agra 2009)

Let a liquid of density ρ' and depth h' move with velocity V' over another liquid of density ρ and depth h moving in the same direction with velocity V; the liquids being bounded above and below by two fixed horizontal planes AB and $A'B'$.

Let c be the velocity of propagation of oscillatory waves at the interface of two liquids in the direction in which the liquids are moving. Let the x-axis be in this direction in the undisturbed interface (*i.e.* common surface of two liquids) and y-axis vertically upwards. As in Art. 12.9, we make the motion steady by superimposing on the whole mass the velocity $-c$. Thus the wave profile is reduced to rest in space and the new velocities of liquids becomes $V'-c$ and $V-c$ as shown in figure.



The velocity potential and stream function for the lower liquid moving with $-(V-c)$ in the negative direction of x-axis are given by

$$\phi = -(V-c)\,x + D\cosh m\,(y+h)\cos mx \qquad \text{...(1)}$$

and $$\psi = -(V-c)\,y - D\sinh m\,(y+h)\sin mx \qquad \text{...(2)}$$

Similar expression for the upper liquid may be deduced from (1) and (2) by replacing V by V' and h by h'. Thus, we get

$$\phi' = -(V'-c)\,x + D'\cosh m\,(y-h')\cos mx \qquad \text{...(3)}$$

and $$\psi' = -(V'-c)\,y - D'\sinh m\,(y-h')\sin mx \qquad \text{...(4)}$$

Clearly the above expression for ψ and ψ' make the boundaries $y=-h$, $y=h'$ streamlines.

Let $$\eta = a\sin mx \qquad \text{...(5)}$$

represent the displacement of the interface. If the liquids do not separate, then (5) must be a streamline for both surfaces. This condition is satisfied by assuming the streamline to be $\psi=\psi'=0$. Neglecting the squares of small quantities (*e.g.* a^2), we thus obtain

$$-(V-c)\,a - D\sin mh = 0 \qquad \text{... (6)}$$

and $$-(V'-c)\,a + D'\sin mh' = 0. \qquad \text{...(7)}$$

From Bernoulli's equations, we obtain

$$p/\rho + gy + (1/2)\times\{(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2\} = \text{const.} \qquad \text{...(8)}$$

and $$p'/\rho + gy + (1/2)\times\{(\partial\phi'/\partial x)^2 + (\partial\phi'/\partial y)^2\} = \text{const.} \qquad \text{...(9)}$$

But at the interface $y=\eta=a\sin mx$. Hence neglecting a^2, (8) and (9) give

$$p/\rho + ga\sin mx + (1/2)\times(V-c)^2\,(1-2am\coth mh\sin mx) = \text{cont.}$$

$$p'/\rho + ga\sin mx + (1/2)\times(V'-c)^2\,(1+2am\coth mh'\sin mx) = \text{cont.}$$

Since the pressure is continuous across the interface, putting $p=p'$ in above equations, subtracting and then equating to zero the coefficient of sin mx, we obtain

$$g(\rho-\rho') = (V-c)^2 m\rho\coth mh + (V'-c)^2 m\rho'\coth mh' \qquad \text{...(10)}$$

Equation (10) determines the velocity of propagation c of waves of wave length $2\pi/m$ at the interface. We can also treat (10) as the condition for stationary waves at the interface of two streams whose velocities are $V-c$ and $V'-c$.

Cor 1. When the liquids are at rest (*i.e.* $V=V'=0$), the wave velocity is given by

$$c^2 = \frac{g}{m}\,\frac{\rho-\rho'}{\rho\coth mh + \rho'\coth mh'}. \qquad \text{...(11)}$$

Since there is no real value of c when $\rho' > \rho$, (11) shows that when $\rho' > \rho$ the equilibrium position is unstable.

Cor. 2. Let the liquids be at rest and the depths of both liquids be so large compared to the wave length that we may take $\coth mh = \coth mh' = 1$. Then we have

$$c^2 = \frac{g}{m}\,\frac{\rho-\rho'}{\rho+\rho'} \qquad \text{or} \qquad c^2 = \frac{g\lambda}{2\pi}\,\frac{\rho-\rho'}{\rho+\rho'} \qquad ...(12)$$

Cor. 3. Let the liquids be at rest and the upper fluid be air.

Let $s = \rho'/\rho$ be the specific gravity of air. Suppose the depth of air be infinite so that $\coth mh' \to 1$ as $h' \to \infty$. Then (11) reduces to

$$c^2 = \frac{g}{m}\,\frac{\rho-\rho'}{\rho\coth mh+\rho'} = \frac{g}{m}\,\frac{1-s}{\coth mh+s}$$

$$= \frac{g\,(1-s)}{m\coth mh}(1+s\tanh mh)^{-1} = \frac{g\tan mh}{m}(1-s)\,(1-s\tanh mh)$$

(expanding by Binomial theorem and neglecting $s^2$and higher powers of s).

Simplifying, neglecting s^2 and writing $s = \rho'/\rho$ we get

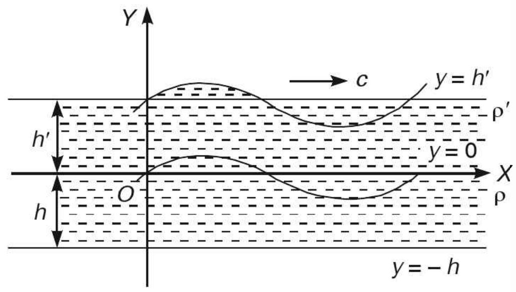
$$c^2 = (g/m)\tanh mh\,\{1-(\rho'/\rho)(1+\tanh mh)\} \text{ approximately} \qquad ...(13)$$

Cor. 4. Let the velocities V, V' make angles α, α' with the direction of c. But the components $v'\sin\alpha'$ and $v\sin\alpha$ (perpendicular to the direction of c) do not affect the value of c. Hence the required value of c can be obtained by replacing $V$and V' by $V\cos\alpha$ and $V'\cos\alpha'$ in (10). Thus, we get

$$g\,(\rho-\rho') = m\rho\,(V\cos\alpha-c)^2\coth mh + m\rho'\,(V'\cos\alpha'-c)^2\coth mh' \qquad ...(14)$$

12.11. Waves at the interface of two liquids with upper surface free.

Let a liquid of density ρ' and depth h' lie over another liquid of density ρ and depth h and let both the liquids to be at rest save for wave motion. Let there be a common velocity of wave propagation c at the free surface of the upper liquid and at the common surface. In order to make the motion steady, impose on the whole mass a velocity equal and opposite to that of propagation of waves. Then both the liquids begin to flow with velocity c in the negative direction of x-axis. With axes as shown in the adjoining figure and using notations of Art. 12.10 we may take



$$\psi = cy - D\sinh m\,(y+h)\sin mx \qquad \text{in the lower liquid} \qquad ...(1)$$

and

$$\psi' = cy - (A\cosh my + B\sin my)\sin mx \qquad \text{in the upper liquid.} \qquad ...(2)$$

From (1), it easily follows that the bottom $y = -h$ is a stream surface $\psi = -ch$. Let the common surface be given by $\eta = a\sin mx$ ...(3)

Then (3) is the stream surface $\psi = \psi' = 0$, if

and
$$\left.\begin{aligned} ca - D\sinh mh &= 0 \\ ca - A &= 0 \end{aligned}\right\} \qquad ...(4)$$

Finally, the free surface $y = h' + b\sin mx$ ...(5)

is a stream surface $\psi' = \text{const.}$ if $cb - (A\cosh mh' + B\sin mh') = 0.$ ...(6)

From Bernoulli's equation for the lower and upper liquids respectively, we have

$$p/\rho + gy + (1/2)\times\{(\partial\psi/\partial x)^2 + (\partial\psi/\partial y)^2\} = \text{const.} \qquad ...(7)$$

$$p'/\rho' + gy + (1/2)\times\{(\partial\psi'/\partial x)^2 + (\partial\psi'/\partial y)^2\} = \text{const.} \qquad ...(8)$$

As the pressure is to be continuous across the interface, we have $p = p'$ when $y = a\sin mx$. Hence on subtracting (8) from (7), we have

$$g(\rho-\rho')\sin mx + (\rho/2)\{(\partial\psi/\partial x)^2 + (\partial\psi/\partial y)^2\} - (\rho'/2)\{(\partial\psi'/\partial x)^2 + (\partial\psi'/\partial y)^2\} = 0$$

Substituting for ψ and ψ' from (1) and (2), noting that A, B and D are of order a, neglecting squares of small quantities, the above equation reduces to

$$g\,a(\rho-\rho') - cm(\rho D\cosh mh - \rho' B) = 0. \qquad ...(9)$$

Solving (4) and (5) for D and B and substituting their values in (9), we get

$$g(\rho-\rho') = c^2 m\{\rho\coth mh + \rho'\coth mh' - (\rho' b/a)\operatorname{cosech} mh'\} \qquad ...(10)$$

Since $p' = \text{const.}$ at the free surface (5), (8) gives

$$gb\sin mx + (1/2)\times\{(\partial\psi'/\partial x)^2 + (\partial\psi'/\partial y^2)\} = \text{const.} \qquad ...(11)$$

Substituting for ψ' from (2) and neglecting squares of small quantities as before, (11) gives

$$gb = cm(A\sinh mh' + B\cosh mh') \qquad ...(12)$$

Solving (4) and (5) for A and B and substituting their values in (12), we have

$$g = c^2 m\{\coth mh' - (a/b)\times\operatorname{cosech} mh'\} \qquad ...(13)$$

Eliminating a/b from (10) and (13), the desired equation for c is given by

$$c^4 m^2(\rho\coth mh\coth mh' + \rho') - c^2 mg\rho(\coth mh + \coth mh') + g^2(\rho-\rho') = 0 \qquad ...(14)$$

Again the ratio of the amplitudes of waves at free surface and interface is given from (13) by

$$\frac{b}{a} = \frac{c^2 m}{c^2 m\cosh mh' - g\sinh mh'} \qquad ...(15)$$

Equation (14) shows that there exist two possible velocities of propagation for a given wave length, provided $\rho > \rho'$.

Cor. When the lower liquid is 'deep' we may take coth mh = 1 approximately. Then (14) reduces to

$$c^4 m^2(\rho\coth mh' + \rho') - c^2 mg\rho(1+\coth mh') + g^2(\rho-\rho') = 0$$

or

$$(mc^2 - g)[mc^2(\rho\coth mh' - \rho') - g(\rho-\rho')] = 0$$

so that

$$c^2 = \frac{g}{m} \quad \text{or} \quad c^2 = \frac{g}{m}\,\frac{\rho-\rho'}{\rho\coth mh' + \rho'}, \qquad ...(16)$$

which are in full agreement with the results already obtained in case II of Art. 12.9. The ratios of amplitudes of the upper and lower waves in the two cases are

$$c^{mh'} \quad \text{and} \quad -(\rho/\rho' - 1)\,e^{-mh'}$$

12.12. Capillary waves or Ripples [Himachal 2003, 06]

Let there be an interface between two fluids, like water in contact with air. Then the interface will not be a constant pressure surface unless it is a plane surface. Since free surface is, in general, a curved surface, so waves would be affected due to surface tension.

In the present article we consider the flow to be two-dimensional, irrotational and incompressible. Let δp and $\delta p'$ denote the variable parts of the pressure below and above the surface. Let η be the vertical displacement of the interface at a point x at any time t and let T be the surface tension. Then we have

$$T(d^2\eta/dx^2) + \delta p - \delta p' = 0, \qquad \text{as the surface condition.} \qquad ...(1)$$

We now illustrate the above theory with the help of the following two cases:

Case I. Capillary waves on a canal of uniform depth.

Consider the problem discussed in case I of Art. 12.6 and Art 12.9. Using the method of Art. 12.9, reduce the problem to one of steady motion by superposing a velocity $-c$ on the whole liquid, where c is the velocity of propagation. As in Art. 12.9, we obtain

$$\phi = cx + D\cosh m\,(y + h)\cos mx \qquad ...(2)$$

and $$\psi = cy - D\sinh m\,(y + h)\sin mx \qquad ...(3)$$

and for the free surface $$\eta = a\sin mx \qquad ...(4)$$

provided $$ca - D\sinh mh = 0. \qquad ...(5)$$

Using these in the Bernoulli's equation, the variable part of the pressure is given by

$$\delta p/\rho + ga\sin mx + (c^2/2)\times(1 - 2ma\coth mh\sin mx) = \text{const.}, \qquad ...(6)$$

where the terms containing a^2 have been neglected. Now if we suppose that pressure (air pressure) on the upper side of the interface (between air and liquid) is constant, then $\delta p' = 0$ in (1) and so

(1) reduces to $$\delta p = -T(d^2\eta/dx^2) = Tam^2\sin mx. \qquad ...(7)$$

Substituting the above value of δp in (6) and then equating to zero the coefficient of sin mx,

$$c^2 = (g/m + Tm/\rho)\tanh mh. \qquad ...(8)$$

When h is large compared to the wave length, (8) reduces to

$$c^2 = g/m + Tm/\rho \qquad ...(9)$$

Case II. Capillary waves at the common surface of two liquids.

Proceed as in Art. 12.10 until we arrive at the equations for pressure on either side of the interface. Writing δp and $\delta p'$ for and p and p', we now get

$$\delta p/\rho + ga\sin mx + (1/2)\times(V - c)^2(1 - 2am\coth mh\sin mx) = \text{const.} \qquad ..(10)$$

and $$\delta p'/\rho' + ga\sin mx + (1/2)\times(V' - c)^2(1 + 2am\coth mh'\sin mx) = \text{const.} \qquad ...(11)$$

From (1) and (4), we have $$\delta p - \delta p' = Tam^2\sin mx \qquad ...(12)$$

Eliminating δp and $\delta p'$ from (10), (11) and (12), we get

$$Tm^2 + g(\rho - \rho') = (V - c)^2 m\rho\coth mh + (V' - c)^2 m\rho'\coth mh' \qquad ...(13)$$

Cor. 1. Let the liquids be so deep compared to the wave length that $\coth mh = \coth mh' = 1$ approximately. Let the liquids be at rest save for the wave motion so that $V = V' = 0$. The velocity of propagation c_0 is given from (13) by

$$c_0^2 = \frac{g}{m}\,\frac{\rho - \rho'}{\rho + \rho'} + \frac{Tm}{\rho + \rho'} \qquad ...(14)$$

Cor. 2. Treating air as incompressible, we get the case of the effect of wind on deep water by taking $V = 0$ in (13). Thus, we have

$$Tm + (\rho - \rho')(g/m) = c^2\rho + (V' - c)^2\rho'$$

or
$$c^2 - \frac{2\rho'}{\rho+\rho'}V'c + \frac{\rho}{\rho+\rho'}V^2 - c_0^2 = 0, \qquad \text{...(15)}$$

where c_0 is the velocity of propagation when there is no wind. Solving (15), we get

$$c = \frac{\rho' V'}{\rho+\rho'} \pm \left\{c_0^2 - \frac{\rho\rho' V'^2}{(\rho+\rho')^2}\right\}^{1/2}, \qquad \text{...(16)}$$

showing that the velocities of propagation along positive and negative x-axis are different.

12.13. Group velocity. (Agra 2007, 2012)

When a stone is dropped into a pond or a boat starts moving on the surface of still water, a local disturbance takes place. This gives rise to a wave which can be analysed into a set of simple harmonic components each of different wave length. Since the velocity of propagation depends upon the wave length, the waves of different wave lengths will be gradually sorted out into groups of waves of approximately the same wave length. Since the waves in front pass out of the group and new waves enter the group from behind, the energy within the group does not change.

We now study the properties of such a group. To this end we examine the disturbance due to the superposition of two simple harmonic waves of the same amplitude and slightly different wave lengths.

$$\eta_1 = a\sin(mx - nt) \quad \text{and} \quad \eta_2 = a\sin\{(m+\delta m)x - (n+\delta n)t\} \qquad \text{...(1)}$$

The resulting disturbance is given by $\eta = \eta_1 + \eta_2$ *i.e.*

$$\eta = 2a\cos\{(1/2)\times(x\delta m - t\delta n)\}\sin(mx - nt) = A\sin(mx - nt) \qquad \text{...(2)}$$

where
$$A = 2a\cos\{(1/2)\times(x\delta m - t\delta n)\} \qquad \text{...(3)}$$

Equation (2) shows that the resulting disturbance is a progressive sine wave whose amplitude A is not constant but is itself varying as a wave of velocity $c_g = \delta n/\delta m$. This velocity is known as the *group velocity*.

Since the velocity of propagation of a single wave is $c = n/m$, we have

$$c_g = \frac{dn}{dm} = \frac{d(cm)}{dm} = c + m\frac{dc}{dm} \qquad \text{...(4)}$$

But $\lambda = \dfrac{2\pi}{m}$ so that $\dfrac{d\lambda}{dm} = -\dfrac{2\pi}{m^2}$ ...(5)

Using (5), (4) may be re-written as

$$c_g = c + m\frac{dc}{d\lambda}\cdot\frac{d\lambda}{dm} = c - \lambda\frac{dc}{d\lambda} \qquad \text{...(6)}$$

For the case of waves on the surface of liquid of depth h, as in Case I of Art. 12.6, we have

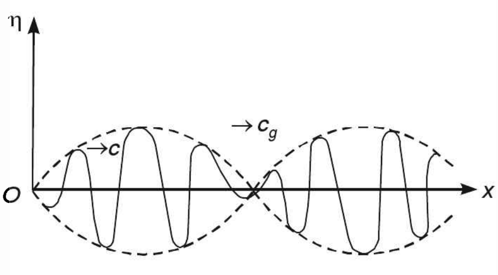
$$c^2 = (g/m)\tanh mh \qquad \text{...(7)}$$

$\therefore$ From (4) and (7), we have

$$c_g = c\left(1 + \frac{m}{2c^2}\frac{dc^2}{dm}\right) = \frac{1}{2}c\left(1 + \frac{2mh}{\sin 2mh}\right) \qquad \text{...(8)}$$

so that the ratio of the group velocity to the wave velocity is given by

$$\frac{c_g}{c} = \frac{1}{2} + \frac{mh}{\sin 2mh}$$



giving $$c_g = \frac{1}{2}(1 + 2mh \text{ cosech } mh) \qquad ...(9)$$

When h is small (*e.g.* consider shallow water) compared with the wave length, $c_g/c = 1$ so that *group velocity for shallow water is equal to the wave velocity.* Again, when $h \to \infty$ (*e.g.*, consider deep sea waves), $c_g/c = 1/2$ *i.e.* $c_g = c/2$ Thus, *the group velocity for deep sea wave is half the wave velocity.*

Note that we have a wave within a wave, as shown in the figure. The full line wave form advances with c whereas the dotted line form advances with the group velocity c_g.

12.14. Rate of transmission of energy in simple harmonic surface waves. Dynamical significance of group velocity.

In a simple harmonic train of surface waves, energy crosses a fixed vertical plane perpendicular to the direction of propagation at an average rate equal to group velocity.

Proof. Consider a vertical section of the liquid (of depth h as in case I of Art 12.6) at right angles to the direction of propagation. Then the rate of transmission of energy is calculated by determining the rate at which the pressure on one side of the chosen section is doing work on the liquid on the other side. Now, the velocity potential is given by

$$\phi = \frac{ga}{n} \frac{\cosh m(y+h)}{\cos mh} \cos(mx - nt). \qquad ...(1)$$

Again neglecting squares of small quantities the variable part of the pressure is given by

$$\delta p = \rho(\partial\phi/\partial t) \qquad ...(2)$$

and the horizontal velocity u is given by $$u = -\partial\phi/\partial x. \qquad ...(3)$$

Hence the rate at which work is being done on the fluid to the right of x is given by

$$W = \int_{-h}^{0} \delta p\, u dy = -\int_{-h}^{0} \delta p \frac{\partial\phi}{\partial y} dy = \frac{g^2 \rho a^2 m}{n} \frac{\sin^2(mx - nt)}{\cosh^2 mh} \int_{-h}^{0} \cosh^2 m(y+h)\, dy, \text{ by (1), (2) and (3)}$$

Then $$W = \frac{g^2 \rho a^2 m}{n} \frac{\sin^2(mx - nt)}{\cosh^2 mh} \left(\frac{\sinh 2mh}{4m} + \frac{h}{2} \right) \qquad ...(5)$$

From (23) and (24) of Art. 12.6, $n^2 = gm \tanh mh$. Hence (5) reduces to

$$W = (1/2) \times g\rho a^2 (n/m)(1 + 2mh \text{ cosech } 2mh) \sin^2(mx - nt) \qquad ...(6)$$

The average value of $\sin^2(mx - nt)$ over a period is 1/2. Hence the average rate of work done is given from (6) by

$$W = (1/4) \times g\rho a^2 (n/m)(1 + 2mh \text{ cosech } 2mh) \qquad ...(7)$$

But group velocity c_g from Art. 12.13 is given by

$$c_g = (c/2) \times (1 + 2mh \text{ cosech } 2mh) \qquad ...(8)$$

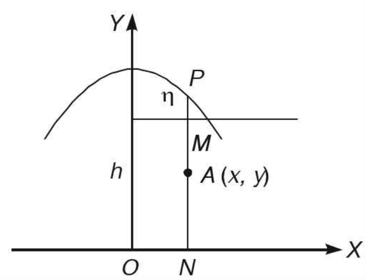
Since $n/m = c$, (7) and (8) give $$W = (g\rho a^2/2) \times c_g \qquad ...(9)$$

Since $(g\rho a^2)/2$ is the whole energy per unit length (refer Art. 10.9), (9) shows that the *energy is transmitted at a rate equal to the group velocity.*

12.15. Long waves or gravity waves [Meerut 1997]

The surface waves which have been studied in the preceding articles were not restricted as to wave length. We shall now study waves whose wave length is large compared with the depth of the liquid. In long waves the vertical acceleration can be neglected as compared to the horizontal acceleration and hence as far as vertical forces are concerned we may treat the liquid as in equilibrium. Accordingly, we take for the pressure at any point the statical pressure due to the depth below the free surface.

Let the x-axis be horizontal and parallel to the side of the canal, y-axis vertically upwards, the height h of the undisturbed surface and η the elevation above the undisturbed surface at a point where abscissa is x. In figure $PM = \eta$, $PN = h + \eta$, $PA = h + \eta - y$, p = pressure at A (x, y), p_0 = constant, pressure at P (above the liquid). Then. we have



$$p - p_0 = g\rho(y_0 + \eta - y) \qquad \text{...(1)}$$

so that $$\partial p / \partial x = g\rho(\partial \eta / \partial x) \qquad \text{...(2)}$$

which is independent of y. Again the horizontal acceleration of an element depends on the difference of pressure at its ends *i.e.* on $(\partial p / \partial x)$, which is independent of y Hence the horizontal acceleration of all points on the same vertical plane is the same and so all particles which lie once on a vertical plane will always lie on a vertical plane. Thus the whole vertical plane will move to and fro horizontally.

Consider a small horizontal cylindrical liquid element AA' of length $\delta x'$, which in equilibrium is situated at x and whose displacement at time t is ξ. Then the position of this element at time t' is given by

$$x' = x + \xi \qquad \text{...(3)}$$

Let α be the cross-section of the cylinder AA' and the horizontal acceleration of PP' is $\partial^2 \xi / \partial t^2$. Then the equation of motion of AA' gives

$$\rho\alpha\delta x' \frac{\partial^2 \xi}{\partial t^2} = p\alpha - \alpha\left(p + \frac{\partial p}{\partial x'}\partial x'\right) \quad \text{or} \quad \frac{\partial^2 \xi}{\partial t^2} = -\frac{1}{\rho}\frac{\partial p}{\partial x'} \qquad \text{...(4)}$$

But $$\frac{\partial p}{\partial x'} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial x'} = \frac{\partial p}{\partial x}\left(1 - \frac{\partial \xi}{\partial x'}\right), \text{ by (3)}$$

or $$\partial p / \partial x' = \partial p / \partial x, \text{ negelecting the second term}$$

Thus, $$\partial p / \partial x' = g\rho(\partial \eta / \partial x), \text{ using (2)}$$

Hence (4) reduces to $$\partial^2 \xi / \partial t^2 = -g\,(\partial \eta / \partial x). \qquad \text{...(5)}$$

We now write down the equation of continuity. Let S be area of cross-section of the canal, and b the breadth at the surface. In the position of equilibrium the volume of liquid between the planes x and $x + \delta x$ is $S\delta x$. At any time t the distance between the bounding planes of these liquid will be $\delta x + (\partial \xi / \partial x)\delta x$, and the area of the cross-section of the liquid will be $S + b\eta$. Then from principle of conservation of mass, we get

$$\rho(S + b\eta)\left(\delta x + \frac{\partial \xi}{\partial x}\,\delta x\right) = \rho S\,\delta x \qquad \text{...(6)}$$

Omitting the product of small quantities, (6) gives $$S\,(\partial \xi / \partial x) + b\eta = 0 \qquad \text{...(7)}$$

Substituting the value of η from (7) in (5), we get

$$\partial^2 \xi / \partial t^2 = (gS / b)\,(\partial^2 \xi / \partial x^2) \qquad \text{...(8)}$$

or $$\partial^2 \xi / \partial t^2 = c^2(\partial^2 \xi / \partial x^2), \qquad \text{...(9)}$$

where $c^2 = gS / b$ or $c = \sqrt{gS / b}$ ...(10)

Equation (9) is well known one dimensional wave equation. To solve (9), we assume that

$$x - ct = x_1 \qquad \text{and} \qquad x + ct = x_2 \qquad ...(11)$$

Then
$$\frac{\partial \xi}{\partial x} = \frac{\partial \xi}{\partial x_1}\frac{\partial x_1}{\partial x} + \frac{\partial \xi}{\partial x_2}\frac{\partial x_2}{\partial x} = \frac{\partial \xi}{\partial x_1} + \frac{\partial \xi}{\partial x_2} = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\xi \qquad ...(12)$$

and
$$\frac{\partial \xi}{\partial t} = \frac{\partial \xi}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial \xi}{\partial x_2}\frac{\partial x_2}{\partial t} = -c\left(\frac{\partial \xi}{\partial x_1} + \frac{\partial \xi}{\partial x_2}\right) = -c\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\xi \qquad ...(13)$$

$\therefore$
$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\right) = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\left(\frac{\partial \xi}{\partial x_1} + \frac{\partial \xi}{\partial x_2}\right), \text{ by (12)}$$

Thus,
$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial x_1^2} + 2\frac{\partial^2 \xi}{\partial x_1 \partial x_2} + \frac{\partial^2 \xi}{\partial x_2^2} \qquad ...(14)$$

and
$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial t}\right) = c^2\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\left(\frac{\partial \xi}{\partial x_1} - \frac{\partial \xi}{\partial x_2}\right)$$

Thus
$$\frac{\partial^2 \xi}{\partial t^2} = c^2\left(\frac{\partial^2 \xi}{\partial x_1^2} - 2\frac{\partial^2 \xi}{\partial x_1 \partial x_2} + \frac{\partial^2 \xi}{\partial x_2^2}\right) \qquad ...(15)$$

Using (14) and (15), (9) reduces to on simplification

$$\frac{\partial^2 \xi}{\partial x_1 \partial x_2} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial x_1}\left(\frac{\partial \xi}{\partial x_2}\right) = 0 \qquad ...(16)$$

Integrating (16) w.r.t. x_1, we get $\qquad \partial \xi / \partial x_2 = \phi(x_2)$. $\qquad$ ...(17)

Integrating (17) w.r.t. x_2, we get

$$\xi = f(x_1) + F(x_2) = f(x - ct) + F(x + ct), \qquad ...(18)$$

where f and F are arbitrary functions. Hence solution of (8) is given by (18), which represents two waves travelling in opposite directions along the x-axis with velocity $c = \sqrt{gS/b} = \sqrt{gh}$. Here h is depth of the canal of rectangular section so that $S = bh$..

Differentiating (8) particularly with respect to x and treating A and b to be independent of x we get

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial \xi}{\partial x}\right) = \frac{gS}{b}\frac{\partial^2}{\partial x^2}\left(\frac{\partial \xi}{\partial x}\right) \qquad ...(19)$$

Substituting the value of $\partial \xi / \partial x$ from (7) in (19), we get

$$\partial^2 \eta / \partial t^2 = (gS/b)\,(\partial^2 \eta / \partial x^2), \qquad ...(20)$$

showing that the long waves are transverse waves as η is the displacement along y-axis and the propagation of waves takes place along x-axis.

To find the motion of surface particle, we take $\qquad \xi = f(x - ct)$ $\qquad$ ...(21)

The velocity of the particle $= \dot{\xi} = -cf'(x - ct)$, $\qquad$ ...(22)

where dot denotes differentiation will respect of 't'

From (7) and (21), $\eta = -(S/b) \times (\partial \xi / \partial x) = -(S/b)f'(x - ct) = -hf'(x - ct)$, as $S = bh$ $\quad$ ...(23)

$\therefore$ From (22) and (23), $\qquad \dot{\xi} = c\eta / h.$ $\qquad$ ...(24)

12.16. Conditions for long waves.

Consider the motion of a wave consisting of a single elevation of wave length λ and maximum elevation η. Then the time taken by it to pass a particular particle is λ / c, where c is the velocity, so that the vertical velocity will be of order $\eta c/\lambda$, and the vertical acceleration of order $\eta c^2/\lambda^2$. From Art. 12.15, the maximum horizontal velocity is $\eta c/h$. Taking $c^2 = gh$, the ratio of the maximum vertical and horizontal velocities is of order h/λ, and the vertical acceleration being of order $g\eta h/\lambda^2$ can be neglected provided h/λ is a small quantity. Thus the wave length is very great compared with the depth of the fluid. For this reason these are known as *long waves*.

Again the results of Art 12.15 have been obtained under the conditions that the vertical acceleration $u(\partial u/\partial x)$ is small compared to the horizontal acceleration $\partial u/\partial t$, *i.e.*

$$\frac{u(\partial u/\partial x)}{\partial u/\partial t} \text{ must be small} \qquad \text{...(1)}$$

But $\quad \xi = f(x-ct) \quad$ so that $\quad u = \dot{\xi} = -cf'(x-ct)$

$\therefore \quad \partial u/\partial t = c^2 f''(x-ct) \quad$ and $\quad \partial u/\partial x = -cf''(x-ct)$

so that $\quad \dfrac{\partial u}{\partial x} = -c\dfrac{\partial u}{\partial x} \quad$ and $\quad$ hence $\quad \dfrac{u(\partial u/\partial x)}{\partial u/\partial t} = -\dfrac{u}{c}.$

So (1) shows that u/c must be small.

But from Art. 12.15, $\dot{\xi}/c = u/c = \eta/h$. Hence, η/h must be small.

Thus the elevation above the mean level must be small compared to the depth of the liquid.

12.17. Energy of long wave.

For a wave in a canal of rectangular section the potential energy V is due to the elevation or depression of the liquid above the mean level, and for a unit breadth of the wave, we have

$$V = \frac{1}{4} g\rho \int \eta^2 dx, \qquad \text{...(1)}$$

where η is the elevation at x and integration is performed over one wave length. Again, for the same range of integration, the kinetic energy T is given by

$$T = \frac{1}{2}\rho h \int \dot{\xi}^2 dx = \frac{1}{2}\rho h \int \frac{c^2}{h^2}\eta^2 dx, \quad \text{as} \quad \dot{\xi} = \frac{c\eta}{h} \text{ from Art. 12.15}$$

Thus, $$T = \frac{1}{2} g\rho \int \eta^2 dx, \quad \text{as} \quad c^2 = gh \qquad \text{...(2)}$$

From (1) and (2) we find that *kinetic energy is equal to the potential energy and each is equal to half the total energy.*

12.18. Long waves reduced to a case of steady motion.

An alternative approach to determine the velocity of propagation is to apply technique outlined in Art. 12.9. The problem of a long wave travelling in one direction without change of profile can be reduced to steady motion by imposing on the whole system a velocity equal and opposite to the velocity c of propagation.

Neglecting the vertical velocity, let u be the small additional velocity due to the wave motion at points where the elevation is η. Let A be the area of the cross-section and b the breadth at the surface. The equation of continuity is then

$$\rho(A + b\eta)(c + u) = \rho Ac. \qquad \text{...(1)}$$

If the pressure difference between the sections x and $x + \delta x$ be δp then for a streamline passing through these points the Bernoulli's equation gives

$$\frac{\delta p}{\rho}+g\eta+\frac{1}{2}(u+c)^2=\frac{1}{2}c^2 \qquad ...(2)$$

Re-writing (2) and using (1), we have

$$\delta p=\frac{1}{2}\rho c^2\left\{1-\frac{A^2}{(A+b\eta)^2}\right\}-g\rho\eta \qquad \text{or} \qquad \delta p=\rho\eta\left\{\frac{1}{2}c^2\frac{2Ab+b^2\eta}{(A+b\eta)^2}-g\right\} \qquad ...(3)$$

If η be small compared to A/b, (3) reduces to

$$\delta p=\rho\eta\,(c^2b/A-g) \qquad ...(4)$$

If the streamline lies on the free surface, then $\delta p=0$ and hence (4) gives $c=\sqrt{gA/b}$ for non-zero η. This value of c gives the velocity of propagation of a long wave in still water, or the velocity of the stream for a stationary long wave. Now, re-writing (3), we have

$$\frac{\delta p}{\eta\rho}=\frac{1}{2}c^2\frac{2Ab}{A^2}\left(1+\frac{b\eta}{2A}\right)\left(1+\frac{b\eta}{A}\right)^{-2}-g \qquad ...(5)$$

$$=\frac{1}{2}\cdot\frac{Ag}{b}\cdot\frac{2b}{A}\left(1+\frac{b\eta}{2A}\right)\left(1-\frac{2b\eta}{A}\right)-g, \quad \text{as} \quad c^2=\frac{gA}{b}$$

[to second order of approximation]

$$\frac{\delta p}{\eta\rho}=g\left(1-\frac{3}{2}\frac{b\eta}{A}\right)-g \qquad \text{so that} \qquad \delta p=-\frac{3g\rho b\eta^2}{2A}, \qquad ...(6)$$

showing that the pressure is defective at all parts of the wave at which η is not zero. *So, unless* η^2 *can be neglected, we cannot have a free surface for a stationary long wave; that is, it is impossible for a long wave whose height is not small compared to depth of the liquid to be propagated in still water without change of form.*

From (5) we find that $\delta p=0$, when $\quad c^2=\dfrac{gA}{b}\left(1+\dfrac{b\eta}{2A}\right)^{-1}\left(1+\dfrac{b\eta}{A}\right)^{2}$

or $$c^2-\frac{gA}{b}=\frac{gA}{b}\left[\left(1+\frac{b\eta}{2A}\right)^{-1}\left(1+\frac{b\eta}{A}\right)^{2}-1\right]=\frac{gA}{b}\left(1+\frac{b\eta}{2A}\right)^{-1}\left\{\left(1+\frac{b\eta}{A}\right)^{2}-\left(1+\frac{b\eta}{2A}\right)\right\}$$

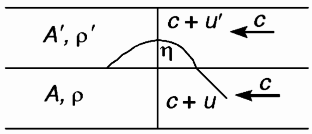
$$=\frac{gA}{b}\left(1+\frac{b\eta}{2A}\right)^{-1}\left(\frac{3}{2}\times\frac{b\eta}{A}\right)=\frac{3g\eta}{2}\left(1+\frac{b\eta}{2A}\right)^{-1}$$

showing that $\eta>0 \Rightarrow$ R.H.S. $>0 \Rightarrow c^2>gA/b$. Similarly, $\eta<0 \Rightarrow c^2<gA/b$.

Thus an elevation in the surface travels rather faster then a depression.

12.19. Long waves at the common surface of two liquids bounded above and below by two fixed horizontal planes.

Let ρ, ρ' be the densities of the liquids, A, A' the cross sections of the two liquid streams and b the breadth of the common surface. The problem of a long wave travelling in one direction without change of profile can be reduced to steady motion by impressing on the whole system a velocity equal and opposite to the velocity c of propagation. Let u, u' be the small additional velocities due the wave motions in the two liquids and η be the elevation of the common surface.



Then the equations of continuity in the two liquids are

$$\rho(A+b\eta)(c+u)=\rho Ac \qquad ...(1)$$

and
$$\rho'(A'-b\eta)(c+u')=\rho' A' c. \qquad ...(2)$$

Neglecting quantities of second order of smallness, (1) and (2) respectively reduce to

$$Au+bc\eta=0 \qquad ...(3)$$

and
$$A'u'-bc\eta=0. \qquad ...(4)$$

Let δp and $\delta p'$ be the increments of pressure close to the common surface in the two liquids due to the waves. Then from Bernoulli's equation, we have

$$\delta p/\rho+g\eta+(1/2)\times(c+u)^2=c^2/2 \qquad ...(5)$$

and
$$\delta p'/\rho'+g\eta+(1/2)\times(c+u')^2=c^2/2. \qquad ...(6)$$

Neglecting the surface-tension, we have $\delta p=\delta p'$ and hence (5) and (6) give

$$g(\rho-\rho')\eta=(\rho' u'-\rho u)c$$

or
$$g(\rho-\rho')\eta=\left(\rho'\cdot\frac{bc\eta}{A'}+\rho\cdot\frac{bc\eta}{A}\right)c, \text{ using (3) and (4)}$$

or
$$g(\rho-\rho')=(\rho'/A'+\rho/A)bc^2 \quad \text{or} \quad c^2=\frac{g(\rho-\rho')}{b(\rho/A+\rho'/A')} \qquad ...(7)$$

Cor. *The space between two infinity horizontal planes is filled with two fluids, one density* ρ *and depth h and the other of density* ρ' *and depth* h'. *Show that the velocity of a long wave on the surface of separation is* $\sqrt{\dfrac{g(\rho-\rho')hh'}{h'\rho+h\rho'}}$

Proof. Here $h=A/b$ and $h'=A'/b'$ and so (7) reducs to

$$c^2=\frac{g(\rho-\rho')}{\rho/h+\rho'/h'}=\frac{g(\rho-\rho')hh'}{h'\rho+h\rho'} \quad \text{so that} \quad c^2=\sqrt{\frac{g(\rho-\rho')hh'}{h'\rho+h\rho'}}.$$

12.20. Illustrative solved examples.

Ex. 1. *If a canal of rectangular section contains a depth h of liquid of density* ρ, *on which is superposed a depth* h' *of liquid of density* ρ', *the free surface of the latter being exposed to a constant atmospheric pressure, prove that the velocities of propagation of waves of length* $2\pi/m$ *are given by* $c^2=gu/m$, *where* $\rho(u\coth mh-1)(u\coth mh'-1)=\rho'(1-u^2)$.

[Bangalore 2004; Lucknow 2003, 06, Pune 2001, 05]

Sol. Proceeding as in Art. 12.11 upto equation (14), we get

$$c^4m^2(\rho\coth mh\coth mh'+\rho')-c^2gm\rho(\coth mh+\coth mh')+g^2(\rho-\rho')=0$$

Putting $c^2=gu/m$ in the above equation, we get

$$g^2u^2(\rho\coth mh\coth mh'+\rho')-g^2u\rho(\coth mh+\coth mh')+g^2(\rho-\rho')=0$$

or
$$\rho[u^2\coth mh\coth mh'-u(\coth mh+\coth mh')+1]=\rho'(1-u^2)$$

or
$$\rho(u\coth mh-1)(u\coth mh'-1)=\rho'(1-u^2).$$

Ex. 2. *If there be two liquids in a straight canal of uniform section of densities* σ_1, σ_2 *and depths* l_1, l_2, *show that the velocity c of propagation of long waves is given by*

$$\left(\frac{c^2}{l_1 g}-1\right)\left(\frac{c^2}{l_2 g}-1\right)=\frac{\sigma_1}{\sigma_2},$$ *where* $\sigma_2 > \sigma_1$ *and it is assumed that liquids do not mix.*

[Garhwal 2000, 02; Nagpur 2001, 06]

Sol. Proceeding as in Art. 12.11 with $\rho = \sigma_2$, $\rho' = \sigma_1$, $h = l_2$, $h' = l_1$, we get from equation (14), of Art. 12.11

$$c^4 m^2 (\sigma_2 \coth ml_2 \coth ml_1 + \sigma_1) - c^2 gm\sigma_2 (\coth ml_2 + \coth ml_1) + g^2 (\sigma_2 - \sigma_1) = 0 \quad \ldots (i)$$

But for long waves, m is small and so we have

$\coth ml_1 = 1/ml_1$ and $\coth ml_2 = 1/ml_2$, approximately.

Hence, (*i*) reduces to $$\frac{c^4\sigma_2}{l_1 l_2} + \sigma_1 m^2 c^4 - c^2 g\sigma_2\left(\frac{1}{l_1}+\frac{1}{l_2}\right) + g^2(\sigma_2 - \sigma_1) = 0.$$

But for log waves, $m = 2\pi/\lambda$ is small. So neglecting m^2, we get

$$\frac{c^4}{l_1 l_2 g^2} - \frac{c^2}{g}\left(\frac{1}{l_1}+\frac{1}{l_2}\right) + 1 = \frac{\sigma_1}{\sigma_2} \quad \text{or} \quad \left(\frac{c^2}{l_1 g}-1\right)\left(\frac{c^2}{l_2 g}-1\right)=\frac{\sigma_1}{\sigma_2}.$$

Ex. 3. *Let a shallow trough be filled with oil and water, and let the depth of the water be k and its density* σ *and the depth of the oil h and its density* ρ. *Then show that if g be gravity, and v the velocity of propagation of long waves* $$\frac{v^2}{g} = \frac{1}{2}(h+k) + \frac{1}{2}\left[(h-k)^2 + 4hk\frac{\rho}{\sigma}\right]^{1/2},$$ *assuming that there may be slipping between the two fluids.*

Sol. Proceeding as in Art. 12.11 with $\rho = \sigma$, $\rho' = \rho$, $h = k$, $h' = h$, $c = v$, we obtain from equation (14), of Art. 12.11

$$v^4 m^2 (\sigma \coth mk \coth mh + \rho) - v^2 mg\sigma (\coth mk + \coth mh) + g^2 (\rho - \sigma) = 0. \quad \ldots(i)$$

But for long waves m is small and so we have

$\coth mk = 1/mk$, and $\coth mh = 1/mh$, approximately.

$\therefore$ (*i*) gives $$v^4 m^2\left(\frac{\sigma}{m^2 hk}+\rho\right) - v^2 mg\sigma\left(\frac{1}{mk}+\frac{1}{mh}\right) + g^2(\sigma-\rho) = 0$$

or $$v^4\left(\frac{\sigma + \rho m^2 hk}{hk}\right) - v^2 g\sigma\frac{h+k}{hk} + g^2(\sigma - \rho) = 0$$

or $$v^4\sigma - v^2 g\sigma(h+k) + g^2 hk(\sigma-\rho) = 0, \text{ neglecting } m^2$$

$\therefore$ $$v^2 = [g\sigma(h+k) \pm \{g^2\sigma^2(h+k)^2 - 4g^2 hk\sigma(\sigma-\rho)\}^{1/2}]/2\sigma$$

$$\frac{v^2}{g} = \frac{h+k}{2} \pm \frac{1}{2}\left\{(h+k)^2 - \frac{4hk}{\sigma}(\sigma-\rho)\right\}^{1/2} \quad \text{or} \quad \frac{v^2}{g} = \frac{h+k}{2} \pm \frac{1}{2}\left\{(k-h)^2 + 4hk\frac{\rho}{\sigma}\right\}^{1/2}$$

For long waves velocity is large and ρ/σ is approximately equal to unity. Hence we omit negative sign before the radical sign in the above equation and obtain the required result.

Ex. 4. *When simple harmonic waves of length* λ *are propagated over the surface of deep water, prove that, at a point whose depth below the undisturbed surface is h, the pressure at the instants when the disturbed depth of the point is* $h + \eta$ *bears to the undisturbed pressure at the same point the ratio* $1 + (\eta/h)e^{-2\pi h/\lambda} : 1$, *atmospheric pressure and surface tension being neglected.*

Sol. For deep water, the velocity potential (as in case II of Art. 12.6) is given by

$$\phi = (na/m)e^{my}\cos(mx - nt) \qquad ..(1)$$

$$\therefore \qquad \partial\phi/\partial t = (an^2/m)\, e^{my}\sin(mx - nt) \qquad ..(2)$$

Also $\eta = \sin(mx - nt)$ and $c^2 = n^2/m^2 = g/m$. So (2) becomes

$$\partial\phi/\partial t = g\eta e^{my} \qquad ...(3)$$

Pressure at any point within the water is given by

$$p/\rho - (\partial\phi/\partial t) + gy = c \text{ (a constant)} \qquad ...(4)$$

When $y = 0$, $p = 0$, $\partial\phi/\partial t = 0$ so $c = 0$ and hence (4) gives

$$\therefore \quad p = \rho(\partial\phi/\partial t) - g\rho y \qquad \text{or} \qquad p = \rho g\eta e^{my} - g\rho y, \text{ by (3)} \qquad ...(5)$$

$\therefore$ Disturbed pressure p_1 when $y = -h$ is given by

$$p_1 = \rho g\eta e^{-mh} + g\rho y = \rho gh\{1 + (\eta/h)\times e^{-mh}\} \qquad ...(6)$$

and undisturbed pressure p_2 at a depth h is given by $\qquad p_2 = \rho gh \qquad ...(7)$

$$\therefore \quad p_1 : p_2 = \left(1 + \frac{\eta}{h}e^{-mh}\right) : 1 \qquad \text{or} \qquad p_1 : p_2 = \left(1 + \frac{\eta}{h}e^{-2\pi h/\lambda}\right) : 1, \qquad \text{as } m = \frac{2\pi}{\lambda}$$

Ex. 5. *A fixed buoy in deep water is observed to rise and fall twenty times in a minute, prove that the velocity of the wave is about* 12.5 *miles per hour.*

Sol. As in Case II. Art. 12.6, we have $\qquad c^2 = g\lambda/2\pi \qquad ...(1)$

Frequency of the wave = 20 per minute = 1/3 per second

$\therefore$ $\lambda = 3c$ and so (1) gives $\qquad c = 3g/2\pi\ ft/\text{sec} = 12.5$ miles/hour.

Ex. 6. *The crests of roller which are directly following a ship* 220 *ft. long are observed to overtake it at an interval of* 33/2 *seconds and it takes a crest* 6 *seconds to run along the ship. Find the length of the ship.* **(Agra 2006)**

Sol. As in case II, Art, 12.6, $\qquad c^2 = g\lambda/2\pi \qquad ...(1)$

Let u be the velocity of the ship. Then, we have

$$(33/2)\times(c - u) = \lambda \qquad \text{and} \qquad 6(c - u) = 220 \qquad ...(2)$$

From (2), $\qquad \dfrac{\lambda}{220} = \dfrac{33}{2\times 6} \qquad$ or $\qquad \lambda = 605$ ft.

$\therefore$ From (1), $\qquad c = \sqrt{\dfrac{32\times 605\times 7}{2\times 22}}$ ft./sec. = 55 ft./sec.approx. = $37\dfrac{1}{2}$ miles/hour.

Hence from (2), $\qquad u = \left(c - \dfrac{110}{3}\right)$ ft./sec. = $\dfrac{55}{3}$ ft./sec = $12\dfrac{1}{2}$ miles/hour.

Ex. 7. *Two fluids of densities* ρ_1, ρ_2, *have horizontal surface of separation but are otherwise unbounded. Show that when waves of small amplitude are propagated at their common surface, the particles of the two fluids describe circles about their mean position and that at any point of the surface of separation, where the elevation is* η, *the particles on either side have a relative velocity* $4\pi c\eta/\lambda$.

Sol. Let the wave profile at the common surface be

$$\eta = a \sin(mx - nt) \qquad ...(1)$$

where $\quad m = 2\pi/\lambda \quad$ and $\quad a << 1 \qquad ...(2)$

Let ϕ and ϕ' be velocity potentials in the lower and upper fluids. Then they must satisfy Laplace's equation and the conditions that

$$\partial\phi/\partial y \to 0 \quad \text{as} \quad y \to -\infty \qquad \text{(for lower fluid)}$$

and $\quad \partial\phi'/\partial y \to 0 \quad$ as $\quad y \to \infty. \qquad$ (for upper fluid)

Hence ϕ and ϕ' may be taken as*

$$\phi = Ae^{my}\cos(mx - nt) \qquad ...(3)$$

and $$\phi' = Be^{-my}\cos(mx - nt) \qquad ...(4)$$

Since the normal velocity at common surface must be continuous, we have

$$\partial\eta/\partial t = -(\partial\phi/\partial y) = -(\partial\phi'/\partial y) \quad \text{at} \quad y = \eta. \qquad ...(5)$$

If η be small, (5) reduces to

$$\partial\eta/\partial t = -(\partial\phi/\partial y) = -(\partial\phi'/\partial y) \quad \text{at} \quad y = 0. \qquad ...(6)$$

Using (1), (3) and (4), (6) reduces

$$-an\cos(mx - nt) = -Am\cos(mx - nt) = Bm\cos(mx - nt)$$

or $\quad A = an/m \quad$ and $\quad B = -an/m$

$\therefore$ From (3), $$\phi = (an/m)\,e^{my}\cos(mx - nt) \qquad ...(7)$$

and from (4), $$\phi' = -(an/m)\,e^{-my}\cos(mx - nt). \qquad ...(8)$$

We now determine the path of a particle in the lower fluid. Let (x_0, y_0) be the equilibrium position of the chosen particle and (x, y) at any time t. Then the coordinates (X, Y) of the fluid particle relative to its mean position are given by

$$x = x_0 + X \quad \text{and} \quad y = y_0 + Y \qquad ...(9)$$

$\therefore$ $$\dot{X} = \dot{x} = -(\partial\phi/\partial x) = ane^{my_0}\sin(mx - nt) \qquad ...(10)$$

and $$\dot{Y} = \dot{y} = -(\partial\phi/\partial y) = -ane^{my_0}\cos(mx - nt), \qquad ...(11)$$

where we have omitted small quantities after writing y_0 for y in the exponential terms.

Initially $X = Y = 0$. So integrating (10) and (11), we get

$$X = ae^{my_0}\cos(mx - nt) \qquad ...(12)$$

and $$Y = ae^{my_0}\sin(mx - nt). \qquad ...(13)$$

From (12) and (13), $$X^2 + Y^2 = a^2e^{2my_0}$$

or $$(x - x_0)^2 + (y - y_0)^2 = (ae^{my_0})^2, \qquad ...(14)$$

which is a circle with centre at (x_0, y_0) Similarly, we can prove that the path of a particle in the upper fluid is also a circle.

For the relative velocity at the common surface, we have

$$\left[\left(-\frac{\partial\phi}{\partial x}\right) - \left(-\frac{\partial\phi'}{\partial x}\right)\right]_{y=0} = [ane^{my}\sin(mx - nt) + ane^{-my}\sin(mx - nt)]_{y=0} = 2m\eta.$$

* You may refer case II in Art 12.6 for more details.

$$= 2cm\eta = 4\pi c\eta/\lambda \qquad [\because\ c = n/m \text{ and } m = 2\pi/\lambda].$$

and
$$\left[\left(-\frac{\partial\phi}{\partial y}\right) - \left(-\frac{\partial\phi'}{\partial y}\right)\right]_{y=0} = 0$$

Therefore the desired relative velocity is $4\pi c\eta/\lambda$.

Ex. 8. *Two dimensional waves of length $2\pi/m$ are produced at the surface of separation of two liquids which are of densities ρ, ρ' $(\rho > \rho')$ and depths h, h′ confined between two fixed horizontal planes. Prove that, if the potential energy is reckoned zero in the position of equilibrium, the total energy of the lower liquid to that of the upper is in the ratio*

$$\rho\{(2\rho - \rho')\coth mh + \rho' \coth mh'\} : \rho'\{(\rho - 2\rho')\coth mh' - \rho\coth mh\}.$$

Sol. Replacing both V and V' by zero, (1), (3), (6), (7) and (8) of Art. 12.10 give the following results :

Corresponding to wave profile
$$\eta = a\sin(mx - nt), \qquad \text{...(1)}$$

for lower liquid
$$\phi = \frac{ac}{\sinh mh}\cosh m(y+h)\cos(mx - nt) \qquad \text{...(2)}$$

and for upper liquid
$$\phi' = -\frac{ac}{\sinh mh}\cosh m(y+h')\cos(mx - nt) \qquad \text{...(3)}$$

where
$$g(\rho - \rho') = c^2 m(\rho\coth mh + \rho'\cot mh'). \qquad \text{...(4)}$$

Kinetic energy of the lower liquid per wave length

$$T_1 = \frac{1}{2}\rho\int_0^\lambda \left(\phi\frac{\partial\phi}{\partial y}\right)_{y=0} dx = \frac{1}{2}\rho a^2c^2 m\coth mh\int_0^\lambda \cos^2(mx - nt)\,dx$$

Thus,
$$T_1 = \frac{1}{2}\rho a^2 c^2 m\coth mh\cdot\frac{\lambda}{2} = \frac{1}{4}\lambda\rho a^2c^2 m\coth mh.$$

Potential energy of the lower liquid per wave length

$$V_1 = \frac{1}{2}g\rho\int_0^\lambda \eta^2 dx = \frac{1}{2}g\rho a^2\int_0^\lambda \sin^2(mx - nt)\,dx = \frac{1}{2}g\rho a^2 g\lambda$$

So the total energy of the lower liquid is given by

$$E_1 = T_1 + V_1 = \frac{1}{4}ga^2\rho\lambda\left[1 + \frac{c^2 m\coth mh}{g}\right] = \frac{1}{4}ga^2\rho\lambda\left[1 + \frac{(\rho-\rho')\coth mh}{\rho\coth mh + \rho'\coth mh'}\right], \text{ by (4)}$$

Thus,
$$E = \frac{1}{4}ga^2\rho\lambda\,\frac{(2\rho - \rho')\coth mh + \rho'\coth mh'}{\rho\coth mh + \rho'\coth mh'} \qquad \text{...(5)}$$

Similarly the kinetic and potential energies T_2 and V_2 for the upper fluid are given by

$$T_2 = (1/4)\times\lambda\rho' a^2c^2 m\coth mh', \qquad \text{and} \qquad V_2 = -(1/4)\times g\rho' a^2\lambda$$

So the total energy of the upper fluid is given by

$$E_2 = T_2 + V_2 = \frac{1}{4}ga^2\rho'\lambda\left[\frac{(\rho - \rho')\coth mh'}{\rho\coth mh + \rho'\coth mh'} - 1\right]$$

$$= \frac{1}{4}ga^2\rho'\lambda\,\frac{(\rho - 2\rho')\coth mh' - \rho'\coth mh}{\rho\coth mh + \rho'\coth mh'} \qquad \text{...(6)}$$

From (5) and (6), the required ratio is given by $E_1 : E_2$.

Ex. 9. *Show that if velocity of the wind is just great enough to prevent the propagation of wave of length* λ *against it, the velocity of propagation of waves of wind is* $2c\sqrt{\sigma/(1+\sigma)}$, *where* σ *is the specific gravity of the air and c is the wave velocity when no wind is present.*

[Agra 2011; I.A.S. 1987]

Sol. From Art. 12.10, if V, V' be the velocities of the lower and upper of two liquids of densities ρ, ρ' and depths h, h' then (taking c_1 as the velocity of wave)

$$g(\rho-\rho') = m[(V-c_1)^2 \rho \coth mh + (V'-c_1)^2 \rho' \coth mh'] \qquad \text{...(1)}$$

Given $\rho'/\rho = \sigma$. Since the sea is as rest, $V = 0$ and h and h' both $\to \infty$. Hence (1) reduces to

$$g(1-\sigma) = m[c_1^2 + (V'-c_1)^2 \sigma]. \qquad \text{...(2)}$$

If no wind is present, $\quad V' = 0, \quad$ then $\quad c_1 = c$

$\therefore$ From (2), $$g(1-\sigma) = m(c^2 + c^2\sigma) = mc^2(1+\sigma) \qquad \text{...(3)}$$

When there is no wave, $c_1 = 0$. Hence (2) reduces to

$\therefore$ $$g(1-\sigma) = mV'^2\sigma \qquad \text{...(4)}$$

Now from (2), $$g(1-\sigma) = m(c_1^2 + V'^2\sigma + c_1^2\sigma - 2V'c_1\sigma)$$

or $$mV'^2\sigma = m(c_1^2 + V'^2\sigma + c_1^2\sigma - 2V'c_1\sigma), \text{ using (4)}$$

or $$0 = c_1^2(1+\sigma) - 2V'c_1\sigma \qquad \text{or} \qquad V' = \{c_1(1+\sigma)\}/2\sigma \qquad \text{...(5)}$$

Putting this value of V' in (4), we get

$$g(1-\sigma) = m\sigma \cdot \frac{c_1^2(1+\sigma)^2}{4\sigma^2} \qquad \text{or} \qquad mc^2(1+\sigma) = m\sigma \cdot \frac{c_1^2(1+\sigma)^2}{4\sigma^2}, \text{ using (3)}$$

or $$c_1^2 = \frac{4\sigma}{(1+\sigma)}c^2 \qquad \text{so that} \qquad c_1 = 2c\left(\frac{\sigma}{1+\sigma}\right)^{1/2}.$$

Ex. 10. *A canal of infinite length and rectangular section is of uniform depth h and breadth b in one part and changes gradually to uniform depth* h' *and breadth* b' *in another part. An infinite train of simple harmonic waves travelling in one direction only is propagated along the canal. Prove that, if a,* a' *are heights and* $2\pi/m$, $2\pi/m'$ *the lengths of the waves in the two uniform portions*

$$m \tanh mh = m' \tanh m'h'$$

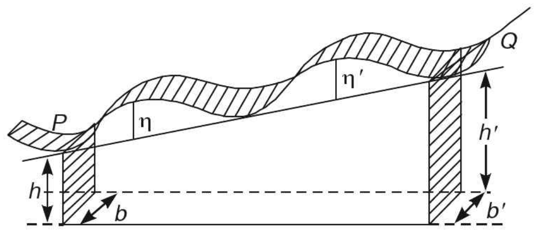
and

$$a^2 b \operatorname{sech}^2 mh (\sinh 2mh + 2mh) = a'^2 b' \operatorname{sech}^2 m'h' (\sinh 2m'h' + 2m'h').$$

Sol. If the mean profiles of the wave in the two parts be given by

$$\eta = a\sin(mx - nt) \qquad \text{and} \qquad \eta' = a'\sin(m'x - n't),$$

then we have [Refer Case I, Art 12.6]



$$c^2 = (g/m)\tanh mh \qquad \text{and} \qquad c'^2 = (g/m')\tanh m'h' \qquad ...(1)$$

Since the period of simple harmonic wave must remain unchanged all along the canal,

$$\frac{2\pi}{n} = \frac{2\pi}{n'} \qquad \text{or} \qquad \frac{m}{n}\cdot\frac{1}{m} = \frac{m'}{n'}\cdot\frac{1}{m'} \qquad \text{or} \qquad \frac{c}{c'} = \frac{m'}{m} \qquad ...(2)$$

$[\because\ c = n/m \text{ and } c' = n'/m']$

From (1) and (2), $\dfrac{m'^2}{m^2} = \dfrac{c^2}{c'^2} = \dfrac{\tanh mh}{m}\cdot\dfrac{m'}{\tanh m'h'}$ or $m\tan mh = m'\tanh m'h'$

Now the average energy transmitted in the portion of depth of the canal

$$= \frac{1}{4}g\rho a^2 bc\left(1+\frac{2mh}{\sinh 2mh}\right)$$

and for the other part of the canal $= \dfrac{1}{4}g\rho a'^2 b'c'\left(1+\dfrac{2m'h'}{\sinh 2m'h'}\right)$

Since the energy transmitted must be the same in both parts of the canal, we have

$$\frac{a^2bc(\sinh 2mh + 2mh)}{\sin 2mh} = \frac{a'^2b'c'(\sinh 2m'h' + 2m'h')}{\sinh 2m'h'}$$

or $\dfrac{a^2bg}{mc}\dfrac{\tanh mh}{\sinh 2mh}(\sin 2mh + 2mh) = \dfrac{a'^2b'g\tanh m'h'}{m'c'\sin 2m'h'}(\sinh 2m'h' + 2m'h')$, using (1)

Since $mc = m'c'$, the above equation reduces to

$$a^2b\,\text{sech}^2 mh\,(\sinh 2mh + 2mh) = a'^2b'\,\text{sech}^2 m'h'\,(\sinh 2m'h' + 2m'h')$$

Ex. 11. *Prove that* $w = A\cos(2\pi/\lambda)(z+ih-ct)$ *is complex potential for the propagation of simple harmonic surface waves of small height on water of depth h, the origin being in the undisturbed free surface. Express A in terms of the amplitude of the surface oscillations. Also prove that* $c^2 = (g\lambda/2\pi)\tanh(2\pi h/\lambda)$.

Sol. From Art. 12.6, Case I, we have

$$\phi = \frac{ag}{n}\frac{\cosh m(y+h)}{\cosh mh}\cos(mx - nt) \qquad ...(1)$$

Since cos $m(x + iy)$ = cos mx cosh my – i sin mx sinh my, we take

$$\psi = -\frac{ag}{n}\frac{\sinh m(y+h)}{\cosh mh}\sin(mx - nt) \qquad ...(2)$$

Hence $$w = \phi + i\psi = \frac{ag}{n}\frac{\cos\{(mx-nt)+im(y+h)\}}{\cosh mh}$$

or $$w = \frac{ag}{n}\frac{\cos\{m(x+iy)+imh-nt\}}{\cosh mh} = \frac{ag}{n}\frac{\cos m(z+ih-nt/m)}{\cosh mh}$$

$$= \frac{ag}{n}\frac{\cos\{(2\pi/\lambda)\ (z+ih-ct)\}}{\cosh mh}, \quad \text{as} \quad m = \frac{2\pi}{\lambda} \quad \text{and} \quad c = \frac{n}{m}$$

$$w = A\cos\left\{\frac{2\pi}{\lambda}(z+ih-ct)\right\},$$

where $$A = \frac{ag}{n\cosh mh} = \frac{ag}{mc\cosh mh}, \quad \text{as} \quad c = \frac{n}{m}$$

For the last part refer case I, Art. 12.6.

Ex. 12. *If a horizontal rectangular canal of great depth has two vertical barriers at a distance l apart, prove that the periods of oscillations of water are* $2\sqrt{\pi l/sg}$ *where s is the positive integer; and that corresponding to any node, all the particles of fluid oscillate in straight lines of length inversely proportional to* $\exp(\pi sz/l)$ *where z is the depth.*

Sol. Let η be the elevation of the stationary waves at the surface, then

$$\eta = a\sin mx\cos nt \qquad ...(1)$$

and $$\phi = (an/m)\, e^{my}\cos mx\sin nt \qquad ...(2)$$

where $$n^2 = mg \qquad ...(3)$$

Now, $-\partial\phi/\partial x = 0$ at $x = 0$ and $x = l$ ...(4)

$\therefore$ (2) gives $\sin ml = 0$ so that $ml = s\pi$

or $$m = s\pi/l, \; s \text{ being } a \text{ positive integer.} \qquad ...(5)$$

$\therefore$ From (3), $n^2 = s\pi g/l$. Hence, we get period of oscillation $= \dfrac{2\pi}{n} = 2\pi\left(\dfrac{\pi l}{gs}\right)^{1/2}$

Let (x_0, y_0) be the equilibrium position of a particle and (x, y) at any time t. Then the coordinates (X, Y) of the fluid particle relative to its mean position are given by

$$x = x_0 + X \quad \text{and} \quad y = y_0 + Y.$$

$$\left.\begin{aligned} \text{Then} \quad & \dot{X} = \dot{x} = -\frac{\partial\phi}{\partial x} = \frac{agm}{n}e^{my_0}\sin mx_0\sin nt \\ \text{and} \quad & \dot{Y} = \dot{y} = -\frac{\partial\phi}{\partial y} = -\frac{agm}{n}e^{my_0}\cos mx_0\sin nt \end{aligned}\right\} \qquad ...(6)$$

where we have neglected quantities of small order.

Integrating (6), $X = A - \dfrac{agm}{n^2}e^{my_0}\sin mx_0\cos nt = A - ae^{my_0}\sin mx_0\cos nt$, as $n^2 = mg$

and similarly, $$Y = B + ae^{my_0}\cos mx_0\cos nt$$

$$\therefore \quad (Y - B)/(X - A) = -\cot mx_0 = \text{constant},$$

showing that the path is a straight line. Again, we have

$$\sqrt{\{(X - A)^2 + (Y - B)^2\}} = ae^{my_0}\cot nt. \qquad ...(7)$$

$\therefore$ Maximum value of L.H.S. of (7) $= ae^{my_0} = ae^{-mz}$, as $y_0 = -z$

$= a/e^{mz} = a/e^{(\pi sz/l)}$, by (5)

$= a/\exp(\pi sz/l)$, where $e^x = \exp x$

So all the particles of fluid oscillate in straight lines of length proportional to $\exp(\pi sz/l)$.

Ex. 13. *Two liquids, which do not mix, occupy the region between two fixed horizontal planes. The upper, of density* ρ' *and mean depth* h'*, is flowing with the general velocity U over* *lower, which is of density* ρ *and mean depth h, and is at rest except for wave motion. Prove, neglecting viscosity, that the velocity V of waves of length* $(2\pi/k)$, *travelling, over the common surface in the direction of U, is given by* $\rho V^2 \coth kh + \rho' (U-V)^2 \coth kh' = Tk + g(\rho - \rho')/k$, *where T is the tension.*

Sol. In equation (13) case II of Art 12.12, replace m by k, c by V, V' by U and V by 0. Then

$$Tk^2 + g(\rho - \rho') = k\rho V^2 \coth kh + k\rho' (U-V)^2 \coth kh'$$

or
$$\rho V^2 \coth kh + \rho' (U-V)^2 \coth kh' = Tk + g(\rho - \rho')/k.$$

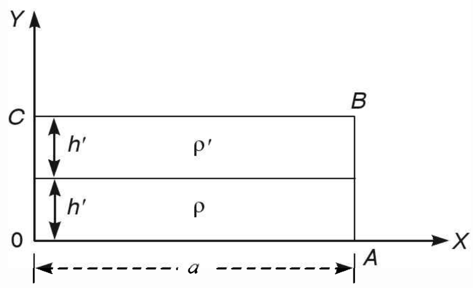
Ex. 14. *An open rectangular box of length a contains two liquids of densities* ρ, ρ' *and depths h, h' respectively, that of density* ρ *being at the bottom. Prove that the periods of oscillations when the liquids are slightly disturbed so that there is no motion perpendicular to the sides of the box, are determined by the type*

$$\left(p^2 \coth\frac{n\pi h}{a} - \frac{gn\pi}{a}\right)\left(p^2 \coth\frac{n\pi h'}{a} - \frac{gn\pi}{a}\right) + \frac{\rho'}{\rho}\left(p^4 - \frac{g^2 n^2 \pi^2}{a^2}\right) = 0,$$

where n is an integer. **[I.A.S. 1990; Meerut 1998]**

Sol. Since the liquid is confined in a rectangular box, with closed vertical ends $x = 0$ and $x = a$ as shown in figure, the wave length is restricted. Hence the stationary waves must be the principal modes of free oscillations of the restricted system. Let a wave profile be given by

$$\eta = a \sin mx \cos pt \qquad ...(1)$$



Since there must be loops at the end $x = a$,

$$ma = n\pi \quad \text{that} \quad m = n\pi/a, \qquad ...(2)$$

where n is any positive integer.

Again, the velocity of waves $= c = p/m$ ...(3)

From Art. 12.10, we have

$$c^4 m^2 (\rho \coth mh \coth mh' + \rho') - c^2 mg\rho (\coth mh + \coth mh') + g^2 (\rho - \rho') = 0.$$

Using (2) and (3), the above equation gives

$$\frac{p^4}{m^2}\left(\rho \coth\frac{n\pi h}{a}\coth\frac{n\pi h'}{a} + \rho'\right) - \frac{p^2}{m} g\rho\left(\coth\frac{n\pi h}{a} + \coth\frac{n\pi h'}{a}\right) + g^2(\rho - \rho') = 0$$

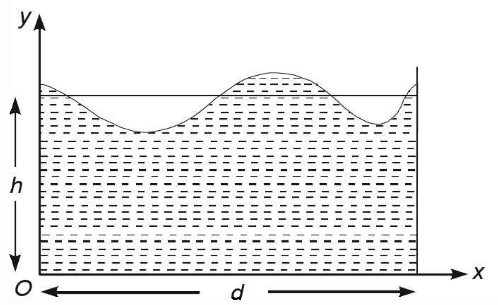
or
$$p^4 \coth\frac{n\pi h}{a}\coth\frac{n\pi h'}{a} - gp^2\frac{n\pi}{a}\left(\coth\frac{n\pi h}{a} + \coth\frac{n\pi h'}{a}\right) + g^2\left(1 - \frac{\rho'}{\rho}\right)\frac{n^2\pi^2}{a^2} + \frac{\rho'}{\rho}p^4 = 0$$

or
$$\left(p^2 \coth\frac{n\pi h}{a} - \frac{g\pi n}{a}\right)\left(p^2 \coth\frac{n\pi h'}{a} - \frac{g\pi n}{a}\right) + \frac{\rho'}{\rho}\left(p^4 - \frac{\pi^2 n^2 g^2}{a^2}\right) = 0.$$

Ex. 15. *Prove that in a fluid of depth h, limited by two vertical barriers distant d apart, at right angles to the direction of propagation of straight crested, irrotational waves, the periods of waves are found by giving r postitive integral values in the formula* $2[(\pi d/rg)\coth(\pi rh/d)]^{1/2}$.

[Kolkata 2000; Kurukahetra 2002, 04, 06; Garhwal 2003]

Sol. For restricted waves; *i.e.*, liquid oscillating in a tank of finite dimensions, the possible wave lengths of the oscillations are given by



$$\lambda = (2d)/1, (2d)/2., (2d)/3, \ldots (2d)/r,$$

where r assumes only positive integral values. This follows from the fact that since $x = 0$ has to be a loop, the surface distrubance must be of the form $\eta = a \cos mx \cos nt$, and there must be a loop at $x = d$, then we must have $md = 2\pi$, where r is any integer; for it gives the same distrubance.

Thus, we have $\lambda = (2\pi)/m = (2\pi d)/r\pi = (2d)/r$

The equation connecting the frequency with wave length is $n^2/m^2 = c^2 = (g/m) \tanh mh$

giving $$n^2 = gm \tanh mh = (r\pi g/d) \tanh(r\pi h/d) \quad \ldots(1)$$

Therefore, the time period T is given by

$$T = \frac{2\pi}{n} = 2\pi\left\{\frac{d}{r\pi g}\coth\left(\frac{r\pi h}{d}\right)\right\}^{1/2} = 2\left\{\frac{\pi d}{rg}\coth\left(\frac{r\pi h}{d}\right)\right\}^{1/2}$$

EXERCISES

1. If water flows along a rectangular canal which consists of two uniform portions of slightly different breadth with a gradual transition, the free surface will be lower where the canal is narrower or contrariwise according as $U^2 \leq gh$ or $U^2 \geq gh$, where U is the mean velocity and h the mean depth (motion to be supposed steadily).

2. The cross-section of a canal is semi-circle of radius a. Prove that the velocity of propagation of long waves is $(1/2) \times \sqrt{\pi a g}$, the banks of the canal being supposed vertical. **(Agra 2005, 10)**

3. Show that when irrotational waves of length λ are propagated in water of infinite depth, the pressure at any point of water is the same as it was in the equilibrium position of the particle when the water was at rest.

4. If water of depth h be flowing with velocity proportional to the distance from the bottom, V being the velocity of the stream at the surface, prove that velocity c of propagation of waves in the direction of the stream is given by $(c - V)^2 + V(c - V) \times (W^2/gh) - W^2 = 0$, where W is the velocity of propagation in still water. **(Agra 2010)**

5. The bottom of a straight uniform canal of rectangular cross-section has its vertical longitudinal section in the form $y = b \sin mx$, where b is small compared with the mean depth of the liquid in the canal. If the liquid is moving horizontally with a mean velocity V in the direction of the axis of x show that the free surface has the form

$$\eta = b\frac{\sinh mh'}{\sinh m(h' - h)}\sin mx, \qquad \text{where} \qquad mV^2 = g \tanh mh'.$$

6. Find the type of waves that would travel on deep water at 30 knots. How much is the velocity of wave affected by the presence of atmosphere above the water, its density being .0013?

[**Hint** : 30 nautical miles/hour = (30 × 6082)/(60 × 60) feet/sec.]

7. Give the theory of long waves in a canal of uniform width and depth h, proving that the velocity of free waves is $\sqrt{gh}$. Discuss briefly the hypothesis on which this result is obtained justifying the use of the term ‘long waves’.

8. Incompressible fluid of density ρ and depth h rests upon a horizontal plane surface $y = -h$, and above it rests incompressible fluid of density $\rho'(<\rho)$ and depth h'. The pressure on the upper surface of the upper fluid is constant. Show that a wave motion in which the form of the interface is $y = m\ (x - ct)$ (where a is small), is possible provided that

$$\rho(g - mc^2 \coth mh)\,(g - mc^2 \coth mh') = \rho'(g^2 - m^2 c^4).$$

9. A layer of inviscid incompressible liquid of density ρ and depth h lies over another inviscid incompressible liquid of infinite depth and density $\sigma (> \rho)$. Its upper surface is free. Neglecting surface tension, show that two possible types of waves of length $2\pi/m$ can be propagated along the layer, with velocity V given by $m\,V^2 = g$ and $m(\sigma \coth mh + \rho)V^2 = g(\sigma - \rho)$.

10. Derive the equation for long waves travelling along a straight canal of uniform cross-section A and breadth b in the form $\partial^2\xi/\partial t^2 = (gA/b)\,(\partial^2\xi/\partial x^2)$ where ξ is the horizontal displacement of the particles in a vertical plane. Find the solution given by $\xi(x, 0) = f(x)$, $\xi_t(x, 0) = g(x)$.

[**Hint :** For derivation refer Art. 12.15 and for the desired solution refer Part III in author's Advanced Differential Equations, published by S. Chand & Co. Delhi.]

11. Find the velocity of propagation of two-dimensional waves of given length propagated over infinitely deep liquid. Prove that the paths of the particles are approximately circles.

12. Investigate the wave motion occurring at a horizontal interface between two fluids of which the upper one of density ρ_2 has a general stream velocity V, and the lower one of density ρ_1 is at rest except for the small motion, the fluids being otherwise unlimited.

Show that the wave velocity c of waves of length λ is given by

$$g(\rho_1 - \rho_2) = (2\pi/\lambda)[\rho_1 c^2 + \rho_2 (c - V)^2]$$

and prove that for a given value of V, waves below a certain wave length cannot propagate.

13. In the case of surface waves, show that the condition to be satisfied at the free surface by the velocity potential is $\partial^2\phi/\partial t^2 + g(\partial\phi/\partial y) = 0$

14. An Infinite liquid of density σ lies above an infinite liquid of density ρ, the two liquids being separated by a horizontal plane surface. Show that the velocity v of propagation of waves of length λ along the interface is given by $v^2 = \dfrac{g\lambda}{2\pi} \tanh \dfrac{\rho - \sigma}{\rho + \sigma}$. Prove that, for any group of such waves, the group velocity is equal to one-half the wave velocity.

15. Obtain the equation for long waves travelling along a straight canal of uniform cross-section; and find the velocity of the waves.

16. Show that the energy of a progressive train of waves over the surface of water of depth h is half kinetic and half potential.

17. Prove that the velocity of propagation of surface wave of length λ on a rectangular coaxial of depth h is given by the formula $c^2 = (g\lambda/2\pi)\tanh(2\pi h/\lambda)$

Deduce from this an expression for the velocity of long waves in shallow water.

[**Hint.** For first part, refer case I, Art. 12.6. For deduction, note that $h/\lambda \to 0$ for long waves. So $\lim \dfrac{\tanh(2\pi h/\lambda)}{(2\pi h/\lambda)} \to 1$ and hence $c^2 = \dfrac{g\lambda}{2\pi}\cdot\dfrac{2\pi h}{\lambda}$ so that $c = \sqrt{gh}$.]

18. Prove that the kinetic energy of a progressive sine wave is equal to the potential energy due to elevation above the undisturbed level. Assume the depth of water to be finite.

19. Prove that in a uniform heavy liquid of depth k, there is not more than one wave length corresponding to any given velocity, and that any velocity less than $\sqrt{2g}$ is the velocity of the same wave,

20. Discuss the motion of surface waves in the case of a liquid of depth h in an infinitely long rectangular tank, the motion being along the length of the tank, and derive as a particular case the formula $c^2 = gh$ for long waves in shallow water.

21. Find the equations that completely characterize waves at the surface of water of depth h, and show that (*i*) for small values of λ / h, (λ = constant wave length), the speed of propagation is proportional to $\sqrt{\lambda}$ and (*ii*) for large λ / h the speed tends to constant value $\sqrt{gh}$.

22. Tidal waves are occurring in a square tank of depth h and side a. Find the normal modes and calculate the kinetic and potential energies for each of them. Show that when more than one such mode is present, the total energy is just the sum of the separate energies of each normal mode.

23. A layer of fluid of density ρ_3 and thickness h separates two fluids of densities ρ_1 and ρ_2, extending to infinity in opposite directions. If waves of length λ, large compared with h, be set up in the fluid, show that their velocity of propagation is either

$$\left\{\frac{g\lambda}{2\pi}\,\frac{\rho_3-\rho_1}{\rho_3+\rho_1}\right\}^{1/2} \quad \text{or} \quad \left\{gh\,\frac{(\rho_2-\rho_1)\,(\rho_3-\rho_2)}{\rho_2\,(\rho_3-\rho_1)}\right\}^{1/2}$$

24. Show that the complex potential for the simple harmonic progressive wave of small elevation η is $\quad w = \dfrac{ac\cos(mz - nt)}{\sinh mh}$.

25. Show that the complex potential for the simple harmonic progressive wave on the surface of a canal of uniform depth h is $\quad w = \dfrac{ac\cos(z + ih - ct)}{\sinh mh}$.

26. Show that the potential for a progressive wave in a deep canal of uniform depth is

$$w = ace^{-i(mz - nt)}.$$

27. Write short note on (*i*) progressive wave (*ii*) standing waves (*iii*) group velocity.

28. Show that the speed of long waves in a canal is given by $c = (gS/b)^{1/2}$ where S is the area of the cross-section of the canal and b the breadth at the surface.

29. Explain what is meant by the group velocity of a train of progressive waves in a dispersive medium. Obtain the expression $c^2 = (g/k)\tanh(kh)$ for the wave velocity of simple harmonic progressive wave of length $2\pi / k$ on the surface of a canal of uniform depth h and show that the group velocity is

$$\frac{1}{2}c\left[1+\frac{2kh}{\sinh(2kh)}\right],$$

Hence prove that the group velocity for deep sea waves is half the wave velocity.

30. Define group velocity and show that for waves on deep water, the group velocity is half the wave velocity and that on shallow water the group velocity is equal to the wave velocity.

31. What do you understand by standing wave. Prove that for a train of progressive wave on the surface of water, depth h, given by $\eta = a\sin(mx - nt)$, the total energy per wave length is $(1/2) \times g\rho a^2\lambda$.

32. What are surface waves? Show that in the progressive waves on the surface of a canal particle describes the ellipse about its mean position. **[Garhwal 2005]**

[**Hints :** Refer Art. 12.6 and its case I.]

33. Show that for the simple progressive wave $\eta = a\sin(mx - nt)$, the complex potential w, and the velocity of propagation c are given by $w = \{ac \cos(mz - nt)\}/\sinh m\pi$, $c^2 = (g/m) \tanh mh$, h being the depth of water below the undisturbed level and the axis of x being taken along the bottom in the direction of propogation, but for the above progressive wave in deep water, the complex potential is $w = ac \exp\{-i(mz + nt)\}$, the origin and the x-axis being taken in the undisturbed level.

Calculate the kinetic and potential energies associated with a single train of progressive wave on deep water, and from the condition that these energies are equal, obtain the relation $c^2 = (g\lambda / 2\pi)$, λ being the wave length. **[Agra 1996, 98, 2000]**

34. Prove that path of particle in case of progressive waves, when the canal is deep is circular. **[Agra 2006]**

35. Show that in case of standing waves $T + V = (\rho g a^2 \lambda)/2$. **[Agra 2006, 07]**

36. Find the speed of propagation of surface waves in deep water. **[Agra 2007]**

37. Explain the concept of group velocity. **[Agra 2007]**

38. Discuss the propagation of waves at the interface of two liquids. **[Agra 2008]**

39. Elevation above the mean level must be small compared to the depth of the liquid, the waves are called

(*a*) surface waves (*b*) stationary waves

(*c*) long waves (*d*) fast waves **(Agra 2008)**

40. For surface waves in deep water

(*a*) $\dfrac{c^2}{gh} = \dfrac{\lambda}{2\pi}$ (*b*) $\dfrac{c^2}{g} = \dfrac{\lambda}{2\pi}$ (*c*) $\dfrac{c^2}{\lambda} = \dfrac{g}{\pi}$ (*d*) $\dfrac{c^2}{2\pi} = g$ **[Agra 2007]**

41. The energy density of the water waves is

(*a*) $\rho g a^2$ (*b*) $(\rho g a)/2$ (*c*) $\rho g a$ (*d*) $(\rho g a^2)/2$ **[Agra 2007, 09]**

42. Fill up the gaps:

(*i*) The velocity of propagation in case of standing waves is given by......... **[Agra 2006]**

(*ii*) In the case of stationary waves, $T + V =$ .... **[Agra 2007]**

43. Find the speed of propogation of surface waves in deep water. **[Agra 2009]**

44. Discuss the propagation of waves at the interface of two liquids. **[Agra 2009]**

45. Fill up the gaps :

(*i*) For shallow water, the group velocity is equal to .... **[Agra 2010]**

(*ii*) The energy is transmitted (in waves) at a rate equal to .... **[Agra 2010, 2012]**

(*iii*) Let an equation of waves be $y = a \sin(mx - xt)$, then velocity of propagation will be .... **[Agra 2010, 2012]**

Ans. (*i*) wave velocity (*iii*) n/m

46. Distinguish between group velocity and wave velocity and establish the relation between them. When simple harmonic surface waves of small height are propagated over a surface of sheet of water, show that the ratio of the group velocity to wave velocity lies between 1/2 and 1.

Hint. Refer Art. 12.13, page 12.15. **[Agra 2012]**

47. Show that the path of particles relative to mean position in a deep canal is a circle. **[Agra 2010]**

CHAPTER

13 General Theory of Stress and Rate of Strain

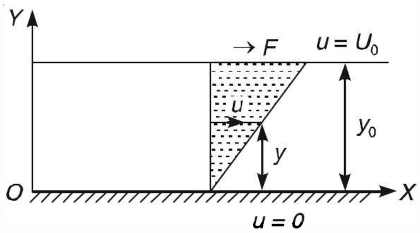
13.1. Introduction.

So far we have discussed situations which are based on the concept of a perfect (*i.e.* frictionless on non-viscous) fluid. In such a flow problem, we assume that two contacting layers experience no tangential forces (shearing stresses) but act on each other with normal forces (pressures) only. It, therefore, follows that a perfect fluid offers no internal resistance to a change in shape. The theory of perfect fluids is simple and it explains many real situations satisfactory. But the theory of perfect fluids fails completely to account for the drag of a body. In fact it leads to well known *D*' Alembert's paradox (refer Art. 4.6). This and many such real situations cannot be explained with help of theory of perfect fluids developed so far. We now propose to develop theory to explain drag of a body and other problems which cannot be solved with the help of theory of perfect fluids.

13.2(A). Newton's law of viscosity.

[Himanchal 2001, 03, 07; Meerut 2001; G. N. D. U. Amritsar 2003]

Energy losses or the need to supply energy to maintain flow leads to the conclusion that the deformation of real fluids is resisted by forces caused by internal friction or viscosity. Viscosity is that property of a real fluid which generates shear stress between two fluid elements. These tangential or shear forces are also known as *fluid friction*. A relationship between the shear stress and the velocity field was first stated by Newton. He concluded that the internal friction between two adjustment fluid particles should be independent of the normal pressure between them but proportional to the difference in their velocities.



Consider a layer of liquid between two parallel plates lying at a distance y_0 from each other. Let the lower plate be fixed, while the upper is moving with a velocity U_0 uniformly and parallel to the lower one. A resistance F is experienced, and to a first approximation is given by the formula

$$F = \mu A_0 \,(U_0 / y_0), \qquad \text{...(1)}$$

where A_0 is the area of upper plate and μ is a constant of proportionality, which is called the *coefficient of viscosity or the coefficient of dynamic viscosity*. The real fluids have a zero velocity at the walls and hence they cannot slip at the boundary wall. This is known as *no slip condition*.

Since there is no slip on the walls for real fluids, the velocity u of a layer at a distance y from the lower plate is given by

$$u = U_0\,(y/y_0) \qquad \text{so that} \qquad U_0/y_0 = u/y \qquad \text{...(2)}$$

$\therefore$ From (1) and (2), $\quad F = \mu A_0\,(u/y) \quad$ or $\quad F/A_0 = \mu(u/y)$

Thus $$\tau = \mu\,(u/y), \qquad ...(3)$$

where $\tau = F/A_0$ is the *friction (tangential) force per unit area or the shear stress*. The term u/y is called the velocity gradient. In its differential form, the velocity gradient is re-written as du/dy. Then (3) becomes $$\tau = \mu\,(du/dy), \qquad ...(4)$$

which is known as *Newton's law of viscosity*. Equation (4) can be regarded as the definition of viscosity.

It should be noted carefully that μ depends on the pressure and temperature but is independent of the pressure for gases at ordinary temperature.

From (4) above we easily find the dimensions of μ. Thus, we have

$$[\mu] = \frac{[\tau]}{[du/dy]} = \frac{(MLT^{-2})/L^2}{(LT^{-1})/L} = ML^{-1}T^{-1}, \qquad ...(5)$$

where M, L, T signify mass, length and time. In *C.G.S.* units, the unit of is poise, such that

$$1\,poise = \frac{dyne - \sec}{cm^2}$$

Sometimes a small unit centipoise is also used, which is (1/100)th of a poise.

The ratio of μ to the density ρ of the fluid is called the *kinematic coefficient of viscosity* (or *kinematic viscosity*) and is denoted by v. Thus, we have

$$v = \mu/\rho \qquad ...(6)$$

From (5) and (6) we readily find the dimensions of v. Thus, we have

$$[v] = \frac{[\mu]}{[\rho]} = \frac{ML^{-1}T^{-1}}{ML^{-3}} = L^2T^{-1} \qquad ...(7)$$

In *C.G.S.* units, the unit of kinematic viscosity is stoke such that 1 stoke = cm^2/sec

We now present the values of the coefficient of viscosity of some important fluids at 20°*C*

(*i*) Water : 1.005 centipeoise (*ii*) Air : 0.0181 centipoise

(*iii*) Glycerine : 8.5 poise (*iv*) Gastor oil : 9.86 poise

Remark. The existence of tangential (shearing) stresses and the condition of no slip near solid walls constitute the essential difference between a perfect (non-viscous) and real (viscus) fluids. Thus for a perfect or nonviscous fluids, $\mu = 0$

13.2(B). Newtonian and non-Newtonian fluids. **(Kanpur 2003)**

Fluids which obey Newton's law of viscosity are known as *Newtonian fluids*. Common fluids like water, air and mercury are all Newtonian fluids. This book will deal with such fluids only.

Fluids which do not obey Newton's law of viscosity are known as *non-Newtonian* fluids. Thus, for such fluids the shear stress is not proportional to the velocity gradient. Fluids like paints, coal tar and polymer solutions are all non-Newtonian fluids.

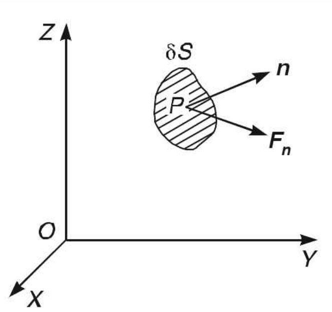
13.3. Body and surface forces.

In the study of fluid dynamics we distinguish between two types of forces acting on a fluids element, namely, body forces and surface forces. The body forces are distributed throughout the body, and these are usually expressed as 'force per unit mass of the element'. Examples are gravity and intertia forces Morever such forces may arise from other physical reasons also, such as electric and magenetic. Surface forces, on the other hand, arise due to the action of surrounding fluid on the element under consideration through direct contact. Thus it a boundary or surface action. These forces are expressed as 'force per unit surface area of the element'.

13.4 Definitions of stress, stress vector and components of stress tensor.

(Himancha 1998, 2003, Meerut 1998 Garhwal 2000, Kuruhshera 1997)

Consider a small area δS, includig a point P, on the surfce of a fluid body. Let (x, y, z) be the coordinates of P referred to a set of fixed axes, OX. OY, OZ. Let $\mathbf{n}$ be taken as the unit vector to specify the normal at P to δS on its $R.H.S.$ Then the chosen side of δS will be acted on by forces (exerted by the surrounding fluid) at its various points and various directions. These, by the principle of statics, can be combined into a single force $\delta \boldsymbol{F}$ through P and a single couple $\delta \boldsymbol{C}$ about some axis Let $\delta S \to 0$ in such a way as to always include the point P. In most of the fluids when $\delta S \to 0$, the couple $\delta \boldsymbol{C}$ must disappear, becaused in such a limit the force distribution across the area becomes uniform and parallel and therefore can be replaced by a sinle force. We shall, therefore, delete the couple $\delta \boldsymbol{C}$ from our further discussion*



We know that in the case of inviscid fluid, $\delta \boldsymbol{F}$ is along the direction of $\boldsymbol{n}$ so that there is only normal stress. On the other hand, in the case of viscous fluid, frictional forces are called into play between the surface and the fluid so that $\delta \boldsymbol{F}$ will now possess normal and tangential components δF_{nn} and δF_{ns}. Then normal and shear stresses are defined as follows :

$$\text{The normal stress} = \lim_{\delta S \to 0} \frac{\delta F_{nn}}{\delta S} \qquad ...(1)$$

$$\text{The shear stress} = \lim_{\delta S \to 0} \frac{\delta F_{ns}}{\delta S} \qquad ...(2)$$

Now $\delta \boldsymbol{F}/\delta S$ tends to a definite number as $\delta S \to 0$. This number will depend not only on the position of the point P but also on the orientation of the area δS. Hence it is presented by the vector symbol $\mathbf{F}_n$, the subscript n indicating the direction of the normal to δS at P as discussed before. $\mathbf{F}_n$ so defined, is called the *stress vector* or *surface traction* at P corresponding to the orientation $\mathbf{n}$ of the area. Thus, we haves

$$\mathbf{F}_n = \lim_{\delta S \to 0} \frac{\delta \mathbf{F}}{\delta S}. \qquad ...(3)$$

Let $\sigma_{nx}, \sigma_{ny}, \sigma_{nz}$ be the cartesian components of $\mathbf{F}_n$ and let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be the unit vectors parallel to the axes. Then we have

$$\mathbf{F}_n = \mathbf{i}\,\sigma_{nx} + \mathbf{j}\,\sigma_{ny} + \mathbf{k}\,\sigma_{nz} \qquad ...(4)$$

In particular, if the direction of $\mathbf{n}$ is parallel to x-axis (which may be obtained by re-orienting δS), we get

$$\text{Similarly,} \quad \left.\begin{aligned} \mathbf{F}_x &= \mathbf{i}\sigma_{xx} + \mathbf{j}\sigma_{xy} + \mathbf{k}\,\sigma_{xz} \\ \mathbf{F}_y &= \mathbf{i}\sigma_{yx} + \mathbf{j}\sigma_{yy} + \mathbf{k}\,\sigma_{yz} \\ \mathbf{F}_z &= \mathbf{i}\sigma_{zx} + \mathbf{j}\sigma_{zy} + \mathbf{k}\,\sigma_{zz} \end{aligned}\right\} \qquad ...(5)$$

and

In this way nine quantities are defined at a point, which may be arranged as follows :

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \qquad ...(6)$$

---

* In modern literature an attempt has been made to discuss fluids for which $\delta C/\delta S \to$ a finite non-zero value as $\delta S \to 0$. Such fluids are said to be *polar fluids* (*i.e.* fluids with couple stresses). These will not be considered in this book.

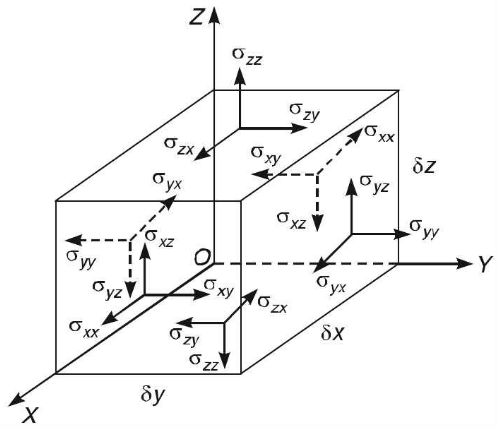
The above mentioned nine quantities** σ_{ii} $(i, j = x, y, z)$ constitute the components of the *stress tensor** of order two. We have used the double-subscript notation for stress components. The first subscript denotes the direction of the normal to the plane on which the stress acts, and the second subscript denotes the direction of the force producing the stress. It follows that normal stresses have repeated subscripts.

The diagonal elements $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ in (6) are said to be *normal stresses*. The remaining six elements are said to be *shearing stresses*. For any non-viscous fluid

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p;$$

and $$\sigma_{xy} = \sigma_{yx} = \sigma_{xz} = \sigma_{zx} = \sigma_{yz} = \sigma_{zy} = 0,$$

where p is the hydrostatic pressure. The matrix given by (6) is said to be the *stress matrix*. Knowledge of its components is employed to obtain the total force on any desired point of fluid. Stresses on a fluid element (which is an infinitesimal rectangular parallelepiped) are shown in adjoining figure.



We shall employ the following sign convention for the stress components : First we agree that an area vector $\delta \mathbf{S}$ has a sense pointing outward from the enclosed volume. Then we say that a stress component is positive if area vector of the surface on which the stress acts and the stress itself both have senses in either the positive or the negative direction of the reference axes. A mixed combination of senses of these two quantities then means a negative stress component. By this scheme, tensile normal stresses are positive, and shear stresses on faces farthest from the reference planes are positive if they point in the positive direction of the reference axes. Finally, shear stresses on faces nearest the reference are positive if they point in the negative direction of the reference. It can be seen that all the stresses shown in the above figure are positive according to the sign convention adopted here.

13.5. State of stress at a point. **(Himanchal 2003, 01, 03, 06; Meerut 1998, 2000)**

The state of stress at a point in a fluid is said to be completely known if the direction and magnitude of the stress vector at that point is known, or can be determined from the known data, for every possible orientation of area. We now prove the following useful theorem :

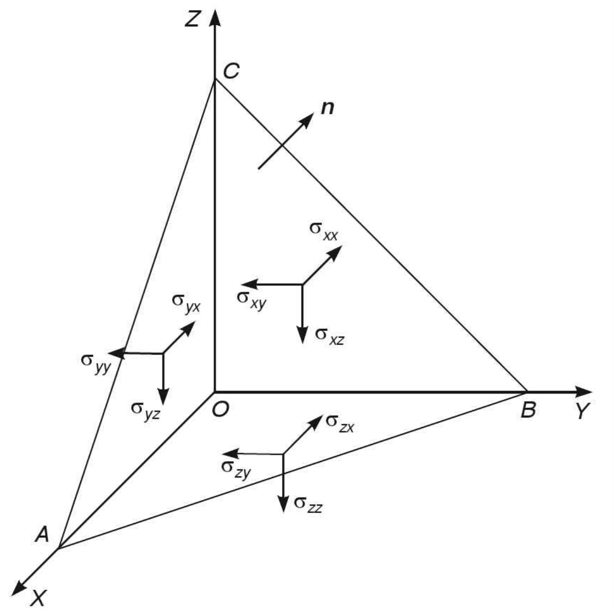
The state of stress at a point is completely known if the nine components of stress tenser at that point are known. **(Meerut 2007)**

Consider the motion of a small tetrahedron $OABC$. Three of its faces lie along the co-ordinate planes and the fourth face ABC has area Δ, say. Let l, m, n be the direction cosines of normal to ABC drawn outwards. Figure shows all the possible known stresses on the fluid element of viscous fluid. Since the tetrahedron is small, the stress across every face may be taken to be uniform. Let the stress vectors on faces OBC, OCA, OAB and ABC be F_x, F_y, F_z and F_n respectively. Let $\boldsymbol{B}$ the body force per unit mass acting on the fluid element, and $\mathbf{a}$ the acceleration of the element. Then using Newton's second law of motion, the equation of motion of the tetrahedron gives

$$\Delta \mathbf{F}_n - l\Delta \mathbf{F}_x - m\Delta \mathbf{F}_y - n\Delta \mathbf{F}_z + (1/3)\times \rho \Delta p \mathbf{B} = (1/3)\times \rho \Delta p \, \mathbf{a}, \qquad \ldots(1)$$

* A tenser of order two is a physical quantity which is completely specified by magnitude and two directions. For more details refer Art. 1.8.

** Notation τ_{ii} is also used in place of σ_{ij} by some authors.



where p is the perpendicular from O on ABC, and $(1/3)\times\rho\Delta p$ is the mass of the tetrahedron. $-l\Delta, -m\Delta, -n\Delta$ are the areas of the faces BOC, COA, AOB respectively. Since the outward normals on the faces BOC, COA and AOB are in the negative directions of the axes, it follows that the direction cosines of the outward normal $\boldsymbol{n}$ with respect to the other three outward normals are $-l, -m, -n$. This explains the minus sign attached with areas of BOC, COA and AOB.

Dividing (1) by Δ and supposing that the plane ABC approach 0 moving parallel to itself so that $p \to 0$ and $\Delta \to 0$, we obtain in the limit

$$\mathbf{F}_n = l\mathbf{F}_x + m\mathbf{F}_y + n\mathbf{F}_z \qquad ...(2)$$

But we know that (refer Art 13.4) $\qquad \mathbf{F}_n = \mathbf{i}\sigma_{nx} + \mathbf{j}\sigma_{ny} + \mathbf{k}\sigma_{nz} \qquad ...(3)$

and

$$\left.\begin{aligned} \mathbf{F}_x &= \mathbf{i}\,\sigma_{xx} + \mathbf{j}\,\sigma_{xy} + \mathbf{k}\,\sigma_{xz} \\ \mathbf{F}_y &= \mathbf{i}\,\sigma_{yx} + \mathbf{j}\,\sigma_{yy} + \mathbf{k}\,\sigma_{yz} \\ \mathbf{F}_z &= \mathbf{i}\,\sigma_{zx} + \mathbf{j}\,\sigma_{zy} + \mathbf{k}\,\sigma_{zz} \end{aligned}\right\} \qquad ...(4)$$

Substitution of the values of $\mathbf{F}_n$, $\mathbf{F}_x$, $\mathbf{F}_y$, $\mathbf{F}_z$ given by (3) and (4) in (2) and comparison of coefficients of **i**, **j** and **k** gives

$$\left.\begin{aligned} \sigma_{nx} &= l\sigma_{xx} + m\,\sigma_{yx} + n\,\sigma_{zx} \\ \sigma_{ny} &= l\sigma_{xy} + m\,\sigma_{yy} + n\,\sigma_{zy} \\ \sigma_{nz} &= l\,\sigma_{xz} + m\,\sigma_{yz} + n\,\sigma_{zz} \end{aligned}\right\} \qquad ...(5)$$

From (3) and (5), it follows that given the nine components $\sigma_{ij}\,(i, j = x, y, z)$ of stress at any point, the stress vector $\mathbf{F}_n$ at that point is known for arbitrary values of l, m, n. In other words, the state of stress at a point is completely known if the nine stress tensor components are known.

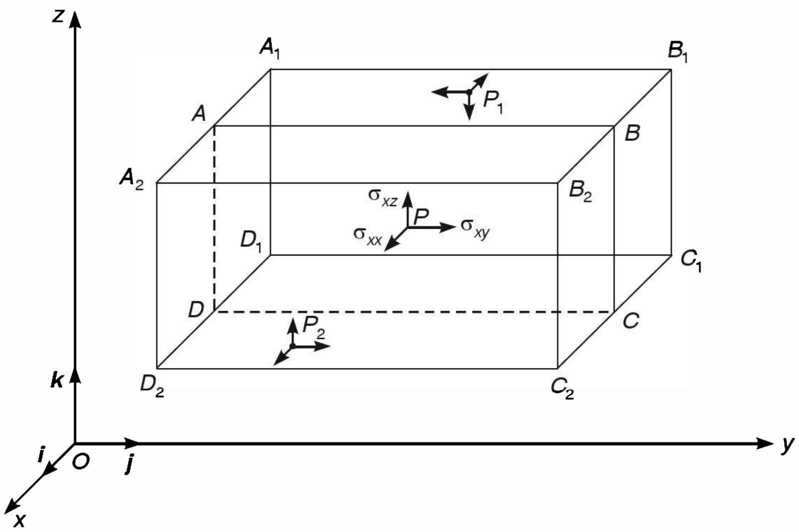
Remark. Equations (5) are also often expressed in the following matrix form :

$$\begin{bmatrix} \sigma_{nx} \\ \sigma_{ny} \\ \sigma_{nz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} \qquad ...(6)$$

13.6. Symmetry of stress tensor.

(Himanchal 1998, 2000, Kuruhshera 1999, Meerut 2008)

With $P(x, y, z)$ as centre and edges of lengths $\delta x, \delta y, \delta z$ parallel to fixed coordinate axes, construct an elementary rectangular parallelepiped as shown in the given figure. We consider the motion of the above mentioned parallelepiped of viscous fluid. We suppose that the element is moving with the fluid and mass $\rho \delta x \delta y \delta z$ of the fluid element remains constant.



Let coordinates of P_1 and P_2 be $(x - \delta x/2, y, z)$ and $(x + \delta x/2, y, z)$ respectively.

At P, the force components parallel to OX, OY, OZ on the rectangular surface $ABCD$ of area $\delta y \delta z$ through P and having $\boldsymbol{i}$ as unit normal are $\qquad [\sigma_{xx}\, \delta y\, \delta z,\ \ \sigma_{xy}\, \delta y\, \delta z,\ \ \sigma_{xz}\, \delta y\, \delta z]$

At P_2, since $\boldsymbol{i}$ is the unit normal measured outward from the fluid, the corresponding force components on the rectangular surface $A_2B_2C_2D_2$ (parallel to $ABCD$) of area $\delta y \delta z$ are

$$\left[\left(\sigma_{xx} + \frac{\delta x}{2} \cdot \frac{\partial \sigma_{xx}}{\partial x}\right)\delta y\, \delta z,\ \left(\sigma_{xy} + \frac{\delta x}{2} \cdot \frac{\partial \sigma_{xy}}{\partial x}\right)\delta y\, \delta z,\ \left(\sigma_{xz} + \frac{\delta x}{2} \cdot \frac{\partial \sigma_{xz}}{\partial x}\right)\delta y\, \delta z\right].$$

At P_1, since $-\mathbf{i}$ is the unit normal measured outwards from the fluid, the corresponding force components on the rectangular surface $A_1B_1C_1D_1$ (parallel to $ABCD$) of area $\delta y \delta z$ are

$$\left[-\left(\sigma_{xx} - \frac{\delta x}{2} \cdot \frac{\partial \sigma_{xx}}{\partial x}\right)\delta y\, \delta z,\ -\left(\sigma_{xy} - \frac{\delta x}{2} \cdot \frac{\partial \sigma_{xy}}{\partial x}\right)\delta y\, \delta z,\ -\left(\sigma_{xz} - \frac{\delta x}{2} \cdot \frac{\partial \sigma_{xz}}{\partial x}\right)\delta y\, \delta z\right]$$

Hence, the forces on the parallel planes $A_2\, B_2\, C_2\, D_2$ and $A_1B_1C_1D_1$, passing through P_2 and P_1 are equivalent to a single force at P with components

$$\left[\frac{\partial \sigma_{xx}}{\partial x}\delta x \delta y \delta z,\ \frac{\partial \sigma_{xy}}{\partial x}\delta x \delta y \delta z,\ \frac{\partial \sigma_{xz}}{\partial x}\delta x \delta y \delta z\right]$$

together with couples whose moments* (to the third order of smallness) are

$$-\sigma_{xz}\delta x \delta y \delta z \text{ about } OY \qquad \text{and} \qquad \sigma_{xy}\delta x \delta y \delta z \text{ about } OZ.$$

Similarly, the forces on the parallel planes perpendicular to the y-axis are equivalent to a

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* **Convention of sign of a couple :** If a couple in the plane XOY causes rotation from OX towards OY, then it shall be represented by a positive length along OZ. Similarly, a couple in the plane YOZ which would cause rotation from OY towards OZ will be represented by a positive length along OX and a couple in the plane ZOX causing rotation from OZ towards OX will be represented by a positive length along OY.

single force at P with components

$$\left[\frac{\partial \sigma_{yx}}{\partial y}\delta x\delta y\delta z, \frac{\partial \sigma_{yy}}{\partial y}\delta x\delta y\delta z, \frac{\partial \sigma_{yz}}{\partial y}\delta x\delta y\delta z\right]$$

together with couples whose moments are

$-\sigma_{yx}\delta x\delta y\delta z$ about OZ and $\sigma_{yz}\delta x\delta y\delta z$ about OX.

Again, the forces on the parallel planes perpendicular to the z-axis are equivalent to a single force at P with components

$$\left[\frac{\partial \sigma_{zx}}{\partial z}\delta x\delta y\delta z, \frac{\partial \sigma_{zy}}{\partial z}\delta x\delta y\delta z, \frac{\partial \sigma_{zz}}{\partial z}\delta x\delta y\delta z\right]$$

together with couples whose moments are

$-\sigma_{zy}\delta x\delta y\delta z$ about OX and $\sigma_{zx}\delta x\delta y\delta z$ about OY.

Thus, the surface forces on all the six faces of the rectangular parallelepiped ($A_1 B_1 C_1 D_1$, $A_2 B_2 C_2 D_2$) are equivalent to a single force at P having components

$$\left[\left(\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}\right)\delta x\delta y\delta z, \left(\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{zy}}{\partial z}\right)\delta x\delta y\delta z, \left(\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}\right)\delta x\delta y\delta z\right]$$

together with a vector couple having components

$$[(\sigma_{yz}-\sigma_{zy})\delta x\delta y\delta z,\ (\sigma_{zx}-\sigma_{xz})\delta x\delta y\delta z,\ (\sigma_{xy}-\sigma_{yx})\ \delta x\delta y\delta z].$$

Let the components of external body forces at P per unit mass be X, Y, Z. So the total body force on the elementary rectangular parallelepiped has components

$$X\rho\,\delta x\delta y\delta z, \qquad Y\rho\,\delta x\delta y\delta z, \qquad Z\rho\,\delta x\delta y\delta z.$$

Taking moments about the **i**-direction through P, we get

Total moment of forces = (moment of inertia about OX) × (angular acceleration)

or $$(\sigma_{yz}-\sigma_{zy})\delta x\delta y\delta z + O_4 = O_5, \qquad \text{...(1)}$$

where O_4 and O_5 represent quantities of fourth and fifth order of smallness in $\delta x, \delta y, \delta z$, Hence, to the third order of smallness in $\delta x, \delta y, \delta z$, (1) reduces to

$$(\sigma_{yz}-\sigma_{zy})\delta x\delta y\delta z = 0 \qquad \text{...(2)}$$

Since $\delta x, \delta y, \delta z$ are arbitrary but small, so in (2) the coefficient of $\delta x\delta y\delta z$ must vanish and hence, we get

$$\sigma_{yz}-\sigma_{zy}=0 \qquad \text{or} \qquad \sigma_{yz}=\sigma_{zy}.$$

Similarly, $\sigma_{zx}=\sigma_{xz}$ and $\sigma_{xy}=\sigma_{yx}$.

Thus, it follows that the stress tensor is symmetric.

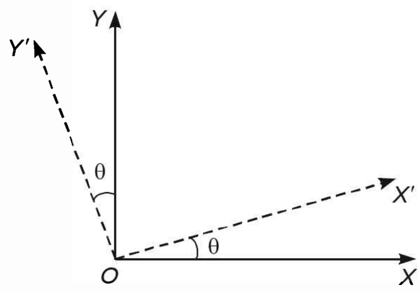
13.7. To Show that only six components suffice to determine the state of stress at a point. (Meerut 2008)

As shown in Art 13.5, the state of stress at a point is completely known if the nine components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yx}, \sigma_{yz}, \sigma_{zy}, \sigma_{zx}, \sigma_{xz},$ of stress tensor are known. Again from Art. 13.6, it is known that the stress tensor is symmetric, *i.e.*, $\sigma_{xy} = \sigma_{yx}, \sigma_{yz}=\sigma_{zy}$ and $\sigma_{xz}=\sigma_{zx}$. Hence, it follows that six, rather than nine, $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}$ suffice to determine the state of stress at a point.

13.8. Transformation of stress components.

Case I. Two-dimensional stress-components. **(Garhwal 2001, 2003, 05)**

Consider a state of stress in which $\sigma_{yz} = \sigma_{xz} = \sigma_{zz} = 0.$



Then such a state of stress is said to be the *plane stress*. Let the two-dimensional stress components σ_{xx}, σ_{yy} and σ_{xy} at O with respect to coordinate axes OX and OY be known. Let OX', OY' be another set of orthogonal axes as shown in the figure. Then we require the stress components $\sigma_{x'x'}, \sigma_{y'y'}$, and $\sigma_{x'y'}$ with respect to the new axes OX', OY',. The direction cosines of one set of axes with respect to the other are shown in the following table.

	OX	OY
OX′	$l_1 = \cos\theta$	$m_1 = \cos(\pi/2 - \theta) = \sin\theta$
OY′	$l_2 = \cos(\pi/2 + \theta) = -\sin\theta$	$m_2 = \cos\theta$

From (5) of Art 13.5, the stress components in x, y directions on the elementary area (in xy – plane and of thickness unity) normal to OX' are given by

$$\sigma_{x'x} = l_1\sigma_{xx} + m_1\sigma_{yx} \qquad \text{...(1)}$$

$$\sigma_{x'y} = l_1\sigma_{xy} + m_1\sigma_{yy} \qquad \text{...(2)}$$

with well known relation $\sigma_{xy} = \sigma_{yx}.$

Resolving the above stress components in x' – and y' – directions, we get

$$\sigma_{x'x'} = l_1\sigma_{x'x} + m_1\sigma_{x'y} = l_1^2\sigma_{xx} + m_1^2\sigma_{yy} + 2l_1m_1\sigma_{xy} \qquad \text{...(3)}$$

and
$$\sigma_{x'y'} = l_2\sigma_{x'x} + m_2\sigma_{x'y} = l_1l_2\sigma_{xx} + m_1m_2\sigma_{yy} + (l_1m_2 + l_2m_1)\sigma_{xy} \qquad \text{...(4)}$$

As before, the stress components in x-and y-directions on the element of area normal to OY′ are given by

$$\sigma_{y'x} = l_2\sigma_{xx} + m_2\sigma_{yx} \qquad \text{...(5)}$$

and
$$\sigma_{y'y} = l_2\sigma_{xy} + m_2\sigma_{yy} \qquad \text{...(6)}$$

Resolving the above stress components in y' directions, we get

$$\sigma_{y'y'} = l_2\sigma_{y'x} + m_2\sigma_{y'y} = l_2^2\sigma_{xx} + m_2^2\sigma_{yy} + 2l_2m_2\sigma_{xy} \qquad \text{...(7)}$$

Using the above mentioned table, (3), (4) and (7) can be expressed in terms of angle θ as follows:

$$\sigma_{x'x'} = \cos^2\theta\,\sigma_{xx} + \sin^2\theta\,\sigma_{yy} + 2\sin\theta\cos\theta\,\sigma_{xy}$$

$$\sigma_{y'y'} = \sin^2\theta\,\sigma_{xx} + \cos^2\theta\,\sigma_{yy} - 2\sin\theta\cos\theta\,\sigma_{xy}$$

$$\sigma_{x'y'} = -\sin\theta\cos\theta\,\sigma_{xx} + \sin\theta\cos\theta\,\sigma_{yy} + (\cos^2\theta - \sin^2\theta)\,\sigma_{xy}$$

But $2\sin\theta\cos\theta = \sin 2\theta$, $\sin^2\theta = (1/2)\times(1 - \cos 2\theta)$, $\cos^2\theta = (1/2)\times(1 + \cos 2\theta)$.

Hence the above results can be re-written as

$$\left.\begin{aligned}\sigma_{x'x'} &= (1/2)\times(\sigma_{xx}+\sigma_{yy})+(1/2)\times(\sigma_{xx}-\sigma_{yy})\cos 2\theta+\sigma_{xy}\sin 2\theta\\ \sigma_{y'y'} &= (1/2)\times(\sigma_{xx}+\sigma_{yy})-(1/2)\times(\sigma_{xx}-\sigma_{yy})\cos 2\theta-\sigma_{xy}\sin 2\theta\\ \sigma_{x'y'} &= -(1/2)\times(\sigma_{xx}-\sigma_{yy})\sin 2\theta+\sigma_{xy}\cos 2\theta\end{aligned}\right\} \quad \ldots(8)$$

Stress invariants for a two-dimensional case.

Basic invariants for two-dimensional stress-components are given by

$$\sigma_{x'x'}+\sigma_{y'y'}=\sigma_{xx}+\sigma_{yy}=\sigma_1+\sigma_2 \quad \ldots(9)$$

and $$\sigma_{x'x'}\sigma_{y'y'}-\sigma^2{}_{x'x}=\sigma_{xx}\sigma_{yy}-\sigma_{xy}^2=\sigma_1\sigma_2, \quad \ldots(10)$$

where σ_1 and σ_2 are the principal stresses*. These results can be easily proved by using relations (8). The stress invariants may be defined as those functions of the stress components which remain invarinant under a transformation of coordinates.

Case II. Three-dimensional stress-components

Suppose OX, OY, OZ and OX', OY', OZ' determine two sets of rectangular cartesian coordinate axes through O. Further, suppose that the direction cosines of each axis of the one system *w.r.t.* the three axes of the other are as shown in the following table.

	OX	OY	OZ
OX′	l_1	m_1	n_1
OY′	l_2	m_2	n_2
OZ′	l_3	m_3	n_3

The table means that, *e.g.* (l_1, m_1, n_1)) are the direction cosines of OX' with respect to the axes OX, OY, OZ and that (l_1, l_2, l_3) are the direction cosines of OX with respect to the axes OX', OY', OZ'.

These direction cosines satisfy the following twelve relations ;

$$\left.\begin{aligned} l_1^2+m_1^2+n_1^2 &= 1 & \qquad l_1^2+l_2^2+l_3^2 &= 1\\ l_2^2+m_2^2+n_2^2 &= 1, & m_1^2+m_2^2+m_3^2 &= 1\\ l_3^2+m_3^2+n_3^2 &= 1, & n_1^2+n_2^2+n_3^2 &= 1\end{aligned}\right\} \quad \ldots(11)$$

$$\left.\begin{aligned} l_1m_1+l_2m_2+l_3m_3 &= 0, & \qquad l_1l_2+m_1m_2+n_1n_2 &= 0\\ m_1n_1+m_2n_2+m_3n_3 &= 0, & l_2l_3+m_2m_3+n_2n_3 &= 0\\ n_1l_1+n_2l_2+n_2l_3 &= 0, & l_3l_1+m_3m_1+n_3n_1 &= 0\end{aligned}\right\} \quad \ldots(12)$$

Let the stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}$ at O with respect to the axes *OX, OY, OZ* be given. Our aim is to obtain new expressions for stress components $\sigma_{x'x'}, \sigma_{y'y'}, \sigma_{z'z'}, \sigma_{x'y'}, \sigma_{y'z'}, \sigma_{z'x'}$ at O with respect to the axes. OX', OY', OZ'.

Then as explained in (5) of Art. 13.5, the stress components in *x*, *y*, *z* directions on the element of area normal to OX′ are given by

$$\left.\begin{aligned}\sigma_{x'x} &= l_1\sigma_{xx}+m_1\sigma_{yx}+n_1\sigma_{zx}\\ \sigma_{x'y} &= l_1\sigma_{xy}+m_1\sigma_{yy}+n_1\sigma_{zy}\\ \sigma_{x'z} &= l_1\sigma_{xz}+m_1\sigma_{yz}+n_1\sigma_{zz}\end{aligned}\right\} \quad \ldots(13)$$

with well-known relations $\sigma_{xy}=\sigma_{yx}$, $\sigma_{xz}=\sigma_{zx}$, $\sigma_{yz}=\sigma_{zy}$. ...(14)

* Refer next Art. 13.9 for principal stresses

Now resolving the stress components given by (13) in OX' direction and using (14), we get

$$\sigma_{x'x'} = l_1\sigma_{x'x} + m_1\sigma_{x'y} + n_1\sigma_{x'z} = l_1^2\sigma_{xx} + m_1^2\sigma_{yy} + n_1^2\sigma_{zz} + 2l_1m_1\sigma_{xy} + 2m_1n_1\sigma_{yz} + 2n_1l_1\sigma_{zx} \quad \ldots(15\text{A})$$

and similarly resoving the stress components given by (13) in OY' direction, we get

$$\sigma_{x'y'} = l_2\sigma_{x'x} + m_2\sigma_{x'y} + n_2\sigma_{x'z}$$

$$= l_1l_2\sigma_{xx} + m_1m_2\sigma_{yy} + n_1n_2\sigma_{zz} + (l_2m_2 + l_2m_1)\sigma_{xy} + (m_1n_2 + n_1m_2)\sigma_{yz} + (l_1n_2 + n_1l_2)\sigma_{zx} \quad \ldots(15\text{ B})$$

In a like manner similar expressions can be obtained for other components, namely

$$\sigma_{y'y'} = l_2^2\sigma_{xx} + m_2^2\sigma_{yy} + n_2^2\sigma_{zz} + 2l_2m_2\sigma_{xy} + 2m_2n_2\sigma_{yz} + 2n_2l_2\sigma_{zx} \quad \ldots(15\text{ C})$$

$$\sigma_{y'z'} = l_2l_3\sigma_{xx} + m_2m_3\sigma_{yy} + n_2n_3\sigma_{zz} + (l_2m_3 + m_2l_3)\sigma_{xy}$$
$$+ (m_2n_3 + m_3n_2)\sigma_{yz} + (n_2l_3 + l_2n_3)\sigma_{zx} \quad \ldots(15\text{ D})$$

$$\sigma_{z'z'} = l_3^2\sigma_{xx} + m_3^2\sigma_{yy} + n_3^2\sigma_{zz} + 2l_3m_3\sigma_{xy} + 2m_3n_3\sigma_{yz} + 2n_3l_3\sigma_{zx} \quad \ldots(15\text{E})$$

$$\sigma_{z'x'} = l_1l_3\sigma_{xx} + m_1m_3\sigma_{yy} + n_1n_3\sigma_{zz} + (l_1m_3 + m_1l_3)\sigma_{xy}$$
$$+ (m_1n_3 + n_1m_3)\sigma_{yz} + (l_3n_1 + n_3l_1)\sigma_{zx} \quad \ldots(15\text{ F})$$

Stress invariants for a three dimensional case. **(Himanchal 2003)**

The invariants for three-dimensional stress components are given by

$$\sigma_{x'x'} + \sigma_{y'y'} + \sigma_{z'z'} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \quad \ldots(16)$$

$$\sigma_{x'x'}\sigma_{y'y'} + \sigma_{y'y'}\sigma_{z'z'} + \sigma_{z'z'}\sigma_{x'x'} - \sigma_{x'y'}^2 - \sigma_{y'z'}^2 - \sigma_{z'x'}^2$$
$$= \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{zx}^2 \quad \ldots(17)$$

$$\sigma_{x'x'}\sigma_{y'y'}\sigma_{z'z'} + 2\sigma_{x'y'}\sigma_{y'z'}\sigma_{z'x'} - \sigma_{x'x'}\sigma_{y'z'}^2 - \sigma_{y'y'}\sigma_{z'x'}^2 - \sigma_{z'z'}\sigma_{x'y'}^2$$
$$= \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2 \quad \ldots(18)$$

Using (11) and (12), (15*A*) to (15 *F*) lead to the above mentioned relations (16) to (18). The proof is left as an exercise for the reader. If I_1, I_2, I_3, denote the *R.H.S.* of (16), (17) and (18), then observe that

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \quad \ldots(16)'$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{zy} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{zz} & \sigma_{zx} \\ \sigma_{xz} & \sigma_{xx} \end{vmatrix} \quad \ldots(17)'$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix} \quad \ldots(18)'$$

These are known respectively as the *first, second and third stress invariants*. For an alternative method of geting invariants, refer Art. 13.10

To show that stress at a point is a tensor of order two. **(Himanchal 1998, 2003)**

The equations of transformation (15A) to (15F) characterize the mathematical nature of stress. A set of nine quantities σ_{ij} $(i, j = x, y, z)$ and defined with respect to x, y, z axes are transformed by the equaftions of the type (15A) to (15F) into the corresponding set of nine quantites with respect to x', y', z' axes. A set of such nine quantities is said to be a tensor* of the second order. Notice that stress is a symmeric tensor of order two because

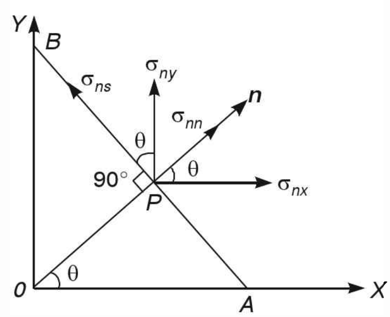
$$\sigma_{xy} = \sigma_{yx}, \qquad \sigma_{yz} = \sigma_{yz} \qquad \text{and} \qquad \sigma_{zx} = \sigma_{xz}$$

* Reader is advised to read Art 1.8, chapter 1 for more details.

13.9. Plane stress. Principal stresses and principal directions.

Consider a state of stress in which

$$\sigma_{yz} = \sigma_{zx} = \sigma_{zz} = 0.$$



Then a state of stress is known as the plane stress and relations (5) of Art. 13.5 reduce to

$$\sigma_{nx} = l\sigma_{xx} + m\,\sigma_{yx}$$

$$\sigma_{ny} = l\sigma_{xy} + m\,\sigma_{yy}$$

$$\sigma_{nz} = 0$$

Let the element of area AB with normal $\boldsymbol{n}$ be perpendicular to xy-plane. Let $\boldsymbol{n}$ make an angle θ with x-axis as shown in figure. Then $l = \cos\theta$ and m $= \sin\theta$ and so we get

$$\left.\begin{aligned} \sigma_{nx} &= \sigma_{xx}\cos\theta + \sigma_{yx}\sin\theta \\ \sigma_{ny} &= \sigma_{xy}\cos\theta + \sigma_{yy}\sin\theta \\ \sigma_{nz} &= 0 \end{aligned}\right\} \qquad \text{...(1)}$$

Then resolving the above stress components along and perpendicular to the normal $\boldsymbol{n}$ of AB, the normal and tangential components of stress are gives by

$$\sigma_{nn} = \sigma_{nx}\cos\theta + \sigma_{ny}\sin\theta = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta, \text{ using (1)}$$

$$= (1/2)\times(\sigma_{xx} + \sigma_{yy}) + (1/2)\times(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \sigma_{xy}\sin 2\theta \qquad \text{...(2)}$$

and

$$\sigma_{ns} = \sigma_{ny}\cos\theta - \sigma_{nx}\sin\theta = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \sigma_{xy}\cos^2\theta - \sigma_{yx}\sin^2\theta, \text{ by (1)}$$

$$= -(1/2)\times(\sigma_{xx} - \sigma_{yy})\sin 2\theta + \sigma_{xy}\cos 2\theta \qquad \text{...(3)}$$

where s denotes the direction normal to $\boldsymbol{n}$ and also to the z-axis, the sense of n, s being the same as of x, y. Relations (2) and (3) give the stress components at any point P in terms of three stress components $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$. From (3), we find that

$$\sigma_{ns} = 0 \qquad \Rightarrow \qquad \tan 2\theta = 2\sigma_{xy}/(\sigma_{xx} - \sigma_{yy}). \qquad \text{...(4)}$$

If $\sigma_{xy} = 0$ and $\sigma_{xx} = \sigma_{yy}$, then θ is indeterminate from (4). For such a situation $\sigma_{nn} = \sigma_{xx} = \sigma_{yy}$ and $\sigma_{ns} = 0$ for all directions. Thus for all values of **n** the stress is normal. Such state is known as *uniform plane stress*.

Next consider the case when $\sigma_{xy} = 0$ and $\sigma_{xx} = \sigma_{yy}$ do not hold simultaneously. Since $\tan 2\theta = \tan 2\ (\pi/2 + \theta)$ is true, (4) holds for two values of θ. Hence we see that for any state of stress at a point, there are two mutually perpendicular directions, corresponding to which the tangential component of stress vanishes. Again for these directions stress is wholly normal and the values of these normal stresses can be obtained by substituting the two values of θ given by the equation (4).

The two directions given by (4) are known as the *principal directions* of stress at that point and the normal stress corresponding to them are known as the *principal stresses.* The normal stresses are denoted by σ_1 and $\sigma_2, (\sigma_1 > \sigma_2)$. It follows that a plane state of stress at a point can always be completely given in terms of σ_1 and σ_2. For the case of uniform plane stress, each pair of perpendicular directions are principal directions, whereas in the remaining situations the principal directions are uniquely determined.

Since the choice of the coordinate axes is at our disposal, we may take them to coincide with the principal directions. Then with respect to such axes, we have

$$\sigma_{xx} = \sigma_1, \qquad \sigma_{yy} = \sigma_2 \qquad \text{and} \qquad \sigma_{xy} = 0. \qquad ...(5)$$

Making use of (5), (2) and (3) now take the form

$$\sigma_{nn} = (1/2) \times (\sigma_1 + \sigma_2) + (1/2) \times (\sigma_1 - \sigma_2) \cos 2\theta \qquad ...(6)$$

and
$$\sigma_{ns} = (1/2) \times (\sigma_2 - \sigma_1) \sin 2\theta. \qquad ...(7)$$

13.10. Principal stresses, principal directions of stress tensor.

(Garhwal 2003; Kurukshetra 1998; Meerut 2003, 05)

The analysis of Art. 13.9 can be easily extended to discuss the general state of stress. In what follows we propose to prove the following theorem.

Theorem : *Whatever the state of stress at a point, there always exist three mutually perpendicular planes for which the shear stresses vanish, and consequently the resultant stress components are normal.*

The planes are called the *principal planes,* the directions of their normals the *principal directions*, and the corresponding normal stresses the *principal stresses.* The principal stresses will be denoted by $\sigma_1, \sigma_2, \sigma_3$.

Proof of the theorem. Refer Art 13.5. If a given area *ABC* is a principal plane, the total stress σ_{nn} acting on it is directed along the normal ***n*** and is a principal stress. Let it be denoted by σ. Then its components in the *x*, *y*, *z* directions are given by

$$\sigma_{nx} = \sigma l, \qquad \sigma_{ny} = \sigma m \qquad \text{and} \qquad \sigma_{nz} = \sigma n. \qquad ...(1)$$

Again, we known that

$$\left.\begin{aligned} \sigma_{nx} &= l\sigma_{xx} + m\,\sigma_{yx} + n\,\sigma_{zx} \\ \sigma_{ny} &= l\sigma_{xy} + m\,\sigma_{yy} + n\,\sigma_{zy} \\ \sigma_{nz} &= l\sigma_{xz} + m\,\sigma_{yz} + n\,\sigma_{zz} \end{aligned}\right\} \qquad ...(2)$$

From (1) and (2), we have

$$l\sigma_{xx} + m\,\sigma_{xy} + n\,\sigma_{zx} = \sigma\, l, \qquad ...(3)$$

$$l\sigma_{xy} + m\,\sigma_{yy} + n\,\sigma_{yz} = \sigma\, m \qquad ...(4)$$

and
$$l\sigma_{zx} + m\,\sigma_{yz} + n\,\sigma_{zz} = \sigma\, n \qquad ...(5)$$

wherein we have used the relations $\sigma_{yx} = \sigma_{xy}, \quad \sigma_{zy} = \sigma_{yz}, \quad \sigma_{xz} = \sigma_{zx}.$

Re-writing (3), (4) and (5), we have

$$l(\sigma_{xx} - \sigma) + m\sigma_{xy} + n\sigma_{zx} = 0 \qquad ...(6)$$

$$l\sigma_{xy} + m(\sigma_{yy} - \sigma) + n\sigma_{yz} = 0 \qquad ...(7)$$

and
$$l\sigma_{zx} + m\sigma_{yz} + n\,(\sigma_{zz} - \sigma) = 0. \qquad ...(8)$$

Since, *l*, *m*, *n* are the direction cosines, we have $l^2 + m^2 + n^2 = l.$ ...(9)

Thus we have four equations [namely, (6) to (9)] for the determination of the principal stress σ and the corresponding principal plane *i.e.*, the direction cosines of its normal, *l*, *m*, *n*. The system of homogeneous equations (6) to (8) does not admit a trivial solution *l* = *m* = *n* = 0 because it is contrary to (9). Again for the existence of other solutions of this system (in which at least one the *l*, *m*, *n* have a value different from zero) it is necessary that its determinant should vanish. Thus we arrive at the condition

$$\begin{vmatrix} \sigma_{xx} - \sigma & \sigma_{xy} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} - \sigma & \sigma_{yz} \\ \sigma_{zx} & \sigma_{yz} & \sigma_{zz} - \sigma \end{vmatrix} = 0. \qquad ...(10)$$

Writing out the determinant in the left-hand member, we obtain the cubic equation (also known as the *characteristic equation*)

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \qquad ...(11)$$

in which the coefficients have the following values :

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \qquad ...(12)$$

$$I_1 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{zy} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{zz} & \sigma_{zx} \\ \sigma_{xz} & \sigma_{xx} \end{vmatrix} \qquad ...(13)$$

and

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix} \qquad ...(14)$$

The roots of (11) must not depend on the system of coordinates *x*, *y*, *z*. If follows that the coefficients I_1, I_2 and I_3 do not depend on the choice of a system of coordinates. Thus we conclude that formulae (12), (13) and (14) give three functions of the components of the stress tensor σ_{ij} $(i, j = x, y, z)$ which are invariants under a transformation of coordinates. I_1, I_2 and I_3 are called the *first, second and third stress invariants.**

To show that the roots of the cubic (11) are all real and hence *l, m, n* corresponding to each σ are real.

Let σ be any root of (11) and *l*, *m*, *n* any non-zero set of values of *l*, *m*, *n* satisfying (6) to (8) and hence (3) to (5). Note that we cannot treat *l*, *m*, *n* as real, for λ is not yet proved to be real. In what follows, the complex conjugate of any number will be expressed by putting a bar over the same.

Multiplying (3) to (5) by $\bar{l}, \bar{m}, \bar{n}$ respectively and adding, we get

$$l\bar{l}\sigma_{xx} + m\bar{m}\sigma_{yy} + n\bar{n}\sigma_{zz} + (l\bar{m} + \bar{l}m)\sigma_{xy} + (m\bar{n} + m\bar{n})\sigma_{yz} + (n\bar{l} + \bar{n}l)\sigma_{zx} = \sigma(l\bar{l} + m\bar{m} + n\bar{n})$$

i.e.

$$\Sigma l\bar{l}\sigma_{xx} + \Sigma(l\bar{m} + \bar{l}m)\sigma_{xy} = \sigma\Sigma l\bar{l} \qquad ...(15)$$

Now $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}\sigma_{yz}, \sigma_{zx}$ are all real. Also $l\bar{l}, m\bar{m}, n\bar{n}$ being the products of pairs of conjugate complex numbers are real. Further note that $\bar{m}n$ is the conjugate complex of $m\bar{n}$ so that $m\bar{n} + \bar{m}n$ is also real. Similarly, $n\bar{l} + \bar{n}l, \bar{l}m + l\bar{m}$ are real. Finally $\Sigma l\bar{l}$ is a non-zero real number. Thus σ, being the ratio of two real numbers from (15), is necessarily a real number.

Hence the roots of (11) are all real and so *l*, *m*, *n*, corresponding to each σ are also real.

To show that the two principal directions corresponding to any two distinct principal stresses are orthogonal. **(Meerut 2000, 04)**

Let σ_1 and σ_2 be two principal stresses (*i.e.* two distinct roots of the cubic (11)) and let $l_1, m_1, n_1, ; l_2, m_2, n_2$ be the two corresponding principal directions. Then (3), (4) and (5) respectively reduce to

$$l_1\,\sigma_{xx} + m_1\,\sigma_{xy} + n_1\,\sigma_{zx} = \sigma_1\,l_1 \qquad ...(3A)$$

$$l_1\,\sigma_{xy} + m_1\,\sigma_{yy} + n_1\,\sigma_{yz} = \sigma_1\,m_1 \qquad ...(4A)$$

$$l_1\,\sigma_{zx} + m_1\,\sigma_{yz} + n_1\,\sigma_{zz} = \sigma_1\,n_1 \qquad ...(5A)$$

$$l_2\,\sigma_{xx} + m_2\,\sigma_{xy} + n_2\,\sigma_{zx} = \sigma_2\,l_2 \qquad ...(3B)$$

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* For alternative method, refer Art 13.8.

$$l_2\,\sigma_{xy} + m_2\,\sigma_{yy} + n_2\,\sigma_{yz} \;=\; \sigma_2\,m_2 \qquad \text{...(4B)}$$

$$l_2\,\sigma_{zx} + m_2\,\sigma_{yz} + n_2\,\sigma_{zz} \;=\; \sigma_2\,n_2 \qquad \text{...(5B)}$$

Multiplying (3*A*), (4*A*), (5*A*) by l_2, m_2, m_2 respectively and adding we obtain

$$\Sigma l_1\,l_2\,\sigma_{xx} + \Sigma\,(l_1\,m_2 + l_2\,m_1)\,\sigma_{xy} = \sigma_1\,\Sigma l_1\,l_2. \qquad \text{...(16)}$$

Also multiplying (3*B*), (4*B*), (5*B*) by l_1, m_1, n_1 by respectively and adding, we obtain

$$\Sigma l_1\,l_2\,\sigma_{xx} + \Sigma\,(l_1\,m_2 + l_2\,m_1)\,\sigma_{xy} = \sigma_2\,\Sigma l_1\,l_2. \qquad \text{...(17)}$$

From (16) and (17), we obtain $\sigma_1 \Sigma l_2 l_2 = \sigma_2 \Sigma l_1 l_2$, *i.e.*, $(\sigma_1 - \sigma_2)\Sigma l_1 l_2 = 0.$

Hence $\Sigma l_1 l_2 = 0,$ as $\sigma_1 \neq \sigma_2 \;\Rightarrow\; (\sigma_1 - \sigma_2) \neq 0$]

i.e. $l_1\, l_2 + m_1\, m_2 + n_1\, n_2 = 0,$

showing thereby that the two directions are orthogonal. This proves the required result.

To show that for each state of stress at a point, there exists at least one set of three mutually perpendicular principal directions.

We have to consider the following three cases :

(*i*) When all the roots $\sigma_1, \sigma_2, \sigma_3$ are distinct.

(*ii*) When two roots are equal and the third is different from these.

(*iii*) When the three roots are all equal.

Let us transform the axes from x, y, z to x', y', z', taking the x' axis in a chosen principal direction of principal stress σ_1, and y', z' axes arbitrarily chosen, subject of course, to x', y', z' being orthogonal. Since x' axis coincides with a principal direction, the stress components with respect to the new axes are such the $\sigma_{x'y'} = \sigma_{x'z'} = 0$. With respect to the new axes equations (6), (7) and (8) reduce to

$$l\,(\sigma_{x'x'} - \sigma) = 0, \qquad \text{...(6A)}$$

$$m\,(\sigma_{y'y'} - \sigma) + n\,\sigma_{y'z'} = 0, \qquad \text{...(7A)}$$

and

$$m\,\sigma_{y'z'} + n\,(\sigma_{z'z'} - \sigma) = 0. \qquad \text{...(8A)}$$

Its possible solutions are given by $l = 0,$ $m^2 + n^2 = 1.$ ...(9A)

Let us now consider the various cases one by one.

Case (*i*). Let $\sigma_1, \sigma_2, \sigma_3$ be all distinct. Then for each of the two roots σ_2 and σ_3 it is possible to solve any one of the equations (7A) and (8A) to get the values of *m* and *n* subject to $m^2 + n^2 = 1$, thus fixing uniquely the other two principal directions. Hence the three principal directions are unique in this case.

Case (*ii*). Let $\sigma_2 = \sigma_3 \neq \sigma_1$. Then $\sigma_{y'y'} = \sigma_{z'z'}$ and $\sigma_{y'z'} = 0$ and so (6*A*), (7*A*) and (8*A*) reduce to

$$l\,(\sigma_{x'x'} - \sigma) = m\,(\sigma_{y'y'} - \sigma) = n\,(\sigma_{z'z'} - \sigma) = 0 \qquad \text{...(18)}$$

With $l = 0$, the only solution to the above is $\sigma = \sigma_{y'y'} = \sigma_{z'z'}$, leaving *m* and *n* arbitrary. Thus the corresponding principal directions are arbitrary except that they are both perpendicular to the first principal direction fixed by the x' axis.

Case (*iii*). Let $\sigma_1 = \sigma_2 = \sigma_3$. Then as before we see that every direction is a principal direction.

Thus, if all the principal stresses are distinct, there exist a unique set of principal directions. On the other hand, if two or three of the principal values are equal, then there may be an infinite number of principal directions.

Remarks. The stress invariants in terms of principal stresses $\sigma_1, \sigma_2, \sigma_3$, are given by **(Meerut 2003 05)**

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3, \qquad I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \qquad \text{and} \qquad I_3 = \sigma_1\sigma_2\sigma_3.$$

13.11. Illustrative solved examples based on stress.

Ex. 1. *The stress tensor at a point P is given by* $\sigma_{ij} = \begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{pmatrix}$

Determine the stress vector on the plane at P whose unit normal is

$\mathbf{n} = (2/3)\,\mathbf{i} - (2/3)\,\mathbf{j} + (1/3)\mathbf{k}$. **(Agra 2011; Garhwal 2000. 02)**

Sol. Refer Art. 13.5 Since $n = li + mj + nk$, we have $l = 2/3$, $m = -2/3$, $n = 1/3$. Then

$$\begin{bmatrix} \sigma_{nx} \\ \sigma_{ny} \\ \sigma_{nz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10/3 \\ 0 \end{bmatrix}$$

Hence, $\sigma_{nx} = 4,$ $\qquad \sigma_{ny} = -10/3$ $\qquad$ and $\qquad \sigma_{nz} = 0$.

Hence the stress vector F_n is given by $\qquad \mathbf{F}_n = \mathbf{i}\,\sigma_{ny} + \mathbf{j}\,\sigma_{ny} + \mathbf{k}\,\sigma_{nz} = 4\,\mathbf{i} - (10/3)\,\mathbf{j}$.

Ex. 2. *The state of stress at a point is given by the stress lensor*

$\sigma_{ij} = \begin{pmatrix} \sigma & a\sigma & b\sigma \\ a\sigma & \sigma & c\sigma \\ b\sigma & c\sigma & \sigma \end{pmatrix}$ *where a, b, c are constants and* σ *is some stress value.*

Determine the constants a, b and c so that the stress vector on the octahedral plane $(\mathbf{n} = (1/\sqrt{3})\mathbf{i} + (1/\sqrt{3})\mathbf{j} + (1/\sqrt{3})\mathbf{k})$ vanishes.

Sol. Here $\boldsymbol{F}_n = \mathbf{0}$ so that $\sigma_{nx} = \sigma_{ny} = \sigma_{nz} = 0$. Then we have

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma & a\sigma & b\sigma \\ a\sigma & \sigma & c\sigma \\ b\sigma & \sigma c & \sigma \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

so that $\quad a + b = -1, \qquad a + c = -1, \qquad b + c = -1$. These give $\qquad a = b = c = -1/2$.

Ex. 3. The stress tensor at a point is given by $\sigma_{ij} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

Determine the principal stress values and the principal stress directions. **(Meerut 200, 02)**

Sol. From condition (10) of Art. 13.10, the principal values are given by the charcteristic

equation $$\begin{vmatrix} 3-\sigma & 1 & 1 \\ 1 & -\sigma & 2 \\ 1 & 2 & -\sigma \end{vmatrix} = 0.$$

On expanding and solving, it gives $(\sigma+2)(\sigma-4)(\sigma-1)=0.$

Hence the principal stress values are $\sigma_1=-2, \sigma_2=1$, and $\sigma_3=4$. Let the $x'-$ axis be the direction of σ_1, and let l_1, m_1, n_1 be the direction cosines of this axis. Then from (6), (7) (8) of Art 13.10, we have

$$(3+2)l_1+m_1+n_1=0$$
$$l_1+2m_1+2n_1=0$$
$$l_1+2m_1+2n_1=0$$

giving $l_1=0$ and $m_1=-n_1$. Since $l_1^2+m_1^2+n_1^2=1$, we get $2m_1^2=1$ or $m_1=1/\sqrt{2}$.

Thus, $l_1=0$, $m_1=1/\sqrt{2}$, $n_1=-1/\sqrt{2}$.

Similarly, let $y'-$ axis be associated with σ_2 and let l_2, m_2, n_2 be the direction cosines of this axis. Then, as before

$$2l_2+m_2+n_2=0 \qquad l_2-m_2+2n_2=0, \qquad l_2+2m_2-n_2=0$$

Solving these as before gives $l_2=1/\sqrt{3}$, $m_2=-1/\sqrt{3}$, $n_2=-1/\sqrt{3}$.

Finally, let $z'-$ axis be associated with σ_3 and let l_3, m_3, n_3 be the direction cosines of this axis. Then as before

$$-l_3+m_3+n_3=0 \qquad l_3-4m_3+2n_3=0 \qquad l_3+2m_3-4n_3=0$$

giving as before $l_3=-2/\sqrt{6}$, $m_3=-1/\sqrt{6}$, $n_3=-1/\sqrt{6}$.

Ex. 4. *Determine the principal stresses and principal directions for the stress tensor*

$$\sigma_{ij}=\begin{pmatrix} T & T & T \\ T & T & T \\ T & T & T \end{pmatrix}.$$

(Agra 2005, 06.Garhwal 2001)

Sol. Let σ be a princijpal stress.Then the characteristic equation (10) of art. 13.10 yields

$$\begin{vmatrix} T-\sigma & T & T \\ T & T-\sigma & T \\ T & T & T-\sigma \end{vmatrix}=0 \quad \text{or} \quad (3T-\sigma)\sigma^2=0, \quad \text{on simplification.}$$

Hence the principal stresses are given by $\sigma_1=0$, $\sigma_2=0$, $\sigma_3=3T$.

For $\sigma_3=3T$, let l_3, m_3, n_3 be the direction cosines of the corresponding principal direction. Then (6), (7) (8) of Art. 13.10 reduce to

$$\left.\begin{aligned} -2l_3+m_3+n_3&=0 \\ l_3-2m_3+n_3&=0 \\ l_3+m_3-2n_3&=0 \end{aligned}\right\} \qquad \dots(1)$$

Also we have $$l_3^2+m_3^2+n_3^2=1 \qquad \dots(2)$$

Solving (1) and (2) gives $l_3=m_3=n_3=1/\sqrt{3}$.

For $\sigma_1=0$, let l_1, m_1, n_1 be the direction cosines of corresponding principal axis. Then (6), (7), (8) of Art.13.10 reduce to

$$l_1+m_1+n_1=0, \qquad l_1+m_1+n_1=0, \qquad l_1+m_1+n_1=0$$

which together with $l_1^2+m_1^2+n_1^2=1$ are not sufficient to obtain uniquely the values of l_1, m_1, n_1. The same arguments show that l_2, m_2, n_2 for $\sigma_2=0$ also cannot be uniquely determined. Thus any pair of axes perpendicular to the direction of σ_1 and perpendicular to each other may be taken as principal axes.

Ex. 5. *Evaluate directly the invariants I_1, I_2, I_3 for the stress tensor*

$$\sigma_{ij} = \begin{pmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Determine the principal stress values for this state of stress and show that the diagonal form of the stress tensor yields the same values of the stress invariants

(Himanchal 2000; Meerut 2003, 05, 08; Garhwal 2003)

Sol. Here $\sigma_{xx} = 6, \quad \sigma_{xy} = -3, \quad \sigma_{zx} = 0, \quad \sigma_{yy} = 6, \quad \sigma_{yz} = 0, \quad \sigma_{zz} = 8.$

Hence using (12), (13), (14) of Art. of 13.10, we have

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 6 + 6 + 8 = 20.$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{zy} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{zz} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{xx} \end{vmatrix} = \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} + \begin{vmatrix} 6 & 0 \\ 0 & 8 \end{vmatrix} + \begin{vmatrix} 8 & 0 \\ 0 & 6 \end{vmatrix}$$

$$= 36 - 9 + 48 + 48 = 123.$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} = \begin{vmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 8 \end{vmatrix} = 8 \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 216.$$

Again let σ be a principal stress. Then the characteristic equation (10) of Art. 13.10 yields

$$\begin{vmatrix} 6-\sigma & -3 & 0 \\ -3 & 6-\sigma & 0 \\ 0 & 0 & 8-\sigma \end{vmatrix} = 0, \quad \text{giving} \quad \sigma_1 = 3, \quad \sigma_2 = 8, \quad \sigma_3 = 9 \quad \text{(on simplification)}$$

Hence using remark of Art. 13.10, the stress invariants in terms of principal values are given by

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 = 3 + 8 + 9 = 20,$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = 24 + 72 + 27 = 123,$$

and $$I_3 = \sigma_1\sigma_2\sigma_3 = 3 \times 8 \times 9 = 216.$$

Ex. 6. *Find the principal stresses and principal directions of stress at a point (l, l, l) if the components of the stress tensor are given by*

$$\sigma_{ij} = \begin{pmatrix} 0 & 2y_3 & 0 \\ 2y_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ **(Meerut 2006)**

Sol. The stress tensor at the point (1, 1, 1) is given by

$$\sigma_{ij} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let σ be the principal stress. Then σ is given by the characteristic equation

$$\begin{vmatrix} 0-\sigma & 2 & 0 \\ 2 & 0-\sigma & 0 \\ 0 & 0 & 0-\sigma \end{vmatrix} = 0 \quad \text{or} \quad \sigma(\sigma^2 - 4) = 0, \quad \text{on expanding the determinant.}$$

Hence the principal stresses are given by

$$\sigma_1 = 0, \qquad \sigma_2 = -2, \qquad \sigma_3 = 2. \qquad \ldots(1)$$

For $\sigma_1 = 0$, let l_1, m_1, n_1, be the direction cosines of the corresponding principal direction. These are given by

$$\left.\begin{aligned} 0\cdot l_1 + 2m_1 + 0\cdot n_1 &= 0 \\ 2l_1 + 0\cdot m_1 + 0\cdot n_1 &= 0 \\ 0\cdot l_1 + 0\cdot m_1 + 0\cdot n_1 &= 1 \\ l_1^2 + m_1^2 + n_1^2 &= 1 \end{aligned}\right\} \qquad \ldots(2)$$

From (2), l_1, m_1, n_1, cannot be determined uniquely.

For $\sigma_2 = -2$, let l_2, m_2, n_2 be direction cosines of the corresponding principal direction. Thes are given by

$$\left.\begin{aligned} 2l_2 + 2m_2 &= 0 \\ 2l_2 + 2m_2 &= 0 \\ n_2 &= 0 \\ l_2^2 + m_2^2 + n_2^2 &= 1 \end{aligned}\right\} \qquad \ldots(3)$$

Solving (3) $\qquad l_2 = 1/\sqrt{2}, \qquad m_2 = -1/\sqrt{2}, \qquad n_2 = 0.$

For $\sigma_3 = 2$, let l_3, m_3, n_3 be the direction cosines of the corresponding principal direction. These are given by

$$\left.\begin{aligned} -2l_3 + 2m_3 &= 0 \\ 2l_3 - 2m_3 &= 0 \\ n_3 &= 0 \\ l_3^2 + m_3^2 + n_3^2 &= 1 \end{aligned}\right\} \qquad \ldots(4)$$

Solving (4), $\qquad l_3 = m_3 = 1/\sqrt{2}, \qquad n_3 = 0$

Ex. 7. *Verify the following invariant for two dimensional stress-components:*

$$(\sigma_{x'x'} - \sigma_{y'y'})^2 + 4\sigma_{x'y'}^2 = (\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2.$$

Give a physical meaning of this invariant. **(Garhwal 1997, 98)**

Sol. We know that (refer relations given by (8) in Art. 13.8)

$$\sigma_{x'x'} = (1/2)\times(\sigma_{xx} + \sigma_{yy}) + (1/2)\times(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \sigma_{xy}\sin 2\theta \qquad \ldots(1)$$

$$\sigma_{y'y'} = (1/2)\times(\sigma_{xx} + \sigma_{yy}) - (1/2)\times(\sigma_{xx} - \sigma_{yy})\cos 2\theta - \sigma_{xy}\sin 2\theta \qquad \ldots(2)$$

$$2\sigma_{x'y'} = -(\sigma_{xx} - \sigma_{yy})\sin 2\theta + 2\sigma_{xy}\cos 2\theta. \qquad \ldots(3)$$

Subtracting (2) from (1), we have

$$\sigma_{x'x'} - \sigma_{y'y'} = (\sigma_{xx} - \sigma_{yy})\cos 2\theta + 2\sigma_{xy}\sin 2\theta. \qquad \ldots(4)$$

Squaring both sides of (4), we have

$$(\sigma_{x'x'} - \sigma_{y'y'})^2 = (\sigma_{xx} - \sigma_{yy})^2\cos^2 2\theta + 4\sigma_{xy}^2\sin^2 2\theta + 4(\sigma_{xx} - \sigma_{yy})\sigma_{xy}\sin 2\theta\cos 2\theta \qquad \ldots(5)$$

Squaring both sides of (3), we have

$$4\sigma_{x'y'}^2 = (\sigma_{xx} - \sigma_{yy})^2\sin^2 2\theta + 4\sigma_{xy}^2\cos^2 2\theta - 4(\sigma_{xx} - \sigma_{yy})\sigma_{xy}\sin 2\theta\cos 2\theta \qquad \ldots(6)$$

Adding (5) and (6), we have

$$(\sigma_{x'y'} - \sigma_{y'y'})^2 + 4\sigma_{x'y'}^2 = (\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2. \qquad \ldots(7)$$

Physical meaning of invariant (7) can be interpreted as follows :

Suppose θ be the angle of an arbitrary coordinate system relative to the principal axis. Then the principal stresses σ_{11} and σ_{22} relative to these coordinates are given by

$$\sigma_{xx} = \sigma_1, \qquad \sigma_{yy} = \sigma_2, \qquad \sigma_{xy} = 0. \qquad \ldots(8)$$

Using (8), (7) reduces to $\quad (\sigma_{x'x'} - \sigma_{y'y'})^2 + 4\sigma_{x'y'}^2 = (\sigma_1 - \sigma_2)^2 \qquad \ldots(9)$

Using (8), (5) reduces to $\quad (\sigma_{x'x'} - \sigma_{y'y'})^2 = (\sigma_1 - \sigma_2)^2 \cos^2 2\theta. \qquad \ldots(10)$

Subtracting (10) from (9), we have $\quad 4\sigma_{x'y'}^2 = (\sigma_1 - \sigma_2)^2 \sin^2 2\theta.$

Hence, $\quad 4(\sigma_{x'y'}^2)_{\max} = (\sigma_1 - \sigma_2)^2. \qquad \ldots(11)$

From (7),(9) and (11) we have

$$(\sigma_{x'x'} - \sigma_{y'y'})^2 + 4\sigma_{x'y'}^2 = (\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2 = (\sigma_1 - \sigma_2)^2 = 4(\sigma_{x'y'}^2)_{\max} \qquad \ldots(12)$$

(12) shows that the invariant for stress components given in (7) has a physical meaning, being four times the square of the maximum shearing stress.

Ex. 8. *The stress matrix at a point P is given by* $\begin{pmatrix} 2 & 1 & -3 \\ 1 & 1 & 2 \\ -3 & 2 & 1 \end{pmatrix}$.

Find the stress vector on the plane passing through P and parallel to the plane whose unit ***normal is* (3/7) *i* + (6/7)*j* + (2/7)*k*.** **(Purvanchal 2000, 05)**

Sol. Proceed as in Ex.1. **Ans. (6/7)*i* + (13/7)*j* + (5/7)*k*.**

Ex. 9. *The stress vector at a point P is given by* $\sigma_{ij} = \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & l \\ 0 & l & 2 \end{bmatrix}$.

Determine the stress vector on the plane passing through the P and having for its equation $(x/4) + (y/2) + (z/6) = 1$ **(Meerut 2007, 09)**

Sol. The given plane is $\quad (x/4) + (y/2) + (z/6) = 1 \quad$ or $\quad 3x + 6y + 2z = 12 \quad \ldots(1)$

Then the direction rations of normal to the plane (1) are 3, 6, 2. Hence the direction cosines of the normal to the plane (1) are

$$\frac{3}{(3^2+6^2+2^2)^{1/2}}, \frac{6}{(3^2+6^2+2^2)^{1/2}}, \frac{2}{(3^2+6^2+2^2)^{1/2}} \qquad i.e., \qquad \frac{3}{7}, \frac{6}{7}, \frac{2}{7}.$$

Thus, here, $\quad l = 3/7, \qquad m = 6/7, \qquad n = 2/7.$

Given $$\sigma_{ij} = \begin{vmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix}, \text{ say} \qquad \ldots(2)$$

$$\Rightarrow \quad \sigma_{xx} = 7, \sigma_{xy} = -5, \sigma_{xz} = 0, \sigma_{yx} = -5, \ \sigma_{yy} = 3, \sigma_{yz} = 1, \sigma_{zx} = 0, \sigma_{zy} = 1, \sigma_{zz} = 2 \qquad \ldots(3)$$

The required stress vector $\mathbf{F_n}$ is given by $\quad \mathbf{F}_n = \mathbf{i}\sigma_{nx} + \mathbf{j}\sigma_{ny} + \mathbf{k}\sigma_{nz}, \qquad \ldots(4)$

where σ_{nx}, σ_{ny} and σ_{nz} are given by the matrix equation

$$\begin{bmatrix} \sigma_{nx} \\ \sigma_{ny} \\ \sigma_{nz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{vmatrix} 3/7 \\ 6/7 \\ 2/7 \end{vmatrix}$$

or
$$\begin{bmatrix} \sigma_{nx} \\ \sigma_{ny} \\ \sigma_{nz} \end{bmatrix} = \begin{vmatrix} 7\cdot(3/7)+(-5)\cdot(6/7)+0\cdot(2/7) \\ (-5)\cdot(3/7)+3\cdot(6/7)+1\cdot(2/7) \\ 0\cdot(3/7)+2\cdot(6/7)+2\cdot(2/7) \end{vmatrix} = \begin{vmatrix} -9/7 \\ 5/7 \\ 10/7 \end{vmatrix}$$

$\Rightarrow \qquad \sigma_{nx} = -9/7, \qquad \sigma_{ny} = 5/7 \qquad$ and $\qquad \sigma_{nz} = 10/7.$

Then, (4) gives $\qquad \mathbf{F}_n = -(9/7)\mathbf{i} + (5/7)\mathbf{j} + (10/7)\mathbf{k}.$

Ex. 10. *Let the stress tensor at a point in the fluid be* $\begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & l \\ 2 & l & 2 \end{bmatrix}.$

Obtain the principal stresses and the corresponding principal directions.

Sol. Here, we have
$$\sigma_{ij} = \begin{vmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}, \text{ say}$$

$\Rightarrow \; \sigma_{xx} = 5, \quad \sigma_{yy} = 2, \quad \sigma_{zz} = 2, \quad \sigma_{xy} = \sigma_{yx} = 2, \quad \sigma_{yz} = \sigma_{zy} = 1, \quad \sigma_{zx} = \sigma_{xz} = 2.$

Let σ be a principal stress. Then σ is given by the characteristic equation

$$\begin{bmatrix} \sigma_{xx}-\sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}-\sigma & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}-\sigma \end{bmatrix} = 0 \qquad \text{or} \qquad \begin{vmatrix} 5-\sigma & 2 & 2 \\ 2 & 2-\sigma & 1 \\ 2 & 1 & 2-\sigma \end{vmatrix} = 0$$

or
$$(5-\sigma)[(2-\sigma)^2 - 1] - 2\times[2(2-\sigma)-2] + 2\times[2-2(2-\sigma)] = 0$$

or
$$\sigma^3 - 9\sigma^2 + 15\sigma - 7 = 0 \qquad\qquad (\sigma-1)^2(\sigma-7) = 0.$$

Hence the principal stresses are given by $\qquad \sigma_1 = 7, \qquad \sigma_2 = 1, \qquad \sigma_3 = 1.$

To find the principal direction corresponding corresponding to $\boldsymbol{\sigma_1 = 7}$.

Here l_1, m_1, n_1, the direction cosines of the corresponding principal direction, are given by the following system of equations

$$l_1(\sigma_{xx} - \sigma_1) + m_1\sigma_{xy} + n_1\sigma_{xz} = 0,$$
$$l_1\sigma_{yx} + m_1(\sigma_{yy} - \sigma_1) + n_1\sigma_{yz} = 0,$$
$$l_1\sigma_{zx} + m_1\sigma_{zy} + n_1(\sigma_{zz} - \sigma_1) = 0$$

and
$$l_1^2 + m_1^2 + n_1^2 = 1.$$

i.e.
$$-2l_1 + 2m_1 + 2n_1 = 0, \qquad \ldots(1)$$
$$2l_1 - 5m_1 + n_1 = 0 \qquad \ldots(2)$$
$$2l_1 + m_1 - 5n_1 = 0 \qquad \ldots(3)$$

and
$$l_1^2 + m_1^2 + n_1^2 = 1. \qquad \ldots(4)$$

Adding (1) and (2), $-3m_1 + 3n_1 = 0$ or $m_1 = n_1$. ...(5)

Putting $n_1 = m_1$ in (1), $-2l_1 + 4n_1 = 0$ or $l_1 = 2n_1$. ...(6)

From (5) and (6),
$$\frac{l_1}{2} = \frac{m_1}{1} = \frac{n_1}{1} = \frac{(l_1^2 + m_1^2 + n_1^2)^{1/2}}{(2^2 + 1^2 + 1^2)^{1/2}} = \frac{1}{\sqrt{6}}$$

$\Rightarrow$ $l_1 = 2/\sqrt{6}$, $m_1 = 1/\sqrt{6}$, $n_1 = 1/\sqrt{6}$.

To find the principal direction corresponding to $\sigma_2 = 1$.

Here l_2, m_2, n_2, the direction cosines of the corresponding principal direction, are given by the following system of equations

$$l_2(\sigma_{xx} - \sigma_2) + m_2\sigma_{xy} + n_2\sigma_{xz} = 0,$$
$$l_2\sigma_{yx} + m_2(\sigma_{yy} - \sigma_2) + n_2\sigma_{yz} = 0,$$
$$l_2\sigma_{zx} + m_2\sigma_{zy} + n_2(\sigma_{zz} - \sigma_2) = 0,$$

and $$l_2^2 + m_2^2 + n_2^2 = 1.$$

i.e. $4l_2 + 2m_2 + 2n_2 = 0$ or $2l_2 + m_2 + n_2 = 0$, ...(7)

$$2l_2 + m_2 + n_2 = 0, \quad \text{...(8)}$$
$$2l_2 + m_2 + n_2 = 0 \quad \text{...(9)}$$

and $$l_2^2 + m_2^2 + n_2^2 = 1. \quad \text{...(10)}$$

Here equations (7), (8) and (9) are the same and hence we have only two equations (7) and (10) to find three unknowns l_2, m_2, n_2. In such conditions, we cannot obtain unique values of l_2, m_2, n_2 as before. Similarly, the directions cosines l_3, m_3, n_3 corresponding to $\sigma_3 = 1$ cannot be uniquely determined. In such a situation we know that any pair of axes perpendicular to the direction of σ_1 and perpendicular to each other may be taken as principal axes. [Refer result (*ii*) of Art. 13.10]

Ex. 11. *Determine the principal stresses and principal axes of the state of stress given by the stress tensor* $\tau_{ij} = \alpha(\lambda_i\lambda'_j + \lambda'_i\lambda_j)$, *where* α *is a scalar and* λ_i, λ'_j *are unit vectors.*

(Agra 2005; Meerut 1999, 2002, 05)

Sol. Here the state of stress tensor is given by $\tau_{ij} = \alpha(\lambda_i\lambda'_j + \lambda'_i\lambda_j)$.

For simplification, we assume that x_1-axis is along λ_i and x_3-axis is normal to both λ_i and λ_j'. Then the components of λ_i and λ'_i are given by (1, 0, 0) and $(\lambda'_1, \lambda'_2, 0)$ respectively.

Here
$$\tau_{ij} = \begin{bmatrix} 2\lambda'_1\alpha & \alpha\lambda'_2 & 0 \\ \alpha\lambda'_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix}$$

$\Rightarrow$ $\tau_{11} = 2\lambda'_1\alpha$, $\tau_{22} = \tau_{33} = 0$, $\tau_{12} = \tau_{21} = \alpha\lambda'_2$, $\tau_{13} = \tau_{31} = 0$, $\tau_{32} = \tau_{23} = 0$.

Let τ be a principal stress. then τ is given by the characteristic equation

$$\begin{vmatrix} \tau_{11} - \tau & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} - \tau & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} - \tau \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 2\lambda'_i\alpha - \tau & \alpha\lambda'_2 & 0 \\ \alpha\lambda'_2 & 0 - \tau & 0 \\ 0 & 0 & 0 - \tau \end{vmatrix} = 0$$

or $\quad -\tau\,[-\tau(2\lambda_i'\alpha-\tau)-\alpha^2\lambda_2'^2]=0 \quad$ or $\quad \tau\,[(\tau/\alpha)^2-2\lambda_1'(\tau/\alpha)-\lambda_2'^2]=0$

$\Rightarrow \quad \tau=0 \quad$ or $\quad (\tau/\alpha)^2-2\lambda_1'(\tau/\alpha)+\lambda_1'^2-(\lambda_1'^2+\lambda_2'^2)=0$

$\Rightarrow \quad \tau=0 \quad$ or $\quad [(\tau/\alpha)-\lambda_1']^2=1, \quad$ as $\quad \lambda_1'^2+\lambda_2'^2=1$

$\Rightarrow \quad \tau=0 \quad$ or $\quad \tau=\alpha(\lambda_1'\pm 1).$

Hence the principal stresses are $\quad \tau_1=0, \quad \tau_2=\alpha(\lambda_1'+1), \quad \tau_3=\alpha(\lambda_1'-1).$

Principal direction corresponding to $\tau_1=0$. Here l_1, m_1, n_1, the direction cosines of the corresponding principal direction, are given by the following system of equations:

$$l_1(\tau_{11}-\tau_1)+m_1\tau_{12}+n_1\tau_{13}=0,$$

$$l_1\tau_{21}+m_1(\tau_{22}-\tau_1)+n_1\tau_{23}=0,$$

$$l_1\tau_{31}+m_1\tau_{32}+n_1(\tau_{33}-\tau_1)=0$$

and
$$l_1^2+m_1^2+n_1^2=1.$$

i.e., $2\lambda_i'\alpha l_1+2\lambda_2' m_1=0, \quad \lambda_2'\alpha l_1=0, \quad 0=0 \quad$ and $\quad l_1^2+m_1^2+n_1^2=1.$

Solving these, $\quad l_1=0, \quad m_1=0 \quad$ and $\quad n_1=1.$

Principal direction coresponding to $\tau_2=\alpha(\lambda_1'+1)$. Here l_2, m_2, n_2, the direcition cosines of the correspoding principal direcition are given by the following system of equations:

$$l_2(\tau_{11}-\tau_2)+m_2\tau_{12}+n_2\tau_{13}=0,$$

$$l_2\tau_{21}+m_2(\tau_{22}-\tau_2)+n_2\tau_{23}=0,$$

$$l_2\tau_{31}+m_2\tau_{32}+n_2(\tau_{33}-\tau_2)=0$$

and
$$l_2^2+m_2^2+n_2^2=1.$$

i.e.
$$\alpha(\lambda_1'-1)l_2+\alpha\lambda_2' m_2=0 \qquad \ldots(1)$$

$$\alpha\lambda_2' l_2-\alpha(\lambda_1'+1)m_2=0 \qquad \ldots(2)$$

$$-\alpha(\lambda_1'+1)m_2=0 \qquad \ldots(3)$$

and
$$l_2^2+m_2^2+n_2^2=1. \qquad \ldots(4)$$

Now, (1) and (3) $\Rightarrow$
$$\frac{l_2}{\lambda_2'}=\frac{-m_2}{\lambda_1'-1}=\frac{n_2}{0}. \qquad \ldots(5)$$

Again, (2) and (3) $\Rightarrow$
$$\frac{l_2}{\lambda_1'+1}=\frac{m_2}{\lambda_2'}=\frac{n_2}{0}. \qquad \ldots(6)$$

Multiplying the coresponding sides of (5) and (6), we get

$$\frac{l_2^2}{\lambda_2'(\lambda_1'+1)}=\frac{m_2^2}{-\lambda_2'(\lambda_1'-1)}=\frac{n_2^2}{0} \quad \text{or} \quad \frac{l_2^2}{\lambda_1'+1}=\frac{m_2^2}{-(\lambda_1'-1)}=\frac{n_2^2}{0}. \qquad \ldots(7)$$

Now, (7) $\Rightarrow$
$$\frac{l_2^2}{\lambda_1'+1}=\frac{m_2^2}{1-\lambda_1'}=\frac{n_2^2}{0}=\frac{l_2^2+m_2^2+n_2^2}{\lambda_1'+1+1-\lambda_1'+0}=\frac{1}{2}$$

Hence, $\quad l_2=\{(\lambda_1'+1)/2\}^{1/2}, \quad m_2=\{(1-\lambda_1')/2\}^{1/2}, \quad n_2=0.$

Principal direction corresponding to $\tau_3 = \alpha(\lambda_1' - 1)$. Here the direction cosines l_3, m_3, n_3 of the corresponding principal direction can be obtain a before and thus,

$$l_3 = \{(1-\lambda_1')/2\}^{1/2}, \qquad m_3 = \{(1+\lambda_1')/2\}^{1/2}, \qquad n_3 = 0.$$

Ex. 12. *A plate of 0.0025 cm distant from a fixed plate moves at 60 cm/sec and requires a force of 0.2kg/cm*^{2} *to maintain this speed. Determine the fluid viscosity of the substance between the plates.* **[Allahabad 2001; Nagpur 2003]**

Sol. Refer formula (3) of Art. 13.1, namely $T = \mu(u/y)$ giving $\mu = (Ty)/u$ ...(1)

Here $T = 0.2$ kg/ $cm^2 = (0.2 \times 1000)$ $gm/cm^2 = 200$ gm/ cm^2, $y = 0.0025 cm$, $u = 60$ cm/sec.

Hence, (1) reduces to $\mu = (200 \times 0.0025)/60 = (1/120)$ poise

EXERCISES 13-A

1. Define the principal stresses and principal directions. Obtain the principal stresses at a point where the stress is given by

$$\begin{pmatrix} 6 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

(Aligarh 2000, 06; Bangalore 2003, 06; Nagpur 2002)

[**Hint.** Proceeding like solved example 3. Art 13.11, you will get $\sigma_1 = 2, \sigma_2 = 4, \sigma_3 = 7$. Corresponding Principal directions are $1/\sqrt{5}, \; -2/\sqrt{5}, 0$; $2/\sqrt{5}, \; 1/\sqrt{5}, 0$ and $1/\sqrt{5}, \; 2/\sqrt{5}, 0.$]

2. Obtain the rates of strain components in two dimensional case in fluid dynamics.

3. The stress matrix at a point is given by

$$\begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Determine the stress vector on the plane passing through P and having for its equation $x/4 + y/2 + z/6 = 1$. **(G.N.D.U Amritar 2000, 04, 06; Himanchal 2003; Patna 2003)**

[**Hint.** The equation of the plane is $3x + 6y + 2z = 12$ so that the direction cosines l, m, n of its normal are (3/7,6/7, 2/7). Now proceed like solved example 1of Art 13.11]

4. Determine the principal stresses for

$$(i)\ \sigma_{ij} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad (ii)\ \sigma_{ij} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

and show that both have the same principal directions.

Ans. (*i*) $\sigma_1 = 2, \sigma_2 = \sigma_3 = -1$ (*ii*) $\sigma_1 = 4, \sigma_2 = \sigma_3 = 1$.

5. State and prove stress invariant in two dimensions.

6. Define and obtain invariants for three dimenssional stress components.

7. Define principal stresses and principal directions of a stress tensor. Prove that for a symmetric tensor there exist three principal directions which are mutually orthogonal provided principal stresses are distinct.

8. Explain what you understand by principal stresses at a point of a body. Show that if two of three of the principal stresses are equal, then there may be an infinite number of principal directions.

9. Describe the nature of stresses on an element of a continuous medium in general. Prove that only six components suffice to determine the state of stress at a point.

10. Two horizontal plates are placed 1.25 cm apart in the space between them being filled with oil of viscosity of 14 poises. Calculate the shear stress in the oil, if the upper plate is move with a velocity of 2.5 m/sec. **(Bangalore 1998)**

[**Hint:** Ditance between the plates $dy = 1.25$ cm, viscoity of oil $= \mu = 14$ poises, velocity of the upper plate $dv = 2.5$ m/sec $=$ 250 cm/sec. Let τ be the shear stress in the oil. Then by relation $\tau = \mu(dv/dy)$ we get, $\tau = 14 \times (250/1.25) = 2{,}800$ dynes/cm^2.]

11. Define stress at a point. Find the principal stresses and principal directions at a point at which stress tensor is given by

$$\begin{bmatrix} 11 & 2 & 8 \\ 2 & 2 & -10 \\ 8 & -10 & 5 \end{bmatrix}$$

[**Ans.** Principal stresses are – 9, 9, 8 and the corresponding principal directions are 1/3, 2/3, 2/3; 2/3, 2/3, – 1/3; 2/3,– 1/3, 2/3 respectively]

12. Let the stress tensor at a point in the fluid be $\begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

Obtain the principal stresses and the corresponding principal directions. Are the principal directions unique ? **(Meerut 2001, 04)**

[**Ans.** Principal stresses are $\sigma_1 = 7, \sigma_1 = 1, \sigma_2 = 1$ and principal direction corresponding to principal stress $\sigma_1(=7)$ is given by $2/\sqrt{6},\ 1/\sqrt{6}, 1/\sqrt{6}$. Any pair of axes perpendicular to the direction of $\sigma_1(=7)$ and perpendicular to each other may be taken as principal axes corresponding to $\sigma_2 = 1, \sigma_3 = 1$]

13. Determine invariants of a stress tensor. Show that $\begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{bmatrix}$, where $T_1 \neq T_2 \neq T_3$ cannot be stress tensor of a fluid at rest **(Himachal 1998)**

14. Obtain the principal stresses and principal directions of stress tensor σ_{ij} given by

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(Himachal 2003)

15. Show that the state of stress at a point is completely known if the nine components of stress tensor at the point are known. **(Meerut 2000, 11)**

16. Define stress at a point and prove that it is a symmetric tensor of order two **(Himachal 2003)**

17. Show the principal stresses may be expressed as

$$\sigma_{11} = \sigma_{\max} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

$$\sigma_{12} = \sigma_{\min} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

18. What are principal stresses, Derive stress invariants in terms of principal stresses. **(Meerut 2003, 05)**

19. Prove that the stress matrix is diagonally symmetric and contains only six unknowns. **(Meerut 2008)**

20. Define stress at a point. Define stress quadratic at a point and discuss its properties. **(Himanchal 2006, 09)**

13.12 Nature of strain (Himanchal 1998, Meeruth 2004, 05)

Strain may be defined as a non-dimensional deformation which measures the change of relative positions of the parts of a body under any cause.

Strain can be divided into the following two types :

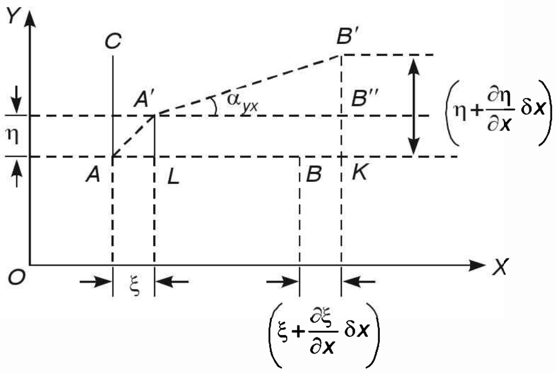
(*i*) **Normal (or direct) strain.**

The ratio of the change in length to the original length of a linear element is known as the *normal strain*.

(*ii*) **Shearing strain.**

It is measured in terms of the change in the angle between two linear elements from the unstrained state to the strained state.

In order to investigate the state of strain at a point P, we isolate in the elastic body an infinitesimal parallelepiped with edges $PP_1 = \delta x$, $PP_2 = \delta y$, $PP_3 = \delta z$ (not shown in figure, you may draw a figure yourself). During the deformation of the body it will displace and deform,*viz.*, the lengths of its edges will get elongated or contracted and the initially right angles between the faces will change. To estimate the required deformation at the given point P we must evaluate the elongations (normal strains) of the edges $\delta x, \delta y, \delta z$ of the isolated parallelepiped and the distortions of the angles P_1PP_2, P_1PP_3, P_2PP_3 (shearing strains). To this end, we consider the projections of the parallelepiped on the coordinate planes.Then the desired state of strain is said to be known completely if the strain of the above mentionned three projections is known.



Take, for example, the projection of the elementary parallelepiped $PP_1P_2P_3$ on the xy- plane as shown in figure. Thus we begin with the discussion of strain in a two dimensional case. Let AB and AC be the adjacent sides of the projection under consideration

Before strain the length of the edges are $AB = \delta x, AC = \delta y$ and $\angle BAC = 90°$. After very small strain, let AB occupy new position $A'B'$. Let coordinates of A be (x, y) and those of A' be $(x+\xi, y+\eta)$. From the figure,the displacement of A along the axis Ox is ξ *i.e.*, $AL = \xi$. Hence the corresponding displacement of the point B, namely $B'K$, is given by $B'K = \xi + (\partial\xi/\partial x)\delta x$.

Likewise, since, the displacement of the point A along the y-axis is η, the displacement of the point B along the same axis is $\eta + (\partial\eta/\partial x)\delta x$.

The normal strain component in the x direction is defined as

$$(\in_{xx})_s = (A'B' - AB)/AB, \qquad \ldots(1)$$

where the subscript $(\ )_s$ is used to indicate the the quantity being obtained is related to the elastic solid body.

Now, $A'B' = A'B''$, as $\angle B'A'B''$ is very small

$$= AB + BK - AL = \delta x + \xi + (\partial \xi / \partial x)\delta x - \xi$$

Thus, $$A'B' = \delta x + (\partial \xi / \partial x)\,\delta x \qquad \text{... (2)}$$

Using (2) and the fact that $AB = \delta x$, (1) yields $$(\in_{xx})_s = \partial \xi / \partial x. \qquad \text{... (3)}$$

By reasoning analogously, the normal strain component in the y-direction is given by

$$(\in_{yy})_s = \partial \eta / \partial y. \qquad \text{... (4)}$$

The shering strain $(\gamma_{xy})_s$ at the point A is the change of the angle between AB and AC. The angle of rotation α_{yx} of the edge AB is given by

$\alpha_{yx} \approx \tan \alpha_{yx}$, to first order of approximation

$$= \frac{B'B''}{A'B''} = \frac{B'K - B''K}{A'B''} = \frac{\eta + (\partial \eta / \partial x)\delta x - \eta}{\delta x + (\partial \xi / \partial x)\delta x}, \quad = \frac{\partial \eta / \partial x}{1 + \partial \xi / \partial x}.$$

Since we have confined ourselves to the case of very small strain, we may omit the equantiy $\partial \xi / \partial x = (\in_{xx})_s$ in the denominator of the above fraction as negligibly small cormpared with unity;hence we obtain $$\alpha_{yx} = \partial \eta / \partial x.$$

Similarly , considering the angle of rotation of the edge AC, we obtain

$$\alpha_{xy} = \partial \xi / \partial y.$$

Then the shearing strain at the point A is given by

$$(\gamma_{xy})_s = \alpha_{yx} + \alpha_{xy} = \partial \eta / \partial x + \partial \xi / \partial y \qquad \text{... (5)}$$

Similarly, we obtain the expressions of normal strains and sheaing strains in the other two coordinate planes. If ξ, η, ζ be the components of displacements of P in three-dimensional case,we have

(*i*) Normal strains : $(\in_{xx})_s = \partial \xi / \partial x, \quad (\in_{yy})_s = \partial \eta / \partial y, \quad (\in_{zz})_s = \partial \zeta / \partial z$ ... (6)

(*ii*) Shearing strains :

$$\left.\begin{aligned} (\gamma_{xy})_s &= \partial \eta / \partial x + \partial \xi / \partial y = (\gamma_{yx})_s \\ (\gamma_{yz})_s &= \partial \zeta / \partial y + \partial \eta / \partial z = (\gamma_{zy})_s \\ (\gamma_{zx})_s &= \partial \xi / \partial z + \partial \zeta / \partial x = (\gamma_{xz})_s \end{aligned}\right\} \qquad \text{... (7)}$$

In elasticity we deal with actual displafcement in a unit length. On the other hand, we are intersted in rate of strain while studying fluid dynamics. Let u, v, w be the fluid velocities in the x, y, z directions respectively. Then the rates of strain in fluid dynamics corresponding to (6) and (7) in elasticity are defined and given by

$$\in_{xx} = \frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial t}\right) = \frac{\partial u}{\partial x} \text{ etc.} \qquad \text{and} \qquad \gamma_{xx} = \frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right) = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \text{ etc.}$$

Thus the rates of strain components in the general three dimensional case are given by

$$\left.\begin{aligned} &\in_{xx} = \partial u / \partial x, \quad \in_{yy} = \partial v / \partial y, \quad \in_{zz} = \partial w / \partial z \\ &\gamma_{xy} = \partial u / \partial y + \partial v / \partial x = \gamma_{yx} \\ &\gamma_{yz} = \partial v / \partial z + \partial w / \partial y = \gamma_{zy} \\ &\gamma_{zx} = \partial w / \partial x + \partial u / \partial z = \gamma_{xz} \end{aligned}\right\} \qquad \text{... (8)}$$

Sometimes the halves of the shear angles are introduced in formulae (8) :

$$\in_{xy} = (1/2)\times\gamma_{xy}, \qquad \in_{yz} = (1/2)\times\gamma_{yz}, \qquad \in_{zx} = (1/2)\times\gamma_{zx}$$

Then the rates strain components (8) may be re-written as

$$\left.\begin{aligned} \in_{xx} &= \partial u/\partial x, \qquad \in_{yy} = \partial v/\partial y, \qquad \in_{zz} = \partial w/\partial z \\ \in_{xy} &= (1/2)\times(\partial u/\partial y + \partial v/\partial x) = \in_{yx} \\ \in_{yz} &= (1/2)\times(\partial v/\partial z + \partial w/\partial y) = \in_{zy} \\ \in_{zx} &= (1/2)\times(\partial w/\partial x + \partial u/\partial z) = \in_{xz} \end{aligned}\right\} \qquad \ldots(9)$$

Thus the nine quantities $\in_{ij}$ $(i, j = x, y, z)$ may be arranged as follows*;

$$\in_{ij} = \begin{bmatrix} \in_{xx} & \in_{xy} & \in_{xz} \\ \in_{yx} & \in_{yy} & \in_{yz} \\ \in_{zx} & \in_{zy} & \in_{zz} \end{bmatrix} \qquad \ldots(10)$$

The above mentioned nine quantities constitute the components of the rate of strain tensor of order two. Since the strains have the nature of a change in displacement in a given unit length in a given direction, the rate of strain is clearly a tensor of order two. Since $\in_{xy} = \in_{yx}, \in_{xz} = \in_{zx}, \in_{yz} = \in_{zy}$, it follows that the rate of strain tensor is symmetric tensor.

13.13. Transformation of the rates of strain components.

Case I. Two-dimensional rates of strain components

(Garwhal 1996; Kolkata 2001, 04; Meerut 2009, 2012)

Let the two-dimensional rates of strain components $\in_{xx}, \in_{yy}$ and $\in_{xy}$ at O with respect to coordinate axes OX and OY be known. Let OX', OY' be another set of orthogonal axes as shown in the figure of case I Art. 13.8. Then we require rates of strain components with respect to the new axes OX', OY' The direction cosines of one set of axes with respect to the other are shown in the table of case I, Art. 13.8. Let the velocity components in the new coordinate system (x', y') be u', v'. Then, we have

$$\left.\begin{aligned} x &= l_1 x' + l_2 y' \\ y &= m_1 x' + m_2 y' \end{aligned}\right\} \qquad \ldots(1)$$

and

$$\left.\begin{aligned} u' &= l_1 u + m_1 v \\ v' &= l_2 u + m_2 v \end{aligned}\right\} \qquad \ldots(2)$$

Using (1) and (2), the rates of strain components $\in_{x'x'}, \in_{y'y'}$ and $\gamma_{x'y'}$ in the new coordinate system (x', y') are given by

$$\in_{x'x'} = \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'}$$

$$= \left(l_1\frac{\partial u}{\partial x} + m_1\frac{\partial v}{\partial x}\right)l_1 + \left(l_1\frac{\partial u}{\partial y} + m_1\frac{\partial v}{\partial y}\right)m_1 = l_1^2\frac{\partial u}{\partial x} + m_1^2\frac{\partial v}{\partial y} + l_1 m_1\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

$$= l_1^2 \in_{xx} + m_1^2 \in_{yy} + l_1 m_1 \gamma_{xy}, \text{ using relations in (8) of Art 13.12} \qquad \ldots(3)$$

Similarly , we have $$\in_{y'y'} = \partial v'/\partial y' = l_2^2 \in_{xx} + m_2^2 \in_{yy} + l_2 m_2 \gamma_{xy} \qquad \ldots(4)$$

*The reader is advised to compare (10) with (6) of Art. 13.4

and $$\gamma_{x'y'} = \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} = \frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'}$$

$$= \left(l_2 \frac{\partial u}{\partial x} + m_2 \frac{\partial v}{\partial x}\right) l_1 + \left(l_2 \frac{\partial u}{\partial y} + m_2 \frac{\partial v}{\partial y}\right) m_1 + \left(l_1 \frac{\partial u}{\partial x} + m_1 \frac{\partial v}{\partial x}\right) l_2 + \left(l_1 \frac{\partial u}{\partial y} + m_1 \frac{\partial v}{\partial y}\right) m_2$$

$$= 2l_1 l_2 \frac{\partial u}{\partial x} + 2m_1 m_2 \frac{\partial v}{\partial y} + l_1 m_2 \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + l_2 m_1 \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$= 2l_1 l_2 \in_{xx} + 2m_1 m_2 \in_{yy} + (l_1 m_2 + l_2 m_1)\gamma_{xy}, \quad \text{using relations in (8)of Art . 13.12}$$

Using the table of case I, Art. 13.8, (3), (4) and (5) may be re-written as follows :

$$\in_{x'x'} = \cos^2\theta \in_{xx} + \sin^2\theta \in_{yy} + \sin\theta\cos\theta\, \gamma_{xy}$$

$$\in_{y'y'} = \sin^2\theta \in_{xx} + \cos^2\theta \in_{yy} - \sin\theta\cos\theta\, \gamma_{xy}$$

$$\gamma_{x'y'} = -2\sin\theta\cos\theta \in_{xx} + 2\sin\theta\cos\theta \in_{yy} + (\cos^2\theta - \sin^2\theta)\,\gamma_{xy}$$

Since $\sin\theta\cos\theta = (1/2)\times\sin 2\theta$, $\sin^2\theta = (1/2)\times(1-\cos 2\theta)$, $\cos^2\theta = (1/2)\times(1+\cos 2\theta)$, the above results may be further simplified as follows :

$$\left.\begin{aligned} \in_{x'x'} &= \frac{\in_{xx} + \in_{yy}}{2} + \frac{\in_{xx} - \in_{yy}}{2}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta \\ \in_{y'y'} &= \frac{\in_{xx} + \in_{yy}}{2} - \frac{\in_{xx} - \in_{yy}}{2}\cos 2\theta - \frac{1}{2}\gamma_{xy}\sin 2\theta \\ \gamma_{x'y'} &= (\in_{yy} - \in_{xx})\sin 2\theta + \gamma_{xy}\cos 2\theta \end{aligned}\right\} \quad \ldots(6)$$

Invariants of the rates of strain components for a two-dimensional case. [Meerut 2009]

The strain invariants may be defined as those functions of the strain components which remain invariant under a transformation of coordinates. Since the rate of strain is a tensor of order two, there must exist at least two invariants of the rate of strain. The two basic invariants for two-dimensional rates of strain components are given by

$$\in_{x'x'} + \in_{y'y'} = \in_{xx} + \in_{yy} \quad \ldots(7)$$

and $$\in_{x'x'}\in_{y'y'} - (1/4)\times\gamma_{x'y'}^2 = \in_{xx}\in_{yy} - (1/4)\times\gamma_{xy}^2 \quad \ldots(8)$$

These results may be easily proved by using (6).

Case II. Three dimensional rates of strain components.

(Meerut 1998; Garwhal 2000; Himanchal 1999)

For discussion of two sets of axes OX, OY, OZ and OX', OY', OZ', refer Case II, Art 13.8. Let the rates of strain components $\in_{xx}, \in_{yy}, \in_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ be given at O with respect to the axes OX, OY, OZ. Our aim is to obtain new expressions of the rates of strain components $\in_{x'x'}, \in_{y'y'}, \in_{z'z'}, \gamma_{x'y'}, \gamma_{y'z'}, \gamma_{z'x'}$ at O with respect the axes OX', OY', OZ',. Let u', v', w' be the velocity components in the new coordinate system. Then, we have

$$\left.\begin{aligned} x &= l_1 x' + l_2 y' + l_3 z' \\ y &= m_1 x' + m_2 y' + m_3 z' \\ z &= n_1 x' + n_2 y' + n_3 z' \end{aligned}\right\} \quad \ldots(9)$$

and $$\left.\begin{aligned} u' &= l_1 u + m_1 v + n_1 w \\ v' &= l_2 u + m_2 v + n_2 w \\ w' &= l_3 u + m_3 v + n_3 w \end{aligned}\right\} \quad \ldots(10)$$

Using (9) and (10), the rates of strain components in the new coordinate system (x', y', z') are given by

$$\in_{x'x'} = \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial u'}{\partial z}\frac{\partial z}{\partial x'}$$

$$= \left(l_1\frac{\partial u}{\partial x} + m_1\frac{\partial v}{\partial x} + n_1\frac{\partial w}{\partial x}\right)l_1 + \left(l_1\frac{\partial u}{\partial y} + m_1\frac{\partial v}{\partial y} + n_1\frac{\partial w}{\partial y}\right)m_1 + \left(l_1\frac{\partial u}{\partial z} + m_1\frac{\partial v}{\partial z} + n_1\frac{\partial w}{\partial z}\right)n_1$$

$$= l_1^2\frac{\partial u}{\partial x} + m_1^2\frac{\partial v}{\partial y} + n_1^2\frac{\partial w}{\partial z} + l_1 m_1\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + m_1 n_1\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + n_1 l_1\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$\therefore \quad \in_{x'x'} = l_1^2 \in_{xx} + m_1^2 \in_{yy} + n_1^2 \in_{zz} + l_1 m_1 \gamma_{xy} + m_1 n_1 \gamma_{yz} + n_1 l_1 \gamma_{zx}$, by relations (8) of Art. 13.12 ...(11)

Similarly, we have

$$\in_{y'y'} = \partial v'/\partial y' = l_2^2 \in_{xx} + m_2^2 \in_{yy} + n_2^2 \in_{zz} + l_2 m_2 \gamma_{xy} + m_2 n_2 \gamma_{yz} + n_2 l_2 \gamma_{zx} \qquad \ldots(12)$$

$$\in_{z'z'} = \partial w'/\partial z' = l_3^2 \in_{xx} + m_3^2 \in_{yy} + n_3^2 \in_{zz} + l_3 m_3 \gamma_{xy} \; m_3 n_3 \gamma_{yz} + n_3 l_3 \gamma_{zx} \qquad \ldots(13)$$

$$\gamma_{x'y'} = \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} = \frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial v'}{\partial z}\frac{\partial z}{\partial x'} + \frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'} + \frac{\partial u'}{\partial z}\frac{\partial z}{\partial y'}$$

$$= \left(l_2\frac{\partial u}{\partial x} + m_2\frac{\partial v}{\partial x} + n_2\frac{\partial w}{\partial x}\right)l_1 + \left(l_2\frac{\partial u}{\partial y} + m_2\frac{\partial v}{\partial y} + n_2\frac{\partial w}{\partial y}\right)m_1 + \left(l_2\frac{\partial u}{\partial z} + m_2\frac{\partial v}{\partial z} + n_2\frac{\partial w}{\partial z}\right)n_1$$

$$+ \left(l_1\frac{\partial u}{\partial x} + m_1\frac{\partial v}{\partial x} + n_1\frac{\partial w}{\partial x}\right)l_2 + \left(l_1\frac{\partial u}{\partial y} + m_1\frac{\partial v}{\partial y} + n_1\frac{\partial w}{\partial y}\right)m_2 + \left(l_1\frac{\partial u}{\partial z} + m_1\frac{\partial v}{\partial z} + n_1\frac{\partial w}{\partial z}\right)n_2$$

$$= 2l_1 l_2\frac{\partial u}{\partial x} + 2m_1 m_2\frac{\partial v}{\partial y} + 2n_1 n_2\frac{\partial w}{\partial z} + (l_1 m_2 + m_1 l_2)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$+ (m_1 n_2 + m_2 n_1)\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + (n_1 l_2 + l_1 n_2)\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$= 2l_1 l_2 \in_{xx} + 2m_1 m_2 \in_{yy} + 2n_1 n_2 \in_{zz} + (l_1 m_2 + m_1 l_2)\gamma_{xy} + (m_1 n_2 + n_1 m_2)\gamma_{yz}$$

$+ (n_1 l_2 + l_1 n_2)\gamma_{zx}$, using relations (8) of Art. 13.12 ...(14)

Similarly,
$$\gamma_{y'z'} = \partial w'/\partial y' + \partial v'/\partial z' = 2l_2 l_3 \in_{xx} + 2m_2 m_3 \in_{yy} + 2n_2 n_3 \in_{zz}$$
$$+ (l_2 m_3 + m_2 l_3)\gamma_{xy} + (m_2 n_2 + n_2 m_3)\gamma_{yz} + (n_2 l_3 + l_2 n_3)\gamma_{zx} \qquad \ldots(15)$$

and
$$\gamma_{z'x'} = \partial u'/\partial z' + \partial w'/\partial x' = 2l_3 l_1 \in_{xx} + 2m_3 m_1 \in_{yy} + 2n_3 n_1 \in_{zz}$$
$$+ (l_3 m_1 + m_3 l_1)\gamma_{xy} + (m_3 n_1 + n_3 m_1)\gamma_{yz} + (n_3 l_1 + l_3 n_1)\gamma_{zx} \qquad \ldots(16)$$

Invariants of the rates of strain components for a three-dimensional case.
These are defined and given by

$$\in_{x'x'} + \in_{y'y'} + \in_{z'z'} = \in_{xx} + \in_{yy} + \in_{zz} \qquad \ldots(17)$$

$$\in_{x'x'}\in_{y'y'} + \in_{y'y'}\in_{z'z'} + \in_{z'z'}\in_{x'x'} - (1/4)\times(\gamma_{x'y'}^2 + \gamma_{y'z'}^2 + \gamma_{z'x'}^2)$$
$$= \in_{xx}\in_{yy} + \in_{yy}\in_{zz} + \in_{zz}\in_{xx} - (1/4)\times(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2) \qquad \ldots(18)$$

$$\in_{x'x'}\in_{y'y'}\in_{z'z'} + (1/4)\times(\gamma_{x'y'}\gamma_{y'z'}\gamma_{z'x'} - \in_{x'x'}\gamma_{y'z'}^2 - \in_{y'y'}\gamma_{z'x'}^2 - \in_{z'z'}\gamma_{x'y'}^2)$$
$$= \in_{xx}\in_{yy}\in_{zz} + (1/4)\times(\gamma_{xy}\gamma_{yz}\gamma_{zx} - \in_{xx}\gamma_{yz}^2 - \in_{yy}\gamma_{zx}^2 - \in_{zz}\gamma_{xy}^2) \qquad \ldots(19)$$

Using (11) and (12) of case II, Art 13.8. and (11) to (16) of the present article, the above results can be easily proved. The proof of the same is left as exercise for the reader.

To show that the rate of strain is a tensor of order two.

The equations of transformation (11) to (16) characterize the mathematical nature of rate of strain. A set of nine quantities σ_{ij} $(i, j, = x, y, z)$ defined with respect to x, y, z axes are transformed by equations of the type (11) to (16) into the corresponding set of nine equantities with respect to x', y', z', axes. A set of such nine quantites is said to be a tensor* of the second order. Further the rate of strain is a symmetric tensor becauses $\gamma_{xy} = \gamma_{yx}, \gamma_{yz} = \gamma_{zy}, \gamma_{zx} = \gamma_{xz}$. **[Meerut 2010]**

13.14. The constitutive equations for a compressible Newtonian viscous fluid. Relation between stress and rates of strain. Stokes's law of viscosity. Stokes's hypothesis.

(Garwhal 2005; Kuruhshetre 2000, G.N.D.U. Amritsar 2000, 02, 03, 05; Himanchal 2001, 02, 03, 06, 07, 09; Kanpur 2003; Meerut 2010, 12)

In the next article, we shall show that the fluid flow in the vicinity of a point P may be convenienty analysed into three parts as follows : (*i*) a rigid-body tranlation at a velocity equal to the fluid velocity at the point P; (*ii*) rigid-body rotations about axes through P ; (*iii*) straining motion of distortion, characterized by the rates of strain. Since stresses will be generated by either the translation or rigid-body rotation, if follows that the stress tensor is entirely determined by the rate of strain tensor. The physical law connecting them will be obtained by making the following three as-sumptions :

(I) The stress-components may be expressed as a linear function of the rates of strain components.

(II) The relations between stress-components and rates of strain components must be invariant to a coordinate transformation consisting of either a rotation or a mirror reflection of axes.

(III) The stress-components must reduce to the hydrostatic pressure p when all the gradients of velocity are zero.

The first assumption is well borne out experimentally. the second assumption ensures that the law connectiong stress and rate of strain must be a real physical law which should not change with change of the coordinate system. The third assumption ensures that the continuity condition is fulfilled with the hydrostatic case.

With these assumptions we now propose to find the relations between stress and rate of strain in the two-dimensional case. The relations so obtained will be further extended to three-dimensional flows. Such relations are known as the *constitutive equations*.

Relation between stress and rate of strain in two-dimensional case.

In view of assumption (I), we take

$$\sigma_{xx} = A_1 \in_{xx} + B_1 \in_{yy} + C_1 \gamma_{xy} + D_1 \qquad \ldots(1a)$$

$$\sigma_{yy} = A_2 \in_{xx} + B_2 \in_{yy} + C_2 \gamma_{xy} + D_2 \qquad \ldots(1b)$$

$$\sigma_{xy} = A_3 \in_{xx} + B_3 \in_{yy} + C_3 \gamma_{xy} + D_3 \qquad \ldots(1c)$$

where the A' s, B's, C's, D's are constants to be determined with the help of the assumptions (II) and (III). Let $\sigma_{x'x'}$ $\sigma_{y'y'}$ $\sigma_{x'y'}$ be the stress components with respect to the new axes OX', OY' as shown in figure of case I, Art. 13.8. Let $\in_{x'x'}, \in_{y'y'}, \gamma_{x'y'}$ be the rates of strain components with respect to the new axes. In view of the assumption (II), the stress-rate of strain relations must remain unaltered with respect to the new corrdinate system, *i.e.*,

* Reader is advised to read Art 1.6, chapter 1 for further details.

$$\sigma_{x'x'} = A_1 \in_{x'x'} + B_1 \in_{y'y'} + C_1 \gamma_{x'y'} + D_1 \qquad \ldots(2a)$$

$$\sigma_{y'y'} = A_2 \in_{x'x'} + B_2 \in_{y'y'} + C_2 \gamma_{x'y'} + D_2 \qquad \ldots(2b)$$

$$\sigma_{x'y'} = A_3 \in_{x'x'} + B_3 \in_{y'y'} + C_3 \gamma_{x'y'} + D_3 \qquad \ldots(2c)$$

Now from (8) of Art13.8, we wave

$$\sigma_{x'y'} = (1/2)\times(1+\cos 2\theta)\sigma_{xx} + (1/2)\times(1-\cos 2\theta)\sigma_{yy} + \sigma_{xy}\sin 2\theta \qquad \ldots(3)$$

Subsituting values of $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ froom (1a), (1b), (1c) in (3), we get

$$\begin{aligned}\sigma_{x'x'} &= \left[(A_1/2)\times(1+\cos 2\theta) + (A_2/2)\times(1-\cos 2\theta) + A_3 \sin 2\theta\right]\in_{xx} \\ &\quad + \left[(B_1/2)\times(1+\cos 2\theta) + (B_2/2)\times(1-\cos 2\theta) + B_3 \sin 2\theta\right]\in_{yy} \\ &\quad + \left[(C_1/2)\times(1+\cos 2\theta) + (C_2/2)\times(1-\cos 2\theta) + C_3 \sin 2\theta\right]\gamma_{xy} \\ &\quad + \left[(D_1/2)\times(1+\cos 2\theta) + (D_2/2)\times(1-\cos 2\theta) + D_3 \sin 2\theta\right] \qquad \ldots(4)\end{aligned}$$

Now from (6) of Art 13.13, we have

$$\left.\begin{aligned}\in_{x'x'} &= \frac{\in_{xx} + \in_{yy}}{2} + \frac{\in_{xx} - \in_{yy}}{2}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta \\ \in_{y'y'} &= \frac{\in_{xx} + \in_{yy}}{2} - \frac{\in_{xx} - \in_{yy}}{2}\cos 2\theta - \frac{1}{2}\gamma_{xy}\sin 2\theta \\ \gamma_{x'y'} &= (\in_{yy} - \in_{xx})\sin 2\theta + \gamma_{xy}\cos 2\theta\end{aligned}\right\} \qquad \ldots(5)$$

Substituting the values of $\in_{x'x'}, \in_{y'y'}, \gamma_{x'y'}$ from (5) in (2*a*), we obtain

$$\begin{aligned}\sigma_{x'x'} &= \left[(A_1/2)\times(1+\cos 2\theta) + (B_1/2)\times(1-\cos 2\theta) - C_1 \sin 2\theta\right]\in_{xx} \\ &\quad + \left[(A_1/2)\times(1-\cos 2\theta) + (B_1/2)\times(1+\cos 2\theta) + C_1 \sin 2\theta\right]\in_{yy} \\ &\quad + \left[(A_1/2)\times\sin 2\theta - (B_1/2)\times\sin 2\theta + C_1 \cos 2\theta\right]\gamma_{xy} + D_1 \qquad \ldots(6)\end{aligned}$$

Since (4) and (6) are identical, it follows that the coefficients of $\in_{xx}, \in_{yy}, \gamma_{xy}$ and constant terms in these equations must be the same for all values θ. *i.e.*,

$$\frac{A_1}{2}(1+\cos 2\theta) + \frac{A_2}{2}(1-\cos 2\theta) + A_3 \sin 2\theta = \frac{A_1}{2}(1+\cos 2\theta) + \frac{B_1}{2}(1-\cos 2\theta) - C_1 \sin 2\theta \qquad \ldots(7)$$

$$\frac{B_1}{2}(1+\cos 2\theta) + \frac{B_2}{2}(1-\cos 2\theta) + B_3 \sin 2\theta = \frac{A_1}{2}(1-\cos 2\theta) + \frac{B_1}{2}(1+\cos 2\theta) + C_1 \sin 2\theta \qquad \ldots(8)$$

$$\frac{C_1}{2}(1+\cos 2\theta) + \frac{C_2}{2}(1-\cos 2\theta) + C_3 \sin 2\theta = \frac{A_1}{2}\sin 2\theta - \frac{B_1}{2}\sin 2\theta - C_1 \cos 2\theta \qquad \ldots(9)$$

and $$(D_1/2)\times(1+\cos 2\theta) + (D_2/2)\times(1-\cos 2\theta) + D_3 \sin 2\theta = D_1 \qquad \ldots(10)$$

Since (7), (8) (9), (10) are all true for all θ, equating coefficients of $\cos 2\theta$,$\sin 2\theta$ and constant terms, these give

$$(1/2)\times(A_1 + A_2) = (1/2)\times(A_1 + B_1) \qquad \ldots(11)$$

$$(1/2)\times(A_1 - A_2) = (1/2)\times(A_1 - B_1) \qquad \ldots(12)$$

$$A_3 = -C_1 \qquad \ldots(13)$$

$$(1/2)\times(B_1 + B_2) = (1/2)\times(A_1 + B_1) \qquad \ldots(14)$$

$$(1/2)\times(B_1 - B_2) = (1/2)\times(-A_1 + B_1) \quad \ldots(15)$$
$$B_3 = C_1 \quad \ldots(16)$$
$$(1/2)\times(C_1 + C_2) = 0 \quad \ldots(17)$$
$$(1/2)\times(C_1 - C_2) = C_1 \quad \ldots(18)$$
$$C_3 = (1/2)\times(A_1 - B_1) \quad \ldots(19)$$
$$(1/2)\times(D_1 + D_2) = D_1 \quad \ldots(20)$$
$$(1/2)\times(D_1 - D_2) = 0 \quad \ldots(21)$$
$$D_3 = 0 \quad \ldots(22)$$
$$(11) \text{ and } (12) \quad \Rightarrow \quad A_2 = B_1 = B, \text{ say} \quad \ldots(23)$$
$$(14) \text{ and } (15) \quad \Rightarrow \quad B_2 = A_1 = A, \text{ say} \quad \ldots(24)$$
$$(17) \quad \Rightarrow \quad C_1 = -C_2 \quad \ldots(25)$$

$\therefore$ From (13), (16), (25), we have

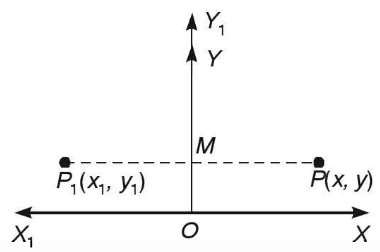
$$C_1 = -A_3 = B_3 = -C_2 = C, \text{ say} \quad \ldots(26)$$
$$(20) \text{ and } (21) \quad \Rightarrow \quad D_1 = D_2 = D, \text{ say} \quad \ldots(27)$$

Using (22), (23),(24), (26) and (27), the assumed linear relations (*Ia*) to (*Ic*) reduce to

$$\sigma_{xx} = A\in_{xx} + B\in_{yy} + C\,\gamma_{xy} + D, \quad \ldots(28a)$$
$$\sigma_{yy} = B\in_{xx} + A\in_{yy} - C\,\gamma_{xy} + D \quad \ldots(28b)$$

and
$$\sigma_{xy} = C\,(\in_{yy} - \in_{xx}) + (1/2)\times(A - B)\,\gamma_{xy} \quad \ldots(28c)$$

We now proceed to fulfil the second part of the assumption (II),namely, the proposed relations must be invariant to a new coordinate system OX_1, OY_1 which is a mirror reflection of the original system with respect to the y-axis as shown in the figure,



Here $\quad x_1 = -x \quad$ and $\quad y_1 = y$

and $\quad u_1 = -u \quad$ and $\quad v_1 = v.$

Also, we have $\quad PM = P_1M$

Hence $\quad \in_{x_1x_1} = \partial u_1/\partial x_1 = \partial u/\partial x = \in_{xx},$

$$\in_{y_1y_1} = \partial v_1/\partial y_1 = \partial v/\partial y = \in_{yy}$$

$$\gamma_{x_1y_1} = \partial v_1/\partial x_1 + \partial u_1/\partial y_1 = -(\partial v/\partial x + \partial u/\partial y) = -\gamma_{xy}$$

and $\quad \sigma_{x_1x_1} = \sigma_{xx}, \qquad \sigma_{y_1y_1} = \sigma_{yy}, \qquad \sigma_{x_1y_1} = -\sigma_{xy}$

Using these values, (28*a*) to (28*c*) reduce to

$$\sigma_{x_1x_1} = A\in_{x_1x_1} + B\in_{y_1y_1} - C\,\gamma_{x_1y_1} + D, \quad \ldots(29a)$$
$$\sigma_{y_1y_1} = B\in_{x_1x_1} + A\in_{y_1y_1} + C\,\gamma_{x_1y_1} + D \quad \ldots(29b)$$

and
$$\sigma_{x_1y_1} = C(\in_{x_1x_1} - \in_{y_1y_1}) + (1/2)\times(A - B)\gamma_{x_1y_1} \quad \ldots(29b)$$

In view of the remarks just made, the form of equations (28*a*) to (28*c*) must be the same as that of equation (29*a*) to (29*c*). Accordingly, we must choose $C = 0$. Again in view of the assumption (III), (28*a*) and (28*b*) show that $D = -p$. Hence (28*a*) to (28c) reduce to

$$\sigma_{xx} = A\in_{xx} + B\in_{yy} - p \quad \ldots(30a)$$
$$\sigma_{yy} = B\in_{x\,x} + A\in_{yy} - p \quad \ldots(30b)$$
$$\sigma_{xy} = (1/2)\times(A - B)\,\gamma_{xy} \quad \ldots(30c)$$

The constant of proportionality (1/2) × $(A - B)$ in (30 C) is taken as μ.

Thus, (1/2) × $(A - B)$ = μ. so that $A = 2\mu + B$. Then (30c) to (30c) reduce to

$$\sigma_{xx} = 2\mu \in_{xx} + B\,(\in_{xx} + \in_{yy}) - p \qquad \ldots(31a)$$

$$\sigma_{yy} = 2\mu \in_{yy} + B\,(\in_{xx} + \in_{yy}) - p \qquad \ldots(31b)$$

$$\sigma_{xy} = \mu\, \gamma_{xy} = 2\mu \in_{xy}, \qquad \text{as} \qquad \gamma_{xy}/2 = \in_{xy} \qquad \ldots(31c)$$

where $B = -2\mu/3$ as shown in the three dimensional case which is discusses below.

Extending this stress-rate of strain relations given in (31a) to (31c) to three-dimensional flows, we have

$$\sigma_{xx} = 2\mu \in_{xx} + B\, \nabla \cdot \mathbf{q} - p \qquad \ldots(32a)$$

$$\sigma_{yy} = 2\mu \in_{yy} + B\, \nabla \cdot \mathbf{q} - p \qquad \ldots(32b)$$

$$\sigma_{zz} = 2\mu \in_{zz} + B\, \nabla \cdot \mathbf{q} - p \qquad \ldots(32c)$$

$$\sigma_{xy} = 2\mu \in_{xy} = \sigma_{yx} \qquad \ldots(32d)$$

$$\sigma_{yz} = 2\mu \in_{yz} = \sigma_{zy} \qquad \ldots(21e)$$

$$\sigma_{zx} = 2\mu \in_{zx} = \sigma_{xz} \qquad \ldots(32f)$$

wherein we have written

$$\in_{xx} + \in_{yy} + \in_{zz} = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = \nabla \cdot \mathbf{q} \qquad \ldots(33)$$

where $\nabla \equiv \mathbf{i}\,(\partial/\partial x) + \mathbf{j}\,(\partial/\partial y) + \mathbf{k}\,(\partial/\partial z)$, and $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. ...(34)

Adding (32a) to (32c) and using (33),we get

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 2\mu \nabla \cdot \mathbf{q} + 3B\, \nabla \cdot \mathbf{q} - 3p. \qquad \ldots(35)$$

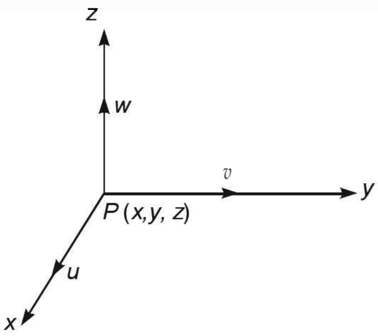
Let us assume, as is usually permissible, that the pressure is equal to the mean of the three normal stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$, *i.e.*, $(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3 = -p$.

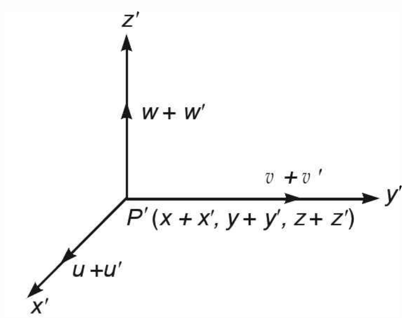
Then (35) reduces to $(2\mu + 3B)\nabla \cdot \mathbf{q} = 0$. Since $\nabla \cdot \boldsymbol{q} \neq 0$ for compressible fluids, we have $2\mu + 3B = 0$ and hence $B = -2\mu/3$. With this value of B, (32a) to (32f) are the desired relations.

$2\mu + 3B = 0$ is known as *Stokes's relation*

13.15. The rate of strain quadratic. (Himanchal 2005, 07, 09)

Let there be a fluid particle at $P(x, y, z)$ and let u, v, w be the velocfity components at P parallel to the rectangular coordinate axes. Let the fluid particle at P move to a neighbouring point P' $(x + x', y + y', z + z')$ and let $u + u'$, $v + v'$, $w + w'$ be the velocity components at P' parallel to the coordinate axes.





The rate of strain quadratic for P is given by

$$\in_{xx} x'^2 + \in_{yy} y'^2 + \in_{zz} z'^2 + 2 \in_{yz} y'z' + 2 \in_{zx} z'x' + 2 \in_{xy} x'y' = \text{const.} \qquad ...(1)$$

When the rate of strain quadratic is referred to a set of axes through P parallel to the axes $P'X'$, $P'Y'$, $P'Z'$, then the coefficients in the transformed equation of the strain quadric will be $\in_{x'x'}$, $\in_{y'y'}$, $\in_{z'z'}$, $\in_{x'y'}$, $\in_{y'z'}$, $\in_{z'x'}$. Let $P'X'$, $P'Y'$, $P'Z'$ be taken along the principal axes of the rate of strain quadratic. Then we know that $\in_{x'y'}, \in_{y'z'}$ and $\in_{z'x'}$ must vanish and the rate of strain quadric takes the form

$$\in_{x'x'} X'^2 + \in_{y'y'} Y'^2 + \in_{z'z'} Z'^2 = \text{const.} \qquad ...(2)$$

It follows that the rate of shearing strain along the axes are zero. Hence at any point of a fluid in motion there is a set of mutually perpendicular straight lines such that, if these lines move with the fluid, after a small time δt the angle between them continues to be right angles to the first order in δt. These lines are known as the *principal axes of the rate of strains*.

13.16. Illustrative solved examples.

Ex. 1. *Find the shear strain for the fluid flow described by the velocity field* $\mathbf{q} = 5x^3\mathbf{i} - 15x^2y\mathbf{j}$.

Sol. Here $\quad \mathbf{q} = u\mathbf{i} + v\mathbf{j} = 5x^3\mathbf{i} - 15x^2y\mathbf{j} \quad \Rightarrow \quad u = 5x^3, \quad v = -15x^2y$ ....(1)

For the given two-dimensional motion, the vorticity is given by

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(-15x^2y) - \frac{\partial}{\partial y}(5x^3) = -30xy. \qquad ...(2)$$

Since $\Omega_z \neq 0$, it follows that the given motion is rotational except at the origin where $x = 0 = y$ so that $\Omega_z = 0$, by (2). Also the shear strain rate in given by

$$\in_{xy} = (1/2) \times (\partial v/\partial x - \partial u/\partial y) = (1/2) \times (-30xy) = -15xy.$$

Ex. 2. *Consider the rectangular flow* $\boldsymbol{q} = \{0, 0, \phi(x_1, x_2)\}$ *of an isotropic incompressible fluid, show that strain rate tensor has non-zero components*

[Allahabad 2001, Himanchal 200; Kolkata 2006, Kurukshetra 2001]

Sol. The rates of strain components are given by (taking $x = x_1, y = x_2, z = x_3$)

$\in_{xx} = \in_{x_1x_1} = \in_{11}$; $\quad \in_{yy} = \in_{x_2x_2} = \in_{22}$, $\quad \in_{zz} = \in_{x_3x_3} = \in_{33}$, $\quad \in_{xy} = \in_{x_1x_2} = \in_{12}$,

$\in_{yx} = \in_{x_2x_1} = \in_{21}$, $\quad \in_{yz} = \in_{x_2x_3} = \in_{23}$, $\quad \in_{zy} = \in_{x_3x_2} = \in_{32}$, $\quad \in_{zx} = \in_{x_3x_1} = \in_{31}$,

$\in_{xz} = \in_{x_1x_3} = \in_{13}$ Then, using relations (9) of Art. 13.12, we have

$$\left.\begin{array}{l} \in_{11} = \partial u/\partial x_1, \qquad \in_{22} = \partial v/\partial x_2, \qquad \in_{33} = \partial w/\partial x_3 \\ \in_{12} = \in_{21} = (1/2) \times (\partial u/\partial x_2 + \partial v/\partial x_1) \\ \in_{23} = \in_{32} = (1/2) \times (\partial v/\partial x_3 + \partial w/\partial x_2) \\ \in_{31} = \partial_{13} = (1/2) \times (\partial w/\partial x_1 + \partial u/\partial x_3) \end{array}\right\} \qquad ...(1)$$

Given $\qquad \mathbf{q} = (u, v, w) = \{0, 0, \phi(x_1, x_2)\}$

$\Rightarrow \qquad u = 0, \qquad v = 0 \qquad$ and $\qquad w = \phi(x_1, x_2) \quad ...(2)$

Using (2), $\quad$ (1) $\Rightarrow \qquad \in_{11} = \in_{22} = \in_{33} = \in_{12} = 0$

$\in_{23} = \in_{32} = (1/2) \times (\partial\phi/\partial x_2) \qquad$ and $\qquad \in_{31} = \in_{13} = (1/2) \times (\partial\phi/\partial x_1)$,

showing that the strain-rate tensor has non-zero components $\in_{23}$ $(= \in_{32})$ and $\in_{31}$ $(= \in_{13})$.

Ex. 3. *The flow field of a fluid is given by* $\mathbf{q} = xy\mathbf{i} + 2yz\mathbf{j} - (yz + z^2)\mathbf{k}$.

(*i*) *Show that it represents a possible three-dimensional steady incompressible continuous flow.*

(*ii*) *Is this flow rotational or irrotational? If rotational, determine at point P(2, 4, 6) :*

(*a*) *Angular velocity* (*b*) *Vorticity* (*c*) *Shear strain* (*d*) *Dilatency (linear strains).*

Sol. Here $\mathbf{q} = xy\mathbf{i} + 2yz\mathbf{j} - (yz + z^2)\mathbf{k} = u\mathbf{i} + v\,\mathbf{j} + w\mathbf{k}$, say

Hence, $u = xy,\quad v = 2yz \quad \text{and} \quad w = -(yz+ z^2)$

Part (i). Here $(\partial u/\partial x) + (\partial v/\partial x) + (\partial w/\partial z) = y + 2z - (y + 2z)$, using (1).

Thus, $(\partial u/\partial x) + (\partial v/\partial y) + (\partial w/\partial z) = 0$, showing that continuity equation is satisfied. Hence the given flow is a possible three-dimensional steady incompressible flow

Part (*ii*). Here $\partial u/\partial y = y$ and $\partial v/\partial x = 0$ so that $\partial u/\partial y \neq \partial v/\partial x$

Hence the flow is rotational.

(a) Angular velocity ω is given by

$$\omega = (1/2) \times [(\partial w/\partial y - \partial v/\partial z)\mathbf{i} + (\partial u/\partial z - \partial w/\partial x)\mathbf{j} + (\partial v/\partial x - \partial u/\partial y)\mathbf{k}]$$
$$= (1/2) \times \{(-z - 2y)\mathbf{i} + (0 - 0)\mathbf{j} + (0 - x)\mathbf{k}]$$

Hence at $P(2, 4, 6)$, $\omega = (1/2) \times [(-6 - 8)\mathbf{i} - 2\mathbf{k}] = -7\mathbf{i} - \mathbf{k}$.

(b) The vorticity vector Ω is given by $\Omega = 2\,\omega = 2(-7\mathbf{i} - \mathbf{k}) = -14\mathbf{i} - 2\mathbf{k}$.

(c) To determine shear strain. We have

$$\in_{xy} = (1/2)\times(\partial u/\partial y + \partial v/\partial x) = (1/2)\times(x + 0) = 1, \text{ at } P(2, 4, 6)$$
$$\in_{yz} = (1/2)\times(\partial v/\partial z + \partial w/\partial y) = (1/2)\times(2y - z) = 1, \text{ at } P(2, 4, 6)$$
$$\in_{zx} = (1/2)\times(\partial w/\partial x + \partial u/\partial z) = (1/2)\times(0 + 0) = 0, \text{ at } P(2, 4, 6)$$

(d) To determine dilatency (linear strains). We have $\in_{xx} = \partial u/\partial x = y = 4$, at $P(2, 4, 6)$

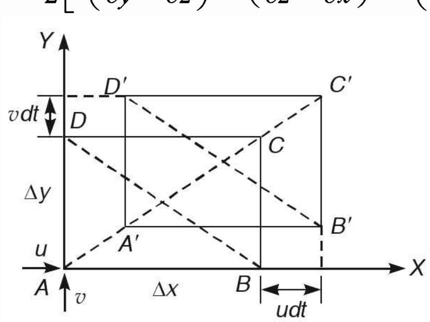
$\in_{yy} = \partial v/\partial y = 2z = 12$, at $P(2, 4, 6)$ and $\in_{zz} = \partial w/\partial z = -(y + 2z) = -16$, at $P(2, 4, 6)$

11.17. Translation, rotation and rate of deformation. [Meerut 2009]

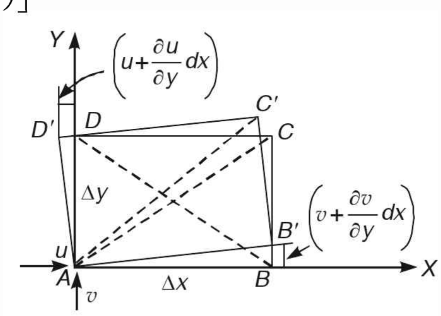
To show that the general motion of a fluid element is made up of three parts, namely, pure translation, pare rotation and pure deformation. (Garhwal 2000, Himanchal 1998)

We begin with the geometrical interpretation of translation, rotation and deformation of fluid element as shown in the following figures (*i*) to (*iv*). If a fluid element moves without changing its shape, then that fluid element is said to undergo *translation*. Figure (*i*) illustrates the translatory motion of a rectangular fluid element from *ABCD* to new position *A'B'C'D'*. Again a fluid element is said to undergo *rotation* if it rotates about a certain axis without altering its shape as shown in figure (*ii*). For more details, refer Art 2.31 of chapter 2 wherein we have shown that angular velocity ω is given by (taking velocity vector $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$)

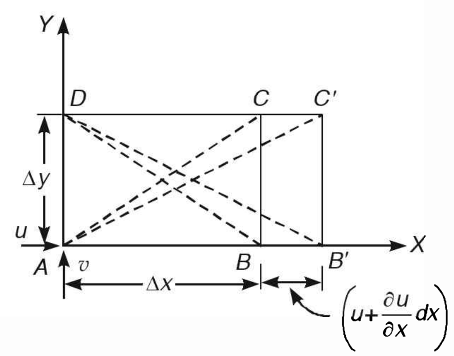
$$\omega = \frac{1}{2}\left[i\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathrm{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathrm{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] \quad \ldots(1)$$



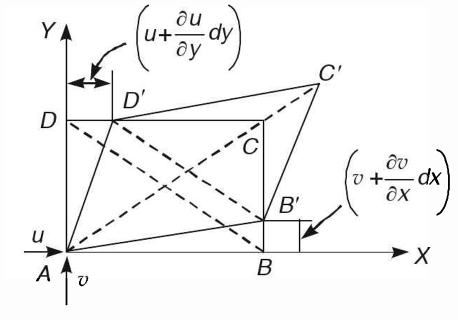
(*i*) Translation



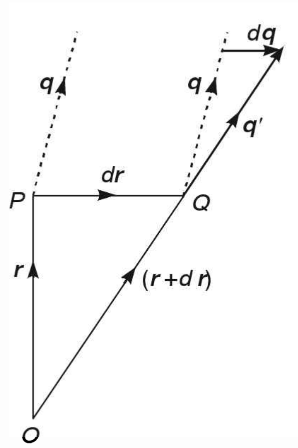
(*ii*) Rotation



(*iii*) Rate of linear deformation



(*iv*) Rate of angular deformation



A fluid element is said to undergo deformation if the distance between two neighbouring fluid particles changes.There are two types of deformations namely linear deformation and angular deformation. The rates of such deformations have been shown in figures (*iii*) and (*iv*) above. For more details, refer Art. 13.12

We have just discussed three different possible modes of motion of fluid element. In what follows we propose to establish that in actual flow the movement of a fluid element, in general, may consist of a translation, a rotation and a rate of deformation.

Consider the small movement of a fluid particle from P to Q. Let the velocity at P (x, y, z) be $\boldsymbol{q}$ and that at a neighbouring point Q be $\boldsymbol{q}' = \boldsymbol{q} + d\boldsymbol{q}$. Let the position vectors of P and Q be $\boldsymbol{r}$ and $\boldsymbol{r} + d\,\boldsymbol{r}$ respectively.

Then, $\boldsymbol{q}' = \boldsymbol{q} + d\boldsymbol{q} = \mathbf{q} + \dfrac{\partial \mathbf{q}}{\partial x}dx + \dfrac{\partial \mathbf{q}}{\partial y}dy + \dfrac{\partial \mathbf{q}}{\partial z}dz$

$$= \mathbf{q} + \mathbf{i}\left[\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right] + \mathbf{j}\left[\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz\right] + \mathbf{k}\left[\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz\right]$$

$$[\because\ \mathbf{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w\,]$$

$$= \mathbf{q} + \mathbf{i}\left[\left\{\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dy\right\} + \left\{\frac{\partial u}{\partial x}dx + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dy + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)dz\right\}\right]$$

$$+\mathbf{j}\left[\left\{\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx - \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dz\right\} + \left\{\frac{\partial v}{\partial y}dy + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dx + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)dz\right\}\right]$$

$$+\mathbf{k}\left[\left\{\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dy - \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dx\right\} + \left\{\frac{\partial w}{\partial z}dz + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)dx + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)dy\right\}\right]$$

Thus
$$\mathbf{q}' = \mathbf{q} + \omega \times d\mathbf{r} + \mathbf{D}, \qquad \ldots(2)$$

where
$$\omega \times d\boldsymbol{r} = \mathbf{i}\left\{\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dy\right\}$$

$$+\mathbf{j}\left\{\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx - \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dz\right\} + \mathbf{k}\left\{\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dy - \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dx\right\} \qquad \ldots(3)$$

[Using (1) and $d\,\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz,$ as $\mathbf{r} = \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z$]

and $\mathbf{D} = \mathbf{i}\left[\frac{\partial u}{\partial x}dx + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dy + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)dz\right]$

$$+\mathbf{j}\left[\frac{\partial v}{\partial y}dy + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dx + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)dz\right] + \mathbf{k}\left[\frac{\partial w}{\partial z}dz + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)dx + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)dy\right]$$

$$= \mathbf{i}\,(\in_{xx}\,dy + \in_{xy}\,dy + \in_{xz}\,dz) + \mathbf{j}\,(\in_{yy}\,dy + \in_{yx}\,dx + \in_{zy}\,dz) + \mathbf{k}\,(\in_{zz}\,dz + \in_{zx}\,dx + \in_{zy}\,dy)$$

Then $$\mathbf{D} = \mathbf{i}\,(\in_x \cdot d\mathbf{r}) + \mathbf{j}\,(\in_y \cdot d\mathbf{r}) + \mathbf{k}\,(\in_z \cdot d\mathbf{r}) \qquad \ldots(4)$$

where $\in_x = \mathbf{i}\in_{xx} + \mathbf{j}\in_{xy} + \mathbf{k}\in_{xz}$, $\in_y = \mathbf{i}\in_{yx} + \mathbf{j}\in_{yy} + \mathbf{k}\in_{yz}$ and $\in_z = \mathbf{i}\in_{zx} + j\in_{zy} + k\in_{zz}$ are the strain-rate tractions of the fluid elements in the x-, y-, and z- directions respectively.

Equation (2) represents the most general mode of motion of a fluid element. The first term $\boldsymbol{q}$ represents the linear motion of all parts of the fluid element without changing the shape of the element. Hence the first term represents the *pure translatory* part of the motion. The second term $\omega \times d\mathbf{r}$ represents the *pure rotation* of the fluid element. The third term **D** represents the rate of strain term (or rate of deformation tensor) and so the third term **D** gives the deformation of the fluid element. Thus we see that the most general motion of a fluid element can be expressed as the combination of translation, rotation and deformation of the fluid element.

13.18. Illustrative solved examples.

Ex. 1. *Given a velocity field with components*

$$u = cx + 2\omega_0 y + u_0, \qquad v = cy + v_0, \qquad w = -2cz + \omega_0,$$

where c, u_0, v_0 *and* ω_0 *are constants. With the above velocity components at a point P(x, y, z) determine the velocity comoponents at a neighbourig point Q(x +dx , y + dy, z + dz) and determine the different types of motion wich are involved.*

(Kanpur 2000; Himanchal 2004: Meerut 1999, 2001)

Sol. Given $u = cx + 2\omega_0 y + u_0,\quad v = cy + v_0,\quad w = -2cz + \omega_0 \qquad \ldots(1)$

Refer Art. 13.17. Let $\mathbf{q}'$ be the velocity at Q. Then, we have

$$\mathbf{q}' = \mathbf{q} + \omega \times d\mathbf{r} + \mathbf{D} \qquad \ldots(2)$$

(*i*) **Translation velocity q** is given by $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

or $$\mathbf{q} = (cx + 2\omega_0 y + u_0)\mathbf{i} + (cy + v_0)\mathbf{j} + (\omega_0 - 2cz)\mathbf{k} \qquad \ldots(3)$$

(*ii*) **Ratational velocity** ω is given by

$$\omega = \frac{1}{2}\left[\mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] = -\omega_0 k, \text{ using (1) and simplifying.}$$

This represents a constant angular velocity about the z-axis. Again, we have

$$\omega \times \mathbf{d}r = -\omega_0\mathbf{k} \times (\mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz) = (\omega_0 dy)\mathbf{i} - (\omega_0 dx)\mathbf{j} \qquad \ldots(4)$$

(*iii*) **Rate of strain. Deformation.**

$$\in_{xx} = \partial u/\partial x = c, \qquad \in_{yy} = \partial v/\partial y = c, \qquad \in_{zz} = \partial w/\partial y = -2c \quad \ldots(5)$$

$$\in_{xx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \omega_0, \qquad \in_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0, \qquad \in_{zy} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0 \quad \ldots(6)$$

These show that there is uniform rate of normal and shearing strain. Further, we have

$$\mathbf{D} = \mathbf{i}\,(\in_{xx}\,dx + \in_{xy}\,dy + \in_{xz}\,dz) + \mathbf{j}\,(\in_{yy}\,dy + \in_{yx}\,dx + \in_{yz}\,dz) + \mathbf{k}(\in_{zz}\,dz + \in_{zx}\,dx + \in_{zy}\,dy)$$

Thus, $$\mathbf{D} = \mathbf{i}\,(cdx + \omega_0 dy) + \mathbf{j}\,(\omega_0 dx + cdy) - \mathbf{k}(2cdz). \qquad \ldots(7)$$

Using (3),and (7),the required velocity $\mathbf{q}'$ is given by (2), namely,

$$\mathbf{q}' = (cx + 2\omega_0 y + u_0)\mathbf{i} + (cy + v_0)\mathbf{j} + (\omega_0 - 2cz)\mathbf{k} + (\omega_0 dy)\mathbf{i} - (\omega_0 dx)\mathbf{j}$$
$$+\mathbf{i}\,(cdx + \omega_0 dy) + \mathbf{j}(\omega_0 dx + cdy) + \mathbf{k}(-2cdz)$$

or $$\mathbf{q}' = \mathbf{i}\,(cx + 2\omega_0 y + u_0 + cdx + 2\omega_0 dy) + \mathbf{j}\,(cy + v_0 + cdy) + \mathbf{k}\,(\omega_0 - 2cz - 2cdz) \qquad \ldots(8)$$

Hence, the motion at the point Q has the following three contributions:

(*i*) Translational velocity given by (3); (*ii*) Rotational velocity given by (4);

(*iii*) Rate of strain velocity given by (7).

Ex. 2. *What type of the motion do the following velocity components constitute ?*

$$u = a + by - cz, \qquad v = d - bx + ez, \qquad w = f + cx - ey.$$

where a, b, c, d, e,f are arbitrary constants. **(Garhwal 2002; Meerut 2006; Agra 2012)**

Sol. Let $\mathbf{q} = \mathbf{i}u + \mathbf{j}\,v + \mathbf{k}w$ be velocity at a point **P** and $\mathbf{q}' = \mathbf{q} + d\mathbf{q}$ be velocity at a neighbourig point Q. Then, we known that (Refer Art 13.17)

$$\mathbf{q}' = \mathbf{q} + \omega \times d\mathbf{r} + \mathbf{D} \qquad \ldots(1)$$

Given $$u = a + by - cz, \qquad v = d - bx + ez, \qquad w = f + cx - ey \qquad \ldots(2)$$

Then $$\omega = \frac{1}{2}\left[\mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]$$

$$= (1/2) \times [\mathbf{i}\,(-2e) + \mathbf{j}\,(-2c) + \mathbf{k}(-2b)] = -(e\mathbf{i} + c\mathbf{j} + b\mathbf{k})$$

$$\therefore \qquad \omega \times d\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -e & -c & -b \\ dx & dy & dz \end{vmatrix} = \mathbf{i}\,(bdy - cdz) + \mathbf{j}\,(edz - bdx) + \mathbf{k}\,(cdx - edy) \qquad \ldots(3)$$

Now, $$\in_{xx} = \partial u/\partial x = 0, \qquad \in_{yy} = \partial v/\partial y = 0 \qquad \in_{zz} = \partial w/\partial z = 0$$

$$\in_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}(b - b) = 0, \qquad \in_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}(e - e) = 0,$$

$$\in_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}(c - c) = 0$$

Hence, $\mathbf{D} = 0$, using result (4) of Art. 13.17

Hence the motion of fluid given by (2) consists of a translation with translatory velocity given by $\mathbf{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$ and a rotation with rotational velocity given by (3). Also the motion is free from deformation ($\because$ $\mathbf{D} = \mathbf{0}$). Hence the general motion of fluid element is made up of only two parts, namely, pure translation and pure rotation. Thus (2) represents a rigid-body motion.

Ex. 3. *Velocity field at point is given by 1 + 2y – 3z, 4 – 2x + 5z, 6 + 3x – 5y, Show that it represents a rigid body motion* **(Grawhal, 2003; Himanchal 2000, 03)**

Sol. Proceed just like Ex. 1. above.

Here $$u = 1 + 2y - 3z, \qquad v = 4 - 2x + 5z, \qquad w = 6 + 3x - 5y \qquad \ldots(1)$$

$$\omega = \frac{1}{2}\left[\mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]$$

$$= (1/2) \times [\mathbf{i}\,(-10) + \mathbf{j}\,(-6) + \mathbf{k}\,(-4)] = -(5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$$

$$\therefore \qquad \omega \times d\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -3 & -2 \\ dx & dy & dz \end{vmatrix} = \mathbf{i}\,(2dy - 3dz) + \mathbf{j}\,(5dz - 2dx) + \mathbf{k}\,(3dx - 5dy) \qquad \ldots(2)$$

Also, $\quad \in_{xx} = \partial u/\partial x = 0, \qquad \in_{yy} = \partial v/\partial y = 0, \qquad \in_{zz} = \partial w/\partial z = 0$

$$\left.\begin{aligned} \in_{xy} &= (1/2)\times(du/\partial y + \partial v/\partial x) = (1/2)\times(2-2) = 0 \\ \in_{yz} &= (1/2)\times(\partial v/\partial z + \partial w/\partial y) = (1/2)\times(5-5) = 0 \\ \in_{zx} &= (1/2)\times(\partial w/\partial x + \partial u/\partial z) = (1/2)\times(3-3) = 0 \end{aligned}\right\}$$

$\therefore \qquad$ **D** = **0**, using result (4) of Art. 13.17

Hence the motion of a fluid element is made up of only two parts, namely, pure translation and pure rotation (*i.e.* without any deformation). Hence the given distribution of velocity (1) represents a rigid body motion.

Ex. 4. *Let the new coordinate system (x', y') be obtained from the original coordinate system (x, y) by rotating through an angle of 45°. Verify the invariants of the rates of strain for a rectilinear flow with a linear profile, i.e., $u = ay$, $v = 0$.*

Sol. Given $\qquad u = ay \qquad$ and $\qquad v = 0 \qquad \ldots(1)$

The rates of strain with respect to the original coordinate system (x, y) are given by

$$\in_{xx} = \partial u/\partial x = 0, \qquad \in_{yy} = \partial v/\partial y = 0, \qquad \gamma_{xy} = \partial u/\partial y + \partial y/\partial x = a \qquad \ldots(2)$$

The direction cosines of OX, OY and OX', OY' are shown in the following table :

	OX	OY
OX′	$l_1 = \cos(\pi/4) = (1/\sqrt{2})$	$m_1 = \cos(\pi/2 - \pi/4) = (1/\sqrt{2})$
OY′	$l_2 = \cos(\pi/2 + \pi/4) = -(1/\sqrt{2})$	$m_2 = \cos(\pi/4) = (1/\sqrt{2})$

Now, $\qquad x = l_1 x' + l_2 y' = (x' - y')/\sqrt{2}, \qquad y = m_1 x' + m_2 y' = (x' + y')/\sqrt{2} \qquad \ldots(3)$

and $\qquad u' = l_1 u + m_1 v = ay/\sqrt{2}, \qquad v' = l_2 u + m_2 v = -ay/\sqrt{2} \qquad \ldots(4)$

Usinng (3) and (4), the rates of strain in the new coordinate system (x', y') are given by

$$\in_{x'x'} = \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'} = 0 + \frac{a}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{a}{2}$$

$$\in_{y'y'} = \frac{\partial v'}{\partial y'} = \frac{\partial v'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial y'} = 0 + \left(-\frac{a}{\sqrt{2}}\right) \times \frac{1}{\sqrt{2}} = -\frac{a}{2}$$

$$\in_{x'y'} = \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} = \left(\frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'}\right) + \left(\frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'}\right) = \left(0 + \frac{a}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) + \left(0 - \frac{a}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) = 0.$$

Here $\qquad \in_{xx} + \in_{yy} = 0 \qquad$ and $\qquad \in_{x'x'} + \in_{y'y'} = 0 \qquad \ldots(5)$

Also, $\quad \in_{xx}\in_{yy} - (1/4)\times\gamma_{xy}^2 = -(1/4)\times a^2, \qquad \in_{x'x'}\in_{y'y'} - (1/4)\times\gamma_{x'y'}^2 = -(1/4)\times a^2 \quad \ldots(6)$

From (5) and (6), we have

$$\in_{xx} + \in_{yy} = \in_{x'x'} + \in_{y'y'} \qquad \text{and} \qquad \in_{xx}\in_{yy} - (1/4)\times\gamma_{xy}^2 = \in_{x'x'}\in_{y'y'} - (1/4)\times\gamma_{x'y'}^2 \qquad \ldots(7)$$

This verifies the invariants of the rates of strain.

EXERCISE 11-(B)

1. Show that the vorticity is an invariant *i.e.* $\Omega_z = \partial v/\partial x - \partial u/\partial y = \partial v'/\partial x' - \partial u'/\partial y'$

2. Show that the following velociy components represent a rigid body motion : $u = a + by - cz, v = d - bx + ez, w = f + cx - ey,$ where a, b, c, d, e and f are arbitrary constants.

3. If a new coordinate system (x', y') is obtained from the original coordinate system (x, y) by a rotation through an angle of 30°, verify the following invariants of the rates of strain for the flow : $u = ay, v = 0.$

$$\epsilon_{x'x'} + \epsilon_{y'y'} = \epsilon_{xx} + \epsilon_{yy} \qquad \text{and} \qquad \epsilon_{x'x'}\epsilon_{y'y'} - (1/4)\times\gamma_{x'y'}^2 = \epsilon_{xx}\epsilon_{yy} - (1/4)\times\gamma_{xy}^2.$$

4. Determine the rates of strain and explain the nature of the rates of strain for following velociy components:

(*i*) $u = cx, v = cy, w = -2cz$ (*ii*) $u = c, v = w = 0$

(*iii*) $u = 2cy, v = w = 0$ (*iv*) $u = u\,(x, y), v = v\,(x, y), w = 0,$

where c is an arbitrary constant .

5. For a Newtonian isotopic and incompressible fluid, obtain a relation beween stress and strain rate tensors for three dimensional flow.

[**Hint :** Refer (32a) to (32f) of Art. 13.14. Since fluid is incompressible, $\nabla \cdot \mathbf{q} = 0$ and hence

$$\sigma_{xx} = 2\mu\,\epsilon_{xx} - p, \qquad \sigma_{yy} = 2\mu\,\epsilon_{yy} - p, \qquad \sigma_{zz} = 2\mu\,\epsilon_{zz} - p$$

$$\sigma_{xy} = 2\mu\,\epsilon_{xy}, \qquad \sigma_{yz} = 2\mu\,\epsilon_{yz}, \qquad \sigma_{zx} = 2\mu\,\epsilon_{zx}$$

These may be put in tensor form as follows :

$$\sigma_{ij} = 2\mu\,\epsilon_{ij} - p\delta_{ij}, \qquad \ldots(A)$$

where $\delta_{ij} = \text{Kronecker delta} = \begin{cases} 0 \text{ when } i \neq j \\ 1 \text{ when } i = j \end{cases}$ and $i, j = x, y, z.$

Equation (A) gives the desired result]

6. What is a constitutive equation ? Deduce the constitutive equation for Newtonian fluid. **(Meerut 1999)**

7. Define strain rate tensor e_{ij} at a point and find e_{xx} and e_{xy}. **(Meerut 2000)**

[**Hint :** In the present book e_{xx} and e_{xy} have been denoted by ϵ_{xx} and γ_{xy} respectively. Refer Art. 13.12]

8. Establish the relation between the stresses and the rate of strain in a viscous compressible fluid in motion.

9. Give the geometrical interpretation of translation, rotation and deformation of fluid element. If in a two-dimensional flow the velocity components are $u = x + y + 2t, v = 2y + t,$ then find the translation, rotation and deformation of fluid particles.

10. Define a constitutive equation. Obtain the constitutive equation for an isotopic Newton fluid. **(Himanchal 1999; 2003)**

11. Derive the relations between stress and rate of strain in the two-dimensional case. **(Garhwal 2005)**

12. Define principal stresses and determine first, second and third invariants in terms of principal stresses.

13. A fluid moving with velocity $V\,(axy^2, bx^2y, cxyz)$ where a, b, c are constants. Obtain translation, fluid angular velocity and deformation of fluid particles.

14. Explain Stokes hypothesis. **(Himanchal 2001, 02, 10)**

15. Establish a linear relation between the components of stresses and rates of strain for an isotropic fluid medium. **(Himanchal 1999, 2002)**

16. Define stress and rate of strain. Establish relation between stress and rate of strain components. Give interpretation of the constants involved. **(Himanchal 2001)**

17. Discuss the strain analysis for the fluid motion consisting of (*i*) tranlation (*ii*) rotation (*iii*) deformation. **(Himanchal 1999)**

18. Show that the two dimensions rates of strain are given by $\in_{xx} = \partial u / \partial x$, $\in_{yy} = \partial v / \partial y$, $\gamma_{xy} = \partial v / \partial x + \partial u / \partial y$ **(Allahabad 2003, Garwhal 19998, Lucknow 1997)**

19. Define rate of strain quadratic and discuss its properties. **(Himanchal 2005, 07, 10)**

20. Establish relation between stress and rate of strain components. **(Himanchal 2007, 10)**

21. Derive stress matrix in a fluid at rest. **(Himanchal 2007)**

22. Write a short note on Stokes law of friction. **(Himanchal 2006)**

23. Define σ_{ij}, stress quadratic at a point and discuss the properties of stress quadratic. **(Himanchal 2006)**

24. Define relation between stress and rate of strain components (law of friction). **(Himanchal 2007)**

25. The ratio of the change to the original length of a linear element is known as

(*a*) Normal stress (*b*) Normal strain

(*c*) Shear stress (*d*) None of these **(Agra 2005)**

[Hint: Ans (b). See Art 13.12.]

26. The stress invariants of principal stresses σ_1, σ_2 and σ_3 are given by

(*a*) $I_1 = \sigma_1 + \sigma_2 + \sigma_3$, $I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$, $I_3 = \sigma_1 - \sigma_2 - \sigma_3$

$\in_{x'x'} \in_{y'y'} -(1/4) \times y^2{}_{x'y'} = \in_{xx} \in_{yy} -(1/4) \times y^2{}_{xy}$

(*b*) $I_1 = \sigma_1 + \sigma_2 + \sigma_3$, $I_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1$, $I_3 = \sigma_1 \sigma_2 \sigma_3$

(*c*) $I_1 = \sigma_1(\sigma_2 + \sigma_3)$, $I_2 = \sigma_1 \sigma_3 - \sigma_2\sigma_3 - \sigma_1^2$, $I_3 = \sigma_1 \sigma_2 \sigma_3$

(*d*) None of these **(Agra 2005)**

[Hint: Ans. (*b*) See Art. 13.10)

27. If in two dimensions , x-axis and y-axis are rotated through an angle θ, then find the expressions for new rates of strain components in terms of old rates of strain components, and prove that $\in_{x'x'} + \in_{y'y'} = \in_{xx} + \in_{yy}$. **[Meerut 2009, 10]**

[Hint: Refer case I, Art. 13.13]

28. Obtain relations between stress and rates of strain for a compressible Newtonian viscous fluid in two dimensions. **[Meerut 2010]**

29. The dimensions of coefficient of viscosity is

(a) L^2/T (b) M/LT^2 (c) M/LT (d) ML/T **[Agra 2009, 10, 12]**

Sol. Ans. (c). Refer equation (5) of Art. 13. 2A.

30. When x-axis and y-axis are rotated through an angle θ in two-dimensions, then find the values of new rates of strain components in terms of old rates of strain components, and prove that

$$\in_{x'x'} \in_{y'y'} -(1/4) \times \gamma^2_{x'y'} = \in_{xx} \in_{yy} -(1/4) \times \gamma^2{}_{xy}.$$ **[Meerut 2012]**

Hint : Refer case I of Art 13.13, page 13.27.

31. Fill up gap : The unit of kinematic viscosity is ..... **[Agra 2011]**

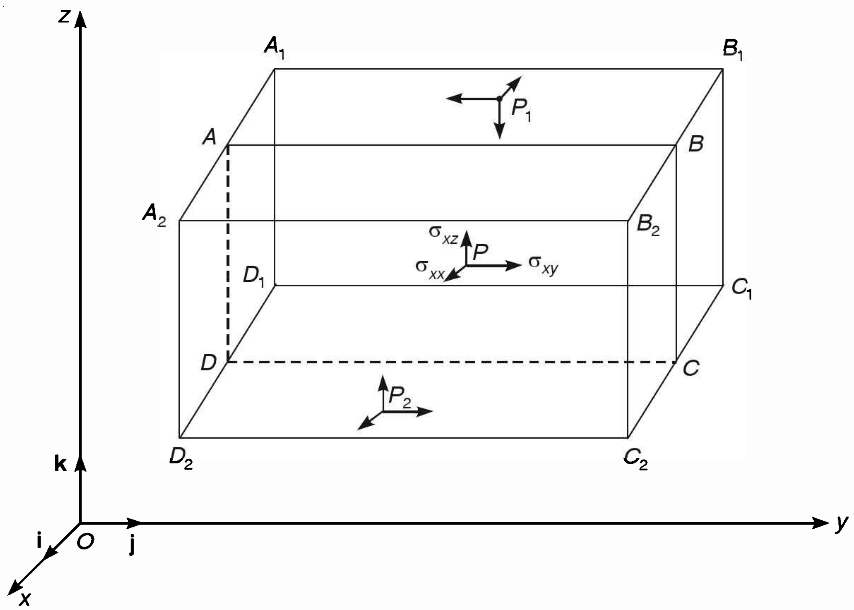
Hint : Refer page 13.2. In C.G.S. units, the unit of kinematic viscosity is stoke such that 1 stoke = cm^2/sec.

CHAPTER

14 The Navier-Stokes Equations And The Energy Equation

14.1. The Navier-Stokes equations of motion of a viscous fluid. (Agra 2005, 07, 08, 09, 10, Garhwal 2005; Himanchal 2000, 01, 02, 03, 09; Kanpur 2004, 09; Meerut 2000, 01, 08, 09, 10, 12)

With $P(x, y, z)$ as centre and edges of lengths δx, δy, δz parallel to fixed coordinate axes, construct an elementary rectangular parallelepiped as shown in the figure. We consider the motion of above mentioned parallelepiped of viscous fluid. We suppose that the element is moving with the fluid and mass $\rho\, \delta x \delta y \delta z$ of the fluid remains constant. Let coordinates of points P_1 and P_2 be $(x - \delta x/2, y, z)$ and $(x + \delta x/2, y, z)$ respectively.



At P, the force components parallel to OX, OY, OZ on the rectangular surface $ABCD$ of area $\delta y \delta z$ through P and having $\mathbf{i}$ as unit normal are

$$[\sigma_{xx}\delta y \delta z, \qquad \sigma_{xy}\delta y \delta z, \qquad \sigma_{xz}\delta y \delta z].$$

At P_2, since $\mathbf{i}$ is the unit normal measured outwards from the fluid, the corresponding force components on the rectangular surface $A_2B_2C_2D_2$ (parallel to $ABCD$) of area $\delta y \delta z$ are

$$\left[\left(\sigma_{xx} + \frac{\delta x}{2}\frac{\partial \sigma_{xx}}{\partial x}\right)\delta y \delta z, \qquad \left(\sigma_{xy} + \frac{\delta x}{2}\frac{\partial \sigma_{xy}}{\partial x}\right)\delta y \delta z, \qquad \left(\sigma_{xz} + \frac{\delta x}{2}\frac{\partial \sigma_{xz}}{\partial x}\right)\delta y \delta z\right] \quad ...(1)$$

At P_1, since $-\mathbf{i}$ is the unit normal measured outwards from the fluid, the corresponding force components on the rectangular surface $A_1B_1C_1D_1$ (parallel to $ABCD$) of area $\delta y\delta z$ are.

$$\left[-\left(\sigma_{xx}-\frac{\delta x}{2}\frac{\partial\sigma_{xx}}{\partial x}\right)\delta y\delta z,\qquad -\left(\sigma_{xy}-\frac{\delta x}{2}\frac{\partial\sigma_{xy}}{\partial x}\right)\delta y\delta z,\qquad -\left(\sigma_{xz}-\frac{\delta x}{2}\frac{\partial\sigma_{xz}}{\partial x}\right)\delta y\delta z\right] \quad ...(2)$$

Hence the forces on the parallel planes $A_2B_2C_2D_2$ and $A_1B_1C_1D_1$ passing through P_1 and P_2 are equivalent to a single force at P with components

$$\left[\frac{\partial\sigma_{xx}}{\partial x}\delta x\delta y\delta z,\qquad \frac{\partial\sigma_{xy}}{\partial x}\delta x\delta y\delta z,\qquad \frac{\partial\sigma_{xz}}{\partial x}\delta x\delta y\delta z\right] \quad ...(3)$$

together with couples whose moments*. (to the third order of smallness) are

$$-\sigma_{xz}\delta x\delta y\delta z \text{ about } OY \qquad \text{and} \qquad \sigma_{xy}\delta x\delta y\delta z \text{ about } OZ. \quad ...(4)$$

Similarly, the forces on the parallel planes perpendicular to the y-axis are equivalent to a single force at P with components

$$\left[\frac{\partial\sigma_{yx}}{\partial y}\delta x\delta y\delta z,\qquad \frac{\partial\sigma_{yy}}{\partial y}\delta x\delta y\delta z,\qquad \frac{\partial\sigma_{yz}}{\partial y}\delta x\delta y\delta z\right] \quad ...(5)$$

together with couples whose moments are

$$-\sigma_{yx}\delta x\delta y\delta z \text{ about } OZ \qquad \text{and} \qquad \sigma_{yz}\delta x\delta y\delta z \text{ about } OX. \quad ...(6)$$

Again, the forces on the parallel planes perpendicular to the z-axis are equivalent to a single force at P with components

$$\left[\frac{\partial\sigma_{zx}}{\partial z}\delta x\delta y\delta z,\qquad \frac{\partial\sigma_{zy}}{\partial z}\delta x\delta y\delta z,\qquad \frac{\partial\sigma_{zz}}{\partial z}\delta x\delta y\delta z\right] \quad ...(7)$$

together with couples whose moments are

$$-\sigma_{zy}\delta x\delta y\delta z \text{ about } OX \qquad \text{and} \qquad \sigma_{zx}\delta x\delta y\delta z \text{ about } OY. \quad ...(8)$$

Thus, the surface forces on all the six faces of the rectangular parallelepiped ($A_1B_1C_1D_1$, $A_2B_2C_2D_2$) are equivalent to a single force at P having components

$$\left[\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}\right)\delta x\delta y\delta z,\left(\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}\right)\delta x\delta y\delta z,\left(\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}\right)\delta x\delta y\delta z\right] \quad ...(9)$$

together with a vector couple having components

$$\left[(\sigma_{yz}-\sigma_{zy})\,\delta x\delta y\delta z,\qquad (\sigma_{zx}-\sigma_{xz})\,\delta x\delta y\delta z,\qquad (\sigma_{xy}-\sigma_{yx})\,\delta x\delta y\delta z\right]. \quad ...(10)$$

Let $\quad \mathbf{q}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k} \quad$ and $\quad \mathbf{B}=B_x\,\mathbf{i}+B_y\,\mathbf{j}+B_x\,\mathbf{k} \quad ...(11)$

be the velocity of the fluid at $P(x, y, z)$ at any time t and external body force at P per unit mass respectively.

Clearly the total body force on the elementary rectangular parallelepiped has components

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* **Convention of sign of a couple.** If a couple in the plane XOY causes rotation from OX towards OY, then it shall be represented by a positive length along OZ. Similarly, a couple in the plane YOZ which would cause rotation from OY towards OZ will be represented by a positive length along OX and a couple in the plane ZOX causing rotation from OZ towards OX will be represented by a positive length along OY.

$$(B_x\ \rho\ \delta x\delta y\delta z, \qquad B_y\ \rho\ \delta x\delta y\delta z, \qquad B_z\ \rho\ \delta x\delta y\delta z).$$

Taking account of surface forces and body forces, we find that the total force component in the **i**-direction on the element of fluid under consideration is

$$\left(\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}\right)\delta x\delta y\delta z + B_x\ \rho\ \delta x\delta y\delta z.$$

Since the mass $\rho\ \delta x\delta y\delta z$ of the element is treated to be constant, the equation of motion of the element in the **i**-direction (*i.e.* *OX*) is

$$(\rho\ \delta x\delta y\delta z)\frac{Du}{Dt} = \left(\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}\right)\delta x\delta y\delta z + B_x\ \rho\ \delta x\delta y\delta z$$

or
$$\rho\frac{Du}{Dt} = \rho\, B_x + \frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{xz}}{\partial z}.$$

Thus by cyclic permutation we obtain three equations of motion in the **i**, **j**, **k** directions (*i.e.* *OX*, *OY*, *OZ*) :

$$\rho\frac{Du}{Dt} = \rho\, B_x + \frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{xz}}{\partial z} \qquad \text{...(12}a\text{)}$$

$$\rho\frac{Dv}{Dt} = \rho\, B_y + \frac{\partial \sigma_{yx}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{yz}}{\partial z} \qquad \text{...(12}b\text{)}$$

$$\rho\frac{Dw}{Dt} = \rho\, B_z + \frac{\partial \sigma_{zx}}{\partial x}+\frac{\partial \sigma_{zy}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z} \qquad \text{...(12}c\text{)}$$

The constitutive equations for a Newtonian (viscous) compressible fluid are given by [refer equation (32*a*) to (32*f*) in Art. 13.14]

$$\left.\begin{aligned}
\sigma_{xx} &= 2\mu(\partial u/\partial x) - (2\mu/3)\times\nabla\cdot\mathbf{q} - p\\
\sigma_{yy} &= 2\mu(\partial v/\partial y) - (2\mu/3)\times\nabla\cdot\mathbf{q} - p\\
\sigma_{zz} &= 2\mu(\partial w/\partial z) - (2\mu/3)\times\nabla\cdot\mathbf{q} - p\\
\sigma_{xy} &= \sigma_{yx} = \mu\left(\partial u/\partial y + \partial v/\partial x\right)\\
\sigma_{yz} &= \sigma_{zy} = \mu\left(\partial v/\partial z + \partial w/\partial y\right)\\
\sigma_{zx} &= \sigma_{xz} = \mu\left(\partial w/\partial x + \partial u/\partial z\right)
\end{aligned}\right\} \qquad \text{...(13)}$$

Using (13), equations (12*a*) to (12*c*) may be expressed in terms of the velocity derivatives as follows :

$$\rho\frac{Du}{Dt} = \rho\, B_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left\{2\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla\cdot\mathbf{q})\right\}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right] \qquad \text{...(14}a\text{)}$$

$$\rho\frac{Dv}{Dt} = \rho\, B_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left\{2\frac{\partial v}{\partial y} - \frac{2}{3}(\nabla\cdot\mathbf{q})\right\}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] \qquad \text{...(14}b\text{)}$$

$$\rho\frac{Dw}{Dt} = \rho\, B_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left\{2\frac{\partial w}{\partial z} - \frac{2}{3}(\nabla\cdot\mathbf{q})\right\}\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right] \qquad \text{...(14}c\text{)}$$

The above three equations are called the *Navier-Stokes equations of motion for a viscous compressible fluid in cartesian coordinates.*

Particular Case: Incompressible viscous fluid flow. **(Himanchal 2007)**

The above system of equation (14*a*), (14*b*) and (14*c*) become further simplified in the case of incompressible fluids (ρ = constant) even if the temperature in not constant. First, as already shown in Art.2.8 we have $\nabla \cdot \mathbf{q} = 0$. Secondly, since temperature variation are, generally, speaking, small in this case, the viscosity may be taken to be constant. Writing the acceteration terms in full, the equations of motion for incompressible flow are

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho\, B_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad ...(14a')$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho\, B_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad ...(14b')$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho\, B_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \qquad ...(14c')$$

Deduction of equations of motion for some particular cases :

In what follows, we shall use the following result :

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\mathbf{q}\cdot\nabla=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z} \qquad ...(15)$$

(*i*) Viscous compressible fluid with constant viscosity (Vector form) (Meerut 2000)

Let the coefficient of viscosity, μ, be constant. Then equations (14*a*) to (14*c*) may be expressed in vector form

$$\rho[\partial \mathbf{q}/dt+(\mathbf{q}\cdot\nabla)\mathbf{q}]=\rho\mathbf{B}-\nabla p+\mu\nabla^2\mathbf{q}+(\mu/3)\times\nabla(\nabla\cdot\mathbf{q}) \qquad ...(16)$$

Now, $\mathbf{q}\times(\nabla\times\mathbf{q})=\nabla(\mathbf{q}\cdot\mathbf{q})-(\mathbf{q}\cdot\nabla)\mathbf{q}$ or $\mathbf{q}\times(\nabla\times\mathbf{q})=\nabla(\mathbf{q}^2/2)-(\mathbf{q}\cdot\nabla)\mathbf{q}$

$\therefore$
$$(\mathbf{q}\cdot\nabla)\mathbf{q}=\nabla(\mathbf{q}^2/2)-\mathbf{q}\times(\nabla\times\mathbf{q}) \qquad ...(i)$$

Again, $\nabla\times(\nabla\times\mathbf{q})=\nabla(\nabla\cdot\mathbf{q})-(\nabla\cdot\nabla)\mathbf{q}$ or $\nabla\times(\nabla\times\mathbf{q})=\nabla(\nabla\cdot\mathbf{q})-\nabla^2\mathbf{q}$

$\therefore$
$$\nabla^2\mathbf{q}=\nabla(\nabla\cdot\mathbf{q})=-\nabla\times(\nabla\times\mathbf{q}) \qquad ...(ii)$$

Using (*i*) and (*ii*), (16) may be re-written as

$$\rho\left[\frac{\partial\mathbf{q}}{\partial t}+\nabla\left(\frac{1}{2}\mathbf{q}^2\right)-\mathbf{q}\times(\nabla\times\mathbf{q})\right]=\rho\mathbf{B}-\nabla p+\mu[\nabla(\nabla\cdot\mathbf{q})-\nabla\times(\nabla\times\mathbf{q})]+\frac{\mu}{3}\nabla(\nabla\cdot\mathbf{q})$$

or
$$\rho\left[\frac{\partial\mathbf{q}}{\partial t}+\nabla\left(\frac{1}{2}\mathbf{q}^2\right)-\mathbf{q}\times(\nabla\times\mathbf{q})\right]=\rho\mathbf{B}-\nabla p+\frac{4}{3}\mu\nabla(\nabla\cdot\mathbf{q})-\mu\nabla\times(\nabla\times\mathbf{q}). \qquad ...(16)'$$

(*ii*) Viscous incompressible fluid with constant viscosity. Let ρ and μ be constants for the given incompressible fluid. Further, for such a fluid $\nabla\cdot\mathbf{q}=0$. If $\nu=\mu/\rho$ be the kinematic viscosity, then (16) reduces to

$$\partial\mathbf{q}/\partial t+(\mathbf{q}\cdot\nabla)\mathbf{q}=\mathbf{B}-(1/\rho)\times\nabla p+\nu\nabla^2\mathbf{q}. \qquad ...(17)$$

(*iii*) Non-viscous incompressible fluid. For such fluid $\mu = 0$ and hence (16) further reduces to

$$\partial\mathbf{q}/\partial t+(\mathbf{q}\cdot\nabla)\mathbf{q}=\mathbf{B}-(1/\rho)\times\nabla p, \qquad ...(18)$$

which is the well known Euler's equation* Note that (18) is valid for both incompressible flows and compressible flows. For incompressible flows ρ is constant while for compressible flows, ρ is usually a function of both pressure and temperature.

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* It was obtained independently in Art. 3.1 of chapter 3. Refer equation (7) of that article.

For viscous incompressible fluid, $\nabla \cdot \mathbf{q} = 0$ and so (16)' reduces to

$$\rho[\partial \mathbf{q}/\partial t + (\mathbf{q} \cdot \nabla)\mathbf{q}] = \rho \mathbf{B} - \nabla p - \mu \nabla \times (\nabla \times \mathbf{q}). \qquad \text{...(18a)}$$

where we have also used relation (*i*).

Comparing the above equation (18*a*) with (18), we find that for incompressible flow the equation of motion differs from Euler's equation of motion for non-viscous flow by the term $-\mu \nabla \times (\nabla \times \mathbf{q})$. This term, due to viscosity, increases the complexity by increasing the order of the differential equation of the motion. Hence an additional boundary condition is required. This is provided by the condition that there must be no slip between a viscous fluid and its boundary. It follows that we cannot arrive at the solution of the corresponding non-viscous flow problem by solving (18*a*) and then letting $\mu \to 0$.

(*iv*) **Plane two-dimensional flow of incompressible viscous fluid.**

Here we have $w = 0$ and $\partial/\partial z = 0$. Then (14*a*) to (14*c*) reduce to

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho B_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad \text{...(19a)}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho B_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad \text{...(19a)}$$

and

$$0 = \rho B_z. \qquad \text{...(19c)}$$

14.2. The energy equation-conservation of energy. [Himanchal 1999, 2000, 03, 06]

Consider motion of a viscous compressible (Newtonian) fluid. We propose to consider conservation of energy on the basis of the first law of thermodynamics. According to this law the total energy added to the system (both by heat and by work done on the fluid) increases the internal energy per unit mass of the fluid. Let Q be the heat added per unit mass of fluid through conduction and E be internal energy per unit mass of fluid. Then the rate of work done W by the normal and shearing stresses on a unit volume of the fluid is given by

$$W = \sigma_{xx} \in_{xx} + \sigma_{yy} \in_{yy} + \sigma_{zz} \in_{zz} + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{zx}\gamma_{zx} \qquad \text{...(1)}$$

Then the first law of thermodynamics (in terms of variation of energy) may be re-written as

$$\rho(dQ/dt) + W = \rho(dE/dt). \qquad \text{...(2)}$$

The relations between the stresses and the rates of strain (constitutive equations) are given by [see (32*a*) to (32 *f*) in Art. 13.14]

$$\left.\begin{aligned} \sigma_{xx} &= 2\mu \in_{xx} -(2/3)\times \mu \nabla \cdot \mathbf{q} - p \\ \sigma_{yy} &= 2\mu \in_{yy} -(2/3)\times \mu \nabla \cdot \mathbf{q} - p \\ \sigma_{zz} &= 2\mu \in_{zz} -(2/3)\times \mu \nabla \cdot \mathbf{q} - p \end{aligned}\right\} \qquad \text{...(3a)}$$

and

$$\sigma_{xy} = \mu\,\gamma_{xy}, \qquad \sigma_{yz} = \mu\,\gamma_{yz}, \qquad \sigma_{zx} = \mu\,\gamma_{zx} \qquad \text{...(3b)}$$

Also

$$\in_{xx} = \partial u/\partial x, \qquad \in_{yy} = \partial v/\partial y, \qquad \in_{zz} = \partial w/\partial z \qquad \text{...(4a)}$$

and

$$\left.\begin{aligned} \gamma_{xy} &= \partial u/\partial y + \partial v/\partial x \\ \gamma_{yz} &= \partial v/\partial z + \partial w/\partial y \\ \gamma_{zx} &= \partial w/\partial x + \partial u/\partial z \end{aligned}\right\} \qquad \text{...(4b)}$$

Using results (3*a*) and (3*b*), (1) reduces to

$$W = \{2\mu \in_{xx} - (2/3)\times\mu\nabla\cdot\mathbf{q} - p\}\in_{xx} \text{ + two similar terms + } \mu\gamma_{xy}\cdot\gamma_{xy} \text{ + two similar terms}$$

$$= 2\mu(\in_{xx}^2 + \in_{yy}^2 + \in_{zz}^2) - (2/3)\times\mu(\nabla\cdot\mathbf{q})(\in_{xx} + \in_{yy} + \in_{zz}) - p(\in_{xx} + \in_{yy} + \in_{zz}) + \mu(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)$$

$$= -p(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z) + \mu[2\{(\partial u/\partial x)^2 + (\partial v/\partial y)^2 + (\partial w/\partial z)^2\}$$

$$-(2/3)\times(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z)^2 + (\partial u/\partial y + \partial v/\partial x)^2 + (\partial v/\partial z + \partial w/\partial y)^2 + (\partial w/\partial x + \partial u/\partial z)^2\}$$

[Using 4 (*a*) and 4 (*b*) and the fact that $\nabla\cdot\mathbf{q} = \left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}\right)\cdot(\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}$]

Thus, $$W = -p\nabla\cdot\mathbf{q} + \mathbf{\Phi}, \qquad \text{...(5)}$$

where $\mathbf{\Phi}$ denotes the *dissipation function* and it represents the time rate at which energy is being dissipated per unit volume through the action of viscosity. Hence we have

$$\mathbf{\Phi} = \mu[\,2\{(\partial u/\partial x)^2 + (\partial v/\partial y)^2 + (\partial w/\partial z)^2\} - (2/3)\times(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z)^2$$

$$+(\partial u/\partial y + \partial v/\partial x)^2 + (\partial v/\partial z + \partial w/\partial y)^2 + (\partial w/\partial x + \partial u/\partial z)^2\,] \qquad \text{... (6)}$$

Using (5), (2) reduces to $$\rho(dQ/dt) + \mathbf{\Phi} = \rho(dE/dt) + p\nabla\cdot\mathbf{q} \qquad \text{...(7)}$$

The equation of continuity, for compressible (viscous) fluid is given by [Refer Equation (5) in Art. 2.7]

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{q} = 0 \qquad \text{so that} \qquad \frac{p}{\rho}\nabla\cdot\mathbf{q} = -\frac{p}{\rho^2}\frac{D\rho}{Dt} \qquad \text{...(8)}$$

Now, $$\frac{D}{Dt}\left(\frac{p}{\rho}\right) = \frac{1}{\rho}\frac{Dp}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}, \qquad \text{so that} \qquad -\frac{p}{\rho^2}\frac{D\rho}{Dt} = \frac{D}{Dt}\left(\frac{p}{\rho}\right) - \frac{1}{\rho}\frac{Dp}{Dt} \qquad \text{...(9)}$$

From (8) and (9), we have

$$\frac{p}{\rho}\nabla\cdot\mathbf{q} = \frac{D}{Dt}\left(\frac{p}{\rho}\right) - \frac{1}{\rho}\frac{Dp}{Dt} \qquad \text{or} \qquad p\nabla\cdot\mathbf{q} = \rho\frac{D}{Dt}\left(\frac{p}{\rho}\right) - \frac{Dp}{Dt} \qquad \text{...(10)}$$

Using (10), (7) reduces to

$$\rho\frac{dQ}{dt} + \mathbf{\Phi} = \rho\frac{dE}{dt} + \rho\frac{D}{Dt}\left(\frac{p}{\rho}\right) - \frac{Dp}{Dt} = \rho\frac{D}{Dt}\left(E + \frac{p}{\rho}\right) - \frac{Dp}{Dt} = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt}, \qquad \text{...(11)}$$

where $h = E + p/\rho$ is the *enthalpy** of the fluid per unit mass.

We now evaluate Q. According to the *Fourier's heat-conduction law, heat* flux f crossing an area (*i.e.*, quantity of heat per unit time) is proportional to the temperature gradient along the surface. Hence, $$f = -k(\partial T/\partial n),$$

where k is the thermal conductivity of the fluid, and the negative sign signifies that the direction of the flux is opposite to that of the temperature gradient.

Refer figure of Art 2.9, Chapter 2. Let there be a fluid particle at $P(x, y, z)$. Let T and ρ be the temperature and density of the fluid at P. Construct a small parallelepiped with edges of length parallel to their respective coordinate axes, having P at one of the angular points as shown in the figure just referred. Then we have

The heat flow through the face $PQRS$ per unit time $= -k(\partial T/\partial x)\delta y\delta z = f(x, y, z)$, ...(12)

* It is also known as the total heat content (heat introduced into the system).

$\therefore$ The heat flow through the opposite face $P'Q'R'S'$ per unit time

$$= f(x+\delta x, y, z) = f(x,y,z) + \delta x \frac{\partial}{\partial x} f(x,y,z) + \ldots, \text{ by Taylor's theorem} \qquad \ldots(13)$$

Hence the net gain in energy per unit time within the fluid element in the x-direction (due to flow through faces $PQRS$ and $P'Q'R'S'$) from (12) and (13) is

$$= f(x,y,z) - \left[f(x,y,z) + \delta x \frac{\partial}{\partial x} f(x,y,z) + \ldots \right] = -\delta x \frac{\partial}{\partial x} f(x,y,z)$$

[to the first order of approximation]

$$= -\delta x \frac{\partial}{\partial x}\left(-k \delta y \delta z \frac{\partial T}{\partial x} \right) = \delta x \delta y \delta z \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right)$$

Similarly, the net gains in energy per unit time within the fluid element in y- and z-directions are given by $\delta x \delta y \delta z \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} \right)$ and $\delta x \delta y \delta z \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z} \right)$ respectively.

Hence the total quantity of heat introduced in the fluid element during time δt is

$$\delta t \delta x \delta y \delta z \left[\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z} \right) \right].$$

Hence the rate of heat added by conduction per unit volume is given by

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z} \right) \qquad \ldots(14a)$$

i.e.

$$\nabla \cdot (k \nabla T) \qquad \ldots(14b)$$

Thus,

$$\rho(dQ/dt) = \nabla \cdot (k \nabla T) \qquad \ldots(15)$$

Using (15) and assuming that there is no direct heating from chemical reaction and radiation heating, the required energy equation from (11) is given by

$$\nabla \cdot (k \nabla T) + \mathbf{\Phi} = \rho(Dh/Dt) - (Dp/Dt). \qquad \ldots(16)$$

In cartesian coordinates the energy equation for viscous compressible fluid reduces to

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z} \right) + \mathbf{\Phi} = \rho \frac{D(C_p T)}{Dt} - \frac{Dp}{Dt}, \qquad \ldots(17)$$

where $k = c_p\, T$ and c_p is specific heat at constant pressure. Using the kinetic theory of gases together with experiments, μ and k are found to be functions of the temperature only for gases having ordinary densities.

Energy equation for special cases :

(*i*) **Viscous incompressible fluid.** **[Himanchal 2003, 09]**

When the fluid is taken as incompressible viscous fluid, then k = constant and μ = constant. Further-more, the equation of continuity for such a fluid is given by

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0. \qquad \ldots(18)$$

Hence the dissipation function $\mathbf{\Phi}'$ for the present problem is given by [on using (6)]

$$\mathbf{\Phi}' = \mu \left[2\left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right\} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \right] \qquad \ldots(19)$$

If c_v be the specific heat at constant volume, then $c_p = c_v = c$ for an incompressible fluid. Here c is the specific heat of the fluid. With the above mentioned discussion, the energy equation

(17) assumes the form

$$k(\partial^2 T/dx^2 + \partial^2 T/\partial y^2 + \partial^2 T/\partial z^2) + \mathbf{\Phi}' = \rho c(DT/Dt) - (Dp/Dt) \qquad \text{...(20)}$$

(*ii*) **Non-viscous fluid.** Since $\mu = 0$ for such fluids, $\mathbf{\Phi} = 0$ by (6). Then (17) yields

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = \rho\frac{D(C_p T)}{Dt} - \frac{Dp}{Dt} \qquad \text{...(21)}$$

(*iii*) **Non-viscous incompressible fluid.** As before k = constant and $\mu = 0$ (hence $\mathbf{\Phi} = 0$). Also $c_p = c_v = c$. Hence the energy equation (21) assumes the following form

$$k(\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2 + \partial^2 T/\partial z^2) = \rho c\,(DT/Dt) - (Dp/Dt) \qquad \text{...(22)}$$

14.3. Equation of state for perfect fluid.

The equation of state of a substance is a relation between its pressure, temperature and specific volume. There exist an equation of state corresponding to a given homogeneous substance, solid, liquid or gas. The relationship may be expresses as

$$f(\rho, p, T) = 0, \qquad \text{...(1)}$$

which is known as the *equation of state*. The exact nature of the function f is, in general, very complicated and varies from fluid to fluid. However, for a perfect gas or an ideal gas the equation of state is given by

$$p = \rho R T \qquad \text{...(2)}$$

or

$$p = (c_p - c_v)\,\rho T, \qquad \text{...(3)}$$

where R is called the *gas constant* and c_p and c_v are specific heats at constant pressure and volume respectively. Relation (2) is also known as *Boyle's law*.

14.4. Diffusion of vorticity. [Agra 2005, 06; Kolkata 2006; Himanchal 2001]

The Navier-stokes equation for viscous incompressible fluid is given by

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{q}^2\right) - \mathbf{q}\times\Omega = \mathbf{B} - \nabla\int\frac{dp}{\rho} + \nu\nabla^2\mathbf{q} \qquad \text{...(1)}$$

where

$$\Omega = \text{vorticity vector} = \nabla\times\mathbf{q}. \qquad \text{...(2)}$$

Let the body forces be conservative so that $\nabla\times\mathbf{B} = \mathbf{0}$. ...(3)

On taking the curl of both sides of (1) and using (3), we obtain

$$\nabla\times(\partial\mathbf{q}/\partial t) - \nabla\times(\mathbf{q}\times\Omega) = \nu\nabla\times(\nabla^2\mathbf{q})$$

or

$$(\partial/\partial t)\,(\nabla\times\mathbf{q}) - [(\Omega\cdot\nabla)\,\mathbf{q} - (\mathbf{q}\cdot\nabla)\,\Omega] = \nu\nabla^2(\nabla\times\mathbf{q})$$

or

$$\partial\Omega/\partial t + (\mathbf{q}\cdot\nabla)\,\Omega = (\Omega\cdot\nabla)\,\mathbf{q} + \nu\,(\nabla^2\,\Omega), \text{ using (2)}$$

or

$$D\Omega/Dt = (\Omega\cdot\nabla)\,\mathbf{q} + \nu\nabla^2\Omega, \qquad \text{...(4)}$$

which is known as *vorticity transport equation*.

The first term on *R.H.S* of (4) represents the rate at which Ω varies for a given particle when the vortex lines move with the fluid, strengths of the vortices remaining constant. Since this term is negligible for slow motion, approximate form of (4) is

$$D\Omega/Dt = \nu\nabla^2\Omega. \qquad \text{...(5)}$$

In the special case of two-dimensional flow, with reference to fixes axes, we have

$$\mathbf{q} = u(x, y)\,\mathbf{i} + v(x, y)\,\mathbf{j}$$

Then $\Omega = \nabla\times\mathbf{q} = (\partial v/\partial x - \partial u/\partial y)\mathbf{k}$ and $(\Omega\cdot\nabla)\,\mathbf{q} = (\partial v/dx - \partial u/\partial y)(d\mathbf{q}/\partial z) = \mathbf{0}$, showing that (4) reduces to (5) for a two dimensional case.

It follows that for slow three-dimensional motion, or for two-dimensional motion, (3) describes the manner in which vorticity is transmitted throughout a viscous fluid.

Remark Equation (5) is of the same form as the equation of heat conduction in a liquid. Hence vorticity diffuses through a liquid in almost the same way as heat does. By analogy it follows that vorticity cannot be generated within the interior of a viscous fluid. In fact it is transmitted from the boundaries into the fluid. As an example, a sailing ship will generate vortices in its wake, arising from the hull which is a moving boundary. As time passes, the disturbance is soon damped out as the vortices diffuse through the water.

14.5. Equations for vorticity and circulation. To prove that $d\Gamma/dt = \nu\nabla^2\Gamma$

(Himanchal 2003, 09; Meerut 1998)

The equations of motion for viscous fluid are given by

$$\left.\begin{aligned}\frac{Du}{Dt} &= -\frac{\partial Q}{\partial x} + \frac{1}{3}\nu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \nu\nabla^2 u \\ \frac{Dv}{Dt} &= -\frac{\partial Q}{\partial y} + \frac{1}{3}\nu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \nu\nabla^2 v \\ \frac{Dw}{Dt} &= -\frac{\partial Q}{\partial z} + \frac{1}{3}\nu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \nu\nabla^2 w\end{aligned}\right\} \quad \text{...(1)}$$

where $$Q = V + \int \frac{dp}{\rho}.$$

The above equations (1) can also be re-written as

$$Du/Dt - 2(v\xi - w\eta) = -(\partial\chi/\partial x) + \nu\,\nabla^2 u \quad \text{...(2)}$$

$$Dv/Dt - 2(w\xi - u\zeta) = -(\partial\chi/\partial y) + \nu\,\nabla^2 v \quad \text{...(3)}$$

$$Dw/Dt - 2(u\eta - v\xi) = -(\partial\chi/\partial z) + \nu\,\nabla^2 w \quad \text{...(4)}$$

where $$\chi = p/\rho + q^2/2 + V.$$

Differentiating (3) and (4) partially *w.r.t.*'z', and 'y' respectively and subtracting, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - 2\left[\frac{\partial}{\partial y}(u\eta - v\xi) - \frac{\partial}{\partial z}(w\xi - u\zeta)\right] = \nu\nabla^2\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

or $$\frac{\partial\xi}{\partial t} - \left[\eta\frac{\partial u}{\partial y} + u\frac{\partial\eta}{\partial y} - \xi\frac{\partial v}{\partial y} - v\frac{\partial\xi}{\partial y} - \xi\frac{\partial w}{\partial z} - w\frac{\partial\xi}{\partial z} + \zeta\frac{\partial u}{\partial z} + u\frac{\partial\zeta}{\partial z}\right] = \nu\nabla^2\xi$$

or $$\frac{\partial\xi}{\partial t} + \xi\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \eta\frac{\partial u}{\partial y} - \zeta\frac{\partial u}{\partial z} - u\left(\frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z}\right) + v\frac{\partial\xi}{\partial y} + w\frac{\partial\xi}{\partial z} = \nu\nabla^2\xi$$

or $$\frac{\partial\xi}{\partial t} + u\frac{\partial\xi}{\partial x} + v\frac{\partial\xi}{\partial y} + w\frac{\partial\xi}{\partial z} - \xi\frac{\partial u}{\partial x} - \eta\frac{\partial u}{\partial y} - \zeta\frac{\partial u}{\partial z} = \nu\nabla^2\xi$$

or $$\frac{D\xi}{Dt} = \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} + \nu\,\nabla^2\xi \quad \text{...(5)}$$

Similarly, $$\frac{D\eta}{Dt} = \xi\frac{\partial v}{\partial x} + \eta\frac{\partial v}{\partial y} + \zeta\frac{\partial v}{\partial z} + \nu\,\nabla^2\eta \quad \text{...(6)}$$

$$\frac{D\zeta}{Dt} = \xi\frac{\partial w}{\partial x} + \eta\frac{\partial w}{\partial y} + \zeta\frac{\partial w}{\partial z} + \nu\,\nabla^2\zeta \qquad ...(7)$$

The first three terms on the R.H.S. in equations (5), (6) and (7) represent the rates at which ξ, η, ζ vary for a given particle, when the vortex lines move with the fluid and their strengths remain constant. When the motion is very slow, these terms can be neglected and the remaining terms give the variations of vorticity. Since the resulting equations are the same in form as standard equation of conduction of heat, hence as in conduction of heat, we can say that vortex-motion cannot originate in the interior of a viscous liquid but must be diffused inwards from the boundary.

Let Γ be the circulation round a closed circuit moving with the fluid. Then, we have

$$\Gamma = \int_C (udx + vdy + wdz)$$

$$\therefore \qquad \frac{D\Gamma}{Dt} = \int_C \left(\frac{Du}{Dt}dx + \frac{Dv}{Dt}dy + \frac{Dw}{Dt}dz\right) + \int_C (u\,du + v\,dv + w\,dw) \qquad ...(8)$$

Since circulation is taken round a closed circuit, the second integral on R.H.S. in (8) is zero

$\therefore$ Hence,
$$\frac{D\Gamma}{Dt} = \int_C \left(\frac{Du}{Dt}dx + \frac{Dv}{Dt}dy + \frac{Dw}{Dt}dz\right) \qquad ...(9)$$

Now, from equation (1), $\qquad D\mathbf{q}/Dt = -\nabla(p/\rho + V) + \nu\,\nabla^2\mathbf{q} \qquad ...(10)$

From (9) and (10), we get

$$\frac{D\Gamma}{Dt} = \int_C \left\{-\frac{\partial}{\partial x}\left(\frac{p}{\rho} + V\right)dx - \frac{\partial}{\partial y}\left(\frac{p}{\rho} + V\right)dy - \frac{\partial}{\partial z}\left(\frac{p}{\rho} + V\right)dz + \nu\,(\nabla^2 u\,dx + \nabla^2 v\,dy + \nabla^2 w\,dz)\right]$$

or
$$\frac{D\Gamma}{Dt} = -\int_C d\left(\frac{p}{\rho} + V\right) + \nu\int_C \nabla^2 (udx + vdy + wdz)$$

$$D\Gamma/Dt = \nu\,\nabla^2\Gamma. \qquad [\because \text{ the first integral is zero for a closed circuit}]$$

14.6A. Dissipation of energy. definition.

Dissipation of energy is that energy which is dissipated in a viscous liquid in motion on account of the internal friction.

Determination of the rate of dissipation of energy of a fluid due to viscosity.

[Agra 2007; Meerut 2005]

Suppose we follow a particle of viscous incompressible fluid of fixed mass $\rho\,\delta V$ and moving with velocity $\mathbf{q}$ at any time t. Then its kinetic energy is $(1/2) \times (\rho\delta V)\mathbf{q}^2$. Hence the rate of gain of kinetic energy at time t as we follow the particle is given by

$$\frac{D}{Dt}\left(\frac{1}{2}\rho\,\delta V\,\mathbf{q}^2\right) = \rho\delta V\,\mathbf{q}\cdot\frac{D\mathbf{q}}{Dt}.$$

Let the total volume be V and S be the total surface enclosing the volume V.

Hence the total rate of gain of kinetic energy *dT/dt*. (say), of the total volume V, is given by

$$\frac{dT}{dt} = \int_V \rho\,\mathbf{q}\cdot\frac{D\mathbf{q}}{Dt}dV = \rho\int_V \mathbf{q}\cdot\frac{D\mathbf{q}}{Dt}dV. \qquad ...(1)$$

The Navier-Stokes equation for viscous incompressible fluid is given by [refer equation

(17) in Art. 14.1) $$\frac{D\mathbf{q}}{Dt} = \mathbf{B} - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\mathbf{q},$$

where **B** is the body force. Using this relation, (1) reduces to

$$\frac{dT}{dt} = \rho\int_V \mathbf{q}\cdot\left[\mathbf{B} - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\mathbf{q}\right]dV$$

or $$\frac{dT}{dt} = \int_V \mathbf{q}\cdot(\rho\mathbf{B})\,dV - \int_S p\,\mathbf{q}\cdot\mathbf{n}\,dS + \rho\int_V (\mathbf{q}\cdot\frac{\mu}{\rho}\nabla^2\mathbf{q})\,dV \qquad ...(2)$$

[On using Gauss divergence theorem in the second term]

The first term in R.H.S. of (2) represents the rate at which the external force **B** is doing work throughout the mass of the liquid while the second term on R.H.S. of (2) represents the rate at which the pressure is doing work on the boundary. It follows that for an ideal fluid ($\mu = 0$), the rate of increase of kinetic energy equals the rate at which work is done by the body forces and pressures at the boundary. Hence, if D is the rate of dissipation of energy due to viscosity, then by virtue of (2), we have $$D = -\mu\int_V (\mathbf{q}\cdot\nabla^2\mathbf{q})\,dV. \qquad ...(3)$$

Let Ω denote the vorticity vector. Then $$\Omega = \nabla\times\mathbf{q} \qquad ...(4)$$

Now, consider the vector identity $$\nabla\times(\nabla\times\mathbf{q}) = \nabla(\nabla\cdot\mathbf{q}) - \nabla^2\mathbf{q}. \qquad ...(5)$$

Since the fluid is incompressible, we have $$\nabla\cdot\mathbf{q} = 0 \qquad ...(6)$$

Using (4) and (6), (5) reduces to

$$\nabla\times\Omega = -\nabla^2\mathbf{q} \quad \text{so that} \quad \nabla^2\mathbf{q} = -\nabla\times\Omega \qquad ...(7)$$

From (7), $$\mathbf{q}\cdot\nabla^2\mathbf{q} = -\mathbf{q}\cdot(\nabla\times\Omega) \qquad ...(8)$$

Now, $$\nabla\cdot(\mathbf{q}\times\Omega) = \Omega\cdot(\nabla\times\mathbf{q}) - \mathbf{q}\cdot(\nabla\times\Omega) = \Omega\cdot\Omega - \mathbf{q}\cdot(\nabla\times\Omega), \quad \text{by (4)}$$

$$\Rightarrow \nabla\cdot(\mathbf{q}\times\Omega) = \Omega^2 - \mathbf{q}\cdot(\nabla\times\Omega) \quad \text{or} \quad -\mathbf{q}\cdot(\nabla\times\Omega) = \nabla\cdot(\mathbf{q}\times\Omega) - \Omega^2. \qquad ...(9)$$

From (8) and (9), we have $$\mathbf{q}\cdot\nabla^2\mathbf{q} = \nabla\cdot(\mathbf{q}\times\Omega) - \Omega^2. \qquad ...(10)$$

Using (10), (3) reduces to

$$D = -\mu\int_V [\nabla\cdot(\mathbf{q}\cdot\Omega) - \Omega^2]\,dV \quad \text{or} \quad D = \mu\int_V \Omega^2 dV - \mu\int_V \nabla\cdot(\mathbf{q}\cdot\Omega)\,dV$$

$$\therefore \quad D = \mu\int_V \Omega^2 dV - \mu\int_S \mathbf{n}\cdot(\mathbf{q}\cdot\Omega)\,dS, \text{ by Gauss divergence theorem} \qquad ...(11)$$

In case the boundary is at rest and there is no slip between fluid and boundary so that $\mathbf{q} = \mathbf{0}$ on S, then (11) reduces to

$$D = \mu\int_V \Omega^2 dV \quad \text{or} \quad D = 4\mu\int_V (\xi^2 + \eta^2 + \zeta^2)\,dxdydz,$$

where ξ, η, ζ are components of vorticity vector Ω, *i.e.*,

$$\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

14.6 B. Dissipation of energy (cartesian form).

[Agra 1999; 2006; Allahabad 2002; Garhwal 2000; Kanpur 1997; 1998; Kolkata 2000]

The kinetic energy T at time t of a portion of fluid bounded by S is given by

$$T = \frac{1}{2}\iiint \rho(u^2 + v^2 + w^2)\, dx\, dy\, dz \qquad ...(1)$$

Hence, differentiating following the motion of the same portion, we have

$$\frac{DT}{Dt} = \iiint \rho\left(u\frac{Du}{Dt} + v\frac{Dv}{Dt} + w\frac{Dw}{Dt}\right) dx\, dy\, dz \qquad ...(2)$$

Navier-Stokes equations of motion (refer Art. 14.1) are given by

$$\left.\begin{aligned}
\rho\frac{Du}{Dt} &= \rho B_x + \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} \\
\rho\frac{Dv}{Dt} &= \rho B_y + \frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} \\
\rho\frac{Dw}{Dt} &= \rho B_z + \frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}
\end{aligned}\right\} \qquad ...(3)$$

Using (3), (2) reduces to

$$\frac{DT}{Dt} = \iiint \rho(uB_x + vB_y + wB_z)\, dxdydz$$

$$+\iiint\left[u\left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z}\right) + v\left(\frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z}\right) + w\left(\frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}\right)\right] dxdydz \qquad ...(4)$$

The first term on R.H.S. of (4) represents the rate at which the external forces are doing work throughout the mass of the fluid. The second term on R.H.S. of (4) may be re-written as

$$\iiint\left[\frac{\partial}{\partial x}(u\,\sigma_{xx}) + \frac{\partial}{\partial y}(u\,\sigma_{xy}) + \frac{\partial}{\partial z}(u\,\sigma_{xz}) + \ldots - \left\{\frac{\partial u}{\partial x}\sigma_{xx} + \frac{\partial u}{\partial y}\sigma_{xy} + \frac{\partial u}{\partial z}\sigma_{xz} + \ldots\right\}\right] dxdydz$$

$$= -\iint [u(l\sigma_{xx} + m\sigma_{xy} + n\sigma_{xz}) + v(l\sigma_{yx} + m\sigma_{yy} + n\sigma_{yz}) + w(l\sigma_{zx} + m\sigma_{zy} + n\sigma_{zz})]dS$$

$$-\iiint\left[\frac{\partial u}{\partial x}\sigma_{xx} + \frac{\partial v}{\partial y}\sigma_{yy} + \frac{\partial w}{\partial z}\sigma_{zz} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sigma_{xy} + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\sigma_{yz} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\sigma_{zx}\right] dxdydz, \qquad ...(5)$$

where l, m, n are the direction cosines of the inward drawn normal to dS.

Now, we know that (refer Art. 13.5)

$$\left.\begin{aligned}
\sigma_{nx} &= l\sigma_{xx} + m\sigma_{xy} + n\sigma_{xz} \\
\sigma_{ny} &= l\sigma_{yx} + m\sigma_{yy} + n\sigma_{yz} \\
\sigma_{nz} &= l\sigma_{zx} + m\sigma_{zy} + n\sigma_{zz}
\end{aligned}\right\} \qquad ...(6)$$

Using (6), the first integral in (5) may be re-written as

$$-\iint (u\sigma_{nx} + v\sigma_{ny} + w\sigma_{nz})\, dS, \qquad ...(7)$$

where the suffix n indicates a normal to dS, and this integral represents the rate at which the kinetic energy is being increased by the action of the stresses on the boundary of the fluid.

Using the constitutive equations for an incompressible viscous fluid $(\nabla \cdot \mathbf{q} = 0)$, we have (refer Art. 13.12 and Art 13.14).

$$\left.\begin{aligned}\sigma_{xx} &= -p + 2\mu \in_{xx} = -p + 2\mu(\partial u/\partial x)\\ \sigma_{yy} &= -p + 2\mu \in_{yy} = -p + 2\mu(\partial v/\partial y)\\ \sigma_{zz} &= -p + 2\mu \in_{zz} = -p + 2\mu(\partial w/\partial z)\\ \sigma_{xy} &= 2\mu \in_{xy} = \partial u/\partial y + \partial v/\partial x\\ \sigma_{yz} &= 2\mu \in_{yz} = \partial v/\partial z + \partial w/\partial y\\ \sigma_{zx} &= 2\mu \in_{zx} = \partial w/\partial x + \partial u/\partial z\end{aligned}\right\} \quad ...(8)$$

Again, the equation of continuity $\nabla \cdot \mathbf{q} = 0$ may be written as

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0. \quad ...(9)$$

Using (8), the second term on R.H.S. of (5)

$$= \iiint p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) - \mu\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2\right.$$

$$\left. + \left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right] dxdydz$$

$$= -\iiint \mu\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right\} dxdydz, \text{ by (9)}$$

$$= -\iiint \mathbf{\Phi}\, dxdydz, \quad ...(10)$$

$$\text{where } \mathbf{\Phi} = \mu\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right\}, \quad ...(11)$$

$\mathbf{\Phi}$ is called the *dissipation function.*

$\therefore$ The rate of dissipation of energy $= \iiint \mathbf{\Phi}\, dxdydz$

$$= \mu\iiint\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right\} dxdydz, \text{ by (11)}$$

$$= \mu\iiint\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right.$$

$$\left. -2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2\right\} dxdydz, \text{ using (9),}$$

$$= \mu\iiint\left\{\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)^2\right.$$

$$+ 4\left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial z}\frac{\partial v}{\partial y}\right) + 4\left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial w}{\partial z}\right) + 4\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial x}\right) dxdydz \quad ...(12)$$

Now, $\iiint\left(\dfrac{\partial w}{\partial y}\dfrac{\partial v}{\partial z}-\dfrac{\partial w}{\partial z}\dfrac{\partial v}{\partial y}\right)dxdydz$

$$=\iiint\left\{\frac{\partial}{\partial y}\left(w\frac{\partial v}{\partial z}\right)-w\frac{\partial^2 v}{\partial y\partial z}-\frac{\partial}{\partial z}\left(w\frac{\partial v}{\partial y}\right)+w\frac{\partial^2 v}{\partial y\partial z}\right\}dxdydz=-\iint\left(mw\frac{\partial v}{\partial z}-nw\frac{\partial v}{\partial y}\right)dS$$

If $w = 0$ on the boundary, then $$\iiint\left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial z}\frac{\partial v}{\partial y}\right)dxdydz=0 \qquad ...(13)$$

Similarly, $$\iiint\left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}\right)dxdydz=0 \qquad ...(14)$$

and $$\iiint\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial x}\right)dxdydz=0 \qquad ...(15)$$

If $,\xi,\eta,\zeta,$ are the components of the vorticity vector, then we know that

$$\xi=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),\qquad \eta=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right),\qquad \zeta=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \qquad ...(16)$$

Using (13), (14), (15) and (16), (12) reduces to

The rate of dissipation of energy $=4\mu\iiint(\xi^2+\eta^2+\zeta^2)\,dxdydz.$ ...(17)

This is the rate of dissipation of energy for a liquid filling a closed vessel.

Remark As an application of the Art. 14.6. B, consider the following example.

Illustrative solved example. *Prove that for a liquid filing up a vessel in the form of a surface of revolution which is rotating about its axis (z-axis) with the angular velocity* ω *the rate of dissipation of energy has an additional term*

$$2\mu\omega\iiint(l\,Du+m\,Dv)\,dS, \qquad \text{where} \qquad D=\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)$$

and l, m, n are the direction cosines of the inward drawn normal at the element dS of the surface of the vessel. **[Meerut 2004]**

Sol. First do the whole Art. 14.6.B. Next, we have, here

$$u=-\omega y, \qquad v=\omega x, \qquad w=0. \qquad ...(18)$$

Hence as in the above article 12.7 B, the integral on **L.H.S.** of (15) will not vanish. So we have an additional term which will be calculated as follows.

The additional term

$$=4\mu\iiint\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)dxdydz=4\mu\iiint\left\{\frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial x}\right)\right\}dxdydz$$

$$=-4\mu\iint\left(lv\frac{\partial u}{\partial y}-mv\frac{\partial u}{\partial x}\right)dS=-4\mu\iint v\left(l\frac{\partial u}{\partial y}+m\frac{\partial v}{\partial y}\right)dS$$

[$\because$ For two-dimensional case, equation of continuity is $\partial u/\partial x+\partial v/\partial y=0$]

$$=-4\mu\omega\iint x\left(l\frac{\partial u}{\partial y}+m\frac{\partial v}{\partial y}\right)dS, \text{ using (18)} \qquad ...(19)$$

Similarly, the same expression

$$= 4\mu \iiint \left\{ \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right)\right\} dxdydz = -4\mu \iint \left(mu\frac{\partial v}{\partial x} - lu\frac{\partial v}{\partial y}\right) dS$$

$$= -4\mu \iint u\left(m\frac{\partial v}{\partial x} + l\frac{\partial u}{\partial x}\right) dS \qquad [\because\ \partial u/\partial x + \partial v/\partial y = 0 \text{ as before }]$$

$$= 4\mu\omega \iint y\left(m\frac{\partial v}{\partial x} + l\frac{\partial u}{\partial x}\right) dS, \text{ using (18)} \qquad ...(20)$$

Taking the mean of the above two expressions (19) and (20), the additional term

$$= 2\mu\omega \iint \left\{ y\left(m\frac{\partial v}{\partial x} + l\frac{\partial u}{\partial x}\right) - x\left(l\frac{\partial u}{\partial y} + m\frac{\partial v}{\partial y}\right)\right\} dS$$

$$= 2\mu\omega \iint \left\{ l\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) u + m\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) v\right\} dS$$

$$= 2\mu\omega \iint (l\,Du + m\,Dv)\,dS, \text{ since } D = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \text{ (given)}$$

14.7 Illustrative solved examples.

The reader is advised to study carefully and remember all results of articles 14.10, 14.11 and 14.12. These results can be used directly while solving problems.

Ex. 1. *Show that for an incompressible steady flow with constant viscosity, the velocity components* $u(y) = y\dfrac{U}{h} + \dfrac{h^2}{2\mu}\left(-\dfrac{dp}{dx}\right)\dfrac{y}{h}\left(1 - \dfrac{y}{h}\right)$, $v = w = 0$ *satisfy the equation of motion, when the body force is neglected. h, U, dp/dx are constants and* $p = p(x)$. **(Meerut 2007, 11)**

Sol. Given $$u(y) = (yU/h) + (h^2/2\mu)(-dp/dx)(y/h)(1 - y/h) \qquad ...(1)$$

$$v = 0 \quad \text{and} \quad w = 0 \qquad ...(2)$$

The equation of motion for viscous incompressible fluid is given by

$$\partial \mathbf{q}/\partial t + (\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{B} - (1/\rho)\times\nabla p + \nu\,\nabla^2\mathbf{q} \qquad ...(3)$$

Here $\partial \mathbf{q}/\partial t = \mathbf{0}$, the motion being steady. ...(4)

and $\mathbf{B} = \mathbf{0}$, as the body force is neglected. ...(5)

Since $v = w = 0$, we have $\mathbf{q} = \mathbf{i}u$

$$\therefore \qquad \nabla^2\mathbf{q} = \mathbf{i}\nabla^2 u \qquad ...(6)$$

Given that $p = p(x)$ so that $$\nabla p = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) p = \mathbf{i}\frac{dp}{dx} \qquad ...(7)$$

Also $$\mathbf{q}\cdot\nabla = (\mathbf{i}u)\cdot\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) = u\frac{\partial}{\partial x}$$

$$\therefore \qquad (\mathbf{q}\cdot\nabla)\mathbf{q} = \left(u\frac{\partial}{\partial x}\right)(\mathbf{i}u) = \mathbf{i}u\frac{\partial u}{\partial x} = 0, \text{ as } u = u(y), \text{ given} \qquad ...(8)$$

Substituting (4), (5), (6), (7) and (8) into (3), we have

$$0 = -\frac{1}{p}\frac{dp}{dx} + \nu\nabla^2 u \quad \text{or} \quad \frac{1}{p}\frac{dp}{dx} = \frac{\mu}{\rho}\frac{d^2u}{dy^2} \quad \text{or} \quad \frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}. \qquad ...(9)$$

$$\left[\because \nu = \mu/\rho \text{ and } \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2\right]$$

Now from (1), $$\frac{du}{dy}=\frac{U}{h}-\frac{h}{2\mu}\frac{dp}{dx}\left(1-\frac{2y}{h}\right),$$ as dp/dx is given to be constant

$$\therefore \qquad \frac{d^2u}{dy^2}=0-\frac{h}{2\mu}\frac{dp}{dx}\left(-\frac{2}{h}\right)=\frac{1}{\mu}\frac{dp}{dx},$$

which is the same as (9). This proves that the equation of motion is satisfied.

Ex. 2. *Consider an inviscid, incompressible, steady flow with negligible body force whose velocity components are* $q_r=U(1-R^3/r^3)\cos\theta,\ q_\theta=-U(1+R^3/2r^3)\sin\theta,\ q_\phi=0$ *in spherical coordinates where R is a constant. Is the equation of motion satisfied.*

Sol. Here
$$q_r=U(1-R^3/r^3)\cos\theta \qquad \ldots(1)$$
$$q_\theta=-U(1+2R^3/r^3)\sin\theta \qquad \ldots(2)$$
and
$$q_\phi=0 \qquad \ldots(3)$$

Equations of motion for non-viscous fluid in absence of body forces are given by [Refer equations (11a) to (11c) and equation (2) of Art. 14.12 of this chapter]

$$\frac{Dq_r}{Dt}-\frac{q_\theta^2+q_\phi^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots(4)$$

$$\frac{Dq_\theta}{Dt}+\frac{q_rq_\theta}{r}-\frac{q_\phi^2\cot\theta}{r}=-\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \qquad \ldots(5)$$

$$\frac{Dq_\phi}{Dt}+\frac{q_\phi q_r}{r}+\frac{q_\theta q_\phi\cot\theta}{r}=-\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\phi} \qquad \ldots(6)$$

Here
$$\frac{D}{Dt}=\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial r}+\frac{q_\theta}{r}\frac{\partial}{\partial\theta}+\frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \qquad \ldots(7)$$

For steady flow, $\partial/\partial t=0,$ Also, here $q_\phi=0.$ Hence (4), (5) and (6) may be rewritten as

$$q_r\frac{\partial q_r}{\partial r}+\frac{q_\theta}{r}\frac{\partial q_r}{\partial\theta}-\frac{q_\theta^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots(8)$$

$$q_r\frac{\partial q_\theta}{\partial r}+\frac{q_\theta}{r}\frac{\partial q_\theta}{\partial\theta}+\frac{q_rq_\theta}{r}=-\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \qquad \ldots(9)$$

$$0=-\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\phi} \qquad \ldots(10)$$

Equation (10) shows that p is function of r and θ alone. From (1) and (2), we have

$$\frac{\partial q_r}{\partial r}=\frac{3UR^3}{r^4}\cos\theta \quad \text{and} \quad \frac{\partial q_r}{\partial\theta}=-U\left(1-\frac{R^3}{r^3}\right)\sin\theta \qquad \ldots(11)$$

$$\frac{\partial q_\theta}{\partial r}=\frac{3UR^3}{2r^4}\cos\theta \quad \text{and} \quad \frac{\partial q_\theta}{\partial\theta}=-U\left(1-\frac{R^3}{2r^3}\right)\cos\theta \qquad \ldots(12)$$

Using (1), (2) and (11), (8), reduces to

$$\frac{3U^2R^3}{r^4}\left(1-\frac{R^3}{r^3}\right)\cos^2\theta+\frac{U^2}{r}\left(1-\frac{R^3}{r^3}\right)\left(1+\frac{R^3}{2r^3}\right)\sin^2\theta-\frac{U^2}{r}\left(1+\frac{R^3}{2r^3}\right)^2\sin^2\theta=-\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or $$\frac{3U^2R^3}{r^4}\left(1-\frac{R^3}{r^3}\right)\cos^2\theta+\frac{U^2}{r}\sin^2\theta\left[\left(1-\frac{R^3}{r^3}\right)\left(1+\frac{R^3}{2r^3}\right)-\left(1+\frac{R^3}{2r^3}\right)^2\right]=-\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or $$\frac{3U^2R^3}{r^4}\left(1-\frac{R^3}{r^3}\right)\cos^2\theta-\frac{3U^2R^3}{2r^4}\left(1+\frac{R^3}{2r^3}\right)\sin^2\theta=-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad ...(13)$$

Using (1), (2) and (12) in (9) and proceeding as above, the reader can verify that

$$\frac{3U^2R^3}{2r^3}\left(1-\frac{R^3}{r^3}\right)\sin\theta\cos\theta+\frac{3U^3R^3}{2r^3}\left(1+\frac{R^3}{2r^3}\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad ...(14)$$

Differentiating (13) partially with respect to θ, we get

$$-\frac{6U^2R^3}{r^4}\left(1-\frac{R^3}{r^3}\right)\sin\theta\cos\theta-\frac{3U^2R^3}{r^4}\left(1+\frac{R^3}{2r^3}\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\partial\theta}$$

or $$\frac{9U^2R^3}{r^4}\left(\frac{R^3}{2r^3}-1\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\partial\theta} \qquad ... (15)$$

Finally, differentiating (14) partially with respect to r, we get

$$\frac{3}{2}U^2R^3\left(-\frac{3}{r^4}+\frac{6R^3}{r^7}\right)\sin\theta\cos\theta+\frac{3}{2}U^2R^3\left(-\frac{3}{r^4}-\frac{6R^3}{2r^7}\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\partial\theta}$$

or $$-\frac{9}{2}\frac{U^2R^3}{r^4}\left(1-\frac{2R^3}{r^3}\right)\sin\theta\cos\theta-\frac{9}{2}\frac{U^2R^3}{r^4}\left(1+\frac{R^3}{r^3}\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\partial\theta}$$

or $$\frac{9U^2R^3}{r^4}\left(\frac{R^3}{2r^3}-1\right)\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial^2 p}{\partial r\partial\theta} \qquad ...(16)$$

Since (15) and (16) are identical, it follows that the equations of motion are satisfied.

Ex. 3 (*a*) *Define circulation. Show that the time rate of change of circulation in a closed circuit, drawn in a viscous incompressible fluid under the action of conservative forces, moving with the fluid depends only on the kinematic viscosity and the space rate of change of vorticity components at the the contour. Hence state and prove Kelvin's circulation theorem.*

[Himanchal 1998; 2003]

(*b*) *Derive the time rate of change of circulation of a closed curve drawn in a viscous incmpressible fluid, moving with the fluid.* **[Himanchal 2002, 05, 07]**

Sol. (*a*) Let C be a closed circuit moving with the fluid so that C always consists of the same fluid particles. Let $\boldsymbol{q}$ be the fluid velocity at any point P of circuit and let $\boldsymbol{r}$ be its position vector. Then the circulation along the closed C is given by

$$\Gamma=\oint_C \mathbf{q}\cdot d\boldsymbol{r} \qquad \text{so that} \qquad \frac{D\Gamma}{Dt}=\frac{D}{Dt}\oint_C \mathbf{q}\cdot d\boldsymbol{r} \qquad ...(1)$$

Since the above integration is performed at constant time, reversing the order of integration and differentiation is justified. Then (1) may be re-written as

$$\frac{D\Gamma}{Dt}=\oint_C \frac{D}{Dt}(\mathbf{q}\cdot d\boldsymbol{r}) \qquad ...(2)$$

But $$\frac{D}{Dt}(\mathbf{q}\cdot d\boldsymbol{r})=\frac{D\mathbf{q}}{Dt}\cdot d\boldsymbol{r}+\mathbf{q}\cdot\frac{D}{Dt}d\boldsymbol{r}=\frac{D\mathbf{q}}{Dt}\cdot d\boldsymbol{r}+\mathbf{q}\cdot d\mathbf{q} \qquad ...(3)$$

Equation of motion for viscous incompressible fluid with constant viscosity in vector form (refer equation (17) of Art. 14.1) is given by

$$D\mathbf{q}/Dt = \mathbf{B} - (1/\rho) \times \nabla p + \nu\, \nabla^2 \mathbf{q} \qquad ...(4)$$

Let the external forces be conservative and derivable from a single valued potential V. Then $\mathbf{B} = -\nabla V$ and hence (4) reduces to

$$D\mathbf{q}/Dt = -\nabla V - (1/\rho) \times \nabla p + \nu\, \nabla^2 \mathbf{q} \qquad ...(5)$$

$$\therefore \qquad (D\mathbf{q}/Dt) \cdot d\mathbf{r} = -\nabla V \cdot d\mathbf{r} - (1/\rho) \times \nabla p \cdot d\mathbf{r} + \nu\, \nabla^2 \mathbf{q} \cdot d\mathbf{r} \qquad ...(6)$$

If Ω is the vorticity vector, then with help of a vector identity, we obtain

$$\text{curl } \Omega = \nabla \times (\nabla \times \mathbf{q}) = \nabla (\nabla \cdot \mathbf{q}) - \nabla^2 \mathbf{q}$$

or $\qquad \text{curl } \Omega = -\nabla^2 \mathbf{q}, \qquad$ as $\nabla \cdot \mathbf{q} = 0$ for incompresible fluid

Hence (6) can be re-written as

$$(D\mathbf{q}/Dt) \cdot d\mathbf{r} = -dV - (1/\rho)\, dp - \nu\, (\text{curl } \Omega) \cdot d\mathbf{r} \qquad ...(7)$$

Also, we have $\qquad \mathbf{q} \cdot d\mathbf{q} = (1/2) \times d\,(\mathbf{q} \cdot \mathbf{q}) = (1/2) \times d\mathbf{q}^2 \qquad ...(8)$

Using (7) and (8), (3) reduces to

$$\frac{D}{Dt}(\mathbf{q} \cdot d\mathbf{r}) = -dV - \frac{1}{\rho} dp - \nu\, (\text{curl } \Omega) \cdot d\mathbf{r} + \frac{1}{2} d\mathbf{q}^2 \qquad ...(9)$$

Using (9) and assuming that ρ is a single valued function of p only, (2) reduces to

$$\frac{D\Gamma}{Dt} = \oint_C \left\{ \left(\frac{1}{2} d\mathbf{q}^2 - dV - \frac{1}{\rho} dp \right) - \nu\, (\text{curl } \Omega) \cdot d\mathbf{r} \right\}$$

or

$$\frac{D\Gamma}{Dt} = \left[\frac{1}{2}\mathbf{q}^2 - V - \oint_C \frac{1}{\rho} dp \right]_c - \nu \oint_C (\text{curl } \Omega) \cdot d\mathbf{r}, \qquad ...(10)$$

where the symbol $[\ ]_C$ denotes change in the quantity enclosed within brackets on moving once round C. Since $\boldsymbol{q}$, V and p are single-valued functions of $\boldsymbol{r}$, it follows that the first term on the R.H.S. of (10) vanishes.

Then, (10) reduces to
$$\frac{D\Gamma}{Dt} = -\nu \oint_C (\text{curl } \Omega) \cdot d\mathbf{r}, \qquad ...(11)$$

showing that the rate of change of circulation in a closed circuit, drawn in a viscous incompressible fluid, moving with the fluid depends only on the kinematic vicosity ν and on the space rate of change of the vorticity components at the contour.

As a particular case, let $\nu = 0$, *i.e.*, let the fluid be inviscid. Then (11), reduces to $D\Gamma/Dt = 0$, which is well known Kelvin's circulation theorem, namely, *the circulation round any closed circuit moving with the fluid does not change with the time, provided the fluid is inviscid, the field of force is conservative and density is a single valued function of pressure only.*

(*b*) Refer part (*a*). Omit the particular case given at the end.

Ex. 4. *Write Navier-Stokes equations in cartesian co-ordinates. Simplify the equations when*

(*a*) *Fluid is incompressible and dynamic vicosity is constant*

(*b*) *The fluid is incompressible and viscous effects are neligible.*

[Andhra 2002, 03, 06; Kanpur 2003, Meerut 1996]

Sol. (*a*) For incompressible fluid, $\nabla \cdot \mathbf{q} = 0$ *i.e.*, $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$.

Also, given that μ = dynamic viscosity = constant.

Re-writing equation (14*a*) of Art 14.1, we have

$$\rho \frac{Du}{Dt} = \rho B_x - \frac{\partial p}{\partial x} + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 v}{\partial x \partial y} + \mu \frac{\partial^2 w}{\partial x \partial z} + \mu \frac{\partial^2 u}{\partial z^2}$$

or
$$\rho \frac{Du}{Dt} = \rho B_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \mu \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

Since $D/Dt = \partial/\partial t + u(\partial/\partial x) + v(\partial/\partial y) + w(\partial/\partial z)$, and $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$, the above equation reduces to

$$\rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) = \rho B_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Similarly, equations (14b) and (14c) of Art. 14.1 yield

$$\rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) = \rho B_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

and
$$\rho \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) = \rho B_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

(*b*) For incompressible fluid, $\nabla \cdot \mathbf{q} = 0$. Also, if viscous effects are negligible, then setting $\mu = 0$ in equations of part (a), the required equations are

$$\rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) = \rho B_x - \frac{\partial p}{\partial x},$$

$$\rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) = \rho B_y - \frac{\partial p}{\partial y}$$

and
$$\rho \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) = \rho B_z - \frac{\partial p}{\partial z}$$

Ex. 5. *Consider the case of simple Couette (see Art. 16.3A of chapter* 16) *flow with the velocity and temperature distributions as follows* :

$$u = Uy/h, \qquad v = 0, \qquad p = \text{constant} \qquad \ldots(1)$$

$$\frac{T - T_\omega}{T_\infty - T_\omega} = \frac{y}{h} + \frac{\mu U^2}{2k(T_\infty - T_\omega)} \left(\frac{y}{h} \right) \left(1 - \frac{y}{h} \right), \qquad \ldots(2)$$

where T_ω *and* T *are temperatures (constant in value) of stationary and moving plates, respectively, and* μ, h *and* k *are constants. Verify that* (1) *and* (2) *are the solutions of the energy equation for steady viscous incompressible fluid.* **[Garhwal 1996, 98, Meerut 1998]**

Sol. The energy equation for a two-dimensional viscous incompressible fluid is [Refer equation (20) in Art. 14.3]

$$k(\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2) + \mathbf{\Phi}' = \rho C(DT/Dt) - (Dp/Dt) \qquad \ldots(3)$$

where
$$\mathbf{\Phi}' = \mu[2\{(\partial u/\partial x)^2 + (\partial v/\partial y)^2\} + (\partial u/\partial y + \partial v/\partial x)^2] \qquad \ldots(4)$$

Given that u and T are functions of y alone, v = 0 and p = constant. Also for steady motion, $\partial/\partial t = 0$. Hence

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = 0 \qquad \text{and} \qquad \Phi' = \mu\left(\frac{\partial u}{\partial y}\right)^2.$$

Hence (3) reduces $\qquad k(d^2T/dy^2) + \mu(d^2u/dy^2) = 0 \qquad$ ...(5)

From (1), $\qquad \partial u/dy = U/h \qquad$ ...(6)

From (2),
$$\frac{dT}{dy} = (T_\infty - T_\infty)\frac{1}{h} + \frac{\mu U^2}{2kh}\left(1 - \frac{2y}{h}\right)$$

and hence
$$\frac{d^2T}{dy^2} = 0 + \frac{\mu U^2}{2kh}\left(-\frac{2}{h}\right) = -\frac{\mu U^2}{kh^2}. \qquad \text{...(7)}$$

Using (6) and (7) in (5), we have

$$k\left(-\frac{\mu U^2}{kh^2}\right) + \mu\frac{U^2}{h^2} = 0 \qquad i.e. \qquad 0 = 0,$$

showing that (1) and (2) satisfy the energy equation (3) for steady flow.

Ex. 6. *Consider a two-dimensional viscous incompressible steady flow with velocity components* $\qquad q_r = q_\theta = 0, \qquad q_z = (1/4\mu)\times(dp/dz)\,r^2 + A\log r + B \qquad$ ...(1)

and $\qquad p = p(z), \qquad$ ...(2)

where A, B, and μ *are constants and* $0 \le r_0 \le r$. *Is the equation of motion with negligible body force satisfied* ? **[Kolkota 2001]**

Sol. The equations of motion for viscous incompressible flow in absence of body forces are [refer equations (2), (10), (11*a*), (11b) and (11*c*) of Art. 14.11 of this chapter]

$$\rho\left(\frac{Dq_r}{Dt} - \frac{q_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 q_r - \frac{q_r}{r^2} - \frac{2}{r^2}\frac{\partial q_\theta}{\partial \theta}\right) \qquad \text{...(3)}$$

$$\rho\left(\frac{Dq_\theta}{Dt} - \frac{q_r q_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial r} + \mu\left(\nabla^2 q_\theta + \frac{2}{r^2}\frac{\partial q_r}{\partial \theta} - \frac{q_\theta}{r}\right) \qquad \text{...(4)}$$

$$\rho\frac{Dq_z}{Dt} = -\frac{\partial p}{\partial z} + \mu\nabla^2 q_z \qquad \text{...(5)}$$

with
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + q_z\frac{\partial}{\partial r} \qquad \text{...(6)}$$

and
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \qquad \text{...(7)}$$

For a two dimensional flow, $\qquad \partial/\partial\theta = 0, \qquad \partial/\partial z = 0$

and for steady motion $\partial/\partial t = 0$. With $q_r = q_\theta = 0$ and $p = p(z)$, (3) and (4) are identically equal to zero. Furthermore, (5) reduces to

$$0 = -\frac{dp}{dz} + \mu\left(\frac{d^2 q_z}{dr^2} + \frac{1}{r}\frac{dq_z}{\partial r}\right) \qquad \text{...(8)}$$

From (1), $$\frac{dq_z}{dr} = \frac{1}{2\mu}\left(\frac{dp}{dz}\right)r + \frac{A}{r} \qquad \text{...(9)}$$

and $$\frac{d^2q_z}{dr^2} = \frac{1}{2\mu}\left(\frac{dp}{dz}\right) - \frac{A}{r^2} \qquad \text{...(10)}$$

Using (9) and (10) in (8), we get

$$0 = -\frac{dp}{dz} + \mu\left[\frac{1}{2\mu}\frac{dp}{dz} - \frac{A}{r^2} + \frac{1}{r}\left(\frac{r}{2\mu}\frac{dp}{dz} + \frac{A}{r}\right)\right], \qquad i.e. \qquad 0 = 0.$$

Thus we find that the equations of motion are satisfied.

14.8. Vorticity equation or vorticity transport equation.

Theorem. *Show that the vorticity vector* Ω *of an incompressible viscous fluid moving under no external forces satisfies the differential equation*

$D\Omega/Dt = (\Omega\cdot\nabla)q + \nu\nabla^2\Omega$ where ν is the kinematic coefficient of viscosity.

[Agra 2000, 05, 06; Kolkata 2006; Himanchal 2000, 02, 03, 09; Meerut 2011]

Proof. Navier-Stokes equation for incompressible viscous fluid with constant viscosity (refer equation (17) in Art. 14.1) is

$$\partial\mathbf{q}/\partial t + (\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{B} - (1/\rho)\times\nabla p + \nu\,\nabla^2\mathbf{q}. \qquad \text{...(1)}$$

Let the forces be conservative. Then there exists a force potential V such that $\mathbf{B} = -\nabla V$.

Again, by vector calculus $\qquad \nabla\mathbf{q}^2 = \nabla(\mathbf{q}\cdot\mathbf{q}) = 2[(\mathbf{q}\cdot\nabla)\mathbf{q} + \mathbf{q}\times\text{curl}\,\mathbf{q}.]$

or $$(\mathbf{q}\cdot\nabla)\mathbf{q} = \nabla(q^2/2) - \mathbf{q}\times\text{curl}\,\mathbf{q}$$

or $$(\mathbf{q}\cdot\nabla)\mathbf{q} = \nabla(q^2/2) - 2\mathbf{q}\times\Omega \qquad [\text{Taking } \Omega = (1/2)\times\text{curl}\,\mathbf{q}]$$

Then (1) reduces to $$\partial\mathbf{q}/dt + \nabla(q^2/2) - 2\mathbf{q}\times\Omega = -\nabla V - (1/\rho)\times\nabla p + \nu\,\nabla^2\mathbf{q}$$

or $$\partial\mathbf{q}/\partial t - 2\mathbf{q}\times\Omega = -\nabla(V + p/\rho + q^2/2) + \nu\,\nabla^2\mathbf{q}$$

Taking curl of both sides and using the results curl grad $\equiv 0$ and curl $(\partial\mathbf{q}/\partial t) = \partial(\text{curl}\,\mathbf{q})/\partial t$ $= 2(\partial\Omega/\partial t)$ and curl $\nabla^2\mathbf{q} \equiv \nabla^2\text{curl}\,\mathbf{q} = 2\nabla^2\Omega,$ we obtain

$$\partial\Omega/\partial t - \text{curl}\,(\mathbf{q}\times\Omega) = \nu\,\nabla^2\Omega$$

or $$\partial\Omega/\partial t - [\mathbf{q}\,div\,\Omega - \Omega\,div\,\mathbf{q} + (\Omega\cdot\nabla)\mathbf{q} - (\mathbf{q}\cdot\nabla)\Omega] = \nu\,\nabla^2\Omega$$

or $$\partial\Omega/\partial t + (\mathbf{q}\cdot\nabla)\Omega = (\Omega\cdot\nabla)\mathbf{q} + \nu\nabla^2\Omega$$

[$\because$ Equation of continuity is *div* $\mathbf{q} = 0$ Also *div* Ω = *div* curl $\mathbf{q} = 0$]

or $$D\Omega/Dt = (\Omega\cdot\nabla)\mathbf{q} + \nu\,\nabla^2\Omega, \qquad \text{...(2)}$$

which is known as *vorticity equation or vorticity transport equation.*

Remark. Let $\Omega = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$, $\mathbf{q} = u\,\mathbf{i} + v\,\mathbf{j} + w\,\mathbf{k}$. Then the above vector equation (2) in Cartesian form reduces to

$$\left.\begin{aligned}\frac{D\xi}{Dt} &= \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)u + \nu\nabla^2\xi \\ \frac{D\eta}{Dt} &= \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)v + \nu\nabla^2\eta \\ \frac{D\zeta}{Dt} &= \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)w + \nu\nabla^2\zeta\end{aligned}\right\} \qquad \text{...(3)}$$

where $\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$, $\eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$, $\zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$.

14.9. Diffusion of a vortex filament.

Let there be a vortex filament of strength k along the axis of z in an infinite liquid. The motion will be in circles about the z-axis, the vorticity at distance r from the axis being a function of r only. We have, therefore $w = 0$, $\xi = \eta = 0$, ...(1)

and u, v are independent of z.

We know that (refer remark of Art. 14.8)

$$\left.\begin{aligned} \frac{D\xi}{Dt} &= \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)u + \nu\nabla^2\xi \\ \frac{D\eta}{Dt} &= \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)v + \nu\nabla^2\eta \\ \frac{D\zeta}{Dt} &= \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)w + \nu\nabla^2\zeta \end{aligned}\right\} \qquad ...(2)$$

Using (1), (2) reduces to $D\zeta / Dt = \nu\,\nabla^2\zeta$ ...(3)

Let u, v be of the form $u = -(y/r)\times f(r)$, $v = (x/r)\times f(r)$, ...(4)

where $r^2 = x^2 + y^2$ ...(5)

$$\therefore \quad \frac{D\zeta}{Dt} = \frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} = \frac{\partial\zeta}{\partial t} - \frac{y\,f(r)}{r}\left(\frac{x}{r}\frac{\partial\zeta}{\partial r}\right) + \frac{x\,f(r)}{r}\left(\frac{y}{r}\frac{\partial\zeta}{\partial y}\right), \text{ by (4)}$$

Thus, $D\zeta / Dt = \partial\zeta / \partial t$ ...(6)

Also, $$\nabla^2\zeta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\zeta = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\zeta \qquad ...(7)$$

Using (6) and (7), (3) reduces to $$\frac{\partial\zeta}{\partial t} = \nu\left(\frac{\partial^2\zeta}{\partial r^2} + \frac{1}{r}\frac{\partial\zeta}{\partial r}\right) \qquad ...(8)$$

To solve (8), we assume that $\zeta = (1/t)\times f(\chi)$, ...(9)

where $\chi = r/2\sqrt{\nu t}$ ...(10)

Then, $$\frac{\partial\zeta}{\partial t} = -\frac{1}{t^2}f(\chi) - \frac{r}{2t^2\sqrt{\nu t}}\frac{df}{d\chi} \quad \text{and} \quad \frac{\partial\zeta}{\partial r} = \frac{1}{2\sqrt{\nu t}}\frac{df}{d\chi}$$

Using the above relations, (8) reduces to

$$\chi\frac{d^2 f}{d\chi^2} + \frac{df}{d\chi}(1 + 2\chi^2) + 4\chi f = 0 \quad \text{or} \quad \frac{d}{d\chi}\left(\chi\frac{df}{d\chi}\right) + \frac{d}{d\chi}(2f\chi^2) = 0.$$

Integrating, $\chi(df/\partial\chi) + 2\chi^2 f = C$, where C is constant of integration.

Hence, $df/d\chi + 2\chi f = C/\chi$. ...(11)

When $\chi = 0$ *i.e.*, $r = 0$, f and f' are both infinite, therefore, we have $C = 0$ and (11) reduces to

$$df/d\chi + 2\chi f = 0 \quad \text{or} \quad (1/f)df + 2\chi d\chi = 0$$

Integrating, $\log f + \chi^2 = \log A,$ where A is constant of integration ...(12)

From (12), $$f = Ae^{-\chi^2} \qquad ...(13)$$

From (9), (10) and (13), we have $$\zeta = (1/t) \times Ae^{-\chi^2} = (1/t) \times Ae^{-(r^2/4\nu t)} \qquad ...(14)$$

By Stokes's theorem, circulation Γ round a circle of radius r is given by

$$\Gamma = \int_0^r 2\zeta\, dS \text{ over the circle } = \int_0^r 2\zeta \cdot 2\pi r\, dr = \frac{4\pi A}{t}\int_0^r re^{-(r^2/4\nu t)}\, dr, \quad \text{by (14)}$$

$\therefore$ $$\Gamma = 8\pi A\nu(1 - e^{-r^2/4\nu t}). \qquad ...(15)$$

Let $\Gamma \to \Gamma_1$ as $t \to 0$. So (15) gives

$\Gamma_1 = 8\pi A\nu$ so that $A = \Gamma_1/8\pi\nu$

$\therefore$ From (15), $$\Gamma = \Gamma_1(1 - e^{-r^2/4\nu t}) \qquad ...(16)$$

and from (14), $$\zeta = (\Gamma_1/8\pi\nu\, t) \times e^{-r^2/4\nu t} \qquad ...(17)$$

Also, if v be the velocity, then $\Gamma = 2\pi rv$

$\therefore$ from (16), $\Gamma_1(1 - e^{-r^2/4\nu t}) = 2\pi rv$ or $v = (\Gamma_1/2\pi r) \times (1 - e^{-r^2/4\nu t}).$ ...(18)

For small values of r, (18) reduces to

$$v = \frac{\Gamma_1}{2\pi r}\left[1 - \left\{1 - \left(-\frac{r^2}{4\nu t}\right) + \frac{1}{2!}\left(-\frac{r^2}{4\nu t}\right)^2 - \ldots\right\}\right] = \frac{\Gamma_1}{2\pi}\left\{\frac{r}{4\nu t} - \frac{r^2}{2 \times (4\nu t)^2} - \ldots\right\} \qquad ...(19)$$

From (19), we see that as $r \to 0$, then $v \to 0$.

Again on the axis $r = 0$, so from (17), $$\zeta = \zeta_0 = \Gamma_1/(8\pi\nu t). \qquad ...(20)$$

Hence for very small values of r, from (19) $$v = \frac{\Gamma_1}{2\pi} \cdot \frac{r}{4\nu t} = \frac{\Gamma_1 r}{8\pi\nu t}. \qquad ...(21)$$

From (20) and (21), $v = \zeta_0 r$.

From (18), it follows that as t increases from 0 to ∞, v decreases from $\Gamma_1/2\pi r$ to 0.

14.10. Summary of basic equations governing the flow of viscous fluid in certesian *co-ordinates* (*x*, *y*, *z*) :

Case I For flow of viscous compressible fluid

Equation of continuity : [Refer equation (8) of Art. 2.9]

$$\partial\rho/\partial t + \partial(\rho u)/\partial x + \partial(\rho v)/\partial y + \partial(\rho w)/\partial z = 0 \qquad ...(1)$$

The Navier-Stokes equation: [Refer equations (14*a*), (14*b*), (14*c*), Art 14.1]

$$\rho\{\partial u/\partial t + u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z)\} = \rho B_x - \partial p/\partial x$$

$$+\frac{\partial}{\partial x}\left[\mu\left\{2\frac{\partial u}{\partial x} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\}\right] + \frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\} + \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\right)\right\} \qquad ...(2a)$$

$$\rho\{\partial v/\partial t + u\,(\partial v/\partial x) + v\,(\partial v/\partial y) + w\,(\partial v/\partial z)\} = \rho B_y - \partial p/\partial y$$

$$+\frac{\partial}{\partial y}\left[\mu\left\{2\frac{\partial v}{\partial y} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\}\right] + \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right\} + \frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\} \qquad ...(2b)$$

$$\rho\{\partial w/\partial t + u\,(\partial w/\partial x) + v\,(\partial w/\partial y) + w\,(\partial w/\partial z)\} = \rho B_z - \partial p/\partial z$$

$$+\frac{\partial}{\partial z}\left[\mu\left\{2\frac{\partial w}{\partial z} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\}\right] + \frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right\} + \frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right\} \qquad ...(2c)$$

Energy equation : [Refer equation (17), Art 14.2]

$$\rho\left[\frac{\partial (C_pT)}{\partial t} + u\frac{\partial (C_pT)}{\partial x} + v\frac{\partial (C_pT)}{\partial y} + w\frac{\partial (C_pT)}{\partial z}\right] = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}$$

$$+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\Bigg\} + \mu\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\}\right.$$

$$\left. - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right] \qquad ...(3)$$

Equation of state: $p = \rho R T$ ...(4)

Thus, we have six equations. But for a the flow of compressible fluids, the coefficient of viscosity μ and the coefficient of thermal conductivity k are not constants but depend on temperature. Therefore, we have eight unknowns (u, v, w, p, ρ, T, μ and k) instead of six and we require two additional equations to solve a problem of flow of viscous compressible fluid. Let these two additional equations, in general froms, be given by

$\mu = \mu\,(T)$ and $k = k\ (T)$ ...(5)

For air, the variation of viscosity μ with absolute temperature T is given by the following Sutherland's formula,

$$\frac{\mu}{\mu_\infty} = \left(\frac{T}{T_\infty}\right)^{3/2} \frac{T_\infty + S_1}{T + S_1}, \text{ approximately} \qquad ...(6)$$

where μ_∞ denotes the viscosity at a reference temperature T_∞ and S_1 is a constant. For air, S_1 = 110° K.

The above formula (6) is quite complicated. From either the simple kinetic theory of gases, of empirical data, the coefficient of viscosity μ may, be expressed quite accurately as a power of the absolute temperature, $\mu/\mu_\infty = (T/T_\infty)^m,\ 0.5 \le m \le 1$ ...(7)

For air at ordinary temperature, we take m = 0.76. As the temperature increases, m decreases, towards 0.5

It has been shown that at high temperatures the relation (6) can be well approximated by (7), when $0.5 \le m \le 0.75$ and at low temperature the appropriate value of m is 1.

Now, non dimensional number P_r (See Art. 15.7) is defined as

P_r = Prandial number = $(\mu C_p)/k$ ...(8)

It has been shown that P_r is constant for air even at large temperature differences. Since C_p is also nearly constant for a wide range of temperatures around ordinary temperatures, it follows from (8) that the coefficient of heat conductivity k is directly proportional to μ. Therefore, the dependence of the coefficient of heat conductivity k on temperature is of similar nature as that of viscosity.

The components of stress at any point (x, y, z).

$\sigma_{xx} = 2\mu(\partial u/\partial x) - (2\mu/3)\underline{\nabla}\cdot\mathbf{q} - p,$ $\sigma_{yy} = 2\mu(\partial v/\partial y) - (2\mu/3)\underline{\nabla}\cdot\mathbf{q} - p,$

$\sigma_{zz} = 2\mu(\partial w/\partial z) - (2\mu/3)\underline{\nabla}\cdot\mathbf{q} - p,$ $\sigma_{xy} = \sigma_{yx} = \mu(\partial u/\partial y + \partial v/\partial x),$

$\sigma_{yz} = \sigma_{zy} = \mu(\partial v/\partial z + \partial w/\partial y),$ $\sigma_{zx} = \sigma_{xz} = \mu(\partial w/\partial x + \partial u/\partial z)$

The components of the heat-flux vector

$Q_x = -k(\partial T/\partial x),$ $Q_y = -k(\partial T/\partial y),$ $Q_z = -k(\partial T/\partial z)$

Case II For flow of viscous incompressible fluid

While dealing with incompressible fluid flow, we suppose that fluid properties such as density ρ, coefficient of viscosity μ and coefficient of heat conductivity k are nearly constant. Accordingly, the number of unknown quantities reduce to five (u, v, w, p and T), which are obtained with the help of the following fundamental equations

Equation of continuity: $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$ ...(1)'

The Narier-Stokes equations:

$$\rho\{\partial u/\partial t + u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z)\}$$
$$= \rho B_x - \partial p/\partial x + \mu(\partial^2 u/\partial x^2 + \partial^2 v/\partial y^2 + \partial^2 w/\partial z^2) \quad \text{...(2a)'}$$

$$\rho\{\partial v/\partial t + u(\partial v/\partial x) + v(\partial v/\partial y) + w(\partial v/\partial z)\}$$
$$= \rho B_y - \partial p/\partial y + \mu(\partial^2 u/\partial x^2 + \partial^2 v/\partial y^2 + \partial^2 w/\partial z^2) \quad \text{...(2b)'}$$

$$\rho\{\partial w/\partial t + u(\partial w/\partial x) + v(\partial w/\partial y) + w(\partial w/\partial z)\}$$
$$= \rho B_z - \partial p/\partial z + \mu(\partial^2 u/\partial x^2 + \partial^2 v/\partial y^2 + \partial^2 w/\partial z^2) \quad \text{...(2c)'}$$

The energy equation

$$C_p T\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} + k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$
$$+\mu\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right] \quad \text{...(3)'}$$

The components of stress at any point (x, y, z)

$\sigma_{xx} = 2\mu(\partial u/\partial x) - p,$ $\sigma_{yy} = 2\mu(\partial v/\partial y) - p,$ $\sigma_{zz} = 2\mu(\partial w/\partial z) - p$

$\sigma_{xy} = \sigma_{yx} = \mu(\partial u/\partial y + \partial v/\partial x),$ $\sigma_{yz} = \sigma_{zy} = \mu(\partial v/\partial z + \partial w/\partial y),$ $\sigma_{zx} = \sigma_{xz} = \mu(\partial w/\partial x + \partial u/\partial z)$

The components of the heat flux vector

$Q_x = -k(\partial T/\partial x),$ $Q_y = -k(\partial T/\partial y),$ $Q_z = -k(\partial T/\partial z)$

Main difference between the compressible fluid flow and incompressible fluid flow

Observing carefully the above fundamental equations of compressible fluid flow and incompressible fluid flow, we see that, in compressible fluid flow, the equations of motion and energy are coupled whereas in an incompressible fluid flow, with constant fluid properties ρ, μ, k, the equations of motion and energy are uncoupled. Accordingly, while dealing with flow of incompressible fluid flow, the equation of continuity and equations of motion are first solved for u, v, w and finally the equation of energy is solved for the temperature.

Remark The above fundamental equations are solved subject to given initial and boundary conditions. The boundary condition are those required by geometrical considerations, together with the no-slip condition which states that on a wall the tangential component of relative velocity must be zero. To solve energy equation some conditions must be imposed on the temperature on the boundary and will be provided by the given problem.

14.11. Summary of basic equations governing the flow of viscous fluid in cylindrical co-ordinates (r, θ, z).

Equation of continuity (Refer Art. 2.10)

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r\, q_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta)+\frac{\partial}{\partial z}(\rho q_z)=0 \qquad ...(1)$$

(*a*) **Cylinrical coordinate system.** The Navier – stokes equations of motion of viscous compressible fluids in cylindrical coordinates (r,θ,z) are given by

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2}{r}\right)=\rho B_r+\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{\partial \sigma_{zr}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} \qquad ...(1a)$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}\right)=\rho B_\theta+\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{\partial \sigma_{\theta z}}{\partial z}+\frac{2\sigma_{r\theta}}{r} \qquad ...(1b)$$

$$\rho\frac{Dq_z}{Dt}=\rho B_z+\frac{\partial \sigma_{zr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{\sigma_{zr}}{r} \qquad ...(1c)$$

where
$$\frac{D}{Dt}=\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial r}+\frac{q_\theta}{r}\frac{\partial}{\partial \theta}+q_z\frac{\partial}{\partial z}. \qquad ...(2)$$

Constitutive equations are given by

$$\left.\begin{aligned}\sigma_{rr}&=-p+2\mu\in_{rr}-(2/3)\times\mu\nabla\cdot\mathbf{q}\\ \sigma_{\theta\theta}&=-p+2\mu\in_{\theta\theta}-(2/3)\times\mu\nabla\cdot\mathbf{q}\\ \sigma_{zz}&=-p+2\mu\in_{zz}-(2/3)\times\mu\nabla\cdot\mathbf{q}\end{aligned}\right\} \qquad ...(3)$$

$$\sigma_{r\theta}=\mu\gamma_{r\theta},\qquad \sigma_{\theta z}=\mu\gamma_{\theta z},\qquad \sigma_{zr}=\mu\gamma_{zr} \qquad ...(4)$$

Again the components of the rates of strain are given by

$$\in_{rr}=\frac{\partial q_r}{\partial r},\qquad \in_{\theta\theta}=\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{q_r}{r},\qquad \in_{zz}=\frac{\partial q_z}{\partial z} \qquad ...(5)$$

$$\left.\begin{aligned}\gamma_{r\theta}&=\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}+\frac{1}{r}\frac{\partial q_r}{\partial \theta}\\ \gamma_{\theta z}&=\frac{1}{r}\frac{\partial q_z}{\partial \theta}+\frac{\partial q_\theta}{\partial z}\\ \gamma_{zr}&=\frac{\partial q_r}{\partial z}+\frac{\partial q_z}{\partial r}\end{aligned}\right\} \qquad ...(6)$$

Using (3), (4), (5) and (6), the equations of motion (1*a*) to (1*c*) may be re-written as follows:

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2}{r}\right)=\rho B_r-\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left[\mu\left(2\frac{\partial q_r}{\partial r}-\frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]$$
$$+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{1}{r}\frac{\partial q_r}{\partial \theta}+\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial q_r}{\partial z}+\frac{\partial q_z}{\partial r}\right)\right]+\frac{2\mu}{r}\left(\frac{\partial q_r}{\partial r}-\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}-\frac{q_r}{r}\right) \qquad ...(7a)$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}\right)=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{2}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{2q_\theta}{r}-\frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]$$

$$+\frac{\partial}{\partial r}\left[\mu\left(\frac{1}{r}\frac{\partial q_r}{\partial\theta}+\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{1}{r}\frac{\partial q_z}{\partial\theta}+\frac{\partial q_\theta}{\partial z}\right)\right]+\frac{2\mu}{r}\left(\frac{1}{r}\frac{\partial q_r}{\partial\theta}+\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}\right), \quad ...(7b)$$

$$\rho\frac{Dq_z}{Dt}=\rho B_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial q_z}{\partial z}-\frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{1}{r}\frac{\partial q_z}{\partial\theta}+\frac{\partial q_\theta}{\partial z}\right)\right]+\frac{\partial}{\partial r}\left[\mu\left(\frac{\partial q_r}{\partial z}+\frac{\partial q_z}{\partial r}\right)\right]+\frac{\mu}{r}\left(\frac{\partial q_r}{\partial z}+\frac{\partial q_z}{\partial r}\right) \quad ...(7c)$$

where
$$\nabla\cdot\mathbf{q}=\frac{\partial q_r}{\partial r}+\frac{1}{r}\frac{\partial q_\theta}{\partial\theta}+\frac{\partial q_z}{\partial z}+\frac{q_r}{r}. \quad ...(8)$$

For some particular flows, equations (7*a*) to (7*c*) take the following forms:

(*i*) **Viscous compressible fluid with constant viscosity.**

Let μ be constant. Then (7*a*) to (7*c*) reduce as follows.

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2}{r}\right)=\rho B_r-\frac{\partial p}{\partial r}+\mu\left[\nabla^2 q_r-\frac{q_r}{r^2}-\frac{2}{r^2}\frac{\partial q_\theta}{\partial\theta}+\frac{1}{3}\frac{\partial}{\partial r}(\nabla\cdot\mathbf{q})\right], \quad ...(9a)$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}\right)=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left[\nabla^2 q_\theta+\frac{2}{r^2}\frac{\partial q_r}{\partial\theta}-\frac{q_\theta}{r^2}+\frac{1}{3r}\frac{\partial}{\partial\theta}(\nabla\cdot\mathbf{q})\right] \quad ...(9b)$$

and
$$\rho\frac{Dq_z}{Dt}=\rho B_z-\frac{\partial p}{\partial z}+\mu\left[\nabla^2 q_z+\frac{1}{3}\frac{\partial}{\partial z}(\nabla\cdot\mathbf{q})\right], \quad ...(9c)$$

where
$$\nabla^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial}{\partial z^2}. \quad ...(10)$$

(*ii*) **Viscous incompressible flow.** Let ρ and μ be both constant. Also $\nabla\cdot\mathbf{q}=0$ for incompressible fuids. Then (9*a*) to (9*c*) become

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2}{r}\right)=\rho B_r-\frac{\partial p}{\partial r}+\mu\left(\nabla^2 q_r-\frac{q_r}{r^2}-\frac{2}{r^2}\frac{\partial q_\theta}{\partial\theta}\right), \quad ...(11a)$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}\right)=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\nabla^2 q_\theta+\frac{2}{r^2}\frac{\partial q_r}{\partial\theta}-\frac{q_\theta}{r}\right) \quad ...(11b)$$

$$\rho\frac{Dq_z}{Dt}=\rho B_z-\frac{\partial p}{\partial z}+\mu\nabla^2 q_z \quad ...(11c)$$

(*iii*) **Non viscous fluid.** With $\mu=0$, equations (9*a*) to (9*c*) reduce to

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2}{r}\right)=\rho B_r-\frac{\partial p}{\partial r} \quad ...(12a)$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}\right)=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial\theta} \quad ...(12b)$$

$$\rho\frac{Dq_z}{Dt}=\rho B_z-\frac{\partial p}{\partial z} \quad ...(12c)$$

(*iv*) **Axi-symmetric flow of incompressible fluids** $(\partial/\partial\theta=0)$:

$$\rho\left\{\frac{\partial q_r}{\partial t}+q_r\frac{\partial q_r}{\partial r}+q_z\frac{\partial q_r}{\partial z}+\frac{q_\theta^2}{r}\right\}=\rho B_r-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rq_r)\right\}+\frac{\partial^2 q_r}{\partial z^2}\right] \quad ...(13a)$$

$$\rho\left\{\frac{\partial q_\theta}{\partial t}+q_r\frac{\partial q_\theta}{\partial r}+q_z\frac{\partial q_\theta}{\partial z}+\frac{q_r q_\theta}{r}\right\}=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rq_\theta)\right\}+\frac{\partial^2 q_\theta}{\partial z^2}\right] \qquad ...(13b)$$

$$\rho\left\{\frac{\partial q_z}{\partial t}+q_r\frac{\partial q_z}{\partial r}+q_z\frac{\partial q_z}{\partial z}\right\}=\rho B_z-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r}\right)+\frac{\partial^2 q_z}{\partial z^2}\right] \qquad ...(13c)$$

Energy eqution:

(*i*) **For viscous compressible fluid:** Equation of energy of a viscous compressible fluid in cylinderical polar coordinates (r,θ,z) is given by

$$\rho\frac{D}{Dt}(C_pT)=\frac{Dp}{Dt}+\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\mathbf{\Phi}, \qquad ...(14)$$

where
$$\frac{D}{Dt}=\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial r}+\frac{q_\theta}{r}\frac{\partial}{\partial \theta}+q_z\frac{\partial}{\partial z},$$

$$\mathbf{\Phi}=2\mu\left[\left(\frac{\partial q_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{q_r}{r}\right)^2+\left(\frac{\partial q_z}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}+\frac{1}{r}\frac{\partial q_r}{\partial \theta}\right)^2\right.$$
$$\left.+\frac{1}{2}\left(\frac{1}{r}\frac{\partial q_z}{\partial \theta}+\frac{\partial q_\theta}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial q_r}{\partial z}+\frac{\partial q_z}{\partial r}\right)^2-\frac{1}{3}(\nabla\cdot\mathbf{q})^2\right] \qquad ...(15)$$

and
$$\nabla\cdot\mathbf{q}=\frac{\partial q_r}{\partial r}+\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{\partial q_z}{\partial z}+\frac{q_r}{r}. \qquad ...(16)$$

(*ii*) **For viscous incompressible fluid:** Equation of energy of a viscous incompressible fluid for which k, ρ and μ are constants in cylindrical polar coordinates (r,θ,z) is given by

$$\rho C_v\frac{DT}{Dt}=k\nabla^2 T+\Phi, \qquad \text{where} \qquad \frac{D}{Dt}=\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial r}+\frac{q_\theta}{r}\frac{\partial}{\partial \theta}+q_z\frac{\partial}{\partial z} \qquad ...(17)$$

and
$$\Phi=2\mu\left[\left(\frac{\partial q_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{q_r}{r}\right)^2+\left(\frac{\partial q_z}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}+\frac{1}{r}\frac{\partial q_r}{\partial \theta}\right)^2\right.$$
$$\left.+\frac{1}{2}\left(\frac{1}{r}\frac{\partial q_z}{\partial \theta}+\frac{\partial q_\theta}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial q_r}{\partial z}+\frac{\partial q_z}{\partial r}\right)^2\right]. \qquad ...(18)$$

Equation of state : $p=\rho RT$ ...(19)

The components of stress at any point (r, θ, z)

(*i*) **For compressible viscous fluid**

$$\sigma_{rr}=2\mu(\partial q_r/\partial r)-(2/3)\times\mu\nabla\cdot\mathbf{q}-p \qquad ...(20a)$$

$$\sigma_{\theta\theta}=2\mu\left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{q_r}{r}\right)-\frac{2}{3}\mu\nabla\cdot\mathbf{q}-p \qquad ...(20b)$$

$$\sigma_{zz}=2\mu(\partial q_z/\partial z)-(2/3)\times\mu\nabla\cdot\mathbf{q}-p \qquad ...(20c)$$

$$\sigma_{r\theta}=\sigma_{\theta r}=\mu\left(\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}+\frac{1}{r}\frac{\partial q_r}{\partial \theta}\right)=\mu\left\{r\frac{d}{dr}\left(\frac{q_\theta}{r}\right)+\frac{1}{r}\frac{\partial q_r}{\partial \theta}\right\} \qquad ...(20d)$$

$$\sigma_{\theta z}=\sigma_{z\theta}=\mu\left(\frac{1}{r}\frac{\partial q_z}{\partial\theta}+\frac{\partial q_\theta}{\partial z}\right),\qquad \sigma_{zr}=\sigma_{rz}=\mu\left(\frac{\partial q_r}{\partial z}+\frac{\partial q_z}{\partial r}\right) \qquad ...(20e)$$

(*ii*) **For incompressible visocus fluid**

$$\sigma_{rr}=2\mu\left(\frac{\partial q_r}{\partial r}\right),\qquad \sigma_{\theta\theta}=2\mu\left(\frac{1}{r}\frac{\partial q_\theta}{\partial\theta}+\frac{q_r}{r}\right),\qquad \sigma_{zz}=2\mu\frac{\partial q_z}{\partial z} \qquad ...(21a)$$

$$\sigma_{r\theta}=\sigma_{\theta r}=\mu\left(\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}+\frac{1}{r}\frac{\partial q_r}{\partial\theta}\right)=\mu\left\{r\frac{d}{dr}\left(\frac{q_\theta}{r}\right)+\frac{1}{r}\frac{\partial q_r}{\partial\theta}\right\} \qquad ...(21b)$$

$$\sigma_{\theta z}=\sigma_{z\theta}=\mu\left(\frac{1}{r}\frac{\partial q_z}{\partial\theta}+\frac{\partial q_\theta}{\partial r}\right),\qquad \sigma_{zr}=\sigma_{rz}=\mu\left(\frac{\partial q_r}{\partial z}+\frac{\partial q_z}{\partial r}\right) \qquad ...(21c)$$

The components of heat-flux vector are

$$Q_r=-k\,(\partial T/\partial r),\qquad Q_\theta=-(k/r)\times(\partial T/\partial\theta),\qquad Q_z=-k\,(\partial T/\partial z) \qquad ...(22)$$

14.12. Summary of basic equations geverning the flow of viscous fluid in spherical coordinates (r,θ,ϕ).

Equation of continutiy (Refer Art. 2.11)

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\sin\theta\, q_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho\, q_\phi)=0 \qquad ...(1)$$

(*b*) **Spherical coordinate system.** The Navier–Stokes equations of motion of viscous compressible fluids in spherical coordinates (r,θ,ϕ) are given by

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2+q_\phi^2}{r}\right)=\rho B_r+\frac{1}{r\sin\theta}\left[\frac{\sin\theta}{r}\frac{\partial(r^2\sigma_{rr})}{\partial r}+\frac{\partial(\sigma_{r\theta}\sin\theta)}{\partial\theta}+\frac{\partial\sigma_{\phi r}}{\partial\theta}\right]-\frac{\sigma_{\theta\theta}+\sigma_{\phi\phi}}{r} \qquad ...(1a)$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}-\frac{q_\phi^2\cot\theta}{r}\right)=\rho B_\theta+\frac{1}{r\sin\theta}\left[\frac{\sin\theta}{r}\frac{\partial(r^2\sigma_{r\theta})}{\partial r}+\frac{\partial(\sigma_{\theta\theta}\sin\theta)}{\partial\theta}+\frac{\partial\sigma_{\theta\phi}}{\partial\phi}\right]$$
$$+\frac{\sigma_{r\theta}}{r}-\frac{\sigma_{\phi\phi}\cot\theta}{r} \qquad ...(1b)$$

$$\rho\left(\frac{Dq_\phi}{Dt}+\frac{q_\phi q_r}{r}-\frac{q_\phi q_\theta\cot\theta}{r}\right)=\rho B_z+\frac{1}{r\sin\theta}\left[\frac{\sin\theta}{r}\frac{\partial(r^2\sigma_{\phi r})}{\partial r}+\frac{\partial(\sigma_{\theta\phi}\sin\theta)}{\partial\theta}+\frac{\partial\sigma_{\phi\phi}}{\partial\phi}\right]$$
$$+\frac{\partial_{\phi r}}{r}-\frac{\sigma_{\theta\phi}\cot\theta}{r} \qquad ...(1c)$$

where
$$\frac{D}{Dt}=\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial r}+\frac{q_\theta}{r}\frac{\partial}{\partial\theta}+\frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}. \qquad ...(2)$$

The constitutive equations are given by

$$\left.\begin{aligned}\sigma_{rr}&=-p+2\mu\,\epsilon_{rr}-(2\mu/3)\times\nabla\cdot\mathbf{q}\\ \sigma_{\theta\theta}&=-p+2\mu\,\epsilon_{\theta\theta}-(2\mu/3)\times\nabla\cdot\mathbf{q}\\ \sigma_{\phi\phi}&=-p+2\mu\,\epsilon_{\phi\phi}-(2\mu/3)\times\nabla\cdot\mathbf{q}\end{aligned}\right\} \qquad ...(3)$$

$$\sigma_{r\theta}=\mu\,\gamma_{r\theta},\qquad \sigma_{\theta\phi}=\mu\,\gamma_{\theta\phi},\qquad \sigma_{\phi r}=\mu\,\gamma_{\phi r} \qquad ...(4)$$

Again, the components of the rates of strain are given by

$$\left.\begin{aligned}
&\epsilon_{rr}=\frac{\partial q_r}{\partial r}, \qquad \epsilon_{\theta\theta}=\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{q_r}{r}, \qquad \epsilon_{\phi\phi}=\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial \phi}+\frac{q_r}{r}+\frac{q_\theta\cot\theta}{r}\\
&\gamma_{r\theta}=r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)+\frac{1}{r}\frac{\partial q_r}{\partial \theta},\ \gamma_{\theta\phi}=\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{q_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial \phi},\ \gamma_{\phi r}=\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial \phi}+r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)
\end{aligned}\right\} \quad ...(5)$$

Using (3), (4) and (5), the equations of motion (1*a*) to (1*b*) and (1*c*) may be re-written as:

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2+q_\phi^2}{r}\right)=\rho B_r-\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left[\mu\left(2\frac{\partial q_r}{\partial r}-\frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)+\frac{1}{r}\frac{\partial q_r}{\partial \theta}\right\}\right]$$

$$+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left[\mu\left\{\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial \phi}+r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)\right\}\right]$$

$$+\frac{\mu}{r}\left[4\frac{\partial q_r}{\partial r}-\frac{2}{r}\frac{\partial q_\theta}{\partial \theta}-\frac{4q_r}{r}-\frac{2}{r\sin\theta}\frac{\partial q_\phi}{\partial \phi}-\frac{2q_\theta\cot\theta}{r}+r\cot\theta\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)+\frac{\cot\theta}{r}\frac{\partial q_r}{\partial \theta}\right] \quad ...(6a)$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}-\frac{q_\phi^2\cot\theta}{r}\right)=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{2}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{2q_r}{r}-\frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]$$

$$+\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left[\mu\left\{\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{q_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial \phi}\right\}\right]+\frac{\partial}{\partial r}\left[\mu\left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)+\frac{1}{r}\frac{\partial q_r}{\partial \theta}\right\}\right]$$

$$+\frac{\mu}{r}\left[2\left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}-\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial \phi}-\frac{q_\theta\cot\theta}{r}\right)\cot\theta+3\left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}+\frac{1}{r}\frac{\partial q_r}{\partial \theta}\right)\right\}\right] \quad ...(6b)$$

$$\rho\left(\frac{Dq_\phi}{Dt}+\frac{q_\phi q_r}{r}+\frac{q_\theta q_\phi\cot\theta}{r}\right)=\rho B_\phi-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}$$

$$+\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left[\mu\left(\frac{2}{r\sin\theta}\frac{\partial q_\phi}{\partial \phi}+\frac{2q_r}{r}+\frac{2q_\theta\cot\theta}{r}-\frac{2}{3}\nabla\cdot\mathbf{q}\right)\right]$$

$$+\frac{\partial}{\partial r}\left[\mu\left\{\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial \phi}+r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)\right\}\right]+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left\{\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{q_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial \phi}\right\}\right]$$

$$+\frac{\mu}{r}\left[3\left\{\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial \phi}+r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)\right\}+2\cot\theta\left\{\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{q_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial \phi}\right\}\right] \quad ...(6c)$$

where
$$\nabla\cdot\mathbf{q}=\frac{1}{r}\frac{\partial}{\partial r}(r^2 q_r)+\frac{1}{r\sin\theta}\frac{\partial(q_\theta\sin\theta)}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial \phi}. \quad ...(7)$$

For some particular flows, equations (6*a*), (6*b*) and (6*c*) take the following forms:

(*i*) **Viscous compressible fluid with constant viscosity i.e. with μ = constant.**

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2+q_\phi^2}{r}\right)=\rho B_r-\frac{\partial p}{\partial r}+\mu\left[\nabla^2 q_r-\frac{2q_r}{r^2}-\frac{2}{r^2}\frac{\partial q_\theta}{\partial \theta}-\frac{2q_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial q_\phi}{\partial \phi}\right.$$

$$+\frac{1}{3}\left\{\frac{\partial^2 q_r}{\partial r^2}+\frac{2}{r}\left(\frac{\partial q_r}{\partial r}-\frac{q_r}{r}\right)+\frac{1}{r}\frac{\partial^2 q_\theta}{\partial r\partial\theta}-\frac{1}{r^2}\left(\frac{\partial q_\theta}{\partial\theta}+q_\theta\cot\theta\right)+\frac{\cot\theta}{r}\frac{\partial q_\theta}{\partial r}+\frac{1}{r\sin\theta}\left(\frac{\partial^2 q_\phi}{\partial r\partial\phi}-\frac{1}{r}\frac{\partial q_\phi}{\partial\phi}\right)\right\}\Bigg] \quad ...(8a)$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}-\frac{q_\phi^2\cot\theta}{r^2}\right)=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\Bigg[\nabla^2 q_\theta+\frac{2}{r^2}\frac{\partial q_r}{\partial\theta}-\frac{q_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\phi}{\partial\phi}$$

$$+\frac{1}{3}\left\{\frac{1}{r^2}\frac{\partial^2 q_\theta}{\partial\theta^2}+\frac{1}{r}\left(\frac{\partial^2 q_r}{\partial\theta\partial r}+\frac{2}{r}\frac{\partial q_r}{\partial\theta}+\frac{\cot\theta}{r}\frac{\partial q_\theta}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\left(\frac{\partial^2 q_\phi}{\partial\theta\partial\phi}-\cot\theta\frac{\partial q_\phi}{\partial\phi}-q_\theta\right)\right\}\Bigg] \quad ...(8b)$$

$$\rho\left(\frac{Dq_\phi}{Dt}+\frac{q_\phi q_r}{r}+\frac{q_\theta q_\phi\cot\theta}{r}\right)=\rho B_\phi-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}+\mu\Bigg[\nabla^2 q_\phi-\frac{q_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial q_r}{\partial\phi}$$

$$+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\theta}{\partial\phi}+\frac{1}{3r\sin\theta}\left\{\frac{1}{r\sin\theta}\frac{\partial^2 q_\phi}{\partial\phi^2}+\frac{\partial^2 q_r}{\partial\phi\partial r}+\frac{2}{r}\frac{\partial q_r}{\partial\phi}+\frac{1}{r}\frac{\partial^2 q_\theta}{\partial\phi\partial\theta}+\frac{\cos\theta}{r\sin\theta}\frac{\partial q_\theta}{\partial\varphi}\right\}\Bigg] \quad ...(8c)$$

(*ii*) **Viscous incompressible fluid.** For such fluids, ρ = constant, μ = constant and $\nabla\cdot\mathbf{q}=0$. Hence (8*a*) to (8*c*) become

$$\frac{Dq_r}{Dt}-\frac{q_\theta^2+q_\phi^2}{r}=B_r-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 q_r-\frac{2q_r}{r^2}-\frac{2}{r^2}\frac{\partial q_\theta}{\partial\theta}-\frac{2q_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial q_\phi}{\partial\phi}\right) \quad ...(9a)$$

$$\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}-\frac{q_\phi^2\cot\theta}{r^2}=B_\theta-\frac{1}{\rho r}\frac{\partial p}{\partial\theta}+\nu\left(\nabla^2 q_\theta+\frac{2}{r^2}\frac{\partial q_r}{\partial\theta}-\frac{q_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\phi}{\partial\phi}\right) \quad ...(9b)$$

$$\frac{Dq_\phi}{Dt}+\frac{q_\phi q_r}{r}+\frac{q_\theta q_\phi\cot\theta}{r}=B_\phi-\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\phi}$$

$$+\nu\left(\nabla^2 q_\phi-\frac{q_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial q_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\theta}{\partial\varphi}\right) \quad ...(9c)$$

where $$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\theta^2} \quad ...(10)$$

(*iii*) **Non-viscous fluid (with $\mu = 0$).** Then (8*a*) to (8*c*) become

$$\rho\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2+q_\phi^2}{r}\right)=\rho B_r-\frac{\partial p}{\partial r} \quad ...(11a)$$

$$\rho\left(\frac{Dq_\theta}{Dt}+\frac{q_r q_\theta}{r}-\frac{q_\phi^2\cot\theta}{r}\right)=\rho B_\theta-\frac{1}{r}\frac{\partial p}{\partial\theta} \quad ...(11b)$$

$$\rho\left(\frac{Dq_\phi}{Dt}+\frac{q_\phi q_r}{r}+\frac{q_\theta q_\phi\cot\theta}{r}\right)=\rho B_\phi-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} \quad ...(11c)$$

Energy Equation

(*i*) **For viscous compressible fluid:** Equation of energy of viscous compressible fluid in spherical polar coordinates (r,θ,ϕ) is

$$\rho\frac{D}{Dt}(C_pT)=\frac{Dp}{Dt}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(k\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(k\frac{\partial T}{\partial\phi}\right)+\mathbf{\Phi},\quad ...(12)$$

where
$$\frac{D}{Dt}=\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial r}+\frac{q_\theta}{r}\frac{\partial}{\partial\theta}+\frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial\phi},\quad ...(13)$$

$$\mathbf{\Phi}=\mu\left[\left\{2\left(\frac{\partial q_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial q_\theta}{\partial\theta}+\frac{q_r}{r}\right)^2+\left(\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi}+\frac{q_r}{r}+\frac{q_\theta\cot\theta}{r}\right)^2\right\}+\left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)+\frac{1}{r}\frac{\partial q_r}{\partial\theta}\right\}^2\right.$$

$$\left.+\left\{\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{q_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial\phi}\right\}^2+\left\{\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)\right\}^2\right]-\frac{2}{3}\mu(\nabla\cdot\mathbf{q})^2\quad ...(14)$$

and
$$\nabla\cdot\mathbf{q}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2q_r)+\frac{1}{\sin\theta}\frac{\partial}{\partial r}(q_\theta\sin\theta)\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi}.\quad ...(15)$$

(*ii*) **For viscous incompressible fluid:** Equation of energy of a viscous incompressible fluid in spherical polar coordinates (r,θ,z) is given by

$$\rho C_v\frac{DT}{Dt}=k\nabla^2T+\Phi,\quad ...(16)$$

where
$$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial r}+\frac{q_\theta}{r}\frac{\partial}{\partial\theta}+\frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial\phi},\quad ...(17)$$

$$\nabla^2\equiv\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\quad ...(18)$$

and
$$\Phi\equiv\mu\left[2\left\{\left(\frac{\partial q_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial q_\theta}{\partial\theta}+\frac{q_r}{r}\right)^2+\left(\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi}+\frac{q_r}{r}+\frac{q_\theta\cot\theta}{r}\right)^2\right\}\right.$$

$$\left.+\left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)+\frac{1}{r}\frac{\partial q_r}{\partial\theta}\right\}^2+\left\{\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{q_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial\phi}\right\}^2+\left\{\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\phi}+r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)\right\}^2\right].\quad ...(19)$$

Equation of state : $p=\rho RT$ ...(20)

The components of stress at any point (r,θ,ϕ)

(*i*) **For compressible viscous fluid**

$$\sigma_{rr}=2\mu\left(\frac{\partial q_r}{\partial r}\right)-\frac{2}{3}\mu\nabla\cdot\mathbf{q},\quad ...(21a)$$

$$\sigma_{\theta\theta}=2\mu\left(\frac{1}{r}\frac{\partial q_\theta}{\partial_\theta}+\frac{q_r}{r}\right)-\frac{2}{3}\mu\nabla\cdot\mathbf{q}\quad ...(21b)$$

$$\sigma_{\phi\phi}=2\mu\left(\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi}+\frac{q_r}{r}+\frac{q_\theta\cot\theta}{r}\right)-\frac{2}{3}\mu\nabla\cdot\mathbf{q}\quad ...(21c)$$

$$\sigma_{r\theta}=\sigma_{\theta r}=\mu\left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)+\frac{1}{r}\frac{\partial q_r}{\partial\theta}\right\}=\mu\left(\frac{\partial q_\theta}{\partial r}-\frac{q_\theta}{r}+\frac{1}{r}\frac{\partial q_r}{\partial\theta}\right)\quad ...(21d)$$

...(21*d*)

$$\sigma_{\theta\phi} = \sigma_{\phi\theta} = \mu\left\{\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{q_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial\phi}\right\}, \qquad \text{...(21}e\text{)}$$

$$\sigma_{\phi r} = \sigma_{r\phi} = \mu\left\{\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)\right\} \qquad \text{...(21}f\text{)}$$

(*ii*) **For incompressible viscous fluid**

$$\sigma_{rr} = 2\mu\frac{\partial q_r}{\partial r}, \qquad \sigma_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial q_\theta}{\partial\theta} + \frac{q_r}{r}\right), \qquad \text{...(22}a\text{)}$$

$$\sigma_{\phi\phi} = 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi} + \frac{q_r}{r} + \frac{q_\theta\cot\theta}{r}\right) \qquad \text{...(22}b\text{)}$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \mu\left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right) + \frac{1}{r}\frac{\partial q_r}{\partial\theta}\right\} \qquad \text{...(22}c\text{)}$$

$$\sigma_{\theta\phi} = \sigma_{\phi\theta} = \mu\left\{\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{q_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial\phi}\right\} \qquad \text{...(22}d\text{)}$$

$$\sigma_{\phi r} = \sigma_{r\phi} = \mu\left\{\frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)\right\} \quad . \qquad \text{..(22}e\text{)}$$

The components of heat-flux vector are

$$Q_r = -k\frac{\partial T}{\partial r}, \qquad Q_\theta = -\frac{k}{r}\frac{\partial T}{\partial\theta}, \qquad Q_\phi = -\frac{k}{r\sin\theta}\frac{\partial T}{\partial\phi} \quad \text{...(23)}$$

EXERCISES

1. Find an expression for the rate of dissipation of energy of a liquid due to viscosity. Discuss the motion of a viscous liquid for which there is no dissipation of a viscosity.

Prove that for a liquid filling a closed vessel which is at rest, the rate of dissipation of energy due to viscosity is $\mu\iiint(\text{curl}\,\mathbf{q})^2\,dxdydz,$

where μ is the coefficient of viscosity and $\mathbf{q}$ the velocity vector.

2. Show that the velocity field $u(y) = -(h^2/8\mu)(dp/dx)\{1-4(y/h)^2\},\ v = w = 0,$

satisfies the equation of motion for the two-dimensional steady flow a viscous incompressible fluid with constant viscosity and constant pressure gradient.

3. The velocity components $q_r(r,\theta) = -U(1-a^2/r^2)\cos\theta,\quad q_\theta(r,\theta) = U(1+a^2/r^2)\sin\theta$

satisfy the equation of motion for a two-dimensional inviscid incompressible flow. Find the pressure associated with this velocity field. U and a are constants.

4. Derive Navier Stokes equations of motion for viscous compressible fluid and also deduce the equation for viscous compressible fluid with constant viscosity.

[Himanchal 2001; 03, 10; Meerut 2000, 01, 02]

5. Derive the hydrodymical equations of motion of viscous and incompressible fluid in cartesian form as obtained by Navier and Stokes. **[Garhwal 2005]**

6. Starting from the Navier-Stokes equation for the motion of an incompressible fluid moving under conservative forces, prove that the vorticity Ω satisfies the differential equation $D\Omega/dt = (\Omega\cdot\nabla)\mathbf{q} + \nu\nabla^2\Omega,$ ν being coefficient of kinetic viscosity. **[Himanchal 2000, 02]**

7. Define the principle of energy conservation. Derive energy equation for a compressible fluid and deduce it for incompressible fluids. **[Himanchal 1999, 99, 2000, 01]**

8. Derive vorticity transport equation and show that vorticity cannot originate within the interior of a viscous fluid but must be diffused from the boundary into the fluid.

Hint : Refer Art. 14.4 and its remark. **[Himanchal 1998, 2001]**

9. Derive the equation of energy for an incompressible fluid motion with constant fluid properties. **[Himanchal 2003, 09]**

10. Sate the constitutive equations for an isotropic Newtonian fluid and use it to derive the Navier-Stokes equations of motion for a viscous compressible fluid. **[Himanchal 1999]**

11. Find an expression for the rate of dissipation of energy of a liquid due to viscosity. **[Kanpur 2005]**

12. Derive Navier – Stokes equations of motion of a viscous fluid **(Himanchal 2009; Meerut 2008)**

13. Derive Navier – Stokes equations of motion of an incompressible fluid. **(Himanchal 2007)**

14. Write a short note on viscosity in a viscous incompressible fluid motion" **(Himanchal 2007)**

15. Derive the equation of energy with constant viscosity and heat conductivity of fluid. **(Himanchal 2007)**

16. Define the law of conservation of energy. Derive equation of energy and deduce it for flow of a viscous incompressible fluid. **(Himanchal 2006)**

17. Find the Navier – Stokes equations of motion for the flow of incompressible viscous fluid in cartesian coordinates. State the Green' theorem. **(Agra 2005, 06)**

18. For a non-viscous incompressible fluid the Navier – Stokes equations of motion are

(*a*) $\dfrac{\partial \mathbf{q}}{\partial t} - (\mathbf{q} \cdot \nabla)\mathbf{q} = \mathbf{B} + \dfrac{1}{\rho}\nabla p + \nu\nabla^2 \mathbf{q}$ (*b*) $\dfrac{\partial \mathbf{q}}{\partial t} - (\mathbf{q} \cdot \nabla)\mathbf{q} = \mathbf{B} - \dfrac{1}{\rho}\nabla p + \nu\nabla^2 \mathbf{q}$

(*c*) $\dfrac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{q} = \mathbf{B} + \dfrac{1}{\rho}\nabla p + \nu^2\nabla^2 \mathbf{q}$ (*d*) None of these **[Agra 2003, 05]**

Hint: Ans. (*d*). See equation (17), Art. 14.1.

19. Rate of dissipation of energy when there is no slip of the boundary is:

(*a*) $\mu\iiint(\xi^2 + \eta^2 + \zeta^2)\,dV$ (*b*) $2\mu\iiint(\xi^2 + \eta^2 + \zeta^2)\,dV$

(*c*) $3\mu\iiint(\xi^2 + \eta^2 + \zeta^2)\,dV$ (*d*) $4\mu\iiint(\xi^2 + \eta^2 + \zeta^2)\,dV$ **(Agra 2006)**

Hint: Ans. (*d*). Refer equation (17), Art. 14.6*B*.

20. Obtain Navier–Stokes equations of motion for viscous fluid in cartesian coordinates. **[Agra 2009; 10]**

CHAPTER

15 Dynamical Similarity Inspection Analysis and Dimensional Analysis

15.1. Dimensional homogeneity.

An equation is said to be dimensionally homogeneous if the powers of the fundamental dimensions are identical on both sides. Obviously such an equation do not depend on any system of measurement. The principle of dimensional homogeneity has the following main applications:

(*i*) *To obtain the dimension of a physical quantity.*

(*ii*) *To check the dimensional homogeneity of an equation.*

(*iii*) *To obtain the new coefficient of an equation when we employ other system of units.*

(*iv*) *Dimensional analysis.*

Ex. *Check the dimensional homogeneity of the following equaitons* :

(*i*) $p/\rho + q^2/2g + z = \text{const.}$

(*ii*) $\partial u/\partial x + \partial v/\partial y = 0$ **[Himanchal 2000; Meerut 1994]**

Sol. (*i*) The dimensions of each term of the given equation are :

$$\frac{p}{\rho} : \frac{ML^{-1}T^{-2}}{ML^{-3}} = L^2T^{-2}, \qquad \frac{q^2}{2g} : \frac{(LT^{-1})^2}{LT^{-1}} = L; \qquad z : L,$$

showing that dimensions of the second and third terms are L whereas the dimensions of the first term is not L. Hence the given equation is not dimensionally homogeneous.

(*ii*) The dimensions of each of the given equation are,

$$\partial u/\partial x : (LT^{-1})/L = T^{-1} \qquad \text{and} \qquad \partial v/\partial y : (LT^{-1})/L = T^{-1},$$

showing that the dimensions of each term of the given equation are same and hence the given equation is dimensionally homogeneous.

15.2. Model analysis.

In Chapter 14, we have obtained the fundamental equations governing the flow of a viscous compressive fluid. Since these equations are non-linear, there do not exist general methods for solving such equations. In chapter 16, we shall solve such equations in few particular cases under restricted conditions which will be valid for all ranges of viscosity. We shall further simplify these equations for two extreme cases of viscosity, very large and very small, and the theories corresponding to these extreme cases will be referred to as "*theory of slow motion*" and '*theory of boundary layers*". However, the cases of moderate viscosities cannot be interpolated from these two theories. Even in the above-mentioned two cases, we are faced with lot of mathematical difficulties. Hence the most of the research work on the behavior of viscous fluid is done by performing experiments.

In order to know about the performance of ships, aeroplanes, etc., before actually constructing them, their models are made and tested to obtain the necessary information. The actual structure or machine is known as *prototype* and model is geometrically similar but smaller (or larger) than prototype. The dimensions of the model should be such that the testing may become economical and convenient. It may be noted that design, construction and operation of a model may be changed several times if necessary, without increasing much expenditure, till we achieve the necessary required results. For example, before constructing an aeroplane, we create its model of a convenient size to test the effect of air resistance, vibrations, speed etc. of the proposed aeroplane.

15.3. Similitude.

In order to constuct the model which should have all the characteristics of the actual object (*i.e.* prototype) and should give the required information about the prototype, the following three similarities must be ensured between the model and the prototype :

(*i*) Geometrical similarity (*ii*) Kinematic similarity (*iii*) Dynamic similarity.

(*i*) **Geometrical similarly.** For existence of *geometrical similarity* between the model and the prototype, the ratios of the corresponding lengths in the model and in the prototype must be the same and the included angles between two corresponding sides must be the same. Let L_m, B_m, H_m, D_m, A_m and V_m be length, breath, height, diameter, area and volume of the model and let L_p, B_p, H_p, D_p, A_p and V_p be the corresponding values of the prototype. Then, for geometrical similarity, we must have

$$L_m / L_p = B_m / B_p = H_m / H_p = D_m / D_p = L_r$$

where L_r is called the scale ratio or the scale factor.

Similarly, A_r = area ratio $= A_m / A_p = L_r^2$ and V_r = volume ratio $V_m / V_p = L_r^3$.

(*ii*) **Kinematic similarity.** Kinematic similarity is connected with similarity of motion. If at the corresponding points in the model and in the prototype, the velocity or acceleration ratios are same and velocity or acceleration vectors have the same direction, the two flows are said to be *kinematically similar*. It follows that streamline patterns will be similar in the two flows.

Let $(V_1)_m$ = velocity of fluid at point 1 in the model,

$(V_2)_m$ = velocity of fluid at point 2 in the model,

$(a_1)_m$ = acceleration of fluid at point 1 in the model,

$(a_2)_m$ = acceleration of fluid at point 2 in the model.

Let $(V_1)_p$, $(V_2)_p$, $(a_1)_p$, $(a_2)_p$ be the corresponding values at the corresponding points of fluid velocity and acceleration in the prototype. Then for kinematic similarity, we must have

$$(V_1)_m / (V_1)_p = (V_2)_m / (V_2)_p = V_r \text{ (= velocity ratio)}$$

$$(a_1)_m / (a_1)_p = (a_2)_m / (a_2)_p = a_r \text{ (= acceleration ratio).}$$

Also, the directions of the velocities (or accelerations) in the model and in the prototype should be same. The geometrical similarity is a pre-requisite for kinematic similarity.

(*iii*) **Reynold' law of dynamic similarity.**

[Himanchal 1999, 2002; Kanpur 2005; Meerut 2002, 03, 04, 05]

Dynamic similarity is the similarity of forces. The flows in the model and in the prototype are dynamically similar if at all the corresponding points, identical types of forces are parallel and bear the same ratio.

Let $(F_i)_m$, $(F_v)_m$, $(F_g)_m$ be inertia, viscous and gravity forces respectively at a point in the model and let $(F_i)_p$, $(F_v)_p$, $(F_g)_p$ be corresponding values at the corresponding point in the prototype. Then, for dynamic similarity, we must have

$$\frac{(F_i)_m}{(F_i)_p} = \frac{(F_v)_m}{(F_v)_p} = \frac{(F_g)_m}{(F_g)_p} = F_r \text{ (= force ratio)}$$

Also, the directions of the corresponding forces at the corresponding points in the model and prototype should be the same.

15.4. Dynamical similarity. (Meerut 2009; Himanchal 2005, 06)

In general, analytical solutions of non-linear Navier-Stokes equations are difficult to obtain, except in a few simple situations (refer chapter 16). Even solutions for simple situations are based on idealizations such as infinite plates, infinitely long cylinders and so on. Obviously, then the research work in viscous fluids was undertaken by performing experiments. Moreover, in design of ships, aircraft, underwater projectiles, etc., it is necessary to carry out experiments on models and to investigate as to how far the results obtained with the models can be employed to predict the behaviour of prototype (full scale body). Such model analysis is based on the concept of *dynamical similarity or similar, flow.* Two fluid motions are said to be dynamically similar if, with similar geometrical boundaries, the velocity fields are geometrically similar, *i.e.*, if they have geometrically similar stream-lines. For two flows about similar geometrical bodies with different fluids, different velocities and different linear dimensions to be dynamically similar, the following condition must be fulfilled : At all geometrically similar points the forces acting on a fluid particle must bear a fixed ratio at every instant of time.

Osborne Reynolds was the first to consider the laws of similar flows. Evidently various laws of similar flows will result, depenting on what kinds of forces are being considered important in a particular situation.

We now proceed to discuss conditions under which the fluid motions are dynamically similar. In other words we shall obtain controlling parameters in fluid flow. There are two methods for getting these parameters. (*i*) Inspection analysis (*ii*) Dimensional analysis.

In the inspection analysis, we reduce the fundamental equation of fluid dynamics to a non-dimensional form and arrive at the non-dimensional numbers from the resulting equations. In dimensional analysis non-dimensional numbers are obtained from the physical quantities occurring in a problem, even when the knowledge of the governing equations is missing.

15.5. Inspection analysis in the case of incompressible viscous fluid flow. Reynold's principle of similarity. Reynold's number [Himanchel 2009]

The Navier-Stokes equations of motion of a viscous incompressible fluid in the absence of body forces are given by

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad ...(1a)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad ...(1b)$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{p}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \qquad ...(1c)$$

Let us consider flows about two similar geometrical bodies of different linear dimensions in streams of different velocities. Then these two flows will be dynamically similar if with a suitable choice of units of length, time and force (1*a*) to (1*c*) are so transformed that they completely coincide for the two different flows, This is done by using dimensionless quantities. Let L, V, P denote a characteristic* length, velocity and pressure respectively. Let $x', y', z', u', v', w', p'$ be dimensionless numbers such that

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* *e.g.*, the free stream velocity and the diameter of the sphere (if geometrical body happens to be a sphere) can be taken as characteristic velocity and length respectively, and so on.

$$x' = x/\mathrm{L}, \qquad y' = y/\mathrm{L}, \qquad z' = z/\mathrm{L} \qquad ...(2a)$$

$$u' = u/\mathrm{V}, \qquad v' = v/\mathrm{V}, \qquad w' = w/\mathrm{V} \qquad ...(2b)$$

and $$p' = p/\mathrm{P} \qquad ...(2c)$$

Using (2*a*) to (2*c*), we have

$$u\frac{\partial u}{\partial x} = (\mathrm{V}u')\frac{\partial(\mathrm{V}u')}{\partial(\mathrm{L}x')} = \frac{\mathrm{V}^2}{\mathrm{L}}u'\frac{\partial u'}{\partial x'}, \text{ etc.;} \qquad \frac{\partial p}{\partial x} = \frac{\partial\,(\mathrm{P}p')}{\partial\,(\mathrm{L}x')} = \frac{\mathrm{P}}{\mathrm{L}}\frac{\partial p'}{\partial x'}, \quad \text{etc.;}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial(\mathrm{L}x')}\left\{\frac{\partial\,(\mathrm{V}u')}{\partial(\mathrm{L}x')}\right\} = \frac{\mathrm{V}}{\mathrm{L}^2}\frac{\partial^2 u'}{\partial x'^2}, \text{ etc.}$$

Substituting these results, (1*a*) becomes

$$\frac{\mathrm{V}^2}{\mathrm{L}}\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'}\right) = -\frac{\mathrm{P}}{\rho\mathrm{L}}\frac{\partial p'}{\partial x'} + \frac{\nu\,\mathrm{V}}{\mathrm{L}^2}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\right)$$

or $$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'} = -\frac{\mathrm{P}}{\rho\mathrm{V}^2}\frac{\partial p'}{\partial x'} + \frac{\nu}{\mathrm{VL}}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\right) \qquad ...(3a)$$

Similarly, (1*b*) and (1*c*) transform into

$$u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} + w'\frac{\partial v'}{\partial z'} = -\frac{\mathrm{P}}{\rho\mathrm{V}^2}\frac{\partial p'}{\partial y'} + \frac{\nu}{\mathrm{VL}}\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2} + \frac{\partial^2 v'}{\partial z'^2}\right) \qquad ...(3b)$$

$$u'\frac{\partial w'}{\partial x'} + v'\frac{\partial w'}{\partial y'} + w'\frac{\partial w'}{\partial z'} = -\frac{\mathrm{P}}{\rho\mathrm{V}^2}\frac{\partial p'}{\partial z'} + \frac{\nu}{\mathrm{VL}}\left(\frac{\partial^2 w'}{\partial x'^2} + \frac{\partial^2 w'}{\partial y'^2} + \frac{\partial^2 w'}{\partial z'^2}\right) \qquad ...(3c)$$

In each of equations 3(*a*) to 3(*c*), the L.H.S. is entirely dimensionless. Hence the R.H.S. must likewise be so. It follows that the two quantities $\mathrm{P}/\rho\mathrm{V}^2, \nu/\mathrm{VL}$ must be dimensionless quantities. In order to creat a reliable model of a given incompressible flow it is essential to keep these two numbers constant.

The first number $(\mathrm{P}/\rho\mathrm{V}^2)$ ensures dynamic similarity in the two flows at points where viscosity is not important. Clearly such points exist at points situated far away from the boundaries.

The second number (ν/VL) ensures dynamic similarity at corresponding points near the boundaries where viscous effects are most important. Its reciprocal is called the *Reynold's number* and is denoted by Re so that $$\mathrm{Re} = (\mathrm{VL})/\nu = (\mathrm{VL}\rho)/\mu, \qquad \text{as } \nu = \mu/\rho \qquad ...(4)$$

Remark. In actual practice, in general, we do preserve geometrical similarily and we do not care for keeping both the numbers constant. In each simulation, we have to fix the main aims for achievement. For example, if wish to study the viscous drag on sailing ships, then we must keep the Reynold's number constant when modelling the full-scale body. Simultaneously, suppose we are unable to keep $(\mathrm{P}/\rho\mathrm{V}^2)$ constant then it means that the model will not be reliable reproduction of the prototype at points far away from the boundaries where the viscous effects are not important. Clearly this would not matter from the point of the aim of the proposed simulation.

15.6. Significance of Reynold's number. [Meerut 2005, 09, 10, 11; Himanchal 2003]

(*i*) Two flows of incompressible viscous fluid about similar geometrical bodies are dynamically similar when the Raynold's numbers for the flows are equal.

(*ii*) The Reynold's number throws light on important features of a given flow. Thus, for example, a small Reynold's number implies that viscosity is predominant whereas a large Reynold's number implies that viscosity is small.

(*iii*) It is experimentally shown that if the values of Reynold's number exceeds a certain critical value (namely 2,800) the flow ceases to be laminar and the flow becomes turbulent. When $\mathrm{Re} < 2,000$, the flow is laminar.

(*iv*) Concept of laminar boundary layer was developed by examining the flow for which Reynold's number is very large.

(*v*)Concept of very slow motion or creeping motion was developed by examining the flow for which Reynold number is very small.

15.7. Inspection analysis in case of flow of viscous compressible fluid. Theory of similarity in heat transfer. Controlling parameters in compressible flow. (Hemanchal 2005, 06)

Before proceeding to the dynamics of compressible fluids, it is important to obtain an orientation of the subject by considering the controlling parameters involved in the problem. These parameters are expressed in terms of flow properties which characterize the physical situation under consideration.

In what follows, we assume that μ, C_p and k are constants. Then the basic equations governing the unsteady flow of a viscous compressible fluid are given by

The equation of continuity [Refer equation (8), Art. 2.9]

$$\partial\rho/\partial t+\partial(\rho u)/\partial x+\partial(\rho v)/\partial y+\partial(\rho w)/\partial z=0 \qquad ...(1)$$

The Navier-Stokes equation (Refer equations (14a), (14b), (14c) of Art 14.1)

$$\rho\{\partial u/\partial t+u(\partial u/\partial x)+v(\partial u/\partial y)+w(\partial u/\partial z)\}=\rho B_x-\partial p/\partial x$$
$$+\mu\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}+\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right\} \qquad ...(2a)$$

$$\rho\{\partial v/\partial t+u(\partial v/\partial x)+v(\partial v/\partial y)+w(\partial v/\partial z)\}=\rho B_y-\partial p/\partial y$$
$$+\mu\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}+\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right\} \qquad ...(2b)$$

$$\rho\{\partial w/\partial t+u(\partial w/\partial x)+v(\partial w/\partial y)+w(\partial w/\partial z)\}=\rho B_z-\partial p/\partial z$$
$$+\mu\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}+\frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right\} \qquad ...(2c)$$

The energy equation : (Refer equation (17), Art 14.2]

$$C_p\rho\{\partial T/\partial t+u(\partial T/\partial x)+v(\partial T/\partial y)+w(\partial T/\partial z)\}$$
$$=\partial p/\partial t+u(\partial p/\partial x)+v(\partial p/\partial y)+w(\partial p/\partial z)+k(\partial^2T/\partial x^2+\partial^2T/\partial y^2+\partial^2T/\partial z^2)$$
$$+\mu\Big[2\{(\partial u/\partial x)^2+(\partial v/\partial y)^2+(\partial w/\partial z)^2\}-(2/3)\times(\partial u/\partial x+\partial v/\partial y+\partial w/\partial z)^2$$
$$+(\partial u/\partial y+\partial v/\partial x)^2+(\partial v/\partial z+\partial w/\partial y)^2+(\partial w/\partial x+\partial u/\partial z)^2\Big] \qquad ...(3)$$

The equation of state : $\quad p=\rho RT \qquad ...(4)$

In order to transform the above equations in a non-dimensional form we introduce non-dimensional quantities as follows :

$$\left.\begin{array}{lllll} x'=x/L_0, & y'=y/L_0, & z'=z/L_0, & u'=u/U_0, & v'=v/U_0, \\ w'=w/U_0, & p'=p/p_0, & \rho'=\rho/\rho_0, & T'=T/T_0, & B'_x=B_x/F_0, \\ B'_y=B_y/F_0, & B'_z=B_z/F_0, & t'=(tU_0)/L_0 & \text{and} & R'=(R\rho_0T_0)/p_0, \end{array}\right\}$$

where the quantities with dash denote dimensionless quantities and the quantities with subscript 'o' are certain reference values associated with the flow.

Substituting the above non-dimensional quantities into the above equations, we obtain

$$\partial\rho'/\partial t' + \partial(\rho' u')/\partial x' + \partial(\rho' v')/\partial y' + \partial(\rho' w')/\partial z' = 0 \qquad \text{(1)'}$$

$$\rho'\{\partial u'/\partial t' + u'(\partial u'/\partial x') + v'(\partial u'/\partial y') + w'(\partial u'/\partial z')\} = (F_0 L_0/U_0^2)\times\rho' B_x' - (p_0/\rho_0 U_0^2)\times(\partial p'/\partial x')$$

$$+\frac{\mu}{\rho_0 U_0 L_0}\left\{\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2}+\frac{\partial^2 w'}{\partial z'^2}+\frac{1}{3}\frac{\partial}{\partial x'}\left(\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}+\frac{\partial w'}{\partial z'}\right)\right\} \qquad \text{...(2a)'}$$

$$\rho'\{\partial v'/\partial t' + u'(\partial v'/\partial x') + v'(\partial v'/\partial y') + w'(\partial v'/\partial z')\} = (F_0 L_0/U_0^2)\times\rho' B_y' - (p_0/\rho_0 U_0^2)\times(\partial p'/\partial y')$$

$$+\frac{\mu}{\rho_0 U_0 L_0}\left\{\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2}+\frac{\partial^2 w'}{\partial z'^2}+\frac{1}{3}\frac{\partial}{\partial y'}\left(\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}+\frac{\partial w'}{\partial z'}\right)\right\} \qquad \text{...(2b)'}$$

$$\rho'\{\partial w'/\partial t' + u'(\partial w'/\partial x') + v'(\partial w'/\partial y') + w'(\partial w'/\partial z')\} = (F_0 L_0/U_0^2)\times\rho' B_z' - (p_0/\rho_0 U_0^2)\times(\partial p'/\partial z')$$

$$+\frac{\mu}{\rho_0 U_0 L_0}\left\{\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2}+\frac{\partial^2 w'}{\partial z'^2}+\frac{1}{3}\frac{\partial}{\partial z'}\left(\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}+\frac{\partial w'}{\partial z'}\right)\right\} \qquad \text{...(2c)'}$$

$$\rho' C_p\{\partial T'/\partial t' + u'(\partial T'/\partial x') + v'(\partial T'/\partial y') + w'(\partial T'/dz')\}$$

$$=\left(\frac{p_0}{\rho_0 U_0^2}\right)\left(\frac{U_0^2}{C_p T_0}\right)\left(\frac{\partial p'}{\partial t'}+u'\frac{\partial p'}{\partial x'}+v'\frac{\partial p'}{\partial y'}+w'\frac{\partial p'}{\partial z'}\right)+\frac{k}{C_p\rho_0 U_0 L_0}\left(\frac{\partial^2 T'}{\partial x'^2}+\frac{\partial^2 T'}{\partial y'^2}+\frac{\partial^2 T'}{\partial z'^2}\right)^2$$

$$+\left(\frac{\mu}{\rho_0 U_0 L_0}\right)\left(\frac{U_0^2}{C_p T_0}\right)\left[2\left\{\left(\frac{\partial u'}{\partial x'}\right)^2+\left(\frac{\partial v'}{\partial y'}\right)^2+\left(\frac{\partial w'}{\partial z'}\right)^2\right\}-\frac{2}{3}\left(\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}+\frac{\partial w'}{\partial z'}\right)^2\right.$$

$$\left.+(\partial u'/\partial y'+\partial v'/\partial x')^2+(\partial v'/\partial z'+\partial w'/\partial y')^2+(\partial w'/\partial x'+\partial u'/\partial z')^2\right] \qquad \text{...(3)'}$$

$$p' = \rho' R T' \qquad \text{...(4)'}$$

It is recognized that the solutions of equations (1)', (2a)', (2b)', (2c)', (3)' and (4)' depend on the following five dimensionless groups :

$$\frac{\mu}{\rho_0 U_0 L_0},\quad \frac{F_0 L_0}{U_0^2},\quad \frac{p_0}{\rho_0 U_0^2},\quad \frac{k}{C_{p_0}\rho_0 U_0 L_0}\quad\text{and}\quad \frac{U_0^2}{C_{p_0} T_0}, \qquad \text{...(5)}$$

We now re-write these dimensional groups in terms of some well know dimensionless numbers.

(*i*) $\mathrm{Re} = (\rho_0 U_0 L_0)/\mu$ (Reynolds number) ...(6)

which is the already familiar dimensionless number discussed in Art. 15.5.

(*ii*) $F_r = U_0^2/F_0 L_0 = U_0^2/g_0 L_0$ (Froude number) ...(7)

where we have taken the body force as gravitational force so that $F_0 = g_0$.

(*iii*) $\gamma = C_p/C_v$ (Ratio of specific heats) ...(8)

(*iv*) $Ma = U_0/c_0$ (Mach number) ...(9)

where c_0 denotes the velocity of sound at the reference values and is given by (see equation (6) of Art. 20.8 B, Chapter 20), $c_0^2 = \gamma p_0/\rho_0$ ...(10)

(*v*) $P_r = \mu C_p / k$ (Prandtl number) ...(11)

In heat transfer theory the peclet number

$$P_e = (U_0 L_0 \rho_0 C_p)/k \qquad ...(12)$$

is sometimes used. It is related to the Prandtl number and Reynolds number by the relation $P_e = P_r Re$

In some situations, in place of Mach number or ratio of specific heats, we use another dimensionless number, namely, Eckert number, which is denoted and defined as follows

$$Ec = U_0^2 / C_p T_0 \qquad ...(13)$$

We now re-write Ec in terms of Ma and γ.

We know that (see equation (7) of Art. 20.2 of chapter 20) $\quad R = C_p - C_v$ ...(14)

From equation of state, $\quad p_0 = \rho_0 R T_0 = \rho_0 T_0 (C_p - C_v)$, using (14)

Hence, $\quad p_0 = \rho_0 T_0 C_p (1 - C_v / C_p) = \rho_0 T_0 C_p (1 - 1/\gamma)$, using (8)

Thus, $\quad \gamma p_0 / \rho_0 = T_0 C_p (\gamma - 1) \quad$ or $\quad c_0^2 = T_0 C_p (\gamma - 1)$, by (10) ...(15)

Then, (13) and (15), $\quad Ec = (\gamma - 1) \times (U_0^2 / c_0^2) = (\gamma - 1) M_a^2$, by (9) ...(16)

Now, $$\frac{p_0}{\rho U_0^2} = \frac{p_0}{\rho_0 c_0^2 M_a^2} = \frac{p_0}{\rho_0 \times (\gamma p_0 / \rho_0) \times M_a^2}, \text{ using (9) and (10)}$$

Thus, $$p_0 / \rho U_0^2 = 1/(\gamma M_a^2)$$

Also, $$\frac{k}{C_p \rho_0 U_0 L_0} = \frac{k}{\mu C_p} \times \frac{\mu}{\rho U_0 L_0} = \frac{1}{P_r \,\mathrm{Re}} = \frac{1}{P_e}, \text{ by (6) and (11)} \qquad [\because P_e = P_r R_e]$$

From the above discussion, it follows that the dimensionless groups (5) can be re-written in terms of commonly used dimensionless numbers as follows :

$$\mu/(\rho_0 U_0 L_0) = 1/\mathrm{Re}, \quad F_0 L_0 / U_0^2 = (g_0 L_0)/U_0^2 = Fr, \quad \text{taking } F_0 = g_0$$

$$p_0 / \rho_0 U_0^2 = 1/(\gamma M_a^2), \quad U_0^2 / C_p T_0 = (\gamma - 1) M_a^2 = Ec, \qquad k/(C_p \rho_0 U_0 L_0) = 1/P_r \,\mathrm{Re} = 1/P_e$$

Thus, we conclude that for the complete dynamical similarity of the flows of viscous compressible fluids past geometrical similar bodies, when the body force is the gravitational force only, we must have the same Raynolds number, same Froude number, same ratio of specific heats, same Mach number and same Prandtl number.

Remark. The significance of the five parameters arrived at in the above discussion can be seen in the requirement of dynamical similarity of flows of a viscous compressible fluid. Note that in order to have flows around two bodies to be completely similar, not only must the bodies be geometrically similar, but also the five parameters (Reynold's numbers, Prandtl number, Froude number, Mach number and ratio of specific heats) in the two flow patterns must be equal in magnitude. On the other hand, for flows of an inviscid fluid we need not be concerned about the Reynolds number and the Prandtl number; however, the Mach number, ratio of specific heats and Froude number are important in this case.

15.8 Some useful dimensionless numbers.

We know that inertia force (*i.e.*, the product of mass and acceleration) always exists in all flow problems. Besides the inertia force, there always exist some additional forces which are responsible for fluid motion. The required conditions for dynamic similarity can always be obtained by considering the ratio of the inertia force and any one of the remaining forces (*e.g.*, viscous force, gravity force, pressure force, elastic force and so on). Since ratios of two forces will be considered, we obtain some dimensionless numbers as discussed below :

(*i*) **Reynold 's number.** **[Agra 2007, Kolkata 1995, 2001, 04; C.S.I.R. 2003 [Himanchal 2000, 01, 02, 03, 04, 06, 07, 09, 10; Kanpur 1997; Meerut 2002, 03, 09, 10, 11]**

The Reynold's number Re is defined as

$$\text{Re} = \frac{\text{inertia force}}{\text{Viscous force}} = \frac{\text{Mass} \times \text{Acceleration}}{\text{Shear Stress} \times \text{Cross sectional area}}$$

$$= \frac{\text{Volume} \times \text{Density} \times (\text{Velocity/Time})}{\text{Shear Stress} \times \text{Cross sectional area}}$$

$$= \frac{\text{Cross section area} \times \textit{Linear dinmension} \times \rho \times (\text{Velocity/Time})}{\text{Shear Stress} \times \text{Cross sectional area}}$$

$$= \frac{(\text{Velocity})^2 \times \rho}{\mu\,(du/dy)} = \frac{V^2\rho}{\mu(V/L)} = \frac{VL\rho}{\mu} = \frac{VL}{\nu}, \quad \left(\text{as } \nu = \frac{\mu}{\rho}\right)$$

where L and V denote the characteristic length and characteristic velocity respectively so that velocity will be proportional to V and du/dy will be proportional to V/L. If for any flow problem Re is small then we can ignore the inertia force, whereas if Re is large then we can neglect the effect of the viscous force and consequently the fluid may be treatd as non-viscous fluid. When the viscous-force is the prodominating force. Reynolds number must be the same for dynamic similarity of two flows.

(*ii*) **Froude number. [Agra 2007, Himanchal 2002, Meerut, 2005, 09, 10, 11; Mumbai 2006]**

The Froude number Fr is defined by

$$Fr = \frac{\text{Inertia force}}{\text{Gravity force}} = \frac{\text{Mass} \times \text{Acceleration}}{\text{Mass} \times \text{Accleration due to gravity}} = \frac{(\text{Velocity/time})}{g} = \frac{\text{Velocity}}{g \times \text{Time}}$$

$$= \frac{\text{Velocity}}{g \times (\text{Linear dimension/Velocity})} = \frac{(\text{Velocity})^2}{g \times \text{Linear dimension}} = \frac{V^2}{gL},$$

where L and V denote the characteristic length and characteristic velocity respectively so that velocity will be proportional to V and linear dimension will be proportional to L. When the gravity force is the predominating, Froude number must be the same for dynamic simikarity of two flows.

(*iii*) **Euler number. (or pressure coefficient)** **[Himanchal 2001; Meerut 2005, 09, 10]**

The Euler number Eu is defined by

$$\frac{1}{\text{Eu}} = \frac{\text{Inertia froce}}{\text{Pressure force}} = \frac{\text{Mass} \times \text{Acceleration}}{\text{Pressure} \times \text{Cross sectional area}}$$

$$= \frac{\text{Volume} \times \text{Density} \times (\text{Velocity/Time})}{\text{P} \times \text{Cross sectional area}}$$

$$= \frac{\text{Cross section area} \times \text{Linear dimension} \times \rho \times (\text{Velocity / Time})}{\text{P} \times \text{cross sectional area}} = \frac{(\text{Velocity})^2 \times \rho}{\text{P}} = \frac{V^2\rho}{P}$$

so that
$$Eu = P/V^2\rho,$$

where P, V are the characteristic pressure and characteristic veloicty respectively. When the pressure force is the predominating force, Euler's number must be the same for dynamic similarity of two flows.

(*iv*) **Mach number.** **(Himanchal 1999, 2002)**

The Mach number M or Ma is defined by $Ma = M = q/c$,

where q is the velocity of flow and c is the velocity of sound. Mach number is also expressed in terms of the ratio of inertia force and the elastic force. When the Mach number is small (*i.e.*, $M << 1$), the fluid can be taken as incompressible. On the other hand, if Mach number is nearly one or greater than one, the fluid will be taken as compressible.

(*v*) **Prandtl number.**

[Himanchal 2002, 06; Meerut 2000 , 05; Garhwal 2000; Mumbai 2006]

The Prandtl number P_r is defined by $P_r = (\mu C_p) / k$

where C_p is the specific heat at constant pressure and k is the conductivity. Evidently P_r depends only on the properties of the fluid. For air $P_r = 0.7$ approx. and for water (at 60°F) $P_r = 7$ approximately, whereas for oils it is of the order of 1000 due to large values of μ for oils. Prandtl number is the ratio of viscous force to the thermal force. It throws light on the relative importance of viscous dissipation to the thermal dissipation.

(*vi*) **Eckert number.** **[Himanchal 2000, 07]**

The Eckert Ec is defined by, $Ec = V^2 / C_p T_0$,

where T_0 = temperature difference between the wall and the fluid at a large distance from the body. Alternatively, Eckert number is also defind (see Art 15.7)

$$Ec = U_0^2 / C_{p_0} T_0 = (\gamma - 1) M_a^2$$

(*vii*) **Peclet number.** **[Mumbai 2000; Meerut 2005]**

The Peclet number P_e is defined by, $P_e = (VL\rho C_p)/k$

Notice that $P_e = P_r Re$. The Peclet number plays an important role when the viscous force is small while thermal force is large as compared to inertia force.

(*viii*) **Weber number. [Himanchal 2005, 03, 04, 06; Kolkata 95, 2001, 04; C.S.I.R. 2003]**

The Weber number We is defined as the ratio of inertia force $\rho V^2/L$ to the force of surface tension σ/L^2. Thus, we have

$$\text{We} = \frac{\rho V^2 / L}{\sigma / L^2} = \frac{\rho V^2 L}{\sigma}$$

The Weber number plays an important role while dealing with problems involving a free surface or an interface such as flows in rivers, canals etc.

(*ix*) **Grashoff number** **[Himanchal 2002, 07 Meerut 1998]**

The dimensionless quantity Gr which characterizes the free convection (refer Art. 18.2of chapter 18) is known as *Grashoff number* and is defined as

$$Gr = \left\{ gL^3 (T_w - T_\infty) \right\} / \nu^2 T_\infty,$$

where g is the acceleration due to gravity and T_w and T_∞ are two representative temperatures.

15.9 Some dimensionless coefficients employed in the study of flow of viscous fluids.

(*i*) **Local skin-friction coefficient** **[Himanchal 2000, 05, 06, 10]**

The dimensionless shearing stress on the surface of a body, due to a fluid motion, is known

as *local skin-friction coefficient* and is defined is $C_f = \tau_w / (\rho U^2 / 2)$,
where τ_w is the local shearing stress on the surface of the body.

(*ii*) Lift and drag coefficient **[Himanchal 1998, 2000]**

Refer Art. 18.5 of chapter 18 for definitions of drag and lift. Let the drag force and lift force be denoted by D and L respectively. Then the expressions

$$C_D = D / (\rho U^2 A / 2) \qquad \text{and} \qquad C_L = L / (\rho U^2 A / 2)$$

where A is the typical area associated with the solid body, are known as the *drag coefficient* and *lift coefficient* respectively.

(*iii*) Nusselt number **[Himanchal 2000, 01, 07]**

In the dynamics of fluids, we have to find out the quantity of heat exchanged between the body and the fluid. This quantity of heat transfer is obtained by using a coefficient of heat transfer $\alpha(x)$, which is defined by Newton's law of cooling as follows : $q(x) = \alpha(x)\,(T_w - T_\infty)$, where $(T_w - T_\infty)$ is the difference between the temperature of the wall and that of the fluid.

But at the boundary the heat exchange between the body and the fluid takes place only due to conduction. Hence, according to Fourier's law, we get $q(x) = -k\,(\partial T / \partial n)_{n=0}$, where n is the direction of the normal to the surface of the body. From these two laws, dimensionless coefficient of heat transfer, known as *Nusselt number*, is denoted and defined as

$$Nu = \frac{L\,\alpha(x)}{k} = -\frac{L}{(T_w - T_\infty)}\left(\frac{\partial T}{\partial n}\right)_{n=0}, \text{ where } k \text{ is the conductivity.}$$

where L is some characteristic length in the problem.

(*iv*) Temperature recovery factor (or recovery factor) **[Himanchal 1998, 2000]**

The temperature which a surface assumes under the influence of friction is called the *recovery temperature* (or *adiabatic wall temperature*). Let T_r and T_∞ be the recovery temperature and the temperature of the stream. If C_p be the specific heat at constant pressure, then the *temperature* recovery factor is denoted by r and is defined as $r = (T_r - T_\infty)/(U^2 / 2C_p)$

15.10 Illustrative solved examples.

Ex. 1. *An oil of specific gravity* 0.85 *is flowing through a pipe of* 5 cm *diameter at the rate of* 3 *liters/sec. Find the type of flow, if the viscosity for the oil is* 3.8 *poise.*

Sol. $\text{V} = \text{velocity of oil} = \dfrac{\text{Discharge}}{\text{Area}} = \dfrac{3000}{\pi \times (5/2)^2} = 152.8 \text{ cm/sec}$

$L = \text{diameter} = 5 \text{ cm.}, \qquad \mu = 3.8 \text{ poise} \qquad \text{and} \qquad \rho = 0.85$

$$\therefore \qquad Re = \frac{\text{V}L\rho}{\mu} = \frac{152.8 \times 5 \times 0.85}{3.8} = 171.$$

Since $Re < 2{,}000$, it follows that the flow must be laminar.

Ex. 2. *A one-fourth scale model of a helicopter is to be tested in a wind-tunnel such that kinematic similarity is attained. The helicopter is designed to fly in level flight* 40 m/sec *in a region where* $\nu = 1.4 \times 10^{-5}\ \text{m}^2/\text{sec}$. *Further* $\nu = 2 \times 10^{-5}\ \text{m}^2/\text{sec}$. *in the wind-tunnel. Determine the velocity of air in the tunnel. Is dynamic similarity attained in the test.*

Sol. The kinetic similarity is said to exist between the prototype and the model if the ratio of the corresponding velocities at corresponding points are equal. It can be show that for kinematic similarity, the Reynold's numbers must be the same. Let the subscripts p and m be used for the prototype and the model respectively. Then we must have

$$\frac{V_p L_p}{\nu_p} = \frac{V_m L_m}{\nu_m} \qquad \text{so that} \qquad V_m = \frac{L_p}{L_m} \times \frac{\nu_m}{\nu_p} \times V_p = \frac{4}{1} \times \frac{2}{1.4} \times 40 = 228 \,\text{m/sec.}$$

giving the velocity of air over the model in the wind tunnel.

Further, for dynamic similarity the Froude numbers must be the same (in addition to the

Reynolds number) for the two situations.

Now, $$\frac{(F_r)_m}{(F_r)_p} = \frac{V_m^2 / gL_m}{L_p^2 / gL_p} = \left(\frac{V_m}{V_p}\right)^2 \times \left(\frac{L_p}{L_m}\right) = \left(\frac{228}{40}\right)^2 \times \frac{4}{1} = 130,$$

showing that $(F_r)_m \neq (F_r)_p$. It follows that dynamic similarity is not attained in the test being conducted.

Remark. If influence of gravity is neglected, then Froude number need not be considered; and so dynamic similarity is said to exist in the test.

Ex. 3. (*a*) *A model of an aeroplane built to* $(1/10)$*th scale is said to be tested in a wind tunnel which operates at a pressure of* 20 *atmosphere. The aeroplane is expected to fly at a speed of* 500 *km/h. At what speed should the wind tunnel operate to give dynamic similarity of model and prototype.*

(*b*) 1 : 20 model of an air-duct is to be tested in water which is 45 times more viscous and 850 times more dense than air. What should be the pressure drop in the prototype if the pressure drop is 3 kg/cm^2 in the model when tested under hydrodynamically similar conditions.

Sol. (*a*) The dynamic similarity is said to exist between the prototype and model if the Reynold's numbers are the same for the model and prototype. Let the subscripts p and m be used for the prototype and the model respectively. Then, we have

$$(\text{Re})_m = (\text{Re})_p \quad \Rightarrow \quad \frac{V_m L_m \rho_m}{\mu_m} = \frac{V_p L_p \rho_p}{\mu_p} \quad \Rightarrow \quad V_m = \frac{L_p}{L_m} \times \frac{\rho_p}{\rho_m} \times \frac{\mu_m}{\mu_p} \times V_p$$

Here V_p = 500 km/hr, L_p/L_m = 10 and $\rho_m/\rho_p = 20$. Also, we take $\mu_m = \mu_p$ because the coefficient of dynamic viscosity does not significantly change due to pressure changes unless the pressure change is very large. Hence, form (1), we have

$$V_m = (1/20) \times 10 \times 500 = 250 \text{ km/h}$$

(*b*) Let the subscripts *p* and *m* be used for the prototype and the model respectively. Since the pressure drop is due to viscous effects, hence for dynamic similarity, Reynold number must be the same in the model and the prototype that is,

$$(\text{Re})_p = (\text{Re})_m \quad \Rightarrow \quad \frac{V_p L_p \rho_p}{\mu_p} = \frac{V_m L_m \rho_m}{\mu_m} \quad \Rightarrow \quad \frac{V_p}{V_m} = \frac{\rho_m}{\rho_p} \times \frac{L_m}{L_p} \times \frac{\mu_p}{\mu_m} \qquad \ldots(1)$$

For, dynamic similarity, Euler's number must be the same in the model and the prototype that is,

$$(Eu)_p = (Eu)_m \quad \Rightarrow \quad \frac{P_p}{\rho_p V_p^2} = \frac{P_m}{\rho_m V_m^2} \quad \Rightarrow \quad P_p = P_m \times \frac{\rho_p}{\rho_m} \times \left(\frac{V_p}{V_m}\right)^2 \qquad \ldots(2)$$

Given that $L_m/L_p = 1/20$, $\quad \rho_m/\rho_p = 850 \quad$ and $\quad \mu_p/\mu_p = 1/45 \qquad \ldots(3)$

Using (3), (1) $\Rightarrow$ $V_p / V_m = (850)/(2o \times 45) = 17/18 \qquad \ldots(4)$

Using (3) and (4), (2) $\Rightarrow$ $P_p = 3 \times (1/850) \times (17/18)^2 = 3.4 \times 10^{-3}$ kg/cm^2

which gives the required pressure drop.

Ex. 4. *The drag of a small submarine hull is desired when it is moving far below the surface of water. A* 1/10 *scale model is to be tested. What dimensionless group should be duplicated between the modal and prototype? If the drag of the prototype at* 1 *knot is desired at what speed should the model be moved to give the drag to be expected by the prototype*?

Sol. We know that a small submarine is a wholly submarged body in an infinite mass of fluid at rest. The dimensionless group Reynolds number should be duplicated for the model and prototype. Let the subscripts *p* and *m* be used for the prototype and the model respectively. Since the Reynold's number must be the same in the model and the prototype, we have

$$(Re)_p = (Re)_m \quad \Rightarrow \quad \frac{V_p L_p \rho_p}{\mu_p} = \frac{V_m L_m \rho_m}{\mu_m}$$

$$\Rightarrow \qquad V_m = \frac{L_p}{L_m} \times \frac{\rho_p}{\rho_m} \times \frac{\mu_m}{\mu_p} \times V_p \qquad \text{...(1)}$$

Given $L_p/L_m = 10$, and for some fluid (water), $\rho_p = \rho_m$ and $\mu_m = \mu_p$. Also, if $V_p = 1$ knot.
Hence, (1) reduces to $V_m = 10 \times 1 \times 1 \times 1 = 10$ knots.

The drag coefficient $D/(\rho U^2 L^2)$ should be duplicated in order to compute the drag experienced by the prototype

$$\therefore \qquad \frac{D_p}{\rho_p U_p^2 L_p^2} = \frac{D_m}{\rho_m U_m^2 L_m^2} \Rightarrow D_p = \frac{\rho_p}{\rho_m} \times \left(\frac{U_p}{U_m}\right)^2 \times \left(\frac{L_p}{L_m}\right)^2 \times D_m = 1 \times (10)^2 \times \left(\frac{1}{10}\right)^2 \times D_m$$

so that $D_p = D_m$, showing that if the 1/10 model is moved at 10 knots the drag experienced by it would be the same as that by the prototype moving with speed of 1 knot.

EXERCISE 15(A)

1. Define Reynold's number and indicate its significance

[Himanchal 2000, 01, 02, 03; Kanpur 1997; Meerut 2000, 03, 05, 06, 07]

2. State and prove Buckingham π – theorem.

[Himanchal 1999, 2001, 03, 07; Meerut 2000, 03]

3. Explain Reynold's law of dynamic similarity. **[Himanchal 1999; Meerut 2002]**

4. Show that in the dynamics of compressible fluids there are only five independent dimensionless groups **[Himanchal 1998, 2000]**

Ex. 5 *Discuss similarity of flows. Give the importance of Reynold's number, Froude number, Prandtl number and Peclet number.* **[Mumbai 2006]**

Ex. 6. *A one-tenth scale torpedo model was tested in a wind tunnel using compressed air having density of* 25.5 kg/m^3 *and kinematic viscosity of* 0.67×10^6 m^2/s. *The resistance was found to be* 8*N*, *when the air velocity was* 30 m/s. *Find the resistance and speed of the full scale terpedo, when moving under dynamically similar conditions, through sea-water having density of* 1 kg/m^3 *and kinematic viscosity of* 1.58 m^2/s. **Ans.** 7.08 m/s; 1756 N

7. Define and give physical importance of the following non-dimensional parameters: (*i*) Grashoff number (*ii*) Echert number (*iii*) Reynold's number **[Himanchal 2007]**

8. Define and give physical importance of the following nondimessional quantities: (*i*) Reynold's number (*ii*) local skin friction coefficient (*iii*) Prandlt number. **[Himanchal 2008]**

9. Define dynamic similarity, Write short note on inspection analysis. **[Himanchal 2006]**

10. Write a note on Reynold's number and its uses. **[Agra 2007]**

11. Consider the statements: (*i*) Mach number is a measure of compressibility of the fluid due to flow speed (*ii*) Reynolds number is a measure of the ratio of the inertia force to viscous force (*iii*) Prandtl number is a measure of heat conductance and viscosity of the fluid. The number of true statements is (*a*) Zero (*b*) one (*c*) two (*d*) all **[(Agra 2005]**

Sol. Am (*c*). See Art. 15.8

12. Coefficient of drag is

(*a*) $\dfrac{F_D}{(\rho A U^2)/2}$ (*b*) $\dfrac{\tau_o}{(\rho U^2)/2}$ (*c*) $\dfrac{F_D}{(\rho U^2)/2}$ (*d*) $\dfrac{\tau_o}{(\rho A U^2)/2}$ **[Agra 2008]**

Sol. Ans. (*a*). See (*ii*) Art 15.9

13. Fill up the gap:

(*i*) The ratio (inertia force)/(gravity force) is .... **[Agra 2007]**

(*ii*) The ratio (inerertic force)/pressure force) is .... **[Agra 2008]**

Sol. (*i*) Froude number, see Art 15.8. (*ii*) Euler number, See Art 15.8

15.11 Dimensional analysis. [Himanchal 1999, 2000, 08, 10]

The dimensional analysis is a mathematical technique, which enables us to obtain the dimensionless numbers of the variables of a physical problem without the knowledge of the governing equation. This analysis is based on the assumption that each physical phenomenon is expressible in terms of a dimensionally homogeneous equation. It is used in developing experiments with several variables, in interpreting experimental data and in establishing link between the scale model (or simply model) and the actual structure (also known as prototype) after performing suitable experiments.

15.12 Technique of dimensional analysis.

We now propose to discuss the following two techniques of dimensional analysis:

(*i*) Rayleigh's technique.

(*ii*) Buckingham π – theorem.

15.13 Rayleigh's technique (working rule).

Step 1. An arbitrary functional relationship is assumed among the variables of the problem. The choice of the dependent and independent variables is made as required by the problem.

Step 2. The equation of step 1 is now re-written by using a constant k and raising the independent variables to powers $a, b, c, \ldots$ etc.

Step 3. The principle of dimensional homogeneity is used to compare the powers of fundamental units on both sides. This leads to some simultaneous equations to determine $a, b, c \ldots$etc.

Step 4. The substitution of the values of $a, b, c, \ldots$ in the proposed relation of step 2 leads to the desired result.

15.14 Illustrative sloved examples.

Ex 1. *Show that the resistance (R) to the motion of a sphere of diameter (D) moving with a uniform velocity (V) through a real fluid having densty* ρ *and viscosity* (μ) *is given by*

$$R = \rho D^2 V^2 f(\mu/\rho V D).$$

Sol. Let $$R = F(D, V, \rho, \mu). \quad \ldots(1)$$

Let k be a dimensionless constant. Then (1) can be re-written as

$$R = k[D^a \cdot V^b \cdot \rho^c \cdot \mu^d] \quad \ldots(2)$$

Substituting the dimensios of each physical quantity, (2) reduces to

$$[MLT^{-2}] = k\,[(L)^a\,(LT^{-1})^b\,(ML^{-3})^c\,(ML^{-1}\,T^{-1})^d]$$

or $$[MLT^{-2}] = k\,[M^{c+d}\,L^{a+b-3c-d}\,T^{-b-d}] \quad \ldots(3)$$

Since (3) must be dimensionally homogeneous, we equate the powers of M, L and T and obtain

$$c + d = 1 \quad \ldots(4)$$

$$a + b - 3c - d = 1 \quad \ldots(5)$$

$$-b - d = -2 \quad \ldots(6)$$

From (4) and (6), $c = 1 - d$ and $b = 2 - d.$ ...(7)

Substituting the values of c and b in (5), we get

$$a = 1 - b + 3c + d = 1 - (2 - d) + 3\,(1 - d) + d = 2 - d. \quad ...(8)$$

Using (7) and (8) in (2), we get

$$R = k \cdot D^{2-d}\, V^{2-d}\, \rho^{1-d}\, \mu^{d} = \rho D^2 V^2 k(\mu/\rho VD)^d. \quad ...(9)$$

Since d and k are arbitrary constants, we may take

$$k(\mu/\rho VD)^d = f(\mu/\rho VD), \quad ...(10)$$

where f is an arbitrary function, Using (10), (9) yields to the required result

$$R = \rho D^2 V^2 f\,(\mu/\rho VD)$$

Ex. 2. *The resistance force R of a supersonic plane during flight can be considered as dependent upon the length of the aircraft l, velocity V, air velocity* μ, *air density* ρ *and bulk modulus of air K. Find an expression for R.*

Sol. Here $R = f\,(l, V, \mu, \rho, K)$ or $R = k\,(l^a\, V^b \mu^c \rho^d K^e),$ ...(1)

where k is a non-dimensional constant.

Substituting the dimensions of each physical quantity, (1) reduces to

$$[MLT^{-2}] = k\,[L^a \cdot (LT^{-1})^b \cdot (ML^{-1}T^{-1})^c \cdot (ML^{-3})^d \cdot (ML^{-1}T^{-2})^e]. \quad ...(2)$$

Since (2) must be dimensionally homogeneous, the exponents of each dimension on both sides of (2) must be identical. Thus, we have

For M: $1 = c + d + e$ ...(3)

For L: $1 = a + b - c - 3d - e$ ...(4)

For T: $-2 = -b - c - 2e$ ...(5)

Note that we have five unknowns a, b, c, d, e and we have only the above three equations. Here we express the three unknowns a, b, d in terms of c and e (exponents of viscosity and bulk modulus which are more important).

From (3), $d = 1 - c - e$ ...(6)

From (5), $b = 2 - c - 2e$ ...(7)

From (4), $a = 1 - b + c + 3d + e$

or $a = 1 - (2 - c - 2e) + c + 3(1 - c - e) + e$, by (6) and (7)

Thus, $a = 2 - c.$ ...(8)

Using (6), (7) and (8) in (1), we have

$$R = k\,(l^{2-c}\, V^{2-c-2e}\, \mu^c\, \rho^{1-c-e}\, K^e) = k\, l^2 V^2 \rho \cdot (l^{-c} V^{-c} \mu^c \rho^{-c})\,(V^{-2e} \rho^{-e} K^e)$$

or $$R = kl^2V^2\rho(\mu/lV\rho)^c (K/V^2\rho)^e = l^2V^2\rho\phi(\mu/lV\rho, K/V^2\rho)$$

where ϕ is an arbitrary function.

Ex. 3. *The pressure drop* ΔP *in a pipe of diameter D and length l depends on the density* ρ *and* μ *of fluid flowing, mean velocity V of flow and average height of protuberance t. Show that the pressure drop can be expressed in the form :* $\Delta p = \rho V^2 \phi\,(l/D, \mu/VD\rho, t/D)$.

Sol. Here $\Delta p = f\,(D, l, \rho, \mu, V, t)$ or $\Delta p = k\,(D^a l^b \rho^c \mu^d V^e t^f),$ ...(1)

where k is a non-dimensional constant.

Substituting the dimensions of each physical quantity (1) reduces to

$$[ML^{-1}T^{-2}] = k\,[L^a \cdot L^b \cdot (ML^{-3})^c \cdot (ML^{-1}T^{-1})^d \cdot (LT^{-1})^e \cdot L^f] \quad ...(2)$$

Since (2) must be dimensionally homogeneous, the exponents of each dimension on both sides of (2) must be identical. Thus, we have

For M: $$1 = c + d \qquad ...(3)$$

For L: $$-1 = a + b - 3c - d + e + f \qquad ...(4)$$

For T: $$-2 = -d - e. \qquad ...(5)$$

Note that we have six unknowns a, b, c, d, e, f and we have only the above three equations. Here we express the three unknowns, a, c and e in terms of the unknowns b, d, f

From (3) and (5), $c = 1 - d$ and $e = 2 - d$. ...(6)

Then, (4) $\Rightarrow$ $a = -1 - b + 3c + d - e - f$

or $$a = -1 - b + 3(1 - d) + d - (2 - d) - f, \text{ using (6)}$$

or $$a = -(b + d + f). \qquad ...(7)$$

Using (6) and (7) in (1), we have

$$\Delta p = k[D^{-(b+d+f)} \cdot l^b \cdot \rho^{2-d} \cdot \mu^d \cdot V^{2-d} \cdot t^f) = k[\rho \cdot V^2 \cdot (D^{-b} l^b)(D^{-d} \cdot \rho^{-d} \cdot \mu^d V^{-d}) \cdot (D^{-f} \cdot t^f)]$$

or $$\Delta p = k[\rho V^2 \cdot (l/D)^b \cdot (\mu/VD\rho)^d \cdot (t/D)^f]$$

$\therefore$ $$\Delta p = \rho V^2 \phi(l/D, \mu/VD\rho, t/D),\quad \phi \text{ being an arbitrary function,}$$

Ex. 4. *Show by means of dimensional analysis that the thrust T of a screw propeller is given by $T = \rho d^2 V^2 \phi(Vd\rho/\mu, dn/V)$, where ρ is the fluid density, μ its viscosity, d is the diameter of the propeller, V is the speed of advance and n is the revolution per second.*

[Nagpur 2004]

Soluiton. Let $T = F(\rho, \mu, d, V, n)$, F being an arbitrary function ...(1)

Let k be a dimensionless constant. Then (1) can be re-written as

$$T = k[\rho^a \cdot \mu^b \cdot d^c \cdot V^d \cdot n^e] \qquad ...(2)$$

Substituting the dimensions of each physical quantity, (2) reduces to

$$[MLT^{-2}] = k\ \{(ML^{-3})^a\ (ML^{-1}\ T^{-1})^b\ (L)^c\ (LT^{-1})^d\ (T^{-1})^e]$$

or $$[MLT^{-2}] = k\ [M^{a+b}\ L^{-3a-b+c+d}\ T^{-b-d-e}] \qquad ...(3)$$

Since (3) must be dimensionally homogeneous, we equate the powers of M, L, T and obtain.

$1 = a + b$, $\quad 1 = -3a - b + c + d \quad$ and $\quad -2 = -b - d - e$

Solving these, we have $a = 1 - b$, $\quad c = 2 + e - b$, $\quad d = 2 - b - e$.

Substituting the above values of a, c and d is (2), we get

$$T = k\rho^{1-b}\ \mu^b\ d^{2+c-b}\ V^{2-b-e}\ n^e = k\rho d^2 V^2 (Vd\rho/\mu)^{-b} (dn/V)^e \qquad ...(4)$$

Since k, b and e are arbitrary constants, we write $k(Vd\rho/\mu)^{-b}(dn/V)^e = \phi(Vd\rho/\mu, dn/V)$, ...(5)

where ϕ is an arbitrary function. Using (5), (4) yields the required result

$$T = \rho d^2 V^2 \phi(Vd\rho/\mu, dn/V).$$

Ex. 5. *For the flow of a fluid through similar pipes, show that $P = (\rho l V^2/d)\phi(\rho Vd/\mu)$, where P is the drop in pressure over the length l of the pipe, d is diameter of the pipe, ρ is the density, μ is viscosity of the fluid and V is mean velocity of flow through the pipe.*

Sol. Let $$P = F(\rho, \mu, V, d, l) \qquad ...(1)$$

Let k be a dimemsionless constant. Then (1) can be re-written as

$$P = k[\rho^a\ \mu^b\ V^c\ d^e\ l^f] \qquad ...(2)$$

Substituting the dimensions of each physical quantity, (2) reduces to

$$ML^{-1}T^{-2} = k\,[(ML^{-3})^a\,(ML^{-1}T^{-1})^b\,(LT^{-1})^c\,(L)^e\,(L)^f]$$

or $$ML^{-1}\,T^{-2} = k\,[M^{a+b}\,L^{-3a-b-c+e+f}\,T^{-b-c}] \qquad ...(3)$$

Since (3) must be dimensionally homogeneous, we equal the powers of M, L, T and obtain

$$1 = a + b, \qquad -1 = -3a - b - c + e + f \quad \text{and} \quad -2 = -b - c$$

Solving these equations for the powers of ρ, V and d, we get

$$a = 1 - b, \qquad c = 2 - b \qquad \text{and} \qquad e = -b - f \qquad ...(4)$$

$\therefore$ From (2) and (4), $$P = k\rho^{1-b}\mu^b V^{2-b} d^{-b-f} l^f = k\rho V^2 (l/d)^f (\rho V d/\mu)^{-b}$$

or $$P = k\,(\rho l V^2/d)\,(l/d)^{f-1}(\rho V d/\mu)^{-b} \qquad ...(5)$$

Now, for geometrically similar pipes, (l/d) is a constant. Since $(l/d)^{f-1}$ and k are arbitrary constants, we write $$k\,(l/d)^{f-1}(\rho V d/\mu)^{-b} = \phi\,(\rho V d/\mu), \qquad ...(6)$$

where ϕ is an arbitrary function. Using (6), (5) takes the required form

$$P = (\rho l V^2/d)\,\phi(\rho V d/\mu)$$

Ex. 6. *The losses $\Delta h/l$ per unit length of pipe in a fluid flowing through a smooth pipe depend upon velocity V, diameter D, gravity g, dynamic viscosity μ and density ρ. With dimensional analysis, determine the general form of the euqation.* **[Meerut 2005]**

Sol. Since $\Delta h/l$ depends on V, D, ρ, μ and g, so we assumes that

$$\Delta h/l = kV^a \cdot D^b \cdot \rho^c \cdot \mu^d \cdot g^e, \text{ where } k \text{ is a non-dimensional constant.} \qquad ...(1)$$

Substituting the dimensions on both sides of (1), we have

$$[M^0L^0T^0] = [k\,(LT^{-1})^a \cdot L^b \cdot (ML^{-3})^c \cdot (ML^{-1}\,T^{-1})^d \cdot (LT^{-2})^e]$$

or $$[M^0L^0T^0] = [k\,M^{c+d} \cdot L^{a+b-3c-d+e} \cdot T^{-a-d-2e}] \qquad ...(2)$$

For dimensional homogeneity the exponents of each dimension on both sides of the equation (2) must be identical. Thus, we have

For M : $$c + d = 0 \qquad ...(3)$$

For L : $$a + b - 3c - d + e = 0 \qquad ...(4)$$

For T: $$-a - d - 2e = 0 \qquad ...(5)$$

There are five unknowns but there are only three equations involving them. Hence, we express these three unknowns in terms of the other two unknowns which are more important. Viscosity and gravity are more important in the given problem. Hence expressing a, b and c in terms of d and e, (3) and (5) yield.

$$a = -d - 2e, \qquad b = e - d \qquad \text{and} \qquad c = -d$$

Substituting these values of a, b and c in (1), we have

$$\Delta h/l = kV^{-d-2e} \cdot D^{e-d} \cdot \rho^{-d} \cdot \mu^d \cdot g^e$$

or $$\Delta h/l = k(\mu/\rho DV)^d \times (gD/V^2)^e = k(\rho DV/\mu)^{-d} \times (V^2/gD)^{-e}$$

or $$\Delta h/l = k(\text{Re})^{-d} \times (V^2/gD)^{-e}, \text{ where } \text{Re} = \rho DV/\mu = \text{Reynolds number}$$

Thus, $\Delta h/l = f(\text{Re}, V^2/gD)$, where f is an arbitary function.

This is the general form of the required equation.

Ex. 7. *The discharge through a horizontal capillary tube depends upon the pressure drop per unit length, the diameter, and the viscosity. Find the form of the equation.*

[Meerut 2001, 02, 04, 05]

Sol. Since the discharge Q depends upon the pressure drop per unit lingth $\Delta p/l$, the diameter D and the viscostiy μ, so we take

$Q = k(\Delta p/l)^a \cdot D^b \cdot \mu^c$, where k is a non-dimensions constant. ...(1)

Substituting the dimensions on both sides of (1), we get

$$[M^0 L^3 T^{-1}] = k\,[(ML^{-2}T^{-2})^a \cdot L^b \cdot (ML^{-1}T^{-1})^c]$$

$$[M^0 L^3 T^{-1}] = k\,[M^{a+c} \cdot L^{-2a+b-c} \cdot T^{-2a-c}] \qquad ...(2)$$

For dimensional homogeneity the exponents of each dimension on both sides of the equation (2) must be identical. Thus, we have

For M : $a + c = 0$ ...(3)

For L : $-2a + b - c = 3$ ...(4)

For T : $-2a - c = -1$ ...(5)

Solving (3), (4) and (5), $a = 1$ $b = 4$ and $c = -1$, Substituting these values in (1), we have

$$Q = k(\Delta p/l) D^4 \mu^{-1}, \qquad i.e., \qquad Q = k(\Delta p/l) \times (D^4/\mu),$$

which gives the general form of the desired equation, k being a dimensionless arbitrary constant.

Remark. While dealing with a large number of variables in a physical problem, the Rayleigh's technique is obviously very lengthy in practice. To face such situations, we now discuss an easy alternative method in the article 15.17.

15.15. Some useful results.

Result I. Complete set. Definition. A set of dimensionless products of given physical quantities is said to be complete if each product in the set is independent of the others and every other dimensionless product, besides the complete set, formed out of the given physical quantities can be expressed in terms of the dimensionless products of the complete set.

Result II. An important theorem of Matrix Algebra. If we have m homogeneous equations in n unknowns, then the number of independent solutions is $n - r$, where r is the rank of the matrix of coefficients and any other solution can be expressed as a linear combination of these linearly independent solutions. Moreover, there will be only r independent solutions.

15.16. Buckingham π – theorem or simply π – theorem.

[Himanchal 2001. 03, 07; 09, 10; Meerut 2002, 03, 06]

Statement. *If Q_1, Q_2, ... Q_n be n physical quantities involved in a physical phenomenon and if there are m independent fundamental units in this system then a relation*

$\phi(Q_1, Q_2, \ldots, Q_n) = 0$ *is equivalent to the relation* $f(\pi_1, \pi_2, \ldots \pi_{n-r}) = 0$,

where $\pi_1, \pi_2, \ldots \pi_{n-r}$ are the dimensionless quantities formed by Q_1, Q_2, ... Q_n and r is the rank of the dimensional matrix of the given physical quantities.

An alteruative statement of π-theorem

An equation in physical variables which is dimensionally homogeneous, can be reduced to a relationship among a complete set of dimensionless products.

Proof. The π – theorem is based on the following assumptions:

(*i*) It is always possible to select m independent fundamental units in a physical phenomenon. (For example, in the case of viscous incompressible fluid, $m = 3$, *i.e.* mass, length and time. Again for the case of viscous compressible fluid, $m = 4$, *i.e.* mass, length, time and temperature).

(*ii*) There exists n quantities say Q_1, Q_2, ... Q_n involved in a physical phenomenon whose dimensional formulae may be expressed in terms of m fundamental units.

(*iii*) There exists a fundamental relationship between n dimensional quantities Q_1, Q_2, ... Q_n, say $\qquad \phi(Q_1, Q_2, \ldots, Q_n) = 0$ ...(1)

and this equation is independent of the types of units and is dimensionally homogeneous.

Let dimensions of Q_1, Q_2, ... Q_n be expressed in terms of m fundamental units $f_1, f_2 \ldots, f_m$, as follows:

$$\left.\begin{aligned} Q_1 &= f_1^{a_{11}} f_2^{a_{21}} \ldots f_m^{a_{m1}} \\ Q_2 &= f_1^{a_{12}} f_2^{a_{22}} \ldots f_m^{a_{m2}} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \text{and} \qquad Q_n &= f_1^{a_{1n}} f_2^{a_{2n}} \ldots f_m^{a_{mn}} \end{aligned}\right\}$$

The matrix of dimensions (*i.e.*, dimensional matrix) of the given physical n quantities Q_1, Q_2, ... Q_n may expressed as follows :

$$\begin{array}{c c} & \begin{matrix} Q_1: & Q_2: & \ldots & Q_n: \end{matrix} \\ \begin{matrix} f_1: \\ f_2: \\ \\ f_m: \end{matrix} & \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \end{array} \qquad \ldots(3)$$

Let the above $m \times n$ matrix be denoted by A.

Let us form a dimensionless product π of powers of Q_1, Q_2, ... Q_n as follows :

$$\pi = Q_1^{x_1} Q_1^{x_2} \ldots Q_n^{x_n}. \qquad \ldots(4)$$

Since product π is dimensionless, we have

$$\pi = f_1^0 f_2^0 \ldots f_m^0. \qquad \ldots(5)$$

Substituting the dimensions of π, Q_1, Q_2, ... Q_n from (5) and (2) in (4), we get

$$f_1^0 f_2^0 \ldots f_m^0 = (f_1^{a_{11}} f_2^{a_{21}} \ldots f_m^{a_{m1}})^{x_1} \cdot (f_1^{a_{12}} f_2^{a_{22}} \ldots f_m^{a_{m2}})^{x_2} \ldots (f_1^{a_{1n}} f_2^{a_{2n}} \ldots f_m^{a_{mn}})^{x_n}$$

Since the above equaiton must be dimensionally homogeneous, the exponents of each dimension on both sides of (2) must be identical. Thus, we have

$$\left.\begin{aligned} \text{For } f_1: \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ \text{For } f_2: \quad & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \text{For } f_m: \quad & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{aligned}\right\} \qquad \ldots(6)$$

Rewriting (6) in matrix form, we have $\qquad AX = 0,$ $\qquad \ldots(7)$

$$\text{where} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m2} & a_{m2} & \ldots & a_{mn} \end{bmatrix}. \text{ and } \qquad O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \ldots(8)$$

Now, (6) is a set of m homogeneous equations in n unknows. Let r be the rank of the dimensional matrix A. Then we conclude that (Refer Result II Art. 15.15) the number of linearly independent solutions of the equations are $n - r$. It follows that corresponding to each independent solution of X we will have a dimensionless product π and hence the number of dimensionless products in a complete set will be $n - r$.

15.17. Working rule for solving problems by Buckingham π– theorem.

Statement of Buckingham's π– theorem. *If there are n variables (dependent and independent variables) in a dimensionally homogeneous equaiton and if these variables contain* *m fundamental units, then the variables are arranged into (n – m) dimensionless numbers known as π – numbers.*

Thus, if any variable Q_1 depends on independent variables, Q_2, $Q_3 \ldots Q_n$ so that

$$Q_1 = f(Q_2, Q_3, \ldots, Q_n). \qquad \ldots(1)$$

or

$$f_1(Q_1, Q_2, \ldots, Q_n) = 0. \qquad \ldots(2)$$

which is dimensionally homogeneous and contains n variables. If there are m fundamental units, then according to Buckingham's π – theorem, (2) can be written in terms of $n - m$ dimensionless number as

$$f_1(\pi_1, \pi_2, \pi_3, \ldots, \pi_{n-m}) = 0. \qquad \ldots(3)$$

In (1), (2) and (3), f and f_1 are arbitrary functions.

Each dimensionless π – number is formed by combining m variables out of the total n variables with one of the remaining $(n - m)$ variables *i.e.* each π – term contains $(m + 1)$ variables. These m variables which appear repeatedly in each of π – number are known as repeated variables and are chosen from among the variables such that they together involve all the fundamental units and they themselves do not form a dimensionless number. To illustrate the working rule, suppose that Q_2, Q_3, Q_4 be the repeating variables. Then each π – number can be written as

$$\left.\begin{aligned} \pi_1 &= Q_2^{a_1}\, Q_3^{b_1}\, Q_4^{c_1}\, Q_1 \\ \pi_2 &= Q_2^{a_2}\, Q_3^{b_2}\, Q_4^{c_2}\, Q_5 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \pi_{n-m} &= Q_2^{a_{n-m}}\, Q_3^{b_{n-m}}\, Q_4^{c_{n-m}}\, Q_n \end{aligned}\right\} \qquad \ldots(4)$$

where $a_1, b_1, c_1, a_2, b_2, c_2, \ldots, a_{n-m}, b_{n-m}, c_{n-m}$ are constants. These constants can be obtained by considering dimensional homogeneity of equations (4). With these values of constants, (4) gives the values of $\pi_1, \pi_2, \ldots, \pi_{n-m}$. These values of π's are substituted in (3). Finally, we may express anyone of the π number as function of the remaining π numbers as

$$\pi_1 = \phi(\pi_2, \pi_3, \ldots, \pi_{n-m}), \qquad \pi_2 = \psi(\pi_1, \pi_3, \ldots, \pi_{n-m}) \qquad \text{and so on.}$$

Remark. While selecting repeating variables, note the follwing points :

(*i*) As far as possible, the dependent variable should not be selected as repeating variable.

(*ii*) No two repeating variables should have the same dimensions.

(*iii*) The repeating variables must not form the non-dimensionless number among themselves.

(*iv*) The repeating variables should be chosen in such a way that one variable contains geometric property (*e.g.* length, l, diameter, d, height, H etc.) the other variable contains flow property (*e.g.*, velocity, V, acceleration, a etc.) and third variable contains fluid property (*e.g.*, mass density, ρ, weight density, w; dynamic viscosity, μ, etc.).

(*v*) m repeating variables must contain jointly all the fundamental units involved in the phenomenon. Usually the fundamental units are M, L and T. However, if only two units are involved, there will be 2 repeating variables and they must contain together the two units involved.

The entire procedure for solving problems by Buckingham's π – theorem is outlined in the solved examples of Art. 15.19.

15.18. An Application of π – theorem to viscous compressible fluid flow.

To show that in the dynamics of compressible fluids, there are only five independent dimensionless groups. **[Himanchal 1998, 2000]**

Sol. In the study of dynamics of viscous compressible fluid the physical quantities involved are L, U, ρ, μ, k, g, p, C_p, T and the fundamental units in which the dimensions of all these quantities can be expressed are length (L), mass (M), time (t) temperature (θ). We now use working rule given in Art. 15.17.

Step I: The dimensional matrix in the present problem is

	L	U	ρ	μ	k	g	p	C_p	T
L:	1	1	–3	–1	1	1	–1	2	0
M:	0	0	1	1	1	0	1	0	0
t:	0	–1	0	–1	–3	–2	–2	–2	0
θ:	0	0	0	0	–1	0	0	–1	1

Step II: *As usual, the rank, of the above matrix is 4. Accordingly the number of independent dimensionless products will be 5.

Step III: Let L, U, ρ and k be chosen as base quantities.

Step IV: Assume that $\qquad \pi_1 = L^{a_1} U^{b_1} \rho^{c_1} k^{d_1} \mu \qquad$ ...(1)

$$\pi_2 = L^{a_2} U^{b_2} \rho^{c_2} k^{d_2} g, \qquad \pi_3 = L^{a_3} U^{b_3} \rho^{c_3} k^{d_3} p \qquad ...(2)$$

$$\pi_4 = L^{a_4} U^{b_4} \rho^{c_4} k^{d_4} C_p \quad \text{and} \quad \pi_5 = L^{a_5} U^{b_5} \rho^{c_5} k^{d_5} T, \qquad ...(3)$$

where a_i, b_i, c_i, d_i, (i = 1, 2, 3, 4, 5) are constants:

Step V : Determination of π_1. Substituting the dimensions of each physical quantity in (1) and noting that π_1 is a dimensionless number, we get

$$M^0 L^0 t^0 \theta^0 = L^{a_1} (Lt^{-1})^{b_1} (L^{-3}M)^{c_1} (LMt^{-3}\theta^{-1})^{d_1} (L^{-1}Mt^{-1}) \qquad ...(4)$$

Since (4) must be dimensionally homogeneous, we equate the exponents of M, L and t on both sides and obtain

$$c_1 + d_1 + 1 = 0, \quad a_1 + b_1 - 3c_1 + d_1 - 1 = 0, \quad - b_1 - 3d_1 - 1 = 0, \quad - d_1 = 0$$

Solving these, we get $\quad a_1 = -1, \quad b_1 = -1, \quad c_1 = -1 \quad$ and $\quad d_1 = 0$

Hence, from (1), $\qquad \pi_1 = L^{-1} U^{-1} \rho^{-1} \mu = \mu/(UL\rho)$

Proceeding likewise, it can be shown that

$$\pi_2 = (Lg)/U^2, \quad \pi_3 = p/(\rho U^2), \quad \pi_4 = (LU\rho C_p)/k, \quad \pi_5 = (kT)/(LU^3\rho)$$

With help the above dimensionless products, we can easily construct the following five dimensionless numbers:

$$Re = 1/\pi_1, \quad Fr = 1/\pi_2, \quad Pr = \pi_1 \pi_4, \quad (\gamma - 1)/\gamma = \pi_3/(\pi_4 \pi_5), \quad M_a^2 = 1/\gamma\pi_3$$

Hence, it follows that in the dynamics of compressible fluids there are only five independent dimensionless groups. This is in complete agreement with the results obtained by using inspection analysis (See Art 15.7).

15.19. Solved examples based on Art 15.17.

Ex. 1. *The resistance R experienced by a partially submerged body depends upon the velocity V, length of the body l, viscosity of the fluid* μ, *density of the fluid* ρ *and gravitational acceleration g. Obtain a dimensionless expression for R.*

Sol. Step I Here $\quad R = f(V, l, \mu, \rho, g) \quad$ or $\quad f_1(R, V, l, \mu, \rho, g) = 0, \qquad$ ...(1)

where f and f_1 are arbitrary functions.

* Refer any book on "Matrices" for rule of finding rank of a matrix.

Here, n = total number of variables = 6.

Now, writing dimensions of each variable, we have $R = MLT^{-2}$, $V = LT^{-1}$, $\mu = ML^{-1}T^{-1}$, $\rho = ML^{-3}$, $g = LT^{-2}$. Hence the fundamental units in the present problem are M, L, T.

Hence, m = total number of fundamental units = 3.

So according to Buckingham's $\pi-$ theorem, the number of dimensionless numbers = $n - m$ = 6 – 3 = 3. Let π_1, π_2, π_3 be dimensionless numbers. Then (1) can be rewrtten as

$$f_1(\pi_1, \pi_2, \pi_3) = 0. \qquad \ldots(2)$$

Step II. We propose to select repeating variables. Since m = 3, so three variables out of six variables R, V, l, μ, ρ, g are to be chosen as repeating variables. Since R is dependent variable, so we do not select R as a repeating variable. Again, out of the remaining five variables V, l, μ, ρ, g, repeating variables should be so chosen that one variable contains geometric property, other variable contains flow property and the remaining variable contains fluid property. Hence we select l, V and ρ as repeating variables. Now, dimensions of l, V, ρ are L, LT^{-1}, ML^{-3}. Hence the three fundamental units M, L, T exist in l, V and ρ. Again, it can be verified that no dimensionless number can be formed by l, V and ρ.

Step III. We now form each dimensionless $\pi-$ number by combining m (= 3) repeating variables l, V and ρ with one of the remaining variables, R, μ and g. Note that each number contains m + 1 (= 3 + 1 = 4) variables.

$$\pi_1 = l^{a_1} \cdot V^{b_1} \cdot \rho^{c_1} \cdot R, \qquad \ldots(3)$$

$$\pi_2 = l^{a_2} \cdot V^{b_2} \cdot \rho^{c_2} \cdot \mu \qquad \ldots(4)$$

and

$$\pi_3 = l^{a_3} \cdot V^{b_3} \cdot \rho^{c_3} \cdot g, \qquad \ldots(5)$$

where a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 are constants.

Step IV. We now determine the constants a_1, b_1, c_1, etc. involved in (3), (4), (5) and hence obtain π_1, π_2, π_3.

Determination of π_1. Substituting the dimensions of each physical quantity in (3) and noting that π_1 is a dimensionless number, we have

$$M^0L^0T^0 = L^{a_1} \cdot (LT^{-1})^{b_1} \cdot (ML^{-3})^{c_1} \cdot (MLT^{-2}). \qquad \ldots(6)$$

Since (6) must be dimensionally homogeneous, we equate the exponents of M, L, T on both sides and obtain $0 = c_1 + 1$, $0 = a_1 + b_1 - 3c_1 + 1$ and $0 = -b_1 - 2$

Solving these, $a_1 = -1$, $b_1 = -2$ and $c_1 = -1$.

$\therefore$ (3) gives,

$$\pi_1 = l^{-2} \cdot V^{-2} \cdot \rho^{-1} \cdot R = R/(l^2V^2\rho). \qquad \ldots(7)$$

Determination of π_2. Substituting the dimensions of each physical quantity in (4) and noting that π_2 is a dimensionless number, we have

$$M^0L^0T^0 = L^{a_2} \cdot (LT^{-1})^{b_2} \cdot (ML^{-3})^{c_2} \cdot (ML^{-1}T^{-1}). \qquad \ldots(8)$$

(8) $\Rightarrow$ $0 = c_2 + 1$, $0 = a_2 + b_2 - 3c_2 - 1$ and $0 = -b_2 - 1$.

Solving these, $a_2 = -1$, $b_2 = -1$, $c_2 = -1$.

$\therefore$ (4) gives

$$\pi_2 = l^{-1} \cdot V^{-1} \cdot \rho^{-1} \cdot \mu = \mu/(lV\rho). \qquad \ldots(9)$$

Determination of π_3. Substituting the dimenions of each physical quantity in (5) and noting that π_3 is a dimensionless number, we have

$$M^0L^0T^0 = L^{a_3} \cdot (LT^{-1})^{b_3} \cdot (ML^{-3})^{c_3} \cdot (MT^{-2}). \qquad \ldots(10)$$

(10) $\Rightarrow$ $0 = c_3$, $0 = a_3 + b_3 - 3c_3 + 1$, and $0 = -b_3 - 2$

Solving these, $a_3 = 1$, $b_3 = -2$ and $c_3 = 0$

$\therefore$ (5) gives, $\pi_3 = l^1 \cdot V^{-2} \cdot \rho^0 \cdot g = (lg)/V^2$. ...(11)

Using (7), (9) and (11), (2) reduces to

$$f_1\left(\frac{R}{l^2V^2\rho}, \frac{\mu}{lV\rho}, \frac{lg}{V^2}\right) = 0 \quad \text{or} \quad \frac{R}{l^2V^2\rho} = \phi\left(\frac{\mu}{lV\rho}, \frac{lg}{V^2}\right). \qquad ...(12)$$

Since the reciprocal of $\pi-$ number and its square root are non dimensional, so (12) can be rewritten as

$$\frac{R}{l^2V^2\rho} = \phi\left(\frac{lV\rho}{\mu}, \frac{V}{\sqrt{lg}}\right) \quad \text{or} \quad R = l^2V^2\rho\,\phi\left(\frac{lV\rho}{\mu}, \frac{V}{\sqrt{lg}}\right),$$

showing that the resistance R is a function of Reynold's number $(\rho Vl)/\mu$ and Froude's number $V/\sqrt{lg}$.

Ex. 2. *The pressure difference Δp in a pipe of diameter D and length l due to turbulent flow depends on the velcity V, viscosity μ, density ρ and roughness k. Using Buckingham's theorem, obtain an expression for Δp.*

Sol. Here $\Delta p = f(D, l, V, \mu, \rho, k)$ or $f(\Delta p, D, l, V, \mu, \rho, k) = 0$. ...(1)

where f and f_1 are arbitrary functions.

Hence n = total number of variables = 7.

Here, writing dimensions of each variable, we have $\Delta p = ML^{-1}T^{-2}$, $D = L$, $l = L$, $V = LT^{-1}$, $\mu = ML^{-1}T^{-1}$, $\rho = ML^{-3}$ and $k = L$.

Hence, m = total number of fundamental units = 3.

So according to Buckingham's π–theorem, the number of dimensionless numbers = $n - m$ = 7 – 3 = 4. Let π_1, π_2, π_3 and π_4 be dimensionless numbers. Then (1) can be rewritten as

$$f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0. \qquad ...(2)$$

We select D, V and ρ as m (=3) repeating variables.

We now form each dimensionless π–number by combining repeating variables D, V and ρ with one of the remaining variables Δp, l, μ, k. Note that each π–number contains $m + 1$ (= 3 + 1 = 4) variables.

$$\pi_1 = D^{a_1} \cdot V^{b_1} \cdot \rho^{c_1} \cdot \Delta p, \qquad ...(3)$$

$$\pi_2 = D^{a_2} \cdot V^{b_2} \cdot \rho^{c_2} \cdot l, \qquad ...(4)$$

$$\pi_3 = D^{a_3} \cdot V^{b_3} \cdot \rho^{c_3} \cdot \mu, \qquad ...(5)$$

and

$$\pi_4 = D^{a_4} \cdot V^{b_4} \cdot \rho^{c_4} \cdot k, \qquad ...(6)$$

where $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, a_4, b_4, c_4$ are constants.

Determination of π_1. Substituting the dimensions of each physical quantity in (3) and noting that π_1 is a dimensionless number, we get

$$M^0L^0T^0 = L^{a_1} \cdot (LT^{-1})^{b_1} \cdot (ML^{-3})^{c_1} \cdot (ML^{-1}T^{-2}). \qquad ...(7)$$

Since (7) must be dimensionally homogeneous, we equate the exponents of M, L, T on both sides and obtain $0 = c_1 + 1$, $0 = a_1 + b_1 - 3c_1 - 1$ and $0 = -b_1 - 2$.

Solving these, $a_1 = 0$, $b_1 = -2$ and $c_1 = -1$.

$\therefore$ (3) gives $\pi_1 = D^0 \cdot V^{-2} \cdot \rho^{-1} \cdot \Delta p = (\Delta p)/(\rho V^2)$. ...(8)

Determination of π_2. As before (4) gives

$$M^0L^0T^0 = L^{a_2} \cdot (LT^{-1})^{b_2} \cdot (ML^{-3})^{c_2} \cdot L. \qquad ...(9)$$

(9) $\Rightarrow$ $\quad 0 = c_2, \quad 0 = a_2 + b_2 - 3c_2 + 1 \quad$ and $\quad 0 = -b_2$.

Solving these, $\quad a_2 = -1, \quad b_2 = 0, \quad c_2 = 0$.

$\therefore$ (4) gives

$$\pi_2 = D^{-1} \cdot V^0 \cdot \rho^0 \cdot l = l/D. \qquad ...(10)$$

Determination of π_3. As before (5) gives

$$M^0L^0T^0 = L^{a_3} \cdot (LT^{-1})^{b_3} \cdot (ML^{-3})^{c_3} \cdot (ML^{-1}T^{-1}). \qquad ...(11)$$

(11) $\Rightarrow$ $\quad 0 = c_3 + 1, \quad 0 = a_3 + b_3 - 3c_3 - 1 \quad$ and $\quad 0 = -b_3 - 1$.

Solving these, $\quad a_3 = -1, \quad b_3 = -1 \quad$ and $\quad c_3 = -1$.

(5) gives

$$\pi_3 = D^{-1} \cdot V^{-1} \cdot \rho^{-1} \cdot \mu = \mu/(DV\rho). \qquad ...(12)$$

Determination of π_4. As before (6) gives

$$M^0L^0T^0 = L^{a_4} \cdot (LT^{-1})^{b_4} \cdot (ML^{-3})^{c_4} \cdot L. \qquad ...(13)$$

(13) $\Rightarrow$ $\quad 0 = c_4, \quad 0 = a_4 + b_4 - 3c_4 + 1 \quad$ and $\quad 0 = -b_4$.

Solving these, $\quad a_4 = -1, \quad b_4 = 0 \quad$ and $\quad c_4 = 0$.

$\therefore$ (6) gives,

$$\pi_4 = D^{-1} \cdot V^0 \cdot \rho^0 \cdot k = k/D. \qquad ...(14)$$

Using (8), (10) (12) and (14), (2) reduces to

$$f_1\left(\frac{\Delta p}{\rho V^2}, \frac{l}{D}, \frac{\mu}{DV\rho}, \frac{k}{D}\right) = 0 \qquad \text{or} \qquad \frac{\Delta p}{\rho V^2} = \phi\left(\frac{l}{D}, \frac{\mu}{DV\rho}, \frac{k}{D}\right).$$

so that

$$\Delta p = \rho V^2 \phi (l/D, \mu/(DV\rho), k/D).$$

Ex. 3 *Prove that the discharge over a spillway is given by the relation*

$$Q = VD^2 f(\sqrt{gD}/V, H/D),$$

where V = velocity of flow, D = depth at throat, H = Head of water, g = acceleration due to gravity. **[Meerut 2001, 06, 07, 08; Agra 1997, 2001]**

Sol. The physical probem can be expressed by the relation

$$F(Q, V, D, H, g) = 0. \qquad ...(1)$$

Thus there are five significant physical quantities in the given problem. The dimensional representation of these quantites is given by

$Q = [L^3 T^{-1}]; \quad V = [LT^{-1}], \quad D = [L], \quad H = [L], \quad g = [LT^{-2}]$.

Here L and T are two fundamental dimensions. Hence we choose two physical quantities (*i.e.*, as many as the number of fundamental dimensions) such that no product or ratio involving two (or more of these if there be more than two fundamental dimensions) can lead to a dimensionless number. Here we choose V and D as repeating variables. According to Buckingham π–theorem, the number of independent dimensionless numbers = 5 – 2 = 3. Let π_1, π_2, π_3 be these numbers. These are obtained in following manner :

We express π_1, π_2, π_3 in terms repeating quantities V, D and one of the remaining three quantities (namely, Q, H, g) by turn as follows* :

$$\pi_1 = (V)^{a_1} (D)^{b_1} Q \qquad ...(2a)$$

so that

$$M^0 L^0 T^0 = [LT^{-1}]^{a_1} [L]^{b_1} [L^3 T^{-1}] \qquad ...(2b)$$

$$\pi_2 = (V)^{a_2} (D)^{b_2} H \qquad ...(3a)$$

so that

$$M^0 L^0 T^0 = [LT^{-1}]^{a_2} [L]^{b_2} [L] \qquad ...(3b)$$

and

$$\pi_3 = (V)^{a_3} (D)^{b_3} g \qquad ...(4a)$$

so that

$$M^0 L^0 T^0 = [LT^{-1}]^{a_3} [L]^{b_3} [LT^{-2}] \qquad ...(4b)$$

Since ($2b$) must be dimensionally homogeneous, equate the powers of L and T and obtain $a_1 + b_1 + 3 = 0$ and $- a_1 - 1 = 0$, giving $a_1 = -1$ and $b_1 = -2$. Then ($2a$) reduces to

$$\pi_1 = Q/VD^2 \qquad ...(5)$$

Similarly, (36) gives $\qquad \pi_2 = H/D \qquad ...(6)$

and ($4b$) gives $\qquad \pi_3 = gD/V^2 = \sqrt{gD}/V \qquad$ [Note]** ...(7)

Hence the physical problem may be expressed as $\qquad \phi(\pi_1, \pi_2, \pi_3) = 0. \qquad ...(8)$

(8) may be re-written as $\qquad \pi_1 = f(\pi_3, \pi_2)$, f being an arbitrary function

or $\qquad Q/VD^2 = f(\sqrt{gD}/V, H/D) \qquad$ or $\qquad Q = VD^2 f(\sqrt{gD}/V, H/D).$

Ex. 4. *A cylindrical tank of diameter D discharges water through a short pipe of diameter d into the atmosphere. The water surface is at a distance L above the level of the pipe. The mass flow rate* $\dot{m}$ *(i.e. dm/dt) is known to depend on the lengths D, d and L ; the acceleration due to gravity g ; density of water* ρ. *Determine the dimensionless mumbers of the physical problem just described.*

Sol. The physical problem can be expressed by the relation

$$F(\dot{m}, D, d, L, g, \rho) = 0 \qquad ...(1)$$

The dimensional represenation of the physical quantities is given by

$\dot{m} = [MT^{-1}]$, $\quad D = [L]$, $\quad d = [L]$, $\quad L = [L]$, $\quad g = [LT^{-2}]$, $\quad \rho = [ML^{-3}]$.

Since the number of significant quantities = n = 6, and the number of fundamental dimensions (M, L, T) = m = 3, hence the number of repeating quantites = m = 3. We choose them as d, g, and ρ. Notice that no product or ratio of two or more of d, g and ρ can give rise to a dimensionless number. According to Buckingham π–theorem, there are $n - m$ *i.e.* 3 independent dimensionless numbers π_1, π_2, π_3, say.

We express π_1, π_2, π_3 in terms of the repeating quantities (*viz.*, d, g, ρ) and one of the remaining three quantities (*viz.*, $\dot{m}$, D, L) by turn as follows :

$$\pi_1 = (d)^{a_1} (g)^{b_1} (\rho)^{c_1} \dot{m} \qquad ...(2a)$$

so that

$$M^0 L^0 T^0 = [L]^{a_1} [LT^{-2}]^{b_1} [ML^{-3}]^{c_1} [ML^{-1}] \qquad ...(2b)$$

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* Note that on R.H.S. only V and D have been raised to powers.

** If gD/V^2 is dimensionless number, then $[gD/V^2]^{1/2}$ *i.e.* $\sqrt{gD}/V$ is also dimensionless number.

$$\pi_2 = (d)^{a_2} (g)^{b_2} (\rho)^{c_2} D \qquad ...(3a)$$

so that $$M^0L^0T^0 = [L]^{a_2} [LT^{-2}]^{b_2} [ML^{-3}]^{c_2} [L] \qquad ...(3b)$$

$$\pi_3 = (d)^{a_3} (g)^{b_3} (\rho)^{c_3} L \qquad ...(4a)$$

so that $$M^0L^0T^0 = [L]^{a_3} [LT^{-2}]^{b_3} [ML^{-3}]^{c_3} [L] \qquad ...(4b)$$

Since (2*b*) must be dimensionally homogeneous, equate powers of M, L and T and obtain

$$c_1 + 1 = 0, \qquad a_1 + b_1 - 3c_1 = 0, \qquad -2b_1 - 1 = 0,$$

giving $c_1 = -1$, $b_1 = -1/2$, $a_1 = -5/2$. Then (2*a*) gives

$$\pi_1 = \dot{m}/(\rho d^{5/3} g^{1/2}) \qquad ...(5)$$

Similarly, $\pi_2 = D/d$ and $\pi_3 = L/d$. ...(6)

Thus, the required dimensionless numbers are π_1, π_2, π_3 given by (5) and (6).

Ex. 5. *The viscous force F_D exerted by the fluid on a sphere of diameter D depends on viscosity μ, mass density of fluid ρ and velocity of the sphere U. Show that the drag coefficient C_D $(= F_D/\rho U^2 D^2)$ is a funciton of Reynolds number Re.*

Sol. The given physical problem can be expressed by the relation

$$f(F_D, \mu, \rho, U) = 0, \qquad ...(1)$$

where f is an arbitrary function. Thus there are five significant physical quantities in the given problem. The dimensional representation of these quantities is given by

$$F_D = [MLT^{-2}], \qquad \mu = [ML^{-1}T^{-1}], \qquad \rho = [ML^{-3}] \quad \text{and} \quad U = [LT^{-1}]$$

Since the number of significant quantities = n = 5 and the number of fundamental dimensions (M, L, T) = m = 3, hence the repeating quantities = m = 3. We choose U, D and ρ as repeating variables. Note that no product or ratio of two or more of U, D and ρ can give rise to a dimensionless number. According to Buckingham π–theorem, there are $n - m$, *i.e.* 5 – 3, *i.e.*, two independent dimensionless numbers π_1 and π_2, say.

We express π_1 and π_2 in terms of the repeating quantities U, D and ρ and one of the remaining two quantities F_D and μ by turn as follows:

$$\pi_1 = U^{a_1} D^{b_1} \rho^{c_1} F_D \qquad ...(2a)$$

so that $$M^0L^0T^0 = [LT^{-1}]^{a_1} [L]^{b_1} [ML^{-3}]^{c_1} [MLT^{-2}] \qquad ...(2b)$$

and $$\pi_2 = U^{a_2} D^{b_2} \rho^{c_2} \mu \qquad ...(3a)$$

so that $$M^0L^0T^0 = [LT^{-1}]^{a_2} [L]^{b_2} [ML^{-3}]^{c_2} [ML^{-1}T^{-2}] \qquad ...(3b)$$

Since (2*b*) must be dimensionally homogeneous, equate powers of M, L, T and obtain

$$c_1 + 1 = 0, \quad a_1 + b_1 - 3c_1 + 1 = 0, \quad -a_1 - 2 = 0 \Rightarrow a_1 = -2, \; c_1 = -1, \quad b_1 = -2$$

$\therefore$ (2*a*) yields $$\pi_1 = U^{-2} D^{-2} \rho^{-1} F_D = F_D/(\rho U^2 D^2) \qquad ...(4)$$

Since (3*b*) must be dimensionally homogeneous, equate powers of M, L, T and obtain

$$c_2 + 1 = 0, \quad a_2 + b_2 - 3c_2 - 1 = 0, \quad -a_2 - 1 = 0 \Rightarrow a_2 = -1, \quad b_2 = -1, \; c_2 = -1$$

$\therefore$ (2*a*) yields $$\pi_2 = U^{-1} D^{-1} \rho^{-1} \mu = \mu/(\rho UD) \qquad ...(5)$$

Hence the given physical problem may be expressed as

$$\pi_1 = \phi(\pi_2) \quad \text{or} \quad F_D/(\rho U^2 D^2) = \phi(\mu/\rho UD)$$

or $$F_D/(\rho U^2 D^2) = f(\rho UD/\mu) \quad \text{or} \quad C_D = f(\text{Re}), \qquad ...(6)$$

where $C_D = F_D/(\rho U^2 D^2)$ = drag cedfficient and $\text{Re} = (\rho UD)/\mu$ = Reynold's number. While getting equation (6), we have used the fact that the dimensionless number $\pi_2 \ (= \mu/\rho UD)$ may be inverted without affecting its dimensionless nature.

Ex. 6 *A V-notch weir is a vertical plate with a notch of angle ϕ cut into the top of it and placed across an open channel. The liquid in the channel is backed up and forced to blow through the notch. The discharge Q is some function of the elevation H of the stream liquid surface above the bottom of notch. In addition the discharge depends upon gravity and upon the velocity of approach V_0 to the weir. Determine the form of discharge equation.*

Sol. The given physical problem can be expressed by the relation

$$F(Q, H, g, V_0, \phi) = 0, \qquad \text{...(1)}$$

where F is an arbitrary function. Here ϕ is dimensionless and the dimensional representations of the remaining four significant physical quantities Q, H, g and V_c are given by

$$Q = [L^3\, T^{-1}], \qquad H = [L], \qquad g = \{LT^{-2}], \qquad V_0 = [LT^{-1}]$$

Since the number of significant quantities = n = 4 and the number of fundamental dimension (L and T) = m = 2, hence the number of repeating quantities = m = 2. We choose H and V_0 as repeating variables. Note that no product or ratio of these repeating variables can give rise to a dimensionless number. According to Buckingham π–theorem, there are $n - m$ (= 4 – 2) *i.e.*, 2 independent dimensionless number π_1 and π_2, say.

We express π_1 and π_2 in terms of the repeating quantities H and V_0 by turn as follows :

$$\pi_1 = H^{a_1} V_0^{b_1} Q \qquad \text{...(2a)}$$

so that

$$L^0 T^0 = L^{a_1} [LT^{-1}]^{b_1} [L^3 T^{-1}] \qquad \text{...(2b)}$$

and

$$\pi_2 = H^{a_2} V_0^{b_2} g \qquad \text{...(3a)}$$

so that

$$L^0 T^0 = L^{a_2} [LT^{-1}]^{b_2} [LT^{-2}] \qquad \text{...(3b)}$$

Since (2*b*) must be dimensionally homogeneous, we equate the powers of L, T, and obtain

$$a_1 + b_1 + 3 = 0 \quad \text{and} \quad -b_1 - 1 = 0 \quad \Rightarrow \quad a_1 = -2 \quad \text{and} \quad b_1 = -1$$

$\therefore$ (2*a*) yields

$$\pi_1 = H^{-2} V_0^{-1} Q = Q/H^2 V_0 \qquad \text{...(4)}$$

Since (3*b*) must be dimensionally homogeneous, we equate the powers of L, T and obtain

$$a_2 + b_2 + 1 = 0, \quad -b_2 - 2 = 0 \quad \Rightarrow \quad a_2 = 1 \quad \text{and} \quad b_2 = -2$$

$\therefore$ (3*a*) yields

$$\pi_2 = H V_0^{-2} g = Hg/V_0^2 \qquad \text{...(5)}$$

Hence the given physical problem may be expressed as

$$Q/H^2 V_0 = f(gH/V_0^2, \phi) \qquad \text{or} \qquad Q = H^2 V_0 f(V_0/\sqrt{gH}, \phi),$$

where f is an arbitrary function. While getting equation (6), we have used the fact that the dimensionless number $\pi_2 \ (= Hg/V_0^2)$ may be inverted or raised to any power without affecting its dimensionless nature.

Ex.7. *A fluid flow situation depends upon the velocity V, the density ρ, several linear dimensions l. l_1. l_2., pressure drop Δp, gravity g, viscosity μ, surface tension σ, and bulk modulus of elasticity k. Apply dimensional analysis to these variables to find a set of π parameters and an expression for Δp.*

Sol. The given physical problem can be expressed by the relation

$$F(V, \rho, l, l_1, l_2, \Delta p, g, \mu, \sigma, k) = 0, \qquad \text{...(1)}$$

where F is an arbitrary function. Since the number of significant quantities = n = 10 and the number of fundamental dimensions (M, L, T) = m = 3, hence the number of repeating quantities = m = 3. We choose V, ρ and l as repeating variables. Note that no product or ratio of two or more of V, ρ and l can give rise to dimensionless number. According to Buckingham π–theorem, there are n–m *i.e.*, 7 independent dimensionless numbers $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6$ and π_7, say,

We express $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6$ and π_7 in terms of the repeatring quantites V, ρ, l by turn as follows:

$$\pi_1 = V^{a_1}\rho^{b_1}l^{c_1}\Delta p \quad \Rightarrow \quad M^0L^0T^0 = [LT^{-1}]^{a_1}[ML^{-3}]^{b_1}[L]^{c_1}[ML^{-1}T^{-2}]$$

$$\pi_2 = V^{a_2}\rho^{b_2}l^{c_2}g \quad \Rightarrow \quad M^0L^0T^0 = [LT^{-1}]^{a_2}\ [ML^{-3}]^{b_2}[L]^{c_2}[LT^{-2}]$$

$$\pi_3 = V^{a_3}\rho^{b_3}l^{c_3}\mu \quad \Rightarrow \quad M^0L^0T^0 = [LT^{-1}]^{a_3}\ [ML^{-3}]^{b_3}[L]^{c_3}[ML^{-1}T^{-1}]$$

$$\pi_4 = V^{a_4}\rho^{b_4}l^{c_4}\sigma \quad \Rightarrow \quad M^0L^0T^0 = [LT^{-1}]^{a_4}\ [ML^{-3}]^{b_4}[L]^{c_4}[ML^{-2}]$$

$$\pi_5 = V^{a_5}\rho^{b_5}l^{c_5}k \quad \Rightarrow \quad M^0L^0T^0 = [LT^{-1}]^{a_5}\ [ML^{-3}]^{b_5}[L]^{c_5}[ML^{-1}T^{-2}]$$

$$\pi_6 = l/l_1 \qquad \text{and} \qquad \pi_2 = l/l_2$$

Since the above equations must be dimensionally homogeneous, as usual equating the powers of M, L, T, obtain a_1, b_1 , c_1, a_2, b_2, c_2, … , a_5, b_5, c_5 and thus we finally get the required set of seven π parameters given by

$$\pi_1 = \Delta p/\rho V^2,\ \pi_2 = gl/V^2,\ \pi_3 = \mu/Vl\rho,\ \pi_4 = \sigma/V^2\rho l,\ \pi_5 = k/\rho V^2,\ \pi_6 = l/l_1, \text{and } \pi_7 = l/l_2.$$

Since any of the π parameters may be inverted or raised to any power without affecting their dimensionless nature, hence we may write

$$\Delta p/\rho V^2 = f(V^2/gl, Vl\rho/\mu, V^2\rho l/\sigma, V/\sqrt{k/\rho},\ l/l_1, l/l_2)$$

or
$$\Delta p = \rho V^2 f(Fr, Re, We, Ma, l/l_1, l/l_2),$$

where f is an arbitary funciton and *Fr*, *Re*, *We* and *Ma* are the Froude number, the Reynolds number, the Weber number and the Mach number respectively.

EXERCISE 15 (B)

1. Check the dimensional homogeneity of the following common equations in the field of hydraulics (*i*) $Q = c_d a\sqrt{2gh}$. (*ii*) $v = c\sqrt{mi}$.

[**Ans.** (*i*) Homogeneous, (*ii*) Non-homogeneous]

2. The following equation is applicable in MKS system of units $v = 50\sqrt{mi}$, where v is the velocity of water, m hydraulic mean depth and i the longitudinal slope of the channel, Find the corresponding equation in FPS system of units. [**Ans.** $v = 90.5\sqrt{mi}$]

3. In a certain pipe, the pressure difference Δp across its ends is known to depend on the length l of the pipe ; its diameter D; density ρ and viscosity μ of the fluid ; the velocity U ; the roughness of the pipe wall given in terms of a length ε which is the mean height of the roughness elements on the inside of the pipe. Show that the Euler number $\nabla p/\rho U^2$, the Reynolds number $\rho UD/\mu$. and the length ratios l/D and ε/D are functionally related.

4. The thrust (P) of a propeller depends upon the diameter (D), speed (V), mass density (ρ), revolutions per minute (N) and coefficient of viscosity (μ). Show that

$$\mathrm{P} = \rho \mathrm{D}^2\mathrm{V}^2 f(\mu/\rho \mathrm{DV}, \mathrm{DN/V}).$$ **[Meerut 1998]**

5. Using Rayleigh's method, determine the rational formula for discharge through a sharp-edged orifice freely into the atmosphere in terms of constant head (H), diameter of orifice d, mass density (ρ), dynamic viscosity (μ). and acceleration due to gravity (g).

6. Show that the discharge of a centrifugal pump is given by

$$Q = ND^3 f(gH / N^2 D^2, \mu / ND^2 \rho),$$

where N is the speed of the pump in r.p.m., D the diameter of the impeller, g acceleration due to gravity, H manometric head, μ viscosity of fluid and ρ the density of the fluid.

7. Show that the shear stress in a fluid flowing through a pipe is given by

$$\tau = \rho V^2 f(\mu / DV, k / D),$$

where D is the diameter of the pipe, ρ is the mean density, V the velocity and μ the viscosity of the fluid and k the roughness projection.

8. Show by the use of Buckingham's π–theorem that velocity through a circular orifice is given by $\mathrm{V} = \sqrt{2g\mathrm{H}}\, f(\mathrm{D/H}, \mu / \rho \mathrm{VH})$, where H is the head causing flow, D diameter of the orifice, μ the coefficient of viscosity, ρ the mass density and g the gravitational acceleration.

9. Show that the pressure drop due to an obstruction in a pipe is given by $\Delta p = \rho V^2 f(\mathrm{DV}\rho / \mu)$, where D is the diameter of the pipe, V the velocity, ρ the mass density and μ the dynamic viscosity of the fluid.

MISCELLANEOUS PROBLEM ON CHAPTER 15

1. Write a short note on inspection analysis. **[Himanchal 2009]**

2. State and prove Buckingham π-theorem in dimensional form. **[Himanchal 2009]**

3. Write a short note on dimensional analysis. **[Himanchal 2008]**

4. Indicate the correct answer :

The ratio (Inertia force / Viscous force) is said to be (*a*) Magnetic number

(*b*) Reynolds number (*c*) Kirchoff number (*d*) Picklet number. **[Agra 2009, 10]**

Hint. Ans. (b). Refer Art. 15.8.

5. Define Reynold's number in the context of Dynamical Similarity and mention the significance of Reynold's number. **(Meerut 2012)**

Hint. Refer Art. 15.5 and Art. 15.6

6. Fill up gap : The ratio q/c is said to be ... number . (Agra 2011)

Hint. Ans. March number. Refer page 15.9

7. The Reynold's number Re is defined as

(*a*) Intertia force / pressure force (*b*) Inertia force / gravity force

(*c*) Inertia force / viscous force (*d*) None of these. **(Agra 2012)**

Sol. Ans. (c). Refer page 15.8.

CHAPTER

16 Laminar Flow of Viscous Incompressible Fluids

16.1. The main limitations of the Navier-Stokes equations.

[Kanpur 2009; Meerut 2003, 10, 11, 12]

In discussing these limitations we shall refer to equations (14*a*), (14*b*), (14*c*) of Art. 14.1, chapter 14. The limitations are :

1. They are unable to throw light on flow of non-Newtonian fluids. Derivation of the Navier-Stokes equations is based on Stokes' law of viscosity which holds for most common fluids (known as the Newtonian fluids). Since Stokes' law is not applicable to non Newtonian fluids (such as slurries, drilling muds, oil paints, tooth paste, sewage sludge, pitch, coal-tar, flour doughs, high polymer solutions, colloidal suspensions, clay in water, paper pulp in water, lime in water etc.), the Navier-Stokes equations cannot be applied to study non-Newtonian fluids.

2. In derivation of the Navier-Stokes equations we regarded fluid as a continuum. The continuum hypothesis though simplifies mathematical work, but it is unable to explain the inner structure of the fluid. Hence for the concept of viscosity, we have to depend on the empirical formulation.

3. These equations are non-linear in nature and hence prevent us from getting a single solution in which convective terms interact in a general manner with viscous terms. Due to presence of terms such as $u\,(\partial u/\partial x)$ etc. in these equations, they are non-linear in nature. Hence the solutions of these equations even in the restricted case of flow which is incompressible and steady, is extremely difficult.

4. Due to idealizations such as infinite plates, fully developed parallel flow in a pipe, even limited number of exact solutions of these equations are valid only in a particular region in a real situation.

16.2. Some exact solutions of the Navier-Stokes equations.

The Navier-Stokes equations are second order non-linear partial differential equations. Until the present day there exist no general method for solving these equations. Analytical (exact) solutions have therefore been attempted only for flows with relatively simple geometry. Even such solutions are based on idealizations such as infinite plates, infinitely long cylinders, fully developed parallel flow in pipe etc. Obviously then these exact solutions hold good in a particular region of a real problem. Restricted though they are, the exact solutions are very useful and add greatly to our knowledge of the flow of real (Newtonian) fluids.

In the present chapter we propose to discuss some useful real problems for which exact solutions are possible. We shall take the geometry in such a manner that the convective term (and hence non-linearity) disappears.

Before proceeding for exact solutions we make the following observation. Equations (14*a*) to (14*c*) of Art. 14.1, chapter 14 show that for incompressible flow the equation of motions differ from Euler's equations of motion (refer Art 3.1) having terms of second order attached with μ. This increases the order of the Navier-Stokes equations by 1. Consequently an additional boundary condition will be required. This is produced by the condition that there must be no slip between a viscous fluid and its boundary. It follows that the solution to the corresponding non-viscous flow problem cannot be obtained by first solving (14*a*) to (14*c*) and then taking the limit as $\mu \to 0$. We now present some solvable viscous flow problems by analytical methods.

For convenience we shall divide problems of this chapter into the following seven types of problems:

Type 1: *Determination of velocity distribution in steady laminar flow of viscous incompressible fluid with constant fluid properties*

Type 2: *Determination of temperature distribution in steady laminar flow of viscous incompressible fluid with constant fluid properties*

Type 3: *Flow of two immiscible fluids*

Type 4: *Steady incompressible flow fluid section/injection on the boundaries*

Type 5: *Unsteady incompressible flow with constant fluid properties*

Type 6: *Steady incompressible flow with variable viscosity*

Type 7: *Stagnation in viscous incompressible flow*

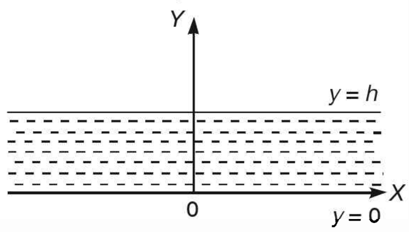
We now proceed to discuss the above types of flow problems

Type 1: Determination of velocity distribution in steady laminar flow of viscous incompressible fluid with constant fluid properties

16.3A. Steady laminar flow between two parallel plates. Plane Couette flow.

[Agra 2006, 09, 10; Meerut 1999 Garwhal 2002 Himanchal 2009; Kanpur 2003, 04]

Consider the steady laminar flow of viscous incompressible fluid between two infinite parallel plates separated by a distance h. Let x be the direction of flow, y the direction perpendicular to the flow, and the width of the plates parallel to the z-direction. Here the word `infinite' implies that the width of the plates is large compared with h and hence the flow may be treated as two-dimensional (*i.e.* $\partial/\partial z = 0$). Let the plates be long enough in the x-direction for the flow to be parallel*. Here we take the velocity components v and w to be zero everywhere. Moreover the flow being steady, the flow variables are independent of time $(\partial/\partial t = 0)$. Furthermore, the equation of continuity [namely, $(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0]$ reduces to $\partial u/\partial x = 0$ so that $u = u\ (y)$. Thus for the present problem, we have



$$u = u\ (y), \qquad v = 0, \qquad w = 0, \qquad \partial/\partial z = 0, \qquad \partial/\partial t = 0. \qquad \text{...(1)}$$

For the present two-dimensional flow in absence of body forces, the Navier-Stokes equations for x and y-directions (Refer (14*a*) and (14*b*) in Art 14.1 of chapter 14 keeping the above equation (1) in mind) are :

$$0 = -(\partial p/\partial x) + \mu(d^2 u/dy^2) \qquad \text{...(2)}$$

$$0 = -\partial p/\partial y \qquad \text{...(3)}$$

Equation (3) show that the pressure does not depend on y. Hence p is function of x alone and so (2) reduces to

$$d^2 u/dy^2 = (1/\mu) \times (dp/dx). \qquad \text{...(4)}$$

Differentiating both sides of (4) *w.r.t.* `x',
$$0 = \frac{1}{\mu}\frac{d^2 p}{dx^2} \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{dp}{dx}\right) = 0$$

* A flow is said to be parallel if only one velocity component is nonzero, all fluid particles moving in one direction.

so that $$dp/dx = \text{const.} = P \text{ (say)}. \qquad ...(5)$$

Then, (4) reduces to $$d^2u/dy^2 = P/\mu \qquad ...(6)$$

Integrating (6), $$du/dy = (Py)/\mu + A \qquad ...(7)$$

Integrating (7), $$u = Ay + B + (P/2\mu) \times y^2 \qquad ...(8)$$

where A and B are arbitrary constants to be determined by the boundary conditions of the problem under consideration.

For the plane Couette flow, $P = 0$. Again, the plate $y = 0$ is kept at rest and the plate $y = h$ is allowed to move with velocity U. Then the no slip condition gives rise to the boundary conditions.

$$u = 0 \quad \text{at} \quad y = 0; \qquad \text{and} \qquad u = U \quad \text{at} \quad y = h. \qquad ...(9)$$

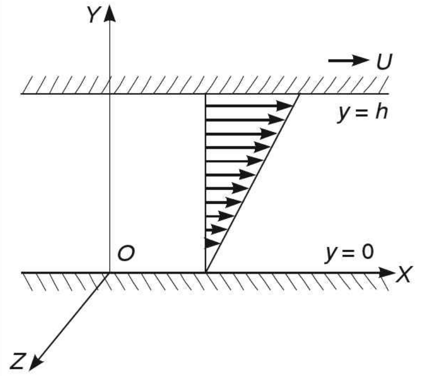
Using (9), (8) yields

$$O = B \quad \text{and} \quad U = Ah + B$$

so that $$B = 0 \quad \text{and} \quad A = U/h \qquad ...(10)$$

Using (10) in (8), we obtain

$$u = Uy/h \qquad ...(11)$$



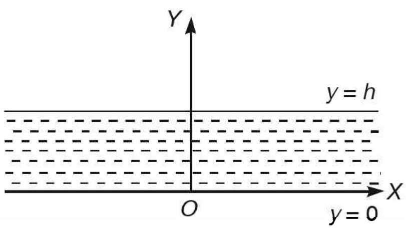
The velocity distribution is linear as shown in the adjoining figure. Now the skin friction (or drag per unit area, i.e., the shearing stress at the plates) σ_{yx} is given by

$$\sigma_{yx} = \mu\,(du/dy) = \mu U/h, \text{ using (11)}$$

16.3B. Generalized plane Couette flow.

(Agra 2008, Kanpur 2002, Meerut 2002,08,09, 10; Himanchal 2010)

Consider the steady laminar flow of viscous incompressible fluid between two infinite parallel plates separated by a distance h. Let x be the direction of the flow, and the width of the plates parallel to the z-direction. Here the word 'infinite' implies that the width of the plates is large compared with h and hence the flow may be treated as two-dimensional (*i.e.* $\partial/\partial z = 0$). Let the plates be long enough in the x-direction for the flow to be parallel. Here we take the velocity components v and w to be zero everywhere. Moreover the flow being steady, the flow variables are independent of time ($\partial/\partial t = 0$). Furthermore, the equation of continuity $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$ reduces to $\partial u/\partial x = 0$ so that



$u = u\,(y)$. Thus for the present problem

$$u = u\,(y), \qquad v = 0, \qquad w = 0. \qquad \partial u/\partial z = 0, \qquad \partial/\partial t = 0. \qquad ...(1)$$

For the present two-dimensional flow in absence of body forces, the Navier-Stokes equations for x and y-directions take the form :

$$0 = -(\partial p/\partial x) + \mu\,(d^2u/\partial y^2) \qquad ...(2)$$

and $$0 = -\partial p/\partial y \qquad ...(3)$$

Equation (3) shows that the pressure does not depend on y. Hence p is function of x alone and so (2) reduces to

$$d^2u/\partial y^2 = (1/\mu) \times (dp/dx) \qquad ...(4)$$

Differentiating both sides of (4) *w.r.t.* 'x', we find

$$0 = \frac{1}{\mu}\frac{d^2p}{dx^2} \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{dp}{dx}\right) = 0$$

so that $$dp/dx = \text{const.} = P \text{ (say)}. \qquad \text{...(5)}$$

Then, (4) reduces to $$d^2u/dy^2 = P/\mu \qquad \text{...(6)}$$

Integrating (6), $$du/dy = Py/\mu + A \qquad \text{...(7)}$$

Integrating (7), $$u = Ay + B + Py^2/2\mu, \qquad \text{...(8)}$$

where A and B are arbitrary constants to be determined by the boundary conditions of the problem under consideration.

For the so-called generalized plane Couette flow, the plate $y = 0$ is kept at rest and the plate $y = h$ is allowed to move with velocity U. Then the no slip condition gives rise to the following boundary conditions : $u = 0$ at $y = 0$, and $u = U$ at $y = h$.

Using these conditions, (8) gives $O = B$ and $U = Ah + B + Ph^2/2\mu$

so that $$B = 0 \quad \text{and} \quad A = (U/h) - (Ph/2\mu) \qquad \text{...(9)}$$

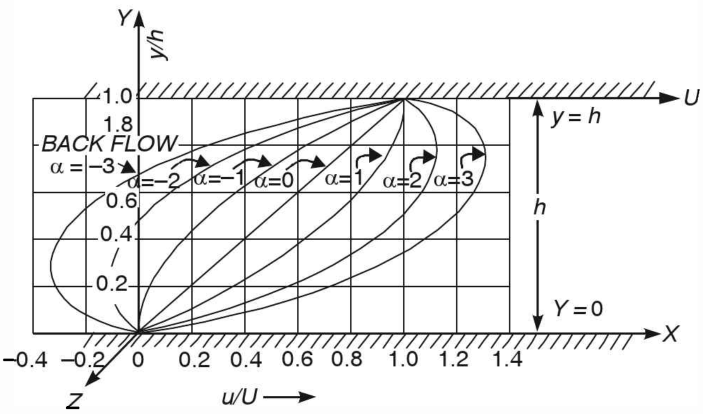
Using (9) in (8), we get

$$u = \frac{Uy}{h} - \frac{Phy}{2\mu} + \frac{Py^2}{2\mu} \quad \text{or} \quad u = U\frac{y}{h} - \frac{h^2P}{2\mu} \times \frac{y}{h}\left(1 - \frac{y}{h}\right) \qquad \text{...(10)}$$

or $$\frac{u}{U} = \frac{y}{h} + \alpha\frac{y}{h}\left(1 - \frac{y}{h}\right), \qquad \text{...(11)}$$

where $$\alpha = -h^2P/2\mu U \qquad \text{...(12)}$$

is the dimensionless pressure gradient. For $\alpha > 0$, *i.e.* for a pressure decreasing in the direction of motion, (11) shows that the velocity is positive over the entire width between the plates. For



$\alpha < 0$, *i.e.* for a pressure increasing in the direction of motion, (11) shows that the velocity over a portion of the channel width can become negative, that is back-flow may occur near the wall which is at rest. The figure given above shows that this occurs when $\alpha < -1$. The reverse flow near the stationary wall takes place because the influence of the adverse pressure gradient surpasses the action of the viscous force in that region. The velocity distribution as a function of the distance from the stationary wall for various values of α is shown in the figure. The figure also indicates the arrangement of plates with coordinate axes. u/U and y/h have been plotted along x-and y-axes respectively.

(*i*) **To determine average and maximum velocities.** **[Meerut 2010]**

The average velocity distribution for the present flow is given by

$$u_a = \frac{1}{h}\int_0^h u\,dy = \frac{1}{h}\int_0^h \left[\frac{y}{h}U + \alpha U(y/h - y^2/h^2)\right]dy, \quad \text{using (11)}$$

$$= (1/2 + \alpha/6)\,U, \text{on simplification}$$

Thus,
$$u_a = (1/6)\times(\alpha+3)\,U. \qquad \text{...(13)}$$

The volumetric flow Q per unit time per unit width of the channel is given by

$$Q = hu_a = (1/6)\times(\alpha+3)hU.$$

From (11),
$$\frac{du}{dy} = \frac{U}{h} + \frac{\alpha U}{h}\left(1 - \frac{2y}{h}\right) \qquad \text{...(14)}$$

For the maximum or minimum velocity, $du/dy = 0$

i.e. $\dfrac{U}{h} + \dfrac{\alpha U}{h}\left(1 - \dfrac{2y}{h}\right) = 0$ giving $\dfrac{y}{h} = \dfrac{1}{2}\left(1 + \dfrac{1}{\alpha}\right).$ ...(15)

From (15), it follows that the maximum velocity for $\alpha = 1$ occurs at $y/h = 1$ (*i.e.* $y = h$) and the minimum velocity for $\alpha = -1$ at $y/h = 0$ (*i.e.* $y = 0$). This further shows that for $\alpha = -1$ the velocity gradient at the stationary wall is zero and it becomes negative for some value of $\alpha < -1$. Thus the reverse flow takes place when $\alpha < -1$. Equation (15) breaks down when $-1 < \alpha < 1$ because the maximum and minimum values of y/h have already been reached at $\alpha = 1$ and $\alpha = -1$ respectively. Using (15) in (14), the maximum and minimum velocities are given by

$$\left.\begin{aligned} U_{\max} &= \{U(1+\alpha)^2\}/4\alpha, \quad \text{when } \alpha \geq 1 \\ U_{\min} &= \{U(1+\alpha)^2\}/4\alpha, \quad \text{when } \alpha \leq -1 \end{aligned}\right\} \qquad \text{...(16)}$$

(*ii*) **To determine shearing stress, skin friction and the coefficient of friction.**
[Meerut 2009, 10]

Using (14), the shearing stress distribution in the flow is given by

$$\sigma_{yx} = \mu\frac{du}{dy} = \frac{\mu U}{h}\left\{1 + \alpha\left(1 - \frac{2y}{h}\right)\right\} \qquad \text{...(17)}$$

Using (13) and (17), the skin frictions at the plates $y = 0$ and $y = h$ are given by

$$\left[\sigma_{yx}\right]_{y=0} = \frac{\mu U}{h}(1+\alpha) = \frac{6\mu\,(1+\alpha)}{(3+\alpha)\,h}u_a \qquad \text{...(18)}$$

$$\left[\sigma_{yx}\right]_{y=h} = \frac{\mu U}{h}(1-\alpha) = \frac{6\mu\,(1-\alpha)}{(3+\alpha)h}u_a \qquad \text{...(19)}$$

The coefficient of friction (or the drag coefficient) corresponding to $(\sigma_{yx})_{y=0}$ is given by

$$C_f = \frac{\left[\sigma_{yx}\right]_{y=0}}{(\rho u_a^2)/2} = \frac{12\mu\,(1+\alpha)}{\rho h\,(\alpha+3)\,u_a}, \quad \text{using (18)}$$

If Reynold's number $= \text{Re} = \dfrac{\rho h u_a}{\mu} = \dfrac{h u_a}{\nu}$, then $C_f = \dfrac{12(1+\alpha)}{\text{Re}\,(\alpha+3)}$ ...(20)

Similarly, the coefficient of friction corresponding to $(\sigma_{yx})_{y=h}$ is given by

$$C_f' = 12\,(1-\alpha)/\text{Re}\,(\alpha+3). \qquad \text{...(21)}$$

In practical applications, the mean of C_f and C_f' is employed to estimate the energy losses in channels.

16.3C. Plane poiseuille flow. **[Agra 2007, Himanchal 1999, Meerut 2001, 05, 12; Kanpur 2009; Kurukshetra 1993]**

Consider the steady laminar flow of viscous incompressible fluid between two infinite parallel plates separated by a distance h. Let axis of x be taken in the middle of the channel parallel to the direction of flow, y the direction perpendicular to the flow, and the width of the plates parallel to the z-direction. Here the word `infinite' implies that the width of the plates is large compared with h and hence the flow may be treated as two-dimensional ($i.e.\ \partial/\partial z = 0$). Let the plates be long enough in the x-direction for the flow to be parallel. Here we take the velocity components v and w to be zero everywhere. Moreover the flow being steady, the flow variables are independent of time $(\partial/\partial t = 0)$. Furthermore, the equation of continuity $(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z) = 0$ reduces to $\partial u/\partial x = 0$ so that $u = u(y)$. Thus for the present problem

$$u = u\,(y), \qquad v = 0, \qquad w = 0, \qquad \partial/\partial z = 0, \qquad \partial/\partial t = 0. \qquad \ldots(1)$$

For the present two-dimensional flow in absence of body forces, the Navier-Stokes equations for x and y-directions take the form :

$$0 = -(\partial p/\partial x) + \mu\,(d^2u/dy^2) \qquad \ldots(2)$$

$$0 = -\partial p/\partial y \qquad \ldots(3)$$

Equation (3) shows that the pressure does not depend on y. Hence p is function of x alone and so (2) reduces to
$$d^2u/\partial y^2 = (1/\mu) \times (dp/dx) \qquad \ldots(4)$$

Differentiating both sides of (4) $w.r.t.$ `x', we find

$$0 = \frac{1}{\mu}\frac{d^2p}{dx^2} \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{dp}{dx}\right) = 0$$

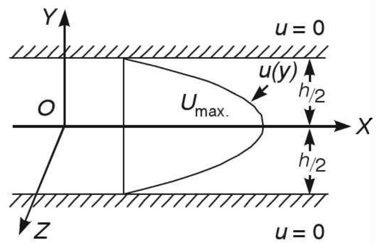
so that
$$dp/dx = \text{const. } P \text{ (say)} \qquad \ldots(5)$$

Then, (4) reduces to
$$d^2u/dy^2 = P/\mu \qquad \ldots(6)$$

Integrating (6),
$$du/dy = (py/\mu) + A \qquad \ldots(7)$$

Integrating (7),
$$u = Ay + B + Py^2/2\mu \qquad \ldots(8)$$

where A and B are arbitrary constants to be determined by the boundary conditions of the flow problem under consideration.



For the so called *plane Poiseuille flow* the plates are kept at rest and the fluid is kept in motion by a pressure gradient P. Let the two plates be situated at $y = -\ h/2$ and $y = h/2$ as shown in the adjoining figure. The axis of x is along the centre between two plates.

Using the no-slip condition, the boundary conditions for the problem are :

$$u = 0 \quad \text{at} \quad y = -\,h/2 \qquad \text{and} \qquad u = 0 \quad \text{at} \quad y = h/2. \qquad \ldots(9)$$

Using (9), (8) yields
$$0 = -\frac{Ah}{2} + B + \frac{Ph^2}{8\mu}, \qquad \text{and} \qquad 0 = \frac{Ah}{2} + B + \frac{Ph^2}{8\mu}$$

so that
$$A = 0 \qquad \text{and} \qquad B = -(h^2P/8\mu).$$

With these values, (8) reduces to
$$u = -\frac{h^2P}{8\mu}\left[1 - 4\left(\frac{y}{h}\right)^2\right], \qquad \ldots(10)$$

showing that the velocity distribution for the flow is parabolic as shown in the above figure.

(*i*) **To determine the maximum and average velocities.**

Equation (10) shows that the maximum velocity, $u_{\max}$, for the plane Poiseuille flow can be obtained by writing $y = 0$. Thus $\quad u_{\max} = -(h^2 P/8\mu)$ ...(11)

Using (10), the average velocity distribution for the present flow is given by

$$u_a = \frac{1}{h}\int_{-h/2}^{h/2} u\,dy = -\frac{1}{h}\times\frac{h^2 P}{8\mu}\int_{-h/2}^{h/2}\left(1-\frac{4y^2}{h^2}\right)dy = \frac{1}{h}u_{\max}\left[\frac{y}{2}-\frac{4y^3}{3h^2}\right]_{-h/2}^{h/2}, \text{ using (11)}$$

Thus, $\quad u_a = (2/3)\times u_{\max}$, on simplification ...(12)

Combining (11) and (12), we have $\quad P = -12\mu\times(u_a/h^2)$ ...(13)

Using (10), the shearing stress distribution in the flow is given by

$$\sigma_{yx} = -\mu\frac{du}{dy} = -\mu\times\frac{h^2 P}{8\mu}\times\frac{4}{h^2}\times 2y = -yP. \qquad ...(14)$$

Then using (11), (12) and (14), the skin frictions at $y = h/2$ is given by

$$[\sigma_{yx}]_{y=h/2} = -\frac{hP}{2} = 4\mu\frac{u_{\max}}{h} = \frac{6\mu u_a}{h}. \qquad ...(15)$$

Hence using (15), the frictional coefficient for laminar flow between two stationary plates is given by

$$C_f = \frac{[\sigma_{yx}]_{y=h/2}}{(1/2)\times\rho u_a^2} = \frac{6\mu u_a}{h}\times\frac{2}{\rho u_a^2} = 12\times\frac{\mu}{\rho h u_a} = \frac{12}{\text{Re}}, \qquad ...(16)$$

where $\quad$ Reynold's number $= \text{Re} = \dfrac{\rho h u_a}{\mu} = \dfrac{h u_a}{\nu}$

16.3D. Illustrative solved examples

Ex. 1. *Water at 20°C flows between two large parallel plates at a distance 1.5 mm apart. If the average velocity is 0.15 m/sec. Evaluate*

(*a*) *the maximum velocity* $\quad$ (*b*) *the pressure drop*

(*c*) *the wall shearing stress* $\quad$ (*d*) the frictional coefficient,

given that $\mu = 1.01$ g/m sec. **[Garwhal 1997, Kunpur 2000]**

Sol. (*a*) Refer equation (12) in Art. 16.3C. The maximum velocity $u_{\max}$ is given by

$$u_{\max} = (3/2)\times u_a = (3/2)\times 0.15 = 0.225 m/\text{sec}.$$

(*b*) Refer equation (13) in Art. 16.3*C*. The pressure drop *dp*/*dx* or *P* is given by

$$\frac{dp}{dx} = P = -\frac{12\mu\, u_a}{h^2} = -\frac{12\times(1.01\times10^{-3})\times 0.15}{(0.0015)^2} = -808\ N/m^3$$

(*c*) Refer equation (15) in Art. 16.3*C*. The wall shearing stress is given by

$$(\sigma_{yx})_{y=h/2} = -\frac{1}{2}hP = -\frac{1}{2}h\frac{dp}{dx} = -\frac{0.0015\times(-808)}{2} = 0.606\ N/m^2.$$

(*d*) Refer equation (16) in Art 16.3C. The frictional coefficient C_f is given by

$$C_f = \frac{12}{R_e} = \frac{12\mu}{u_a h\rho} = \frac{12\times(1.01\times10^{-3})}{0.15\times0.0015\times1\times10^3} = \frac{12}{223} = 0.0538, \text{ as Re} = \frac{u_a h\rho}{\mu}$$

Ex. 2. *Water at 70°C flows between two large parallel plates at a distance 1/16 inch apart.*

If the average velocity is 1/2 ft/sec. Evaluate ;

(*a*) *the maximum velocity* (*b*) *the pressure drop*

(*c*) *the wall shearing stress* (*d*) *the frictional coefficient.*

Sol. Proceed exactly as in Ex. 1. Here we take $\mu = 2.05 \times 10^{-5}\, lbf - \sec/ ft^2,\ \nu = 1.059 \times 10^{-5}$

(*a*) The maximum velocity $= u_{\max} = \dfrac{3}{2}u_a = \dfrac{3}{2} \times \dfrac{1}{2} = 0.75$ ft / sec

(*b*) The pressure drop $= \dfrac{dp}{dx} = P = -\dfrac{12\mu u_a}{h^2} = -\dfrac{12 \times (2.05 \times 10^{-5}) \times (1/2)}{\{1/(16 \times 12)\}^2} = -4.54$ *lbf/ft*^{3}

(*c*) The shearing stress $= (\sigma_{yx})_{y=h/2} = -\dfrac{1}{2}hP = -\dfrac{1}{2}h\dfrac{dp}{\partial x} = -\dfrac{1}{32 \times 12} \times (-4.54) = 0.1745$ *lbf/*ft^2

(*d*) The frictional coefficient $= C_f = \dfrac{12}{\text{Re}} = \dfrac{12\mu}{u_a h\rho} = \dfrac{12\nu}{u_a h} = \dfrac{12 \times 1.059 \times 10^{-5}}{0.5 \times \{1/(16 \times 12)\}} = 0.0118$

Ex. 3. *Water at 70°F flows between two parallel plates, one of which is at rest and the other moving with a velocity U. If the volumetric flow Q per unit width is zero, find the relation between U and dp/dx in terms of the dynamic viscosity* μ *and the spacing of the plates, h. Calculate the pressure gradient for U = 1/2 ft/sec and h = 1/2 in.* **[Garhwal 1996]**

Sol. From Art. 16.3B, we have $$Q = (1/6) \times (\alpha + 3)hU \qquad ...(1)$$

where $$\alpha = -\frac{h^2 P}{2\mu U} = -\frac{h^2}{2\mu U}\frac{dp}{dx}. \qquad ...(2)$$

$\therefore$ If $Q = 0$ (given), (1) reduces to $\alpha = -3$. Then (2) gives the desired relation as

$$dp/dx = 6\mu U/h^2 \qquad ...(3)$$

Numerical Part. For water, we know that $\mu = 2.05 \times 10^2\, lbf -$ sec / ft^2

Also, given that $U = \dfrac{1}{2}$ ft/sec and $h = \dfrac{1}{2}\text{in} = \dfrac{1}{2 \times 12} ft$

$\therefore$ From (3), $\dfrac{dp}{dx} = \dfrac{6 \times 2.05 \times 10^{-5} \times (1/2)}{(1/24)^2} = 3 \times (24)^2 \times 2.05 \times 10^{-5} = 0.0354$ *lbf/*ft^3

Ex. 4. *Oil flows between two parallel plates, one of which is at rest and the other moves with a velocity U. If the pressure is decreasing in the direction of the flow at a rate of 0.10 lbf/ft*^{3}*, the dynamic viscosity is* 10^{-3} *lbf-sec/ft*^{2}*, the spacing of the plates is 2 in. and the volumetric flow Q per unit width is 0.15 ft*^{2}*/sec, what is the value of U?*

Sol. From Art. 16.3*B*, we have $$Q = (1/6) \times (\alpha + 3)hU \qquad ...(1)$$

where $$\alpha = -\frac{h^2 P}{2\mu U} = -\frac{h^2}{2\mu U}\frac{dp}{dx}. \qquad ...(2)$$

From (1) and (2), we have

$$6Q = \left(3 - \frac{h^2}{2\mu U}\frac{dp}{dx}\right)hU \qquad \text{or} \qquad U = \frac{1}{3h}\left(6Q + \frac{h^3}{2\mu}\frac{dp}{dx}\right).$$

Here $h = 2$ inches $= (1/6)$ ft, $Q = 0.15\, ft.$/sec, $dp/dx = -0.10$lbf/*ft*^{3} and $\mu = 10^{-3}\, lbf - \sec/ft^2$

$\therefore \qquad U = \dfrac{1}{3 \times (1/6)}\left[6 \times 0.15 - \dfrac{(1/6)}{2 \times 10^{-3}} \times (0.10)\right] = 1.333\,ft/\text{sec}.$

Ex. 5. *Let there be a laminar flow of water at 54° F between two parallel plates separated by a distance of 1 inch. If the pressure drop per foot of channel is recorded to be 0.003 inch of water, find the maximum velocity, the shearing stress at the wall, and the velocity distribution between the plates. The viscosity of water at 50°F is $\mu = 2.74 \times 10^{-5}$ lbf-sec/ft².* **[Agra 2001]**

Sol. Here $\qquad \dfrac{dp}{dx} = -\dfrac{0.003 \times 62.5}{12} = -0.0156\,lbf/ft^3$

$\therefore$ The maximum velocity $u_{\max}$ [Refer Art. 16.3*C*]

$$= -\frac{h^2 P}{8\mu} = \frac{(1/12)^2 \times 0.0156}{8 \times 2.74 \times 10^{-5}} = 0.494\,ft/\text{sec}.$$

The shearing stress at the wall $= \dfrac{4\mu u_{\max}}{h} = \dfrac{4 \times 2.74 \times 10^{-5} \times 0.494}{(1/12)} = 0.000649\,lbf/ft^2.$

Finally the velocity distribution is given by

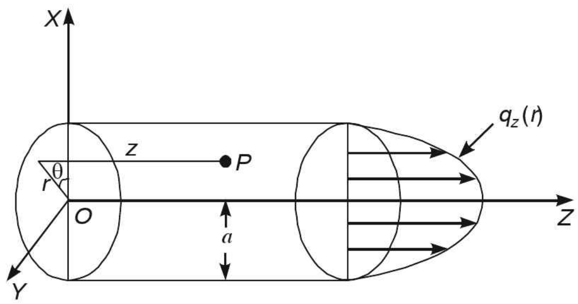
$u = u_{\max}(1 - 4y^2/h^2) \qquad$ or $\qquad u = 0.494\,(1 - 4y^2),$

where h = 1 inch and y is being measured in inches.

16.4*A*. Flow through a circular pipe-The Hagen-Poiseuille flow.

[Himanchal 2000, 01, 02, 03, 07; 09, 10; Meerut 2003, 10, 12; Garhwal 1999; Kanpur 2002; Kurukshetra 1999]

Consider the laminar steady flow, without body forces of an incompressible fluid through an infinite circular pipe of radius a with axial symmetry as shown in the following figure.



For the present problem, we consider all basic equations in cylindrical coordinates (r, θ, z). Let z be the direction of flow along the axis of the pipe. Clearly, the radial and tangential velocity components are zero, *i.e.* $q_r = q_\theta = 0$. Due to axial symmetry of flow, q_z will be independent of θ. Further, the equation of continuity for steady flow, namely, [Refer Art. 2.10, chapter 2]

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0 \qquad \text{reduces to} \qquad \frac{\partial q_z}{\partial z} = 0,$$

showing that q_z is independent of z also. Hence q_z is function of r alone, *i.e.* $\quad q_z = q_z(r)$

Thus, for the given problem, $\quad q_r = 0, \quad q_\theta = 0 \quad$ and $\quad q_z = q_z(r) \quad$ ...(1)

For the present steady axi-symmetric flow of incompressible fluid with velocity components (1), the equations of motion [refer 11 (*a*) to 11 (*c*) in Art. 14.11 of chapter 14] in cylindrical coordinates reduce to

$$0 = -(\partial p/\partial r) \qquad ...(2)$$

$$0 = -(1/r) \times (\partial p/\partial \theta) \qquad ...(3)$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r}\right)\right] \qquad ...(4)$$

(2) and (3) show that p is independent of r and θ. Thus p is function of z alone. Further q_z is function of r alone by (1). Hence (4) may be re-written as

$$\mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dq_z}{dr}\right)\right]=\frac{dp}{dz} \qquad \text{...(5)}$$

Differentiating both sides of (5) *w.r.t.* `z', we find

$$0=\frac{d^2p}{dz^2} \qquad \text{or} \qquad \frac{d}{dz}\left(\frac{dp}{dz}\right)=0$$

so that $\qquad dp/dz = \text{const.} = P$ (say). ...(6)

We may take $\qquad P = (p_2 - p_1)/l,$ ...(7)

where p_1, p_2 denote the values of p, at the ends of a length l of the circular pipe. In what follows, we now write $q_z = u$. Then, using (6), (5) reduces to

$$\frac{d}{dz}\left(r\frac{du}{dr}\right)=\frac{\Pr}{\mu} \qquad \text{...(8)}$$

Integrating (8), $\qquad r\dfrac{du}{dr}=\dfrac{Pr^2}{2\mu}+A \qquad$ or $\qquad \dfrac{du}{dr}=\dfrac{Pr}{2\mu}+\dfrac{A}{r}$ ...(9)

Integrating (9), $\qquad u=(Pr^2/4\mu)+A\log r+B\,,$ ...(10)

where the constants A and B are to be found by using the boundary conditions. Now u must be finite on the axis of the tube (where $r = 0$). So we must take $A = 0$ in (10) because otherwise u would become infinite when $r = 0$. Thus (10) reduces to

$$u=(Pr^2/4\mu)+B. \qquad \text{...(11)}$$

Since the circular boundary of the tube is at rest, the no-slip condition at the wall gives rise to the following boundary condition

$$u = 0 \qquad \text{at} \qquad r = a. \qquad \text{...(12)}$$

Using (12), (11) gives $B=-(Pa^2/4\mu)$. Hence (11) becomes

$$u=-(Pa^2/4\mu)\times\{1-(r/a)^2\}, \qquad \text{...(13)}$$

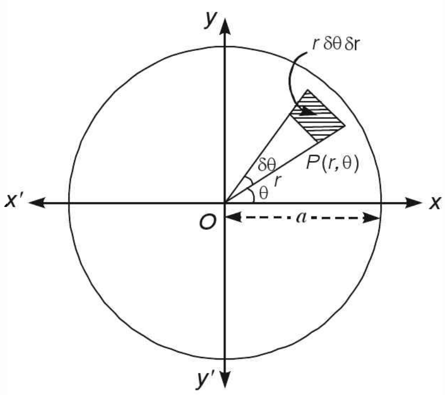
which has the form of a paraboloid of revolution as shown in the figure on page 16.9.

(*i*) **To determine the maximum and average velocities.**

From (13), it follows that the maximum velocity $u_{\max}$ can be obtained by putting $r = 0$ in it. Thus maximum velocity occurs on the axis of the pipe and is given by

$$u_{\max}=-(Pa^2/4\mu) \qquad \text{...(14)}$$

where $P < 0$. The average velocity distribution for the present flow is given by



$$u_a = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a urdrd\theta \;= -\frac{P}{4\pi\mu}\int_0^{2\pi}\int_0^a r(1-r^2/a^2)drd\theta, \text{ by (13)}$$

$$= -\frac{P}{4\pi\mu}\int_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^4}{4a^2}\right]_0^a d\theta = -\frac{P}{4\pi\mu}\times\frac{a^2}{4}\int_0^{2\pi} d\theta = -\frac{Pa^2}{16\pi\mu}\times 2\pi$$

Thus, $$u_a = -(Pa^2/8\mu) = (1/2)\times u_{\max}, \text{ by (14)} \qquad ...(15)$$

The volumetric flow per unit time over any section is given by

$$Q = \pi a^2 \times u_a = -(\pi a^4 P)/8\mu. \qquad ...(16)$$

(*ii*) **To determine shearing stress, skin friction and the coefficient of friction.**

Using (13), the shearing stress distribution for the present flow is given by

$$\sigma_{rz} = -\mu\left(\frac{du}{dr}\right) \quad \text{or} \quad \sigma_{rz} = -\mu\left(\frac{Pa^2}{4\pi}\right)\frac{2r}{a^2} = -\frac{rP}{2}. \qquad ...(17)$$

Then the skin friction (*i.e.* shearing stress at the wall $r = a$) is given by

$$[\sigma_{rz}]_{r=a} = -\frac{aP}{2} = 4\mu\frac{u_a}{a}, \text{ using (14) and (15)} \qquad ...(18)$$

$\therefore$ Drag per unit length of the tube $= 2\pi a\times[\sigma_{rz}]_{r=a} = 2\pi a\times(-aP/2) = -\pi a^2 P.$

The (local) coefficient of friction C_f is given by

$$C_f = \frac{[\sigma_{rz}]_{r=a}}{(1/2)\times\rho u_a^2} = \frac{(4\mu u_a)/a}{(1/2)\times\rho u_a^2} = 16\times\frac{\pi}{2a\rho u_a}. \qquad ...(19)$$

If Re = $(2a\rho u_a)/\mu$ = Reynold's number, then (19) reduces to

$$C_f = 16/\text{Re}, \qquad ...(20)$$

showing that skin friction can be obtained from the knowledge of Re. The above formula is used to determine energy losses in pipe flows.

16.4*B*. Laminar steady flow between two coaxial circular cylinders

[Agra 2008; I.A.S. 2001; Meerut 2003, 04, 09; Garhwal 1993; G.N.D.U. Amritsar 2003]

For the present problem, we consider all basic equations in cylindrical coordinates (r, θ, z). Let z be the direction of flow along the axis of the pipe. Clearly, the radial and tangential velocity components are zero *i.e.* $q_r = q_\theta = 0$. Due to axial symmetry of flow, q_z will be independent of θ. Further the equation of continuity for steady flow, namely,

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0 \quad \text{reduces to} \quad \frac{\partial q_z}{\partial z} = 0,$$

showing that q_z is independent of z also. Hence q_z is function of r alone, *i.e.* $q_z = q_z(r)$.

Thus, $$q_r = 0, \quad q_\theta = 0 \quad \text{and} \quad q_z = q_z(r). \qquad ...(1)$$

For the present steady axi-symmetrical flow of incompressible fluid with velocity components (1), the equations of motion in cylinderical coordinates reduce to

$$0 = -\partial p/\partial r \qquad ...(2)$$

$$0 = -(1/r)\times(\partial p/\partial\theta) \qquad ...(3)$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r}\right)\right] \qquad ...(4)$$

(2) and (3) show that p is independent of r and θ. Thus p is function of z alone. Further q_z is function of r alone by (1). Hence (4) may be re-written as

$$\mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dq_z}{dr}\right)\right]=\frac{dp}{dz}. \qquad ...(5)$$

Differentiating both sides of (5) *w.r.t.* 'z', we find

$$0=\frac{d^2p}{dz^2} \qquad \text{or} \qquad \frac{d}{dz}\left(\frac{dp}{dz}\right)=0$$

so that dp/dz = const. = P, say. ...(6)

We may take $P = (p_2 - p_1)/l$, ...(7)

where p_1, p_2 denote the values of p at the ends of a length l of the tube. In what follows, we now write $q_z = u$. Then using (6), (5) reduces to

$$\frac{d}{dr}\left(r\frac{du}{dr}\right)=\frac{\Pr}{\mu} \qquad ...(8)$$

Integrating (8), $r\dfrac{du}{dr}=\dfrac{\Pr^2}{2\mu}+A$ or $\dfrac{du}{dr}=\dfrac{Pr}{2\mu}+\dfrac{A}{r}$ ...(9)

Integrating (9), $u=(\Pr^2/4\mu)+A\log r+B$, ...(10)

where A and B are arbitrary constants of integration.

Suppose there are two coaxial circular cylinders of radii a and b ($b > a$) through which laminar steady flow without body forces of an incompressible fluid takes place along the axial direction as shown in the adjoining figure. Since the circular boundaries of both the tubes are at rest, the no-slip conditions at their walls give rise to the following boundary conditions.

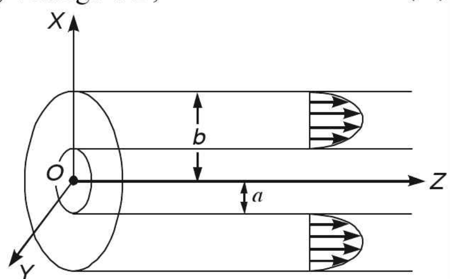


Fig (*i*)

$u = 0$ at $r = a$; and $u = 0$ at $r = b$. ...(11)

Using (11), (10) gives

$$0=(Pa^2/4\mu)+A\log a+B \qquad \text{and} \qquad 0=(Pb^2/4\mu)+A\log b+B$$

Solving these, $A=-\dfrac{P}{4\mu}\times\dfrac{b^2-a^2}{\log(b/a)}$, and $B=-\dfrac{Pa^2}{4\mu}+\dfrac{P}{4\mu}\times\dfrac{b^2-a^2}{\log(b/a)}\log a.$

Substituting these values in (1), $u=-\dfrac{P}{4\mu}\left[a^2-r^2+(b^2-a^2)\dfrac{\log(r/a)}{\log(b/a)}\right]$ ...(12)

(*i*) **To determine volumetric rate of flow Q and average velocity.** **[Himanchal 2003]**

The flux of the fluid (*i.e.* volumetric flow per unit time over any section of the annulus) Q is given by

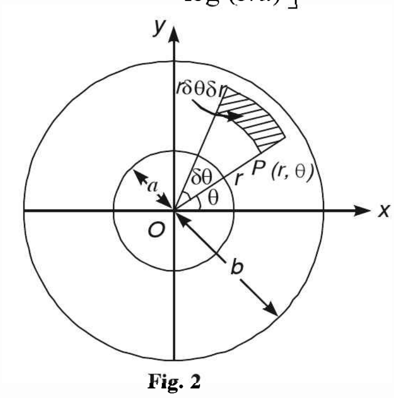


Fig. 2

$$Q = \int_0^{2\pi}\int_a^b u\, r\, dr\, d\theta \;= -\frac{P}{4\mu}\int_0^{2\pi}\int_a^b r\left[a^2 - r^2 + (b^2 - a^2)\frac{\log (r/a)}{\log (b/a)}\right] dr\, d\theta$$

$$= -\frac{P}{4\mu}[\theta]_0^{2\pi} \times \int_a^b r\left[a^2 - r^2 + (b^2 - a^2)\frac{\log (r/a)}{\log (b/a)}\right] dr$$

$$= -\frac{\pi P}{2\mu}\int_a^b \left[r\,(a^2 - r^2) + \frac{b^2 - a^2}{\log (b/a)} \times r \log \frac{r}{a}\right] dr$$

$$= -\frac{\pi P}{2\mu}\left[\left\{\frac{a^2 r^2}{2} - \frac{r^4}{4}\right\}_a^b + \frac{b^2 - a^2}{\log(b/a)}\left\{\left(\frac{r^2}{2}\log\frac{r}{a}\right)_a^b - \int_a^b \frac{r^2}{2} \times \frac{1}{(r/a)} \times \frac{1}{a} dr\right\}\right]$$

$$= -\frac{\pi P}{2\pi}\left[\frac{a^2}{2}(b^2 - a^2) - \frac{1}{2}(b^4 - a^4) + \frac{b^2 - a^2}{\log (b/a)}\left\{\frac{b^2}{2}\log\frac{b}{a} - \frac{1}{4}\left[r^2\right]_a^b\right\}\right]$$

$$= -\frac{\pi P}{2\mu}\left[\frac{a^2}{2}(b^2 - a^2) - \frac{1}{4}(b^4 - a^4) + \frac{b^2 - a^2}{\log (b/a)}\left\{\frac{b^2}{2}\log\frac{b}{a} - \frac{1}{4}(b^2 - a^2)\right\}\right]$$

$$= -\frac{\pi P}{2\mu}\left[\frac{a^2}{2}(b^2 - a^2) - \frac{1}{4}(b^4 - a^4) + \frac{b^2}{2}(b^2 - a^2) - \frac{(b^2 - a^2)^2}{4\log (b/a)}\right]$$

$$= -\frac{\pi P}{2\mu}\left[\frac{1}{2}(b^2 - a^2)(b^2 + a^2) - \frac{1}{4}(b^4 - a^4) - \frac{(b^2 - a^2)^2}{4\log(b/a)}\right]$$

$$= -\frac{\pi P}{2\mu}\left[\frac{1}{2}(b^4 - a^4) - \frac{1}{4}(b^4 - a^4) - \frac{(b^4 - a^2)^2}{4\log (b/a)}\right] = -\frac{\pi P}{8\mu}\left[(b^4 - a^4) - \frac{(b^2 - a^2)^2}{\log (b/a)}\right] \qquad \ldots(13)$$

The average velocity u_a in the annuals is given by

$$u_a = \frac{Q}{\pi\,(b^2 - a^2)} = -\frac{P}{8\mu}\left[b^2 + a^2 - \frac{b^2 - a^2}{\log (b/a)}\right], \text{ using (13)} \qquad .(13)'$$

(*ii*) **To determine the stress and the skin frictions (*i.e.*, the shearing stress at the walls of the inner and outer cylinders).** **[Meerut 2009]**

Using (3), the shearing stress distribution is given by

$$\sigma_{rz} = \mu\frac{du}{dr} = -\frac{P}{4}\left[-2r + \frac{b^2 - a^2}{\log (b/a)} \times \frac{1}{r}\right] = -\frac{P}{4}\left[\frac{b^2 - a^2}{r \log (b/a)} - 2r\right] \qquad \ldots(14)$$

Hence the skin frictions at the inner and outer cylinder are respectively given by

$$(\sigma_{rz})_{r=a} = -\frac{P}{4}\left[\frac{b^2 - a^2}{a \log (b/a)} - 2a\right] \qquad \ldots(15)$$

and
$$(\sigma_{rz})_{r=b} = -\frac{P}{4}\left[\frac{b^2 - a^2}{b \log (b/a)} - 2b\right]. \qquad \ldots(16)$$

From (15) and (16), it follows that skin frictions at both walls are positive ; however, the velocity gradient at the wall of the outer cylinder is negative as shown in the figure (i).

16.5. Laminar steady flow of incompressible viscous fluid in tubes of cross-section other than circular.

In usual practice the pipes of different shapes are employed in order to transport a given fluid. Accordingly, we now study steady flows of viscous incompressible fluids through infinite pipes of various cross-sections.

In such cases we take the only component of velocity, different from zero, to be the velocity parallel to the axis of the tube.

Taking z-axis along the axis of the tube, we take $u = v = 0$ and hence the equation of continuity gives $\partial w/\partial z = 0$ so that

$$w = w\,(x, y), \qquad \text{... } (i)$$

i.e. w is a function of x and y only. The equations of motion are

$$0 = -\,\partial p/\partial x, \qquad \text{...}(ii)$$

$$0 = -\,\partial p/\partial y, \qquad \text{...}(iii)$$

$$0 = -\partial p / \partial z + \mu(\partial^2 w / \partial x^2 + \partial^2 w / \partial y^2) \qquad \text{...}(iv)$$

From (ii) and (iii), we see that p is independent of x and y. Hence (iv) reduces to

$$\mu(\partial^2 w / \partial x^2 + \partial w / \partial y^2) \;=\; dp / dz \qquad \text{... } (iv)'$$

Differentiating both sides of (iv) $w.r.t.$ `z', we get

$$0 = \frac{d}{dz}\left(\frac{dp}{dz}\right), \qquad \text{giving} \qquad \frac{dp}{dz} = \text{ constant } = -\,P, \text{ say}$$

$\therefore$ $(iv)'$ reduces to $\qquad \partial^2 w / \partial x^2 + \partial^2 w / \partial y^2 = -P / \mu, \qquad \text{....}(v)$

with the boundary condition $w = 0$ on the surface of the tube.

Thus the problem reduces to solving Poission's equation (v) with the boundary condition $w = 0$ on the surface of the tube. Direct solution of (v) is not easy. So to simplify the solution we convert (v) into a Laplace equation by the transformation :

$$w = w_1 - (P/4\mu) \times (x^2 + y^2), \qquad \text{...}(vi)$$

then w_1 satisfies the equation $\qquad \partial^2 w_1/\partial x^2 + \partial^2 w_1/\partial y^2 = 0 \qquad \text{...}(vii)$

with the boundary condition $w_1 = (P/4\mu) \times (x^2 + y^2)$ on the surface of the tube.

Thus in order to solve the problem for a particular boundary, we take

$$w = w_1 + B - (P/4\mu) \times (x^2 + y^2), \qquad \text{...}(viii)$$

where B is a constant, w_1 is a suitable solution of the two-dimensional Laplace's equation and apply the condition that $w = 0$ on the surface of the tube, then B is found out.

To illustrate the whole procedure, we shall take the cross-section of the tubes as ellipse, equilateral triangle and rectangle.

Case I. Tube having elliptic cross-section.

[Meerut 2012, Kurukshetra 2000 Himanchal 1998, 2000]

Let the cross section of the tube to be an ellipse $\qquad x^2/a^2 + y^2/b^2 = 1. \qquad \text{...}(1)$

Let $\qquad w = A\,(x^2 - y^2) + B - (P/4\mu) \times (x^2 + y^2). \qquad \text{...}(2)$

On the boundary of the pipe $w = 0$. Hence the boundary is given by

$$0 = A(x^2 - y^2) + B - (P/4\mu) \times (x^2 + y^2)$$

or
$$\frac{1}{B}\left(\frac{P}{4\mu} - A\right)x^2 + \frac{1}{B}\left(\frac{P}{4\mu} + A\right)y^2 = 1 \qquad ...(3)$$

(3) must now be identical to (1) and hence, we have

$$\frac{1}{B}\left(\frac{P}{4\mu} - A\right) = \frac{1}{a^2} \quad \text{and} \quad \frac{1}{B}\left(\frac{P}{4\mu} + A\right) = \frac{1}{b^2}. \qquad ...(4)$$

Solving (4), $\quad A = \dfrac{P}{4\mu} \times \dfrac{a^2 - b^2}{a^2 + b^2} \quad$ and $\quad B = \dfrac{P}{4\mu} \times \dfrac{a^2 b^2}{a^2 + b^2} \qquad ...(5)$

Putting these values of A and B, (2) gives

$$w = \frac{P}{4\mu}\frac{a^2 - b^2}{a^2 + b^2}(x^2 - y^2) + \frac{P}{4\mu}\frac{a^2 b^2}{a^2 + b^2} - \frac{P}{4\mu}(x^2 + y^2) = \frac{P}{2\mu}\frac{a^2 b^2}{a^2 + b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \qquad ...(6)$$

Now the flux Q (*i.e.* the volume discharged through the tube per unit time) can be obtained by double integration over the elliptic section (1) and is given by

$$Q = \iint w\, dxdy = \frac{P}{2\mu}\frac{a^2 b^2}{a^2 + b^2}\iint (1 - x^2/a^2 - y^2/b^2)\, dxdy$$

$$= \frac{P}{2\mu}\frac{a^2 b^2}{a^2 + b^2}\left[\iint dxdy - \frac{1}{a^2}\iint x^2 dxdy - \frac{1}{b^2}\iint y^2\, dxdy\right]$$

$$= \frac{P}{2\mu}\frac{a^2 b^2}{a^2 + b^2}\left[\pi ab - \frac{1}{a^2} \times \pi ab \times \frac{a^2}{4} - \frac{1}{b^2} \times \pi ab \times \frac{b^2}{4}\right],$$ the second and third integrals being moments of inertia,and Ist integral = area of elliptic cross-section = πab]

Thus, $\qquad Q = (\pi P a^3 b^3)/4\mu(a^2 + b^2)$

Case II Tube having equilateral triangular cross-section.

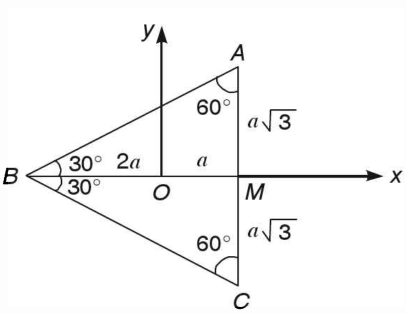
[Kurukshetra 1999, Meerut 2000, Himachal 1998; Kanphur 1998, 2000]

Let $\qquad w = A\,(x^3 - 3xy^2) + B - (P/4\mu) \times (x^2 + y^2). \qquad ..(1)$

On the boundary of the pipe $w = 0$. Hence the boundary is given by

$$A\,(x^3 - 3xy^2) + B - (P/4\mu) \times (x^2 + y^2) = 0 \qquad ...(2)$$

If $x = a$ be a part of the boundary, then



$$\text{A}\,(a^3 - 3ay^2) + B - (P/4\mu) \times (a^2 + y^2) = 0$$

or $\qquad Aa^3 + B - (P/4\mu) \times a^2 - y^2\,(3aA + P/4\mu) = 0$

so that $\qquad Aa^3 + B - Pa^2/4\mu = 0 \;$ and $\; -(3aA + P/4\mu) = 0$

Solving these, $\quad A = -\dfrac{P}{12a\mu}, \quad B = \dfrac{Pa^2}{3\mu}. \qquad ...(3)$

Putting these values of A and B in (2), the boundary is given by

$$-\frac{P}{12a\mu}(x^3 - 3xy^2) + \frac{Pa^2}{3\mu} - \frac{P}{4\mu}(x^2 + y^2) = 0$$

or $x^3 - 3xy^2 + 3ax^2 + 3ay^2 - 4a^3 = 0$ or $(x-a)(x+2a-y\sqrt{3})(x+2a+y\sqrt{3}) = 0$

i.e. the boundary consists of $x = a,\quad y = (x+2a)/\sqrt{3},\quad y = -(x+2a)/\sqrt{3}$

which represent sides AC, BA and BC of an equilateral triangle ABC as shown in the figure. Here $BM = 3a$. Origin of the coordinate system is taken at centre of the triangle, x-axis along BM and y-axis is parallel to AC. Side $AC = 2\,AM = 2 \times BM \tan 30^\circ = 2 \times 3a \times (1/\sqrt{3}) = 2\sqrt{3}a$ Putting values of A and B in (1), we get

$$w = -(P/12a\mu) \times (x^3 - 3xy^2 + 3ax^2 + 3ay^2 - 4a^3). \qquad ...(4)$$

If Q be the flux of the fluid over an area of equilateral triangular cross-section, we have

$$Q = \iint w\,dxdy = -\frac{P}{12a\mu}\int_{x=-2a}^{a}\int_{y=-(x+2a)/\sqrt{3}}^{y=(x+2a)/\sqrt{3}} (x^3 - 3xy^2 + 3ax^2 + 3ay^2 - 4a^3)\,dxdy$$

$$= -\frac{P}{12a\mu}\int_{-2a}^{a}\left\{\left(x^3 + 3ax^2 - 4a^3\right)[y]_{y=-(x+2a)/\sqrt{3}}^{y=(x+2a)/\sqrt{3}} - 3(x-a)\left[\frac{1}{3}y^3\right]_{y=-(x+2a)/\sqrt{3}}^{y=(x+2a)/\sqrt{3}}\right\}dx$$

$$= -\frac{P}{6\sqrt{3}a\mu}\int_{-2a}^{a}\left(\frac{2}{3}x^4 + \frac{10}{3}ax^3 + 4a^2x^2 - \frac{8}{3}a^3x - \frac{10a^4}{3}\right)dx = \frac{27}{20\sqrt{3}} \times \frac{Pa^4}{\mu}$$

$$\text{Average flow} = \frac{\text{Flux}}{\text{Area}} = \frac{27}{20\sqrt{3}} \times \frac{Pa^4}{\mu} \times \frac{1}{(1/2)\times(3a)\times(2a\sqrt{3})} = \frac{3}{20} \times \frac{Pa^2}{\mu}.$$

Case III Tube having rectangular cross-section.

Consider the flow through a rectangular pipe whose corss-section is bounded by the lines $x = \pm a$ and $y = \pm b$.

Take $$w = w_1 - (P/4\mu) \times (x^2 + y^2), \qquad ...(1)$$
where w_1 is a plane harmonic.

Now $w = 0$, on the boundary $\quad x = \pm a, \quad y = \pm b. \qquad ...(2)$

Boundary conditions (2) and (1) show that on the boundary $x = \pm a, y = \pm b.$ we must have

$$w_1 = (P/4\mu) \times (x^2 + y^2)$$

Take again, $$w_1 = w_2 + (P/4\mu) \times (x^2 - y^2) + K, \qquad ...(3)$$
where since $(x^2 - y^2)$ and w_1 are plane harmonic functions, w_2 is also plane harmonic such that on the boundary

$$\frac{P}{4\mu}(x^2 + y^2) = w_2 + \frac{P}{4\mu}(x^2 - y^2) + K, \quad i.e., \quad w_2 = \frac{P}{2\mu}y^2 - K = \frac{P}{2\mu}(y^2 - b^2), \qquad ...(4)$$

where $$K = Pb^2/2\mu. \qquad ...(5)$$

$\therefore$ From (1) and (3), we have

$$w = w_2 + \frac{P}{4\mu}(x^2 - y^2) + K - \frac{P}{4\mu}(x^2 + y^2) = w_2 + \frac{P}{4\mu}(x^2 - y^2) + \frac{Pb^2}{2\mu} - \frac{P}{4\mu}(x^2 + y^2) \text{ by (5)}$$

$\therefore$ $$w = w_2 + (P/2\mu) \times (b^2 - y^2), \qquad ...(6)$$
where w_2 is a plane harmonic such that on the boundary $w = 0$ so that

$$w_2 = (P/2\mu) \times (y^2 - b^2), \quad \text{when} \quad x = \pm a, \quad y = \pm b. \qquad ...(7)$$

Since w_2 is a plane harmonic, it must satisfy the Laplace's equation

$$\partial^2 w_2/\partial x^2 + \partial^2 w_2/\partial y^2 = 0. \qquad ...(8)$$

Let $$w_2 = X\,(x)\,Y\,(y). \qquad ...(9)$$

$\therefore$ From (8), $$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \quad \text{or} \quad \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} \qquad ...(10)$$

L.H.S. of (10) is function of x alone whereas *R.H.S.* of (10) is function of y alone. So (10) is valid only if each side is a constant, say λ^2. So (10) gives

$$\frac{1}{X}\frac{d^2X}{dx^2} = \lambda^2, \quad -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2 \quad \Rightarrow \quad \frac{d^2X}{dx^2} - \lambda^2 X = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} + \lambda^2 Y = 0. \qquad ...(11)$$

Solving (11), we get $$\left.\begin{aligned} X &= A\cosh\lambda x + B\sinh\lambda x \\ Y &= C\cos\lambda y + D\sin\lambda y. \end{aligned}\right\} \qquad ...(12)$$

$\therefore$ A solution of (8) is given by

$$w_2 = \Sigma\,(A\cosh\lambda x + B\sinh\lambda x)\,(C\cos\lambda y + D\sin\lambda y) \qquad ...(13)$$

Since $w_2 = (P/2\mu)\times(y^2 - b^2)$ when $x = \pm a$ and also when $y = \pm b$, the terms containing sinh λx and sin λy must be taken zero in (13), so we have

$w_2 = \Sigma E_\lambda \cosh\lambda x \cos\lambda y$, where $E_\lambda\,(= AC)$ are new arbitrary constants ...(14)

When $y = \pm b, w_2 = 0$ from (7). H $\qquad \cos\lambda b = 0,$

giving $\lambda b = \{(2m+1)\pi\}/2$, m being an integer so that $\qquad \lambda = \{(2m+1)\pi\}/2b$

$\therefore$ From (14), $$w_2 = \sum_{m=0}^{\infty} E_{2m+1}\cosh\left\{(2m+1)\frac{\pi x}{2b}\right\}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\}. \qquad ...(15)$$

Using boundary condition (7), namely, when $x = \pm a, w_2 = (P/2\mu)\times(y^2 - b^2)$, (15) gives

$$\frac{P}{2\mu}(y^2 - b^2) = \sum_{m=0}^{\infty} E_{2m+1}\cosh\left\{(2m+1)\frac{\pi a}{2b}\right\}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\} \qquad ...(16)$$

Multiplying both sides of (16) by $\cos\left\{(2m+1)\dfrac{\pi y}{2b}\right\}$ and integrating between the limits $-b$

and b and noting that $$\int_{-b}^{b}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\}\cos\left\{(2n+1)\frac{\pi y}{2b}\right\}dy = \begin{cases} 0, & n \neq m \\ b, & n = m \end{cases}$$

and $$\int_{-b}^{b}(y^2 - b^2)\cos\left\{(2m+1)\frac{\pi y}{2b}\right\}dy$$

$$= \left[(y^2 - b^2)\times\frac{2b}{(2m+1)\pi}\sin\left\{(2m+1)\frac{\pi y}{2b}\right\}\right]_{-b}^{b} - \int_{-b}^{b} 2y\times\frac{2b}{(2m+1)\pi}\sin\left\{(2m+1)\frac{\pi y}{2b}\right\}dy$$

$$= -\frac{4b}{(2m+1)\pi}\int_{-b}^{b} y\sin\left\{(2m+1)\frac{\pi y}{2b}\right\}dy$$

$$= -\frac{4b}{(2m+1)\pi}\left[\left[y\times\frac{-2b}{(2m+1)\pi}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\}\right]_{-b}^{b} - \int_{-b}^{b}\frac{-2b}{(2m+1)\pi}\cos\left\{(2m+1\frac{\pi y}{2b}\right\}dy\right]$$

$$= -\frac{8b^2}{(2m+1)^2\pi^2}\int_{-b}^{b}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\}dy = -\frac{8b^2}{(2m+1)^2\pi^2}\times\frac{2b}{(2m+1)\pi}\left[\sin\left\{(2m+1)\frac{\pi y}{2b}\right\}\right]_{-b}^{b}$$

$$= -\frac{16b^3}{(2m+1)^3\pi^3}\times 2\sin\left\{(2m+1)\frac{\pi}{2}\right\} = -\frac{32b^3(-1)^m}{(2m+1)^3\pi^3},$$

we obtain
$$\frac{P}{2\mu}\times\frac{-32b^3(-1)^m}{(2m+1)^3\pi^3} = b\,E_{2m+1}\cosh\left\{(2m+1)\frac{\pi a}{2b}\right\}$$

giving
$$E_{2m+1} = -\frac{P}{\mu}\times\frac{16b^2}{(2m+1)^3\pi^3}\times\frac{(-1)^m}{\cosh\{(2m+1)\pi a/2b\}}$$

$\therefore$ From (15),
$$w_2 = -\sum_{m=0}^{\infty}\frac{P}{\mu}\times\frac{16b^2(-1)^m}{(2m+1)^3\pi^3}\times\frac{\cosh\{(2m+1)\pi x/2b\}}{\cosh\{(2m+1)\pi a/2b\}}\cos\left\{2m+1)\frac{\pi y}{2b}\right\}$$

Putting this value of ω_2 in (6), we have

$$w = \frac{P}{2\mu}(b^2-y^2) - \frac{P}{\mu}\times\frac{16b^2}{\pi^3}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3}\frac{\cosh\{(2m+1)\pi x/2b\}}{\cosh\{(2m+1)\pi a/2b\}}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\} \quad \ldots(17)$$

Flux Q of the fluid over an area of rectangular cross-section, is given by

$$Q = \int_{y=-b}^{b}\int_{x=-a}^{a} w\,dxdy = \frac{P}{2\mu}\int_{-b}^{b}\int_{-a}^{a}(b^2-y^2)\,dxdy - \frac{P}{\mu}\times\frac{16b^2}{\pi^3}$$

$$\times\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3\cosh\{(2m+1)\pi a/2b\}}\times\int_{-b}^{b}\int_{-a}^{a}\cosh\left\{(2m+1)\frac{\pi x}{2b}\right\}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\}dxdy$$

$$= \frac{P}{2\mu}\int_{-b}^{b}(b^2-y^2)\,dy\times\int_{-a}^{a}dx - \frac{P}{\mu}\times\frac{16b^2}{\pi^3}$$

$$\times\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3\cosh\{(2m+1)\pi a/2b\}}\times\int_{-b}^{b}\cos\left\{(2m+1)\frac{\pi y}{2b}\right\}dy\times\int_{-a}^{a}\cosh\left\{(2m+1)\frac{\pi x}{2b}\right\}dx$$

$$= \frac{P}{2\mu}\left(2b^3-\frac{2b^3}{3}\right)\times 2a - \frac{p}{\mu}\times\frac{16b^2}{\pi^3}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3\cosh\{(2m+1)\pi a/2b\}}$$

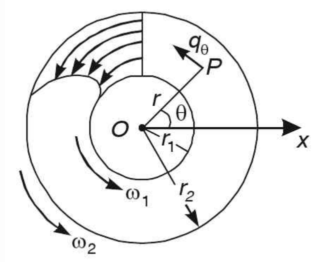
$$\times\frac{2b}{(2m+1)\pi}\left[\sin\left\{(2m+1)\frac{\pi y}{2b}\right\}\right]_{-b}^{b}\times\frac{2b}{(2m+1)\pi}\left[\sinh\left\{(2m+1)\frac{\pi x}{2b}\right\}\right]_{-a}^{a}$$

$$= \frac{4Pab^3}{3\mu} - \frac{64Pb^4}{\mu\pi^5} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^5 \cosh\{(2m+1)\pi a/2b\}} \times 2\,(-1)^m \times 2 \sinh\left\{(2m+1)\frac{\pi a}{2b}\right\}$$

$$= \frac{4Pab^3}{3\mu} - \frac{256Pb^4}{\mu\pi^5} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^5} \tanh\left\{(2m+1)\frac{\pi a}{2b}\right\}$$

16.6. Laminar flow between two concentric rotating cylinders- couette flow.

[Meerut 2011, 12; Himachal 2001, 06, 07, 10]



Consider two infinitely long, concentric circular cylinders of radii r_1 and r_2 rotating with constant angular velocities ω_1 and ω_2. Let there be viscous incompressible fluid in the annular space. Then the cylinders induce a steady, axi-symmetric, tangential motion in the fluid. Let z-axis be taken along the axis of the cylinders. Since the motion is only tangential, we have $q_r = 0$, $q_z = 0$. Then the continuity equation in cylindrical coordinates reduces to $\partial q_\theta/\partial\theta = 0$, so that q_θ depends on r and z only.

Again, the cylinder being very long, the flow will not depend on z. Hence $q_\theta = q_\theta(r)$. Hence the equations of motion (refer 13 (a) to 13 (c), Art. 14.11 of chapter 14) in cylindrical coordinates for the present problem reduce to

$$\rho\,(q_\theta^2/r) = \partial p/\partial r \qquad ...(1)$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left[\frac{\partial^2 q_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r^2}\right] \quad \text{or} \quad 0 = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left\{\frac{d^2 q_\theta}{dr^2} + \frac{d}{dr}\left(\frac{q_\theta}{r}\right)\right\} \qquad ...(2)$$

$$0 = -(\partial p/\partial z) \qquad ...(3)$$

Equation (3) shows that p is independent of z. Since q_θ is function of r only and the flow is axially symmetric, it follows that p must be either a function of r or a constant. Hence we take $\partial p/\partial\theta = 0$ in (2) and so it may be re-written as

$$\frac{d^2 q_\theta}{dr^2} + \frac{d}{dr}\left(\frac{q_\theta}{r}\right) = 0. \qquad ...(4)$$

Integrating (4), $\quad dq_\theta/dr + q_\theta/r = 2A$, A being an arbitrary constant.

or $\quad \dfrac{1}{r}\dfrac{d}{dr}(rq_\theta) = 2A \quad$ or $\quad \dfrac{d}{dr}(rq_\theta) = 2Ar.$

Integrating it, $\quad rq_\theta = Ar^2 + B \quad$ or $\quad q_\theta = Ar + B/r, \qquad ...(5)$

where A and B are constants of integration to be determined. These constants are obtained from the boundary conditions :

$$\left.\begin{aligned} q_\theta &= r_1\omega_1 \quad at \quad r = r_1 \\ q_\theta &= r_2\omega_2 \quad at \quad r = r_2 \end{aligned}\right\} \qquad ...(6)$$

Using (6), (5) yields $\quad r_1\omega_1 = Ar_1 + B/r_1 \quad$ and $\quad r_2\omega_2 = Ar_2 + B/r_2 \qquad ...(7)$

On solving (7), $\quad A = \dfrac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2}, \quad$ and $\quad B = -\dfrac{r_1^2 r_2^2}{r_2^2 - r_1^2}(\omega_2 - \omega_1).$

Substituting these values into (5), we obtain

$$q_\theta = \frac{1}{r_2^2 - r_1^2}\left[(\omega_2 r_2^2 - \omega_1 r_1^2)\, r - \frac{r_1^2 r_2^2}{r}(\omega_2 - \omega_1) \right] \qquad ...(8)$$

As explained earlier, equation (1) may be written as

$$\frac{dp}{dr} = \frac{\rho}{r} q_\theta^2 \qquad \text{or} \qquad \frac{dp}{dr} = \frac{\rho}{r\,(r_2^2 - r_1^2)^2}\left[(\omega_2 r_2^2 - \omega_1 r_1^2) r - \frac{r_1^2 r_2^2}{r}(\omega_2 - \omega_1) \right]^2$$

or $$\frac{dp}{dr} = \frac{\rho}{(r_2^2 - r_1^2)^2}\left[(\omega_2 r_2^2 - \omega_1 r_1^2)^2 r - \frac{2 r_1^2 r_2^2}{r}(\omega_2 r_2^2 - \omega_1 r_1^2)(\omega_2 - \omega_1) + \frac{r_1^4 r_2^4}{r^3}(\omega_2 - \omega_1)^2 \right] \qquad ...(9)$$

Integration of (9) gives

$$p = \frac{\rho}{(r_2^2 - r_1^2)^2}\left[(\omega_2 r_2^2 - \omega_1 r_1^2)^2 \frac{r^2}{2} - 2 r_1^2 r_2^2 (\omega r_2^2 - \omega_1 r_1^2)(\omega_2 - \omega_1)\log r - \frac{r_1^4 r_2^4}{2r^2}.(\omega_2 - \omega_1)^2 \right] + C, \qquad ...(10)$$

where C is constant of integration to be determined. Suppose that $p = p_1$ at $r = r_1$ Then (10) gives

$$p = \frac{\rho}{(r_2^2 - r_1^2)^2}\left[(\omega_2 r_2^2 - \omega_1 r_1^2)^2 \frac{r_1^2}{2} - 2 r_1^2 r_2^2 (\omega_2 r_2^2 - \omega_1 r_1^2)(\omega_2 - \omega_1)\log r_1 - \frac{r_1^4 r_2^4}{2r_1^2}.(\omega_2 - \omega_1)^2 + C \right] \qquad ...(11)$$

Subtracting (11) from (10) and re-writing the resulting equation, we have

$$p = p_1 + \frac{\rho}{(r_2^2 - r_1^2)^2}\left[(\omega_2 r_2^2 - \omega_1 r_1^2)^2 \left(\frac{r^2 - r_1^2}{2} \right) - 2 r_1^2 r_2^2 (\omega_2 r_2^2 - \omega_1^2 r_1^2)(\omega_2 - \omega_1)\log \frac{r}{r_1} \right.$$
$$\left. - (r_1^4 r_2^4 / 2) \times (\omega_2 - \omega_1)^2 (1/r^2 - 1/r_1^2) \right] \qquad ...(12)$$

The shearing stress for the present problem is given by*

$$\sigma_{r\theta} = \mu(dq_\theta / dr - q_\theta / r) \qquad ...(13)$$

Substituting the value of q_θ given by (8) into (13), we have

$$\sigma_{r\theta} = \frac{2\mu}{r_2^2 - r_1^2}\,\frac{r_1^2 r_2^2}{r^2}(\omega_2 - \omega_1) \qquad ...(14)$$

Hence the shearing stress at the walls of the outer and inner cylinders are given by

$$(\sigma_{r\theta})_{r = r_2} = \frac{2\mu r_1^2}{r_2^2 - r_1^2}(\omega_2 - \omega_1) \qquad ...(15)$$

and $$(\sigma_{r\theta})_{r = r_1} = \frac{2\mu r_2^2}{r_2^2 - r_1^2}(\omega_2 - \omega_1). \qquad ...(16)$$

Deduction. Let the inner cylinder be at rest, *i.e.*, $\omega_1 = 0$. Then, writing $q_\theta = r\omega$, (8) gives

$$\omega = \frac{\omega_2 r_2^2 (r^2 - r_1^2)}{r^2 (r_2^2 - r_1^2)} = \frac{\omega_2 r_2^2}{r_2^2 - r_1^2}\left(1 - \frac{r_1^2}{r^2} \right) \qquad ...(17)$$

where ω is the angular velocity of the fluid at any point $P(r, \theta)$.

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* Refer results (4) and (6) of Art. 14.11 in chapter 14

For the present problem there would be only tangential stress $\sigma_{r\theta}$ given by

$$\sigma_{r\theta} = \mu\left(\frac{dq_\theta}{dr} - \frac{q_\theta}{r}\right) = \mu\left(\omega + r\frac{d\omega}{dr} - \omega\right) = \mu r\frac{d\omega}{dr}$$

$\therefore$ Moment of $\sigma_{r\theta}$ about the common axis of the cylinders

$$= (\sigma_{r\theta} \times 2\pi r) \times r = 2\pi\mu r^3 \frac{d\omega}{dr} = 4\pi\omega_2\mu\frac{r_1^2 r_2^2}{r_2^2 - r_1^2}, \text{ using (7)}$$

16.7 Illustrative solved examples

Ex. 1. *Incompressible liquid is flowing steadily through a circular pipe. Prove that the mean pressure is constant over the cross section and that the rate of flow is* $\pi a^4 (p_1 - p_2)/8\mu l$, *where* p_1 *and* p_2 *are the pressures over sections at distance* l *apart.* **[Himanchal 1999; Agra 2000, 06; 09; 11, Meerut 2004; Nagpur 2003, 06; Mumbai 2005; Patna 2003]**

Sol. Refer Art. 16.4 A, We have $\qquad P = (p_2 - p_1)/l = -(p_1 - p_2)/l \qquad$ ...(1)

and so
$$Q = -\frac{\pi a^4 P}{8\mu} = \frac{\pi a^4 (p_1 - p_2)}{8\mu l}, \text{ using (1)}$$

Ex. 2. *The space between two co-axial cylinders of radii a and b is filled with viscous fluid, and the cylinders are made to rotate with angular velocities* ω_1, ω_2. *Prove that in steady motion the angular velocity of the fluid is given by*

$$\omega = \{a^2 (b^2 - r^2)\omega_1 + b^2 (r^2 - a^2)\omega_2\}/r^2(b^2 - a^2)$$

[Agra 1998; Meerut 2001; Himanchal 1999]

Sol. Setting $r_1 = a$ and $r_2 = b$ in equation (8) of Art. 16.6, we have

$$q_\theta = \frac{1}{b^2 - a^2}\left[(\omega_2 b^2 - \omega_1 a^2) r - \frac{a^2 b^2}{r}(\omega_2 - \omega_1)\right]$$

or
$$\omega r = [(\omega_2 b^2 - \omega_1 a^2) r^2 - a^2 b^2 (\omega_2 - \omega_1)]/r(b^2 - a^2)$$

or
$$\omega = \{a^2 (b^2 - r^2)\omega_1 + b^2 (r^2 - a^2)\omega_2\}/r^2(b^2 - a^2)$$

Ex. 3. *A viscous liquid flows steadily parallel to the axis in the annular space between two co-axial cylinders of radii a, na (n > 1). Show that the rate of discharge is*

$$\frac{\pi P a^4}{8\mu}\left[n^4 - 1 - \frac{(n^2 - 1)^2}{\log n}\right],$$ where *P is the pressure gradient.* **(Meerut 2007)**

Sol. From equation (13) of Art. 16.4*B*, we have

$$Q = -\frac{\pi P}{8\mu}\left[b^4 - a^4 - \frac{(b^2 - a^2)^2}{\log(b/a)}\right] \qquad ...(i)$$

Here $b = an$. Also replacing P by $-P$ in (*i*) for the present problem, we have

$$Q = \frac{\pi P}{8\mu}\left[a^4 (n^4 - 1) - \frac{a^4 (n^2 - 1)^2}{\log n}\right] = \frac{\pi P a^4}{8\mu}\left[n^4 - 1 - \frac{(n^2 - 1)^2}{\log n}\right].$$

Ex. 4. (*a*) *Determine the maximum value of the velocity profile in the annular space between two coaxial cylinders.*

(*b*) If $a = 50\ mm$, $b = 75\ mm$, and the volumetric flow of water, $Q = 0.006\ m^3/s$, calculate (*i*) the pressure drop (*ii*) the maximum value of u and (*iii*) the shearing stress at the wall of both cylinders. Assume that $\mu = 1.01\ g/ms$. **[Garwhal 1995, 97]**

Sol. Part (a). Refer Art. 16.4*B*. We have

$$u = -\frac{P}{4\mu}\left[a^2 - r^2 + (b^2 - a^2)\frac{\log(r/a)}{\log(b/a)}\right]. \qquad \text{...(1)}$$

From (1),
$$\frac{du}{dr} = -\frac{P}{4\mu}\left[-2r + \frac{b^2 - a^2}{\log(b/a)}\frac{1}{r}\right]. \qquad \text{...(2)}$$

For the maximum value of u, we must have $du/dr = 0$, and so

$$-2r + \frac{b^2 - a^2}{\log(b/a)}\frac{1}{r} = 0 \quad \text{so that} \quad r = \pm\left[\frac{b^2 - a^2}{2\log(b/a)}\right]^{1/2}, \qquad \text{...(3)}$$

which gives the values of r for which u will be maximum. Putting this value of r in (1), the required maximum velocity is given by

$$u_{\max} = -\frac{P}{4\mu}\left[a^2 - \frac{b^2 - a^2}{2\log(b/a)} + \frac{b^2 - a^2}{\log(b/a)}\log\frac{1}{a}\left\{\frac{b^2 - a^2}{2\log(b/a)}\right\}^{1/2}\right]$$

$$= -\frac{Pa^2}{4\mu}\left[1 - \frac{n^2 - 1}{2\log n}\left\{1 - \log\frac{n^2 - 1}{2\log n}\right\}\right], \qquad \text{...(4)}$$

where $n = b/a = 75/50 = 1.5.$

Part (*b*). (*i*). From Art 16.4 B, we have

$$Q = -\frac{\pi P}{8\mu}\left[(b^4 - a^4) - \frac{(b^2 - a^2)^2}{\log(b/a)}\right] \quad \text{or} \quad Q = -\frac{\pi P a^4}{8\mu}\left[n^4 - 1 - \frac{(n^2 - 1)^2}{\log n}\right].$$

$$0.006 = -\frac{3.14 \times (0.05)^4 P}{8 \times (1.01 \times 10^{-3})}\left[(1.5)^4 - 1 - \frac{\{(1.5)^2 - 1\}^2}{\log 1.5}\right]$$

or $0.006 = -0.000486P$ so that $P = dp/dz = -12.3\ N/m^3$

(*ii*) The mamimum velocity is given by (4) as follows

$$u_{\max} = \frac{12.3 \times (0.05)^2}{4 \times (1.01 \times 10^{-3})}\left[1 - \frac{(1.5)^2 - 1}{2\log 1.5}\left\{1 - \log\frac{(1.5)^2 - 1}{2\log 1.5}\right\}\right] = 0.96\, m/s.$$

(*iii*) The shearing stress at the walls of the two cylinders are given by [Refer equations (15) and (16) in Art. 16.4*B*]

$$(\sigma_{rz})_{r=a} = -\frac{P}{4}\left[\frac{b^2 - a^2}{a\log(b/a)} - 2a\right] = -\frac{Pa}{4}\left[\frac{n^2 - 1}{\log n} - 2\right] = \frac{12.3 \times 0.05}{4}\left[\frac{(1.5)^2 - 1}{\log 1.5} - 2\right] = 0.184\, N/m^2.$$

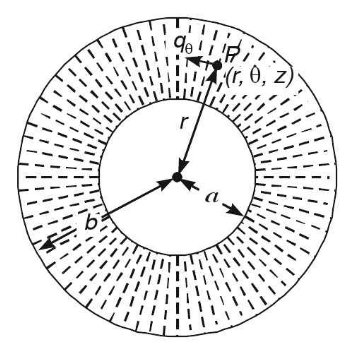
$$(\sigma_{rz})_{r=b} = -\frac{P}{4}\left[\frac{b^2 - a^2}{b\log(b/a)} - 2b\right] = -\frac{Pb}{4}\left[\frac{1 - (1/n)^2}{\log n} - 2\right] = \frac{12.3 \times 0.05}{4}\left[\frac{1 - (1/1.5)^2}{\log 1.5} - 2\right] = 0.145\, N/m^2$$

Ex. 5. *A liquid occupying the space between two co-axial circular cylinders is acted upon by a force c/r per unit mass, where r is the distance from the axis, the lines of force being circles around the axis. Prove that in the steady motion the velocity at any point is given by the formula*

$$\frac{c}{2\nu}\left\{\frac{b^2}{r}\frac{r^2 - a^2}{b^2 - a^2}\log\frac{b}{a} - r\log\left(\frac{r}{a}\right)\right\},$$

where a, b are the two radii and ν is the coefficient of kinematic viscosity.

[Agra, 2000, 02, 06; Kanpur 2002, Kolkata 2006, Rajasthan 2001]



Sol. Consider two infinitely, long, concentric circular cylinders of radii a and b ($b > a$). Let there be viscous incompressible fluid in the annular space. Since the lines of force are circles around the axis of the cylinders, this will produce steady, axi-symmetric, tangential motion in the fluid. Let z-axis be taken along the axis of the cylinders. Since the motion is only tangential, we have

$$q_r = 0 \qquad \text{and} \qquad q_z = 0 \qquad \ldots(1)$$

Hence the continuity equation in cylindrical coordinates reduces to

$$\partial q_\theta / \partial \theta = 0,$$

showing that q_θ depends on r and z only. Furthermore, the cylinders being very long, the flow will not depend on z. Hence, we suppose that

$$q_\theta = q_\theta(r) = r\omega, \qquad \ldots(2)$$

where ω is the angular velocity of the liquid at any point $P(r, \mathbf{0}, z)$.

Here, body force $\mathbf{B} = \mathbf{B}\,(\mathbf{B}_r, \mathbf{B}_\theta, \mathbf{B}_z) = \mathbf{B}\,(0, c/r, 0)$. Hence $\mathbf{B}_\theta = c/r$ ...(3)

The Navier-Stokes's equation in θ-direction for axi-symmetric flow of incompressible fluid is given by (Refer equation 13 (b) in Art. 14.11, chapter 14)

$$\rho\left(\frac{\partial q_\theta}{\partial t} + q_r \frac{\partial q_\theta}{\partial r} + q_z \frac{\partial q_\theta}{\partial z} + \frac{q_r q_\theta}{r}\right) = \rho B_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r} \times \frac{\partial}{\partial r}(r q_\theta)\right\} + \frac{\partial^2 q_\theta}{\partial z^2}\right] \qquad \ldots(4)$$

For the present steady ($\partial/\partial t = 0$) and axi-symmetric ($\partial/\partial\theta = 0$) flow, using (1), (2) and (3), (4) reduces to

$$0 = \frac{\rho c}{r} + \mu \frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(r^2\omega)\right\} \qquad \text{or} \qquad \frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(r^2\omega)\right\} = -\frac{\rho c}{\mu r} = -\frac{c}{r\nu}, \quad \text{as} \quad \nu = \frac{\mu}{\rho}$$

or
$$\frac{d}{dr}\left\{\frac{1}{r}\left(2r\omega + r^2\frac{d\omega}{dr}\right)\right\} = -\frac{c}{r\nu} \qquad \text{or} \qquad \frac{d}{dr}\left(2\omega + r\frac{d\omega}{dr}\right) = -\frac{c}{r\nu}$$

or
$$2(d\omega/dr) + \{d\omega/dr + r(d^2\omega/dr^2)\} = -(c/r\nu)$$

or
$$r\frac{d^2\omega}{dr^2} + 3\frac{d\omega}{dr} = -\frac{c}{r\nu} \qquad \text{or} \qquad r^3\frac{d^2\omega}{dr^2} + 3r^2\frac{d\omega}{dr} = -\frac{cr}{\nu}$$

or
$$\frac{d}{dr}\left(r^3\frac{d\omega}{dr}\right) = -\frac{cr}{\nu}.$$

Integrating, $$r^3\frac{d\omega}{dr} = -\frac{cr^2}{2\nu} + A \qquad \text{or} \qquad \frac{d\omega}{dr} = -\frac{c}{2r\nu} + \frac{A}{r^3}$$

or
$$d\omega = \left(-\frac{c}{2r\nu} + \frac{A}{r^3}\right)dr.$$

Integrating, $$\omega = -\frac{c}{2\nu}\log r - \frac{A}{2r^2} + B, \qquad \ldots(5)$$

where A and B are constants of integration to be determined. To determine A and B, we use the boundary conditions :

$$\omega = 0 \qquad \text{at} \qquad r = 0 \qquad \ldots(6A)$$

$$\omega = 0 \qquad \text{at} \qquad r = b \qquad \ldots(6B)$$

Using (6*A*), (5) gives $$0 = -\frac{c}{2\nu}\log a - \frac{A}{2a^2} + B \qquad ...(7A)$$

Using (6B), (5) gives $$0 = -\frac{c}{2\nu}\log b - \frac{A}{2b^2} + B \qquad ...(7B)$$

Subtracting (7*A*) from (5), $$\omega = -\frac{A}{2}\left(\frac{1}{r^2} - \frac{1}{a^2}\right) - \frac{c}{2\nu}\log\left(\frac{r}{a}\right) \qquad ...(8)$$

Subtracting (7*B*) from (7*A*), we have

$$0 = -\frac{A}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \frac{c}{2\nu}\log\left(\frac{b}{a}\right) \quad \text{so that} \quad A = \frac{ca^2b^2}{\nu\,(b^2 - a^2)}\log\left(\frac{b}{a}\right)$$

Substituting this value of A in (8), we have

$$\omega = -\frac{ca^2b^2 \log (b/a)}{2\nu\,(b^2 - a^2)}\left(\frac{1}{r^2} - \frac{1}{a^2}\right) - \frac{c}{2\nu}\log\left(\frac{r}{a}\right) \quad \text{or} \quad \frac{q_\theta}{r} = \frac{c}{2\nu}\frac{r^2 - a^2}{b^2 - a^2}\frac{b^2}{r^2}\log\frac{b}{a} - \frac{c}{2\nu}\log\frac{r}{a}, \text{ using (2)}$$

$$\therefore \qquad q_\theta = \frac{c}{2\nu}\left\{\frac{b^2}{r}\frac{r^2 - a^2}{b^2 - a^2}\log\frac{b}{a} - r\log\left(\frac{r}{a}\right)\right\}.$$

Ex. 6. *Oil is filled between two concentric rotating cylinders with radii 5 in. and (11/2) in. Assume that* $\mu = 0.005$ *lbf-sec/ft*^{3}. *The inner cylinder rotates at a speed of 5 rpm, while the outer cylinder is at rest. Calculate the stress at the wall of the inner cylinder.*

Sol. Refer Art 16.6 Here, $r_2 = \frac{11}{2} \times \frac{1}{12} = \frac{11}{24}\,ft,$ $\qquad r_1 = 5 \times \frac{1}{12} = \frac{5}{12}\,ft,$

$\omega_1 = 5/60 = (1/12)\,rps,$ $\qquad \omega_2 = 0$ $\qquad$ and $\qquad \mu = 0.005\ lbf - \sec/ft^2$

Shear stress at the wall of the inner cylinder $= (\sigma_{r\theta})_{r = r_1} = \dfrac{2\mu r_2^2}{r_2^2 - r_1^2}(\omega_2 - \omega_1)$

$$\therefore \qquad (\sigma_{r\theta})_{r = 5/12} = \frac{2 \times 0.005 \times (11/24)^2}{(11/24)^2 - (5/12)^2} \times \left(0 - \frac{1}{12}\right) = -\ 0.0048\ lbf/ft^2$$

Ex. 7. *In the case of steady flow of compressible liquid flowing steadily through a circular pipe of radius a, show that the mass which crosses any section per unit time is*

$$\pi a^4\,(p_1 - p_2)\,(\rho_1 + \rho_2)/16\mu l,$$

where ρ_1 *and* ρ_2 *are the densities at two sections at distance l apart. It is assumed that the temperature is constant, and the velocity gradient in the direction of the axis may be neglected in comparison with its gradient in the direction of a radius.* **[Agra 2008, Meerut 2006]**

Sol. Using cylindrical coordinates with z-axis along the axis of the pipe, we have $q_r = 0 = q_\theta$.

The continuity equation reduces to $$\frac{\partial}{\partial z}(\rho q_z) = 0. \qquad ...(1)$$

Since temperature is constant, equation of state is given by $$p = k\rho. \qquad ...(2)$$

Using (2), (1) may be written as $$\frac{\partial}{\partial z}(p q_z) = 0. \qquad ...(3)$$

For present steady motion in absence of body forces, the Navier-Stokes equations reduce to

$$0 = -(\partial p/\partial r), \qquad ...(4)$$

$$0 = -(1/r) \times (\partial p / \partial \theta), \qquad \text{...(5)}$$

and $$0 = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dq_z}{dr} \right) \right]. \qquad \text{...(6)}$$

Equations (4) and (5) show that p is function of z alone. Hence (6) reduces to

$$\frac{\mu}{r} \frac{d}{dr} \left\{ r \frac{d\,(pq_z)}{dr} \right\} = p \frac{dp}{dz}. \qquad \text{...(7)}$$

Since the *L.H.S.* of (7) is a function of r alone while the *R.H.S.* of (7) is a function of z alone, (7) is true if its each side is a constant, A, say.

$\therefore$ $$\frac{\mu}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} (pq_z) \right\} = A \qquad \text{...(8)}$$

and $\qquad p(dp/dz) = A. \qquad$ or $\qquad 2pdp = 2Adz. \qquad$ ...(9)

Integrating (9), $\qquad p^2 = 2Az + B$, where B is an arbitrary constant ...(10)

Now $p = p_1$ when $z = 0$ and $p = p_2$ when $z = l$. Hence (10) gives

$p_1^2 = B \qquad$ and $\qquad p_2^2 = 2Al + B \qquad$ so that $\qquad A = (p_2^2 - p_1^2)/2l \qquad$ and $\qquad B = p_1^2$. ...(11)

$\therefore$ From (10), $$p^2 = (p_2^2 - p_1^2) \times (z/l) + p_1^2 \qquad \text{...(12)}$$

Using (11), (8) reduces to

$$\frac{\mu}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} (pq_z) \right\} = \frac{p_2^2 - p_1^2}{2l} \qquad \text{or} \qquad \frac{d}{dr} \left\{ r \frac{d}{dr} (pq_z) \right\} = \frac{p_2^2 - p_1^2}{2l\mu} r$$

Integrating it, $$r \frac{d}{dr} (pq_z) = \frac{p_2^2 - p_1^2}{4l\mu} r^2 + C \qquad \text{or} \qquad \frac{d\,(pq_z)}{dr} = \frac{p_2^2 - p_1^2}{4l\mu} r + \frac{C}{r}$$

Integrating it, $$pq_z = (p_2^2 - p_1^2) \times (r^2 / 8l\mu) + C \log r + D, \qquad \text{...(13)}$$

where C and D are constants of integration. On the axis (where $r = 0$) q_z must be finite. Hence we take $C = 0$ in (13). Further, $q_z = 0$, when $r = a$. Hence (13) gives

$$0 = \frac{p_2^2 - p_1^2}{8l\mu} a^2 + 0 + D \qquad \text{so that} \qquad D = -\frac{p_2^2 - p_1^2}{8l\mu} a^2$$

Thus, (13) reduces to $$pq_z = \frac{p_2^2 - p_1^2}{8l\mu} r^2 - \frac{p_2^2 - p_1^2}{8l\mu} a^2$$

or $$q_z = \frac{p_2^2 - p_1^2}{8l\mu p} (r^2 - a^2). \qquad \text{...(14)}$$

$\therefore$ The mass of the liquid crossing any section per unit time

$$= \int_0^a 2\pi r\,(q_z \rho)\,dr = \frac{p_2^2 - p_1^2}{8l\mu} \times 2\pi \int_0^a r\,(r^2 - a^2) \frac{\rho}{p} dr = \frac{(p_2^2 - p_1^2)\pi}{4l\mu} \frac{\rho_1 + \rho_2}{p_1 + p_2} \int_0^a (r^3 - a^2 r)\,dr$$

$$\left[\because \text{ from (2), } \frac{\rho}{p} = \frac{1}{k} = \frac{\rho_1}{p_1} = \frac{\rho_2}{p_2} = \frac{\rho_1 + \rho_2}{p_1 + p_2} \right]$$

$$= \frac{\pi\,(p_2 - p_1)\,(\rho_1 + \rho_2)}{4l\mu} \left(-\frac{a^4}{4} \right) = \frac{\pi a^4\,(p_1 - p_2)\,(\rho_1 + \rho_2)}{16\mu l}.$$

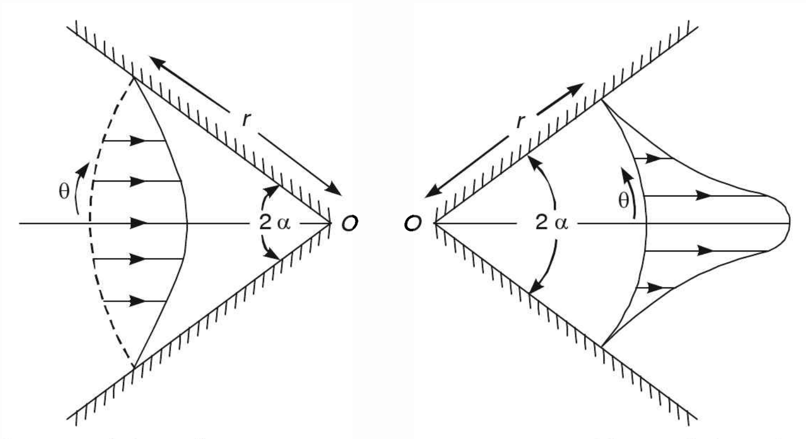
16.8 Flow in convergent and divergent channels (or Jeffery-Hamel flow)

[Garwhal 1998; Himanchal 2002, 03 Meerut 1997]

Consider steady flow of an incompressible fluid through a convergent or a divergent channel. Assume that the family of straight lines passing through a point O in a plane constitute the streamlines of the flow under consideration. Let the velocity vary from streamline to streamline so the velocity is function of the polar angle θ. The rays along which the velocity varies will be treated as the solid walls of given convergent or divergent channel.

We assume that the velocity is only in radial direction and depends on r and θ only. Thus,

$$q_r = q_r\,(r, \theta), \qquad q_\theta = 0, \qquad q_z = 0,$$



Convergent channel Divergent channel

Hence the continutiy equation in cylindrical polar coordinates (Refer Art. 2.10) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}(rq_r) = 0. \qquad \ldots(1)$$

Again, Navier-Stokes equations in cylindrical polar coordinates reduce to (Refer Art. 14.11)

$$q_r \frac{\partial q_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 q_r}{\partial r^2} + \frac{1}{r}\frac{\partial q_r}{\partial r} - \frac{q_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 q_r}{\partial \theta^2}\right] \qquad \ldots(2)$$

and

$$0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \frac{2\nu}{r^2}\frac{\partial q_r}{\partial \theta} \qquad \ldots(3)$$

Intergrating (1), $\quad rq_r = \nu F(\theta)$, F being an arbitrary function $\quad \ldots(4)$

where ν has been so asjusted that the new shape of (2) and (3) may be more compact.

From (4), $\quad q_r\,(r, \theta) = \{\nu F(\theta)\}/r. \quad \ldots(5)$

Using (5), (2) and (3) reduce respectively to

$$-\frac{\nu^2 F^2}{r^3} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\nu^2 F''}{r^3} \qquad \ldots(6)$$

and $\quad 0 = -\dfrac{1}{\rho r}\dfrac{\partial p}{\partial \theta} + \dfrac{2\nu^2 F'}{r^3} \quad$ or $\quad 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial \theta} + \dfrac{2\nu^2 F'}{r^2} \quad \ldots(7)$

Differentiating both sides of (6) *w.r.t* 'θ', we get

$$-\frac{\nu^2}{r^3} \times 2FF' = -\frac{1}{\rho}\frac{\partial^2 p}{\partial \theta \partial r} + \frac{\nu^2 F'''}{r^3} \qquad \ldots(8)$$

Differentiating both side of (7) *w.r.t.* 'r', we get

$$0 = -\frac{1}{\rho}\frac{\partial^2 p}{\partial r \partial \theta} - \frac{4\nu^2 F'}{r^3}$$

or $$0 = -\frac{1}{\rho}\frac{\partial^2 p}{\partial \theta \partial r} - \frac{4\nu^2 F'}{r^3}, \qquad \text{as} \qquad \frac{\partial^2 p}{\partial r \partial \theta} = \frac{\partial^2 p}{\partial \theta \partial r}$$

Subtraction (9) from (8), we have

$$-\frac{\nu^2}{r^3} \times 2FF' = \frac{\nu^2 F'''}{r^3} + \frac{4\nu^2 F'}{r^3}$$

or $$F''' + 4F' + 2FF' = 0 \qquad \text{or} \qquad d(F'' + 4F - F^2)/d\theta = 0. \qquad \ldots(10)$$

Integrating (10) *w.r.t.* 'θ', $F'' + 4F - F^2 = C_1$, where C_1 is a constant of integration. ...(11)

Multiplying both sides of (11) by F', we get

$$F'F'' + 4FF' + F^2F' = C_1F' \qquad \text{or} \qquad \frac{d}{d\theta}\left[\frac{1}{2}F'^2 + 2F^2 + \frac{1}{3}F^3\right] = C_1\frac{dF}{d\theta}. \qquad \ldots(12)$$

Integrating (12) *w.r.t.* 'θ', we have

$(1/2) \times F'^2 + 2F^2 + (1/3) \times F^3 = C_1F + (1/3) \times C_2$, where C_2 is a constant of integration. ...(13)

Re-writing (13), we have $$3F'^2 = 2\,(C_2 + 3C_1F - 6F^2 - F^3)$$

or $$dF/d\theta = (2/3)^{1/2} \times (C_2 + 3C_1F - 6F^2 - F^3)^{1/2}$$

Intergrating, $$\theta = \sqrt{\frac{3}{2}}\int_0^F \frac{dF}{\sqrt{(C_2 + 3C_1F - 6F^2 - F^3}} \qquad \ldots(14)$$

which is a solution expressed in terms of an *elliptic integral*.

Since the order of the differential equation (10) is three, it follows that three boundary conditions are need to be satisfied by (10). These are

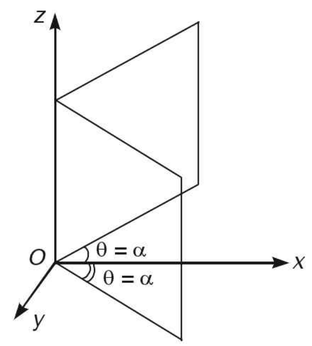
For convergent channel: $$\left.\begin{aligned} q_r\,(\pi + \alpha) = q_r\,(\pi - \alpha) &= 0 \\ \left(\frac{\partial q_r}{\partial \theta}\right)_{(r,\,\pi)} &= 0 \end{aligned}\right\} \qquad \ldots(15)$$

For divergent channel : $$\left.\begin{aligned} q_r\,(\alpha) = q_r\,(-\alpha) &= 0 \\ \left(\frac{\partial q_r}{\partial \theta}\right)_{(r,\,0)} &= 0 \end{aligned}\right\} \qquad \ldots(16)$$

Using the above boundary conditions, the constants C_1 and C_2 can be determined and hence the desired solution can be obtained.

16.9 Flow in convergent and divergent channels. Jeffery-Hamel flow.

(Alteruative detailes solution) (Himanchal 2004)



Consider the steady flow of a viscous incompressible fluid, in the absence of external forces, between two non-parallel plane walls. In order to solve the problem, we employ cylindrical coordinates (r, θ, z), in which the line of internection of the planes is taken as the z-axes and r is the distance from this line. The wales are in the planes $\theta = \pm\,\alpha$.

If the mution in purely radial, we have $q_\theta = q_z = 0$.

Then the equation of continuity (refer Art.2.10) and Navier Stokes equations (refer Art. 14.11) are given by

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) = 0 \qquad \text{or} \qquad \frac{\partial}{\partial r}(r q_r) = 0 \qquad \ldots(1)$$

$$\rho q_r \frac{\partial q_r}{\partial r} = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 q_r}{\partial r^2} + \frac{1}{r}\frac{\partial q_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 q_r}{\partial \theta^2} - \frac{q_r}{r^2}\right) \qquad \ldots(2)$$

and

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{2\mu}{r^2}\frac{\partial q_r}{\partial \theta} \qquad \ldots(3)$$

The boundary condition are : at $\theta = \pm\alpha$, $q_r = 0$ ...(4)

Integrating (1), $rq_r = f(\theta)$ or $q_r = f(\theta)/r$ ...(5)

when $f(\theta)$ is an arbitrary function of θ to be determined

Using (5), (2) and (3) reduce respectively to

$$-\rho\frac{f^2}{r^3} = -\frac{\partial p}{\partial r} + \mu\frac{f''}{r^3} \qquad \ldots(6)$$

$$0 = -\frac{\partial p}{\partial \theta} + 2\mu\frac{f'}{r^2}, \qquad \ldots(7)$$

where a prime denotes differentiation with respect to 'θ'.

The corresponding modified boundary conditions are : at $\theta = \pm\alpha$, $f = 0$ ...(8)

We now proceed to eliminate p between (6) and (7). To this end, differentiating (6) partially *w.r.t.*'θ', and (7) partially *w.r.t.* 'r' we have

$$-\frac{\rho}{r^3}\times 2ff' = -\frac{\partial^2 p}{\partial\theta\partial r} + \frac{\mu}{r^3}f''' \qquad \text{and} \qquad 0 = -\frac{\partial^2 p}{\partial r\partial\theta} - \frac{4\mu}{r^3}f' \qquad \ldots(9)$$

But $\partial^2 p/\partial\theta\partial r = \partial^2 p/\partial r\partial\theta$. So, on subtracting the second equation of (9) from the first equation of (9), we get

$$-\frac{\rho}{r^3}\times 2ff' = \frac{\mu}{r^3}(f''' + \mu f') \qquad \text{or} \qquad 2ff' + \frac{\mu}{\rho}(f''' + 4f') = 0$$

or

$$2ff' + \nu(f''' + 4f') = 0 \qquad \ldots(10)$$

where $\nu = \mu/\rho =$ kinematic viscosity of the fluid

Integrating (10), $f^2 + \nu(f'' + 4f) = c$, c being an arbitrary constant ...(11)

Multiplying both sides of (11) by $2f'$ and then re-writing it, we obtain

$$2f^2 f' + 4\nu\times(2ff') + \nu\times(2f'f'') = 2cf' \qquad \ldots(12)$$

Integrating (12), $(2/3)\times f^3 + 4\nu f^2 + \nu f'^2 = 2cf + 2h/3,$ ...(13)

where we have taken, h as an arbitrary constant.

Re-wring (13), we get $\nu f'^2 = 2h/3 + 2cf - 4\nu f^2 - (2/3)\times f^3$

or

$$f'^2 = (2/3\nu)\times(h + 3cf - 6\nu f^2 - f^3) \qquad \ldots(14)$$

Here we have to solve (14) with help of the boundary conditions (8), which are two in number. But the solution of (14) will involve three arbitrary constants (note that two arbitrary constants h and c already occur in (14) and third arbitrary constant will occur on integrating (14) to get $f(\theta)$). Consequently, we need three boundary conditions to compute three arbitrary constants. Thus, we must have an additional boundary condition. When the flow is purely divergent or purely convergent the function f must be symmetrical about $\theta = 0$ and in such as case the value of f at $\theta = 0$ may be prescribed. Thus,

Third boundary condition is, $\quad f(0) = f_0$, say ... (15)

Let $$h + 3cf - 6\nu f^2 - f^3 = (f_1 - f)(f_2 - f)(f_3 - f) \qquad \ldots(16)$$

or $$h + 3cf - 6\nu f^2 - f^3 = f_1 f_2 f_3 - f(f_1 f_2 + f_2 f_3 + f_3 f_1) + f^2 (f_1 + f_2 + f_3) - f^3 \qquad \ldots(17)$$

where f_1, f_2, and f_3 are constants. Using theory of quadratic equations, (17) yields

$$f_1 + f_2 + f_3 = -6\nu \qquad \ldots(18)$$

$$f_1 f_2 + f_2 f_3 + f_3 f_1 = -3c \qquad \ldots(19)$$

$$f_1 f_2 f_3 = h \qquad \ldots(20)$$

From (14) and (16), $$df/d\theta = \pm (2/3\nu)^{1/2} \sqrt{(f_1 - f)(f_2 - f)(f_3 - f)} \qquad \ldots(21)$$

$$d\theta = \pm \left(\frac{3}{2}\nu\right)^{1/2} \frac{df}{\{(f_1 - f)(f_2 - f)(f_3 - f)\}^{1/2}} \qquad \ldots(21)'$$

Integrating (21)′ between the corresponding limits $\theta = -\alpha$ to $\theta = 0$ and $f = 0$ to $f = f_0$, we get

$$\alpha = \pm \left(\frac{3}{2}\nu\right)^{1/2} \int_0^{f_0} \frac{df}{\{(f_1 - f)(f_2 - f)(f_3 - f)\}^{1/2}} \qquad \ldots(22)$$

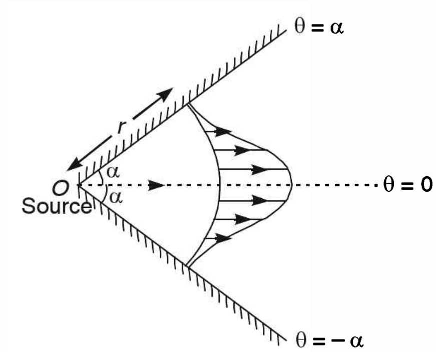
Following Millsaps and Pohlhausen*, we shall proceed to find the solution involving elliptic function. In what follows, we shall consider only the flows with large Reynolds number.

Case I: Flow in a divergent channel

For the flow in a divergent channel the radical velocity q_r must be positive Hence (5) implies that $f > 0$. Again, since in the middle of the channel $df/d\theta = 0$, (21) shows that $f = f_1$ or $f = f_2$ or $f = f_3$.

Let $f = f_1 = f_0$ Then, using (18) and (20), (22) may be re-written as

$$\alpha = \left(\frac{3\nu}{2}\right)^{1/2} \int_0^{f_1} \frac{df}{[f_1 - f)\{f^2 + (6\nu + f_1)f + h/f_1\}]^{1/2}} \qquad \ldots(23)$$



Since on the walls $f = 0$ (see condition (8)), (14) inplies that

$$f'^2 = 2h/3\nu, \qquad \ldots(24)$$

showing that $h > 0$. It follows that α has its greatest permissible value for a given value of f_1 when $h = 0$, i.e., either f_2 or f_3 is zero.

* Millsaps, K and Pohlhausen K: Thermal distribution in Jeffery—Hamel flows between non-parallel walls. Jou. Aero. Sc. 20, Page 187–196 (1953)

Let $\quad f = f_1 \cos^3 \psi$

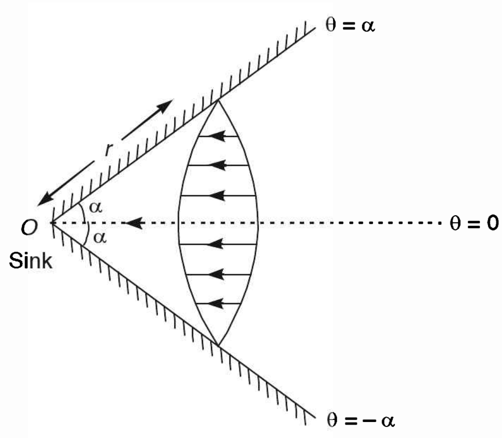
and $\quad \text{Re} = \text{Reynolds's number} = \dfrac{f_1}{\nu} = \dfrac{r(q_r)_{\max}}{\nu}$ $\quad \ldots(25)$

Substituting (25) in (23) and neglecting the terms containing 6ν when Re is large, we get

$$\alpha = \left(\frac{3\nu}{f_1}\right)^{1/2} \int_0^{\pi/2} \frac{d\psi}{\{1-(1/2)\times \sin^2\psi\}^{1/2}}$$

or
$$\alpha\sqrt{\text{Re}} = \sqrt{3}\int_0^{\pi/2} \frac{d\psi}{\{1-(1/2)\times \sin^2\psi\}^{1/2}} = 3.211, \qquad \ldots(26)$$

showing that $\alpha\sqrt{\text{Re}}$ has an upper limit when Re is large and α is small. Thus if α and Re are given, then the velocity profile is given by (16) by assuming that $f_2 = 0$ or $f_3 = 0$



Case II: Flow a convergent channel For the flow in a convergent channel the radial velocity q_r must he negative. Then, (5) $\Rightarrow f < 0$ Hence, from (20) and (24) it follows that one of the root should be positive and the other two must be negative. Let $f_1 > 0$ and $f_2 < 0, f_3 < 0$. Again, we observe that $f_2 \le f \le 0$ (it is only in the middle of the channel where $f = f_2$) and $f_3 \le f_2$.

Let $\quad \text{Re} = -f_2/\nu, \quad f/f_2 = \omega, \quad -f_1/f_2 = \omega_1 \quad$ and $\quad f_3/f_2 = \omega_3$ ...(27)

so that ω's are positive and $0 < \omega_3 < 1$.

Using (27), (28) reduces to $\quad 1 - \omega_1 + \omega_3 = 6/\text{Re}$ $\quad \ldots(28)$

Hence, (14) yields

$$\theta = \left(\frac{3}{2\text{Re}}\right)^{1/2} \int_\omega^1 \frac{d\omega}{\{(1-\omega)(\omega_3-\omega)(\omega_1+\omega)\}^{1/2}} \qquad \ldots(29)$$

$$\therefore \quad \alpha\sqrt{\text{Re}} = \left(\frac{3}{2}\right)^{1/2} \int_0^1 \frac{d\omega}{\{(1-\omega)(\omega_3-\omega)(\omega_1+\omega)\}^{1/2}} \qquad \ldots(30)$$

Note that there in no restriction on the upper limit of $\alpha\sqrt{\text{Re}}$. But $\alpha\sqrt{\text{Re}}$. is large if ω_3 is nearly equal to 1. Hence, if 6/Re is neglected from (28), we see that ω_1 is nearly equal to 2. With these value of ω_1 and ω_3, we find from (29) and (30), that

$$\alpha - \theta = \left(\frac{3}{2\text{Re}}\right)^{1/2} \int_0^\omega \frac{d\omega}{(1-\omega)(2+\omega)^{1/2}} \qquad \ldots(31)$$

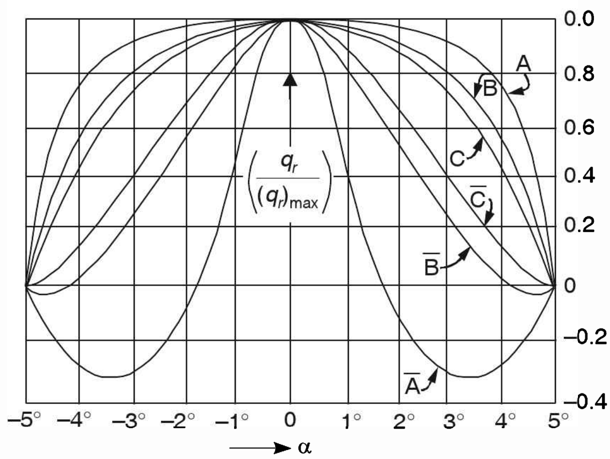
Putting $2 + \omega = 3\tanh^2\phi$ and integrating (31), we get

$$\omega = \frac{f}{f_2} = \frac{q_r}{(q_r)_{\max}} = 3\tanh^2\{(\text{Re}/2)^{1/2}(\alpha-\theta) + 1.46\} - 2, \qquad \ldots(32)$$

satisfying the boundary conditions (18). Note that $\tanh^2(1.146) = 2/3$ from standard tables.

We know that tanh x is close to unity when x is close to 2.5. Hence from (32), it follows that for large Re the velocity $q_r = (q_r)_{\max}$ everywhere except in a thin layer near each wall of thickness proportional to $(Re)^{-1/2}$

We now proceed to study the character of the velocity distribution, as calculated by Millsaps and Pohlhausen, which differ clearly for convergent and divergent channels and for different values of the Reynolds number . A total angle of 10° has been chosen for the convergence or divergence of the fixed walls. In a convergent channel the velocity distribution for a sufficiently large Reynolds number (Re = 5000) remains almost constant everywhere except the walls where it decrease rapidly to the value zero, exhibiting a boundary layer phenomenon (see chapter 18)



Convergent channel	Devergent channel
(A) Re = 5000	($\bar{A}$) Re = 5000
(B) Re = 1342	($\bar{B}$) Re = 1342
(C) Re = 684	($\bar{C}$) Re = 684

Fig. Velocity distribution in a convergent and a divergent channel

In a divergent channel the velocity is positive over the whole cross-section if the Reynolds number is small .However, for large Reynolds number (Re = 5000) it becomes negative near the walls and a back flow occurs

TYPE 2. DETERMINATION OF TEMPERATURE DISTRIBUTION IN STEADY INCOMPRESSIBLE FLOW WITH CONSTANT FLUID PROPERTIES

16.10 Introduction.

If all quantities (such as velocity, acceleration, pressure, density) associated with the flow field do not change with time at all points of the flow field, motion is called *steady*: otherwise it is called *unsteady*. Accordingly, various quantities of the flow field become functions of the space coordinates only because time drops out of the independent variables. A fluid is called *incompressible* if the density of the given fluid is constant at various points of the flow field. Such an assumption is possible for liquid and also for gases at low speeds ($M << 1$), where M is Mach number. In general , the viscosity of a fluid depends on the temperature. It is known that for most of the incompressible fluids the viscosity can be treated as a constant. This assumption is of great significance because for such situations the velocity field does not depend on the temperature field. Accordingly, the equation of continuity and equations of motions can be first solved for the three velocity components and the pressure p and the results so obtained can be used to solve the equation of energy to determine the desired temperature field.

Remark 1. One of the main difference between the compressible and incompressible flow is that, in compressible flow the equations of motion and energy are coupled whereas in an incompressible flow , they are uncoupled. Hence, in case of incompressible fluids, we can easily find temperature distribution as explained above.

Remark 2. For detailed description and particular cases of energy equation, reader is advised to read carefully Art. 14.2, Art 14.11 and Art 14.12. For definition of mach number M, refer Art 15.8.

16.11. Temperature distribution in steady laminar flow of an in-compressible fluid flow between two parallel plates. Plane Couette flow. [Himanchal 2005]

First of all give the entire matter of Art. 16.3A upto equation (11). Thus velocity distribution for the present case is given by

$$u = Uy/h. \quad \ldots(11)$$

The energy equation, for the steady flow and without heat addition, for the present problem is given by

$$\rho C_v \, u(\partial T/\partial x) = k(\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2) + \mu(\partial u/\partial y)^2, \quad \ldots(12)$$

where k and μ are regarded as constants.

Since we have taken the velocity only along x-axis and all the variables depend only on y, we have $\partial T/\partial x = 0$ and $\partial^2 T/\partial x^2 = 0$. Then (12) reduces to

$$k\,(d^2T/dy^2) = -\mu\,(du/dy)^2. \quad \ldots(13)$$

Substitutiong the value of u from (11), (13) becomes

$$d^2T/dy^2 = -(\mu/k) \times (U^2/h^2). \quad \ldots(14)$$

Integrating (14),

$$dT/dy = -(\mu/k) \times (U^2/h^2) y + C_1 \quad \ldots(15)$$

Integrating (15),

$$T = -(\mu/2k) \times (U^2/h^2) y^2 + C_1 y + C_2, \quad \ldots(16)$$

where C_1 and C_2 are arbitrary constants. We now find temperature distribution in the following three situations.

Case I. When the plates are kept at different termperatures. [Himanchal 2005]

Refer figure (*i*) of art 16.3A Suppose the lower plate ($y = 0$) and upper plate ($y = h$) be kept at constant temperatures T_0 and T_1 respectively, where $T_1 > T_0$. Then, for the present problem the boundary conditions are:

$$T = T_0, \quad \text{when} \quad y = 0 \quad \ldots(17)$$

and

$$T = T_1, \quad \text{when} \quad y = h. \quad \ldots(18)$$

Putting $y = 0$, $T = T_0$ and $y = h$, $T = T_1$ by turn in (16), we have

$$T_0 = C_2 \quad \text{and} \quad T_1 = -\frac{\mu}{2k} \times \frac{U^2}{h^2} \times h^2 + C_1 h + C_2. \quad \ldots(19)$$

Solving (19),

$$C_1 = \frac{T_1 - T_0}{h} + \frac{\mu U^2}{2kh} \quad \text{and} \quad C_2 = T_0. \quad \ldots(20)$$

Subsituting the values of C_1 and C_2 from (20) in (18), we get

$$T = T_0 + \frac{T_1 - T_0}{h} y + \frac{\mu U^2}{2k} \times \frac{y}{h} - \frac{\mu U^2}{2k} \times \frac{y^2}{h^2} \quad \text{or} \quad \frac{T - T_0}{T_1 - T_0} = \frac{y}{h} + \frac{\mu U^2}{2k\,(T_1 - T_0)} \frac{y}{h}\left(1 - \frac{y}{h}\right)$$

or

$$\frac{T - T_0}{T_1 - T_0} = \frac{y}{h} + \frac{1}{2} E_c \times P_r \frac{y}{h}\left(1 - \frac{y}{h}\right), \quad \ldots(21)$$

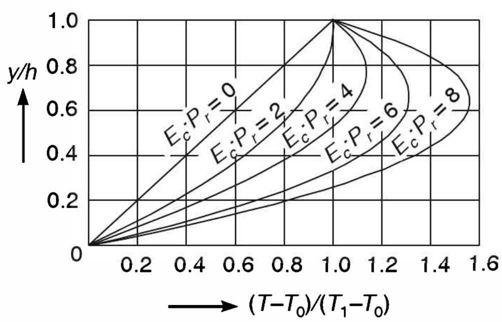
where

$$E_c = \text{Eckert number} = U_2 / \{ C_p\,(T_1 - T_0)\} \quad \ldots(22)$$

and

$$P_r = \text{Prandtl number} = \mu C_p / k. \quad \ldots(23)$$

From equation (21), it follows that the temperature distribution depends on the product $E_c\, P_r$ of Eckert and Prandtl numbers. In the following figure, for different values of $E_c \cdot P_r$, we have shown how dimensionless temperature $(T - T_0)/(T_1 - T_0)$ varies with dimensionless distance y/h between the plates.



We now propose to compute heat transfer at the upper plate. to this end, we calculate the dimensionless coefficient of heat transfer, namely, Nusselt Number defined as

$$N_u = -\frac{h}{(T_1 - T_0)}\left(\frac{dT}{dy}\right)_{y=h} \quad \ldots(24)$$

Differentiating (21) *w.r.t ;y'*, we have

$$\frac{1}{T_1 - T_0}\frac{dT}{dy} = \frac{1}{h} + \frac{1}{2}E_c \times P_r\left(\frac{1}{h} - \frac{2y}{h^2}\right) \quad \ldots(25)$$

Puting $y = h$ in (25),we have $\quad \dfrac{1}{T_1 - T_0}\left(\dfrac{dT}{dy}\right)_{y=h} = \dfrac{1}{h} + \dfrac{1}{2}E_c \times P_r\left(\dfrac{1}{h} - \dfrac{2}{h}\right)$

or $$\left(\frac{dT}{dy}\right)_{y=h} = (T_1 - T_0)\left[\frac{1}{h} - \frac{E_c \cdot P_r}{2h}\right] \quad \ldots(26)$$

Using (26), (24) reduces to

$$N_u = -\frac{h}{(T_1 - T_0)}\cdot(T_1 - T_0)\left[\frac{1}{h} - \frac{E_c \cdot P_r}{2h}\right] = \frac{1}{2}E_c \cdot P_r - 1. \quad \ldots(27)$$

From (27), we observe that

(*i*) If $E_c \cdot P_r > 2,$ then N_u will be positive and in this case the heat will be transferred from fluid to the upper plate.

(*ii*) If $E_c \cdot P_r < 2,$ then N_u will be negative and in this case the heat will be tranferred from upper plate to the fluid.

(*iii*) If $E_c \cdot P_r = 2,$ then Nu = 0 and in this case there will be no transfer of heat between the fluid nd the upper plate.

In order to compute heat transfer at the lower plate, we can calculate the Nusselt number

$$N_u = -\frac{h}{(T_1 - T_0)}\left(\frac{dT}{dy}\right)_{y=0} \quad \ldots(28)$$

Proceeding as before, we can easily show that at the lower stationary plate the heat is always transferred from fluid to plate irrespective of the range of $E_c \cdot P_r$.

Case II. When both the plates are kept at the same constant temperature T_0.

Refer figure (*i*) of Art 16.3A. Hence the boundary conditions for the present case are :

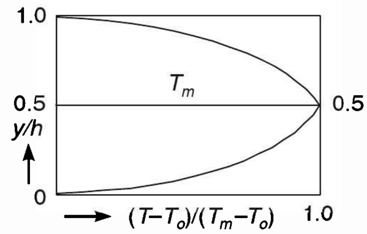
$$T = T_0, \quad \text{when} \quad y = 0 \quad \ldots(29)$$
$$T = T_0, \quad \text{when} \quad y = h \quad \ldots(30)$$

Putting $y = 0$, $T = T_0$ and $y = h$, $T = T_0$ by turn in (16) ,we get

$$T_0 = C_2 \quad \text{and} \quad T_0 = -\frac{\mu}{2k}\frac{U^2}{h^2}h^2 + C_1 h + C_2. \quad \ldots(31)$$

Solving (31), $\quad C_1 = -(\mu U^2)/2kh \quad$ and $\quad C_2 = T_0 \quad \ldots(32)$

Substituting the values of C_1 and C_2 from (32) in (16), we get



$$T - T_0 = \frac{\mu U^2}{2k}\frac{y}{h}\left(1 - \frac{y}{h}\right). \qquad \ldots(33)$$

Equation (33) shows that the temperature distribution is parabolic as shown in the adjoining figure. Let T_m denote the temperature in the middle of the channel so the $T = T_m$ when $y = h/2$. So putting $y = h/2$ in (33), we have

$$T_m - T_0 = \mu U^2/(8k). \qquad \ldots(34)$$

For the present case at the lower plate, theNusselt number is defined as

$$N_u = -\frac{h}{(T_0 - T_m)}\left(\frac{dT}{dy}\right)_{y=0} \qquad \ldots (35)$$

Differentiating (33) *w.r.t.* 'y', we get $\quad \dfrac{dT}{dy} = \dfrac{\mu U^2}{2k}\left(\dfrac{1}{h} - \dfrac{2y}{h^2}\right). \qquad \ldots(36)$

Putting y = 0 in (36), we get $\quad \left(\dfrac{dT}{dy}\right)_{y=0} = \dfrac{\mu U^2}{2lh}. \qquad \ldots(37)$

Using (34) and (37), (35) gives $\quad N_u = \dfrac{8kh}{\mu U^2} \times \dfrac{\mu U^2}{2kh} = 4,$

showing that the Nusselt number for the lower plates has a constant value 4.

Case III. When at one of the plate, say the lower stationary plate, no heat transfer takes place (adiabatic wall) and the other moving wall is kept at temperature T_1.

Refer Fig (i) of Art .16.3A Here the boundary conditions for the present case are :

$$dT/dy = 0 \quad \text{when} \quad y = 0 \qquad \ldots(38)$$

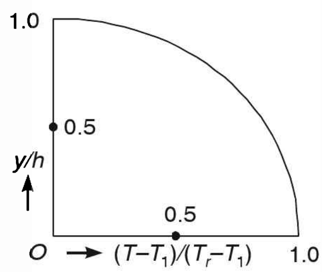
$$T = T_1 \quad \text{when} \quad y = h. \qquad \ldots(39)$$

From (15), we have

$$dT/dy = -(\mu/k) \times (U^2/h^2)y + C_1. \qquad \ldots(40)$$

Putting $y = 0$ and $dT/dy = 0$ in (40), gives $C_1 = 0$.

Next, putting $y = h$ and $T = T_1$in (16), we have



$$T_1 = -(\mu U^2/2k) + C_1 h + C_2$$

$\Rightarrow \qquad C_2 = T_1 + (\mu U^2/2k), \quad$ as $\quad C_1 = 0.$

Substituting the above values of C_1 and C_2 in (16), we get

$$T - T_1 = \frac{\mu U^2}{2k}\left(1 - \frac{y^2}{h^2}\right). \qquad \ldots(41)$$

The temperature which an insutated surface assumes under the influence of internal friction is known as *recovery temperature* T_r. From (41), we have

$$T_r = (T)_{y=0} = T_1 + (\mu U^2)/(2k). \quad \Rightarrow \quad T_r - T_1 = (\mu U^2)/(2k). \qquad \ldots(42)$$

Hence the recovery factor r in a plane Couette flow is given by

$$r = \frac{T_r - T_1}{U^2/2C_p} = \frac{\mu U^2}{2k} \times \frac{2C_p}{U^2} = \frac{\mu C_p}{k} = P_r \quad \text{(Prandtl number).} \qquad \ldots(43)$$

Temperature distribution for the present case is shown in the above figure.

16.12. Temperature distribution in steady laminar flow of an incompressibnle fluid between two parallel plates. Generalized plane Couette flow, (Himanchal 2009)

First of all give the entire matter of Art.16.3B upto equation (12). Thus the velocity distribution for the present case is given by
$$\frac{u}{U} = \frac{y}{h} + \alpha \frac{y}{h}\left(1 - \frac{y}{h}\right), \qquad \ldots(11)$$
where
$$\alpha = -h^2 P / 2\mu U.$$

The energy equation, for the steady flow and without heat addition, is given by
$$\rho C_v u \frac{\partial T}{\partial x} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2, \text{ where k and } \mu \text{ are regarded as constants.} \qquad \ldots(12)$$

Since we have taken the velocity only along x-axis and all the variables depend only on y, we have $\partial T/\partial x = 0$ and $\partial^2 T/\partial x^2 = 0$. Then (12) reduces to
$$k\,(d^2T/dy^2) = -\mu\,(du/dy)^2. \qquad \ldots(13)$$

From (11),
$$\frac{du}{dy} = U\left[\frac{1}{h} + \frac{\alpha}{h} - \frac{2y\alpha}{h^2}\right] = \frac{U}{h}\left[(1+\alpha) - \frac{2\alpha y}{h}\right].$$

Substituting the above value of du/dy in (13), we get
$$\frac{d^2T}{dy^2} = -\frac{\mu U^2}{kh^2}\left[(1+\alpha)^2 - 4\alpha\,(1+\alpha)\frac{y}{h} + 4\alpha^2\left(\frac{y}{h}\right)^2\right]. \qquad \ldots(14)$$

Integrating (14) twice in succession *w.r.t.* 'y',we have
$$\frac{dT}{dy} = -\frac{\mu U^2}{kh^2}\left[(1+\alpha)^2 y - 2\alpha\,(1+\alpha)\frac{y^2}{h} + \frac{4}{3}\times\frac{\alpha^2 y^3}{h^2}\right] + C_1 \qquad \ldots(15)$$

and
$$T = -\frac{\mu U^2}{kh^2}\left[\frac{1}{2}(1+\alpha)^2 y^2 - \frac{2}{3}\alpha\,(1+\alpha)\frac{y^3}{h} + \frac{1}{3}\alpha^2\frac{y^4}{h^2}\right] + C_1 y + C_2, \qquad \ldots(16)$$

where C_1 and C_2 are arbitrary constants.

Suppose that both the plates are kept at the same constant temperature T_0 (refer figure (ii) of Art 16.3B). Then, for the present problem, the boundary conditions are :
$$T = T_0, \quad \text{when} \quad y = 0 \qquad \ldots(17)$$
and
$$T = T_0, \quad \text{when} \quad y = h. \qquad \ldots(18)$$

Putting $y = 0$, $T = T_0$ and $y = h$, $T = T_0$ by turn in (16), we get
$$T_0 = C_2 \quad \text{and} \quad T_0 = -\frac{\mu U^2}{kh^2}\left[\frac{1}{2}(1+\alpha)^2 h^2 - \frac{2}{3}\alpha\,(1+\alpha)h^2 + \frac{1}{3}\alpha^2 h^2\right] + C_1 h + C_2.$$

Solving these,
$$C_1 = \frac{\mu U^2}{kh^2}\left[\frac{1}{2}(1+\alpha)^2 - \frac{2}{3}\alpha\,(1+\alpha) + \frac{\alpha^2}{3}\right] \quad \text{and} \quad C_2 = T_0.$$

Substituting these values of C_1 and C_2 in (16), we get
$$T - T_0 = \frac{\mu U^2}{6k}\times\frac{y}{h}\left[3\,(1+\alpha)^2\left(1 - \frac{y}{h}\right) - 4\alpha\,(1+\alpha)\left(1 - \frac{y^2}{h^2}\right) + 2\alpha^2\left(1 - \frac{y^3}{h^3}\right)\right] \qquad \ldots(19)$$

Differentiating (19). *w.r.t.* 'y',we have

$$\frac{dT}{dy}=\frac{\mu U^2}{6kh}\left[3\,(1+\alpha)^2\left(1-\frac{y}{h}\right)-4\alpha\,(1+\alpha)\left(1-\frac{y^2}{h^2}\right)+2\alpha^2\left(1-\frac{y^3}{h^3}\right)\right]$$

$$+\frac{\mu U^2}{6k}\times\frac{y}{h}\left[-3\,(1+\alpha)^2\times\frac{1}{h}+4\alpha\,(1+\alpha)\times\frac{2y}{h^2}-2\alpha^2\times\frac{3y^2}{h^3}\right]$$

Hence the temperature gradient at the lower plate is given by

$$\left(\frac{dT}{dy}\right)_{y=0}=\frac{\mu U^2}{6kh}[3\,(1+\alpha)^2-4\alpha\,(1+\alpha)+2\alpha^2]=\frac{\mu U^2}{6kh}[2+(1+\alpha)^2],$$

showing that heat will always be transferred from the fluid to the lower plate irrespective to the sign of α.

16.13 Temperature distribution in steady laminar flow of an incompressible fluid between two plates. Plane Poiseuille flow. [Himanchel 2009]

First of all give the entire matter of Art. 16.3C. upto equation (11). Thus the velocity distribution of the present case is given by

$$u=-(h^2P/8\mu)\{1-4(y/h)^2\} \qquad \dots (10)$$

Also maximum velocity u_{max} is given by $\quad u_{max}=-h^2P/(8\mu). \qquad \dots(11)$

Using (11), (10) reduces to $\quad u=u_{max}\,[1-4(y/h)^2]. \qquad \dots(12)$

The energy equation for the steady flow and without heat addition, is given by

$$\rho C_v u\,(\partial T/\partial x)=k(\partial^2 T/\partial x^2+\partial^2 T/\partial y^2)+\mu(\partial u/\partial y)^2,$$

where k and μ are regarded as constants. Since we have taken the velocity only along x-axis and all the variables depend on y, we have $\partial T/\partial x=\partial^2 T/\partial x^2=0$ Then, the above equation becomes

$$k\,(d^2T/dy^2)=-\mu\,(du/dy)^2 \qquad \dots(13)$$

From (12), $\quad du/dy=u_{max}\times(-8y/h^2).$

$\therefore$ (13) becomes $\quad d^2T/dy^2=-(64\mu u_{max}^2 y^2)/(kh^4). \qquad \dots(14)$

Integrating (14) twice in succession *w.r.t* 'y', we get

$$dT/dy=-(64\mu u_{max}^2 y^3)/(3kh^4)+C_1$$

and $\quad T=-(16\mu u_{max}^2 y^4)/(3kh^4)+C_1 y+C_2, \qquad \dots(15)$

where C_1 and C_2 are arbitrary constants.

Suppose that both the plates are kept at the same constant temperature T_0 (Refer figure (*iii*) of Art16.3C). Then, for the present problem, the boundary conditions are:

$T=T_0,\quad$ when $\quad y=-h/2 \qquad \dots(16)$

and $\quad T=T_0,\quad$ when $\quad y=h/2. \qquad \dots(17)$

Putting $y=-h/2,\ T=T_0$ and $y=h/2, T=T_0$ by turn in (15), we get

$$T_0=-\frac{\mu u_{max}^2}{3k}+\frac{C_1 h}{2}+C_2 \quad \text{and} \quad T_0=-\frac{\mu u_{max}^2}{3k}-\frac{C_1 h}{2}+C_2.$$

Solving these, $\quad C_1=0 \quad$ and $\quad C_2=T_0+(\mu/3k)\times u_{max}^2.$

Substituting these values of C_1 and C_2 in (15), we get

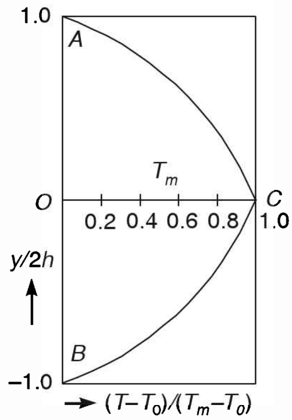
$$T - T_0 = \frac{\mu u_{\max}^2}{3k}\left(1 - \frac{16y^4}{h^4}\right). \qquad \ldots(18)$$

The maximum temperature T_m exists in the middle of the channel and is given by putting $y = 0$ and $T = T_m$ is (18).

$$\therefore \qquad T_m - T_0 = (\mu u_{\max}^2)/3k. \qquad \ldots(19)$$

From (18) and (19), we have

$$\frac{T - T_0}{T_m - T_0} = 1 - 16\left(\frac{y}{h}\right)^4, \qquad \ldots(20)$$



giving the dimensionless temperature distribution $(T - T_0)/(T_m - T_0)$ as function of the dimensionless distance $y/2h$ from the middle of the channel. The same is shown in the above figure.

16.14. Temperature distribution in steady laminar flow of an incompressible fluid through a circular pipe. The Hagen-Poiseuille flow. [Himanchal 2002, 03, 06, 10]

First of all given the entire matter of Art. 16.4.A upto equation (14). Thus the velocity distribution for the present case is given by

$$q_z = u = -(Pa^2/4\mu)\{1 - (r/a)^2\} \qquad \ldots(13)$$

Again the maximum velocity $u_{\max}$ is given by $\qquad u_{\max} = -(Pa^2)/(4\mu). \qquad \ldots(14)$

Using (14), (13) becomes $\qquad u = -u_{\max}\{1 - (r/a)^2\} \qquad \ldots(15)$

Here $q_r = q_\theta = 0$. Due to axial symmetry of flow, q_z will be independent of θ.

Hence the energy equation for the steady flow and without heat addition reduces to

$$\rho c_v u \frac{\partial T}{\partial z} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \mu\left(\frac{\partial u}{\partial r}\right)^2, \qquad \ldots(16)$$

where k and μ are regarded as constants.

We now discuss solution of (16) under two different situations as given below.

Case I. When wall of the pipe is kept at a constant temperature.

[Himanchal 2000; 01, 03, 09]

Since the wall of the pipe is kept at a constant temperature, $\partial T/\partial z = 0$. So, (16) reduces to

$$k\left(\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right) = -\mu\left(\frac{du}{dr}\right)^2. \qquad \ldots(17)$$

Substituting the value of u from (15) in (17) we get

$$k\left(\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right) = -\mu\frac{4(u_{\max})^2 r^2}{a^4}$$

or $\quad r\dfrac{d^2T}{dr^2} + \dfrac{dT}{dr} = -\dfrac{4\mu\,(u_{\max})^2 r^3}{a^4 k} \quad$ or $\quad \dfrac{d}{dr}\left(r\dfrac{dT}{dr}\right) = -\dfrac{4\mu\,(u_{\max})^2 r^3}{a^4 k}.$

Integrating, $\quad r\dfrac{dT}{dr} = -\dfrac{\mu\,(u_{\max})^2 r^4}{a^4 k} + C_1 \quad$ so that $\quad \dfrac{dT}{dr} = -\dfrac{\mu\,(u_{\max})^2 r^3}{a^4 k} + \dfrac{C_1}{r}$

Integrating, $T = -\dfrac{\mu\,(u_{\max})^2 r^4}{4a^4 k} + C_1 \log r + C_2$, where C_1 and C_2 are arbitrary constants ...(18)

For the present problem, the boundary conditions are;

T = finite, when $r = 0$; $T = T_0$, when $r = a$ ...(19)

Since T = finite when $r = 0$, (18) shows that we must assume that $C_1 = 0$. Then (18) reduces to

$$T = -(\mu/4a^4 k) \times (u_{\max})^2 r^4 + C_2. \qquad ...(20)$$

Since $T = T_0$ when $r = a$, (20) reduces to

$$T_0 = -(\mu/4k) \times (u_{\max})^2 + C_2 \qquad \text{so that} \qquad C_2 = T_0 + (\mu/4k) \times (u_{\max})^2.$$

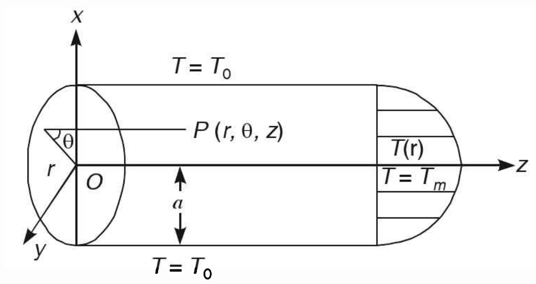
Substituting the above values of C_1 and C_2 in (18),we gets

$$T - T_0 = \frac{\mu\,(u_{\max})^2}{4k}\left(1 - \frac{r^4}{a^4}\right). \qquad ...(21)$$

The maximum temperature T_m exists on the axis of pipe (where $r = 0$). Hence, from (6) we have

$$T_m - T_0 = (\mu/4k) \times (u_{\max})^2 \qquad ...(22)$$

Temperature distribution in Hagen-Poiseuille flow is shown in the adjoining diagram.



From (21) and (22), the dimensionless temperature distribution in Hagen-poiseuille flow is given by $(T - T_0) / (T_m - T_0) = 1 - (r/a)^4$. ...(23)

The mean temperature T_{mean} over a cross-section is given by

$$T_{mean} = \frac{\int_0^a T \cdot 2\pi r\, dr}{\pi a^2} = \frac{2\int_0^a r[T_0 + (\mu/4k) \times (u_{\max})^2 \{1 - (r^4/a^4)\}]\, dr}{a^2}$$

$$= \frac{2[T_0 \times (r^2/2) + (\mu/4k) \times (u_{\max})^2 \times (r^2/2) - (\mu/4ka^4) \times (u_{\max})^2 \times (r^6/6)]_0^a}{a^2}$$

$$= \frac{a^2[T_0 + (\mu/4k) \times (u_{\max})^2 - (\mu/12k) \times (u_{\max})^2]}{a^2} = T_0 + \frac{\mu\,(u_{\max})^2}{6k}.$$

$$\therefore \qquad T_{mean} - T_0 = (\mu/6k) \times (u_{\max})^2. \qquad ...(24)$$

The Nusselt number N_u for the present case is given by

$$N_u = -\frac{2a}{T_{mean} - T_0}\left(\frac{dT}{dr}\right)_{r=a}. \qquad ...(25)$$

From (21), $dT/dr = (\mu/4k) \times (u_{\max})^2 \times (-4r^3/a^4)$.

Hence, $(dT/dr)_{r=a} = (\mu/4k) \times (u_{\max})^2 \times (-4/a) = -(\mu/ka) \times (u_{\max})^2$ ...(26)

Using (24) and (26),(25) reduces to

$$\therefore \qquad N_u = -\frac{2a}{(\mu/6k) \times (u_{\max})^2} \times \left[-\frac{\mu}{ka}(u_{\max})^2\right] = 12.$$

Case II. when the wall of the pipe is kept at a constatu temperature gradient.
[Himanchal 2001]

Let the wall of the pipe be kept at a constant temperature gradient A.

Since the conditions will be similar at each section of the pipe, we propose to find the solution of (16) in the form $$T = Az + f(r). \qquad \ldots(27)$$

Neglecting the last term in (16), *i.e.*, the heat due to dissipation, the energy equation for the present case is
$$\rho C_v u \frac{\partial T}{\partial z} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right). \qquad \ldots(28)$$

Substituting the values of u and T from (15) and (27) in (28), we gett
$$k\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right) = \rho C_v A\, u_{\max}\left\{1 - \left(\frac{r}{a}\right)^2\right\}$$

or $$r\frac{d^2 f}{dr^2} + \frac{df}{dr} = \frac{\rho C_v A u_{\max}}{k}\left[r - \frac{r^3}{a^2}\right] \quad \text{or} \quad \frac{d}{dr}\left(r\frac{df}{dr}\right) = \frac{\rho C_v A u_{\max}}{k}\left(r - \frac{r^3}{a^2}\right).$$

Integrating, $$r\frac{df}{dr} = \frac{\rho C_v A u_{\max}}{k}\left(\frac{r^2}{2} - \frac{r^4}{4a^2}\right) + C_1 \quad \text{or} \quad \frac{df}{dr} = \frac{\rho C_v A u_{\max}}{k}\left(\frac{r}{2} - \frac{r^3}{4a^2}\right) + \frac{C_1}{r}.$$

Integrating, $$f = \frac{\rho C_v A u_{\max}}{k}\left(\frac{r^2}{4} - \frac{r^4}{16a^2}\right) + C_1 \log r + C_2, \qquad \ldots(29)$$

where C_1 and C_2 are arbitrary constants of integration.

We wish to find required solution subject to the following two conditions:
$$T = \text{finite} \quad \text{when} \quad r = 0\,; \quad \text{and} \quad f(r) = 0 \quad \text{when} \quad r = a. \qquad \ldots(30)$$

Since T = finite when $r = 0$, hence (27) $\Rightarrow$ $f(r)$ = finite when $r = 0$. Then (29) shows that we must take $C_1 = 0$.

Again,since $f(r) = 0$ when $r = a$, (29) reduces to
$$0 = \frac{\rho C_v A u_{\max}}{k}\left(\frac{a^2}{4} - \frac{a^2}{16}\right) + C_2 \quad \text{so that} \quad C_2 = -\frac{3a^2 \rho C_v A u_{\max}}{16k}.$$

Substituting the above values of C_1 and C_2 in (29), we get
$$f(r) = \frac{\rho C_v A u_{\max}}{k}\left(\frac{r^2}{4} - \frac{r^4}{16a^2} - \frac{3a^2}{16}\right)$$

or $$f(r) = -\frac{\rho C_v A u_{\max} a^2}{k}\left\{\frac{3}{16} - \frac{1}{4}\left(\frac{r}{a}\right)^2 + \frac{1}{16}\left(\frac{r}{a}\right)^4\right\}. \qquad \ldots(31)$$

Substituting the above value of $f(r)$ in (27), we get
$$T = Az - \frac{\rho C_v A\, u_{\max} a^2}{k}\left\{\frac{3}{16} - \frac{1}{4}\left(\frac{r}{a}\right)^2 + \frac{1}{16}\left(\frac{r}{a}\right)^4\right\} \qquad \ldots(32)$$

The maximum temperature T_m exists on the axis of the pipe (where $r = 0$) and hence it is given by putting $r = 0$ and $T = T_m$ in (32).

$$\therefore \qquad T_m = Az - (3/16k) \times \rho C_v A u_{\max} a^2. \qquad \ldots(33)$$

We now determine mean temperatures T_{mean} and T_{Mean}, namely, the unweighted mean and mean weighted with respect to the velocity *i.e.*, the temperature which is measured in fluid which is mixed after passing through the pipe, respectively as follows,

$$T_{mean} = \frac{\int_0^a (T \times 2\pi r)\, dr}{\pi a^2} = \frac{2\int_0^a r\left[Az - \frac{\rho C_u A u_{\max} a^2}{k}\left\{\frac{3}{16} - \frac{1}{4}\left(\frac{r}{a}\right)^2 + \frac{1}{16}\left(\frac{r}{a}\right)^4\right\}\right] dr}{a^2} \text{ ,using (32)}$$

$$= \frac{2}{a^2}\left[Az\frac{r^2}{2} - \frac{\rho C_v A u_{\max} a^2}{k}\left\{\frac{3}{32}r^2 - \frac{r^4}{16a^2} + \frac{r^6}{96a^4}\right\}\right]_0^a = \frac{2}{a^2}\left[\frac{Aza^2}{2} - \frac{\rho C_v A u_{\max} a^2}{k}\left(\frac{3a^2}{32} - \frac{a^2}{16} + \frac{a^2}{96}\right)\right]$$

$\therefore$
$$T_{mean} = Az - (1/12k) \times \rho C_v A a^2 u_{\max}. \qquad \ldots(34)$$

and
$$T_{Mean} = \frac{\int_0^a (Tu \times 2\pi r)\, dr}{\int_0^a (u \times 2\pi r)\, dr} = Az - \frac{11}{96} \times \frac{\rho C_v A a^2 u_{\max}}{k} \qquad \ldots(35)$$

[Using (15) and (32) and simplifying as before]

The Nusselt number based on T_{mean} is given by $N_u = -\frac{2a}{T_{mean} - T_w}\left(\frac{dT}{dr}\right)_{r=a}$

Here $T_w = Az$. Using (23) and (34), $N_u = 6$.

The Nusselt number based on T_{Mean} is given by $N_u = -\frac{2a}{T_{Mean} - T_w}\left(\frac{dT}{dr}\right)_{r=a}$

Here $T_w = Az$. Using (23) and (35), $N_u = 48/11$.

16.15 Temperature distribution in steady laminar flow of an incompressible fluid between two concentric rotating cylinders. Couette flow.

First of all give the entire matter of Art. 16.6 upto equation (8). Thus the velocity distribution for the present caseis given by

$$q_\theta = \frac{1}{r_2^2 - r_1^2}\left\{(\omega_2 r_2^2 - \omega_1 r_1^2)r - \frac{r_1^2 r_2^2}{r}(\omega_2 - \omega_1)\right\}. \qquad \ldots(8)$$

Here $q_r = q_z = 0$ and q_θ is function of r. The energy equation in the cylindrical polar coordinates for the present case is given by (Refer equation (16) of Art. 14.16)

$$0 = \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \mu\left\{r\frac{d}{dr}\left(\frac{q_\theta}{r}\right)\right\}^2. \qquad \ldots(9)$$

We propose to solve (9) subject to the boundary conditions:

$T = T_1$ when $r = r_1$ and $T = T_2$ when $r = r_2$. ...(9)

Substituting the value of q_θ given by (8) in (9), we have

$$0 = \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \mu\left[r\frac{d}{dr}\left\{\frac{1}{r_2^2 - r_1^2}\left((\omega_2 r_2^2 - \omega_1 r_1^2) - \frac{r_1^2 r_2^2}{r^2}(\omega_2 - \omega_1)\right)\right\}\right]^2.$$

or
$$\frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\mu\left[\frac{r}{(r_2^2 - r_1^2)}\left\{0 - \frac{2r_1^2 r_2^2(\omega_2 - \omega_1)}{r^3}\right\}\right]^2$$

or $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)=-\mu\frac{4\,(\omega_2-\omega_1)^2 r_1^4 r_2^4}{k(r_2^2-r_1^2)^2 r^4} \quad \text{or} \quad \frac{d}{dr}\left(r\frac{dT}{dr}\right)=-\frac{4\mu\,(\omega_2-\omega_1)^2 r_1^4 r_2^4}{k(r_2^2-r_1^2)^2}\times\frac{1}{r^3}. \qquad \ldots(11)$$

Integrating (11), $$r\frac{dT}{dr}=\frac{2\mu\,(\omega_2-\omega_1)^2 r_1^4 r_2^4}{k(r_2^2-r_1^2)^2}\times\frac{1}{r^2}+C_1$$

or $$\frac{dT}{dr}=\frac{2\mu\,(\omega_2-\omega_1)^2 r_1^4 r_2^4}{k(r_2^2-r_1^2)^2}\times\frac{1}{r^3}+\frac{C_1}{r}. \qquad \ldots(12)$$

Integrating (12), $$T=-\frac{\mu\,(\omega_2-\omega_1)^2 r_1^4 r_2^4}{k(r_2^2-r_1^2)^2}\times\frac{1}{r^2}+C_1\log r+C_2, \qquad \ldots(13)$$

where C_1 and C_2 are arbitrary constants.

In view of boundery conditions (10), putting $T = T_1$, $r = r_1$ and $T = T_2$, $r = r_2$ by turn in (13), we have $$T_1=-\frac{\mu\,(\omega_2-\omega_1)^2 r_1^4 r_2^4}{(r_2^2-r_1^2)^2}\times\frac{1}{r_1^2}+C_1\log r_1+C_2 \qquad \ldots(14)$$

and $$T_2=-\frac{\mu\,(\omega_2-\omega_1)^2 r_1^4 r_2^4}{(r_2^2-r_1^2)^2}\times\frac{1}{r_2^2}+C_1\log r_2+C_2. \qquad \ldots(15)$$

Solving (14) and (15) for C_1 and C_2, we have

$$C_1=\frac{T_1-T_2}{\log(r_1/r_2)}+\frac{\mu\,(\omega_2-\omega_1)^2 r_1^2 r_2^2}{k(r_2^2-r_1^2)\log(r_1/r_2)}$$

and $$C_2=T_1+\frac{\mu\,(\omega_2-\omega_1)^2 r_1^4 r_2^4}{k(r_2^2-r_1^2)r_1^2}-\frac{(T_1-T_2)\log r_1}{\log(r_1/r_2)}-\frac{\mu\,(\omega_2-\omega_1)^2 r_1^2 r_2^2\log r_1}{k\,(r_2^2-r_1^2)\log(r_1/r_2)}.$$

Substituing the above values of C_1 and C_2 in (13),we get

$$\frac{T-T_1}{T_2-T_1}=N\frac{(r^2-r_1^2)r_2^2}{(r_2^2-r_1^2)r^2}+(1-N)\frac{\log(r/r_1)}{\log(r_2/r)}, \qquad \ldots(16)$$

where $$N=\frac{\mu\,(\omega_2-\omega_1)^2 r_1^2 r_2^2}{k\,(r_2^2-r_1^2)\,(T_2-T_1)},$$ which is a non-dimensional parameter.

EXERCISES

1. Discuss the temperature distribution in the plane Couette flow when the moving plate is at a higher temperature than the stationary plate. **[Himanchal 1999, 2002]**

2. (*a*) Discuss the temperature distribution in the Hagen-Poiseuille flow in a circular pipe when the wall of the pipe is kept at a constant temperature gradient. **[H imanchal 2003]**

(*b*) Derive temperature distribution of steady incompressible fluid in a circular pipe when (*i*) wall is at constant temperature and (*ii*) wall is at uniform temperature gradient. **[Himanchal 2001]**

(*c*) Derive the temperature distribution of Hagen-Poiseuille flow taking wall at constant temperature. Also find the Nusselt number at the wall. **[Himanchal 2000]**

TYPE 3. FLOW OF TWO IMMISCIBLE FLUIDS

16.16. Flow of two immiscible viscous fluids between two parallel plates

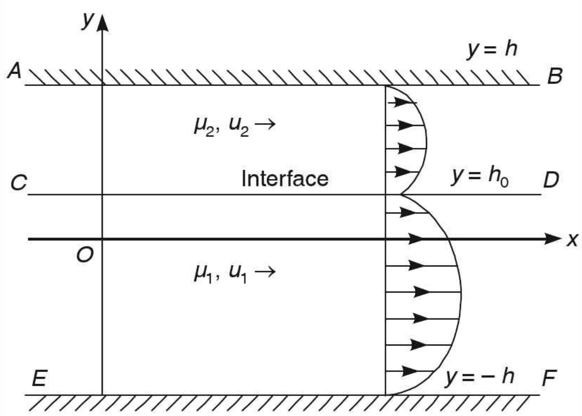
Consider the flow of two immiscible fluids between two parallel fixed horizontal plates under a constant presents gradient $P\,(=-dp/dx)$.

Let the fluid with coefficient of viscosity μ_1 extend from y $= -h$ to $y = h_0$ and fluid with coefficient of viscosity μ_2 extend from $y = h_0$ to $y = h$.

Assuming the fluids to be of constant densities and the flow to be steady uni-directional and depending on y alone, the Navier-Stokes equation in x - dinection is given by

$$0 = -(dp/dx) + \mu(du^2/dy^2) \qquad \text{or} \qquad d^2u/dy^2 = -(P/\mu) \quad \ldots(1)$$

Integrating (1), $\quad du/dy = -\ (P/\mu)y + c_1, c_1,$ being an arbitrary constant $\quad \ldots(2)$



Integrating (2), $\quad u = -(P/2\mu)y^2 + c_1y + c_2, c_2$ being an arbitrary constant. $\quad \ldots(3)$

Let u_1 be the velocity in the region $-\ h < y < h_0$ and u_2 be the velocity in the region $h_0 < y < h$. Then, from (3), we get

$$u_1 = -(P/2\mu_1)y^2 + Ay + B, \qquad \text{where} \qquad -h < y < h_0 \qquad \ldots(4)$$

$$u_2 = -(P/2\mu_2)y^2 + Cy + D, \qquad \text{where} \qquad h_0 < y < h \qquad \ldots(5)$$

where the arbitrary constants A, B, C and D are to be obtained from the boundary and interface conditions. The boundary conditions for u_1 and u_2 *are* given by

$$u_1 = 0, \qquad \text{where} \qquad y = -h \qquad \ldots(6)$$

and $$u_2 = 0, \qquad \text{where} \qquad y = h \qquad \ldots(7)$$

Since both the fluids are taken to be viscous, the fluids cannot slip over each other and consequently the velocity must be continuous at the interface. Thus,

$$u_1 = u_2, \qquad \text{at} \qquad y = h_0 \qquad \ldots(8)$$

By balancing the forces on a fluid element partly in the first fluid and partly in the other, it follows that the shearing stress must be continuous at the interface. Thus, we have

$$\mu_1(du_1/dy) = \mu_2(du_2/dy), \qquad \text{at} \qquad y = h_0 \qquad \ldots(9)$$

In view of (6), putting $u_1 = 0$ and $y = -\ h$ in (4), we have

$$0 = -\ (P/2\mu_1) \times h^2 - Ah + B \qquad \ldots(10)$$

Sultracting (10) from (4), we have

$$u_1 = -(P/2\mu_1) \times (y^2 - h^2) + A(y + h), \qquad \text{where} \ -h < y < h_0 \qquad \ldots(11)$$

In view of (7), putting $u_2 = 0$ and $y = h$ in (5) , we have

$$0 = -(P/2\mu_2) \times h^2 + Ch + D \qquad \ldots(12)$$

Subtracting (12) from (5), we have

$$u_2 = -(P/2\mu_2) \times (y^2 - h^2) + C(y - h), \quad \text{where} \quad h_0 < y < h \qquad \ldots(13)$$

Putting $y = d_0$ in (11) and (13) and equating the values so obtained by virtue of (8), we get

$$-(P/2\mu_1)\times(h_0^2-h^2)+A(h_0+h)=-(P/2\mu_2)\times(h_0^2-h^2)+C(h_0-h)$$

or $$C\,(h_0-h)-A(h_0+h)=\{P(h^2-h_0^2)/2\}\times(1/\mu_1-1/\mu_2) \qquad \ldots(14)$$

From (11) and (13), we have

$$du_1/dy=-(P/2\mu_1)\times 2y+A, \qquad \text{where} \qquad -h<y<h_0 \qquad \ldots(15)$$

and $$du_2/dy=-(P/2\mu_2)\times 2y+C, \qquad \text{where} \qquad h_0<y<h \qquad \ldots(16)$$

Putting $y = h_0$ in (15) and (16) and substituting the values so obtained in condition (9), we obtain

$$\mu_1\{-(P/2\mu_1)\times 2h_0+A\}=\mu_2\{-(P/2\mu)\times 2h_0+C\} \qquad \text{so that} \qquad \mu_1 A=\mu_2 C \quad \ldots(17)$$

Eliminating C from (14),and (17), we get

$$(\mu_1/\mu_2)\times A(h_0-h)-A(h_0+h)=\{P(h^2-h_0^2)/2\}\times(1/\mu_1-1/\mu_2)$$

ro $$\frac{A\mu_1(h_0-h)-A\mu_2(h_0+h)}{\mu_2}=\frac{P\,(h^2-h_0^2)}{2}\times\frac{\mu_2-\mu_1}{\mu_1\mu_2}$$

or $$A\{-h_0\,(\mu_2-\mu_1)-h\,(\mu_1+\mu_2)\}=\frac{P\,(h^2-h_0^2)}{2}\times\frac{\mu_2-\mu_1}{\mu_1}$$

or $$-A\left\{h+h_0\frac{\mu_2-\mu_1}{\mu_2+\mu_2}\right\}=\frac{P(h^2-h_0^2)}{2}\times\frac{\mu_2-\mu_1}{\mu_2\times\mu_1}$$

or $$A=-\{(P/2\mu_1)\times(h^2-h_0^2)\alpha\}/(h+h_0\alpha), \qquad \ldots(18)$$

where $$\alpha=(\mu_2-\mu_1)/(\mu_2-\mu_1) \qquad \ldots(19)$$

Substituting the value of A given by (18) in (11), we have

$$u_1=-\frac{P}{2\mu_1}\left\{y^2-h^2+\frac{\alpha\,(h^2-h_0^2)\,(y+h)}{h+\alpha\,h_0}\right\}, \quad \text{where} \quad -h<y<h_0 \qquad \ldots(20)$$

Substituting the value of A given by (18) in (17), $$C=-\frac{P}{2\mu_2}\times\frac{(h^2-h_0^2)\alpha}{h+h_0\alpha} \qquad \ldots(21)$$

Substituting the above value of C in (13), we have

$$u_2=-\frac{P}{2\mu_2}\left\{y^2-h^2+\frac{\alpha(h^2-h_0^2)\,(y-h)}{h+\alpha\,h_0}\right\}, \quad \text{where} \quad h_0<y<h \qquad \ldots(22)$$

The required velocity is given by (20) and (22), The velocity distribution is plotted in figure on page 16.42. Observe that at the interface CD the slope of the velocity profile is discontinuous. This is so due to change in the coefficents of viscosty of fluids on the two side of the interface.

The flux Q is given by $$Q=\int_{-h}^{h_0}u_1\,dy+\int_{h_0}^{h}u_2\,dy \qquad \ldots(23)$$

Substituting the values of u_1 and u_2 given by (20) and (22), respectively in (23) and simplifying, we have

$$Q=\frac{P(\mu_1+\mu_2)}{6\mu_1\mu_2}\{2h^3-\alpha(h_0^3-2h^2h_0)-\beta(2h\,h_0+\alpha h^2+\alpha h_0^2\}, \qquad \ldots(24)$$

where $$\beta=3\alpha(h^2-h_0^2)/2(h+\alpha h_0) \qquad \ldots(25)$$

Particular case I: If $\mu_1 = \mu_2 = \mu$ (say),then (19) and(25),yield $\alpha = \beta = 0$. Hence (24) reduces to

$$Q = (2Ph^3)/3\mu,$$

which is the flux obtained in the case of Poiseuille flow.

Particular Case II If we assume that the whole space is filled with fluid of coefficient of viscosity μ_1 so that $h_1 = h$. Then we get $Q = (2Ph^3)/3\mu_1$.

Similarly, if we take $h_0 = -h$, we get $Q = (2Ph^3)/3\mu_2$

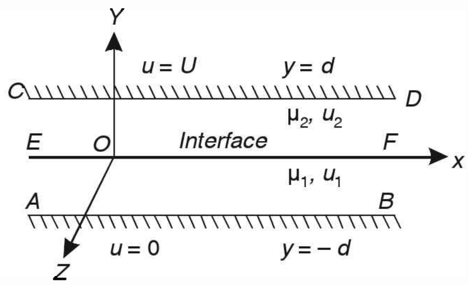
An illustrative Solved Example

Ex. 1. *Two fluids of coefficient of viscosities* μ_1 *and* μ_2 *confined in region* $-d < y < 0$ *and* $0 < y < d$ *respectively, are flowing between two parallel plates under a constant pressure gradient* $P\ (= -\partial p/\partial x)$. *Show that when the plate at* $y = d$ *is moving with constant velocity U, then the velocity distribution is given by*

$$u = \frac{P}{2\mu_1}(d^2 - y^2) + \frac{\varepsilon\mu_2 U}{(\mu_1+\mu_2)d}(y+d), -d < y < 0$$

$$u = U + \frac{P}{2\mu_2}(d^2 - y^2) + \frac{\varepsilon\mu_1 U}{(\mu_1+\mu_2)d}(y-d), 0 < y < d, \text{ where } \varepsilon = 1 + \frac{1}{2}\frac{Pd^2}{\mu_1\mu_2 U}(\mu_1 - \mu_2).$$ **[Meerut 1997]**

Sol. Here we consider the flow of two immiscible fluids between two parallel plates *AB* ($y = -d$) at rest and *CD* ($y = d$) *moving* with constant velocity U under a constant pressure gradient $P(= -\partial p/\partial x)$.



Let x be the direction of flow, y the direction perpendicular to the flow, and the width of the plates parallel to the z-direction. Here the word infinite implies that the width of the plates is large compared with $2d$ and the flow may be treated as two-dimensional (*i.e.*, $\partial/\partial z = 0$). Let the plates be long enough in the x-direction for the flow to be parallel. Here we take the velocity components v and w to be zero everywhere. Moreover the flow being steady, variables are independent of time $(\partial/\partial t = 0)$. Furthermore the equation of continuity (namely $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$) reduces to $\partial u/\partial x = 0$ so that $u = u(y)$. Thus for the present problem, we have

$$u = u(y), \qquad v = 0, \qquad w = 0, \qquad \partial/\partial z = 0, \qquad \partial/\partial t = 0 \qquad \ldots(1)$$

For the present two dimensional flow in absence of body forces, the Nevier-Stokes equations for x and y directions are given by

$$0 = -\partial p/\partial x + \mu\,(\partial^2 u/\partial y^2) \qquad \ldots(2)$$

and
$$0 = -\partial p/\partial y. \qquad \ldots(3)$$

Equation (3) shows that the pressure does not depend on y. Hence p is function of x alone and so ((2) reduces to $d^2u/dy^2 = (1/\mu) \times (dp/dx)$.

Since $P = -\,dp/dx$, the above equation reduces to $d^2u/dy^2 = -(P/\mu)$. ...(4)

Integrating (4), $du/dy = -(P/\mu)\,y + C_1$, C_1 being an arbitrary constant. ...(5)

Integrating (5), $u = -(P/2\mu)y^2 + C_1 y + C_2$. C_2 being an arbitrary constant. ...(6)

Let u_1 be the velocity in the region $-d < y < 0$ and u_2 be the velocity in the region $0 < y < d$, then (6) gives

$$u_1 = -(P/2\mu_1)y^2 + Ay + B, \qquad \text{when } -d < y < 0 \qquad \ldots(7)$$

and $$u_2 = -(P/2\mu_2)y^2 + Cy + D, \qquad \text{when } 0 < y < d \qquad \ldots(8)$$

Here A, B, D and D are arbitrary constants. We now proceed to calculate these constant with help of boundary conditions and interface conditions.

Now, boundary conditions for u_1 and u_2 are given by

$$u_1 = 0, \qquad \text{when} \qquad y = -d \qquad \ldots(9)$$

and $$u_2 = U, \qquad \text{when} \qquad y = d. \qquad \ldots(10)$$

Putting $u_1 = 0$ and $y = -d$ in (7), we have

$$0 = -(P/2\mu_1)d^2 - Ad + B, \qquad \ldots(11)$$

Subtracting (11)from (7), we obtain

$$u_1 = -(P/2\mu_1)(y^2 - d^2) + A(y + d), \qquad \text{when} \quad -d < y < 0. \qquad \ldots(12)$$

Next, putting $u_2 = U$ and $y = d$ in (8), we have

$$U = -(P/2\mu_2)d^2 + Cd + D. \qquad \ldots(13)$$

Subtracting (13) from (8), we obtain

$$u_2 - U = -(P/2\mu_2) \times (y^2 - d^2) + C(y - d), \qquad \text{when} \qquad 0 < y < d. \qquad \ldots(14)$$

Since both the fluids are taken to be viscous, the fluids cannot slip over each other and hence the velocity has to be continues at the interface EF

Thus, $$u_1 = u_2 \qquad \text{at} \qquad y = 0. \qquad \ldots(15)$$

Putting $y = 0$ in (12) and (14) and equating the values so obtained by virtue of (15) gives

$$(P/2\mu_1) \times d^2 + Ad = (P/2\mu_2) \times d^2 - Cd + U \qquad \text{or} \qquad d(A + C) = (Pd^2/2) \times (1/\mu_2 - 1/\mu_1) + U$$

or $$A + C = \frac{U}{d} + \frac{Pd}{2}\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right). \qquad \ldots(16)$$

Further, by balancing the forces on a fluid element partly lying in one fluid and partly in the other, we find that the shearing stress has to be continuous at the interface EF. Thus,

$$\mu_1(du_1/dy) = \mu_2(du_2/dy) \qquad \text{at} \qquad y = 0. \qquad \ldots(17)$$

From (12), $$du_1/dy = -(Py/\mu_1) + A.$$

$\therefore$ When $y = 0$, $$du_1/dy = A. \qquad \ldots(18)$$

Now from(14), $$du_2/dy = -(Py/\mu_2) + C.$$

$\therefore$ When $y = 0$, $$du_2/dy = C. \qquad \ldots(19)$$

Using (18) and (19), $\quad$ (17) $\Rightarrow \quad \mu_1 A = \mu_2 C \quad \Rightarrow \quad A/\mu_2 = C/\mu_1$

$\therefore$ $$\frac{A}{\mu_2} = \frac{C}{\mu_1} = \frac{A + C}{\mu_2 + \mu_1} = \frac{1}{\mu_2 + \mu_1}\left\{\frac{U}{d} + \frac{Pd}{2}\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right)\right\}, \text{ using (16)}$$

$\Rightarrow$ $$\frac{A}{\mu_2} = \frac{C}{\mu_1} = \frac{U}{d(\mu_2 + \mu_1)} + \frac{Pd(\mu_1 - \mu_2)}{2(\mu_2 + \mu_1)\mu_1\mu_2}$$

$$\Rightarrow \qquad A = \frac{U\mu_2}{d\,(\mu_2+\mu_1)} + \frac{Pd\,(\mu_1-\mu_2)}{2\mu_1\,(\mu_1+\mu_2)}, \qquad \text{and} \qquad C = \frac{U\mu_1}{d\,(\mu_2+\mu_1)} + \frac{Pd\,(\mu_1-\mu_2)}{2\mu_2\,(\mu_1+\mu_2)}$$

Substituing these values of A and C in (12) and (14), the required velociy distribution is

$$u_1 = -\frac{P}{2\mu_1}(y^2-d^2) + (y+d)\left\{\frac{U\mu_2}{d\,(\mu_1+\mu_2)} + \frac{Pd\,(\mu_1-\mu_2)}{2\mu_1(\mu_1+\mu_2)}\right\}, \quad \text{when } -d<y<0$$

and
$$u_2 = U - \frac{P}{2\mu_2}(y^2-d^2) + (y-d)\left\{\frac{U\mu_1}{d\,(\mu_1+\mu_2)} + \frac{Pd\,(\mu_1-\mu_2)}{2\mu_2(\mu_1+\mu_2)}\right\}, \quad \text{when } 0<y<d$$

Rewriting the above results, we have

$$u_1 = \frac{P}{2\mu_1}(d^2-y^2) + \left[1 + \frac{Pd^2\,(\mu_1-\mu_2)}{2\mu_1\mu_2 U}\right]\frac{U\mu_2\,(y+d)}{d\,(\mu_1+\mu_2)} \qquad \ldots(20)$$

and
$$u_2 = U + \frac{P}{2\mu_2}(d^2-y^2) + \left[1 + \frac{Pd^2\,(\mu_1-\mu_2)}{2\mu_1\mu_2 U}\right]\frac{U\mu_1\,(y-d)}{d\,(\mu_1+\mu_2)} \qquad \ldots(21)$$

Given that $\varepsilon = 1 + \{Pd^2(\mu_1-\mu_2)\}/(2\mu_1\mu_2 U)$. Hence the required velociy distribution given by (20) and (21) can be re-written as

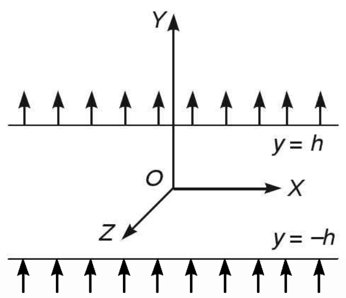
$$u = \frac{P}{2\mu_1}(d^2-y^2) + \frac{\varepsilon\mu_2 U}{(\mu_1+\mu_2)d}(y+d), \qquad -d<y<0$$

$$= U + \frac{P}{2\mu_2}(d^2-y^2) + \frac{\varepsilon\mu_1 U}{(\mu_1+\mu_2)d}(y-d), \qquad 0<y<d.$$

TYPE 4: STEADY INCOMPRESSIBLE FLUID FLOW. SUCTION/INJECTION ON THE BOUNDARIES

Modern investigation in the field of fluid dynamics are characterized by a very close relation between theory and experiment. Recently many research papers have been published dealing with the flow of an incompressible viscous fluid through porous channels and pipes. Such problems are known as *transpiration cooling* which is very effective process in reducing the heat transfer between the fluid and the boundary layer. This has enabled us to understand the problem of cooling rocket and jet. In this technique, a flow in the direction, perpendicular to the main direction of flow is created by suction or injection of the fluid at the boundaries.

16.17 A. Steady flow of viscous incompressible fluid between two porous parallel plates.



Consider the steady laminar flow of viscous incompressible fluid between two infinite parallel porous plates separated by a distance $2h$ as shown in the adjoining figure. By porous plates we mean that plates possess very fine holes distributed uniformly over the entire surface of the plates through which fluid can flow freely and continuously. The plate from which the fluid enters the flow region is known as the *plate with injection* and the plate from which the fluid leaves the flow region is known as the *plate with suction*. Let x be the direction of the main flow, y the direction perpendicular to the flow , and the width of the plates parallel to the z-direction. We take the velocity component w to be zero everywhere and u as function of y alone. Then the equation of continuity reduces to $\partial v/\partial y = 0$ so that v does not vary with y. This implies that the fluid enters the flow region through one plate (say, the plate situated at $y = -h$) at the same constant velocity v_0, say and it leaves through the other plate (*i.e.*, the plate situated at $y = h$) as shown in the figure. Hence there is constant velocity component v_0 along y –direction.

For the present steady flow in the absence of body forces, the Navier Stokes equations for x – and y – directions (Refer (14a) and (14b) in Art. 14.1 of chapter 14) are given by

$$v_0 \frac{du}{dy} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{d^2 u}{dy^2} \qquad \ldots(1)$$

and

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}. \qquad \ldots(2)$$

(2) shows that the pressure does no depend on y. *Hence p must be function of x alone and* so (1) reduces to

$$\frac{dp}{dx} = \rho\left[\nu \frac{d^2 u}{dy^2} - v_0 \frac{du}{dy}\right]. \qquad \ldots(3)$$

Differentiating (3) $w.r.t.$'x', we have $\dfrac{d^2 p}{dx^2} = 0$ or $\dfrac{d}{dx}\left(\dfrac{dp}{dx}\right) = 0.$

Integrating, $dp/dx = \text{constant} = -P$ (say), $\qquad \ldots(4)$

where the minus sign has been taken as we expect p to decrease as x increases. Then (3) gives

$$\frac{d^2 u}{dy^2} - \frac{v_0}{\nu}\frac{du}{dy} = -\frac{P}{\rho\nu}. \qquad \ldots(5)$$

Integrating (5), we have $\dfrac{du}{dy} - \dfrac{v_0}{\nu}u = A - \dfrac{Py}{\rho\nu}$, $\qquad \ldots(6)$

which is linear differential equation of first order and first degree.

Integrating factor of (6) $= e^{-\int (v_0/\nu)dy} = e^{-(v_0 y/\nu)}$. Hence the solution of (6) is given by

$$ue^{-(v_0 y/\nu)} = \int\left(A - \frac{Py}{\rho\nu}\right)e^{-(v_0 y/\nu)}dy + B = \left(A - \frac{Py}{\rho\nu}\right)\left(-\frac{\nu}{v_0}e^{-(v_0 y/\nu)}\right) - \left(-\frac{P}{\rho\nu}\right)\left(\frac{\nu^2}{v_0^2}e^{-(v_0 y/\nu)}\right) + B$$

$$\therefore \quad u = -\frac{\nu}{v_0}\left(A - \frac{Py}{\rho\nu}\right) + \frac{P\nu}{\rho v_0^2} + Be^{(v_0 y/\nu)} \quad \text{or} \quad u = C + \frac{P}{\rho v_0}y + Be^{(v_0 y/\nu)} \qquad \ldots(7)$$

where C = – $(A\nu/v_0)$. Here B and C are arbitrary constants to be determined. Let the plate situated at y = –h be at rest and the plate at y = h be moving with a constant velocity U. Then, B and C will be determined from the boundary conditions :

$u = 0$ at $y = -h$ and $u = U$ at $y = h$ $\qquad \ldots(8)$

Using (8), (7) gives

$$0 = C - \frac{Ph}{\rho u_0} + Be^{-(v_0 h/\nu)} \quad \text{and} \quad U = C + \frac{Ph}{\rho u_0} + Be^{(v_0 h/\nu)} \qquad \ldots(9)$$

Solving (9) for B and C and substituting the values so obtained in (7), we have

$$u = \left(U - \frac{2Ph}{\rho v_0}\right)\frac{e^{(v_0 y/\nu) - e^{-(v_0 h/\nu)}}}{2\sinh(v_0 h/\nu)} + \frac{P}{\rho v_0}(y+h) \qquad \ldots(10)$$

Let $\text{Re} = (v_0 h)/\nu$ and $\eta = y/h.$ $\qquad \ldots(11)$

Then (10) reduces to

$$u = \left(U - \frac{2Ph^2}{\mu \text{Re}}\right)\frac{e^{\eta \text{Re}} - e^{-\text{Re}}}{2\sinh \text{Re}} + \frac{Ph^2}{\mu \text{Re}}(1+\eta), \qquad \ldots(12)$$

which gives the velocity distribution in terms of non-dimensional quantities Re (Reynold;s number) and η. Notice that plates are situated at $\eta = \pm 1$.

We now consider two particlar cases.

Case I. Plane Couette Flow. In this case there is no pressure gradient *i.e.*, $P = 0$. Then (13) reduces to

$$u = (1/2) \times U\,(e^{\eta \mathrm{Re}} - e^{-\mathrm{Re}})\ \mathrm{cosec}h\ \mathrm{Re} \qquad \ldots(13)$$

The shearing stress at any point is given by $$\sigma_{yx} = \mu \frac{du}{dy} = \frac{\mu\,\mathrm{Re}\,U e^{\eta \mathrm{Re}}}{2h \sinh \mathrm{Re}}. \qquad \ldots(14)$$

Hence the skin friction at the plates $\eta = \pm 1$ are given by

$$[\sigma_{yx}]_{\eta=1} = \frac{\mu\,\mathrm{Re}\,U}{2h} \times \frac{e^{\mathrm{Re}}}{\sinh \mathrm{Re}} \qquad \ldots(15)$$

and $$[\sigma_{yx}]_{\eta=-1} = \frac{\mu\,\mathrm{Re}\,\mathrm{U}}{2h} \times \frac{e^{-\mathrm{Re}}}{\sinh \mathrm{Re}}. \qquad \ldots(16)$$

Case II. Plane Poiseuille flow. In this case both the plates are taken at rest. Hence the velocity distribution may be deduced from (12) by writing $U = 0$.Thus we obtain

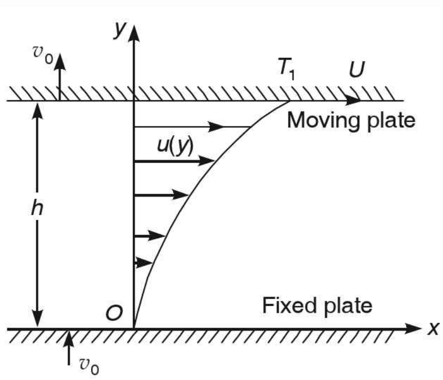
$$u = \frac{Ph^2}{\mu\,\mathrm{Re}} \left(1 + \eta - \frac{e^{\eta \mathrm{Re}} - e^{-\mathrm{Re}}}{2 \sinh Re} \right). \qquad \ldots(17)$$

It can be easily established that the maximum velocity occurs when

$$\eta = \frac{1}{\mathrm{Re}} \log \frac{\sinh Re}{\mathrm{Re}} \qquad \ldots(18)$$

and the skin friction is given by $$\sigma_{yx} = -\frac{Ph^2}{\mathrm{Re}} \left(\frac{\mathrm{Re}\, e^{\eta \mathrm{Re}}}{\sinh Re} - 1 \right). \qquad \ldots(19)$$

16.17 B. Plane Couette flow with transpiration cooling [Himanchal 2003, 06, 07]



Consider a two dimensional steady laminar flow of a viscous incompressible fluid between two parallel plates, one in uniform motion and the other at rest with uniform injection of the same fluid at the fixed plate and a corresponding uniform suction at the moving plate as shown in the adjoining figure.

Let the x-axis be taken along the fixed plate and y-axis in the direction perpendicular to the flow. Let the plates be long enough in the x-direction for the flow to be parallel. For the given problem, the pressure gradient is taken to be zero and all the variables of the fluid motion are treated as function of y alone.

For the problem under consideration, the equation of continuity and the equation of motion, in the absence of body forces, respectively reduce to

$$dv/dy = 0 \qquad \ldots(1)$$

and $$\rho v\,(du/dy) = \mu\,(d^2u/dy^2) \qquad \ldots(2)$$

The boundary conditions are:

At $y = 0$; $u = 0$, $v = v_0$; At $y = h$; $u = U$, $v = v_0$ ...(3)

where h is the distance between the plates. U is the velocity of the upper moving plate and v_0 is the uniform injection and suction velocity at the lower and upper plates respectively.

Integrating (11), we get $\quad v = \text{constant} = v_0 \quad$ ...(4)

Using (4),(2) may be re-written as

$$\rho v_0 \frac{du}{dy} = \mu \frac{d^2u}{dy^2} \quad \text{or} \quad \frac{d^2u/dy^2}{du/dy} = \frac{v_0}{\nu}, \quad \text{where} \quad \frac{\mu}{\rho} = \nu \qquad ...(5)$$

Integrating (5), $\quad \log(du/dy) - \log A = \; v_0 y/\nu$, A being an arbitrary constant

or $$du/dy = Ae^{v_0 y/\nu} \qquad ...(6)$$

Integrating (6), $\quad u = (A\nu/v_0) \times e^{v_0 y/\nu} + B$, being an arbitray constant ...(7)

Using the boundary condition (3), (7) gives

$$0 = (A\nu/v_0) + B \quad \text{and} \quad U = (A\nu/v_0) \times e^{v_0 h/\nu} + B$$

Solving these, $\quad A = (Uv_0/\nu) \times (e^{v_0 h/\nu} - 1)^{-1} \quad$ and $\quad B = -U(e^{v_0 h/\nu} - 1)^{-1}$

Substituting the above values in (7), we have

$$u = \frac{Ue^{v_0 y/\nu}}{e^{v_0 h/\nu} - 1} - \frac{U}{e^{v_0 h/\nu} - 1} \quad \text{or} \quad \frac{u}{U} = \frac{e^{v_0 y/\nu} - 1}{e^{v_0 h/\nu} - 1} \qquad ...(8)$$

We now define the following non-dimensional quantities :

$$\eta = y/h \quad \text{and} \quad \lambda = v_0 h/2 \quad \text{(Injection perameter)} \qquad ...(9)$$

Using (9), the above equation (8) for the velocity distribution in the dimensionless form is

$$\mu/U = (e^{\lambda\eta} - 1)/(e^{\lambda} - 1) \qquad ...(10)$$

showing that the velocity ceases to be linear and it decreases with increasing value of λ.

Determination of temperature distribution. Let the fixed and the moving plates posses temperatures T_0 and T_1 respectively. Since the temperatures of the plates are constant, the temperature distribution in the fluid will be function of y only. Assume that C_p, p and ρ are constants. Then the energy equation in the present case reduces to

$$\rho C_p v_0 (dT/dy) = k\,(d^2T/dy^2) + \mu(d\mu/dy)^2 \qquad ...(11)$$

The boundary conditions are:

At $\quad y = 0\,;\quad T = T_0 \quad$ and at $\quad y = h\,;\quad T = T_1 \quad$ ...(12)

In order to reduce (11) in non-dimensionless form, we introduce the following additional four dimensionless quantities:

E_c = Eckert number = $U^2/C_p(T_1 - T_0)$, $\qquad P_r = \mu C_p/k$ = Prandtl number

P'_e = Peclet number = $\lambda P_r \qquad$ and $\qquad T^* = (T - T_0)/(T_1 - T_0)$

Using the above dimensionless quantities, (9) and (10), the energy equation (11), in the dimensionless form reduces to

$$\frac{d^2T^*}{d\eta^2} - P'_e \frac{dT^*}{d\eta} = -E_c P'_e \frac{e^{2\lambda\eta}}{(e^{\lambda} - 1)^2}$$

or $$(D^2 - P'_e D)T^* = -Ec\; P'_e \frac{e^{2\lambda\eta}}{(e^{\lambda} - 1)^2}, \text{ where } D \equiv d/d\eta \qquad ...(13)$$

and the corresponded boundary conditions are : At $\eta = 0,\ T^* = 0;\quad$ at $\eta = 1, T^* = 1$ ...(14)

The auxiliary equation for (13) is $D^2 - P_e' D = 0$

giving $D(D - P_e') = 0$ so that $D = 0,\ P_e'$

$\therefore$ $C.F. = C_1 + C_2\, e^{P_e' \eta}$, C_1 and C_2 being arbitrary constants

and
$$\text{P.I.} = \frac{1}{D(D - P_e')}\left\{-\frac{E_c P_e'}{(e^{\lambda}-1)^2} e^{2\lambda\eta}\right\} = -\frac{E_c P_e'}{2\lambda(2\lambda - P_e')} \times \frac{e^{2\lambda\eta}}{(e^{\lambda}-1)^2}$$

Hence the general solution of (13) is given by

$$T^* = C_1 + C_2 e^{P_e' \eta} - \frac{E_c P_e' e^{2\lambda\eta}}{2\lambda(2\lambda - P_e')(e^{\lambda}-1)^2} \qquad \ldots(15)$$

Using the boundary conditions (14) in (15), we get two equations to compute C_1 *and* C_2 Substituting these values of C_1 and C_2 in (15), we finally arrive at

$$T^* = \frac{E_c P_e'}{(e^{\lambda}-1)^2}\left[\frac{e^{P_e'}\left\{1 - e^{(2\lambda - P_e')\eta}\right\}}{(2\lambda - P_e')} - \frac{P_e'\left\{1 - e^{(2\lambda - P_e')}\right\}}{(2\lambda - P_e')} \times \frac{e^{\eta P_e'} - 1}{e^{P_e'} - 1}\right] + \frac{e^{\eta P_e'} - 1}{e^{P_e'} - 1} \qquad \ldots(16)$$

Suppose we neglect the dissipation term (heat generated due to internal friction), *i.e.*, we take $E_c = 0$, then (16), reduces to $T^* = (e^{\eta P_e'} - 1)/(e^{P_e'} - 1)$ ...(17)

In order to compute the heat transfer at the fixed plate, we compute the dimensionless coefficient of heat transfer, namely, Nusselt number, defined as

$$Nu = -\frac{h}{(T_0 - T_1)}\left(\frac{dT}{dy}\right)_{y=0} \qquad \ldots(18)$$

Re-writting (18) in term of T* and η, we get $Nu = (dT^*/d\eta)_{\eta=0}$ ...(19)

Substituting the value of $dT^*/d\eta$ from (17) in (19), we have

$$Nu = P_e'/(e^{P_e'} - 1) \qquad \ldots(20)$$

When $\lambda = 0$, *i.e.*, $P_e' = 0$, then we see that $Nu = 1$ and Nu goes on decreasing as the value of P_e' increases, which indicates cooling of the fixed plate with the ejection process.

An illustrative solved example. *Incompressible viscous liquid is moving steadily under pressure between planes y = 0, y = h. The plane y = 0 has a constant velocity U in direction of the axis of x and the plane y = h is fixed. The planes are porous and liquid is sucked uniformly over one and ejected uniformly over the other. Show that a possible solution is given by*

$$u = \frac{Ue^{h/a} + Ah - (U + Ah)e^{y/a}}{e^{h/a} - 1} \quad \text{and} \quad v = \frac{\nu}{a},$$

where ν is the coefficient of viscosity. Interpret the constants A and a.

[Agra 2004; Garwhal 2001 Meerut 1999; Rohilkhand 1998]

OR

Incompressible viscous liquid is moving steadily under pressure between planes y = 0, y = h. The plane y = 0 has a constant velocity U in the direction of the axis of x, and the plane y = h is fixed. The planes are porous, and the liquid is sucked in uniformly over one and ejected uniformly over the other. Obtain a possible solution.

Solution. Here x-axis is the direction of the main flow. Let y be the direction perpendicular to the flow and width of the planes parallel to the z-direction. We take the velocity component w to be zero everywhere and u as function of y alone. Since the plates are infinite, velocity components u and v will be independent of x and z. *Then* the equation of continuity reduces to $\partial v/\partial y = 0$ so that v does not vary with y. Hence v must be a constant, ν/a (given). Thus

$$v \quad \nu/a \qquad \ldots(1)$$

For the present steady flow in the absence of body forces the Navier Stokes equations for x- and y-directions [refer equations (14*a*) and (14*b*) in Art. 14.1 of Chapter 14] are given by

$$\frac{du}{dy} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{d^2 u}{dy^2} \qquad \ldots(2)$$

and
$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}. \qquad \ldots(3)$$

The boundary conditions of the problem are

$$u = U \quad \text{at} \quad y = 0; \qquad \text{and} \qquad u = 0 \quad \text{at} \quad y = h. \qquad \ldots(4)$$

Using (1), (2) may be re-written as
$$\frac{\nu}{a}\frac{du}{dy} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{d^2u}{dy^2}. \qquad \ldots(5)$$

(3) shows that p does not depend on y, Since plates are infinite, p must not depend on z. So p must be function of x alone. Then, (5) may be re-written as

$$\frac{\nu}{a}\frac{du}{dy} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{d^2u}{dy^2}. \qquad \ldots(6)$$

Differentiating (6) $w.r.t$'x', we have
$$\frac{d^2 p}{dx^2} = 0 \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{dp}{dx}\right) = 0.$$

Integrating, $\quad dp/dx =$ costant $= -\ P$ (say), $\qquad \ldots(7)$

where the minus sign has been taken as we expect. p to decrease as x increases.

Using (7), (6) reduces to

$$\frac{\nu}{a}\frac{du}{dy} = \frac{P}{\rho} + \nu\frac{d^2u}{dy^2} \qquad \text{or} \qquad \frac{d^2u}{dy^2} - \frac{1}{a}\frac{du}{dy} = -\frac{P}{\rho\nu}. \qquad \ldots(8)$$

Integrating (8),
$$\frac{du}{dy} - \frac{1}{a}u = -\frac{Py}{\rho\nu} + c_1, \quad c_1 \text{ being an arbitrary constant} \qquad \ldots(9)$$

which is linear differential equation of first order and first degree. Its integrating factor (I.F.) is given by
$$I.F = e^{\int(-1/a)dy} = e^{-y/a}$$

and solution of (9) is given by

$$ue^{-y/a} = c_2 + \int e^{-y/a}\left(c_1 - \frac{Py}{\rho\nu}\right)dy = c_2 - ac_1 e^{-y/a} - \frac{P}{\rho\nu}\int ye^{-y/a}dy$$

or
$$ue^{-y/a} = c_2 - ac_1 e^{-y/a} - \frac{P}{\rho\nu}\left[y\,(-ae^{-y/a}) - \int 1\cdot(-ae^{-y/a})dy\right] = c_2 - ac_1 e^{-y/a} + \frac{Pa}{\rho\nu}(y+a)e^{-y/a}$$

or
$$u = c_2 e^{y/a} - ac_1 + \frac{Pa}{\rho\nu}(y+a). \qquad \ldots(10)$$

Using (4), (10) gives $$U = c_2 - ac_1 + (Pa^2/\rho v) \qquad \ldots(11)$$

and $$0 = c_2 e^{h/a} - ac_1 + (Pa/\rho v) \times (h+a). \qquad \ldots(12)$$

Solving (11) and (12), we have

$$c_1 = \frac{\left(\dfrac{Pa^2}{\rho v} - U\right) e^{-h/a} - \dfrac{Pa}{\rho v}(h+a)}{a\,(e^{h/a} - 1)}, \qquad \text{and} \qquad c_2 = \frac{-\left(U + \dfrac{Pah}{\rho v}\right)}{e^{h/a} - 1}$$

Substituting these values in (10), we have

$$u = \frac{\left(U + \dfrac{Pah}{\rho v}\right) e^{y/a} + \left(\dfrac{Pa^2}{\rho v} - U\right) e^{h/a} - \dfrac{Pa}{\rho v}(h+a)}{e^{h/a} - 1} + \frac{Pa^2}{\rho v} + \frac{Pay}{\rho v} \qquad \ldots(13)$$

Let $$A = \frac{Pa}{\rho v} = -\frac{a}{\rho v}\frac{dp}{dx}. \qquad \ldots(14)$$

Then, (13) may be re-written as

$$u = -\frac{(U + Ah)\, e^{y/a} + (aA - U)\, e^{h/a} - A\,(h+a)}{e^{h/a} - 1} + Aa + Ay$$

or $$u = \frac{Aa\,(a^{h/a} - 1) - (U + Ah)\, e^{y/a} - (aA - U)\, e^{h/a} + A\,(h+a)}{a^{h/a} - 1} + Ay$$

ro $$u = \frac{Ue^{h/a} + Ah - (U + Ah)e^{y/a}}{e^{h/a} - 1} + Ay. \qquad \ldots(15)$$

Hence the required velocity components are given by (1) and (15).

Interpretation of A and a : Suppose that m is the mass of the liquid sucked per unit area per unit time at the plate $y = 0$. Then, we have

$$m = \rho v = \rho \times \frac{v}{a} = \frac{\rho}{a} \times \frac{\mu}{\rho} = \frac{\mu}{a}$$

so that $$a = \mu/m. \qquad \ldots(16)$$

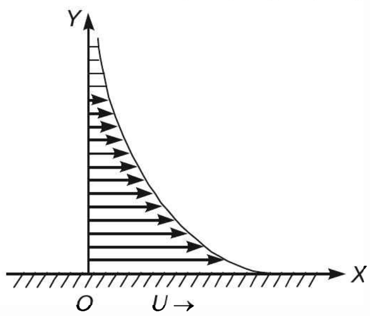
Substituting this value of time a in (14), we have

$$A = -\frac{\mu}{m\rho v}\frac{dp}{dx} = -\frac{1}{m}\frac{dp}{dx}, \qquad \text{as} \qquad v = \frac{\mu}{\rho} \qquad \ldots(17)$$

(16) and (17) determine the constants a and A in terms of μ, m and dp/dx.

TYPE 5. UNSTEADY INCOMPRESSIBLE FLOW WITH CONSTANT FLUID PROPERTIES

16.18A. Unsteady flow of viscous incompressible fluid over a suddenly accelerated flat plae. Flow over a plane wall suddenly set in motion [Kanpur 2009, 10; Garwhal 1999; Himanchal 2000, 01, 03, 04, 06, 09 Meerut 2003, 04, 09, 10, 12]



Let there be a flat plate as shown in the adjoining figure, extending to large distance in the x and z-directions. Let there be an incompressible viscous fluid over a half plane $y = 0$ (*i.e.*, $x\,z$-plane). Let the fluid in contact with the plate be infinite in extent and let it be at rest at time t <0. At $t = 0$, the plate is suddenly set in motion at a constant velocity U in the x-direction. This generates a two-dimensional parallel flow near the plate,

i.e., $\quad v = 0, \qquad w = 0, \qquad \partial/\partial z = 0. \qquad \ldots(1)$

The continuity equation reduces to $\partial u/\partial x = 0$ so that $u = u\ (y,\ t)$ ...(2)

Since the plate is situated in an infinite fluid , the pressure must be constant everywhere. Hence the Navier-Stokes equations in absence of body forces reduce to (Refer equations (14a) to (14c) of Art. 14.1).

$$\partial u / \partial t = \nu(\partial^2 u / \partial y^2) \qquad \text{...(3)}$$

which is to be solved under the follwing initial and boundary conditions of the problem :

$$u = 0 \quad \text{when} \quad t \le 0 \quad \text{for all } y \qquad \text{...(4a)}$$

$$\left.\begin{array}{l} u = U \quad \text{at} \quad y = 0 \\ u = 0 \quad \text{at} \quad y = \infty \end{array}\right\} \quad \text{when} \quad t > 0 \qquad \text{...(4b)}$$

To obtain the desired solution. the partial differential equation (3) is first reduced to an ordinary differential equation by the substitutions

$$\eta = \frac{y}{2\sqrt{\nu t}} = \frac{y}{2\sqrt{\nu}} t^{-1/2} \qquad \text{...(5)}$$

and

$$u = U\ f(\eta). \qquad \text{...(6)}$$

From(5) and(6), we have

$$\frac{\partial u}{\partial t} = Uf'(\eta)\frac{\partial \eta}{\partial t} = Uf'(\eta)\left[-\frac{y}{4\sqrt{\nu}} t^{-3/2}\right] = -\frac{U}{2t}\frac{y}{2\sqrt{\nu t}} f'(\eta) = -\frac{U}{2t}\eta\, f'(\eta). \qquad \text{...(7)}$$

Also,

$$\frac{\partial u}{\partial y} = Uf'(\eta)\frac{\partial \eta}{\partial y} = Uf'\frac{1}{2\sqrt{\nu t}} \qquad \text{...(8)}$$

$$\therefore \quad \frac{\partial^2 u}{\partial y^2} = \frac{U}{2\sqrt{\nu t}} f''(\eta)\frac{\partial \eta}{\partial y} = \frac{U}{2\sqrt{\nu t}} f''(\eta) \times \frac{1}{2\sqrt{\nu t}} \quad \text{or} \quad \frac{\partial^2 u}{\partial y^2} = \frac{U}{4\nu t} f''(\eta). \qquad \text{...(9)}$$

Substituting (7) and (9) into (3), we get

$$-\frac{U}{2t}\eta\, f'(\eta) = \nu \frac{U}{4\nu t} f''(\eta) \quad \text{or} \quad \frac{f''(\eta)}{f'(\eta)} = -2\eta \qquad \text{...(10)}$$

Inegrating (10) w.r.t. 'η' ,we have

$$\log f'(\eta) - \log A = -\eta^2 \quad \text{or} \quad f'(\eta) = Ae^{-\eta^2} \qquad \text{...(11)}$$

i.e.,

$$df(\eta)/d\eta = Ae^{-\eta^2}$$

Integrating it, $f\ (\eta) = A\int_0^{\eta} e^{-\eta^2} d\eta + B,$ B being an arbitrary constant ...(12)

We now re-write the boundary conditions 4 (b) in terms of η and the function $f\ (\eta)$.

Now, $y = 0 \Rightarrow \eta = 0$, by (5)

Hence, $u = U$ at $y = 0 \Rightarrow f\,(0) = 1$, by (6)

Further, $y = \infty \Rightarrow \eta = \infty$, by (5)

Hence, $u = 0$ at $y = \infty \Rightarrow f(\infty) = 0$, by (6)

Thus, the boundary conditions 4(b) may be re-written as

$$f\,(0) = 1 \qquad \text{...(13)}$$

and

$$f\,(\infty) = 0. \qquad \text{...(14)}$$

Setting $\eta = \infty$ in (12) and using (14), we get $$0 = A\int_0^\infty e^{-\eta^2} d\eta + B. \qquad \ldots(15)$$

But it is known that $$\int_0^\infty e^{-\eta^2} d\eta = \frac{\sqrt{\pi}}{2}. \qquad \ldots(16)$$

$\therefore$ From (15) and (16),we have $$B = -(A\sqrt{\pi}/2)$$

Substituting this value of B into (12), $$f(\eta) = A\left[\int_0^\eta e^{-\eta^2} d\eta - \frac{\sqrt{\pi}}{2}\right] \qquad \ldots(17)$$

Now putting $\eta = 0$ in (17) and using(13), we have

$1 = -A \times (\sqrt{\pi}/2)$ so that $A = -(2/\sqrt{\pi})$ ...(18)

With this value of A, (17) reduces to $$f(\eta) = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2} d\eta. \qquad \ldots(19)$$

Substituting (19) into (6), we obtainthe desired velocity ditribution

$$u = U\left[1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2} d\eta\right]. \qquad \ldots(20)$$

The result (20) is sometimes re-written in terms of the well known *error function* which is defined as follows :

$$erf\,(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta} d\eta. \qquad \ldots(21)$$

Using (21), (20) nmay be expressed as $u = U[1 - erf\,(\eta)]$. ...(22)

The velocity distribution (20) or (22) has been represented in the figure. The shear stress at the plate is given by

$$\sigma_{yx} = [\mu U f'(0)]/2\sqrt{\nu t}, \text{ by (8)}$$
$$= (\mu U A)/2\sqrt{\nu t}, \text{ by (11)}$$
$$= -(\mu U)/2\sqrt{\nu \pi t}, \text{ by (18)}$$

the negative sign shows that the force on the fluid adjacent to the plate is in the positive x-direction.

16.18B. Unsteady flow of viscous incompressible fluid between two parallel plates. [Meerut 1999; 2003, 05]

Let there be two infinite parallel plates separated by a distance h as shown in the figure (*i*) of Art. 16.3*A*. Let the plates extend to infinity in x and z directions and let y-direction be perpendicular to these plates. Let there be a viscous incompressible fluid in the space between the plates. Let both the plates be at rest at time $t < 0$. At $t = 0$, plate situated at $y = 0$ is suddenly set in motion at a constant velocity U in the x-direction. This generates a two-dimensional parallel flow between two plates, *i.e.*,

$$v = 0, \qquad w = 0, \qquad \partial/\partial z = 0. \qquad \ldots(1)$$

The equation of continuity reduces to $\partial/u\partial x = 0$. so that $u = u\,(y, t)$ ...(2)

Let the pressure be constant throughout the flow region. Hence the Navier-Stokes equations in absence of body forces reduce to

$$\partial u / \partial t = \nu \, (\partial^2 u / \partial y^2) \qquad \ldots(3)$$

which is to be solved under the following initial and boundary conditions of the problem :

$$u = 0 \quad \text{when} \quad t \le 0 \text{ for all } y \qquad \ldots(4a)$$

$$\left.\begin{aligned} u &= U \text{ at } y = 0 \\ u &= 0 \text{ at } y = h \end{aligned}\right\} \quad \text{when} \quad t > 0 \qquad \ldots(4b)$$

To obtain the desired solution, the partial differential equation (3) is first reduced to an ordinary differential equation by the substitutions

$$\eta = \frac{y}{2\sqrt{\nu t}} = \frac{y}{2\sqrt{\nu}} t^{-1/2} \qquad \ldots(5)$$

and

$$u = U f(\eta). \qquad \ldots(6)$$

From (5) and (6), we have

$$\frac{\partial u}{\partial t} = U f'(\eta) \frac{\partial \eta}{\partial t} = U f'(\eta) \left[-\frac{y}{4\sqrt{\nu}} t^{-3/2} \right] = -\frac{U}{2t} \frac{y}{2\sqrt{\nu t}} f'(\eta) = -\frac{U}{2t} \eta f'(\eta) \qquad \ldots(7)$$

Also,

$$\frac{\partial u}{\partial y} = U f'(\eta) \frac{\partial \eta}{\partial y} = U f'(\eta) \frac{1}{2\sqrt{\nu t}} \qquad \ldots(8)$$

$$\therefore \quad \frac{\partial^2 u}{\partial y^2} = \frac{U}{2\sqrt{\nu t}} f''(\eta) \frac{\partial \eta}{\partial y} = \frac{U}{2\sqrt{\nu t}} f''(\eta) . \frac{1}{2\sqrt{\nu t}} = \frac{U}{4\nu t} f''(\eta). \qquad \ldots(9)$$

Substituting (7) and (9) into (3), we have

$$-\frac{U}{2t} \eta f'(\eta) = \nu \times \frac{U}{4\nu t} f''(\eta) \quad \text{or} \quad \frac{f''(\eta)}{f'(\eta)} = -2\eta$$

Integrating the above equation $w.r.t.$ 'η', we have

$$-\log f'(\eta) - \log A = -\eta^2 \quad \text{or} \quad f'(\eta) = A e^{-\eta^2} \quad \text{or} \quad df(\eta)/d\eta = A e^{-\eta^2}$$

Integrating it, $\quad f(\eta) = A \int_0^{\eta} e^{-\eta^2} d\eta + B.$ B being an arbitrary constant. ...(10)

We now re-write the boundary condition 4(b) in terms of η and the function $f(\eta)$.

Now, $\quad y = 0 \quad \Rightarrow \quad \eta = 0$, by (5)

$\therefore \quad u = U$ *at* $y = 0 \quad \Rightarrow \quad f(0) = 1$, by (6)

Further, $\quad y = h \quad \Rightarrow \quad \eta = \dfrac{h}{2\sqrt{\nu t}}$ $(= \eta_0, \text{say})$,

$\therefore \quad u = 0$ *at* $y = h \quad \Rightarrow \quad f(\eta_0) = 0$, by (6).

Thus the boundary conditions 4 (b) may re-written as

$$f(0) = 1 \qquad \ldots(11)$$

and

$$f(\eta_0) = 0. \qquad \ldots(12)$$

Putting $\eta = 0$ in (10) and using (11), we have $\quad 1 = 0 + B \quad$ so that $\quad B = 1$.

$\therefore$ From (10), $$f(\eta) = A\int_0^{\eta} e^{-\eta^2} d\eta + 1. \qquad \ldots(13)$$

Putting $\eta = \eta_0$ in (13) and using (12), we have

$$0 = A\int_0^{\eta_0} e^{-\eta} d\eta + 1 \qquad \text{so that} \qquad A = -1\Big/\int_0^{\eta_0} e^{-\eta^2} d\eta.$$

With this value of A, (13) reduces to

$$f(\eta) = 1 - \left[\int_0^{\eta} e^{-\eta^2} d\eta \Big/ \int_0^{\eta_0} e^{-\eta^2} d\eta\right] = \left[\int_0^{\eta_0} e^{-\eta^2} d\eta - \int_0^{\eta} e^{-\eta^2} d\eta\right] \Big/ \int_0^{\eta_0} e^{-\eta^2} d\eta$$

$$= \left[\int_{\eta}^{0} e^{-\eta^2} d\eta + \int_0^{\eta_0} e^{-\eta^2} d\eta\right] \Big/ \int_0^{\eta_0} e^{-\eta^2} d\eta = \left[\int_{\eta}^{\eta_0} e^{-\eta^2} d\eta\right]\left[\int_0^{\eta_0} e^{-\eta^2} d\eta\right]^{-1}$$

Substituting the above value of $f(\eta)$ ino (6), the desired velocity distribution is given by

$$u = U\left[\int_{\eta}^{\eta_0} e^{-\eta^2} d\eta\right]\left[\int_0^{\eta_0} e^{-\eta^2} d\eta\right]^{-1} \qquad \ldots(14)$$

Thus shearing stress at any point is given by

$$\sigma_{yx} = \mu\frac{\partial u}{\partial y} = \frac{\mu U}{2\sqrt{\nu t}} f'(\eta), \text{ using (8)} \qquad \ldots(15)$$

Hence the skin friction at the lower and upper plates are given by

$$(\sigma_{yx})_{y=0} = \{\mu U / 2\sqrt{\nu t}\} \times f''(0) \qquad \ldots(16)$$

and $$(\sigma_{yx})_{y=h} = \{\mu U / 2\sqrt{\nu h}\} \times f''(\eta_0) \qquad \ldots(17)$$

respectively. These values indicate that shearing stresses at the plates are infinite when $t = 0$. This explains why we initially require a large force to set the plates in motion.

16.18C. Pulsalite flow between parallel surfaces. [Meerut 2000, 01]

Let there be two parallel surfaces situated at $y = \pm h$. Let oscillatory pressure gradient act in the x-direction. Let the surfaces extend to infinity in x-and z-directions. Let there be a viscous incompressible fluid in the space between the plates. Let both the plates be kept fixed throughout the motion. For the present two- dimensional flow between the two plates $v = 0, w = 0, \partial/\partial z = 0$

The continuity equation reduces to $\partial u/\partial x = 0$ so that $u = u\,(y, t)$.

The Navier Stokes equations in the absence of body forces reduce to

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \qquad \ldots(1)$$

which is to be solved under the boundary conditions

$$u\,(h, t) = u\,(-h, t) = 0. \qquad \ldots(2)$$

If p_x be constant denoting the magnitude of the pressure gradinet, then the pressure gradinet $\partial p/\partial x$ may be assumed in the form $\partial p/\partial x = P_x \cos kt$, where k is a constant. $\ldots(3)$

In order to obtain the desired solution keeping in mind that velocity and pressure gradient are oscillatory in nature, we suppose that

$$u\,(y, t) = R\left\{f(y)e^{ikt}\right\} \qquad \ldots(4)$$

and $$\partial p / \partial x = R\{p_x e^{ikt}\}, \qquad \ldots(5)$$

where R denotes the real part of the expression that follows.

Substituting (4) and (5) in (1), we have

$$ikf(y)\cos kt = -\frac{1}{\rho} p_x \cos kt + \nu \frac{d^2 f}{dy^2}\cos kt$$

or $$\frac{d^2 f}{dy^2} - \frac{ik}{\nu} f = \frac{p_x}{\rho\nu} \quad \text{or} \quad \left(D^2 - \frac{ik}{\nu}\right) f = \frac{p_x}{\rho\nu}, \text{ where } D \equiv d/dy \qquad \ldots(6)$$

The auxiliary equation of (6) is $D^2 - (ik/\nu) = 0$ giving $D = \pm\sqrt{ik/\nu} = \pm\sqrt{k/\nu}\, i^{1/2}$

But $(1+i)^2 = 1 + 2i - 1 \quad \Rightarrow \quad i = (1+i)^2/2 \quad \Rightarrow \quad i^{1/2} = (1+i)/\sqrt{2}$

$$\therefore \qquad D = \pm\{(1+i)/\sqrt{2}\} \times \sqrt{k/\nu} = \pm(1+i)\sqrt{k/2\nu}$$

$\therefore$ Complementary function $= A\cosh[(1+i)y\sqrt{k/2\nu}] + B\sinh[(1+i)y\sqrt{k/2\nu}]$

Now, particular Integral $= \dfrac{1}{D^2 - (ik/\nu)} \dfrac{p_x}{\rho\nu} e^{0.y} = \dfrac{1}{0 - (ki/\nu)} \dfrac{p_x}{\rho\nu} = \dfrac{ip_x}{\rho k}$, as $\nu = \dfrac{\mu}{\rho}$

Hence the general solution of (6) is

$$f(y) = A\cosh[(1+i)y\sqrt{k/2\nu}] + B\sinh[(1+i)y\sqrt{k/2\nu}] + (ip_x)/\rho k \qquad \ldots(7)$$

Using (4), (2) reduces to $\qquad f(h) = f(-h) = 0 \qquad \ldots(8)$

Using boundary conditions (8), (7) gives

$$0 = A\cosh[(1+i)h\sqrt{k/2\nu}] + B\sinh[(1+i)h\sqrt{k/2\nu}] + (ip_x)/\rho k \qquad \ldots(9)$$

$$0 = A\cosh[(1+i)h\sqrt{k/2\nu}] - B\sinh[(1+i)h\sqrt{k/2\nu}] + (ip_x)/\rho k \qquad \ldots(10)$$

[Note that $\cosh(-x) = \cosh x$ and $\sinh(-x) = -\sinh x$]

Solving (9) and (10), $\quad B = 0 \quad$ and $\quad A = -\dfrac{ip_x}{\rho k \cosh[(1+i)h\sqrt{k/2\nu}]}$.

$\therefore$ From (7), $$f(y) = \frac{ip_x}{\rho k}\left[1 - \frac{\cosh\{(1+i)y\sqrt{k/2\nu}\}}{\cosh\{(1+i)h\sqrt{k/2\nu}\}}\right] \qquad \ldots(11)$$

Hence, from (4 and (11), the velocity distribution of the fluid is given by

$$u(y,t) = R\left[\frac{ip_x}{\rho k} e^{ikt}\left\{1 - \frac{\cosh[(1+i)y\sqrt{k/2\nu}]}{\cosh[(1+i)h\sqrt{k/2\nu}]}\right\}\right],$$

showing that the velocity oscillates with same frequency as the pressure gradient but with a phase lag depending upon y exists.

16.18D. Unsteady flow of a viscous incompressible fluid over as oscillating plate. [Himachal 2001, 02, 03, 10; Meerut 2002, 05, 07, 08, 11]

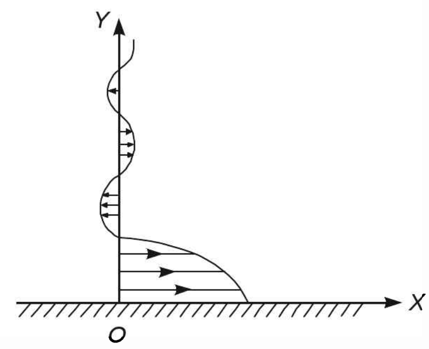
OR

Flow of a viscous incompressible fluid due to an oscillating plane wall.

(Himanchal 2006, 07, 09)

Let there be a flat plate as shown in the following figure extending to large distances in the x-and z-directions. Let there be an incompressible viscous fluid over a half plane $y = 0$ (*i .e.*, x z-plane). Let the fluid extend to infinity and let it be at rest there. Further , let the plate be oscillating with a constant amplitude and frequency with velocity $U \cos nt$. *This* generates a two dimensional parallel flow near the plate, *i.e.*,

$$v = 0, \qquad w = 0, \qquad \partial/\partial z = 0. \qquad \ldots(1)$$



The continutity equation reduces to

$$\partial u/\partial x = 0 \quad \text{so that} \quad u = u\,(y, t) \qquad \ldots(2)$$

Since the plate is situated in an infinite fluid, the pressure must be constant everywhere. Hence the Navire-Stokes equations in absence of body forces reduce to

$$\partial u/\partial t = \nu\,(\partial^2 u/\partial y^2) \qquad \ldots(3)$$

which is to be solved under the following initial and boundary conditions of the problem:

$$u = U\cos nt \quad \text{at} \quad y = 0 \qquad \ldots(4a)$$

$$u = 0 \quad \text{at} \quad y = \infty \qquad \ldots(4b)$$

In order to obtain the desired solution, we make use of complex variables. Suppose that

$$u\,(y, t) = R\,\{e^{int} f\,(y)\}, \qquad \ldots(5)$$

where R denotes the real part of the expression that follows.

Putting (5)in (3), we have

$$R\,\{f\,(y) \times in\, e^{int}\} = R\left\{\nu e^{int}\,(d^2 f/dy^2)\right\}$$

so that $$\frac{d^2 f}{dy^2} - \frac{in}{\nu} f = 0 \quad \text{or} \quad \left(D^2 - \frac{in}{\nu}\right) f = 0, \qquad \ldots(6)$$

where $D \equiv d/dy$. The auxiliary equation of (6) is

$$D^2 - \frac{in}{\nu} = 0 \quad \text{giving} \quad D = \pm\sqrt{\frac{in}{\nu}} = \pm\sqrt{\frac{n}{\nu}}\, i^{1/2}$$

Now, $(1+i)^2 = 1 + 2i - 1 = 2i \quad \Rightarrow \quad i = (1+i)^2/2 \quad \Rightarrow \quad i^{1/2} = (1+i)/\sqrt{2}$

Thus, $D = \pm\,(1+i)\sqrt{n/2\nu}$. Hence the general solution for (6) is

$$f\,(y) = Ae^{(1+i)y\sqrt{n/2\nu}} + Be^{-(1+i)y\sqrt{n/2\nu}} \qquad \ldots(7)$$

Since $u = 0$ at $y = \infty$ by 4 (b), so from (5) we see that $f\,(\infty) = 0$. To satisfy this boundary condition we must take $A = 0$ in (7). Then (7) reduces to

$$f\,(y) = Be^{-(1+i)y\sqrt{n/2\nu}}. \qquad \ldots(8)$$

Further, from 4 (a) and (5) we have

$U\cos nt = R\{(e^{int} f\,(0)\} = R\{(\cos nt + i\sin nt)\, f\,(0)\}$ so that $U\cos nt = f\,(0)\cos nt$.

i.e. $$f\,(0) = U. \qquad \ldots(9)$$

Putting $y = 0$ in (8) and using (9), we obtain $B = U$. Then (8) reduces

$$f\,(y) = Ue^{-(1+i)y\sqrt{n/2\nu}} \qquad \ldots(10)$$

Hence the required velocity distribution (5) becomes

$$u\,(y, t) = R\,\{e^{int} Ue^{-(1+i)y\sqrt{n/2\nu}}\} = Ue^{-y\sqrt{n/2\nu}} R\{e^{i(nt - y\sqrt{n/2\nu})}\} = Ue^{-y\sqrt{n/2\nu}} \cos(nt - y\sqrt{n/2\nu}). \qquad \ldots(11)$$

The drag of the fluid on the boundary per unit area (i.e., skin friction at $y = 0$) is given by

$$-\mu(\partial u/\partial y)_{y=0} = -U\mu\Big[-\sqrt{n/2\nu}\, e^{-y\sqrt{n/2\nu}} \cos(nt - y\sqrt{n/2\nu})$$

$$-\sqrt{n/2\nu}\, e^{-y\sqrt{n/2\nu}} \sin(nt - y\sqrt{n/2\nu})\Big]_{y=0}$$

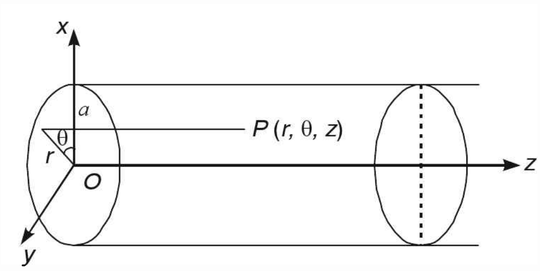
$$= U\mu\sqrt{n/2\nu}\,(\cos nt - \sin nt) \;= U\mu\sqrt{n/2\nu}\,\cos(nt + \pi/4). \qquad \ldots(12)$$

From (11), we notice that the velocity profile $u\ (y,\ t)$ has the form of a damped harmonic oscillation.

16.18E. Flow in pipe, starting from rest [Himanchal 2002; 03, 05]

Suppose that the viscous fluid in an infinitely long pipe of circular cross-section of radius a is at rest for $t < 0$. At the instant $t = 0$ constant pressure gradient dp/dz, which is constant in time, begins to act along it. The fluid will begin to move under the influence of viscous and inertia forces.

For the present problem, we consider all basic equations in cylindrical coordinates. Let the axis of the pipe be taken as z-axis along which the flow takes place and r denotes the radial distance measured outward from the z-axis. Due to axial symmetry $(\partial/\partial\theta = 0)$ all variables will be independent of θ. Clearly, the radial and tangential velocity components are zero, *i.e.*, $q_r = q_\theta = 0$. Due to axial symmetry q_z will be independent of θ. Further the equation of continuity for incompressible flow (Refer equation (8), Art. 2.10 with ρ = constant)



$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial r}(\rho q_z) = 0 \qquad \text{reduces to} \qquad \frac{\partial q_z}{\partial z} = 0,$$

showing that q_z is independent of z also. Hence q_z is function of r and t alone

Thus, $\qquad q_r = o \qquad q_\theta = 0 \qquad$ and $\qquad q_z = q_z(r,t) \qquad \ldots(1)$

For the present axi-symmetrical flow of incompressible flow in absence of body forces, the equation governing the fluid motion is (Refer Art. 14.11)

$$\rho\frac{\partial q_z}{\partial t} = P + \mu\left(\frac{\partial^2 q_z}{\partial r^2} + \frac{1}{r}\frac{\partial q_z}{\partial r}\right), \qquad \ldots(2)$$

where $P\ (= -\,dp\ /\ dz)$ is the constant pressure gradient

The initial and boundary conditions are

Initial condition: $\quad t \le 0, \quad q_z = 0, \quad$ for all $\quad 0 \le r \le a \qquad \ldots(3)$

Boundary conditions: $\quad t > 0, \quad$ at $r = 0$, q_z = finite; $\quad$ at $r = a$, $q_r = 0 \qquad \ldots(4)$

In what follows, we shall use the method of solution given by F. Szymanski

In order to make (2) homogeneous, we introduce a new variable $u(r,\ t)$ as follows

$$u\,(r, t) = (P/4\mu)\times(a^2 - r^2) - q_z(r, t), \qquad \ldots(5)$$

which is the deviation of velocity from its steady asymptotic value (refer equation (13) of Art 16.4A)

Using the value of $q_z\ (r,\ t)$ given by (5) in (2), we get

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \qquad \ldots(6)$$

and the corresponding initial and boundary conditions are

Initial condition: $t = 0$, $u = (P/4\mu) \times (a^2 - r^2)$ ...(7)

Boundary conditions: $t > 0$, at $r = 0$, u = finite and at $r = a$, $u = 0$ ...(8)

We shall now solve (6) by the usual the method of separation of variables.* Let a solution of (6) be of the form

$$u(r, t) = R(r)\,T(t) \quad \text{...(9)}$$

where R is a function of r alone and T that of t.

From (9), $\partial u/\partial t = RT'$, $\partial u/\partial r = R'T$ and $\partial^2 u/\partial r^2 = R''T$,

where the dashes denote the derivative with respect to relevant variable. Hence, (6) reduces to

$$RT' = \nu\left(R''T + \frac{1}{r}R'T\right) \qquad \text{or} \qquad \frac{1}{\nu}\frac{T'}{T} = \frac{1}{R}\left(R'' + \frac{1}{r}R'\right), \quad \text{...(10)}$$

Clearly the L.H.S. of (10) is a function of t alone and the R.H.S. is a function of r alone. Since r and t are independent variables, (10) can hold good if each side is equal to a constant, say $-(\alpha_n^2/a^2)$. Then, (10) leads to the following ordinary differential equations

$$\frac{1}{\nu}\frac{1}{T}\frac{dT}{dt} = -\frac{\alpha_n^2}{a^2} \qquad \textit{i.e.,} \qquad \frac{dT}{T} = -\frac{\alpha_n^2 \nu}{a^2}dt \quad \text{...(11)}$$

and $$\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) = -\frac{\alpha_n^2}{a^2} \qquad \text{or} \qquad r^2 R'' + rR' + \left(\frac{\alpha_n^2}{a^2}\right)r^2 R = 0 \quad \text{...(12)}$$

Integrating (11), $\log T - \log A = -(\alpha_n^2 \nu t/a^2)$, A being an arbitrary constant

or $$T(t) = A\,e^{-(\alpha_n^2 \nu t/a^2)} \quad \text{...(13)}$$

Solution of (12), involving Bessel functions** is given by

$$R(r) = B\,J_0(\alpha_n r/a) + C\,Y_0(\alpha_n r/a) \quad \text{...(14)}$$

where J_0 and Y_0 are Bessel functions of first and second kind respectively.

We take $C = 0$ in (14), since otherwise $R(r) \to -\infty$ as $r \to 0$ ($\because Y_0 \to -\infty$ as $r \to 0$) and so $u(r, t)$ becomes infinite which contradicts the fact that $u(r, t)$ is finite at $r = 0$. Hence (14) becomes

$$R(r) = B\,J_0(\alpha_n r/a), \qquad \text{where} \qquad B \neq 0 \quad \text{...(15)}$$

Putting $r = a$ in (9) and using the boundary condition (8), we get

$$R(a)\,T(t) = 0 \qquad \text{so that} \qquad R(a) = 0 \quad \text{...(16)}$$

where we have taken $T(t) \neq 0$ since otherwise $u(r, t) = 0$ which is a contradiction as before

Putting $r = a$ in (15) and using (16), we have

$$B\,J_0(\alpha_n) = 0 \qquad \text{or} \qquad J_0(\alpha_n) = 0 \quad \text{...(17)}$$

Let α_n ($n = 1, 2, 3, \ldots$) denote the nth positive root of (17) .Thus there are infinite number of particular solutions of $u(r, t)$ which satisfy the boundary conditions (8). In order to obtain a

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* Refer chapter 1 in part III of author's "Advanced differential equations", published by S.Chand and Co., New Delhi.

** Refer chapter 11 of Part I of author's "Advanced differential equations", published by S.Chand and Co., New Delhi.

solution also satisfying the initial condition (7), we consider a move general solution (by superposition principle)

$$u(r, t) = \sum_{n=1}^{\infty} D_n J_0(\alpha_n r/a) e^{-(\alpha_n^2 \nu t/a^2)} \qquad \ldots(18)$$

where D_n (= AB) are new arbitrary coustants.

Putting t = 0 in (18) and using the intial condition (7), we get

$$(P/4\mu)\times(a^2 - r^2) = \sum_{n=1}^{\infty} D_n\, J_0(\alpha_n r/a), \qquad 0 \le r \le a \qquad \ldots(19)$$

We introduce a new variable u given by $\quad r/a = u \quad$ so that $\quad dr = a\, du \quad \ldots(20)$

Using (20), (19) reduces to

$$(Pa^2/4\mu)\times(1-u^2) = \sum_{n=1}^{\infty} D_n\; J_0\;(\alpha_n u), \qquad 0 \le u \le 1 \qquad \ldots(21)$$

which in standard Bessel series* and hence the coefficients D_n are given by

$$D_n = \frac{2}{\{J_1(\alpha_n)\}^2}\int_0^1 \frac{Pa^2 u}{4\mu}(1-u^2) J_0(\alpha_n u)\, du$$

or

$$D_n = \frac{Pa^2}{2\mu\{J_1(\alpha_n\}^2}\left\{\int_0^1 u\, J_0(\alpha_n u)\; du - \int_0^1 u^3 J_0(\alpha_n u) du\right\} \qquad \ldots(22)$$

We now introduce another new variable η given by

$$\alpha_n\, u = \eta \qquad \text{so that} \qquad du = (1/\alpha_n)\times d\eta \qquad \ldots(23)$$

Using (23), (22) veduces to

$$D_n = \frac{Pa^2}{2\mu\{J_1(\alpha_n)\}^2}\left\{\frac{1}{\alpha_n^2}\int_0^{\alpha_n} \eta\, J_0(\eta)\; d\eta - \frac{1}{\alpha_n^4}\int_0^{\alpha_n} \eta^3\; J_0(\eta)\; d\eta\right\} \qquad \ldots(24)$$

From recurrence relations of Bessel functions*, we have

$$\frac{d}{d\eta}\{\eta^n J_n(\eta)\} = \eta^n\; J_{n-1}(\eta) \qquad \text{so that} \qquad \int_0^{\alpha_n} \eta^n\; J_{n-1}(\eta)\; d\eta = \left[\eta^n J_n(\eta)\right]_0^{\alpha_n} \quad \ldots(25)$$

For n = 1, (25) reduces to

$$\int_0^{\alpha_n} \eta\; J_0(\eta)\; d\eta = [\eta J_1(\eta)]_0^{\alpha_n} = \alpha_n\; J_1(\alpha_n) \qquad \ldots(26)$$

$\therefore$

$$\left[\because\; J_1(\eta) = \frac{\eta}{2} - \frac{\eta^3}{2^2\times 3} + \frac{\eta^5}{2^2\times 4^2\times 6} \quad \Rightarrow \quad J_1(0) = 0\right]$$

Now,

$$\int_0^{\alpha_n} \eta^3 J_0(\eta)\; d\eta = \int_0^{\alpha_n} \{\eta^2 \times \eta J_0(\eta)\} d\eta$$

$$= [\eta^2 \times \eta\; J_1(\eta)]_0^{\alpha_n} - \int_0^{\alpha_n} \{2\eta \times \eta\; J_1(\eta)\}\; d\eta$$

$$\left[\because \text{ For } n = 1, (25) \;\Rightarrow\; \int_0^{\alpha_n} \eta\; J_0(\eta)\; d\eta = \eta J_1(\eta)\right]$$

$$= \alpha_n^3 J_1(\alpha_n) - 2\int_0^{\alpha_n} \eta^2 J_1(\eta) d\eta, \qquad \text{as} \qquad J_1\;(0) = 0$$

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* Refer chapter 11 of Part I of author's "Advanced differential equations", published by S.Chand and Co., New Delhi.

$$= \alpha_n^3 J_1(\alpha_n) - 2\{\eta^2 J_2(\eta)\}_0^{\alpha_n}$$

$$\left[\because \text{ For } n = 2, (25) \quad \Rightarrow \quad \int_0^{\alpha_n} \eta^2 J_1(\eta)\, d\eta = \eta^2 J_2(\eta)\right]$$

$$= \alpha_n^3 J_1(\alpha_n) - 2\alpha_n^2 J_2(\alpha_n) \qquad \ldots(27)$$

From recurrence relations of Bessel functions, we have

$$\eta J_{n+1}(\eta) + \eta J_{n-1}(\eta) = 2n J_n(\eta) \quad \Rightarrow \quad J_{n+1}(\eta) = \{2n J_n(\eta) - \eta J_{n-1}(\eta)\}/\eta \qquad \ldots(28)$$

Putting $\eta = \alpha_n$ and $n = 1$ in (28), we get

$$J_2(\alpha_n) = \{2J_1(\alpha_n) - \alpha_n J_0(\alpha_n)\}/\alpha_n \quad \text{or} \quad J_2(\alpha_n) = (2/\alpha_n) \times J_1(\alpha_n) \qquad \ldots(29)$$

$$[\because \text{ Front (17), } J_0(\alpha_n) = 0]$$

Using (29), (27) reduce to
$$\int_0^{\alpha_n} \eta^3 J_0(\eta)\, d\eta = \alpha_n^3 J_1(\alpha_n) - 4\alpha_n J_1(\alpha_n) \qquad \ldots(30)$$

Using (26) and (30), (24) reduces to

$$D_n = \frac{Pa^2}{2\mu\{J_1(\alpha_n)\}^2}\left[\frac{1}{\alpha_n^2} \times \alpha_n J_1(\alpha_n) - \frac{1}{\alpha_n^4} \times \{\alpha_n^3 J_1(\alpha_n) - 4\alpha_n J_1(\alpha_n)\}\right]$$

Thus,
$$D_n = 2Pa^2/\mu\alpha_n^3 J_1(\alpha_n)$$

Substituting the above value of D_n in (18), we have

$$u(r, t) = \frac{2Pa^2}{\mu}\sum \frac{J_0(\alpha_n r/a)}{\alpha_n^3 J_1(\alpha_n)} e^{-(\alpha_n^2 \nu t/a^2)} \qquad \ldots(31)$$

From (5),
$$q_z = (P/4\mu) \times (a^2 - r^2) - u(r, t)$$

or
$$q_z = \frac{P(a^2 - r^2)}{4\mu} - \frac{2Pa^2}{\mu}\sum \frac{J_0(\alpha_n r/a)}{\alpha_n^3 J_1(\alpha_n)} e^{-(\alpha_n^2 \nu t/a^2)} \qquad \ldots(32)$$

which is the required velocity distribution where α_n are the positive roots of (17).

The velocity profile is shown in the adjoining figure for various instants. It is interesting to observe that in the early stages the velocity near the axis is approximately constant over the radius and that viscosity makes itself felt in a narrow layer near the wall. We also observe that as time progresses the viscosity also effects the central portion of the fluid and the velocity profile tends asymptotically to the parabolic distribution for steady flow.

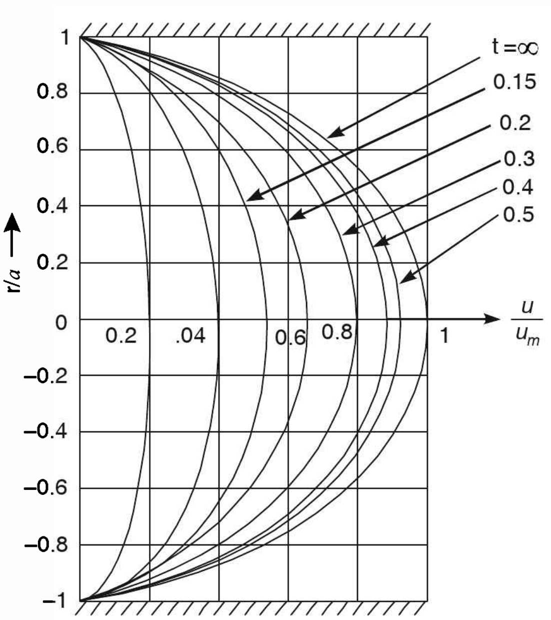

Fig. Velocity distribution for unsteady (starting) flow in a circular pipe

Type 6: Steady incompressible flow with variable viscosity

16.19A. Basic equations of steady incompressible flow with variable viscosity

We know that lubricating oils have practical applications in engineering problems. In such situations, the temperature rise due to friction, even at moderate velocity, is so large that dependence of viscosity on temperature becomes important and it can no longer be treated as constant as we have so far done in this chapter. When viscosity is variable, the velocity field in not independent of the temperature field. Consequently the situation is more difficult in comparison to the flow with constant fluid properties even when we wish to study flow pattern in simple cases.

In the present chapter, we shall study the case of a two dimensional ($\partial/\partial z \equiv 0$ and $w = 0$) steady incompressible flow in absence of body forces and with variable viscosity. For ready reference , the complete list of basic equations governing the flow under consideration (using results of Art. 14.10 with ρ = constant for incompressible fluid) are given by

The equation of continuty $\qquad \partial u/\partial x + \partial v/\partial y = 0 \qquad$ ...(1)

The Navier-Stokes equations of motion

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\} \qquad \text{...(2a)}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\} \qquad \text{...(2b)}$$

The energy equaion

$$\rho C_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+2\mu\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right\}+\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2 \qquad \text{...(3)}$$

16.19.B. Plane Couette flow of viscous incompressible fluid with variable viscosity

(Himanchal 2004)

Consider the steady laminar flow of viscous incompressible fluid between two infinite plates separated by a distance h. Let the upper plate move with uniform velocity U and the lower plate be at rest. Let x be the direction of flow, y the direction perpendicular to the flow, and the width of the plates parallel to the z-direction. Here the word infinite imples that the width of the plates is large compared with h and hence the flow may be treated as two dimensional (*i.e.*, $\partial/\partial z = 0$). Let the plates be long enough in the x-direction for the flow to be parallel so that both the velocity components v and w vanish everywhere. Let both the plates be kept at the same constant temperature T_0.

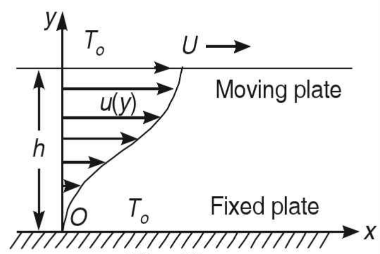

Fig. (*i*)

For the present problem the equation of continuity is $\quad \partial u/\partial x = 0 \quad$ so that $\quad u = u\,(y)$.

Hence , for the flow under consideration, we have

$$u = u\,(y), \qquad v = w = 0, \qquad \partial/\partial z = 0, \qquad \partial/\partial t = 0 \qquad p = \text{constant} \qquad \text{...(1)}$$

In the absence of by body forces keeping (1) in mind, the Navier-Stokes equations and energy equation for steady flow of incompresible viscous fluid with variable viscosity are given by (refer equations (2*a*) (2*b*) and (3) of Art 16.9A)

$$\frac{d}{dy}\left(\mu\frac{du}{dy}\right)=0 \qquad \text{...(2)}$$

and $$k\frac{d^2T}{dx^2}+\mu\left(\frac{du}{dy}\right)^2=0 \qquad \ldots(3)$$

The boundary conditions are

$y=0;$ $u=0$; $T=T_0$; $y=h$: $u=U,$ $T=T_0$ ...(4)

Let the quantity with dash denote a non-dimensional quantity. The non-dimensional quantities may now be introduced ;

$u'=u/U$, $T'=(T-T_0)/T_0$, $\mu'=\mu/\mu_0$, $\eta=y/h$, P_r = Prandtl number $=\mu_0 C_p/k$ and E_c = *Eckert number* = U^2/C_pT_0, where μ_0 is the viscosity of the fluid at temperature T_0

Substituting the above non-dimensional quantities into (2) and (3), we obtain

$$\frac{d}{d\eta}\left(\mu'\frac{du'}{d\eta}\right)=0 \qquad \ldots(5)$$

and $$\frac{d^2T'}{d\eta^2}+E_c\,P_r\,\mu'\left(\frac{du'}{d\eta}\right)^2=0 \qquad \ldots(6)$$

Again the corresponding boundary conditions are

$\eta=0$: $u'=0,$ $T'=0;$ $\eta=1$: $u'=1,$ $T'=0$ ...(7)

Integrating (5), $\mu'\,(du'/d\eta)=A$, A being an arbitrary constant ...(8)

Let τ'_w = dimensionless shearing stress at the fixed plate $=[\mu'(du'/d\eta)]_{\eta=0}$ ...(9)

Putting $\eta=0$, in (8) and using (9), we get A = τ'_w

Thus, (8) becomes $\mu'(du'/d\eta)=\tau'_w$ ...(10)

Using (10) and (6) may be re-written as $d^2T'/d\eta^2+(E_cP_r\,\tau'^2_w)/\mu'=0$ ...(11)

On the basis of experimental results by Prandtl*, we shall use the following empirical relation between viscosity and temperature $1/\mu'=e^{\beta T'}$, ...(12)

where $\beta=b/T_0$. b has the dimension of temperature and depends on the nature of fluid. Using (12), (11) reduces to $d^2T'/d\eta^2+E_c\,P_r\,(\tau'_w)^2e^{\beta T'}$ ...(13)

An exact solution of (13) was given by Jain and Bansal** in the following form

$$T'=\{\log_e C_1-2\log_e \cosh\,(C_2\eta+C_3)\}/\beta \qquad \ldots(14)$$

where $$C_1=\{2/(\beta\,E_cP_r\,\tau'^2_w)\}\times C_2^2 \qquad \ldots(15)$$

Using (12) and (14) in (10) and on integration, we get

$$u'=(\tau'_w\,C_1/C_2)\,\tanh\,(C_2\eta+C_3)+C_4, \qquad \ldots(16)$$

where C_2, C_3, C_4 and τ'_w are unknown constants to be determined. Using the boundary condition (7), equations (14) and (16) yield the required unknown constants as given below

$$\tau'_w=\tanh^{-1}\{u'_h/(1+u'^2_h)^{1/2}\},\ C_2=2\tau'_wu'_h(1+u'^2_h)^{1/2},\ C_3=-\tau'_w\,u'_h\,(1+u'^2_h)^{1/2},\ C_4=1/2 \qquad \ldots(17)$$

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* Prandtl, L. ZAMM, 8, page 85 (1928)

** Jain N.C and Bansal J.L.: Couette flow with transpiration cooling when the viscosity of the fluid depends on temperature. Proc. Ind. Aca. Sci, Volume 77, Sec. A, No. 4, pages 184–200 (1973).

where $$u'_h = \{(\beta\, E_c P_r)/8\}^{1/2} \qquad \ldots(18)$$

Using (18), C_1 given by (15) reduces to $$C_1 = 1 + u'^2_h \qquad \ldots(19)$$

Note carefully that for the case of constant viscosity $\beta = 0$ and hence $u'_h = 0$ and then we have,

$$C_1 = 1, \qquad C_2 = C_3 = 0, \qquad C_4 = 1/2, \qquad \lim_{u'_h \to 0} \tau'_w = 1,\ \lim_{u'_h \to 0} (C_2/C_3) = -2 \qquad \ldots(20)$$

Consequently the expressions in (14) and (16) can be simplified to yield

$$T' = E_c P_r\, \eta\, (1-\eta)/2 \qquad \text{and} \qquad u' = \eta$$

which are in agreement with results already obtained in Art. 16. 11 and Art 16. 3*A*.

The velocity profiles are shown in the following figure (ii) for different values of u'_h. In the present case the velocity ceases to be linear and the shearing stress goes on increasing as u'_h increases in comparison to the case of viscosity flow.

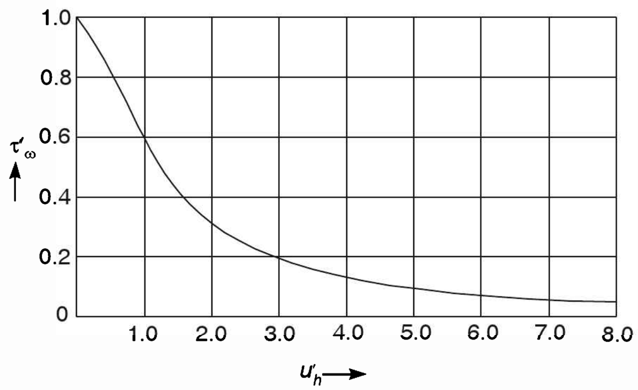


Fig.(ii) Dimensionless shearing stress on the stationary plate in variable viscosity. Plane Couette flow.

16.19C. Plane Poiseuille flow of viscous incompresele fluid with variable viscosity. (Himanchal 2005)

Consider the steady laminar viscous incompressible fluid with variable viscosity between two infinite parallel fixed plates separated by a distance 2b. Let axis of x be taken in the middle of the channel parallel be the direction of flow, y the direction perpendicular to the flow, and width of the plates parallel to the z-direction. Here the word infinite implies that the width of the plates is large compared to b and hence the flow may be treated as two-dimensional (*i.e.* $\partial/\partial z = 0$). Let the plates be long enough in the x-direction for the flow to be parallel. Here we take the velocity components v and w to be zero everywhere. Moreover the flow being steady, the flow variables are independent of time $(\partial/\partial t = 0)$

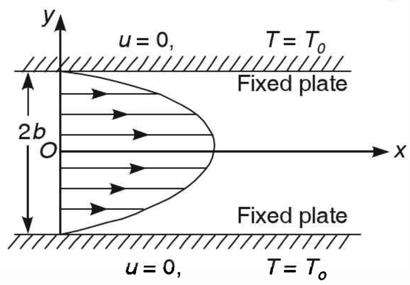


Fig. (*i*)

For the present problem the equation of continuity $\partial u/\partial x + \partial v/\partial y = 0$ reduces to $\partial u/\partial x = 0$ giving $u = u(y)$. Note that for the present motion the pressure is not constant.

Hence, for the flow under consideration, we have

$$u = u\,(y), \qquad v = w = 0, \qquad \partial/\partial z = 0, \qquad \partial/\partial t = 0 \qquad \ldots(1)$$

In the absence of body forces, keeping (1) in mind the Navier-Stokes equations and energy equation for steady flow of incomprentible viscous fluid with variable viscosity are given by (refer equations (2*a*), (2*b*) and (3) of Art. 16.19A).

$$0 = -\frac{\partial p}{\partial x} + \frac{d}{dy}\left(\mu \frac{du}{dy}\right) \qquad \ldots(1)$$

$$0 = -(\partial p/\partial y) \quad \ldots(2)$$

and $$k\,(d^2T/dy^2) + \mu\,(du/dy)^2 = 0 \quad \ldots(3)$$

Equation (2) shows that the pressure does not depend on y. Hence p is function of x alone and hence (1) reduces to $$\frac{d}{dy}\left(\mu\frac{du}{dy}\right) = \frac{dp}{dx} \quad \ldots(4)$$

Assume that both the plates are kept at the same constant temperature T_o

The boundary conditions are $y = \pm h$: $u = 0$ and $T = T_0 \ldots(5)$

Let u_m denote the velocity in the middle of the channel. When viscosity is regarded as constant, then as shown in equation (11) of Art, 16.3C with h = 2b and $P = dp/dx$, we have

$$u_m = -(b^2/2\mu_0)\times(dp/dx) \quad \ldots(6)$$

where μ_0 is the coefficient of viscosity of the fluid at temperature T_o

Solution of the above system of equations (3), (4), (5) was given by Hausenblas. Accordingly, we introduce the following non-dimensional quantities. In what follows, the quantity with dash will denote a non-dimensional quantity.

$$\left.\begin{array}{l} u' = u/u_m, \quad x' = x/b, \quad \mu' = \mu/\mu_{0,} \quad T' = (T - T_0)/T_{0,} \\ \eta = y/b,\ P_r = \text{Prandtl number} = \mu_0 C_p/k,\ \text{Ec} = \text{Eckert number} = u_m^2/C_pT_0, \end{array}\right\} \quad \ldots(7)$$

Substituting the above non-dimensional quantities into (3) and (4), we obtain

$$d^2T'/d\eta^2 + E_cP_r\,\mu'\,(du'/d\eta)^2 = 0 \quad \ldots(8)$$

and $$\frac{d}{d\eta}\left(\mu'\frac{du'}{d\eta}\right) = -2 \quad \ldots(9)$$

and the corresponding boundary condition are

$$\eta = \pm 1: \qquad u' = 0 \qquad \text{and} \qquad T' = 0 \quad \ldots(10)$$

Integrating (9), $$\mu'\,(du'/d\eta) = -2\eta, \quad \ldots(11)$$

where the constant of integration vanishes from symmetry consideration.

Using (11), (8) gives $$d^2T'/d\eta^2 + (4E_cP_r\,\eta^2)/\mu' = 0 \quad \ldots(12)$$

Following Hausenblas, (12) is solved by assuming the following relation between viscosity and temperature $$1/\mu' = 1 + \beta T' = \theta',\ \text{say} \quad \ldots(13)$$

where $\beta = b/T_0$. Using (13), (12) be re-written as

$$d^2\theta'/d\eta^2 + 4N\,\eta^2\theta' = 0, \quad \ldots(14)$$

where $$N = \beta\,E_cP_r \quad \ldots(15)$$

The boundary conditions on θ' are $\eta = \pm 1$: $\theta' = 1$ $\ldots(16)$

Now, The series solution of (14), satisfying the boundary condition (16) is given by

$$\theta' = A\sum_{n=0}^{\infty} a_n\eta^{4n}, \quad \ldots(17)$$

where $$a_n = \frac{1}{n!}\times\frac{(-1/4)!}{(-1/4+n)!}\times(i\sqrt{N}/2)^2 \quad \ldots(18)$$

and
$$A = \left(\sum_{n=0}^{\infty} a_n \right)^{-1} \qquad \ldots(19)$$

Using (13) and (17), (11) reduces to

$$\frac{du'}{d\eta} = -2\eta\theta' = -2A\eta \sum_{n=0}^{\infty} a_n \eta^{4n} \qquad \ldots(20)$$

The solution of (20) satisfying the boundary condition (10) in given by

$$u' = 2A \sum_{n=0}^{\infty} \frac{a_n}{4n+2} (1 - \eta^{4n+2}) \qquad \ldots(21)$$

For the case of constant viscosity $\beta = 0$ so that N = 0, and we can easily verify that

$$a_n = 0, \text{ if } n \neq 0; \qquad a_n = 0, \text{ if } n = 0 \qquad \text{and} \qquad A = 1 \qquad \ldots(22)$$

Hence (21) yields velocity distributon given by $\qquad u' = 1 - \eta^2,$

which is of the same form as obtained in Art. 16.3*C*

For the temperature, we evaluate the following limit

$$T' = \lim_{\beta \to 0} \frac{\theta' - 1}{\beta} = \frac{1}{3} E_c P_r \, (1 - \eta^4),$$

which is of the same form as obtained in Art 16.13.

In figure (*ii*) the velocity profiles for different values of *N* are plotted. The curve corresponding to *N* = 0 is for constant viscosity flow and is a parabola. It is clear from the graph that the velocity at any section on increases as *N* increases and is always greater than the velocity in constant viscosity flow. Dotted curve represents constant viscosity flow.

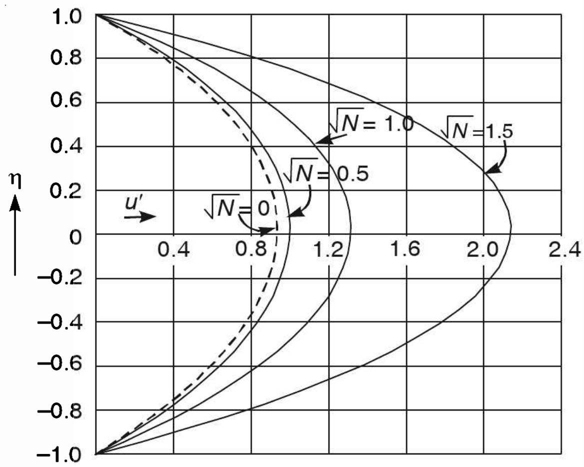


Fig. (ii) Velocity distribution in variable viscosity plane Poiseuille flow ($N = \beta\, Ec\, P_r$)

TYPE 7 : EXACT SOLUTION OF THE FLOW IN THE NEIGHBOURHOD OF A STAGNATION PAINT

16.20 Stagnation in two dimensional flow (Hiemenz flow)

Recall that a stagnation point is the point where the velocity is zero in the potential flow (*i,e*., the flow of inviscid fluid)

In order to obtain an exact solution of the flow of a viscous incompressible fluid in the neighbourhood of a stagnation point in a plane, we consider the steady flow at large distance from the stagnation point to be potential flow.

The velocity distribution in frictionless flow in the neighborhood of the stagnation paint at $x = y = 0$ is given by $\qquad U = ax, \qquad V = -ay, \qquad \ldots(1)$
where a denotes a constant. This in an example of a plane potential flow which arrives from the y-axis and impinges on a flat wall placed at $y = 0$, divides into two streams on the wall and leaves in both directions. The viscous flow must adhere to the wall, whereas the potential flow slides along it. In potential flow the pressure is given by Bernoulli's equation. If p_0 denotes the pressure at the stagnation point and p in the pressure at an arbitrary point in potential flow we get

$$p/\rho + (U^2 + V^2)/2 = p_0/\rho \qquad \text{giving} \qquad p_0 - p = \rho\, a^2\,(x^2 + y^2)/2, \text{ by (1)} \qquad \ldots(2)$$

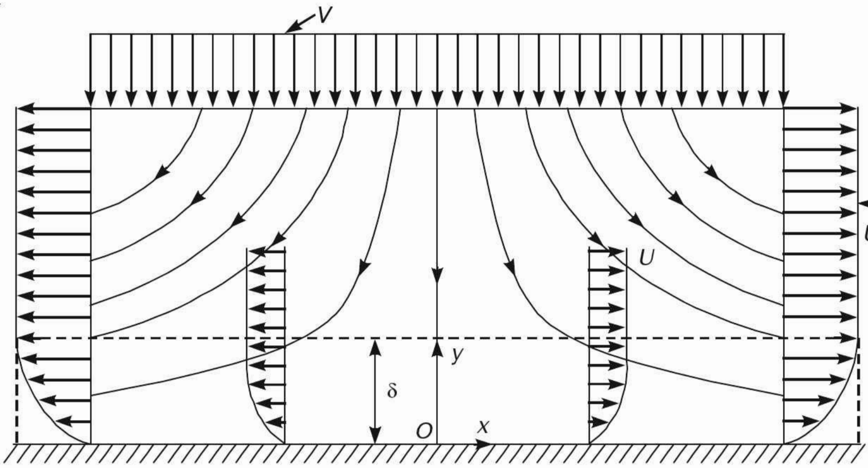


Fig. Stagnation in plane flow

For viscous flow, we now make the assumptions

$$u = x\, f'(y), \qquad v = -f(y) \qquad \ldots(3)$$

and

$$p_0 - p = \rho\, a^2 \{x^2 + F(y)\}/2 \qquad \ldots(4)$$

where the prime denotes differentition with respect to y Equation of continuity for steady incompressible two dimensional flow is given by (refer quation (8), Art. 2.9)

$$\partial u/\partial x + \partial v/\partial y = 0. \qquad \ldots(5)$$

From (3), $\qquad \partial u/\partial x = f'(y) \qquad$ and $\qquad \partial v/\partial y = -f'(y) \qquad \ldots(6)$

In view of (6), we see that the equation of continuity (5) in satisfied. Again , the Navier Stokes equations for steady incompressible flow in xy-plane are given by (refer equations $(2a)'$ and $(2b)'$ of Art 14.10)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \ldots(7)$$

and

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad \ldots(8)$$

Substituting equations (3) and (4) into equations (7) and (8) we obtain two ordinary differential equations for determination of f and F

$$f'^2 - f\, f'' = a^2 + \nu\, f''' \qquad \ldots(9)$$

and

$$f\, f' = a^2 F'/2 - \nu\, f'' \qquad \ldots(10)$$

The boundary conditions for f and F are obtained from the consideration that at the wall $y = 0$, $u = v = 0$ and at stagnation point $x = y = 0, p = p_0$ and at a large distance from the wall (*i.e.*, as $y \to \infty$), $u = U$. Thus, we have

When $y = 0: \ f = 0, \quad f' = 0, \quad F = 0;$ when $y = \infty: \ f' = a$ ...(11)

Equations (9) and (10) are two differential equations containing $f(y)$ and $F(y)$ which determine the velocity and pressure distributions. Since $F(y)$ does not appear in the first equation, it is possible to begin by finding $f(y)$ and then to proceed to determine $F(y)$ from (10). The non-linear differential equation (9) cannot be solved in closed form. In order to solve it numerically it is convenient to remove the constant a^2 and ν by putting

$$\eta = ay \qquad \text{and} \qquad f(y) = A\phi(\eta), \qquad \text{...(12)}$$

Using (12), (9) can be *re*-written as

$$\alpha A^2 (\phi'^2 - \phi\phi'') = a^2 + \nu A \alpha^3 \phi''',$$

where the prime now denotes differentiation with respect to η. The coefficients of the equation becomes all identically equal to unity if we put

$$\alpha^2 A^2 = a^2 \quad \text{and} \quad \nu A \alpha^3 = a^2 \quad \text{so that} \quad A = (\nu a)^{1/2} \quad \text{and} \quad \alpha = (a/\nu)^{1/2}$$

Hence, (12) reduces to $\eta = (a/\nu)^{1/2} y$ and $f(y) = (a\nu)^{1/2}\phi(\eta)$ ...(13)

Using (13), (9) yields $\phi''' + \phi\phi'' - \phi'^2 + 1 = 0$ ...(14)

and the corresponding boundary conditions for ϕ are

$\eta = 0: \quad \phi = 0, \quad \phi' = 0;$ and $\eta = \infty: \quad \phi' = 1.$ ...(15)

Equation (14) was first solved numerically by *K*. Hiemenz and later his solution was improved by *L.Howarth*. Some values of ϕ, ϕ' and ϕ'' for different values of η are tabulated in the following table

$\eta = (a/\nu)^{1/2} y$	ϕ	$\phi' = u/U$	ϕ''
0	0	0	1.2326
0.2	0.0233	0.2266	1.0345
0.6	0.1867	0.5663	0.6752
1.0	0.4592	0.7779	0.3980
1.4	0.7967	0.8968	0.2110
1.8	1.1689	0.9568	0.1000
2.0	1.3620	09732	0.0658
2.4	1.7553	0.9905	0.0260
2.8	2.1530	0.9970	0.0090
3.0	2.3526	0.9984	0.0051

The dimensionles velocity in the *x*-direction is given by

$$u/U = f'(y)/a = \phi'(\eta) \qquad \text{...(16)}$$

After determination of the value of $\phi(\eta)$ or in other words value of f (y), $F(y)$ can be oftained by integrating (10). Thus , from (10) we have

$$a^2 F' = 2f f' + 2\nu f'' \qquad \text{giving} \qquad a^2 F = f^2 + 2\nu f', \text{ on integration} \quad \text{...(17)}$$

or

$$F = (\nu/a) \times (\phi^2 + 2\phi'), \qquad \text{...(18)}$$

where the constant of integration vanishes with help of the boundary conditions (11). Substituting the value of F given by (17) in (4), we obtain the required pressure distribution.

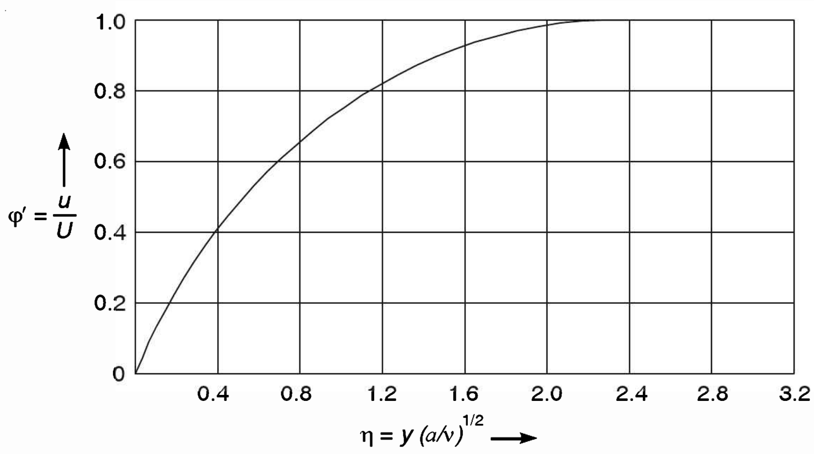


Fig. Velocity distribution in plane stagnation point flow

The dimensionless velocity is shown in the above figure. The curve $\phi'(\eta)$ begins to increase linearly at $\eta = 0$ and tends approximately to unity. At approximately $\eta = 2.4$, we here $\phi' = 0.99$, *i.e.*, the final value is reached there with an accuracy of 1 per cent. If we consider the corresponding distance from the wall denoted by $y = \delta$, the boundary layer thickness, we have

$$\delta = 2.4 \times (\nu/a)^{1/2}, \qquad \ldots(19)$$

showing that the boundary layer thickness which is influened by viscosity is small at low kinematic viscosities and proportional to $\sqrt{\nu}$. Again we can easily show that the pressure gradient is proportional to $\rho a(\nu / a)^{1/2}$ and is also very small for small kinematic viscosities.

Also observe that the dimensionaless velocity distribution u/U and the boundary layer thickness given by (19) are independent of x, *i.e.*, they do not vary along the wall.

Remark It will be seen in chapter 18 that the above two results confirm the basic assumptions of the Prandtl boundary layer theory.

16.21 Miscellaneous solved examples on chapter 16

Ex.1. *Prove that* $$\left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi = \frac{\partial\,(\psi, \nabla^2\psi)}{\partial\,(x, y)}$$

where ψ *is the stream function for a two-dimensional motion of a viscous fluid*

[Agra 2010; Kanpur 2002, 04; Allahabad 2001; Garwhal 200; Kolkata 2004]

Sol. In the absence of body forces, the Navier-Stokes' equation of a viscous incompressible fluid are given by

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\frac{1}{\rho}\nabla p + \nu\,\nabla^2\mathbf{q} \qquad \ldots(1)$$

But $$\nabla\,(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{b}\cdot\nabla)\mathbf{a} + (\mathbf{a}\cdot\nabla)\mathbf{b} + \mathbf{b}\times \text{curl}\,\mathbf{a} + \mathbf{a}\times\text{curl}\,\mathbf{b}$$

Hence, $\nabla\,(\mathbf{q}\cdot\mathbf{q}) = 2(\mathbf{q}\cdot\nabla)\,\mathbf{q} + 2\mathbf{q}\times\text{curl}\,\mathbf{q}$ or $(\mathbf{q}\cdot\nabla)\mathbf{q} = (1/2)\times\nabla\,(\mathbf{q}\cdot\mathbf{q}) - \mathbf{q}\times\text{curl}\,\mathbf{q}$

Thus, $$(\mathbf{q}\cdot\nabla)\mathbf{q} = \nabla\,(\mathbf{q}^2/2) - \mathbf{q}\times\text{curl}\,\mathbf{q} \qquad \ldots.(2)$$

Using (2), (1) reduces to $\dfrac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \times \text{curl } \mathbf{q} = -\nabla(p/\rho + q^2/2) + \nu \nabla^2 \mathbf{q}$ ...(3)

If Ω be the vorticity vector, we have $\Omega = \text{curl } \mathbf{q}$. ...(4)

Taking curl of (3) and using (4), we get

$$\frac{\partial \Omega}{\partial t} - \mathbf{curl}(\mathbf{q} \times \Omega) = \nu\, \mathbf{curl}\, \nabla^2 \mathbf{q} \qquad [\because \text{for any scalar function } \phi, \text{curl } \nabla\phi = \mathbf{0}]$$

or
$$\partial \mathbf{\Omega}/\partial t - (\mathbf{q} \cdot \nabla)\mathbf{\Omega} - (\mathbf{\Omega} \cdot \nabla)\mathbf{q} = \nu \nabla^2 \mathbf{\Omega} \qquad \ldots(5)$$

For the present two-dimensional flow, $\mathbf{q} = (u, v, 0)$ and $\Omega = (0, 0, \zeta)$ ...(6)

Using (6), (5) reduces to
$$\left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\zeta = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta \qquad \ldots(7)$$

Also $u = -(\partial\psi/\partial y)$, $v = \partial\psi/\partial x$, and $\zeta = \nabla^2\psi$ ...(8)

Using (8), (7) reduces to
$$\left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi = \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi$$

Thus,
$$\left(\nu\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi = \frac{\partial\,(\psi, \nabla^2\psi)}{\partial\,(x, y)}. \qquad \ldots(9)$$

Note. while writing (9),we have sued the following definition of jacobian of functions f_1, f_2 which respect x_1, x_2 :
$$\frac{\partial\,(f_1, f_2)}{\partial\,(x_1, x_2)} = \begin{vmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{vmatrix} = \frac{df_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}.$$

Ex. 2. *prove that in the steady motion of a viscous liquid in two dimensions*

$\nu\, \nabla^4\psi = dX/\partial y - dY/\partial x$ *where (X, Y) is the impressed force per unit area.*

[Meerut 1997, Kanpur 2005,06, 09; Agra 2002, 06 Allahabad 2003; Kolkata 2005, G.N.D.U. 2001, 03, 06]

Sol. The Navier-Stokes equation of a viscous incompressible fluid is given by

$$\partial \mathbf{q}/\partial t + (\mathbf{q} \cdot \nabla)\mathbf{q} = -(1/\rho) \times \nabla p + \mathbf{B} + \nu\nabla^2\mathbf{q}. \qquad \ldots(1)$$

For the steady motion $\partial\mathbf{q}/\partial t = \mathbf{0}$. For the fluid flowing at very slow speeds, the inertia-force terms (namely, $(\mathbf{q}.\nabla)\,\mathbf{q}$ on the left- hand side of (1) can be neglected in comparison with the firction-force terms. Thus, for the steady slow motion (1) takes the form

$$\nu\nabla^2\,\mathbf{q} = -(1/\rho) \times \nabla p + \mathbf{B}. \qquad \ldots(2)$$

Let Ω be the vorticity vector. Then $\Omega = \text{curl } \mathbf{q}$. ...(3)

Taking curl of (2) and using (3), we get $\nu\nabla^2\Omega = \text{curl } \mathbf{B}$ ...(4)

$[\because \text{For any scalar function } \phi, \text{ curl } \nabla\phi = \mathbf{0}]$

For the present two dimensional flow , we have

$$\mathbf{B} = (X, Y, 0), \qquad \mathbf{q} = (u, v, 0), \qquad \Omega = (0, 0, \zeta).$$

Then, (4) reduces to $\nu\nabla^2\zeta = \partial X/\partial y - \partial Y/\partial x.$ ...(5)

But $\zeta = \nabla^2\psi.$ ...(6)

Using (6), (5) reduces to $\nu\nabla^4\psi = \partial X/\partial Y - \partial Y/\partial x$

Ex. 3. *Viscous incompressible fluid is in steady two-dimensional radial motion between two non-parallel plane walls, r and* θ *are polar coordinates, r being the distance from the line of intersection of the walls, which are* $\theta = \pm\alpha$. *Show that the velocity is given by*

$$q_r = \frac{f(\theta)}{r}, \quad \text{where} \quad \left(\frac{df}{d\theta}\right)^2 = \frac{2}{3\nu}(h - 2\nu kf - 6\nu f^2 - f^3), \text{ where } h \text{ and } k \text{ are constants.}$$

Sol. To solve the problem, we employ cylindircal coordinates with z-axix normal to the plane of the flow, Since the motion is purely radial, $q_\theta = q_z = 0$ and the continuity equation reduces to

$$\frac{\partial}{\partial r}(rq_r) = 0 \qquad \text{so that} \qquad q_r = \frac{f(\theta)}{r}, \qquad \ldots(1)$$

where $f(\theta)$ in an arbitrary function of θ to be determined

For the present steady motion in absence of body force, Navier-Stokes equations reduce to

$$q_r \frac{\partial q_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 q_r}{\partial r^2} + \frac{1}{r}\frac{\partial q_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 q_r}{\partial \theta^2} - \frac{q_r}{r^2}\right) \qquad \ldots(2)$$

and

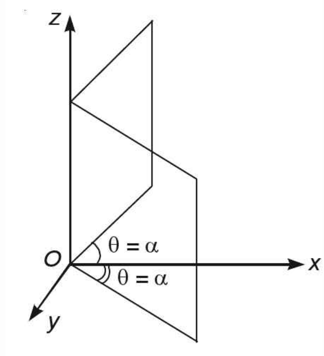
$$0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{2}{r^2}\frac{\partial q_r}{\partial \theta}\right) \qquad \ldots(3)$$

Subsituting the value of q_r given by (1)into (2) and (3), we obtain

$$\frac{f}{r} \times \left(-\frac{f}{r^2}\right) = -\frac{1}{r}\frac{\partial p}{\partial r} + \nu\left(\frac{2f}{r^3} - \frac{f}{r^3} + \frac{f''}{r^3} - \frac{f}{r^3}\right) \quad \text{or} \quad -\frac{f^2}{r^3} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\nu f''}{r^3} \qquad \ldots(4)$$

and

$$0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \frac{2\nu f'}{r^3} \qquad \ldots(5)$$



Differentiating (4) $w.r.t.$ 'θ', and (5) $w.r.t.$ 'r', we have

$$\frac{2ff''}{r^3} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial r \partial \theta} + \frac{\nu f'''}{r^3} \qquad \ldots(6)$$

and

$$0 = -\frac{1}{\rho}\frac{\partial^2 p}{\partial r \partial \theta} - \frac{4\nu f'}{r^3} \qquad \ldots(7)$$

Subtracting (7) from (6), we get

$$\nu(f''' + 4f') + 2ff' = 0.$$

Integrating it, $\quad \nu(f'' + 4f) + f^2 + \nu k = 0, \qquad \ldots(8)$

where k is a constant of integration. Multiplying by $2f'$ and then integrating, (8) gives

$$\nu(f'^2 + 4f^2) + (2/3) \times f^3 + 2\nu kf = (2/3) \times h, \qquad \ldots(9)$$

where h is a constant of integration. From (9), we have

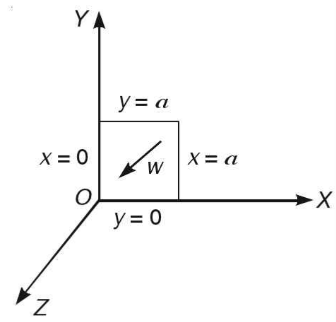
$$3\nu f'^2 = h - 6\nu f^2 - 3\nu kf - f^3, \qquad \text{giving} \qquad (df/d\theta)^2 = (2/3\nu) \times (h - 3\nu kf - 6\nu f^2 - f^3).$$

Ex. 5. *Viscous liquid is flowing steadily under pressure through an infinitely long rectangular tube whose axis is parallel to the axis of z. The sides x = 0, x = a are smooth and the sides y = 0, y = a do not permit of slipping of liquid in contact with them. The pressure gradient maintaining the motion is suddenly annulled, show that the total flux across any section is* $Qa^2/10\nu$, *where Q is the flux per unit time across a section in the initial steady motion. In obtaining the above result it may be assumed that* $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960}$. **[Garhwal 2001]**

Sol. The tube being infinitely long and the sides $x = 0$, $x = a$ being smooth, we have

$$u = v = 0 \quad \text{and} \quad w = w(y) \qquad \ldots(1)$$

The continuity equation is satisfied identically by (1) and the Navier-Stokes equations of motion for steady flow and in absence of body forces reduce to



$$\mu(d^2w/dy^2) = \partial p/\partial z \qquad \ldots(2)$$

$$\partial p/\partial x = 0 \quad \text{and} \quad \partial p/\partial y = 0 \qquad \ldots(3)$$

(3) shows that p is function of z only. So (1) reduces to

$$\mu(d^2w/dy^2) = dp/dz. \qquad \ldots(4)$$

Differentiation of (4) *w.r.t* '*z*' gives $\dfrac{d^2p}{dz^2} = 0$ or $\dfrac{dp}{dz}\left(\dfrac{dp}{dz}\right) = 0$

so that dp/dz = const. = P, say Then (4) gives $\mu(d^2w/dy^2) = P$ ...(5)

Integrating (5) twice *w.r.t.* '*y*', $\mu w = (1/2) \times Py^2 + Ay + B,$ ...(6)

where A and B are constants of integration

Now the boundary conditions are : $w = 0$ on $y = 0$; $w = 0$ on $y = a$. ...(7)

Using (7),(6) gives B = 0 and $0 = (Pa^2)/2 + Aa + B$

so that $A = -(Pa)/2$ and $B = 0$.

With these values (6) reduces to $w = (P/2\mu) \times (y^2 - ay)$ ...(8)

$$\therefore \quad Q = \int_0^a wady = \frac{aP}{2\mu}\int_0^a (y^2 - ay)dy = -\frac{a^4P}{12\mu} \quad \Rightarrow \quad P = -\frac{12\mu Q}{a^4}. \qquad \ldots(9)$$

$\therefore$ From (8), $w = (6Q/a^4) \times y\ (a - y)$.

When the pressure gradient is annulled, the equation of motion reduces to

$$\partial w/\partial t = \nu(\partial^2 w/\partial y^2) \qquad \ldots(11)$$

Also $u = v = 0$. In order to solve (11), we take

$$w = f(y)e^{-\nu k^2 t} \qquad \ldots(12)$$

$\therefore$ From (11), $d^2f/dy^2 = -k^2f$, showing that $f(y)$ is of the form of $\sin ky$ or $\cos ky$

$\therefore$ we take

$$w = \sum Ae^{-k^2\nu t} \left.\begin{matrix}\cos ky \\ \sin ky\end{matrix}\right\} \qquad \ldots(13)$$

But $w = (6Q/a^4) \times y\,(a - y)$ at $t = 0$. ...(14)

Expanding y $(a - y)$ in terms of Fourier series*, we have

$$y\,(a - y) = \frac{8a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\{(2n+1)\times(\pi y/a)\}}{(2n+1)^3}.$$

---

* Refer chapter 1 in author's "Integral Transforms" published by S. Chand and Co., New Delhi

$\therefore$ we choose $k = (2n+1) \times (\pi/a)$. So at any time t, we have

$$w = \frac{6Q}{a^4} \times \frac{8a^2}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin\{(2n+1)\times(\pi y/a)\}}{(2n+1)^3} e^{-\nu t\{(2n+1)\pi/a\}^2}.$$

The required flux

$$= \int_{t=0}^{\infty}\int_{y=0}^{a} wa\,dy\,dt = \frac{48Q}{\pi^3 a}\sum \frac{1}{(2n+1)^3}\int_0^{\infty} e^{-\nu t\{(2n+1)\pi/a\}^2} dt \times \int_0^a \sin\left\{(2n+1)\frac{\pi y}{a}\right\} dy$$

$$= \frac{48Q}{\pi^3 a}\sum \frac{1}{(2n+1)^3}\frac{1}{\nu\{(2n+1)\pi/a\}^2}\frac{2}{(2n+1)\pi/a} = \frac{48Q}{\pi^3 a}\times\frac{2a^3}{\nu\pi^3}\sum\frac{1}{(2n+1)^6} = \frac{96Qa^2}{\nu\pi^6}\times\frac{\pi^6}{960} = \frac{Qa^2}{10\nu}$$

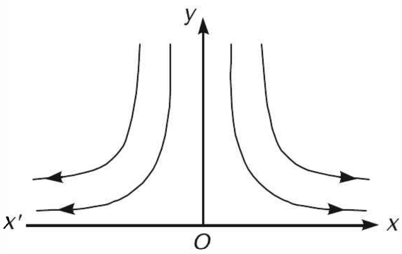
Ex.6. *Two-dimensional potential flow of an inviscid and incompressible fluid near the stangation point at the origin at a fixed point taken as y = 0, is given by u = bx. Show that the coresponding problem for a viscous liquid has the solution*

$$u = bx\frac{\partial\phi}{\partial\eta}, \qquad v = \sqrt{b\nu}\,\phi(\eta), \qquad \eta = y\sqrt{\frac{b}{\nu}}, \quad \textit{where} \quad \frac{d^3\phi}{d\eta^3} + \phi\frac{d^2\phi}{d\eta^2} - \left(\frac{d\phi}{d\eta}\right)^2 + 1 = 0,$$

with the boundary conditions : when $\eta = 0, \phi = 0 = d\phi/d\eta$ *and when* $\eta = \infty, d\phi/d\eta = 1.$

Sol. The potential flow for inviscid liquid is given by

$$u = bx, \qquad v = -by$$



By Bernoulli's equation, $p = p_0 - (1/2)\times\rho b^2(x^2+y^2)$,
where p_0 is the stagnation pressure.

For viscous liquid, we assume that

$$u = xf'(y), \qquad \text{where} \qquad f'(y), = df/dy. \qquad \ldots(1)$$

Hence from equation of continuity, we have $\qquad v = -f(y). \qquad \ldots(2)$

Also, $\qquad p = p_0 - (1/2)\times\rho b^2[x^2 + F(y)] \qquad \ldots(3)$

The Navier-Stokes equations of motion are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

and

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

so that $\qquad x f' - x f f'' = b^2 x + \nu x f''' \qquad \ldots(4)$

and $\qquad ff'' = (1/2)\times b^2 F' - \nu f'' \qquad \ldots(5)$

Boundary conditions are : $\qquad y = 0, \qquad u = v = 0 \qquad \ldots(6)$

i.e., $\qquad f = f' = 0$, using (1) and (2) $\qquad \ldots(7)$

and at orgin, $p = p_0$ hence $F = 0$. $\qquad \ldots(8)$

On $y = \infty$, $\qquad u = bx$, so $\qquad f' = b \qquad \ldots(9)$

We write $\qquad f(y) = A\phi(\eta), \qquad \ldots(10)$

where $\qquad \eta = \alpha y. \qquad \ldots(11)$

Then (4), after removing x from both sides, reduces to

$$A^2\alpha^2\phi'^2 - A^2\alpha^2\phi\phi'' = b^2 + \nu A\alpha^3\phi''' \qquad \ldots(12)$$

Assume that $$A^2\alpha^2 = b^2 + \nu A\alpha^3 \qquad \ldots(13)$$

Then, $A = \sqrt{\nu b}$ and $\alpha = \sqrt{b/\nu}$ ...(14)

$\therefore$ From (11), $$\eta = y\sqrt{b/\nu} \qquad \ldots(15)$$

and from (10), $$f(y) = \sqrt{b\nu}\,\phi(\eta). \qquad \ldots(16)$$

Using (13), (12) reduces to $$\phi''' + \phi\phi'' - \phi'^2 + 1 = 0$$

i.e., $$\frac{d^3\phi}{d\eta^2} + \phi\frac{d^2\phi}{d\eta^2} - \left(\frac{d\phi}{d\eta}\right)^2 + 1 = 0. \qquad \ldots(17)$$

Boundary conditions become :

when $\eta = 0$ $\quad \phi = 0 = \phi'$ $\quad$ *i.e.*, $\quad \phi = d\phi/d\eta = 0.$

when $\eta = \infty$ $\quad \phi' = 1$ $\quad$ *i.e.*, $\quad d\phi/d\eta = 1.$

Ex.7. *The liquid fills the space $z > 0$, being bounded by the plane $z = 0$ only, and this plane is rotating with constant angular velocity ω about the axis $r = 0$. verify that the steady motion is given by* $u = \omega r F(z_1)$, $\quad v = \omega r G(z_1)$, $\quad \omega = (\nu\omega)^{1/2} H(z_1)$,

$p = \rho\nu\omega P(z_1)$, where $z = (\nu/\omega)^{1/2} z_1$, *with the boundary conditions:*

$F(0) = 0$, $\quad G(0) = 1$, $\quad H(0) = 0$, $\quad F(\infty) = 0$, $\quad G(\infty) = 0$. **[Meerut 2006]**

Sol. The Navier-Stokes equations in cylindrical polar coordinates are (taking $q_r = u$, $q_\theta = v$ and $q_z = w$ and using results of Art. 14.11)

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2}\right) \qquad \ldots(1A)$$

$$u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2}\right) \qquad \ldots(2A)$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right) \qquad \ldots(3A)$$

Also, the equation of continuity in cylindrical polar coordinates is given by

$$(\partial u/\partial r) + (u/r) + (\partial w/\partial z) = 0 \qquad \ldots(4A)$$

Using the given substitutions, (1A) to (4A) reduce to

$$r\omega F\omega F + (\nu\omega)^{1/2} Hr\omega F'\left(\frac{\omega}{\nu}\right)^{1/2} - r\omega^2 G^2 = \nu\left[\frac{\omega}{r}F + r\omega F''\frac{\omega}{\nu} - \frac{\omega}{r}F\right]$$

i.e., $$F^2 + HF' - G^2 = F''. \qquad \ldots(1B)$$

$$r\omega F\omega G + (\nu\omega)^{1/2} Hr\omega G'\left(\frac{\omega}{\nu}\right)^{1/2} + r\omega FG = \nu\left(\frac{\omega}{r}G + r\omega G''\frac{\omega}{r} - \frac{\omega}{r}G\right)$$

i.e., $$2FG + HG' = G''. \qquad \ldots(2B)$$

$$(\nu\omega)^{1/2} H (\nu\omega)^{1/2} H' \left(\frac{\omega}{\nu}\right)^{1/2} = -\nu\omega P' \left(\frac{\omega}{\nu}\right)^{1/2} + \nu(\nu\omega)^{1/2} H'' \left(\frac{\omega}{\nu}\right)$$

i.e., $$HH' = -P' + H''. \qquad \ldots(2C)$$

and $$2\omega F + (\nu\omega)^{1/2} H' \times (\omega/\nu)^{1/2} = 0$$

i.e., $$2F + H' = 0. \qquad \ldots(2D)$$

The boundary conditions are

on	$z = 0$,	$u = 0$,	$v = Wr$,	$w = 0$,
at	$z = \infty$,	$u = 0$,	$v = 0$,	$w \neq 0$,
$\therefore$ on	$z_1 = 0$,	$F = 0$,	$G = 1$,	$H = 0$,
at	$z_1 = \infty$,	$G \to 0$,	and	$F \to 0$,

Also, ω will not vanish at $z = \infty$. In fact ω must tend to a finite negative limit.

The shearing stress on the plane

$$= \mu\left(\frac{dv}{\partial z}\right)_{z=0} = \rho\, \nu\omega r\, G'(z_1)\left(\frac{\omega}{\nu}\right)^{1/2} = \rho\,(\nu\omega^3)^{1/2} r\, G'(0)$$

and the moment $$= -2\int_0^a 2\pi r^2 \rho\,(\nu\omega^3)^{1/2} r G'(0)\, dr = -\pi a^4 \rho\,(\nu\omega^3)^{1/2} G'(0).$$

Exercises

1. Discuss the flow of an incompressible viscous fluid between two rotating concentric cylinders. **[Himanchal 2001]**

2. Determine velocity distribution for rotating Couette flow between two rotating cylinders. What is the magnitude and direction of the shearing stress at the inner cylinder.

3. Obtain the solution of Couette flow with a pressure gradient. Discuss the conditions under which the back-flow may occur near the wall which is a rest. What are its practical applications ?

4. (*a*) Obtain velocity profile for plane Couette flow. **[Kanpur 2004]**

(*b*) Discuss the steady flow a viscous incompressible fluid between parallel plates. Derive the expressions of skin friction at the plates. **[Himanchal 2001]**

5. (*a*) Obtain expression for velocity in case of generalized Couette flow. **[Kanpur 2002]**

(*b*) Define laminar flow. Discuss generalized plane Couette flow between parallel plates. Sketch the velocity profile. Determine the volumetric flow rate, shear stress and coeffcient of friction. **[Meerut 1999]**

6. (*a*) Determine the average velocity distribution, shear stress and drag coefficient for laminar flow in a plane Poiseuille flow. Sketch the velocity profile and name it. **[Meerut 2005]**

(*b*) Prove that the steady, laminar flow of a viscous incompressible fluid between two fixed infinite parallel plates fully developed flow is given by $u = -(h^2/2\mu) \times (dp/dx)\,(1 - y^2/h^2)$, when y is measured from the mid point. Also draw a sketch of the velocity profile. Determine the maximum and average velocity. **[Meerut 1998]**

(*c*) Discuss plane Poiseuille flow for parallel plates. **[Meerut 2001]**

7. Find the velocity distribution in the steady flow of a viscous incompressible fluid along an infinitely long circular pipe due to an applied pressure gradient. Also determine the expressions of volumetric rate of flow and coefficient of friction at the wall of the pipe. **[Himachal 2001, 02]**

8. (*a*) Discuss the steady flow between co-axial circular pipes. Also, determine the average velocity in the annulus, shear stress at the walls of the inner and outer cylinders. **[Meerut 2003]**

(*b*) A viscous incompressible fluid moves in a steady flow under constant presser gradient parallel to the axis in the annular space between the two co-axial cylinders of radii *a*, *b* (*b* < *a*). Find the volume rate of flow. **[Himanchal 2003]**

9. (*b*) Discuss the laminar flow problem of a viscous incompressible fluid, through infinite pipe of equilateral triangular cross section. **[Meerut 1998, 2000]**

(*b*) Discuss the steady flow of a viscous incompressible fluid through a tube of uniform equilateral triangular cross-section. Determine volume rate of flow and compare it with a circular cross-section of equal area. **[Himachal 1998]**

10. (*a*) Show that the velocity distribution in the steady flow of a viscous incompressible fluid through a tube of elliptic cross-section is given by

$$w = \frac{P}{2\mu}\frac{a^2b^2}{a^2+b^2}\left(1 - x^2/a^2 - y^2/b^2\right),$$

where the symbols have their usual meaning **[Himanchal 2000]**

(*b*) Discuss the steady flow of a viscous incompressible fluid through a tube of uniform elliptic cross-section. Show that the volume rate of flow is less as compared to a tube of circular cross-section of equal area. **[Himanchal 1998]**

11. Discuss the flow of an incompressible fluid through convergent and divergent channels. **[Himanchal 2000, 2001, 2003]**

12. (*a*) Discuss the flow due to an oscillating plate. Sketch the velocity profile. **[Meerut 2005]**

(b) Prove that $u\,(y,t) = Ue^{-y\sqrt{n/2v}}\cos\,(nt - y\sqrt{n/2v})$ for oscillating plate of unsteady flow, **[Meerut 2001, 02]**

(*c*) Viscous incompressible fluid occupies the region $y > 0$ on one side of an infinite plate $y = 0$. The plate oscillates with velocity *U* in the *x*-direction. Find the velocity distribution of the flow. **[Himanchal 2002]**

13. Discuss the flow due to plane wall suddenly set in motion in its plane in an infinite mass of viscous incompressible fluid. **[Himanchal 2000; 03, 06]**

14. Derive and discuss the velocity distribution for unsteady incompressible flow due to (*i*) a plane wall suddenly set in motion and (*ii*) an oscillating plane wall. **[Himanchal 2001]**

15. Discuss the unsteady flow of viscous incompressible fluid between two parallel plates. Also, determine the shear stress at any point at the lower and upper plates. Under what conditions the shear stress at the plates are infinite. **[Meerut 1998; 2003]**

16. Determine the velocity distribution in the pulsatile flow between two infinite parallel surfaces at distances $y = \pm h$ apart. **[Meerut 2000]**

17. (*a*) Water flows between two very large flat plates 1/2 cm apart. If the average velocity of the water is 1/2m/sec, what is the shearing stress at the lower plate, 1/8 cm and 1/4 cm from the lower plate

(*b*) Oil of dynamic viscosity 0.07 Kg/ms flows through a 20mm diameter pipe with a mean velocity of 0.5 m/s. Calculate the pressure drop which occurs over a length of 50 m of the pipe.

18. Discuss the steady flow problem of a viscous incompressible fluid through an infinite pipe of rectangular cross-section.

19. The space above the plane $y = 0$ is filled with liquid of kinematic viscosity ν. Initially the plane and the liquid are at rest. At $t = 0$, the plane begins to move parallel to itself with velocity U and the liquid also moves in the same direction. Show that equation of motion is $\partial u/\partial t = \nu(\partial^2 u/\partial t^2)$,

and that the appropriate solution is $u = U - (2U/\sqrt{\pi})\int_0^{\eta} e^{-t^2}\, dt,$ where $\eta = y/2\sqrt{\nu t}.$

20. Derive the following in case of steady laminar flow of an incompressible viscous fluid through a circular pipe: (*i*) Velocity distribution (*ii*) Volumetric rate of flow (*iii*) Coefficient of skin friction. **(Himanchal 2007)**

21. Derive velocity distribution, volume rate of flow and skin friction coefficient of the pipe in Hagen-Poiseuille flow. **(Himanchal 2007)**

22. Derive velocity distribution and torque required to turn the outer cylinder for flow between two concentric rotating cylinders. **(Himanchal 2006, 07)**

23. Derive velocity distribution in plane Couette flow, plane Poiseuille flow and generalized plane Couette flow and discuss in brief. **(Himanchal 2007)**

24. Write short notes on

(*i*) Flow due to plane wall suddenly set in motion **(Himanchal 2006, 09)**

(*ii*) Flow due to an oscillating plane wall **(Himanchal 2006)**

25. Discuss the generalized Couette flow. Draw the velocity profile. Determine volumetric flow rate, shear stress and coefficient of friction. **(Agra 2008, Meerut 2008)**

26. Discuss unsteady flow of a viscous incompressible flow over an oscillating plate and show that the velocity profile has the form of a damped harmonic oscillation. **(Meerut 2007, 08)**

27. Discuss the steady flow of a viscous incompressible fluid (in absence of external forces) in a divergent channel. **(Himanchal 2004)**

28. Derive and discuss velocity distribution in variable viscosity plane Couette flow. **(Himanchal 2003)**

29. Derive and discuss velocity and temperature distribution in variable viscosity plane Poiseuille flow of a viscous incompressible flow. **(Himanchal 2005).**

30. Discuss the steady flow of a viscous incompressible fluid (in absence of external forces) in a convergent channel. **(Himanchal 2004)**

31. Discuss plane Couette flow between two parallel plates. **(Agra 2006)**

32. Show that the velocity profile of steady flow of viscous fluid between the parallel plates is parabolic. **(Agra 2007)**

33. Discuss the steady flow between coaxial circular pipes. Also, determine the average velocity in the annulus, shear stress at the walls of the inner and outer cylinder. **(Agra 2006)**

34. Show that in the steady motion of a viscous fluid, $\left(\nabla^2 - \dfrac{1}{\nu} q \dfrac{\partial}{\partial s}\right)\left(\dfrac{p}{\rho} + V + \dfrac{1}{2}q^2\right) = (\text{curl}\, \boldsymbol{q})^2$,

where s is taken along a streamline, $\boldsymbol{q}$ is the fluid velocity, $\boldsymbol{q} = |\, q\,|$ and V is the force potential. **(Agra 2006)**

35. $dp/dr = 0$ is taken in (*a*) plane Couette flow.

(*b*) Plane-Poiseuille flow (*c*) Generalized planed Couette flow (*d*) Harmonic flow **(Agra 2006)**

Sol. Ans. (*a*) Refer. 16.3 A.

36. If viscous incompressible fluid is placed over an infinite flat plate which is suddenly accelerated, then discuss the unsteady motion of the fluid and obtain the expression of velocity in terms of error function. **[Meerut 2009, 10]**

CHAPTER

17 Theory of Very Slow Motion

17.1 Introduction.

Since the Navier-Stokes equations are non-linear, their exact solutions, in general, are not very easy. In what follows, we propose to discuss some approximate solutions of the Navier-Stokes equations which are valid in the limiting case when the viscous forces are considerably greater than the inertia forces. It can be easily seen from the Navier-Stokes equations of motion that the inertia forces (terms like $\rho U(\partial u/\partial x)$) are of the order $\rho U^2/L$ and viscous forces (terms like $\mu(\partial^2 u/\partial x^2)$) are of the order $\mu U/L^2$. Thus, we have

$$\frac{\text{Inrtia forces}}{\text{Viscous forces}} \sim \frac{(\rho U^2)/L}{(\mu U)/L^2} = \frac{UL}{\nu} = \text{Re}, \qquad \ldots(1)$$

where $\qquad \text{Re} = \text{Reynolds number} = (UL\rho)/\mu = (UL)/\nu, \qquad \ldots(2)$

where U, L, ρ, μ, ν are respectively some characteristic value of velocity, some characteristic value of length, density, coefficient of viscosity and kinematic viscosity of the fluid.

Form (1), it follows that when the Reynolds number is very small (Re << 1), *i.e.*, the characteristic length L and the characteristic velocity U of the body are small or the kinemetic viscosity ν of the fluid is large, then the viscous forces will be considerably greater than the inertia forces and as a first approximation the inertia terms may be neglected altogether from the Navier-Stokes equations. The equations so obtained are called Stokes' equations and, being linear, are easier to solve than the full Navier Stokes' equations. Since the order of Stokes' equations remains the same as that of full Navier-Stokes' equations, it follows that Stokes' equation must be supplemented with the same boundary conditions as the full Navier-Stokes equations, namely those expressing the slip in the fluid at the walls, *i.e.*, the vanishing of the normal and tangential components of velocity:

$$q_n = 0: \qquad \text{and} \qquad q_t = 0 \qquad \text{at walls} \qquad \ldots(3)$$

Fortunately, apart form special cases, motions at *very low Reynolds numbers*, sometimes also *known as creeping motion*, do not occur too often in practical applications.

Solution of Stokes' equations for the flow past a sphere was first obtained by Stokes.

In 1910, an improvement of Stokes solution was later presented by Oseen, who took the inertial terms in the Navier-Stokes equations partly into account and improved the picture of the flow field. We shall discuss Oseen's equations and Oseen approximation in Art. 17.4 and Art. 17.5.

In the end, we shall discuss application of the theory of slow motion in the hydrodynamic theory of lubrication in Art. 17.10 and 17.11 and 17.11 A.

17.2 Stokes' equations and Stokes approximation

[Himanchal 1999, 2000, 03, 07; Meerut 2007]

In the absence of the body forces, the Navier-Stokes equation, for steady flow of an incompressible fluid, is given by (refer equation (17), Art. 14.1)

$$(\mathbf{q}\cdot\nabla)\,\mathbf{q} = -(1/\rho)\,\nabla p + \nu\nabla^2\mathbf{q}$$

When inertia terms are neglected altogether from the above Navier-Stokes equations, we get

$$0 = -(1/\rho)\nabla p + (\mu/\rho)\nabla^2 \mathbf{q} \qquad \text{or} \qquad \mu\nabla^2\mathbf{q} = \nabla p \quad ...(1)$$

Re-written (1) in terms of certesian coordinates, we get

$$\mu(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2) = \partial p/\partial x \quad ...(1a)$$

$$\mu(\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 + \partial^2 v/\partial z^2) = \partial p/\partial y \quad ...(1b)$$

$$\mu(\partial^2 w/\partial x^2 + \partial^2 w/\partial y^2 + \partial^2 w/\partial z^2) = \partial p/\partial z \quad ...(1c)$$

Equation (1) and equations (1*a*) – (1*c*) are called *Stokes' equations* and the approximation involved is called *Stokes approximation*

Equation of continuity for incompressible fluid is given by

$$\text{div}\ \mathbf{q} = 0, \qquad i.e., \qquad \nabla\cdot\mathbf{q} = 0 \quad ...(2)$$

ie.

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \quad ...(2a)$$

Taking the divergence of both sides of (1), we have

$$div\ (\nabla p) = div(\mu\nabla^2\mathbf{q}) \qquad \text{or} \qquad \nabla\cdot(\nabla p) = \mu\nabla^2(div\,\mathbf{q}), \quad ...(3)$$

where we have used the fact that the operations div and ∇^2 on the right hand side may be performed in the reversed order.

Form (2) and (3), we have

$$\nabla^2 p = 0, \quad ...(4)$$

showing that for very slow motion the pressure p satisfies *Laplace's equation* (*potential equation*), and is, therefore a *harmonic function*.

17.3 Stokes' flow past a sphere

[Agra 2005, 07; Himanchel 2000, 01, 02, 03, 04, 05, 06, 07; Meerut 2000, 05, 07]

As shown in the figure (*i*), let a solid sphere of radius '*a*' be held fixed in a uniform stream flowing past it with velocity U_0. Let the flow be steady and let there be no body forces. Then Stokes' equations (refer (1*a*) to (1*c*) of Art. 17.2) and the continuity equation (refer (2a) of Art 17.2) are given by

$$\mu\nabla^2 u = \partial p/\partial x \quad ...(1a)$$

$$\mu\nabla^2 v = \partial p/\partial y \quad ...(1b)$$

$$\mu\nabla^2 w = \partial p/\partial z \quad ...(1c)$$

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0, \quad ...(1d)$$

where

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2. \quad ...(1e)$$

Let U_0 and p_0 denote the velocity and the fluid pressure at infinity. Then we are to solve (1*a*) to (1*d*) under the following boundary conditions:

$$u = v = w = 0 \qquad \text{at} \qquad r = a \quad ...(2a)$$

$$u = U_0, \quad v = w = 0 \quad p = p_0 \quad \text{at} \quad r = \infty \quad ...(2b)$$

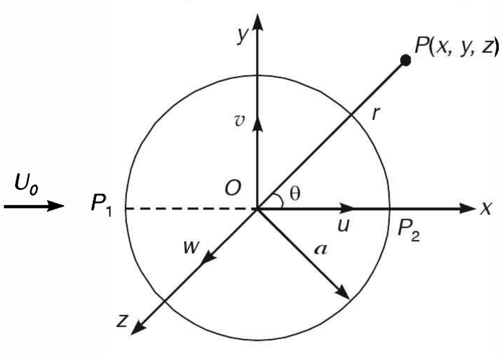


Fig. (i)

For Stokes's equations, we know that p is a harmonic function satisfying the Laplace's equation

$$\nabla^2 p = 0,\ i.e.,\ \partial^2 p/\partial x^2 + \partial^2 p/\partial y^2 + \partial^2 p/\partial z^2 = 0 \quad ...(3)$$

Let the centre O of the sphere be taken as the origin of the coordinate system, and let x-axis be in the direction of the uniform stream far away the sphere. Also, let

$$r = (x^2 + y^2 + z^2)^{1/2} \quad ...(4)$$

be the distance of an arbitrary point $P(x, y, z)$ from the origin O.

The pressure on the side where the fluid flows towards the sphere (*i.e*, negative x-axis) will be greater, and on the other side where it flows away from the sphere (*i.e*, positive x-axis) will be less than the undisturbed pressure p_0, *i.e.*, pressure disturbance, p, must be positive for negative values of x and vice versa. Furthermore, the pressure disturbances must die away at infinity. Clearly, the appropriate harmonic function satisfying the above conditions may be taken as

$$p = p_0 - (Ax / r^3), \qquad \text{...(5)}$$

where A is a constant to be determined.

Let F be any function of r, where r is given by (4).

Then, $$\frac{\partial F}{\partial x} = \frac{dF}{dr}\frac{\partial r}{\partial x} = \frac{dF}{dr} \times \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \times 2x = \frac{x}{r}\frac{dF}{dr}$$

Thus, $$\frac{\partial}{\partial x} \equiv \frac{x}{r}\frac{d}{dr}. \quad \text{Similarly,} \quad \frac{\partial}{\partial y} \equiv \frac{y}{r}\frac{d}{dr}, \quad \frac{\partial}{\partial z} \equiv \frac{z}{r}\frac{d}{dr} \qquad \text{...(6)}$$

From (5), $$\frac{\partial p}{\partial x} = -A\frac{\partial}{\partial x}(xr^{-3}) = -A\left(r^{-3} + x\frac{\partial r^{-3}}{\partial x}\right) = -A\left(\frac{1}{r^3} + x \times \frac{x}{r}\frac{dr^{-3}}{dr}\right), \text{ by (6)}$$

Thus, $$\frac{\partial p}{\partial x} = -A\left\{\frac{1}{r^3} + \frac{x^2}{r} \times (-3r^{-4})\right\} = A\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right) \qquad \text{...(7}a\text{)}$$

Again, from (5), $$\frac{\partial p}{\partial y} = -Ax\frac{\partial r^{-3}}{\partial y} = -Ax \times \frac{y}{r}\frac{dr^{-3}}{dr}, \text{ by (6)}$$

Thus, $$\partial p / \partial y = -Ax \times (y / r) \times (-3r^{-4}) = (3Axy)/r^5 \qquad \text{...(7}b\text{)}$$

Similarly, $$\partial p / \partial z = (3Axz) / r^5 \qquad \text{...(7}c\text{)}$$

Using (7a) to (7c), (1a) to (1c) may be rewritten as

$$\nabla^2 u = (A / \mu) \times (3x^2 / r^5 - 1 / r^3) \qquad \text{...(8}a\text{)}$$

$$\nabla^2 v = (A / \mu) \times (3xy / r^5) \qquad \text{...(8}b\text{)}$$

$$\nabla^2 w = (A / \mu) \times (3xz / r^5) \qquad \text{...(8}c\text{)}$$

Equations (8a) to (8c) are linear differential equations, so we proceed to obtain their particular integrals keeping the following points of symmetry:

We observe that the velocity component u, starting as the undisturbed velocity U_0 far away from the sphere and then starts from zero and tends to U_0 at a large distance from it. It follows that the flow must be symmetrical about yz-plane and also about x-axis. Consequently, u must be even function of x, y and z. Again, the velocity component v, for positive y, must be positive on the inflow side and negative on the outflow side, and for negative y, it must be exactly opposite. Hence v must be even in z but odd in x and y. Likewise, w must be even in y but odd in x and z.

The following dimensional consideration will be employed to obtain the particular intergrals of (8a) to (8c). Observe that the operator ∇^2 changes the dimension of the expression, on which it operates, in th same way as division by $1/r^2$. Since the expressions on the right hand sides of (8a) to (8c) are of dimensions $1/r^3$, it follows that the particular integrals of u, v and w must be expression of dimensions $1/r$. To start with, we consider the simplest such expression such expression as $1/r$. But $\nabla^2(1/r) = 0$ and hence $1/r$ cannot be taken as a particular integral. The next simplest expression having the desired symmetry for u is x^2/r^3.

Now, $$\frac{\partial}{\partial x}\left(\frac{x^2}{r^3}\right)=\frac{2x}{r^3}+x^2\frac{\partial r^{-3}}{\partial x}=\frac{2x}{r^3}+x^2\times\frac{x}{r}\frac{dr^{-3}}{dr},\ \text{by (6)}$$
$$= 2x/r^3 + (x^3/r) \times (-3r^{-4}) = 2x/r^3 - 3x^3/r^5 \qquad \text{...(9}a\text{)}$$

$$\therefore \quad \frac{\partial^2}{\partial x^2}\left(\frac{x^2}{r^3}\right)=\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\left(\frac{x^2}{r^3}\right)\right\}=\frac{\partial}{\partial x}\left(\frac{2x}{r^3}-\frac{3x^3}{r^5}\right),\ \text{by (9}a\text{)}$$
$$=2\frac{\partial(xr^{-3})}{\partial x}-3\frac{\partial(x^3r^{-5})}{\partial x}=2\left(r^{-3}+x\frac{\partial r^{-3}}{\partial x}\right)-3\left(3x^2r^{-5}+x^3\frac{\partial r^{-5}}{\partial x}\right)$$
$$=2\left(r^{-3}+x\times\frac{x}{r}\frac{dr^{-3}}{dr}\right)-3\left(3x^2r^{-5}+x^3\times\frac{x}{r}\frac{dr^{-5}}{dr}\right),\ \text{by (6)}$$
$$=\frac{2}{r^3}-\frac{6x^2}{r^5}-\frac{9x^2}{r^5}+\frac{15x^4}{r^7}=\frac{2}{r^3}-\frac{15x^2}{r^5}+\frac{15x^4}{r^7} \qquad \text{...(9}b\text{)}$$

Again, $$\frac{\partial}{\partial y}\left(\frac{x^2}{r^3}\right)=x^2\frac{\partial r^{-3}}{\partial y}=x^2\times\frac{y}{r}\frac{dr^{-3}}{dr},\ \text{by (6)}$$
$$= x^2 \times (y/r) \times (-3r^{-4}) = -(3x^2y/r^5) \qquad \text{...(9}c\text{)}$$

$$\therefore \quad \frac{\partial^2}{\partial y^2}\left(\frac{x^2}{r^3}\right)=\frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}\left(\frac{x^2}{r^3}\right)\right\}=\frac{\partial}{\partial y}(-3x^2yr^{-5}),\ \text{by (9}c\text{)}$$
$$=-3x^2\left(r^{-5}+y\frac{\partial r^{-5}}{\partial y}\right)=-3x^2\left(r^{-5}+y\times\frac{y}{r}\frac{dr^{-5}}{dr}\right),\ \text{by (6)}$$
$$=-3x^2\left(\frac{1}{r^5}-\frac{5y^2}{r^7}\right)=-\frac{3x^2}{r^5}+\frac{15x^2y^2}{r^7} \qquad \text{...(9}d\text{)}$$

Similarly, $$\frac{\partial^2}{\partial z^2}\left(\frac{x^2}{r^3}\right)=-\frac{3x^2}{r^5}+\frac{15x^2z^2}{r^7} \qquad \text{...(9}c\text{)}$$

$\therefore$ Thus, $$\nabla^2\left(\frac{x^2}{r^3}\right)=\frac{\partial}{\partial x^2}\left(\frac{x^2}{r^3}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{x^2}{r^3}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{x^2}{r^3}\right)$$
$$=\frac{2}{r^3}-\frac{15x^2}{r^5}+\frac{15x^4}{r^7}-\frac{3x^2}{r^5}+\frac{15x^2y^2}{r^7}-\frac{3x^2}{r^5}+\frac{15x^2z^2}{r^7}$$
[Using (9*b*), (9*d*) and (9*e*)]
$$=\frac{2}{r^3}-\frac{21x^2}{r^5}+\frac{15x^2(x^2+y^2+z^2)}{r^7}=\frac{2}{r^3}-\frac{21x^2}{r^5}+\frac{15x^2r^2}{r^7}$$

Thus, $$\nabla^2(x^2/r^3)=-2(3x^2/r^5-1/r^3),$$
which when multiplied by $-(A/2\mu)$ yields exactly the terms in $\nabla^2 u$.

As before a simple expression for v, having the required symmetry and dimensions is xy/r^3

Now, $$\frac{\partial}{\partial x}\left(\frac{xy}{r^3}\right)=y\frac{\partial(xr^{-3})}{\partial x}=y\left[r^{-3}+x\frac{\partial r^{-3}}{\partial x}\right]=y\left[r^{-3}+x\times\frac{x}{r}\frac{dr^{-3}}{dr}\right],\ \text{by (6)}$$
$$= y\,(1/r^3 - 3x^2/r^5) \qquad \text{...(10}a\text{)}$$

$$\therefore \qquad \frac{\partial^2}{\partial x^2}\left(\frac{xy}{r^3}\right)=\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\left(\frac{xy}{r^3}\right)\right\}=\frac{\partial}{\partial x}y\left(\frac{1}{r^3}-\frac{3x^2}{r^5}\right),\ \text{by (10}a\text{)}$$

$$=y\left(\frac{\partial r^{-3}}{\partial x}-3\frac{\partial(x^2r^{-5})}{\partial x}\right)=y\frac{\partial r^{-3}}{\partial x}-3y\frac{\partial(x^2r^{-5})}{\partial x}=y\times\frac{x}{r}\frac{dr^{-3}}{dr}-3y\left(2xr^{-5}+x^2\times\frac{x}{r}\frac{dr^{-5}}{dr}\right)$$

$$=y\times(x/r)\times(-3r^{-4})-3y\{2xr^{-5}+x^2\times(x/r)\times(-5r^{-6})\}=-(9xy)/r^5+(15x^3y)/r^7 \qquad \text{...(10}b\text{)}$$

Similarly,
$$\frac{\partial^2}{\partial y^2}\left(\frac{xy}{r^3}\right)=-\frac{9xy}{r^5}+\frac{15y^3x}{r^7} \qquad \text{...(10}c\text{)}$$

Finally,
$$\frac{\partial}{\partial z}\left(\frac{xy}{r^3}\right)=xy\frac{\partial r^{-3}}{\partial z}=xy\times\frac{z}{r}\frac{dr^{-3}}{dr},\ \text{by (6)}$$

$$=(xy)\times(z/r)\times(-3r^{-4})=-(3xyz)/r^5 \qquad \text{...(10}d\text{)}$$

$$\therefore \qquad \frac{\partial^2}{\partial z^2}\left(\frac{xy}{r^3}\right)=\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial z}\left(\frac{xy}{r^3}\right)\right\}=\frac{\partial}{\partial z}\left(-\frac{3xyz}{r^5}\right)=-3xy\frac{\partial(zr^{-5})}{\partial z}=-3xy\left(r^{-5}+z\frac{\partial r^{-5}}{\partial z}\right),\ \text{by (10d)}$$

$$=-3xy\left(r^{-5}+z\times\frac{z}{r}\frac{dr^{-5}}{dr}\right)=-3xy\left(\frac{1}{r^5}-\frac{5z^2}{r^7}\right)=-\frac{3xy}{r^5}+\frac{15xyz^2}{r^7} \qquad \text{... (10}e\text{)}$$

Hence,
$$\nabla^2\left(\frac{xy}{r^3}\right)=\frac{\partial^2}{\partial x^2}\left(\frac{xy}{r^3}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{xy}{r^3}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{xy}{r^3}\right)$$

$$=\frac{15x^3y}{r^7}-\frac{9xy}{r^5}+\frac{15y^3x}{r^7}-\frac{9xy}{r^5}-\frac{3xy}{r^5}+\frac{15xyz^2}{r^7}$$

[Using (10b), (10c) and (10e)]

$$=\frac{15xy(x^2+y^2+z^2)}{r^7}-\frac{21xy}{r^5}=\frac{15xyr^2}{r^7}-\frac{21xy}{r^5}=-\frac{6xy}{r^5},$$

which when multiplied by $-(A/2\mu)$ yields exactly the terms in $\nabla^2 v$. Likewise, we can show that a simple expression for w, having the required symmetry and dimensions is xz/r^3. As before, we can prove that
$$\nabla^2(xz/r^3)=-(6xz)/r^5,$$
which when multiplied by $-(A/2\mu)$ yields exactly the terms in $\nabla^2 w$.

From the above discussion, it follows that a particular solution for u, v, w can be taken as

$$-(A/2\mu)\times(x^2/r^3), \qquad -(A/2\mu)\times(xy/r^3) \qquad \text{and} \qquad -(A/2\mu)\times(xz/r^3) \qquad \text{...(11)}$$

To these solutions (11), we must now add appropriate solutions (harmonic function) of the auxiliary equations $\nabla^2 u=0$, $\nabla^2 v=0$ and $\nabla^2 w=0$, so as to satisfy the boundary conditions (2a).

For u, we add U_0, $\quad \dfrac{1}{r} \quad$ and $\quad \dfrac{\partial}{\partial x^2}\left(\dfrac{1}{r}\right)$, $\quad$ *i.e.*, $\quad \dfrac{3x^2}{r^5}-\dfrac{1}{r^3} \qquad$ ...(12a)

$$\therefore \qquad \left[\because\ \nabla^2\left(\frac{1}{r}\right)=0 \quad \text{and} \quad \nabla^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)=\frac{\partial^2}{\partial x^2}\nabla^2\left(\frac{1}{r}\right)=0\right]$$

for v, we add $\dfrac{\partial^2}{\partial x \partial y}\left(\dfrac{1}{r}\right)$, *i.e.*, $\dfrac{3xy}{r^5}$ ...(12*b*)

and for w, we add $\dfrac{\partial^2}{\partial x \partial z}\left(\dfrac{1}{r}\right)$, *i.e.*, $\dfrac{3xz}{r^5}$ ...(12*c*)

$$\left[\therefore\ \nabla^2 \frac{\partial^2}{\partial x \partial y}\left(\frac{1}{r}\right) = \frac{\partial^2}{\partial x \partial y}\nabla^2\left(\frac{1}{r}\right) = 0 \quad \text{and} \quad \nabla^2 \frac{\partial^2}{\partial x \partial z}\left(\frac{1}{r}\right) = \frac{\partial^2}{\partial x \partial z}\nabla^2\left(\frac{1}{r}\right) = 0\right]$$

In view of the facts (11) and (12*a*) to (12*c*), the solutions of (8*a*) to (8*c*) are given by

$$u = -\frac{A}{2\mu}\frac{x^2}{r^3} + U_0 + \frac{B}{r} + C\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right) \qquad \text{...(13}a\text{)}$$

$$v = -\frac{A}{2\mu}\frac{xy}{r^3} + 3D\frac{xy}{r^5} \qquad \text{...(13}b\text{)}$$

$$w = -\frac{A}{2\mu}\frac{xz}{r^3} + 3D\frac{xz}{r^5}, \qquad \text{...(13}c\text{)}$$

where B, C and D are constants to be determined.

Substituting the above values of u, v, w in (1*d*), we get

$$-\frac{A}{2\mu}\left(\frac{\partial}{\partial x}\frac{x^2}{r^3} + \frac{\partial}{\partial y}\frac{xy}{r^3} + \frac{\partial}{\partial z}\frac{xz}{r^3}\right) + B\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + 3C\frac{\partial}{\partial x}\left(\frac{x^2}{r^5}\right) - C\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) + 3D\left(\frac{\partial}{\partial y}\frac{xy}{r^5} + \frac{\partial}{\partial z}\frac{xz}{r^5}\right) = 0 \quad \text{...(14)}$$

Using (9*a*), $$\frac{\partial}{\partial x}\left(\frac{x^2}{r^3}\right) = \frac{2x}{r^3} - \frac{3x^3}{r^5} \qquad \text{...(15}a\text{)}$$

Again, $$\frac{\partial}{\partial y}\left(\frac{xy}{r^3}\right) = x\frac{\partial(yr^{-3})}{\partial y} = x\left(r^{-3} + y\frac{\partial r^{-3}}{\partial y}\right) = x\left(r^{-3} + y \times \frac{y}{r}\frac{d}{dr}r^{-3}\right)$$

$$= x\{r^{-3} + (y^2/r) \times (-3r^{-4})\} = x/r^3 - (3xy^2)/r^5 \qquad \text{...(15}b\text{)}$$

Similarly, $$\frac{\partial}{\partial z}\left(\frac{xz}{r^3}\right) = \frac{x}{r^3} - \frac{3xz^2}{r^5} \qquad \text{...(15}c\text{)}$$

Also, $$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial r^{-1}}{\partial x} = \frac{x}{r}\frac{dr^{-1}}{dr} = -\frac{x}{r^3} \qquad \text{..(15}d\text{)}$$

$$\frac{\partial}{\partial x}\left(\frac{x^2}{r^5}\right) = \frac{2x}{r^5} + x^2\frac{\partial r^{-5}}{\partial x} = \frac{2x}{r^5} + x^2 \times \frac{x}{r}\frac{dr^{-5}}{dr} = \frac{2x}{r^5} - \frac{5x^3}{r^7} \qquad \text{...(15}e\text{)}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) = \frac{x}{r}\frac{dr^{-3}}{dr} = \frac{x}{r} \times (-3r^{-4}) = -\frac{3x}{r^5} \qquad \text{...(15}f\text{)}$$

and $$\frac{\partial}{\partial y}\left(\frac{xy}{r^5}\right) = x\frac{\partial(yr^{-5})}{\partial y} = x\left(r^{-5} + y \times \frac{y}{r}\frac{dr^{-5}}{dr}\right) = \frac{x}{r^5} - \frac{5xy^2}{r^7} \qquad \text{...(15}g\text{)}$$

Similarly, $$\frac{\partial}{\partial y}\left(\frac{xz}{r^5}\right) = \frac{x}{r^5} - \frac{5xz^2}{r^7} \qquad \text{... (15h)}$$

Using (15*a*) to (15*h*) in (14), we obtain

$$-\frac{A}{2\mu}\left(\frac{2x}{r^3}-\frac{3x^3}{r^5}+\frac{x}{r^3}-\frac{3xy^2}{r^5}+\frac{x}{r^3}-\frac{3xz^2}{r^5}\right)-\frac{Bx}{r^3}+3C\left(\frac{2x}{r^5}-\frac{5x^3}{r^7}\right)+\frac{3Cx}{r^5}$$

$$+3D\left(\frac{x}{r^5}-\frac{5xy^2}{r^7}+\frac{x}{r^5}-\frac{5xz^2}{r^7}\right)=0$$

or $$-\frac{A}{2\mu}\left\{\frac{4x}{r^3}-\frac{3x(x^2+y^2+z^2)}{r^5}\right\}-\frac{Bx}{r^3}+(9C+6D)\frac{x}{r^5}-[15Cx^2+15D(y^2+z^2)]\times\frac{x}{r^7}=0$$

or $-(Ax)/(2\mu r^3)-(Bx)/r^3+(9C+6D)\times(x/r^5)-[15Cx^2+15D(y^2+z^2)]\times(x/r^7)=0,$ ...(16)

where we have used the fact that $r^2 = x^2 + y^2 + z^2$.

Putting $C = D$ in (16) and noting that $r^2 = x^2 + y^2 + z^2$, we see that the last two terms in (16) cancel. Furthermore, if we take $B=-(A/2\mu)$ we find that the first two terms in (16) also cancel. Thus, the equation of continuity is satisfied if we assume that

$$C = D \qquad \text{and} \qquad B=-A/2\mu \qquad \text{...(17)}$$

Substituting the above value of B in (13*a*) to (13*c*) and then using the boundary conditions (2a), namely, $u = v = w = 0$ when $r = a$, we obtain

$$0=-\frac{A}{2\mu}\frac{x^2}{a^3}+U_0-\frac{A}{2\mu a}+C\left(\frac{3x^2}{a^5}-\frac{1}{a^3}\right) \qquad \text{...(18a)}$$

$$0=-\frac{A}{2\mu}\frac{xy}{a^3}+3D\frac{xy}{a^5} \qquad \text{...(18b)}$$

$$0=-\frac{A}{2\mu}\frac{xz}{a^3}+3D\frac{xz}{a^5} \qquad \text{...(18c)}$$

Then, (18a) and 18*c*) $\Rightarrow$ $D=(Aa^2)/6\mu$ ...(18*d*)

Hence, by (17), $C=(Aa^2)/6\mu$ ... (18e)

Substituting the above value of C in (18*a*), we have

$$0=-\frac{Ax^2}{2\mu a^3}+U_0-\frac{A}{2\mu a}+\frac{Aa^2}{6\mu}\left(\frac{3x^2}{a^5}-\frac{1}{a^3}\right)$$

or $0=U_0-(2A)/3\mu a$ and $A=(3/2)\times\mu a U_0$ ...(18*f*)

Also, $B=-A/2\mu=-(3/4)\times aU_0$ ...(18*g*)

Thus, $A=(3/2)\times\mu a U_0$ and $C=D=(Aa^2)/6\mu$ ...(19)

Substituting these values of A, B, C and D in (5) and (13*a*) to (13*c*), the velocity components and the pressure distribution is given by

$$u=U_0\left\{\frac{3}{4}\frac{ax^2}{r^3}\left(\frac{a^2}{r^2}-1\right)+1-\frac{a}{4r}\left(3+\frac{a^2}{r^2}\right)\right\} \qquad \text{...(20a)}$$

$$v=U_0\frac{3}{4}\frac{axy}{r^3}\left(\frac{a^2}{r^2}-1\right) \qquad \text{...(20b)}$$

$$w = U_0 \frac{3}{4} \frac{axz}{r^3} \left(\frac{a^2}{r^2} - 1 \right) \qquad \text{...(20c)}$$

and
$$p = p_0 - \frac{3}{2} \frac{\mu U_0 ax}{r^3} \qquad \text{...(21)}$$

From (21), it follows that the pressure on the surface of the given sphere ($r = a$) is given by

$$p = p_0 - \frac{3\mu U_0 x}{2a^2} \qquad \text{...(22)}$$

Clearly, the maximum and minimum pressures occur at points $P_1(x = -a)$ and P_2 $(x = a)$ respectively, their values being

$$(p)_{\max} = p_0 + (3\mu U_0)/2a; \qquad \text{and} \qquad (p)_{\min} = p_0 - (3\mu U_0)/2a \qquad \text{...(23)}$$

From the above fact, it follows that the fluid exerts a force in the x-direction. The total force (drag) D acting on the sphere will be equal to the surface integral of the normal and tangential stresses acting on it in a meridian plane.

Let $OP = r$. Refer figure (*ii*). Let PM be perpendicular on yz plane. Let PN be parallel to Ox. Let ML and MK be parallel to Oz and Oy respectively.

Let $\angle PON = \theta$ and $\angle MOL = \phi$. Let $OM = \tilde{\omega}$. Also. $ON = x$, $OL = y$ and $OK = z$.

$$\left.\begin{aligned} &\text{From } \Delta MOL, \quad \tilde{\omega}^2 = y^2 + z^2. \qquad \text{Also,} \quad \tilde{\omega} = r\sin\theta \\ &\text{Also, we have} \quad x = r\cos\theta, \quad y = r\sin\theta\cos\phi, \quad z = r\sin\theta\sin\phi \end{aligned}\right\} \qquad \text{...(24)}$$

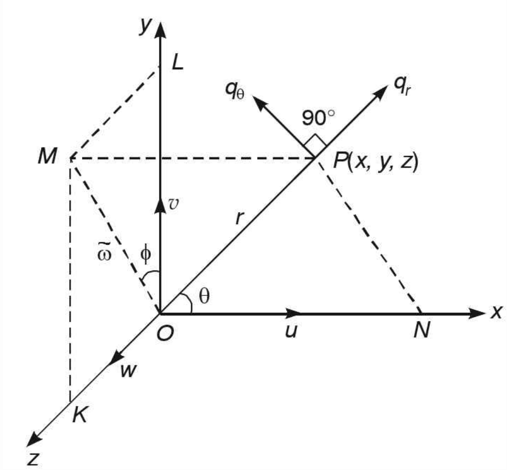


Fig.(ii) Relation between cartesian and polar components of velocity for Stokes' flow past a sphere

The normal stress on the surface of the sphere (using results (4) and (5) of Art. 14.12 and noting that $\nabla \cdot q = 0$) is given by

$$(\sigma_{rr})_{r=a} = [-(p - p_0) + 2\mu \in_{rr}]_{r=a} = [-p + p_0 + 2\mu(\partial q_r / \partial r)]_{r=a} \qquad \text{...(25)}$$

Again the shearing stress on the surface of the sphere (using results (4) and (5) of Art 14.12 and noting that $\nabla \cdot \mathbf{q} = 0$) is given by

$$(\sigma_{r\theta})_{r=a} = (\mu\gamma_{r\theta})_{r=a} = \mu \left[r \frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right) + \frac{1}{r}\frac{\partial q_r}{\partial \theta} \right]_{r=a} \qquad \text{...(26)}$$

In (25) and (26), q_r and q_θ are the radial and tangential components of the velocity in a meridian plane. As shown in the figure (*ii*), we now proceed to compute q_r and q_θ in terms of u, v, w. We have,

$$q_r = u\cos\theta + (v\cos\phi + w\sin\phi)\sin\theta = u\,(x/r) + v(y/r) + w(z/r), \text{ using (24)}$$

$$= U_0\left\{\frac{3}{4}\frac{ax^3}{r^4}\left(\frac{a^2}{r^2}-1\right)+\frac{x}{r}-\frac{ax}{4r^2}\left(3+\frac{a^2}{r^2}\right)\right\} + U_0\frac{3}{4}\frac{axy^2}{r^4}\left(\frac{a^2}{r^2}-1\right)+U_0\frac{3}{4}\frac{axz^2}{r^4}\left(\frac{a^2}{r^2}-1\right),$$

[using (20*a*), (20*b*) and (20*c*)]

$$= \frac{3U_0ax}{4r^4}\left(\frac{a^2}{r^2}-1\right)(x^2+y^2+z^2)+\frac{U_0x}{r}-\frac{U_0ax}{4r^2}\left(3+\frac{a^2}{r^2}\right)$$

$$= \frac{3U_0ar\cos\theta}{4r^4}\left(\frac{a^2}{r^2}-1\right)r^2+\frac{U_0r\cos\theta}{r}-\frac{U_0ar\cos\theta}{4r^2}\left(3+\frac{a^2}{r^2}\right)$$

[$\because\ x^2+y^2+z^2=r^2$ and $x = r\cos\theta$]

$$= U_0\cos\theta\left\{\frac{3a}{4r}\left(\frac{a^2}{r^2}-1\right)+1-\frac{a}{4r}\left(3+\frac{a^2}{r^2}\right)\right\}$$

Thus, $$q_r = U_0\cos\theta\,(1-3a/2r+a^3/2r^3) \qquad ...(27)$$

Similarly, $q_\theta = (v\cos\phi + w\sin\phi)\cos\theta - u\sin\theta$

$= v\,(xy/\tilde{\omega}r) + w\,(zx/\tilde{\omega}r) - u\,(\tilde{\omega}/r)$, using (24)

$$= \frac{3U_0x^2y^2}{4r^3\tilde{\omega}}\left(\frac{a^2}{r^2}-1\right)+\frac{3aU_0x^2z^2}{4r^3\tilde{\omega}}\left(\frac{a^2}{r^2}-1\right)-U_0\left\{\frac{3ax^2\tilde{\omega}}{4r^4}\left(\frac{a^2}{r^2}-1\right)+\frac{\tilde{\omega}}{r}-\left(\frac{a\tilde{\omega}}{4r^2}\right)\times\left(3+\frac{a^2}{r^2}\right)\right\},$$

[using (20*a*) (20*b*) and (20*c*)]

$$= \frac{3aU_0x^2}{4r^3\tilde{\omega}}\left(\frac{a^2}{r^2}-1\right)(y^2+z^2)-\frac{3aU_0x^2\tilde{\omega}}{4r^2}\left(\frac{a^2}{r^2}-1\right)-\left(\frac{U_0\tilde{\omega}}{r}\right)+\frac{U_0a\tilde{\omega}}{4r^2}\left(3+\frac{a^2}{r^2}\right)$$

$$= \frac{3aU_0x^2}{4r^3\tilde{\omega}}\left(\frac{a^2}{r^2}-1\right)\tilde{\omega}^2-\frac{3aU_0x^2\tilde{\omega}}{4r^2}\left(\frac{a^2}{r^2}-1\right)-U_0\sin\theta+\{(U_0a\sin\theta)/4r\}\times(3+a^2/r^2),$$

[$\because$ from (24) $y^2+z^2=\tilde{\omega}^2$ and $\tilde{\omega} = r\sin\theta$]

Thus, $$q_\theta = -U_0\sin\theta\,(1-3a/4r-a^3/4r^3) \qquad ...(28)$$

Form (27) and (28), it follows that q_r and q_θ are both independent of the azimuthal angle ϕ.

Now, (27) $\Rightarrow$ $\dfrac{\partial q_r}{\partial r} = U_0\left(\dfrac{3a}{2r^2}-\dfrac{3a^3}{2r^4}\right)$ and $\dfrac{\partial q_r}{\partial\theta} = -U_0\sin\theta\left(1-\dfrac{3a}{2r}+\dfrac{a^3}{2r^3}\right)$

The above results $\Rightarrow$ $(\partial q_r/\partial r)_{r=a} = 0$ and $(\partial q_r/\partial\theta)_{r=a} = 0$ ...(29)

Now, (28) $\Rightarrow$ $$\frac{q_\theta}{r} = -U_0\sin\theta\left(\frac{1}{r}-\frac{3a}{4r^2}-\frac{a^3}{4r^4}\right)$$

and, hence $$\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right) = -U_0\sin\theta\left(-\frac{1}{r^2}+\frac{3a}{2r^3}+\frac{a^3}{r^5}\right)$$

Thus, $$\left[r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)\right]_{r=a}=\left[-U_0\sin\theta\left(-\frac{1}{r}+\frac{3a}{2r^2}+\frac{a^3}{r^4}\right)\right]_{r=a}=-\frac{3U_0\sin\theta}{2a} \qquad ...(30)$$

Also, (21) $\Rightarrow$ $$(p_0-p)_{r=a}=\left[\frac{3\mu U_0 ar\cos\theta}{2r^3}\right]_{r=a}, \quad \text{since} \quad x=r\cos\theta$$

Thus, $$(p_0-p)_{r=a}=(3\mu U_0\cos\theta)/2a \qquad ...(31)$$

Using (29) and (31), (25) reduces to

$$(\sigma_{rr})_{r=a}=(p_0-p)_{r=a}+0=(3\mu U_0\cos\theta)/2a \qquad ...(32)$$

Using (29) and (30), (26), reduces to $$(\sigma_{rr})_{r=a}=-(3\mu U_0\sin\theta)/2a \qquad ...(33)$$

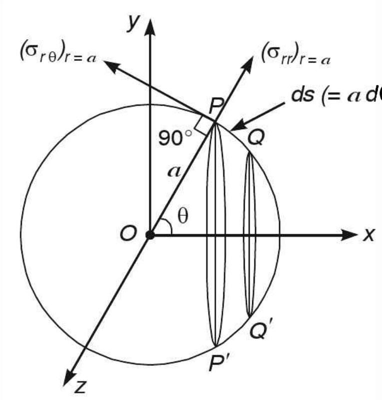


Fig. (iii) Computing the drag in Stokes flow past a sphere

In the figure (*iii*), let $PP'QQ'$ be an elementary surface of the given sphere of radius 'a'. Then, arc $PQ = ds = a\,d\theta$. and so the area of the elementary surface $PP'QQ' = 2\pi a\sin\theta ds$ $= 2\pi a\sin\theta\times a d\theta$. The required total drag D on the sphere can be obtained by integrating the x-component of the normal and shear stresses over surface of the sphere. Thus we have

$$D=\int_0^\pi\left\{(\sigma_{rr})_{r=a}\cos\theta\times 2\pi a^2\sin\theta\right\}d\theta+\int_0^\pi\left\{(\sigma_{r\theta})_{r=a}\cos\left(\theta+\frac{\pi}{2}\right)\times 2\pi a^2\sin\theta\right\}d\theta$$

$$=\int_0^\pi\left\{\frac{3\mu U_0\cos^2\theta}{2a}\times 2\pi a^2\sin\theta\right\}d\theta+\int_0^\pi\left\{\frac{3\mu U_0\sin^2\theta}{2a}\times 2\pi a^2\sin\theta\right\}d\theta$$

$$=3\mu\pi U_0 a\int_0^\pi\cos^2\theta\sin\theta\,d\theta+3\mu U_0 a\pi\int_0^\pi\sin^3\theta\,d\theta$$

$$=-3\mu\pi U_0 a\int_0^\pi\cos^2\theta(-\sin\theta)\,d\theta+3\mu U_0 a\pi\times 2\int_0^{\pi/2}\sin^3\theta\,d\theta$$

$$=-3\mu\pi U_0 a\left[(-1/3)\times\cos^3\theta\right]_0^\pi+6\mu U_0 a\pi\times(2/3)=2\pi\mu U_0 a+4\pi\mu U_0 a$$

Thus $$D=6\mu\pi U_0 a, \qquad ...(34)$$

which is known as *Stokes' formula for the drag* on the sphere in which one third of the value arises from the normal stress (pressure drag) and two-thirds from shearing stress (frictional drag). It is further remarkable that the drag is proportional to the first power of velocity U_0.

The drag coefficient C_D is defined as (refer result (*ii*) of Art 15.9)

$$C_D = \frac{D}{(\rho U_0^2 A)/2} \qquad ...(35)$$

Taking, A = frontal area = πa^2 and $\text{Re} = (2aU_0)/\nu$ = Reynolds number, the coefficient given by (35) reduces to $$C_D = 24/\text{Re} \qquad ...(36)$$

A comparision between Stokes' drag coefficient in equation (36) and experimental results is shown in figure (*iv*). It is observed that Stokes formula is valid only for cases when Re < 1.

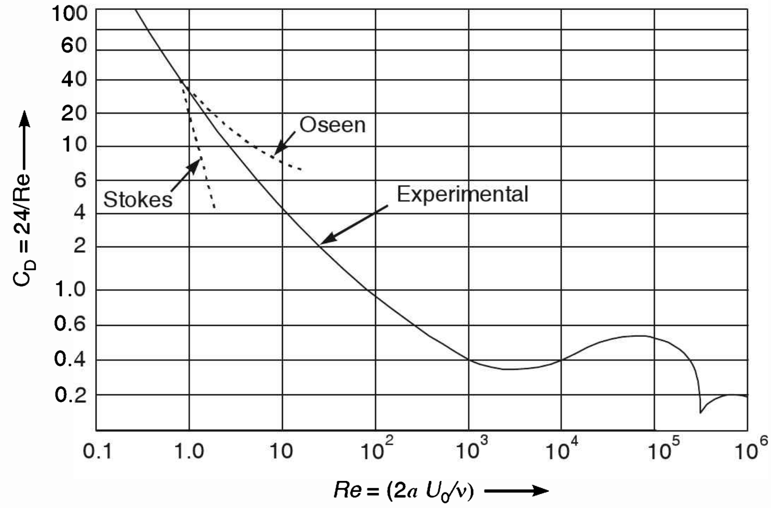


Figure (iv)

Finally, we propose to discuss the character of the motion which is most concisely expressed in terms of Stokes stream function ψ. Since the velocity at any point in the flow field is independent of the azimuthal angle ϕ, the equation of continuity in spherical polar coordinates is given by (refer equation (8) in Art. 2.11 and note that ρ is constant)

$$\frac{1}{r^2}\cdot\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\sin\theta q_\theta) = 0 \qquad \text{or} \qquad \frac{\partial}{\partial r}(r^2\sin\theta q_r) + \frac{\partial}{\partial\theta}(r\sin\theta q_\theta) = 0$$

which is identically satisfied if we define a function ψ such that

$$q_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \qquad \text{and} \qquad q_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \qquad ...(37)$$

The function ψ thus defined is known as *Stokes stream function.*

Now, (37) $\Rightarrow$ $$\psi = -\int_0^\theta q_r\, r^2\sin\theta\, d\theta$$

or, $$\psi = -\int_0^\theta U_0\cos\theta\{1 - 3a/2r + a^3/2r^3\}r^2\sin\theta d\theta, \text{ using (27)}$$

or $$\psi = -U_0\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)r^2\left[\frac{1}{2}\sin^2\theta\right]_0^\theta$$

Thus, $$\psi = -\frac{1}{2}U_0\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)r^2\sin^2\theta \qquad ...(38)$$

For the sake of clarity we choose a system of reference in which the undisturbed fluid is at rest and the sphere is moving through it. Accordingly if we superimpose on the flow field a velocity $-U_0$ in the direction of x, we obtain the case of a sphere moving steadily through a viscous fluid, which is at rest at infinity. The stream function for the uniform superimposed flow is given by

$$\psi = (1/2) \times U_0 r^2 \sin^2 \theta \qquad \text{... (39)}$$

Therefore, the stream function for a sphere moving through the viscous fluid is the sum of equations (38) and (39) and is given by

$$\psi = \frac{3}{4} \times U_0 a r \left(1 - \frac{a^2}{2r^2}\right) \sin^2 \theta \qquad \text{...(40)}$$

The pattern of streamlines in front and behind the sphere must be the same, as by changing the sign of the velocity components and the pressure in equations (1*a*) to (1*d*) the system is tranformed into itself. The streamlines in viscous flow past a sphere are shown in figure (*v*). They were drawn as they would appear to an observer in front of whom the sphere is dragged with a constant velocity U_0. The figure contains also velocity profiles at several cross-sections.

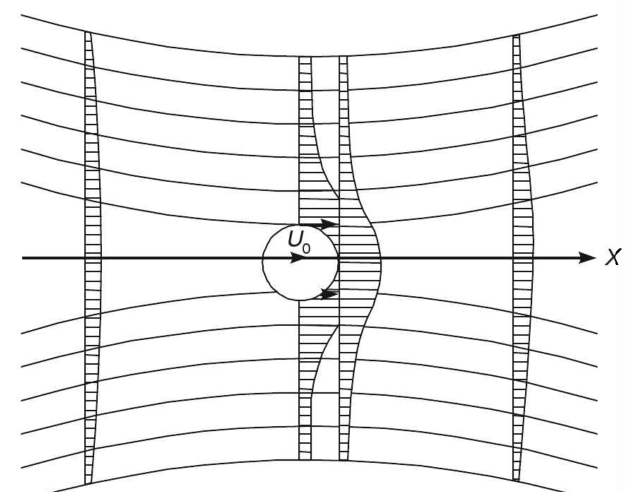


Figure (V)

We observe that the sphere drags with it a very wide layer of fluid which extends over about one diameter on both sides. Note that at very high Reynolds number this boundary layer becomes very thin.

17.3A. Slow motion of a sphere in an incompressible viscous fluid.
Stokes' flow past a sphere. (Alteruetive method)

Due to slow motion of sphere, the velocity of the liquid is very small. Since the non-linear inertia terms contain velocities to the second order for steady motion and since the frictional terms contain velocities to the first order, it follows that the inertia terms decrease more rapidly than the frictional terms and hence they may be omitted while writting equations of motion. Let the flow be steady and let there be no body force. Then the Navier-Stokes equations and the continuity equations reduce, respectively,

$$\partial p / \partial x = \mu (\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 + \partial^2 u / \partial z^2) \qquad \text{...(1)}$$

$$\partial p / \partial y = (\partial^2 v / \partial x^2 + \partial^2 v / \partial y^2 + \partial^2 v / \partial z^2) \qquad \text{...(2)}$$

$$\partial p / \partial z = \mu (\partial^2 w / \partial x^2 + \partial^2 w / \partial y^2 + \partial^2 w / \partial z^2) \qquad \text{...(3)}$$

and

$$\partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 0 \qquad \text{...(4)}$$

Let U_0 and p_0 denote the velocity and the fluid pressure at infinity. Then we are to solve (1) to (4) under the following boundary conditions:

$$u = v = w = 0 \qquad \text{at} \qquad r = a \qquad \text{...(5a)}$$

$$u = U_0, \qquad v = w = 0, \qquad p = p_0 \quad \text{at} \quad r = \infty \qquad \ldots(5b)$$

where a is the radius of the sphere.

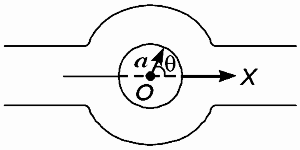
Differentiating (1), (2), (3) with respect to x, y, z respectively and adding, we have

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}=\mu\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right].$$

Making use of (4), the above equaiton reduces to

$$\partial^2 p/\partial x^2 + \partial^2 p/\partial y^2 + \partial^2 p/\partial z^2 = 0 \qquad \ldots(6)$$

showing that the pressure p satisfies the Laplace equation. Thus, for a very slow motion, the pressure must be a harmonic function.



We take the x-axis as the direction of flow over a sphere. The pressure on the side where the liquid flows towards the sphere (i.e. negative x-axis) will be greater, and the side where it flows away form it, less than than the undisturbed pressure. Furthermore, the pressure disturbances must die away at infinity. Evidently the appropriate harmonic function satisfying the above conditions may be taken as

$$p = p_0 - (Ax/r^3), \qquad \ldots(7)$$

where A is a constant to be determined later and $\qquad r = (x^2 + y^2 + z^2)^{1/2}$. $\qquad \ldots(8)$

Let F be any function of r, where r is given by (8). Then, we have

$$\frac{\partial F}{\partial x}=\frac{dF}{dr}\frac{\partial r}{\partial x}=\frac{dF}{dr}\cdot\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x$$

or $$\frac{\partial F}{\partial x}=\frac{x}{r}\cdot\frac{dF}{dr} \quad \text{so that} \quad \frac{\partial}{\partial x}=\frac{x}{r}\frac{d}{dr} \qquad \ldots(9)$$

Similarly, $$\frac{\partial}{\partial y}=\frac{y}{r}\frac{d}{dr} \quad \text{and} \quad \frac{\partial}{\partial z}=\frac{z}{r}\frac{d}{dr}. \qquad \ldots(10)$$

$$\therefore \quad \frac{\partial p}{\partial x}=\frac{\partial}{\partial x}\left(-\frac{Ax}{r^3}\right)=-A\frac{\partial}{\partial x}(xr^{-3})=-A\left[r^{-3}+x\frac{\partial}{\partial x}r^{-3}\right]=-A\left[r^{-3}+x\cdot\frac{x}{r}\frac{d}{dr}r^{-3}\right], \text{ by (9)}$$

Thus, $$\frac{\partial p}{\partial x}=-A\left[r^{-3}-\frac{x^2}{r}\times 3r^{-4}\right]=A\left(\frac{3x^2}{r^5}-\frac{1}{r^3}\right). \qquad \ldots(11)$$

Also, $$\frac{\partial p}{\partial y}=\frac{\partial}{\partial y}\left(-\frac{Ax}{r^3}\right)=-Ax\cdot\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-3/2}$$

$$=-Ax\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}\times 2y=A\left(\frac{3xy}{r^5}\right). \qquad \ldots(12)$$

Similarly, $$\partial p/\partial z = (3Axz)/r^5 \qquad \ldots(13)$$

Using (11), (12) and (13), equations (1), (2) and (3) may be re-written as

$$\nabla^2 u = \frac{A}{\mu}\left(\frac{3x^2}{r^5}-\frac{1}{r^3}\right) \qquad \ldots(14)$$

$$\nabla^2 v = \frac{3A}{\mu}\frac{xy}{r^5}. \qquad \ldots(15)$$

$$\nabla^2 w = \frac{3A}{\mu}\frac{xz}{r^5} \qquad ...(16)$$

where $$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2. \qquad ...(17)$$

From the symmetry considrations (14) to (16) can be integrated to give, respectively,

$$u = -\frac{A}{2\mu}\frac{x^2}{r^3} + U_0 + \frac{B}{r} + C\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right) \qquad ...(18)$$

$$v = -\frac{A}{2\mu}\frac{xy}{r^3} + 3D\frac{xy}{r^5} \qquad ...(19)$$

$$w = -\frac{A}{2\mu}\frac{xz}{r^3} + 3D\frac{xz}{r^5}. \qquad ...(20)$$

where B, C, and D are constants to be determined. It can be easily seen that the velocity components (18) to (20) satisfy (14) to (16). Substituting the above values of u, v and w in (4), we get

$$-\frac{A}{2\mu}\left(\frac{\partial}{\partial x}\frac{x^2}{r^3} + \frac{\partial}{\partial y}\frac{xy}{r^3} + \frac{\partial}{\partial z}\frac{xz}{r^3}\right) + B\frac{\partial}{\partial x}\frac{1}{r} + 3C\frac{\partial}{\partial x}\left(\frac{x^2}{r^5}\right) - C\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) + 3D\left[\frac{\partial}{\partial y}\frac{xy}{r^5} + \frac{\partial}{\partial z}\frac{xz}{r^5}\right] = 0.$$

Making use of (9) and (10), the above equation can be further simplified as

$$-\frac{A}{2\mu}\left(\frac{2x}{r^3} - \frac{2x^2}{r^5} + \frac{x}{r^3} - \frac{3xy^2}{r^5} + \frac{x}{r^3} - \frac{3xz^2}{r^5}\right) + \frac{Bx}{r^3} + 3C\left(\frac{2x}{r^5} - \frac{5x^3}{r^7}\right) + 3C\times\frac{x}{r^5}$$

$$+3D\left(\frac{x}{r^5} - \frac{xy^2}{r^7} + \frac{x}{r^5} - \frac{xz^2}{r^7}\right) = 0$$

or $$-\frac{Ax}{2\mu r^3} - \frac{Bx}{r^3} + (9C + 6D)\frac{x}{r^5} - [15Cx^2 + 15D(y^2 + z^2)]\frac{x}{r^7} = 0. \qquad ...(21)$$

Putting $C = D$ in (21) and noting that $r^2 = x^2 + y^2 + z^2$, we see that last two terms in (21) cancel. Further, if we take $B = -A/2\mu$, we find that the first two terms also cancel. Thus the continuity equation (4) is satisfied if we take

$$C = D \qquad \text{and} \qquad B = -A/2\mu. \qquad ...(22)$$

In order to satisfy boundary conditions $5(a)$ *i.e.* $v = w = 0$ at $r = a$, (19) and (20) show that

$$0 = -\frac{Axy}{2\mu a^3} + \frac{3Dxy}{a^5} \qquad \text{and} \qquad 0 = -\frac{Axz}{2\mu a^3} + \frac{3Dxz}{a^5}$$

Both of these show that $$D = (Aa^2)/6\mu = C, \text{ using (22)} \qquad ...(23)$$

Put the values of B and C in (18). Then setting $u = 0$ at $r = a$ in order to satisfy the boundary condition $5(a)$, we obtain

$$0 = -\frac{Ax^2}{2\mu a^3} + U_0 - \frac{A}{2\mu a} + \frac{Aa^2}{6\mu}\left(\frac{3x^2}{a^5} - \frac{1}{a^3}\right)$$

so that $$A = (3/2)\times\mu a U_0. \qquad ...(24)$$

Using (22), (23) and (24) in (18), (19), (20) and (7), the required solution of (1), (2), (3) and (4) under boundary conditions $5(a)$ and $5(b)$ is given by

$$u = U_0 \left[\frac{3ax^2}{4r^3} \left(\frac{a^2}{r^2} - 1 \right) + 1 + \frac{a}{4r} \left(3 + \frac{a^2}{r^2} \right) \right] \qquad ...(25)$$

$$v = U_0 \cdot \frac{3axy}{r^3} \left(\frac{a^2}{r^2} - 1 \right) \qquad ...(26)$$

$$w = U_0 \cdot \frac{3axz}{r^3} \left(\frac{a^2}{r^2} - 1 \right) \qquad ...(27)$$

and

$$p = p_0 - \frac{3\mu U_0 ax}{2r^3}. \qquad ...(28)$$

This is the well known *Stokes' solution for the slow motion* of a sphere in a viscous fluid. This problem is often referred to as *Stokes's law*.

Let the components of stress across a sphere of radius r be $\sigma_{rx}, \sigma_{ry}, \sigma_{rz}$, where

$$\sigma_{rx} = (x/r) \times \sigma_{xx} + (y/r) \times \sigma_{yx} + (z/r) \times \sigma_{zx}$$

$$= \frac{x}{r}\left(-p + 2\mu \frac{\partial u}{\partial x} \right) + \frac{y}{r} \cdot u \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{z}{r} \cdot \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \text{ (sec chapter 13)}$$

$$= -\frac{xp}{r} + \frac{\mu x}{r}\frac{\partial u}{\partial x} + \frac{\mu}{r}\left(x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x} + z\frac{\partial w}{\partial x} \right) + \frac{\mu}{r}\left(y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} \right)$$

$$= -\frac{xp}{r} + \frac{\mu x}{r}\frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\mu}{r}\frac{\partial}{\partial x}(xy + yv + zw) - \frac{\mu u}{r} + \frac{\mu}{r}\left(y\frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + z\frac{\partial u}{\partial r}\frac{\partial r}{\partial z} \right)$$

$$= -\frac{xp}{r} + \frac{\mu}{r^2}\frac{\partial u}{\partial r}(x^2 + y^2 + z^2) + \frac{\mu}{r}\frac{\partial}{\partial x}(xu + yv + zw) - \frac{\mu u}{r}$$

$$\left[\because \text{ From (18), } \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r}, \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r} \right]$$

$$= -\frac{xp}{r} + \mu\left(\frac{\partial u}{\partial r} - \frac{u}{r} \right) + \frac{\mu}{r}\frac{\partial}{\partial x}(xu + yv + zw) \qquad ...(29)$$

Similarly, we have

$$\sigma_{ry} = -\frac{yp}{r} + \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) + \frac{\mu}{r}\frac{\partial}{\partial y}(xu + yv + zw) \qquad ...(30)$$

and

$$\sigma_{rz} = -\frac{zp}{r} + \mu\left(\frac{\partial w}{\partial r} - \frac{w}{r} \right) + \frac{\mu}{r}\frac{\partial}{\partial z}(xu + yv + zw). \qquad ...(31)$$

Using (25), (26) and (27), we have

$$xu + yv + zw = U_0\left[\frac{3ax^3}{4r^3}\left(\frac{a^2}{r^2} - 1 \right) + x - \frac{ax}{4r}\left(3 + \frac{a^2}{r^2} \right) + \frac{3axy^2}{4r^3}\left(\frac{a^2}{r^2} - 1 \right) + \frac{3axz^2}{4r^3}\left(\frac{a^2}{r^2} - 1 \right) \right]$$

$$= U_0\left[\frac{3ax}{4r^3}\left(\frac{a^2}{r^2} - 1 \right) + x - \frac{ax}{4r}\left(3 + \frac{a^2}{r^2} \right) \right] \qquad ...(32)$$

$$\therefore \quad \frac{\partial}{\partial x}(xu+yv+zw)$$

$$= U_0\left[\frac{3a}{4r}\left(\frac{a^2}{r^2}-1\right)+\frac{3ax}{4}\left(-\frac{3a^2}{r^4}+\frac{1}{r^2}\right)+1-\frac{a}{4r}\left(3+\frac{a^2}{r^2}\right)-\frac{ax}{4}\left(-\frac{3}{r^2}-\frac{3a^2}{r^4}\right)\frac{x}{r}\right]$$

[using (9) and (10)]

Hence, $$\frac{\partial}{\partial x}(xu+yv+zw)=0 \quad \text{on} \quad r=a. \qquad \ldots(33)$$

Similarly, on $r=a$, $$\frac{\partial}{\partial y}(xu+yv+zw)=0, \qquad \frac{\partial}{\partial z}(xu+yv+zw)=0 \qquad \ldots(34)$$

Again, $$\mu\left(\frac{\partial u}{\partial r}-\frac{u}{r}\right)$$

$$=\mu U_0\left[\frac{3ax^2}{4}\left(-\frac{5a^2}{r^6}+\frac{3}{r^4}\right)-\frac{a}{4}\left(-\frac{3}{r^2}-\frac{3a^2}{r^4}\right)-\frac{3ax^2}{4}\left(\frac{a^2}{r^6}-\frac{1}{r^4}\right)-\frac{1}{r}+\frac{a}{4}\left(\frac{3}{r^2}+\frac{a^2}{r^4}\right)\right]$$

On $r=a$, we have $$\mu\left(\frac{\partial u}{\partial r}-\frac{u}{r}\right)=\frac{3\mu U_0}{2a}\left(1-\frac{x^2}{a^2}\right). \qquad \ldots(35)$$

Again on $r=a$, (28) reduces to $$p=p_0-\frac{3\mu U_0 x}{2a^2}. \qquad \ldots(36)$$

Using (33), (35) and (36), on $r=a$, (29) reduces to

$$\sigma_{rx}=-\frac{x}{a}\left(p_0-\frac{3\mu U_0 x}{2a^2}\right)+0+0=-\frac{xp_0}{a}+\frac{3\mu U_0}{2a} \qquad \ldots(37)$$

Similarly, on $r=a$ $$\sigma_{ry}=-\frac{yp_0}{a} \quad \text{and} \quad \sigma_{rz}=-\frac{zp_0}{a}. \qquad \ldots(38)$$

Since (0, 0, 0) are the coordinates $(\bar{x},\bar{y},\bar{z})$ of the centre of gravity of the surface of the sphere, we have

$$0=\bar{x}=\frac{\iint xdS}{\iint dS} \quad \text{so that} \quad \iint xdS=0. \qquad \ldots(39)$$

Hence the force on the surface of the sphere due to σ_{rx} is given by

$$\iint \sigma_{rx}dS=\iint\left[-\frac{xp_0}{a}+\frac{3\mu U_0}{2a}\right]dS, \text{ by (37)}$$

$$=-\frac{p_0}{a}\iint xdS+\frac{3\mu U_0}{2a}\iint dS=0+\frac{3\mu U_0}{2a}\times 4\pi a^2, \text{ by (39)}$$

$$=6\pi\mu a U_0$$

Similarly, $$\iint \sigma_{ry}dS=0 \quad \text{and} \quad \iint \sigma_{rz}dS=0.$$

Hence the total drag D on the sphere is $6\pi\mu a U_0$, in the positive direction of x-axis.

Thus, $$D=6\pi\mu a U_0. \qquad \ldots(40)$$

which is known as *Stokes' formula for the drag of a sphere*. It can be verified that $D/3$ arises from the normal pressure forces and $2D/3$ arises form frictional forces. Result (40) hold only if Re < 1.

Remark. Suppose a soild sphere of radius a is held with its centre at the origin in a uniform stream of viscous incompressible liquid having constant velocity U_0 in the negative direction of x-axis. Then, clearly the same formula (40) holds for the viscous drag.

Particular case. *To find the terminal velocity of a sphere for a vertical fall in liquid* **[Himanchal 2003]**

Let a solid sphere of radius a and density σ fall vertically through liquid of density $\rho(<\sigma)$ and let U_0 be the terminal velocity produced by the viscous drag. Then the weight of the sphere must be equal to the sum of the upthrust and viscous drag D, *i.e.*,

$$(4/3)\times\pi a^2\sigma g = (4/3)\times\pi a^2\rho g + 6\pi a\mu U_0 \quad \text{so that} \quad U_0 = (2/9\mu)\times(\sigma-\rho)a^2 g.$$

17.3 B. Small Reynold's number flows **[Meerut 2002]**

Since Navter-Stokes equations are non-linear, their solution in general case in not simple. The main difficulty arises due to presence of non-linear convertive terms. These non-linear terms are unimportant when we consider situation with very small Reynolds number. The Reynolds number Ul/ν can be small by reason of the typical velocity U being small or the typical length l being small, or by the kinematic viscosity ν being large. When U is small we have *slow motion* or *creeping motion*; when l is small we have the motion of minute objects, for example *Brownian motion*. Theoretically in creeping flow the Reynolds number is taken to be much less than one. However, it has been seen that the solutions obtained by this process hold good even when Reynolds number is merely less than one.

17.3 C. Flow past a sphere. Stokes flow. **[Himanchal 2000, 01]**

Let a solid sphere of radius a be held fixed in a uniform stream U flowing steadily in the positive direction of the x. Let the fluid be viscous incompressible. Let the flow be steady and axi-symmetric at small Reynolds number. As a first approximation, Stokes neglected the convetive terms in the Navier-Stokes equations because they are quadratic in the velocity. Now, the pressure must be balanced by viscous forces alone. Hence the equations of motion and continuity reduce to

$$0 = -\nabla p + \mu\nabla^2\mathbf{q} \qquad ...(1)$$

and

$$\nabla\cdot\mathbf{q} = 0 \qquad ...(2)$$

with boundary conditions:

$$\mathbf{q} = 0 \quad \text{at} \quad r = a; \qquad \mathbf{q} = (U, 0, 0) \quad \text{at} \quad r = \infty. \qquad ...(3)$$

Taking the divergence of (1) and using (2), we get

$$\nabla^2 p = 0, \qquad ...(4)$$

showing that the pressure satisfies the Laplace equation and so the pressure is a harmonic function for small Reynolds number flows.

In spherical polar coordinates (r,θ,ϕ), we choose the axis $\theta = 0$ to lie in the direction of the free stream U. Then the equation of continuity (2) is satisfied if the velocity components are given in terms of stream function ψ by

$$q_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial r}, \quad \text{and} \quad q_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \qquad ...(5)$$

Using (5), (1) yields $E^4\psi = 0$, where $E^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{\sin\theta}{r^2}\dfrac{\partial}{\partial\theta}\left(\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\right)$ ... (6)

or

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0. \qquad ...(7)$$

The boundary conditions at the surface of the sphere take the new form:

$$\psi_\theta(1,\theta) = 0 \quad \text{and} \quad \psi_r(1,\theta) = 0, \qquad ...(8)$$

Since the flow is uniform upstream, $\psi(r,\theta) \sim (1/2)\times r^2\sin^2\theta$, as $r\to\infty$ ...(9)

and this suggests the trial solution $\psi = f(r)\sin^2\theta.$ ...(10)

Substituting this in (7) gives successively

$$\left[\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\left[\left(\frac{d^2 f(r)}{dr^2}-\frac{2f(r)}{r^2}\right)\sin^2\theta\right]=0$$

$$\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)f(r)=0, \qquad \text{...(11)}$$

which is linear homogeneous differential equation of the fourth order. To satisfy (11) by a sum of terms of the Ar^n, we find

$[(n-2)(n-3)-2]\,[n(n-1)-2]=0,$ so that $n=-1, 1, 2, 4,$ and hence

The general solution of (11) is $\quad f(r)=(A/r)+Br+Cr^2+Dr^4. \qquad$ ...(12)

where A, B, C and D are arbitrary constants.

Condition (9) shows we must take $D = 0$. Again conditions (8) show that

$A=1/4, \qquad B=(-3/4) \qquad$ and $\qquad C=1/2.$

$\therefore$ From (10) and (12), $\quad \psi=\dfrac{1}{4}\left(2r^2-3r+\dfrac{1}{r}\right)\sin^2\theta \qquad$ ...(13)

so that $\quad q_r=U\left(1-\dfrac{3}{2r}+\dfrac{1}{2r^2}\right)\cos\theta \qquad$ ...(14)

and $\quad q_\theta=-U\left(1-\dfrac{3}{4r}-\dfrac{1}{4r^3}\right)\sin\theta. \qquad$ ...(15)

The solution (13) was obtained by Stokes. The first term is the uniform stream and the third is a dipole at the centre of the sphere, both representing the irrotational flows. The second term, which contains all the vorticity, is known as *Stokeslet*. For non-viscous fluid flow, the Stokeslet is not present and the coefficient of dipole is –1/2 in place of 1/4. The solution satisfies the surface boundary conditions of the problem. On the other hand it fails to satisfy the boundary conditon at infinity. It follows that this expansion breaks down for large r and this breakdown is known as *Whitehead's paradox*.

Let F be the resultant force (drag force) exerted by the fluid on the surface of the sphere in the z-direction. Then $\quad F=2\pi a^2\displaystyle\int_0^\pi F_{rz}\sin\theta\, d\theta, \qquad$ ... (16)

where F_{rz} is the force per unit area of the spherical surface in the z-direction and is given by

$$F_{rz}=\sigma_{rr}\cos\theta-\sigma_{r\theta}\sin\theta \;=-p\cos\theta+2\mu\cos\theta\frac{\partial q_r}{\partial r}-\mu\sin\theta\left(\frac{\partial q_\theta}{\partial r}+\frac{1}{r}\frac{\partial q_r}{\partial\theta}-\frac{q_\theta}{r}\right)$$

[Using relations (21*a*) and (21d) of Art. 14.12]

Substituting this in (16) and integraing. $\quad F=6\pi\mu Ua. \qquad$ ...(17)

This result was first obtain by Stokes and is known as *Stokes formula for drag on a sphere*.

17.3 D. Steady motion of viscous fluid due to a slowly rotating sphere.

[Garwhal 2000; I.A.S. 1984; Kanpur 2005; Meerut 20001, 02]

For the present problem, the velocity components are :

$u=-\omega y, \qquad v=\omega x, \qquad$ and $\qquad w=0, \qquad$ ...(1)

where ω, the angular velocity, is function of r alone. Also, we have

$r^2=x^2+y^2+z^2 \qquad$ or $\qquad r=(x^2+y^2+z^2)^{1/2} \qquad$ ...(2)

It is easily seen that (1) satisfy the continuity equation.

Since $\partial r/\partial x = (1/2) \times (x^2+y^2+z^2)^{-1/2} \times (2x) = x/r$,so $\partial w/\partial x = (dw/dr)(\partial r/\partial x) = (x/r)(dw/dr)$

Similarly, $\partial w/\partial y = (y/r)(dw/dr)$ and $\partial w/\partial z = (z/r)(dw/dr)$ ...(2a)

Since ω is a function of r only, we have

$$\partial^2 \omega / \partial x^2 + \partial^2 \omega / \partial y^2 + \partial^2 \omega / \partial z^2 = d^2 w / dr^2 + (2/r)(dw/dr) \qquad \text{... (2b)}$$

Let the flow be steady and let there be no body forces. then for the slow rotation of the sphere, Stokes equations (Refer equations (1*a*), (1*b*), (1*c*) of Art. 17.3) are given by

$$0 = -(\partial p / x) + \mu \nabla^2 u \qquad \text{...(3)}$$

$$0 = -(\partial p / \partial y) + \mu \nabla^2 v \qquad \text{...(4)}$$

$$0 = -(\partial p / \partial z) \qquad \text{...(5)}$$

From (1), we obtain

$$\frac{\partial u}{\partial x} = -y\frac{\partial \omega}{\partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = -y\frac{\partial^2 \omega}{\partial x^2} \qquad \text{...(6)}$$

$$\frac{\partial u}{\partial y} = -y\frac{\partial \omega}{\partial y} - \omega \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -y\frac{\partial^2 \omega}{\partial y^2} - 2\frac{\partial \omega}{\partial y} \qquad \text{...(7)}$$

$$\frac{\partial u}{\partial z} = -y\frac{\partial \omega}{\partial z} \quad \text{and} \quad \frac{\partial^2 u}{\partial z^2} = -y\frac{\partial^2 \omega}{\partial z^2} \qquad \text{...(8)}$$

Using (6), (7) and (8), we have

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -y\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2}\right) - 2\frac{\partial \omega}{\partial y}$$

$$= -y\left[\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} + \frac{2}{y}\frac{\partial \omega}{\partial y}\right] = -y\left[\frac{d^2 \omega}{dr^2} + \frac{2}{r}\frac{d\omega}{dr} + \frac{2}{r}\frac{d\omega}{dr}\right], \qquad \text{... (9)}$$

[Using (2*a*) and (2*b*)]

Similarly, $$\nabla^2 v = x\left(\frac{d^2 \omega}{dr^2} + \frac{4}{r}\frac{d\omega}{dr}\right) \qquad \text{...(10)}$$

Using (9) and (10), the equations of motion (3) to (5) reduce to

$$0 = -\frac{\partial p}{\partial x} - \mu y\left(\frac{d^2 \omega}{dr^2} + \frac{4}{r}\frac{d\omega}{dr}\right) \qquad \text{...(11)}$$

$$0 = -\frac{\partial p}{\partial x} + \mu x\left(\frac{d^2 \omega}{dr^2} + \frac{4}{r}\frac{d\omega}{dr}\right) \qquad \text{...(12)}$$

$$0 = -(\partial p/\partial z) \qquad \text{...(13)}$$

All the equations (11) to (13) are satisfied by taking p = constant and

$$\frac{d^2 \omega}{dr^2} + \frac{4}{r}\frac{d\omega}{dr} = 0 \qquad \textit{i.e.} \qquad r^4\frac{d^2 \omega}{dr^2} + 4r^3\frac{d\omega}{dr} = 0$$

or $$\frac{d}{dr}\left(r^4\frac{d\omega}{dr}\right) = 0. \qquad \text{...(14)}$$

Integrating (14), $r^4(d\omega/dr) = A$ or $d\omega/dr = A/r^4$ ... (15)

Integrating (15), $\omega = -(A/3r^3) + B$, *i.e.*, $\omega = (C/r^3) + B$, ...(16)

where $C = -A/3$. Here B and C are arbitrary constants of integration to be determined.

Suppose the motion is generated by a solid sphere of radius a rotating with angular velocity ω' and the fluid extends to infinity and is at rest there. Then we shall determine ω under the boundary conditions :

$$\omega = 0 \quad \text{at} \quad r = \infty; \qquad \omega = \omega' \quad \text{at} \quad r = a. \qquad \text{...(17)}$$

Using (17), (16) gives $B = 0$ and $\omega' = C/a^3$ Then, (16) reduces to

$$\omega = (a^3/r^3) \times \omega' \qquad \text{...(18)}$$

Next, suppose that there is an outer fixed concentric sphere of radius b which is fixed. Then we shall determine ω under the boundary conditions:

$$\omega = \omega' \quad \text{at} \quad r = a; \qquad \omega = 0 \quad \text{at} \quad r = b \qquad \text{...(19)}$$

Using (19), (16) gives $\qquad \omega' = (C/a^3) + B \quad$ and $\quad 0 = (C/b^3) + B$

so that $$B = -\frac{a^3\omega'}{b^3 - a^3} \quad \text{and} \quad C = -\frac{a^3b^3\omega'}{b^3 - a^3}.$$

Substituting these values into (16), we obtain $$\omega = \frac{a^3\omega'}{r^3}\frac{b^3 - r^3}{b^3 - a^3}. \qquad \text{...(20)}$$

Suppose N is the couple which must be applied to the moving sphere to maintain the rotation. Then we know that

$N\omega'$ = Rate of dissipation of energy

$$= \mu\iiint\left\{2(\partial u/\partial x)^2 + 2(\partial v/\partial y)^2 + (\partial u/\partial z)^2 + (\partial v/\partial z)^2 + (\partial u/\partial y + \partial v/\partial x)^2\right\} dxdydz \;\text{...(21)}$$

Now using (1) and (2), we have

$$\frac{\partial u}{\partial x} = -y\frac{\partial \omega}{\partial x} = -\frac{xy}{r}\frac{d\omega}{dr}, \qquad \frac{\partial v}{\partial y} = x\frac{\partial \omega}{\partial y} = \frac{xy}{r}\frac{d\omega}{dr}$$

$$\frac{\partial v}{\partial z} = x\frac{\partial \omega}{\partial z} = \frac{xz}{r}\frac{d\omega}{dr}, \qquad \frac{\partial u}{\partial z} = -\frac{yz}{r}\frac{d\omega}{dr}$$

and $$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \omega + x\frac{\partial \omega}{\partial x} - \omega - y\frac{\partial \omega}{\partial y} = \frac{x^2 - y^2}{r^2}\frac{d\omega}{dr}$$

Using the above results, (21) reduces to

$$N\omega' = \mu\iiint\left[\frac{2x^2y^2 + x^2z^2 + y^2z^2 + x^4 + y^4}{r^2}\left(\frac{d\omega}{dr}\right)^2\right]dxdydz = \mu\iiint (x^2 + y^2)\left(\frac{d\omega}{dr}\right)dxdydz, \text{ by (2)}$$

$$= \mu\int_{\theta=0}^{\pi}\int_{r=a}^{b} r^2\sin^2\theta\left(\frac{d\omega}{dr}\right)^2 \cdot 2\pi r^2 \sin\theta\, dr\, d\theta = 8\pi\mu\omega'^2\frac{a^3b^3}{b^3 - a^3}, \text{ using (20) and simplifying}$$

Thus, $$N = 8\pi\mu\omega'\frac{a^3b^3}{b^3 - a^3}. \qquad \text{...(22)}$$

If there be an infinite liquid outside a sphere of radius a so that $b \to \infty$, (22) reduces to

$$N = 8\pi\mu\omega'\frac{a^3}{1 - a^3/b^3} \quad \text{where} \quad b \to \infty \qquad i.e., \qquad N = 8\pi\mu a^3\omega'. \qquad \text{...(23)}$$

17.3E. Motion of a viscous fluid due to slowly rotating sphere (Alteruative proof). (Meerut 1999, 2001, 02)

Let a sphere of radius a rotate with very small velocity $\mathbf{q}$ given by $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$,, where

$$u = -\omega y, \qquad v = \omega x, \qquad w = 0, \qquad \text{...(1)}$$

where W is angular velocity of the sphere such that W is a function of r alone, where

$$r^2 = x^2 + y^2 + z^2 \qquad \text{...(2)}$$

The Navier-Stokes equations in the absence of the body forces is given by

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{q}. \qquad \text{...(3)}$$

Since the motion is steady, so $\partial\mathbf{q}/\partial t = 0$. Again, since $\mathbf{q}$ is very small, so neglecting squares of velocities, we have $(\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{0}$. Hence (3) reduces to

$$0 = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\mathbf{q} \qquad \text{or} \qquad \mu\nabla^2\mathbf{q} = \nabla p$$

or
$$\mu\,(\mathbf{i}\nabla^2 u + \mathbf{j}\nabla^2 v + \mathbf{k}\nabla^2 w) = \mathbf{i}\frac{\partial p}{\partial x} + \mathbf{j}\frac{\partial p}{\partial y} + \mathbf{k}\frac{\partial p}{\partial z}$$

or
$$\mu\,[-\mathbf{i}\nabla^2(\omega y) + \mathbf{j}\nabla^2(\omega x) + (0)\mathbf{k}] = \mathbf{i}\frac{\partial p}{\partial x} + \mathbf{j}\frac{\partial p}{\partial y} + \mathbf{k}\frac{\partial p}{\partial z}$$

Therefore,
$$-\mu\nabla^2(\omega y) = \partial p/\partial x \qquad \text{...(4)}$$

$$\mu\nabla^2(\omega x) = \partial p/\partial y \qquad \text{...(5)}$$

and
$$0 = \partial p/\partial z \qquad \text{...(6)}$$

Now, $\nabla^2(\omega y) = (\partial^2/\partial x^2 + \partial/\partial y^2 + \partial^2/\partial z^2)(\omega y)$

$$= \frac{\partial^2}{\partial x^2}(\omega y) + \frac{\partial^2}{\partial y^2}(\omega y) + \frac{\partial^2}{\partial z^2}(\omega y) = y\frac{\partial^2\omega}{\partial x^2} + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(\omega y)\right] + y\frac{\partial^2\omega}{\partial z^2}$$

$$= y\frac{\partial^2\omega}{\partial x^2} + \frac{\partial}{\partial y}\left(\omega + y\frac{\partial\omega}{\partial y}\right) + y\frac{\partial^2\omega}{\partial z^2} = y\frac{\partial^2\omega}{\partial x^2} + \left(\frac{\partial\omega}{\partial y} + \frac{\partial\omega}{\partial y} + y\frac{\partial^2\omega}{\partial y^2}\right) + y\frac{\partial^2\omega}{\partial z^2}$$

$$\therefore \qquad \nabla^2\,(\omega y) = y\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} + \frac{\partial^2\omega}{\partial z^2}\right) + 2\frac{\partial\omega}{\partial y} \qquad \text{...(7)}$$

Since ω is function of r alone, we have
$$\frac{\partial\omega}{\partial y} = \frac{d\omega}{dr}\frac{\partial r}{\partial y} = \frac{y}{r}\frac{d\omega}{dr}. \qquad \text{...(8)}$$

[$\therefore$ From (2), $2r\,(\partial r/\partial y) = 2y$ so that $(\partial r/\partial y) = y/r$]

Also, from (8), we get
$$\frac{\partial}{\partial y} = \frac{y}{r}\frac{d}{dr} \qquad \text{...(9)}$$

Now, $\dfrac{\partial^2\omega}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial\omega}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{y}{r}\dfrac{d\omega}{dr}\right)$, by (8)

$$= \frac{1}{r}\frac{d\omega}{dr} + y\frac{\partial}{\partial y}\left(\frac{1}{r}\frac{d\omega}{dr}\right) = \frac{1}{r}\frac{d\omega}{dr} + y \times \frac{y}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{d\omega}{dr}\right), \text{ by (9)}$$

$$= \frac{1}{r}\frac{d\omega}{dr} + \frac{y^2}{r}\left(-\frac{1}{r^2}\frac{d\omega}{dr} + \frac{1}{r}\frac{d^2\omega}{dr^2}\right)$$

Thus, $$\frac{\partial^2 \omega}{\partial y^2} = \frac{r^2 - y^2}{r^3}\frac{d\omega}{dr} + \frac{y^2}{r^2}\frac{d^2\omega}{dr^2} \qquad \ldots(10)$$

Similarly, we have $$\frac{\partial^2 \omega}{dx^2} = \frac{r^2 - x^2}{r^3}\frac{d\omega}{dr} + \frac{x^2}{r^2}\frac{d^2\omega}{dr^2} \qquad \ldots(11)$$

and $$\frac{\partial^2 \omega}{dz^2} = \frac{r^2 - z^2}{r^3}\frac{d\omega}{dr} + \frac{z^2}{r^2}\frac{d^2\omega}{dr^2} \qquad \ldots(12)$$

Adding (10), (11), (12), we have

$$\frac{\partial^2 \omega}{dx^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} = \frac{3r^2 - (x^2 + y^2 + z^2)}{r^3}\frac{d\omega}{dr} + \frac{x^2 + y^2 + z^2}{r^2}\frac{d^2\omega}{dr^2} = \frac{3r^2 - r^2}{r^3}\frac{d\omega}{dr} + \frac{d^2\omega}{dr^2},$$

Thus, $$\frac{\partial^2 \omega}{dx^2} + \frac{\partial^2 \omega}{dy^2} + \frac{\partial^2 \omega}{dz^2} = \frac{2}{r}\frac{d\omega}{dr} + \frac{d^2\omega}{dr^2}. \qquad \ldots(13)$$

Using (8) and (13), (7) reduces to

$$\nabla^2 (\omega y) = y\left(\frac{2}{r}\frac{d\omega}{dr} + \frac{d^2\omega}{dr^2}\right) + \frac{2y}{r}\frac{d\omega}{dr} = y\left(\frac{d^2W}{dr^2} + \frac{4}{r}\frac{dW}{dr}\right)$$

$$\left.\begin{aligned}
&\therefore \quad \text{From (4)} && -\mu y\left(\frac{d^2\omega}{dr^2} + \frac{4}{r}\frac{d\omega}{dr}\right) = \frac{\partial p}{\partial x} \\
&\text{Similarly from (5)} && \mu x\left(\frac{d^2\omega}{dr^2} + \frac{4}{r}\frac{d\omega}{dr}\right) = \frac{\partial p}{\partial y} \\
&\text{Also, from (6),} && 0 = \partial p / \partial z
\end{aligned}\right\} \qquad \ldots(14)$$

Equations in (14) are all satisfied by taking p = constant and

$$\frac{d^2\omega}{dr^2} + \frac{4}{r}\frac{d\omega}{dr} = 0. \qquad \ldots(15)$$

Re-writing (15), $\quad r^4\dfrac{d^2\omega}{dr^2} + 4r^3\dfrac{d\omega}{dr} = 0 \quad$ or $\quad \dfrac{d}{dr}\left(r^4\dfrac{d\omega}{dr}\right) = 0.$

Integrating, $\quad r^4\dfrac{d\omega}{dr} = A \quad$ or $\quad d\omega = \dfrac{A}{r^4}dr.$

Integrating, $\quad \omega = -(A/3r^3) + B$, A and B being arbitrary constants.

or $\quad \omega = C/r^3 + B$, taking $C = -A/3 \qquad \ldots(16)$

Let the motion be produced by a solid sphere of radius a rotating with angular velocity Ω and the fluid extends to infinity and is at rest there so that

$$\omega = 0 \quad \text{at} \quad r \to \infty \quad \text{and} \quad \omega = \Omega \quad \text{at} \quad r = a \qquad \ldots(17)$$

Using (17), (16) gives $\quad B = 0 \quad$ and $\quad \Omega = C/a^3 \quad$ so that $\quad B = 0, \quad C = a^3\Omega$

Using these values of B and C, (16) reduces to $\quad \omega = (a^3/r^3) \times \Omega. \qquad \ldots(18)$

Again if there is an outer fixed concentric spherical boundary of radius b, then we have

$$\omega = 0 \quad \text{at} \quad r = b \quad \text{and} \quad \omega = \Omega \quad \text{at} \quad r = a. \qquad \ldots(19)$$

Using (19), (16) gives $\quad 0 = C/b^3 + B \quad$ and $\quad \Omega = C/a^3 + B. \quad \ldots(20)$

Solving (20), $B = -\dfrac{a^3\Omega}{b^3 - a^3}$ and $C = \dfrac{a^3 b^3}{b^3 - a^3}\Omega.$

Putting these values is (16), we get $$\omega = \frac{a^3}{r^3}\frac{b^3 - r^3}{b^3 - a^3}\Omega. \qquad ...(21)$$

The couple N on the sphere will be obtained by using formulae for the stresses [Refer Art. 14.12].

Here $q_r = 0,$ $q_\theta = 0$ and $q_\phi = \omega r \sin\theta$

and the only stress which has a moment about the axis is $\sigma_{r\phi} = \mu r \sin\theta\,(d\omega/dr)$

and its moment is $\mu r^2 \sin\theta (d\omega/dr).$

So, the couple N on the sphere of radius a is given by

$$N = \int_0^\pi \mu a^2 \sin^2\theta \left(\frac{d\omega}{dr}\right)_{r=a} \cdot 2\pi a^2 \sin\theta\, d\theta \qquad ...(22)$$

From (21), $$\frac{d\omega}{dr} = \frac{a^3\Omega}{b^3 - a^3}\left(-\frac{3b^3}{r^4} - 0\right) \Rightarrow \left(\frac{d\omega}{dr}\right)_{r=a} = -\frac{3b^3\Omega}{(b^3 - a^3)a}. \qquad ...(23)$$

Using (23), (22) reduces to

$$N = -\frac{6\pi\mu\Omega a^3 b^3}{b^3 - a^3}\int_0^\pi \sin^3\theta\, d\theta = -\frac{6\pi\mu\Omega a^3 b^3}{b^3 - a^3} \times 2\int_0^{\pi/2} \sin^3 d\theta$$

$$= -\frac{6\pi\mu\Omega a^3 b^3}{b^3 - a^3} \times 2 \times \frac{2}{3}, \text{ by Wallis' formula}$$

$$= -\frac{8\pi\mu\Omega a^3 b^3}{b^3 - a^3} \qquad ...(24)$$

$\therefore$ Rate of dissipation of energy $= N\Omega = -\dfrac{8\pi\mu\Omega^2 a^3 b^3}{b^3 - a^3}$, by (24)

For an infinite liquid outside a sphere of radius a, we have

$$\text{Rate of dissipation of energy} = \lim_{b \to \infty}\left[-\frac{8\pi\mu\Omega^2 a^3 b^3}{b^3 - a^3}\right] = \lim_{b \to \infty}\left[-\frac{8\pi\Omega^2 a^3}{1 - (a/b)^3}\right] = -8\pi\Omega^2 a^3.$$

17.3F. Flow past a circular cylinder. [Meerut 1999]

We propose to solve the Stokes equations for uniform flow past a circular cylinder of radius a. For steady flow, the Stokes equations reduce to

$$0 = -\nabla p + \mu\nabla^2 \mathbf{q} \qquad ...(1)$$

and $$\nabla\cdot\mathbf{q} = 0 \qquad ...(2)$$

Taking the curl of (1), we obtain $$\nabla^2\Omega = 0. \qquad ...(3)$$

where $$\Omega = \nabla \times \mathbf{q} = \text{vorticity vector} \qquad ...(4)$$

Since in two dimensions the only non-zero component of Ω is ζ (whcih is the vorticity in the z-direction), we have $$\nabla^2 \zeta = 0 \qquad ...(5)$$

where $$\zeta = \partial v / \partial x - \partial u / \partial y = -(\partial^2 \psi / \partial x^2 + \partial^2 \psi / \partial y^2), \qquad ...(6)$$
where ψ is the stream function satisfying the continuity equation (2).

Now, the vorticity must satisfy the equaton $\quad (\partial^2 / \partial x^2 + \partial^2 / \partial y^2)^2 \psi = 0. \qquad ...(7)$

Transforming (7) into cylindrical polar coordinates, we get

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2} + \frac{\partial^2}{\partial \theta^2} \right)^2 \psi = 0 \qquad ... (8)$$

Since the flow is uniform upstream, take $\quad \Psi(r, \theta) \sim r \sin\theta$ as $r \to \infty \qquad ... (9)$

and this suggests the trail solution $\quad \psi(r,\theta) = f(r) \sin\theta. \qquad ...(10)$

Substituting this in (8) gives $$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\frac{1}{r^2} \right)^2 f(r) = 0,$$

whose general solution is given by $\quad f(r) = Ar^3 + Br \log r + Cr + D/r.$

$\therefore$ Form (10), we get $\quad \psi(r,\theta) = (Ar^3 + Br\log r + Cr + D/r)\sin\theta. \qquad ...(11)$

in which the second term is a two-dimensional Stokeslet containing the vorticity, the third term is the uniform parallel stream and the fourth term is a dipole at the origin.

Since we want a uniform flow far from the cylinder , $\psi(r,\theta) \to Ur\sin\theta$ as $r \to \infty$ and **therefore** $A = B = 0$ and $C = U$. With these values. (11) reduces to

$$\psi(r, \theta) = (Ur + D/r)\sin\theta \qquad ... (12)$$

The boundary conditions of the present problem are

$$\partial\psi/\partial r = 0 \quad \text{and} \quad \partial\psi/\partial\theta = 0 \quad \text{when} \quad r = a. \qquad ...(13)$$

But $\partial\psi/\partial\theta = 0$ for all values of θ and the tangential component is zero. It follows that $\psi(a, \theta)$ = constant, where the constant may be chosen to be zero. Thus the non-slip boundary conditions on the surface of the cylinder take the form:

$$\psi(a, \theta) = 0, \qquad \text{and} \qquad \psi_r(a, \theta) = 0. \qquad ...(14)$$

These two boundary conditions cannot be satisfied by any appropriate choice of D. It follows that there is no solution to the two dimensional Stokes equations which can satisfy both the near and the far boundary conditions. The absence of such a solution is called *Stokes's paradox*.

17.4 Oseen's equations and Oseen approximations [Himanchal 1998, 2005, 06]

In the absence of body forces, the Navier Stokes equation. for steady flow of an incompressible fluid, is given by (refer equation (17) in Art. 14.1, chapter 14)

$$(\mathbf{q} \cdot \nabla)\mathbf{q} = -(1/\rho)\nabla p + \nu\nabla^2\mathbf{q} \qquad ...(1)$$

The corresponding equation continuity is $\quad \nabla \cdot \mathbf{q} = 0. \qquad ...(2)$

Recall that, while getting Stokes equations (refer Art. 17.2), the inertia terms $(\mathbf{q} \cdot \nabla)\mathbf{q}$, which are of the order U^2/r at a distance r from the centre of the body were neglected altogether from (1), whereas the viscous terms $\nu\nabla^2\mathbf{q}$, which are of the order $\nu U^2/r$, were retained. Now, the ratio of intertia terms to the viscous terms at distance r is given by

$$\frac{Ur}{\nu} = \frac{UL}{\nu} \times \frac{r}{L} = \text{Re} \times \frac{r}{L}, \qquad ...(3)$$

where $$\text{Re} = \frac{UL}{\nu} = \frac{UL\rho}{\mu} = \text{Reynolds number}, \qquad ...(4)$$

where U and L are characteristic velocity and characteristic length respectively.

The ratio defined by (3) will be small (i.e., the inertia forces will be small compared to the viscous forces) only when Re and r/L are both small, Hence, the Stokes' equations will give a true picture of the flow, at very small Reynolds number, only in the neighborhood of the body. At sufficiently large distances, no matter how small Reynolds number may be, Oseen in 1910 pointed out that ratio defined by (3) can be made as large as desired and in that case the intertia and viscous forces will be of comparable magnitude. Thus, for large values of r, the Stokes approximation breaks down. Consequently, if we desire that the solution should be valid even for large values of r, then we cannot neglect the inertia terms altogether. On the other hand, if we retain the non-linear inertia terms, the problem becomes complicated and the solution is not easy.

Oseen provided an improvement to Stokes solution by partly accounting for the inertia terms at large distances. He made the substitutions

$$u = U_0 + u', \qquad v = v', \qquad w = w', \qquad ...(5a)$$

where u', v', w' are the perturbation terms, and as such, small with respect to U_0. It is to be noted, however, that this is not true in the immediate neighbourhood of the given body.

If $\mathbf{q} = (u, v, w)$, $\mathbf{q}' = (u', v', w')$ and $\mathbf{U}_0 = (U_0, 0, 0)$, then (5$a$) may be re-written as

$$\mathbf{q} = \mathbf{U}_0 + \mathbf{q}' \qquad ...(5b)$$

Substituting (5) into (1), we have

$$\big((\mathbf{U}_0 + \mathbf{q}')\cdot\nabla\big)(\mathbf{U}_0 + \mathbf{q}') = -(1/\rho)\nabla p + \nu\nabla^2(\mathbf{U}_0 + \mathbf{q}')$$

or
$$(\mathbf{U}_0\cdot\nabla + \mathbf{q}'\cdot\nabla)(\mathbf{U}_0 + \mathbf{q}') = -(1/\rho)\nabla p + \nu\nabla^2\mathbf{U}_0 + \nu\nabla^2\mathbf{q}'.$$

or
$$(\mathbf{U}_0\cdot\nabla)\mathbf{U}_0 + (\mathbf{U}_0\cdot\nabla)\mathbf{q}' + (\mathbf{q}'\cdot\nabla)\mathbf{U}_0 + (\mathbf{q}'\cdot\nabla)\mathbf{q}' = -(1/\rho)\nabla p + \nu\nabla^2\mathbf{U}_0 + \nu\nabla^2\mathbf{q}' \qquad ...(5c)$$

Since $\mathbf{U}_0$ is constant vector, we have $(\mathbf{U}_0\cdot\nabla)\mathbf{U}_0 = (\mathbf{q}'\cdot\nabla)\mathbf{U}_0 = \nu\nabla^2\mathbf{U}_0 = \mathbf{0}$.
Hence the equation (5c) reduces to

$$(\mathbf{U}_0\cdot\nabla)\mathbf{q}' + (\mathbf{q}'\cdot\nabla)\mathbf{q}' = -(1/\rho)\nabla p + \nu\nabla^2\mathbf{q}' \qquad ...(6)$$

The second term on the left hand side of (6) is neglected as it is small of the second order compared with the first term on the left hand side of (6). Thus, (6) reduces to the linear equation

$$(\mathbf{U}_0\cdot\nabla)\mathbf{q}' = -(1/\rho)\nabla p + (\mu/\rho)\nabla^2\mathbf{q}', \qquad \text{as} \qquad \nu = \mu/\rho$$

or
$$\rho(\mathbf{U}_0\cdot\nabla)\mathbf{q}' = -\nabla p + \mu\nabla^2\mathbf{q}' \qquad ...(7)$$

Substituting (5) into (2), we have

$$\nabla\cdot(\mathbf{U}_0 + \mathbf{q}') = 0 \qquad \text{or} \qquad \nabla\cdot\mathbf{q}' = 0, \qquad ...(8)$$

since $\nabla\cdot\mathbf{U}_0 = 0$, $\mathbf{U}_0$ being a constant vector.

Equations (7) and (8) are called *Oseen's equations*, and the approximations involved is known as *Oseen approximations*. In essence. the Oseen approximation linearizes the inertia terms, whereas the Stokes approximation drops inertia terms altogether.

Since $\mathbf{q}' = (u', v', w')$, (7) and (8) can be re-written as

$$\rho U_0(\partial u'/\partial x) + \partial p/\partial x = \mu\nabla^2 u' \qquad ...(9a)$$

$$\rho U_0(\partial v'/\partial x) + \partial p/\partial y = \mu\nabla^2 v' \qquad ...(9b)$$

$$\rho U_0(\partial w'/\partial x) + \partial p/\partial z = \mu\nabla^2 w' \qquad ...(9c)$$

and
$$\partial u'/\partial x + \partial v'/\partial y + \partial w'/\partial z = 0 \qquad ...(9d)$$

Observe that the Oseen's equations are valid at large distances from the body for any Reynolds number, however large. When the Reynolds number is small (Re < 1), the Oseen equations will be valid in the whole region of the flow, because they are valid at large distance while in the finite region they differ from the Navier-Stokes equations by only negligible inertia terms. Futhermore, in the neighbourhood of the body the term $(\mathbf{U}_0 \cdot \nabla)\mathbf{q}'$ will be negligible in comparision to the viscous term and so Oseen's equations then coincides with Stokes' equations

Remark. Near the body both Stokes and Oseen approximations have the same order of accuracy. However, the Oseen approximation is better in the far field of flow where the velocity is only slightly different than U_0. The Oseen equations provide a lowest order solution that is uniformly vaild everywhere in the flow field.

17.5 Oseen's solution of Stokes problem. Oseen's flow past a sphere.

[Meerut 2007; Himanchal 2001, 02, 04, 05, 06]

Refer figure (*i*) of Art. 17.3 As shown in that figure, let a solid sphere of radius '*a*' be held fixed in a uniform stream flowing past it with velocity U_0. Let the flow be steady and let there be no body forces. Then Oseen's equations (refer (9*a*) to (9*d*) of Art 17.4) are given by

$$\rho U_0 (\partial u'/\partial x) + \partial p/\partial x = \mu \nabla^2 u' \qquad \text{...(1}a\text{)}$$

$$\rho U_0 (\partial v'/\partial x) + \partial p/\partial y = \mu \nabla^2 v' \qquad \text{...(1}b\text{)}$$

$$\rho U_0 (\partial w/\partial x) + \partial p/\partial z = \mu \nabla^2 w' \qquad \text{...(1}c\text{)}$$

$$\partial u'/\partial x + \partial v'/\partial y + \partial w'/\partial z = 0, \qquad \text{...(1}d\text{)}$$

where $\quad u = U_0 + u', \qquad v = v', \qquad w = w' \qquad$ ...(2)

and u', v', w' are the perturbation terms, and as such small, with respect to the free stream velocity U_0. Here (u, v, w) are components of the velocity of the fluid at any point (x, y, z).

Differentiating (1*a*), (1*b*) and (1*c*) partially with respect to *x*, *y* and *z* respectively and the adding, we have

$$\rho U_0 \left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 v'}{\partial y \partial x} + \frac{\partial^2 w'}{\partial z \partial x} \right) + \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \mu \left(\nabla \frac{\partial u'}{\partial x} + \nabla \frac{\partial v'}{\partial y} + \nabla \frac{\partial w'}{\partial z} \right)$$

or
$$\rho U_0 \frac{\partial}{\partial x} \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right) + \nabla^2 p = \mu \nabla \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right)$$

or
$$\nabla^2 p = 0, \text{ using } (1d) \qquad \text{...(3)}$$

showing that for Oseen flow the pressure satisfies Laplce's equation, and is, therefore a harmonic function. Therefore, a particular solution of (1*a*) to (1*c*) is obtained if we write

$$p = \rho U_0 (\partial \phi/\partial x) \qquad \text{...(4)}$$

and $\quad u' = -\partial \phi/\partial x, \qquad v' = -\partial \phi/\partial y \quad$ and $\quad w' = -\partial \phi/\partial z \qquad$ ...(5)

Substituting the above value of u', v', w' in (1*d*), we get

$$-\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right) = 0 \qquad \text{or} \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

so that
$$\nabla^2 \phi = 0 \qquad \text{...(7)}$$

The solutions of the equations (1*a*), (1*b*) and (1*c*) will be complete if we write

$u' = u_0 - \partial \phi/\partial x, \qquad v' = v_0 - \partial \phi/\partial y, \qquad w' = w_0 - \partial \phi/\partial z, \qquad$ ...(7)

where u_0, v_0 and w_0 are the solutions of the equations

$$\mu U_0(\partial u_0/\partial x) = \mu\nabla^2 u_0 \qquad ...(7a)$$

$$\mu U_0(\partial v_0/\partial x) = \mu\nabla^2 v_0 \qquad ...(7b)$$

and

$$\mu U_0(\partial w_0/\partial x) = \mu\nabla^2 w_0 \qquad ...(7c)$$

respectively, Substituting the values u', v', w' given by (7) in (1*d*), we get

$$\frac{\partial}{\partial x}\left(u_0 - \frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(v_0 - \frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(w_0 - \frac{\partial\phi}{\partial z}\right) = 0$$

or

$$(\partial u_0/\partial x + \partial v_0/\partial y + \partial w_0/\partial z) - \nabla^2\phi = 0$$

or

$$\partial u_0/\partial x + \partial v_0/\partial y + \partial w_0/\partial z) = 0, \text{ using (6)} \qquad ...(7d)$$

showing that u_0, v_0, w_0 satisfy the continuity equation.

The solutions of (7*a*), 7(*b*) and (7*c*) can be written as

$$u_0 = \frac{\partial\chi}{\partial x} - \frac{U_0}{\nu}\chi, \qquad v_0 = \frac{\partial\chi}{\partial y} \qquad \text{and} \qquad w_0 = \frac{\partial\chi}{\partial z} \qquad ...(8)$$

Since u_0, v_0, w_0 must satisfy the continuity equation (7d), the function χ satisfies the equation

$$\frac{\partial}{\partial x}\left(\frac{\partial\chi}{\partial x} - \frac{U_0}{\nu}\chi\right) + \frac{\partial}{\partial y}\left(\frac{\partial\chi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\chi}{\partial z}\right) = 0$$

or

$$\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} + \frac{\partial^2\chi}{\partial z^2} - \frac{U_0}{\nu}\frac{\partial\chi}{\partial x} = 0 \qquad \text{or} \qquad \nabla^2\chi = \frac{U_0}{\nu}\frac{\partial\chi}{\partial x}$$

or

$$\partial\chi/\partial x = (\nu/U_0)\nabla^2\chi \qquad ...(9)$$

Let

$$k = U_0/2\nu \qquad ...(10)$$

Then, (9) may be re-written as

$$(\nabla^2 - k^2)e^{-kx}\chi = 0, \qquad ...(11)$$

the solution of which is well known, the simplest type being

$$e^{-kx}\chi = (A/r)e^{-kr} \qquad \text{so that} \qquad \chi = (A/r)e^{-k(r-x)}, \qquad ...(12)$$

where A is an arbitrary constant

From (7), (8) and (10), we have

$$u' = -\partial\phi/\partial x + \partial\chi/\partial x - 2k\chi \qquad ...(13a)$$

$$v' = -\partial\phi/\partial y + \partial\chi/\partial y \qquad ..(13b)$$

$$w' = -\partial\phi/\partial z + \partial\chi/\partial z \qquad ...(13c)$$

Since ϕ must involve only zonal harmonics of negative degree, we take

$$\phi = \frac{A_0}{r} + A_1\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + A_2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + ... \qquad ...(14)$$

Again, for small values of *kr*, from equation (12), we get

$$\chi = A\left(\frac{1}{r} - k + \frac{kx}{r} + \dots\right) \qquad ...(15)$$

Substituting (14) and (15) is (13*a*), we have

$$u' = -\left\{A_0\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + A_1\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \dots\right\} + A\left\{\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + k\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \dots\right\} - 2kA\left(\frac{1}{r} - k + \frac{kx}{r} + \dots\right) \qquad ...(16)$$

In what follows, we shall use the following results:

$$\frac{\partial}{\partial x} \equiv \frac{x}{r}\frac{d}{dr}, \qquad \frac{\partial}{\partial y} \equiv \frac{y}{r}\frac{d}{dr}, \qquad \frac{\partial}{\partial z} \equiv \frac{z}{r}\frac{d}{dr}, \qquad ...(17)$$

where $r^2 = x^2 + y^2 + z^2.$

We have,
$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{x}{r}\frac{d}{dr}\left(\frac{1}{r}\right) = -\frac{x}{r^3} \qquad ...(17a)$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right\} = -\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = -\left[\frac{1}{r^3} + x\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)\right]$$

$$= -\left[\frac{1}{r^3} + x \times \frac{x}{r}\frac{d}{dr}\left(\frac{1}{r^3}\right)\right] = -\left[\frac{1}{r^3} - \frac{3x^2}{r^5}\right] \qquad ...(17b)$$

and
$$\frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{1}{r} + x\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{1}{r} + x\left(-\frac{x}{r^3}\right) = \frac{1}{r} - \frac{x^2}{r^3} \qquad ...(17c)$$

Hence (16) reduces to

$$u' = -A_0\left(-\frac{x}{r^3}\right) - A_1\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right) + \ldots + A\left\{-\frac{x}{r^3} + k\left(\frac{1}{r} - \frac{x^2}{r^3}\right) + \ldots - \frac{2k}{r} + \ldots\ldots\right\}$$

or
$$u' = (A_0 - A)\frac{x}{r^3} - \left(\frac{3A_1}{r^5} + \frac{Ak}{r^3}\right)x^2 + \frac{A_1}{r^3} - \frac{Ak}{r} + \ldots\ldots.. \qquad ...(18)$$

Substituting (14) and (15) in (13*b*), we have

$$v' = -\left\{A_0\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + A_1\frac{\partial^2}{\partial y \partial x}\left(\frac{1}{r}\right) + \ldots\right\} + A\left\{\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k\frac{\partial}{\partial y}\left(\frac{x}{r}\right) + \ldots\right\} \qquad ...(19)$$

We have
$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \frac{y}{r}\frac{d}{dr}\left(\frac{1}{r}\right) = -\frac{y}{r^3} \qquad ...(20a)$$

$$\frac{\partial^2}{\partial y \partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right) = \frac{\partial}{\partial y}\left(-\frac{x}{r^3}\right), \text{ by (17}a\text{)}$$

$$= -x\frac{\partial}{\partial y}\left(\frac{1}{r^3}\right) = -x \times \frac{y}{r}\frac{d}{dr}\left(\frac{1}{r^3}\right) = \frac{3xy}{r^5} \qquad ...(20b)$$

Also,
$$\frac{\partial}{\partial y}\left(\frac{x}{r}\right) = x\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = x \times \left(-\frac{y}{r^3}\right) = -\frac{xy}{r^3}, \text{ by (20}a\text{)} \qquad ...(20c)$$

Hence (19) reduce to

$$v' = \frac{A_0 y}{r^3} - \frac{3A_1 xy}{r^5} + \ldots + A\left(-\frac{y}{r^3} - \frac{kxy}{r^3} + \ldots\right) = (A_0 - A)\frac{y}{r^3} - \left(\frac{3A_1}{r^5} + \frac{kA}{r^3}\right)xy + \ldots \qquad ...(21)$$

Proceeding likewise, (13*c*) reduces to
$$w' = (A_0 - A)\frac{z}{r^3} - \left(\frac{3A_1}{r^5} + \frac{kA}{r^3}\right)xz + \ldots \qquad ...(22)$$

In order to get the desired solution, the values of u', v', w' given by (18), (21) and (22) must satisfy the boundary conditions on the surface of the sphere, namely $u = v = w = 0$ for $r = a$

i.e.,
$$u' = -U_0, \qquad v' = 0 \quad \text{and} \quad w' = 0, \quad \text{when} \quad r = a \qquad ...(23)$$

Putting $u' = -U_0$ and $r = a$ in (18), we find

$$-U_0 = (A_0 - A)\frac{x}{a^3} - \left(\frac{3A_1}{a^5} + \frac{Ak}{a^3}\right)x^2 + \frac{A_1}{a^3} - \frac{Ak}{a} \qquad \text{...(24)}$$

Equating to zero the coefficients of different powers of x on both sides of the identity (24),

$$(A_0 - A)/a^3 = 0 \qquad \text{so that} \qquad A_0 = A \qquad \text{...(25a)}$$

$$3A_1/a^5 + Ak/a^3 = 0 \qquad \text{...(25b)}$$

and

$$A_1/a^3 - Ak/a = -U_0 \qquad \text{...(25c)}$$

Multiplying both sides of (25c) by $1/a^2$, $\quad A_1/a^5 - Ak/a^3 = -U_0/a^2$ ...(25d)

Adding (25b) and (25c), $\quad 4A_1/a^5 = -U_0/a^2$ giving $\quad A_1 = (1/4) \times U_0 a^3$ ...(25e)

From (25b), $A = -\dfrac{3A_1}{ka^2} = \dfrac{3}{ka^2} \times \dfrac{U_0 a^3}{4} = \dfrac{3U_0 a}{4} \times \dfrac{2\nu}{U_0}$, by (10)

Thus, $\quad A = (3/2) \times \nu a = A_0$, by (25$a$) ...(25$f$)

where ak, *i.e.*, $U_0 a/2\nu$ or Re/4 (taking Re $= 2U_0 a/\nu$ = Reynolds number) is taken to be small

Substituting the values of A_0, A_1 and A as given by (25e) and (25f) in (18), (21), (22) and then using (2), we finally obtain

$$u = U_0 + u' = U_0 \left\{\frac{3}{4}\frac{ax^2}{r^3}\left(\frac{a^2}{r^2} - 1\right) + 1 - \frac{a}{4r}\left(3 + \frac{a^2}{r^2}\right)\right\} \qquad \text{...(26a)}$$

$$v = v' = U_0 \frac{3}{4}\frac{axy}{r^3}\left(\frac{a^2}{r^2} - 1\right) \qquad \text{...(26b)}$$

$$w = w' = U_0 \frac{3}{4}\frac{axz}{r^3}\left(\frac{a^2}{r^2} - 1\right), \qquad \text{...(26c)}$$

which are exactly the same as obtained in Stokes flow (refer (20a) to (20c) of Art. 17.3).

The drag coefficient will therefore be the same as obtained in Stokes flow, namely,

$$C_D = 24/\text{Re} \qquad \text{...(27)}$$

However, by considering a more general solution of the equation (11), Oseen correction to the Stokes drag coefficient is given by

$$C_D = (24/\text{Re}) \times (1 + 3\,\text{Re}/16) = 24/\text{Re} + 4.5 \qquad \text{...(28)}$$

Equation (27) is plotted in figure (*iv*) of Art. 17.3 together with experimental results and Stokes formula. It is seen that Oseen formula is good upto Re = 5 whereas Stokes formula is accurate upto only when Re < 0.5.

Re-writing the equations (13a) to (13c) in vector form, we have

$$\mathbf{q}' = -\nabla\phi + \nabla\chi - 2k\chi\,\mathbf{i}, \qquad \text{...(29)}$$

where $\mathbf{q}' = (u', v', w')$, Keeping (29) in view, when the sphere is regarded as in motion, and the fluid at rest at infinity, the radial velocity q'_r is given by

$$q'_r = -\partial\phi/\partial r + (1/2k) \times (\partial\chi/\partial r) - \chi\cos\theta, \qquad \text{...(30)}$$

where θ denotes the inclination of the radius vector to the axis of x.

Therefore, the stream function ψ is given by

$$\psi = -\int_0^\theta r^2 \sin\theta\, q'_r\, d\theta = r^2 \int_0^\theta \left(\frac{\partial\phi}{\partial r} - \frac{1}{2k}\frac{\partial\chi}{\partial r} + \chi\cos\theta\right)\sin\theta\, d\theta \qquad \text{...(31)}$$

Substituting values of χ and ϕ as given by (12) and (14), with the values of constants A_1, A_0 and A as given by (25*e*) and (25*f*), and performing the indicated integration, we have

$$\psi = (3/2)\times \nu a(1+\cos\theta)\left\{1-e^{-kr(1-\cos\theta)}\right\}-(U_0 a^3/4r)\times \sin^2\theta \qquad ...(32)$$

For small values of kr, (32) reduces to $$\psi = \frac{3U_0 ar}{4}\left(1-\frac{a^2}{2r^2}\right)\sin^2\theta, \qquad ...(33)$$

which is in agreement with result (40*a*) of Art. 17.3

For Oseen flow the flow pattern of streamlines is now no longer the same in front of and behind the sphere. This fact can be easily understood with help of equations (1*a*) to (1*d*), because if we change the sign of the velocities and the pressure, the equations do not transform into themselves, whereas the Stokes equations (see (1*a*) to (1*d*) of Art. 17.3) did. The streamlines of the Oseen flow are given in the following figure, and the observer is assumed to be at rest with respect to the sphere; it is assumed that the sphere is dragged with constant velocity U_0. Observe that the flow in front of the sphere is very similar to that given by Stokes, but behind the sphere the streamlines are closer together which implies that the velocity is larger than in the former case.

Thus, Oseen flow has a wake where the streamlines are closer together than in the Stokes flow. The velocities in the wake are larger than in the front the sphere. Relative to the sphere, the flow is slower in the wake than in front of the sphere.

Furthermore, note that the velocity distribution in Oseen flow at a large distance from the sphere differs from that of Stokes flow.

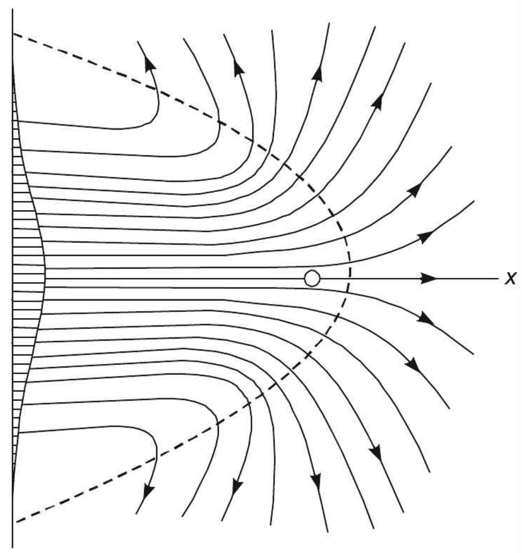


Figure showing streamlines in the flow past a sphere from Oseen's solution

17.6 Oseen's solution for the motion of a circular cylinder. [Meerut 1999]

If we proceed to find the steady motion produced by the translation of a cylinder with constant velocity U_0 through an infinite mass of fluid on the basis of the usual Stoke's equations (1*a*)-(1*c*) of Art. 17.2, then we have already seen (refer Art. 17.3*F*) that it is impossible to satisfy all the boundary conditions of the problem. This fact was pointed out by Stokes, who gave the following explanation: "*The pressure of the cylinder on the fluid continually tends to increase the quantity of fluid which it carries with it, while the friction of the fluid at a distance from the cylinder continually tends to diminish it. In the case of a sphere, these two causes eventually counteract each other, and motion becomes uniform. But in the case of a cylinder, the increase in the quantity of fluid carried continually gains on the decrease due to the friction of the surrounding fluid, and the quantity carried increases indefinitely as the cylinder moves on*"

In what follows, we shall use Oseen's equations to study the steady motion produced by the translation of cylinder with constant velocity U_0, through an infinite mass of viscous fluid. We shall show that the above conclusion gets modified, and a definite value for the resistance is obtained.

Let a solid circular cylinder of radius a be fixed. Let x-axis be taken in the direction of uniform stream U_0 flowing past it. Let z-axis be taken along the axis of the cylinder and centre of a circular section of it be taken as origin of coordinate system. Let (u, v) and p be the components of velocity and pressure at an arbitrary point $P(x, y)$ of the fluid. Let the flow be steady and let there be no body forces. Then Oseen's equations (refer (9a) to (9d) Art 17.4) for the two-dimensional motion in xy-plane are given by

$$\rho U_0(\partial u'/\partial x)+\partial p/\partial x=\mu\nabla_1^2 u' \qquad ...(1a)$$

$$\rho U_0(\partial v'/\partial x)+\partial p/\partial y=\mu\nabla_1^2 v' \qquad ...(1b)$$

$$\partial u'/\partial x+\partial v'/\partial y=0 \qquad ...(1c)$$

where $$\nabla_1^2=\partial^2/\partial x^2+\partial^2/\partial y^2, \qquad ...(2)$$

and $$u=U_0+u' \qquad \text{and} \qquad v=v' \qquad ...\ (3)$$

where u', v' are the perturbation terms, and as such small, with respect to free stream velocity U_0.

Differentiating (1a) and (1b) partially with respect to 'x' and y respectively and then adding, we obtain

$$\rho U_0\left(\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 v'}{\partial y\partial x}\right)+\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=\mu\nabla_1^2\frac{\partial u'}{\partial x}+\mu\nabla_1^2\frac{\partial v'}{\partial y}$$

or $$\rho U_0\frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)+\nabla_1^2 p=\mu\nabla_1^2\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right), \text{ using (2)}$$

or $$\nabla_1^2 p=0, \text{ using } (1c) \qquad ...(4)$$

showing that for Oseen flow the pressure satisfies Laplace's eauation, and is, therefore a harmonic function. Therefore, a particular solution of (1a) and (1b) is obtained if we write

$$p=\rho U_0(\partial\phi/\partial x) \qquad ...(5)$$

and $$u'=-\partial\phi/\partial x \qquad \text{and} \qquad v'=-\partial\phi/\partial y \qquad ...(6)$$

Substituting these values in (1c), we have

$$-\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)=0 \qquad \text{or} \qquad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

so that $$\nabla_1^2\,\phi=0 \qquad ...(7)$$

The solution of the equations (1a) and (1d) will be complete if we write

$$u'=u_0-\partial\phi/\partial x \qquad \text{and} \qquad v'=v_0-\partial\phi/\partial y \qquad ...(8)$$

where u_0 and v_0 are the solutions of the equations

$$\mu U_0(\partial u_0/\partial x)=\mu\nabla^2 u_0 \qquad ...(9a)$$

and $$\mu U_0(\partial v_0/\partial x)=\mu\nabla^2 v_0 \qquad ...(9b)$$

respectively. Substituting the values of u' and v' given by (8) in (1c), we have

$$\frac{\partial}{\partial x}\left(u_0-\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial x}\left(v_0-\frac{\partial\phi}{\partial y}\right)=0 \qquad \text{or} \qquad \frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial x}-\nabla_1^2\,\phi=0$$

or $$\partial u_0 / \partial x + \partial v_0 / \partial y = 0, \text{ using (7)} \qquad ...(10)$$
showing that u_0 and v_0 satisfy the continuity equation.

The solutions of (9*a*) and (9*b*) can be written as
$$u_0 = \partial \chi / \partial x - (U_0 / \nu)\chi \qquad \text{and} \qquad v_0 = \partial \chi / \partial y \qquad ...(11)$$
Substituting these values of u_0 and v_0 in (10), we get
$$\frac{\partial}{\partial x}\left(\frac{\partial \chi}{\partial x} - \frac{U_0}{\nu}\chi\right) + \frac{\partial}{\partial y}\left(\frac{\partial \chi}{\partial y}\right) = 0 \qquad \text{or} \qquad \nabla_1^2 \chi = \frac{U_0}{\nu}\frac{\partial \chi}{\partial x}$$
or $$\partial \chi / \partial x = (\nu / U_0)\nabla_1^2 \chi \qquad ...(12)$$
Let $$k = U_0 / 2\nu \qquad ...(13)$$
Then (12) can be re-written as $$\left(\nabla_1^2 - 2k\frac{\partial}{\partial x}\right)\chi = 0 \qquad ...(14)$$

Transforming (14) in terms of polar coordinates (r, θ), we have
$$\frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r}\frac{\partial \chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \chi}{\partial \theta^2} - 2k\cos\theta \frac{\partial \chi}{\partial \theta} = 0. \qquad ...(15)$$
We now solve (15) by using the know method of separation of variables*. Accordingly, suppose that $$\chi = X(x) R(r) \qquad ...(16)$$
where X (x) is function of x alone whereas $R(r)$ is a function of r alone.

We take $$X(x) = e^{kx} = e^{kr\cos\theta}, \qquad ...(17)$$
since in polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$ ...(18)

Then, from (16), $$\chi = e^{kr\cos\theta} R(r). \qquad ...(19)$$
From (19), $$\partial \chi / \partial r = k\cos\theta\, e^{kr\cos\theta} R + e^{kr\cos\theta}(dR/dr) \qquad ...(20)$$
$\therefore$ (20) $\Rightarrow$ $$\partial^2 \chi / \partial r^2 = k^2 \cos^2\theta\, e^{kr\cos\theta} R + 2k\cos\theta\, e^{kr\cos\theta}(dR/dr) + e^{kr\cos\theta}(d^2R/dr^2) \qquad ...(21)$$
Again, (19) $\Rightarrow$ $$\partial \chi / \partial \theta = -kr\sin\theta e^{kr\cos\theta} R = -kr\, R(\sin\theta\, e^{kr\cos\theta})$$
and so $$\partial^2 \chi / \partial \theta^2 = -kr\, R(\cos\theta\, e^{kr\cos\theta} - kr\sin^2\theta\, e^{kr\cos\theta}) \qquad ...(22)$$
Substituting the values of $\partial \chi / \partial r, \partial^2 \chi / \partial r^2$ and $\partial \chi / \partial \theta$ as given by (20), (21) and (22) in (15) and simplifying, we finally obtain
$$d^2R/dr^2 + (1/r) \times (dR/dr) - k^2 R = 0, \qquad ...(23)$$
which is Bessel's equation with imaginary argument of zeroth order. We shall choose the solution which vanishes at infinity. Clearly, such solution is
$$R(r) = K_0(kr) \qquad ...(24)$$
From (16), (17) and (24), it follow that $$\chi = Ae^{kr} K_0(k\,r), \qquad ...(25)$$
where A is an arbitrary constant to be determined.

Take $$\phi = A_0 \log r + A_1 \partial(\log r)/\partial x + \ldots \qquad ...(26)$$
Now, $$K_0(kr) = \int_0^\infty e^{-kr\cosh w} dw \qquad ...(27)$$

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* Refer "Boundary value problems" in author's book "Advanced Differential equations", published by S. Chand and Co., New Delhi

or $$K_0(kr) = -\left(\gamma + \log\frac{1}{2}kr\right)\left\{1 + \frac{1}{2^2}(kr)^2 + \frac{1}{2^2\cdot 4^2}(kr)^4 + \ldots\right\} + \frac{1}{2^2}(kr)^2$$
$$+\left(1+\frac{1}{2}\right)(kr)^4 + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{1}{2^2\cdot 4^2\cdot 6^2}(kr)^6 + \ldots, \qquad \ldots(28)$$

where $$\gamma = 0.577 \text{ approximately.} \qquad \ldots(29)$$

From small values of kr, (28) shows that $$K_0(kr) \sim -\left(\gamma + \log\frac{1}{2}kr\right) \qquad \ldots(30)$$

Also, we know that $$dK_0(x)/dx = -K_1(x) \qquad \ldots(31)$$

and $$K_1(kr) = \frac{1}{kr} + \frac{1}{2}kr\left\{\left(\gamma + \log\frac{1}{2}kr\right) - \frac{1}{2} + \ldots\right\}$$

so that for small values of kr, $$K_1(kr) \sim 1/kr \qquad \ldots(32)$$

Using (11), (8) reduces to

$$u' = -\frac{\partial\phi}{\partial x} + \frac{\partial\chi}{\partial x} - \frac{U}{\nu}\chi = -\frac{\partial}{\partial x}(\phi - \chi) - 2k\chi, \text{ by (13)} \qquad \ldots(33a)$$

and $$v' = -\partial\phi/\partial y + \partial\chi/\partial y = -\partial(\phi - \chi)/\partial y \qquad \ldots(33b)$$

Using (25) and (26), (33*a*) reduces to

$$u' = -\frac{\partial}{\partial x}\left\{A_0 \log r + A_1\frac{\partial}{\partial x}\log r + \ldots - Ae^{kx}K_0(kr)\right\} - 2kAe^{kx}K_0(kr)$$

or $$u' = -A_0\frac{\partial}{\partial x}\log r - A_1\frac{\partial^2}{\partial x^2}\log r + \ldots + A\frac{\partial}{\partial x}\left(e^{kx}K_0(kr)\right) - 2kAe^{kx}K_0(kr) \qquad \ldots(34)$$

In what follows, we shall use the following results.

$$\frac{\partial}{\partial x} = \frac{x}{r}\frac{d}{dr} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{y}{r}\frac{d}{dr}, \quad \text{where for xy-plane} \quad r^2 = x^2 + y^2 \qquad \ldots(35)$$

Now, $$\frac{\partial}{\partial x}\log r = \frac{x}{r}\frac{d}{dr}\log r = \frac{x}{r^2} \qquad \ldots(36a)$$

$$\therefore \quad \frac{\partial^2}{\partial x^2}\log r = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\log r\right) = \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) = \frac{1}{r^2} + x\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right)$$

$$= \frac{1}{r^2} + x \times \frac{x}{r}\frac{d}{dr}\left(\frac{1}{r^2}\right) = \frac{1}{r^2} - \frac{2x^2}{r^4} \qquad \ldots(36b)$$

$$\frac{\partial}{\partial x}e^{kx}K_0(kr) = ke^{kx}K_0(kr) + e^{kx}\frac{\partial}{\partial x}K_0(kr) = ke^{kx}K_0(kr) + e^{kx}\frac{x}{r}\frac{d}{dr}K_0(kr)$$

$$= ke^{kx}K_0(kr) + e^{kx} \times (x/r) \times K_0'(kr) \times k$$

$$= ke^{kx}K_0(kr) - kx\,e^{kx}(1/r)K_1(kr), \text{ using (31)} \qquad \ldots(36c)$$

Using the results given by(36*a*), (36*b*) and (36*c*) in (34), we obtain

$$u' = -\frac{A_0 x}{r^2} - A_1\left(\frac{1}{r^2} - \frac{2x^2}{r^4}\right) + \ldots + Ake^{kx}K_0(kr) - Akxe^{kx}\frac{1}{r}K_1(kr) - 2Ak\,e^{kx}K_0(kr)$$

or $$u' = -\frac{A_0 x}{r^2} - A_1\left(\frac{1}{r^2} - \frac{2x^2}{r^4}\right) - Ak\, e^{kx} K_0(kr) - \frac{Akx}{r} e^{kx} K_1(kr) \qquad ...(37)$$

Using (25) and (26), (33*b*) reduces to

$$v' = -\frac{\partial}{\partial y}\left\{A_0 \log r + A_1 \frac{\partial}{\partial x}\log r + \ldots - Ae^{kx} K_0(kr)\right\}$$

or $$v' = -A_0 \frac{\partial \log r}{\partial y} - A_1 \frac{\partial^2}{\partial y \partial x}\log r + \ldots + Ae^{kx}\frac{\partial}{\partial y} K_0(kr) \qquad ...(38)$$

Now, $$\frac{\partial \log r}{\partial y} = \frac{y}{r}\frac{d \log r}{dr} = \frac{y}{r^2} \qquad ...(39a)$$

$\therefore$ $$\frac{\partial^2 \log r}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\log r\right) = \frac{\partial}{\partial x}\left(\frac{y}{r^2}\right) = y\frac{\partial}{\partial x}\left(\frac{1}{r^2}\right) = y \times \frac{x}{r}\frac{d}{dr}\left(\frac{1}{r^2}\right) = -\frac{2xy}{r^4} \qquad ...(39b)$$

Also, $$\frac{\partial}{\partial y} K_0(kr) = \frac{y}{r}\frac{d}{dr} K_0(kr) = \frac{y}{r} K_0'(kr) \times k = -\frac{ky}{r} K_0(kr) \qquad ...(39c)$$

Using the results given by (39*a*), (39*b*) and (39*c*) in (38), we get

$$v' = -\frac{A_0 y}{r^2} + \frac{2A_1 xy}{r^4} + \ldots - Ae^{kx} \times \left(\frac{ky}{r}\right) K_1(kr) \qquad ...(40)$$

We now proceed to find the values of u' and v' near the cylinder. From (30) and (32), we have

$$K_0(kr) \sim -\left(\gamma + \log\frac{1}{2}kr\right) \quad \text{and} \quad K_1(kr) \sim \frac{1}{kr} \qquad ...(41)$$

Also, near the cylinder, we take $e^{kx} = 1 + kx$ ...(42)

Using (41) and (42), near the cylinder (38) and (40) reduces to

$$u' = -\frac{A_0 x}{r^2} - A_1\left(\frac{1}{r^2} - \frac{2x^2}{r^4}\right) + Ak(1+kx)\left(\gamma + \log\frac{1}{2}kr\right) - \frac{Akx}{r}(1+kx) \times \frac{1}{kr}$$

or $$u' = -\frac{A_0 x}{r^2} - A_1\left(\frac{1}{r^2} - \frac{2x^2}{r^4}\right) + Ak\left(\gamma + \log\frac{1}{2}kr\right) - \frac{Ax}{r^2}(1+kx) \qquad ...(43a)$$

and $$v' = -\frac{A_0 y}{r^2} + \frac{2A_1 xy}{r^4} - \frac{Aky}{r}(1+kx) \times \frac{1}{kr} = -\frac{A_0 y}{r^2} + \frac{2A_1 xy}{r^4} - \frac{Ay}{r^2} - \frac{Akxy}{r^2}$$

or $$v' = -(A_0 + A)\frac{y}{r^2} + \left(\frac{2A_1}{r^2} - Ak\right)\frac{xy}{r^2} \qquad ...(43b)$$

The boundary conditions of the problems are:

$u = U_0 + u' = 0$ and $v = v' = 0$ when $r = a$

i.e., $u' = -U_0$ and $v' = 0$ when $r = a$ ...(44)

$\therefore$ Substituting $r = a$ and $u' = -U_0$ in (43*a*), we have

$$-U_0 = -\frac{A_0 x}{a^2} - A_1\left(\frac{1}{a^2} - \frac{2x^2}{a^4}\right) + Ak(\gamma + \log\frac{1}{2}kr) - \frac{Ax}{a^2}(1+kx)$$

or $$-U_0 = -(A_1/a^2) + ak\{\gamma + \log(kr/2)\} - (A_0 + A) \times (x/a^2) + (2A_1/a^4 - Ak/a^2) \times x^2$$

Equating the coefficients of like powers on both sides, we get

$$-(A_1/a^2) + Ak\{\gamma + \log(kr/2)\} = -U_0 \qquad \text{...(45}a\text{)}$$

$$(A_0 + A)/a^2 = 0 \quad \text{so that} \quad A_0 = -A \qquad \text{...(45}b\text{)}$$

and $$2A_1/a^4 - Ak/a^2 = 0 \quad \text{so that} \quad A_1 = (1/2) \times Aka^2 \qquad \text{...(45}c\text{)}$$

(45*a*) and (45*c*) $\Rightarrow$ $$-\frac{Ak}{2} + Ak\left(\gamma + \log\frac{1}{2}kr\right) = -U_0$$

or $$Ak\left\{\frac{1}{2} - \left(\gamma + \log\frac{1}{2}kr\right)\right\} = U_0 \quad \text{or} \quad Ak = \frac{U_0}{1/2 - \gamma - \log(kr/2)} \qquad \text{...(45}d\text{)}$$

In *xy*-pleane, we have, $x = r\cos\theta$ and $y = r\sin\theta$ ...(45*e*)

Using (45*b*), (45*c*) and (45*e*), (43*a*) reduces to

$$u' = \frac{Ax}{r^2} - A_1\left(\frac{1}{r^2} - \frac{2r^2\cos^2\theta}{r^4}\right) + Ak\left(\gamma + \log\frac{1}{2}kr\right) - \frac{Ax}{r^2} - \frac{Akr^2\cos^2\theta}{r^2}$$

or $$u' = -\frac{Aka^2}{2}\left(\frac{1}{r^2} - \frac{2\cos^2\theta}{r^2}\right) + Ak\left(\gamma + \log\frac{1}{2}kr - \cos^2\theta\right)$$

or $$u' = -\frac{Aka^2}{2r^2}(1 - 2\cos^2\theta) + Ak\left(\gamma + \log\frac{1}{2}kr - \cos^2\theta\right) \qquad \text{...(46}a\text{)}$$

Similarly, using (45b), (45c) and (45e), (43b) reduces to

$$v' = \left(\frac{2}{r^2} \times \frac{Aka^2}{2} - Ak\right)\sin\theta\cos\theta = Ak\left(\frac{a^2}{r^2} - 1\right)\sin\theta\cos\theta \qquad \text{... (46b)}$$

(5) and (26) $\Rightarrow$ $$p = \rho U_0\frac{\partial}{\partial x}\left(A_0\log r + A_1\frac{\partial}{\partial x}\log r + \dots\right) = \rho U_0\left(A_0\frac{\partial}{\partial x}\log r + A_1\frac{\partial^2}{\partial x^2}\log r + \dots\right)$$

or $$p = \rho U_0\left\{\frac{A_0 x}{r^2} + A_1\left(\frac{1}{r^2} - \frac{2x^2}{r^4}\right) + \dots\right\}, \text{ by (36}a\text{) and (36}b\text{)}$$

or $$p = \rho U_0\left\{-\frac{Ax}{r^2} + \frac{Aka^2}{2}\left(\frac{1}{r^2} - \frac{2r^2\cos^2\theta}{r^4}\right) + \dots\right\}, \text{ by (45}b\text{), (45}c\text{) and (45}d\text{)}$$

or $$p = A\rho U_0\left\{-\frac{\cos\theta}{r} + \frac{ka^2(1 - 2\cos^2\theta)}{2r^2} + \dots\right\}, \quad \text{as} \quad x = r\cos\theta \qquad \text{...(47)}$$

Let the components of stress across a sphere of radius r be σ_{rx} and σ_{ry}. Then, we have

$$\sigma_{rx} = (x/r)\sigma_{xx} + (y/r)\sigma_{yx} = \left(-p + 2\mu\frac{\partial u'}{\partial x}\right)\cos\theta + \mu\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)\sin\theta \qquad \text{...(48)}$$

[Using (45*e*) and results of Art 14.10 of chapter 14]

From (46*a*), $$\frac{\partial u'}{\partial x} = -\frac{Aka^2(1 - 2\cos^2\theta)}{2}\frac{\partial r^{-2}}{\partial x} + Ak\frac{\partial\log(kr/2)}{\partial x}$$

or $$\frac{\partial u'}{\partial x} = -\frac{Aka^2(1 - 2\cos^2\theta)}{2} \times \frac{x}{r}\frac{dr^{-2}}{dr} + Ak \times \frac{x}{r}\frac{d\log(kr/2)}{dr}$$

$$= -\frac{Aka^2(1-2\cos^2\theta)}{2} \times \frac{x}{r} \times \left(-\frac{2}{r^3}\right) + \frac{Akx}{r} \times \frac{k/2}{(kr)/2}$$

$$= -\frac{Aka^2(1-2\cos^2\theta)}{2} \times \left(-\frac{2r\cos\theta}{r^4}\right) + \frac{Akr\cos\theta}{r} \times \frac{1}{r}, \text{ as } x = r\cos\theta$$

$$\therefore \qquad \left(\frac{\partial u'}{\partial x}\right)_{r=a} = \frac{Ak\{(1-2\cos^2\theta)\cos\theta + \cos\theta\}}{a} \qquad \text{...(49}a\text{)}$$

From (46*a*),
$$\frac{\partial u'}{\partial y} = -\frac{Aka^2(1-2\cos^2\theta)}{2} \frac{\partial r^{-2}}{\partial y} + AK\frac{\partial \log(kr/2)}{\partial y}$$

or
$$\frac{\partial u'}{\partial y} = -\frac{Aka^2(1-2\cos^2\theta)}{2} \times \frac{y}{r}\frac{dr^{-2}}{dr} + Ak \times \frac{y}{r}\frac{d\log(kr/2)}{dr}$$

$$= -\frac{Aka^2(1-2\cos^2\theta)}{2} \times \frac{y}{r}\left(-\frac{2}{r^3}\right) + \frac{Aky}{r} \times \frac{kr/2}{(kr)/2}$$

$$= -\frac{Aka^2(1-2\cos^2\theta)}{2} \times \left(-\frac{2r\sin\theta}{r^4}\right) + \frac{Akr\sin\theta}{r} \times \frac{1}{r}, \text{ as } \quad y = r\sin\theta$$

$$\therefore \qquad \left(\frac{\partial u'}{\partial y}\right)_{r=a} = \frac{Ak\{(1-2\cos^2\theta)\sin\theta + \sin\theta\}}{a} \qquad \text{...(49}b\text{)}$$

From (46*b*),
$$\frac{\partial v'}{\partial x} = Aka^2\sin\theta\cos\theta\frac{\partial r^{-2}}{\partial x} = Aka^2\sin\theta\cos\theta \times \frac{x}{r} \times \frac{dr^{-2}}{dr} = \frac{Aka^2 r\sin\theta\cos^2\theta}{r} \times \left(-\frac{2}{r^3}\right)$$

or
$$(\partial v'/\partial x)_{r=a} = -(2Ak\sin\theta\cos^2\theta)/a \qquad \text{...(49}c\text{)}$$

Hence
$$\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)_{r=a} = \frac{2Ak}{a}(\sin\theta - 2\cos^2\theta\sin\theta), \text{ by (49b) and (49}c\text{)} \qquad \text{...(49}d\text{)}$$

From (48),
$$(\sigma_{rx})_{r=a} = -(p\cos\theta)_{r=a} + 2\mu\cos\theta\left(\frac{\partial u'}{\partial x}\right)_{r=a} + \mu\sin\theta\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)_{r=a}$$

$$\therefore \qquad (\sigma_{rx})_{r=a} = -A\rho U_0\cos\theta\left\{-\frac{\cos\theta}{a} + \frac{k}{2}(1-2\cos^2\theta)\right\} + \frac{4\mu Ak\cos\theta}{a}(\cos\theta - \cos^3\theta)$$

$$+(2Ak\mu/a)\,(\sin\theta - 2\cos^2\theta\sin\theta)\sin\theta, \text{ by (47), (49}a\text{) and (49}d\text{)}$$

$$= A\rho U_0\left\{(1/a)\cos^2\theta - (k/2)\cos\theta + k\cos^3\theta\right\}$$

$$+(2Ak\mu/a)\,\{2\cos^2\theta - 2\cos^4\theta + \sin^2\theta - 2\cos^2(1-\cos^2\theta)\}$$

Then,
$$(\sigma_{rx})_{r=a} = A\rho U_0\left\{(1/a)\cos^2\theta - (k/2)\cos\theta + k\cos^3\theta\right\} + (2Ak\mu/a) \times \sin^2\theta \qquad \text{...(50)}$$

This gives the force exerted by the fluid on a unit area of the cylinder along the x-axis. In getting the value of $(\sigma_{rx})_{r=a}$, we have neglected the terms of the order higher than k.

Let D be drag on the unit length of the cylinder. Then, we have

$$D = 2\int_0^{\pi} \sigma_{rx}\, a\, d\theta = 2a\int_0^{\pi}\Big[A\rho U_0 \big\{(1/a)\cos^2\theta - (k/2)\cos\theta + k\cos^3\theta\big\} + \{(2kA\mu)/a\}\times \sin^2\theta\Big]d\theta$$

or $D = 2A\rho U_0 \int_0^{\pi}\cos^2\theta d\theta - A\rho U_0 ak\int_0^{\pi}\cos\theta\, d\theta + 2A\rho U_0\, ak\int_0^{\pi}\cos^3\theta d\theta + 4kA\mu\int_0^{\pi}\sin^2\theta\, d\theta$ ...(51)

Now, $$\int_0^{\pi}\cos^2\theta\, d\theta = \int_0^{\pi}\frac{1-\cos 2\theta}{2}d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \frac{\pi}{2}$$

$$\int_0^{\pi}\sin^2\theta\, d\theta = \int_0^{\pi}\frac{1-\cos 2\theta}{2}d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \frac{\pi}{2}, \qquad \int_0^{\pi}\cos\theta\, d\theta = [\sin\theta]_0^{\pi} = 0$$

and $$\int_0^{\pi}\cos^3\theta\, d\theta = \frac{1}{4}\int_0^{\pi}(\cos 3\theta + \cos\theta)\, d\theta = \frac{1}{4}\left[\frac{1}{3}\sin 3\theta + \sin\theta\right]_0^{\pi} = 0$$

$$[\because \cos 3\theta = 4\cos^3\theta - \cos\theta \;\Rightarrow\; \cos^3\theta = (\cos 3\theta + \cos\theta)/4]$$

Hence (51) reduces to $D = A\rho U_0\pi + 2kA\mu\pi = 2Ak\mu\pi + 2Ak\mu\pi$

$$[\because k = U_0/2\nu \;\Rightarrow\; U_0 = 2k\nu = 2k(\mu/\rho)\,]$$

Thus, $D = (4\pi\mu)\times(Ak) = \dfrac{4\pi\mu U_0}{1/2 - \gamma - \log(kr/2)}$, using relation (45*d*) ...(52)

The velocity is given by

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = Ae^{kx}\frac{\partial}{\partial y}\int_0^{\infty} e^{-kr\cosh w}dw$$

which for large values of kr takes the form

$$\zeta = -kA\times(y/r)\times(\pi/2kr)^{1/2}e^{-k(r-x)} \qquad \text{...(53)}$$

The above investigtion is subject to the condition that ka, or $U_0 a/2\nu$ is to be small. Observe that the value of the expression in (52) does not vary repidly with a. Thus, for example, for $ka = 1/10$, (52) yields $D = 4.31\mu U_0$. whereas for $ka = 1/20$, (52) yields $D = 3.48\mu U_0$.

17.7 Reynold's hydrodynamic theory of lubrication [Himanchal 2003, 05, 07, 09]

It is a matter of common experience that if we have two metals which are in contact with each other and are moving with some relative velocity then besides wear and tear in the metal the friction is very high and hence it results in a great loss of energy. This is avoided as follows. At high velocities the clearance between two machine elements which are in relative motion (*e.g.* journal and bearing) is filled by a viscous fluid in which extremely large pressure differences may be created. As a consequence the revolving journal is lifted somewhat by the viscous fluid and metallic contact between the moving parts is prevented. This method reduces both the wear and tear in the metals and the friction. This device is generally known as *bearing*. The fluid used in the bearing usually has high viscosity and is called *lubricant*. Oils and greases are commonly used as lubricants. The theory of lubrication was initiated by Osborne Reynolds.

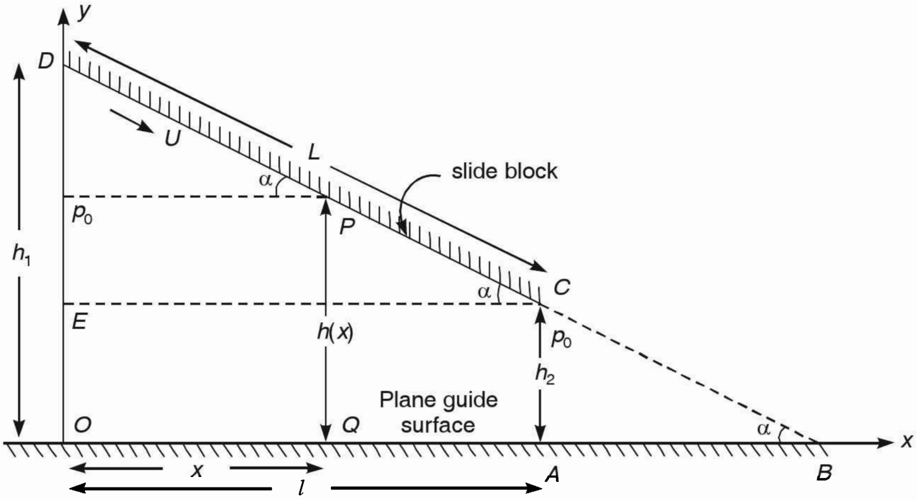
Thus, the lubrication theory states that two solid bodies can slide over one another very easily when there is a thin layer of fluid between them and the under certain conditions a high positive pressure is set up in the fluid layer. For example, a sheet of paper dropped on to a smooth floor will often float on a film of air between it and the floor and thereby will be able to glide horizontally for some distance before coming to rest.

With the advancement of industry and complicated machines of high speeds, the theory of lubrication has become very important. An extensive work has been done in this direction in recent years. In the next article, we propose to discuss the theory of lubrication with help of the example of a side block (or slipper) moving on a plane guide surface.

17.8 An illustration of the hydrodynamic theory of lubrication

[Himanchal 2000, 03, 05]

The essential features of theory of lubrication can be understood on the example of a slide block (or slipper) DC of length L moving with velocity U over a plane guide surface OA of length l as shown in the diagram, it being important that they are inclined at a small angle α to each other. In what follows we shall assume that the sliding surfaces are very large in a transverse direction with respect to the motion so that the problem is one in two dimensions. Let the x-axis be in the direction of motion and y-axis perpendicular to the plane guide surface. Let $h(x)$, which is a linear function of x, for simplicity, be the height of the wedge between the block and the plane guide surface.



On the basis of estimation of the viscous and inertia forces, it can be seen that in all cases of pratical importance the viscous forces are predominant. The largest viscous term in the equation of motion for the x-direction is equal to $\mu\,(d^2u/dy^2)$. We can make the following estimate:

$$\frac{\text{Inertia forces}}{\text{Viscous forces}} = \frac{\rho u\,(\partial u/\partial x)}{\mu\,(\partial^2 u/\partial y^2)} = \frac{\rho U^2/L}{\mu U/h^2} = \text{Re}\left(\frac{h}{L}\right)^2 << 1,$$

which is true when the Reynolds number $\text{Re}\,(=\rho UL/\mu)$ is very small and $h(x)$ very small compared with L. The problem is, therefore, essentially of slow motion and we may use Stokes' equations and continuity equation ((1a) to (1c) and (2) of Art 17.2 for steady flow of incompressible viscous fluid in xy-plane, namely,

$$\rho\,(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2) = \partial p/\partial x \qquad ...(1a)$$

$$\rho\,(\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2) = \partial p/\partial y \qquad ...(1b)$$

$$\partial u/\partial x + \partial v/\partial y = 0 \qquad ...(2)$$

The above Stokes' equations can be further simplified for the case under consideration. Since the component v is very small with respect to u, (1b) reduces to $\partial p/\partial y = 0$ and so p is independent of y. Hence p is function of x alone and so we can write $\qquad \partial p/\partial x = dp/dx.$

Futhermore, in the equation (2a) for the x-direction $\partial^2 u/\partial x^2$ can be neglected with respect to $\partial^2 u/\partial y^2$, because the former is smaller than the latter by a factor of the order $(h/L)^2$. The pressure distribution must satisfy the condition that $p = p_0$ at both ends of slide block DC, *i.e.* $p = p_0$ when $h = h_1$ and $h = h_2$, where $h_1 = OD$ and $h_2 = AC$:

As already assumed, v being very small with respect to u, (2) reduces to $\partial u/\partial x = 0$ so that u is independent of x and so u is function of y alone. Hence we can write $\partial^2 u/\partial y^2 = d^2u/dy^2$. Hence (1*a*) may be re-written as

$$\mu(d^2u/dy^2) = dp/dx \qquad \text{...(3)}$$

We now solve (3) subject to the boundary conditions:

When $y = 0$, $u = 0$; when $h = h_1$, $p = p_0$ ...(4*a*)

When $y = h$, $u = U$; when $h = h_2$, $p = p_0$ ...(4*b*)

Differentiating both sides of (3) with respect to x, we get

$$\frac{d}{dx}\left(\frac{dp}{dx}\right) = \frac{d}{dx}\left(\mu\frac{d^2u}{dy^2}\right) = \mu\frac{d^2}{dy^2}\left(\frac{\partial u}{\partial x}\right) \quad \text{or} \quad \frac{d}{dx}\left(\frac{dp}{dx}\right) = 0, \qquad \text{...(5)}'$$

as u is function of y only and so $\partial u/\partial x = 0$

Integration (5)', dp/dx = constant = K, say ...(5)

Thus, (3) reduces to $d^2u/dy^2 = K/\mu$ ...(6)

Integrating (6), $du/dy = Ky/\mu + A$ ...(7)

Integrating (7), $u = Ky^2/2\mu + Ay + B$, ...(8)

where A and B are arbitrary constants to be determined.

Using boundary condition (4*a*), namely $u = 0$ when $y = 0$, (8) gives B = 0. Next using boundary condition (4*b*), namely, $u = U$ when $y = h$, (8) gives $U = Kh^2/2\mu + Ah$ so that $A = U/h - Kh/2\mu$. With the above values of A and B. (8) reduces to

$$u = \frac{Ky^2}{2\mu} + \left(\frac{U}{h} - \frac{Kh}{2\mu}\right)y \quad \text{or} \quad u = \frac{Uy}{h} - \frac{yK}{2\mu}(h-y), \qquad \text{...(9)}$$

where K and h are both functions of x alone.

Form the equation of continuity, it follows that the volume of flow in every section must be constant. Hence at given x, we must have

$$\int_0^h u\,dy = \text{const.} \quad \text{or} \quad \int_0^h \left\{\frac{Uy}{h} - \frac{K}{2\mu}(yh - y^2)\right\}dx = \text{const., by (9)}$$

or
$$\left[\frac{Uy^2}{2h} - \frac{K}{2\mu}\left(\frac{hy^2}{2} - \frac{h^3}{3}\right)\right]_0^h = \text{const.} = \frac{Uh_0}{2}, \text{ say}$$

or
$$\frac{Uh}{2} - \frac{K}{2\mu}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = \frac{Uh_0}{2} \quad \text{or} \quad \frac{Uh}{2} - \frac{Uh_0}{2} = \frac{Kh^3}{2\mu}$$

or
$$K = dp/dx = \{6\mu U(h - h_0)\}/h^3, \qquad \text{...(10)}$$

where the value of the constant h_0 is still to be determined.

Now, the coordinates of points D, P and C are $(0, h_1)$ (x, h) and (l, h_2) respectively. From the coordinate geometry, we have

the slope of PD = the slope of CD $\Rightarrow$ $(h - h_1)/(x - 0) = (h_2 - h_1)/(l - 0)$

$\Rightarrow$ $h = h_1 + (h_2 - h_1) \times (x/l)$ $\Rightarrow$ $dh/dx = (h_2 - h_1)/l$ ...(11)

Dividing (10) by (11), we have

$$\frac{dp/dx}{dh/dx} = \frac{6\mu Ul(h-h_0)}{h^3(h_2-h_1)} \qquad \text{or} \qquad dp = \frac{6\mu Ul}{h_2-h_1}\left(\frac{1}{h^2}-\frac{h_0}{h^3}\right)dh$$

Intergrating, $$p = \frac{6\mu Ul}{h_2-h_1}\left(-\frac{1}{h}+\frac{h_0}{2h^2}\right)+C, \; C \text{ being an arbitrary constant} \qquad \ldots(12)$$

Using boundary condition (4*a*), namely, $p = p_0$ when $h = h_1$ (12) yields

$$p_0 = \frac{6\mu Ul}{h_2-h_1}\left(-\frac{1}{h_1}+\frac{h_0}{2h_1^2}\right)+C \qquad \ldots(13)$$

Subtracting (13) from (12), we have

$$p-p_0 = \frac{6\mu Ul}{h_2-h_1}\left\{\frac{h_0}{2}\left(\frac{1}{h^2}-\frac{1}{h_1^2}\right)-\left(\frac{1}{h}-\frac{1}{h_1}\right)\right\} = \frac{6\mu Ul}{h_2-h_1}\left\{\frac{h_0}{2}\left(\frac{1}{h}-\frac{1}{h_1}\right)\left(\frac{1}{h}+\frac{1}{h_1}\right)-\left(\frac{1}{h}-\frac{1}{h_1}\right)\right\}$$

or $$p-p_0 = \frac{6\mu Ul}{h_2-h_1}\left(\frac{1}{h}-\frac{1}{h_1}\right)\left\{\frac{h_0}{2}\left(\frac{1}{h}+\frac{1}{h_1}\right)-1\right\} \qquad \ldots(14)$$

Now, using the boundary condition (4*b*), namely, $p = p_0$ when $h = h_2$, (14) yields

$$0 = \frac{6\mu Ul}{h_2-h_1}\left(\frac{1}{h_2}-\frac{1}{h_1}\right)\left\{\frac{h_0}{2}\left(\frac{1}{h_2}+\frac{1}{h_1}\right)-1\right\} \qquad \text{or} \qquad \frac{h_0}{2}\left(\frac{1}{h_2}+\frac{1}{h_1}\right)-1=0,$$

since $h_1 \neq h_2$. Simplifying this relation, we obtain

$$h_0(h_1+h_2) = 2h_1h_2 \qquad \text{or} \qquad h_0 = (2h_1h_2)/(h_1+h_2) \qquad \ldots(15)$$

Substituting the above value of h_0 in (14), we get

$$p-p_0 = \frac{6\mu Ul}{h_2-h_1}\left(\frac{1}{h}-\frac{1}{h_1}\right)\left(\frac{h_1h_2}{h_1+h_2}\times\frac{h_1+h}{hh_1}-1\right)$$

or $$p-p_0 = \frac{6\mu Ul}{h_2-h_1}\times\frac{h_1-h}{hh_1}\times\frac{h_2(h_1+h)-h(h_1+h_2)}{h(h_1+h_2)} = \frac{6\mu Ul}{h_2^2-h_1^2}\times\frac{(h_1-h)(h_2-h)}{h^2}$$

or $$p-p_0 = \frac{6\mu Ul}{h_1^2-h_2^2}\times\frac{(h_1-h)(h-h_2)}{h^2} \qquad \ldots(16)$$

From (16), it follows that the condition for positive pressure excess (*i.e.*, $p-p_0$ to be positive) requires $h_1 > h_2$, or in other words, the channel should be convergent.

Since α in very small, we take $\alpha = \tan\alpha = (h_2-h_1)/l$ approximately.

Then (16) may be re-written in the form

$$p-p_0 = \frac{6\mu U}{\alpha}\times\frac{(h_1-h)(h-h_2)}{h^2(h_1+h_2)} \qquad \ldots(16)'$$

We now proceed to compute the total thrust P and the frictional resistance F experienced by the inclined bolck DC. Now, we have

$$P = \int_0^l (p-p_0)\,dx = \frac{l}{h_2-h_1}\int_{h_1}^{h_2}(p-p_0)\,dh, \text{ using (11)}$$

$$= -\frac{6\mu Ul^2}{(h_1^2-h_2^2)(h_2-h_1)}\int_{h_1}^{h_2}\frac{(h-h_1)(h-h_2)}{h^2}dh, \text{ using (16)}$$

$$= \frac{6\mu U l^2}{(h_2 - h_1)^2\,(h_1 + h_2)} \int_{h_1}^{h_2} \left\{ 1 - \frac{(h_1 + h_2)}{h} + \frac{h_1 h_2}{h^2} \right\} dh$$

$$= \frac{6\mu U l^2}{(h_2 - h_1)^2\,(h_1 + h_2)} \left[h - (h_1 + h_2) \log h - \frac{h_1 h_2}{h} \right]_{h_1}^{h_2}$$

$$= \frac{6\mu U l^2}{(h_2 - h_1)^2\,(h_1 + h_2)} \left\{ h_2 - h_1 - (h_1 + h_2) \log \frac{h_2}{h_1} - h_1 h_2 \left(\frac{1}{h_2} - \frac{1}{h_1} \right) \right\}$$

$$= \frac{6\mu U l^2}{(h_2 - h_1)^2\,(h_1 + h_2)} \left\{ -(h_1 - h_2) + (h_1 + h_2) \log \frac{h_1}{h_2} - (h_1 - h_2) \right\}$$

Thus, $$P = \frac{6\mu U l^2}{(h_2 - h_1)^2} \left\{ \log \frac{h_1}{h_2} - 2 \frac{(h_1 - h_2)}{h_1 + h_2} \right\} \qquad ...(17)$$

Let $k = h_1/h_2$ so that $h_1 = k\,h_2$. Then, (17) yields

$$P = \frac{6\mu U l^2}{(k-1)^2 h_2^2} \left\{ \log k - 2 \frac{(k-1)}{k+1} \right\} \qquad ...(18)$$

Now, $$F = \int_0^l \left(\mu \frac{du}{dy} \right)_{y=h} dx = \int_{h_1}^{h_2} \left(\mu \frac{du}{dy} \right)_{y=h} \frac{l\,dh}{h_2 - h_1}, \text{ by (11)} \qquad ...(19)$$

From (9), $\dfrac{du}{dy} = \dfrac{U}{h} - \dfrac{K}{2\mu}(h - 2y)$ and hence $\left(\dfrac{du}{dy} \right)_{y=h} = \dfrac{U}{h} + \dfrac{hK}{2\mu}$

$\therefore$ By (19), $$F = \int_{h_1}^{h_2} \mu \left(\frac{U}{h} + \frac{hK}{2\mu} \right) \frac{l\,dh}{h_2 - h_1}$$

or $$F = \frac{\mu U l}{h_2 - h_1} \int_{h_1}^{h_2} \frac{dh}{h} + \frac{l}{2(h_2 - h_1)} \int_{h_1}^{h_2} \left\{ h \times \frac{6\mu U\,(h - h_0)}{h^3} \right\} dh, \text{ using (10)}$$

$$= \frac{\mu U l}{h_2 - h_1} [\log h]_{h_1}^{h_2} + \frac{3\mu U l}{h_2 - h_1} \int_{h_1}^{h_2} \left(\frac{1}{h} - \frac{h_0}{h^2} \right) dh = \frac{\mu U l}{h_2 - h_1} \log \frac{h_2}{h_1} + \frac{3\mu U l}{h_2 - h_1} \left[\log h + \frac{h_0}{h} \right]_{h_1}^{h_2}$$

$$= \frac{\mu U l}{h_2 - h_1} \log \frac{h_2}{h_1} + \frac{3\mu U l}{h_2 - h_1} \left\{ \log \frac{h_2}{h_1} + h_0 \left(\frac{1}{h_2} - \frac{1}{h_1} \right) \right\} = \frac{4\mu U l}{h_2 - h_1} \log \frac{h_2}{h_1} + \frac{3\mu U l h_0}{h_2 - h_1} \times \frac{h_1 - h_2}{h_1 h_2}$$

$$= \frac{4\mu U l}{h_1 - h_2} \log \frac{h_1}{h_2} - \frac{3\mu U l}{h_1 h_2} \times \frac{2 h_1 h_2}{h_1 + h_2} = \frac{2\mu U l}{h_1 - h_2} \left\{ 2 \log \frac{h_1}{h_2} - \frac{3(h_1 - h_2)}{h_1 + h_2} \right\}, \text{ using (15)}$$

Thus, $$F = \frac{2\mu U l}{(k-1) h_2} \left\{ 2 \log k - \frac{3(k-1)}{k+1} \right\}, \quad \text{as} \quad k = \frac{h_1}{h_2} \qquad ...(20)$$

Considering P as a function of k, it can be easily seen from equation (18) that P is maximum when $k = 2.2$ approximately. Hence, we have

$$P_{\max} = (0.16\,\mu U l^2)/h_2^2 \qquad ...(21)$$

and from (20), $$F = (0.75\,\mu U l)/h_2 \qquad ...(22)$$

The ratio of the frictional resistance to the maximum thrust is given by

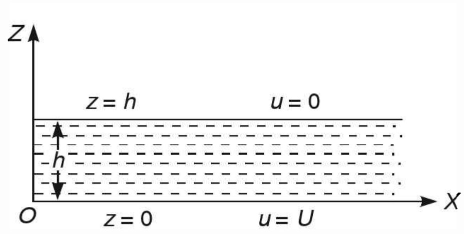
$$F/P_{\max} = (4.7 h_2)/l \qquad ...(23)$$

For good lubricaton, the above ratio (23) can be made very small by taking h_2 much small compared with l.

17.9 The hydrodynamic Theory of lubrication. (Alternative method)

It is a well known fact that two nearly parallel planes can slide one over the other without much resistance, if a flow of viscous liquid is kept between them. A necessary condition is that the opposing surfaces should be slightly inclined to one another and that the relative motion should tend to drag the fluid from the wider to the narrower part of the intervening space. In what follows, we shall need some results, which we now prove.

Steady liminar flow between two parallel plates.



Consider the steady laminar flow of viscous incompressible fluid between two parallel plates separated by a distance h. Let x be the direction of flow, z the direction perpendicular to the flow, and the width of the plates parallel to y-direction. Let the plates be so wide in the y-driection that the problem may be treated as two-dimensional (*i.e.* $\partial/\partial y = 0$). Here we take the velocity components v and w to be zero everywhere. Moreover the flow being steady, the flow variables are independent of time $(\partial/\partial t = 0)$ Furthermore the equation of continuity, $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$

reduces to $\quad \partial u/\partial x = 0 \quad$ so that $\quad u = u\ (z)$.

For the present two-dimensional flow in absence of body forces, the Navier-Stoke's equation for x and z-directions are (Refer Art. 14.10)

$$0 = -(\partial p/\partial x) + \mu(d^2u/dz^2) \qquad \text{...(1)}$$

and
$$0 = -(\partial p/\partial z) \qquad \text{...(2)}$$

Equation (2) shows that p does not depend on z. So p is function of x alone and hence (1) reduces to

$$d^2u/dz^2 = (1/\mu) \times (dp/dx) \qquad \text{...(3)}$$

Differentiating both sides of (3) *w.r.t.* 'x', we get

$$0 = \frac{1}{\mu}\frac{d^2 p}{dx^2} \quad \text{or} \quad \frac{d}{dx}\left(\frac{dp}{dx}\right) = 0 \quad \Rightarrow \quad \frac{dp}{dx} = \text{const.} = P, \text{ say}$$

$\therefore$ (3) reduces to
$$d^2u/dz^2 = P/\mu \qquad \text{...(4)}$$

Integrating (4), $\quad \dfrac{du}{dz} = \dfrac{Pz}{\mu} + A,\ A$ being an arbitrary constant $\qquad$ ...(5)

Integrating (5), $\quad u = Az + B + \dfrac{Pz^2}{2\mu},\ B$ being an arbitrary constant $\qquad$ ...(6)

Let the upper plate ($z = h$) be at rest while the lower plate ($z = 0$) move with a uniform velocity U, so that the boundary conditions are

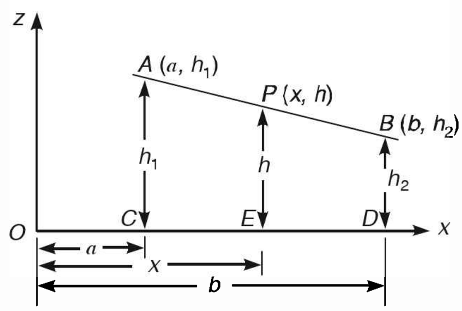
$$u = 0 \quad \text{at} \quad z = h; \qquad u = U \quad \text{at} \quad z = 0. \qquad \text{...(7)}$$

Using (7), (6) yields $\quad 0 = Ah + B + (Ph^2)/2\mu \quad$ and $\quad U = B$,

so that $\quad$ B = U $\quad$ and $\quad A = -\dfrac{U}{h} - \dfrac{Ph}{2\mu} \qquad$ ...(8)

Using (8), (6) reduces to
$$u = -\left(\frac{U}{h} + \frac{Ph}{2\mu}\right)z + U + \frac{Pz^2}{2\mu}. \qquad \text{...(9)}$$

Discussion of theory of lubrication.



Consider a fixed block with face AB nearly parallel to another plane CD ($z = 0$), which has a uniform velocity U in the x-direction. Let the block be so wide in the y-direction that the problem may be regarded as two-dimensional. Let P be any point on AB. Let coordinates of A, P, B be (a, h_1), (x, h) and (b, h_2) respectively. Since the inclination of the plane face is small, the velocity u given by (9) may be used for the present problem also.

Let Q be the total flux in x-direction. Then, we have

$$Q = \int_0^h u dz = \int_0^h \left[-\left(\frac{U}{h} + \frac{Ph}{2\mu}\right) z + U + \frac{Pz^2}{2\mu} \right] dz = -\left(\frac{U}{h} + \frac{Ph}{2\mu}\right)\frac{h^2}{2} + Uh + \frac{P}{2\mu} \times \frac{h^3}{3} = \frac{1}{2} hU - \frac{h^3 P}{12\mu}.$$

The condition of continuity requires that the total flux Q must be independent of x. Hence,

$$\frac{1}{2} hU - \frac{h^3 P}{12\mu} = \text{cosntant} = \frac{1}{2} h_0 U, \quad \text{say} \quad \Rightarrow \quad P = \frac{dp}{dx} = 6\mu U \left(\frac{h - h_0}{h^3}\right), \qquad ...(10)$$

where h_0 is the value of h at points of maximum pressure.

$$\text{The slope of } AB = \frac{h_2 - h_1}{b - a} = \frac{h_2 - h}{b - x} \quad \Rightarrow \quad (h_2 - h_2)(b - x) = (b - a)(h_2 - h) \qquad ...(11)$$

Differentiation both sides of (11) *w.r.t.* 'x', we get

$$-(h_2 - h_1) = -(b - a)\frac{dh}{dx} \qquad \text{so that} \qquad \frac{dh}{dx} = \frac{h_2 - h_1}{b - a} = \frac{h_2 - h_1}{l}, \qquad ...(12)$$

where $b - a = l$ = length of the block.

Now, $$\frac{dp}{dh} = \frac{dp}{dx} \cdot \frac{dx}{dh} = 6\mu U \left(\frac{h - h_0}{h^3}\right) \cdot \frac{l}{h_2 - h_1}, \text{ by (10) and (12)}$$

or $$dp = \frac{6\mu U l}{h_2 - h_1}\left(\frac{1}{h^2} - \frac{h_0}{h^3}\right) dh$$

Integrating, $$p = \frac{6\mu U l}{h_2 - h_1}\left(-\frac{1}{h} + \frac{h_0}{2h^2}\right) + C = \frac{3\mu U l}{h_2 - h_1}\left(\frac{h_0 - 2h}{h^2}\right) + C \qquad ...(13)$$

Since the addition of a constant pressure throughout the fluid will make no difference to the solution, for the sake of simplicity, we suppose that $p = 0$ beyond the ends of block. Accordingly,

$$p = 0 \quad \text{at} \quad h = h_1; \qquad p = 0 \quad \text{at} \quad h = h_2. \qquad ...(14)$$

Using (14), (13) reductes to

$$\frac{3\mu U l}{h_2 - h_1}\left(\frac{h_0 - 2h_1}{h_1^2}\right) + C = 0 \qquad ...(15A)$$

and $$\frac{3\mu U l}{h_2 - h_1}\left(\frac{h_0 - 2h_2}{h_2^2}\right) + C = 0. \qquad ...(15B)$$

Solving (15*A*) and (15*B*), $$h_0 = \frac{2h_1 h_2}{h_1 + h_2} \quad \text{and} \quad C = \frac{6\mu U l}{(h_2 - h_1)(h_2 + h_1)}. \qquad ...(16)$$

Putting these values in (13), we have

$$p = \frac{3\mu Ul}{h_2 - h_1}\left[\frac{1}{h^2}\left(\frac{2h_1h_2}{h_1+h_2} - 2h\right)\right] + \frac{6\mu Ul}{(h_2-h_1)(h_2+h_1)}$$

or $$p = \frac{6\mu Ul(h-h_1)(h-h_2)}{h^2(h_2^2-h_1^2)} \qquad \text{or} \qquad p = \frac{6\mu Ul(h_1-h)(h-h_2)}{h^2(h_1^2-h_2^2)} \qquad ...(17)$$

Form (17), we find that p cannot be positive unless $h_1 > h_2$ *i.e.* there is contraction in the direction of motion. Let P' be the total pressure. Then, we have

$$\therefore \qquad P' = \int_{h_1}^{h_2} p dx = \int_a^h p\frac{dx}{dh}dh = \frac{l}{h_2-h_1}\int_{h_1}^{h_2} p\, dh \quad \text{using (12)}$$

$$= \frac{6\mu Ul^2}{(h_1^2-h_2^2)(h_2-h_1)}\int_{h_1}^{h_2}\frac{(h_1-h)(h-h_2)}{h^2}dh, \text{ using (17)}$$

Thus, $$P' = \frac{6\mu Ul^2}{(h_1-h_2)^2(h_1+h_2)}\int_{h_1}^{h_2}\frac{(h-h_1)(h-h_2)}{h^2}dh \qquad ...(18)$$

Now, $$\int_{h_1}^{h_2}\frac{(h-h_1)(h-h_2)}{h^2}dh = \int_{h_1}^{h_2}\left(1+\frac{h_1h_2}{h^2}-\frac{h_1}{h}-\frac{h_2}{h}\right)dh$$

$$= \left[h - \frac{h_1h_2}{h} - h_1\log h - h_2\log h\right]_{h_1}^{h_2} = -2(h_1-h_2)+(h_1+h_2)\log(h_1/h_2).$$

Hence (18) reduces to

$$P' = \frac{6\mu Ul^2}{(h_1-h_2)^2(h_1+h_2)}\left\{-2(h_1-h_2)+(h_1+h_2)\log\frac{h_1}{h_2}\right\}$$

or $$P' = \frac{6\mu Ul^2}{(k-1)^2h_2^2}\left\{\log k - 2\frac{(k-1)}{k+1}\right\}, \qquad ...(19)$$

where $k = h_1/h_2$.

Now, using (9), the tangential stress σ_{zx} is given by

$$\sigma_{zx} = \mu\frac{du}{dz} = -\mu\left(\frac{U}{h}+\frac{Ph}{2\mu}\right)+\frac{Pz}{\mu}.$$

Hence the tangential stress on either surface is given by

$$(\sigma_{zx})_{z=0} = -\mu\left(\frac{U}{h}+\frac{Ph}{2\mu}\right) = -\left(\frac{\mu U}{h}+\frac{Ph}{2}\right) \qquad ...(20)$$

Hence the total frictional force F is given by

$$F = \int_a^b(-\sigma_{zx})_{z=0}\,dx = \int_{h_1}^{h_2}\left(\frac{\mu U}{h}+\frac{Ph}{2}\right)\frac{dx}{dh}dh, \text{ by (20)}$$

$$= \int_{h_1}^{h_2}\left[\frac{\mu U}{h}+\frac{h}{2}\times 6\mu U\left(\frac{h-h_0}{h^3}\right)\right]\cdot\frac{l}{h_2-h_1}dh, \text{ using (10) and (12)}$$

$$= \frac{\mu Ul}{h_2-h_1}\int_{h_1}^{h_2}\left(\frac{4}{h}-\frac{3h_0}{h^3}\right)dh = \frac{\mu Ul}{h_2-h_1}\int_{h_1}^{h_2}\left(\frac{4}{h}-\frac{3}{h^3}\times\frac{2h_1h_2}{h_1+h_2}\right)dh, \text{ by (16)}$$

$$\therefore \qquad F = \frac{2\mu Ul}{(k-1)h_2}\left\{2\log k - \frac{3(k-1)}{k+1}\right\}, \text{ on simplification} \qquad ...(21)$$

Comparing (19) and (21), it follows that the ratio F/P' of the total frictional force to the total load is independent of both μ and U, but proportinal to h if the scale of h is altered.

It has been found by Reynold and Rayleigh that P', considered as a function of k, is maximum for $k = 2.2$ approximately. This gives

$$P' = (0.1602) \times (\mu U l^2 / h_2) \qquad \text{and} \qquad F = (0.75) \times (\mu U l / h_2)$$

$$\therefore \qquad P'/F = (4.7) \times (h_2 / l).$$

Hence by taking h_2 small enough compared to l, we can ensure a small frictional drag *i.e.* fair good lubrication.

Remark. When there is flow in the direction of y with velocity V along with the flow in the direction of x with velocity U, then we have

$$\text{total flow in } x\text{-direction} = \frac{1}{2} hU - \frac{h^3}{12\mu} \frac{\partial p}{\partial x}$$

and

$$\text{total flow in } y\text{-direction} = \frac{1}{2} hV - \frac{h^3}{12\mu} \frac{\partial p}{\partial y}.$$

Hence the equation of continuity is

$$\frac{\partial}{\partial x}\left(\frac{1}{2} hU - \frac{h^3}{12\mu} \frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{2} hV - \frac{h^3}{12\mu} \frac{\partial p}{\partial y}\right) = 0$$

or

$$\frac{\partial}{\partial x}\left(h^3 \frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3 \frac{\partial p}{\partial y}\right) = 6\mu\left[\frac{\partial}{\partial x}(hU) + \frac{\partial}{\partial y}(hV)\right].$$

An Illustrative solved example.

Ex.1. *One surface (nearly plane) is fixed and another surface (plane) rotates with angular velocity* ω *about an axis perpendicular to is plane and there is a film of viscous fluid between them. Prove that the pressure p satisfies the equation*

$$h^2\left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2}\right) + \frac{\partial h^3}{\partial r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial h^3}{\partial \theta}\frac{\partial p}{\partial \theta} = 6\mu\omega\frac{\partial h}{\partial \theta},$$

where (r, θ) *are polar coordinates in the plane of the film, the origin being in the axis of the film of rotation, and h is the thickness of the film.*

Solution. We know that equation of continuity is (refer remark in Art. 17.9)

$$\frac{\partial}{\partial x}\left(h^3 \frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3 \frac{\partial p}{\partial y}\right) = 6\mu\left[\frac{\partial}{\partial x}(hU) + \frac{\partial}{\partial y}(hV)\right]. \qquad \text{...(1)}$$

At any point (x, y) on the upper surface, $\quad U = -\omega y, \qquad V = \omega x. \qquad$ ...(2)

Using (2), (1) reduces to

$$\frac{\partial}{\partial x}\left(h^3 \frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3 \frac{\partial p}{\partial y}\right) = 6\mu\omega\left[\frac{\partial}{\partial y}(xh) - \frac{\partial}{\partial x}(yh)\right]$$

or

$$h^3\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) + \frac{\partial h^3}{\partial x}\frac{\partial p}{\partial x} + \frac{\partial h^3}{\partial y}\frac{\partial p}{\partial y} = 6\mu\omega\left(x\frac{\partial h}{\partial y} - y\frac{\partial h}{\partial x}\right) \qquad \text{...(3)}$$

We now transform (3) into polar coordinates by taking

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}(y/x). \qquad \text{...(4)}$$

Also, we know that (Refer any standard textbook of calculus for partial differention)

$$\left.\begin{aligned} \frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \qquad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \\ \text{and} \qquad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \end{aligned}\right\} \qquad \text{...(5)}$$

Using (4) and (5), (3) reduces to

$$h^3\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)p + \left(\cos\theta\frac{\partial h^3}{\partial r} - \frac{\sin\theta}{r}\frac{\partial h^3}{\partial\theta}\right)\left(\cos\theta\frac{\partial p}{\partial r} - \frac{\sin\theta}{r}\frac{\partial p}{\partial\theta}\right)$$

$$+\left(\sin\theta\frac{\partial h^3}{\partial r} + \frac{\cos\theta}{r}\frac{\partial h^3}{\partial\theta}\right)\left(\sin\theta\frac{\partial p}{\partial r} + \frac{\cos\theta}{r}\frac{\partial p}{\partial\theta}\right)$$

$$= 6\mu\omega\left[r\cos\theta\left(\sin\theta\frac{\partial h}{\partial r} + \frac{\cos\theta}{r}\frac{\partial h}{\partial\theta}\right) - r\sin\theta\left(\cos\theta\frac{\partial h}{\partial r} - \frac{\sin\theta}{r}\frac{\partial h}{\partial\theta}\right)\right]$$

Simplifying, we finally arrive at

$$h^3\left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2}\right) + \frac{\partial h^3}{\partial r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial h^3}{\partial\theta}\frac{\partial p}{\partial\theta} = 6\mu\omega\frac{\partial h}{\partial\theta}.$$

EXERCISE

1. Derive Stokes' equation for very slow motion and discuss Stokes' flow past a sphere. **[Himanchal 1997, 99, 2000]**

2. Derive the expressions of velocity components, pressure distribution and drag coefficient for flow past a sphere (Stokes' flow) **[Himanchal 2001, 02, 04]**

3. Find the drag on a sphere in Stokes' flow. Determine the terminal velocity of a sphere for a vertical fall in a liquid. **[Himanchal 2003]**

Hint: Refer particular case on page 17.17 of Art. 17.3 A

4. Show that stream function for two dimensional steady very slow viscous flow is a biharmonic function. Also, discuss the motion of a sphere in a viscous fluid to derive Stokes' formula for drag at very small Reynolds number. **[Himanchal 1998]**

5. Derve Oseen's equations for very slow motion and discuss Oseen's flow past a sphere. **[Himanchal 2005]**

6. Derive the velocity distribution in the neighbourhood of the sphere fixed in a uniform stream flowing steadily past it (Oseen's flow). **[Himanchal 2001, 04]**

7. Derive the expressions of the velocity components and drag coefficient for steady flow of an incompressible viscous fluid past a sphere (Oseen's flow). **[Himanchal 2002, 09]**

8. Discuss Oseen's improvement to Stokes' theory and hence determine the drag coefficient on a sphere moving in a viscous fluid **[Himanchal 1998]**

9. A sphere of radius a moving with a constant velocity U_0 along the x-axis through a viscous liquid at rest at infinity. Verify that, on Oseen's hypothesis the stream function is

$$\psi = \frac{U_0 a^3 \sin^2\theta}{4r} - \frac{3\nu a}{2}(1+\cos\theta)(1-e^{-k(r-x)}), \text{ where } k = U_0/2\nu$$ **[Himanchal 1999]**

Hint: Refer result (32) of Art. 17.5

10. Discuss the principle of lubrication theory, treating a thin liquid film between two inclined planes as an example. **[Himanchal 2003, 05]**

11. Incompressible viscous fluid under no body forces moves in a thin film between the fixed plane $y = 0$, and a rigid moving plane $y = h(x)$, inclined at a small angle α to the plane $y = 0$, and the leading and trailing edges are at heights h_1 and h_2 respectively. Determine the pressure at points in the section whose thickness is h in terms of the velocity of the moving plane and the pressures at the edges. **[Himanchal 2003]**

12. Let the pressure distribution p in a thin film between the fixed plane $y = 0$ and a rigid moving plane $y = h(x)$ inclined at a small angle α to the plane $y = 0$ be given by

$$p - p_0 = \frac{6\mu U}{\alpha} \times \frac{(h_1 - h)(h - h_2)}{(h_1 + h_2) h^2},$$

where p_0 denotes the pressure at the edges which are at heights h_1 and h_2 and U denote the velocity of the moving plane.

Calculate total thrust and the frictional resistance experienced by the inclined block. Obtain the condition for ensuring good lubrication.

Hint: Refer result (16) Art 17.11 **[Himanchal 1997, 99, 2000]**

13. Derive Reynolds equation of lubrication by order of magnitude analysis. **[Himanchal 1998]**

14. Discuss the steady, axi-symmetric uniform flow of a viscous incompressible fluid around a sphere at rest. Obtain the Stokes' solution of the problem. Show that the total drag on the force is $6\pi a\mu U$, where a is the radius of the sphere, U is uniform streaming and μ is the coefficient of viscosity of the fluid. Point out the range of validity of the solution. **[Meerut 2000, 05; Agra 2005]**

15. (*a*) Discuss the steady motion of viscous fluid due to a slowly rotating sphere. **[Meerut 2002]**

(*b*) Discuss the steady motion of a viscous fluid due to slowly rotating sphere and determine the dissipation of energy. **[Meerut 1999]**

16. Discuss Stokes' flow and derive an expression for Stokes formula for drag on a sphere **(Meerut 2007)**

17. Discuss Stokes flow for a viscous incompressible fluid and derive (*i*) velocity components and pressure distribution (*ii*) Stokes formula for the drag on a sphere. **(Himanchal 2007)**

18. Write a short note on lubrication theory **(Himanchal 2007)**

19. Derive the expressions of the velocity components and pressure distribution in Stokes' flow case. **(Himanchal 2007)**

20. Derive the expressions of velocity components and drag coefficient in Oseen's flow. **(Himanchal 2006, 09)**

21. Discuss theory of slow motion of sphere and derive expression for the coefficient of drag $C_D = (24/\text{Re}) \times (1 + 3\,\text{Re}/16)$. **(Meerut 2007)**

22. Find the drag on the sphere of radius a, which is being held with its centre at the origin in a uniform stream of viscous incompressible liquid having undisturbed velocity $\vec{U} = -U\,\vec{i}$ **(Agra 2007)**

23. A solid sphere of radius a is rotating with small angular velocity Ω in a mass of viscous liquid which is at rest at infinity. Calculate the couple which must be applied to the sphere to maintain its rotation. **(Meerut 1999, 2001, 02)**

24. Write a short note on lubrication theory. **(Himanchal 2009)**

25. Discuss the motion of a sphere moving with velocity U through an infinite mass of ideal fluid at rest at infinity. **(Meerut 2011)**

CHAPTER

18 Boundary Layer Theory

18.1. Introduction.

The Navier-Stokes equations were obtained in Art. 14.1. In Art. 16.1, it was observed that a complete solution of these equations has not been accomplished to date. This is particularly true when friction and inertia forces are of the same order of magnitude in the entire flow system, so that neither can be neglected. In chapter 16, we discussed some very special cases of flow problems for which exact solutions of the Navier-Stokes equations are possible. In those cases, the equations were made linear by taking a simple geometry of flow and assuming the fluid to be incompressible.

In chapter 17 we dealt with a case of the approximate solutions of the Navier-Stokes equations for very small Reynold's number. In that chapter the friction forces far over-shadowed the inertia forces, and the equations became linear by omitting the convective acceleration. The present chapter discusses the opposite, *i.e.*, flow characterized by very large Reynold's numbers.

18.2. The main limitations of ideal (non-viscous) fluid dynamics.

(*i*) *The theory is unable to predict flow separation.*

In other words, by neglecting viscosity of the fluid, we are not in a position to explain why fluid which flows close to the cylinder's surface around the upstream side would tend to move away from the surface on the downstream side.

(*ii*) *The theory is unable to explain the existence of a wake.*

A non-viscous theory applied to flow over a symmetrical blunt-nosed body like a cylinder predicts a symmetrical flow pattern. In other words, the flow downstream of the cylinder is similar to the flow upstream which is contrary to the observed facts.

(*iii*) *The theory predicts that the pressure distribution will produce no total force.*

Since the flow pattern is symmetrical, the pressure forces on the upstream side exactly balance those on the downstream side and so the net force on the cylinder due to fluid pressure must vanish. This is contrary to actual observations.

(*iv*) *The theory assumes that viscous forces are absent.*

Since ideal fluid theory assumes to possess no viscosity, ideal fluid exerts no force on the cylinder, which is again contrary to experience.

(*v*) *The theory makes the analysis of heat transfer unsatisfactory.*

We consider the predictions of heat transfer from the following two points of view:
(*a*) An ideal fluid provides no mechanism for heat transfer. Hence the theory predicts zero rates of heat transfer,
(*b*) since there is nothing between the fluid and the solid surface which could give rise to a resistance to heat transfer, we would expect infinite transfer rates. But these both facts are contrary to our experience.

(*vi*) *The solutions based on the theory are unable to fulfil the desired boundary conditions.*

Since the ideal fluid has zero coefficient of viscosity, the order of the momentum equation of a viscous fluid is decreased by omitting the viscous terms. Now the order of a differential equation determines the number of conditions to be satisfied by its solution. Hence we conclude that the desired solution of an ideal fluid can satisfy fewer boundary conditions than that of a viscous fluid.

We now propose to modify the theory of ideal fluids so that it may explain the most casual observations. An important contribution to fluid dynamics was made by L. Prandtl in 1904 by introducing the concept of boundary layer. He clarified the essential influence of viscosity in flows at high Reynold's numbers by showing how the Navier-Stokes equations could be simplified to yield approximate solutions for overcoming the limitation just listed.

Remarks. Note that the solution for an ideal fluid is not the limiting form of the solution for a viscous fluid when $\mu \to 0$ (Re $\to \infty$) because the order of the differential equation is higher than that of ideal fluid and so requires more boundary conditions to be satisfied for its solution. Moreover, there exists an essential difference in the boundary conditions in the two cases, namely an ideal fluid slips freely over a solid boundary whereas a viscous fluid adheres to the boundary and has no velocity relative to it.

18.3. Prandtl's boundary layer theory. [Agra 2008; Himanchal 2007; Meerut 2003, 04]

For convenience, consider laminar two-dimensional flow of fluid of small viscosity (large Reynold's number) over a fixed semi-infinite plate. It is observed that, unlike an ideal (non-viscous) fluid flow, the fluid does not slide over the plate, but "sticks" to it. Since the plate is at rest, the fluid in contact with it will also be at rest. As we move outwards along the normal, the velocity of the fluid will gradually increase and at a distance far from the plate the full stream velocity U is attained. Strictly speaking this is approached asymptotically. However, it will be assumed that the transition from zero velocity at the plate to the full magnitude U takes place within a thin layer of fluid in contact with the plate. This is known as the *boundary layer.*

There is no definite line between the potential flow region where friction is negligible and the boundary layer. Therefore, in practice, we define the boundary layer as that region where the fluid velocity, parallel to the surface, is less than 99% of the free stream velocity which is described by potential flow theory. The thickness of the boundary layer, δ, grows along a surface (over which fluid is flowing) from the leading edge.

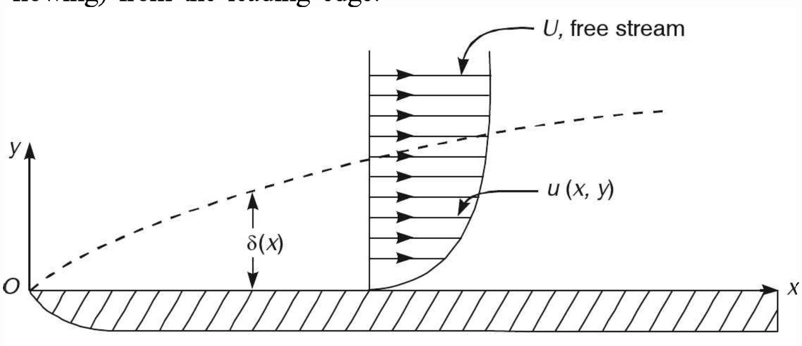


Fig. (i)

The shape of the velocity profile and the rate of increase of the boundary layer thickness, depend on the pressure gradient, $\partial p / \partial x$ Thus, if the pressure increases in the direction of flow, the boundary layer thickness increases rapidly and the velocity profiles will take the form as shown in Fig. (ii). When this adverse pressure gradient is large, then separation will occur

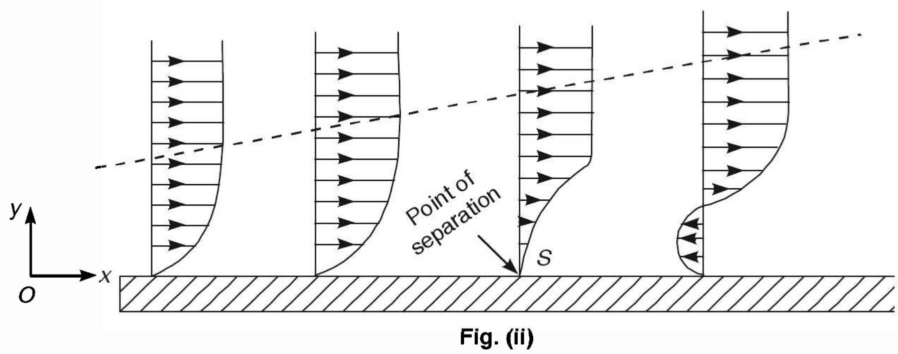


Fig. (ii)

followed by a region of *reversed flow*. The separation point S is defined as the point where

$$(\partial u / \partial y)_{y=0} = 0, \qquad \text{(separation)}$$

where u is the velocity parallel to the wall in the x direction and y is the coordinate normal to the wall. Due to the reversal of flow there is a considerable thickening of the boundary layer, and associated with it, there is a flow of boundary layer fluid into the outside region. The exact location of the *point of separation* can be determined only with the help of integration of the boundary layer equations.

The method of dividing the fluid in two regions was first proposed by Prandtl in 1904. He suggested that the entire field of flow can be divided, for the sake of mathematical analysis, into the following regions :

(*i*) A very thin layer (boundary layer) in the vicinity of the plate in which the velocity gradient normal to the wall (*i.e.* $\partial u/\partial y$) is very large. Accordingly the viscous stress $\mu\,(\partial u/\partial y)$ becomes important even when μ is small. Thus the viscous and inertial forces are of the same order within the boundary layer.

(*ii*) In the remaining region (*i.e.* outside the boundary layer) $\partial u/\partial y$ is very small and so the viscous forces may be ignored completely. Outside the boundary layer, the flow can be regarded non-viscous and hence the theory of non-viscous fluids offers a very good approximation there.

Remark. 1. The above discussion equally holds even if a blunt body (*i.e.*, a body with large radius of curvature such as aerofoil *etc.*) is considered in place of a flat plate.

Remark. 2. The following three conditions must be satisfied by any velocity distribution in boundary layer: (*i*) At $y = 0$, $u = 0$ and du/dy has some finite value

(*ii*) At $y = \delta$, $u = U$: (*iii*) At $y = \delta$, $du/dy = 0$.

18.4. Importance of Prandtl's boundary layer theory in fluid dynamics.

Although the boundary layer is thin, it plays a vital role in fluid dynamics. It has become a very powerful method of analysing the complex behaviour of real fluids. The concept of a boundary layer can be utilized to simplify the Navier-Stokes equations to such an extent that it becomes possible to tackle many practical problems of great importance. The drag on ships and missiles, the efficiency of compressors and turbines in jet engines, the effectiveness of air intakes for ram- and turbojets and so on depend on the concept of the boundary layer and its effects on the main flow.

While solving any equation and depending on the arguments in physical terms, the boundary layer theory is capable of explaining the difficulties encountered by ideal fluid dynamics [refer Art. 18.2]. The boundary layer theory is able to predict flow separation. It can explain the existence of a wake. The pressure distribution produces a net force in the direction in which the stream flows. There exist a viscous stress on the boundary region and it acts in the direction of flow. The analysis of heat transfer is satisfactory. The solution of a particular problem is able to satisfy the required boundary conditions. We shall impose the following conditions on the distribution of velocity in the boundary layer : (*i*) the no-slip condition, (*ii*) that no mass shall flow through the wall, and (*iii*) that the velocity at the outer edge of the boundary layer shall approach that predicted by an appropriate non-viscous theory.

18.5. Some basic definitions.

(*a*) **Boundary layer thickness.** **[Himanchal 2005; Meerut 2003]**

In a qualitative manner, the boundary layer thickness is defined as the elevation above the boundary which covers a region of flow where there is a large velocity gradient and consequently non-negligible viscous effects. Since transition from velocity in the boundary to that outside it takes place asymptotically, there is no obvious demarcation for permitting the measurement of a boundary layer thickness in a simple quantitative manner. For mathematical convenience, the thickness of the boundary layer is generally defined as that distance from the solid boundary where the velocity differs by 1 per cent from the external velocity U (*i.e.* free-flow velocity). It is easily seen that the above definition of the boundary layer is to a certain extent arbitrary. Because of this arbitrary and somewhat ambiguous definition of δ, we employ following three other types of thicknesses which are based on physically meaningful measurements.

(*i*) **Displacement thickness.** **[Himanchal 2001, 06; Meerut 2003]**

Because of viscosity the velocity on the vicinity of the plate is smaller than in the free-flow region. The reduction in total flow rate caused by this action is $\int_0^\infty (U-u)\,dy$.

If this integral is equated to a quantity $U\delta_1, \delta_1$ can be considered as the amount by which the potential flow has been displaced from the plate. Thus, for displacement thickness δ_1 we have the definition

$$U\delta_1 = \int_0^\infty (U-u)\,dy \qquad \text{...(1)}$$

or

$$\delta_1 = \int_0^\infty (1-u/U)\,dy. \qquad \text{...(2)}$$

(*ii*) **Momentum thickness.** **[Himanchal . 2001; 06 Meerut 2003; 04]**

It is defined by comparing the loss of momentum due to wall-friction in the boundary to the momentum in the free flow region. Thus, for the momentum thickness δ_2 we have the definition

$$\rho U^2 \delta_2 = \rho \int_0^\infty u\,(U-u)\,dy \qquad \text{...(3)}$$

or

$$\delta_2 = \int_0^\infty \frac{u}{U}(1-u/U)\,dy. \qquad \text{...(4)}$$

(*iii*) **Energy thickness or dissipation energy thickness or kinetic energy thickness.** **[Himanchal 2000; Meerut 2004; Kanpur 2002]**

There is always a loss in energy because of the viscosity of the fluid. Now the loss of kinetic energy in the boundary layer at a distance y from the plate is $(\rho/2)\times(U^2-u^2)$ and consequently the total rate at which the kinetic energy is being lost is

$$\frac{1}{2}\rho\int_0^\infty (U^2-u^2)\,u\,dy.$$

If this integral is equated to a quantity $(\rho/2)\times U^3\delta_3, \delta_3$ can be considered as thickness of a layer which has the same kinetic energy flux as the rate at which the kinetic energy is being lost in the boundary region. Thus, for the energy thickness δ_3 we have the definition

$$\frac{1}{2}\rho U^3\delta_3 = \frac{1}{2}\rho\int_0^\infty (U^2-u^2)\,u\,dy \qquad \text{...(5)}$$

or

$$\delta_3 = \int_0^\infty \frac{u}{U}(1-u^2/U^2)\,dy. \qquad \text{...(6)}$$

(*b*) **Drag and Lift.** **[Himanchal 2001; Kanpur 2002]**

Let a solid body be present in the flow region of a fluid flow. Then the body will experience two types of forces. The force component exerted on the body from a moving fluid in the direction of free-stream of the fluid far from the body is defined as *drag* and the force component on the body normal to the free-stream of the fluid far from the body is defined as *lift*.

Since the measurement of drag D and lift L depend on the transition in boundary layer, separation of the boundary layer and so on, it is a very difficult task to measure them. We, therefore, employ experimental data and define them as follows :

$$D = \text{Drag} = (C_D A\,\rho\,U^2)/2 \qquad \text{...(7)}$$

and $$L = \text{Lift} = (C_L A \rho U^2)/2, \quad ...(8)$$

where C_D is the *coefficient* of drag, C_L is the *coefficient of lift*, A is the projected area in the direction of flow and U is the free-flow velocity, Notice that the coefficients C_D and C_F are both dimensionless quantities.

(*c*) **Local skin coefficient**

The sheaning stress on the plane boundadry layer is given by $\tau_w = \mu\,(du/dy)_{y=0}$. Then, the dimensionless shear stress, which is known as *local skin coefficient* is denoted and defined as

$$C_f = \tau_w/(\rho U^2/2)$$

18.5A. Derivation of different types of thicknesses

Boundary layer thickness (δ).

The velocity u within the boundary layer increases from zero at the boundary surface to the velocity U of the main stream asymptotically. So the boundary layer thickness is arbitrarily defined as that distance from the boundary in which the velocity reaches 99 per cent of the velocity of the free stream (u = 0.99 U). It is denoted by δ.

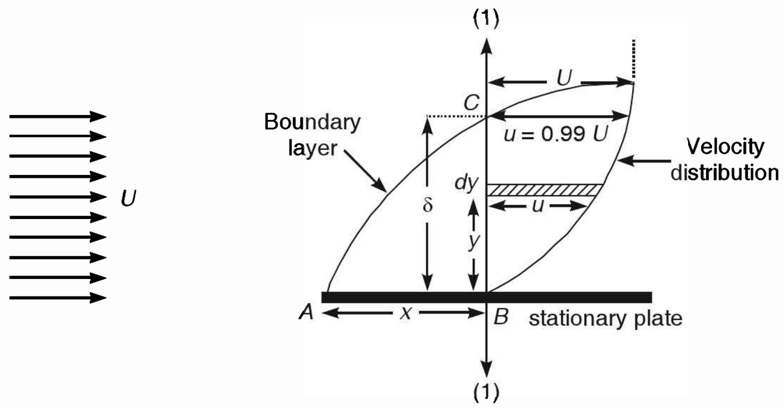
The above definition gives an approximate value of the boundaary layer thickness. For better accuracy, the boundary layer thickness is defined in terms of certain mathematical expressions which are the measure of the boundary layer of the flow. We now give three commonly used definitions of the boundary layer thickness.

(*i*) **Displacement thickness. Definition.** It is the distance measured perpendicular to the boundary of the solid body, by which the main/ free stream is displaced on account of boundary layer.

Another definition. It is an additional "wall thickness" that would have to be added to compensate for the reduction in flow rate on account of boundary layer formation.

The displacement thickness is denoted by δ_1 or δ^*.

Derivation of expression for δ_1



Let an incompressible fluid of density ρ flow past a stationary plate with velocity U as shown in the above figure. At a distance x from the leading edge,consider a section 1-1. The velocity of fluid at B is zero and at C, which lies on the boundary layer, is U. Thus, here BC = boundary layer thickness = δ. At the section 1-1, consider an elementary strip. Let y = distance of the elementry strip from the plate, dy = thickness of the elementry strip, u = velocity of fluid at the elementary strip and b = width of plate.

Mass flow per second through the elementary strip

$$= \rho \times \text{velocity} \times \text{area of elementary strip} = \rho \times u \times (bdy). \qquad \ldots(1)$$

If there had been no plate then the fluid would have been flowing with a constant velocity equal to free stream velocity U at the section 1-1

Hence, mass flow per second through the elementary strip if the plate were not there

$$= \rho \times \text{velocity} \times \text{area} = \rho \times U \times (bdy). \qquad \ldots(2)$$

Since $U > u$, it follows that due to the presence of the plate and consequently due to the formation of the boundary layer, there will be a reduction in mass flowing per second through the elementary strip.

$\therefore$ The reduction of mass flow rate through the elementary strip

= mass/sec given by equation (2) – mass/sec given by equation (1)

$$= \rho\, Ub\, dy - \rho\, ub\, dy = \rho b\,(U - u)\, dy.$$

So, total reduction of mass flow rate through BC due to introduction of plate

$$= \int_0^{\delta} \rho b\,(U-u)\,dy = \rho b \int_0^{\delta} (U-u)\,dy. \qquad \ldots(3)$$

[$\therefore$ fluid is incompressible $\Rightarrow$ ρ is constant]

Let the plate be displaced by a distanced δ_1 and velocity of flow for the distance δ be equal to the main/free stram velocity (*i.e.*, U). Then,

loss of mass of the fluid/sec flowing through the distance δ_1.

$$= \rho \times \text{velocity} \times \text{area} = \rho \times U \times (\delta_1\, b). \qquad \ldots(4)$$

Equating the expressions obtained in (3) and (4), we have

$$\rho U \delta_1 b = \rho b \int_0^{\delta} (U-u)\,dy$$

or

$$\delta_1 = \frac{1}{U}\int_0^{\delta} (U-u)\,dy = \int_0^{\delta} \frac{(U-u)\,dy}{U}, \qquad \text{as } U \text{ is constant}$$

or

$$\delta_1 = \int_0^{\delta} (1 - u/U)\,dy \qquad \ldots(5)$$

Another modified expression for δ_1

From boundary layer theory, $u = U$ for $y > \delta$.

$$\therefore \quad \int_{\delta}^{\infty} \left(1 - \frac{u}{U}\right) dy = 0, \qquad \text{as} \qquad 1 - \frac{u}{U} = 1 - \frac{U}{U} = 0 \ \text{ for } \ y \ge \delta$$

Thus,

$$\delta_1 = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy + \int_{\delta}^{\infty} \left(1 - \frac{u}{U}\right) dy = \int_0^{\infty} \left(1 - \frac{u}{U}\right) dy. \qquad \ldots(6)$$

(*ii*) **Momentum thickness. Definition.** It is the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensated for reduction in momentum of the flowing fluid on account of boundary layer formation.

Another definition. It is the distance through which the total loss of momentum per second be equal to it if it were passing a stationary plate.

The momentum thickness is denoted by δ_2 or θ.

Derivation of expression for δ_2. Refer figure given on page 18.5. Let an incompressible fluid of density ρ flow past a stationary plate with velocity U as shown in the figure. At a distance x from the leading edge, consider a section 1-1. The velocity of fluid at B is zero and at C, which lies on the boundary layer, is U. Thus, here BC = boundary layer thickness = δ. At the section 1-1 consider an elementary strip. Let y = distance of the elementary strip from the plate, dy = thickness of the elementary strip, u = velocity of fluid at the elementary strip, and b = width of plate.

Mass flow per second through the elementary strip

$$= \rho \times \text{velocity} \times \text{area of elementary strip} = \rho \times u \times (b\,dy).$$

Momentum/sec of this fluid inside the boundary layer $= (\rho u b dy) \times u = \rho u^2 b dy$.

Momentum/sec. of the same fluid in the absence of bounday layer $= (\rho u b dy) \times U = \rho u U b dy$.

$\therefore$ Loss of momentum/sec through elementary strip $= \rho u^2 b dy - \rho u U b dy = \rho b u (U - u) dy$.

So total loss of momentum/sec through BC due to introduction of plate

$$= \int_0^\delta \rho b u\,(U - u) dy = \rho b \int_0^\delta u\,(U - u)\,dy. \qquad \ldots(1)$$

[$\because$ fluid is incompressible $\Rightarrow$ ρ is constant]

Let δ_2 = distance by which plate is displaced when the fluid is flowing with constant velocity U of main/free stream.

$\therefore$ Loss of momentum/sec of fluid flowing though distance δ_2 with velocity U

$$= (\rho \times \text{area} \times \text{velocity}) \times \text{velocity} = [\rho \times (\delta_2 b) \times U] \times U = \rho \delta_2 b U^2 \qquad \ldots(2)$$

Equating the expressions obtained in (1) and (2), we have

$$\rho \delta_2 b U^2 = \rho b \int_0^\delta u\,(U - u)\,dy$$

or
$$\delta_2 = \frac{1}{U^2}\int_0^\delta u\,(U - u)\,dy = \int_0^\delta \frac{u\,(U - u)}{U^2} dy, \quad \text{as } U \text{ is constant.}$$

or
$$\delta_2 = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy. \qquad \ldots(3)$$

Another modified expression for δ_2

From boundary layer theory, u = U for $y > \delta$.

Hence, $\displaystyle\int_\delta^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = 0, \quad \text{as} \quad 1 - \frac{u}{U} = 1 - \frac{U}{U} = 0, \quad \text{for } y \geq \delta.$

Thus,
$$\delta_2 = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy + \int_\delta^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy. \qquad \ldots(4)$$

(*iii*) **Energy thickness (or dissipation energy thickness or kinetic energy thickness).**

It is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundadry should be displaced to compensate for the reduction in kinetic energy of the flowing fluid on account of boundary layer formation. It is denoted by δ_3 or δ^{**} or δ_e.

Derivation of expression for δ_3.

Refer figure of Art 18.5A given on pag 18.5. Let an incompressible fluid of density ρ flow past a stationary plate with velocity U as shown in the figure. At a distance x from the leading edge, consider a section 1-1. The velocity of fluid at B is zero and at C, *which* lines on the boundary layer, is U. Thus, here BC = boundary layer thickness = δ. At the section 1-1, consider an elementary strip. Let y = distance of the elementary strip from the plate, dy = thickness of the elementary strip, u = velocity of fluid at the elementary strip and b = width of plate.

Mass flow per second through the elementary strip

$$= \rho \times \text{velocity} \times \text{area of elementary strip} = \rho \times u \times (b\, dy).$$

Then, kinetic energy of this fluid inside the boundary layer

$$= (1/2) \times (\text{mass}) \times (\text{velocity})^2 = (1/2) \times (\rho\, ub\, dy) \times u^2.$$

Again, kinetic energy of the same fluid in the absence of boundary layer

$$= (1/2) \times (\text{mass}) \times (\text{velocity})^2 = (1/2) \times (\rho\, ub\, dy) \times U^2.$$

$\therefore$ Loss of kinetic energy through elementary strip

$$= (1/2) \times (\rho\, ub\, dy)\, U^2 - (1/2) \times (\rho\, ub\, dy) u^2 = (1/2) \times \rho\, ub\, (U^2 - u^2)\, dy.$$

$\therefore$ Total loss of kinetic energy through BC due to introduction of plate

$$= \int_0^{\delta} \frac{1}{2} \rho ub\, (U^2 - u^2)\, dy = \frac{1}{2} \rho b \int_0^{\delta} u\, (U^2 - u^2)\, dy. \qquad \ldots(1)$$

[$\therefore$ fluid is incompressible $\Rightarrow$ ρ is constant]

Let δ_3 = distance by which the plate is displaced to compensate for the reduction in kinetic energy. Hence, loss of kinetic energy through δ_3 of flowing with velocity U

$$= (1/2) \times \text{mass} \times (\text{velocity})^2 = (1/2) \times (\rho \times \text{area} \times \text{velocity}) \times (\text{velocity})^2$$

$$= (1/2) \times [\rho \times (b \times \delta_3) \times U] \times U^2 = (1/2) \times \rho b \delta_3 U^2. \qquad \ldots(2)$$

Equating the expressions obtained in (1) and (2), we have

$$\frac{1}{2} \rho b \delta_3 U^3 = \rho b \int_0^{\delta} u\, (u^2 - u^2)\, dy$$

or $$\delta_3 = \frac{1}{U^3} \int_0^{\delta} u\, (U^2 - u^2)\, dy = \int_0^{\delta} \frac{u\, (U^2 - u^2)}{U^3}\, dy, \qquad \text{as } U \text{ is constant.}$$

or $$\delta_3 = \int_0^{\delta} \frac{u}{U} \left(1 - \frac{u^2}{U^2}\right) dy \qquad \ldots(3)$$

Another modified expression for δ_3.

From boundary layer thoery, we have $u = U$ for $y \geq \delta$.

$$\therefore \quad \int_{\delta}^{\infty} \frac{u}{U} \left(1 - \frac{u^2}{U^2}\right) dy = 0, \quad \text{as} \quad 1 - \frac{u^2}{U^2} = 1 - \frac{U^2}{U^2} = 0, \quad \text{for } y \geq \delta.$$

Thus, $$\delta_3 = \int_0^{\delta} \frac{u}{U} \left(1 - \frac{u^2}{U^2}\right) dy + \int_{\delta}^{\infty} \frac{u}{U} \left(1 - \frac{u^2}{U^2}\right) dy = \int_0^{\infty} \frac{u}{U} \left(1 - \frac{u^2}{U^2}\right) dy. \qquad \ldots(4)$$

18.5 B. Displacement, momentum and energy thicknesses for axially symmetric flow.

In Art 18.5A we have presented definitions of displacement, momentum and energy thicknesses for two-dimensional boundary layer flows. In the present article, we wish to define displacement, momentum and energy thicknesses for axially symmetric boundary layer flows. In order to achieve these thicknesses, it can be shown that we have to replace dy by $2\pi\, r\, dn$ in expressions for δ_1, δ_2 and δ_3 as obtained in Art. 18.5A. Here r is the axial distance and dn is the line element along the normal to the surface of the body. From dimensional consideration we drive the resultant expressions by $2\pi a$, where a is a reference radius, which may be a function of the axial distance. Accordingly, for axially symmetric boundary layer flows, we have

(*i*) Displacement thickness = $\delta_1 = \int_0^\infty \left(1-\frac{u}{U}\right)\frac{r}{a}dn$

(*ii*) Momentum thickness = $\delta_2 = \int_0^\infty \left(1-\frac{u}{U}\right)\frac{ru}{aU}dn$

(*iii*) Energy thickness = $\delta_3 = \int_0^\infty \left(1-\frac{u^2}{U^2}\right)\frac{ru}{aU}dn$

In above results, n is the normal distance from the surface of the body.

18.6. The boundary layer equations in two-dimensional flow.

In what follows, we shall obtain the boundary layer equations in two-dimensional flow by using the following two methods: (*i*) Order of magnitude approach (*ii*) asymptotic approach.

Method I. Order of magnitude approcah

[Agra 2008, 11; Himanchal 2004, 05, 07, 09, 10; Meerut 1998, 2002]

For simplicity we shall first derive the boundary layer equations for the flow over a semi-infinite flat plate. We take rectangular cartesian coordinates (x, y) with x measured in the plate in the direction of the two-dimensional laminar incompressible flow, and y measured normal to the plate, and (u, v) are the velocity components. Let viscosity of the fluid be small and let δ be small thickness of the boundary layer. Let U be the velocity in the main stream just outside the boundary layer. Then the Navier-Stokes equations, without body forces for two dimensional flow are:

$$x\text{–direction}: \quad \underset{1}{\frac{\partial u}{\partial t}}+\underset{1}{u}\,\underset{1}{\frac{\partial u}{\partial x}}+\underset{\delta}{v}\,\underset{1/\delta}{\frac{\partial u}{\partial y}}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\underset{1}{\frac{\partial^2 u}{\partial x^2}}+\underset{1/\delta^2}{\frac{\partial^2 u}{\partial y^2}}\right) \qquad ...(1)$$

$$y\text{–direction}: \quad \underset{\delta}{\frac{\partial v}{\partial t}}+\underset{1}{u}\,\underset{\delta}{\frac{\partial v}{\partial x}}+\underset{\delta}{v}\,\underset{1}{\frac{\partial v}{\partial y}}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\underset{\delta}{\frac{\partial^2 v}{\partial x^2}}+\underset{1/\delta}{\frac{\partial^2 v}{\partial y^2}}\right) \qquad ...(2)$$

The continuity equation is $\qquad \underset{|}{\partial u/\partial x} + \underset{|}{\partial v/\partial y} = 0.$ ...(3)

The boundary conditions are : $u = 0, v = 0$ when $y = 0$; $u = U(x, t)$ when $y = \infty$ ...(4)

We now determine the order of magnitude of each term in (1), (2) and (3) to enable us to drop small terms and thus to arrive at the simplified boundary layer equations. We shall designate the order of any quantity (q, say) by $O(q)$. The orders of magnitude are shown in (1) to (3) under the individual terms. Let $O(u) = 1$ and $O(\partial u/\partial x) = 1$ within the boundary layer. Then $\partial u/\partial y$ is large as $O(\partial u/\partial y) = 1/\delta$, u decreasing from a finite value U at the outer boundary of the layer to zero at the flat plate. Again $O(\partial^2 u/\partial y^2) = 1/\delta^2$ in the boundary layer. Further $O(\partial u/\partial t) = O(\partial^2 u/\partial x^2) = 1$. Since $O(\partial u/\partial x) = 1$, (3) shows that $O(\partial v/\partial y) = 1$.

Since $v = 0$ when $y = 0$, $O(v) = \delta$. Again, $O(\partial v/\partial t) = O(\partial v/\partial x) = O(\partial^2 v/\partial x^2) = \delta$ whereas $O(\partial^2 v/\partial y^2) = \delta/\delta^2 = 1/\delta$. All these values have been inserted into (1) to (3).

Since the viscous force is taken as of the same order as the inertia forces within the boundary layer, (1) implies that we must have

$$O(\nu/\delta^2) = 1 \qquad \text{so that} \qquad O(\delta) = \sqrt{\nu} = \sqrt{\mu/\rho}, \qquad ...(5)$$

showing that smaller the viscosity of the fluid, the thinner the boundary layer.

By neglecting the terms of the order δ and smaller from (1) to (3), we obtain

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial x^2} \qquad ...(6)$$

$$\partial p/\partial y=0 \qquad ...(7)$$

$$\partial u/\partial x+\partial v/\partial y=0 \qquad ...(8)$$

The above equations (6) and (8) are known as Prandtl's boundary layer equations with boundary conditions (4).

Now (7) shows that the pressure distribution is a function of x only, *i.e.*, for a given x, p is constant throughout the boundary layer. Thus the pressure throughout the boundary layer possesses the same value of p as in the potential flow. It follows that we can now regard u and v as unknown variables in (6) and (8) instead of three unknown variables u, v and p in (1), (2) and (3). This simplifies the process of finding solution to a great extent because we now have a system of two simultaneous equations for the two unknowns u and v.

At the outer edge of the boundary layer, we have $u = U(x, t)$. Since large velocity gradients are absent there, the viscous terms in (1) vanish for small values of $\nu(\nu=\mu/\rho$ is small because μ is small by assumption). Hence for the flow outside the boundary, we have

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}=-\frac{1}{\rho}\frac{dp}{dx}. \qquad ...(9)$$

For the case of steady flow, (9) reduces to

$$U\frac{dU}{dx}=-\frac{1}{\rho}\frac{dp}{dx}, \qquad \text{so that} \qquad \frac{dp}{dx}=-\rho U\frac{dU}{dx}. \qquad ...(10)$$

Further, for the steady flow boundary layer equations (6) and (8) reduce to still more simplified form :

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \qquad ...(11)$$

$$\partial u/\partial x+\partial v/\partial y=0, \qquad ...(12)$$

where dp/dx is gives by (10).

Remark 1. Since the same argument can be carried out at all places along the boundary layer (except at the front edge), the equations deduced so far are general and hold for any two-dimensional boundary layer flow. It is also to be pointed out that, for a slightly curved boundary, these equations are valid if we use curvilinear coordinates fitting the boundary.

Method II. Asymptotic approach. **[Himanchal 2003, 04]**

In what follows, we shall prove that Prandtl boundary layer equations may be treted as the asymptotic form of the Navier-Stokes equations at large Reynold's number. Let t',x',y',u',v' and p' be dimensionless quantities such that

$$t'=t/T, \quad x'=x/X, \quad y'=y/Y, \quad u'=u/U, \quad v'=v/V \text{ and } p'=p/P, \qquad ...(1)$$

where T, X, Y, U, V and P are the units of measurement of the corresponding quantities.

The Navier-Stokes equations of motion for a viscous incompressible fluid in the absence of body forces, in a two-dimensional flow are given by

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad ...(2)$$

and $$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right). \qquad ...(3)$$

The continuity equation is $$\partial u/\partial x+\partial v/\partial y=0. \qquad ...(4)$$

Here the xy-plane is the plane of motion, the x-axis is along the wall of the plate and y-axis perpendicular to it. Substituting (1) in equations (2), (3) and (4), we get

$$\frac{X}{UT}\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+\frac{XV}{YU}v'\frac{\partial u'}{\partial y'}=-\frac{P}{\rho U^2}\frac{\partial p'}{\partial x'}+\frac{\nu}{XU}\frac{\partial^2 u'}{\partial x'^2}+\frac{\nu X}{Y^2 U}\frac{\partial^2 u'}{\partial y'^2}, \qquad ...(5)$$

$$\frac{X}{UT}\frac{\partial v'}{\partial t'}+u'\frac{\partial v'}{\partial x'}+\frac{XV}{YU}v'\frac{\partial v'}{\partial y'}=-\frac{PX}{\rho YUV}\frac{\partial p'}{\partial y'}+\frac{\nu}{XU}\frac{\partial^2 v'}{\partial x'^2}+\frac{\nu X}{Y^2 U}\frac{\partial^2 v'}{\partial y'^2} \qquad ...(6)$$

and $$\frac{\partial u'}{\partial x'}+\frac{XV}{YU}\frac{\partial v'}{\partial y'}=0. \qquad ...(7)$$

Treating X and U as the fundamental units, the Reynolds number for the given flow is taken as

$$\mathrm{Re}=(XU)/\nu. \qquad ...(8)$$

Then the units of time and pressure can be expressed in terms of X and U as follows:

$$T=X/U \qquad \text{and} \qquad P=\rho U^2. \qquad ...(9)$$

We now derive the units of measurement of Y and V by applying the condition that the system of equations (5), (6) and (7) must possess only a single flow parameter Re (given by (8)). Accordingly, we must have

$$\frac{XV}{YU}=1 \qquad \text{and} \qquad \frac{\nu X}{Y^2 U}=1 \qquad \text{so that} \qquad Y=\frac{X}{\sqrt{\mathrm{Re}}} \qquad \text{and} \qquad V=\frac{U}{\sqrt{\mathrm{Re}}}. \qquad ...(10)$$

Hence equation (5), (6) and (7) may be re-wrtten as

$$\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}=-\frac{\partial p'}{\partial x'}+\frac{1}{\mathrm{Re}}\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}, \qquad ...(11)$$

$$\frac{1}{\mathrm{Re}}\left(\frac{\partial v'}{\partial t'}+u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}\right)=-\frac{\partial p'}{\partial y'}+\frac{1}{(\mathrm{Re})^2}\frac{\partial^2 v'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2} \qquad ...(12)$$

$$\partial u'/\partial x'+\partial v'/\partial y'=0. \qquad ...(13)$$

If Re is large, then $1/\sqrt{\mathrm{Re}}$ can be treated as small parameter. We now obtain the solution of (11), (12) and (13) with help of a power series containing 1/Re as follows:

$$\left.\begin{aligned} u' &= u_0+(1/\sqrt{\mathrm{Re}})u_1+(1/\sqrt{\mathrm{Re}})^2 u_2+\dots, \\ v' &= v_0+(1/\sqrt{\mathrm{Re}})v_1+(1/\sqrt{\mathrm{Re}})^2 v_2+\dots, \\ \text{and} \quad p' &= p_0+(1/\sqrt{\mathrm{Re}})p_1+(1/\sqrt{\mathrm{Re}})^2 p_2+\dots \end{aligned}\right\} \qquad ...(14)$$

Substituting (14) in equations (11), (12) and (13) and then comparing the coefficient of the zeroth power of Re, we get

$$\frac{\partial u_0}{\partial t'}+u_0\frac{\partial u_0}{\partial x'}+v_0\frac{\partial u_0}{\partial y'}=-\frac{\partial p_0}{\partial x'}+\frac{\partial^2 u_0}{\partial y'^2} \qquad ...(15)$$

$$0 = -\partial p_0 / \partial y' \qquad ...(16)$$

and
$$\partial u_0 / \partial x' + \partial v_0 / \partial y' = 0. \qquad ...(17)$$

Equations (15), (16) and (17) are the same resulting equations as if we would have allowed directly in equations (11), (12) and (13) the Reynolds number tending to infininity.

Reverting to the dimensional form and dropping the index zero in equations (15), (16) and (17), we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \qquad ...(18)$$

$$0 = -\partial p / \partial y \qquad ...(19)$$

and
$$\partial u / \partial x + \partial v / \partial y = 0. \qquad ...(20)$$

Now equation (19) shows that p is a function of x only. Hence the pressure is constant in a direction normal to the boundary layer and may be equal to that at the outer edge of the boundary layer where it is determined by the inviscid flow (potential flow). Thus, we have

$$-\frac{1}{\rho}\frac{dp}{dx} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}, \qquad ...(21)$$

where U is the potential flow velocity

Hence the Prandtl boundary layer equations for a two dimensional unsteady incompressible flow are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{\partial^2 u}{\partial y^2} \qquad ...(22)$$

and
$$\partial u / \partial x + \partial v / \partial y = 0. \qquad ...(23)$$

Now due to no-slip condition and the wall being solid (non-porous), both u and v must vanish at $y = 0$. Hence equations (22) and (23) are solved under the following boundary conditions:

$u = v = 0$ when $y = 0$ and $u = U$ when $y \to \infty$ ...(24)

18.7. Boundary layer flow over a flat plate. Blasius-Topfer solution. (or Simply Blasius Solution) [Meerut 2000, 02; Himanchal 2000, 01]

In the present article we propose to discuss an application of the boundary layer equations by considering what is geometrically the simplest possible configuration. Consider a thin infinite flat plate submerged in steady incompressible plane parallel flow, whose undisturbed velocity is U. The fluid has low viscosity, and the plate is at rest in such a way that its plane coincides with the direction of U. Since the plate is of infinite length, the flow may be regarded as two-dimensional. Let the origin of the coordinate system coincide with the front edge of the plate, the x-axis lying along the plate parallel to U and the y-axis normal to the plate. The velocity U of the potential flow is constant in this case so that $dU/dx = 0$ and hence $dp/dx = 0$ [$\because$ $dp/dx = -\rho U(dU/dx)$ by (10) of Art 18.6]. Thus the Prandtl boundary layer equations in the case under consideration are (refer (11) and (12) in At. 18.6)

$$u(\partial u / \partial x) + v(\partial u/\partial y) = \nu(\partial^2 u / \partial y^2) \qquad ...(1)$$

$$\partial u / \partial x + \partial v / \partial y = 0 \qquad ...(2)$$

The boundary conditions to be satisfied by u and v are :

$u = v = 0$ when $y = 0$ and $u = U$ when $y = \infty$ ...(3)

The integration of (1) and (2) can be simplified by reducing the number of unknowns with the help of the stream function ψ defined by

$$u = \partial\psi/\partial y \qquad \text{and} \qquad v = -(d\psi/dx) \qquad \text{...(4)}$$

Then, (2) is satisfied automatically by (4).

The order of the boundary layer thickness is $(\nu x/U)^{1/2}$. Hence we take the new dimensionless distance parameter $\eta = y/\delta$ so that

$$\eta = y(U/\nu x)^{1/2} = yx^{-1/2}(U/\nu)^{1/2} \qquad \text{...(5)}$$

In accordance with the procedure of the law of similarity, let the velocity profile be

$$u/U = F(\eta). \qquad \text{...(6)}$$

Using (4), (5) and (6) the stream function ψ is given by

$$\psi = \int u\, dy = \frac{U}{\sqrt{U/\nu x}} \int F(\eta)\, d\eta = \sqrt{U\nu x}\; f(\eta), \qquad \text{...(7)}$$

where $f(\eta) = \int F(\eta)\, d\eta$. Then using (4), (5) and (7), we have

$$u = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial y} = \sqrt{U\nu x} f'(\eta) \times \sqrt{\frac{U}{\nu x}} = Uf'(\eta), \qquad \text{...(8)}$$

$$v = -\frac{\partial\psi}{\partial x} = -\left[f(\eta)\frac{\partial}{\partial x}(\sqrt{U\nu x}) + \sqrt{U\nu x}\cdot\frac{\partial}{\partial x} f(\eta)\right], \text{ using (7)}$$

$$= -\left[f(\eta)\,\sqrt{U\nu}\,\frac{1}{2\sqrt{x}} + \sqrt{U\nu x}\;\frac{df(\eta)}{d\eta}\cdot\frac{\partial\eta}{\partial x}\right]$$

$$= -\frac{1}{2}\sqrt{U\nu}\; x^{-1/2} f(\eta) - \sqrt{U\nu x}\; f'(\eta)\left(-\frac{1}{2}\right)x^{-3/2} y\sqrt{U/\nu}$$

$$= -\frac{1}{2}\sqrt{\frac{U\nu}{x}}\; f(\eta) + \frac{1}{2}\sqrt{\frac{U\nu}{x}}\; \eta f'(\eta), \text{ using (5)}$$

$$= \frac{1}{2}\sqrt{\frac{\nu U}{x}}[\eta f'(\eta) - f(\eta)], \qquad \text{...(9)}$$

$$\frac{\partial u}{\partial x} = Uf''(\eta)\cdot\left(-\frac{1}{2}x^{-3/2}\right) y\sqrt{\frac{U}{\nu}} = -\frac{U}{2x}\cdot y\sqrt{\frac{U}{\nu x}}\cdot f''(\eta) = -\frac{U\eta}{2x} f''(\eta), \text{ using (5)} \qquad \text{...(10)}$$

$$\frac{\partial u}{\partial y} = Uf''(\eta) \times \sqrt{\frac{U}{\nu x}} \qquad \text{...(11)}$$

and

$$\frac{\partial^2 u}{\partial y^2} = Uf'''(\eta) \times \left(\sqrt{\frac{U}{\nu x}}\right)^2 = Uf'''(\eta)\cdot\frac{\mathrm{U}}{\nu x} \qquad \text{...(12)}$$

Substituting (8) to (12) into (1), we get

$$-\frac{\mathrm{U}^2\eta}{2x} f' f'' - \frac{\mathrm{U}^2}{2x}(\eta f' - f) f'' = \frac{\mathrm{U}^2 f'''}{x}.$$

After simplification, the following ordinary differential equation is obtained :

$$f f'' + 2f''' = 0, \qquad \text{...(13)}$$

which is known as the *Blasius equation.*

Using (5), we see that $y = 0 \Rightarrow \eta = 0$ and $y = \infty \Rightarrow \eta = \infty$. Then from (8) and (9), we find that $u = 0$, $v = 0$ at $y = 0 \Rightarrow f = 0$, $f' = 0$ at $\eta = 0$ Furthermore, (8) shows that $u = U \Rightarrow f' = 1$. Hence the boundary conditions (3) may be re-written as

$$f = 0, \quad f' = 0 \quad \text{when} \quad \eta = 0 \quad \text{and} \quad f' = 1 \quad \text{when} \quad \eta = \infty \qquad \text{...(14)}$$

Since (13) is a third-order non-linear equation, the three boundary conditions (14) are sufficient to determine the solution completely. but the general solution of (13) has not been possible in closed form. H. Blasius obtained the solution in the form of a power series expansion about $\eta = 0$. The power series near $\eta = 0$ is assumed to be of the form

$$f(\eta) = A_0 + A_1\eta + \frac{A_1}{2!}\eta^2 + \frac{A_3}{3!}\eta^3 + \ldots \qquad \text{...(15)}$$

$$f'(\eta) = A_1 + A_2\eta + \frac{A_3}{2!}\eta^2 + \frac{A_4}{3!}\eta^3 + \ldots \qquad \text{...(16)}$$

$$f''(\eta) = A_2 + A_3\eta + \frac{A_4}{2!}\eta^2 + \frac{A_5}{3!}\eta^3 + \ldots \qquad \text{...(17)}$$

$$f'''(\eta) = A_3 + A_4\eta + \frac{A_5}{2!}\eta^2 + \frac{A_6}{3!}\eta^3 + \ldots \qquad \text{...(18)}$$

From (14), $f(\eta) = f'(\eta) = 0$ at $\eta = 0$. Hence (15) and (16) give

$$A_0 = 0 \quad \text{and} \quad A_1 = 0. \qquad \text{...(19)}$$

Substituting (15), (17) and (18) into (13), we get

$$2A_3 + 2A_4\eta + (A_2^2 + 2A_5)\frac{\eta^2}{2!} + (4A_2A_3 + 2A_6)\frac{\eta^3}{3!} + \ldots = 0,$$

which is an identity, and hence all coefficients of the various powers of η must vanish identically. Thus, we obtain $A_3 = A_4 = A_6 = A_7 = 0$, $A_5 = -(A_2^2/2)$ and $A_8 = (11A_2^2/4)$.

With these values, (15) reduces to

$$f(\eta) = \frac{A_2}{2!}\eta^2 - \frac{1}{2}\frac{A_2^2}{5!}\eta^5 + \frac{1}{4}\frac{11A_2^3}{8!}\eta^8 - \frac{1}{8}\frac{375A_2^4}{(11)!}\eta^{11} + \ldots, \qquad \text{...(20)}$$

where the constant A_2 is, for the time being, undetermined, because the third boundary condition at $\eta = \infty$ still remains to be satisfied. Re-writing (20), we obtain

$$f(\eta) = A_2^{1/3}\left[\frac{(A_2^{1/3}\eta)^2}{2!} - \frac{1}{2}\frac{(a_2^{1/3}\eta)^5}{5!} + \frac{1}{4}\frac{11(A_2^{1/3}\eta)^8}{8!} - \frac{1}{8}\frac{375(A_2^{1/3}\eta)^{11}}{(11)!} + \ldots\right]$$

Thus, $$f(\eta) = A_2^{1/3}F(A_2^{1/3}\eta), \text{ say} \qquad \text{...(21)}$$

$\therefore$ $$f'(\eta) = A_2^{1/3} \times F'(A_2^{1/3}\eta)\cdot A_2^{1/3} = A_2^{2/3}F'(A_2^{1/3}\eta). \qquad \text{...(22)}$$

From (14), $f'(\eta) = 1$ when $\eta = \infty$. Hence (22) gives

$$\lim_{\eta\to\infty}[A_2^{2/3}F'(A_2^{1/3}\eta) = f'(\infty) = 1 \quad \text{so that} \quad A_2 = \left[\frac{1}{\lim_{\eta\to\infty} F'(\eta)}\right]^{3/2} \qquad \text{...(23)}$$

The value of A_2 can be obtained numerically from (23) to any desired approximation. Howarth found that $A_2 = 0.332$. This completes the solution which is also known as *Blasius solution* Using (17), we find $$f''(0) = A_2 = 0.332.$$

Determination of shearing stress and boundary layer thickness.

Using (11) and the facts $y = 0 \Rightarrow \eta = 0$ and $\mu = \nu\rho$, shearing stress τ_0 at the plate is given by

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu U f''(0)\left(\frac{U}{\nu x}\right)^{1/2} = \frac{\nu\rho U^2 f''(0)}{(U\nu x)^{1/2}} = \frac{f''(0)\rho U^2}{(Ux/\nu)^{1/2}} = \frac{0.332\rho U^2}{(\mathrm{Re}_x)^{1/2}} \qquad \text{... (24)}$$

where $Re_x = Ux/\nu$ = Reynold's number.

The *local skin friction coefficient* or *friction drag coefficient* is given by

$$C_f = \tau_0/(\rho U^2/2) = (0.664)/(\mathrm{Re}_x)^{1/2}, \text{ using (24)} \qquad \text{... (25)}$$

Let D denote the total drag on one side of the plate of length l and breadth b. Then, using (24), we have

$$D = \int_0^b b\tau_0 . dx = b\int_0^l \frac{0.332\rho U^2}{(U/\nu)^{1/2}}\, x^{-1/2} dx = \frac{0.664\,\rho U^2 b l^{1/2}}{(U/\nu)^{1/2}} = 0.664\rho U^2 b\left(\frac{\nu l}{U}\right)^{1/2} \qquad \text{... (26)}$$

From (26), it follows that the total drag is proportional to 3/2 power of the free stream velocity U. Let C_D denote the *coeffiucient of drag*. Then, using (26) and the fact that A = area of plate = lb, we have

$$C_D = \frac{D}{(\rho U^2 A)/2} = \frac{0.664\rho U^2 b \times (\nu l/U)^{1/2}}{(\rho U^2 lb)/2} = \frac{1.328}{(Ul/\nu)^{1/2}} = \frac{1{,}328}{(\mathrm{Re}_l)^{1/2}}, \qquad \text{... (27)}$$

where $\mathrm{Re}_l = Ul/\nu$ = Reynold's number

18.8. 'Similar solutions' of the boundary layer equaitons [Himanchal 2000, 04, 05]

Boundary layer equations are simpler than the general momentum equation, yet they are non-linear partial differential equations. We, therefore, simplify them further by reducing them into ordinary differential equations. To this end, we propose to change, if possible, the independent and dependent variables in such way so as to transform the partial differential equations of the boundary layer equations into an ordinary differential equation. Whenever such a transformation exists, we say that 'similar solutions' exists. In general, by a similar solution we mean that the dimensionless velocity $u/U(x)$, $U(x)$ being the potential flow velocity, should be a function of only one variable, namely, $y/g(x)$, where $g(x)$ is proportional to the boundary layer thickness. It follows that for any two streamwise coordinates x_1 and x_2, we have

$$\frac{u[x_1, y/g(x_1)]}{U(x_1)} = \frac{u[x_2, y/g(x_2)]}{U(x_2)}. \qquad \text{...(1)}$$

As an example of the existence of a similarity solution, we consider a steady boundary layer flow over a plate. Then boundary layer equations for such a flow are given by

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial x^2} \qquad ...(2)$$

$$\partial u/\partial x+\partial v/\partial y=0 \qquad ...(3)$$

and the boundary conditions are:

$$u=v=0 \text{ when } y=0 \qquad \text{and} \qquad u=U(x) \text{ when } y\to\infty. \qquad ...(4)$$

Introduce the stream function $\psi(x, y)$ such that

$$u=\partial\psi/\partial y \qquad \text{and} \qquad v=-\partial\psi/\partial x. \qquad ...(5)$$

Then the equation of continuity (3) is identically satisfied and the equation of motion (2) reduces to

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=U\frac{dU}{dx}+\nu\frac{\partial^3\psi}{\partial y^3} \qquad ...(6)$$

with the boundary conditions:

$$\psi=0 \quad \text{and} \quad \partial\psi/\partial y=0 \quad \text{when } y=0 \quad \text{and} \quad \partial\psi/\partial y=U(x) \quad \text{when } y\to\infty. \qquad ...(7)$$

We now introduce three non-dimensional quantities as follows:

$$\xi=\frac{x}{L}, \qquad \eta=\frac{y\sqrt{\mathrm{Re}_l}}{Lg(x)}, \qquad \text{and} \qquad f(\xi,\eta)=\frac{\psi(x,y)\sqrt{\mathrm{Re}_l}}{LU(x)\,g(x)}, \qquad ...(8)$$

where $\mathrm{Re}_l=(LU_\infty)/\nu$ (Reynolds number). ...(9)

Here L and U_∞ are some characteristic length and velocity of the flow and $g(x)$ is a dimensionless function of x to be determined. Using (8), from (5), we have

$$u=\frac{\partial\psi}{\partial y}=U\frac{\partial f}{\partial\eta}=Uf' \qquad ...(10)$$

$$-v=\frac{\partial\psi}{\partial x}=\frac{L}{\sqrt{\mathrm{Re}_l}}\left\{f\frac{d}{dx}(Ug)+Ug\left(\frac{1}{L}\frac{\partial f}{\partial\xi}+f'\frac{\partial\eta}{\partial x}\right)\right\}, \qquad ...(11)$$

$$\frac{\partial u}{\partial x}=\frac{\partial^2\psi}{\partial x\partial y}=f'\frac{dU}{dx}+U\left(\frac{1}{L}\frac{\partial f'}{\partial\xi}+f''\frac{\partial\eta}{\partial x}\right), \qquad ...(12)$$

$$\frac{\partial u}{\partial y}=\frac{\partial^2\psi}{\partial y^2}=\frac{U\sqrt{\mathrm{Re}_1}}{Lg}f'' \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2}=\frac{\partial^3\psi}{\partial y^3}=\frac{U\,\mathrm{Re}_1}{L^2g^2}f''', \qquad ...(13)$$

where a prime denotes differentiation with respect to η.

Substituting the values of $\partial\psi/\partial y$, $\partial\psi/\partial x$, $\partial^2\psi/\partial x\partial y$, $\partial^2\psi/\partial y^2$ and $\partial^3\psi/\partial y^3$ given by (10), (11), (12) and (13) in (6) and noting that the terms containing $\partial\eta/\partial x$ cancel out, we obtain (after simplification and rearranging)

$$f'''+\alpha ff''+\beta(1-f'^2)=\frac{Ug^2}{U_\infty}\left(f'\frac{\partial f'}{\partial\xi}-f''\frac{\partial f}{\partial\xi}\right), \qquad ...(14)$$

where α and β are, in general, functions of x given by

$$\alpha=\frac{Lg}{U_\infty}\frac{d}{dx}(Ug) \qquad \text{and} \qquad \beta=\frac{Lg^2}{U_\infty}\frac{dU}{dx}. \qquad ...(15)$$

The boundary conditions (7) take the new form:

$$f = f' = 0 \quad \text{when} \quad \eta = 0 \quad \text{and} \quad f' = 1 \quad \text{when} \quad \eta \to \infty. \qquad ...(16)$$

From (14), it follows that a similar solution will exist if both f and f' are independent of ξ *i.e.* they are functions of η only. Again α and β must be independent of x *i.e.* they are constants. When similar solutions exist the stream function $f(\eta)$ satisfies the equation

$$f''' + \alpha\, ff'' + \beta(1 - f'^2) = 0 \qquad ...(17)$$

with the boundary condition (16).

We now proceed to find out the froms of $g(x)$ and $U(x)$ for which the similar solutions exist. From (15),

$$2\alpha - \beta = \frac{2Lg}{U_\infty}\frac{d}{dx}(Ug) - \frac{Lg^2}{U_\infty}\frac{dU}{dx} = \frac{L}{U_\infty}\frac{d}{dx}(g^2U). \qquad ...\ (18)$$

Two cases arise:

Case 1. when $(2\alpha - \beta) \neq 0$. Then, from (18), we have

$$\frac{1}{U_\infty} d(g^2U) = \frac{2\alpha - \beta}{L} dx \qquad \text{so that} \qquad \frac{g^2U}{U_\infty} = \frac{(2\alpha - \beta)x}{L} + C, \qquad ...(19)$$

where C is an arbitrary constant.

Assuming that $g^2U/U_\infty = 0$ when $x = 0$, (19) gives $C = 0$. Hence (19) yields,

$$(g^2U)/U_\infty = (2\alpha - \beta)x/L \qquad ...(20)$$

Again, from (15),

$$\alpha - \beta = \frac{Lg}{U_\infty}\frac{d}{dx}(Ug) - \frac{Lg^2}{U_\infty}\frac{dU}{dx}$$

or

$$\alpha - \beta = \frac{Lg}{U_\infty}\left\{\frac{dU}{dx}g + U\frac{dg}{dx}\right\} - \frac{Lg^2}{U_\infty}\frac{dU}{dx} = \frac{LgU}{U_\infty}\frac{dg}{dx}$$

or

$$\alpha - \beta\frac{1}{U}\frac{dU}{dx} = \frac{Lg}{U_\infty}\frac{dg}{dx}\frac{dU}{dx} = \frac{\beta}{g}\frac{dg}{dx}, \text{ using (15)}$$

$\therefore$

$$(\alpha - \beta)\,(1/U)dU = \beta(1/g)\,dg.$$

Integrating,

$$(\alpha - \beta)\log U - (\alpha - \beta)\log U_\infty = \beta \log g + \log D,$$

where D is a constant of integration

or

$$(\alpha - \beta)\log\left(\frac{U}{U_\infty}\right) = \log g^\beta + \log D \qquad \text{or} \qquad \left(\frac{U}{U_\infty}\right)^{\alpha-\beta} = Dg^\beta. \qquad ...(21)$$

We now eliminate g from (20) and (21).

From (20),

$$g = \left[\frac{(2\alpha - \beta)x}{L}\frac{U_\infty}{U}\right]^{1/2}. \qquad ...(22)$$

Substituting this values of g in (21), we get

$$\left(\frac{U}{U_\infty}\right)^{\alpha-\beta} = D\left[\frac{(2\alpha-\beta)x}{L}\frac{U_\infty}{U}\right]^{\beta/2} = D\left[\frac{(2\alpha-\beta)x}{L}\right]^{\beta/2} \times \left(\frac{U_\infty}{U}\right)^{\beta/2}$$

$$\Rightarrow \qquad \left(\frac{U}{U_\infty}\right)^{\alpha-\beta} \times \left(\frac{U}{U_\infty}\right)^{\beta/2} = D\left[\frac{(2\alpha-\beta)x}{L}\right]^{\beta/2} \qquad \text{or} \qquad \left(\frac{U}{U_\infty}\right)^{(2\alpha-\beta)/2} = D\left[\frac{(2\alpha-\beta)x}{L}\right]^{\beta/2}$$

$$\Rightarrow \qquad \frac{U}{U_\infty} = \left[D\left\{\frac{(2\alpha-\beta)x}{L}\right\}^{\beta/2}\right]^{2/(2\alpha-\beta)} = D^{2/(2\alpha-\beta)}\left\{\frac{(2\alpha-\beta)x}{L}\right\}^{\beta/(2\alpha-\beta)}. \qquad \text{...(23)}$$

Again, using (8), (9) and (22), the similarity variable η is given by

$$\eta = \frac{y\sqrt{\mathrm{Re}_l}}{Lg(x)} = \frac{y}{L}\left(\frac{LU_\infty}{\nu}\right)^{1/2}\left\{\frac{L}{(2\alpha-\beta)x}\frac{U}{U_\infty}\right\}^{1/2} = y\left\{\frac{1}{2\alpha-\beta}\frac{U}{\nu x}\right\}^{1/2}. \qquad \text{...(24)}$$

Thus (23) gives the desired potential flow velocity distribution for which the similar solutions exist and in such case the function $g(x)$, which is proportional to the boundary layer thickness, and the similarity variable η are given by (22) and (24) respectively.

The equation (23) is normalized in the following two situations as follows:

Situation I. When $\alpha \neq 0$. From (15), it follows that if g is multiplied by a constant K, then both α and β increase by a factor K^2. Therefore, when $\alpha \neq 0$, we may adjust g, without loss of generally, such that $\alpha = 1$.

For convience, we write $\qquad \beta/(2-\beta) = m. \qquad$ ...(25)

Then the equations (22), (23) and (24), in the normalised form, may be re-written as

$$g(x) = \left(\frac{2}{1+m}\frac{U_\infty}{U}\frac{x}{L}\right)^{1/2} \qquad \text{...(26)}$$

$$\frac{U(x)}{U_\infty} = D^{1+m}\left(\frac{2}{1+m}\frac{x}{L}\right)^m \qquad \text{...(27)}$$

and

$$\eta = y\left(\frac{1+m}{2}\frac{U}{\nu x}\right)^{1/2} \qquad \text{...(28)}$$

From (26), we concude that the similar solution exist when the potential flow velocity varies as some power of x, *i.e*, $\qquad U(x) \sim x^m. \qquad$ ...(29)

Situation II. When $\alpha = 0$. Then from (23), it follows that $U(x)$ is proportional to $1/x$ for all values of β. Then equation (23) may be normalised by taking $\beta = \pm 1$.

Case II. When $2\alpha - \beta = 0$. Then (18) reduces to

$$\frac{d}{dx}(g^2 U) = 0 \qquad \text{so that} \qquad g^2 U(x) = \text{constant}. \qquad \text{...(30)}$$

From (15), $\qquad \beta = \dfrac{Lg^2}{U_\infty}\dfrac{dU}{dx} \qquad$ so that $\qquad g^2 = \dfrac{\beta U_\infty}{L}\dfrac{dx}{dU}.$

Substituting this values of g^2 in (30), we get

$$\frac{\beta U_\infty}{L}\frac{dx}{dU}U = \text{constant} \qquad \text{or} \qquad \frac{1}{U}\frac{dU}{dx} = \text{constant}.$$

Integrating, $\qquad U(x) = e^{px}$, where p is a positive constant.

This solution may also be treated as a limiting form when $\beta \to 2$ *i.e* $m \to \infty$ of situation *I* of case-I.

18.9. Separation of boundary layer flow. [Himanchal 2000, 04, 05]

(*a*) **Physical approach.** When a solid body is immersed in a flowing fluid, a thin layer of fluid called the boundary layer is formed adjacent to the solid body. In this layer of fluid, the velocity varies from zero to free stream velocity in the direction normal to the solid body. Again, along the length of the solid body, the thickness of the boundary layer increases. The decelerated fluid particles in the boundary layer do not always remain in the thin boundary layer along the whole wetted length of the wall. Sometimes the boundary layer thickness increase considerably in the downstream direction and a point on the body comes after that the flow in the boundary layer takes place in reverse direction. Hence the decelerated fluid particles are forced to move outwards and consequently the boundary layer is separated from the wall. This phenomenan is known as the "*boundary layer separation*". The point on the body at which the boundary layer is on the verge of separation from the surface is known as "*point of separation*".

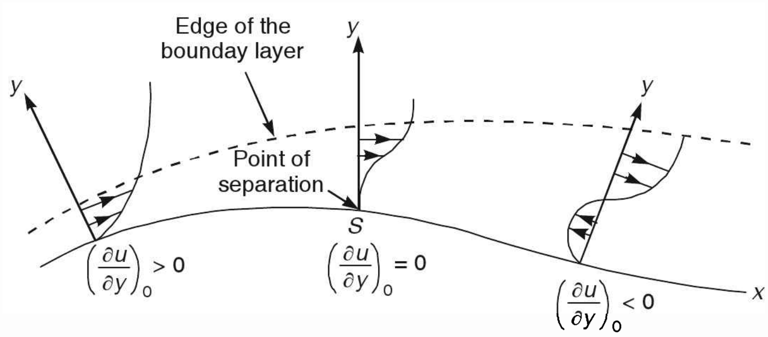


Fig. (i)

We now study the course of events which follow when a fluid is moving against a pressure gradient parallel to the wall. We know that the pressure over the width of the boundary layer has the same magnitude as outside the boundary layer at any cross-section, this pressure gradient is transmitted without change through the boundary layer to the surface. Again, the motion within the boundary layer is affected by three factors

(*i*) it is pulled forward by the stream above it through the action of viscosity,

(*ii*) it is retarded by friction at the boundary wall, and

(*iii*) it is retarded by the adverse pressure gradient ($dp/dx > 0$).

When the pressure gradient in the direction of flow is negative ($dp/dx < 0$) *i.e.* when the pressure decreases in the direction of flow, the flow is accelerated. In this case, the force (*i*) and the pressure force add together and jointly tend to reduce the effect of force (*ii*) in the boundary layer This results in the decrease in the thickness of boundary layer in the direction of flow.

When the pressure gradient in the direction of flow is positive ($dp/dx > 0$) *i.e.* when the pressure forces act opposite to the direction of flow, the flow is decelerated. In this case, the forces (*ii*) and (*iii*) together try to increase the retarding effect of the viscous force (*i*). Subsequently, the thickness of the boundary layer increases rapidly in the direction of flow. Whenever such forces act over a long stretch, the forward flow is then brought to rest and farther on a back flow in the direction of the pressure gradient sets in, which causes the boundary layer separation. Fig (*i*) shows the course of events described so far.

(*b*) **Mathematical criterian (or analytical approach)** **[Himanchal 2001]**

In order to explain the phenomenon of the separation of boundary layer flow, we shall apply the Prandtl boundary layer equations for steady flow

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \qquad ...(1)$$

and $$\partial p/\partial y = 0. \qquad ...(2)$$

both outside the boundary layer and at the wall. Outside the bounary layer, (1) applied to steady flow is given by

$$U\frac{dU}{dx}=-\frac{1}{\rho}\frac{dp}{dx}. \qquad ...(3)$$

Equation (2) shows that the pressure distribution is determined by the nature of the flow outside the boundary layer. Equation (2) also shows that this pressure is transmitted without change through the boundary layer to the surface.

From (2), we find that p is function of x alone. So (1) becomes

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \qquad ...(4)$$

At the wall, where $y = 0$, we have $u = v = 0$ and (4) becomes

$$\mu(\partial^2 u/\partial y^2)_{y=0} = dp/dx. \qquad ...(5)$$

From equation (5), we conclude that the curvature of the velocity profiles in the immediate neighbourhood of the wall depends only on the pressure gradient. Hence we shall discuss the following three situations:

Case (i) When dp/dx = 0 (region of zero pressure gradient).

Then from (5), $(\partial^2 u/\partial y^2)_0 = 0$ and hence the velocity gradient $\partial u/\partial y$ decreases steadity from a positive value at the wall to zero at the outer edge of the boundary layer. So the velocity profile must have a steadily decreasing form, *i.e.*, it must be without a point of inflexion. The fluid particles continue to move forward and so the phenomenon of boundary layer separation does not take place. Refer figure (*ii*) for velocity distribution in the case of zero pressure.

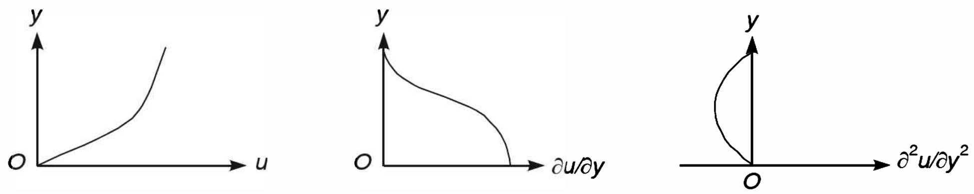


Fig. (*ii*)

Case (*ii*) *dp*/*dx* < 0 (region of favourable pressure gradient)

When $dp/dx < 0$, the flow is accelerated. Again (5) gives $(\partial^2 u/\partial y^2)_0 < 0$ and this negative value of $(\partial^2 u/\partial y^2)_0$ at $y = 0$ increases steadity to zero at $y = \delta$ as shown in figure (*iii*).

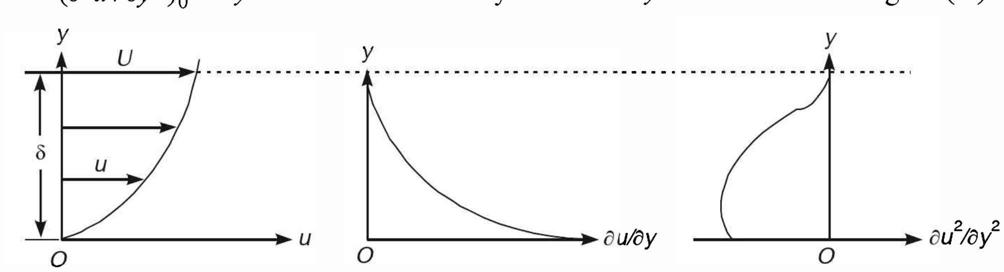


Fig. (*iii*)

The velocity gradient $\partial u/\partial y$ again decreases continually from a positive value at the wall to the value zero at $y = \delta$. The velocity profiles do not exibit any point of inflexion and has a form similar to the case of zero pressure gradient. In this case also the fluid particles continue to move forward and so the phenomenon of boundary layer separation does not take place.

Case (*iii*). *dp/dx* > 0 (region of adverse pressure gradient)

When $dp/dx > 0$, the flow is accelerated. Again (5) gives $(\partial^2 u/\partial y^2)_0 > 0$. In order to have a positive value of $\partial^2 u/\partial y^2$ at $y = 0$, the slope of the velocity gradient, $\partial u/\partial y$ at $y = 0$ must be positive. But the boundary condition required $\partial u/\partial y = 0$ at $y = \delta$. Hence the slope of the veloctiy gradient must change signs from positive to negative in the boundary layer. Consequently we have a point of inflexion I of the velocity profile in the boudary layer as shown in figure (*iv*).

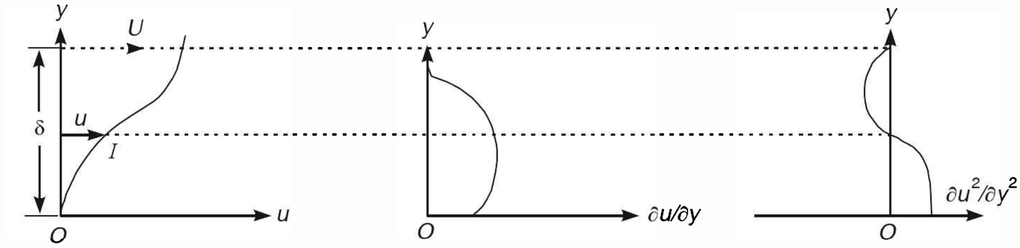


Fig. (*iv*)

As the adverse pressure gradient increases further the velocity profile may become increasingly distorted until the velocity gradient at the wall $(\partial u/\partial y)_0$ is zero as shown in fig. (*iv*). At this point separation of flow from the wall begins. Still further downstream from the point of separation there will be a back-flow region as the velocity component u is negative near the wall.

Remark. For a given velocity profile, it can be determined whether the boundary layer has separated, or on the verge of separation or will not separate with the help of the following conditions:

(*i*) $(\partial u/\partial y)_{y=0} = 0 \Rightarrow$ the flow is on the verge of separation

(*ii*) $(\partial u/\partial y)_{y=0} < 0 \Rightarrow$ the flow has separted

(*iii*) $(\partial u/\partial y)_{y=0} > 0 \Rightarrow$ the flow will not separate or flow will remain attached with the surface

An illustrative example. For the following velocity profiles determine whether flow is attached or detached or on the verge of separation:

(*i*) $u/U = 2(y/\delta) - (y/\delta)^2$. (*ii*) $u/U = -2(y/\delta) + (y/\delta)^3 + 2(y/\delta)^4$.

(*iii*) $u/U = 2(y/\delta)^2 + (y/\delta)^3 - 2(y/\delta)^4$.

Solution. (*i*) Given $$u = 2U(y/\delta) - U(y/\delta)^2$$

Hence, $$\partial u/\partial y = 2U(1/\delta) - 2U(y/\delta)\,(1/\delta).$$

Hence at $y = 0$, $(\partial u/\partial y)_{y=0} = (2U)/\delta$, which is positive

Therefore, the given flow is attached.

(*ii*) Given $$u = -2U(y/\delta) + U(y/\delta)^3 + 2U(y/\delta)^4.$$

Hence, $$\partial u/\partial y = -2U(1/\delta) + 3U(y/\delta)^2\,(1/\delta) + 8U(y/\delta)^3\,(1/\delta).$$

Hence at $y = 0$, $(\partial u/\partial y)_{y=0} = -2U(1/\delta)$, which is negative.

Therefore, the given flow is detached.

(*iii*) Given $$u = 2U(y/\delta)^2 + U(y/\delta)^3 - 2U(y/\delta)^4.$$

Hence, $$\partial u/\partial y = 4U(y/\delta)\,(1/\delta) + 3U(y/\delta)^2(1/\delta) - 8U(y/\delta)^3(1/\delta).$$

Hence at $y = 0$, $(\partial u/\partial y)_{y=0} = 0$.

Therefore, the given flow is on the verge of separation.

18.10. Boundary layer flow over a wedge. [Himachal 2000, 01, 02, 03, 04]

OR

Boundary layer on a surface with pressure gradient.

Let us consider the flow around a wedge submerged in a fluid of very small viscosity. At the leading stagnation point O, the thickness of the boundary layer is zero and it grows slowly towards the rear of the wedge (see fig. (i)). Within a very thin boundary layer of thickness δ a large velocity gradient exists, *i.e.* the velocity increases from zero at the wall to the value of potential flow at the edge of the boundary layer. If the x-axis of the coordinates coincides with the wall of the wedge and the y-axis is perpendicular to it, the Prandtl boundary layer equations for steady flow in the situation under consideration are given by

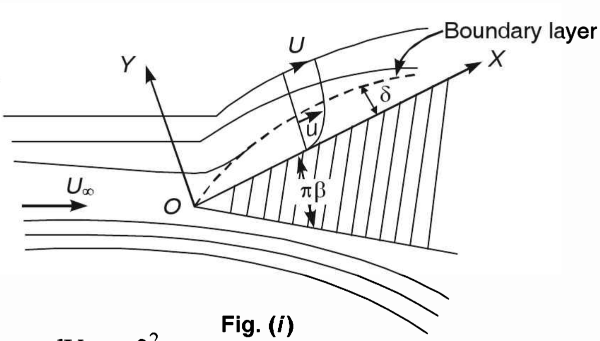


Fig. (i)

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \qquad \text{...(1)}$$

and $$\partial u/\partial x+\partial v/\partial y=0 \qquad \text{...(2)}$$

with the boundary conditions :

$$\text{On} \quad y=0, \qquad y=v=0 \qquad \text{...(3)}$$

$$\text{At} \quad y=\infty, \qquad u=U(x). \qquad \text{...(4)}$$

The integration of (1) and (2) can be simplified if we can reduce the number of variables by introducing the stream function

$$u=\partial\psi/\partial y \qquad \text{and} \qquad v=-\partial\psi/\partial x. \qquad \text{...(5)}$$

The continuity equation (2) is satisfied automatically by equations (5). In terms of the variable ψ the boundary layer equation (1) reduces to

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=U\frac{dU}{dx}+\nu\frac{\partial^3\psi}{\partial y^3}. \qquad \text{...(6)}$$

The boundary conditions (3) and (4), in terms of ψ may be re-written as

$$\text{On} \quad y=0, \qquad \partial\psi/\partial x=\partial\psi/\partial y=0 \qquad \text{...(7)}$$

$$\text{At} \quad y=\infty, \qquad \partial\psi/\partial y=U(x). \qquad \text{...(8)}$$

We want to solve (6) if the velocity outside the boundary layer is assumed to be proportional to a power of distance along the wall. This is the case for flow past a wedge and the velocity of the potential flow is given by $$U(x)=U_1\,x^m, \qquad \text{...(9)}$$

where U_1 is a constant and the wedge angle is denote by $\pi\beta$.

Since the order of the boundary layer thickness is $(\nu x/U)^{1/2}$, hence we take a new dimensionless distance parameter $\eta=y/\delta$ so that

$$\eta=yC(U/\nu x)^{1/2}=yC(U_1/\nu)^{1/2}\,x^{(m-1)/2}, \text{ using (9)} \qquad \text{...(10)}$$

where the arbitrary constant C is to be determined.

In accordance with the procedure of the law of similarity, let the velocity profile be given by

$$u/U=F(\eta). \qquad \text{...(11)}$$

From (5), $\psi = \int u dy = \int UF(\eta) \cdot \dfrac{d\eta}{C(U_1/\nu)^{1/2} x^{(m-1)/2}}$, using (10)and (11)

$$= \frac{U}{C(U_1/\nu)^{1/2} x^{(m-1)/2}} \int F(\eta)\, d\eta$$

$$= \frac{U_1 x^m}{C(U_1/\nu)^{1/2}\, x^{(m-1)/2}} f(\eta), \qquad \text{taking } f(\eta) = \int F(\eta)\, d\eta$$

Hence,
$$\psi = (1/C) \times \sqrt{\nu U_1}\, x^{(m+1)/2} f(\eta) \qquad \text{...(12)}$$

Suppose that
$$C^2 = (m+1)/2 \qquad \text{...(13)}$$

$$\therefore \quad \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{C}\sqrt{\nu U_1} x^{(m+1)/2} f'(\eta) \cdot C \sqrt{\frac{U_1}{\nu}}\, x^{(m-1)/2}, \text{ using (10) and (12)}$$

$$= U_1 x^m f'(\eta) = Uf'(\eta) \text{ using (9)}$$

Thus,
$$\partial \psi / \partial y = Uf'(\eta) \qquad \text{...(14)}$$

$$\frac{\partial \psi}{\partial x} = \frac{1}{C}\sqrt{\nu U_1}\, \frac{m-1}{2} x^{(m-1)/2} f(\eta) + \frac{1}{C}\sqrt{\nu U_1}\, x^{(m+1)/2} f'(\eta)\, y\, C \sqrt{\frac{U_1}{\nu}}\, \frac{m-1}{2} x^{(m-3)/2}$$

$$= \frac{1}{C}\sqrt{\frac{\nu U x^{m-1}}{x^m}}\, C^2 f(\eta) + \frac{1}{C}\sqrt{\frac{\nu U}{x^m}}\, x^{(m-1)/2} f'(\eta)\, y C \sqrt{\frac{U_1}{\nu}} \left(\frac{m+1}{2} - 1\right) x^{(m-3)/2} \cdot x$$

$$= \sqrt{\frac{\nu U}{x}}\, Cf(\eta) + \frac{1}{C}\sqrt{\frac{\nu U x^{m-1}}{x^m}}\, f'(\eta)\, (C^2 - 1)\eta \text{ using (10)}$$

Thus,
$$\frac{\partial \psi}{\partial x} = C\sqrt{\frac{\nu U}{x}} \left[f(\eta) + \frac{C^2 - 1}{C^2} \eta f'(\eta) \right] \qquad \text{...(15)}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial \psi}{\partial y}\right) \cdot \frac{\partial \eta}{\partial y} = \frac{\partial}{\partial \eta}[Uf'(\eta)] \cdot \frac{\partial \eta}{\partial y}, \text{ using (14)}$$

$$= Uf''(\eta) \cdot C \sqrt{\frac{U_1}{\nu}}\, x^{(m-1)/2} = CUf''(\eta) \sqrt{\frac{U}{x^m \nu} x^{m-1}}, \text{ using (10)}$$

Thus,
$$\partial^2 \psi / \partial y^2 = CU(U/\nu x)^{1/2}\, f''(\eta) \qquad \text{...(16)}$$

Now,
$$\frac{\partial^3 \psi}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 \psi}{\partial y^2}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial^2 \psi}{\partial y^2}\right) \cdot \frac{\partial \eta}{\partial y}$$

$$= \frac{\partial}{\partial \eta}\left[CU \sqrt{\frac{U}{\nu x}}\, f''(\eta) \right] \cdot C \sqrt{\frac{U_1}{\nu}}\, x^{(m-1)/2}, \text{ using (10) and (16)}$$

$$= CU \sqrt{\frac{U}{\nu x}} f'''(\eta) \cdot C \sqrt{\frac{U}{x\nu}} = U\left(\frac{C^2 U}{\nu x}\right) f'''(\eta) \qquad \text{...(17)}$$

Finally, $\dfrac{\partial^2 \psi}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial \psi}{\partial y}\right) = \dfrac{\partial}{\partial x}\{U f'(\eta)\}$, using (14)

$$= \frac{\partial}{\partial x}\left[U_1 x^m f'(\eta)\right] = f'(\eta)\frac{\partial}{\partial x}(U_1 x^m) + U_1 x^m \frac{d}{d\eta} f'(\eta)\cdot\frac{\partial \eta}{\partial x}, \text{ using (9)}$$

$$= U_1 m x^{m-1} f'(\eta) + U f''(\eta)\cdot yC\sqrt{\frac{U_1}{\nu}}\frac{m-1}{2}x^{(m-3)/2} \text{ using (9) and (10),}$$

$$= \frac{mU}{x}f'(\eta) + Uf''(\eta)\left\{\frac{m+1}{2}-1\right\}\cdot yC\sqrt{\frac{U_1}{\nu}}x^{(m-1)/2}\cdot\frac{1}{x} = \frac{2C^2-1}{x}Uf'(\eta) + Uf''(\eta)\frac{(C^2-1)\eta}{x}$$

$$[\because C^2 = (m+1)/2 \Rightarrow m = 2C^2 - 1]$$

Thus,
$$\frac{\partial^2 \psi}{\partial x \partial y} = (2C^2-1)\frac{U}{x}\left[f' + \frac{C^2-1}{2C^2-1}\eta f''\right] \qquad \text{...(18)}$$

Using (14), (15), (16), (17) and (18), (6) reduces to

$$Uf'\frac{(2C^2-1)U}{x}\left[f' + \frac{C^2-1}{2C^2-1}\eta f''\right] - C\sqrt{\frac{\nu U}{x}}\left[f + \frac{C^2-1}{C^2}\eta f'\right] CU\sqrt{\frac{U}{\nu x}}f''$$

$$= U\cdot U_1 m x^{m-1} + \nu U\left(\frac{C^2 U}{\nu x}\right)f'''(\eta)$$

or
$$f''' + f f'' - \left(\frac{2C^2-1}{C^2}\right)f'^2 + \frac{m}{C^2} = 0$$

or
$$f''' + f f'' - \frac{2m}{m+1}(f'^2 - 1) = 0, \quad \text{as} \quad C^2 = (m+1)/2 \qquad \text{...(19)}$$

The boundary conditions (7) and (8) take the form

on $\eta = 0$, $f = 0$, $f' = 0$ ...(20)

and on $\eta = \infty$, $f' = 1$. ...(21)

The solution of (19) was given by Hartree by setting $(2m)/(m+1) = \beta$, ...(22)
where $\pi\beta$ is the *wedge angle*. Hence (19) reduces to

$$f''' + ff'' + \beta(1 - f'^2) = 0 \qquad \text{...(23)}$$

and the boundary conditions (20) and (21) may be re-written as

$f(0) = f'(0) = 0$ and $f'(\infty) = 1$. ...(24)

Equation (23) in known as *Hartree's equation*

The solution of equation (23) under boundary conditions (24) is shown in the figure (*ii*). From this figure we find that when $m > 0$ $(\beta > 0)$, the flow is accelerated and the velocity profiles have no pint of inflexion. On the other hand, if $m < 0$ $(\beta < 0)$ the flow in decelerated (with adverse pressure gradient) and a point of inflexion occurs in the velocity profile. As m reaches a value -0.09 $(\beta = -0.199)$, the separation of flow from the wall occurs.

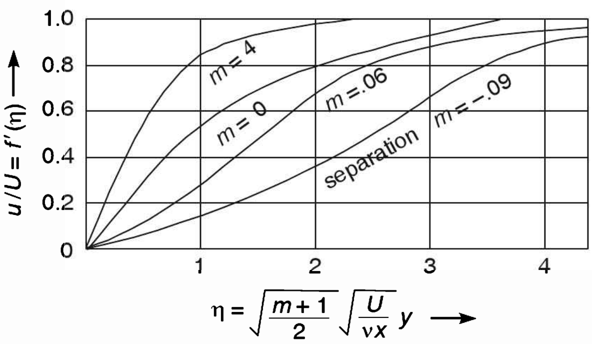


Fig (*ii*)

18.11. The spread of a jet.

We have already studied the boundary layer assumptions which are valid for regions near a solid wall (refer Art. 18.3). These assumptions are also valid when two layers of fluid with different velocities meet, for example, when a fluid is being discharged through an orifice and is allowed to mix with the surroundings. We now propose to discuss three such examples of a jet, namely, (*i*) two dimensional motion (*ii*) the flow of jet bounded on one side by a wall and on the other by fluid at rest and (*iii*) motion symmetrical about an axis.

18.12. Plane free jet (two dimensional jet) [Himanchal 2001, 02, 04, 05]

As shown in the adjoining figure, consider the flow of an incompressible fluid passing through a slit O in the wall AB and then mixing with the surrounding fluid at rest. Due to such mixing the jet spreads. We propose to find the velocity distribution across the jet. With the origin in the slit, we choose the co-ordinate axes OX and OY along the axis of the jet and normal to it respectively. Then the usual boundary layer equations are given by

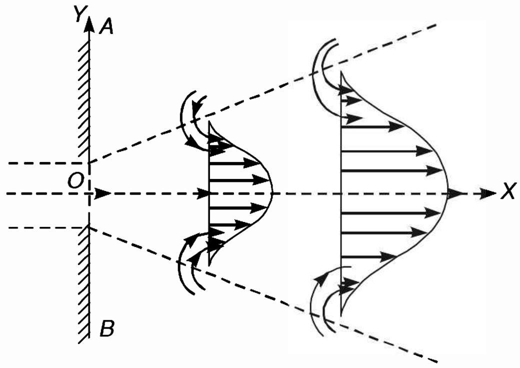


Fig. Spread of free jet

$$u\,(\partial u/\partial x) + v\,(\partial u/\partial y) = \nu\,(\partial u^2/\partial y^2) \quad ...(1)$$

and
$$\partial u/\partial x + \partial v/\partial y = 0. \quad ...(2)$$

Since the constant pressure in the surrounding fluid impress itself on the jet, hence we have taken $dp/dx = 0$ while writing equation (1).

The boundary conditions for the present problem are:

$$\left.\begin{array}{llll} \partial u/\partial y = 0 \text{ (due to symmetry)} & \text{and} \quad v = 0 & \text{when} & y = 0 \\ \text{and} \quad u = 0, & \text{when} & & y = \pm\infty \end{array}\right\} \quad ...(3)$$

Re-writing (2), $\quad u\,(\partial u/\partial x) + u\,(\partial v/\partial y) = 0. \quad ...(4)$

Adding (1) and (4), $\quad 2u\,(\partial u/\partial x) + v\,(\partial u/\partial y) + u\,(\partial v/\partial y) = \nu\,(\partial^2 u/\partial^2 y)$

or
$$\rho\frac{\partial u^2}{\partial x} + \rho\frac{\partial (uv)}{\partial y} = \mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \qquad \text{noting that} \qquad \nu = \frac{\mu}{\rho} \quad ...(5)$$

Integrating (5) *w.r.t.* 'y' between the limits $-\infty$ to ∞, we get

$$\frac{d}{dx}\int_{-\infty}^{\infty} \rho u^2 dy + \rho\left[uv\right]_{-\infty}^{\infty} = \mu\left[\frac{\partial u}{\partial y}\right]_{-\infty}^{\infty} \quad ...(6)$$

Now, at the edge of the boundary layer, $\partial u/\partial y = 0$. Hence, using boundary conditions (3), (6) reduces to

$$\frac{d}{dx}\int_{-\infty}^{\infty} \rho u^2 dy = 0 \quad ...(7)$$

Integrating (7), $\quad \rho\int_{-\infty}^{\infty} u^2 dy = \text{constant} = J_0, \text{ (say)} \quad ...(8)$

wherein the value of J_0 should be prescribed.

Let ψ be the stream function such that

$$u = \partial\psi/\partial y \qquad \text{and} \qquad v = -(\partial\psi/\partial x) \qquad ...(9)$$

Then we find that the equation of continuity (2) is identically satisfied. Again, using (9), the equation of motion (1) can be re-written as

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \nu\left(\frac{\partial^3\psi}{\partial y^3}\right) \qquad ...(10)$$

We now proceed to find a similar solution of the given problem by taking

$$\psi = Ax^p f(y/x^q) \qquad \text{or} \qquad \psi = Ax^p f(\eta), \qquad ...(11)$$

where
$$\eta = y/x^q. \qquad ...(12)$$

Here p and q are two unknown exponents and A is an unknown constant. We now proceed to evaluate p, q *and* A by satisfying the following three conditions:

(*i*) In order that a similar solution may exist, the equation of motion must reduce to an ordinary differential equation having η as its independent variable, that is, each term in equation of motion must contain same degree in x.

(*ii*) J_0 is independent of x.

(*iii*) The equation of motion should take a new form, which should be independent of fluid properties, that is, it should be universal in character.

From (11) and (12), we have

$$\left.\begin{aligned} &\partial\psi/\partial y = Ax^p \times (1/x^q) f'(y/x^q) = Ax^{p-q} f' \\ \Rightarrow \quad &\partial^2\psi/\partial y^2 = Ax^{p-2q} f'' \qquad \text{and} \qquad \partial^3\psi/\partial y^3 = Ax^{p-3q} f''' \end{aligned}\right\} \qquad ...(13)$$

$$\left.\begin{aligned} &\partial\psi/\partial x = Apx^{p-1} f + Ax^p(-yqx^{-q-1}) f' = Apx^{p-1} f - Ayqx^{p-q-1} f' \\ \text{and} \quad &\partial^2\psi/\partial x\partial y = A(p-q)x^{p-q-1} f' + Ax^{p-q}(-yqx^{-q-1}) f'' \end{aligned}\right\} \qquad ...(14)$$

Using (13) and (14), (10) reduces to

$$Ax^{p-q} f'\{A(p-q)x^{p-q-1} f' - Ayqx^{p-2q-1} f''\}$$
$$-(Apx^{p-1} f - Aypx^{p-q-1}) \times (Ax^{p-2q} f'') = \nu Ax^{p-3q} f'''$$

or
$$(A/\nu) \times x^{p+q-1}\{(p-q)f'^2 - p f f'\} = f''', \text{ on simplification,} \qquad ...(15)$$

where a prime denotes the differentiation with respect to η.

Using condition (*i*), (15) holds if we choose $\qquad p + q - 1 = 0 \qquad ...(16)$

From (9) and (13), $\qquad u = \partial\psi/\partial y = Ax^{p-q} f' \qquad ...(17)$

From (12), $\qquad y = \eta x^q \qquad$ so that $\qquad dy = x^q d\eta \qquad ...(18)$

Using (17) and (18), (8) reduces to

$$\rho\int_{-\infty}^{\infty} A^2 x^{2p-2q} f'^2\, x^q dy = J_0 \qquad \text{or} \qquad \rho A^2 x^{2p-q}\int_{-\infty}^{\infty} f'^2 d\eta = J_0 \qquad ...(19)$$

Using condition (*ii*), (19) will be true it we choose $\qquad 2p - q = 0 \qquad ...(20)$

Solving (16) and (20), $\qquad p = 1/3 \qquad$ and $\qquad q = 2/3. \qquad ...(21)$

Using (21), (15), gives $\qquad (A/3\nu) \times (f'^2 + f f'') + f''' = 0 \qquad ...(22)$

In order to satisfy the condition (*iii*), (22) shows that we must take

$$A/3\nu = 1 \qquad \text{so that} \qquad A = 3\nu \qquad \text{...(23)}$$

Using (21) and (23), (12) reduces to

$$\psi = 3\nu x^{1/3} f(\eta), \qquad \text{where} \qquad \eta = y x^{-2/3} \qquad \text{...(24)}$$

From (9) and (24) we have

$$u = \partial\psi/\partial y = 3\nu x^{1/3} f'(\eta) \times (\partial\eta/\partial y) = 3\nu x^{1/3} f'(\eta) x^{-2/3} = 3\nu x^{-1/3} f'$$

and $v = -(\partial\psi/\partial x) = -3\nu\left\{(1/3)\times x^{-2/3} f(\eta) + 3\nu x^{1/3} f'(\eta)\times(\partial\eta/\partial x)\right\}$

$$= -3\nu\left\{(1/3)\times x^{-2/3} f(\eta) + 3\nu x^{1/3} f'(\eta)\times y \times(-2/3)\times x^{-5/3}\right\} = \nu x^{-2/3}(2\eta f' - f)$$

Thus, $\quad u = 3\nu x^{-1/3} f' \qquad$ and $\qquad v = \nu x^{-2/3}(2\eta f' - f). \qquad$ ...(25)

Using (23), (22) reduces to

$$f''' + f f'' + f'^2 = 0 \qquad \text{or} \qquad f''' + (f f')' = 0 \qquad \text{...(26)}$$

Using (25), the boundary conditions (3) reduces to

$$f = 0 \text{ and } f'' = 0, \text{ when } \eta = 0\ ; \qquad f' = 0, \qquad \text{when} \qquad \eta = \pm\infty. \qquad \text{...(27)}$$

Using (21) and (23), condition (19) reduces to

$$9\rho\nu^2 \int_{-\infty}^{\infty} f'^2 d\eta = J_0 \qquad \text{...(28)}$$

In order to solve the given problem, we proceed to solve (26) satisfying the conditions (27) and (28).

Integrating (26), $\qquad f'' + f f' = C,\ C$ being an arbitrary constant. $\qquad$ ...(29)

Using condition $f = f'' = 0$ when $\eta = 0$ given by (27), (29) gives $C = 0$.

Hence, (29) reduces to $\qquad f'' + f f' = 0. \qquad$ ...(30)

In order to solve (30), we introduce the following transformation of independent and dependent variables

$$\xi = \alpha\eta \qquad \text{and} \qquad f = 2\alpha F(\xi), \qquad \text{...(31)}$$

where α is a free constant to be evaluated.

From (31), we obtain

$$f' = \frac{df}{d\eta} = 2\alpha\frac{dF(\xi)}{d\eta} = 2\alpha\frac{dF(\xi)}{d\xi}\frac{d\xi}{d\eta} = 2\alpha^2 F'(\xi) \qquad \text{...(31)'}$$

and
$$f'' = \frac{df'}{d\eta} = \frac{d}{d\eta}\{2\alpha^2 F'(\xi)\} = 2\alpha^2\frac{dF'(\xi)}{d\xi}\frac{d\xi}{d\eta} = 2\alpha^3 F''(\xi) \qquad \text{...(31)''}$$

Using (31), (31)' and (31)'', (30) reduces to

$$2\alpha^3 F''(\xi) + 4\alpha^3 F(\xi) F'(\xi) = 0 \qquad \text{or} \qquad F'' + 2FF' = 0, \qquad \text{...(32)}$$

where a prime now denotes differentiation with respect to ξ.

Using (31), (27) gives $F = 0$ when $\xi = 0 \qquad$ and $\qquad F' = 0$ when $\xi = \infty \qquad$ ...(33)

Integrating (32), $\qquad F' + F^2 = D,\ D$ being an arbitrary constant. $\qquad$ ...(34)

Since (31) involves a free constant α, without loss of generality, we may take $F'(0) = 1$. Then, putting $\xi = 0$ in (34) and using (33), we obtain.

$$F'(0) + \{F(0)\}^2 = D \qquad \text{or} \qquad 1 + 0 = D \qquad \text{so that} \qquad D = 0.$$

Then, (34) gives $\qquad F' + F^2 = 1 \qquad$ or $\qquad dF/d\xi = 1 - F^2$

or $\qquad \{1/(1-F^2)\}dF = d\xi \qquad$ so that $\tanh^{-1} F = \xi + E$ ...(35)

From condition (33), $\qquad F = 0$ when $\xi = 0$. So (35) gives $E = 0$

Hence, $\qquad$ (35) $\Rightarrow$ $\quad$ F = tanh ζ ... (36)

We now produced to evaluate the value of α. Using (31) and (31)', (28) reduces to

$$9\rho\nu^2 \int_{-\infty}^{\infty} \{4\alpha^4 F'^2) \times (1/\alpha)\} d\xi = J_0 \qquad \text{or} \qquad 36\rho\nu^2\alpha \int_{-\infty}^{\infty} F'^2 d\xi = J_0$$

or
$$36\rho\nu^2\alpha^3 \int_{-\infty}^{\infty} \operatorname{sech}^4\xi \; d\xi = J_0, \text{ using (36)}$$

or
$$72\rho\nu^2\alpha^3 \int_0^{\infty} \operatorname{sech}^4\xi \, d\xi = J_0 \qquad \text{...(37)}$$

Now, $\displaystyle\int_0^{\infty} \operatorname{sech}^4\xi\, d\xi = \int_0^{\infty} \operatorname{sech}^2\xi \operatorname{sech}^2\xi \, d\xi = \int_0^{\infty} (1 - \tanh^2\xi)\operatorname{sech}^2\xi \, d\xi$

$$= \int_0^1 (1 - t^2)\, dt, \text{ putting } \tanh\xi = t \text{ and } \operatorname{sech}^2\xi \; d\xi = dt$$

$$= [t - t^3/3]_0^1 = 1 - (1/3) = 2/3.$$

Hence, (37) becomes $\qquad 48\rho\nu^2\alpha^3 = J_0 \qquad$ or $\qquad \alpha = (J_0/48\rho\nu^2)^{1/3}$ ...(38)

Using (31), (36) and (38), (24) reduces to

$$\psi = (3\nu x^{1/3}) \times (2\alpha F) = 6\nu x^{1/3} \times (J_0/48\rho\nu^2)^{1/3} \tanh\xi$$

or
$$\psi = 6 \times (J_0 \nu x/48\rho)^{-1/3} \tanh\xi = 1.6510 \times (J_0\nu x/\rho)^{1/3} \tanh\xi \qquad \text{...(39)}$$

Using (31)', (36) and (38), (25) gives

$$u = (3\nu x^{-1/3}) \times (2\alpha^2 F') = (6\nu x^{-1/3}) \times (J_0/48\rho\nu^2)^{1/3} \operatorname{sech}^2\xi$$

or
$$u = (1/4) \times (6J_0^2/\nu\rho^2 x)^{1/3} \operatorname{sech}^2\xi = 0.4543\,(J_0^2/\nu\rho^2 x)^{1/3} \operatorname{sech}^2\xi \qquad \text{...(40)}$$

Again, using (31), (31)', and (38), (25) gives

$$v = \nu x^{-2/3}\{(\xi/\alpha) \times (2\alpha^3 F') - 2\alpha F\} = 2\nu x^{-2/3}\alpha\,(2\xi F' - F)$$

or
$$v = 2\nu x^{-2/3}(J_0/48\rho\nu^2)^{1/3}(2\xi \operatorname{sech}^2\xi - \tanh\xi)$$

Thus
$$v = 0.5503\,(J_0\nu/\rho x^2)^{1/3}(2\xi \operatorname{sech}^2\xi - \tanh\xi) \qquad \text{...(41)}$$

Using (24) and (38), (31) gives

$$\xi = \alpha\eta = (J_0/48\rho\nu^2)^{1/3} \times (yx^{-2/3}) = 0.2752\,(J_0/\rho\nu^2)^{1/3} yx^{-2/3} \qquad \text{...(42)}$$

The required velocity distribution is given by (40) and (41), where ξ is given by (42).

Let Q be the volume rate of discharge per unit height of the slit at a distance x from the slit and normal to the jet. Then, we have

$$Q = \int_{-\infty}^{\infty} \rho u \, dy = 2\rho \int_{0}^{\infty} u \, dy = 2\rho[\psi]_0^{\infty} = 3.3019\,(J_0 \nu \rho^2 x)^{1/3} \qquad \text{...(43)}$$

showing that Q increases in the downstream as $x^{1/3}$.

From (40), it follows that the maximum value u_m of the velocity component u exists on the axis of the jet and is given by

$$u_m = [u]_{\eta=0} = [u]_{\xi=0} = 0.4543(J_0^2 / \nu\rho^2 x)^{1/3} \qquad \text{...(44)}$$

From (40) and (44), $\qquad u/u_m = \operatorname{sech}^2 \xi. \qquad$ ...(45)

The velocity distribution is displayed in the following figure.

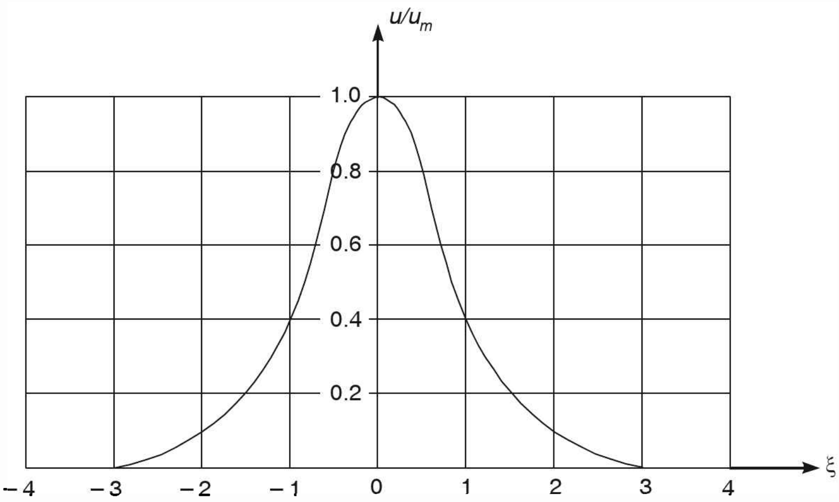


Fig. Velocity distribution in a two-dimensional free get.

From the above figure, we find the $u \to 0$ at $\xi = 3$ (approximately).

Therefore, if the semi-width of the jet be $b(x)$, then we get

$$b(x) = [y]_{u \to 0} = 10.91(\rho\nu^2 / J_0)^{1/3} x^{2/3}, \qquad \text{...(46)}$$

showing that the width of the jet varies as $x^{2/3}$.

Measurements performed by E. N. Andrade* for the two dimensional laminar jet confirm the preceding theoretical theoretical argument very well. The jet remains laminar upto R = 30 approximately, where the Reynolds number R is referred to the efflux velocity and to the width of the slit.

18.13 Plane wall jet. [Himanchal 2000, 02]

The flow in a viscous jet bounded on one side by a wall and on the other by fluid at rest is know as a *plane wall jet*.

Let an incompressible viscous fluid be discharged through a norrow slit at O in half space along a plane and mix with the same surrounding fluid which is at rest. With O as origin and

* Andrade, E.N : The plane jet. Proc. Phys. Soc. London, 51, 784 – 793. (1939)

x-axis and y-axis along and normal to the plane wall respectively, the boundary layer equations for the present problem reduce to

$$u(\partial u/\partial x)+v(\partial u/\partial y)=\nu(\partial^2 u/\partial y^2) \qquad ...(1)$$

and $$\partial u/\partial x+\partial v/\partial y=0, \qquad ...(2)$$

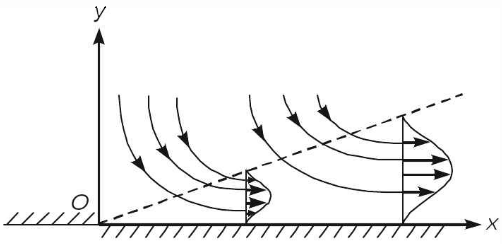


Fig. Spread of a plane wall jet

Since the constant pressure in the surrounding fluid impress itself on the jet, hence we have taken $dp/dx = 0$ while writing equation (1). The boundary conditions are

$$u=0, \quad v=0 \quad \text{when} \quad y=0 \qquad \text{and} \qquad u=0 \quad \text{when} \quad y=\infty. \qquad ...(3)$$

Besides these boundary conditions, we must have a law of conservation for the motion under consideration which may provide a non-trivial solution. Akatnow suggested the follwoing procedure for getting the proposed law in the form of an integral condition.

Re-writing (2), $$u(\partial u/\partial x)+u(\partial v/\partial y)=0 \qquad ...(4)$$

Adding (2) and (4), $$2u(\partial u/\partial x)+v(\partial u/\partial y)+u(\partial v/\partial y)=\nu(\partial^2 u/\partial y^2)$$

or $$\partial u^2/\partial x+\partial(uv)/\partial y=\nu(\partial^2 u/\partial y^2) \qquad ...(5)$$

Integrating (5) *w.r.t.* 'y' between the limits y to ∞, we get

$$\int_y^\infty \frac{\partial u^2}{\partial x}dy+[uv]_y^\infty=\nu\left[\frac{\partial u}{\partial y}\right]_y^\infty \qquad \text{or} \qquad \frac{\partial}{\partial x}\int_y^\infty u^2 dy+uv=\nu\left[\left(\frac{\partial u}{\partial y}\right)_{y=\infty}-\frac{\partial u}{\partial y}\right]$$

or $$\frac{\partial}{\partial x}\int_y^\infty u^2 dy-uv=\nu\frac{\partial u}{\partial y}, \qquad ...(6)$$

where we have used the boundary conditions (3) and the fact that at edge of the boundary layer, $\partial u/\partial y=0$.

Multiplying both sides of (6) by u and then integrating *w.r.t.* 'y' between the limits 0 to ∞, we have

$$\int_0^\infty u\left\{\frac{\partial}{\partial x}\int_y^\infty u^2 dy\right\}dy-\int_0^\infty u^2 v\,dy=-\nu\left[u^2\right]_0^\infty=0, \qquad ...(7)$$

since $u = 0$ when $y = 0$ and $y=\infty$ by boundary conditions (3).

Re-writing the first term and integrating by parts the second term on L.H.S. of (7), we have

$$\frac{d}{dx}\int_0^\infty u\left(\int_y^\infty u^2 dy\right)dy-\int_0^\infty \frac{\partial u}{\partial x}\left(\int_y^\infty u^2 dy\right)dy-\left[\left[-v\int_y^\infty u^2 dy\right]_0^\infty-\int_0^\infty\left(-\frac{\partial v}{\partial y}\right)\left(\int_y^\infty u^2 dy\right)dy\right]=0$$

From (2), $\partial u/\partial x=-(\partial v/\partial y)$. Hence the second and fourth terms cancel out in the above equation. Furthermore, the third term vanishes with help of bounary condition (3). So finally,

$$\frac{d}{dx}\int_0^\infty u\left(\int_y^\infty u^2 dy\right)dy=0 \qquad \text{so that} \qquad \int_0^\infty u\left(\int_y^\infty u^2 dy\right)dy=\text{const.}=E.\ \text{(say)} \qquad ....(8)$$

Since $u\,dy=d\left(\int_0^y u\,dy\right)$, (8) may be re-written as

$$E=\int_0^\infty\left(\int_y^\infty u^2\,dy\right)d\left(\int_0^y u\,dy\right)=\left[\left(\int_y^\infty u^2\,dy\right)\int_0^y u\,dy\right]_0^\infty-\int_0^\infty(-u^2)\left(\int_0^y u\,dy\right)dy$$

$$\left[\because\ \frac{d}{dy}\int_y^\infty u^2\,dy=\left[u^2\right]_y^\infty=-u^2,\text{ using(3)}\right]$$

or
$$\int_0^\infty u^2\left(\int_0^y u\,dy\right)dy=E. \qquad ...(9)$$

We now proceed to find a similar solution of the given problem be taking stream function in the form

$$\psi=(\nu Ex)^{1/4}f(\eta) \qquad \text{where} \qquad \eta=(E/\nu^3)^{1/4}yx^{-3/4} \qquad ...(10)$$

Hence,
$$u=\partial\psi/\partial y=(E/\nu x)^{1/2}f'$$

and
$$v=-(\partial\psi/\partial x)=(1/4)\times(\nu E/x^3)^{1/4}(3\eta f'-f) \qquad ...(11)$$

Here a prime denotes differentiation with respect to η. With the above values of u and v, (2) is identically satisfied and (1) reduces to

$$4f'''+ff''+2f'^2=0. \qquad ...(12)$$

Boundary conditions (3) become; $f=f'=0$, when $\eta=0$ and $f'=0$ when $\eta\to\infty$ ... (13)

and the integral condition (9) reduces to $\displaystyle\int_0^\infty f f'^2 d\eta=1.$ ...(14)

Multiplying both sides of (12) by f, we get $\quad 4ff'''+f^2f''+2ff'^2=0$

or $\quad (4ff'''+4f'f'')-4f'f''+(f^2f''+2ff'^2)=0$ or $d(4ff'')-d(2f'^2)+d(f^2f')=0$

Integrating, $\quad 4ff''-2f'^2+f^2f'=C$, C being an arbitrary constant ...(15)

Putting $\eta=0$ in (15) and using B.C. (13), we get $C=0$ Then (15) reduces to

$$4ff''-2f'^2+f^2f'=0 \qquad ...(16)$$

The above non-linear equation (16) will be linearised by taking f as the independent variable and ϕ as new dependent variable given by

$$df/d\eta=f'=\phi \qquad ...(17)$$

$$\therefore \qquad f''=\frac{d^2f}{d\eta^2}=\frac{d}{d\eta}\left(\frac{df}{d\eta}\right)=\frac{d\phi}{d\eta}=\frac{d\phi}{df}\frac{df}{d\eta}=\phi\frac{d\phi}{df}. \qquad ...(17A)$$

Using (17) and (17A), (16) reduces to

$$4f\phi\frac{d\phi}{df}-2\phi^2+f^2\phi=0 \qquad \text{or} \qquad \frac{d\phi}{df}-\frac{1}{2f}\phi=-\frac{f}{4} \qquad ...(17B)$$

whose I.F. $=e^{-\int(1/2f)df}=e^{-(1/2)\times\log f}=e^{\log f^{-1/2}}=f^{-1/2}$. and solution is

$$\phi f^{-1/2}=\int(-f/4)\times f^{-1/2}df+C'=-(1/6)\times f^{3/2}+C'$$

or $\quad f' f^{-1/2} = C' - (1/6) \times f^{3/2} \quad$ or $\quad f' = C'\sqrt{f} - (1/6) \times f^2 \quad$ ...(18)

where C' is an arbitrary constant. To evaluate C', we assume that $f = f_\infty$ when $\eta = \infty$. Also from (13), $f' = 0$ when $\eta = \infty$. Hence making $\eta \to \infty$ in (18) and using the above facts, we get

$0 = C'\sqrt{f_\infty} - (1/6) \times f_\infty^2$ so that $C' = (1/6) \times f_\infty^{3/2}$. ...(19)

Then (18) becomes $\quad f' = (1/6) \times \left[f_\infty^{3/2} f^{1/2} - f^2 \right]$. ...(20)

Again to determine the value of f_∞, re-write (14) as $\quad \int_0^{f_\infty} f\, f'\, df = 1.$ ...(21)

or $$\frac{1}{6}\int_0^{f_\infty} \left(f_\infty^{3/2} f^{1/2} - f^2 \right) f\, df = 1, \text{ using (20)}$$

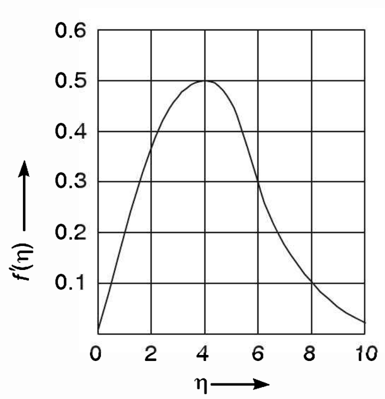
or $$\frac{1}{6}\left[f_\infty^{3/2} \times (2/5) \times f^{5/2} - (1/4) \times f^4 \right]_0^{f_\infty} = 1 \quad \text{or} \quad f_\infty = (40)^{1/4} = 2.515 \qquad \text{...(22)}$$

To solve (20), we put $\quad F = f / f_\infty$ ...(23)

Then, (20) becomes $$\frac{dF}{d\eta} = \frac{f_\infty}{6}(\sqrt{F} - F^2) \quad \text{or} \quad d\eta = \frac{6}{f_\infty} \times \frac{dF}{\sqrt{F} - F^2}$$

Integrating it, $$\eta = \frac{2}{f_\infty}\left\{ \log_e \frac{1 + \sqrt{F} + F}{(1 - \sqrt{F})^2} + 2\sqrt{3} \tan^{-1} \frac{\sqrt{3F}}{2 + \sqrt{F}} \right\}. \qquad \text{...(24)}$$

The function $f'(\eta)$, which is proportional to the velocity distribution u, has been plotted against η in the adjoining figure. From the figure we find that the velocity distribution near the plate is of Blasius type and away from it is like that of a plane free jet.



The characteristic quantities of the flow, namely, the volume flux Q and momentum flux K through a cross-section of the jet are given by

$$\left. \begin{aligned} Q &= \int_0^\infty u\, dy = f_\infty (\nu E x)^{1/4}, \\ K &= \int_0^\infty u^2\, dy = \frac{20}{9 f_\infty}\left(\frac{E^3}{\nu x} \right)^{1/4} \end{aligned} \right\} \qquad \text{...(25)}$$

Hence, $\quad QK = (20E)/9,$

showing that the product of the volume flux and momentum flux through any cross section of the bounday layer remains constant and this may be taken as the physical meaning of the integral condition (9).

Finally, the shearing stress τ_w at the plane wall is given by

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} = 0.221\mu \left(\frac{E^3}{\nu^5 x^5} \right)^{1/4} \qquad \text{...(26)}$$

Note: The students are advised to provide the missing steps of results (24), (25) and (26) while writing the complete solution.

18.14. Circular jet (axially symmetrical jet) [Himanchal 2000, 02, 03, 04]

The boundary layer equations in cylindrical polar co-ordinates for the given axially -symmetrical flow (for which $q_\theta = 0$ and all variables are independent of θ), in the absence of body forces and for steady flow, are given by (Refer Art 14.11)

$$\frac{\partial}{\partial r}(r\,q_r)+\frac{\partial}{\partial z}(r\,q_z)=0$$

$$\partial p/\partial r=0$$

$$q_z\frac{\partial q_z}{\partial z}+q_r+\frac{\partial q_z}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r}\right)$$

If a fluid is issuing from a small circular orifice in a wall and is mixing with the same surrounding fluid at rest, then the above equations reduces to

$$\frac{\partial}{\partial r}(r\,q_r)+\frac{\partial}{\partial z}(r\,q_z)=0 \qquad ...(1)$$

and
$$q_z\frac{\partial q_z}{\partial z}+q_r\frac{\partial q_z}{\partial r}=\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r}\right), \qquad ...(2)$$

where the pressure gradient $\partial p/\partial r$ is zero, because the constant pressure in the surrounding fluid impress itself on the jet. For the present discussion, we take z-axis along the jet and denote distance from z-axis by r. The boudary conditions for the present problem are

$q_r = 0$, $\quad \partial q_z/\partial r=0 \quad$ when $\quad r=0 \quad$ and $\quad q_z = 0 \quad$ when $\quad r=\infty$, ...(3)

Since the pressure is constant in the flow field and the motion is steady, the rate at which momentum flows across a section z = constant must be constant, that is,

$$\int_0^\infty (\rho\,q_z^2)\times(2\pi\,r\,dr)=\text{constant}=M_0, \text{ say} \qquad ...(4)$$

The value of M_0 should be prescribed. Let ψ be the stream function such that

$q_z=(1/r)\times(\partial\psi/\partial r) \qquad$ and $\qquad q_r=-(1/r)\times(\partial\psi/\partial z)$ ...(5)

We now proceed to find a similar solution of the given problem by taking

$\psi=\nu z\,g(\eta) \qquad$ when $\qquad \eta=(1/\sqrt{\nu})\times(r/z)$ ...(6)

(5) and (6) $\Rightarrow \qquad q_z=(g'/\eta z) \qquad$ and $\qquad q_r=(\sqrt{\nu}/z)\times(g'-g/\eta)$, ...(7)

where a prime denots differentiation with respect to η.

Then the equaiton of continuity (1) is identically satisfied and the momentum equation (2) reduces to

$$-\frac{d}{d\eta}\left(\frac{gg'}{\eta}\right)=\frac{d}{d\eta}\left(g''-\frac{g'}{\eta}\right), \qquad ...(8)$$

with the boundary conditions: $\quad g = 0,\ g'=0$ when $\eta=0 \quad$ and $\quad g'=0\ \ \eta=\infty$. ...(9)

The first two boundary conditions follow from the consideration that at $\eta=0$, g'/η must remain finite (*i.e.*, axial velocity muat be finite) and both $(g'-g/\eta)$ and $d(g'/\eta)/d\eta$ i.e. $(g''/\eta-g'/\eta^2)$ must vanish.

Integrating (8), $\qquad g''-g'/\eta=-(gg'/\eta)+C$, C being an arbitrary constant ...(10)

Since $\quad (g''-g'/\eta)=0 \quad$ and $\quad gg'/\eta=0 \quad$ at $\eta=0$, (10) gives $\quad C = 0$.

Then (10) becomes $\qquad \eta g''-g'+gg'=0$

or
$$\eta\frac{d^2g}{d\eta^2}+\frac{dg}{d\eta}-2\frac{dg}{d\eta}+g\frac{dg}{d\eta}=0 \qquad \text{or} \qquad \frac{d}{d\eta}\left(\eta\frac{dg}{d\eta}\right)-2\frac{dg}{d\eta}+\frac{1}{2}\frac{dg^2}{d\eta}=0$$

Integrating, $\eta g' - 2g + (1/2) \times g^2 = C'$, C' being an arbitrary constant ...(11)

Since $g = g' = 0$ at $\eta = 0$ by (9), (11) gives $C' = 0$ Then (11) becomes

$$2\eta \frac{dg}{d\eta} = 4g - g^2 \qquad \text{or} \qquad 2\frac{d\eta}{\eta} = \frac{4dg}{2^2 - (g-2)^2}$$

Integrating, $$2\log\eta + 2\log(\alpha/2) = 4 \times \frac{1}{(2\times 2)} \log\frac{2+(g-2)}{2-(g-2)} = \log\frac{g}{4-g}$$

or $$\frac{\alpha^2\eta^2}{4} = \frac{g}{4-g} \qquad \text{or} \qquad \frac{\xi^2}{4} = \frac{g}{4-g}, \qquad ...(12)$$

taking $\xi = \alpha\eta$, α being an arbitrary constant ...(13)

Re-writing (12), $g = \xi^2/(1 + \xi^2/4)$ ...(14)

We now proceed to determine value of α using (6) and (7), (4) reduces to

$$\int_0^\infty \rho\left(\frac{g'}{\eta z}\right)^2 (2\pi\sqrt{\nu} z\eta)\sqrt{\nu}\, z\, d\eta = M_0 \qquad \text{or} \qquad \int_0^\infty \frac{g'^2}{\eta} d\eta = \frac{M_0}{2\pi\mu} \qquad ...(15)$$

where we have used the fact that $\nu = \mu/\rho$.

Now, (13) and (14) $\Rightarrow$ $g = (\alpha^2\eta^2)/(1+\alpha^2\eta^2/4)$. ...(16)

Differentiating, (16) w.r.t. 'η', we get

$$g' = \frac{2\alpha^2\eta(1+\alpha^2\eta^2/4) - (\alpha^2\eta/2)\times(\alpha^2\eta^2)}{(1+\alpha^2\eta^2/4)^2} = \frac{2\alpha\eta}{(1+\alpha^2\eta^2/4)^2} \qquad ...(17)$$

Substituting the above value of g' in (15), we get

$$\frac{M_0}{2\pi\mu} = \int_0^\infty \frac{1}{\eta} \times \frac{4\alpha^4\eta^2 d\eta}{(1+\alpha^2\eta^2/4)^4} = 8\alpha^2 \int_0^\infty \frac{(1/2)\times\alpha^2\eta\, d\eta}{(1+\alpha^2\eta^2/4)^4}$$

$$= 8\alpha^2 \int_0^\infty \frac{dt}{(1+t)^4} = \frac{8\alpha^2}{3}\left[-\frac{1}{(1+t)^3}\right]_0^\infty = \frac{8\alpha^2}{3}$$

[Putting $\alpha^2\eta^2/4 = t$ so that $(1/2)\times\alpha^2\eta\, d\eta = dt$]

Thus, $\alpha^2 = 3M_0/16\pi\mu$ or $\alpha = (3M_0/16\pi\mu)^{1/2}$ ...(18)

Using (16) and (17) in (7), we have

$$q_z = \frac{2\alpha^2}{z(1+\xi^2/4)^2} \qquad \text{and} \qquad q_r = \frac{\alpha\sqrt{\nu}}{z} \times \frac{\xi(1-\xi^2/4)}{(1+\xi^2/4)^2} \qquad ...(19)$$

The maximum velocity exists on the axis of the jet and is given by setting $\xi = 0$ in (19).

$$(q_z)_{\max} = 2\alpha^2/z = 3M_0/8\pi\mu z, \text{ using (18).}$$

Finally, the volume rate of discharge Q across any section of the jet is given by

$$Q = 2\pi\int_0^\infty q_z\, r\, dr = 8\pi\nu z,$$

which is independent of the flux of momentum M_0 in the jet *i.e.* of the pressure under which the fluid is forced through the opening in the wall.

18.15. Approximate solutions of boundary layer equations.

In general, the process of finding a complete solution of the boundary layer equations is very tedious and time consuming, and hence it cannot be employed in practice. The solutions obtained so far represent only very special cases. For engineering problems, it is often desirable to possess at least approximate methods of solution, to be applied in cases when an exact solution of the boundary layer equations consumes a lot of time. The basic concept of such approximate method is that the solutions are allowed to satisfy the differential equations on the average. Here we do not require that the differential equations to be satisfied by each fluid particle. For this we use well known Von Karman's integral equation (Refer Art. 18.16), which is obtained by integrating the boundary layer equations over the boundary layer thickness. For approximate solution we insist on satisfying the Von Karman's integral equation and the boundary conditions.

Remarks. Approximate methods are easy to apply. Although there are less accurate, yet this technique yields acceptable results.

18.16. Von Karman's integral equation (or condition) The momentum integral equation of the boundary layer. [Himanchal 2000; Meerut 1996; Agra 2010]

Prandtl's boundary layer equations for two-dimensional flow of incompressible fluid over a semi-infinite flat plate (refer Art. 18.6) are given by

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \qquad ...(1)$$

and

$$\partial u/\partial x+\partial v/\partial y=0. \qquad ...(2)$$

with boundary conditions: $u=v=0$ when $y=0$ and $u=U(x,t)$ when $y=\infty$ ...(3)

Moreover, we have

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}=-\frac{1}{\rho}\frac{dp}{dx}. \qquad ...(4)$$

Let δ be the boundary layer thickness. Then u varies from 0 to U as y varies from 0 to δ.

Let τ_δ = shearing stress (at $y=\delta$) $=\left(\mu\dfrac{\partial u}{\partial y}\right)_{y=\delta}$

Since the fluid is regarded as non-viscous outside the boundary layer, $\tau_\delta=0$ so that $\partial u/\partial y=0$ at $y=\delta$. Hence the boundary conditions may be written as

$u=v=0$, when $y=0$ and $u=U$ and $\partial u/\partial y=0$, when $y=\delta$ ...(5)

Now,

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u\frac{\partial u}{\partial x}+\frac{\partial(uv)}{\partial y}-u\frac{\partial v}{\partial y}=u\frac{\partial u}{\partial x}+\frac{\partial(uv)}{\partial y}-u\left(-\frac{\partial u}{\partial x}\right), \text{ using (2)}$$

Thus,

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=2u\frac{\partial u}{\partial y}+\frac{\partial(uv)}{\partial y} \quad \text{or} \quad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial u^2}{\partial x}+\frac{\partial(uv)}{\partial y}. \qquad ...(6)$$

Using (6), (1) reduces to

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial(uv)}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \qquad ...(7)$$

Integrating (7) *w.r.t.* y from $y=0$ to $y=\delta$, we get

$$\int_0^\delta \frac{\delta u}{\delta t}dy+\int_0^\delta \frac{\partial u^2}{\partial x}dy+\left|uv\right|_0^\delta=-\frac{1}{\rho}\frac{dp}{dx}\int_0^\delta dy+\nu\left[\frac{\partial u}{\partial y}\right]_0^\delta$$

where we have used the fact that dp/dx is constant across any section. Re-writing and using the boundary conditions (5), the above equation reduces to

$$\frac{\partial}{\partial t}\int_0^\delta u dy + \frac{\partial}{\partial x}\int_0^\delta u^2\, dy + Uv_\delta = -\frac{\delta}{\rho}\frac{dp}{dx} - \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)_0 \qquad ...(8)$$

where the suffixes δ and 0 signify that the indicated quantities are to be evaluated at $y = \delta$ and $y = 0$ respectively.

Since $\displaystyle\int_0^\delta \frac{\partial v}{\partial y}dy = [uv]_0^\delta = v_\delta$ by using (5), we have

$$Uv_\delta = U\int_0^\delta \frac{\partial v}{\partial y}dy = -U\int_0^\delta \frac{\partial u}{\partial x}dy, \quad \text{using (2)}$$

$$= -U\frac{\partial}{\partial x}\int_0^\delta u dy \qquad ...(9)$$

Also, let $\tau_0 = \mu\,(\partial u/\partial y)_{y=0}$ = shearing stress at the wall ...(10)

Using (9) and (10), (8) becomes

$$\frac{\partial}{\partial t}\int_0^\delta u dy + \frac{\partial}{\partial x}\int_0^\delta u^2 dy - U\int_0^\delta \frac{\partial u}{\partial x}dy = -\frac{\delta}{\rho}\frac{dp}{dx} - \frac{\tau_0}{\rho}$$

$$\frac{\partial}{\partial t}\int_0^\delta u dy + \frac{\partial}{\partial x}\int_0^\delta u^2 dy - U\frac{\partial}{\partial x}\int_0^\delta u dy = -\frac{\delta}{\rho}\frac{dp}{dx} - \frac{\tau_0}{\rho}, \qquad ...(11)$$

which is one form of the Von Karman's integral equation.

Alternative forms of the Von Karman's integral equation:

Using (4), we have $\displaystyle -\frac{\delta}{\rho}\frac{dp}{dx} = \frac{\partial U}{\partial t}\delta + U\delta\frac{\partial U}{\partial x} = \frac{\partial U}{\partial t}\int_0^\delta dy + U\frac{\partial U}{\delta x}\int_0^\delta dy$

$$= \frac{\partial}{\partial t}\int_0^\delta U dy + \int_0^\delta U\frac{\partial U}{\partial x}dy \qquad ...(12)$$

Using (12), (11) reduces to

$$\frac{\partial}{\partial t}\int_0^\delta u\, dy + \frac{\partial}{\partial t}\int_0^\delta U\, dy - \frac{\partial}{\partial x}\int_0^\delta u^2 dy + U\frac{\partial}{\partial x}\int_0^\delta u\, dy + \int_0^\delta U\frac{\partial U}{\partial x}dy = \frac{\tau_0}{\rho}$$

or

$$\frac{\partial}{\partial t}\int_0^\delta (U-u)\,dy - \frac{\partial}{\partial x}\int_0^\delta u^2 dy + U\frac{\partial}{\partial x}\int_0^\delta u dy + \frac{\partial U}{\partial x}\int_0^\delta U dy = \frac{\tau_0}{\rho}. \qquad ...(13)$$

But $\displaystyle\frac{\partial}{\partial x}\int_0^\delta Uu\, dy = \int_0^\delta \frac{\partial}{\partial x}(Uu)dy = \int_0^\delta\left(u\frac{\partial U}{\partial x} + U\frac{\partial u}{\partial x}\right)dy = \frac{\partial U}{\partial x}\int_0^\delta u\, dy + U\frac{\partial}{\partial x}\int_0^\delta u\, dy$

Hence

$$U\frac{\partial}{\partial x}\int_0^\delta u dy = \frac{\partial}{\partial x}\int_0^\delta Uu\, dy - \frac{\partial U}{\partial x}\int_0^\delta u\, dy. \qquad ...(14)$$

Using (14), (13), reduces to

$$\frac{\partial}{\partial t}\int_0^\delta (U-u)dy - \frac{\partial}{\partial x}\int_0^\delta u^2 dy + \frac{\partial}{\partial x}\int_0^\delta Uu\, dy - \frac{\partial}{\partial x}\int_0^\delta u\, dy + \frac{\partial U}{\partial x}\int_0^\delta U dy = \frac{\tau_0}{\rho}$$

or

$$\frac{\partial}{\partial t}\int_0^\delta (U-u)\,dy + \frac{\partial U}{\partial x}\int_0^\delta u(U-u)\,dy + \frac{\partial U}{\partial x}\int_0^\delta (U-u)\,dy = \frac{\tau_0}{\rho}.$$

The above form permits us to take the limit $\delta \to \infty$ and obtain

$$\frac{\partial}{\partial t}\int_0^\infty (U-u)\,dy + \frac{\partial}{\partial x}\int_0^\infty u(U-u)\,dy + \frac{\partial U}{\partial x}\int_0^\infty (U-u)\,dy = \frac{\tau_0}{\rho}. \qquad ...(15)$$

The usual displacement thickness and the momentum thickness are given by

$$U\delta_1 = \int_0^\infty (U-u)\,dy \qquad ...(16)$$

and

$$U^2\delta_2 = \int_0^\infty u\,(U-u)\,dy. \qquad ...(17)$$

Using (16) and (17), (15) reduces to

$$\frac{\partial(U\delta_1)}{\partial t} + \frac{\partial(U^2\delta_2)}{\partial x} + U\delta_1\frac{\partial U}{\partial x} = \frac{\tau_0}{\rho} \quad \text{or} \quad \frac{\partial(U\delta_1)}{\partial t} + U^2\frac{\partial \delta_2}{\partial t} + 2U\delta_2\frac{\partial U}{\partial x} + U\delta_1\frac{\partial U}{\partial x} = \frac{\tau_0}{\rho}$$

or

$$\frac{1}{U^2}\frac{\partial(U\delta_1)}{\partial t} + \frac{\partial \delta_2}{\partial x} + \frac{1}{U}\frac{\partial U}{\partial x}(2\delta_2+\delta_1) = \frac{\tau_0}{\rho U^2}, \qquad ...(18)$$

which is used to determine boundary layer thickness.

Remark 1. Since the Von Karman's integral equation has one order less than the momentum equation, it follows that the calculation of boundary layer equations based on Von Karman's integral equations is quite simple. For this we assume an appropriate expression for the velocity function which must satisfy the no-slip condition at the wall and the conditions at the point where the potential flow solution exists.

Remark 2. Von Karman's integral equation for steady flow under no pressure gradient

Let $dp/dx = 0$. Then for steady flow (4) reduces to $U(\partial U/\partial x) = 0$ so that $\partial U/\partial x = 0$. Hence (15) reduces to

$$\tau_0 = \rho\frac{d}{dx}\int_0^\infty u(U-u)\,dy, \qquad ...(18)$$

which is the required *Von Karman's integral equation.*

18.16A. Momentum equation for boundary layer by Von Karman. Wall shear and drag force on a flat plate due to boundary layer (alternative method)

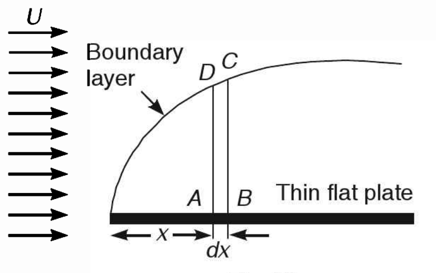


Fig. (*i*)

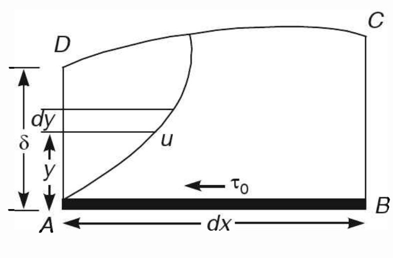


Fig. (*ii*)

Let an incompressible fluid of density ρ flow past a thin flat stationary plate of width b with free steam velocity equal to U as shown in figure (*i*). In figure (*i*), we have taken a small length dx of the plate at a distance x from the leading edge. The enlarged view of the small length of the plate is shown in figure (*ii*).

Consider the flow over a small distance dx. Let $ABCD$ be the control volume of the fluid over the distance dx as shown in figure (*ii*). The edge DC represents the outer edge of the boundary layer. Let u be velocity at any point within the boundary layer. Then, we have

Mass rate of fluid entering through the side AD

$$= \int_0^\delta (\rho \times \text{velocity} \times \text{area of strip of thickness } dy) = \int_0^\delta \rho u b\, dy = \rho b \int_0^\delta u\, dy$$

($\because$ fluid is incompressible $\Rightarrow$ ρ is constant)

Hence, mass rate of fluid leaving through the side BC $= \rho b \int_0^\delta u\, dy + \dfrac{d}{dx}\left\{\rho b \int_0^\delta u\, dy\right\} dx.$

From continuity equation for a steady incompressible fluid flow, we have

Mass rate of fluid entering through DC + Mass rate of fluid entering through AD

= Mass rate of fluid leaving through BC.

Hence, Mass rate of fluid entering through DC

= Mass rate of fluid leaving BC – Mass rate of fluid entering AD

$$= \rho b \int_0^\delta u\, dy + \frac{d}{dx}\left\{\rho b \int_0^\delta u\, dy\right\} dx - \rho b \int_0^\delta u\, dy = \frac{d}{dx}\left\{\rho b \int_0^\delta u\, dy\right\} dx. \quad \text{...(1)}$$

The fluid is entering through DC with a uniform velocity U. Hence using (1), we have

Momentum rate of fluid entering the control volume through DC in x-direction

$= \dfrac{d}{dx}\left\{\rho b \int_0^\delta u\, dy\right\} dx \times U = \rho b \dfrac{d}{dx}\left\{\int_0^\delta U u\, dy\right\} dx$, noting that ρ, b, U are constant quantities.

Momentum rate of fluid entering the control volume in x-direction through AD

$$= \int_0^\delta (\text{mass of strip} \times \text{velocity} = \int_0^\delta (\rho u b\, dy) \times u = \rho b \int_0^\delta u^2\, dy.$$

$\therefore$ Momentum rate of fluid leaving the control volume in x-direction through BC

$$= \rho b \int_0^\delta u^2\, dy + \frac{d}{dx}\left\{\rho b \int_0^\delta u^2\, dy\right\} dx = \rho b \int_0^\delta u^2\, dy + \rho b \frac{d}{dx}\left\{\int_0^\delta u^2\, dy\right\} dx$$

Thus, rate of change of momentum of control volume

= Momentum rate of fluid leaving through BC – Momentum rate of fluid entering through AD

– Momentum rate of fluid entering through DC

$$= \rho b \int_0^\delta u^2\, dy + \rho b \frac{d}{dx}\left\{\int_0^\delta u^2\, dy\right\} dx - \rho b \int_0^\delta u^2\, dy - \rho b \frac{d}{dx}\left\{\int_0^\delta U u\, dy\right\} dx$$

$$= \rho b \frac{d}{dx}\left\{\int_0^\delta u^2\, dy - \int_0^\delta U u\, dy\right\} dx = \rho b \frac{d}{dx}\left\{\int_0^\delta (u^2 - uU)\, dy\right\} dx. \quad \text{...(2)}$$

According to the momentum principle, we have

The rate of change of momentum on control volume $ABCD$ in any direction

= the total force on the control volume in the same direction. ...(3)

The only external force acting on the control volume is the shear force acting on the side AB in the direction from B to A as shown in figure (*ii*).

Then drag force (or shear force) ΔF_D on a small distance dx is given by $\Delta F_D = \tau_0 \times b \times dx$.
Hence, the total external force in the direction of rate of change of momentum = $-\tau_0\, b dx$. ...(4)

As per momentum principle given in (3), the two values given by equations (2) and (4) must be the same. Hence,

$$\rho b \frac{d}{dx}\left\{\int_0^\delta (u^2 - uU)\, dy\right\} dx = -\tau_0\, b dx$$

or $$\tau_0 = -\rho \frac{d}{dx}\left\{\int_0^\delta (u^2 - uU)\,dy\right\} = \rho \frac{d}{dx}\left\{\int_0^\delta (uU - u^2)\,dy\right\}$$

or $$\tau_0 = \rho \frac{d}{dx}\left\{\int_0^\delta U^2\left(\frac{u}{U} - \frac{u^2}{U^2}\right)dy\right\} = \rho U^2 \frac{d}{dx}\left\{\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy\right\}$$

or $$\frac{\tau_0}{\rho U^2} = \frac{d}{dx}\left\{\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy\right\}. \qquad ...(5)$$

Also $$\delta_2 = \text{momentum thickness} = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy.$$

$\therefore$ (5) can be rewritten as $$\frac{\tau_0}{\rho U^2} = \frac{d\delta_2}{dx}. \qquad ...(6)$$

Equation (6) is known as *Von Karman momentum integral equation* for boundary layer flow.

For a given verlocity profile in boundary layer, the shear stress τ_0 is given by (5) or (6). Then, drag force ΔF_D on a small distance dx of the plate in given by $\Delta F_D = \tau_0\, b dx$

$\therefore$ Total drag F_D on the plate of length L on one side is given by

$$F_D = \int \Delta F_D = \int_0^L \tau_0 b\, dx.$$

Some useful definitions:

(*i*) **Local coefficient of drag or coefficient of skin friction.** It is denotes by C_f and is defined as the ratio of the shear stress τ_0 to the quantity $(\rho U^2)/2$.

Thus, $$C_f = \frac{\tau_0}{(\rho U^2/2)}. \qquad ...(7)$$

(*ii*) **Co-efficient of drag.** It is denoted by C_D and is defined as the ratio of the total drag force to the quantity $(\rho A U^2)/2$.

Thus, $$C_D = \frac{F_D}{(\rho A U^2/2)}, \qquad ...(8)$$

where A = area of surface of the plate

18.17. Energy-integral equation for two-dimensional laminar boundary layers in incompressible flow. [Meerut. 2002, 03]

Prandtl's boundary layer equations for two-dimensional steady laminar flow of incompressible fluid over a semi-infinite plate are given by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad ...(1)$$

and $$\partial u/\partial x + \partial v/\partial y = 0. \qquad ...(2)$$

with boundary conditions : $u = v = 0$, when $y = 0$ and $u = U(x)$, when $y = \infty$ ...(3)

Moreover, we have $$U(dU/dx) = -(I/\rho) \times (dp/dx) \qquad ...(4)$$

Let δ be the boundary layer thickness. Then u varies from 0 to U as y varies from 0 to δ.

Let $$\tau_0 = \text{Shear stress (at } y = \delta) = \left(\mu \frac{\partial u}{\partial y}\right)_{y=\delta}$$

Since the fluid is regarded as non-viscous outside the boundary layer, $\tau_0 = 0$ so that $\partial u / \partial y = 0$ at $y = \delta$. Hence the boundary conditions may be written as

$$u = v = 0, \text{ when } y = 0 \qquad \text{and} \qquad u = U, \partial u / \partial y = 0, \text{ when } y = \delta \quad \text{...(5)}$$

Using (4), (1) may be re-written as $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad \text{...(6)}$$

Multiplying (6) by u and re-arranging, $$u\left(u\frac{\partial u}{\partial x} - U\frac{dU}{dx}\right) + uv\frac{\partial u}{\partial y} = \nu u\frac{\partial^2 u}{\partial y^2} \quad \text{...(7)}$$

Integrating (7) *w.r.t.* 'y', from $y = 0$ to $y = \delta$, we get

$$\int_0^\delta u\left(u\frac{\partial u}{\partial x} - U\frac{dU}{dx}\right) dy + \int_0^\delta uv\frac{\partial u}{\partial y} dy = \nu\int_0^\delta u\frac{\partial^2 u}{\partial y^2} dy \quad \text{...(8)}$$

Now, $$\int_0^\delta uv\frac{\partial u}{\partial y} dy = \int_0^\delta v\frac{\partial}{\partial y}\left(\frac{1}{2}u^2\right) dy = \frac{1}{2}\int_0^\delta v\frac{\partial}{\partial y}(u^2 - U^2)\, dy$$

$$= \frac{1}{2}\left[v(u^2 - U^2)\right]_0^\delta - \frac{1}{2}\int_0^\delta (u^2 - U^2)\frac{\partial v}{\partial y} dy, \text{ ntegrating by parts}$$

$$= -\frac{1}{2}\int_0^\delta (u^2 - U^2)\frac{\partial v}{\partial y} dy, \text{ using boundary condition (5)}$$

Thus, $$\int_0^\delta uv\frac{\partial u}{\partial y} dy = \frac{1}{2}\int_0^\delta (u^2 - U^2)\frac{\partial u}{\partial x} dy, \text{ using (2)} \quad \text{...(9)}$$

Again, $$\int_0^\delta u\frac{\partial^2 u}{\partial y^2} dy = \left[u\frac{\partial u}{\partial y}\right]_0^\delta - \int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy, \text{ integrating by parts}$$

Thus, $$\int_0^\delta u\frac{\partial^2 u}{\partial y^2} dy = -\int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy, \text{ using (5)} \quad \text{...(10)}$$

$\therefore$ Using (9) and (10), (8) reduces to

$$\int_0^\delta U\left(u\frac{\partial u}{\partial x} - U\frac{dU}{dx}\right) dy + \frac{1}{2}\int_0^\delta (u^2 - U^2)\frac{\partial u}{\partial x} dy = -\nu\int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy$$

Multiplying both sides by (– 2), we get

$$2\nu\int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy = -2\int_0^\delta u\left(u\frac{\partial u}{\partial x} - U\frac{\partial U}{\partial x}\right) dy - \int_0^\delta (u^2 - U^2)\frac{\partial u}{\partial x} dy$$

$$= \int_0^\delta \left(-2u^2\frac{\partial u}{\partial x} + 2Uu\frac{dU}{dx} - u^2\frac{\partial u}{\partial x} + U^2\frac{\partial u}{\partial x}\right) dy = \int_0^\delta \left(2Uu\frac{dU}{dx} + U^2\frac{\partial u}{\partial x} - 3u^2\frac{\partial u}{\partial x}\right) dy$$

$$= \int_0^\delta \frac{\partial}{\partial x}(U^2u - U^3)\,dy = \int_0^\delta \frac{\partial}{\partial x} U^3\left(\frac{u}{U} - \frac{u^3}{U^3}\right)dy = \frac{d}{dx}\left[U^3 \int_0^\delta \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right)dy\right] = \frac{d}{dx}(U^2\delta_3),$$

where δ_3 = the dissipation-energy thickness $= \int_0^\delta \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right)dy$

Thus,
$$\frac{d}{dx}(U^3\delta_3) = 2\nu \int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy, \qquad \text{...(11)}$$

which may be also re-written as
$$\frac{d}{dx}\left(\frac{1}{2}\rho U^3\delta_3\right) = \mu \int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy. \qquad \text{...(12)}$$

The upper limit of integration can be taken as $y = \infty$ in place of $y = \delta$ because the integrals reduce to zero outside the boundary layer. Hence (11) and (12) may be re-written as

$$\frac{d}{dx}(U^3\delta_3) = 2\nu \int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{1}{2}\rho U^3\delta_3\right) = \mu \int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy.$$

This equation expresses the physical fact that the rate of change of the flux of kinetic energy defect within the boundary layer is equal to the rate at which the kinetic energy is dissipated by viscosity. This equation is known as the *kinetic-energy integral equation.*

18.18. Application of the Von Karman's Integral equations to boundary Layer in absence of pressure-gradient. [Meerut 2004]

Consider a steady boundary layer flow over a semi-infinite plate. We take rectangular cartesian coordinates (x, y) with x measured in the plate in the direction of the two-dimensional laminar incompressible flow, and y measured normal to the plate. Let U be the velocity in the main stream and τ_0 be shearing stress at the plate. Then, in absence of pressure gradient, the Von Karman's integral equation gives

$$\tau_0 = \rho \frac{d}{dx}\int_0^\delta u\,(U - u)\,dy. \qquad \text{...(1)}$$

The displacement thickness δ_1 and the momentum thickness δ_2 are given by

$$U\delta_1 = \int_0^\delta (U - u)\,dy \qquad \text{...(2)}$$

and
$$U^2\delta_2 = \int_0^\delta u(U - u)\,dy. \qquad \text{...(3)}$$

Using (3), (1) reduce to
$$U^2\frac{d\delta_2}{dx} = \frac{\tau_0}{\rho}. \qquad \text{...(4)}$$

Let a suitable velocity distribution in the boundary layer be of the form

$$u = Uf(y/\delta) = Uf(\eta), \qquad \text{...(5)}$$

where
$$\eta = y/\delta. \qquad \text{...(6)}$$

From (3), (5) and (6), we have

$$U^2\delta_2 = \int_0^\delta u\,(U - u)\,dy = \int_0^1 Uf\,(U - Uf)\,\delta d\eta = U^2\delta\int_0^1 f(1 - f)\,d\eta$$

Hence, $U^2\delta_2 = U^2\delta\alpha$, taking $\alpha = \int_0^1 f(1 - f)d\eta$

Thus, we have $\delta_2 = \alpha\delta.$ ...(7)

Similarly, (2), (5) and (6) give

$$U\delta_1 = \int_0^\delta (U-u)\,dy = \int_0^1 (U-uf)\,\delta d\eta = \delta U \int_0^1 (1-f)\,d\eta$$

Hence, $U\delta_1 = \delta U \alpha_1$, taking $\alpha_1 = \int_0^1 (1-f)d\eta$

Thus, $\delta_1 = \alpha_1 \delta.$ ...(8)

Also, $\tau_0 = \mu\left(\dfrac{\partial u}{\partial y}\right)_{y=0} = \mu\left[\dfrac{U}{\delta} f'(\eta)\right]_{\eta=0}$

Again, $\tau_0 = \mu\left(\dfrac{\partial u}{\partial y}\right)_{y=0} = \mu\left[\dfrac{du}{d\eta}\cdot\dfrac{\partial \eta}{\partial y}\right]_{\eta=0}$, as $y=0 \Rightarrow \eta = 0$ by (6)

$$= \mu\left[Uf'(\eta)\cdot\frac{1}{\delta}\right]_{\eta=0} = \frac{\mu U}{\delta} f'(0).$$

Thus, $\tau_0 = (\mu U \beta)/\delta$, taking $f'(0) = \beta$ ...(9)

Using (7) and (9), (4) becomes

$$U^2 \frac{d(\alpha\delta)}{dx} = \frac{1}{\rho}\cdot\frac{\mu U\beta}{\delta} \qquad \text{or} \qquad \delta\frac{d\delta}{dx} = \frac{\beta\nu}{\alpha U} \qquad \left[\because \nu = \frac{\mu}{\rho}\right]$$

Integrating, $\dfrac{1}{2}\delta^2 = \dfrac{\beta\nu}{\alpha U}x + C$, C being an arbitrary constant ...(10)

Since $\delta = 0$ when $x = 0$, (10) gives $C = 0$. Thus, we obtain

$$\delta^2 = \frac{2\beta\nu}{\alpha U}x \qquad \text{so that} \qquad \delta = \sqrt{\frac{2\beta\nu}{\alpha U}}\sqrt{x} \qquad ...(11)$$

$\therefore$ From (9), $\tau_0 = \dfrac{\beta\mu U}{\sqrt{x}}\sqrt{\dfrac{\alpha U}{2\beta\nu}} = \dfrac{\mu U}{\nu\sqrt{x}}\sqrt{\dfrac{\alpha U \nu\beta}{2}} = \rho\left(\dfrac{\alpha\beta\nu U^3}{2x}\right)^{1/2}$ ...(12)

Now, the drag D of the plate of unit width and of length l is given by

$$D = \int_0^l \tau_0 dx = \int_0^l \rho\left(\frac{\alpha\beta\mu U^3}{2\rho}\right)^{1/2}\cdot x^{-1/2}dx = \left(\frac{2\alpha\beta\mu\rho}{U^3}\right)^{1/2} \qquad ...(13)$$

Hence the total drag of the plate wetted on both sides is $2D$.

Deduction I. Let the velocity distribution be linear in the boundary layer, that is

$f(\eta) = \eta$ for $\eta < 1$ and $f(\eta) = 1$, for $\eta \geq 1$. ...(14)

Then $\alpha = \int_0^1 \eta(1-\eta)\,d\eta = \dfrac{1}{6}$, $\alpha_1 = \int_0^1 (1-\eta)\,d\eta = \dfrac{1}{2}$ and $\beta = f'(0) = 1.$

$\therefore$ So, using (11), we get $\delta = 2\sqrt{3}\,(\nu x/U)^{1/2} = 3.464 \times (\nu x/U)^{1/2}$

$$\delta_1 = \alpha_1\delta = \frac{1}{2}\times 3.464\sqrt{\frac{\nu x}{U}} = 1.732\sqrt{\frac{\nu x}{U}}, \qquad \delta_2 = \alpha\delta = \frac{1}{6}\times 3.464\sqrt{\frac{\nu x}{U}} = 0.578\sqrt{\frac{\nu x}{U}}.$$

Deduction II. Taking a third degree polynomial in η with the conditions

$f(\eta) = 0$ when $\eta = 0$; $\qquad f(\eta) = 1$ and $f'(\eta) = 0$ when $\eta = 1$.

we write
$$f(\eta) = \begin{cases} 3\eta/2 - \eta^3/2, & \text{for } 0 \le \eta \le 1 \\ 1, & \text{for } \eta \ge 1 \end{cases}$$

Then as before find α, α_1 and β. Thus, we obtain

$\delta = 4.64\,(\nu x/U)^{1/2}$, $\qquad \delta_1 = 1.74(\nu x/U)^{1/2}$ $\qquad$ and $\qquad \delta_2 = 0.645\,(\nu x/U)^{1/2}$

Deduction III. Taking $f(\eta) = \sin(\pi\eta/2)$ which satisfies all the necessary conditions, verify that

$\alpha = (4 - \pi)/2\pi$, $\qquad \beta = \pi/2$, $\qquad \delta = 4.8\,(\nu x/U)^{1/2}$

18.19. Application of the Von Karman's Integral equation to boundary layer with pressure gradient. Von Karman-Pohlhausen method.

Von Karman's integral equation of steady flow is given by [see Eq. (18) in Art. 18.16]

$$U^2\frac{d\delta_2}{dx} + (2\delta_2 + \delta_1)U\frac{dU}{dx} = \frac{\tau_0}{\rho}, \qquad \text{...(1)}$$

where ρ is the density of the fluid, τ_0 is the shearing stress at the plate, U is the potential flow velocity and δ_1, δ_2 are respectively the displacement and momentum thickness. Let δ be the boundary layer thickness, and let $\eta = y/\delta$ be a new variable. At Karman's suggestion, Pohlhausen expressed the velocity distribution in the boundary layer as a fourth-degree polynomial, namely,

$$u/U = f(\eta) = a\eta + b\eta^2 + c\eta^3 + d\eta^4, \qquad \text{...(2)}$$

$0 \le \eta \le 1$. The boundary condition which are satisfied by the desired velocity distribution are as follows : At the edge of the boundary layer, $y \to \delta$, we have $u \to U$ and hence $\partial u/\partial y = \partial^2 u/\partial y^2 = 0$.

At the wall $y = 0$, the primary boundary conditions are $\qquad u = v = 0$. $\qquad$ ...(3)

Prandtl's boundary layer equation for steady flow [See Eqs. (10) and (11) in Art. 18.6]

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad \text{...(4)}$$

with
$$\frac{1}{\rho}\frac{dp}{dx} = -U\frac{dU}{dx}. \qquad \text{...(5)}$$

Using boundary conditions (3) at $y = 0$, (4) gives

$$0 = -\frac{1}{\rho}\frac{dp}{dx} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} \qquad \text{or} \qquad 0 = U\frac{dU}{dx} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}, \quad \text{using (5)}$$

Thus,
$$\frac{\partial^2 u}{\partial y^2} = -\frac{U}{\nu}\frac{dU}{dx}, \quad \text{at } y = 0$$

Thus Pohlhausen chose the four coefficients in (2) to satisfy the following four boundary conditions :

$$\frac{\partial^2 u}{\partial y^2} = -\frac{U}{\nu}\frac{dU}{dx} \quad \text{when} \quad y = 0 \qquad \text{and} \qquad u = U, \frac{\partial u}{\partial y} = 0. \frac{\partial^2 u}{\partial y^2} = 0 \text{ when } y = \delta \qquad \text{...(6)}$$

Since $\eta = y/\delta$, $y = 0 \;\Rightarrow\; \eta = 0$ and $y = \delta \;\Rightarrow\; \eta = 1$.

Furthermore, $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \eta}\cdot\frac{1}{\delta}, \quad \text{as} \quad \eta = \frac{y}{\delta}$$

and $$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\delta}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \eta}\right) = \frac{1}{\delta}\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)\cdot\frac{\partial \eta}{\partial y} = \frac{1}{\delta^2}\frac{\partial^2 u}{\partial \eta^2}.$$

Using the above results, (6) may be re-written as

$$\frac{\partial^2 u}{\partial \eta^2} = -\frac{U\delta^2}{\nu}\frac{dU}{dx} \quad \text{when} \quad \eta = 0 \qquad \text{...(7)}$$

$$u = U \quad \text{when} \quad \eta = 1, \qquad \text{...(8)}$$

$$\partial u/\partial \eta = 0 \quad \text{when} \quad \eta = 1, \qquad \text{...(9)}$$

and $$\partial^2 u/\partial \eta^2 = 0 \quad \text{when} \quad \eta = 1. \qquad \text{...(10)}$$

From (2), $$\partial u/\partial \eta = U(a + 2b\eta + 3c\eta^2 + 4d\eta^3) \qquad \text{...(11)}$$

and $$\partial^2 u/\partial \eta^2 = U(2b + 6c\eta + 12d\eta^2). \qquad \text{...(12)}$$

$\therefore$ (7) and (12) $\Rightarrow$ $$2bU = -\frac{U\delta^2}{\nu}\frac{dU}{dx} \;\Rightarrow\; 2b = -\frac{\delta^2}{\nu}\frac{dU}{dx} = -\lambda \text{ (say)} \qquad \text{...(13)}$$

(8) and (2) $\Rightarrow$ $a + b + c + d = 1$ ...(14)

(9) and (11) $\Rightarrow$ $a + 2b + 3c + 4d = 0$ ...(15)

(10) and (12) $\Rightarrow$ $2b + 6c + 12d = 0$ ...(16)

Solving (13) to (16), we obtain

$$a = 2 + \lambda/6, \qquad b = -(\lambda/2), \qquad c = -2 + \lambda/2, \qquad d = 1 - (\lambda/6) \qquad \text{...(17)}$$

With these values, (2) reduces to $$u/U = f(\eta) = F(\eta) + \lambda G(\eta), \qquad \text{...(18)}$$

where $$F(\eta) = 2\eta - 2\eta^3 + \eta^4 \quad \text{and} \quad G(\eta) = (1/6)\times\eta(1-\eta)^3. \qquad \text{...(19)}$$

Then, $$U\delta_1 = \int_0^\delta (U - u)\,dy \quad \text{or} \quad \delta_1 = \int_0^1 \left(1 - \frac{u}{U}\right)\delta\, d\eta, \quad \text{as} \quad \eta = \frac{y}{\delta}$$

or $$\delta_1 = \int_0^1 [1 - F(\eta) - \lambda G(\eta)], \text{ using (18)}$$

or $$\delta_1 = \int_0^1 \left[1 - 2\eta + 2\eta^3 - \eta^4 - (1/6)\times\lambda\eta(1-\eta)^3\right] d\eta, \text{ using (19)}$$

or $$\delta_1/\delta = (3/16 - \lambda/120) = a, \text{ say,} \quad \text{[on simplification]} \qquad \text{...(20)}$$

and $$U^2\delta_2 = \int_0^\delta (U-u)\,u\,dy \qquad \text{or} \qquad \delta_2 = \int_0^1 \left(1-\frac{u}{U}\right)\cdot\frac{u}{U}\,\delta\,dy$$

or $$\delta_2 = \delta\int_0^1 [1-F(\eta)-\lambda G(\eta)]\,[F(\eta)+\lambda G(\eta)]\,d\eta$$

or $$\frac{\delta_2}{\delta} = \left(\frac{37}{315}-\frac{\lambda}{945}-\frac{\lambda^2}{9072}\right) = b, \text{ say} \qquad \text{...(21)}$$

[on putting values of $F(\eta)$ and $G(\eta)$ and then integrating and simplifying]

Finally $$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu\left(\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)_{\eta=0}, \quad \text{as} \quad \eta = \frac{y}{\delta}$$

$$= \mu\left(\frac{\partial u}{\partial \eta}\cdot\frac{1}{\delta}\right)_{\eta=0} = \frac{\mu}{\delta}\times aU, \text{ using (11)}$$

Thus, $$\tau_0 = \frac{\mu U}{\delta}\left(2+\frac{\lambda}{6}\right), \text{ using (17)} \qquad \text{...(22)}$$

Now multiplying (1) by $\delta_2/\nu U$, we obtain

$$\frac{U}{\nu}\delta_2\frac{d\delta_2}{dx} + \left(2+\frac{\delta_1}{\delta_2}\right)\frac{dU}{dx}\frac{\delta_2^2}{\nu} = \frac{\tau_0\delta_2}{\mu U} \qquad \text{or} \qquad \frac{U}{2}\frac{d}{dx}\left(\frac{\delta_2^2}{\nu}\right) + \left(2+\frac{\delta_1}{\delta_2}\right)\frac{\delta_2^2}{\nu}\frac{dU}{dx} = \frac{\tau_0\delta_2}{\mu U}. \quad \text{...(23)}$$

But $$\frac{\delta_2^2}{\nu}\frac{dU}{dx} = \frac{b^2\delta^2}{\nu}\frac{dU}{dx}, \text{ using (21)}$$

$$= b^2\lambda, \text{ using (13)} \qquad \text{...(24)}$$

and $$\frac{\tau_0\delta_2}{\mu U} = \frac{\tau_0\delta b}{\mu U}, \quad \text{by (21)}$$

$$= (1/6)\times b(12+\lambda), \quad \text{by (22)} \qquad \text{...(25)}$$

Let $z = \delta_2^2/\nu$. Then using (24) and (25), (23) reduces to

$$dz/dx = (1/U)\times F(\lambda), \qquad \text{...(26)}$$

where $$F(\lambda) = 2\left(\frac{37}{915}-\frac{\lambda}{945}-\frac{\lambda^2}{9072}\right)\left\{2-\frac{116\lambda}{315}+\left(\frac{2}{945}+\frac{1}{120}\right)\lambda^2+\frac{\lambda^3}{9072}\right\}. \qquad \text{...(27)}$$

Since λ depends on x and the potential flow, (26) can be numerically integrated.

Particular Case. Consider boundary layer flow over a flat plate. Then U is constant and hence $dU/dx = 0$. Then (13) $\Rightarrow$ $\lambda = 0$. So from (26), we have

$$U(dz/dx) = F(0) = 0.469 \qquad \text{...(28)}$$

or $$U\frac{d}{dx}\left(\frac{\delta_2^2}{\nu}\right) = 0.469 \qquad \text{or} \qquad \frac{d\delta_2^2}{dx} = \frac{\nu}{U}\times 0.469$$

Integrating, $\delta_2^2 = 0.469 \times (\nu x / U) + A$, A being an arbitrary constant ...(29)

But $\delta_2 = 0$ when $x = 0$. Hence $A = 0$ from (29) and then we obtain

$$\delta_2^2 = 0.469 \times (\nu x / U) \quad \text{or} \quad \delta_2 = 0.685 \times (\nu x / U)^{1/2} \qquad ...(30)$$

For $\lambda = 0$, (20) and (21) respectively reduce to

$$\delta_1 = \frac{3\delta}{16} \quad \text{and} \quad \delta_2 = \frac{37\delta}{315} \quad \Rightarrow \quad \frac{\delta_1}{\delta_2} = \frac{3}{16} \times \frac{315}{37} \quad \text{or} \quad \delta_1 = \frac{3 \times 315}{16 \times 37} \delta_2$$

Using (30), $$\delta_1 = \frac{3 \times 315}{16 \times 37} \times 0.685 \sqrt{\frac{\nu x}{U}} = 1.75 \sqrt{\frac{\nu x}{U}}, \qquad ...(31)$$

Finally, $$\frac{\tau_0}{\rho U^2} = \frac{d\delta_2}{dx}, \quad \text{as} \quad \frac{dU}{dx} = 0 \quad \text{in} \quad (1)$$

Thus, $$\tau_0 = \rho U^2 \times 0.685 \sqrt{\frac{\nu}{U}} \times \frac{1}{2} x^{-1/2}, \quad \text{by (30)}$$

or $$\tau_0 = 0.343\, \rho \nu U \times (U / \nu x)^{1/2} = 0.343\, \mu U \times (U / \nu x)^{1/2} \qquad ...(32)$$

These values are in good agreement with the values obtained from the exact solution.

18.20. Illustrative solved examples

Ex. 1. *Show that*

$$(i) \int_0^\delta \frac{u}{U} dy = \delta - \delta_1 \qquad (ii) \int_0^\delta \left(\frac{u}{U}\right)^2 dy = \delta - \delta_1 - \delta_2 \qquad (iii) \int_0^\delta \left(\frac{u}{U}\right)^3 dy = \delta - \delta_1 - \delta_3,$$

where symbols have their usual meanings. **[Meerut 2002]**

Sol. (*i*) By definition of δ_1, we have

$$\delta_1 = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta dy - \int_0^\delta \frac{u}{U} dy = \delta - \int_0^\delta \frac{u}{U} dy.$$

Thus, $$\int_0^\delta \frac{u}{U} dy = \delta - \delta_1. \qquad ...(1)$$

(*ii*) By definition of δ_2, we have

$$\delta_2 = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^\delta \frac{u}{U} dy - \int_0^\delta \left(\frac{u}{U}\right)^2 dy$$

or $$\delta_2 = \delta - \delta_1 - \int_0^\delta \left(\frac{u}{U}\right)^2 dy, \quad \text{using (1)}$$

Thus, $$\int_0^\delta \left(\frac{u}{U}\right)^2 dy = \delta - \delta_1 - \delta_2. \qquad ...(2)$$

(*iii*) By definition of δ_3, we have

$$\delta_3 = \int_0^\delta \frac{u}{U}\left(1-\frac{u^2}{U^2}\right)dy = \int_0^\delta \frac{u}{U}dy - \int_0^\delta \left(\frac{u}{U}\right)^3 dy$$

or
$$\delta_3 = \delta - \delta_1 - \int_0^\delta \left(\frac{u}{U}\right)^3 dy, \text{ using (1)}$$

Thus,
$$\int_0^\delta \left(\frac{u}{U}\right)^3 dy = \delta - \delta_1 - \delta_3. \qquad ...(3)$$

Ex. 2. *Compute* δ_1, δ_2, *and* δ_3 *when the velocity distribution in the boundary layer is* $u = U(y/\delta)^n$.

Sol. Given that $u = U(y/d)^n$ so that $u/U = (y/\delta)^n$. ...(1)

$$\therefore \quad \delta_1 = \int_0^\delta \left(1-\frac{u}{U}\right)dy = \int_0^\delta \left(1-\frac{y^n}{\delta^n}\right)dy, \text{ using (1)}$$

$$= \left[y - \frac{1}{\delta^n}\frac{y^{n+1}}{n+1}\right]_0^\delta = \delta - \frac{\delta}{n+1} = \frac{n\delta}{n+1},$$

$$\delta_2 = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \int_0^\delta \left[\frac{u}{U} - \left(\frac{u}{U}\right)^2\right]dy = \int_0^\delta \left[\frac{y^n}{\delta^n} - \frac{y^{2n}}{\delta^{2n}}\right]dy, \text{ using (1)}$$

$$= \left[\frac{1}{\delta^n}\frac{y^{n+1}}{n+1} - \frac{1}{\delta^{2n}}\frac{y^{2n+1}}{2n+1}\right]_0^\delta = \frac{\delta}{n+1} - \frac{\delta}{2n+1} = \frac{n\delta}{(n+1)(2n+1)}$$

and
$$\delta_3 = \int_0^\delta \frac{u}{U}\left(1-\frac{u^2}{U^2}\right)dy = \int_0^\delta \left[\frac{u}{U} - \left(\frac{u}{U}\right)^3\right]dy = \int_0^\delta \left[\frac{y^n}{\delta^n} - \frac{y^{3n}}{\delta^{3n}}\right]dy, \text{ using (1)}$$

$$= \left[\frac{1}{\delta^n}\frac{y^{n+1}}{n+1} - \frac{1}{\delta^{3n}}\frac{y^{3n+1}}{3n+1}\right]_0^\delta = \frac{\delta}{n+1} - \frac{\delta}{3n+1} = \frac{2n\delta}{(n+1)(3n+1)}.$$

Ex. 3. *Find the displacement thickness* δ_1, *the momentum thickness* δ_2 *and energy thickness* δ_3 *for the velocity distribution in the boundary layer given by* $u/U = y/\delta$, *where u is the velocity at a distance y from the plate and u = U at* $y = \delta$ *where* δ *= boundary layer thickness. Also calculate* δ_1/δ_2.

Sol. Proceeding as in solved example 2, we obtain

$\delta_1 = \delta/2$, $\delta_2 = \delta/6$ and $\delta_3 = \delta/4$. Also, $\delta_1/\delta_2 = (\delta/2)/(\delta/6) = 3$.

Ex. 4. *Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by* $u/U = 2(y/\delta) - (y/\delta)^2$.

Sol. Given that $u/U = 2(y/\delta) - (y/\delta)^2$ ...(1)

Then, δ_1 = displacement thickness

$$= \int_0^{\delta} (1-u/U)\, dy = \int_0^{\delta} [1-\{2\,(y/\delta)-(y/\delta)^2\}]\, dy, \text{ using (1)}$$

$$= \left[y - \frac{2}{\delta}\cdot\frac{y^2}{2} + \frac{1}{\delta^2}\cdot\frac{y^3}{3} \right]_0^{\delta} = \delta - \delta + \frac{\delta}{3} = \frac{\delta}{3},$$

δ_2 = momentum thickness

$$= \int_0^{\delta} \frac{u}{U}\left(1-\frac{u}{U}\right) dy = \int_0^{\delta} \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right) dy, \text{ using (1)}$$

$$= \int_0^{\delta} \left(\frac{2y}{\delta}-\frac{5y^2}{\delta^2}+\frac{4y^3}{\delta^3}-\frac{y^4}{\delta^4}\right) dy = \left[\frac{2y^2}{2\delta}-\frac{5y^3}{3\delta^2}+\frac{4y^4}{4\delta^3}-\frac{y^5}{5\delta^4}\right]_0^{\delta}$$

$$= \delta - (5\delta/3) + \delta - (\delta/5) = 2\delta/15$$

δ_3 = energy thickness

$$= \int_0^{\delta} \frac{u}{U}\left(1-\frac{u^2}{U^2}\right) dy = \int_0^{\delta} \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left[1-\left(\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right)^2\right] dy, \text{ using (1)}$$

$$= \int_0^{\delta} \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left[1-\left(\frac{4y^2}{\delta^2}+\frac{y^4}{\delta^4}-\frac{4y^3}{\delta^3}\right)\right] dy$$

$$= \int_0^{\delta} \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}-\frac{8y^3}{\delta^3}+\frac{12y^4}{\delta^4}-\frac{6y^5}{\delta^5}+\frac{y^6}{\delta^6}\right) dy = \left[\frac{2y^2}{2\delta}-\frac{y^3}{3\delta^2}-\frac{8y^4}{4\delta^3}+\frac{12y^5}{5\delta^4}-\frac{6y^6}{6\delta^5}+\frac{y^7}{7\delta^6}\right]_0^{\delta}$$

$$= \delta - (\delta/3) - 2\delta + (12\delta/5) - \delta + (\delta/7) = 22\delta/105.$$

Ex.5 *The velocity distribution in the boundary layer is given by $u/U = (3/2) \times (y/\delta) - (1/2) \times (y/\delta)^2$, δ being boundary layer thickness. Calculate*

(*i*) *The ratio of displacement thickness to boundary layer thickness*

(*ii*) *The ratio of momentum thickness to boundary layer thickness*

Sol. Proceed as in Ex. 4 and obtain

(*i*) δ_1 = displacement thickness = $5\delta/12$ and required ratio = δ_1/δ = 5/12

(*ii*) δ_2 = momentum thickness = $19\delta/120$ and required ratio = δ_2/δ = 19/120.

Ex.6. *The velocity distribution in laminar boundary layer over a flat plate is assumed to be given by second order polynomial $u = a + by + cy^2$, determine its form using the necessary conditions.*

Sol. Given velocity distribution : $u = a + by + cy^2$. ...(1)

Then, veloicty distribution (1) must satisfy the following three conditions:

(*i*) At $y = 0$, $u = 0$. Hence (1) gives $0 = a$ so that $a = 0$. ...(2)

(*ii*) At $y = \delta$, $y = U$. Also, from (2), $a = 0$. Then (1) gives $b\delta + c\delta^2 = U$. ...(3)

(*iii*) At $y = \delta$, $dy/dy = 0$, Now from (1), $du/dy = b + 2cy$.

Hence $0 = b + 2c\delta$ or $b + 2c\delta = 0$. ...(4)

Solving (3) and (4), $b = 4U/\delta$ and $c = -U/\delta^2$.

Substituting the above values of a, b, c in (1), we get

$$u = (2U/\delta)y - (U/\delta^2)y^2 \quad \text{or} \quad u/U = 2(y/\delta) - (y/\delta)^2.$$

Ex.7.(*a*) *For the velocity profile for laminar boundary layer flows as* $u/U = 2(y/\delta) - (y/\delta)^2$, *find an expression for boundary layer thickness, shear stress, local co-efficient of drag and co-efficient of drag in terms of Reynold number.*

Sol. Given $u/U = 2(y/\delta) - (y/\delta)^2$ or $u = U\{2(y/\delta) - (y/\delta)^2\}$. ...(1)

(*i*) **To obtain boundary layer thickness (δ)**. We know that

$$\frac{\tau_0}{\rho U^2} = \frac{d}{dx}\left\{\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy\right\}. \quad ...(2)$$

According to Newton's law of viscosity, we have

$$\tau_0 = \mu\left(du/dy\right)_{y=0} \quad ...(3)$$

Form (1), $\dfrac{du}{dy} = U\left(\dfrac{2}{\delta} - \dfrac{2y}{\delta^2}\right)$ so that $\left(\dfrac{du}{dy}\right)_{y=0} = \dfrac{2U}{\delta}$.

Then, from (3), $\tau_0 = \mu(2U/\delta) = (2\mu U)/\delta$. ...(4)

Again, using (1), we have

$$\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left\{1 - \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\right\}dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left\{1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right\}dy,$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{4y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^4}{\delta^4}\right)dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{5y^2}{\delta^2} + \frac{4y^3}{\delta^3} - \frac{y^4}{\delta^4}\right)dy$$

$$= \left[\frac{2y^2}{2\delta} - \frac{5y^3}{3\delta^2} + \frac{4y^4}{4\delta^3} - \frac{y^5}{5\delta^4}\right]_0^\delta = \delta - \frac{\delta}{3} + \delta - \frac{\delta}{3} = \frac{2\delta}{15}$$

Then, (2) $\Rightarrow$ $\dfrac{\tau_0}{\rho U^2} = \dfrac{d}{dx}\left(\dfrac{2\delta}{15}\right)$ or $\tau_0 = \dfrac{2\rho U^2}{15}\dfrac{d\delta}{dx}$. ...(5)

Equating the two values of τ_0 given by (4) and (5), we get

$$\frac{2\rho U^2}{15}\frac{d\delta}{dx} = \frac{2\mu U}{\delta} \quad \text{or} \quad 2\delta d\delta = \frac{30\mu}{\rho U}dx.$$

Integrating, $\delta^2 = (30\mu/\rho U)x + C$, where C is an arbitrary constant. ....(6)

Since $\delta = 0$ when $x = 0$, hence (6) gives $0 = 0 + C$ so that $C = 0$. Then (6) reduces to

$$\delta^2 = (30\mu/\rho U)x, \qquad \text{giving} \qquad \delta = \sqrt{30}\,(\mu x/\rho U)^{1/2} \qquad ...(7)$$

or
$$\delta = 5.48\left(\frac{\mu x^2}{\rho U x}\right)^{1/2} = 5.48\left(\frac{x^2}{\text{Re}_x}\right)^{1/2} = 5.48\frac{x}{\sqrt{\text{Re}_x}}, \qquad ...(8)$$

where
$$\text{Re}_x = (\rho U x)/\mu = \text{Reynolds number}. \qquad ...(9)$$

From (7), we find that boundary layer thickness is proportional to the square root of the distance x from the leading edge. Again, (8) gives the boundary layer thickness in terms of Reynold number.

(*ii*) **To obtain shear stress τ_0.** Substituting the values of δ given by (8) in (4), we have

$$\tau_0 = \frac{2\mu U}{(5.18x)/\sqrt{\text{Re}_x}} = \frac{2\mu U\sqrt{\text{Re}_x}}{5.48x} = \frac{0.365\mu U}{x}\sqrt{\text{Re}_x}. \qquad ...(10)$$

(*iii*) **To obtain local co- efficient of drag (*i.e.* C_f).** It is given by

$$C_f = \frac{\tau_0}{(\rho U^2/2)} = \frac{(1/x)\times 0.365\mu U\sqrt{\text{Re}_x}}{(\rho U^2/2)}, \text{ using (10)}$$

$$= \frac{0.365\times 2}{(\rho U x)/\mu}\sqrt{\text{Re}_x} = \frac{0.73}{\text{Re}_x}\sqrt{\text{Re}_x}, \text{ using (9)}$$

Thus,
$$C_f = (0.73)/\sqrt{\text{Re}_x}. \qquad ...(11)$$

(*iv*) **To obtain co-efficient of drag (*i.e.* C_D).** It is given by

$$C_D = \frac{F_D}{(\rho A U^2)/2}, \text{ } A \text{ being area of the plate} \qquad ...(12)$$

Now, F_D = drag force on one side of the plate

$$= \int_0^L \tau_0 b\, dx, \text{ } L \text{ and } b \text{ being length and width respectively of the plate}$$

$$= \int_0^L \frac{0.365\mu U\sqrt{\text{Re}_x}}{x}\times b\times dx, \text{ using (10)} = 0.365\mu U b\int_0^L \sqrt{\frac{\rho U x}{\mu}}\frac{dx}{x}, \text{ using (9)}$$

$$= 0.365\mu U b\,(\rho U/\mu)^{1/2}\int_0^L x^{-1/2}dx = 0.365\mu U b\,(\rho U/\mu)^{1/2}\left[\frac{x^{1/2}}{1/2}\right]_0^L$$

$$= 0.365\mu U b\,(\rho U/\mu)^{1/2}\times 2L^{1/2} = 0.73\mu U b\,(\rho U L/\mu)^{1/2} \qquad ...(13)$$

Then, (12) $\Rightarrow$
$$C_D = \frac{0.73\mu U b\,(\rho U L/\mu)^{1/2}}{(\rho A U^2)/2} = \frac{1.46\mu b\,(\rho U L/\mu)^{1/2}}{\rho\times L b\times U^2}$$

or
$$C_D = \frac{1.46}{(\rho U L/\mu)}(\rho U L/\mu)^{1/2} = \frac{1.46}{(\rho U L/\mu)^{1/2}} = \frac{1.46}{\sqrt{\text{Re}_L}},$$

where
$$\text{Re}_L = (\rho U L)/\mu \text{ (Reynold number)} \qquad ...(15)$$

Ex. 7. (*b*) *For the velocity profile for laminar boundary layer flows as* $u/U = 2(y/\delta) - (y/\delta)^2$. *find thickness of boundary layer at the end of the plate and the drag force on one side of a plate 1 m long and 0.8 m wide when placed in water flowing with a velocity of 150 mm per second. Calculate the value of co-efficient of drag also. Take* μ *for water = 0.01 poise..*

Sol. Proceed is in Ex. 7(*a*) upto equaiton (15). For the given problem, length of plane = L = 1 m, width of plate = 0.8m, free-stream velocity = U = 150 mm/sec = 0.15 m/sec and μ = 0.01 pose = (0.01/10) (N – s)/m^2 = 0.001 (N – s)/m^2. Let Re_L be Reynold number at the end of the plate *i.e.*, at a distance of 1 m from the leading edge. Then, we have

$$\mathrm{Re}_L = \frac{\rho UL}{\mu} = \frac{1000 \times 0.15 \times 1.0}{.001} = 150000, \text{ as } \rho = 1000$$

Since $Re_L < 5 \times 10^6$, it follows that in the present case we have laminar boundary layer. Boundary layer thickness δ at x = 1.0 m is given by equaiton (8) of Ex. 7(a)

$$\therefore \qquad \delta = 5.48\frac{x}{\sqrt{\mathrm{Re}_x}} = \frac{5.48 \times 1.0}{\sqrt{150000}} = 0.01415 \text{ m} = 14.15 \text{ mm.}$$

Drag force F_D on one side of the plate is given by equation (13) of Ex. 7(*a*). Hence, we have

$$F_D = 0.73\mu Ub\,(\rho UL/\mu)^{1/2} = 0.73\mu Ub\sqrt{\mathrm{Re}_L}, \text{ using (15)}$$

$$= 0.73 \times 0.001 \times 0.15 \times 0.8 \times \sqrt{150000} = 0.0338 \text{ N.}$$

Co-efficient of drag C_D is given by equation (14) of Ex. 7(a). Thus, we have

$$C_D = \frac{1.46}{\sqrt{\mathrm{Re}_L}} = \frac{1.46}{\sqrt{150000}} = 0.00376.$$

Ex. 7. (*c*). *The velocity profile for laminar boundary is given by* $u/U = 2(y/\delta) - (y/\delta)^2$. *Find the thickness of boundary layer at the end of the plate and the drag force on one side of a plate 1.5 m long and 1 m wide when placed in water flowing with a velocity of 0.12 m/s. Calculate the value* of co-efficient of drag also. Take μ for water = 0.001 *N-s*/m^2.

Sol. Proceed as in Ex. 7(*b*) and obtain $\delta = 19.37$mm, F_D = 0.0372 N and C_D = 0.00344.

Ex. 8. (*a*) For the velocity profile for laminar boundary layer $u/U = (3/2) \times (y/\delta) - (1/2) \times (y/\delta)^3$, determine the boundary layer thickness, shear stress, drag force, local co-efficient of drag and co-efficient of drag in terms of Reynold number.

Sol. Here, $\quad u/U = (3/2) \times (y/\delta) - (1/2) \times (y/\delta)^3$

or $$u = U\{(3/2) \times (y/\delta) - (1/2) \times (y/\delta)^3\} \qquad ...(1)$$

(*i*) **To obtain boundary layer thickness δ.** We know that

$$\frac{\tau_0}{\rho U^2} = \frac{d}{dx}\left\{\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy\right\}. \qquad ...(2)$$

According to Newton's law of viscosity, $\qquad \tau_0 = \mu\,(du/dy)_{y=0} \qquad ...(3)$

From (1), $\qquad \dfrac{du}{dy} = U\left(\dfrac{3}{2\delta} - \dfrac{3y^2}{2\delta^3}\right) \qquad$ so that $\qquad \left(\dfrac{du}{dy}\right)_{y=0} = \dfrac{3U}{2\delta}.$

Then, from (3), $$\tau_0 = \mu(3U/2\delta) = (3\mu U)/2\delta \qquad ...(4)$$

$$\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \int_0^\delta \left(\frac{3y}{2\delta}-\frac{y^3}{2\delta^3}\right)\left\{1-\left(\frac{3y}{2\delta}-\frac{y^3}{2\delta^3}\right)\right\}dy = \int_0^\delta \left(\frac{3y}{2\delta}-\frac{y^3}{2\delta^3}\right)\left(1-\frac{3y}{2\delta}+\frac{y^3}{2\delta^3}\right)dy$$

$$= \int_0^\delta \left(\frac{3y}{2\delta}-\frac{9y^2}{4\delta^2}+\frac{3y^4}{4\delta^4}-\frac{y^3}{2\delta^3}+\frac{3y^4}{4\delta^4}-\frac{y^6}{4\delta^6}\right)dy$$

$$= \left[\frac{3y^2}{4\delta}-\frac{9y^3}{12\delta^2}+\frac{3y^5}{20\delta^4}-\frac{y^4}{8\delta^3}+\frac{3y^5}{20\delta^4}-\frac{y^7}{28\delta^6}\right]_0^\delta = \frac{3\delta}{4}-\frac{3\delta}{4}+\frac{3\delta}{20}+\frac{3\delta}{20}-\frac{\delta}{8}-\frac{\delta}{28}=\frac{39\delta}{280}.$$

Then, (2) $\Rightarrow$ $$\frac{\tau_0}{\rho U^2}=\frac{d}{dx}\left(\frac{39\delta}{280}\right) \quad \text{or} \quad \tau_0=\frac{39\rho U^2}{280}\frac{d\delta}{dx}. \qquad ...(5)$$

Equating the two values of τ_0 given by (4) and (5), we get

$$\frac{39\rho U^2}{280}\frac{d\delta}{dx}=\frac{3\mu U}{2\delta} \quad \text{or} \quad 2\delta\, d\delta=\frac{280\mu}{13\rho U}dx.$$

Integrating, $\delta^2=(280\mu/13\rho U)x+C$, C being an arbitrary constant ...(6)

Since $\delta=0$ when $x = 0$, hence (6) gives $C = 0$. Then, (6) reduces to

$$\delta^2=(280\mu/13\rho U)x \quad \text{or} \quad \delta=(280/13)^{1/2}(\mu x/\rho U)^{1/2} \qquad ...(7)$$

or $$\delta=4.64\left(\frac{\mu x^2}{\rho U x}\right)^{1/2}=4.64\left(\frac{x^2}{\mathrm{Re}_x}\right)^{1/2}=4.64\frac{x}{\sqrt{\mathrm{Re}_x}}, \qquad ...(8)$$

were $\mathrm{Re}_x=(\rho U x)/\mu$ = Reynold number. ...(9)

(*ii*) **To obtain shear stress τ_0.** Substituting the value of δ given by (8) in (4), we have

$$\tau_0=\frac{3\mu U}{2\times(4.64x)/\sqrt{\mathrm{Re}_x}}=\frac{0.323\mu U}{x}\sqrt{\mathrm{Re}_x}. \qquad ...(10)$$

(*iii*) **To obtain local co-efficient of drag *i.e.* C_f).** It is given by

$$C_f=\frac{\tau_0}{(\rho U^2/2)}=\frac{(1/x)\times 0.323\mu U\sqrt{\mathrm{Re}_x}}{(\rho U^2/2)}, \text{ using (10)}$$

$$=\frac{0.646}{(\rho U x)/\mu}\sqrt{\mathrm{Re}_x}=\frac{0.646}{\mathrm{Re}_x}\sqrt{\mathrm{Re}_x}=\frac{0.646}{\sqrt{\mathrm{Re}_x}}. \qquad ...(11)$$

(*iv*) **To obtain co-efficient of drag (*i.e.*, C_D).** It is given by

$$C_D=\frac{F_D}{(\rho A U^2)/2}, \; A \text{ being area of the plate} \qquad ...(12)$$

Now F_D = drag force on one side of the plate

$$=\int_0^L \tau_0 b\,dx, \; L \text{ and } b \text{ being length and width respectively of the plate}$$

$$= \int_0^L \frac{0.323\mu U\sqrt{\text{Re}_x}}{x} \times b \times dx, \text{ using (10) } = 0.323\mu Ub \int_0^L \sqrt{\frac{\rho Ux}{\mu}} \frac{dx}{x}, \text{ using (9)}$$

$$= 0.323\mu Ub\,(\rho U/\mu)^{1/2} \int_0^L x^{-1/2} dx = 0.323\mu Ub\,(\rho U/\mu)^{1/2} \left[\frac{x^{1/2}}{1/2}\right]_0^L$$

$$= 0.323\mu Ub\,(\rho U/\mu)^{1/2} \times 2L^{1/2} = 0.646\mu Ub\,(\rho UL/\mu)^{1/2}. \qquad ...(13)$$

Then, (12) $\Rightarrow$ $$C_D = \frac{0.646\mu Ub(\rho UL/\mu)^{1/2}}{(\rho AU^2/2)} = \frac{1.292\mu b(\rho UL/\mu)^{1/2}}{\rho \times Lb \times U^2}$$

or $$C_D = \frac{1.292}{(\rho UL/\mu)}\left(\frac{\rho UL}{\mu}\right)^{1/2} = \frac{1.292}{(\rho UL/\mu)^{1/2}} = \frac{1.292}{\sqrt{\text{Re}_L}},$$

where $\text{Re}_L = (\rho UL)/\mu$ (Reynold number) ...(15)

Ex. 8(*b*). *For the velocity profile in laminar boundary layer as, u/U=* $(3/2) \times (y/\delta) - (1/2) \times (y/\delta)^3$, *find the thickness of the boundary layer and the shear stress 1.5 m from the leading edge of a plate. The plate is 2 m long and 1.4 m wide and is placed in water which is moving with a velocity of 200 mm per second. Find the total force on the plate if* μ *for water = 0.01 pose.*

Sol. This problem is a particular case of Ex. 8(*a*). Here, we have x = distance form leading edge = 1.5 m, length of plate = L = 2m, width of plate = b = 1.4 m, free-steam velocity = U = 200 mm/s = 0.2 m/s, viscosity of fluid (water) = μ = 0.01 poise = 0.01/ 10 (N-s)/m^2 = 0.001 (N-s)/m^2.

From equation (9) of Ex. 8(a), we have

$$\text{Re}_x = \frac{\rho Ux}{\mu} = \frac{1000 \times 0.2 \times 1.5}{0.001} = 300000.$$

From equation (8) of Ex. 8(a), boundary layer thickness δ is given by

$$\delta = \frac{4.64x}{\sqrt{\text{Re}_x}} = \frac{4.64 \times 1.5}{\sqrt{300000}} = 0.0127\text{m} = 12.7\text{mm}.$$

Shear stress τ_0 is given by equaiton (10) of Ex. 8(a). Thus, we have

$$\therefore \quad \tau_0 = \frac{0.323\mu U}{x}\sqrt{\text{Re}_x} = \frac{0.323 \times 0.001 \times 0.2}{1.5}\sqrt{300000} = 0.0235\text{N/m}^2$$

Finally, drag force F_D on one side of the plate is given by equation (13) of Ex. 8(*a*). Thus,

$$F_D = 0.646\mu Ub\sqrt{\frac{\rho UL}{\mu}} = 0.646 \times 0.001 \times 0.2 \times 1.4 \times \sqrt{\frac{1000 \times 0.2 \times 0.2}{0.001}}$$

$$= 0.646 \times 0.001 \times 0.2 \times 1.4 \times \sqrt{400000} = 0.1138 \text{ N}.$$

$\therefore$ Total drag force = Drag force on both sides of the plate $= 2 \times F_D = 2 \times 0.1138 = 0.2276$ N.

Ex.9. *For the velocity profile for laminar boundary layer* $u/U = 2(y/\delta) - 2(y/\delta)^3 + (y/\delta)^4$, *find an expression for boundary layer thickness, shear stress, local co-efficient of drag, drag force on one side of the plate and co-efficient of drag in terms of Reynold number.*

Sol. Hint: Try yourself as in Ex. 7(*a*) or Ex. 8(*a*).

Ans. If $\mathrm{Re}_x = (\rho U x/\mu)^{1/2}$ and $\mathrm{Re}_L = (\rho U L/\mu)^{1/2}$. Then, we have $\delta = (5.84x)/\sqrt{\mathrm{Re}_x}$, $\tau_0 = \{(0.343\mu U)/x\}\sqrt{\mathrm{Re}_x}$, $C_f = 0.686/\sqrt{\mathrm{Re}_x}$, $F_D = 0.686\rho U b\,\mathrm{Re}_L$ and $C_D = (1.372)/\sqrt{\mathrm{Re}_L}$.

Ex.10. (*a*). *For the velocity profile for laminar boudary layer* $u/U = \sin(\pi y/2\delta)$, *find an expression for boundary layer thickness, shear stress, local coefficient of drag, drag force on one side of the plate and coefficient of drag in terms of Reynold number.*

Sol. Given $\quad u/U = \sin(\pi y/2\delta) \quad$ or $\quad u = U\sin(\pi y/2\delta)$. ...(1)

(*i*) **To obtain boundary layer thickness.** We know that

$$\frac{\tau_0}{\rho U^2} = \frac{d}{dx}\left\{\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy\right\}. \qquad ...(2)$$

According to Newton's law of viscosity, we have

$$\tau_0 = \mu\,(du/dy)_{y=0} \qquad ...(3)$$

From (1), $\quad \dfrac{du}{dy} = \left(U \times \dfrac{\pi}{2\delta}\right)\cos\dfrac{\pi y}{2\delta} \quad$ so that $\quad \left(\dfrac{du}{dy}\right)_{y=0} = \dfrac{\pi U}{2\delta}$.

Then, from (3), $\quad \tau_0 = \mu(\pi U/2\delta) = (\mu\pi U)/2\delta$. ...(4)

Again, $\quad \displaystyle\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \int_0^\delta \sin\frac{\pi y}{2\delta}\left(1-\sin\frac{\pi y}{2\delta}\right)dy = \int_0^\delta\left(\sin\frac{\pi y}{2\delta} - \sin^2\frac{\pi y}{2\delta}\right)dy$

$$= \int_0^\delta\left[\sin\frac{\pi y}{2\delta} - \frac{1}{2}\left\{1-\cos\frac{2\pi y}{2\delta}\right\}\right]dy, \text{ as } \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$= \left[-\frac{\cos(\pi y/2\delta)}{(\pi/2\delta)} - \frac{y}{2} + \frac{\sin(\pi y/\delta)}{(\pi/\delta)}\right]_0^\delta = -\frac{\delta}{2} + \frac{1}{(\pi/2\delta)} = \frac{4-\pi}{2\pi}\delta.$$

Then, (2) $\Rightarrow \quad \dfrac{\tau_0}{\rho U^2} = \dfrac{d}{dx}\left\{\dfrac{(4-\pi)\delta}{2\pi}\right\} \quad \Rightarrow \quad \tau_0 = \dfrac{4-\pi}{2\pi}\rho U^2\dfrac{d\delta}{dx}$. ...(5)

Equating the two values of τ_0 given by (4) and (5), we get

$$\frac{4-\pi}{2\pi}\rho U^2\frac{d\delta}{dx} = \frac{\mu\pi U}{\delta} \quad \text{or} \quad 2\delta\, d\delta = \frac{2\pi^2}{4-\pi}\frac{\mu}{\rho U}dx.$$

Integrating, $\quad \delta^2 = \dfrac{2\pi^2}{4-\pi}\dfrac{\mu}{\rho U}x + C$, where C is an arbitrary constant. ...(6)

Since $\delta = 0$ when $x = 0$, hence (6) gives $C = 0$. Then, (6) reduces to

$$\delta^2 = \frac{2\pi^2}{4-\pi}\frac{\mu}{\rho U}x \quad \text{or} \quad \delta = \sqrt{\frac{2\pi^2}{4-\pi}}\sqrt{\frac{\mu x}{\rho U}} \qquad ...(7)$$

or

$$\delta = 4.795\sqrt{\frac{\mu x^2}{\rho U x}} = 4.795\sqrt{\frac{\mu}{\rho U x}}\;x = \frac{4.795x}{\sqrt{\mathrm{Re}_x}}, \qquad ...(8)$$

where $\quad \mathrm{Re}_x = (\rho U x)/\mu$ = Reynold number. ...(9)

(*ii*) **To obtain shear stress τ_0.** Substituting the value of δ given by (8) in (4), we have

$$\tau_0 = \frac{\mu\pi U}{2\times\{4.795x/\sqrt{\mathrm{Re}_x}\}} = \frac{\mu U\pi\sqrt{\mathrm{Re}_x}}{2\times 4.795x} = \frac{0.327\mu U}{x}\sqrt{\mathrm{Re}_x}. \qquad \text{...(10)}$$

(*iii*) **To obtain local co-efficient of drag (*i.e.* C_f).** It is given by

$$C_f = \frac{\tau_0}{(\rho U^2/2)} = \frac{(1/x)\times 0.327\mu U\sqrt{\mathrm{Re}_x}}{(\rho U^2/2)}, \text{ using (10)}$$

$$= \frac{0.654}{(\rho Ux/\mu)}\sqrt{\mathrm{Re}_x} = \frac{0.654}{\mathrm{Re}_x}\sqrt{\mathrm{Re}_x} = \frac{0.654}{\sqrt{\mathrm{Re}_x}}. \qquad \text{...(11)}$$

(*iv*) **To find drag force F_D on side of the plate.** It is given by

$$F_D = \int_0^L \tau_0 b\,dx, \ L \text{ and } b \text{ being length and width respectively of the plate.}$$

$$= \int_0^L \frac{0.327\mu U}{x}\sqrt{\mathrm{Re}_x}\times b\times dx, \text{ using (10) } = 0.327\mu Ub\int_0^L \sqrt{\frac{\rho Ux}{\mu}}\frac{dx}{x}, \text{ using (9)}$$

$$= 0.327\mu Ub\,(\rho U/\mu)^{1/2}\int_0^L x^{-1/2}dx = 0.327\mu Ub\,(\rho U/\mu)^{1/2}\left[\frac{x^{-1/2}}{1/2}\right]_0^L$$

$$= 0.327\times 2\mu Ub\,(\rho U/\mu)^{1/2}\times L^{1/2} = 0.654\mu Ub\,(\rho UL/\mu)^{1/2}. \qquad \text{...(12)}$$

Thus, $F_D = 0.654\mu Ub\sqrt{\mathrm{Re}_L}$, where $\mathrm{Re}_L = (\rho UL/\mu)^{1/2}$ = Renold number ...(13)

(*v*) **To obtain coefficient of drag (*i.e.* C_D).** It is given by

$$C_D = \frac{F_D}{(\rho AU^2)/2}, \text{ where } A = \text{area of the plate } (= Lb)$$

$$= \frac{0.654\mu Ub\sqrt{\mathrm{Re}_L}}{(\rho/2)\times Lb\times U^2} = \frac{1.31}{(\rho UL/\mu)}\sqrt{\mathrm{Re}_L} = \frac{1.31\sqrt{\mathrm{Re}_L}}{\mathrm{Re}_L}, \text{ by (12)}$$

Thus, $$C_D = 1.31/\sqrt{\mathrm{Re}_L}. \qquad \text{...(14)}$$

Ex. 10 (*b*). *Air flows over a plate 0.5 m long and 0.6 m wide with a velocity 4 m/s. The velocity profile is in the from $u/U = \sin(\pi y/2\delta)$. If $\rho = 1.24$ kg/m^3 and $\nu = 0.15\times 10^{-4}$ m^2/s, calculate:* (*i*) boundary layer thickness at the end of the plate, (*ii*) shear stress at 250 mm from the leading edge, and (*iii*) drag force on one side of the plate.

Sol. It is a particular case of Ex. 10(*a*). Here, we have length of plate = L = 0.5 m, width of plate = b = 0.6m, free-stream velocity of fluid (air) = U = 4 m/s, density of air = ρ = 1.24 kg/m^3, kinematic viscosity of air = $\nu = \mu/\rho = 0.15\times 10^{-4}$ m^2/s.

(*i*) **To find boundary layer thickness δ at the end of the plate.**

From relation (9) of Ex. 10 (a), we have –

$$\mathrm{Re}_x = \frac{\rho Ux}{\mu} = \frac{Ux}{(\mu/\rho)} = \frac{Ux}{\nu} = \frac{4\times 0.5}{0.15\times 10^{-4}} = 1.33\times 10^5.$$

Since $\mathrm{Re}_x < 5\times 10^5$, it follows that the boundary layer is laminar over the entire length of the plate. Again, by relation (8) of Ex. 10 (a), we have

$$\delta = \frac{4.795x}{\sqrt{\text{Re}_x}} = \frac{4.795 \times 0.5}{\sqrt{1.33 \times 10^5}} = 0.00657 \text{ m} = 6.57 \text{ mm}.$$

(*ii*) **To obtain shear stress τ_0 at x = 250 mm = 0.25 m from leading edge.**

From result (10) of Ex. 10 (a), we have

$$\tau_0 = \frac{0.327\mu U}{x}\sqrt{\text{Re}_x} = \frac{0.327\rho U}{x} \times \left(\frac{\mu}{\rho}\right) \times \sqrt{\frac{\rho U x}{\mu}} = \frac{0.327\rho U}{x} \times \nu \times \sqrt{\frac{Ux}{\nu}}$$

$[\because \ \nu = \text{kinematic viscosity} = \mu/\rho]$

$$= \frac{0.327 \times 1.24 \times 4 \times 0.15 \times 10^{-4}}{0.25}\sqrt{\frac{4 \times 0.25}{0.15 \times 10^{-4}}} = 0.025 \text{ N/m}^2.$$

(*iii*) **To find drag force F_D on one side of the plate.** Using relation (12) of Ex. 10 (a),

$$F_D = 0.654\mu U b\sqrt{\frac{\rho UL}{\mu}} = 0.654\left(\frac{\mu}{\rho}\right)\rho U b\sqrt{\frac{UL}{\mu/\rho}} = 0.654\nu\rho U b\left(\frac{UL}{\nu}\right)^{1/2}$$

$$= 0.654 \times 0.15 \times 10^{-4} \times 1.24 \times 4 \times 0.6 \times \left(\frac{4 \times 0.5}{0.15 \times 10^{-4}}\right)^{1/2} = 0.01069 \text{ N}.$$

Ex. 11. *Show that for a flat plate of length l placed lengthwise in a uniform stream, the assumption* $u = (Uy/\delta^2) \times (2\delta - y)$ *s in Karman's integral conditions leads to* $\delta = \sqrt{(30\nu x/U)}$ *with a frictional resistance* $8\sqrt{(\mu\rho l U^3/30)}$ *where U is the velocity in the uniform stream.*

[Agra 2006, 08; Meerut 1996, 97, 2003]

Sol. The Karman's integral condition is (Refer result (11) of Art. 18.16)

$$\frac{\partial}{\partial t}\int_0^\delta u\,dy + \frac{\partial}{\partial x}\int_0^\delta u^2 dy - U\frac{\partial}{\partial x}\int_0^\delta u\,dy = -\frac{\delta}{\rho}\frac{dp}{dx} - \frac{\tau_0}{\rho} \qquad \text{...(1)}$$

where U is determined by

$$\frac{\partial U}{\partial t} + U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dp}{dx} \qquad \text{...(2)}$$

Also, given

$$u = (Uy/\delta^2) \times (2\delta - y). \qquad \text{... (3)}$$

Then,

$$\frac{\partial}{\partial t}\int_0^\delta u\,dy = \int_0^\delta \frac{\partial u}{\partial t}dy = 0, \quad \text{as} \quad \frac{\partial u}{\partial t} = 0 \ \text{by (3)} \qquad \text{...(4)}$$

Since U is constant, we have $\partial U/\partial t = 0$ and $dU/dx = 0$ so that $dp/dx = 0$ by using (2).

Then (1) reduces to

$$\tau_0 = \rho U\frac{d}{dx}\int_0^\delta u\,dy - \rho\frac{d}{dx}\int_0^\delta u^2 dy. \qquad \text{...(5)}$$

Using (3), (5) reduces to

$$\tau_0 = \rho U^2\frac{d}{dx}\left\{\frac{1}{\delta^2}\int_0^\delta y(2\delta - y)\,dy\right\} - \rho U^2\frac{d}{dx}\left\{\frac{1}{\delta^4}\int_0^\delta y^2(2\delta - y)^2\,dy\right\}$$

$$= \rho U^2\frac{d}{dx}\frac{1}{\delta^2}\left(\delta^3 - \frac{\delta^3}{3}\right) - \rho U^2\frac{d}{dx}\frac{1}{\delta^4}\left(\frac{4}{3}\delta^5 + \frac{\delta^5}{5} - \delta^5\right)$$

Thus, $$\tau_0 = \frac{2}{3}\rho U^2 \frac{d\delta}{dx} - \frac{8}{15}\rho U^2 \frac{d\delta}{dx} = \frac{2}{15}\rho U^2 \frac{d\delta}{dx}. \qquad \text{...(6)}$$

But $$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu\left[\frac{U}{\delta^2}(2\delta - 2y)\right]_{y=0}, \text{ using (3)}$$

Thus, $$\tau_0 = 2\mu U/\delta \qquad \text{...(7)}$$

From (6) and (7), we have

$$\frac{2}{15}\rho U^2 \frac{d\delta}{dx} = \frac{2\mu U}{\delta} \qquad \text{or} \qquad \delta\, d\delta = \frac{15\mu}{\rho U} dx$$

Integrating, $$\frac{1}{2}\delta^2 = \frac{15\mu}{\rho U}x + C,$$ C being an arbitrary constant

But $\delta = 0$ when $x = 0$. So $C = 0$ and hence the above equation gives

$$\frac{1}{2}\delta^2 = \frac{15\mu}{\rho U}x \qquad \text{so that} \qquad \delta = \sqrt{\frac{30\mu x}{\rho U}} \qquad \text{...(8)}$$

Using (8), (7) reduces to $$\tau_0 = 2\mu U\sqrt{\frac{\rho U}{30\mu x}} = 2\sqrt{\frac{\mu U^3 \rho}{30x}}$$

Hence the desired total resistance

$$= 2\int_0^l \tau_0 dx = 2\int_0^l 2\sqrt{\frac{\mu U^3\rho}{30}}\, dx = 4\sqrt{\frac{\mu U^3\rho}{30}}\int_0^l x^{-1/2} dx = 8\sqrt{\frac{\mu U^3 l\rho}{30}}$$

Ex. 12. *Discuss Prandtl's theory of the boundary layer and obtain the differential equations of motion in the layer along a plate.*

Show that at a distance x from the leading edge of a flat plate parallel to a stream of unbounded fluid moving outside the boundary layer with velocity U, the tangential stress on the plate is $(\alpha/4)\times(\rho\mu U^3/x)^{1/2}$, *where* $2\alpha^{-2/3} = \lim_{\eta\to\infty} F'(\eta)$ *and* $F(\eta)$ *is the solution of the equation* $F''' + FF'' = 0$, *for which* $F(0) = F'(0) = 1,\ F''(0) = 1.$

Sol. For the first part refer Art. 18.6.

Second part. Let a thin flat plate be immersed at zero incidence in a uniform stream which flows with speed U. Let there be unbounded fluid and let the origin of coordinates be taken at the leading edge, with x measured downstream along the plate and y perpendicular to it. In the absence of pressure gradient, the Prandtl's boundary layer equations are :

$$u\,(\partial u/\partial x) + v(\partial u/\partial y) + \nu(\partial^2 u/\partial y^2) \qquad \text{...(1)}$$

and $$\partial u/\partial x + \partial v/\partial y = 0 \qquad \text{...(2)}$$

Boundary conditions to be satisfied by u and v are :

$$u = v = 0 \quad \text{when} \quad y = 0 \qquad \text{and} \qquad u = U \quad \text{when} \quad y = \infty \qquad \text{...(3)}$$

The integration of (1) and (2) can be simplified by reducing the number of unknowns with help of the stream function : $u = \partial\psi/\partial y$ and $v = -(\partial\psi/\partial x)$ ...(4)

Then, (2) is satisfied automatically by (4). We now introduce a new dimensionless distance parameter $\eta = y/\delta$. Since the order of the boundary layer thickness is $(\nu x/U)^{1/2}$, we may take

$$\eta = \frac{1}{2}y\sqrt{\frac{U}{\nu x}} = \frac{1}{2}yx^{-1/2}\sqrt{\frac{U}{\nu}}. \qquad \text{...(5)}$$

In accordance with the procedure of the law of similarity, let the velocity distribution be given by
$$u/U = \phi(\eta). \qquad ...(6)$$

Using (4), (5) and (6), the stream function ψ is given by

$$\psi = \int u\,dy = \frac{U}{\sqrt{U/\nu x}}\int \phi(\eta)\,d\eta = \sqrt{U\nu x}\; f(\eta) \qquad ...(7)$$

where $f(\eta) = \int \phi(\eta)\,d\eta$. Then using (4), (5) and (7), we get

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = \sqrt{U\nu x}\; f'(\eta) \times \frac{1}{2}\sqrt{\frac{U}{\nu x}} = \frac{1}{2}Uf'(\eta) \qquad ...(8)$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{1}{2}\sqrt{U\nu}\; x^{-1/2} f(\eta) - \sqrt{U\nu x} f'(\eta)\left(-\frac{x^{-3/2}}{2}\right)\left(\frac{1}{2}\sqrt{\frac{U}{\nu}}\right)$$

or
$$v = \frac{1}{4}\sqrt{\frac{\nu U}{x}}\; y\sqrt{\frac{U}{\nu x}} f'(\eta) - \frac{1}{2}\sqrt{\frac{\nu U}{x}}\; f(\eta) = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\; [\eta f'(\eta) - f(\eta)]. \qquad ...(9)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}Uf''(\eta)\cdot\frac{1}{2}\left(-\frac{1}{2}x^{-3/2}\right)y\sqrt{\frac{U}{\nu}} = -\frac{U}{4x}\cdot\frac{y}{2}\sqrt{\frac{U}{\nu x}}\cdot f''(\eta) = -\frac{U\eta}{4x} f''(\eta) \qquad ...(10)$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}Uf''(\eta) \times \frac{1}{2}\sqrt{\frac{U}{\nu x}} \qquad ...(11)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}Uf'''(\eta) \times \left[\frac{1}{2}\sqrt{\frac{U}{\nu x}}\right]^2 = Uf'''(\eta)\cdot\frac{1}{8}\frac{U}{\nu x} \qquad ...(12)$$

Substituting (8) to (12) into (1) we get

$$\frac{1}{2}Uf'\left(-\frac{U\eta f''}{4x}\right) + \frac{1}{2}\sqrt{\frac{\nu U}{x}}\;(\eta f' - f) \times \frac{f''U}{4}\sqrt{\frac{U}{\nu x}} = \frac{\nu U^2 f'''}{8\nu x}$$

or
$$-\eta f' f'' + (\eta f' - f) f'' = f''' \qquad \text{or} \qquad ff'' + f''' = 0. \qquad ...(13)$$

Using (5), we see that $y = 0 \Rightarrow \eta = 0$ and $y = \infty \Rightarrow \eta = \infty$. Then from (8) and (9), we find that $u = 0$, $v = 0$ at $y = 0 \Rightarrow f = 0$, $f' = 0$ at $\eta = 0$. Furthermore, (8) shows that $u = U \Rightarrow f' = 2$. Hence the boundary conditions (3) may be re-written as

$$f = 0,\ f' = 0 \quad \text{when } \eta = 0 \quad \text{and} \quad f' = 2 \quad \text{when } \eta = \infty \qquad ...(14)$$

The shearing stress τ_0 at the wall is given by

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu\left(\frac{\partial u}{\partial y}\right)_{\eta=0} = \mu U f''(0) \times \frac{1}{4}\sqrt{\frac{U}{\nu x}}, \text{ by (11)} \qquad [\because\ y = 0 \Rightarrow \eta = 0]$$

Thus,
$$\tau_0 = \frac{1}{4}\rho\left(\frac{\nu U^3}{x}\right)^{1/2} f''(0). \qquad ...(15)$$

If $F(\eta)$ is any solution of (13), so also is $f(\eta) = a\ F(a\eta)$ where a is an arbitrary constant.

Then
$$\lim_{\eta\to\infty} f'(\eta) = a^2 \lim_{\eta\to\infty} F'(a\eta) = a^2 \lim_{\eta\to\infty} F'(\eta)$$

Since from (14), $f'(\infty) = 2$, hence we obtain $$2 = a^2 \lim_{\eta\to\infty} F'(\eta)$$

so that $$a = \{2 / \lim_{\eta\to\infty} F'(\eta)\}^{1/2}. \qquad ...(16)$$

Moreover, since $f(0) = f'(0) = 0$ by (14), we also have $F(0) = F'(0) = 0$.

Moreover $$f''(0) = a^3 F''(0).$$

For convenience we take $F''(0) = 1$ and obtain

$$f''(0) = a^3 = \{2 / \lim_{\eta\to\infty} F'(\eta)\}^{3/2}, \text{ using (16)} \qquad ...(17)$$

Using (17), (15) reduces to

$$\tau = \frac{1}{4}\rho\left(\frac{\nu U^3}{x}\right)^{1/2} \cdot \left\{\frac{2}{\lim\limits_{\eta\to\infty} F'(\eta)}\right\}^{3/2} = \frac{1}{4}\rho\left(\frac{\nu U^3}{x}\right)^{1/2} \alpha, \qquad ...(18)$$

where $$\alpha^{2/3} = \frac{2}{\lim\limits_{\eta\to\infty} F'(\eta)}$$ so that $$\lim_{\eta\to\infty} F'(\eta) = 2\alpha^{-2/3}$$

and $F(\eta)$ is a solution of (13) with boundary conditions

$$F(0) = F'(0) = 0 \qquad \text{and} \qquad F''(0) = 1$$

Note. For both sides of the wall of length l, the total drag D per unit breadth is given by

$$D = 2\int_0^l \tau_0\, dx = \frac{1}{2}\rho\alpha \int_0^l (\nu U^3 / x)^{1/2}\, dx, \text{ using (18)}$$

$$= \frac{1}{2}\rho U^2 l\,(Ul/\nu)^{-1/2} = \frac{\alpha\rho l U^2}{\sqrt{\text{Re}}}$$

where $\text{Re} = Ul/\nu$ (Reynold's number).

Ex. 13. *Based on the Von Karman integral relation, determine the local frictional coefficient C_f, for flow over a flat plate.*

Sol. For a flat plate the Von Karman integral equation [refer Art. 18.15] reduces

$$d\delta_2 / dx = \tau_0 / \rho U^2. \qquad ...(1)$$

The momentum thickness may be expressed as

$$\frac{\delta_2}{\delta} = \int_0^l (1 - u/U)\frac{u}{U}\, d\eta = k_1, \text{ say} \qquad ...(2)$$

where $\eta = y/\delta$. Substituting (2) into (1) we get $$k_1 (d\delta/dx) = \tau_0 / \rho U^2, \qquad ...(3)$$

where τ_0 is the shearing stress on the surface of the plate and is given by

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu\frac{U}{\delta}\left[\frac{\partial}{\partial\eta}\left(\frac{u}{U}\right)\right]_{\eta=0} = k_2\frac{\mu U}{\delta}, \qquad ...(4)$$

where $$k_2 = \left[\frac{\partial}{\partial\eta}\left(\frac{u}{U}\right)\right]_{\eta=0} \qquad ...(5)$$

Using (4), (3) reduces to $\quad k_2 \dfrac{d\delta}{dx} = \dfrac{1}{\rho U^2} \times \dfrac{k_2 \mu U}{\delta} \quad$ or $\quad 2\delta\, d\delta = \dfrac{2k_2}{k_1} \dfrac{\nu}{U} dx$

Integrating, $\quad \delta^2 = \dfrac{2k_2}{k_1}\left(\dfrac{\nu x}{U}\right) + C$, C being an arbitrary constant ...(6)

But when $x = 0$, $\delta = 0$ and hence $C = 0$. Then, (6) yields

$$\delta = \sqrt{\frac{2k_2}{k_1}\left(\frac{\nu x}{U}\right)}. \qquad \text{...(7)}$$

Using (7), (4) reduces to $\quad \tau_0 = k_2 \mu U \sqrt{\dfrac{k_1}{2k_2}\left(\dfrac{U}{\nu x}\right)}.$ ...(8)

Hence the local frictional coefficient is given by

$$C_f = \frac{\tau_0}{(\rho U^2/2)} = \frac{2}{\rho U^2} \times k_2 \mu U \sqrt{\frac{k_1}{2k_2}\left(\frac{U}{\nu x}\right)} = \sqrt{2k_1 k_2}\, \frac{\nu}{U} \sqrt{\frac{U}{\nu}}$$

$$= \sqrt{2k_1 k_2} \sqrt{\frac{\nu}{Ux}} = \sqrt{2k_1 k_2} \cdot \frac{1}{\sqrt{\mathrm{Re}_x}} = \sqrt{\frac{2k_1 k_2}{\mathrm{Re}_x}},$$

where $\quad \mathrm{Re}_x = (Ux)/\nu$ = Reynold's number

Ex. 14. *Determine the displacement thickness and momentum thickness for the laminar boundary layer on a flat plate for which the velocity distribution is given by the relation*

$$u/U = 2(y/\delta) - 2(y/\delta)^3 + (y/\delta)^4.$$ **[Meerut 2003]**

Sol. Given $\quad u/U = 2(y/\delta) - 2(y/\delta)^3 + (y/\delta)^4.$ ...(1)

The displacement thickness δ_1 is given by

$$\delta_1 = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left\{1 - 2\left(\frac{y}{\delta}\right) + 2\left(\frac{y}{\delta}\right)^3 - \left(\frac{y}{\delta}\right)^4\right\} dy$$

$$= \left[y - \frac{1}{\delta} y^2 + \frac{1}{2\delta^3} y^4 - \frac{1}{5\delta^4} y^5\right]_0^\delta = \frac{3\delta}{10}.$$

Again, the momentum thickness δ_2 is given by

$$\delta_2 = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left\{2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4\right\}\left\{1 - 2\left(\frac{y}{\delta}\right) + 2\left(\frac{y}{\delta}\right)^3 - \left(\frac{y}{\delta}\right)^4\right\} dy$$

$$= \int_0^\delta \left[2\left(\frac{y}{\delta}\right) - 4\left(\frac{y}{\delta}\right)^2 - 2\left(\frac{y}{\delta}\right)^3 + 9\left(\frac{y}{\delta}\right)^4 - 4\left(\frac{y}{\delta}\right)^5 - 4\left(\frac{y}{\delta}\right)^6 + 4\left(\frac{y}{\delta}\right)^7 - \left(\frac{y}{\delta}\right)^8\right] dy$$

$$= \left[\frac{1}{\delta} y^2 - \frac{4}{3\delta^2} y^3 - \frac{1}{2\delta^3} y^4 + \frac{9}{5\delta^4} y^5 - \frac{2}{3\delta^5} y^6 - \frac{4}{7\delta^6} y^7 + \frac{1}{2\delta^7} y^8 - \frac{1}{9\delta^8} y^9\right]_0^\delta$$

$$= \left(1 - \frac{4}{3} - \frac{1}{2} + \frac{9}{5} - \frac{2}{3} - \frac{4}{7} + \frac{1}{2} - \frac{1}{9}\right)\delta = \frac{37\delta}{315}.$$

Ex. 15. *For a velocity distribution in a boundary layer*

$$u/U = \alpha + \beta(y/\delta) + \gamma(y/\delta)^2 + \varepsilon(y/\delta)^3.$$

determine the constants $\alpha, \beta, \gamma, \varepsilon$. *Also, find displacement thickness and skin friction coefficient.* **[Himanchal 2001; Meerut 2004]**

Sol. Given $$u/U = \alpha + \beta(y/\delta) + \gamma(y/\delta)^2 + \varepsilon(y/\delta)^3. \qquad ...(1)$$

Differentiating (1) *w.r.t.* 'y', we have $$\frac{1}{U}\frac{du}{dy} = \frac{\beta}{\delta} + \frac{2\gamma}{\delta^2}y + \frac{3\varepsilon}{\delta^3}y^2 \qquad ...(2)$$

and $$\frac{1}{U}\frac{d^2u}{dy^2} = \frac{2\gamma}{\delta^2} + \frac{6\varepsilon}{\delta^3}y \qquad ...(3)$$

The boundary conditions are given by

when $y = 0$, $u = 0$ and $d^2u/dy^2 = 0$ ...(4A)

when $y = \delta$, $u = U$ and $du/dy = 0$ ...(4B)

Using (4A) and (4B), (1), (2) and (3) give $\alpha = 0$, $\gamma = 0$, $\beta = 3/2$ and $\varepsilon = -1/2$.

$\therefore$ From (1), $$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3. \qquad ...(5)$$

The displacement thickness is given by

$$\delta_1 = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(1 - \frac{3y}{2\delta} + \frac{y^3}{2\delta^3}\right) dy = \left[y - \frac{3}{4\delta}y^2 + \frac{1}{8\delta^3}y^4\right]_0^\delta = \frac{3\delta}{8} \qquad ...(6)$$

If δ_2 be the momentum thickness, τ_0 be the shearing stress at the plate, then the momentum integral equation is given by

$$\tau_0/\rho U^2 = d\delta_2/dx, \qquad ...(7)$$

where $$\tau_0 = \mu\left(\frac{du}{dy}\right)_{y=0} = \mu\left(\frac{3U}{2\delta}\right), \text{ by (5)} \qquad ...(8)$$

and $$\delta_2 = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(\frac{3y}{2\delta} - \frac{y^3}{2\delta^3}\right)\left(1 - \frac{3y}{2\delta} - \frac{y^3}{2\delta^3}\right) dy$$

Thus, $$\delta_2 = (39\delta/280), \text{ on simplification} \qquad ...(9)$$

$\therefore$ $$\frac{d\delta_2}{dx} = \frac{39}{280}\frac{d\delta}{dx}. \qquad ...(10)$$

From (7), (8) and (10), we have

$$\mu\left(\frac{3U}{2\delta}\right)\cdot\frac{1}{\rho U^2} = \frac{39}{280}\frac{d\delta}{dx}. \qquad \text{so that} \qquad 2\delta\, d\delta = \frac{280}{13}\frac{\nu}{U}dx$$

Integrating, $\delta^2 = (280\nu x)/13U + C$, C being an arbitrary constant. ...(11)

But $\delta = 0$ when $x = 0$. Hence (11) $\Rightarrow$ C = 0. Then, (11) reduces to

$$\delta^2 = (280\nu x)/13U \qquad \text{or} \qquad (\delta/x)^2 = (280\nu)/13Ux.$$

$\therefore$ $$\frac{\delta}{x} = \frac{4.64}{\sqrt{R_x}}, \qquad \text{where} \qquad R_x = \frac{x\rho U}{\mu} = \frac{Ux}{\nu} = \text{Reynold's number} \qquad ...(12)$$

$\therefore$ From (6), the displacement thickness $= \dfrac{3\delta}{8} = \dfrac{3}{8} \times \dfrac{4.64x}{\sqrt{R_x}} = \dfrac{1.74x}{\sqrt{R_x}}$, using (12)

Again, the skin-friction coefficient

$$= C_f = \frac{\tau_0}{(\rho U^2/2)} = \frac{(3\mu U)/2\delta}{(\rho U^2/2)} = \frac{3\mu}{\rho U} \times \frac{\sqrt{R_x}}{4.64x} = \frac{3}{4.64} \times \frac{\sqrt{R_x}}{R_x} = \frac{0.646}{\sqrt{R_x}}, \text{ by (8) and (12)}$$

Ex. 16. *A jet of air issues from a straight narrow slit in a wall and mixes with the surrounding air. If the motion is steady and two dimensional, show that at some distance from the slit the velocity along the axis of the jet is* $(3M^2/32\rho^2 \nu x)^{1/3} \operatorname{sech}^2[(M/48\rho\nu^2 x^2)^{1/3} y]$, *where M is the rate at which the momentum flows across unit length of a section of the jet. The axis of x is along and the axis of y perpendicular, to the axis of the jet.*

Sol. The equation of motion is $$u(\partial u/\partial x) + v(\partial u/dy) = \nu(\partial^2 u/\partial y^2) \qquad ...(1)$$

and the equation of continuity is $$\partial u/\partial x + \partial v/\partial y = 0 \qquad ...(2)$$

The pressure variation in the jet may be neglected, so that the equations of boundary layer holds good with the conditions :

on $y = 0$, $v = 0$, $\partial u/\partial y = 0$ ...(3A)

at $y = \infty$, $u = 0$ ...(3B)

From (2), we get $u = \partial \psi/\partial y$ and $v = -\partial \psi/\partial x$ ...(4)

where ψ is the stream function.

Let us put $$\eta = Ay/x^n \qquad ...(5)$$

and $$\psi = x^m f(\eta). \qquad ...(6)$$

Then, $$u = \partial\psi/\partial y = x^m f'(\eta) \times (A/x^n) = Ax^{m-n} f'(\eta), \text{ using (4), (5) and (6)} \qquad ...(7)$$

$$v = -\partial\psi/\partial x = -\left[m\, x^{m-1} f(\eta) - x^m f(\eta) \times (Any/x^{n+1})\right]$$

or $$v = -x^{m-1}\{mf(\eta) - n\,\eta\, f'(\eta)\}, \text{ using (5)} \qquad ...(8)$$

From (7), $$\partial u/\partial x = A\left[(m-n)x^{m-n-1} f' + x^{m-n} f''\, x - (nAy/x^{n+1})\right]$$

$$= Ax^{m-n-1}[(m-n)f' - n\,\eta\, f''] \qquad ...(9)$$

From (8), $$\partial u/\partial y = Ax^{m-n} f'' \times (A/x^n) = A^2 x^{m-2n} f'' \qquad ...(10)$$

From (10), $$\partial^2 u/\partial y^2 = A^2 x^{m-2n} f''' \times (A/x^n) = A^3 x^{m-3n} f''' \qquad ...(11)$$

Using (7), (8), (9), (10), (11) in (1), we get

$$Ax^{m-n} f' \cdot Ax^{m-n-1}[(m-n)f' - A\,n\,\eta\, f''] \; - x^{m-1}(mf - n\,\eta\, f') \cdot A^2 x^{m-2n} f'' = \nu A^3 x^{m-3} f'''$$

or $$A^2 x^{2m-2n-1} f'\{(m-n)f' - n\,\eta\, f''\} - A^2 x^{2m-2n-1} f''(mf - n\,\eta\, f') = \nu A^3 x^{m-3n} f''' \qquad ...(12)$$

Since all terms must have the same degree in x in (12), we get

$$2m - 2n - 1 = m - 3n \quad \text{so that} \quad m + n = 1. \qquad ...(13)$$

Now, $$M = \rho\int u \cdot (2udy) = 2\rho\int u^2 dy = 2\rho\int u^2 \frac{dy}{d\eta} d\eta = 2\rho A^2 x^{2m-2n} \int\left(f'^2 \frac{x^n}{A}\right) d\eta$$

Thus, $$M = 2\rho A x^{2m-n} \int f'^2 d\eta. \qquad ....(14)$$

Since the rate of flow of momentum M is constant, hence it must be independent of x, and so (14) gives $$2m - n = 0. \qquad ...(15)$$

Solving (13) and (15), $m = 1/3$ and $n = 2/3$. ...(16)

With these values of m and n, the equation (12) now becomes

$$f'\left(-\frac{1}{3}f' - \frac{2}{3}\eta f''\right) - f''\left(\frac{1}{3}f - \frac{2}{3}\eta f'\right) = A\nu f'''$$

or $$-f'^2 - 2\eta f' f'' - ff'' + 2\eta f' f'' = 3A\nu f'''. \qquad ...(17)$$

Let $$3A\nu = 1. \qquad ...(18)$$

Using (18), (17) reduces to

$$f''' + ff'' + f'^2 = 0 \qquad \text{or} \qquad f''' + d(ff')/d\eta = 0.$$

Integrating it, $f'' + ff' = C$, where C is an arbitrary constant. ...(19)

The boundary conditions are :

on $\eta = 0$, $\partial u/\partial y = 0$ so that $f'' = 0$; Also, $v = 0$ so that $f = 0$

at $\eta = \infty$, $f'(\infty) = 0$, *i.e.*, $f(0) = 0 = f''(0)$ and $f'(\infty) = 0$.

So, from (19), we have $C = 0$. Then, (19) becomes

$$f'' + ff' = 0. \qquad ...(20)$$

To solve (3), we put $$\zeta = \alpha\eta = (\alpha/3\nu)\times(y/x^{3/2}) \qquad ...(21)$$

and $$f(\eta) = 2\alpha F(\zeta). \qquad ...(22)$$

Then, $f'(\eta) = 2\alpha F'(\zeta)\,(d\zeta/d\eta) = 2\alpha^2 F'(\zeta)$ and $f''(\eta) = 2\alpha^3 F''(\zeta)$.

Hence (20) can be re-written as $$2\alpha^3 F'' + 2\alpha F\cdot 2\alpha^2 F' = 0$$

or $$F'' + 2FF' = 0 \qquad ...(23)$$

subject to the conditions $F(0) = 0 = F''(0)$ and $F'(\infty) = 0$. ...(24)

Since we have introduced an arbitrary constant α, we can take the solution for which

$$F''(0) = 1. \qquad ...(25)$$

All these conditions are satisfied by the solution $f = \tanh\zeta$. ...(26)

This makes $u = \dfrac{1}{3\nu}x^{-1/3}\cdot 2\alpha^2 F' = \dfrac{2\alpha^2}{3\nu}x^{-1/3}\operatorname{sech}^2\zeta$ and $dy = 3\nu x^{2/3}\dfrac{1}{\alpha}d\zeta$

$$\therefore \quad M = 2\rho\int_0^\infty \frac{4\alpha^2}{9\nu^2}x^{-2/3}\operatorname{sech}^4\zeta\cdot 3\nu x^{2/3}\frac{1}{\alpha}d\zeta = \frac{8\rho}{3\nu}\alpha^3\int_0^\infty \operatorname{sech}^4\zeta\, d\zeta, \text{ using (14)}$$

$$= (8\rho\alpha^3/3\nu)\times(2/3), \text{ on evaluating the definite integral}$$

Thus, $$\alpha^3 = (9\nu M)/16\rho. \qquad ...(27)$$

$\therefore$ From (21), we have

$$\zeta = \left(\frac{9\nu M}{16\rho}\right)^{1/3}\frac{1}{3\nu}\frac{y}{x^{2/3}} = \left(\frac{M}{48\rho\nu^2 x^2}\right)^{1/3} y$$

$$u = \frac{2}{3\nu}\left(\frac{9\nu M}{16\rho}\right)^{2/3} x^{-1/3}\,\mathrm{sech}^2\left\{\left(\frac{M}{48\rho\nu^2 x^2}\right)^{1/3} y\right\}$$

$$= \frac{2\times 3\times 3^{1/3}}{3\times\nu\times 4\times 4^{1/3}}\left(\frac{\nu^2 M^2}{\rho^2}\right)^{1/3} x^{-1/3}\,\mathrm{sech}^2\left\{\left(\frac{M}{48\rho\nu^2 x^2}\right)^{1/3} y\right\}$$

$$= \left(\frac{3M^2\nu^2}{32\rho^2\nu^3}\right)^{1/3} x^{-1/3}\,\mathrm{sech}^2\left\{\left(\frac{M}{48\rho\nu^2 x^2}\right)^{1/3} y\right\} = \left(\frac{3M^2}{32\rho^2\nu x}\right)^{1/3}\mathrm{sech}^2\left\{\left(\frac{M}{48\rho\nu^2 x^2}\right)^{1/3} y\right\}$$

Ex. 17. *Model aeroplane wings are tested in a wind tunnel under atmospheric pressure and with a wind velocity of 2m/s. If the aerofoil wing section can be treated as a flat plate of length* 14 *cm. and width 50 cm, calculate (i) the boundary layer thickness at the trailing edge, and (ii) the drag force exerted by the wind. The kinematic viscosity is* $1.5 \times 10^{-5} m^2/s$ *and the critical Reynolds number* $= \mathrm{Rc} = 5 \times 10^5$.

Sol. The Reynolds number Re at the trailing edge of the aeroplane is given by

$$\mathrm{Re} = \frac{Ux}{\nu} = \frac{2\times 0.15}{1.5\times 10^{-5}} = 2\times 10^4.$$

Since Re < Rc, it follows that the boundary layer is wholly laminar.

Now, δ = boundary layer thickness $= \dfrac{4.91x}{\sqrt{\mathrm{Re}}} = \dfrac{4.91\times 0.15\times 100}{\sqrt{2}\times 10^4} = 0.52\,\mathrm{cm}.$

δ_1 = displacement thickness $= \dfrac{1.729x}{\sqrt{\mathrm{Re}}} = \dfrac{1.729\times 0.15\times 100}{\sqrt{2}\times 10^4} = 0.18\,\mathrm{cm}$

C_f = The average drag coefficient $= \dfrac{1.328x}{\sqrt{\mathrm{Re}}} = \dfrac{1.328}{\sqrt{2}\times 10^4} = 0.0094.$

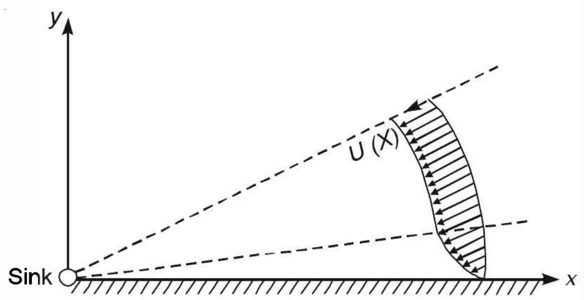
$\therefore$ The required drag force exerted by the wind

$$= C_f \times \frac{1}{2}\rho U^2 A = 0.0094\times\frac{1}{2}\times 1.22\times 2^2\times(2\times 0.5\times 0.15)\,\mathrm{N} = 0.0034\ \mathrm{N}$$

Ex. 18. *Fluid is flowing between two non-parallel plane walls, towards the intersection of the planes so that if x is measured along a wall from the intersection of the planes, U is negative and inversely proportional to x. Verify that a solution of the equations may be obtained in the form* $\dfrac{u}{U} = 3\tanh^2\left\{\beta + \left(\dfrac{|U|}{2\nu x}\right)^{1/2} y\right\} - 2,$ *where* $\tanh^2\beta = \dfrac{2}{3}$. **[Himanchal 1999]**

Sol. Consider the potential flow given by $U = -c/x$, ...(1)

were c is constant. With $c > 0$, (1) represents two dimensional motion in a convergent channel with flat walls (sink) as shown in the following figure.



Prandtl's boundary layer equation for steady motion are given by

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \qquad ...(2)$$

and

$$\partial u/\partial x+\partial v/\partial y=0 \qquad ...(3)$$

Also, we have

$$U\frac{dU}{dx}=-\frac{1}{\rho}\frac{dp}{dx}. \qquad ...(4)$$

From (2) and (4), we have

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \quad \text{or} \quad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{c^3}{x^3}+\nu\frac{\partial^2 u}{\partial y^2}, \text{ using (1)} \qquad ...(5)$$

Introduce the similarity transformation $\eta=(c/\nu)^{1/2}\times(y/x)$. ...(6)

In accordance with the procedure of the law of similarity, let the velocity profile be

$$\frac{u}{U}=f(\eta) \quad \text{so that} \quad u=-\frac{c}{x}f(\eta)\text{, using (1)} \qquad ...(7)$$

From (3),

$$\partial u/\partial x=-(\partial v/\partial y) \qquad ...(8)$$

Now, from (7), we have

$$\frac{\partial u}{\partial x}=\frac{c}{x^2}f(\eta)-\frac{c}{x}f'(\eta)\frac{d\eta}{\partial x}=\frac{c}{x^2}f(\eta)-\frac{c}{x}f'(\eta)\left(\frac{c}{\nu}\right)^{1/2}\left(-\frac{y}{x^2}\right)\text{, by (6)}$$

Thus,

$$\frac{\partial u}{\partial x}=\frac{c}{x^2}(f+\eta f')\text{, using (6)} \qquad ...(9)$$

Next,

$$\frac{\partial v}{\partial y}=\frac{\partial v}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}=\frac{\partial v}{\partial \eta}\left(\frac{c}{\nu}\right)^{1/2}\frac{1}{x}\text{, using (6)}$$

Substituting this value in (8), we get

$$\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial \eta}\left(\frac{c}{\nu}\right)^{1/2}\frac{1}{x}.$$

With this value of $\partial u/\partial x$, (9) reduces to

$$-\frac{\partial v}{\partial \eta}\left(\frac{c}{\nu}\right)^{1/2}\frac{1}{x}=\frac{c}{x^2}(f+\eta f') \quad \text{or} \quad \frac{\partial v}{\partial \eta}=-\frac{(c\nu)^{1/2}}{x}\frac{d}{d\eta}(f\eta).$$

On integration and omitting the constant of integration, the above equation leads to

$$v=-\frac{(c\nu)^{1/2}}{x}(f\eta). \qquad ...(10)$$

From (7), we have

$$\frac{\partial u}{\partial y}=-\frac{c}{x}f'(\eta)\frac{d\eta}{dy}=-\frac{c}{x}f'(\eta)\left(\frac{c}{\nu}\right)^{1/2}\frac{1}{x}\text{, using (6)} \qquad ...(11)$$

From (11),

$$\frac{\partial^2 u}{\partial y^2}=-\frac{c}{x^2}f''(\eta)\left(\frac{c}{\nu}\right)^{1/2}\frac{d\eta}{dy}=-\frac{c}{x^2}f''(\eta)\left(\frac{c}{\nu}\right)^{1/2}\left(\frac{c}{\nu}\right)^{1/2}\frac{1}{x}\text{, using (6)}$$

Thus,

$$\frac{\partial^2 u}{\partial y^2}=-\frac{c^2}{\nu x^3}f''. \qquad ...(12)$$

Using (7), (9), (10), (11) and (12), (5) reduces to

$$\left(-\frac{cf}{x}\right)\left\{\frac{c}{x^2}(f+\eta f')\right\}-\left\{\frac{(c\nu)^{1/2}}{x}(f\eta)\right\}\left\{-\frac{c}{x^2}f'(\eta)\left(\frac{c}{\nu}\right)\right\} = -\frac{c^2}{x^3}+\nu\left(-\frac{c^2}{\nu x^3}f''\right)$$

or $\quad -f(f+\eta f')+\eta ff' = -1-f'' \qquad$ or $\qquad f''-f^2+1=0. \quad$ ....(13)

We are now to solve (13) under the following boundary conditions :

$u = v = 0$ when $y = 0$ and $u = U$ when $y = \infty$ ...(14)

In view of (6) and (7), boundary conditions (14) may be re-written as

$y=0 \;\Rightarrow\; \eta=0;\quad u=0 \;\Rightarrow\; f(\eta)=0$ so that $f(0)=0$ ...(15)

Again, $y=\infty \;\Rightarrow\; \eta=\infty;\quad u=U \;\Rightarrow\; f(\eta)=1$ so that $f(\infty)=1$ ...(16)

Multiplying both sides of (13) by f', we get $\qquad f'f''-f^2f'+f'=0.$

Integrating the above equation, we get

$(1/2)\times f'^2-(1/3)\times f^3+f=A$, A being an arbitrary constant ...(17)

At the edge of the boundary layer, $\quad u = U \quad$ and $\quad \partial u/\partial y=0$

$\therefore$ From (7) and (11), $\quad f = 1 \quad$ and $\quad f'=0.$

Hence (17) gives $\quad 0-(1/3)+1=A \quad$ or $\quad A = 2/3.$

$\therefore$ from (17), $\quad \frac{1}{2}f'^2-\frac{1}{3}f^3+f=\frac{2}{3} \quad$ or $\quad \frac{3}{2}f'^2=2-3f+f^3$

or $\quad \frac{3}{2}\left(\frac{df}{d\eta}\right)^2=(f-1)^2(f+2) \quad$ or $\quad \sqrt{\frac{3}{2}}\,\frac{df}{d\eta}=(1-f)(f+2)^{1/2}$

[Since maximum value of $f(\eta)$ is 1, so $\{(f-1)^2\}^{1/2}=1-f$.]

or
$$d\eta=\sqrt{\frac{3}{2}}\,\frac{df}{(1-f)\sqrt{f+2}} \qquad \text{...(18)}$$

Put $f+2 = 3\tanh^2 t \quad$ so that $\quad df = 6\tanh t\,\operatorname{sech}^2 t\,dt$ ...(19)

and $\quad 1-f = 1-(3\tanh^2 t-2) \quad$ or $\quad 1-f=3(1+\tanh^2 t)=3\operatorname{sech}^2 t.$

(18) reduces to $\quad d\eta=\sqrt{\frac{3}{2}}\,\frac{6\tanh t\,\operatorname{sech}^2 t}{3\operatorname{sech}^2 t\cdot\sqrt{3}\tanh t}\,dt \quad$ or $\quad d\eta=\sqrt{2}\,dt$

Integrating, $\sqrt{2}\,t=\eta+\sqrt{2}\,\beta$, where $\sqrt{2}\,\beta$ is taken as the constant of integration.

or
$$t=\beta+\eta/\sqrt{2}.$$

$\therefore$ (19) reduces to $\quad f+2=3\tanh^2(\beta+\eta/\sqrt{2}) \quad$ or $\quad f=3\tanh^2(\beta+\eta/\sqrt{2})-2$

So by (6), $\quad f=3\tanh^2\left\{\beta+\frac{1}{\sqrt{2}}\left(\frac{c}{\nu}\right)^{1/2}\frac{y}{x}\right\}-2=3\tanh^2\left\{\beta+\frac{c^{1/2}}{\sqrt{2\nu}}\frac{y}{x}\right\}-2.$ ...(20)

From (1), $\quad c = -Ux \quad$ or $\quad c^{1/2}=(-U)^{1/2}x^{1/2}$

$\therefore \quad c^{1/2} = (\,|\,U\,|\,)^{1/2}\, x^{1/2}, \qquad [\because \;\; \text{U is negative} \;\Rightarrow\; |\,U\,| = -\,U]$

$\therefore \quad$ (20) reduces to

$$f = 3\tanh^2\left\{\beta + \frac{(|\,U\,|)^{1/2}x^{1/2}}{(2\nu)^{1/2}}\frac{y}{x}\right\} - 2 \quad \text{or} \quad f = 3\tanh^2\left\{\beta + \left(\frac{|\,U\,|}{(2\nu x)}\right)^{1/2} y\right\} - 2. \qquad \text{...(21)}$$

From boundary condition (15), when $y = 0, f = 0$. So (21) gives

$$0 = 3\tanh^2\beta - 2 \qquad \text{or} \qquad \tanh^2\beta = 2/3. \qquad \text{...(22)}$$

(21) and (22) together prove the required result.

Ex. 19. *Show that for a two-dimensoinally axially symmetric boundary layer flow*

$$\int_0^\infty (1 - u/U)^2 \frac{r}{a} dn = \delta_1 - \delta_2 \qquad \text{and} \qquad \int_0^\infty (1 - u/U)^3 \frac{r}{a} dn = \delta_1 - 3\delta_2 + \delta_3.$$

where n is the normal distance from the surface of the body, r is the axial distance and a is the reference radius, which may be function of the axial distance.

Sol. By definitions, we have (refer Art. 18.5 B)

$$\delta_1 = \int_0^\infty (1 - u/U)\frac{r}{a} dn,$$

$$\delta_2 = \int_0^\infty (1 - u/U)\frac{ru}{aU} dn \qquad \text{...(2)}$$

and

$$\delta_3 = \int_0^\infty \left(1 - u^2/U^2\right)\frac{ru}{aU} dn. \qquad \text{...(3)}$$

$$\therefore \quad \delta_1 - \delta_2 = \int_0^\infty \left(1 - \frac{u}{U}\right)\frac{r}{a} dn - \int_0^\infty \left(1 - \frac{u}{U}\right)\frac{ru}{aU} dn, \quad \text{by (1) and (2)}$$

$$= \int_0^\infty \left\{\left(1 - \frac{u}{U}\right)\frac{r}{a} - \left(1 - \frac{u}{U}\right)\frac{ru}{aU}\right\} dn = \int_0^\infty \left(1 - \frac{u}{U}\right)\frac{r}{a}\left(1 - \frac{u}{U}\right) dn = \int_0^\infty \left(1 - \frac{u}{U}\right)^2 \frac{r}{a} dn.$$

Hence, $$\int_0^\infty (1 - u/U)^2 \frac{r}{a} dn = \delta_1 = \delta_2.$$

Again, $\delta_1 - 3\delta_2 + \delta_3$

$$= \int_0^\infty \left(1 - \frac{u}{U}\right)\frac{r}{a} dn - 3\int_0^\infty \left(1 - \frac{u}{U}\right)\frac{ru}{aU} dn + \int_0^\infty \left(1 - \frac{u^2}{U^2}\right)\frac{ru}{aU} dn, \quad \text{using (1), (2) and (3)}$$

$$= \int_0^\infty \left(1 - \frac{u}{U}\right)\frac{r}{a}\left\{1 - 3\frac{u}{U} + \left(1 - \frac{u}{U}\right)\frac{u}{U}\right\} dn = \int_0^\infty \left(1 - \frac{u}{U}\right)\frac{r}{a}\left(1 - \frac{u}{U}\right)^2 dn$$

Hence, $$\int_0^\infty (1 - u/U)^3 \frac{r}{a} dn = \delta_1 - 3\delta_2 + \delta_3.$$

Ex. 20. *Show that for steady Poiseuille flow in a pipe of radius a,* $\delta_1 = a/4,\; \delta_2 = a/12$ *and* $\delta_3 = a/8.$

Sol. Try yourself.

EXERCISES

1. (*a*) Derive two dimensional boundary layer equations for flow over a plane wall, using order of magnitude approach. Also reduce them for steady case. **[Himanchal 2000, 01, 02, 05, 10]**

(*b*) Derive two dimensional boundary layer equations for flow over a plane using asymptotic approach. Also reduce them for steady case. **[Himanchal 1999, 2002, 03, 04]**

2. (*a*) Describe four parameters, which are characteristics of a boundary layer. Give meaning and utility of each of these parameters. **[Himanchal 2003, 04]**

Hint. Refer Art. 18.5. Boundary layer thickness, displacement thickness, momentum thickness and skin function are known as four characteristic boundary layer parameters.

(*b*) Write short note on the characteristic boundary layer parameters.

3. Discuss the Blasius-Topfer solution for the boundary layer on a flat plate and calculate the coefficient of skin fraction. **[Himanchal 2000]**

4. Describe Hartree's equation for laminar boundary layer flow past a wedge. Discuss velocity distribution and show through graph. **[Himanchal 2002]**

5. Define similar solutions of the boundary layer equations in a steady two-dimensional incompressible flow. Determine all possible forms of the potential flow velocity when similar solutions exist for flow over a plane surface. **[Himanchal 1999, 2001, 02, 04, 05]**

6. Define separation of boundary layer. Discuss it using physical and analytical approach. **[Himanchal 1999, 2001, 02, 04, 05]**

7. Derive the expressions of velocity distribution and flux of mass across any section in laminar boundary layer flow through a circular jet. **[Himanchal 2003, 04]**

8. Derive the expressions of velocity distribution and flux of mass across a plane in laminar boundary layer flow of a plane free jet. **[Himanchal 2001, 02, 04]**

9. A jet of air issues from a small hole in a wall and mixes with the same surrounding air. If the compressibility of the air is neglected and motion is assumed to be laminar and symmetric about an axis, determine the velocity in the jet at some distance from the hole. Show that the flux of the mass across any section of the jet is independent of the pressure under which the air is forced through the opening in the wall. **[Himanchal 2003]**

Hint. Refer Art. 18.13

10. Discuss the spread of a plane free jet. Show that the mass flux increases in the downstream as $x^{1/3}$ and the width of the jet varies as $x^{2/3}$ where x is the distance measured downstream from the slit of the jet **[Himanchal 2000]**

11. Write short notes on the following:

(*i*) Separation of boundary layer **[Himanchal 1998, 2000]**

(*ii*) Similar solutions of boundary layer equations **[Himanchal 2000, 01, 02, 03]**

(*iii*) Displacement thickness. **[Himanchal 1998, 01, 04]**

(*iv*) Momentum thickness **[Himanchal 2000, 01, 04]**

(*v*) Drag coefficient **[Himanchal 2001]**

(*vi*) Energy thickness. **[Himanchal 2000]**

(vii) Boundary layer on flat plate **[Himanchal 2009]**

(*viii*) Two dimensional boundary layer equations for flow of a viscous incompresible fluid on a plane wall. **(Himanchal 2010)**

12. Discuss Prandtl's theory of boundary layer and obtain the differential equations of motion in the layer along *a* plate. **(Agra 2008)**

13. The boundary layer theory was introduced by

(*a*) navier (*b*) Prandtl (*c*) Stokes (*d*) Kelvin **(Agra 2006, 12)**

Sol. Ans. (*b*). Refer Art. 18.2

CHAPTER

19 Thermal Boundary Layer

19.1 Thermal boundary layer. [Himanchal 2000, 04]

When a fluid flows past a heated or cooled bodies the heat is transferred by conduction, convection and radiation. Heat transfer by radiation is negligible unless the temperature is very high. Accordingly, we shall confine our present discussion to heat transfer by conduction and convection only. The conductivity of ordinary fluids is small. For such fluids the heat transport due to conduction is comparable to that due to convection only across a thin layer near the surface of the body. It follows that the temperature field which spreads from the body extends only over a narrow zone in the immediate vicinity of its surface, whereas the fluid at a large distance from the surface is not materially effected by the heated body. This thin layer (narrow region) near the surface of the body is called *thermal boundary layer*. This concept is analogous to the concept of *velocity boundary layer* described in Art. 18.3 of chapter 18.

There are two types of problems of thermal boundary layers, namely,

(*i*) forced convection and (*ii*) free (or natural) convection.

19.2. Forced convection. Definition. [Himanchal 2000]

The forced convection is the flow in which the velocities, arising from variable density (*i.e.*, due to the force of bouyancy) are negligible in comparison with the velocity of the main or forced flow.

Free or natural convection. Definition. [Himanchal 2000]

The free convection is the flow in which the motion is essentially caused by the effect of gravity on the heated fluid of variable density.

19.3. The thermal boundary layer equations in two – dimensional flow [Himanchal 2000, 02, 03, 04]

We propose to derive the thermal-boundary layer equations for the flow over a semi-infinite flat plate. We take rectangular cartesian co-ordinates (x, y) with x measured in the plate in the direction of the two-dimensional laminar incompressible flow, and y measured normal to the plate. Let (u, v) be the velocity components in x-and y-directions. Let viscosity of the fluid be small and let δ be small thickness of the boundary layer.

Let δ_t be small thermal boundary layer. Let U_∞ be the velocity in the main stream just outside the boundary layer. Then the equation of energy for the steady flow of a viscous incompressible fluid in two-dimensional is given by

$$\rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \Phi, \qquad \text{...(1)}$$

$$1 \quad 1 \;\; 1 \qquad \delta \;\; 1/\delta_t \qquad \delta_t^2 \quad 1 \qquad 1/\delta_t^2$$

where the dissipation function Φ is given by

$$\Phi = 2\mu\left\{(\partial u/\partial x)^2 + (\partial v/\partial y)^2\right\} + \mu\left\{(\partial v/\partial x)^2 + (\partial u/\partial y)^2 + 2(\partial v/\partial x)\times(\partial u/\partial y)\right\} \qquad \text{...(2)}$$

$$\delta^2 \qquad 1 \qquad \delta^2/\delta_t^2 \qquad \delta^2 \qquad \delta^2 \qquad 1/\delta_t^2 \qquad \delta\times(1/\delta_t)$$

We shall follow a method similar to that used in Art. 18.6 in order to simplify (1). We now determine the order of magnitude of each term in (1) to enable us to drop small terms and thus to arrive at the simplified thermal boundary layer equations. We shall designate the order of any quantity (q, say) by $O(q)$. The orders of magnitude are shown in (1) under the individual terms. Let δ and δ_t be velocity boundary layer thickness and thermal boundary layer thickness respectively. Let $O(x)$ = O(u) = O(T) = 1 and $O(y)$ = δ_t. As proved in Art. 18.6, $O(v) = \delta$ and $O(\nu) = \delta^2$. From (1), it follows that the term $\partial^2 T/\partial x^2$ may be neglected in comparison with $\partial^2 T/\partial y^2$. Again, the conduction terms become of the same order of magnitude as the convectional term, only if $O(k) = \delta_t^2$ and δ and δ_t are of the same order of magnitude.

$$\therefore \quad O\left(\frac{k/\rho C_p}{\nu}\right) = \frac{\delta_t^2}{\delta^2} \qquad \text{that is,} \qquad O(1/\sqrt{P_r}) = \delta_t/\delta, \qquad \text{as} \qquad O(\rho C_p) = 1,$$

where P_r = Prandtl number = $(\mu C_p)/k$ and $\nu = \mu/\rho$.

Since O(P_r) = 1 for gases, so $O(\delta_t) = O(\delta)$ approximately for gases. On the other hand for liquids, $O(\delta_t) < O(\delta)$.

Suppose that δ_t and δ be of the same order of magnitude. Then observing (1) and (2), we note that the only term which can be retained in Φ is $\mu(\partial u/\partial y)^2$. Hence the equation of the thermal boundary layer for an incompressible constant properties fluid in two-dimensional steady flow is given by

$$\rho C_p \{u(\partial T/\partial x) + v(\partial T/\partial y)\} = k(\partial^2 T/\partial y^2) + \mu(\partial u/\partial y)^2 \qquad \ldots(3)$$

The first term on the *R.H.S.* of (3) is of order $T_w - T_\infty$ whereas the second term is of order U_∞^2 (U_∞ and T_∞ being velocity and temperature of the free steam and T_w the temperature of the plate). Hence both are of the same order when $U_\infty^2/(T_w - T_\infty)$ is of order unity. To make it dimensionless we divide by C_p which is of order unity. Hence the second term on *R.H.S.* of (3) representing heat due to friction is of the same order as the other terms when

$$U_\infty^2/\rho(T_w - T_\infty) = Ec = \text{Eckert number} \qquad \ldots(4)$$

is of order unity. For example, air at a temperature difference of *10°F,* O(Ec) = 1 when U_∞ = 500 ft/sec.

Usually (3) is solved under the following boundary conditions

When $y = 0$, $T = T_w$ or $\partial T/\partial y = 0$; when $y = \infty$, $T = T_\infty$. ...(5)

The temperature of the plate may be constant or be a function of the distance along the wall. The boundary condition $\partial T/\partial y = 0$ when $y = 0$, which is condition of an *adiabatic plate,* is valid only when the frictional heat is taken into consideration.

In Art. 18.6, we have already obtained boundary layer equations (Refer (10), (11) and (12), in the following form :

$$\partial u/\partial x + \partial v/\partial y = 0 \qquad \ldots(6)$$

$$u(\partial u/\partial x) + v(\partial u/\partial y) = U(dU/dx) + \nu(\partial^2 u/\partial y^2) \qquad \ldots(7)$$

subject the boundary conditions :

$u = v = 0$ when $y = 0$ and $u = U(x)$ when $y = \infty$. ...(8)

Thus the basic equations which govern the velocity and temperature distribution in a boundary layer past a solid body, in forced convection, are given by

$$\partial u/\partial x + \partial v/\partial y = 0 \qquad \ldots(i)$$

$$u(\partial u/\partial x) + v(\partial u/\partial y) = U\,(dU/dx) + \nu(\partial^2 u/\partial y^2) \qquad \ldots(ii)$$

$$\rho C_p\,\{u(\partial T/\partial x) + v(\partial T/\partial y)\} = k(\partial^2 T/\partial y^2) + \mu(\partial u/\partial y)^2 \qquad \ldots(iii)$$

with the boundary conditions

$$u = v = 0, \qquad T = T_w(x) \qquad \text{or} \qquad dT/dy = 0 \qquad \text{when} \qquad y = 0 \quad \ldots(iv)$$

$$u = U(x) \qquad \text{and} \qquad T = T_\infty \qquad \text{when} \qquad y = \infty \quad \ldots(v)$$

Note 1. In an incompressible flow the velocity is usually small and the temperature difference is moderate. Hence the second term on the *R.H.S.* of (*iii*) which is the heat due to friction, can be neglected and then (*iii*) reduces to

$$u(\partial T/\partial x) + v(\partial T/\partial y) = (k/\rho C_p) \times (\partial^2 T/\partial y^2) \qquad \ldots(vi)$$

Note 2. The velocity boundary equation (*ii*) is non-linear in character and is independent of the temperature field and hence can be solved independently to the thermal boundary layer equation (*iii*). On the other hand, the thermal boundary layer equation (*iii*) is linear in character and it depends on the velocity field and hence it can be solved only when the velocity field is known. In what fillows we shall use solutions of (*i*) and (*ii*) as already obtained and then solve (*iii*).

Note 3. Since (*iii*) is linear in character hence its solution is easy from Mathematics point of view because the law of superposition of known solutions holds.

19.4. Forced convection in a laminar boundary layer on a flate plate

[Himanchal 1995, 96, 97, 2000]

Let us consider the steady two-dimensional flow of a viscous incompressible fluid past a thin semi-infinite flat plate at a constant temperature T_w (or insulated). Suppose the plate is kept along the direction of a uniform stream of velocity U_∞ and temperature T_∞. Let the origin of co-ordinates be at the leading-edge of the plate, the x-axis along the plate and y-axis normal to it.

For the given problem, $U(x) = U_\infty =$ constant. Hence the equations which govern the velocity and temperature distribution in a boundary layer past the plate, in forced convection, are

$$\partial u/\partial x + \partial v/\partial y = 0 \qquad \ldots(1)$$

$$u(\partial u/\partial x) + v(\partial u/\partial y) = \nu(\partial^2 u/\partial y^2) \qquad \ldots(2)$$

$$u(\partial T/\partial x) + v(\partial T/\partial y) = a(\partial^2 T/\partial y^2) + (\mu/\rho C_p) \times (\partial u/\partial y)^2 \qquad \ldots(3)$$

where $\qquad a = k/\rho C_p =$ Thermal diffusivity

with the boundary conditions

$$\left.\begin{array}{llll} y = 0\,; & u = v = 0\,; & T = T_w \text{ (isothermal)}, & \partial T/\partial y = 0 \text{ (adiabatic)} \\ & y = \infty : & u = U_\infty; & T = T_\infty \end{array}\right\} \qquad \ldots(4)$$

For solutions of (1) and (2), refer Art. 18.7. We now proceed to find the solution of (3) for temperature field.

We consider the following two cases :

Case I : Simple integral of (3) when $P_r = 1$.

Two situations arise :

Situation 1 : If the frictional heat is neglected, then a simple integral of (3) exists. For the present case, the basic equations for velocity and temperature distribution in a boundary layer reduce to

$$u(\partial u/\partial x) + v(\partial u/\partial y) = \nu(\partial^2 u/\partial y^2) \qquad \ldots(5)$$

and

$$u(\partial T/\partial x) + v(\partial T/\partial y) = a(\partial^2 T/\partial y^2) \qquad \ldots(6)$$

with the boundary conditions

$$y=0: \quad u=v=0\,; \quad T=T_w \qquad \text{and} \qquad y=\infty: \quad u=U_\infty; \quad T=T_\infty \quad \ldots(7)$$

When $\nu = a$, then $P_r = (\mu C_p)/k = (\mu/\rho)/(k/\rho C_p) = \nu/a = 1$ and hence (5) and (6) become identical, with boundary conditions, if $(T-T_w)/(T_\infty - T_w)$ is replaced by u/U_∞.

Now, $$\frac{T-T_w}{T_\infty - T_w} = \frac{u}{U_\infty} \quad \Rightarrow \quad 1-\frac{T-T_w}{T_\infty - T_w} = 1-\frac{u}{U_\infty} \quad \ldots(8)$$

$$\therefore \quad (T-T_\infty)/(T_w - T_\infty) = 1-(u/U_\infty) \qquad (P_r = 1) \quad \ldots(9)$$

It is known as *Crocco's first integral.*

From (8), we have $$\frac{\partial}{\partial y}\left(\frac{u}{U_\infty}\right) = \frac{\partial}{\partial y}\left(\frac{T-T_w}{T_\infty - T_w}\right), \quad \ldots(10)$$

showing that the heat-flux and the skin friction are proportional to each other. For getting the exact relationship between the two, we write the value of the local Nusselt number $Nu(x)$, which is given by

$$Nu(x) = -\frac{x(\partial T/\partial x)_0}{T_w - T_\infty} = \frac{x}{U_\infty}\left(\frac{\partial u}{\partial y}\right)_0, \text{ using (10)}$$

$$= \frac{x}{\mu U_\infty} \times T_w = \frac{x}{\mu U_\infty} \times \frac{1}{2}\rho U_\infty^2 C_f = \frac{1}{2}\mathrm{Re}_x C_f.$$

[$\because$ Local skin friction coefficient $C_f = T_w/(\rho U_\infty^2/2) \Rightarrow T_w = (1/2)\times \rho U_\infty^2 C_f$]

where $\mathrm{Re}_x = (U_\infty x)/\nu =$ Reynold's number.

Thus, we have $Nu(x) = (1/2) \times \mathrm{Re}_x C_f$

which is known as *Reynold's analogy.*

Situation 2. If the frictional heat is not neglected but the wall is insulated, then another simple integral of (3) is possible as shown below.

Let $T = T(u)$. Then (9) can be re-written as

$$u\frac{dT}{du}\frac{\partial u}{\partial x} + v\frac{dT}{du}\frac{\partial u}{\partial y} = a\frac{\partial}{\partial y}\left(\frac{dT}{du}\frac{\partial u}{\partial y}\right) + \frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2$$

or $$\left(u\frac{du}{dx} + v\frac{\partial u}{\partial y}\right)\frac{dT}{du} = a\frac{dT}{du}\frac{\partial^2 u}{\partial y^2} + \left(a\frac{d^2T}{du^2} + \frac{\mu}{\rho C_p}\right)\left(\frac{\partial u}{\partial y}\right)^2.$$

or $$\nu\frac{\partial^2 u}{\partial y^2}\frac{dT}{du} = a\frac{dT}{du}\frac{\partial^2 u}{\partial y^2} + \left(a\frac{d^2T}{du^2} + \frac{\mu}{\rho C_p}\right)\left(\frac{\partial u}{\partial y}\right)^2, \text{ using (2)}$$

or $$(\nu - a)\frac{dT}{du}\frac{\partial^2 u}{\partial y^2} = \left(a\frac{d^2T}{du^2} + \frac{\mu}{\rho C_p}\right)\left(\frac{\partial u}{\partial y}\right)^2 \quad \ldots(11)$$

which is identically satisfied, that is, $T = T(u)$ will be a solution of (3), if

$$\nu = a \qquad \text{and} \qquad d^2T/du^2 = -\mu/(a\rho C_p).$$

or $\qquad P_r = 1 \qquad$ and $\qquad d^2T/du^2 = -(1/C_p).$ ...(12)

Integrating (12), $\qquad dT/du = -(u/C_p),$ ...(13)

where the constant of integration vanishes, since at $y = 0$, we have $u = 0$, $\partial u/\partial y \neq 0$ and $\partial T/\partial y = 0$, which implies $dT/du = 0$.

Integrating (13), $\qquad T = -(u^2/2C_p) + C$, C being an arbitrary constant ...(14)

Using the boundary condition at infinity, namely, $u = U_\infty$, $T = T_\infty$, (14) gives

$$T_\infty = -(U_\infty^2/2C_p) + C \qquad \text{...(15)}$$

Subtracting (15) from (14), $\qquad T - T_\infty = (U_\infty^2 - u^2)/2C_p$

or
$$\frac{T - T_\infty}{U_\infty^2/2C_p} = 1 - \left(\frac{u}{U_\infty}\right)^2, \qquad (P_r = 1) \qquad \text{...(16)}$$

It is known as *Crocco's's second integral.*

Case II. Integrals of (3) for arbitrary values of the Prandtl number

For the solution of (3), we shall require velocity distribution given by (1) and (2). For this purpose refer the Blasius solution in Art. 18.7. We have

$$u = U_\infty f'(\eta) \qquad \text{and} \qquad v = (1/2) \times (\nu U_\infty/x)^{1/2}\{\eta f'(\eta) - f(\eta)\} \qquad \text{...(17)}$$

where $\qquad \eta = y(U_\infty/\nu x)^{1/2} \qquad$ (similarity variable) ...(18)

and the function $f(\eta)$ satisfies the equation $\qquad 2f''' + f\,f'' = 0$ ...(19)

with the boundary conditions

$$f = f' = 0 \quad \text{when} \quad \eta = 0 \qquad \text{and} \qquad f' = 1 \quad \text{when} \quad \eta = \infty. \qquad \text{...(20)}$$

In order to find solution of (3) for an isothermal plate, it will be easier to find the solution of (3) when the dissipation term is neglected, that is, the *solution of the cooling problem* with a given value of $(T_w - T_\infty)$ and then another solution of (3) when the friction heat is not neglected but the plate is adiabatic, that is, the problem of *plate thermometer.* Since (3) is linear in character, to get the complete integral of (3), the two solutions may then be properly superimposed.

We now obtain two solutions of (3) as just indicated above.

Situation 1. Solution of the cooling problem, that is, solution of (3) when the dissipation term is neglected. For the present case, (3) reduces to

$$u(\partial T/\partial x) + v(\partial T/\partial y) = a(\partial^2 T/\partial y^2) \qquad \text{...(21)}$$

with the boundary, conditions

$$T = T_w \quad \text{when} \quad y = 0 \quad \text{and} \quad T = T_\infty \quad \text{when} \quad y = \infty. \qquad \text{...(22)}$$

Let $\qquad \theta_1 = (T - T_\infty)/(T_w - T_\infty)$ ...(23)

and let us try to find a solution of (21) in which θ_1 is a function of η only, that is, let us find a similar solution of θ_1. Then, upon substituting for *u*, *v* and *T* from (17) and (23) in (21), we have

$$\theta_1'' + (1/2) \times p_r\, f\, \theta_1' = 0. \qquad \text{...(24)}$$

with the boundary conditions.

$$\theta_1 = 1 \quad \text{when} \quad \eta = 0 \quad \text{and} \quad \theta_1 = 0 \quad \text{when} \quad \eta = \infty. \qquad \text{...(25)}$$

From (19), $f = -(2f'''/f'')$. Hence (24) reduces to

$$\theta_1'' - Pr \times \frac{f'''\theta_1'}{f''} = 0 \qquad \text{or} \qquad \frac{\theta_1''}{\theta_1'} = Pr \times \frac{f'''}{f''} \quad \ldots(26)$$

Integrating (26), $\log\theta_1' - \log C = \Pr\ \log f''$, C being an arbitrary constant

or $\log(\theta_1'/C) = \log(f'')^{\Pr}$ or $d\theta_1/d\eta = C\{f''(\eta)\}^{\Pr}$ ...(27)

Integrating (27), $\theta_1 = -C\int_\eta^\infty \{f''(\eta)\}^{\Pr} d\eta + C'$, C' being an arbitrary constant ...(28)

Putting $\eta = 0$ and $\eta = \infty$ by turn in (28) and using (25), we get

$$1 = -C\int_0^\infty \{f''(\eta)\}^{P_r} d\eta + C' \qquad \text{and} \qquad 0 = 0 + C'.$$

Hence, $C' = 0$ and $C = -1\Big/\int_0^\infty \{f''(\eta)\}^{P_r} d\eta$ ...(29)

With these values (28) reduces to

$$\theta_1(\eta, \Pr) = \int_\eta^\infty \{f''(\eta)\}^{P_r} d\eta \Big/ \int_0^\infty \{f''(\eta)\}^{P_r} d\eta. \quad \ldots(30)$$

When $P_r = 1$ for a given fluid, then (30) becomes

$$\theta_1 = [f'(\eta)]_\eta^\infty \big/ [f'(\eta)]_0^\infty = \{f'(\infty) - f'(\eta)\} \big/ \{f'(\infty) - f'(0)\}$$

or $$\theta_1 = 1 - f'(\eta), \quad \text{using (20)}, \quad \ldots(31)$$

Then, upon substituting for θ_1 and $f'(\eta)$ from (23) and (17) in (31), we have

$$(T - T_\infty)/(T_w - T_\infty) = 1 - (u/U_\infty), \qquad (\Pr = 1)$$

It is known as *Crocco's first integral*

In the following figure the dimensionless temperature distribution θ_1 is plotted against η for various values of *Pr*.

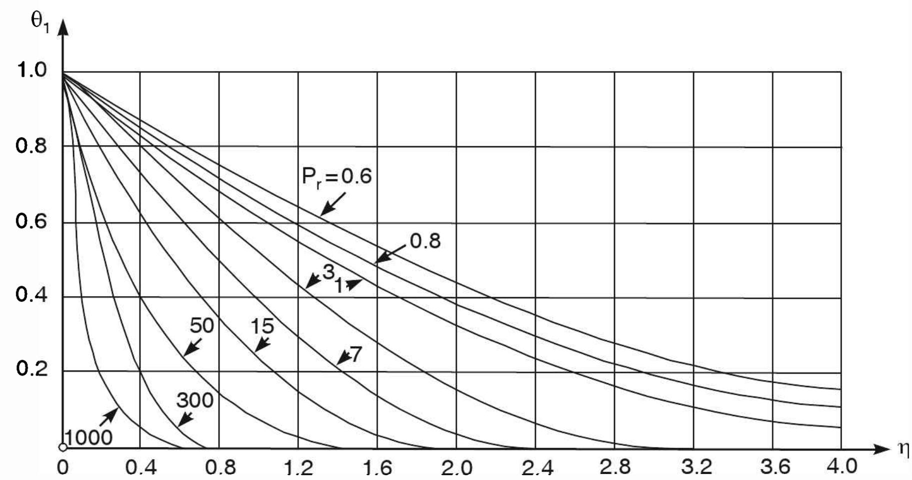


Fig. Temperature distribution in a heated flat plate at zero incidence with small velocity plotted for various Prandtl numbers. *Pr* (frictional heat neglected)

The curve for $Pr = 1$ gives also the velocity distribution. In this case velocity boundary layer thickness δ and thermal boundary layer thickness δ_t are equal. For $Pr < 1$, we find $\delta < \delta_t$ whereas for $Pr > 1$, we find $\delta > \delta_t$.

Refer the following figures for various values of Pr.

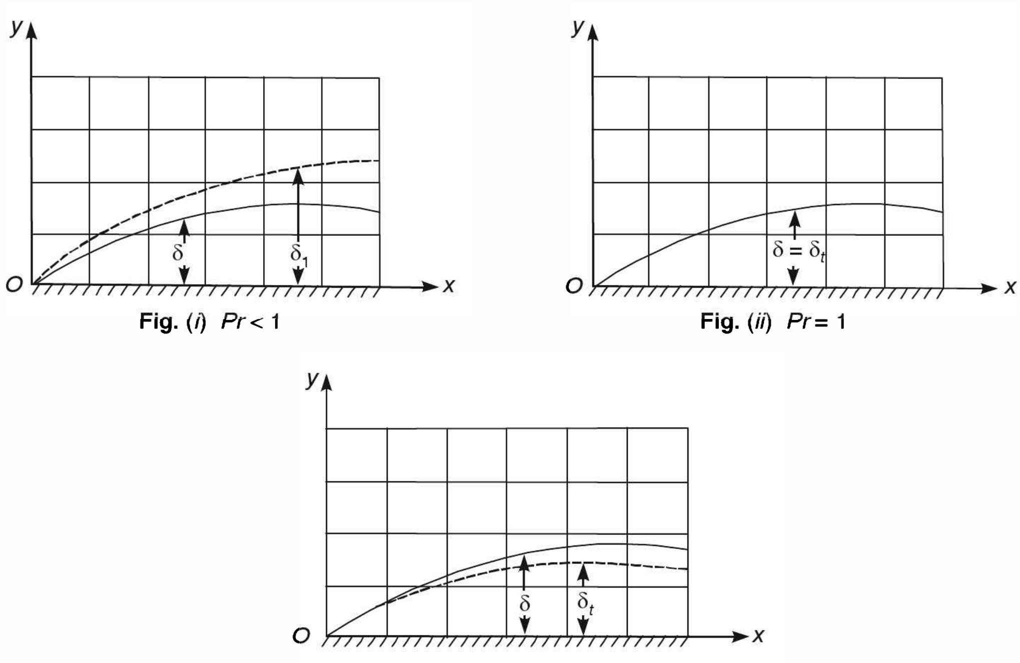


Figure showing comparison between volocity and thermal boundary layers on an isothermal plane wall for various values of Pr

Using (30), the temperature gradient at the plate is given by

$$\left(\frac{d\theta_1}{d\eta}\right)_{\eta=0} = -[f''(0)]^{Pr} \Big/ \int_0^\infty [f''(\eta)]^{Pr} d\eta$$

Since $f''(0) = 0.332$ (refer Art. 18.7), we get

$$-\left(\frac{d\theta}{d\eta}\right)_{\eta=0} = (0.332)^{Pr} \Big/ \int_0^\infty [f''(\eta)]^{Pr} d\eta = a_1(Pr), \text{ say} \qquad \ldots(32)$$

E. Pohlhausen has shown that with good accuracy, a_1 (Pr) may represented approximately by the formula

$$a_1(Pr) = 0.332(Pr)^{1/3}, \qquad \text{for} \qquad 0.6 < Pr < 10. \qquad \ldots(33)$$

We now proceed to derive two separate formulae for extreme values of Pr, namely, $Pr \to 0$ and $Pr \to \infty$.

Limiting Case (*i*) $Pr \to 0$ *which is true approximately for liquid metals (for example, for mercury Pr = 0.044)* **[Himanchal 1997, 99]**

In this case velocity boundary layer is neglected in comparison to the thermal boundary layer. Accordingly the velocity *u may be replaced by* U_∞, that is, $f'(\eta) \sim 1$ and $f(\eta) \sim \eta$. Replacing $f(\eta)$ by η in (24), we have

$$\theta_1'' + (1/2) \times \Pr \eta\, \theta_1' = 0 \qquad \text{or} \qquad \theta_1''/\theta_1' = -(1/2) \times \Pr\, \eta$$

Integrating it, $\quad \log \theta_1' - \log C = -(Pr/4)\eta^2 \quad$ or $\quad \theta_1' = Ce^{-(P_r\eta^2/4)}$.

Integrating it, $\theta_1 = C\int_\eta^\infty e^{-(Pr\,\eta^2/4)}\,d\eta + C'$, C' being an arbitrary constant ...(34)

Putting $\eta = 0$ and $\eta = \infty$ by turn in (34) and using (25), we get

$$1 = C\int_0^\infty e^{-(Pr\,\eta^2/4)}\,d\eta + C' \qquad \text{and} \qquad 0 = 0 + C'$$

$$\Rightarrow \qquad C' = 0 \qquad \text{and} \qquad C = 1\Big/\int_0^\infty e^{-(Pr\,\eta^2/4)}\,d\eta$$

$\therefore$ (34) gives
$$\theta_1 = \int_\eta^\infty e^{-(Pr\eta^2/4)}d\eta \Big/ \int_0^\infty e^{-(Pr\eta^2/4)}\,d\eta \qquad ...(35)$$

$$\therefore \qquad a_1(Pr) = -\left(\frac{d\theta_1}{d\eta}\right)_{\eta=0} = 1\Big/\int_0^\infty e^{-(Pr\eta^2/4)}\,d\eta = (Pr)^{1/2}\sqrt{\pi}$$

Thus, for $Pr \to 0$, we have $a_1(Pr) = 0.564(Pr)^{1/2}$ ...(36)

Limiting case (ii) $Pr \to \infty$, *which is true approximately for lubricating ails (for example, for glycerin* Pr = 7250) **[Himanchal 1997, 99]**

When Pr is large the thermal boundary layer is much thinner than the velocity boundary layer and so rapid change in temperature will take place very near the plate. Hence for the present case, we can replace $f(\eta)$ by its first term in the series expansion near $\eta = 0$, that is,

$$f(\eta) \sim \alpha\eta^2/2, \qquad \text{where} \qquad \alpha = 0.332$$

[Use equation (15) of Art. 18.7 with $A_0 = A_1 = 0$, $A_2 = 0.332$ etc]

Replacing $f(\eta)$ by $\alpha\eta^2/2$ in (24), we have

$$\theta_1'' + (\alpha\eta^2 Pr/4)\theta_1' = 0 \qquad \text{or} \qquad \theta_1''/\theta_1' = -(\alpha^2 Pr/4)\eta^2 .$$

Integrating it, $\log\theta_1' - \log C = -(\alpha Pr/12)\eta^3$ or $\theta_1' = Ce^{-(Pr\,\alpha\eta^3/12)}$

Integrating it, $\theta_1 = C\int_\eta^\infty e^{-(\alpha Pr\,\eta^3/12)}d\eta + C'$, C' being an arbitrary constant ...(37)

Putting $\eta = 0$ and $\eta = \infty$ by turn in (37) and using (25), we get

$$1 = C\int_0^\infty e^{-(\alpha Pr\eta^3)/12}d\eta + C' \qquad \text{and} \qquad 0 = 0 + C'$$

$$\Rightarrow \qquad C' = 0 \qquad \text{and} \qquad C = 1\Big/\int_0^\infty e^{-(\alpha Pr\eta^3)/12}d\eta \qquad ...(38)$$

$\therefore$ (37) gives
$$\theta_1 = \int_\eta^\infty e^{-(\alpha Pr\,\eta^3)/12}d\eta \Big/ \int_0^\infty e^{-(\alpha Pr\,\eta^3)/12}d\eta \qquad ...(38)$$

$$\therefore \qquad a_1(\mathrm{Pr}) = -\left(\frac{d\theta}{d\eta}\right)_{\eta=0} = 1\Big/\int_0^\infty e^{-(\alpha Pr\eta^3)/12}d\eta = \frac{(12)^{2/3}(\alpha\,\mathrm{Pr})^{1/3}}{4\Gamma(1/3)}$$

Thus, for $Pr \to \infty$, we have $a_1(Pr) = 0.339(Pr)^{1/3}$ ...(39)

Evaluation of the local dimensionless coefficient of heat transfer (or the Nusselt number Nu (x)) By definition, we know that

$$Nu(x) = -x(\partial T/\partial y)_0/(T_w - T_\infty) \qquad ...(40)$$

Re-writing in the present notations, (40) becomes

$$Nu(x) = -\left(\frac{\partial \theta_1}{\partial \eta}\right)_0 \frac{U_\infty x}{v} = a_1(Pr)Re_x \quad \ldots(41)$$

where $Re_x = (U_\infty x)/v =$ local Reynolds number

Then, upon substituting for a_1 (Pr) from (33), (36) and (39) by turn in (41), we have

$$Nu(x) = 0.332(Pr)^{1/3}(Re_x)^{1/2} \quad \text{for} \quad 0.6 < Pr < 10$$

$$Nu(x) = 0.564(Pr)^{1/2}(Re_x)^{1/2} \quad \text{for} \quad Pr \to 0,$$

and

$$Nu(x) = 0.339(Pr)^{1/3}(Re_x)^{1/2} \quad \text{for} \quad Pr \to \infty,$$

Situation 2. Solution of the problem of plate thermometer, that is solution when frictional heat is not neglected but the plate is adiabatic.

We now propose to find the solution of the problem of plate thermometer, that is, the temperature which a thermometer in the form of a plate will read when no heat is being transferred to or from it by the fluid, Equation (3), in the present case, is given by

$$u(\partial T/\partial x) + v(\partial T/\partial y) = a(\partial^2 T/\partial y^2) + (\mu/\rho C_p) \times (\partial u/\partial y)^2, \quad \ldots(42)$$

subject to the boundary conditions

$$\partial T/\partial y = 0 \quad \text{when} \quad y = 0 \quad \text{and} \quad T = T_\infty \quad \text{when} \quad y = \infty. \quad \ldots(43)$$

Let

$$\theta_2 = (T - T_\infty)/(U_\infty^2/2C_p) \quad \ldots(44)$$

and let us try to find a solution of (42) in which θ_2 is a function of η only, that is, let us find a similar solution of θ_2. Then, upon substituting for u, v and T from (17) and (44) in (42) we have

$$\theta_2'' + (1/2) \times Pr\, f\, \theta_2' = -2\,\mathrm{Pr}\,(f'')^2, \quad \ldots(45)$$

with the boundary conditions

$$\theta_2' = 0 \quad \text{when} \quad \eta = 0 \quad \text{and} \quad \theta_2 = 0 \quad \text{when} \quad \eta = \infty. \quad \ldots(46)$$

From (19), $f = -(2f'''/f'')$. Hence (45) reduces to

$$\theta_2'' - (\mathrm{Pr}\, f'''/f'')\theta_2' = -2\,\mathrm{Pr}\,(f'')^2 \quad \ldots(47)$$

which is a linear differential equation whose integrating factor is given by

$$I.F. = e^{-\int (P_r f'''/f'')d\eta} = e^{-Pr \log f''} = e^{\log (f'')^{-Pr}} = (f'')^{-Pr} = 1/(f'')^{Pr}.$$

Hence solution of (47) is given by

$$\frac{\theta_2'}{(f'')^{\mathrm{Pr}}} = \int_0^\eta \{-2\,\mathrm{Pr}\,(f'')^2\}\frac{1}{(f'')^{Pr}}d\eta + C \quad \text{or} \quad \theta_2' = -2\,\mathrm{Pr}\,(f'')^{\mathrm{Pr}}\int_0^\eta (f'')^{2-\mathrm{Pr}}d\eta + C$$

Since $\theta_2' = 0$ when $\eta = 0$. the above equation gives $0 = 0 + C$ or $C = 0$

Thus,

$$\theta_2' = -2\,\mathrm{Pr}\,(f'')^{\mathrm{Pr}}\int_0^\eta (f'')^{2-\mathrm{Pr}}d\eta \quad \ldots(48)$$

Integrating (48),

$$\theta_2 = 2\,\mathrm{Pr}\int_\eta^\infty (f'')^{\mathrm{Pr}}\left\{\int_0^\eta (f'')^{2-\mathrm{Pr}}d\eta\right\}d\eta + C'. \quad \ldots(49)$$

Since $\theta_2 = 0$ when $\eta = \infty$, (49) gives $0 = 0 + C'$ or $C' = 0$.

Then, by (49),

$$\theta_2(\eta, \mathrm{Pr}) = 2\,\mathrm{Pr}\int_\eta^\infty (f'')^{\mathrm{Pr}}\left\{\int_0^\eta (f'')^{2-\mathrm{Pr}}d\eta\right\}d\eta \quad \ldots(50)$$

When Pr = 1 for a given fluid, then (50) reduces to

$$\theta_2 = 2\int_\eta^\infty f''\, f'\, d\eta = \left[\{f'(\eta)\}^2\right]_\eta^\infty = [f'(\infty)]^2 - [f'(\eta)]^2 = 1 - [f'(\eta)]^2 \text{ by (20)}$$

Thus, for Pr = 1, $\quad (T - T_\infty)/(U_\infty^2/2C_p) = 1 - (u/U_\infty)^2$, using (17) and (44)

It known as *Crocco's second integral.*

The temperature which the wall assumes under the influence of internal friction is called the *recovery temperature* or *adiabatic wall temperature*. The difference between the recovery temperature $T_r[=(T)_\eta = 0]$ and the temperature of the free stream T_∞, from (44), is given by

$$T_r - T_\infty = (U_\infty^2/2C_p) \times \theta_2(0, \text{Pr}). \qquad \ldots(51)$$

The *temperature recovery factor (or simply the recovery factor)* denote by r, can be obtained from (50) and (51). Thus, we have

$$\therefore \qquad r = \frac{T_r - T_\infty}{U_\infty^2/2C_p} = \theta_2(0, \text{Pr}) = 2\,\text{Pr}\int_0^\infty (f'')^{\text{Pr}} \left\{\int_0^\eta (f'')^{2-\text{Pr}} d\eta\right\} d\eta \qquad \ldots(52)$$

E. Pohlhausen has shown that for moderate values of Pr, the recovery factor r may approximated with good accuracy by the formula

$$r = (Pr)^{1/2}, \quad \text{for moderate value of } Pr. \qquad \ldots(53)$$

We now proceed to derive two separate formulae for extreme values of Pr, namely, $Pr \to 0$ and $\text{Pr} \to \infty$.

Limiting case (i) When $\text{Pr} \to 0$.

In this case the complete adjustment of the temperature takes place in a much wider zone than for the velocity. Hence for solution of (45), we must divide the thermal boundary layer into two zones as described below.

First zone, where η is large. In this zone $f(\eta) \sim \eta$ and (45) becomes

$$\theta_2'' + (1/2) \times \text{Pr}\, \eta\, \theta_2' = 0 \qquad \ldots(54)$$

Let $\quad \xi = (Pr/2)^{1/2}\eta. \qquad \ldots(55)$

Then (54) becomes $\quad (d^2\theta_2/d\xi^2) + \xi(d\theta_2/d\xi) = 0. \qquad \ldots(56)$

Using the boundary condition at infinity only, solution of (56) is given by

$$\theta_2 = C\int_\xi^\infty e^{-\xi^2/2} d\xi = C\left(\int_0^\infty e^{-\xi^2/2} d\xi - \int_0^\xi e^{-\xi^2/2} d\xi\right), \qquad \ldots(57)$$

where C is the a constant of integration to be determined.

For small ξ (*i.e.*, lower limit of η large) in this zone, we have

$$\theta_2 = C\left\{(\pi/2)^{1/2} - \xi\right\} \qquad \ldots(58)$$

Second zone, where η small. In this case rejecting $Pr\, f\theta_2'$ (being a small quantity of higher order), (45) reduces to

$$\theta_2' = -2Pr\, f''. \qquad \ldots(59)$$

Integrating (59), $\quad \theta_2' = C' - 2\,\text{Pr}\int_0^\eta f''^2 d\eta,$ C' being an arbitrary constant $\quad \ldots(60)$

Since $\theta_2' = 0$ when $\eta = 0$, (60) given $C' = 0$. Then 60 reduces to

$$\theta_2' = -2Pr\int_0^{\eta} f''^2 d\eta. \qquad \ldots(61)$$

For large η (*i.e.*upper limit of η small) in this zone, (61) may be approximated by

$$\theta_2' = -2Pr\int_0^{\infty} f''^2 d\eta. \qquad \ldots(62)$$

To find the constant C occuring in (58), we compare the two values of θ_2' given by (58)and (62) and obain

$$-C\left(\frac{\mathrm{Pr}}{2}\right)^{1/2} = -2Pr\int_0^{\infty} f''^2 d\eta \qquad \text{or} \qquad C = 2\times\ (2Pr)^{1/2}\int_0^{\infty} f''^2 d\eta \qquad \ldots(63)$$

So (58) and (63) $\Rightarrow$ $r = \theta_2(0, Pr) = \left(\frac{\pi}{2}\right)^{1/2} C = 2(\pi Pr)^{1/2}\int_0^{\infty} f''^2 d\eta$

or $$r = 0.92(Pr)^{1/2}, \qquad Pr \to 0. \qquad \ldots(64)$$

Limiting case (*ii*) when $Pr \to \infty$.

When Pr is large the thermal boundary layer is much thinner than the velocity boundary layer and therefore the rapid change in temperature will occur very near to the plate. Hence in the present case it is reasonable to replace $f(\eta)$ by its first non-zero term in the series expansion near $\eta = 0$, *i.e.* $f(\eta) \sim \alpha\eta^2/2$ when $\alpha = 0.332$. [Use Eq(15) of Art. 18.7 with $A_0 = A_1 = 0$ and $A_2 = 0.332$ etc.]

Now, $f(\eta) \sim \alpha\eta^2/2 \quad \Rightarrow \quad f''(\eta) \sim \alpha$. Hence (24) reduces to

$$\theta_2'' + (1/4)\times Pr\,\alpha\,\eta^2\theta_2' = -2Pr\,\alpha^2. \qquad \ldots(65)$$

Integrating (65) with the boundary conditions (64), as before, we get

$$\theta_2(\eta, Pr) = 2Pr\,\alpha^2\int_{\eta}^{\infty} e^{-(Pr\,\alpha\,\eta^3/12)}\left\{\int_0^{\eta} e^{(Pr\,\alpha\,\eta^3/12}d\eta\right\}d\eta \qquad \ldots(66)$$

$$\therefore \qquad r = \theta_2(0, Pr) = 2\,Pr\,\alpha^2\int_0^{\infty} e^{-(Pr\,\alpha\,\eta^3/12)}\left\{\int_0^{\eta} e^{(Pr\,\alpha\,\eta^3/12}d\eta\right\}d\eta$$

$$= 2\times(1/3!)^2\times(12\alpha^2)^{2/3}\times(Pr)^{1/3}, \qquad \text{where} \qquad \alpha = 0.332$$

Thus, $$r = 1.92\,(Pr)^{1/3}, \qquad \text{when} \qquad Pr \to \infty \qquad \ldots(67)$$

Thus, the recovery factor r maybe well aproximated by he following formulae for the range of Pr indicated thereat.

$r = (Pr)^{1/2}$, for moderate values of Pr.

$r = 0.92\,(\mathrm{Pr})^{1/2}$, for $\mathrm{Pr} \to 0$

$r = 1.92\,(\mathrm{Pr})^{1/3}$, for $\mathrm{Pr} \to \infty$

Complete solution of thermal boundary layer equation (3) for an isothermal plate when the frictional heat is accounted.

As already discussed, for finding complete solution of (3) we have obtained two solutions $\theta_1(\eta, \mathrm{Pr})$ and $\theta_2(\eta, \mathrm{Pr})$ given by (23) and (44) respectively. Re-writing (23) and (44) we get

$$T - T_\infty = (T_w - T_\infty)\,\theta_1(\eta, Pr) \qquad \ldots(68)$$

and $$T - T_\infty = (U_\infty^2/2C_p)\,\theta_2(\eta, Pr) \qquad \ldots(69)$$

Note that both these solutions θ_1 and θ_2 satisfy one of the common condition at infinity (recall that at $\eta = \infty, \theta_1 = \theta_2 = 0)$. Since (3) is a linear equation, hence a linear combination of two solutions θ_1 and θ_2 is also a solution of (3). So for complete solution of (3) for an isothermal plate, we superimpose the two solutions θ_1 and θ_2 as

$$T - T_\infty = C\theta_1(\eta, \mathrm{Pr}) + (U_\infty^2/2C_p)\theta_2(\eta, Pr) \qquad \ldots(70)$$

where the constant C will be obtained by the condition of isothermal plate, namely, $T = T_w$ when $\eta = 0$. From (25), $\theta_1 = 1$ when $\eta = 0$. Also, when $\eta = 0$ from (51), $(U_\infty^2/2C_p) \times \theta_2(0, Pr) = T_r - T_\infty$. Hence putting $\eta = 0$ in (70) and using the various values just obtained, we get

$$T_w - T_\infty = C + (T_r - T_\infty) \qquad \text{or} \qquad C = T_w - T_r \qquad \ldots(71)$$

$\therefore$ From (70), $$T - T_\infty = (T_w - T_r)\theta_1(\eta, \mathrm{Pr}) + (U_\infty^2/2C_p)\theta_2(\eta, Pr) \qquad \ldots(72)$$

We have, $$\text{Eckert number} = Ec = U_\infty^2/C_p\,(T_w - T_\infty) \qquad \ldots(73)$$

and $$\text{Recovery factor} = r = (T_r - T_\infty)/(U_\infty^2/2C_p) \qquad \ldots(74)$$

For getting the dimensionless temperature distribution in a boundary layer flow over an isothermal plate, we divide both sides of (72) by $(T_w - T_\infty)$ and get

$$\frac{T - T_\infty}{T_w - T_\infty} = \frac{T_w - T_r}{T_w - T_\infty}\theta_1(\eta, Pr) + \frac{U_\infty^2}{2(T_w - T_\infty)C_p}\theta_2(\eta, Pr) \qquad \ldots(75)$$

Now, $$\frac{T_w - T_r}{T_w - T_\infty} = 1 - \frac{T_r - T_\infty}{T_w - T_\infty} = 1 - \frac{1}{2} \times \frac{T_r - T_\infty}{U_\infty^2/2C_p} \times \frac{U_\infty^2}{C_p(T_w - T_\infty)}$$

$\therefore$ $$(T_w - T_r)/(T_w - T_\infty) = 1 - (1/2) \times r\,Ec, \text{ using (73) and (74)} \qquad \ldots(76)$$

Using (73) and (76), (75) may be re-written as

$$(T - T_\infty)/(T_w - T_\infty) = \{1 - (1/2) \times r\,Ec\}\theta_1(\eta, Pr) + (1/2) \times Ec\,\theta_2(\eta, Pr) \qquad \ldots(77)$$

For the present case, the temperature gradient at the plate can be obtained from (72) in the form

$$\left(\frac{\partial T}{\partial \eta}\right)_{\eta=0} = (T_w - T_r)\left(\frac{\partial \theta_1}{\partial \eta}\right)_{\eta=0} = -(T_w - T_r)\; a_1\,(\mathrm{Pr}), \text{ using (32)} \qquad \ldots(78)$$

whereas in the cooling problem, from (23) and (32), we have

$$\left(\frac{\partial T}{\partial \eta}\right)_{\eta=0} = -(T_w - T_\infty)\,a_1(\mathrm{Pr}). \qquad \ldots(79)$$

Comparing (78) and (79), we conclude that heat flow in a high velocity boundary layer (*i.e.* when frictional heat is taken into account) will be given by the same relation as in the low velocity boundary layer (i.e. when frictional heat is neglected) except that the temperature potential determining the heat flow for high velocity is the difference between the actual wall temperature and its recovery temperature. This rule was given by E. R. G Eckert.

19.5. Temperature distribution in the spread of a jet

In chapter 18, we have already studied the velocity distribution in the spread of a jet, for three cases (i) plane free jet in Art. 18.12 (ii) plane wall jet in Art. 18.13 and (iii) circular jet in Art. 18.14. We now propose to study the corresponding temperature distribution, neglecting the heat due to friction. Note that these are those examples, where compact solutions of the thermal boundary layer equations are possible.

19.6. Plane free jet (Two-dimensional jet) **[Himanchal 2000, 03, 04]**

Refer figure of Art. 18.12 in chapter 18 Consider the flow of an incompressible fluid passing through a slit O in the wall AB and the mixing with the surrounding fluid which is at rest and has a temperature T_∞. With the origin in the slit O, we choose x-axis and y-axis along and normal to the plane wall respectively. Neglecting the heat due to friction, the velocity and thermal boundary layer equations are given by

$$\partial u/\partial x + \partial v/\partial y = 0 \qquad \ldots(1)$$

$$u(\partial u/\partial x) + v(\partial u/\partial y) = \nu(\partial^2 u/\partial y^2) \qquad \ldots(2)$$

and
$$u(\partial\theta/\partial x) + v(\partial\theta/\partial y) = (\nu/\mathrm{P}_r) \times (\partial^2\theta/\partial y^2) \qquad \ldots(3)$$

where
$$\theta = (T - T_\infty)/T_\infty.$$

The boundary conditions for the given problem are

$$\left.\begin{array}{llll} y = 0: & \partial u/\partial y = 0, & v = 0; & \partial\theta/\partial y = 0 \ \text{(due to symmetry)} \\ y = \pm\infty: & & u = 0; & \theta = 0 \end{array}\right\} \qquad \ldots(4)$$

In addition to these boundary conditions, for a non-trivial solution, the following integral conditions should also be satisfied :

$$\int_{-\infty}^{\infty} \rho u^2 dy = \text{ const.} = J_0 \quad \text{(say)} \qquad \ldots(5)$$

and
$$\int_{-\infty}^{\infty} \theta\, u\, dy = \text{const.} = \frac{H_0}{T_\infty} \quad \text{(say)}, \qquad \ldots(6)$$

where J_0 and H_0 are to be prescribed.

For proof of (5), refer Art. 18.12. We now prove (6).

From (1), we have
$$\theta(\partial u/\partial x) + \theta(\partial v/\partial y) = 0 \qquad \ldots(7)$$

Adding (3) and (7),
$$\left(\theta\frac{\partial u}{\partial x} + u\frac{\partial\theta}{\partial x}\right) + \left(\theta\frac{\partial v}{\partial y} + v\frac{\partial\theta}{\partial y}\right) = \frac{\nu}{P_r} \times \frac{\partial^2\theta}{\partial y^2}.$$

or
$$\frac{\partial(\theta u)}{\partial x} + \frac{\partial(\theta v)}{\partial y} = \frac{\nu}{\mathrm{Pr}} \times \frac{\partial}{\partial y}\left(\frac{\partial\theta}{\partial y}\right).$$

Integrating it with respect to y between the limits $-\infty$ to ∞, we get

$$\frac{d}{dx}\int_{-\infty}^{\infty} \theta\, u\, dy + [\theta\, v]_{-\infty}^{\infty} = \frac{\nu}{\mathrm{Pr}} \times \left[\frac{\partial\theta}{\partial y}\right]_{-\infty}^{\infty} \qquad \ldots(8)$$

Since at the edge of the boundary $\partial\theta/\partial y = 0$ and $\theta = 0$ when $y = \pm\infty$, (8) reduces to

$$\frac{d}{dx}\int_{-\infty}^{\infty} \theta\, u\, dy = 0 \qquad \text{so that} \qquad \int_{-\infty}^{\infty} \theta\, u\, dy = \text{const} = \frac{H_0}{T_\infty} \quad \text{(say)}.$$

Since the velocity field given by (1) and (2) is independent of the temperature field, therefore we find first the velocity field and use this to find the temperature field from (3). In Art 18.12 we have found an exact solution of the velocity distribution. The necessary results which we shall require are given by

$$\left.\begin{array}{lll} u = 6\nu\alpha^2 x^{-1/3}\,\mathrm{sech}^2\xi & \text{and} & v = 2\alpha\nu x^{-2/3}(2\xi\,\mathrm{sech}^2\xi - \tanh\xi), \\ \text{where } \xi = \alpha\, y\, x^{-1/3} & \text{and} & \alpha = (J_0/48\rho\nu^2)^{1/2} \end{array}\right\} \qquad \ldots(9)$$

A solution of the thermal boundary layer equation (3) can be found by first reducing it into an ordinary differential equation with help of the transformation

$$\theta = (H_0/6\nu\alpha T_\infty) \times x^{-1/3} g(\xi) \qquad \ldots(10)$$

Substituting u, v and θ from (9) and (10) in (3), we get

$$g'' + 2\Pr\{g' \tanh\xi + g\,\text{sech}^2\xi\} = 0 \quad \text{or} \quad \frac{dg'}{d\xi} + 2\Pr\frac{d}{d\xi}(g\tanh\xi) = 0 \qquad \ldots(11)$$

with the boundary conditions $\xi = 0 : g' = 0$ and $\xi = \pm\infty$; $g = 0$. ...(12)

and the integral condition (6) reduces to $\displaystyle\int_{-\infty}^{\infty} g\,\text{sech}^2\xi\, d\xi = 1,$ ...(13)

where the prime denotes differentiation with respect to ξ.

Integrating (11), $g' + 2\Pr g \tanh\xi = C$, C being an arbitrary constant ...(14)

Putting $\xi = 0$ in (14) and using (12), we get $C = 0$. Then, (14) becomes

$$\frac{dg}{d\xi} = -2\Pr\, g\tanh\xi \qquad \text{or} \qquad \frac{dg}{g} = -2\Pr\tanh\xi\, d\xi.$$

Integrating, $\log g - \log C = -2\Pr\log\cosh\xi = 2\,Pr\log(\cosh\xi)^{-1} = 2\,P_r\log\text{sech}\,\xi$

or $\log(g/C) = \log(\text{sech}\,\xi)^{2P_r}$ or $g = C(\text{sech}\,\xi)^{2P_r}$ ...(15)

Relation (15) automatically satisfies the second boundary condition (namely, $g = 0$ when $\xi = \pm\infty$). Hence C can obtained by using (13). This leads to

$$C = \left\{2\int_0^\infty (\text{sech}\,\xi)^{2+2\Pr} d\xi\right\}^{-1} \qquad \ldots(16)$$

As particular case, when Pr = 1, (16) reduces to

$$C = \left\{2\int_0^\infty \text{sech}^4\xi\, d\xi\right\}^{-1} \qquad \ldots(17)$$

But $\displaystyle\int_0^\infty \text{sech}^4\xi\, d\xi = \int_0^\infty \text{sech}^2\xi\,(1-\tanh^2\xi)\, d\xi,$, as $\text{sech}^2\xi = 1 - \tanh^2\xi$

$$= \int_0^1 (1-t^2)dt, \text{ *putting } \tanh\xi = t \text{ and } \text{sech}^2\xi\, d\xi = dt$$

$$= \left[t - (t^3/3)\right]_0^1 = 1 - (1/3) = 2/3.$$

Then, (17) gives $C = [2 \times (2/3)]^{-1} = 3/4$ for $Pr = 1$. ...(18)

Thus the temperature distribution in a plane free jet flow in the absence of frictional heat is given with help of (10) and (15) in the following form

$$\theta = (CH_0/6\nu\alpha T_\infty)\, x^{-1/3}(\text{sech}\,\xi)^{2P_r} \qquad \ldots(19)$$

where C is given by (16).

Note. For Pr = 1, we easily find, from (2) and (3) that the vellocity and temperature distribution will be identical. Hence the Crocco's' integral also exists in the case of jet flow.

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* Since $t = \tan\xi = \dfrac{e^\xi - e^{-\xi}}{e^\xi + e^{-\xi}} = \dfrac{1-e^{-2\xi}}{1-e^{2\xi}}$ $\Rightarrow$ $t = 1$ when $\xi = \infty$ and $t = 0$ when $\xi = 0$.

19.7 The plane wall jet **[Himachel 2001, 02, 03]**

Refer figure of Art. 18.13 in chapter 18. The flow in a viscous jet bounded on one side by a wall and on the other by fluid at rest is known as a plane wall jet. Let an incompressible viscous fluid be discharged through a narrow slit at O in half space along a plane and mix with the surrounding fluid which is at rest and has a temperature T_∞. The wall is also maintained at the same constant temperature T_∞. With the origin in the slit O, we choose x-axis and y-axis along and normal to the plane wall respectively. Neglecting the heat due to friction, the velocity and thermal boundary layer equations are given by

$$\partial u/\partial x + \partial v/\partial y = 0 \qquad \ldots(1)$$

$$u(\partial u/\partial x) + v(\partial u/\partial y) = \nu(\partial^2 u/\partial y^2) \qquad \ldots(2)$$

$$u(\partial \theta/\partial x) + v(\partial \theta/\partial y) = (\nu/Pr) \times (\partial^2 \theta/\partial y^2) \qquad \ldots(3)$$

where $$\theta = (T - T_\infty)/T_\infty. \qquad \ldots(4)$$

The corresponding boundary conditions are

$$y = 0: \quad u = v = 0, \quad \theta = 0 \quad \text{and} \quad y = \infty: \quad u = 0, \quad \theta = 0. \qquad \ldots(5)$$

In addition to these boundary conditions, for a non-trivial solution, the following integral conditions should also be satisfied :

$$\int_0^\infty u^2 \left(\int_0^y u\, dy \right) dy = \text{const.} = E \text{ (say)} \qquad \ldots(6)$$

and $$\int_0^\infty \theta u \left(\int_0^y u\, dy \right) dy = \text{const.} = l \text{ (say)} \qquad \ldots(7)$$

For proof of (6), refer Art. 18.13. Following the similar method, we can easily prove (7).

Since the velocity field given by (1) and (2) is independent of the temperature field, therefore we find first the velocity distribution and use this to find the temperature field from (3). In Art. 18.13, we have found an exact solution of the velocity distribution The necessary results which we shall require are given by

$$\left. \begin{aligned} & u = (E/\nu x)^{1/2} f'(\eta), \quad v = (1/4) \times (\nu E/x^3)^{1/4} \times \{3\eta f'(\eta) - f(\eta)\}, \\ & \text{where} \quad \eta = (E/\nu^3)^{1/4} \times y x^{-3/4} \quad \text{and} \quad f(\eta) = f_\infty F(\eta), \quad f_\infty = (40)^{1/4} \end{aligned} \right\} \qquad \ldots(8)$$

and $F(\eta)$ is given by the equation

$$\eta = \frac{2}{f_\infty} \left\{ \log_e \frac{1 + \sqrt{F} + F}{(1 - \sqrt{F})^2} + 2\sqrt{3} \tan^{-1} \frac{\sqrt{3F}}{2 + \sqrt{F}} \right\}. \qquad \ldots(9)$$

Before solving (3) for arbitrary values of Pr, we find that for $Pr = 1$,

$$\theta = (l/E)u, \qquad \ldots(10)$$

is a solution of (3). It is known as *Crocco's integral.*

In order to solve (3) for arbitrary values of Pr, keeping the Croccos's integral (10) in mind, we take $$\theta = (l/E) \times (E/\nu x)^{1/2} \times h(\eta) \qquad \ldots(11)$$

Substituting u and v from (8) and θ from (11) in (3), we see that (3) reduces to an ordinary differential equation $$4h'' + \Pr(f h' + 2 f' h) = 0, \qquad \ldots(12)$$

with the boundary conditions, $$\eta = 0, \quad h = 0 \quad \text{and} \quad \eta = \infty, \quad h = 0 \qquad \ldots(13)$$

and the integral condition $$\int_0^{\infty} h f f' d\eta = 1, \qquad \ldots(14)$$

where prime denotes the differentiation with respect to η.

In order solve (12), the following transformations of the independent and dependent variables will be used :

$$s = \{F(\eta)\}^{3/2} \qquad \text{and} \qquad H(s) = (2/3) \times f_{\infty}^2 h(\eta). \qquad \ldots(15)$$

Then (12) reduces to

$$s(1-s)\frac{d^2H}{ds^2} + \left\{\frac{2}{3} - \left(\frac{2}{3} - \text{Pr} + 1\right)s\right\}\frac{dH}{ds} + \frac{4}{3}\text{Pr}H = 0, , \qquad \ldots(16)$$

with the boundary conditions : $s = 0, \quad H = 0 \quad$ and $\quad s \to 1, \quad H = 0. \qquad \ldots(17)$

and the integral condition $$\int_0^1 H s^{1/3} ds = 1. \qquad \ldots(18)$$

Solution of the hypergeometric equation (16) is given by

$$H\,(s) = A\ {}_2F_1\,(a, b; c; s) + B\, s^{1/2}\ {}_2F_1(a - c + 1, b - c + 1; 2 - c; s), \qquad \ldots(19)$$

where $\quad a + b = (2 / 3) - Pr, \quad ab = -(4 / 3) \times Pr \quad$ and $\quad c = 2/3. \qquad \ldots(20)$

Since Pr > 0 for a fluid, it follows that the conditions of absolute convergence of the series are satisfied here. Putting $s = 0$ is (19) and using (17), we find $A = 0$.

Now, re-writing the solution of (19), we have

$$s^{1/3}{}_2F_1(a - c + 1, b - c + 1; 2 - c; s) = \frac{s^{1/3}}{(1-s)^{P_r}}(1-s)^{P_r}\ {}_2F_1(1 - a, 1 - b; 2 - c; s) \qquad \ldots(21)$$

From (21), we see that the second boundary condition (namely ; $s \to 0; H = 0$) is identically satisfied by (19). Thus, B is still an unknown constant. Using (18), B is given by

$$B = \left\{\int_0^1 s^{2/3}{}_2F_1(a-c+1, b-c+1; 2-c; s)ds\right\}^{-1}$$

$$= (5/3) \times \left\{{}_3F_2(a-c+1, b-c+1, 5/3; 2-c, 8/3; 1)\right\}^{-1} \qquad \ldots(22)$$

Thus, $$H(s) = Bs^{1/2}\ {}_2F_1\,(a - c + 1, b - c + 1; 2 - c; s), \qquad \ldots(23)$$

where B is given by (22).

The dimensionless temperature gradient at the wall can be obtained by using (11), (15) and (23) and it is given by

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} = \frac{1}{(\nu E\, x)^{1/2}} a_1(Pr), \qquad \ldots(24)$$

where $$a_1(\text{Pr}) = B/8f_{\infty}. \qquad \ldots(25)$$

The heat flux through a across-section of the boundary layer is given by

$$W(x) = \int_0^{\infty} \theta u\, dy = \frac{BI}{(\nu Ex)^{1/4} f_{\infty}}\ {}_3F_2(a - c + 1; b - c + 1, 1; 2 - c, 2; 1) \qquad \ldots(26)$$

Again from Art. 18.13, the volume flux through a cross-section of the jet is given by

$$Q = f_{\infty}(\nu Ex)^{1/4}. \qquad \ldots(27)$$

From (26) and (27), we have $$QW = (5\ l\ /\ 3)\ b_1\ (\text{Pr}), \qquad \ldots(28)$$

where $$b_1(\mathrm{Pr}) = \frac{{}_3F_2(a-c+1, b-c+1,1; 2-c,2; 1)}{{}_3F_2(a-c+1, b-c+1,5/3; 2-c,8/3;1)} \qquad \ldots(29)$$

When Pr = 1, the series on the R.H.S. of (29) terminate and we easily get $QW = (20/9)\, I$, which is in comfirmity with equation $QW = (20/9)\, E$ of Art. 18.13, on noting the relation (10).

The relation (28) I shows that the product of the volume and heat flux through a cross-section of the boundary layer, is constant for a given Prandtl number and this may be taken as the physical meaning of the integral condition (7).

In the following figure we have plotted H, which is proportional to the dimensionless temperature distribution, against the similarity variable η for different values of Pr.

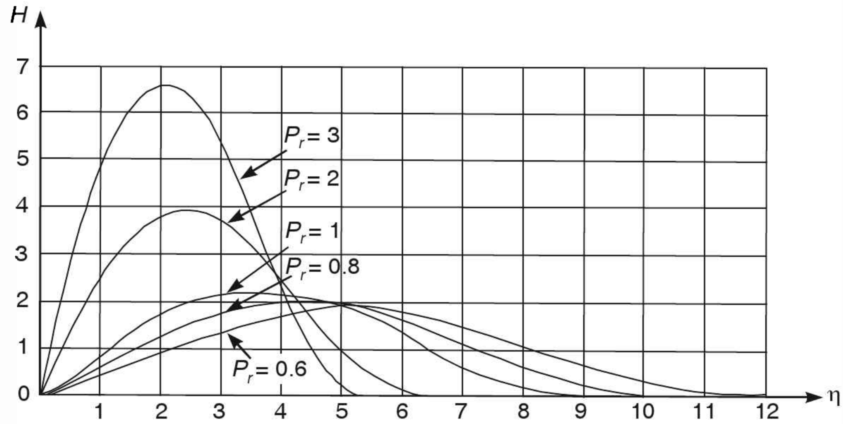


Fig. Tmperature distribution in a plane wall jet.

From the above figure, we find that the curve corresponding to Pr =1 is the same as for the velocity distribution. for Pr > 1, the thermal boundary layer is thinner than the velocity boundary layer whereas for Pr <1, the thermal boundary layer is thicker than the velocity boundary layer.

19.8. Circular jet (axially symmetric jet) [Himanchal, 2002, 04 05]

Suppose a fluid is issuing from a small circular orifice in a wall and is mixing with the same surrounding fluid at rest. Take the origin at the centre of the orifice and z-axis along the jet. Let r denote distance from z-axis. We shall use cylindrical polar co-ordinates for the given axially-symmetrical flow (for which $q_\theta = 0$ and all variables are independent of θ). Then negelecting body forces and heat due to friction, the velocity and thermal boundary layer equations are given by

$$\frac{\partial}{\partial r}(rq_r) + \frac{\partial}{\partial z}(rq_z) = 0 \qquad \ldots(1)$$

$$q_z \frac{\partial q_z}{\partial z} + q_r \frac{\partial q_z}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q_z}{\partial r}\right) \qquad \ldots(2)$$

and $$q_z \frac{\partial \theta}{\partial z} + q_r \frac{\partial \theta}{\partial r} = \frac{\nu}{r\,\mathrm{Pr}}\frac{\partial}{\partial r}\left(\frac{\partial \theta}{\partial r}\right), \qquad \ldots(3)$$

where $$\theta = (T - T_\infty)/T_\infty. \qquad \ldots(4)$$

The corresponding boundary and integral conditions are

$$r = 0: \quad q_r = 0, \quad \partial q_z/\partial r = 0, \quad \partial\theta/\partial r = 0 \quad \text{and} \quad r = \infty: \quad q_z = 0, \quad \theta = 0. \qquad \ldots(5)$$

$$\int_0^\infty \frac{g'^2}{\eta}\, d\eta = \text{const.} = \frac{M_0}{2\pi\mu} \quad \text{(say)} \qquad \ldots(6)$$

and $$\int_0^{\infty} (2\pi r) \times q_z \theta \; dr = \text{const} = \frac{N_0}{T_\infty} \text{ (say)} \qquad \ldots(7)$$

where M_0 and N_0 are to be prescribed. For proof of (6), refer Art. 18.14. Similarly, prove (7).

Since the velocity field given by (1) and (2) is independent of the temperature field, hence we first find the velocity distribution and use this to find the temperature field from (3). In Art. 18.14, we have found an exact solution of the velocity distribution. The necessary results which we shall require are given by

$$\left.\begin{array}{llll} & q_z = g'/\eta z & \text{and} & q_r = (\sqrt{\nu}/z) \times (g' - g/\eta) \\ \text{where} & \eta = \dfrac{r}{z\sqrt{\nu}}, & g(\eta) = \dfrac{\alpha^2 \eta^2}{1 + (1/4) \times \alpha^2 \eta^2}, & \alpha = \left(\dfrac{3M_0}{16\pi\mu}\right)^{1/2} \end{array}\right\} \qquad \ldots(8)$$

A solution of the thermal boundary layer equation (3) can be found by first reducing it into an ordinary differential equation with help of the transformation

$$\theta = (N_0/\nu T_\infty) z^{-1} h(\eta). \qquad \ldots(9)$$

Substituting the value of q_r, q_z and θ from (8) and (9) in (3) ,we get

$$\eta \frac{d^2 h}{d\eta^2} + \frac{dh}{d\eta} + \Pr\left(g \frac{dh}{d\eta} + h \frac{dg}{d\eta}\right) = 0 \qquad \text{or} \qquad \frac{d(\eta h')}{d\eta} + \Pr \frac{d}{d\eta}(gh) = 0 \qquad \ldots(10)$$

with the boundary conditions $\eta = 0$; $\quad h' = 0$; $\quad \eta = \infty$; $\quad h = 0 \qquad \ldots(11)$

and the integral condition $$\int_0^{\infty} g' h \, d\eta = \frac{1}{2\pi}. \qquad \ldots(12)$$

Integrating (10), $\quad \eta h' + Pr\, g\, h = C$, C being an arbitrary constant $\quad \ldots(13)$

Putting $\eta = 0$ and using (8) and (11), we get C = 0. Then, (13) reduces the

$$\eta \frac{dh}{d\eta} = -\Pr g\, h \qquad \text{or} \qquad \frac{h'}{h} = -\frac{\Pr \alpha^2 \eta}{1 + (1/4) \times \alpha^2 \eta^2}, \text{ using (8)}$$

Integrating it, $\quad h(\eta) = C' \{1 + (1/4) \times \alpha^2 \eta^2\}^{-2Pr}$, C' being an arbitrary constant $\quad \ldots(14)$

which automatically satisfies the second condition of (11), namely h = 0 when $\eta \to \infty$. Hence C' is still an unknown constant which is determined by help of (12). Then, we have

$$C = (1 + 2\Pr)/8\pi \qquad \ldots(15)$$

Finally, the required temperature distribution in a circular jet flow, in the absence of frictional heat, can be obtained with help of (9), (14) and (15) and it is given by

$$\theta = \frac{1 + 2\Pr}{8\pi} \frac{N_0}{\nu z T_\infty} \left(1 + \frac{1}{4} \alpha^2 \eta^2\right)^{-2Pr} \qquad \ldots(16)$$

Note that for Pr = 1, the velocity and temperature distribution will be identical.

19.9. Pohlhausen's method of exact solution for the velocity and thermal boundary layers in free convection from a heated vertical plate.

[Himanchal 2000, 01, 02, 03, 04]

Let a flat plate heated to a temperature T_w be placed vertically under gravity in a large body of fluid (air) which is at rest and has temperature T_∞ and density ρ_∞. Let the temperature difference between the plate and the fluid be small. Then the fluid properties may be treated as constant, but the small motion of the fluid in the neighborhood of the heated plate is caused by a buoyancy force due to density variations. We choose origin at the lower edge of the plate, x-axis along the plate and y-axis normal to it.

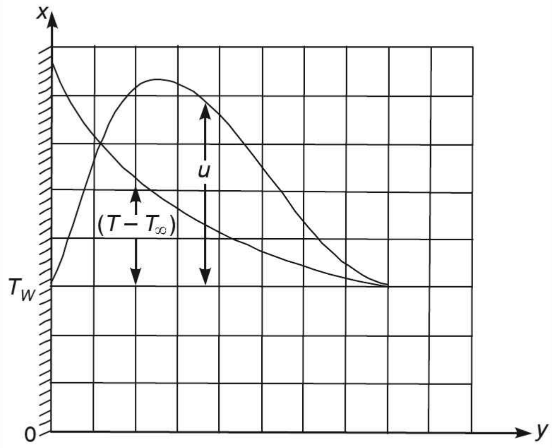


Fig. Temperature and velocity distributions in free convection from a heated vertical plate.

Neglecting the dissipation term in the thermal boundary layer and taking in account a body force due to gravity in the velocity boundary layer equation, the equations governing the motion are given by

$$\partial u/\partial x + \partial v/\partial y = 0, \qquad \ldots(1)$$

$$u(\partial u/\partial x) + v(\partial u/\partial y) = \nu(\partial^2 u/\partial y^2) + (1/\rho) \times Gx \qquad \ldots(2)$$

and

$$u(\partial T/\partial x) + v(\partial T/\partial y) = a(\partial^2 T/\partial y^2), \qquad \ldots(3)$$

where G_x is the resultant body force on the fluid of density ρ due to gravity in a medium of density ρ_∞. Using Archimedes principle, G_x is given by

$$G_x = g\rho_\infty - \rho g = \rho g\{(\rho_\infty/\rho) - 1\} = \rho g \beta(T - T_\infty), \qquad \ldots(4)$$

since for any fluid

$$1/\rho = (1/\rho_\infty) \times \{1 + \beta(T - T_\infty)\}, \qquad \ldots(5)$$

where β is the coefficient of thermal expansion.

In view of boundary layer assumptions, for gases the pressure in the boundary layer is the same as at the outer edge of it. Hence, from equation of state, we obtain

$$\rho_\infty/\rho = T/T_\infty. \qquad \ldots(6)$$

Hence, for gases (5) reduces to $\beta = 1/T_\infty$.

Then (4) gives $G_x = (\rho g/T_\infty) \times (T - T_\infty)$. ...(7)

Let $\theta = (T - T_\infty)/(T_w - T_\infty)$. ...(8)

Then, (2) and (3) may be re-written as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\left(\frac{T_w - T_\infty}{T_\infty}\right)\theta \qquad \ldots(9)$$

and

$$u(\partial\theta/\partial x) + v(\partial\theta/\partial y) = a(\partial^2\theta/\partial y^2). \qquad \ldots(10)$$

with corresponding boundary conditions

$$y = 0: \quad u = v = 0, \quad \theta = 1 \quad \text{and} \quad y = \infty: \quad u = 0, \quad \theta = 0 \quad \ldots(11)$$

We find that the velocity boundary layer equation (9) depends on temperature field and again the thermal boundary layer equation (10) depends on velocity field. In other words, the velocity and thermal boundary layer equations (9) and (10) are coupled and hence we have to solve them simultaneously. E.Pohlhausen solved them by reducing to ordinary differential equations, by making the following substitutions

$$\eta = \{g(T_w - T_\infty)/4\nu^2 T_\infty\}^{1/4} \times (y/x^{1/4}) = C(y/x^{1/4}), \quad \text{say} \qquad \ldots(12)$$

$$\psi(x, y) = 4\nu C\, x^{3/4} f(\eta) \qquad \text{and} \qquad \theta(x, y) = \theta(\eta), \qquad \ldots(13)$$

where ψ is stream function defined by $\quad u = \partial\psi/\partial y, \qquad v = -(\partial\psi/\partial x). \qquad \ldots(14)$

Using (12) and (13), (14) gives

$$u = 4\nu C^2 x^{1/2} f'(\eta) \qquad \text{and} \qquad v = \nu C\, x^{-1/4}(\eta f' - 3f). \qquad \ldots(15)$$

Then equation of continuity (1) is identically satisfed and equations (9) and (10) respectively reduce to

$$f''' + 2f f'' - f'^2 + \theta = 0 \qquad \ldots(16)$$

and

$$\theta'' + 3Pr\, f\theta' = 0. \qquad \ldots(17)$$

Again, the corresponding boundary conditions are

$$\eta = 0: \quad f = f' = 0, \quad \theta = 1 \qquad \text{and} \qquad \eta = \infty: \quad f' = 0, \quad \theta = 0. \qquad \ldots(18)$$

E. Pohlhausen solved(16) and (17) for air in series with $Pr = 0.733$ Using this result, we get

$$(\partial\theta/\partial\eta)_{\eta=0} = -0.508. \qquad \ldots(19)$$

For the present problem, the local Nusselt number $N_u\,(x)$ for heat transfer is given by

$$N_u(x) = \frac{-x(\partial T/\partial y)_{y=0}}{T_w - T_\infty} = -\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} \times Cx^{3/4} = 0.508x^{3/4}\left\{\frac{g(T_w - T_\infty)}{4\nu^2 T_\infty}\right\}^{1/4} \qquad \ldots(20)$$

Since $\quad G_r = \text{Grashoff number} = gx^3(T_w - T_\infty)/\nu^2 T_\infty, \qquad \ldots(21)$

(20) reduces to $\quad N_u\,(x) = 0.359\,(G_r)^{1/4}. \qquad \ldots(22)$

19.10. Thermal – energy integral equation or heat flux equation

[Meerut 2001, 01, 03, 04; Himanchal 2002]

Consider the steady flow of a viscous incompressible fluid past a thin semi-infinite flat plate at a constant temperature T_w (or insulated) placed along the direction of a uniform stream of velocity U_∞ and temperature T_∞. Let the origin of co-ordinates be at the leading edge of the plate, the x-axis along the plate and y-axis normal to it. Then, continuity and thermal boundary layer equations for the present forced convection in a laminar boundary layer on the flate plate are given by

$$\partial u/\partial x + \partial v/\partial y = 0 \qquad \ldots(1)$$

and

$$u(\partial T/\partial x) + v(\partial T/\partial y) = a(\partial^2 T/\partial y^2) + (\mu/\rho C_p) \times (\partial u/\partial y)^2, \qquad \ldots(2)$$

where $\quad a = k/\rho C_p$ (thermal diffusivity)

with boundary conditions

$$\left.\begin{array}{llll} y = 0: & u = v = 0, & T = T_w \text{ (isothermal)}, & \partial T/\partial y = 0 \text{ (adiabatic)} \\ y = \infty: & & u = U_\infty, & T = T_\infty \end{array}\right\} \qquad \ldots(13)$$

Let δ_t be the thermal boundary layer thickness. Then, integrating (2) with respect to y between the limits $y = 0$ to $y = \delta_t$, we have

$$\int_0^{\delta_t} u\frac{\partial T}{\partial x}dx + \int_0^{\delta_t} v\frac{\partial T}{\partial y}dy = -a\left[\frac{\partial T}{\partial y}\right]_0^{\delta_t} + \frac{\mu}{\rho C_p}\int_0^{\delta_t}\left(\frac{\partial u}{\partial y}\right)^2 dy \qquad \ldots(4)$$

We know that $\quad \partial T/\partial y = 0 \quad$ at $\quad y = \delta_t. \qquad \ldots(5)$

Also, $\quad \displaystyle\int_0^{\delta_t} v\frac{\partial T}{\partial y}dy = [vT]_0^{\delta_t} - \int_0^{\delta_t} T\frac{\partial v}{\partial y}dy,\quad$ integrating by parts or $$\int_0^{\delta_t} v\frac{\delta T}{\delta y}dy = [v]_{y=\delta_t}T_\infty - \int_0^{\delta_t} T\left(-\frac{\partial u}{\partial x}\right)dy \qquad \ldots(6)$$

[Using boundary condition (3) and (1). Also note that at $y = \delta_t, T = T_\infty$]

Next, integrating (1) with respect to y between the limits 0 to δ_t, we get

$$\int_0^{\delta_t}\frac{\partial u}{\partial x}dy + \int_0^{\delta_t}\frac{\partial v}{\partial y}dy = 0 \qquad \text{or} \qquad \int_0^{\delta_t}\frac{\partial u}{\partial x}dy + [v]_0^{\delta_t} = 0$$

or $$[v]_{y=\delta_t} = -\int_0^{\delta_t}\frac{\partial u}{\partial x}dy, \text{ using boundary conditon (3)}$$

$\therefore$ $$(6) \text{ and } (7) \Rightarrow \int_0^{\delta_t} v\frac{\partial T}{\partial y}dy = -T_\infty\int_0^{\delta_t}\frac{\partial u}{\partial x}dy + \int_0^{\delta_t} T\frac{\partial u}{\partial x}dy \qquad \ldots(8)$$

Using (5) and (8), (4) reduces to

$$\int_0^{\delta_t} u\frac{\partial T}{\partial x}dx - T_\infty\int_0^{\delta_t}\frac{\partial u}{\partial x}dy + \int_0^{\delta_t} T\frac{\partial u}{\partial x}dy = -a\left(\frac{\partial T}{\partial y}\right)_{y=0} + \frac{\mu}{\rho C_p}\int_0^{\delta_t}\left(\frac{\partial u}{\partial y}\right)^2 dy$$

or $$\int_0^{\delta_t}\left\{u\frac{\partial T}{\partial x} + (T - T_\infty)\frac{\partial u}{\partial x}\right\}dy = -a\left(\frac{\partial T}{\partial y}\right)_{y=0} + \frac{\mu}{\rho C_p}\int_0^{\delta_t}\left(\frac{\partial u}{\partial y}\right)^2 dy$$

or $$\int_0^{\delta_t}\frac{\partial}{\partial x}\{u(T - T_\infty)\}dy = -a\left(\frac{\partial T}{\partial y}\right)_{y=0} + \frac{\mu}{\rho C_p}\int_0^{\delta_t}\left(\frac{\partial u}{\partial y}\right)^2 dy$$

or $$\frac{d}{dx}\int_0^{\delta_t} u(T - T_\infty)dy = -a\left(\frac{\partial T}{\partial y}\right)_{y=0} + \frac{\mu}{\rho C_p}\int_0^{\delta_t}\left(\frac{\partial u}{\partial y}\right)^2 dy,$$

which is the required thermal energy integral equation taking the frictional heat into account.

EXERCISES

1. In the spread of a plane free thermal jet show that the velocity field is independent of the temperature field and obtain the velocity and temperature distribution.
[Himanchal 2000, 2001, 2002, 2003]

2. Obtain the exact solution of problem of the plate thermometer and show that for very large Prandtl numbers the recovery factor is given by $r = 1.92\ (Pr)^{1/3}$.
[Himanchal 2002, 2002, 2003]

3. Derive the temperature distribution of the cooling problem and discuss about temperature gradient at the wall when $0.6 < \Pr < 10$. **[Himanchal 2002]**

4. Derive (*i*) Crocco's fist integral **[Himanchal 2000, 2002, 2005]**
(*ii*) Crocco's second integral **[Himanchal 2000, 2001, 2005]**
(*iii*) Reynolds analogy **[Himanchal 2000, 2001, 2005]**

5. Write short notes an (*i*) Free and forced convection (*ii*) Crocco's integral (*iii*) Prandtl number (*iv*) Nusselt number **[Himanchal 2000]**

6. Discuss laminar free convection flow of an incompressible viscous fluid from a heated vertical plate and derive the expression of the local Nusselt number.
[Himanchal 1999, 2001, 2002, 2003, 2004]

7. Find the exact solution of the cooling problem and show that for very small Prandtl number Pr, the temperature gradinet at the wall is $0.564\ (\Pr)^{1/2}$ **[Himanchal 2004]**

8. Find the exact solution of the cooling problem and show for very large Prandtl number Pr, the temperature gradient is given by $0.339\ (Pr)^{1/3}$. **[Himanchal 2004]**

9. Derive temperature distribution and recovery factor of the problem of plate thermometer (adialatic wall). **[Himanchal 2002]**

CHAPTER

20 Flow of Inviscid Compressible Fluids. Gas Dynamics

20.1 Introduction.

Flow of compressible fluids involves appreciable variation in density throughout a flow field. Compressibility becomes significant at high flow speeds or for large temperature changes. Large changes in velocity produces large pressure changes; for gas flows these pressure changes are produced along with appreciable variations in both density and temperature. Gases are treated as compressible fluids and study of this type of flow is often referred to as '*gas dynamics*'. Some important problems where compressibility effect has to be considered are (*i*) flights of aeroplane at high altitude with high velocities (*ii*) flow of gases through orifices and nozzles (*iii*) flow of gases in compressors.

Another important property of compressible fluids is their ability to support *wave motion*. We shall show that a small disturbance in a compressible fluid is propagated throughout the fluid as a wave motion and at speed known as the *speed of sound*. On the other hand, a small disturbance created at any point of an incompressible fluid is transmitted everywhere in the fluid.

We know that compressions and expansions of the gas involve work done on and by the gas, respectively. This in turn produces changes in temperature of the gas. Hence the concepts of thermodynamics will play an important role to determine the physical quantities involved in a given problem.

In this chapter, large number of problems based on compressible fluid motion will be studied with the assumption of steady one dimensional flow. Hence the variation of fluid properties will occur only in one direction (namely, the direction of motion). It should be noted carefully that this direction of motion need not always be a straight-line motion. It can be the motion along the curved axis of a stream filament. However, in order that fluid properties at each cross-section of the filament to be constant, the curvature of the filament should not be too large. Although many real flows of interest are more complex, the above mentioned restrictions allow us to concentrate on the effects of basic flow process.

20.2 Some thermodynamic relations for a perfect (or ideal) gas

(*i*) **Perfect or ideal gas.** In some cases, the molecules of a gas have but negligible volume and there are almost no mutual attractions between the individual molecules. Such a gas is known as a *perfect (or ideal)* gas.

(*ii*) **Sepcific volume.** The specific volume of as a gas is denoted by v and is defined as the volume occupied by a unit mass of the gas. Thus, the specific volume v is the reciprocal of the density ρ, *i.e.*, $v = 1/\rho$.

(*iii*) **Bulk modulus.** The compressibility of a gas is expressed by the quantity bulk modulus, which is defined as the ratio of volumetric stress to the volumetric strain. If a small increase in pressure dp causes a change dv of the specific volume v, then the bulk modulus is denoted by K and is given by

$$K = \frac{dp}{-(dv/v)} \qquad \text{or} \qquad K = \rho\frac{dp}{d\rho},$$

where we have used the fact that $v = 1/\rho$ so that $dv = -(1/\rho^2)d\rho$.

(*iv*) **Equation of state of a gas.** Equation of state is defined as the equation which gives the relationship between the pressure p, density ρ and temperature T of a gas. For a pefect gas the equation of state is given by

$$p = \rho RT \qquad ...(1)$$

Since $\rho = 1/v$, (1) may be re-written as

$$pv = RT \qquad ...(2)$$

The constant R has different values for various gases. Its unit has the form

$$R = (\text{energy}) / (\text{mass} \times \text{temp}) = \text{J/ kg} - \text{K}$$

The value of R for air is 287 J/kg – K.

Note: Here T is measured in absolute (or kelvin) scale. Thus, if temperature of a body is t°C. then its temperature on the absolute scale is T = (t + 273) K.

(*v*) **Specific heats of a gas.** The specific heat C of a gas is defined as the amount of heat required to raise the temperature of a unit mass of a gas by one degree. Thus $C = \partial Q/\partial T$, where δQ is the amount of heat added to raise the temperature by δT. The value of specific heat depends on two well known processes, namely, the constant volume process and constant pressure process. The specific heats of the above processes are denoted and defined as follows :

Specific heat at constant volume $= C_v = (\partial Q/\partial T)_v$

Specific heat at constant pressure $= C_p = (\partial Q/\partial T)_p$

(*vi*) **Internal energy.** It is denoted by E and is defined as heat stored in a gas. When a certain amoun of heat is supplied to a gas, then temperature of gas may increase or volume of gas may increase thereby doing some external work or both temperature and volume may increase. It has been observed that whenever temperature of gas increases during its heating, its internal energy also increases. For a perfect gas, E is function of temperature T only; *i.e.*, $E = E\ (T)$

(*vii*) **Enthalpy.** The sum of internal energy E and pressure specific volume product pv is denoted by h and is known as *enthalpy* per unit mass of the system. It is also called the *total heat content*. Thus, we have

$$h = E + pv.$$

(*viii*) **Relations between C_v and C_p for a perfect gas.** Let Q be the total thermal energy per unit mass; E the internal energy per unit mass; p the pressure and v the specitfic volume of a perfect gas. Then, by the first law of thermodynamics, we have

$$dQ = dE + p\ dv \qquad ...(1)$$

Note that for a perfect gas, E is function of T only. Hence, from (1), we have

$$C_v = (\partial Q/\partial T)_v = dE/dT \qquad ...(2)$$

and

$$C_p = (\partial Q/\partial T)_p = dE/dT + p(\partial v/\partial T)_p,$$

or

$$C_p = C_v + p(\partial v/\partial T)_p \quad \text{using (2)} \qquad ...(3)$$

Relation (3) shows that $C_p \neq C_v$.

The equation of state for a perfect gas is given by

$$pv = RT \qquad \text{so that} \qquad v = RT/p \qquad ...(4)$$

Form (4),

$$(\partial v/\partial T)_p = R/p \qquad ...(5)$$

From (3) and (5), $$C_p = C_v + R \qquad ...(6)$$

or $$C_p - C_v = R \qquad ...(7)$$

From the kinetic energy theory of a perfect gas we know that the internal energy is directly proportional to the temperature T. Hence (2) shows that C_v must be a constant. Again R is a constant for a particular perfect gas. Hence (5) shown that C_p is also constant.

Since C_p and C_v are both constants, we have

$$C_p/C_v = \gamma \quad \text{or} \quad C_p = \gamma C_v, \qquad ...(8)$$

where γ is a constant known as the *adiabatic constant (or adiabatic index)*. From(6), we have $C_p > C_v$ and hence (8) shows that $\gamma > 1$. Solving (7)and (8), we have

$$C_v = R/(\gamma - 1) \quad \text{and} \quad C_p = \gamma R/(\gamma - 1) \qquad ...(9)$$

Now, from (2), $$E = C_v T + C, \qquad ...(10)$$

where C is constant of integration. If we choose the temperature scale in such a manner that $E = 0$ when $T = 0$. Then (10) given $C = 0$ and hence (10) reduces to

$$E = C_v T. \qquad ...(11)$$

Now, by definition of enthalpy (*vii*), we have

$$h = E + pv \quad \text{or} \quad h = E + RT, \text{ by equation of state}$$

$$\therefore \quad dh/dT = dE/dT + R \quad \text{or} \quad dh/dT = C_v + R, \text{ by (2)}$$

or $$dh/dT = C_p, \text{ using (6).} \qquad ...(12)$$

(*ix*) **Functions of state, entropy and isentropic flow,** Using the equation of state we can express any thermodynamic function ϕ in terms of any two of the measurable quantities p, ρ and T. Since $v = 1/\rho$, we write $\phi(p,v)$ as function of p and v. If, in changing from an initial state P_0 to another state P, the value of $\phi(p,v)$ depends only on the conditions at P_0 and P and not at all on the various paths joining them, then $\phi(p,v)$ is known as a *function of state*.

Let $$\phi(p,v) = M(p,v)\,dp + N(p,v)\,dv, \qquad ...(1)$$

where $M(p,v)$ and $N(p,v)$ are functions of p and v only. If $(\phi_P - \phi_{P_0})$ is independent of the path joining P_0 and P, then $d\phi$ must be an exact differential. The necessary and sufficient condition for $d\phi$ to be exact differential is $$(\partial M/\partial v)_p = (\partial N/\partial p)_v \qquad ...(2)$$

So when (2) holds, $\phi(p,v)$ must be a function of state.

From the first law of thermodynamics, we have $$dQ = dE + p\,dv. \qquad ...(3)$$

For a perfect gas, we know that $$dE = C_v\,dT \qquad ...(4)$$

and $$pv = RT \quad \text{so that} \quad dT = (1/R) \times d\,(pv) \qquad ...(5)$$

From (4) and (5), $$dE = (c_v/R) \times d\,(pv) \qquad ...(6)$$

$\therefore$ From (3) and (6), $$dQ = (c_v/R) \times (p\,dv + v\,dp) + p\,dv$$

or $$dQ = (C_v/R) \times v\,dp + (1 + C_v/R) \times p\,dv$$

or $$dQ = (C_v/R) \times v\,dp + (R + C_v) \times (p/R)\,dv$$

or $$dQ = (C_v/R) \times v\,dp + (C_p/R) \times p\,dv, \quad \text{as} \quad C_p = R + C_v$$

or $$dQ = T(C_v/p) \times dp + T(C_p/v) \times dv, \quad \text{as} \quad R = (pv)/T$$

or $$dS = (C_v/p) \times dp + (C_p/v) \times dv, \qquad ...(7)$$

where $$dS = (1/T) \times dQ. \qquad ...(8)$$

Comparing (7) with (1), here $M(p, v) = C_v/p$ and $N(p, v) = C_p/v$.

Hence $$(\partial M/\partial v)_p = 0 = (\partial N/dp)_v,$$

showing that dS is an exact differential and so S is a function of state.

Integrating (7), $S = C_v \log p + C_P \log v + S_0$, ...(9)

where S_0 is constant of integration. Since $C_p / C_v = \gamma$ so that $C_p = \gamma C_v$, (9) can be re-written as

$$C_v \log p + \gamma C_v \log v = S - S_0 \qquad \text{or} \qquad \log(pv^\gamma) = (S - S_0)/C_v$$

or
$$pv^\gamma = \exp[(S - S_0)/C_v] \qquad ...(10)$$

The quantity S is known as *entropy* per unit mass. Flows for which S is constant are called *isentropic*. Since S_0 and C_v are constants and S is constant for an isentropic flow, it follows that R.H.S. of (10) must also be a constant. Thus, isentropic flows satisfy the relation

$$pv^\gamma = \text{constant}. \qquad ...(11)$$

Since $v = 1/\rho$, (10) gives $S - S_0 = C_v \log(p/\rho^\gamma)$ ...(12)

(*x*) **Isothermal, isentropic, adiabatic and homentropic processes.**

Isothermal process. This is the process in which a gas is compressed or expanded while the temperature T is kept constant. In the case of a perfect gas, we have $vp = RT$. Hence an isothermal process is characterized by pv = constant, which is known as *Boyle's law*.

Isentropic process. If a change from a state P_0 to another state P takes places in such a manner that the entropy of every single particle of gas stays constant, then such a change is known as an *isentropic process*. Hence an *isentropic* process is characterised by pv^γ = constant.

Adiabatic process. If the compression and expansion of a gas takes place in such a way that the gas neither gives heat, nor takes heat from its surroundings, then the process is said to *be adiabatic*. For a reversible *adiabatic* process, we have pv^γ = constant.

Homentropic process. If the entropy of every single quantity of a gas of fixed mass is the same and stays constant in any change, then such a change is known as *homentropic process*.

Remark. Consider relation pv^γ = constant ...(1)

The constant on R.H.S. of (1) is the same for each considered small quantity of a gas in *isentropic flow* but a different constant must be taken corresponding to each such quantity. For homentropic flow, however, the constant on R.H.S. of (1) is always the same throughout the entire volume of gas.

Example. *Find the values of gas constant and adiabatic index for methane, whose values of specific heat at constant pressure and constant volume are* 2.17 *and* 1.65 *respectively.*

Sol. Let R be the required gas constant. Here C_p = 2.17 and C_v = 1.65. Hence, we have

$$R = C_p - C_v = 2.17 - 1.65 = 0.52 \text{ J/kg} - K.$$

Also, adiabatic index $= \gamma = C_p / C_v = 2.17/1.65 = 1.32$

20.3. Basic equations of motion of a gas.

(*i*) **The equation of continuity.** In the case of steady motion, the equation continuity (See Art. 2.8, Chapter 2) is given by $\nabla \cdot (\rho \boldsymbol{q}) = 0$, ...(1)

where $\boldsymbol{q}$ and ρ are velocity vector and density at any point P of a gas.

(*ii*) **The equation of motion.** In the case of steady motion under no body forces, the vector equation of motion (See Art. 3.1, Chapter 3) is given by

$$(\boldsymbol{q} \cdot \nabla)\boldsymbol{q} = -(1/\rho)\nabla p, \qquad ...(2)$$

where p is the pressure at any point P of a gas.

(*iii*) **Bernoulli's equation (Energy equation).** In the case of steady motion under no body forces, the Bernoulli's equation (See Art. 4.1, Chapter 4) is

$$\frac{1}{2}q^2 + \int \frac{dp}{\rho} = \text{constant}\,,\; q = |\boldsymbol{q}| = \text{magnitude of velocity} \qquad ...(3)$$

We now re-write (3) for an isetropic flow – for which the entropy of each particle remains constant along any streamline. For such a flow, we have

$$p/\rho^{\gamma} = \text{constant} = \text{k, say} \qquad \Rightarrow \qquad \rho = (p/k)^{1/\gamma} \qquad ...(4)$$

Substituting the value of ρ from (4), we obtain

$$\int \frac{dp}{\rho} = \int \frac{dp}{(p/k)^{1/\gamma}} = k^{1/\gamma} \int p^{-1/\gamma} dp = k^{1/\gamma} \frac{p^{1-1/\gamma}}{1-1/\gamma} = \left(\frac{p}{\rho^{\gamma}}\right)^{1/y} \frac{\gamma}{\gamma-1} p^{1-1/\gamma} = \frac{\gamma}{\gamma-1} \times \frac{p}{\rho} \qquad ...(5)$$

From (3) and (5),
$$\frac{1}{2}q^2 + \frac{\gamma}{\gamma-1} \times \frac{p}{\rho} = \text{constant}, \qquad ...(6)$$

where the constant is the same along any streamline, but unless the flow is homentropic it will vary from one streamline to another.

(*iv*) **Equation of state.** For a perfect (ideal) gas, the equation of state is given by

$$p = \rho R T, \qquad ...(7)$$

where R is a constant for the particular gas under consideration.

Remark. Equations (1), (2), (6) and (7) can be solved to find p, ρ, q and T. Solutions of these equations can be obtained only by numerical methods. Closed form (analytical) solutions, however, exist for a limited class of problems. Therefore, in what follows, we shall make some simplifying approximations. We shall consider steady one-dimensional inviscid compressible flows.

20.4. Basic equations for one-dimensional flow of a gas.

In this chapter, large number of problems based on compressible motion will be studied with the assumption of one dimensional flow. Accordingly, the variation of fluid properties will occur only in one direction, namely, the direction of motion. It should be noted carefully that this direction of motion need not always be a straight -line motion. It can be the motion along the curved axis of a stream filament. However, in order that fluid properties at each cross-section of the filament to be constant, the curvature of the filament should not be too large. Although the restrictions on one-dimensional flow appears to be too much this physical model is found to be good approximation to many actual flows.

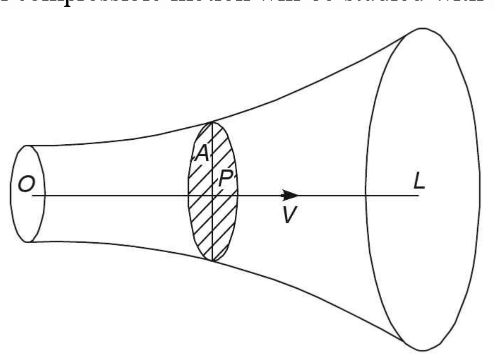


Fig. Flow through stream tube

Let a gas flow parallel to OL through a pipe whose cross-section at any point P on OL be A. Let pressure p, density ρ, temperature T and velocity V of the flowing gas be constant throughout the cross-section A.

(*i*) **The equation of continuity.** Mass of gas moving through the cross-section at P along OL is ρAV. Hence by the law of conservation of mass, the required equation of continuity is

$$\rho AV = \text{constant} = k, \text{ say} \qquad ...(1)$$

Differentiating (1), we get
$$AV\,d\rho + \rho V\,dA + \rho A\,dV = 0$$

Dividing by ρAV, we get
$$(1/\rho)\,d\rho + (1/A)\,dA + (1/V)\,dV = 0 \qquad ...(2)$$

which is known as *equation of continuity* in the differential form.

(*ii*) **The equation of motion.** It is obtained by equating the rate of change of momentum of the flowing gas in the direction of OL to the net force in the direction of OL.

(*iii*) **Bernoull's equation (Energy equation).** Refer Art 20.3 Take $q = V$. Thus, we have

$$\frac{\gamma}{\gamma-1} \times \frac{p}{\rho} + \frac{V^2}{2} = \text{constant}. \qquad ...(3)$$

(*iv*) **Equation of state.** $$p = R\rho T. \qquad ...(4)$$

20.5. The one-dimensional wave equation.

Let x denote a distance associated with a time t, and c a constant. Then consider a function $\phi(x,t)$ defined by $$\phi(x,t) = f(x-ct). \qquad ...(1)$$

In order that (1) may have any physical significance, the dimensions of c must be those of a speed. Again, from (1), we have

$$\phi\,(x + cT, t + T) = f\{(x + cT) - c\,(t + T)\} = f\,(x - ct) \qquad ...(2)$$

From (1) and (2), we get $$\phi\,(x + cT, t + T) = \phi\,(x, t), \qquad ...(3)$$

showing that the values of ϕ at distance x at time t is the same as that at distance $(x + cT)$ at time $(t + T)$.

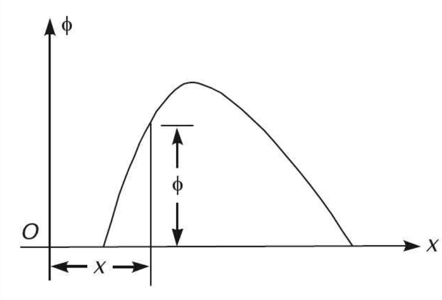


Fig. (*i*) Profitle at t

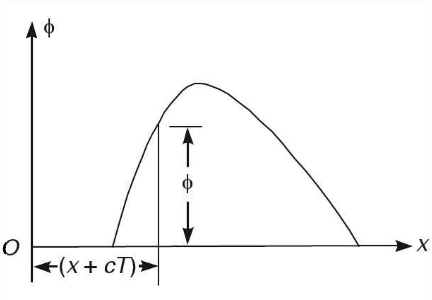


Fig. (*ii*) Profile at $(t + T)$

When ϕ is plotted against x at time t, then an arbitrary shape of graph is drawn as shown in figure (*i*). Similarly the plot of ϕ against x at time $(t + T)$ has been shown in figure (*ii*). We find that the second figure gives a profile congruent to the first but shifted along the positive x-axis a distance cT. As the displacement occurs in time T, we conclude that the profile must be moving with velocity c in the direction of positive x-axis. It follows that a possible physical interpretation of (1) is that $\phi(x,t)$ represents a disturbance moving in the positive x-direction with velocity c, its profile remaining undistorted. Thus (1) provides us with a mathematical model of one-dimensional wave propagation without distortion.

Proceeding likewise we may prove that $$\phi\,(x, t) = g\,(x + ct), \qquad ...(4)$$
represents a disturbance moving without distortion in the negative x-direction with speed c. Combining (1) and (4), it follows that

$$\phi\,(x, t) = f\,(x - ct) + g\,(x + ct) \qquad ...(5)$$

represents the superposition of a forward and a backward travelling wave, each moving with speed c.

Differentiating (5) partially *w.r.t.* 'x' twice, we have

$$\partial\phi / \partial x = f'(x-ct) + g'(x+ct)$$

and $$\partial^2\phi / \partial x^2 = f''(x-ct) + g''(x+ct) \qquad ...(6)$$

Again, differentiating (5) partially *w.r.t.* 't' twice, we get

$$\partial\phi / \partial t = f'(x-ct)\cdot(-c) + g'(x+ct)\cdot c$$

and $$\partial^2\phi/\partial t^2 = f''(x-ct)\cdot(-c)^2 + g'(x+ct)\cdot c^2 \quad ...(7)$$

From (6) and (7), we get $$\partial^2\phi/\partial t^2 = c^2(\partial^2\phi/\partial x^2) \quad ...(8)$$

which is known as one-dimensional wave equation. The form of ϕ *given by (5) is its general solution.*

Solved Example. *Find the profile* ϕ *(x, t) of a dimesnsional wave propagation if at* $t = 0$, $\phi = F(x)$ and $\partial\phi/\partial t = G(x)$.

Sol. One dimensional wave equation is given by

$$\partial^2\phi/\partial t^2 = c^2(\partial^2\phi/\partial x^2). \quad ...(1)$$

Given that $\phi(x, 0) = F(x)$ and $(\partial\phi/\partial t)_{t=0} = G(x)$ ...(2)

We know that the general solution of (1) is given by

$\phi(x, t) = f(x - ct) + g(x + ct)$, where f and g are arbitrary functions. ...(3)

Putting $t = 0$ in (3) and using (2), we have

$$f(x) + g(x) = F(x). \quad ...(4)$$

Differentiating (3) partially *w.r.t.* 't', we get

$$\partial\phi/\partial t = f'(x-ct)\cdot(-c) + g'(x+ct)\cdot c \quad ...(5)$$

Putting $t = 0$ in (5) and using (2), we get

$$G(x) = -c\,f'(x) + c\,g'(x) \qquad \text{or} \qquad -f'(x) + g'(x) = (1/c)G(x) \quad ...(6)$$

Integrating (6), $-f(x) + g(x) = 2A + (1/c)\int_0^x G(x)dx$, where $2A$ is an arbitrary constant` ...(7)

Solving (4) and (7) for $f(x)$ and $g(x)$, we get

$$f(x) = \frac{1}{2}F(x) - A - \frac{1}{2c}\int_0^x G(x)dx, \qquad \text{and} \qquad g(x) = \frac{1}{2}F(x) + A + \frac{1}{2c}\int_0^x G(x)dx$$

From these values of $f(x)$ and $g(x)$, we have

$$f(x-ct) = \frac{1}{2}F(x-ct) - A - \frac{1}{2c}\int_0^{x-ct} G(x)dx, \qquad g(x+ct) = \frac{1}{2}F(x+ct) + A + \frac{1}{2c}\int_0^{x+ct} G(x)dx$$

Substituting the above values of $f(x - ct)$ and $g(x + ct)$ in (3), we get

$$\phi(x,t) = \frac{1}{2}\{F(x-ct) + F(x+ct) + \frac{1}{2c}\int_{x-ct}^{0} G(x)dx + \frac{1}{2c}\int_0^{x+ct} G(x)dx$$

or $$\phi(x,t) = \frac{1}{2}\{F(x-ct) + F(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(x)dx. \quad ...(8)$$

20.6. Wave equations in two and in three dimensions.

Suppose a disturbance occurs in three dimensions in such a manner that the disturbance stays constant over any plane perpendicular to the direction of propagation. Then the wave is known as a *plane wave* and any such plane is know as *wave front*. Let $\boldsymbol{n}$ be unit vector and l, m, n be its direction cosines so that

$$\boldsymbol{n} = l\,\boldsymbol{i} + m\,\boldsymbol{j} + n\,\boldsymbol{k} \qquad \text{where} \qquad l^2 + m^2 + n^2 = 1 \quad ...(1)$$

Suppose a wave front travels with speed c in direction $\boldsymbol{n}$. Then the function $f(lx + my + nz - ct)$ will satisfy the necessary requirements because the wave fronts have equations lx + my + nz = constant, at any considered time. Likewise $g(lx + my + nz + ct)$ will represent a disturbance travelling in the direction $-\boldsymbol{n}$ with the same speed. It follows that the function

$$\phi(x, y, z, t) = f(lx + my + nz - ct) + g(lx + my + nz + ct) \quad ...(2)$$

will represent the superposition of plane waves travelling with speeds c in the directions $\pm$ $\boldsymbol{n}$. Differentiating (2) partially *w.r.t.* $x, y, z,$ and t, twice by turn, we finally obtain

$$\partial^2\phi/\partial x^2 = l^2 f''(lx+my+nz-ct)+l^2 g''(lx+my+nz+ct), \qquad ...(3)$$

$$\partial^2\phi/\partial y^2 = m^2 f''(lx+my+nz-ct)+m^2 g''(lx+my+nz+ct), \qquad ...(4)$$

$$\partial^2\phi/\partial z^2 = n^2 f''(lx+my+nz-ct)+n^2 g''(lx+my+nz+ct) \qquad ...(5)$$

and $$\partial^2\phi/\partial t^2 = c^2\{f''(lx+my+nz-ct)+g''(lx+my+nz+ct)\}$$

i.e., $$(1/c^2)\times(\partial^2\phi/\partial t^2) = f''(lx+my+nz-ct)+g''(lx+my+nz+ct) \qquad ...(6)$$

Adding (3), (4) and (5) and using (1), we get

$$\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2+\partial^2\phi/\partial z^2 = f''(lx+my+nz-ct)+g''(lx+my+nz+ct)$$

Thus, $$\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2+\partial^2\phi/\partial z^2 = (1/c^2)\times(\partial^2\phi/\partial t^2), \text{ using (7)} \qquad ...(7)$$

which is the equation of wave equation is three dimensions.

Equation (2) is the general solution of (7).

Likewise two-dimensional wave equation is given by

$$\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2 = (1/c^2)\times(\partial^2\phi/\partial t^2),$$

whose general solution is given by

$$\phi(x,y,t) = f(lx+my-ct)+g(lx+my+ct), \qquad \text{where} \qquad l^2+m^2 = 1.$$

20.7. Spherical waves

Wave equation $\nabla^2 u = (1/c^2)\times(\partial^2 u/\partial t^2)$ in three dimensions in terms of spherical polar co-ordinates (r,θ,ϕ) is given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2 u}{\partial\phi^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

When there is spherical symmetry, we have $u = u\ (r,\ t)$ and the above equation reduces to

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{or} \qquad \frac{\partial^2(ru)}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2(ru)}{\partial t^2} \qquad ...(1)$$

which is of the from of an one dimensional wave equation (compare equation (1) with equation (8) of Art 20.5, on replacing x by r and ϕ by ru). Hence the general solution of (1) is given by

$$ru = f\ (r-ct)+g\ (r+ct) \qquad \text{or} \qquad u\ (r,\ t) = (1/r)\times\{f\ (r-ct)+g\ (r+ct)\}$$

which represents concentric spherical wave fronts, with centre at origin O and having radii which increase or decrease at speed c. Due to presence of the factor $1/r$ in the solution, the wave profiles will change with time.

20.8A. The speed of sound in a gas.

Consider a small disturbance. produced within a non-viscous gas. Let us assume the following facts :

(*i*) The motion is irrotational and so a velocity potential ϕ exists such that the fluid velociy $\boldsymbol{q}$ at any point P is given by $$\boldsymbol{q} = -\nabla\phi. \qquad ...(1)$$

(*ii*) Let ρ_0 and p_0 respectively denote the equilibrium density and pressure of a mass of fluid at P. Then the squares and products of all distrbances from the equilibrium state are negligible.

Now, if $$\rho = \rho_0(1+s), \qquad ...(2)$$

then s is called the *condensation* and is a small quantity.

(*iii*) While dealing with propagation of sound waves we assume that the velocities of elements of fluid as so small that that their squares may be neglected

(*iv*) The changes take place so rapidly that heat exchanges may be neglected and hence entropy changes may be neglected. Hence the isentropic law holds and so

p/ρ^{γ} = constant = k, say or $p = k\rho^{\gamma}$ ...(3)

Since $\rho = \rho_0$ when $p = p_0$, (3) reduces $p_0 = k\rho_0{}^{\gamma}$ ...(4)

From (3) and (4), $p/p_0 = (\rho/\rho_0)^{\gamma}$ or $p = (p_0/\rho_0^{\gamma})\rho^{\gamma}$ ...(5)

The equation of continuity (See Art. 2.8) is given by

$$\partial\rho/\partial t + \nabla\cdot(\rho\boldsymbol{q}) = 0 \qquad ...(6)$$

or $$\frac{\partial}{\partial t}(\rho_0 + \rho_0 s) = -V\cdot\{\rho_0(1+s)\boldsymbol{q}\} \quad \text{or} \quad \frac{\partial s}{\partial t} = -\nabla\cdot(\boldsymbol{q} + s\boldsymbol{q}) \qquad ...(7)$$

Since s and $\boldsymbol{q}$ are both negligible their product $s\ \boldsymbol{q}$ can be neglencted. Then (7) reduces to

$$\partial s/\partial t = -\nabla\cdot\boldsymbol{q} \quad \text{or} \quad \partial s/\partial t = -\nabla\cdot(-\nabla\phi), \text{ using (1)}$$

Thus, $$\partial s/\partial t = \nabla^2\phi \qquad ...(8)$$

For irrotational motion under no body forces, the Bernoulli's equation (see Art. 4.1) is

$$-\frac{\partial\phi}{\partial t} + \frac{1}{2}q^2 + \int\frac{dp}{\rho} = \text{const.}$$

Since q^2 is negligible, we get $$-\frac{\partial\phi}{\partial t} + \int\frac{dp}{\rho} = \text{const.} \qquad ...(9)$$

Dufferentiating both sides of (5) *w.r.t.* ' ρ ', we have

$$\frac{dp}{d\rho} = \frac{\gamma p_0}{\rho_0^{\gamma}}\rho^{\gamma-1} = \frac{\gamma\ p_0}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = \frac{\gamma\ p_0}{\rho_0}(1+s)^{\gamma-1}, \text{ using (2)} \qquad ...(10)$$

Since s is small, so $$(1+s)^{\gamma-1} = 1, \text{ approximately} \qquad ...(11)$$

Let $$c_0^2 = \gamma\ p_0/\rho_0 \qquad ...(12)$$

Using (11) and (12), (10) reduces to $$dp/d\rho \simeq c_0^2. \qquad ...(13)$$

Now, $$\int\frac{dp}{\rho} = \int\left(\frac{dp}{d\rho}\cdot\frac{d\rho}{\rho}\right) = \int c_0^2\frac{d\rho}{\rho}, \text{ using (13)}$$

or $$\int\frac{dp}{\rho} = c_0^2\log\rho + \text{const.} \qquad ...(14)$$

Using (14), (9) reduces to

$$\partial\phi/\partial t = c_0^2\log\rho + \text{const.} = c_0^2\log\{\rho_0(1+s)\} + \text{const., using (2)}$$

$$= c_0^2\{\log\rho_0 + \log(1+s)\} + \text{const.}$$

$$= c_0^2\log(1+s) + \text{const.} = c_0^2\{s - s^2/2 + s^3/3 - \cdots\} + \text{const.}$$

$$= c_0^2 s + \text{const., to first order of appeoximation s}$$

Thus, $$\partial\phi/\partial t = c_0^2 s, \qquad ...(15)$$

which is obtained by absorbing the constant into s.

Form (15), $\partial^2\phi/\partial t^2 = c_0^2(\partial s/\partial t)$ or $\partial^2\phi/\partial t^2 = c_0^2\nabla^2\phi$, by (8)

which is a wave type equation. This equation shows that small disturbances are propagated in the gas with speed

$$c_0 = (\partial p/\partial\rho)^{1/2} = (\gamma p_0/\rho_0)^{1/2} \qquad ...(16)$$

This speed is known as the *speed of sound in the gas*.

Recall that (16) was obtained by assuming an isentropic flow. Accordingly, (16) is written as

$$c^2 = (\partial p/\partial\rho)_s \qquad ...(17)$$

where the subscript s indicates the partial derivative at constant entropy i.e. isentropic process.

20.8 B. An alternative method for derivation of velocity of sound.

Sound waves are infinitely small pressure disturbances. The speed with which sound propogates in a medium is known as *speed of sound* and is denoted by c.

Consider a one-dimensional flow through a long straight rigid pipe of uniform cross-sectional area A fitted with a frictionless piston at one end as shown in fig. (*i*). Initially, let the gas inside the tube be at rest. The pressure p and density ρ of the gas are initially uniform throughout the pipe.

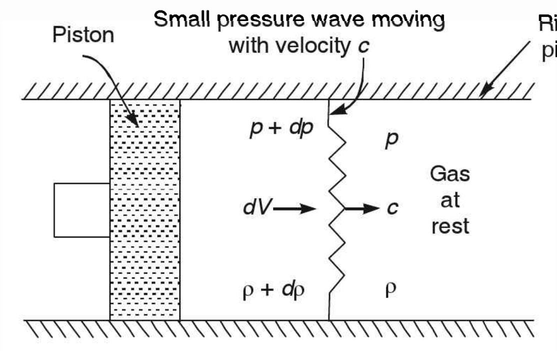


Fig. (*i*) Propagating wave front

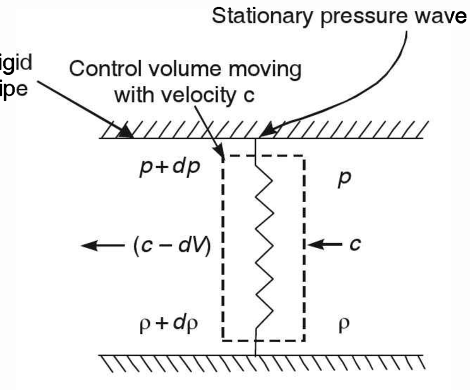


Fig. (*ii*) Stationary wave front

Let the piston be moving with an infinitesimal velocity dV to the right. Due to compressibility of gas, only the gas particles immediately adjacent to the piston will be pushed ahead. At any instant the gas between the piston and the pressure wave is compressed and will be in motion, but ahead of the wave front the gas will continue to remain stationary. Hence pressure and density in the stationary gas ahead of the pressure wave will remain p and ρ respectively whereas the gas in the proximity of the poston will experience an infinitesimal pressure increase dp and density increase $d\rho$. As shown in fig. (*i*), the values of pressure and density in back of the pressure wave will become $p + dp$ and $\rho + d\rho$ respectively.

We shall find the velocity of sound, c, by applying the continuity and momentum equation to a control volume of small length across the pressure wave, as shown in fig. (*ii*), where the pressure wave is brought to rest by superimposing a negative velocity c throughout the gas mass. In the steady-state model just created, note that the static pressure or the density or the temperature jump across the pressure wave will continue to remain the same as for the propagating pressure wave. The velocity approaching the control volume is c and the velocity leaving it is $c - dV$.

The equation of continuity for the control volume is given by

$$A\rho c = A(\rho + d\rho)(c - dV) \quad \text{or} \quad \rho c = \rho c + c d\rho - \rho dV - d\rho \times dV$$

or $\quad \rho dV = c d\rho$, neglecting second order term $d\rho \times dV$ of small quantity. $\qquad$ ...(1)

Again, on neglecting the friction and body forces, the momentum equation for the control volume is given by

$$Ap - A(p+dp) = A\rho c\{(c-dV)-c\} \qquad \text{or} \qquad dp = \rho c\, dV. \qquad ...(2)$$

Using (1), (2) gives $\quad dp = c \times (c d\rho) \quad$ or $\quad c^2 = dp/d\rho$

Thus, $\quad$ velocity of sound $= c = (dp/d\rho)^{1/2}$. $\qquad ...(3)$

Some alternative expressions for velocity of sound

I. Velocity of sound for isentrpic process. The thickness of wave pressure, across which the changes in physical properties occur, is very small. Hence the conducted heat from the solid boundary into the control volume is negligible. Again across the wave pressure on either side of the control surfaces, there is no temperature gradient, and so the conducted heat into the control volume is almost nil. Accordingly, we suppose the propagation of sound wave to be an adiabatic process. Moreover, the changes in temperature and pressure within the control volume being extremely small the adiabatic process may be treated as reversible *i.e.*, isentropic and hence we have

$$p = k\rho^{\gamma} \qquad ...(4)$$

From (4), $\quad \log p = \log k + \gamma \log \rho \quad$ or $\quad (1/p)\times(dp/d\rho) = \gamma/\rho \qquad ...(5)$

From (5), $\; dp/d\rho = \gamma p/\rho\;$ and hence (3) reduces to

$$c = \text{velocity of sund} = (\gamma p/\rho)^{1/2} \qquad ...(6)$$

II. Effect of change in temperature on velocity of sound.

From the equaiton of state, we have $p/\rho = RT$. So (6) reduces to

$$c = \text{velocity of sound} = \sqrt{\gamma RT} \qquad ...(7)$$

III. Velocity sound for an isothermal process.

For an isothermal process, we know that $\quad p = k\rho$, k being a constant $\quad ...(8)$

From (8), $\quad dp/d\rho = k \quad$ or $\quad dp/d\rho = p/\rho$, by (8) $\quad ...(9)$

But from the equation of state, $\quad p/\rho = RT$

Hence (9) becomes $\quad dp/d\rho = RT \qquad ...(10)$

Using (10), (3) gives $\quad$ velocity of sound $= c = \sqrt{RT}$. $\qquad ...(11)$

Remark. Isothermal process is considered for calculation of the velocity of sound waves (or pressure waves) only when it is mentioned in a given problem that process is isothermal. If nothing is mentioned, we assume that the process to be isentropic.

IV. Velocity of sound in terms of bulk modulus.

We know that the bulk modulus K of a gas is given by

$$K = \rho(dp/d\rho) \qquad \text{or} \qquad dp/d\rho = K/\rho$$

Hence, from (3), $\quad c$ = velocity of sound $= (K/\rho)^{1/2} \qquad ...(12)$

V. Effect of change in velocity on velocity of sound.

The energy equation (3) of Art 20.4 is given by

$$\frac{\gamma}{\gamma-1}\,\frac{p}{\rho} + \frac{V^2}{2} = \text{constant} = k, \text{ say} \qquad ...(13)$$

Now, $\quad$ the velocity of sound $= c = (\gamma p/\rho)^{1/2}$

$\therefore \quad$ (13) becomes

$$\frac{c^2}{\gamma-1} + \frac{V^2}{2} = k \qquad ...(14)$$

Let c_0 be the velocity of sound when the gas is at rest, *i.e.*, let $c = c_0$ when $V = 0$. Then (14) gives k = $c_0^2/(\gamma - 1)$. Hence (14) may be re-written as

$$\frac{c^2}{\gamma-1}+\frac{V^2}{2}=\frac{c_0^2}{\gamma-1} \qquad \text{or} \qquad c^2 = c_0^2 - \frac{(\gamma-1)V^2}{2} \qquad ...(15)$$

showing that for an isentropic expansion of a gas initially at rest, the velocity of sound is maximum when the gas is at rest.

20.9 Mach number and its importance.

Let V be the speed of a gas at certain location and c be the local speed of sound given by $c = (dp/d\rho)^{1/2}$. Then the local Mach number M is the dimensionless parameter defined by the relation.

$$M = V/c.$$

The Mach number gives us an important information about the type of compressible flow. In general, the flow is divided into the following three types on the basis of Mach number.

(*i*) **Subsonic flow.** A flow is said to be *subsonic* if $M < 1$, *i.e.*, the speed of flow of gas is less than the local speed of sound in it.

(*ii*) **Sonic flow.** A flow is said to be *sonic* if $M = 1$, *i.e.* the speed of flow of gas is equal to the local speed of sound in it.

(*iii*) **Supersonic flow.** A flow is said to be *supersonic* if $M > 1$, *i.e.*, the speed of flow of gas is greater than the local speed of sound in it.

Remark. When $M > 6$, the flow is known as *hypersonic*.

Subsonic, sonic and supersonic flows have many different physical features. Hence the knowledge of Mach number is important to distinguish clearly the expected flow pattern of a compressible flow.

20.9 A. Illustrative solved examples

Ex.1 *An aeroplane is flying at a height of 14 km where temperature is – 45°. The speed of the plane is corresponding to $M = 2$. find the speed of the plane, given that $R = 287$ J/kg-K and $\gamma = 1.4$.*

Sol. Given : $R = 287$ J/kg-K, $\gamma = 1.4$, Mach number $= M = 2$

and $t = -45°$ so that $T = t + 273° = -45° + 273° = 228$ K.

$$c = \text{Velocity of sound} = (\gamma RT)^{1/2} = (1.4 \times 287 \times 228)^{1/2} = 302.67 \text{ m/s}$$

Now, $M = V/c \Rightarrow V = Mc = 2 \times 302.67 = 605.34$ m/s

$\therefore$ Speed of plane $= V = (605.34 \times 3600) / 1000 = 2179.2$ km/hour.

Ex. 2. *Calculate the Mach number at point on a jet propelled aircraft, which is flying at 1100 km/hour at sea-level where air temperature is 20°, given that $\gamma = 1.4$ and $R = 287$ J/kg-K.*

Sol. Proceed as in Ex. 1 **Ans.** $M = 0.89$.

Ex. 3. *Estimate the percentage error involved in calulating the velocity of sound in air assuming that propagation occurs isothermally.*

Sol. Let c_1 and c_2 be values of the velocity of sound in isentropic and isothermal processes respectively. Then, we know that $c_1 = (\gamma RT)^{1/2}$ and $c_2 = (RT)^{1/2}$.

Hence the required percentage error

$$= \frac{c_1 - c_2}{c_1} \times 100 = \frac{(\gamma RT)^{1/2} - (RT)^{1/2}}{(\gamma RT)^{1/2}} \times 100 = \frac{\gamma^{1/2} - 1}{\gamma^{1/2}} \times 100 = \frac{(1.4)^{1/2} - 1}{(1.4)^{1/2}} \times 100$$

$$= 15.5\,\% \qquad (\because \gamma = 1.4 \text{ for air})$$

Ex. 4 *Find the sonic velocity for the following fluids* :

(*i*) Crude oil of sp. gr. 0.8 and bulk modulus 153036 N/cm^2.

(*ii*) Mercury of sp. gr. 13.6 and bulk modulus 2648700 N/cm^2.

Sol. (*i*) Given : $\rho = 0.8 \times 1000 = 800\, kg/m^3$, K = 153036 × 10^4 N/m^2.

$$\therefore \text{ Required sonic velocity } = \left(\frac{K}{\rho}\right)^{1/2} = \left(\frac{153036 \times 10^4}{800}\right)^{1/2} = 1383\,\text{m/s (approx.)}$$

(*ii*) Try yourself. **Ans.** 1395.55.

20.10. Subsonic, sonic and supersonic flows. Propagation of pressure waves (or disturbance) in a gas:

Whenever any disturbance is produced by a point source in a gas, the disturbance is propagated in all directions with a velocity of sound c. The nature of propagation of the pressure wave depends upon the Mach number. Let us consider the following four situations:

Case I. When a point source *O* omitting sound waves in a gas is at rest.

Refer fig. (*i*) In this case the spherical pressure wave fronts generated there shall move out as concentric spheres with the source O as the centre. S_1, S_2, S_3 ... and so on are the positions of spherical wave fronts of radii ct, $2ct$, $3ct$, ... at times t, $2t$, $3t$, ... seconds after starting from O.

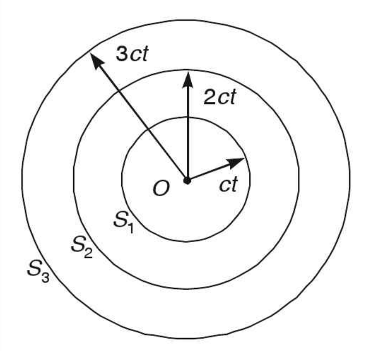


Fig. (*i*) V = 0. Stationary source

Case II. Subsonic flow.

Let a point source O be moving with velocity V from left to right in a gas at rest. Let $V < c$ (*i.e.* $M < 1$) so that the flow is subsonic, M being the Mach number of the point source. Refer figure (ii). Let O be the initial position of the point source. Let A, B, C be the positions of the moving source O at times, t, $2t$, $3t$ seconds after starting from O. Then the point source during a time interval of $3t$ will move from O to C such that $OC = 3\,Vt$. In the same interval of time $3t$, the initial spherical wave front originating at position O would have propagated to radius $3ct$. Now, $V < c \Rightarrow 3\,Vt < 3\,ct \Rightarrow OC < 3\,ct$. This shows that the pressure wave is able to move ahead of the point source at C at time $3t$.

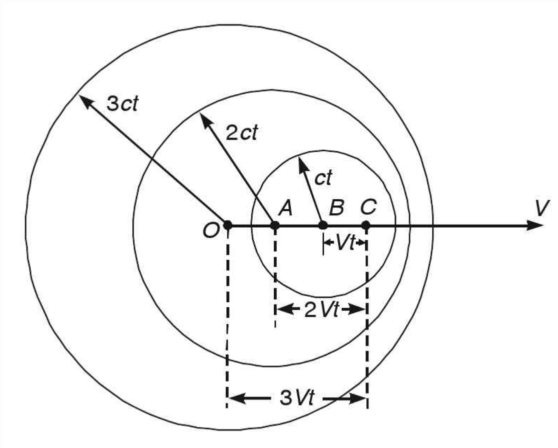


Fig. (*ii*) V < *c*, M < 1. Subsonic flow

Again, the point source during a time interval of t will move from O to A such that $AC = 2Vt$. In the same interval of time 2t, the initial spherical wave front originating at position A would have propagated to radius $2ct$. Since $V < c$, we have $2Vt < 2ct$ so that $AC < 2ct$, showing that the pressure wave is able to move ahead of the point source at A at time t. Similarly, we can show that the pressure wave is able to move ahead of the point source at B at time $2t$ and so on.

As shown in figure (*ii*), we find that when $V < c$, the pressure disturbance preceds the point source and will finally reach the entire fluid space. We also observe that the disturbances, at t, $2t$, $3t$, ... are the boundaries of non-intersecting spheres. Due to the special feature of the subsonic flow just discussed, if an aeroplane is flying at a subsonic speed, the noise created by it can be heard long before the plane reaches directly overhead.

Case III. Supersonic flow.

Let a point source O be moving with velocity V from left to right in a gas at rest. Let $V > c$ (*i.e.* $M > 1$) so that the flow is supersonic, Refer (*iii*). Let O be the initial position of the point source. Let A, B, C be the positions of the moving point source at times t, $2t$, $3t$ seconds after starting form O. Then the point source during a time interval of $3t$ will move from O to C such that $OC = 3\ Vt$. In the same interval of time 3t, the initial spherical wave front originating at position O would have propagated to radius $3ct$. Now $V > c \Rightarrow 3Vt > 3ct \Rightarrow OC > 3ct$. This shows that the pressure wave lags behind the point source at C at time $3t$.

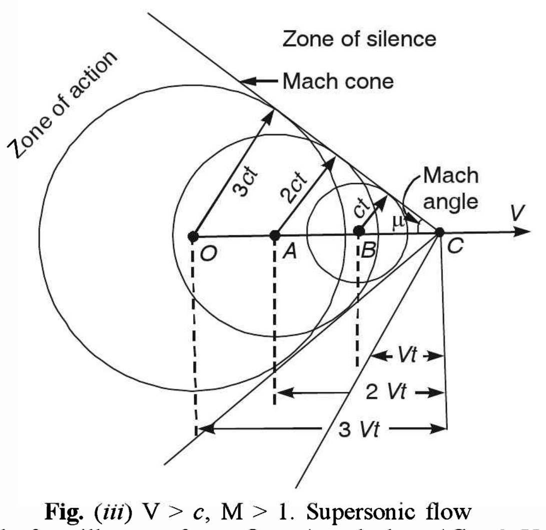


Fig. (*iii*) V > c, M > 1. Supersonic flow

Again, the point source during a time interval of t will move form O to A such that $AC = 2\ Vt$. In the same interval of time $2t$, the initial spherical wave front originating at position A would have propagated to radius $2ct$. Now $V > c \Rightarrow 2\ Vt > 2ct \Rightarrow AC > 2ct$. This shows that the pressure wave lags behind the point source at A at time t. Similarly, we can show that the pressure wave lags behind the point source at B at time $2t$ and so on.

From figure (*iii*), we find that when $V > c$, the pressure wave fronts at times t, $2t$, $3t$, are the boundaries of intersecting spheres. It follows that the effect of the pressure disturbances is restricted to the interior of the envelope of these spheres which is a cone whose vertex is situated at the source of the disturbances. Let μ be the semi-vertical angle of the right circular cone, then from figure, we have

$$\sin\mu = ct/Vt = 2ct/2Vt = 3ct/3Vt = \ldots,$$

$$\therefore \qquad \sin\mu = c/V \qquad \text{or} \qquad \sin\mu = 1/M. \qquad \text{so that} \qquad \mu = \sin^{-1}(1/M) \qquad \ldots(1)$$

This angle is known as *Mach angle* and the resultant cone, the Mach cone. From (1), we find that μ is real only when $M \geq 1$ (*i.e.* for supersonic and sonic flow). On the other hand μ does not exist for $M < 1$ (*i.e.* for subsonic flow). In two dimensional flow the Mach cone reduces to a pair of intersecting lines each of which is known as *Mach line* or a *Mach wave*.

The Mach cone divides the gas into two region*s* : the region inside the cone which is disturbed by the motion of the moving point source, and the region outside the cone, which remains undisturbed. The former region is known as the *zone of action* and the latter as *zone of silence*. Due to the special feature of the supersonic flow just discussed, if an aeroplane is flying at a supersonic speed, an observer cannot hear the noise until the aeroplane is far behind the observer.

Case IV. Sonic flow.

Let a point source O be moving with velocity V from left to right in a gas at rest. Let $V = c$ (*i.e.* $M = 1$) so that the flow is sonic. Refer figure (*iv*). Let O be the initial position of the point source.

Let A, B, C be the positions of the moving point source at times t, $2t$, $3t$ seconds after starting from O. Then the point source during a time interval $3t$ will move O to C such that $OC = 3Vt$. In the same interval of time 3t, the initial spherical wave front originating at position O would have propagated to radius $3ct$. Now $V = c \Rightarrow 3Vt = 3ct$. This shows that both the pressure wave and the point source reach at C simultaneously.

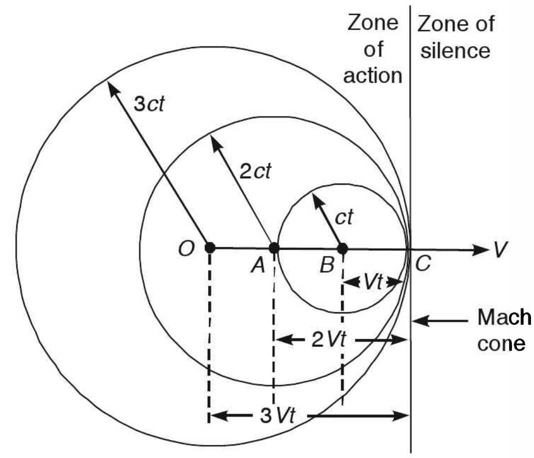


Fig. (*iv*) $V = c$, $M = 1$. Sonic flow

Again the point source during a time interval of t will move from O to A such that $AC = 2\,Vt$. In the same interval of time $2t$, the initial spherical wave front originating at position A would have propagated to radius $2ct$. Now $V = c \Rightarrow 2Vt = 2ct \Rightarrow AC = 2ct$. This shows that both pressure wave and the point source reach at C simultaneously. Similarly, we can show that the pressure wave and the point source reach at C at time $2t$ simultaneously. From figure (*iv*), we find that the disturbances at times t, $2t$, $3t$ always travel with point source. *i.e.*, the disturbances always remain attached to the source.

In this case the Mach angle is 90° and Mach cone is a plane perendicular to the direction of motion. The point in the downstream is unawave of the disturbance, also in this case.

20.10A Illustrative solved examples.

Ex. 1 *Find the velocity of bullet in air. if the Mach angle is 30°. Take $R = 287.1$ J/kg-K, $\gamma = 1.4$ and assume temperature as 15°C.*

Sol. Given : $R = 287.1$ J/kg-K, $\gamma = 1.4$, μ = Mach angle = 30°

and $t = 15°$ so that $T = t + 273° = 15° + 273° = 288$ K.

Then, c = velocity of sound = $(\gamma RT)^{1/2} = (1.4 \times 287.1 \times 288)^{1/2} = 340.2$ m/s

Let V be the required velocity of the bullet. Then

$$\sin\mu = c/V \quad \text{or} \quad V = c/\sin\mu = 340.2 / \sin 30° = 680.4 \text{ m/s}$$

Ex. 2. *If projectile is travelling in air having pressure and temperature 8.829 N/cm^2 and – 2° C. If the Mach angle is 40°, find the velocity of projectile. Take $\gamma = 1.4$ and $R = 287$ J/kg-K and sin 40° = 0.6427.*

Sol. Do as in Ex. 1. You need not use the data for pressure. Then, as before, the required velocity = 513 m/s.

Ex. 3. *A projectile travels in air of pressure 10.1043 N/cm^2 at 10° C at a speed of 1500 km/hour. Find the Mach number and Mach angle. Take $\gamma = 1.4$ and $R = 287$ J/kg-K. (Take $\sin^{-1} 0.8097 = 54.06°$).*

Sol. Given : $\gamma = 1.4$, R = 287 J/kg-K, V = velocity of projectile = 1500 km/hour = {(1500 × 1000)/ (60 × 60)} m/s = 416.67 m/s. Also, $t = 10°$ and so $T = 10° + 273° = 283$ K.

$\therefore$ c = velocity of sound = $(\gamma RT)^{1/2} = (1.4 \times 287 \times 283)^{1/2} = 337.2$ m/s.

$\therefore$ M = Mach number = $V/c = 416.67/337.2 = 1.235$.

Let μ be the Mach angle. Then, we have

$$\mu = \sin^{-1}(1/M) = \sin^{-1} 1/(1.235) = \sin^{-1}(0.8097) = 54.06°.$$

Ex.4. *A supersonic fighter plane moves with a Mach number of 1.5 in atmosphere at an altitude of 500 m above the ground level. What is the time that elapses by which the acoustic disturbance reaches the observer on the ground after it is directly overhead? Assume the atmospheric temperature to be 20°C.*

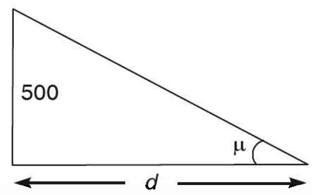
Sol. Here $M = 1.5$, $T = 20° + 273° = 293$ K. Take $R = 287$ J/kg- K and $\gamma = 1.4$. Then,

$$c = \text{velocity of sound} = (\gamma RT)^{1/2} = (1.4 \times 287 \times 293)^{1/2} = 343 \text{ m/s}$$

Let V be the velocity of the fighter plane. Then

$$M = V/c \Rightarrow V = Mc = 1.5 \times 343 = 515 \text{ m/s}$$

Let Mach angle be μ. Then, we have $\sin\mu = 1/M = 1/1.5 = 0.667$ so that $\mu = 41.8°$



Let d be the distance to be covered by the fighter plane before the signal reaches the observer. Then, from figure, we have

$$d = 500 \cot \mu = 500 \cot 41.8° = 500 \times 1.118$$

∴ The required time elapsed = (500 × 1.118)/515 = 1.09 seconds.

20.11 Isentropic gas flow.

Bernoulli's equation for flow of a gas is given by (refer Art. 20.4)

$$\frac{\gamma}{\gamma - 1} \frac{p}{\rho} + \frac{V^2}{2} = \text{constant} = k, \text{ say} \qquad \text{...(1)}$$

For isentropic flow of gas, velocity of sound c is given by $\qquad c^2 = \gamma p / \rho \qquad$ ...(2)

From (1) and (2), we get $\qquad c^2/(\gamma - 1) + V^2/2 = k, \qquad$ ...(3)

where the constant k is the same for a given streamline. From the L.H.S. of (3), it follows that V is maximum when $c = 0$; denote this value by V_{max}. Note that V_{max} is only a theoretical value and it can never be attained in practice. Putting $c = 0$ and $V = V_{max}$, (3) gives

$$k = (1/2) \times V_{max}^2 \qquad \text{...(4)}$$

When $V = c$, let the velocity of sound be denoted by c_*. Then c_* is known as *critical speed of sound*. Thus putting $V = c = c_*$ in (3), we get

$$\frac{c_*^2}{\gamma - 1} + \frac{c_*^2}{2} = k \qquad \text{or} \qquad k = \frac{\gamma + 1}{2(\gamma - 1)} c_*^2 \qquad \text{...(5)}$$

Let c_s, p_s, ρ_s and T_s be the values of c, p, ρ and T on the streamline when $V = 0$. Then c_s, p_s, ρ_s and T_s are known as the stagnation speed of sound, stagnation pressure, stagnation density and stagnation temperature respectively. Putting c = c_s and $V = 0$ in (3), we get

$$k = c_s^2 / (\gamma - 1) \qquad \text{...(6)}$$

Using (2), (4), (5) and (6), Bernoulli's equation (1) for isentropic flow along a streamline is

$$\frac{c^2}{\gamma - 1} + \frac{V^2}{2} = \frac{1}{2} V_{max}^2 = \frac{\gamma + 1}{2(\gamma - 1)} c_*^2 = \frac{c_s^2}{\gamma - 1} \qquad \text{...(7)}$$

Since $c = c_s$, $p = p_s$ and $\rho = \rho_s$ satisfy (2), we have $\qquad c_s^2 = \gamma p_s / \rho_s \qquad$ ...(8)

Using (2) and (8), (7) gives

$$\frac{\gamma p}{\rho(\gamma - 1)} + \frac{V^2}{2} = \frac{\gamma p_s}{\rho_s(\gamma - 1)} \qquad \text{...(9)}$$

We know that (Refer relation (9) in Art. 20.2) $\qquad C_p = (R\gamma)/(\gamma - 1) \qquad$ ...(10)

Also, equation of state for perfect gas is $\qquad p = \rho R T. \qquad$ ...(11)

$$\therefore \left.\begin{aligned} &\text{Hence,} \quad \frac{\gamma p}{\rho(\gamma - 1)} = \frac{\gamma \rho R T}{\rho(\gamma - 1)} = C_p T, \text{ by (10) and (11)} \\ &\text{Similarly,} \quad \gamma p_s / \rho_s (\gamma - 1) = C_p T_s \end{aligned}\right\} \qquad \text{...(12)}$$

Using (12) and (13), (9) can be re-written as $\qquad C_p T + (1/2) \times V^2 = C_p T_s \qquad$ ...(13)

Dividing both sides of (13) by $C_p T$, we get

$$\frac{T_s}{T} = 1 + \frac{V^2}{2} \times \frac{1}{C_p T} = 1 + \frac{V^2}{2T} \times \frac{\gamma - 1}{R\gamma}, \text{ using (10)}$$

or $\dfrac{T_s}{T} = 1 + \dfrac{\gamma - 1}{2}\left(\dfrac{V}{c}\right)^2,$ as $(R\gamma T)^{1/2} = c$

or $T_s / T = 1 + (M^2/2) \times (\gamma - 1)$ ...(14)

as M = Mach number = V/c. ...(15)

Now, (2) and (8) give $c_s^2 / c^2 = (p_s / p) \times (\rho / \rho_s)$ ...(16)

For an isetropic flow, $p / \rho^\gamma = \text{constant} = p_s / \rho_s^\gamma$

Hence, we have $p_s / p = (\rho_s / \rho)^\gamma$ ...(17)

and $\rho_s / \rho = (p_s / p)^{1/\gamma}$ ...(18)

From (16), (17) and (18), $\dfrac{c_s^2}{c^2} = \left(\dfrac{\rho_s}{\rho}\right)^\gamma \times \dfrac{\rho}{\rho_s} = \left(\dfrac{\rho_s}{\rho}\right)^{\gamma-1}$ ...(19)

and $\dfrac{c_s^2}{c^2} = \dfrac{p_s}{p} \times \left(\dfrac{p_s}{p}\right)^{-1/\gamma} = \left(\dfrac{p_s}{p}\right)^{(\gamma-1)/\gamma}$ ...(20)

Since $c^2 = \gamma R T$ and $c_s^2 = \gamma R T_s$, we get $c_s^2 / c^2 = T_s / T$ ...(21)

From (19), (20) and (21), $\dfrac{c_s^2}{c^2} = \left(\dfrac{\rho_s}{\rho}\right)^{\gamma-1} = \left(\dfrac{p_s}{p}\right)^{(\gamma-1)/\gamma} = \dfrac{T_s}{T}$ ...(22)

From (14) and (22), we have

$$\left(\frac{p_s}{p}\right)^{(\gamma-1)/\gamma} = 1 + \frac{(\gamma-1)M^2}{2} \Rightarrow \frac{p_s}{p} = \left\{1 + \frac{(\gamma-1)M^2}{2}\right\}^{\gamma/(\gamma-1)} \quad \text{...(23)}$$

and
$$\left(\frac{\rho_s}{\rho}\right)^{(\gamma-1)} = 1 + \frac{(\gamma-1)M^2}{2} \Rightarrow \frac{\rho_s}{\rho} = \left\{1 + \frac{(\gamma-1)M^2}{2}\right\}^{1/(\gamma-1)} \quad \text{...(24)}$$

Now, dividing both sides of (13) by $C_p\, T_s$, we get as before

$\dfrac{T}{T_s} = 1 - \dfrac{V^2}{2C_p T_s}$ or $\dfrac{T}{T_s} = 1 - \dfrac{V^2}{2T_s} \times \dfrac{\gamma - 1}{R\gamma}$, using (10)

or $\dfrac{T}{T_s} = 1 - \dfrac{V^2(\gamma-1)}{2c_s^2} = 1 - \dfrac{\gamma-1}{2}\left(\dfrac{V}{c_s}\right)^2,$ as $c_s^2 = R\gamma T_s$ ...(25)

From (7), we get $1/c_s^2 = 2/(\gamma+1)c_*^2$ ...(26)

Using (26), (25) becomes $\dfrac{T}{T_s} = 1 - \dfrac{\gamma-1}{\gamma+1}\left(\dfrac{V}{c_*}\right)^2$ ...(27)

With help of (22) and (27), we obtain

$$\left(\frac{p}{p_s}\right)^{(\gamma-1)/\gamma} = \frac{T}{T_s} = 1 - \frac{\gamma-1}{\gamma+1}\left(\frac{V}{c_*}\right)^2 \Rightarrow \frac{p}{p_s} = \left\{1 - \frac{\gamma-1}{\gamma+1}\left(\frac{V}{c_*}\right)^2\right\}^{\gamma/(\gamma-1)} \quad \text{...(28)}$$

and $$\left(\frac{\rho}{\rho_s}\right)^{\gamma-1}=\frac{T}{T_s}=1-\frac{\gamma-1}{\gamma+1}\left(\frac{V}{c_*}\right)^2 \Rightarrow \frac{\rho}{\rho_s}=\left\{1-\frac{\gamma-1}{\gamma+1}\left(\frac{V}{c_*}\right)^2\right\}^{1/(\gamma-1)} \qquad ...(29)$$

At sonic or critical speeds ($M = 1$), when $V = c_*$, p = p_* and $\rho = \rho_*$ (27) and (28) and (29) can be re-wtitten as (taking $\gamma = 1.4$ for air)

$$T/T_s = 2/(\gamma+1) = 2/(1.4+1) = 0.833 \text{ approx.}$$

$$\frac{p}{p_s}=\left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} = 0.528 \text{ approx.} \quad \text{and} \quad \frac{\rho}{\rho_s}=\left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} = 0.630 \text{ approx.}$$

20.11A. Illustrative Solved Examples.

Ex.1. *Find the Mach number when an aeroplane is flying at* 1100 *km/hour through still air havig a pressure of* 7 *N/cm*^{2} *and temp* – 5° *C. Take* $\gamma = 1.4$ *and* R = 287.14 *J/kg–K. Calculate the temperature, pressuree and density of air at stagnation point on the nose of the aeroplane.*

Sol. Here R = 287.14 J/kg-K, $\gamma = 1.4$, T = – 5° + 273° = 268 K and V = velocity of aeroplane = (1100 × 1000)/(60 × 60) = 305.55 m/s. and pressure = p = 7N/cm^2 = 7 × 10^4 N/m^2.

Then c = velocity of sound = $(\gamma RT)^{1/2}$ = (1.4 × 287.14 × 268)$^{1/2}$ = 328.2 m/s. Hence, if M be the required Mach number, Then, $M = V/c$ = 305.55/328.2 = 0.931 (approx.)

To find the stagnation temperature T_s. From relation (14) of Art 20.11, we have

$$T_s = T\left\{1+\frac{M^2(\gamma-1)}{2}\right\} = 268\left\{1+\frac{(0.931)^2(1.4-1)}{2}\right\} = 314.44°\text{K}$$

$$\therefore \qquad t_s = T_s - 273° = 314.44° - 273° = 41.44°\text{C}$$

To find the stagnation pressure p_s. From relation (23) of Art. 20.11, we have

$$p_s = p\left\{1+\frac{(\gamma-1)M^2}{2}\right\}^{\gamma/(\gamma-1)} = 7\times10^4\left\{1+\frac{(1.4-1)}{2}\times(.931)^2\right\}^{(1.4)/(1.4-1)}$$

$$= 7\times10^4\,(1+.1733)^{3.5} = 12.24\times10^4\ \text{N/m}^2 = 12.24\ \text{N/cm}^2.$$

To find the stagnation density ρ_s. We can use the relation (24) of Art. 20.11 or proceed as follows. From equation of state, $p_s = R\rho_s T_s$, we have

$$\rho_s = \frac{p_s}{RT_s} = \frac{12.24\times10^4}{187.14\times314.44} = 1.355\ \text{kg/m}^3.$$

Ex. 2. *An aeroplane is flying at 1000 km/hour through still air having a pressure of 78.5 KN/m*^{2} *(absolute) and temp. – 8° C. Calculate on the stagnation point on the nose of the aeroplane (i) stagnation temperature (ii) stagnation pressure (iii) stagnation density. Take for air, R = 287 J/kg-K and* $\gamma = 1.4$.

Sol. Proceed as in Ex. 1. **Ans.** (*i*) 303.4 K or 30.4° C (*ii*) 126.1 KN/m^2 (*iii*) 1.448 kg/m^3.

20.12. Nozzle and Diffuser.

A nozzle is a duct of short length having variable cross-sectional area for accelerating fluid from a reservoir. Since a nozzle has a short length, the flow through it may be assumed frictionless and isentropic. A nozzle functions as a diffuser when it decelerates a fluid.

20.13 Flow through a nozzle.

In the adjoining figure is shown schematically a convergent – divergent nozzle, known as the Laval – nozzle. We propose to discuss the isentropic flow of a gas through this nozzle with the assumption that the curvature of the tube is very small and the variable cross-sectional area is sufficiently small for the velocity, pressure, density and temperature to be considered constant over each cross-section. Thus we shall assume the flow to be one dimensional.

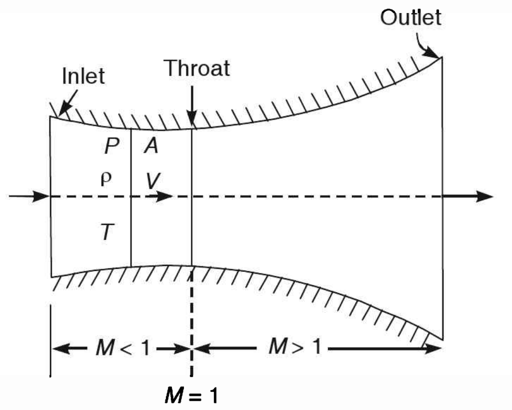


Figure. Laval nozzle

Let p, ρ, V be the pressure, density and velocity at a location of the channel of section A. For a steady flow of gas through the nozzle, the equation of continuity is

ρAV = const. = k = say or $\log\rho + \log A + \log V = 0$

Differentiating it, $(1/\rho)\,d\rho + (1/A)\,dA + (1/V)\,dV = 0$ ...(1)

Bernoulli's equation in absence of body forces for a steady flow of gas is given by

$$\frac{V^2}{2} + \int \frac{dp}{\rho} = \text{const.} \qquad ...(2)$$

Re-writing (2) in differential form, we have $V\,dV + (1/\rho)\,dp = 0$ ...(3)

The velocity of sound c for isentropic flow of gas is given by

$c^2 = dp/d\rho$ or $dp = c^2 d\rho$ ...(4)

From (3) and (4), $V\,dV + (1/\rho)c^2 d\rho = 0$ or $(1/\rho)\,d\rho = (-V/c^2)\,dV$ ...(5)

Substituting the value of $(1/\rho)\,d\rho$ form (5) in (1), we get

$-(V/c^2)\,dV + (1/A)\,dA + (1/V)\,dV = 0$ or $(1/A)\,dA + (1/V) \times (1 - V^2/c^2)\,dV = 0$

or $(1/A)\,dA + (1 - M^2) \times (1/V)\,dV = 0,$ ...(6)

where M = the local Mach number = V/c ...(7)

From (6), $(1 - M^2) \times (1/V)\,dV = -(1/A)\,dA$ ...(8)

The following information can be derived from the area – velocity relation (8) : Three cases arise :

Case (*i*). If $M < 1$, *i.e.*, the flow is subsonic, then (8) shows that a decrease in A produces an increase in V and conversely. Accordingly , to accelerate subsonic flow through a channel it is necessary to decrease the channel section A downstream of the flow.

Case (ii). If $M > 1$, *i.e.*, the flow is supersonic, then (8) shows that A and V increase or decrease simultaneously. Accordingly, to accelerate supersonic flow it is necessary to widen the channel downstream of the flow.

Case (*iii*). When $dA/A = 0$, (8) shows that either $M = 1$ (*i.e.* the flow is sonic) or $dV = 0$. The situation $dV = 0$. occurs in incompressible flow where the speed of flow attains maximum value at the stage when the channel section acquires a minimum area of cross-section. For compressible fluids, M may be unity when the area of cross-section A is minimum. The minimum section is known as the *throat*.

The results of the above discussion can be summarized as follows.

If, starting with subsonic flow in a channel, we decrease the section downstream, then the flow is accelerated until the section has attained a certain minimum at which the Mach number is unity. If beyond this minimum section we now widen the channel ; then the flow can be accelerated downstream of the section to produce supersonic flow.

This is the principle of flow through a *nozzle*.

Derivation of area-density and area-pressure relations

From (5), $dV = -(c^2/\rho V)\,d\rho$. Substituting this values of dV in (8) and using (7), we get

$$(1-M^2)\times\left(-\frac{c^2}{\rho V^2}\right)d\rho = -\frac{dA}{A} \qquad \text{so that} \qquad \frac{d\rho}{\rho} = \frac{M^2}{1-M^2}\times\frac{dA}{A} \qquad ...(9)$$

From (4), $d\rho = (1/c^2)\,dp$, Substituting this value in (9), we get

$$\frac{dp}{\rho c^2} = \frac{M^2}{1-M^2}\times\frac{dA}{A} \qquad \text{or} \qquad \frac{dp}{\rho}\times\frac{\rho}{\gamma p} = \frac{M^2}{1-M^2}\times\frac{dA}{A}, \qquad \text{as} \qquad c = \left(\frac{\gamma p}{\rho}\right)^{1/2}$$

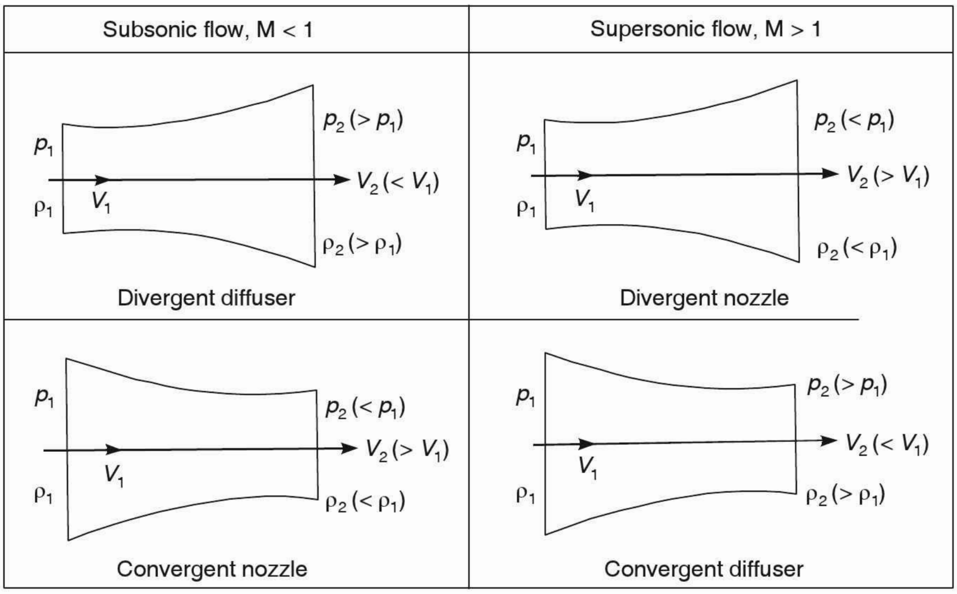
or
$$(1/p)\,dp = \left\{(\gamma M^2)/(1-M^2)\right\}\times(dA/A) \qquad ...(10)$$

The flowing information can be derived from the above relations (9) and (10) :

Case (*i*). If the flow is subsonic, *i.e.* $M < 1$, then the density ρ and pressure p increase or decrease with respective increase or decrease in the area of cross-section A.

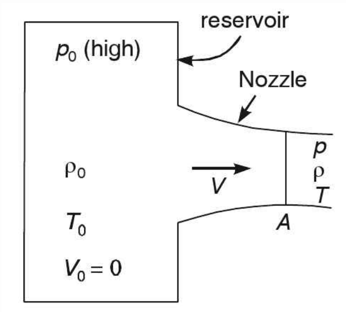
Case (*ii*). If the flow is supersonic, *i.e.* $M > 1$, then the density ρ and pressure p increase or decrease with the respective decrease or increase in the area of cross section A.

The relations (8), (9) and (10) together are illustrated with the help of the following figures.



20.14. To determine maximum mass flow through a nozzle.

Consider a gas filled in a large reservoir. Let the gas be at rest (*i.e.* $V_0 = 0$) and let it be at high pressure p_0, density ρ_0 and temperature T_0. Let an open-ended axially-symmetric nozzle be fitted to the reservoir. Let us consider an isentropic compressible flow from the reservoir where the stagnation conditions exist through the nozzle of varying section. We propose to discuss the flow through the nozzle with the assumption that the curvature of the nozzle is very small and the variable cross-sectional area is sufficiently small for the velocity, pressure, density and temperature to be considered constant over each cross-section. Thus we shall assume the flow to be one-dimensional. Let p, ρ, V be the pressure, density and velocity at a location of the channel of section A.



Applying Bernoulli's equation along a streamline from the reservoir to the section A, we have

$$\frac{V^2}{2} + \frac{\gamma}{\gamma-1} \times \frac{p}{\rho} = \frac{V_0^2}{2} + \frac{\gamma}{\gamma-1} \times \frac{p_0}{\rho_0}, \qquad \text{where} \qquad V_0 = 0.$$

or
$$V^2/2 = \{\gamma/(\gamma-1)\} \times (p_0/\rho_0 - p/\rho) \qquad \text{...(1)}$$

or
$$V^2 = \frac{2\gamma}{\gamma-1} \times \frac{p_0}{\rho_0}\left(1 - \frac{\rho_0}{\rho} \times \frac{p}{p_0}\right) \qquad \text{...(2)}$$

For isentropic flow, $\qquad p_0/\rho_0^\gamma = p/\rho^\gamma \qquad$ or $\qquad \rho_0/\rho = (p_0/p)^{1/\gamma}$ ...(3)

Substituting the value of ρ_0/ρ from (3) in (2), we have

$$V^2 = \frac{2\gamma}{\gamma-1} \times \frac{p_0}{\rho_0}\left\{1 - \left(\frac{p_0}{p}\right)^{1/\gamma} \times \frac{p}{p_0}\right\} = \frac{2\gamma}{\gamma-1} \times \frac{p_0}{\rho_0}\left\{1 - \left(\frac{p}{p_0}\right)^{-1/\gamma} \times \frac{p_0}{p}\right\}$$

Thus,
$$V = \left[\frac{2\gamma}{\gamma-1} \times \frac{p_0}{\rho_0}\left\{1 - \left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}\right\}\right]^{1/2} \qquad \text{...(4)}$$

Now, the mass flux m across the section A per unit time is given by

$$m = \rho V A = \rho_0 V A (p/p_0)^{1/\gamma}, \text{ using (3)}$$

$$= A\rho_0\left(\frac{p}{p_0}\right)^{1/\gamma}\left[\frac{2\gamma}{\gamma-1} \times \frac{p_0}{\rho_0}\left\{1 - \left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}\right\}\right]^{1/2}, \text{ using (4)}$$

Thus,
$$m = A\left[\frac{2\gamma}{\gamma-1} p_0\, \rho_0 \left(\frac{p_0}{\rho_0}\right)^{2/\gamma}\left\{1 - \left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}\right\}\right]^{1/2} \qquad \text{...(5)}$$

Let $p/p_0 = P$. Then (5) may be re-written as

$$m^2 = k P^{2/\gamma}\{1 - P^{(\gamma-1)/\gamma}\} \qquad \text{or} \qquad m^2 = k\{P^{2/\gamma} - P^{(\gamma+1)/\gamma}\}, \qquad \text{...(6)}$$

where $\qquad k = A^2\{2\gamma\, p_0\, \rho_0/(\gamma-1)\} = \text{constant}.$

Equation (6) shows that m^2 is a function of variable P.

To find the value of P (*i.e.* p/p_0) for maximum value of mass rate flow m.

From (6), we have

$$2m\frac{dm}{dP} = k\left[\frac{2}{\gamma}P^{(2/\gamma)-1} - \frac{\gamma+1}{\gamma}P^{1/\gamma}\right] = \frac{k(\gamma+1)}{\gamma}P^{(2-\gamma)/\gamma}\left\{\frac{2}{\gamma+1} - P^{(\gamma-1)/\gamma}\right\} \qquad \text{...(7)}$$

For maximum and minimum value of m, we have

$$\frac{dm}{dP} = 0 \quad \text{so that} \quad \frac{2}{\gamma+1} - P^{(\gamma-1)/\gamma} = 0 \quad \text{giving} \quad P = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}, \text{ using (7)}$$

Again from (7), we observe that (noting that $\gamma > 1$)

$$dm/dP < 0 \quad \text{when} \quad P > \{2/(\gamma+1)\}^{\gamma/(\gamma-1)} \qquad \text{...(8)}$$

and

$$dm/dP > 0 \quad \text{when} \quad P < \{2/(\gamma+1)\}^{\gamma/(\gamma-1)} \qquad \text{...(9)}$$

The above results (8) and (9) show that m is maximum when

$$P = p/p_0 = \{2/(\gamma+1)\}^{\gamma/(\gamma-1)} \qquad \text{...(10)}$$

Substituting the above value of p/p_0 in (5) the maximum valu $m_{\max}$ of m is given by

$$m_{\max} = A\left[\frac{2\gamma}{\gamma-1} p_0 \rho_0 \left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)} \left\{1 - \frac{2}{\gamma+1}\right\}\right]^{1/2}$$

or

$$m_{\max} = A\left\{\frac{2\gamma}{\gamma+1} p_0 \rho_0 \left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)}\right\}^{1/2} \qquad \text{...(11)}$$

To find the value of V for maximum rate of flow of gas.

Substituting the value of p/p_0 from (10) in (4), we have

$$V = \left\{\frac{2\gamma}{\gamma-1} \times \frac{p_0}{\rho_0}\left(1 - \frac{2\gamma}{\gamma+1}\right)\right\}^{1/2} \quad \text{or} \quad V = \left\{\frac{2\gamma}{\gamma+1} \times \frac{p_0}{\rho_0}\right\}^{1/2} \qquad \text{...(12)}$$

20.15. Shock waves.

While analyzing the speed of sound in Art. 20.8 B, the change in flow properties such as pressure, density, or temperature across the pressure wave was considered infinitesimal and it was logical to assume the flow process to be isentropic. When the jump in fluid properties across a pressure wave is of finite magnitude, the wave is known as *shock wave*. Since the flow through a stationary shock is adiabatic and not reversible, hence there is rise in entropy across a shock. In analyzing a shock wave, a shock wave will be treated as a surface across which there is sudden or discontinuous change of flow properties. The change in velocity, pressure, density and temperature etc. of a shock wave usually occur within a very thin layer of fluid film of finite thickness, of the order of 10^{-2} to 10^{-4} mm.

Shock waves waves are of following two types:

(*i*) **Normal shock waves.** A shock wave is said to be a *normal shock*, the plane of which is perpendicular to the direction of flow.

(*ii*) **Oblique shock waves.** A shock wave is said to be *an oblique shock*, the plane of which is inclined at an angle to the direction of flow.

Shock wave may occur in the vicinity of the throat of a supersonic diffuser, in front of a blunt-nosed body, or in the divergent section of a nozzle.

20.16. Elementary analysis of normal shock wave.

As shown in the adjoining figure, let us consider a stationary shock wave separating two uniform gaseous regions 1, 2. The fundamental problem for a normal shock wave is to determine the pressure p_2, density ρ_2, temprature T_2 and velocity V_2 behind the shock when the corresponding values in front of the shock are known.

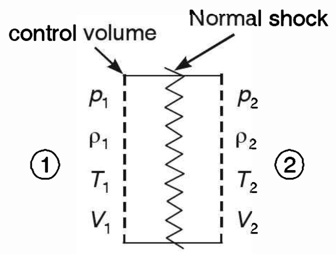


Fig. Flow model for normal shock

In what follows, we shall assume the following facts about the flow model for normal shock: (*i*) The ideal (or perfect) gas (*ii*) The flow is steady (*iii*) The flow is one-dimensional (*iv*) The flow is adiabatic but not reversible (*v*) Area is constant throughout the shock (The channel may vary but not appreciably through the shock thickness).

Based on the figure, the basic equations for one-dimensional flow are given by

The equation of continuity : $$\rho_1 V_1 = V_2 \rho_2 = \text{const.} = m, \text{ say} \qquad ...(1)$$

The momentum equation : $$\rho_1 V_1^2 + p_1 = V_2^2 \rho_2 + p_2 \qquad ...(2A)$$

that is, $$V_1 m + p_1 = V_2 m + p_2, \text{ using } (1) \qquad ...(2B)$$

The energy equation : $$\frac{V_1^2}{2} + \frac{\gamma}{\gamma - 1} \times \frac{p_1}{\rho_1} = \frac{V_2^2}{2} + \frac{\gamma}{\gamma - 1} \times \frac{p_2}{\rho_2} = \text{const.} = k, \text{ say} \qquad ...(3)$$

Also, for a perfect gas, equatin of state is $$p = \rho R T \qquad ...(4)$$

The above equaitons (1), (2A) (or (2B)), and (3) are sufficient to determine p_2, ρ_2 and V_2. The temperature T_2 could then be found by using (4).

From (2A), $$p_1 - p_2 = m\,(V_2 - V_1) \qquad ...(5)$$

and from (1), we get $$V_2 = m/\rho_2 \qquad \text{and} \qquad V_1 = m/\rho_1 \qquad ...(6)$$

From (5) and (6), we get $$p_1 - p_2 = m^2 (1/\rho_2 - 1/\rho_1) \qquad ...(7)$$

From (3), we get $$\frac{\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right) = \frac{1}{2}(V_2^2 - V_1^2)$$

or $$\frac{\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right) = m^2\left(\frac{1}{\rho_2^2} - \frac{1}{\rho_1^2}\right), \text{ by } (6)$$

or $$\frac{\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right) = m^2\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right) \qquad ...(8)$$

Dividing (8) by (7), $$\frac{\gamma}{\gamma - 1} \times \frac{p_1/\rho_1 - p_2/\rho_2}{p_1 - p_2} = \frac{1}{2}\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right) \qquad ...(9)$$

Re-using (9), $$\frac{\rho_2 - \rho_1 (p_2/p_1)}{1 - (p_2/p_1)} \times \frac{1}{\rho_1 \rho_2} = \frac{\gamma - 1}{2\gamma} \times \left(\frac{\rho_1 + \rho_2}{\rho_1 \rho_2}\right)$$

or $$2\gamma \{\rho_2 - \rho_1 (p_2/p_1)\} = (\gamma - 1)\,(\rho_1 + \rho_2)\{1 - (p_2/p_1)\}$$

or $$(p_2/p_1)\{(\gamma - 1)\,(\rho_1 + \rho_2) - 2\gamma \rho_1\} = (\gamma - 1)\,(\rho_1 + \rho_2) - 2\gamma \rho_2$$

or $$\frac{p_2}{p_1} = \frac{(\gamma - 1)\rho_1 - (\gamma + 1)\rho_2}{(\gamma - 1)\rho_2 - (\gamma + 1)\rho_1} \qquad ...(10)$$

From (10), $$\frac{p_2}{p_1} = \frac{(\gamma - 1) - (\gamma + 1)\,(\rho_2/\rho_1)}{(\gamma - 1)\,(\rho_2/\rho_1) - (\gamma + 1)}$$

or $$p_2\{(\gamma - 1)\,(\rho_2/\rho_1) - (\gamma + 1)\} = p_1\{(\gamma - 1) - (\gamma + 1)\,(\rho_2/\rho_1)\}$$

or $$(\rho_2/\rho_1)\{(\gamma + 1)p_1 + (\gamma - 1)p_2\} = (\gamma - 1)p_1 + (\gamma + 1)p_2$$

or $$\frac{\rho_2}{\rho_1} = \frac{(\gamma - 1)p_1 + (\gamma + 1)p_2}{(\gamma + 1)p_1 + (\gamma - 1)p_2} \qquad ...(11)$$

Equations (10) and (11) are known as *Rankine – Hugoniot relations*. Let c_1 and c_2 be local velocities of sound in regious 1 and 2 respectively. Then, we have

$$c_1^2 = \gamma\, p_1/\rho_1 \qquad \text{and} \qquad c_2^2 = \gamma\, p_2/\rho_2 \qquad \text{...(12)}$$

Using (12), (3) may be re-written as

$$\frac{V_1^2}{2} + \frac{c_1^2}{\gamma-1} = \frac{V_2^2}{2} + \frac{c_2^2}{\gamma-1} = k, \qquad \text{...(13)}$$

which is exactly of the same form as energy equation for compressible isentropic flow when there is no shock wave. It follows that the critical speed V_* is the same on both sides of the shock. By definition, at a given instant when the local velocity and local velocity of sound are equal, then the velocity V_1 is said to have attained a critical value V_* *i.e.*, $V_1 = c_1 = V_*$. Then, (13) reduces to

$$\frac{V_*^2}{2} + \frac{V_*^2}{\gamma-1} = k \qquad \text{or} \qquad k = \frac{(\gamma+1)V_*^2}{2(\gamma-1)} \qquad \text{...(14)}$$

From (3) and (14), we get
$$V_1^2 + \frac{\gamma}{\gamma-1}\times\frac{p_1}{\rho_1} = V_2^2 + \frac{\gamma}{\gamma-1}\times\frac{p_2}{\rho_2} = \frac{(\gamma+1)V_*^2}{\gamma-1} \qquad \text{...(15)}$$

From (15), we get
$$\frac{p_1}{\rho_1} = \frac{(\gamma+1)V_*^2}{2\gamma} - \frac{(\gamma-1)V_1^2}{2\gamma}, \qquad \frac{p_2}{\rho_2} = \frac{(\gamma+1)V_*^2}{2\gamma} - \frac{(\gamma-1)V_2^2}{2\gamma} \qquad \text{...(16)}$$

From (2B), we get
$$V_2 - V_1 = \frac{p_1}{m} - \frac{p_2}{m} = \frac{p_1}{\rho_1 V_1} - \frac{p_2}{\rho_2 V_2}, \text{ by (1)}$$

or
$$V_2 - V_1 = \frac{1}{V_1}\left\{\frac{(\gamma+1)V_*^2}{2\gamma} - \frac{(\gamma-1)V_1^2}{2\gamma}\right\} - \frac{1}{V_2}\left\{\frac{(\gamma+1)V_*^2}{2\gamma} - \frac{(\gamma-1)V_2^2}{2\gamma}\right\}, \text{ by (16)}$$

or
$$V_2 - V_1 = \frac{(\gamma+1)V_*^2}{2\gamma}\left(\frac{1}{V_1} - \frac{1}{V_2}\right) + \frac{(\gamma-1)}{2\gamma}(V_2 - V_1) = \frac{(\gamma+1)V_*^2}{2\gamma V_1 V_2}(V_2 - V_1) + \frac{(\gamma-1)}{2\gamma}(V_2 - V_1)$$

or
$$1 = \{(\gamma+1)V_*^2\}/2\gamma V_1 V_2 + (\gamma-1)/2\gamma, \qquad \text{taking} \qquad V_2 \neq V_1$$

or
$$2\gamma V_1 V_2 = (\gamma+1)V_*^2 + (\gamma-1)V_1 V_2 \qquad \text{or} \qquad V_1 V_2 = V_*^2, \qquad \text{...(17)}$$

which is known as *Prandtl's relation*.

From (13) and (14), we get
$$\frac{V_1^2}{2} + \frac{c_1^2}{\gamma-1} = \frac{V_2^2}{2} + \frac{c_2^2}{\gamma-1} = \frac{(\gamma+1)V_*^2}{2(\gamma-1)} \qquad \text{...(18)}$$

From (18),
$$\frac{V_1^2}{2} + \frac{c_1^2}{\gamma-1} = \frac{(\gamma+1)V_*^2}{2(\gamma-1)} \qquad \text{or} \qquad V_1^2\left\{\frac{1}{2} + \frac{1}{(\gamma-1)\,(V_1/c_1)^2}\right\} = \frac{(\gamma+1)V_*^2}{2(\gamma-1)} \qquad \text{...(19)}$$

Let $M_1 = V_1/c_1$ and $M_2 = V_2/c_2$ ...(20)

From (19) and (20),
$$V_1^2\left\{\frac{1}{2} + \frac{1}{(\gamma-1)M_1^2}\right\} = \frac{(\gamma+1)V_*^2}{2(\gamma-1)}$$

or
$$V_1^2\left\{\frac{(\gamma-1)M_1^2+2}{M_1^2}\right\} = (\gamma+1)V_*^2 \qquad \text{or} \qquad \frac{V_1^2}{V_*^2} = \frac{\gamma+1}{(\gamma-1)+(2/M_1^2)} \qquad \text{...(21)}$$

Proceeeding likewise with relations (18) and (20), we have

$$\frac{V_2^2}{V_*^2} = \frac{\gamma+1}{(\gamma-1)+(2/M_2^2)} \qquad ...(22)$$

Multiplying (21) and (22), we have

$$\frac{V_1^2 V_2^2}{V_*^4} = \frac{(\gamma+1)^2}{\{(\gamma-1)+(2/M_1^2)\}\times\{(\gamma-1)+(2/M_2^2)\}}$$

or $$\{(\gamma-1)+(2/M_1^2)\}\times\{(\gamma-1)+(2/M_2^2)\} = (\gamma+1)^2, \text{ using (17)}$$

or $$(\gamma-1)^2+(2/M_1^2)(\gamma-1)+(2/M_2^2)(\gamma-1)+4/M_1^2M_2^2 = (\gamma+1)^2$$

or $$(1/M_1^2)(\gamma-1)+(1/M_2^2)(\gamma-1)+2/M_1^2M_2^2 = 2\gamma$$

or $$(\gamma-1)M_2^2+(\gamma-1)M_1^2+2 = 2\gamma M_1^2M_2^2 \qquad \text{or} \qquad M_2^2\{2\gamma M_1^2-(\gamma-1)\} = 2+(\gamma-1)M_1^2$$

or $$M_2^2 = \frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2-(\gamma-1)} \qquad \text{or} \qquad M_2^2 = \frac{1+(1/2)\times(\gamma-1)M_1^2}{\gamma M_1^2-(1/2)\times(\gamma-1)}, \qquad ...(23)$$

giving the downstream Mach number M_2 in terms of the upstream Mach number M_1.

From (10), $$\frac{p_2}{p_1} = \frac{\{(\gamma+1)/(\gamma-1)\}\times(\rho_2/\rho_1)-1}{(\gamma+1)/(\gamma-1)-(\rho_2/\rho_1)} = \frac{\{(\gamma+1)/(\gamma-1)\}\times(V_1/V_2)-1}{(\gamma+1)/(\gamma-1)-(V_1/V_2)}$$

$[\because$ from (1), $\rho_2/\rho_2 = V_1/V_2]$

or $$\frac{p_2}{p_1} = \frac{\{(\gamma+1)/(\gamma-1)\}\times(V_1^2/V_*^2)-1}{(\gamma+1)/(\gamma-1)-V_1^2/V_*^2}, \qquad \text{as} \qquad \frac{V_1}{V_2} = \frac{V_1^2}{V_1V_2} = \frac{V_1^2}{V_*^2}, \text{ by (17)}$$

$$= \frac{\{(\gamma+1)/(\gamma-1)\}\times\{(\gamma+1)/(\gamma-1)+2/M_1^2\}-1}{(\gamma+1)/(\gamma-1)-\{(\gamma+1)/(\gamma-1+2/M_1^2)\}} \text{ using (21)}$$

$$= \frac{(\gamma+1)^2-\{(\gamma-1)^2+(\gamma-1)\times(2/M_1^2)\}}{(\gamma+1)(\gamma-1+2/M_1^2)-(\gamma+1)(\gamma-1)} = \frac{4\gamma-(\gamma-1)\times(2/M_1^2)}{(\gamma+1)\times(2/M_1^2)}$$

or $$\frac{p_2}{p_1} = \frac{2\gamma M_1^2}{\gamma+1}-\frac{\gamma-1}{\gamma+1} \qquad \text{or} \qquad \frac{p_2}{p_1} = \frac{2\gamma M_1^2-(\gamma-1)}{\gamma+1} \qquad ...(24)$$

From (11), we have

$$\frac{\rho_2}{\rho_1} = \frac{\gamma-1+(\gamma+1)(p_2/p_1)}{\gamma+1+(\gamma-1)(p_2/p_1)} = \frac{\gamma-1+2\gamma M_1^2-(\gamma-1)}{\gamma+1+(\gamma-1)\{(2\gamma M_1^2-\gamma+1)/(\gamma+1)\}}, \text{ by (24)}$$

or $$\frac{\rho_2}{\rho_1} = \frac{2\gamma M_1^2(\gamma+1)}{(\gamma+1)^2+2\gamma(\gamma-1)M_1^2-(\gamma-1)^2} = \frac{2\gamma(\gamma+1)M_1^2}{4\gamma+2\gamma(\gamma-1)M_1^2}$$

or $$\frac{\rho_1}{\rho_2} = \frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2} \qquad \text{or} \qquad \frac{\rho_1}{\rho_2} = \frac{2}{(\gamma+1)M_1^2}+\frac{\gamma-1}{\gamma+1} \qquad ...(25)$$

We now discuss an important implication of relation (17), *i.e.*, $V_1 V_2 = V_*^2$. This relation implies that

(*i*) if $V_1 > V_*$, then $V_2 < V_*$ (*ii*) if $V_1 < V_*$, then $V_2 > V_*$ ...(26)

In what follows we now proceed to decide which, of either, of these alternatives is valid. For this purpose we shall use the fact that there is always gain in entropy across a shock in virtue of the second law of thermodynamics. Let S_1, S_2 be the specific entropies per unit mass in regions 1 and 2 of the shock wave as shown in figure. Then, by definition (refer relation (12) on page 20.4 in part (*ix*) of Art. 20.2), we have

$$S_1 - S_0 = C_v \log (p_1 / \rho_1^\gamma) \qquad \text{and} \qquad S_2 - S_0 = C_v \log (p_2 / \rho_2^\gamma)$$

Subtracting, these give $\qquad S_2 - S_1 = C_v \log\{(p_2/p_1) \times (\rho_1/\rho_2)^\gamma\}$

or
$$\frac{S_2 - S_1}{C_v} = \log\left[\frac{2\gamma M_1^2 - (\gamma-1)}{(\gamma+1)} \times \left\{\frac{(\gamma-1)M_1^2 + 2}{(\gamma+1)M_1^2}\right\}^\gamma\right], \text{ by (24) and (25)} \qquad ...(27)$$

which relates the change in entropy across a normal shock and the Mach number M_1 in front of the shock for any gas.

Let $\qquad M_1^2 = 1 + \varepsilon, \qquad$ where $\qquad |\varepsilon| << 1 \qquad$ ...(28)

Here ε may be positive or negative. Using (28), (27) gives

$$\frac{S_2 - S_1}{C_v} = \log\left[\frac{2\gamma(1+\varepsilon) - \gamma + 1}{\gamma+1} \times \left\{\frac{(\gamma-1)(1+\varepsilon)+2}{(\gamma+1)(1+\varepsilon)}\right\}^\gamma\right]$$

$$= \log\left[\frac{\gamma+1+2\gamma\varepsilon}{\gamma+1} \times \left\{\frac{\gamma+1+(\gamma-1)\varepsilon}{(\gamma+1)(1+\varepsilon)}\right\}^\gamma\right] = \log\left\{\left(1 + \frac{2\gamma\varepsilon}{\gamma+1}\right)\left(1 + \frac{(\gamma-1)}{(\gamma+1)}\varepsilon\right)^\gamma (1+\varepsilon)^{-\gamma}\right\}$$

$$= \log\left(1 + \frac{2\gamma\varepsilon}{\gamma+1}\right) + \gamma \log\left(1 + \frac{\gamma-1}{\gamma+1}\varepsilon\right) - \gamma \log(1+\varepsilon)$$

$$= \left(\frac{2\gamma}{\gamma+1}\right)\varepsilon - \frac{1}{2}\left(\frac{2\gamma}{\gamma+1}\right)^2 \varepsilon^2 + \frac{1}{3}\left(\frac{2\gamma}{\gamma+1}\right)^3 \varepsilon^3 + \ldots + \gamma\left\{\left(\frac{\gamma-1}{\gamma+1}\right)\varepsilon - \frac{1}{2}\left(\frac{\gamma-1}{\gamma+1}\right)^2 \varepsilon^2 + \frac{1}{3}\left(\frac{\gamma-1}{\gamma+1}\right)^3 \varepsilon^3 + \ldots\right\}$$
$$+ \gamma(\varepsilon - \varepsilon^2/2 + \varepsilon^3/3 - \ldots)$$

where we have used to fact that $\gamma > 1$, $|\varepsilon| << 1$ and the expansion

$$\log(1 + x) = x - x^2/2 + x^3/3 \ldots, \text{ where } -1 < x \le 1$$

Thus $(S_2 - S_1)/C_v = (2/3) \times \gamma(\gamma-1)(\gamma+1)^{-2}\varepsilon^3$ = terms containing higher powers of ε.

Since $(S_2 - S_1)/C_v > 0$ and $\gamma > 1$, hence the above relation shows that $\varepsilon > 0$.

Now, from (28), $\qquad \varepsilon > 0 \quad \Rightarrow \quad M_1 > 1$

Now, $\qquad M_1 > 1 \;\Rightarrow\; 2/M_1^2 < 2 \;\Rightarrow\; \gamma - 1 + 2/M_1^2 < \gamma + 1 \qquad$ ...(29)

$\therefore$ From (21), $\qquad \dfrac{V_1^2}{V_*^2} = \dfrac{\gamma+1}{\gamma - 1 + 2/M_1^2} > \dfrac{\gamma+1}{\gamma+1}$, using (29)

Hence $\qquad V_1^2/V_*^2 > 1 \qquad$ or $\qquad V_1 > V_*$ and so by (17), $\quad V_2 < V_*$.

This shows that out of two relations given by (26), relation (*i*) is valid and so relation (*ii*) is false. Solving M_1 in terms of M_2, (23) gives

$$M_1^2 = \{2 + (\gamma - 1)M_2^2\}/(2\gamma M_2^2 - \gamma + 1) \qquad \text{...(30)}$$

Hence, $\quad M_1 > 1 \;\Rightarrow\; M_1^2 > 1 \;\Rightarrow\; 2 + (\gamma - 1)M_2^2 > 2\gamma M_2^2 - \gamma + 1$, by (30)

$\Rightarrow \quad \gamma + 1 > (\gamma + 1)M_2^2 \quad \Rightarrow \quad M_2^2 < 1 \quad \Rightarrow \quad M_2 < 1$

Thus we have shown that $M_1 > 1$ and $M_2 < 1$. Hence, we find that for a stationary normal shock, the upstream flow is supersonic but passage through the shock reduces it to subsonic.

Now, from (24), $\quad \dfrac{p_2}{p_1} = \dfrac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1} > \dfrac{2\gamma - (\gamma - 1)}{\gamma + 1}$, $\quad$ as $\quad M_1 > 1$

Thus, $\quad p_2 / p_1 > 1 \quad$ so that $\quad p_2 > p_1$

From (25), $\quad \rho_2 / \rho_1 = (\gamma + 1)M_1^2 \big/ \{(\gamma - 1)M_1^2 + 2\}$

or $\quad (\rho_2 / \rho_1)(\gamma + 1)M_1^2 + 2(\rho_2 / \rho_1) = (\gamma + 1)M_1^2$

so that
$$M_1^2 = \frac{2(\rho_2 / \rho_1)}{(\gamma + 1) - (\gamma - 1)(\rho_2 / \rho_1)} \qquad \text{...(31)}$$

Now, $\quad M_1 > 1 \;\Rightarrow\; M_1^2 > 1 \;\Rightarrow\; 2(\rho_2 / \rho_1) > (\gamma + 1) - (\gamma - 1)(\rho_2 / \rho_1)$, by (31)

or $\quad (\rho_2 / \rho_1)\{2 + (\gamma - 1)\} > (\gamma + 1) \quad$ or $\quad \rho_2 / \rho_1 > 1 \quad$ or $\quad \rho_2 > \rho_1$

Thus $p_2 > p_1$ and $\rho_2 > \rho_1$, showing that in passing through a shock the gas is compressed.

To find ratio of temperatures on the two sides of a shock

From (4), we have $\quad p_1 = R\rho_1 T_1 \quad$ and $\quad p_2 = R\rho_2 T_2$

Hence, $\quad \dfrac{p_2}{p_1} = \dfrac{\rho_2}{\rho_1} \times \dfrac{T_2}{T_1} \quad$ so that $\quad \dfrac{T_2}{T_1} = \dfrac{p_2}{p_1} \times \dfrac{\rho_1}{\rho_2}$

or
$$\frac{T_2}{T_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1} \times \frac{(\gamma - 1)M_1^2 + 2}{(\gamma + 1)M_1^2}, \text{ by (24) and (25)}$$

or
$$\frac{T_2}{T_1} = \frac{(2\gamma M_1^2 - \gamma + 1)\{(\gamma - 1)M_1^2 + 2\}}{(\gamma + 1)^2 M_1^2} \qquad \text{...(32)}$$

Strength of a shock wave. Definition. The strength P of a shock wave is defined as the ratio of rise in pressure to upstream pressure. Thus, we have

$P = (p_2 - p_1) / p_1 \quad$ or $\quad P = (p_2 / p_1) - 1 \qquad$ ...(33)

From (24) and (33), we get
$$P = \frac{2\gamma M_1^2 - (\gamma + 1)}{\gamma + 1} - 1 = \frac{2\gamma}{\gamma + 1}(M_1^2 - 1) \qquad \text{...(34)}$$

Some dedctions from Rankine – Hugoniot relations (10)

Re-writing (10) $\quad \gamma(p_2 + p_1)/(p_2 - p_1) = (\rho_2 + \rho_1)/(\rho_2 - \rho_1) \qquad$ ...(35)

Deduction I. When the strength of a shock is small, we have $\rho_1 \approx \rho_2$, $p_1 \approx p_2$ and then (35) reduces to $\quad dp / d\rho = \gamma p / \rho$,

showing that the flow through a mild shock is essentially isentropic

Deduction II. Re-writting (2A), we have

$$\Delta p = p_2 - p_1 = \rho_1 V_1^2 - \rho_2 V_2^2 = \rho_1 V_1^2 - \rho_2 \times (\rho_1^2 V_1^2/\rho_2^2), \text{ by (1)}$$

or
$$\Delta p = \rho_1 V_1^2 (1 - \rho_1/\rho_2) = \rho_1 V_1^2 \Delta\rho)/\rho_2 \qquad ...(36)$$

Thus,
$$\Delta p/\Delta\rho = \rho V_1^2/\rho_2$$

Re-writing (35),
$$\Delta p/\Delta\rho = \gamma(p_1 + p_2)/(\rho_1 + \rho_2). \qquad ...(37)$$

From (36) and (37),
$$\frac{\rho_1 V_1^2}{\rho_2} = \frac{\gamma(p_1+p_2)}{\rho_1+\rho_2} \quad \text{or} \quad V_1 = \left\{\frac{\gamma(p_1+p_2)\rho_2}{(\rho_1+\rho_2)\rho_1}\right\}^{1/2} \qquad ...(38)$$

For a weak shock, $\rho_2 \approx \rho_1, \quad p_1 \approx p_2$ then (38) gives

$$V_1 \approx (\gamma\, p_1/\rho_1)^{1/2} = c_1$$

But for a very strong shock, $V_1 \approx (\gamma\, p_2/\rho_1)^{1/2}$, which is greater than $(\gamma\, p_1/\rho_1)^{1/2}$ or $(\gamma\, p_2/\rho_2)^{1/2}$, *i.e.*, the speed of sound in front or behind the shock. This fact shows that the sound of an explosion travels faster and can be heard first.

20.16A Illustrative solved examples.

Ex. 1. *Let the Mach number of a normal shock in air be 2. If the atmospheric pressure and air density are 26.5 KN/m² and 0.413 Kg/m³ respectively, then find all the flow conditions before and after the shock wave. Assume that for air* $\gamma = 1.4$.

Sol. Let M_1, p_1, ρ_1, T_1 and V_1 be the Mach number, pressure, density, temperature and velocity respectively in front of the normal shock whereas the corresponding values behind the shock, be M_2, p_2, ρ_2, T_2 and V_2 respectively

Given that $\quad M_1 = 2, \quad p_1 = 26.5 \text{ KN/m}^2, \quad \rho_1 = 0.413 \text{ Kg/m}^3, \quad \gamma = 1.4.$

Using relation (23) of Art. 20.16, we have

$$M_2^2 = \frac{(\gamma-1)M_1^2 + 2}{2\gamma M_1^2 - \gamma + 1} = \frac{(1.4-1)\times 2^2 + 2}{2\times 1.4\times 2^2 - 1.4 + 1} = \frac{3.6}{10.8} = 0.333 \quad \Rightarrow \quad M_2 = 0.577.$$

Using relation (24) of Art. 20.16, we have

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - \gamma + 1}{\gamma + 1} = \frac{2\times 1.4\times 2^2 - 1.4 + 1}{1.4+1} = \frac{10.8}{2.4} = 4.5$$

$\therefore \qquad p_2 = 4.5 \times p_1 = 4.5 \times 26.5 = 119.25 \text{ KN/m}^2.$

Using relation (25) of Art. 20.16, we have

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2} = \frac{(1.4+1)\times 2^2}{(1.4-1)\times 2^2 + 2} = \frac{9.6}{3.6} = 2.667$$

$\therefore \qquad \rho_2 = 2.667\rho_1 = 2.667 \times 0.413 = 3.101 \text{ Kg/m}^3.$

Now, $\quad p_1 = \rho_1 R T_1 \;\Rightarrow\; T_1 = \dfrac{p_1}{\rho_1 R} = \dfrac{26.5\times 10^3}{0.413\times 287} = 223.6 \text{ K} \quad \text{or} \quad -49.4°\text{C}$

[$\because$ R = Gas constant for air = 287 J/Kg–K]

Using relation (32) of Art. 20.16, we have

$$\frac{T_2}{T_1} = \frac{(2\gamma M_1^2 - \gamma + 1)\{(\gamma-1)M_1^2 + 2\}}{(\gamma+1)^2 M_1^2} = \frac{(2\times 1.4\times 2^2 - 1.4 + 1)[(1.4-1)\times 2^2 + 2]}{(1.4+1)\times 2^2}$$

or $T_2/T_2 = 1.6875 \Rightarrow T_2 = 1.6875 \times T_1 = 1.6875 \times 2236$

Thus, $T_2 = 377.3$ K or $104.3° C$.

Next, $c_1 = (\gamma R T_1)^{1/2} = (1.4 \times 287 \times 223.6)^{1/2} = 299.7$ m/s

Then, $V_1/c_1 = M_1 \Rightarrow V_1 = c_1 M_1 = 299.7 \times 2 = 599.4$ m/s

Similarly $c_2 = (\gamma R T_2)^{1/2} = (1.4 \times 287 \times 377.3)^{1/2} = 389.35$ m/s

Then, $V_2/c_2 = M_2 \Rightarrow V_2 = c_2 M_2 = 389.35 \times 0.577 = 224.6$ m/s

Ex. 2. *In a duct in which air is flowing, a normal shock wave occurs at a Mach number 1.5. The static pressure and temperature upstream of the shock are 170 KN/m^2 and 23° C respectively. Take γ = 1.5 and determine (i) pressure, temperature and Mach number down stream of the shock (ii) srength of shock.*

Sol. Let subscripts 1 and 2 denote the flow conditions upstream and downstream of the shock respectively. Then given that $M_1 = 1.5$, $p_1 = 170$ KN/m^2. $T_1 = 23 + 273 = 296$ *K*.

Part (*i*) : We wish to find p_2, T_2 and M_2. Using relation (24) of Art. 20.16, we get

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - \gamma + 1}{\gamma + 1} = \frac{2 \times 1.4 \times 1.5^2 - 1.4 + 1}{1.4 + 1} = \frac{6.3 - 0.4}{2.4} = 2.458 \quad \text{...(1)}$$

$\therefore$ $p_2 = 2.458\, p_1 = 2.458 \times 170 = 417.86$ KN/m^2.

Again, using relation (32) of Art. 20.16, we have

$$\frac{T_2}{T_1} = \frac{(2\gamma M_1^2 - \gamma + 1)\{(\gamma + 1)M_1^2 + 2\}}{(\gamma + 1)^2 M_1^2} = \frac{(2 \times 1.4 \times 1.5^2 - 1.4 + 1)\{(1.4 - 1) \times 1.5^2 + 2\}}{(1.4 + 1)^2 \times 1.5^2}$$

$\Rightarrow$ $T_2 = 1.32\, T_1 = 1.32 \times 296 = 390.72\, K$ or $117.72\, C$

Finally, using relation (23) of Art. 20.16, we have

$$M_2^2 = \frac{(\gamma - 1)M_1^2 + 2}{2\gamma M_1^2 - \gamma + 1} = \frac{(1.4 - 1) \times 1.5^2 + 2}{2 \times 1.4 \times 1.5^2 - 1.4 + 1} = 0.49 \quad \Rightarrow \quad M_2 = 0.7.$$

Part (*ii*) Strength of shock = $(p_2/p_1) - 1 = 2.458 - 1 = 1.458$, using (1)

Ex. 3. What pressure rise occurs across a blast wave in air when the wave moves at 10 times the speed of sound in air. [**Ans.** $p_2/p_1 = 100$]

Ex. 4. The absolute pressure rise in air across a normal shock is $p_2/p_1 = 3$, where p_1 is the free stream absolute pressure. Find the temperature, density, and velocity ratios across the shock. [**Ans.** 1.421, 2.111, 0.474]

Ex. 5. In a normal shock wave in air, the upstream Mach number, temperature, and pressure are 2, 300 K and 100 KN/m^2, respectively. Find the Mack number, pressure, temperature, and velocity after the shock. [**Ans.** 0.58; 450 KN/m^2, 506 K; 260 m/s]

20.17. Oblique shock.

The discontinuties in supersonic flow do not always exist at normal to the flow. In many practical situations, the discontinuity occurs at an inclination to the flow and in that situation it is called an "*oblique shock*". For example, an oblique shock occurs when a supersonic flow is made to change direction near a corner. It is created when supersonic flow occurs around a body with sharp edge such as a cone or a two-dimensional wedge.

20.18. Elemantary analysis of a two dimensional oblique shock.

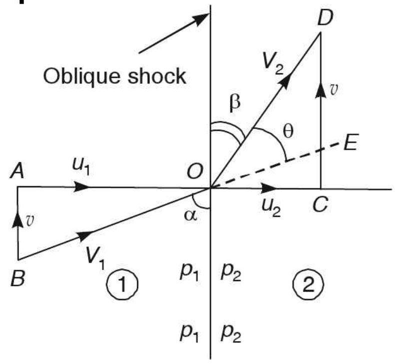


Fig. Two dimensional oblique shock

Let the adjoining figure represent a two-dimensional oblique shock. While discussing the flow through such a shock, we shall assume that it is a normal shock wave on which is superimposed a velocity v parallel to the shock wave. Then changes across the shock front are found in exactly the same way as for the same normal shock wave.

Based on the figure, the basic equations for flow of gas are given by

The equaiton of continuity : $\rho_1 u_1 = \rho_2 u_2 = m$, say ...(1A)

where m is the mass flux per unit time across unit area of the shock. From (1A),

$$u_1 = m/\rho_1, \qquad u_2 = m/\rho_2 \qquad \text{...(1B)}$$

The momentum equaiton is $\quad p_1 + m\,u_1 = p_2 + m\,u_2$ ...(2A)

From (2A), $\quad p_1 - p_2 = m(u_2 - u_1) = m^2(1/\rho_2 - 1/\rho_1)$, by (1B) ...(2B)

The energy equaiton is

$$\frac{V_1^2}{2} + \frac{\gamma}{\gamma-1} \times \frac{p_1}{\rho_1} = \frac{V_2^2}{2} + \frac{\gamma}{\gamma-1} \times \frac{p_2}{\rho_2}$$

or

$$\frac{u_1^2+v^2}{2} + \frac{\gamma}{\gamma-1} \times \frac{p_1}{\rho_1} = \frac{u_2^2+v^2}{2} + \frac{\gamma}{\gamma-1} \times \frac{p_2}{\rho_2}, \quad \text{as} \quad V_1^2 = u_1^2 + v^2, \quad V_2^2 = u_2^2 + v^2$$

or

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma-1} \times \frac{p_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma-1} \times \frac{p_2}{\rho_2} \qquad \text{...(3)}$$

Also, the equation state for gas is $\quad p = R\rho T$ ...(4)

The above equations are analogous to the corresponding equations for normal shock (refer Art. 20.16). Further, if M_1 and M_2 be upstream and downstream Mach numbers respectively, then for an oblique shock wave

$$M_1 = V_1 / c_1 \qquad \text{and} \qquad M_2 = V_2 / c_2 \qquad \text{...(5)}$$

$$\therefore \quad u_1/c_1 = (V_1 \sin\alpha)/c_1 = M_1 \sin\alpha, \qquad \text{and} \qquad u_2/c_2 = (V_2 \sin\beta)/c_2 = M_2 \sin\beta \quad \text{...(6)}$$

As shown in the figure we see that shock wave makes an acute angle α with respect to the stream in front of the shock and hence it known as a *oblique shock*. When $\alpha = \pi/2$, the shock reduces to a normal shock wave. Again, let the shock wave make an angle β with respect to the stream behind the shock. The angle α is called *shock angle*. The angle θ by which the flow turns towards the shock is known as *deflection angle* or *wedge angle*. Clearly, $\theta = \alpha - \beta$.

Now, proceeding exactly as in the case of a normal shock (refer Art. 20.16) and modifying the normal shock relations by writing $M_1 \sin\alpha$ and $M_2 \sin\beta$ in place of M_1 and M_2, the relations (23), (24) and (25) and (32) of Art. 20.16 take the following forms for an oblique shock :

$$M_2^2 \sin^2\beta = \left\{2 + (\gamma-1)M_1^2 \sin^2\alpha)\right\} / (2\gamma M_1^2 \sin^2\alpha - \gamma + 1) \qquad \text{...(7)}$$

$$p_2/p_1 = (2\gamma M_1^2 \sin^2\alpha - \gamma + 1)/(\gamma+1) \qquad \text{...(8)}$$

$$\rho_1/\rho_2 = \{2 + (\gamma-1)M_1^2 \sin^2\alpha\}/(\gamma+1)M_1^2 \sin^2\alpha \qquad \text{...(9)}$$

and

$$\frac{T_2}{T_1} = \frac{[(2\gamma M_1^2 \sin^2\alpha - \gamma + 1) \times \{(\gamma+1)M_1^2 \sin^2\alpha + 2\}]}{(\gamma+1)^2 M_1^2 \sin^2\alpha} \qquad \text{...(10)}$$

Now, using (1), $$\rho_2/\rho_1 = u_2/u_1 = \frac{u_2/v}{u_1/v} = \frac{\tan\beta}{\tan\alpha}, \quad \text{from figure} \qquad ...(11)$$

From (9) and (11), $$\frac{\tan\beta}{\tan\alpha} = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_1^2\sin^2\alpha} \qquad ...(12)$$

We now proceed to find the angle of deflection θ in passing through an oblique shock as follows

$$\tan\theta = \tan(\alpha-\beta) = (\tan\alpha - \tan\beta)/(1+\tan\alpha\tan\beta)$$

$$= \tan\alpha\{1-(\tan\beta)/\tan\alpha\}/\{1+\tan^2\alpha\times(\tan\beta/\tan\alpha)\}$$

$$= \frac{\tan\alpha\{1-(\gamma-1)/(\gamma+1)-2/(\gamma+1)M_1^2\sin^2\alpha\}}{1+\tan^2\alpha\{(\gamma-1)/(\gamma+1)-2/(\gamma+1)M_1^2\sin^2\alpha\}}, \quad \text{using (12)}$$

$$= \frac{\tan\alpha\{(\gamma+1)M_1^2\sin^2\alpha-(\gamma-1)M_1^2\sin^2\alpha-2\}}{(\gamma+1)M_1^2\sin^2\alpha+\tan^2\alpha\{(\gamma-1)M_1^2\sin^2\alpha+2\}}$$

$$= \frac{2\tan\alpha\,(M_1^2\sin^2\alpha-1)}{\gamma M_1^2\sin^2\alpha\,(1+\tan^2\alpha)+M_1^2\sin^2\alpha\,(1-\tan^2\alpha)+2\tan^2\alpha}$$

$$= \frac{2\tan\alpha\,(M_1^2\sin^2\alpha-1)}{\gamma M_1^2\tan^2\alpha+M_1^2\tan^2\alpha\cos2\alpha+2\tan^2\alpha} = \frac{2\cot\alpha\,(M_1^2\sin^2\alpha-1)}{\gamma M_1^2+M_1\cos2\alpha+2}$$

Thus, $$\tan\theta = \frac{M_1^2\sin2\alpha-2\cot\alpha}{M_1^2(\gamma+\cos2\alpha)+2} \qquad ...(13)$$

When the deflection angle θ is zero, then (13) gives

$$M_1^2\sin2\alpha-2\cot\alpha=0 \qquad \text{or} \qquad 2M_1^2\sin\alpha\cos\alpha-2(\cos\alpha/\sin\alpha)=0$$

giving $\cos\alpha=0$ or $\sin\alpha=1/M_1$ *i.e.* $\alpha=\pi/2$ and $\alpha=\sin^{-1}(1/M_1)$

Then $\alpha=\pi/2$ corresponds to the *normal shock* and $\alpha=\sin^{-1}(1/M_1)$ corresponds to the *Mach wave* or a *shock* of *infinitesimal strength.*

It is worth noting from the oblique shock relations (6) – (10) that the ratios of thermodynamic variables depend only on the normal component of velocity ahead of the shock. Again, from our analysis of normal shock (refer Art. 20.16), we know that this component must be supersonic, *i.e.,* $M_1\sin\geq1$ *i.e.* $\alpha\geq\sin^{-1}(1/M_1)$. This requirement imposes the restriction on the wave angle α that it cannot exceed a minimum value for a given M_1. The Maximum value of α is that for a normal shock *i.e.* $\alpha=\pi/2$. It follows that for a given initial Mack number M_1, the possible range of wave angle is $\sin^{-1}(1/M_1)\leq\alpha\leq\pi/2$.

Proceeding as in the case of a normal shock, we can show that $M_2\sin\beta<1$. But inspite of this relation, M_2 may be greater than 1. It follows that the flow behind an oblique shock may be supersonic although the normal component of velocity is subsonic.

20.18A. Illustrative solved examples.

Ex.1. *A supersonic air stream is deflected by a two-dimensional wedge of semi-vertex angle of 10° and shock waves appear at an angle of 30° to the original flow direction. What is Mack number of the air flow if* $\gamma = 1.4$ *for air?*

Solution. Let the required Mack number be M_1. Clearly, here α = shock angle = 30° and θ = deflection angle = 10°. Then we have (refer relation (13) of Art. 20.18).

$$\tan\theta = (M_1^2 \sin 2\alpha - 2\cot\alpha)/\{M_1^2(\gamma + \cos 2\alpha) + 2\}$$

$$\therefore \qquad \tan 10° = \frac{M_1^2 \sin 60° - 2\cot 30°}{M_1^2(1.4 + \cos 60°) + 2} \qquad \text{or} \qquad 0.1763 = \frac{0.866 M_1^2 - 2\times 1.7321}{M_1^2(1.4 + 0.5) + 2}$$

On simplification, we have $M_1^2 = 7.19$ so that $M_1 = 2.68$.

Ex. 2. *A wedge with total angle of 20° is mounted with its axis parallel to the flow of air having a Mack number 2. What is possible inclination of the shock with the initial direction of the streamlines? Find the pressure and density ratios across the oblique wave and the Mack number behind the wave.*

Sol. Try yourself. [**Ans.** 39.3° ; 1.705 ; 1.458 ; 1.64]

CHAPTER

21 Flow of a Compressible Viscous Fluids.

21.1. Introduction.

While discussing flow of an incompressible viscous fluid, we determine only the velocity and pressure at each point of the fluid. On the other hand, while discussing flow of a compressible viscous fluid, we have to determine density and temperature in addition to the velocity and pressure at each point of the fluid. For ready reference, the complete list of basic equations governing the flow of compressible viscous fluid are given below both for cartesian and cylindrical coordinates:

List I. Basic equations in cartesian co-ordinates

The equation of continuity : (Refer equation (8), Art. 2.9)

$$\partial\rho/\partial t + \partial(\rho u)/\partial t + \partial(\rho v)/\partial t + \partial(\rho w)/\partial t = 0 \qquad \text{...(1)}$$

The Navier-Stokes equations (Refer equations (14*a*), (14*b*) and (14*c*), Art. 14.1)

$$\rho\{\partial u/\partial t + u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z) = \rho B_x - \partial p/\partial x$$

$$+\frac{\partial}{\partial x}\left[\mu\left\{2\frac{\partial u}{\partial x} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\}\right] + \frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\} + \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right\} \qquad \text{...(2a)}$$

$$\rho\{\partial v/\partial t + u(\partial v/\partial x) + v(\partial v/\partial y) + w(\partial v/\partial z)\} = \rho B_y - \partial p/\partial y$$

$$+\frac{\partial}{\partial y}\left[\mu\left\{2\frac{\partial v}{\partial y} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\}\right] + \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right\} + \frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\} \qquad \text{...(2b)}$$

$$\rho\{\partial w/\partial t + u(\partial w/\partial x) + v(\partial w/\partial y) + w(\partial w/\partial z)\} = \rho B_z - \partial p/\partial z$$

$$+\frac{\partial}{\partial z}\left[\mu\left\{2\frac{\partial w}{\partial z} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\}\right] + \frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right\} + \frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right\} \qquad \text{...(2c)}$$

The Energy equation [Refer equation (17), Art. 14.2]

$$\rho\left\{\frac{\partial(C_pT)}{\partial t} + u\frac{\partial(C_pT)}{\partial x} + v\frac{\partial(C_pT)}{\partial y} + w\frac{\partial(C_pT)}{\partial z}\right\} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}$$

$$+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \mu\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\}\right.$$

$$\left.-\frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right] \qquad \text{...(3)}$$

Equation of state : $$p = k\rho T \qquad ...(4)$$

We have already discussed the significance of controlling paramenters in compressible viscous flow in Art. 15.7 of chapter 15.

List II. Basic equation in cylindrical co-ordinates

The equation of continuity [Refer equation (8), Art 2.10]

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r q_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho q_\theta)}{\partial \theta} + \frac{\partial(\rho q_z)}{\partial z} = 0 \qquad ...(1)'$$

The Navier-Stokes equations [Refer equations (7*a*), (17*b*) and (7*c*) of Art 14.11]

$$\rho\left(\frac{Dq_r}{Dt} - \frac{q_\theta^2}{r}\right) = \rho B_r - \frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left[\mu\left\{2\frac{\partial q_r}{\partial r} - \frac{2}{3}(\nabla\cdot \boldsymbol{q})\right\}\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left\{\mu\left(\frac{1}{r}\frac{\partial q_r}{\partial \theta} + \frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r}\right)\right\}$$

$$+\frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r}\right)\right\} + \frac{2\mu}{r}\left(\frac{\partial q_r}{\partial r} - \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} - \frac{q_r}{r}\right) \qquad ...(2a)'$$

$$\rho\left(\frac{Dq_\theta}{Dt} + \frac{q_r q_\theta}{r}\right) = \rho B_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left\{\frac{2}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{2q_\theta}{r} - \frac{2}{3}(\nabla\cdot \boldsymbol{q})\right\}\right]$$

$$+\frac{\partial}{\partial r}\left\{\mu\left(\frac{1}{r}\frac{\partial q_r}{\partial \theta} + \frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r}\right)\right\} + \frac{\partial}{\partial z}\left\{\mu\left(\frac{1}{r}\frac{\partial q_z}{\partial \theta} + \frac{\partial q_\theta}{\partial z}\right)\right\} + \frac{2\mu}{r}\left(\frac{1}{r}\frac{\partial q_r}{\partial \theta} + \frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r}\right) \qquad ...(2b)'$$

$$\rho\frac{Dq_z}{Dt} = \rho B_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left\{2\frac{\partial q_z}{\partial z} - \frac{2}{3}(\nabla\cdot \boldsymbol{q})\right\}\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left\{\mu\left(\frac{1}{r}\frac{\partial q_z}{\partial \theta} + \frac{\partial q_\theta}{\partial z}\right)\right\}$$

$$+\frac{\partial}{\partial r}\left\{\mu\left(\frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r}\right)\right\} + \frac{\mu}{r}\left(\frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r}\right) \qquad ...(2c)'$$

The energy equation: [Refer equation (14) of Art. 14.11]

$$\rho\frac{D(C_pT)}{Dt} = \frac{Dp}{Dt} + \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + 2\mu\left[\left(\frac{\partial q_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_r}{r}\right)^2\right.$$

$$\left. + \left(\frac{\partial q_z}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r} + \frac{1}{r}\frac{\partial q_r}{\partial \theta}\right)^2 + \frac{1}{2}\left(\frac{1}{r}\frac{\partial q_z}{\partial \theta} + \frac{\partial q_\theta}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r}\right)^2 - \frac{1}{3}(\nabla\cdot \boldsymbol{q})^2\right] \qquad ...(3)'$$

$$p = \rho R T \qquad ...(4)'$$

where $$\frac{D}{Dt} = \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + q_z\frac{\partial}{\partial z} \qquad ...(5)'$$

and $$\nabla\cdot \boldsymbol{q} = \frac{\partial q_r}{\partial r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z} + \frac{q_r}{r} \qquad ...(6)'$$

Since the basic equations for flow of a compressible viscous fluid are highly non-linear, their solutions are not always easy. Hence, in what follows, we propose to study some simple flows of compressible viscous fluids. The simplicity of the problems will help us to solve complicated basic equations and thus we shall be able to study the effects of compressibility, viscosity, and heat conduction on the motion of compressible viscous fluid.

In addition it is necessary to include the effects of the variation of viscosity with temperature in connection with the large temperature variations which occur in the flow. The relation $\mu(T)$ has been determined by experiment etc. and these lead to the conclusion that the functional relationship is a complicated one and that no single correlation function can be found to apply to all gases. In order not to complicate the already difficult problem of flow of a compressible viscous fluid it is necessary to compromise and to adopt reasonably simple, if not extremely precise, semi-empirical relations. In the case of air, the following formula has been shown:

$$\frac{\mu}{\mu_0} = \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0 + S}{T - S},$$

where μ_0 denotes the viscosity at the reference temperature T_0, and S a constant which for air assumes the value $S = 110°$ K.

Since the above relation is still too complicated, it is customary to approximate it in theoretical calculations by the following simple power law

$$\mu / \mu_0 = (T / T_0)^m, \qquad \text{where} \qquad 0.5 < m < 1$$

The specific heat C_p, and the Prandtl number P_r, can both be assumed to be constant even at large temperature differences approximately.

21.2. One-dimensioned flow of a compressible viscous fluid.

Consider the laminar steady one-dimensional flow, without body forces, of a compressible viscous fluid moving parallel to x-axis. Assume that all the variables are functions of x alone. Under the present flow conditions, we have

$$u = u\,(x), \quad v = 0, \quad w = 0, \quad p = p\,(x), \quad T = T\,(x), \quad \partial/\partial y = 0, \;\; \partial/\partial z = 0 \qquad ...(1)$$

In view of (1), the basic equations for flow of one-dimensional steady compressible viscous fluid are given by (refer equation (1), (2a), (2b), (2c), (3) and (4) of Art. 21.1)

$$d(\rho u)/dx = 0 \qquad ...(1)$$

$$\rho u \frac{du}{dx} = -\frac{dp}{dx} + \frac{4}{3}\frac{d}{dx}\left(\mu \frac{du}{dx}\right) \qquad ...(2)$$

and
$$\rho u C_p \frac{dT}{dx} - u\frac{dp}{dx} - \frac{4}{3}\mu\left(\frac{du}{dx}\right)^2 - \frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \qquad ...(3)$$

$$p = \rho R T \qquad ...(4)$$

Integrating (1), $\qquad \rho u = \rho_0 u_0, \qquad ...(5)$

where ρ_0 is the density at the reference velocity u_0 Using (5), (2) becomes

$$\rho_0 u_0 \frac{du}{dx} + \frac{dp}{dx} - \frac{4}{3}\frac{d}{dx}\left(\mu\frac{du}{dx}\right) = 0$$

Integrating it, $\qquad \rho_0 u_0 u + p - (4\mu/3) \times (du/dx) = C,$ C being an arbitrary constant

or $\qquad \rho u^2 + p - (4\mu/3) \times (du/dx) = C,$ using (5)

Hence, $\qquad p = (4\mu/3) \times (du/dx) - \rho u^2 + C \qquad ...(6)$

or $\qquad \rho R T = (4\mu/3) \times (du/dx) - \rho u^2 + C,$ using (4)

or $\qquad RT = \dfrac{4\mu}{3\rho}\dfrac{du}{dx} - u^2 + \dfrac{C}{\rho} \qquad$ or $\qquad RT = \dfrac{4\mu}{3\rho_0 u_0} u \dfrac{du}{dx} - u^2 + \dfrac{Cu}{\rho_0 u_0},$ using (5) $\qquad ...(7)$

Equations (4) and (7) can be solved numerically by using relationship between μ and T given by

$$\mu / \mu_0 = (T / T_0)^m, \qquad \text{where} \qquad 0.5 < m < 1 \qquad ...(8)$$

21.3. Plane Couette flow of a compressible viscous fluid. [Himanchal 2002]

Consider the steady laminar flow of viscous compressible fluid between two infinite plates separated by a distance h. Let the upper plate move with uniform velocity U and the lower plate be at rest. Let x be the direction of flow, y the direction perpendicular to the flow, and the width of the plates parallel to the z-direction. Here the word 'infinite' implies that the width of the plates is large compared with h and hence the flow may be treated as two-dimensional (*i.e.* $\partial/\partial z = 0$). Let the plates be long enough in the x-direction for the flow to be parallel so that the velocity components v and w are zero everywhere. Moreover the flow being steady, the flow variables are independent of time $(\partial/\partial t = 0)$. Thus, for the flow under consideration, we have

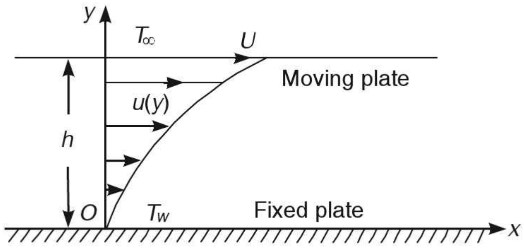


Figure (*i*)

$$u = u(y), \quad v = 0, \quad w = 0, \quad \partial/\partial z = 0, \quad \partial/\partial t = 0, \quad p = \text{constant} \qquad ...(1)$$

In the absence of body forces, keeping (1) in view, the basic equations for steady flow of compressible viscous fluid are given by (refer (1), (2*a*), (2*b*), (2*c*), (3) and (4) of Art 21.1).

$$\frac{d}{dy}\left(\mu\frac{du}{dy}\right) = 0 \qquad ...(2)$$

$$\frac{d}{dy}\left(k\frac{dT}{dy}\right) + \mu\left(\frac{du}{dy}\right)^2 = 0 \qquad ...(3)$$

and
$$p = \rho R T \qquad ...(4)$$

The boundary conditions are given by

$$\left.\begin{array}{llll} y = 0: & u = 0, & T = T_w, & Q = Q_w \\ y = h: & u = U, & T = T_\infty & \end{array}\right\} \qquad ...(5)$$

where Q denotes heat flux, Q_w denotes heat flow to the plate $y = 0$, T_w and T_∞ respectively denote temperatures at moving and stationary plates.

Integrating (2), $\mu(du/dy) = A$, A being an arbitrary constant ...(6)

Let, shearing stress at the lower plate $= [\mu(du/dy)]_{y=0} = \tau_w$, say ...(7)

Putting $y = 0$ in (6) and using (7), we get $A = \tau_w$. Then (6) becomes

$$\mu(du/dy) = \tau_w \qquad \text{so that} \qquad du = \tau_w \times (dy/\mu)$$

Integrating it
$$\int_{u=0}^{u} du = \int_{y=0}^{y} \tau_w \frac{dy}{\mu} \qquad \text{or} \qquad u = \tau_w \int_0^y \frac{dy}{\mu} \qquad ...(8)$$

Substituting the value of du/dy given by (7) in (4), we get

$$\frac{d}{dy}\left(k\frac{dT}{dy}\right) + \mu \times \left(\frac{\tau_w}{\mu}\right)^2 = 0 \qquad \text{or} \qquad \frac{d}{dy}\left(k\frac{dT}{dy}\right) = -\frac{\tau_w^2}{\mu} \qquad ....(9)$$

Integrating (9),
$$k\frac{dT}{dy} = -\tau_w^2 \int_0^y \frac{dy}{\mu} + B, \quad B \text{ being an arbitrary coustant}$$

or
$$k(dT/dy) = -u\,\tau_w + B, \quad \text{using (8)} \qquad ...(10)$$

Let
$$(k\,dT/dy)_{y=0} = -Q_w, \qquad ...(11)$$

wherein the negative sign is conventional, to indicate that Q_w is positive when $(dT/dy)_{y=0}$ is negative. Using the conditions, $y = 0$, $u = 0$ and (11), (10) reduces to $B = -Q_w$. Hence (10) may be re-written as

$$k\frac{dT}{dy} + \mu u\frac{du}{dy} = -Q_w \qquad \text{or} \qquad k\frac{dT}{dy} = -\frac{1}{2}\mu\frac{du^2}{dy} - Q_w \qquad \text{...(12)}$$

Now (12) can be integrated provided we know the relationship between the coefficient of viscosity μ and the coefficient of thermal conductivity k. It has been shown that with sufficient accuracy μ can be expressed as a power of T in the form

$$\mu/\mu_\infty = (T/T_\infty)^m, \qquad \text{where} \qquad 0.5 < m < 1 \qquad \text{...(13)}$$

where μ_∞ is the value of μ at the upper moivng plate. For air at ordinary temperature, we usually take $m = 0.76$. As temperature increases, m decreases towards 0.5

Now, by definition, Prandtl number $= P_r = (C_p \mu)/k$ ...(14)

Since P_r is very nearly constant (of order unity) for all common gases and C_p is also nearly constant for a fairly wide range of temperatures around ordinary temperature, (14) shows that k is proportional to μ.

Re-writing (14), $k = (C_p \mu)/P_r$ ...(15)

Using (15), (12) may be re-written as

$$\frac{C_p\mu}{P_r}\frac{dT}{dy} = -\frac{\mu}{2}\frac{du^2}{dy} - Q_w \qquad \text{or} \qquad \frac{d}{dy}\left(\frac{C_p}{P_r}T + \frac{u^2}{2}\right) = -\frac{Q_w}{\mu} \qquad \text{...(16)}$$

Integrating, $\dfrac{C_pT}{P_r} + \dfrac{u^2}{2} = -Q_w\displaystyle\int_0^y \frac{dy}{\mu} + \frac{C}{P_r}$, C being an arbitrary constant

or

$$C_pT + \frac{1}{2}P_r u^2 = -Q_w P_r \int_0^y \frac{dy}{\mu} + C \qquad \text{...(17)}$$

Using the boundary conditions at the fixed plate, namely,

when $y = 0$, $u = 0$ and $T = T_w$, (17) yields $C = C_p T_w$

Using (8), (17) may be re-written as

$$C_pT + \frac{1}{2}P_r u^2 = -\frac{P_r Q_w}{\tau_w}u + C, \qquad \text{...(18)}$$

which gives relationship between u and T.

Determination of velocity and temperature distribution

Using the boundary conditions : When $y = h$, $u = U$, $T = T_\infty$, (18) gives

$$C_p T_\infty + \frac{1}{2}P_r U^2 = -\frac{P_r Q_w}{\tau_w}U + C \qquad \text{...(19)}$$

Subtracting (19) form (18) and simplifying, we obtain

$$C_p(T - T_\infty) = \frac{P_r Q_w}{\tau_w}(U - u) + \frac{1}{2}P_r(U^2 - u^2) \qquad \text{...(20)}$$

Dividing by $C_p T_\infty$ and re-writting, (20) becomes

$$\frac{T}{T_\infty} = 1 + \frac{P_r Q_w U}{C_p T_\infty \tau_w}\left(1 - \frac{u}{U}\right) + \frac{P_r U^2}{2C_p T_\infty}\left(1 - \frac{u^2}{U^2}\right) \qquad \text{...(21)}$$

Let M_∞ denote the Mach number of the free stream and γ the ratio of specific heats. Then, if Ec denote Eckert number, we have (refer Art. 15.7)

$$Ec = U^2/C_p T_\infty = (\gamma-1)M_\infty^2 \qquad \text{so that} \qquad U/C_p = \{(\gamma-1)M_\infty^2\}/U \qquad \text{...(22)}$$

Using (22), (21) may be re-written as

$$\frac{T}{T_\infty} = 1 + \frac{P_r Q_w}{\tau_w}\frac{\gamma-1}{U}M_\infty^2\left(1-\frac{u}{U}\right) + P_r\frac{(\gamma-1)}{2}M_\infty^2\left(1-\frac{u^2}{U^2}\right) \qquad \text{...(23)}$$

Using the conditions at the fixed plate, namely, when $y = 0$, $u = 0$ and $T = T_w$, (23) yields

$$\frac{T_w}{T_\infty} = 1 + \frac{P_r Q_w(\gamma-1)M_\infty^2}{\tau_w U} + \frac{P_r(\gamma-1)M_\infty^2}{2} \qquad \text{... (24)}$$

Hence,
$$Q_w = \frac{\tau_w U}{P_r M_\infty^2(\gamma-1)}\left\{\frac{T_w}{T_\infty} - 1 - \frac{P_r(\gamma-1)M_\infty^2}{2}\right\} \qquad \text{...(25)}$$

The required velocity distribution is given by (7) in the form

$$\tau_w y = \int_0^u \mu\, du \qquad \text{...(26)}$$

Substituting the value of μ given by (13) and the value of T given by (23) in (26) yields

$$\frac{\tau_w y}{\mu_\infty} = \int_0^u \left\{1 + \frac{P_r Q_w M_\infty^2(\gamma-1)}{\tau_w U}\left(1-\frac{u}{U}\right) + \frac{P_r M_\infty^2(\gamma-1)}{2}\left(1-\frac{u^2}{U^2}\right)\right\}^m du \qquad \text{...(27)}$$

For an arbitrary value of m the integral in (27) is evaluated numerically. However, if $m = 1$, a simple solution of (27) directly yields

$$\frac{\tau_w y}{\mu_\infty U} = \frac{u}{U} + \frac{P_r Q_w M_\infty^2(\gamma-1)}{\tau_w U}\left\{\frac{u}{U} - \frac{1}{2}\left(\frac{u}{U}\right)^2\right\} + \frac{P_r(\gamma-1)M_\infty^2}{2}\left\{\frac{u}{U} - \frac{1}{3}\left(\frac{u}{U}\right)^3\right\} \qquad \text{...(28)}$$

Using the boundary condition at the moving plate, namely, when $y = h$, $u = U$, (28) reduces to

$$\frac{\tau_w}{\mu_\infty} = 1 + \frac{P_r Q_w M_\infty^2(\gamma-1)}{2\tau_w U} + \frac{P_r(\gamma-1)M_\infty^2}{3} \qquad \text{...(29)}$$

Solving (25) and (29), we obtain values of τ_w and Q_w.

For a chosen value of y, we obtain u from (28). Using the value of u so obtained, we can obtain temperature T with the help of (23). Hence, we can obtain μ and ρ. Finally, using (4), pressure p can be obtained

Adiabatic wall : If the heat transfer at the fixed plate is zero, *i.e.*, if wall is adiabatic, we have $Q_w = 0$. Then, (28) reduces to

$$\frac{\tau_w y}{\mu_\infty U} = \frac{u}{U} + \frac{P_r(\gamma-1)M_\infty^2}{2}\left\{\frac{u}{U} - \frac{1}{3}\left(\frac{u}{U}\right)^3\right\} \qquad \text{...(30)}$$

Using the boundary condition at the moving plate, namely, when $y = h$, $u = U$, (30) yields

$$(\tau_w h)/\mu_\infty U = 1 + (1/3) \times P_r M_\infty^2(\gamma-1) \qquad \text{...(31)}$$

Dividing equation (30) by equaiton (31), we have

$$\frac{y}{h} = \frac{1}{1 + P_r M_\infty^2 (\gamma - 1)/3} \left[\frac{u}{U} + \frac{P_r (\gamma - 1) M_\infty^2}{2} \left\{ \frac{u}{U} - \frac{1}{3} \left(\frac{u}{U} \right)^3 \right\} \right] \qquad ...(32)$$

Letting $M_\infty^2 \to \infty$, (32) reduces to

$$y/h = (3/2) \times (u/U) \times \left\{1 - (1/3) \times (u/U)^2 \right\} \qquad ...(33)$$

The velocity distribution is plotted against the distance from the fixed plate (refer fig (*ii*)) for various values of $P_r M_\infty^2$. From the figure, we observe that the effect of the Mach number on the compressible viscous flow is to decrease the velocity gradient at the fixed plate and to increase it at the moving plate, Since in the above discussion Prandtl number appears paired with Mach number, it has the same influence on the velocity distribution as the Mach number. A comparison of the velocity distribution for m = 1 and m = 0.76 has been shown in following figure (*ii*) with γ = 1.4.

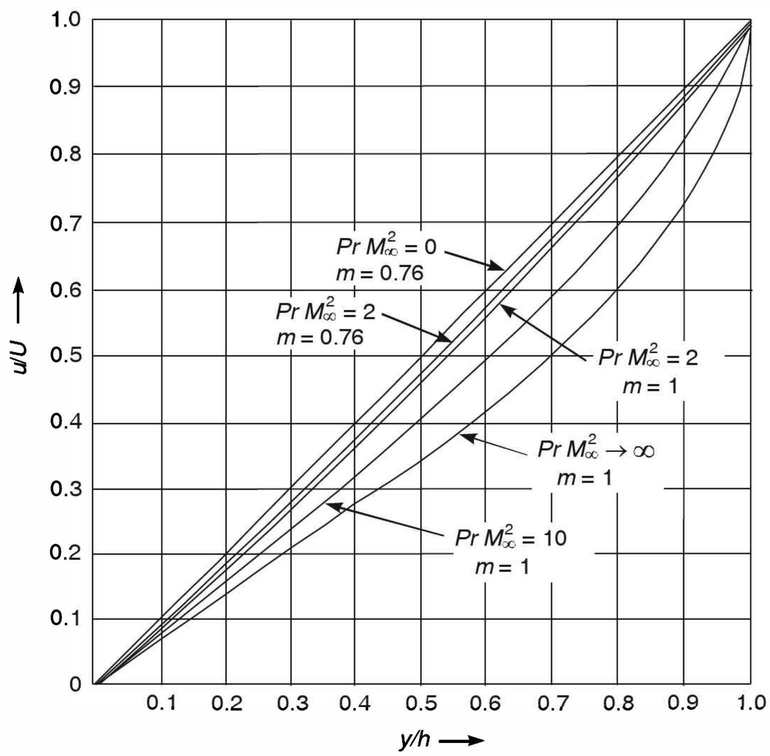


Figure (*ii*)

Typical temperature profiles in compressible viscous flow ($P_r M_\infty^2 = 2$) for an adiabatic wall (Q_w = 0), a heated wall (Q_w < 0), and a cooled wall (Q_w > 0) are shown in the figure (*iii*) on next page, by taking γ = 1.4. Note that the temperature remains unchanged for an incompressible fluid. Again, the temperature gradient becomes zero for a compressible viscous flow with an adiabatic wall. This particular temperature at the fixed insulated wall (adiabatic wall) is known as *recovery temperature*, and is denoted by T_r.

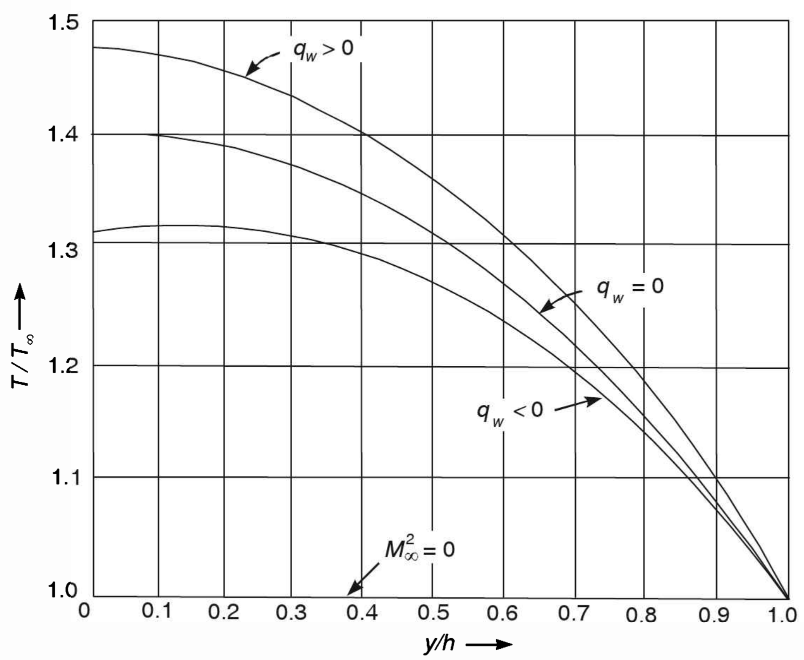


Figure (*iii*)

Putting $u = 0$ and $Q_w = 0$ in (23), T_r is given by

$$T_r / T_\infty = 1 + P_r M_\infty^2 (\gamma - 1)/2 \qquad \text{...(34)}$$

The coefficient of friction at the fixed wall ($Q_w = 0$) can be obtained from (31). Thus,

$$C_f = \frac{\tau_w}{\rho_\infty U^2/2} = \frac{1 + P_r M_\infty^2 (\gamma - 1)/3}{\text{Re}/2}, \qquad \text{...(35)}$$

where Re = Renolds number = Uh/ν_∞ and ν_∞ = kinematic viscosity = μ_∞ / ρ_∞

As shown in figure (*ii*), the velocity gradient for a compressible viscous fluid varies from the fixed wall to the moving wall and the shearing stress given by (31) is constant.

The constnacy of the shearing stress can be proved from the velocity gradient given by

$$\tau_w = \mu_w (du/dy)_w = \mu_\infty (du/dy)_\infty \qquad \text{...(36)}$$

In order to prove (27), we have to prove that $$\frac{\mu_w}{\mu_\infty} = \frac{(du/dy)_\infty}{(du/dy)_w} \qquad \text{...(37)}$$

The viscosity is given by (13) and with help of (23) it may be re-written as ($Q_w = 0$)

$$\mu_w / \mu_\infty = T_w / T_\infty = 1 + P_r M_\infty^2 (\gamma - 1)/2 \qquad \text{...(38)}$$

Differentiating (30) *w. r. t.* '*y*' and applying the limits $y = 0$ and $y = h$, we obtain

$$\left(\frac{du}{dy}\right)_w = \frac{T_w}{\mu_\infty} \times \frac{1}{1 + P_r M_\infty^2 (\gamma - 1)/2} \quad \text{and} \quad \left(\frac{du}{dy}\right)_\infty = \frac{\tau_w}{\mu_\infty} \qquad \text{...(39)}$$

The ratio of the velocity gradients in (39) is given by

$$\frac{(du/dy)_\infty}{(du/dy)_w} = 1 + \frac{P_r M_\infty^2 (\gamma - 1)}{2} \qquad ...(40)$$

Using (40), (37) is verified and so the shearing stress is constant in compressible viscous fluid.

The increase of the skin-friction coefficient at the fixed wall ($Q_w = 0$) with the increase of the Mach number and Prandtl number is illustrated in the following figure (*iv*). Take $\gamma = 1.4$.

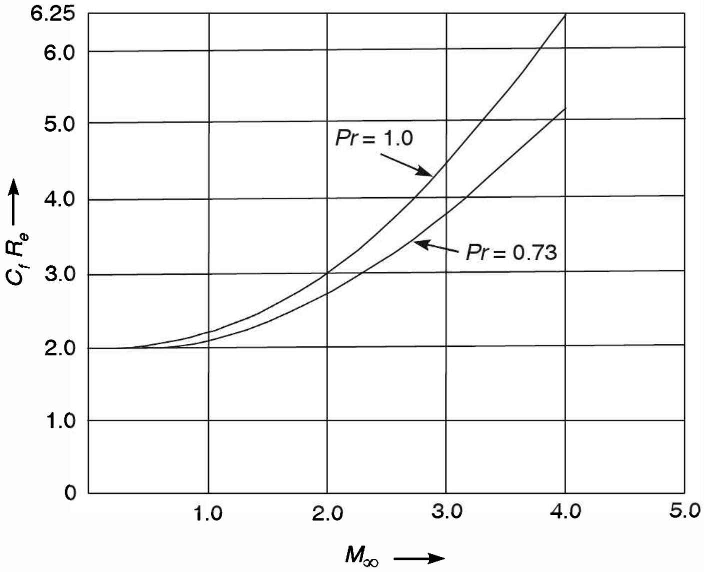
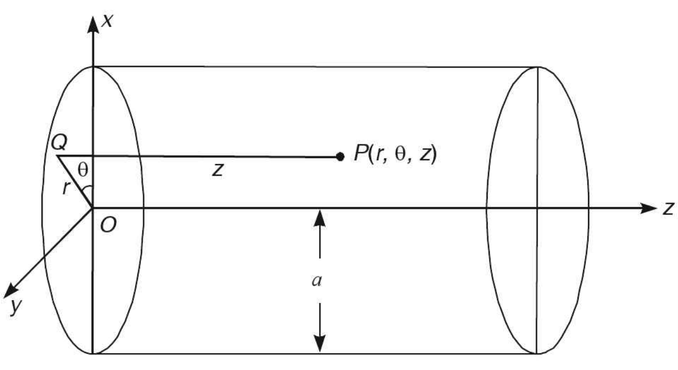


Figure (*iv*)

21.4. Laminar flow of a compressible viscous fluid through a circular pipe.

Consider the laminar steady flow, without body forces of a compressible fluid through an infinite circular pipe of radius a with axial symmetry as shown in the following figure.

For the present problem, we consider all basic equations for compressible fluid flow in cylindrical co-ordinates. Let z be the direction of the flow along the axis of the pipe and r denote the radial direction measured outwards from the z-axis. For the present axi-symmetrical flow ($\partial/\partial\theta = 0$), the velocity components q_r and q_θ in radial and tangential directions, respectively, are zero.



Under these flow conditions the equation of continuity (refer equation (1)' of Art. 21.1) reduces to $\quad \partial(\rho q_z)/\partial z = 0 \quad$ so that $\quad q_z = q_z(r)$

Thus, for the present configuration, we have

$$q_r = 0, \qquad q_\theta = 0, \qquad q_z = q_z(r), \qquad \partial/\partial\theta = 0 \qquad ...(1)$$

In view of (1), the Navier-Stokes equations (refer equation (2*a*)', (2*b*)' and (2*c*)') for the present axi-symmetrical flow are given by

$$0 = -(\partial p/\partial r) \qquad ...(2)$$

$$0 = -(1/r)\times(\partial p/\partial\theta) \qquad ...(3)$$

and
$$0 = -\frac{\partial p}{\partial z} + \frac{d}{dr}\left(\mu\frac{dq_z}{dr}\right) + \frac{\mu}{r}\frac{dq_z}{dr} \qquad ...(4)$$

From (2) and (3), $\partial p/\partial r = 0$ and $\partial p/\partial\theta = 0$

so that the pressure depends on z only. Hence, (4) can be re-written as

$$\frac{dp}{dz} = \frac{1}{r}\frac{d}{dr}\left(\mu\, r\frac{dq_z}{dr}\right) \qquad ...(5)$$

Differentiating both sides of (5) *w. r. t.* 'z', we get

$$\frac{d^2 p}{dz^2} = 0 \qquad \text{so that} \qquad \frac{d}{dz}\left(\frac{dp}{dz}\right) = 0$$

Integrating (12), $dp/dz = P$, where P is a constant ...(6)

From (5) and (6), we have
$$\frac{d}{dr}\left(\mu r\frac{dq_z}{dz}\right) = Pr \qquad ...(7)$$

Integrating (7), $\mu r\,(dq_r/dr) = Pr^2/2 + A$, A being an arbitrary constant ...(8)

Let τ_w be the skin friction at the surface of the cylinder so that when $r = a$, we have

$$[\mu\,(dq_z/dr)]_{r=a} = \tau_w \qquad ...(9)$$

Putting $r = a$ in (8) and using (9), we get

$$a\tau_w = Pa^2/2 + A \qquad \text{so that} \qquad A = a\tau_w - Pa^2/2 \qquad ...(10)$$

Form (8) and (10), $\mu r\,(dq_z/dr) = a\tau_w + P(r^2 - a^2)/2$ ...(11)

Finally, for the present flow satisfying (1), energy equaiton (refer equaiton (3)′ Art 21.1) reduces to

$$0 = \frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right) + \mu\left(\frac{dq_z}{dr}\right)^2 \qquad \text{or} \qquad \frac{d}{dr}\left(rk\frac{dT}{dr}\right) = \left(\mu r\frac{dq_z}{dr}\right)\times\frac{dq_z}{dr}$$

or
$$\frac{d}{dr}\left(rk\frac{dT}{dr}\right) = \left\{a\tau_w + \frac{1}{2}P(r^2 - a^2)\right\}\times\frac{dq_z}{dr}, \text{ using (11)} \qquad ...(12)$$

Integrating (12), we have

$$rk\frac{dT}{dr} = \int_0^r a\tau_w\frac{dq_z}{dr}dr + \frac{P}{2}\int_0^r (r^2 - a^2)\times\frac{dq_z}{dr}dr + B, \ B \text{ being an arbitrary constant} \qquad ...(13)$$

or
$$rk\frac{dT}{dr} + a\tau_w\, q_z = B - \frac{P}{2}\int_0^r (a^2 - r^2)\times\frac{dq_z}{dr}dr \qquad ...(14)$$

While solving the above equation, we assume that Prandtl number P_r ad C_p. are constants. Clearly, (14) can be integrated provided we know the variation of the coefficient of viscosity μ and the coefficient of thermal conductivity k. From either the simple kinetic theory of gases or empirical data, μ is usually expressed with sufficient accuracy as a power of the absolute temperature is the form:

$$\mu/\mu_0 = (T/T_\infty)^m, \qquad 0.5 < m < 1 \qquad ...(15)$$

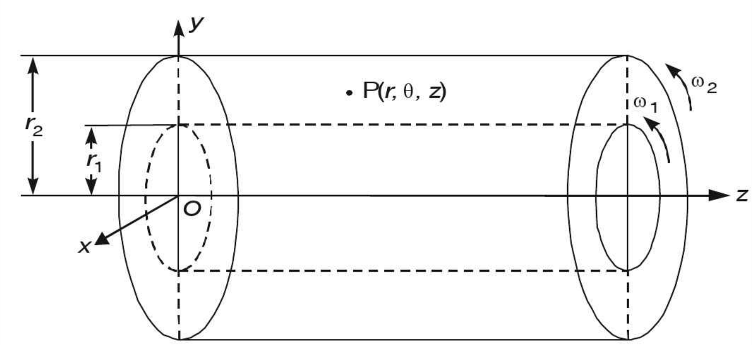
where μ_∞ is the value of μ at the moving plate. For air at ordinary temperature, we usually take $m = 0.76$. As the temperature increases, m decreases towards 0.5.

Now, $$P_r = (C_p \mu)/k \quad \Rightarrow \quad k = (C_p \mu)/P_r \qquad \text{...(16)}$$

Using (15) and (16), equations (11) and (14) can be integrated numerically to compute q_z and T.

Equation of state is given by $$p = k\rho T, \qquad \text{...(17)}$$
from which pressure can be obtained.

21.5. Laminar steady flow of a compressible viscous fluid between two concentric rotating cylinders.



Consider two infinitely long, concentric circular cylinders of radii r_1 and r_2 ($r_1 < r_2$) rotating with constant angular velocities ω_1 and ω_2 respectively. Let there be a compressible viscous fluid in the annular space. Then the cylinders induce a steady, axi-symmetrical, tangential motion in the fluid so that we have only the tangential component of velocity q_θ. Thus here $q_r = q_z = 0$. Let z-axis be taken along the common axis of the cylinders.

Let the flow depend on only r so that for the present flow, we have

$$q_r = 0, \qquad q_\theta = q_\theta(z), \qquad q_z = 0, \qquad p = p\,(r), \qquad T = T\,(r), \qquad \partial/\partial\theta = 0, \qquad \partial/\partial z = 0. \quad \text{...(1)}$$

In view of (1), the equation of continuity (see equation (1)′ of Art 21.1) is satisfied identically and the Navier Stokes equations reduce to (refer equation (2*a*)′ and (2*b*)′ of Art 21.1)

$$-\rho\frac{q_\theta^2}{r} = -\frac{dp}{dr} \qquad \text{so that} \qquad \frac{dp}{dr} = \frac{\rho q_\theta^2}{r} \qquad \text{...(2)}$$

and $$0 = \frac{d}{dr}\left\{\mu\left(\frac{dq_\theta}{dr} - \frac{q_\theta}{r}\right)\right\} + \frac{2\mu}{r}\left(\frac{dq_\theta}{dr} - \frac{q_\theta}{r}\right) \quad \text{or} \quad r^2\frac{d}{dr}\left\{\mu\left(\frac{dq_\theta}{dr} - \frac{q_\theta}{r}\right)\right\} + 2r\mu\left(\frac{dq_\theta}{dr} - \frac{q_\theta}{r}\right) = 0$$

i.e., $$\frac{d}{dr}\left\{r^2\mu\left(\frac{dq_\theta}{dr} - \frac{q_\theta}{r}\right)\right\} = 0 \qquad \text{...(3)}$$

The boundary conditions are

At $r = r_1$: $\quad q_\theta = r_1\omega_1, \quad T = T_1$; $\quad$ at $r = r_2$: $\quad q_\theta = r_2\omega_2, \quad T = T_2 \qquad$ ...(4)

Let ω be angular velocity at point $P(r, \theta, z)$ in the fluid. Then, we have $\quad q_\theta = r\omega$
and hence (3) reduces to

$$\frac{d}{dr}\left[r^2\mu\left\{\frac{\partial(r\omega)}{\partial r} - \omega\right\}\right] = 0 \qquad \text{or} \qquad \frac{d}{dr}\left(r^3\mu\frac{d\omega}{dr}\right) = 0$$

Integrating, $r^3\mu(d\omega/dr) = A$, A being an arbitrary constant ...(5)

Finally, under the assumptions (1), the energy equation for the present flow is given by (refer equaiton (3)' of Art.. 21.1)

$$0 = \frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) + \mu\left(\frac{dq_\theta}{dr} - \frac{q_\theta}{r}\right)^2$$

or
$$0 = \frac{1}{r}\frac{d}{dr}\left(\frac{\mu C_p}{P_r}\, r\, \frac{dT}{dr}\right) + \mu\left(\frac{dq_\theta}{dr} - \frac{q_\theta^2}{r}\right)^2, \quad \text{as} \quad P_r = \frac{\mu C_p}{k} \Rightarrow k = \frac{\mu C_p}{P_r}$$

$$\frac{C_p}{P_r}\frac{1}{r}\frac{d}{dr}\left(r\mu\frac{dT}{dr}\right) + \mu\left(\frac{dq_\theta}{dr} - \frac{q_\theta}{r}\right)^2 = 0$$

or
$$\frac{C_p}{P_r}\frac{d}{dr}\left(r\mu\frac{dT}{dr}\right) + r\mu\left\{\frac{d(r\omega)}{dr} - \omega\right\}^2 = 0, \quad \text{as} \quad q_\theta = r\omega$$

or
$$\frac{C_p}{P_r}\frac{d}{dr}\left(r\mu\frac{dT}{dr}\right) + r\mu\left(r\frac{d\omega}{dr}\right)^2 = 0$$

or
$$\frac{C_p}{P_r}\frac{d}{dr}\left(r\mu\frac{dT}{dr}\right) + A\frac{d\omega}{dr} = 0, \text{ using (5)}$$

Integrating, $(C_p r\mu/P_r) \times (dT/dr) + A\omega = B$, B being an arbitrary constant ...(6)

Equation (6) can be integrated provided we know a relationship between μ and k. From either the simple kinetic theory of gases or empirical data, μ is usually expressed with sufficient accuracy as a power of the absolute temperature in the form

$$\mu/\mu_0 = (T/T_\infty)^m, \quad \text{where} \quad 0.5 < m < 1 \qquad ...(7)$$

where μ_∞ is the value of μ at the moving plate. For air at ordinary temperature, we usually take $m = 0.76$. As the temperature increase, m decreases towards 0.5

Since P_r is very nearly constant for all common gases and C_p is also nearly constant for a fairly wide range of temperatures around ordinary temperature,

$$P_r = (C_p\mu)/k \quad \Rightarrow \quad k \text{ is proportional to } \mu.$$

Using the above facts, equations (5) and (6) can be solved numerically to compute ω and T. Finally, pressure can be obtained by using the equation of state, namely,

$$p = k\rho T. \qquad ...(8)$$

21.6. Laminar boundary layer equtions in compressible viscous fluid flow.

In Art. 18.3, we have already discussed the concept of boundary layer flow of incompressible viscous fluids in the vicinity of a fixed semi-infinite plate The order of magnitude approach for reducing the Navier stokes equations to the boundary layer equations in two-dimensions was explained in Art. 18.6 In incompressible laminar boundary layer the relative orders of magnitude of u, v, x and y were seen to be 1, δ, 1 and δ, respectively. If compressible effects in the boundary layer are to be included, the additional variables ρ and T have to be considered. Now T is of the same order of magnitude of u, and ρ is inversely proportional to T ($\because$ $p = \rho k T$ $\Rightarrow$ $\rho = (p/k) \times (1/T)$ and the pressure p is constant across the boundary layer). Hence ρ and T must be taken as of the order of magnitude unity.

For simplicity, we shall first derive the boundary layer equations for the flow over a semi-infinite flat plate. We take rectangular cartesian co-ordinates (x, y) with x measured in the plate in the direction of the two-dimensional laminar compressible flow, and y measured normal to the plate, and (u, v) are the velocity components. Let δ be small thickness of the boundary and let U be the velocity in the main stream just out side the boundary layer (refer figure (*i*) of Art 18.3). Then the Navier-Stokes equations, without body forces for two-dimensional compressible viscous fluid flow are (refer equations (2*a*) and (2*b*) of Art 21.1)

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left\{2\mu\frac{\partial u}{\partial x}-\frac{2\mu}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}+\frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\} \quad ...(1)$$

and $$\rho\frac{\partial v}{\partial t}+\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\left\{2\mu\frac{\partial v}{\partial y}-\frac{2\mu}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}+\frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\} \quad ...(2)$$

The continuity equaton (refer equation (1) of Art 21.1) reduces to

$$\partial\rho/\partial t+\partial(\rho u)/\partial x+\partial(\rho v)/\partial y=0 \quad ...(3)$$

The energy equation (refer (3) of Art 21.1) reduces to

$$\rho\frac{\partial(C_pT)}{\partial t}+\rho u\frac{\partial(C_pT)}{\partial x}+\rho v\frac{\partial(C_pT)}{\partial y}=\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$$

$$+2\mu\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right\}-\frac{2\mu}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2+\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2 \quad ...(4)$$

Since the Prandtl number $P_r\left(=\mu C_p/k\right)$ of a gas is of the order of unity and C_p is almost constant for ordinary gases at moderate temperatures, μ and k are treated as quantities of the same order. Keeping in mind, the above orders of magnitude and proceeding as in Art. 18.6, the above equations (1), (2), (3) and (4) take the following forms :

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \quad ...(5)$$

$$\partial p/\partial y=0 \quad ...(6)$$

$$\partial\rho/\partial t+\partial(\rho u)/\partial x+\partial(\rho v)/\partial y=0 \quad ...(7)$$

$$\rho\frac{\partial(C_pT)}{\partial t}+\rho u\frac{\partial(C_pT)}{\partial x}+\rho v\frac{\partial(C_pT)}{\partial y}=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x} \quad ...(8)$$

Equations (5), (6), (7) and (8) are known as the *boundary layer equations for a two-dimensional flow of compressible viscous fluid*. The boundary conditions for a steady flow are

$$\left.\begin{array}{llll} y=0: & u=v=0, & T=T_w(x), & p=p_1(x) \\ y=\infty: & u=U(x), & T=T_1(x), & p=p_1(x) \end{array}\right\} \quad ...(9)$$

where the subscripts 1 and w indicate the conditions at the outer edge of the boundary layer and the plate, respectively.

Remark. The above equations were derived without consideration of the curvature of the plate. Now, suppose that the radius of curvature of the wall is large compared with the thickness of the boundary layer. In this situation it can be shown that the equations derived in the above analysis are also valid for curved plates.

21.7. Velocity and temperature relation in laminar boundary layers.

Case I. Boundary layer with pressure gradient

In the present article we propose to discuss an application of the boundary layer equations (refer equations (5). (6), (7) and (8) of Art 21.6) by considering what is geometrically the simplest configuration. Consider a thin infinite flat plate submerged in steady compressible plane parallel flow, whose undisturbed velocity is U (refer figure (*i*) of Art 18.3). Let the plate be at rest in such a way that its plane coincides with the direction of U. Since the plate is of infinite length, the flow may be regarded as two-dimensional. Let the origin of the coordinate system coincide with the front edge of the plate, the x-axis lying along the plate parallel to U and y – axis normal to the plate. Then, the system of boundary layer equations (refer equations (5), (6), (7) and (8) of Art 21.6) reduce to

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \qquad ...(1)$$

$$\partial p / \partial y = 0 \qquad ...(2)$$

$$\partial(\rho u)/\partial x + \partial(\rho v)/\partial y = 0 \qquad ...(3)$$

and
$$\rho u \frac{\partial (C_p T)}{\partial x} + \rho v \frac{\partial (C_p T)}{\partial y} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \mu \left(\frac{\partial u}{\partial y}\right)^2 + u \frac{\partial p}{\partial x} \qquad ...(4)$$

Again, equaiton of state for a perfect gas is given by $\quad p = k\rho T \qquad ...(5)$

Boundary conditions are:

$$\left.\begin{array}{llll} y = 0: & u = v = 0, & T = T_w(x), & p = p_1(x) \\ y = \infty: & u = U(x), & T = T_1(x), & p = p_1(x) \end{array}\right\} \qquad ...(6)$$

where the subscripts 1 and w indicate the conditions at the outer edge of the boundary layer and the wall respectively.

Clearly, in general, the solution of the above system of equations (1) – (5) for given values of U (x) and T_w is quite complicated. In order to solve the above system of equations we take

$P_r = 1 \quad$ so that $\quad \mu C_p / k = 1 \quad$ or $\quad \mu C_p = k \qquad ...(7)$

We also assume that the temperature T is a function of the velocity component u only, so that $\quad C_p T = f(u)$, where f is an arbitrary function $\qquad ...(8)$

Then (4) may be re-written as

$$\left(\rho\, u \frac{\partial u}{\partial x} + \rho\, v \frac{\partial u}{\partial y}\right) \frac{df(u)}{du} = u \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left\{\mu \frac{\partial u}{\partial y} \frac{df(u)}{du}\right\} + \mu \left(\frac{\partial u}{\partial y}\right)^2$$

or
$$\left(\rho\, u \frac{\partial u}{\partial x} + \rho\, v \frac{\partial u}{\partial y}\right) \frac{df(u)}{du} = u \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \frac{df(u)}{du} + \mu \left(\frac{\partial u}{\partial y}\right)^2 \left\{\frac{d^2 f(u)}{du^2} + 1\right\} \qquad ...(9)$$

We easily verify that (9) reduces to (1) when $\quad df(u)/du = -u \qquad ...(10)$

and
$$d^2 f(u)/du^2 = -1 \qquad ...(11)$$

In view of these conditions, (8) becomes

$$C_p\, T = A - u^2/2, \; A \text{ being an arbitrary constant}$$

or
$$C_p T + u^2/2 = A \qquad ... (12)$$

Let T_0 be the absolute temperature at the points where the fluid is at rest *i.e.*, $T = T_0$ when $u = 0$. Hence (1) yields $A = C_p\, T_0$. Therefore, (12) becomes

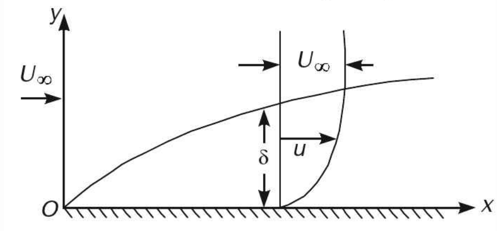
$$C_p\, T + u^2/2 = C_p\, T_0 \quad \text{or} \quad C_p\, T = C_p\, T_0 - u^2/2, \qquad ...(13)$$

showing that the sum of the enthalpy and the kinetic energy is a constant throughout the boundary layer. The heat transfer into or from the wall (refer equation (11) of Art. 21.3) is given by

$$-Q_w \sim \left(\frac{\partial T}{\partial y}\right)_w \sim \left(u\frac{\partial u}{\partial y}\right)_{y=0} = 0 \qquad \text{...(14)}$$

This is the condition of the so called adiabatic wall. Hence, when $P_r = 1$ and $Q_w = 0$, the stagnation temperature is equal to the wall temperature, *i.e.* $T_0 = T_w$ ...(15)

Case II. Boundary layer with zero pressure gradient



When the pressure gradient along the wall is zero (flow along a flat plate as shown in the adjoining figure), the system of boundary layer equaitons (5), (6), (7) and (8) of Art. 21.6) can be further simplified (using the fact $\partial p/\partial x = 0$) and obtain

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \qquad \text{...(15)}$$

$$\partial(\rho u)/\partial x + \partial(\rho v)/\partial y = 0 \qquad \text{...(16)}$$

and $$\rho u\frac{\partial (C_p T)}{\partial x} + \rho v\frac{\partial (C_p T)}{\partial y} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \qquad \text{...(17)}$$

In order to solve the above equations (15), (16) and (17) easily, we take a particular case when Prandtl number $= P_r = 1$ so that $\mu C_p/k = 1$ or $\mu C_p = k$. We also assume that the temperature T is a function of the velocity component u only, Then,

$$C_p T = f(u) \qquad \text{...(18)}$$

Now, (17) may be re-written as

or $$\left(\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y}\right)\frac{df(u)}{du} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\frac{df(u)}{du} + \mu\left(\frac{\partial u}{\partial y}\right)^2\left\{\left(\frac{d^2 f(u)}{du^2}\right) + 1\right\} \qquad \text{...(19)}$$

Equation (19) reduces to (15) only if $d^2 f(u)/du^2 = -1$ ...(20)

Integrating (20) twice, we get $f(u) = c_1 u + c_2 - u^2/2$

or $C_p T = c_1 u + c_2 - u^2/2$, c_1 and c_2 being arbitrary constants ...(21)

The boundary conditions are :

$y = 0$: $u = v = 0$, $T = T_w =$ constant ...(22*a*)

$y = \infty$: $u = U_\infty$, $T = T_\infty =$ constant ...(22*b*)

Using boundary condition (22*a*), (21) $\Rightarrow$ $C_p T_w = c_2$ ...(23)

Using boundary condition (22*b*), (21) $\Rightarrow$ $C_p T_\infty = c_1 U_\infty + c_2 - U_\infty^2/2$ ...(24)

Solving (23) and (24), $c_2 = C_p T_w$ and $c_1 = \left\{C_p(T_\infty - T_w) + U_\infty^2/2\right\}/U_\infty$

Substituting these values of c_1 and c_2 in (24), we get

$$\frac{T}{T_\infty} = \frac{T_w}{T_\infty} + \left(1 - \frac{T_w}{T_\infty}\right)\frac{u}{U_\infty} + \frac{1}{2C_p T_\infty}(U_\infty - u)u \qquad \text{...(25)}$$

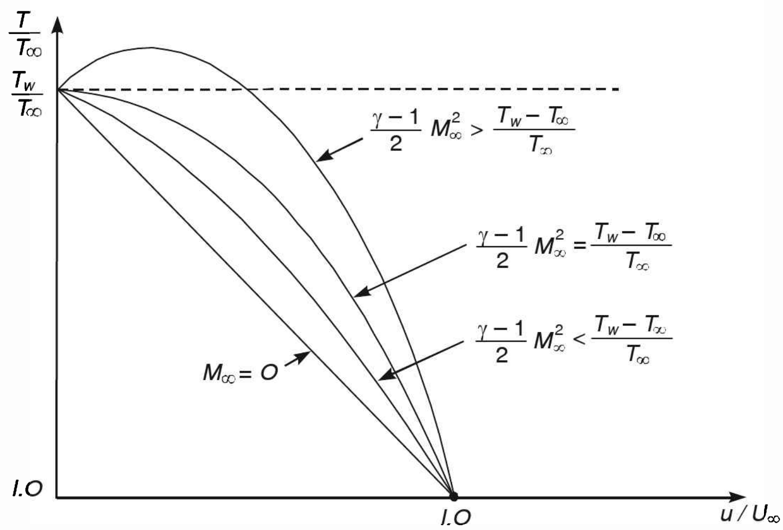
Let M_∞ denote the Mach number of the free stream and γ the ratio of specific heats. Then, if Ec denotes Eckert number, we have (refer Art 15.7)

$$Ec = U_\infty^2 / C_p T_\infty = (\gamma - 1) M_\infty^2 \qquad \Rightarrow \qquad \frac{1}{C_p T_\infty} = \frac{(\gamma-1)M_\infty^2}{U_\infty^2} \qquad \text{...(26)}$$

Using (26), (25) can be re-written as

$$\frac{T}{T_\infty} = \frac{T_w}{T_\infty} + \left(1 - \frac{T_w}{T_\infty}\right)\frac{u}{U_\infty} + \frac{(\gamma-1)M_\infty^2}{2}\left(1 - \frac{u}{U_\infty}\right)\frac{u}{U_\infty} \qquad \text{...(27)}$$

The following figure illustrates the relation between temperature and velocity for various values of Mach number



The shearing stress at the wall and heat conduction to the wall per unit area per unit time are defined, respectively, as

$$\tau_w = [\mu\, (\partial u / \partial y)]_w \qquad \text{...(28)}$$

and

$$Q_w = [k(\partial T / \partial y)]_w \qquad \text{...(29)}$$

Differentiating (27) partially *w.r.t.* 'y' and substituting in (29), we obtain

$$Q_w = k_w T_\infty \left[\left(1 - \frac{T_w}{T_\infty}\right)\frac{1}{U_\infty}\left(\frac{\partial u}{\partial y}\right)_w + \frac{(\gamma-1)M_\infty^2}{2U_\infty}\left(\frac{\partial u}{\partial y}\right)_w\right] \qquad \text{...(30)}$$

Using (7) and (28), the relation between the heat flux and shearing stress at the wall for $P_r = 1$ is given by

$$Q_w = \frac{C_p T_\infty}{U_\infty}\tau_w \left[\left(1 - \frac{T_w}{T_\infty}\right) + \frac{(\gamma-1)}{2}M_\infty^2\right] \qquad \text{...(31)}$$

Let T_0 be the stagnation temperature given by

$$T_0 / T_\infty = 1 + \{(\gamma-1)M_\infty^2\}/2 \qquad \text{...(32)}$$

Using (32), (31) can be re-written by as

$$Q_w = (C_p\, \tau_w / U_\infty)\,(T_0 - T_w), \qquad \text{...(33)}$$

showing that due to dynamic heating of the fluid, the reference temperature which is employed for computing the temperature drop for heat conduction is actually the stagnation temperature and not the static temperature T_∞.

From (33), we also conclude that an adiabatic wall exists ($Q_w = 0$) when $T_0 = T_w$, that heat is conducted into the wall ($Q_w > 0$) when $T_0 > T_w$, and that heat is conducted to the fluid ($Q_w < 0$) when $T_0 < T_w$. The last phenomenon is known as *colling the wall*. The relation (33) is known as the *generalized Reynolds* analogy with heat transfer and skin friction.

For an incompressible fluid ($M = 0$) it can easily be proved that a relation between coefficient of heat transfer and friction exists and is given by

$$C_h = \frac{C_f}{2}, \text{ where } C_h \text{ heat transfer coefficient } = \frac{Q_w}{\rho U_\infty C_p (T_\infty - T_w)} \qquad \text{...(34)}$$

21.8. Approximate soluitons of boundary layer equations.

In general the process of finding a complete solution of the boundary layer equations is very tedious and time consuming, and hence it cannot be employed in practice. The solutions obtained in Art. 21.7 represent only very special cases. For engineering problems, it is often desirable to possess at least approximate methods of solution, to the applied in cases when solution of the boundary layer equations consumes a lot of time. The basic concept of such approximate method is that the solutions are allowed to satisfy the differential equations on the average. Here we do not require that the differential equations to be satisfied by each fluid particle. In the next article, we propose to derive the momentum integral equation and the energy integral equation, These equations will provide approximate solutions of the boundary layer equations.

21.9. Derivation of the momentum integral equation and the energy integral equation of the boundary leyer in compressible viscous fluids.

In Art 18.16, we have discussed the momentum integral equation for incompressible fluids which was obtained by integrating the Prandtl boundary larger equations with respect to y. Proceeding likewise we will obtain the momentum integral equation for compressible fluids.

The system of boundary layer equations for steady two-dimensional of compressible fluid over a semi-infinite flat plate (refer Art. 21.6) are given by

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \qquad \text{...(1)}$$

$$\partial p / \partial y = 0 \qquad \text{...(2)}$$

$$\partial(\rho u)/\partial x + \partial(\rho v)/\partial y = 0 \qquad \text{...(3)}$$

$$\rho u \frac{\partial (C_p T)}{\partial x} + \rho v \frac{\partial (C_p T)}{\partial y} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \mu \left(\frac{\partial u}{\partial y}\right)^2 + u \frac{\partial p}{\partial x} \qquad \text{...(4)}$$

The boundary conditions are:

$$\left.\begin{array}{lll} y = 0: & u = v = 0, & T = T_w(x) \\ y = \infty: & u = U(x), & T = T_1(x) \end{array}\right\} \qquad \text{...(5)}$$

where the subscripts 1 and w indicate the conditions at the outer edge of the boundary layer and the plate, respectively.

According to (2) the pressure distribution is a function of x only, *i.e.*, for a given x, p is constant throughout the boundary layer. In other words the pressure throughout the boundary layer has the same value of p as in the potential flow. Let $U(x)$ be velocity in the potential flow (just outside the boundary layer), then (1) reduces to (noting that $\partial u/\partial y = 0$, $\rho = \rho_1$, $v = 0$ and $\mu = 0$ at $y = \delta$),

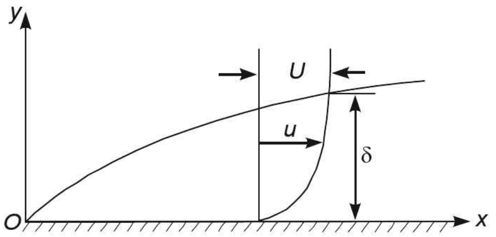


Figure (*i*)

$$\rho_1 U\,(dU/dx) = -(\partial p/\partial x) \qquad ...(6)$$

Integrating (1) with respect to y from $y = 0$ to $y = \delta(x)$, the outer edge of the boundary layer (see figure (i)), we obtain

$$\int_0^\delta \rho u \frac{\partial u}{\partial x} dy + \int_0^\delta \rho v \frac{\partial u}{\partial y} dy = -\int_0^\delta \frac{\partial p}{\partial x} dy - \left[\mu \frac{\partial u}{\partial y}\right]_0^\delta \qquad ...(7)$$

Now, $[\mu(\partial u/\partial y)]$ = shearing stress = τ_{yx}. Then $\tau_{yx} = 0$ at $y = \delta$. Let $\tau_w = (\tau_{yx})_{y=0}$ = shearing stress at fixed plate.

Hence, (7) reduces to
$$\int_0^\delta \rho u \frac{\partial u}{\partial x} dy + \int_0^\delta \rho v \frac{\partial u}{\partial y} dy = -\int_0^\delta \frac{\partial p}{\partial x} dy - \tau_w \qquad ...(8)$$

Integrating by parts the second term on the L.H.S. of (8), we get

$c_f\sqrt{\mathrm{Re}_x} = (\tau_{yx})_{y=0}$

$$\int_0^\delta \rho v \frac{\partial u}{\partial y} dy = [\rho v u]_0^\delta - \int_0^\delta u \frac{\partial(\rho v)}{\partial y} dy = U\int_0^\delta \frac{\partial(\rho v)}{\partial y} dy - \int_0^\delta u \frac{\partial(\rho v)}{\delta y} dy$$

or
$$\int_0^\delta \rho v \frac{\partial u}{\partial y} dy = -U\int_0^\delta \frac{\partial(\rho u)}{\partial x} dy + \int_0^\delta u \frac{\partial(\rho u)}{\partial x} dy, \quad \text{using (3)} \qquad ...(9)$$

Using (6) and (9), (8) can be re-written as

$$\int_0^\delta \rho u \frac{\partial u}{\partial x} dx - U\int_0^\delta \frac{\partial(\rho u)}{\partial x} dy + \int_0^\delta u \frac{\partial(\rho u)}{\partial x} dy = \frac{dU}{dx}\int_0^\delta \rho_1 U dy - \tau_w \qquad ...(10)$$

Now,
$$U\int_0^\delta \frac{\partial(\rho u)}{\partial x} dy = \int_0^\delta \frac{\partial(\rho u U)}{\partial x} dy - \frac{dU}{dx}\int_0^\delta \rho u\, dy \qquad ...(11)$$

and
$$\int_0^\delta u \frac{\partial(\rho u)}{\partial x} dy = \int_0^\delta \frac{\partial(\rho u^2)}{\partial x} dy - \int_0^\delta \rho u \frac{\partial u}{\partial x} dy \qquad ...(12)$$

Re-writing the first term on the R.H.S. of equations (11) and (12) with help of the Leibnitz's rule of differentiation under the integral sign, (11) and (12) can be re-written in the following forms.

$$\int_0^\delta \frac{\partial(\rho u U)}{\partial x} dy = \frac{d}{dx}\int_0^\delta \rho u U\, dy - \rho_1 U^2 \frac{d\delta}{dx} \qquad ...(13)$$

and
$$\int_0^\delta \frac{\partial(\rho u^2)}{\partial x} dy = \frac{d}{dx}\int_0^\delta \rho u^2\, dy - \rho_1 U^2 \frac{d\delta}{\partial x} \qquad ...(14)$$

Hence (10) can be re-written as

$$\frac{d}{dx}\int_0^\delta \rho u(U-u)\,dy + \frac{dU}{dx}\int_0^\delta (\rho_1 U - \rho u)\,dy = \tau_w, \qquad ...(15)$$

which is the *momentum integral equation* of the boundary layer in compressible fluids.

Derivation of the energy-integral equation for two-dimensional compressible flow

In what follows, we shall proceed exactly, as in derivation of the momentum integral equation (15). Integrating (4) with respect to y from $y = 0$ to $y = \delta_t$, the thickness of the thermal boundary layer (refer fig (*ii*)), we obtain

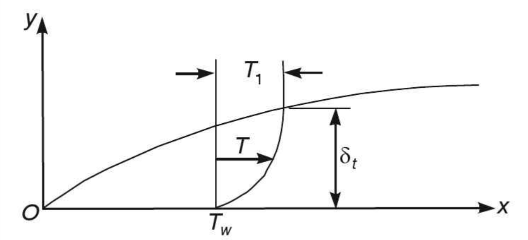


Figure (*ii*)

$$\int_0^{\delta_t} \rho u \frac{\partial (C_p T)}{\partial x} dy + \int_0^{\delta_t} \rho v \frac{\partial (C_p T)}{\partial y} dy = \left[k \frac{\partial T}{\partial x} \right]_{y=0}^{y=\delta_t} + \int_0^{\delta_t} \mu \left(\frac{\partial u}{\partial y} \right)^2 dy + \int_0^{\delta_t} u \frac{\partial p}{\partial x} dx \qquad ...(16)$$

But $\quad -k(\partial T / \partial x)$ = heat flux = Q and hence

$$[k(\partial T / \partial x)]_{y=\delta_t} = 0 \qquad \text{and} \qquad [k(\partial T / \partial x)]_{y=0} = -Q_w$$

where Q_w is the heat flow to the plate. Using the above facts the above equaton (16) reduces to

$$\int_0^{\delta_t} \rho u \frac{\partial (C_p T)}{\partial x} dy + \int_0^{\delta_t} \rho v \frac{\partial (C_p T)}{\partial y} dy = \int_0^{\delta_t} u \frac{\partial p}{\partial x} dy + \int_0^{\delta_t} \mu \left(\frac{\partial u}{\partial y} \right)^2 dy - Q_w \qquad ...(17)$$

The second term on the L.H.S. can be integrated by parts, Thus, we have

$$\int_0^{\delta_t} \rho v \frac{\partial (C_p T)}{\partial y} dy = \left[\rho v C_p T \right]_0^{\delta_t} - \int_0^{\delta_t} C_p T \frac{\partial (\rho v)}{\partial y} dy$$

$$= C_p T_1 \int_0^{\delta_t} \frac{\partial (\rho v)}{\partial y} dy - \int_0^{\delta_t} C_p T \frac{\partial (\rho v)}{\partial y} dy = -C_p T_1 \int_0^{\delta_t} \frac{\partial (\rho u)}{\partial x} dy + \int_0^{\delta_t} C_p T \frac{\partial (\rho u)}{\partial x} dy, \text{ using (3)}$$

$$= -\int_0^{\delta_t} \frac{\partial (C_p T_1 \rho u)}{\partial x} dy + \frac{d(C_p T_1)}{dx} \int_0^{\delta_t} \rho u \, dy + \int_0^{\delta_t} C_p T \frac{\partial (\rho u)}{\partial x} dy \qquad ...(18)$$

We know that the energy equation in the free stream is given by

$$(U^2 + v_1^2)/2 + C_p T_1 = \text{constant}, \qquad ...(19)$$

where v_1 is velocity component in y-direction in free stream

Differentiating (19) *w.r.t.* 'x', we obtain

$$U \frac{dU}{dx} + v_1 \frac{dv_1}{dx} = -\frac{d(C_p T_1)}{dx} \qquad ...(20)$$

Since the flow is irrotational in the free stream, we have $\quad \partial v_1 / \partial x = \partial U / \partial y = 0 \qquad ...(21)$

Using (6) and (21), (20) reduces to $\qquad \dfrac{\partial p}{\partial x} = \rho_1 \dfrac{\partial (C_p T_1)}{\partial x} \qquad ...(22)$

Using (18) and (22), (17) may be rewritten as

$$\int_0^{\delta_t} \rho u \frac{\partial (C_p T)}{\partial x} dy - \int_0^{\delta_t} \frac{\partial (C_p T_1 \rho u)}{\partial x} dy + \frac{d(C_p T_1)}{dx} \int_0^{\delta_t} \rho u \, dy + \int_0^{\delta_t} C_p T_1 \frac{\partial (\rho u)}{\partial x} dy$$

$$= \int_0^{\delta_t} \rho_1 u \frac{\partial (C_p T_1)}{\partial x} dy + \int_0^{\delta_t} \mu \left(\frac{du}{\partial y} \right)^2 dy - Q_w$$

Simplifying and re-writing the above equation, we get

$$\frac{d}{dx}\int_0^{\delta_t} \rho u C_p (T_1 - T)dy + \frac{d(C_p T)}{dx}\int_0^{\delta_t} u(\rho_1 - \rho)\,dy + \int_0^{\delta_t} \mu\left(\frac{\partial u}{\partial y}\right)^2 dy = Q_w\,, \qquad ...(23)$$

which is the *energy-integral equation* of the boundary layer in compressible fluids.

In order to obtain an approximate solution of the boundary layer problem we assume profiles for both the velocity and the temperature in the boundary layer in the following forms:

$$\left.\begin{aligned} u/U &= f(y/\delta) && \text{for} \quad 0 < y < \delta \\ &= 1 && \text{for} \quad y \geq \delta \end{aligned}\right\} \qquad ...(24)$$

and

$$\left.\begin{aligned} T/T_1 &= g(y/\delta_t) && \text{for} \quad 0 < y < \delta_t \\ &= 1 && \text{for} \quad y \geq \delta_t \end{aligned}\right\} \qquad ...(25)$$

The functions f and g are selected to satisfy the following boundary conditions as nearly as possible.

(*i*) *At the outer edges of boundary layer*

$$y = \delta: \qquad \partial u/\partial y = \partial^2 u/\partial y^2 = \partial^3 u/\partial y^3 = \ldots = 0 \qquad ...(26)$$

and

$$y = \delta_t: \qquad \partial T/\partial y = \partial^2 T/\partial y^2 = \partial^2 T/\partial y^3 = \ldots = 0 \qquad ...(27)$$

(*ii*) *At the surface of the fixed plate, y = 0:*

$$u = 0, \qquad \frac{\partial p}{\partial x} = \left[\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\right]_w \qquad ...(28)$$

and

$$T = T_w, \qquad \left[\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right]_w = -\left[\mu\left(\frac{\partial u}{\partial y}\right)^2\right]_w \qquad ...(29)$$

Select appropriate functions f and g occuring in (24) and (25). Substituting the vlaues of u and T so obtained in (15) and (23), respectively, we obtain two simultaneous first - order partial differential equations for determination of δ and δ_t. Solve these equations to compute the boundary layer thicknesses δ and δ_t together with τ_w and Q_w. In this connection, it is to be noted that, in general, the thermal boundary layer thickness δ_t is greater than the boundary layer thickness.

21.10. Application of the momentum integral equation to boundary layers.

In the usual symbols, the mumentum integral equation of the boundary layer in compressible fluids is given by

$$\frac{d}{dx}\int_0^{\delta} \rho u(U-u)dy + \frac{dU}{dx}\int_0^{\delta}(\rho_1 U - \rho u)dy = \tau_w \qquad ...(1)$$

Re-writing (1) in non-dimensional form, we obtain

$$\frac{d}{dx}\int_0^{\delta}\frac{\rho u}{\rho_1 U}\left(1-\frac{u}{U}\right)dy + \left(\frac{1}{\rho_1}\frac{d\rho_1}{dx} + \frac{2}{U}\frac{dU}{dx}\right)\int_0^{\delta}\frac{\rho u}{\rho_1 U}\left(1-\frac{u}{U}\right)dy + \frac{1}{U}\frac{dU}{dx}\int_0^{\delta}\left(1-\frac{\rho u}{\rho_1 U}\right)dy = \frac{\tau_w}{\rho_1 U^2} \qquad ...(2)$$

Let the displacement thickness δ_1 and the momentum thickness δ_2 be defined, respectively by

$$\delta_1 = \int_0^{\delta}\left(1-\frac{\rho u}{\rho_1 U}\right)dy \qquad \text{and} \qquad \delta_2 = \int_0^{\delta}\frac{\rho u}{\rho_1 U}\left(1-\frac{u}{U}\right)dy \qquad ...(3)$$

If c_1 denote the spead of sound in free stream, then we have

$$-\rho_1 U(dU/dx) = dp/dx = c_1^2(d\rho_1/dx) \qquad ...(4)$$

Also, M_1 = Mach number of gas in free - stream = U / c_1 ...(5)

Using (3), (4) and (5), (2) takes the following convenient form

$$\frac{d\delta_2}{dx} + \left(2 - M_1^2 + \frac{\delta_1}{\delta_2}\right)\frac{\delta_2}{U}\frac{dU}{dx} = \frac{\tau_w}{\rho_1 U^2} \qquad ...(6)$$

Since the pressure remains constant across the boundary layer, it follows from the equation of state for perfect gas that

$$p = \rho_1 R T_1 = \rho R T \qquad \text{so that} \qquad p/\rho_1 = T_1/T \qquad ...(7)$$

From either the simple kinetic theory of gases or empirical data, the coefficient of viscosity μ can often be expressed with sufficient accuracy as a power of the absolute temperature. Thus,

$$\mu/\mu_1 = (T/T_1)^m, \qquad \text{where} \qquad 0.5 < m < 1 \qquad ...(8)$$

For the Prandtl number $P_r = 1$ and $dp / dx = 0$, the temperature profile in the boundary can be obtained directly from the corresponding velocity profile. This relation is given by (refer equation (27), Art 21.7)

$$\frac{T}{T_\infty} = \frac{T_w}{T_\infty} + \left(1 - \frac{T_w}{T_\infty}\right)\frac{u}{U_\infty} + \frac{\gamma - 1}{2}M_\infty^2\left(1 - \frac{u}{U_\infty}\right)\frac{u}{U_\infty} \qquad ...(9)$$

For an adibatic wall, we have (refer equaiton (32), Art 21.7)

$$\frac{T}{T_\infty} = \frac{T_0}{T_\infty} = 1 + \frac{\gamma - 1}{2}M_\infty^2 \qquad ...(10)$$

Substituting the value of T_w given by (10) in (9), we have

$$\frac{T}{T_\infty} = 1 + \frac{\gamma - 1}{2}M_\infty^2\left(1 - \frac{u^2}{U^2}\right) \qquad ...(11)$$

In view of (7) and (8), the density and viscosity relations are, respectively

$$\frac{\rho}{\rho_\infty} = \frac{T_\infty}{T} = \frac{1}{1 + \alpha - \alpha(u^2/U_\infty^2)} = \frac{1}{\alpha_1} \times \frac{1}{1 - k(u/U_\infty)^2} \qquad ...(12)$$

and

$$\mu_w/\mu_\infty = (T_w/T_\infty)^m = \alpha_1^m, \qquad ...(13)$$

where $\alpha = (k-1)M_\infty^2/2, \qquad \alpha_1 = 1 + \left\{(k-1)M_\infty^2/2\right\} \qquad$ and $\qquad k = \alpha/\alpha_1$ ...(14)

To simply our calculations, let us assume a linear velocity in the form

$$u/U_\infty = y/\delta = \eta \qquad ...(15)$$

Using (12) and (13), from (3) the momentum thickenss η_2 is given by

$$\eta_2 = \frac{\delta}{\alpha_1}\int_0^1 \frac{\eta(1-\eta)}{1 - k\eta^2}d\eta = \frac{\delta}{\alpha_1}\left[\frac{1}{k}\int_0^{\sqrt{k}} \frac{z\,dz}{1 - z^2} - \frac{1}{k^{3/2}}\int_0^{\sqrt{k}} \frac{z^2}{1 - z^2}dz\right]$$

[Putting $\eta = (1/\sqrt{k})z$ and $d\eta = (1/\sqrt{k})dz$]

Thus,
$$\eta_2 = \frac{\delta}{\alpha_1 k}\left[1 - \frac{1}{2}\log_e(1-k) - \frac{1}{2\sqrt{k}}\log_e\frac{1+\sqrt{k}}{1-\sqrt{k}}\right] = \frac{\delta}{\alpha_1}k_1, \qquad ...(16)$$

where $$k_1 = \frac{1}{k}\left[1 - \frac{1}{2}\log_e(1-k) - \frac{1}{2\sqrt{k}}\log_e\frac{1+\sqrt{k}}{1-\sqrt{k}}\right] \qquad ...(17)$$

Substituting equations (16) and (17) into the momentum integral equation (6), the boundary layer the thickness δ is given by

$$\frac{\delta}{x} = \frac{1}{\sqrt{\text{Re}_x}}\sqrt{\frac{2}{k_1}}\,(\alpha_1)^{(m+1)/2} \qquad ...(18)$$

where $$\text{Re}_x = \text{Reynolds number} = U_\infty x/\nu_\infty \qquad ...(19)$$

Again, the total local skin friction coefficient is given by

$$C_f = \frac{2\tau_w}{(\rho_\infty U^2)/2} = \frac{2\sqrt{2k_1}}{\sqrt{\text{Re}_x}}\,\alpha_1^{(m-1)/2} \qquad ...(20)$$

Von Karman and Tsien* have discuss an exact solution of the boundary layer equations (refer equation (1) of Art. 21.9) for uniform flow passing over a flat plate of constant wall temperature. They took $P_r = 1$ and calculated temperature distribution from (9). In order to compare the coefficient of friction C_f versus the free-stream Mach number M_∞ given in equation (20) with that of Von. Karman-Tsien's result, both cases are based on an insulated plate and $m = 0.76$. From fig (*i*), it follows that the linear profile of velocity approximation is fairly good in the range of high Mach numbers. The reason of this good agreement lies in the fact that the velocity distribution in the boundary layer approaches a linear profile when the Mach number ≥ 6. This can be verified from figure (*ii*) which was based on Von. Karmain - Tsien's, result for an insulated plate. Based on the velocity distribution given in figure (*ii*), the temperature distributions versus the Mach number in the boundary layer are shown in figure (*iii*)

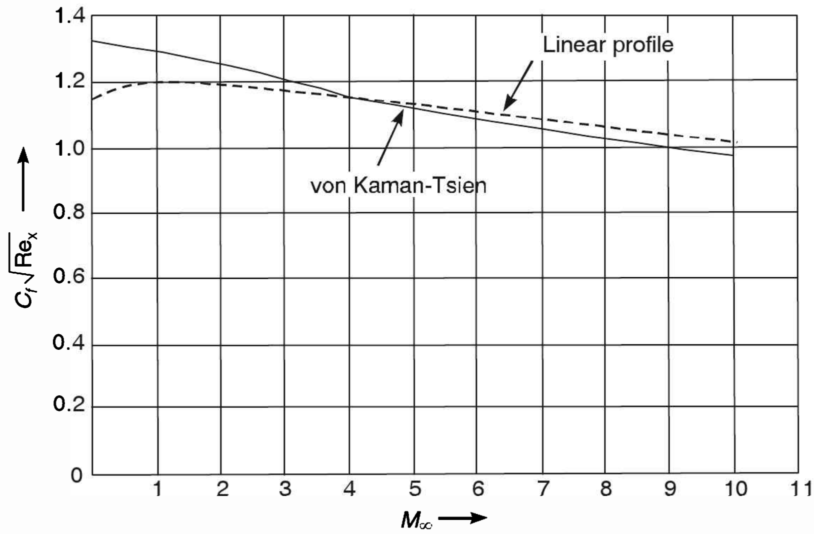


Figure (*i*)

* Karman, Th. Von and H.S. Tsien, "Boundary layer in compressible fluid", Jour Aero Sc., 5, No. 6 (1938), page 227 - 232.

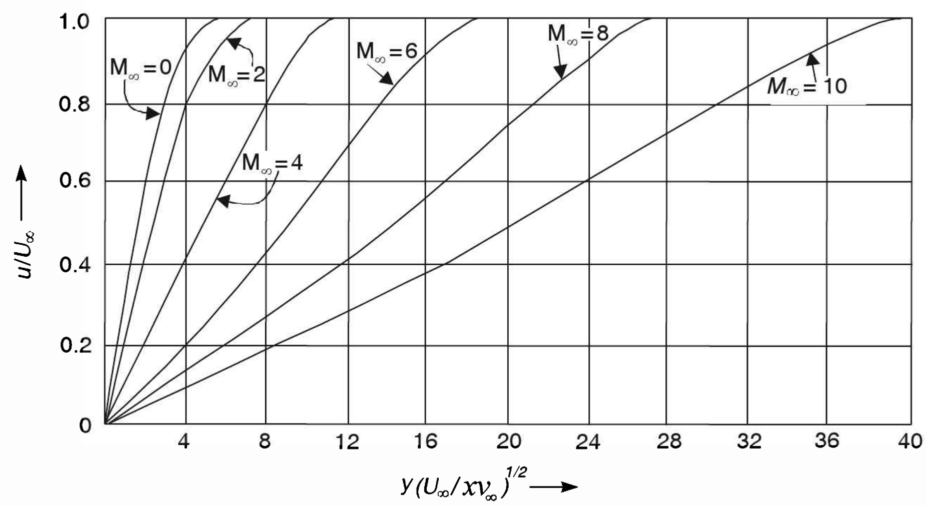


Figure (*ii*)

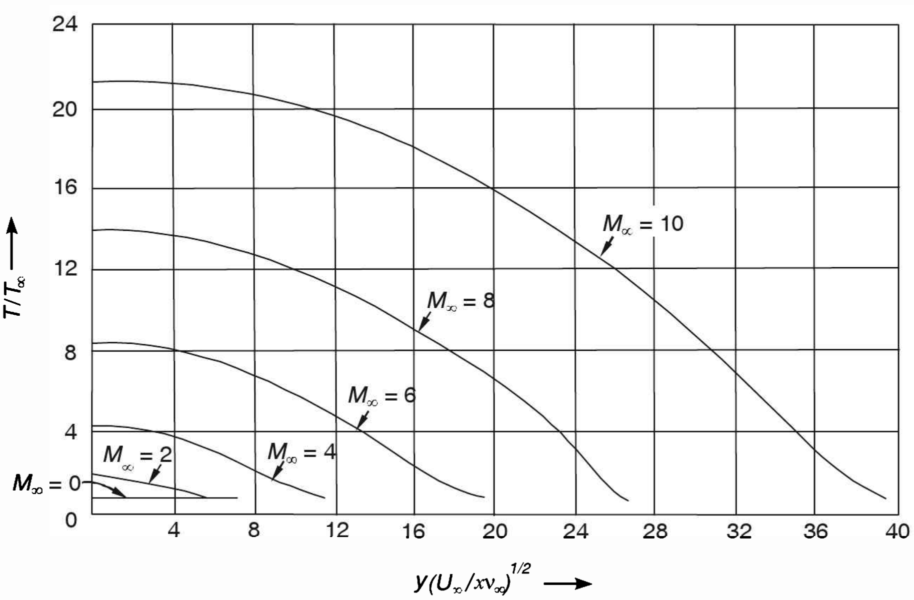


Figure (*iii*)

EXERCISES

1. Calculate and plot the velocity distribution for Couette flow using the following viscosity relation: $\mu/\mu_\infty = (T/T_\infty)^{1/2}$. Compare the result with that shown in Fig (*ii*) of Art. 21.3. For m = 1 and m = 0.76, Q_w = 0 and $\gamma = 1.4$ may be assumed.

2. Calculate and plot the temperature distribution for Couette flow using the viscosity relation $\mu/\mu_\infty = (T/T_\infty)^{1/2}$ (assuming that $P_r M_\infty^2 = 3.0$ and $\gamma = 1.4$) for

(*i*) $Q_w/\tau_w u = 1.0$ (*ii*) $Q_w/\tau_w u = 0$ and (*ii*) $Q_w/\tau_w u = -0.3$

3. Compute the coefficient of skin friction for Couette flow using the following viscosity relation: $\mu/\mu_\infty = (T/T_\infty)^{3/4}$. Plot the coefficient C_f Re versus the Mach number for P_r = 1.0 and $\gamma = 1.4$ and compare the result with that shown in figure (*iv*) of Art 21.3

4. Show that the temperature distribution in a viscous compressible Couette flow over an adiabatic wall (Q_w = 0) the temperature of wall is given by

$$T/T_\infty = 1 + \left\{P_r M_\infty^2 (\gamma+1)\right\}/2$$

5. Show that for viscous hypersonic Couette flow $(M_\infty \to \infty)$ over an adiabatic flat plate the velocity distribution is given by y = $(3u\ d\ /\ 2U) \times (1 - u^3\ /\ 3U^3)$, and the skin-friction at the plate y = 0 is $(2\mu_0 U/3d)$, in which d is the distance from the plate where uniform velocity is U and μ_0 is the coefficient of viscosity at the plate.

6. In a viscous compressible flow through a circular pipe the velocity distribution is given by $u = u_{\max}\left\{1-(r/a)^m\right\}$, where m is a constant. By taking the cylinder to be adiabatic and $\mu \propto T^{1/2}$, find the temperature density distribution and skin friction at the cylinder.

7. In a viscous compressible flow between two rotating co-axial cylinders, assume that $\omega = r^m \omega_0$, where m is a constant and $\mu \propto T^{1/2}$. Obtain the temperature and pressure distributions.

8. Starting with boundary layer equations for steady viscous compressible flow (see equations (1) and (4) of Art. 21.9), derive the following equation

$$\rho u \frac{\partial (C_p T_0)}{\partial x} + \rho v \frac{\partial (C_p T_0)}{\partial y} = \frac{\partial}{\partial y}\left[\mu \frac{\partial (C_p T_0)}{\partial y}\right] + \left(\frac{1}{P_r} - 1\right)\frac{\partial}{\partial y}\left[\mu \frac{\partial (C_p T_0)}{\partial y}\right]$$

where $C_p\ T_0$ is the stagnation enthalpy.

9. The velocity profile in the boundary layer of a flat plate is assumed to have the from

$$u/U_\infty = a\eta + b\eta^3 + c\eta^4;$$

Plot the temperature distribution, T/T_∞ versus η at $M_\infty^2 = 0$, 1.0, 2.0 and 5.0 for the cases: (*i*) $T_w/T_\infty = 1/2$ (*ii*) $T_w/T_\infty = 1.0$ (*iii*) $T_w/T_\infty = 5/4$.

Use equation (27) of Art 21.7 for T/T_∞.

10. Calculate the coefficient of friction factor $C_f\sqrt{\mathrm{Re}_x}$ for flow over a flat plate under the following conditions. (*i*) $u/U_\infty = 2(y/\delta) - 2(y/\delta)^3 + (y/\delta)^4$ (*ii*) P_r = 1.0, $\gamma = 1.4$ (*iii*) $\mu/\mu_\infty = (T/T_\infty)^{1/2}$ (*iv*) adiabatic plate. Plot $C_f\sqrt{\mathrm{Re}}$ versus M_∞.

Ex.1. (Helmoltz vorticity equation) *If the external forces are conservative, density ρ is a function of p only, Ω is the vorticity vector and* $\mathbf{q}$ *is the velocity vector, then prove that* $\dfrac{D}{Dt}\left(\dfrac{\Omega}{\rho}\right) = \left(\dfrac{\Omega}{\rho}\cdot\nabla\right)\mathbf{q}$. **[Garhwal 2003; Kanpur 2000, 06]**

Sol. First method: Proceeding as in Art. 3.13, page 3.60, we obtain

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial v}{\partial x} + \frac{\zeta}{\rho}\frac{\partial w}{\partial x} \qquad \ldots (16\,a)$$

$$\frac{D}{Dt}\left(\frac{\eta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial y} + \frac{\eta}{\rho}\frac{\partial v}{\partial y} + \frac{\zeta}{\rho}\frac{\partial w}{\partial y} \qquad \ldots (16\,a)$$

$$\frac{D}{Dt}\left(\frac{\zeta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial z} + \frac{\eta}{\rho}\frac{\partial v}{\partial z} + \frac{\zeta}{\rho}\frac{\partial w}{\partial z} \qquad \ldots (16\,a)$$

Let $\mathbf{q} = u\,\mathbf{i} + v\,\mathbf{j} + w\,\mathbf{k}$, $\Omega = \xi\,\mathbf{i} + \eta\,\mathbf{j} + \zeta\,\mathbf{k}$ and $\nabla = (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j} + (\partial/\partial z)\mathbf{k}$ ... (17)

Multiplying both sides of (16*a*), (16*b*) and (16*c*) vectorially by **i**, **j**, **k** and then adding, we obtain

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\mathbf{i} + \frac{\eta}{\rho}\mathbf{j} + \frac{\zeta}{\rho}\mathbf{k}\right) = \left(\frac{\xi}{\rho}\frac{\partial}{\partial x} + \frac{\eta}{\rho}\frac{\partial}{\partial y} + \frac{\zeta}{\rho}\frac{\partial}{\partial z}\right)(u\,\mathbf{i} + v\,\mathbf{j} + w\,\mathbf{k}) \qquad \ldots (18)$$

We know that $\qquad D/Dt = u(\partial/\partial x) + v(\partial/\partial y) + w(\partial/\partial z) + \partial/\partial t$

Using (17) and (19), (18) reduces to the required result

$$\frac{D}{Dt}\left(\frac{\mathbf{\Omega}}{\rho}\right) = \left(\frac{\mathbf{\Omega}}{\rho}\cdot\nabla\right)\mathbf{q}$$

Second method (Vector method): Since the external force **F** is conservative, a scalar function **V** must exist such that

$$\mathbf{F} = -\nabla V \qquad \ldots (i)$$

Since ρ is a function of p only, there must exist a relation of the form

$$Q = \int_{p_0}^{p}\frac{dp}{\rho} \qquad \ldots (ii)$$

Hence, $\qquad \nabla Q = \sum \mathbf{i}\frac{\partial Q}{\partial x} = \sum \mathbf{i}\frac{\partial Q}{\partial p}\frac{\partial p}{\partial x} = \sum \mathbf{i}\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{1}{\rho}\nabla p$, using (ii) ... (iii)

Euler's equation of motion is

$$D\mathbf{q}/Dt = \mathbf{F} - (1/\rho)\,\nabla p$$

or $\qquad \partial\mathbf{q}/\partial t + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\nabla V - \nabla Q$, using (i) and (iii) ... (iv)

From vector identities (refer Art. 1.7), we know that

$$\nabla(\mathbf{q}\cdot\mathbf{q}) = 2\,\{\mathbf{q}\times \text{curl}\,\mathbf{q} + (\mathbf{q}\cdot\nabla)\mathbf{q}\}$$

so that $\quad (\mathbf{q}\cdot\nabla)\mathbf{q} = \nabla(q^2/2) - \mathbf{q}\times \text{curl}\,\mathbf{q}, \quad$ where $\quad q = |\mathbf{q}|$ ... (v)

Using (v), (iv) yields $\quad \partial\mathbf{q}/\partial t + \nabla(q^2/2) - \mathbf{q}\times\text{curl}\,\mathbf{q} = -\nabla V - \nabla Q$

or $\quad \partial\mathbf{q}/\partial t + \nabla(q^2/2) - \mathbf{q}\times\boldsymbol{\Omega} = -\nabla V - \nabla Q, \quad$ as $\quad \Omega = \text{curl}\,\mathbf{q}$

Thus, $\quad \partial\mathbf{q}/\partial t + \nabla(V+Q+q^2/2) = \mathbf{q}\times\Omega$ ... (vi)

Taking curl of both sides of (vi) and noting that (refer vector identity on page 1.7) $\nabla(V+Q+q^2/2) = 0$, we obtain

curl $(\partial\mathbf{q}/\partial t) =$ curl $(\mathbf{q}\times\Omega)$ or $\partial(\text{curl}\,\mathbf{q})/\partial t =$ curl $(\mathbf{q}\times\Omega)$

or $\quad \partial\boldsymbol{\Omega}/\partial t = (\nabla\cdot\Omega)\,\mathbf{q} - (\nabla\cdot\mathbf{q})\Omega + (\Omega\cdot\nabla)\mathbf{q} - (\mathbf{q}\cdot\nabla)\Omega$

[Using a vector identity (see page 1.7) and noting that Ω = curl $\mathbf{q}$]

or $\quad \partial\Omega/\partial t + (\boldsymbol{q}\cdot\nabla)\Omega = (\Omega\cdot\nabla)\mathbf{q} - (\nabla\cdot\boldsymbol{q})\,\Omega$ ... (*vii*)

[$\because$ $\nabla\cdot\Omega$ = div Ω = div curl $\boldsymbol{q} = 0$, by a vector identity (see page 1.7)]

From equation of continuity (see page 2.14)

$\mathrm{D}\rho/\mathrm{D}t + \rho(\nabla\cdot\mathbf{q}) = 0 \quad$ so that $\quad \nabla\cdot\mathbf{q} = -(1/\rho)\times(D\rho/Dt)$ ... (viii)

Also, we have $\quad D/Dt = \partial/\partial t + (\boldsymbol{q}\cdot\nabla)$ ... (ix)

Using (viii) and (ix), (vii) reduces to

$$\frac{D\Omega}{Dt} = (\Omega\cdot\nabla)\mathbf{q} + \left(\frac{\Omega}{\rho}\right)\frac{D\rho}{Dt} \quad \text{or} \quad \frac{1}{\rho}\frac{D\Omega}{Dt} - \frac{\Omega}{\rho^2}\frac{D\rho}{Dt} = \frac{\mathbf{q}}{\rho}(\Omega\cdot\nabla)$$

or
$$\frac{D}{Dt}\left(\frac{\Omega}{\rho}\right) = \left(\frac{\Omega}{\rho}\cdot\nabla\right)\mathbf{q},$$

which is known as *Helmoltz vorticity equation*.

Ex.2. *Show that the surfaces exist which cut the streamlines orthogonally whenever velocity potential exist.*

Sol. The Differential equations of streamlines are given by

$$(dx)/u = (dy)/v = (dz)/w \qquad \text{... (1)}$$

The surfaces which cut (1) orthogonally are given by

$$u\,dx + v\,dy + w\,dz = 0$$

The necessary and sufficient condition for the existence of (2) is that (1) must possess a solution of the form

$$\phi(x, y, z) = c, \ c \text{ being an arbitrary constant} \qquad \text{... (3)}$$

The necessary and sufficient condition for the existence of (3) is given by

$$u\,(\partial v/\partial z - \partial w/\partial y) + v(\partial w/\partial x - \partial u/\partial z) + w(\partial u/\partial y - \partial v/\partial x) = 0 \qquad \text{...(4)}$$

Since velocity potential exists, $\quad u = -(\partial\phi/\partial x), \quad v = -(\partial\phi/\partial y), \quad w = -(\partial\phi/\partial z)$ ... (5)

Using (5), L.H.S. of (4)

$= u(-\partial^2\phi/\partial z\partial y + \partial^2\phi/\partial y\partial z) + v(-\partial^2\phi/\partial x\partial z + \partial^2\phi/\partial z\,\partial x) + w(-\partial^2\phi/\partial y\partial x + \partial^2\phi/\partial x\partial y)$

$= 0 =$ R.H.S. of (4).

Therefore, surfaces exist which cut the streamlines orthogonally when velocity potential exists.

Note. The surfaces $\phi\ (x, y, z)$ = constant are known as *equipotential surfaces*.

Ex.3. *If the motion of an idel fluid, for which density is a function of pressure p only, is steady and the external forces are conservative, then prove that there exists a family of surfaces which contain the streamlines and vortex lines*

Sol. The equation of motion for an ideal fluid is given by

$$\partial \mathbf{q}/\partial t + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\frac{1}{\rho}\nabla p + \mathbf{F} \qquad \ldots (1)$$

Since density is a function of pressure only, let $\qquad P = \int_{p_0}^{p} \frac{dp}{\rho} \qquad \ldots (2)$

Hence, $\qquad \nabla P = \sum \mathbf{i}\frac{\partial P}{\partial x} = \sum \mathbf{i}\frac{\partial P}{\partial p}\frac{\partial p}{\partial x} = \frac{1}{\rho}\sum i\frac{\partial p}{\partial x}$, using (2)

Thus, $$\nabla P = \frac{1}{\rho}\,\nabla p \qquad \ldots (3)$$

Since forces are conservative, we have $\qquad \mathbf{F} = -\nabla V, \qquad \ldots (4)$

where V is a scalar point function.

From a vector identity (see page 1.7), we have

$$\nabla(\mathbf{q}\cdot\mathbf{q}) = 2\{\boldsymbol{q}\times \text{curl}\,\boldsymbol{q} + (\boldsymbol{q}\cdot\nabla)\boldsymbol{q}\} \quad \text{giving} \quad (\boldsymbol{q}\cdot\nabla)\boldsymbol{q} = \nabla(q^2/2) - \mathbf{q}\times\text{curl}\,\mathbf{q} \qquad \ldots (5)$$

Given that the motion is steady and so $\qquad \partial\mathbf{q}/\partial t = 0 \qquad \ldots (6)$

Using (3), (4), (5) and (6), (1) reduces to

$$\nabla(\mathbf{q}^2/2) - \mathbf{q}\times\text{curl}\,\mathbf{q} = -\nabla P - \nabla V \qquad \ldots (7)$$

But curl $\boldsymbol{q} = \Omega$ = vorticity vector. Hence, (7) takes the form

$$\nabla(V + P + q^2/2) = \boldsymbol{q}\times\Omega \quad \text{or} \quad \nabla(V + P + q^2/2) = \mathbf{n}, \qquad \ldots (8)$$

where $$\mathbf{n} = \mathbf{q}\times\Omega \qquad \ldots (9)$$

From (9), $\qquad \mathbf{n}\cdot\mathbf{q} = (\mathbf{q}\times\Omega)\cdot\mathbf{q} = 0 \quad$ and $\quad \mathbf{n}\cdot\Omega = (\mathbf{q}\times\Omega)\cdot\Omega = 0,$

showing that $\mathbf{n}$ is perpendicular to both $\mathbf{q}$ and Ω.

From vector calculus, we know that $\nabla\phi$ is perpendicular everywhere to ϕ = constant. From (8),it follows that $\mathbf{n}$ is perpendicular to the family of surfaces

$$\text{V} + \text{P} + q^2/2 = \text{constant} = C, \text{ say} \qquad \ldots (10)$$

Hence the surface (7) is everywhere tangent to the direction of velocity vector $\mathbf{q}$ and vorticity vector Ω. Therefore, the family of surfaces given by (7) contain the streamlines and vertex lines.

Ex.4. *Investigate an expression for the change in an indefinitely short time in the mass of fluid contained within an spherical surface of small radius.*

Prove that the momentum of the mass in the direction of the axis of **x** *is greater that it would be if the whole mass were moving with the velocity at the centre by*

$$\frac{1}{5}\frac{Ma^2}{\rho}\left\{\frac{\partial\rho}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial\rho}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial\rho}{\partial z}\frac{\partial u}{\partial z} + \frac{\rho}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\right\}$$

Sol. Let (x, y, z) be the coordinates of centre of sphere, and let the coordinates of any other point inside the spherical surface be $(x + x', y + y', z + z')$. Let ρ be the density at the centre (x, y, z) and let ρ' be the density at $(x + x', y + y', z + z')$.

Now, total mass $= M = \iiint \rho' dx'\, dy' dz'$

Then, the change in mass in time δt is given by

$$\frac{\partial M}{\partial t}\delta t = \delta t \frac{\partial}{\partial t}\iiint \rho' dx' dy' dz' \quad \ldots (1)$$

Now, neglecting small quantities of third and higher order, we have

$$\rho' = \rho + \left(x'\frac{\partial \rho}{\partial x} + y'\frac{\partial \rho}{\partial y} + z'\frac{\partial \rho}{\partial z}\right) + \frac{1}{2}\left(x'^2\frac{\partial^2 \rho}{\partial x^2} + y'^2\frac{\partial^2 \rho}{\partial y^2} + z'^2\frac{\partial^2 \rho}{\partial z^2}\right.$$

$$\left. + 2y'z'\frac{\partial^2 \rho}{\partial y \partial z} + 2z'x'\frac{\partial^2 \rho}{\partial z \partial x} + 2x'y'\frac{\partial^2 \rho}{\partial x \partial y}\right) \quad \ldots (2)$$

Substituting the value of ρ' given (2) in (1), we obtain

$$\frac{\partial M}{\partial t}\delta t = \delta t \frac{\partial}{\partial t}\iiint \left\{\rho + \left(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z}\right)\rho + \frac{1}{2}\left(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z}\right)^2 \rho\right\} dx' dy' dz' \quad \ldots (3)$$

Integrating throughout the sphere of radius a, we obtain

$$\left.\begin{aligned} &\iiint dx'dy'dz' = \frac{4}{3}\pi a^3, \iiint x'dx'dy'dz' = \iiint y'dx'dy'dz' = \iiint z'dx'dy'dz' = 0 \\ &\iiint x'^2 dx'dy'dz' = \iiint y'^2 dx'dy'dz' = \iiint z'^2 dy'dy'dz' = \frac{4}{15}\pi a^5 \\ \text{or} \quad &\iiint y'z'dx'dy'dz' = \iiint z'x'dx'dy'dz' = \iiint x'y'dx'dy'dz' = 0 \end{aligned}\right\} \quad \ldots (4)$$

Using the above standard results (4) in (3), we obtain

$$\frac{\partial M}{\partial t}\delta t = \delta t\left\{\frac{4}{3}\pi a^3 \rho + \frac{4\pi a^5}{15}\left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2}\right)\right\} \quad \ldots (5)$$

Second part: Total momentum along the axis of $x = \iiint \rho' u'\, dx'\, dy'\, dz'$ ... (6)

Neglecting third and higher powers of the coordinates x', y' and z', and denoting u as the velocity at the centre, we have

$$\rho' u' = \rho u + \left(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z}\right)(\rho u) + \frac{1}{2}\left(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z}\right)^2 (\rho u)$$

Substituting the above value of $\rho' u'$ in (6), we have

$$\iiint \rho' u' dx' dy' dz' = \iiint \rho u\, dx' dy' dz' + \iiint \left(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z}\right)(\rho u)\, dx' dy' dz'$$

$$+ \frac{1}{2}\iiint \left\{\left(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z}\right)^2 (\rho u)\right\} dx' dy' dz' \quad \ldots (7)$$

Using the standard results given by (4) in (7), we obtain

$$\iiint \rho' u' dx' dy' dz' = \rho u \times \frac{4}{3}\pi a^3 + \frac{1}{2}\times\frac{4\pi a^5}{15}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\rho u) \quad \ldots (8)$$

Now, the total momentum if the whole mass were moving with the velocity u at the centre

$$= \iiint \rho' u\, dx' dy' dz' \qquad \ldots (9)$$

Then, using (2) and (4) as before, we obtain,

$$\iiint \rho' u\, dx'dy'dz' = u\iiint \rho'\, dx'dy'dz' = u\left\{\rho \times \frac{4\pi a^3}{3} + \frac{1}{2}\times\frac{4\pi a^5}{15}\left(\frac{\partial^2\rho}{\partial x^2}+\frac{\partial^2\rho}{\partial y^2}+\frac{\partial^2\rho}{\partial z^2}\right)\right\} \ldots (10)$$

Using (6), (8), (9) and (10), the required excess of the momentum (6) over (9)

$$= \iiint \rho' u' dx'dy'dz' - \iiint \rho' u dx'dy'dz'$$

$$= \frac{2\pi a^5}{15}\left[\left\{\frac{\partial^2(\rho u)}{\partial x^2}+\frac{\partial^2(\rho u)}{\partial y^2}+\frac{\partial^2(\rho u)}{\partial z^2}\right\} - u\left(\frac{\partial^2\rho}{\partial x^2}+\frac{\partial^2\rho}{\partial y^2}+\frac{\partial^2\rho}{\partial z^2}\right)\right]$$

$$= \frac{2\pi a^5}{15}\left[2\left(\frac{\partial\rho}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial\rho}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial\rho}{\partial z}\frac{\partial u}{\partial z}\right)+\rho\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\right]$$

$$= \frac{4\pi a^3}{3}\times\frac{a^2}{5}\left\{\frac{\partial\rho}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial\rho}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial\rho}{\partial z}\frac{\partial u}{\partial z}+\frac{\rho}{2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\right\}$$

$$= \frac{Ma^2}{5\rho}\left\{\frac{\partial\rho}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial\rho}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial\rho}{\partial z}\frac{\partial u}{\partial z}+\frac{\rho}{2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\right\}$$

$[\because M = (4\pi\rho a^3)/3 \Rightarrow (4\pi a^3)/3 = M/\rho]$

Ex. 5. *Find the stream function of the two dimensional motion due to two equal sources and an equal sink mid-way between them; sketch the streamlines and find the velocity at any time.*

In a region bounded by a fixed quadrantal arc and its radii, deduce the motion due to a source and an equal sink situated at the ends of one of the bounding radii. Show that the streamline leaving either end at an angle α *with the radius is* $r^2 \sin(\alpha + \theta) = a^2 \sin(\alpha - \theta)$.

Sol. First part: Let A, A' distant $2a$ apart denote the positions of two equal sources of strength m, and let O be the mid-point of AA' where the sink of strength $(-m)$ is situated. Taking O as origin, AA′ as axis of x, the complex potential w for any point P $(z = x + iy)$ is given by

$$w = -m\log(z-a) + m\log z - m\log(z+a) \qquad \ldots (1)$$

or $\quad \phi + i\psi = -m\log\{(x-a)+iy\} - m\log\{(x+a)+iy\} + m\log(x+iy), \quad$ as $\quad z = x+iy$

or $\quad \phi + i\psi = -m\,[\log\{(x-a)^2+y^2\}^{1/2} + i\tan^{-1}\{y/(x-a)\}] - m\,[\log\{x+a\}^2+y^2]^{1/2}$
$\qquad + i\tan^{-1}\{y/(x+a)\}] + m\,[\log(x^2+y^2)^{1/2} + i\tan^{-1}(y/x)] \quad \ldots (2)$

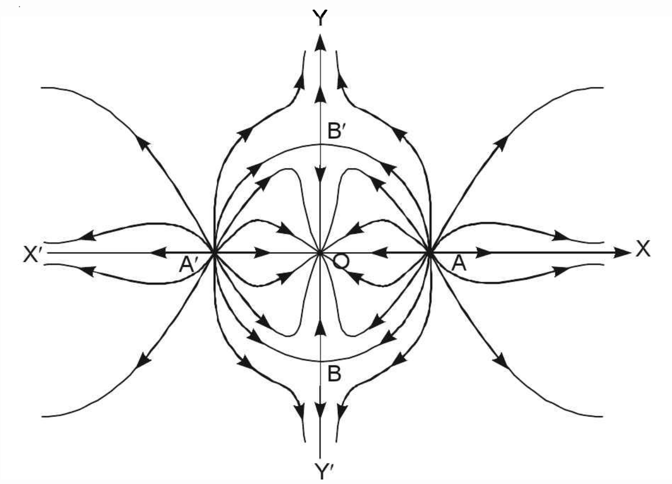


Fig. 1

Equating the imaginary parts on both sides of (2), we have

$$\psi = -m\left(\tan^{-1}\frac{y}{x-a}+\tan^{-1}\frac{y}{x+a}-\tan^{-1}\frac{y}{x}\right) = -m\left[\tan^{-1}\frac{y/(x-a)+y/(x+a)}{1-\{y^2/(x^2-a^2)\}}-\tan^{-1}\frac{y}{x}\right]$$

$$= -m\left[\tan^{-1}\frac{2xy}{x^2-y^2-a^2}-\tan^{-1}\frac{y}{x}\right] = -m\tan^{-1}\frac{(2xy)/(x^2-y^2-a^2)-(y/x)}{1+\{2xy^2/x(x^2-y^2-a^2)\}}$$

Thus, $$\psi = -m\tan^{-1}\frac{x^2y+y^3+a^2y}{x^3+xy^3-a^2x};$$

which gives the stream function. From the above equation, the streamlines are given by

$$y(x^2+y^2+a^2)/x(x^2+y^2-a^2) = C, \; C \text{ being a constant} \qquad \ldots (3)$$

From (3), the streamlines corresponding to $C = 0$ and $C = \infty$ are given by the equations

$$x = 0, \qquad y = 0 \qquad \text{and} \qquad x^2+y^2 = a^2, \qquad \ldots(4)$$

showing that the axes of x and y are the streamlines. Again, the circle with centre O as centre and OA as radius is a streamline. The streamlines given by (4) and other streamlines are sketched in the figure 1.

If q denotes the velocity at any point $P(x, y)$, then

$$q = \left|\frac{dw}{dz}\right| = \left|-\frac{m}{z-a}+\frac{m}{z}-\frac{m}{z+a}\right|, \text{ using (1)}$$

$$= m\left|\frac{z(z+a)-(z-a)(z+a)+z(z-a)}{z(z-a)(z+a)}\right| = m\left|\frac{z^2+a^2}{z(z-a)(z+a)}\right|$$

$$= m\left|\frac{(z+ai)(z-ai)}{z(z-a)(z+a)}\right| = m\frac{PB\times PB'}{OP\times PA\times PA'},$$

where B and B' are the points where the circle $x^2 + y^2 = a^2$ intersects the axis of y (see figure 1)

Second part: In the first part of the solution, we have proved that in two-dimensional motion due to two equal sources and an equal sink midway between them, the axes of x and y and the circle $x^2 + y^2 = a^2$ become the streamlines, that is, the quadrant of the circle OAB is bounded by streamlines, so that if the quadrant OAB is replaced by rigid boundaries the motion inside the quadrant would remain unchanged. In other words, for the motion of fluid bounded by a quadrantal arc OAB (of radius a) due to a source at A and a sink at O on the ends of the radius OA, the image system would be a source at A' (distant '$-a$' from O).

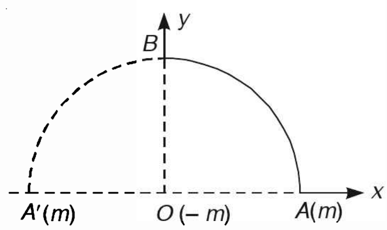


Fig. 2

Therefore, proceeding as in first part, the streamlines for the motion inside the quadrantal arc are given by

$$y(x^2+y^2+a^2)/x(x^2+y^2-a^2) = C \qquad \text{or} \qquad x^2y+y^3+a^2y = Cx^3+Cxy^2-Cxa^2 \qquad \ldots (5)$$

Differentiating (5) w.r.t. 'x', we have

$$2xy + x^2(dy/dx) + 3y^2(dy/dx) + a^2(dy/dx) = 3Cx^2 + Cy^2 + 2\,Cxy\,(dy/dx) - Ca^2$$

so that $$dy/dx = (2xy - 3Cx^2 - Cy^2 + Ca^2)/(2\,Cxy - x^2 - 3y^2 - a^2) \qquad \ldots (6)$$

For the streamline leaving the either end $(\pm a, 0)$ at an angle α, C is given by

$$\tan(\pi-\alpha) = \left(\frac{dy}{dx}\right)_{x=\pm a, y=0} = \frac{-2Ca^2}{-2a^2}, \text{ using (6)}$$

or $\quad -\tan\alpha = C$ $\quad$ or $\quad y(x^2+y^2+a^2)/x(x^2+y^2-a^2) = -(\sin\alpha)/\cos\alpha$ ... (7)

Changing to polars by putting $x = r\cos\theta$ and $y = r\sin\theta$, (7) reduces to

$$\sin\theta\,(r^2+a^2)/\cos\theta\,(r^2-a^2) = -(\sin\alpha)/\cos\alpha$$

or $$(r^2+a^2)\sin\theta\cos\alpha = -(r^2-a^2)\cos\theta\sin\alpha$$

or $r^2(\sin\theta\cos\alpha+\cos\theta\sin\alpha) = a^2(\sin\alpha\cos\theta-\cos\alpha\sin\theta)$ $\quad$ or $\quad r^2\sin(\alpha+\theta) = a^2\sin(\alpha-\theta)$

Ex. 6. *A source of fluid situated in space of two dimensions, is of such strength that $2\pi\rho\mu$ represents the mass of the fluid of density ρ emitted per unit of time. Show that the force necessary to hold a circular disc at rest in the plane of the source is $2\pi\rho\mu^2a^2/r(r^2-a^2)$, where a is the radius of the disc and r the distance of the source from its centre. In what direction is the disc urged by the pressure.*

Sol. Let the given source be situated at S, then since the mass of the fluid emitted per second is $2\pi\rho\mu$, by definition the strength of the source is μ.

Let S' be the inverse point of S with respect to the disc. The image system of the source S, for the circular disc will be a source of strength μ at S' and a sink of strength $-\mu$ at O, the centre of the disc

By Bernoulle's theorem, the pressure at any point $P(a, \theta)$ is given by

$$p/\rho = C - (q^2/2), \qquad \ldots(1)$$

where q is the velocity at any point and C is a constant

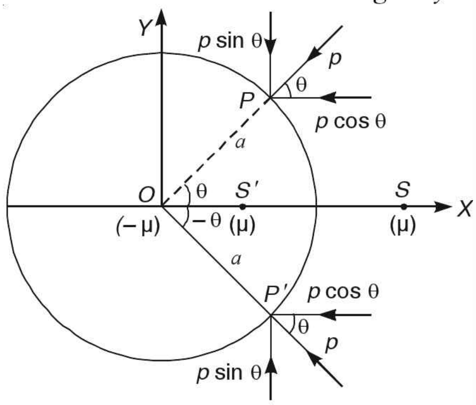


Fig. 3

The pressure p on the disc at P has two components $p\cos\theta$ and $p\sin\theta$ acting respectively along SO and perpendicular to it. By symmetry the component $p\cos\theta$ at $P(a, \theta)$ is balanced by a corresponding component $p\cos\theta$ at $P'(a, -\theta)$, and therefore the net effect of the pressure on the disc is only along SO. Let F be the total force on the disc. Then, we have

$$F = \int_0^{2\pi}(-p\cos\theta)(a d\theta) = -a\rho\int_0^{2\pi}\cos\theta\left(C-\frac{q^2}{2}\right)d\theta, \text{ using (1)}$$

Thus, $$F = \frac{a\rho}{2}\int_0^{2\pi} q^2\cos\theta\, d\theta \qquad \ldots(2)$$

We now proceed to determine the velocity of the fluid q at P. Taking O as origin, and OS as the axis of x, the complex potential for the motion of the fluid element at any point (z. *i.e.* $x + iy$) is given by

$$w = -\mu\log(z-r) - \mu\log(z-a^2/r) + \mu\log z \qquad \ldots(3)$$

[$\because$ S' is inverse point of S, so $OS' = a^2/r$, where $OS = r$]

From (3), $$\frac{dw}{dz} = -\frac{\mu}{z-r} - \frac{\mu}{z-(a^2/r)} + \frac{\mu}{z} = -\mu\frac{z(z-a^2/r)+z(z-r)-(z-r)(z-a^2/r)}{z(z-r)(z-a^2/r)}$$

or $$\frac{dw}{dz} = -\mu\frac{(z^2-a^2)}{z(z-r)(z-a^2/r)} = -\mu\frac{(z-a)(z+a)}{z(z-r)(z-a^2/r)}$$

Hence, $$q = \left|\frac{dw}{dz}\right| = \mu\left|\frac{(z-a)(z+a)}{z(z-r)(z-a^2/r)}\right|$$

Hence, The velocity at only point P (z i.e., $ae^{i\theta}$) is given by

$$q = \mu\left|\frac{(ae^{i\theta}-a)(ae^{i\theta}+a)}{ae^{i\theta}(ae^{i\theta}-r)(ae^{i\theta}-a^2/r)}\right| = \mu r\left|\frac{(e^{i\theta}-1)(e^{i\theta}+1)}{e^{i\theta}(ae^{i\theta}-r)(re^{i\theta}-a)}\right|$$

or
$$q = \mu r \left| \frac{(1-e^{-i\theta})\,(1+e^{i\theta})}{(a-re^{-i\theta})\,(re^{i\theta}-a)} \right| = \frac{2\mu r \sin\theta}{a^2+r^2-2ar\cos\theta} \qquad \ldots (4)$$

Using (4), (2) reduces to

$$F = 2ar^2\mu^2\rho\int_0^{2\pi} \frac{\sin^2\theta\cos\theta}{(a^2+r^2-2ar\cos\theta)^2}d\theta = \frac{2ar^2\mu^2\rho}{r^4}\int_0^{2\pi} \frac{\sin^2\theta\cos\theta}{\{1-(2a/r)\cos\theta+(a/r)^2\}^2}d\theta$$

$$= \frac{2a\mu^2\rho}{r^2}\times 2\int_0^{\pi} \frac{\sin^2\theta\cos\theta}{\{1-(2a/r)\cos\theta+(a/r)^2\}^2}\,d\theta$$

$$\left[\text{Using the well known property: } \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x)=f(x)\right]$$

Thus,
$$F = \frac{2a\mu^2\rho}{r^2}\int_0^{\pi} \sin 2\theta\,\{1-(2a/r)\cos\theta+(a/r)^2\}^{-2}\sin\theta\,d\theta \qquad \ldots (5)$$

Now, $\int \{1-(2a/r)\cos\theta+(a/r)^2\}^{-2}\sin\theta\,d\theta = -\int \{1-(2a/r)u+(a/r)^2\}^{-2}du$

[Putting $\cos\theta = u$ and $\sin\theta\,d\theta = -du$]

$$= \{1-(2a/r)\,u+(a/r)^2\}^{-1}/(-2a/r) = \frac{-(r/2a)}{1-(2a/r)\cos\theta+(a/r)^2} \qquad \ldots (6)$$

Taking $\sin 2\theta$ as first function on L.H.S. of (5) and integrating by parts (also using result (6) just proved), (5) reduces to

$$F = \frac{2a\mu^2\rho}{r^2}\left[\left[\sin 2\theta \times \frac{-(r/2a)}{1-(2a/r)\cos\theta+(a/r)^2}\right]_0^{\pi} - \int_0^{\pi}\left\{2\cos 2\theta\times\frac{-(r/2a)}{1-(2a/r)\cos\theta+(a/r)^2}\right\}d\theta\right]$$

or
$$F = \frac{2a\mu^2\rho}{r^2}\left[0+\frac{r}{a}\int_0^{\pi}\frac{\cos 2\theta}{1-(2a/r)\cos\theta+(a/r)^2}d\theta\right] \qquad \ldots (7)$$

* From expansions of Trigonometrical functions, we know that if $x < 1$, then
$(1-x^2)/(1-2x\cos\theta+x^2) = 1+2x\cos\theta+2x^2\cos 2\theta+2x^3\cos 3\theta+\ldots$ ad. inf.
Since $a < r$, so $a/r < 1$. Replacing x by a/r in the above result, we have
$\{1-(a/r)^2\}/\{1-2(a/r)\cos\theta+\cos^2\theta\} = 1+2(a/r)\cos\theta+2(a/r)^2\cos 2\theta+2(a/r)^3\cos 3\theta+\ldots$
Using the above result, (7) may be re-written as

$$F = \frac{2a\mu^2\rho}{r^2}\times\frac{r}{a}\times\frac{1}{1-(a/r)^2}\int_0^{\pi}\cos 2\theta\left(1+\frac{2a}{r}\cos\theta+\frac{2a^2}{r^2}\cos 2\theta+\frac{2a^3}{r^3}\cos 3\theta+\ldots\right)d\theta \qquad \ldots (8)$$

But
$$\int_0^{\pi}\cos 2\theta\,d\theta = \left[\frac{\sin 2\theta}{2}\right]_0^{\pi} = 0. \qquad \ldots (9)$$

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* Refer page 496 of Author's Trigonometry published by S. Chand and Co., New Delhi

Also, $$\int_0^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, \text{ if } m \neq n \\ \pi/2, \text{ if } m = n \end{cases} \qquad \ldots (10)$$

Using (9) and (10), (8) reduces to

$$F = \frac{2\mu^2 \rho r^2}{r(r^2 - a^2)} \times \frac{2a^2}{r^2} \times \frac{\pi}{2} = \frac{2\pi\rho\mu^2 a^2}{r(r^2 - a^2)},$$

showing that the total force acting on the disc is of magnitude $2\pi\rho\mu^2 a^2 / r\,(r^2 - a^2)$ and is directed along $\overrightarrow{OS}$, and the disc would be urged along OS by this pressure.

Second solution Refer Ex. 7, on page 5.51, chapter 5.

Ex. 7. *In two dimensional irrotational fluid motion, show that, if the streamlines are confocal ellipses $x^2 / (a^2 + \lambda) + y^2 / (b^2 + \lambda) = 1$, then $\psi = A \log (\sqrt{a^2 + \lambda} + \sqrt{b^2 + \lambda}) + B$, and the velocity at any point is inversely proportional to the square root of the rectangle under the focal radii of the point.*

Sol. Let us first investigate a necessary condition which will make a family of curves $f(x, y, \lambda) = 0$ a streamline. We know that on streamline ψ = constant, and on a particular curve of the family $f(x, y, \lambda) = 0$, λ is a constant. Hence, it follows that in order that $f(x, y, \lambda) = 0$ may be a streamline, ψ should be function of λ only. Therefore, ψ may be written as

$$\psi = F(\lambda)$$

so that $$\frac{\partial \psi}{\partial x} = F'(\lambda) \frac{\partial \lambda}{\partial x} \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x^2} = F''(\lambda) \left(\frac{\partial \lambda}{\partial x}\right)^2 + \frac{\partial^2 \lambda}{\partial x^2} F'(\lambda) \qquad \ldots (1)$$

Similarly, $$\frac{\partial \psi}{\partial y} = F'(\lambda) \frac{\partial \lambda}{\partial y} \quad \text{and} \quad \frac{\partial^2 \psi}{\partial y^2} = F''(\lambda) \left(\frac{\partial \lambda}{\partial y}\right)^2 + \frac{\partial^2 \lambda}{\partial y^2} F'(\lambda) \qquad \ldots (2)$$

Since ψ satisfies Laplace's equation, so $\quad \partial^2\psi / \partial x^2 + \partial^2 \psi / \partial y^2 = 0$

or $\quad F''(\lambda) \{(\partial\lambda / \partial x)^2 + (\partial \lambda / \partial y)^2\} + F'(\lambda)\{(\partial^2 \lambda / \partial x^2) + (\partial^2 \lambda / \partial y^2)\} = 0$, using (1) and (2)

Hence, $$\frac{\partial^2 \lambda / \partial x^2 + \partial^2 \lambda / \partial y^2}{(\partial \lambda / \partial x)^2 + (\partial \lambda / \partial \lambda)^2} = -\frac{F''(\lambda)}{F'(\lambda)}, \qquad \ldots (3)$$

which is a function of λ only.

Here, in the given problem, $f(x, y, \lambda) = 0$ is given by

$$x^2 / (a^2 + \lambda) + y^2 / (b^2 + \lambda) = 1 \qquad \ldots (4)$$

Differentiating (4) partially w.r.t. 'x' we get

$$\frac{2x}{a^2 + \lambda} - \left(\frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2}\right) \frac{\partial \lambda}{\partial x} = 0 \qquad \ldots (5)$$

Let $\quad x / (a^2 + \lambda) = X \quad$ and $\quad y / (b^2 + \lambda) = Y \qquad \ldots (6)$

Using (6), (5) reduce to $\quad 2X - (X^2 + Y^2)\,(\partial \lambda / \partial x)\quad$ so that $\quad \partial \lambda / \partial x = 2X / (X^2 + Y^2) \qquad \ldots (7)$

Similarly, $$\partial \lambda / \partial y = 2Y / (X^2 + Y^2) \qquad \ldots (8)$$

Now, differentiating (5) partially w.r.t. 'x', we get

$$\frac{2}{a^2 + \lambda} - \frac{4x(\partial \lambda / \partial x)}{(a^2 + \lambda)^2} + 2\left\{\frac{x^2}{(a^2 + \lambda)^3} + \frac{y^2}{(b^2 + \lambda)^3}\right\}\left(\frac{\partial \lambda}{\partial x}\right)^2 - \left(\frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2}\right)\frac{\partial^2 \lambda}{\partial x^2} = 0$$

or $$\frac{2}{a^2+\lambda}-\frac{8X^2}{(a^2+\lambda)(X^2+Y^2)}+2\left(\frac{X^2}{a^2+\lambda}+\frac{Y^2}{b^2+\lambda}\right)\left(\frac{\partial\lambda}{\partial x}\right)^2=(X^2+Y^2)\,\frac{\partial^2\lambda}{\partial x^2}, \quad \ldots (9)$$

[Using (6) and (7)]

Similarly, $$\frac{2}{b^2+\lambda}-\frac{8Y^2}{(b^2+\lambda)(X^2+Y^2)}+2\left(\frac{X^2}{a^2+\lambda}+\frac{Y^2}{b^2+\lambda}\right)\left(\frac{\partial\lambda}{\partial y}\right)^2=(X^2+Y^2)\frac{\partial^2\lambda}{\partial y^2} \quad \ldots (10)$$

Adding (9) and (10) and re-arranging, we obtain

$$\left(\frac{\partial^2\lambda}{\partial x^2}+\frac{\partial^2\lambda}{\partial y^2}\right)(X^2+Y^2)=\frac{2}{a^2+\lambda}+\frac{2}{b^2+\lambda}-\frac{8}{X^2+Y^2}\left(\frac{X^2}{a^2+\lambda}+\frac{Y^2}{b^2+\lambda}\right)$$

$$+2\left(\frac{X^2}{a^2+\lambda}+\frac{Y^2}{b^2+\lambda}\right)\left\{\left(\frac{\partial\lambda}{\partial x}\right)^2+\left(\frac{\partial\lambda}{\partial y}\right)^2\right\}$$

or $$\left(\frac{\partial^2\lambda}{\partial x^2}+\frac{\partial^2\lambda}{\partial y^2}\right)(X^2+Y^2)=\frac{2}{a^2+\lambda}+\frac{2}{b^2+\lambda}-\frac{8}{X^2+Y^2}\left(\frac{X^2}{a^2+\lambda}+\frac{Y^2}{b^2+\lambda}\right)+8\left(\frac{X^2}{a^2+\lambda}+\frac{Y^2}{b^2+\lambda}\right)\left(\frac{X^2+Y^2}{(X^2+Y^2)^2}\right)$$

[Using (7) and (8)]

$$\Rightarrow \qquad \frac{\partial^2\lambda}{\partial x^2}+\frac{\partial^2\lambda}{\partial y^2}=\frac{2/(a^2+\lambda)+2/(b^2+\lambda)}{X^2+Y^2}$$

$$\Rightarrow \quad \frac{\partial^2\lambda/\partial x^2+\partial^2\lambda/\partial y^2}{(\partial\lambda/\partial x)^2+(\partial\lambda/\partial y)^2}=\frac{2/(a^2+\lambda)+2/(b^2+\lambda)}{(X^2+Y^2)\{(\partial\lambda/\partial x)^2+(\partial\lambda/\partial y)^2\}}=\frac{2/(a^2+\lambda)+2/(b^2+\lambda)}{(X^2+Y^2)\times\{4/(X^2+Y^2)\}}$$

(Using (7)and (8))

$$\Rightarrow \qquad \frac{\partial^2\lambda/\partial x^2+\partial^2\lambda/\partial y^2}{(\partial\lambda/\partial x)^2+(\partial\lambda/\partial y)^2}=\frac{1/(a^2+\lambda)+1/(b^2+\lambda)}{2} \quad \ldots (11)$$

which is a function of λ only.

Therefore the family of confocal ellipses (4) comform to the family of streamlines.

From (I) and (II), $\quad F''(\lambda)/F'(\lambda)=-\{1/(a^2+\lambda)+1/(b^2+\lambda)\}/2 \quad \ldots (12)$

Integrating (12), $\quad F'(\lambda)=A/\{(a^2+\lambda)(b^2+\lambda)\}^{1/2}$, A being an arbitrary constant $\quad \ldots (13)$

Integrating (13) and noting that $\psi=F(\lambda)$, we obtain

$$\psi=F(\lambda)=A\int\frac{d\lambda}{\{(a^2+\lambda)(b^2+\lambda)\}^{1/2}}=A\log\left(\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda}\right)+B,$$ B being a constant.

Now, the velocity q at any point is given by

$$q^2=(\partial\psi/\partial x)^2+(\partial\psi/\partial y)^2=[F'(\lambda)]^2\{(\partial\lambda/\partial x)^2+(\partial\lambda/\partial y)^2\},$$ using (1) and (2)

$$=\frac{A^2}{(a^2+\lambda)(b^2+\lambda)}\times\left\{\frac{4X^2}{(X^2+Y^2)^2}+\frac{4Y^2}{(X^2+Y^2)^2}\right\},$$ using (7), (8) and (13)

$$= \frac{A^2}{(a^2+\lambda)(b^2+\lambda)} \times \frac{4}{X^2+Y^2} = \frac{4A^2}{\dfrac{x^2(b^2+\lambda)}{a^2+\lambda} + \dfrac{y^2(a^2+\lambda)}{b^2+\lambda}}, \text{ using (6)}$$

$$= \frac{4A^2}{\dfrac{x^2(b^2+\lambda)}{a^2+\lambda} + (a^2+\lambda)\left(1 - \dfrac{x^2}{a^2+\lambda}\right)}, \text{ using (4)}$$

Thus, $$q^2 = \frac{4A^2}{(a^2+\lambda) - x^2 + \dfrac{x^2(b^2+\lambda)}{a^2+\lambda}} = \frac{4A^2}{(a^2+\lambda) - \dfrac{x^2(a^2-b^2)}{a^2+\lambda}} \qquad \ldots (4)$$

But, we have $(a^2+\lambda) - (b^2+\lambda) = (a^2+\lambda)e^2$ so that $(a^2-b^2)/(a^2+\lambda) = e^2$.

Hence, from (4), $$q^2 = \frac{4A^2}{(a^2+\lambda) - e^2x^2} \quad \Rightarrow \quad q = \frac{2A}{\{(\sqrt{a^2+\lambda} - ex)(\sqrt{a^2+\lambda} + ex)\}^{1/2}},$$

showing that the velocity is inversely proportional to the square root of the rectangle under the focal radii of the point.

Ex. 8. *Prove that if the velocity potential at any instant to be λxyz, the velocity at any point $(x + \xi, y + \eta, z + \zeta)$ relative to the fluid at the point (x, y, z), where ξ, η, ζ are small, is normal to the quadratic $x\eta\zeta + y\zeta\xi + z\zeta\eta$ = constant, with centre at (x,y, z).*

Sol. Let u, v, w, be the components of velocity of the fluid particle at the point (x, y, z). Since, $\phi = \lambda\, xyz$, we have

$$u = -(\partial\phi/\partial x) = -\lambda\, yz, \qquad v = -(\partial\phi/\partial y) = -\lambda xz, \qquad w = -(\partial\phi/\partial z) = -\lambda xy \qquad \ldots (1)$$

Let u', v', w' be the velocity components at any point $(x+\xi, y+\eta, z+\tau)$ relative to the fluid at the point (x, y, z). Then, we have

$$u' = \frac{\partial u}{\partial x}\xi + \frac{\partial u}{\partial y}\eta + \frac{\partial u}{\partial z}\zeta, \qquad v' = \frac{\partial v}{\partial x}\xi + \frac{\partial v}{\partial y}\eta + \frac{\partial v}{\partial z}\zeta, \qquad w' = \frac{\partial w}{\partial x}\xi + \frac{\partial w}{\partial y}\eta + \frac{\partial w}{\partial z}\zeta,$$

neglecting squares and products of ξ, η, ζ. Using (1), we obtain

$$u' = -\eta\lambda z - \zeta\lambda\, y = -\lambda\,(\eta z + \zeta y), \qquad v' = -\lambda\,(\zeta x + \xi z), \qquad w' = -\lambda\,(\xi y + \eta x), \qquad \ldots (2)$$

We know that the direction cosines of the normal to the quadratic $x\eta\zeta + y\zeta\xi + z\xi\eta$ = constant at $(x+\xi, y+\eta, z+\zeta)$ are proportional to $y\zeta + z\eta, x\zeta + z\xi, x\eta + y\xi$.

Hence from (2), it follows that the velocity at the point $(x+\xi, y+\eta, z+\zeta)$ relative to the fluid at the point (x, y, z) is normal to the given quadratic

Ex. 9. (Spiral vortex). *Define spiral vortex and determine pressure due to spiral vortex.*

Sol. Spiral vortex. Definition. *The combination of a vortex and source is called a spiral vortex*

Let there be a source of strength m and vortex of strength k both at the origin. Then, complex potential w due to them is given by

$$w = -m \log z + (ik/2\pi) \log z \qquad \text{or} \qquad w = (ik/2\pi - m) \log z \qquad \ldots (1)$$

Determination of pressure due to spiral vertex.

From (1), $dw/dz = (ik/2\pi - m) \times (1/z)$ and hence, we have

$$q^2 = \left|\frac{dw}{dz}\right|^2 = \frac{(-m)^2 + (k/2\pi)^2}{|z|^2} = \frac{m^2 + (k^2/4\pi^2)}{r^2}, \quad \text{as} \quad z = re^{i\theta} \Rightarrow |z| = r$$

For steady flow, the pressure is given by

$$\frac{p}{\rho}+\frac{q^2}{2}+V=C,\text{ where } V \text{ is force potential and } C \text{ is a constant}$$

or $$p=\rho\,\{C-V-(m^2+k^2/4\pi^2)/2r^2\},\ \text{using (1)}$$

which gives the pressure due to spiral vortex at any point z $(=re^{i\theta})$

Ex. 10. *Discuss vortex between parallel walls.*

Sol. Consider a vortex of strength k in the liquid between two parallel walls AB and CD at a distance 'a' apart. Let the vortex of strength k be situated at the origin O and let the line through it and parallel to the walls be taken as x-axis.

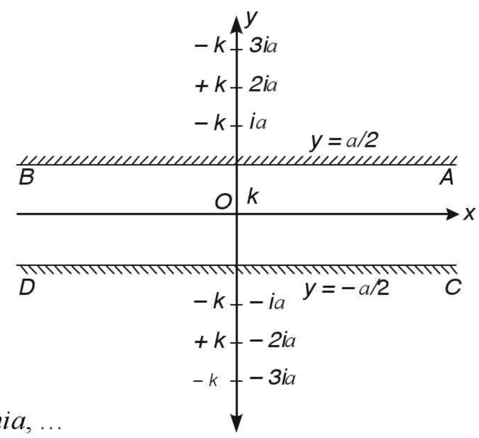


Fig. 4

Then, the image of the vortex in the walls $y=a/2$, $y=-a/2$ are vortices of strength $-k$ at ia, $-ia$ respectively. These vortices will have vortex images of strengths k at $z=2ai, -2ia$ with regard to the two walls AB and CD. It follows that the image system is a row of vortices along y-axis each at a distance 'a' apart, alternating in sign. Thus, we have

(*i*) Vortices of strength k at $z=0, \pm 2\,ia, 4\,ia, \dots, \pm 2\,nia, \dots$

(*ii*) Vortices of strength k at $z=\pm ia, \pm 3ia, \pm 5ia, \dots, \pm(2n-1)\,ia, \dots$

The complex potential w of the system at any point z is given by

$$w=(ik/2\pi)\,[\{\log z+\log(z-2ia)+\log(z+2ia)+\log(z-4ia)+\log(z+4ia)+\dots\}]$$
$$-\{\log(z-ia)+\log(z+ia)+\log(z-3ia)+\log(z+3\,ia)+\dots\}$$

$$=\frac{ik}{2\pi}\log\frac{z(z^2+2^2a^2)(z^2+4^2a^2)\dots}{(z^2+a^2)(z^2+3^2a^2)(z^2+5^2a^2)}=\frac{ik}{2\pi}\log\frac{(z/2a)(1+z^2/2^2a^2)(1+z^2/4^2a^2)+\dots}{(1+z^2/a^2)(1+z^2/3^2a^2)\dots}$$

[Omitting constants]

$$=\frac{ik}{2\pi}\log\frac{\left(\frac{z}{a}\right)\prod_{n=1}^{\infty}\left(1+\frac{(z/2a)^2}{n^2}\right)}{\prod_{n=1}^{\infty}\left(1+\frac{(z/a)^2}{(2n-1)^2}\right)}=\frac{ik}{2\pi}\log\frac{\theta\prod_{n=1}^{\infty}\left(1+\frac{\theta^2}{\pi^2n^2}\right)}{\prod_{n=1}^{\infty}\left(1+\frac{4\theta^2}{(2n-1)\pi^2}\right)}$$

[On putting $z/2a=\theta/\pi$ so that $\theta=\pi z/2a$]

$$=\frac{ik}{2\pi}\log\frac{\sinh\theta}{\cosh\theta},\ \text{using results of Art. 1.10, chapter 1}$$

$= (ik/2\pi)\times\log\tanh\theta=(ik/2\pi)\times\log\tanh(\pi z/2a)$, as $\theta=\pi z/2a$

Thus, $$w=\phi+i\psi=(ik/2\pi)\times\log\tanh(\pi z/2a) \qquad \dots (1)$$

Hence $$\bar{w}=\phi-i\psi=(-ik/2\pi)\times\log\tanh(\pi\bar{z}/2a)$$

Subtracting (2) from (1), $$2i\psi=\frac{ik}{2\pi}\left[\log\tanh\frac{\pi(x+iy)}{2a}+\log\tanh\frac{\pi(x-iy)}{2a}\right]$$

or $$\psi=\frac{k}{4\pi}\log\left[\tanh\frac{\pi(x+iy)}{2a}\tanh\frac{\pi(x-iy)}{2a}\right]=\frac{k}{4\pi}\log\frac{\sinh\frac{\pi(x+iy)}{2a}\sinh\frac{\pi(x-iy)}{2a}}{\cosh\frac{\pi(x+iy)}{2a}\cosh\frac{\pi(x-iy)}{2a}}$$

or $$\psi = \frac{k}{4\pi}\log\frac{(-i)^2 \sin\dfrac{i\pi(x+iy)}{2a} \quad \sin\dfrac{i\pi(x-iy)}{2a}}{\cos\dfrac{i\pi(x+iy)}{2a} \quad \cos\dfrac{i\pi(x-iy)}{2a}}$$, as sinh θ = – sin i θ and cosh θ = cos iθ

or $$\psi = \frac{k}{4\pi}\log\frac{-2\sin\dfrac{\pi(ix-y)}{2a}\sin\dfrac{\pi(ix+y)}{2a}}{2\cos\dfrac{\pi(ix-y)}{2a}\cos\dfrac{\pi(ix+y)}{2a}} = \frac{k}{4\pi}\log\frac{-\left(\cos\dfrac{\pi y}{a}-\cos\dfrac{i\pi x}{a}\right)}{\cos\dfrac{\pi y}{a}+\cos\dfrac{i\pi x}{a}}$$

or $$\psi = \frac{k}{4\pi}\log\frac{\cosh(\pi x/a)-\cos(\pi y/a)}{\cosh(\pi x/a)+\cos(\pi y/a)} \qquad \ldots (3)$$

Streamlines are given by $\psi = \text{constant} = (k/4\pi)\log C'$, say ... (4)

From (3) and (4), $$\frac{\cosh(\pi x/a)-\cos(\pi y/a)}{\cosh(\pi x/a)+\cos(\pi y/a)} = C' \quad \Rightarrow \quad \frac{2\cosh(\pi x/a)}{2\cos(\pi y/a)} = \frac{1+C'}{1-C'} = C, \text{ say}$$

or cosh $(\pi x/a)$ = C cos $(\pi y/a)$, C being a constant

The motion of vortex of strength k at O is given by $(dw_0/dz)_{z=0}$ such that

$$u_0 - iv_0 = -\frac{dw_0}{dz} = -\frac{d}{dz}\left[w - \frac{ik}{2\pi}\log z\right]_{z=0}, \qquad \ldots (5)$$

since the motion is due to other vortices. Thus,we obtain

$$u_0 - iv_0 = -\frac{ik}{2\pi}\left[\frac{2\pi}{2a}\operatorname{cosech}\left(\frac{2\pi z}{2a}\right) - \frac{1}{z}\right]_{z=0}, \text{ using (1)}$$

Thus, $u_0 - iv_0 = 0$, showing that the vortex at O is at rest

Ex. 11 *The space between two fixed coaxial cylinders of radii a and b, and between two planes perpendicular to the axis and distance c apart, is occupied by liquid of density ρ. Show that the velocity potential of a motion whose kinetic energy shall be equal a given quantity T is given by Aθ, where $\pi\rho A^2 c \log(b/a) = T$.*

Workout the same problem for the space between two confical elliptic cylinders.

Sol. If ψ denotes the stream function of the motion of a point whose polar coordinates, with centre of the sections of the cylinders as origin, is (r, θ), then ψ would be independent of θ by symmetry and hence

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} = 0 \qquad \text{so that} \qquad \frac{d^2\psi/dr^2}{d\psi/dr} = -\frac{1}{r}$$

Integrating, $\log(d\psi/dr) = \log C - \log r$ or $d\psi/dr = C/r$ or $d\psi = (C/r)dr$

Integrating, $\psi = C\log r$, (since constant is immaterial)

Since $\phi + i\psi$ is an analytic function, hence we have

$$\phi + i\psi = iC(\log r + i\theta) = iC\log(re^{i\theta}) = iC\log z, \quad \text{as} \quad z = re^{i\theta} \qquad \ldots (1)$$

From (1), we have $\phi = -C\theta = A\theta$, where $A = -C$ ... (2)

Now, $$q^2 = \left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 = 0 + \frac{1}{r^2}\times A^2 = \frac{A^2}{r^2}, \text{ using (2)} \qquad \ldots (3)$$

Then, $$T = \text{kinetic energy} = \frac{1}{2}\int_a^b (2\pi r dr \rho c)\, q^2 = \pi\rho A^2 c \int_a^b \frac{1}{r}\, dr, \text{ using (3)}$$

Thus, $$T = \pi\, \rho A^2\, c \log (b / a) \qquad \ldots (4)$$

Second Part: For the elliptic cylinders, take $x + iy = c \cosh (\xi + i\eta)$

so that $x = c \cosh \xi \cos \eta$ and $y = c \sinh \xi \sin \eta$ $\qquad \ldots (5)$

Then ξ = constant and η = constant, represent confacal ellipses and hyperbolas respectively, namely,

$$x^2/(c^2 \cosh^2 \xi) + y^2 / (c^2 \sinh^2 \xi) = 1 \quad \text{and} \quad x^2 / (c^2 \cos^2\eta) - y^2 / (c^2 \sin^2 \eta) = 1, \qquad \ldots (6)$$

the distance between the foci being $2c$, and in any particular ellipse η denotes the eccentric angle.

Again, since ξ is constant along confocal ellipses, and the streamlines are also confocal ellipses, therefore ψ must be a function of ξ only, so that $\psi = f(\xi)$.

Laplace equation for ψ is $$\partial^2\psi / \partial x^2 + \partial^2\psi / \partial y^2 = 0 \qquad \ldots (7)$$

Changing to ξ, η, (7) reduces to $$\partial^2\psi / \partial \xi^2 + \partial^2\psi / \partial \eta^2 = 0 \qquad \ldots (8)$$

Since ψ is independent of η, (8) reduces to $\partial^2\psi / \partial \xi^2 = 0$ giving $\psi = A\xi$.

Since $\phi + i\psi$ is an analytic function, so $\phi + i\psi = iA\, (\xi + i\eta)$ giving $\phi = -A\eta$

For the present case, the kinetic energy T′ of coaxial cylinders of length l (say) is given by

$$T' = \frac{1}{2}\rho l \int\!\!\int q^2 dxdy = -\frac{1}{2}\rho l \int\!\!\int \frac{\partial(\psi,\phi)}{\partial(x,y)}\, dxdy, \quad \text{as} \quad q^2 = -\frac{\partial(\psi,\phi)}{\partial(x,y)}$$

or $$T' = -\frac{1}{2}\rho l \int\!\!\int d\psi d\phi = -\frac{1}{2}\rho l \int d\psi \int d\phi = \frac{1}{2}\rho l A^2 \int_\alpha^\beta d\xi \int_0^{2\pi} d\eta,$$

where α, β are the values of ξ on the inner and outer cylinders.

Thus, $$T' = (1/2) \times \rho l A^2\, (\beta - \alpha) \times 2\pi = \pi\rho\, l A^2\, (\beta - \alpha) \qquad \ldots (9)$$

Let a_1, b_1 and a_2, b_2 be the semi-major and semi-minor axes of the bounding ellipse. Then, we know that

$b_1/a_1 = \tanh \alpha$ and $b_2 / a_2 = \tanh \beta$ so that $\alpha = \tanh^{-1}(b_1 / a_1)$ and $\beta = \tanh^{-1} (b_2 / a_2)$.

Hence, (9) yields $$T' = \pi\rho l A^2 \left(\tanh^{-1} \frac{b_2}{a_2} - \tanh^{-1} \frac{b_1}{a_1} \right) = \pi\rho l A^2 \log \frac{b_2 + a_2}{b_1 + a_1}$$

Alternative solution for second part. Let a_1, b_1 and a_2, b_2 be the semi major and semi-minor axes of the bounding confocal ellipses

$$x^2 / a_1^2 + y^2 / b_1^2 = 1 \quad \text{and} \quad x^2 / a_2^2 + y^2 / b_2^2 = 1 \qquad \ldots (i)$$

In what follows, we shall use conformal transformations. Consider Joukwaski's transformation given by

$$2z = Z + c^2/Z \qquad \ldots (ii)$$

From (1), $$Z = \{z + (z^2 - c^2)^{1/2}\} \qquad \ldots (iii)$$

Then, we can easily verify that the confocal ellipses (*i*) in the z-plane transform to concentric circles, of radii $(a_1 + b_1)$ and $(a_2 + b_2)$ respectively in the Z-plane, where $2c$ is the distance between the foci of the ellipses.

Re-writing (iii) in terms of elliptic transformation $z = c \cosh \zeta$, we have $Z = e^\zeta$, Again the complex potential in the Z-plane is given by

$$w = iA \log Z \qquad \ldots (iv)$$

Since $Z = e^{\zeta}$, (iv) yields $\quad w = iA \log e^{\zeta} \quad$ or $\quad \phi + i\psi = iA(\xi + i\eta)$ ...(v)

From (v), we have $\quad \phi = -A\eta$

Making use of result (4) of first part, the kinetic energy of the liquid bounded by the two co-oxial circular cylinders of radii $a_1 + b_1$ and $a_2 + b_2$ in the Z-plane is given by

$$T' = \pi\rho l A^2 \log\{(b_2 + a_2)/(b_1 + a_1)\}, \qquad \text{... (vi)}$$

where l is the length of the cylinders.

Since the kinetic energy is unchanged by conformal transformation, it follows that the kinetic energy T' of the fluid bounded by the given elliptic cylinders is given by (vi).

Ex.12. *In the case of steady motion about a solid of an elastic fluid under no forces it is observed that velocities at the point (x, y, z) parallel to the axes are proportional to y + z, z +x, x + y. Prove that surfaces of equal pressure are oblate spheriods, the eccentricity of the generating ellipse being* $\sqrt{3}/2$.

Sol. If u, v, w denote the components of velocity q at (x, y, z), then given that

$$u = k(y+z), \quad v = k(z+x), \quad w = k(x+y), \quad k \text{ being a constant}$$

Hence, $\quad q^2 = k^2\{(y+z)^2 + (z+x)^2 + (x+y)^2\} = 2k^2(x^2+y^2+z^2+xy+yz+zx)$ ... (1)

By Bernouille's theorem, the pressure p at any point (x, y, z) is given by

$$p/\rho = C - q^2/2$$

or $$p/\rho = C - k^2(x^2+y^2+z^2+xy+yz+zx), \text{ using (1)}$$

Hence the equipressure surfaces are given by

$$2x^2 + 2y^2 + 2z^2 + 2xy + 2yz + 2zx = 0 \qquad \text{... (2)}$$

From solid geometry, the length of semi-axis of the quadric (2) are given by

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

or $\quad \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \quad$ or $\quad (\lambda-1)^2(\lambda-4) = 0 \quad$ giving $\quad \lambda = 1, 1, 4.$

Thus, the squares of semi-oxes are 1, 1, 4. Hence, the equation of the quadric, with its principal axes, as axes of coordinates, is given by

$$(x^2 + y^2)/1 + z^2/4 = \text{constant}$$

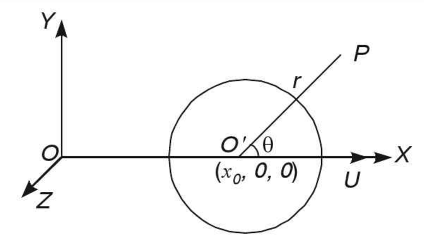
Therefore, The equation of the generating ellipse may be written as

$$y^2/1 + z^2/4 = \text{constant}$$

If e be the eccentricity of (3), then $\quad 1 = 4(1 - e^2) \quad$ giving $\quad e = \sqrt{3}/2$

Ex. 13. *A sphere of radius a is made to move in incompressible perfect fluid with non-uniform velocity U along the x-axis. If the pressure at infinity is zero, prove that at a point x in advance of the centre,* $\quad p = (1/2) \times \rho\, a^3\{\dot{U}/x^2 + U^2(2/x^3 - a^3/x^6)\}$ **(Garhwal 2001, 02; Kanpur 2010)**

Sol. Let O′ be the centre of the given sphere at any time t. Then the coordinates of O' are $(x_0, 0, 0)$, where $OO' = x_0$. Since the given sphere of radius a is moving along x-axis with velocity U, hence velocity potential ϕ at any point P of the fluid is given by (refer result (8) of Art. 10.2)



$$\phi = (Ua^3/2r^2) \times \cos\theta, \qquad \text{...(1)}$$

where $O'P = r$, $\angle PO'X = \theta$, $r^2 = (x - x_0)^2 + y^2 + z^2$ and $\dot{x}_0 = U$ ... (2)

Also, we have $\cos\theta = (x - x_0)/r$ ... (3)

Using (3), (1) reduces to $\phi = (Ua^3/2r^3) \times (x - x_0)$ ... (4)

By pressure equation, $p/p + q^2/2 - (\partial\phi/\partial t) = C$ ... (5)

From (1), $$q^2 = \left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 = \left(\frac{1}{2}Ua^3\right)^2 \left\{\left(\frac{2}{r^3}\cos\theta\right)^2 + \left(\frac{\sin\theta}{r^3}\right)^2\right\}$$

Thus, $q^2 = (U^2a^6/4r^6) \times (4\cos^2\theta + \sin^2\theta)$ ... (6)

From (4), $$\frac{\partial\phi}{\partial t} = \frac{1}{2}\frac{\dot{U}a^3}{r^3}(x - x_0) - \frac{3Ua^3}{2r^4}\dot{r}(x - x_0) + \frac{Ua^3}{2r^3} \times (-\dot{x}_0), \qquad \text{... (7)}$$

where $\dot{U} = dU/dt$, $\dot{r} = dr/dt$ and $\dot{x}_0 = dx_0/dt$

Now, from (2), $2r\dot{r} = 2(x - x_0)(-\dot{x}_0)$ or $2r\dot{r} = -2Ur\cos\theta$, by (2) and (3)

Thus, we have $\dot{r} = -U\cos\theta$ ... (8)

Using (2) and (8), (7) may be re-written as

$$\partial\phi/\partial t = (\dot{U}a^3/2r^2) \times \cos\theta + (3U^2a^3/2r^3) \times \cos^2\theta - (U^2a^3/2r^3) \qquad \text{... (9)}$$

Substituting the values of q^2 and $\partial\phi/\partial t$ given by (6) and (9) in (5), we get

$$\frac{p}{\rho} + \frac{Ua^6}{8r^6}(4\cos^2\theta + \sin^2\theta) - \frac{\dot{U}a^3\cos\theta}{2r^2} - \frac{3U^2a^3\cos^2\theta}{2r^3} + \frac{U^2a^3}{2r^3} = C \qquad \text{... (10)}$$

Given that the pressure at infinity is zero. Hence $p = 0$ when $r = \infty$. Then, from (10), $C = 0$

Hence, (10) yields $$\frac{p}{\rho} + \frac{Ua^6}{8r^6}(4\cos^2\theta + \sin^2\theta) - \frac{\dot{U}a^3\cos\theta}{2r^2} - \frac{3U^2a^3\cos^2\theta}{2r^3} + \frac{U^2a^3}{2r^3} = 0 \qquad \text{... (11)}$$

Pressure at a distance x in advance to the centre is obtained by $\theta = 0$ and $x = x$ in (11). Thus,

$$\frac{p}{\rho} + \frac{U^2a^6}{2x^6} - \frac{\dot{U}a^3}{2x^2} - \frac{3U^2a^3}{2x^3} + \frac{U^2a^3}{2x^3} = 0 \quad \text{or} \quad \frac{p}{\rho} = \frac{a^3}{2}\left\{\frac{\dot{U}}{x^2} + U^2\left(\frac{2}{x^3} - \frac{a^3}{x^6}\right)\right\}$$

Thus, $$p = (\rho a^3/2) \times \left\{\dot{U}/x^2 + U^2(2/x^3 - a^3/x^6)\right\}$$

Ex. 14. *A solid sphere is moving through frictionless liquid. Compare the velocities of slip of the liquid past it at different parts of its surface.*

Prove that when the sphere is in motion with uniform velocity U, the pressure at the part of its surface where the radius makes angle θ with direction of motion is increased on account of the motion by the amount $(\rho U^2/16) \times (9\cos 2\theta - 1)$, *where ρ is the density of the liquid.*

Sol. Let O' (x_0, y_0, z_0) be the position of the centre of the sphere at any time t and let x-axis be the direction of the velocity U of the sphere at that time. Let 'a' be the radius of the sphere. Then velocity potential ϕ of the liquid is given by

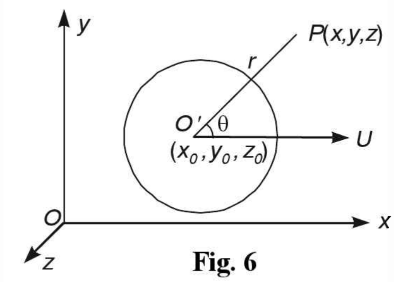


Fig. 6

$$\phi = (\text{U}\, a^3/2r^2)\times\cos\theta \qquad \dots (1)$$

The velocity of slip of the liquid past the sphere at (a, θ)

$$= \left[-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right]_{r=a} = \left[\frac{\text{U}a^3}{2r^2}\sin\theta\right]_{r=a} = \frac{U}{2}\sin\theta \qquad \dots(2)$$

Giving different values to θ in (2), the velocities of slip of the liquid at different parts of the surface of the sphere can be obtained.

Second part: Here $\quad O'P = r. \quad \angle PO'X = \theta, \quad \dot{x}_0 = \text{U}, \quad \dot{y}_0 = 0, \quad \dot{z}_0 = 0, \dots (3)$

since the velocity of the sphere is only along x-axis, and hence $\dot{y}_0 = \dot{z}_0 = 0$

Also, $\quad r^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \qquad$ and $\qquad \cos\theta = (x-x_0)/r \qquad \dots (4)$

Using (4), (1) reduces to $\quad \phi = \dfrac{\text{U}a^3(x-x_0)}{2r^3} = \dfrac{1}{2}\times\dfrac{\text{U}a^3(x-x_0)}{\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\}^{3/2}} \quad \dots (5)$

Now, the pressure equation is $\qquad p/\rho + q^2/2 - (\partial\phi/\partial t) = C \qquad \dots (6)$

$$(1) \Rightarrow \quad q^2 = \left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 = \left(\frac{\text{U}a^3}{2}\right)^2\left\{\left(\frac{2\cos\theta}{r^3}\right)^2 + \left(\frac{\sin\theta}{r^3}\right)^2\right\} = \frac{\text{U}^2a^6}{4r^4}(4\cos^2\theta + \sin^2\theta) \quad \dots (7)$$

From (5), $\quad \dfrac{\partial\phi}{\partial t} = \dfrac{\text{U}a^3\times(-\dot{x}_0)}{2\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\}^{3/2}} + \dfrac{(-3/2)\times\text{U}a^3(x-x_0)\times 2(x-x_0)\times(-\dot{x}_0)}{2\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\}^{5/2}}$

or $\qquad \dfrac{\partial\phi}{\partial t} = -\dfrac{U^2a^3}{2r^3} + \dfrac{3U^2a^3\cos^2\theta}{2r^5}$, using (2) and (3) $\qquad \dots (8)$

Substituting the values of q^2 and $\partial\phi/\partial t$ given by (7) and (8) in (6), we get

$$\frac{p}{\rho} + \frac{\text{U}^2a^6}{8r^6}(4\cos^2\theta + \sin^2\theta) + \frac{\text{U}^2a^3}{2r^3} - \frac{3\text{U}^2a^3\cos^2\theta}{2r^3} = C \qquad \dots (9)$$

On the surface of the sphere at (a, θ), putting $r = a$ in (9), pressure is given by

$$p/\rho + (\text{U}^2/8)\times(4\cos^2\theta + \sin^2\theta) + (\text{U}^2/2) - (3\text{U}^2/2)\times\cos^2\theta = C \qquad \dots (10)$$

Let p_0 denote the presure at (a, θ) on the sphere when there was no motion, i.e., $p = p_0$ when $U = 0$. They (10) gives $C = p_0/\rho$. Putting $C = p_0/\rho$ in (10), we get

$$p/\rho + (U^2/8)\times(4\cos^2\theta + \sin^2\theta + 4 - 12\cos^2\theta) = p_0/\rho$$

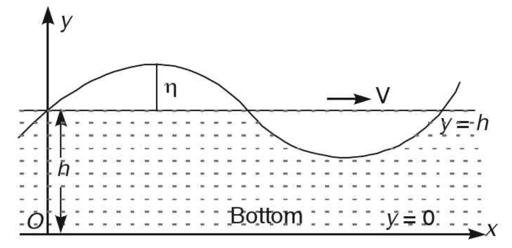
or $\quad (p-p_0)/\rho = (U^2/8)\times(9\cos^2\theta - 5) = (U^2/16)\times\{9\,(1+\cos 2\theta) - 10\} = (U^2/16)\times(9\cos 2\theta - 1)$

Thus, the required increase in pressure $= p - p_0 = (\rho\text{U}^2/16)\times(9\cos 2\theta - 1)$

Ex. 15. *If water of depth h be flowing with velocity proportional to the distance from the bottom, V being the velocity of the stream at the surface, prove that velocity c of propagation of waves in the direction of the stream is given by* $(c - V)^2 + V(c - V) \times (W^2/gh) - W^2 = 0$, *where W is the velocity of propagation in still water.* **(Agra 2010)**

Sol. Let c be the velocity of propagation of waves parallel to x-axis. Now, we have

$$W^2 = (g/m) \times \tanh mh \qquad \ldots (1)$$



Let v be the fluid velocity at any height y from the bottom $y = 0$ Then, given that v is proportional to y. Hence, we have

$$v = ky \qquad \ldots (2)$$

Given that $v = V$ when $y = h$. Hence, (2) gives

$$V = kh \qquad \text{so that} \qquad k = V/h$$

Hence, (2) reduce to $\qquad v = (V/h) \times y \qquad \ldots (3)$

Now, impose a velocity $-c$ to the whole system so that the wave profile reduces to rest and water flows with velocity $v - c$ in steady state. Then, we have

$$-\frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x} = -c \;\Rightarrow\; \psi = cy\,. \text{ Also } \quad -\frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x} = v = \frac{Vy}{b} \quad \Rightarrow \quad \psi = -\frac{Vy^2}{2b}$$

Now the stream function for the reduced motion is given by

$$\psi = -(Vy^2/2h - cy) + (A \cosh my + B \sinh my) \times \sin mx \qquad \ldots (4)$$

Hence, at $y = 0$, $\psi = A \sin mx$

But the bottom $y = 0$ will be the streamline $\psi = 0$ so that

$$(\psi)_{y=0} = 0 \quad \Rightarrow \quad A \sin mx = 0 \quad \Rightarrow \quad A = 0$$

Hence, from (4), $\qquad \psi = -(Vy^2/2h - cy) + \mathrm{B} \sinh my \sin mx \qquad \ldots (5)$

Again, the free surface $y = h + \eta = h + a \sin mx$ will be a streamline $\psi = C$, C being a constant.

Hence, $\qquad -\{(V/2h) \times (h + a \sin mx)^2 - c(h + a \sin mx)\} + B \sinh m(h + a \sin mx) \sin mx = C$

Since $a \ll 1$ and $m \ll 1$, so neglecting m^2, a^2, am, the above equation yields

$$-\{(\mathrm{V}/2h) \times (h^2 + 2ah \sin mx) - c(h + a \sin \mathrm{mx})\} + B \sinh mh \sin mx = C \qquad \ldots (6)$$

Equating the coefficients of $\sin mx$ from both sides of (6), we get

$$-(Va - Ca) + \mathrm{B} \sinh mh = 0 \qquad \text{so that} \qquad B = a(V - c)/\sinh mh \qquad \ldots (7)$$

From (5) and (7), $$\psi = -\left(\frac{Vy^2}{2h} - cy\right) + \frac{V - c}{\sinh mh} \times a \sinh my \sin mx \qquad \ldots (8)$$

Now, $$q^2 = (\partial\psi/\partial x)^2 + (\partial\psi/\partial y)^2$$

or $$q^2 = \left[\frac{V - c}{\sinh mh} am \sinh my \cos mx\right]^2 + \left[-\frac{Vy}{h} + c + \frac{V - c}{\sinh mh} am \cosh my \sin mx\right]^2$$

or $$q^2 = \left[-\frac{Vy}{h} + c + \frac{V - c}{\sinh mh} am \cosh my \sin mx\right]^2, \text{ neglecting } a^2 \qquad \ldots (9)$$

Let $q = q_0$ = fluid velocity at free surface, where $y = h + a \sin mx$. So (9) yields

$$q_0^2 = \left[-V - \frac{Va}{h} \sin mx + c + \frac{V - c}{\sinh mh} am \cosh mh \sin mx\right]^2$$

or $$q_0^2 = [-(V-c) - (Va/h) \times \sin mx + (V-c)\, am \coth mh \sin mx]^2$$

or $$q_0^2 = (V-c)^2 + 2\,(V-c) \times (Va/h) \times \sin mx - 2\,(V-c)^2\, am \coth mh \sin mx \qquad \ldots (10)$$

[On omitting a^2, being very small]

Now, the pressure equation is given by $p/\rho + q^2/2 + gy = C'$, C' being a constant ... (11)

At the free surface, let $p = \Pi$. Also, there $q = q_0$ and $y = h + a \sin mx$.

Hence (11) reduces to $$\Pi/p + q_0^2/2 + g\,(h + a \sin mx) = C' \qquad \ldots (12)$$

Substituting the value of q_0^2 given by (10) in (12), we have

$$\Pi/\rho + (1/2) \times [(V-c)^2 + 2\,(V-c) \times (Va/h) \times \sin mx - 2\,(V-c)^2\, am \coth mh \sin mx]^2 + g(h + a \sin mx) = c' \qquad \ldots (13)$$

Equating the coefficients of $\sin mx$ on both sides of (13), we have

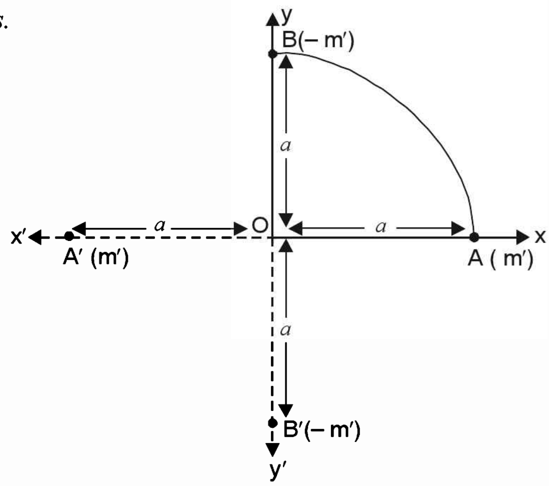
$$ga + (1/2) \times [2\,(V-c) \times (Va/h) - 2\,(V-c)^2\, am \coth mh] = 0$$

or $$(c-V)^2\, m \coth mh + (c-V) \times (V/h) - g = 0$$

or $$(c-V)^2 \times (g/W^2) + (c-V) \times (V/h) - g = 0, \text{ since from (1), } \coth mh = g/mW^2$$

or $$(c-V)^2 + V(c-V) \times (W^2/gh) - W^2 = 0$$

Ex. 16. *In the part of an infinite plane bounded by a circular quadrant AB and the productions of radii OA, OB, there is a two dimensional liquid motion due to the production of liquid at A, and its absorption at B, at the uniform rate m. Find the velocity potential of the motion; and show that the fluid which issues from A in the direction making an angle μ with OA follows the path whose polar equation is* $$r = a\sqrt{\sin 2\theta}\,[\cot\mu + \sqrt{\cot^2\mu + \operatorname{cosec}^2 2\theta}\,]^{1/2},$$
the positive signs being taken for all square roots.



Sol. Let m' be the strength of the source at A. Then, $(\pi/2) \times \rho m' = m$ so that $m' = 2m/\pi\rho$. Also, given that there is a sink of strength $(-m')$ at B. Taking O as origin and OA as the axis of x and OB as the axis of y, let the coordinates of A and B respectively be $(a, 0)$ and $(0, a)$.

Let the image system of the source at A and the sink at B, be a source of strength m' at A' $(-a, 0)$ and a sink of strength $(-m')$ at B' $(0, -a)$ with respect the boundary of the liquid. This image system would be the proper system; provided under both object and image system the boundary of the liquid becomes a streamline.

Now, the complex potential at any point P $(z = x + iy = re^{i\theta})$ is given by

$$w = -m' \log(z-a) - m' \log(z+a) + m' \log(z-ai) + m' \log(z+ai)$$

or $$w = -m' \log(z^2 - a^2) + m' \log(z^2 + a^2) = -m' \log(r^2 e^{2i\theta} - a^2) + m' \log(r^2 e^{2iq} + a^2)$$

or $$w = -m' \log\{r^2(\cos 2\theta + i \sin 2\theta) - a^2\} + m' \log\{r^2(\cos 2\theta + i \sin 2\theta) + a^2\}$$

or $$\phi + i\psi = -m' \log\{(r^2 \cos 2\theta - a^2) + ir^2 \sin 2\theta\} + m' \log\{(r^2 \cos 2\theta + a^2) + ir^2 \sin 2\theta)\} \qquad \ldots (1)$$

Using the result $$\log(a + ib) = \log(a^2 + b^2)^{1/2} + i \tan^{-1}(b/a) \qquad \ldots (2)$$

and then equating imaginary parts on both sides of (1), we obtain

$$\psi = -m'\left(\tan^{-1}\frac{r^2\sin 2\theta}{r^2\cos 2\theta - a^2} - \tan^{-1}\frac{r^2\sin 2\theta}{r^2\cos 2\theta + a^2}\right) = -m'\tan^{-1}\frac{\dfrac{r^2\sin 2\theta}{r^2\cos 2\theta - a^2} - \dfrac{r^2\sin 2\theta}{r^2\cos 2\theta + a^2}}{1+(r^4\sin^2 2\theta)/(r^4\cos^2 2\theta - a^4)}$$

or $$\psi = -m'\tan^{-1}\{(2r^2a^2\sin 2\theta)/(r^4 - a^4)\} \qquad \ldots (3)$$

The streamlines of the motion are given by ψ = constant, i.e.,

$$(2r^2a^2\sin 2\theta)/(r^4 - a^4) = C, \text{ where } C \text{ is a constant} \qquad \ldots (4)$$

When $C = 0$, the streamlines (3) breaks into two factors $r\cos\theta = 0$ and $r\sin\theta = 0$, i.e., $x = 0$ and $y = 0$ which represent axes of y and x respectively. Again, when $C \to \infty$, the streamlines (4) reduces to $r^4 - a^4 = 0$ giving $r = a$ which represent the quadrant AB of circle $r = a$. From these facts, it follows that our image system is the required proper system, which is such that the boundary of the liquid becomes a stream line

The required velocity potential ϕ can be obtained by using formula (2) in (1) and then equating real parts on both sides of (1). Thus, we have

$$\phi = -m'[\log\{(r^2\cos 2\theta - a^2)^2 + r^4\sin^2 2\theta\}^{1/2} - \log\{(r^2\cos 2\theta + a^2)^2 + r^4\sin^2 2\theta\}^{1/2}]$$

or $$\phi = m'\log\left[\frac{(r^2\cos 2\theta + a^2)^2 + r^4\sin^2 2\theta}{(r^2\cos 2\theta - a^2)^2 + r^4\sin^2 2\theta}\right]^{1/2} = \frac{m'}{2}\log\frac{r^4 + 2a^2r^2\cos 2\theta + a^4}{r^4 - 2a^2r^2\cos 2\theta + a^4}$$

Transforming to cartesian coordinates (writing $r\cos\theta = x$, $r\sin\theta = y$ and $x^2 + y^2 = r^2$), the streamlines given by (4) reduce to

$$4a^2(r\cos\theta)(r\sin\theta) = C(r^4 - a^4) \quad \text{or} \quad 4a^2xy = C\{(x^2+y^2)^2 - a^4\} \qquad \ldots (5)$$

From (5), $$\frac{dy}{dx} = -\frac{4a^2x - C\{2(x^2+y^2)\times 2y\}}{4a^2y - C\{2(x^2+y^2)\times 2x\}} = -\frac{a^2x - Cy(x^2+y^2)}{a^2y - Cx(x^2+y^2)} \qquad \ldots (6)$$

The required streamline is given by $\qquad (dy/dx)_{x=a,\, y=0} = \tan\mu$

or $$(a^3/Ca^3) = \tan\mu \quad \text{or} \quad C = \tan\mu \qquad \ldots (7)$$

Substituting the above value of C in (4), the required streamline is given by

$$2a^2r^2\sin 2\theta = (r^4 - a^4)\tan\mu \quad \text{or} \quad r^4 - 2a^2r^2\sin 2\theta\cot\mu - a^4 = 0, \text{ giving}$$

$$r^2 = a^2\{\cot\mu\sin 2\theta + (\cot^2\mu\sin^2 2\theta + 1)^{1/2}\} = a^2\sin 2\theta\{\cot\mu + \sqrt{\cot^2\mu + \operatorname{cosec}^2 2\theta}\}$$

Thus, $r = a\sqrt{\sin 2\theta}\{\cot\mu + \sqrt{\cot^2\mu + \operatorname{cosec}^2 2\theta}\}^{1/2}$, taking positive signs for all square roots.

Ex. 17. *Show that if velocity potential of an irrotational motion is equal to* $A(x^2 + y^2 + z^2)^{-3/2} z\tan^{-1}(y/x)$, *lines of flow lie on the series of surfaces* $x^2 + y^2 + z^2 = c^{2/3}(x^2 + y^2)^{2/3}$.

Sol. If ds denote an element of length on the line of flow at the point (x, y, z), and dx, dy and dz its projections on the axes, the equation to the line of flow is given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{or} \quad \frac{dx}{-(\partial\phi/\partial x)} = \frac{dy}{-(\partial\phi/\partial y)} = \frac{dz}{-(\partial\phi/\partial z)},$$

showing that the element of line of flow lies along the normal to the surface ϕ = constant, (where ϕ denotes the velocity potential) passing through the point (x, y, z), at the point.

Therefore, it follows that lines of flow intersect the surface ϕ = constant orthogonally, in other words the lines of flow would lie on such surfaces which are orthogonal to the surface represented by ϕ = constant. Thus, if the surface

$$F(x, y, z) \equiv x^2 + y^2 + z^2 - c^{2/3}(x^2 + y^2)^{2/3} = 0 \qquad \ldots (1)$$

intersect the surface $\quad \phi(x, y, z) \equiv A(x^2 + y^2 + z^2)^{-3/2} z \tan^{-1}(y/x) = \text{constant} \qquad \ldots (2)$

orthogonally, the lines of flow would lie on it. Now, (1) and (2) will intersect orthogonally if

$$(\partial F/\partial x)(\partial\phi/\partial x) + (\partial F/\partial y)(\partial\phi/\partial y) + (\partial F/\partial z)(\partial\phi/\partial z) = 0 \qquad \ldots (3)$$

From (1), $\dfrac{\partial F}{\partial x} = 2x - c^{2/3} \times \dfrac{2}{3}(x^2 + y^2)^{-1/3} \times 2x = 2x\left\{1 - \dfrac{2c^{2/3}}{3(x^2 + y^2)^{1/3}}\right\}$

$$= 2x \cdot \left\{1 - \frac{2}{3} \times \frac{x^2 + y^2 + z^2}{(x^2 + y^2)^{2/3}(x^2 + y^2)^{1/3}}\right\}, \text{ since from (1), } c^{2/3} = \frac{x^2 + y^2 + z^2}{(x^2 + y^2)^{2/3}}$$

Thus, $\quad \dfrac{\partial F}{\partial x} = 2x\left\{1 - \dfrac{2(x^2 + y^2 + z^2)}{3(x^2 + y^2)}\right\} = \dfrac{2x}{3}\left(\dfrac{x^2 + y^2 - 2z^2}{x^2 + y^2}\right) \qquad \ldots (4)$

Similarly, $\quad \dfrac{\partial F}{\partial y} = \dfrac{2y}{3}\left(\dfrac{x^2 + y^2 - 2z^2}{x^2 + y^2}\right)$. $\qquad$ Also, $\quad \dfrac{\partial F}{\partial z} = 2z \qquad \ldots (5)$

From (2), $\quad \log\phi = \log A - (3/2) \times \log(x^2 + y^2 + z^2) + \log z + \log \tan^{-1}(y/x) \qquad \ldots (6)$

From (6), $\quad \dfrac{1}{\phi}\dfrac{\partial\phi}{\partial x} = -\dfrac{3}{2} \times \dfrac{2x}{x^2 + y^2 + z^2} + \dfrac{1}{\tan^{-1}(y/x)} \times \dfrac{1}{1 + (y/x)^2} \times \left(-\dfrac{y}{x^2}\right)$

Thus, $\quad \dfrac{\partial\phi}{\partial x} = \phi\left[-\dfrac{3x}{x^2 + y^2 + z^2} - \dfrac{y}{\tan^{-1}(y/x)(x^2 + y^2)}\right] \qquad \ldots (7)$

Similarly, $\dfrac{\partial\phi}{\partial y} = \phi\left\{-\dfrac{3y}{x^2 + y^2 + z^2} - \dfrac{x}{\tan^{-1}(y/x)(x^2 + y^2)}\right\}$, $\dfrac{\partial\phi}{\partial z} = \phi\left\{-\dfrac{3z}{x^2 + y^2 + z^2} + \dfrac{1}{z}\right\} \qquad \ldots (8)$

Using (4), (5), (6), (7) and (8), L.H.S. of (3)

$$= \frac{2x\phi}{3} \times \frac{x^2 + y^2 - 2z^2}{x^2 + y^2}\left\{-\frac{3x}{x^2 + y^2 + z^2} - \frac{y}{\tan^{-1}(y/x)(x^2 + y^2)}\right\}$$

$$+ \frac{2y\phi}{3} \times \frac{x^2 + y^2 - 2z^2}{x^2 + y^2}\left\{-\frac{3y}{x^2 + y^2 + z^2} - \frac{x}{\tan^{-1}(y/x)(x^2 + y^2)}\right\} + 2z\phi\left\{-\frac{3z}{x^2 + y^2 + z^2} + \frac{1}{z}\right\}$$

$$= \phi\left[-2 \times \frac{x^2 + y^2 - 2z^2}{x^2 + y^2 + z^2} - \frac{6z^2}{x^2 + y^2 + z^2} + 2\right] = \phi\left[-\frac{(2x^2 + 2y^2 + 2z^2)}{x^2 + y^2 + z^2} + 2\right] = \phi \times 0 = 0 = \text{R.H.S. of (3)}$$

showing that (3) is satisfied and therefore the lines of flow lie on the series of surfaces represented by (1). This is what we wished to prove.

Ex. 18. *A sphere of radius 'a' is in motion in fluid, which is at rest at infinity, the pressure there being* Π, *determine the pressure at any point of the fluid and show that the pressure on the front hemisphere cut off by a plane perpendicular to the direction of motion is the resultant of pressures* $\pi a^2(\Pi - \rho v^2/16)$ *and* $(\pi\rho a^3 f)/3$ *in the direction respectively opposite to those of velocity* v *and acceleration* f, *of the centre of mass.*

Sol. Refer Art. 10.7. Replace θ by θ_2 in figure and entire discussion of Art. 10.7 upto equation (14). Also, note that here $p_0 = \Pi$. Hence, equation (14) reduce to

$$(p - \Pi_0)/\rho = (1/2) \times (a\,\mathrm{f} \cos\theta_1) + (9/8) \times (v \cos\theta_2)^2 - (5/8) \times v^2 \qquad \ldots (14)$$

Setting $r = a$ in (6), $\qquad af\cos\theta_1 = \dot{U}(x - x_0) + \dot{V}(y - y_0) + \dot{W}(z - z_0) \qquad \ldots (15)$

Setting $r = a$ and $\theta = \theta_2$ in (5), $\qquad v\cos\theta_2 = U(x - x_0) + V(y - y_0) + W(z - z_0) \qquad \ldots (16)$

Using (4), (15) and (16), (14) reduces to

$$(p - \Pi)/\rho = (1/2) \times \{\dot{U}(x - x_0) + \dot{V}(y - y_0) + \dot{W}(z - z_0)\} + (9/8a^2) \times \{U(x - x_0) + V(y - y_0) + W(z - z_0)\}^2 - (5/8) \times (U^2 + V^2 + W^2) \qquad \ldots (17)$$

Taking $v = U$, $V = W = 0$, the pressure for the motion of sphere along the x-axis with velocity U, is given by

$$(p - \Pi)/\rho = (1/2) \times \{\dot{U}(x - x_0) + \dot{V}(y - y_0) + \dot{W}(z - z_0)\} + (9U^2/8a^2) \times (x - x_0)^2 - (5U^2/8) \qquad \ldots (18)$$

Spherical polar coordinates of any point on the surface of the sphere are given by $x - x_0 = a\sin\theta\cos\phi$, $y - y_0 = a\sin\theta\sin\phi$, $z - z_0 = a\cos\theta$. So (18) reduces to

$$(p - \Pi)/\rho = (a/2) \times \{\dot{U}\sin\theta\cos\phi + \dot{V}\sin\theta\sin\phi + \dot{W}\cos\theta\} - (5U^2/8) + (9U^2/8) \times \sin^2\theta\cos^2\phi \qquad \ldots (19)$$

Hence the component of the total pressure on the front hemisphere alone x-axis

$$= \int_{\theta=0}^{\pi}\int_{\phi=-\pi/2}^{\phi=\pi/2} (-p)\sin\theta\cos\phi\; a\,d\theta \cdot a\sin\theta\, d\phi$$

$$= -a^2\int_{\theta=0}^{\pi}\int_{\phi=-\pi/2}^{\pi/2}\left[\left(\Pi - \frac{5\rho U^2}{8}\right) + \frac{1}{2}\rho a(\dot{U}\sin\theta\cos\phi + \dot{V}\sin\theta\sin\phi + \dot{W}\cos\theta) + \frac{9\rho U^2}{8}\sin^2\theta\cos^2\phi\right]\sin^2\theta\cos\phi\, d\theta\, d\phi$$

$$= -4a^2\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\left[\left(\Pi - \frac{5\rho U^2}{8}\right)\sin^2\theta\cos\phi + \frac{1}{2}a\rho\dot{U}\sin^3\theta + \frac{9\rho U^2}{8}\sin^4\theta\cos^3\phi\right] d\theta\, d\phi$$

[Using the standard properties of definite integral]

$$= -4a^2\left[\int_{\theta=0}^{\pi/2}\left(\Pi - \frac{5\rho U^2}{8}\right)\sin^2\theta\; d\theta + \int_{\theta=0}^{\pi/2}\left(\frac{1}{2}a\rho\dot{U}\sin^3\theta \times \frac{\pi}{4}\right)d\theta + \frac{9\rho U^2}{8}\int_{\theta=0}^{\pi/2}\frac{\Gamma(2)\Gamma(1/2)}{2\Gamma(5/2)}\sin^4\theta\, d\theta\right]$$

$$= -4a^2\left[\left(\Pi - \frac{5\rho U^2}{8}\right) \times \frac{\pi}{4} + \frac{\pi}{8}a\rho\dot{U} \times \frac{\Gamma(2)\Gamma(1/2)}{2\Gamma(5/2)} + \frac{9\rho U^2}{8} \times \frac{2}{3} \times \frac{\Gamma(5/2)\Gamma((1/2))}{2\Gamma(3)}\right]^*$$

* For these calculation, refer chapter on Beta and Gamma Functions in author's Ordinary and Partial differential equations published by S.Chand and Co., Delhi

$$= -\pi a^2\left[\Pi - \frac{5\rho U^2}{8} + \frac{1}{3}a\rho\dot{U} + \frac{9}{16}\rho U^2\right] = -\pi a^2\left(\Pi - \frac{\rho U^2}{16}\right) - \frac{\pi}{3}\rho a^3\dot{U}$$

Similarly, the component of the total pressure along y-axis

$$= \int_{\theta=0}^{\pi}\int_{\phi=-\pi/2}^{\pi/2}(-p)\sin\theta\sin\phi\, a d\theta\cdot a\sin\theta d\phi = -\frac{\pi}{3}\rho a^3\dot{V}$$

and, the component of total pressure along z-axis $= -\frac{\pi}{3}\rho a^3\dot{W}$

Thus, the total pressure on the front hemisphere is composed of components

$-\pi a^2(\Pi - \rho U^2/16) - (\pi/3)\times\rho a^3\dot{U}$ along x-axis, $-(\pi/3)\times\rho a^3\dot{V}$ along y-axis and $-(\pi/3)\times\rho a^3\dot{W}$ along z-axis.

Therefore, the total pressures may be considered as

(*i*) $\pi a^2(\Pi - \rho U^2/16)$ along the negative direction of x-axis, i.e., opposite to the direction of velocity U

(*ii*) $(\pi/3)\times\rho a^3 f$ opposite to the resultant acceleration f, where $f = (\dot{U}^2 + \dot{V}^2 + \dot{W}^2)^{1/2}$

Thus, we have proved what we wished to prove.

Ex. 19. *The space between two concentric spheres of radii a and b (a > b) is filled with an incompressible fluid of density ρ and the shells suddenly begin to move with velocities U, V in the same direction. Find the velocity potential. Deduce the impulse required to set the outer sphere in motion with velocity V, the masses of the spheres being* M_1, M_2

Sol. As in Art. 10.9, the required velocity potential ϕ is given by

$$\phi = \frac{Ua^3 - Vb^3}{b^3 - a^3}r\cos\theta + \frac{(U-V)a^3b^3}{2(b^3-a^3)}\frac{\cos\theta}{r^2} = \frac{1}{b^3-a^3}\left\{(Ua^3 - Vb^3)r + \frac{a^3b^3(U-V)}{2r^2}\right\}\cos\theta \quad \ldots(1)$$

Let I be the required impulse which imparts velocity V to the outer sphere. Again, suppose that the inner sphere, then, begins to move with velocity U, so that the required velocity potential is given by (1), Now, we have

$$M_2V = I - \iint p\cos\theta\, ds \quad \ldots(2)$$

and

$$M_1U = -\iint p\, ds \quad \ldots(3)$$

where p denotes the impulsive pressure over the element ds of the boundary surfaces.

Since the motion starts from rest, we have $p = \rho\phi$.

or

$$p = \frac{\rho}{b^3-a^3}\left\{(Ua^3 - Vb^3)r + \frac{a^3b^3(U-V)}{2r^2}\right\}\cos\theta \quad \ldots(4)$$

Setting $r = b$ in (4), the value of p on the outer surface is given by

$$p = \{\rho/(b^3 - a^3)\}\times\{(Ua^3 - Vb^3) + (a^3/2)\times(U-V)\}\, b\cos\theta$$

Substituting the above value of p in (2), we have

$$M_2V = I - \frac{\rho}{b^3-a^3}\left\{(Ua^3 - Vb^3) + \frac{a^3}{2}(U-V)\right\}\iint_{r=b} b\cos^2\theta\, ds \text{, on simiplification}$$

or $$M_2 V = I - \frac{\rho}{(b^3 - a^3)}\left\{3a^3 U - V(2b^3 + a^3)\right\} \times \frac{4\pi b^3}{3}, \text{ on simplification}$$

or $$I = M_2 V + \frac{2\pi\rho b^3}{3(b^3 - a^3)}\{3a^3 U - V(2b^3 + a^3)\} \qquad \ldots (5)$$

Setting $r = a$ in (4), the value of p on the inner surface is given by

$$p = \left\{\rho / (b^3 - a^3)\right\} \times \{(Ua^3 - Vb^3) + (b^3/2) \times (U - V)\} a \cos\theta$$

Substituting the above value of p in (3), we have

$$M_1 U = -\frac{\rho}{b^3 - a^3}\left\{(Ua^3 - Vb^3) + \frac{b^3}{2}(U - V)\right\} \int\!\!\int_{r=a} a \cos^2\theta \, ds$$

or $$M_1 U = -\frac{\rho}{2(b^3 - a^3)}\left\{2(Ua^3 - Vb^3) + b^3(U - V)\right\} \times \frac{4\pi a^3}{3}, \text{ on simplification}$$

or $$M_1 U = -\frac{2\pi\rho a^3}{3(b^3 - a^3)} \{U(2a^3 + b^3) - 3b^3 V\}$$

or $$U\left\{M_1 + \frac{2\pi\rho a^3(2a^3 + b^3)}{3(b^3 - a^3)}\right\} = \frac{2\pi\rho V a^3 b^3}{b^3 - a^3}$$

or $$U = (6\pi\rho V a^3 b^3) / \{3M_1 (b^3 - a^3) + 2\pi\rho a^3 (2a^3 + b^3)\} \qquad \ldots (6)$$

Substituting the value of U given by (6) in (5), we get the required impulse.

Ex. 2. *Discuss the motion for which the stream function is given by*

$\psi = (V/2) \times \{(a^4/r^2) \times \cos\theta - r^2\} \sin^2\theta$, where r is the distance from a fixed point and θ is the angle this distance makes with the fixed direction. **(Agra 2002)**

Sol. Given $$\psi = (V/2) \times \{(a^4/r^2) \times \cos\theta - r^2\} \sin^2\theta \qquad \ldots (1)$$

Here $\psi = 0 \Rightarrow \sin^2\theta \{(a^4/r^2) \times \cos\theta - r^2\} = 0 \Rightarrow \theta = 0$ and $a^4 \cos\theta = r^4$, where $\theta = 0$ is the axis of symmetry for Stoke's stream function, and the second equation is $a^4 \cos\theta = r^4$, which is an equation to a meridian curve along which ψ is constant.

Therefore the motion given by (1) seems to be the motion of a fluid streaming past a solid of revolution, the equation to whose meridian curve is $r^4 = a^4 \cos\theta$.

In order to verify the above conjecture, let us consider the velocity components q_r and q_θ along r and θ increasing direction respectively, given by

$$q_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} = -\frac{(V/2)}{r^2 \sin\theta}\left[\{(a^4/r^2) \times \cos\theta - r^2\}\right] \times 2\sin\theta\cos\theta - (a^4/r^2) \times \sin\theta \sin^2\theta]$$

or $$q_r = -V[\{(a^4/r^4) \times \cos\theta - 1\} \cos\theta - (a^4/2r^4) \times \sin^2\theta] \qquad \ldots (2)$$

and $$q_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = \frac{1}{r\sin\theta} \times \frac{V}{2}\left\{\left(-\frac{2a^4}{r^3}\right) \times \cos\theta - 2r\right\}\sin^2\theta = -\mathrm{V}\sin\theta\left(\frac{a^4\cos\theta}{r^4} + 1\right) \qquad \ldots (3)$$

As $r \to \infty$, from (2) and (3), we have $q_r = V\cos\theta$ and $q_\theta = -V\sin\theta$, whose resultant is the velocity V along the axis of symmetry

Let us consider velocity along the normal to the curve

$$r^4 = a^4 \cos\theta \qquad \ldots (4)$$

Let the normal velocity be denoted by V_n. Then, we have

$$V_n = q_r \sin\phi - q_\theta \cos\phi = \sin\phi\,(q_r - q_\theta \cot\phi) \qquad \ldots (5)$$

From (4), $4 \log r = 4 \log a + \log \cos\theta$. Differentiating it w.r.t. 'θ', we get

$(4/r) \times (dr/d\theta) = -\tan\theta$ or $\cot\phi = -(1/4) \times \tan\theta$, as $\cot\phi = (1/r) \times (dr/d\theta)$

Substituting the above value of $\cot\phi$ and using (2) and (3), (5) yields the value of V_n along the boundary of the meridian curve $r^4 = a^4 \cos\theta$. Thus, we have

$$V_n = V \sin\phi \left[-\frac{a^4 \cos^2\theta}{r^4} + \cos\theta + \frac{a^4 \sin^2\theta}{2r^4} - \sin\theta \left(\frac{a^4 \cos\theta}{r^4} + 1 \right) \times \frac{1}{4} \tan\theta \right]$$

$$= V \sin\phi \left\{ -\frac{a^4 \cos^2\theta}{a^4 \cos\theta} + \cos\theta + \frac{a^4 \sin^2\theta}{2a^4 \cos\theta} - \sin\theta \left(\frac{a^4 \cos\theta}{a^4 \cos\theta} + 1 \right) \times \frac{1}{4} \times \frac{\sin\theta}{\cos\theta} \right\}$$

$$= V \sin\phi\,\{-\cos\theta + \cos\theta + (\sin^2\theta/2\cos\theta) - (\sin^2\theta/2\cos\theta)\} = 0,$$

showing that the solid of revolution formed by the revolution of $r^4 = a^4 \cos\theta$ about the axis is at rest in the fluid.

Hence the value of ψ given by (1) determines the motion due to the streaming of the fluid with velocity V past the solid of revolution formed by the revolution of meridian curve (4), which is at rest. Thus, our earlier conjecture is verified.

Ex. 21. *Show that when irrotational waves of length λ are propagated in water of infinite depth, the pressure at any particle of water is the same as it was in the equilibrium position of the particle when it was at rest.*

Sol. We know that the velocity potential for the irrotational motion due to the propagation of the wave profile $\eta = a \sin m(x - ct)$ at the surface of water of uniform depth is given by

$$\phi = (ac/\sinh mh) \times \cos m(x - ct) \cosh m(y + h), \qquad \ldots (1)$$

where x-axis is chosen along the undisturbed level surface and y-axis vertically upwards through the point of the starting point of the wave. In the given problem, $h \to \infty$ and hence (1) reduces to

$$\phi = a\,c \cos m(x - ct) \times \lim_{h \to \infty} \frac{e^{my+mh} + e^{-my-mh}}{e^{mh} - e^{-mh}} = ac \cos m(x - ct) \times \lim_{h \to \infty} \frac{e^{my} + e^{-my-2mh}}{1 - e^{-2mh}}$$

Thus, $\qquad \phi = ac \cos m(x - ct)\, e^{my} \qquad \ldots (2)$

From (2), $\qquad \partial\phi/\partial t = mac^2 e^{my} \sin m(x - ct) \qquad \ldots (3)$

Let (x_0, y_0) be the equilibrium position of a particle at $t = 0$ and at time t, let $x = x_0 + \xi$ and $y = y_0 + \eta$, so that $\dot{x} = \dot{\xi}$, and $\dot{y} = \dot{\eta}$, Then, we have

$$\dot{\eta} = -(\partial\phi/\partial y) = -acm \cos m(x - ct)\, e^{my}, \text{ using (2)} \qquad \ldots (4)$$

Integrating (4), $\qquad \eta = ae^{my} \sin m(x - ct) \qquad \ldots (5)$

The pressure equation is $\qquad (p - p_0)/\rho - \partial\phi/\partial t + g(y_a + \eta) = 0 \qquad \ldots (6)$

But, from (1) and (2), $-(\partial\phi/\partial t) + g\eta = -mac^2 e^{my} \sin m(x - ct) + age^{my} \sin m(x - ct) = 0, \quad \ldots (7)$

where we have used the fact that $g = mc^2$. Using (7), (6) reduces to $p = p_0 + \rho g y_0$, showing that the pressure at particle remains the same as in the equilibrium position. Hence the result.

Ex. 22. *A straight cylindrical vortex column of uniform vorticity ζ is surrounded by an infinite quantity of fluid moving irrotationally which is at rest at infinity, prove that the difference between the kinetic energy included between two planes at right angles to the axis of the cylinder and separated by unit distance when the cross-section of the cylinder is an ellipse and when it is a circle of equal area A is $(\rho/\pi)\times\zeta^2 A^2 \log\{(a+b)/2\sqrt{ab}\}$, where ρ is the density of the fluid and a and b the semi axes of the ellipse .*

Sol. Let R be the radius of the circular cross section. Then, given that $A = \pi R^2 = \pi ab$ *so that* $R = \sqrt{ab}$. The total kinetic energy (K.E.) consists of two parts; one that of the liquid inside the vortex and second that of the liquid outside it.

For the circular cross-section, treating the liquid as rotating like solid, the kinetic energy E_1 of the liquid inside the vortex is given by

$$E_1 = (1/2)\times \pi R^2 \rho \times (1/2)\times R^2\zeta^2 = (\rho A^2\zeta^2)/4\pi, \text{ as } A = \pi R^2 \quad \Rightarrow \quad R^2 = A/\pi$$

Let ψ be stream function for liquid outside the circular cylindrical vortex. Then, we have (refer result (11) of Art. 11.20) $\psi = \zeta R^2 \log (r/R)$.

Hence velocity $= \partial\psi/\partial r = \zeta R^2 \times (R/r)\times(1/R) = \zeta R^2/r$ and hence kinetic energy E_2 of the liquid outside the vortex is given by

$$E_2 = \frac{1}{2}\int_R^\infty (2\pi r\, dr\, \rho)\cdot(\zeta R^2/r)^2 = \frac{\rho\zeta^2 A^2}{\pi}\int_R^\infty \frac{dr}{r}, \text{ as } A = \pi R^2 \Rightarrow R^2 = A/\pi$$

Then, the total K.E. of the circular cross-section = $E_1 + E_2 = E$, say

Thus,
$$E = \frac{\rho A^2\zeta^2}{4\pi} + \frac{\rho A^2\zeta^2}{\pi}\int_R^\infty \frac{dr}{r} \qquad \text{... (1)}$$

Next, ψ for inside the elliptic cross section (refer result (9) on page 11.46) is given by

$$\psi = \{\zeta/(a+b)\}\times(bx^2 + ay^2) \qquad \text{... (2)}$$

$$\therefore \text{ (Velocity)}^2 = \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = \frac{4\zeta^2}{(a+b)^2}(b^2x^2 + a^2y^2), \text{ using (2)}$$

Let E_3 be the kinetic energy of the liquid inside the elliptic cross-section. Then, we have

$$E_3 = \frac{1}{2}\int\int \frac{4\zeta^2 a^2 b^2}{(a+b)^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)(\rho\, dx\, dy) = \frac{\rho\zeta^2 A^2}{\pi}\times\frac{ab}{(a+b)^2}, \text{ as } A = \pi ab$$

*where we have used formula for moment of inertia for points inside the ellipse $x^2/a^2 + y^2/b^2 = 1$

Now, ψ for outside the elliptic section is given by

$$\psi = (\omega/4)\times(a+b)^2 e^{-2\xi}\cos 2\eta + (k\xi/2\pi) = \{(ab\zeta)/2\}\times e^{-2\xi}\cos 2y + \xi\times ab\zeta$$

$$[\because \ \omega = (2ab.\zeta)/(a+b)^2 \text{ and } k = 2\pi\zeta ab, \text{ where } x + iy = c\cosh(\xi + i\eta) \text{ and } c^2 = a^2 - b^2]$$

*Refer chapter 14 in author's "Dynamics" published by S.Chand & Co., New Delhi

Now, $v^2 = (\partial\psi/\partial x)^2 + (\partial\psi/\partial y)^2 = h^2\{(\partial\psi/\partial\xi)^2 + (\partial\psi/\partial\eta)^2\}h^2$, where $h^2 = (\partial\xi/\partial x)^2 + (\partial\xi/\partial y)^2$.

Let E_4 be the kinetic energy of the liquid outside the elliptic cross-section

Then, $E_4 = \dfrac{1}{2}\displaystyle\int_{\xi=\alpha}^{\infty}\int_{\eta=0}^{2\pi} \rho a^2 b^2 \zeta^2 \,(e^{-4\xi} + 1 - e^{-2\xi}\cos 2\eta)\, d\xi\, d\eta$, where $a = c\cosh\alpha,\ b = c\sin\alpha$

Now, $$\frac{1}{2}\int_{\xi=\alpha}^{\infty}\int_{\eta=0}^{2\pi} \rho a^2 b^2 \zeta^2 d\xi\, d\eta = \frac{\rho A^2\zeta^2}{4}\int_{\alpha}^{\infty} d\xi, \text{ where } A = \pi\text{ab}$$

and $$\frac{1}{2}\int_{\xi=\alpha}^{\infty}\int_{\eta=0}^{2\pi} \rho a^2 b^2 \zeta^2 e^{-4\xi}\, d\xi\, d\eta = \frac{\rho A^2\zeta^2}{4\pi}\left(\frac{a-b}{a+b}\right)^2, \text{ where } A = \pi ab$$

Hence, $$E_4 = \frac{\rho A^2\zeta^2}{\pi}\left[\int_{\alpha}^{\infty} d\xi + \frac{1}{4}\left(\frac{a-b}{a+b}\right)^2\right]$$

Let E' denote the total kinetic energy for the elliptic cross-section. Then, we get

$$E' = E_3 + E_4 = \frac{\rho A^2\zeta^2}{4}\left[\frac{ab}{(a+b)^2} + \frac{1}{4}\left(\frac{a-b}{a+b}\right)^2 + \int_{\alpha}^{\infty} d\xi\right] = \frac{\rho A^2\zeta^2}{4}\left[1 + \int_{\alpha}^{\infty} d\xi\right] \quad ...(3)$$

Therefore, the required difference in kinetic energies

$$= E - E' = \frac{\rho A^2\zeta^2}{\pi}\left[\int_{R}^{\infty}\frac{dr}{r} - \int_{\alpha}^{\infty} dJ\right], \text{ using (1) and (3)} \quad ...(4)$$

But $x^2/(c^2\cosh^2\xi) + y^2/(c^2\sinh^2\xi) = 1$ and so $\dfrac{x^2+y^2}{(c^2/4)\times e^{2\xi}} \to 1$, as $\xi \to \infty$

or $4r^2/c^2 \to e^{2\xi}$, as $\xi \to \infty$ and so for large ξ, $\log(2r/c) = \xi$.

Hence, (4) yields, $E - E' = \dfrac{\rho A^2\zeta^2}{\pi}\lim\limits_{r\to\infty}\left[-\displaystyle\int_{\log\left(\frac{a+b}{a-b}\right)^{1/2}}^{\log(2r/c)} d\xi + \int_{\log\sqrt{ab}}^{\log r} dr\right]$, where $A = AR^2 = \pi ab$

$$= \frac{\rho A^2\zeta^2}{\pi}\lim_{r\to\infty}\left[-\log\frac{2r}{c} + \log\left(\frac{a+b}{a-b}\right)^{1/2} + \log r - \log\sqrt{ab}\right]$$

$$= \frac{\rho A^2\zeta^2}{\pi}\lim_{r\to\infty}\left[\log\sqrt{\frac{a+b}{a-b}} - \log\sqrt{ab} + \log\frac{c}{r}\right] = \frac{\rho A^2\zeta^2}{\pi}\log\left\{\sqrt{\frac{a+b}{a-b}}\times\frac{1}{\sqrt{ab}}\times\frac{\sqrt{a^2-b^2}}{2}\right\}$$

$$= \frac{\rho A^2\zeta^2}{\pi}\log\left\{\frac{\sqrt{a+b}}{\sqrt{a-b}}\times\frac{\sqrt{a+b}\sqrt{a-b}}{2\sqrt{ab}}\right\} = \frac{\rho A^2\zeta^2}{\pi}\log\frac{a+b}{2\sqrt{ab}}$$

Ex. 23. Which of the two methods, viz., Lagrangian and Eulerian methods of describing fluid motion is better and why ? Discuss. **(Meerut 2012)**

Hint : Refer Art. 2.1, Chapter 2

Ex. 24. Obtain the Navier – Stoke's equations of motion of a viscous fluid and discuss their limitations. **(Meerut 2012)**

Hint : Refer Art. 14.1 and 16.1.

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