② $y''(x) + y(x) = 0$ with initial conditions $y'(0) = y(0) = 0$

Given, $y''(x) + y(x) = 0$

$\Rightarrow y''(x) = -y(x)$

Integrating from 0 to x,

$$\int_0^x y''(t)\,dt = -\int_0^x y(t)\,dt$$

$$\Rightarrow [y'(t)]_0^x = -\int_0^x y(t)\,dt$$

$$\Rightarrow y'(x) - y'(0) = -\int_0^x y(t)\,dt$$

$$\Rightarrow y'(x) = -\int_0^x y(t)\,dt \qquad [\because y'(0) = 0]$$

again integrating from 0 to x,

$$\int_0^x y'(t)\,dt = -\int_0^x \int_0^{t_1} y(t)\,dt\,dt_1$$

$$\Rightarrow [y(t)]_0^x = -\int_0^x (x-t)\,y(t)\,dt$$

$$\Rightarrow y(x) - y(0) = -\int_0^x (x-t)\,y(t)\,dt$$

$$\Rightarrow y(x) = -\int_0^x (x-t)\,y(t)\,dt \qquad [\because y(0) = 0]$$

- Required Integral Equation

③ $y'' + y = \cos x$ with initial condition $y(0) = 0$, $y'(0) = 1$

Given, $y''(x) + y(x) = \cos x$

$\Rightarrow y''(x) = \cos x - y(x)$

Integrating from 0 to x we get,

$$\int_0^x y''(t)\,dt = \int_0^x (\cos t - y(t))\,dt$$

or, $[y'(t)]_0^x = \int_0^x (\cos t - y(t))\,dt$

or, $y'(x) - y'(0) = \int_0^x [\cos t - y(t)]\,dt$

or, $y'(x) = 1 + \int_0^x (\cos t - y(t))\,dt \quad [\because y'(0) = 1]$

~~again integrating from 0 to x,~~

$$\cancel{\int_0^x y'(t)\,dt = \int_0^x dt + \int_0^x \int_0^{t_1} (\cos t - y(t))\,dt\,dt_1}$$

$$\cancel{\Rightarrow [y(t)]_0^x = x + \int_0^x (x-t)\cos t\,dt} - \int_0^x (x-t)\,y(t)\,dt$$

$$\cancel{\Rightarrow y(t) - y(0) = x + \int_0^x (x-t)\cos t\,dt} - \int_0^x (x-t)\,y(t)\,dt$$

$$\therefore y(t) = x + \int_0^x (x-t)\cos t\,dt - \int_0^x (x-t)\,y(t)\,dt.$$

Required.

$$\Rightarrow y'(x) = 1 + [\sin t]_0^x - \int_0^x y(t)\,dt$$

$$\Rightarrow y'(x) = 1 + \sin x - \int_0^x y(t)\,dt$$

$$\Rightarrow \int_0^x y'(t)\,dt = \int_0^x (1 + \sin t)\,dt - \int_0^x \int_0^{t_1} y(t)\,dt\,dt_1$$

$$\Rightarrow y(x) - y(0) = [t - \cos t]_0^x - \int_0^x (x-t)\,y(t)\,dt$$

$$\Rightarrow y(x) = x - \cos x + 1 - \int_0^x (x-t)\,y(t)\,dt.$$

Required.

④ $y'' - 5y' + 6y = 0$, $y(0) = 0$, $y'(0) = 1$.

$$y''(x) = 5y'(x) - 6y(x)$$

$$\Rightarrow \int_0^x y''(t)\,dt = 5\int_0^x y'(t)\,dt - 6\int_0^x y(t)\,dt$$

$$\Rightarrow y'(x) - y'(0) = 5[y(x) - y(0)] - 6\int_0^x y(t)\,dt$$

$$\Rightarrow y'(x) - 1 = 5y(x) - 6\int_0^x y(t)\,dt$$

$$\Rightarrow \int_0^x y'(t)\,dt = \int_0^x 1\,dt + \int_0^x 5y(t)\,dt - 6\int_0^x\int_0^{t_1} y(t)\,dt\,dt_1$$

$$\Rightarrow y(x) - y(0) = x + \int_0^x 5y(t)\,dt - 6\int_0^x (x-t)\,y(t)\,dt$$

$$\Rightarrow y(x) = x + \int_0^x [5 - 6(x-t)]\,y(t)\,dt.$$

— Required.

⑤ $y'' + \lambda y = 0$ with boundary condition $y(0) = 0$, $y(1) = 0$

$$y''(x) = -\lambda y(x)$$

Integrating from 0 to x,

$$y'(x) - y'(0) = -\lambda\int_0^x y(t)\,dt$$

$$\Rightarrow y'(x) = c - \lambda\int_0^x y(t)\,dt \qquad [c = y'(0)]$$

again, integrating from 0 to x,

$$y(x) - y(0) = cx - \lambda\int_0^x\int_0^{t_1} y(t)\,dt\,dt_1$$

$$\Rightarrow y(x) = cx - \lambda\int_0^x (x-t)\,y(t)\,dt$$

Puts $x = 1$,

$$y(1) = c - \lambda\int_0^1 (x-t)\,y(t)\,dt$$

□ Convert the Integral equation to Differential equn.

(1) $y(x) = 1 + \int_0^x (x+t)\, y(t)\, dt$

Diff. w.r.t. x,

$$y'(x) = 0 + \frac{d}{dx}\int_0^x (x+t)\, y(t)\, dt$$

$$\Rightarrow y'(x) = \int_0^x \frac{d}{dx}(x+t)\, y(t)\cdot dt + (x+x)\, y(x)\cdot 1$$

(By Libnitz Rule)

$$\Rightarrow y'(x) = \int_0^x y(t)\, dt + 2x\, y(x)$$

again diff. w.r.t. x we gets

$$y''(x) = \frac{d}{dx}\int_0^x y(t)\, dt + 2y(x) + 2x\, y'(x)$$

$$\Rightarrow y''(x) = \int_0^x 0\cdot dt + y(x)\cdot 1 + 2y(x) + 2x\, y'(x)$$

$$\Rightarrow y''(x) - 2x\, y'(x) - 3y(x) = 0$$

where, $y(0) = 1 + \int_0^0 (x+t)\, y(t)\, dt = 1$

$$y'(0) = \int_0^0 y(t)\, dt + 2\times 0\; y(0) = 0$$

Thus, $y'' - 2x y' - 3y = 0$ with the initial condition $y(0) = 1,\ y'(0) = 0$

(2) $y(x) = 1 - x - 4\sin x + \int_0^x [3 - 2(x-t)]\, y(t)\, dt$

Diff. w.r.t. x,

$$y'(x) = -1 - 4\cos x + \int_0^x -2\, y(t)\, dt + 3y(x)$$

again Diff. w.r.t. x,

$$y''(x) = 4\sin x - \int_0^x 0\cdot dt - 2y(x) + 3y'(x)$$

$\Rightarrow y''(x) = 4\sin x - 2y(x) + 3y'(x)$.

$\Rightarrow y''(x) - 3y'(x) + 2y(x) = 4\sin x$

where, $y(0) = 1$, $y'(0) = -1 - 4 + 3 = -5+3 = -2$

Hence,
$y'' - 3y' + 2y = 4\sin x$ with the initial conditions $y(0) = 1$, $y'(0) = -5$

(3) $y(x) = x - \cos x + 1 - \int_0^x (x-t)\, y(t)\, dt$

Diff. w.r.t. x,

$y'(x) = 1 + \sin x - \int_0^x y(t)\, dt + 0$

again, Diff. w.r.t. x,

$y''(x) = \cos x - \int_0^x 0\, dt + y(x) \cdot 1$

$\Rightarrow y''(x) - y(x) = \cos x$

where, $y(0) = 0 - \cos 0 + 1 - 0 = -1+1 = 0$

$y'(0) = 1 + \sin 0 - 0 = 1$

Thus, $y'' - y = \cos x$ with initial conditions $y(0) = 0$, $y'(0) = 1$.

(4) $y(x) = \lambda \int_0^1 K(x,t)\, y(t)\, dt$, $K(x,t) = \begin{cases} (x-t)(x-1), & 0 < t < x \\ x(x-t), & x < t < 1 \end{cases}$

$\therefore y(x) = \lambda \left[\int_0^x (x-t)(x-1)\, y(t)\, dt + \int_x^1 x(x-t)\, y(t)\, dt \right]$

Diff. w.r.t. x,

$y'(x) = \lambda \left[\int_0^x (x-1+x-t)\, y(t)\, dt + 0 + \int_x^1 (x-t+x)\, y(t)\, dt + x(x-1)\, y(1) \cdot 0 + 0 \right]$

$y'(x) = \lambda \left[\int_0^x (2x-1-t)\, y(t)\, dt + \int_x^1 (2x-t)\, y(t)\, dt \right]$

$$y''(x) = \lambda\left[\int_0^x 2\,y(t)\,dt + (x-1)\,y(x) + \int_x^1 2\,y(t)\,dt - x\,y(x)\right]$$

$$y''(x) = \lambda\left[2\int_0^1 y(t)\,dt - y(x)\right]$$

(4) $$y(x) = x + \int_0^x [5 - 6(x-t)]\,y(t)\,dt.$$

Diff. w.r.t. x we get,

$$y'(x) = 1 + \int_0^x -6\,y(t)\,dt + 5\,y(x)$$

again Diff. w.r.t. x we get,

$$y''(x) = 0 + \int_0^x 0.\,dt - 6\,y(x) + 5\,y'(x)$$

$$\Rightarrow y''(x) - 5\,y'(x) + 6\,y(x) = 0$$

where, $y(0) = 0$, $y'(0) = 1$

Hence, $y'' - 5y' + 6y = 0$ with initial cond^n $y(0) = 0$, $y'($

(5) $$y(x) = -\int_0^x (x-t)\,y(t)\,dt$$

Diff. w.r.t. x we get,

$$y'(x) = -\int_0^x y(t)\,dt + 0$$

again Diff. w.r.t. x,

$$y''(x) = -\int_0^x 0.\,dt - y(x)$$

$$\Rightarrow y''(x) + y(x) = 0$$

where, $y(0) = 0$, $y'(0) = 0$

Hence, $y'' + y = 0$ with initial condition $y(0) = y'(0) = 0$

again, Diff. w.r.t. x,

$$y''(x) = \lambda\left[\int_0^x 2\,y(t)\,dt + (x-1)y(x) + \int_x^1 2\,y(t)\,dt - x\,y(x)\right]$$

$$y''(x) = \lambda\left[2\int_0^1 y(t)\,dt - y(x)\right]$$

(4) $$y(x) = x + \int_0^x [5 - 6(x-t)]\,y(t)\,dt.$$

Diff. w.r.t. x we get,

$$y'(x) = 1 + \int_0^x -6\,y(t)\,dt + 5\,y(x)$$

again Diff. w.r.t. x we get,

$$y''(x) = 0 + \int_0^x 0.\,dt - 6\,y(x) + 5\,y'(x)$$

$$\Rightarrow y''(x) - 5\,y'(x) + 6\,y(x) = 0$$

where, $y(0) = 0$, $y'(0) = 1$

Hence, $y'' - 5y' + 6y = 0$ with initial condⁿ $y(0) = 0$, $y'(0) = 1$.

$$y(x) = -\int_0^x (x-t)\,y(t)\,dt$$

Diff. w.r.t. x we get,

$$y'(x) = -\int_0^x y(t)\,dt + 0$$

again Diff. w.r.t. x,

$$y''(x) = -\int_0^x 0.\,dt - y(x)$$

$$\Rightarrow y''(x) + y(x) = 0$$

where, $y(0) = 0$, $y'(0) = 0$

Hence, $y'' + y = 0$ with initial condition $y(0) = y'(0) = 0$

1. $g(s) = f(s) + \lambda \int_0^1 K(s,t)\, g(t)\, dt$

(i) $K(s,t) = \pm 1$

$$g(s) = f(s) + \lambda \int_0^1 (\pm 1)\, g(t)\, dt$$

$$= f(s) + \lambda q$$

where, $q = \int_0^1 (\pm 1)\, g(t)\, dt = \int_0^1 (\pm 1)\left[f(t) + \lambda q\right] dt$

$$= (\pm 1)\left[\int_0^1 f(t)\, dt + \lambda \int_0^1 q\, dt\right]$$

$$q = (\pm 1) \int_0^1 f(t)\, dt \pm \lambda q \int_0^1 dt$$

$$q = (\pm 1) \int_0^1 f(t)\, dt \pm \lambda q \,[t]_0^1$$

$$\Rightarrow (1 \mp \lambda)\, q = (\pm 1) \int_0^1 f(t)\, dt$$

so, clearly, $D(\lambda) = 1 \mp \lambda$

(ii) $K(s,t) = st$

$$\therefore g(s) = f(s) + \lambda \int_0^1 st\, g(t)\, dt$$

$$= f(s) + \lambda s \int_0^1 t\, g(t)\, dt$$

$$= f(s) + \lambda s q \quad \text{where, } q = \int_0^1 t\, g(t)\, dt$$

Now, $q = \int_0^1 t\left[f(t) + \lambda t q\right] dt$

$$\Rightarrow q = \int_0^1 t f(t)\, dt + \lambda q \int_0^1 t^2\, dt$$

$$\Rightarrow q = \int_0^1 t f(t)\, dt + \lambda q \left[\frac{t^3}{3}\right]_0^1$$

$$\Rightarrow q = \int_0^1 t f(t)\, dt + \frac{\lambda q}{3}$$

$\Rightarrow \left(1-\frac{\lambda}{3}\right) c_1 = \int_0^1 t\, f(t)\, dt$

So, $D(\lambda) = 1 - \lambda/3$

(ii) $K(s,t) = s^2 + t^2$

$$\therefore g(s) = f(s) + \lambda \int_0^1 (s^2+t^2)\, g(t)\, dt$$

$$= f(s) + \lambda \left[s^2 \int_0^1 g(t)\, dt + \int_0^1 t^2 g(t)\, dt \right]$$

$$= f(s) + \lambda \left[c_1 s^2 + c_2 \right]$$

where, $c_1 = \int_0^1 g(t)\, dt$, $c_2 = \int_0^1 t^2 g(t)\, dt$

$$\therefore c_1 = \int_0^1 \left\{ f(t) + \lambda \left[c_1 t^2 + c_2 \right] \right\} dt$$

$$= \int_0^1 f(t)\, dt + \lambda c_1 \cdot \frac{1}{3} + \lambda c_2$$

$$\Rightarrow \left(1 - \frac{\lambda}{3}\right) c_1 - \lambda c_2 = \int_0^1 f(t)\, dt \quad \text{(A)}$$

Now, $c_2 = \int_0^1 t^2 g(t)\, dt$

$$= \int_0^1 t^2 \left[f(t) + \lambda \left(c_1 t^2 + c_2 \right) \right] dt$$

$$= \int_0^1 t^2 f(t)\, dt + \lambda c_1 \frac{1}{5} + \lambda c_2\, 1/3$$

$$\Rightarrow -\frac{\lambda}{5} c_1 + \left(1 - \lambda/3\right) c_2 = \int_0^1 t^2 f(t)\, dt \quad \text{(B)}$$

From (A) & (B),

$$D(\lambda) = \begin{vmatrix} 1-\lambda/3 & -\lambda \\ -\lambda/5 & 1-\lambda/3 \end{vmatrix} = \left(1 - \lambda/3\right)^2 - \lambda^2/5$$

$$= 1 - \frac{2\lambda}{3} - \frac{4\lambda^2}{45}$$

(2) $g(s) = f(s) + \lambda \int_0^{2\pi} \cos(s+t)\, g(t)\, dt$

$K(s,t) = \cos(s+t) = \cos s \cdot \cos t - \sin s \sin t$

$a_1(s) = \cos s$ $\quad a_2(s) = -\sin s$ $\quad c_1 = \int_0^{2\pi} \cos t\, g(t)\, dt$

$b_1(t) = \cos t$ $\quad b_2(t) = \sin t$ $\quad c_2 = \int_0^{2\pi} \sin t\, g(t)\, dt$

$f_1 = \int_0^{2\pi} \cos t\, f(t)\, dt$ $\quad a_{11} = \int_0^{2\pi} \cos^2 t\, dt = \pi$

$f_2 = \int_0^{2\pi} \sin t\, f(t)\, dt$ $\quad a_{12} = \int_0^{2\pi} \cos t \sin t\, dt = 0$

$a_{21} = \int_0^{2\pi} -\sin t \cos t\, dt = 0$, $\quad a_{22} = \int_0^{2\pi} -\sin^2 t\, dt = -\pi$

$$\therefore D(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} \\ -\lambda a_{21} & 1-\lambda a_{22} \end{vmatrix} = \begin{vmatrix} 1-\lambda\pi & 0 \\ 0 & 1+\lambda\pi \end{vmatrix}$$

$$= 1 - \lambda^2\pi^2$$

Now, $\begin{bmatrix} 1-\lambda\pi & 0 \\ 0 & 1+\lambda\pi \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ ——— (A)

for $\lambda = 1/\pi$, $\quad \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

$\Rightarrow f_1 = 0$

for $\lambda = -1/\pi$, $\quad \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

$\Rightarrow f_2 = 0$

Given, $f(s) = \sin s$

$\therefore f_1 = \int_0^{2\pi} \cos t \sin t\, dt = 0$, $\quad f_2 = \int_0^{2\pi} \sin^2 t\, dt = \pi$

$\therefore$

Now, for $\lambda \neq \pm 1/\pi$ system (A) has unique soln

which is,

$$(1-\lambda\pi)c_1 = f_1 \Rightarrow c_1 = \frac{f_1}{1-\lambda\pi}$$

$$(1+\lambda\pi)c_2 = f_2 \Rightarrow c_2 = \frac{f_2}{1+\lambda\pi}$$

Now, for $f_1 = 0$ & $f_2 = \pi$

$$c_1 = 0, \quad c_2 = \pi/1+\lambda\pi$$

$$\therefore g(s) = \sin s - \frac{\lambda\pi}{1+\lambda\pi}\sin s = \frac{1}{1+\lambda\pi}\sin s$$

(3) $$g(s) = 1 + \lambda \int_0^\pi (\sin s \sin 2t)\, g(t)\, dt$$

$$= 1 + \lambda \sin s \int_0^\pi \sin 2t\, g(t)\, dt$$

$$= 1 + c_1(\lambda \sin s)$$

where, $$c_1 = \int_0^\pi \sin 2t\, g(t)\, dt$$

$$c_1 = \int_0^\pi \sin 2t\,(1 + c_1\lambda \sin t)\, dt$$

$$\Rightarrow c_1 = \int_0^\pi \sin 2t\, dt + \lambda c_1 \int_0^\pi \sin t \sin 2t\, dt$$

$$\Rightarrow c_1 = \left[-\frac{\cos 2t}{2}\right]_0^\pi + \frac{\lambda c_1}{2}\int_0^\pi (\cos t - \cos 3t)\, dt$$

$$= \left[-\frac{\cos 2t}{2}\right]_0^\pi + \frac{\lambda c_1}{2}\left[\sin t - \frac{\sin 3t}{3}\right]_0^\pi$$

$$= 0$$

$\therefore D(\lambda) = 1 \neq 0$ $\therefore$ It has no eigen value.

(7) $g(s) = 1 + \lambda \int_{-\pi}^{\pi} e^{iw(s-t)} g(t)\, dt$

$$= 1 + \lambda \int_{-\pi}^{\pi} e^{iws} \cdot e^{-iwt} g(t)\, dt \qquad = 1 + \lambda e^{iws} q$$

$\therefore a_1(s) = e^{iws}$ $\qquad q = \int_{-\pi}^{\pi} e^{-iwt} g(t)\, dt$

$b_1(t) = e^{-iwt}$

$$a_{11} = \int_{-\pi}^{\pi} e^{iwt} \cdot e^{-iwt}\, dt = 2\pi$$

$$f_1 = \int_{-\pi}^{\pi} e^{-iwt}\, 1 \cdot dt = \left[\frac{e^{-iwt}}{-iw} \right]_{-\pi}^{\pi} = \frac{i}{w}\left[e^{-iw\pi} - e^{iw\pi} \right]$$

$$= \frac{i}{w} \cdot -2i \sin w\pi$$

$$= \frac{2}{w} \sin w\pi$$

$$\therefore D(\lambda) = 1 - \lambda a_{11} = 1 - \lambda \cdot 2\pi$$

$$= 1 - 2\pi\lambda$$

Now, $[1 - \lambda a_{11}][q] = [f_1]$

$$\therefore q = \frac{f_1}{1 - \lambda a_{11}} \qquad \text{for } \lambda \neq 1/2\pi$$

$$\therefore q = \frac{\cancel{2}\; 2/w \,\sin w\pi}{1 - 2\pi\lambda}$$

$$\therefore g(s) = 1 + \lambda q\, e^{iws}$$

$$= 1 + \frac{2\lambda}{w(1 - 2\pi\lambda)} e^{iws} \qquad \text{for } \lambda \neq 1/2\pi$$

If $\lambda = 1/2\pi$ then, $f_1 = 0 \Rightarrow \sin w\pi = 0$, otherwise no solution.

⑤ $g(s) = s^2 + \int_0^1 \sin(st)\, g(t)\, dt$

Given, $\sin st = st - \dfrac{(st)^3}{\lfloor 3} + \cdots$

$$g(s) = s^2 + \int_0^1 st\, g(t)\, dt - \frac{1}{\lfloor 3}\int_0^1 s^3 t^3\, g(t)\, dt$$

$$= s^2 + \left[s c_1 + c_2\frac{(-s^3)}{\lfloor 3}\right]$$

where,

$$c_1 = \int_0^1 t\left(t^2 + c_1 t - \frac{t^3}{6}c_2\right) dt$$

$$= \frac{1}{4} + \frac{c_1}{3} - \frac{c_2}{30}$$

$$\Rightarrow \frac{2}{3}c_1 + \frac{c_2}{30} = 1/4 \quad \text{——— (A)}$$

$$c_2 = \int_0^1 t^3\left(t^2 + c_1 t - \frac{c_2 t^3}{6}\right) dt$$

$$\Rightarrow c_2 = 1/6 + \frac{c_1}{5} - \frac{c_2}{42}$$

$$\Rightarrow -\frac{c_1}{5} + \frac{43}{42}c_2 = 1/6 \quad \text{——— (B)}$$

$$\therefore \quad D(\lambda) = \begin{vmatrix} 2/3 & 1/30 \\ -1/5 & 43/42 \end{vmatrix} = \frac{43}{63} + \frac{1}{150} \neq 0$$

$$\therefore \quad \therefore \quad 2/3\, c_1 + c_2/30 = 1/4$$

$$-\frac{c_1}{5} \quad \frac{43c_2}{42} = 1/6$$

This system has unique sol^n

$$\& \ c_1 = \frac{3155}{2\times 4342}, \quad c_2 = \frac{1015}{4342}$$

$$g(s) = s^2 + \frac{3155}{8684}s - \frac{1015}{26052}s^3$$

① $g(s) = s + \lambda \int_0^1 (st^2 + s^2 t)\, g(t)\, dt$

$= s + \lambda [s c_1 + s^2 c_2]$

where, $c_1 = \int_0^1 t^2 g(t)\, dt$, $c_2 = \int_0^1 t\, g(t)\, dt$

$\therefore c_1 = \int_0^1 t^2 [t + \lambda t c_1 + \lambda t^2 c_2]\, dt$

$\Rightarrow c_1 = \frac{1}{4} + \frac{\lambda c_1}{4} + \frac{\lambda c_2}{5} \Rightarrow (1 - \frac{\lambda}{4}) c_1 - \frac{\lambda}{5} c_2 = 1/4$ —— (A)

$\Rightarrow \cancel{-\frac{\lambda}{3} c_1 + (1 - \frac{\lambda}{4}) c_2 = 1/3}$ —— $\cancel{(A)}$

$c_2 = \int_0^1 t\, [t + \lambda t c_1 + \lambda t^2 c_2]\, dt$

$\Rightarrow c_2 = \frac{1}{3} + \frac{\lambda c_1}{3} + \frac{\lambda c_2}{4}$

$\Rightarrow -\frac{\lambda}{3} c_1 + (1 - \frac{\lambda}{4}) c_2 = 1/3$ —— (B)

$D(\lambda) = \begin{vmatrix} 1 - \frac{\lambda}{4} & -\lambda/5 \\ -\lambda/3 & 1 - \frac{\lambda}{4} \end{vmatrix} = 1 - \lambda/2 + \lambda^2/16 - \lambda^2/15$

$D(\lambda) = 0 \Rightarrow \lambda = \dfrac{-120 \pm \sqrt{(120)^2 + 4 \times 240}}{2}$ are e.v.

for $D(\lambda) \neq 0$, we get unique sol^n of the following system

$(1 - \frac{\lambda}{4}) c_1 - \lambda/5\, c_2 = 1/4$

$-\lambda/3\, c_2 + (1 - \frac{\lambda}{4}) c_2 = 1/3$

$\Delta_1 = \begin{vmatrix} 1/4 & -\lambda/5 \\ 1/3 & 1 - \lambda/4 \end{vmatrix} = \dfrac{60 + \lambda}{240}$, $\Delta_2 = \begin{vmatrix} 1 - \lambda/4 & 1/4 \\ -\lambda/3 & 1/3 \end{vmatrix} = 1/3$

$$\therefore c_1 = \frac{\Delta_1}{\Delta} = \frac{60+\lambda}{-\lambda^2-120\lambda+240}$$

$$c_2 = \frac{\Delta_2}{\Delta} = \frac{80}{240-120\lambda-\lambda^2}$$

$$\therefore g(s) = s + \lambda s c_1 + \lambda s^2 c_2$$

$$g(s) = s + \frac{\lambda s(60+\lambda)}{(-\lambda^2-120\lambda+240)} + \frac{\lambda s^2 80}{240-120\lambda-\lambda^2}$$

② $$g(s) = \lambda \int_1^2 \left[st + \frac{1}{st}\right] g(t)\, dt$$

$$= \lambda\left[s c_1 + \frac{1}{s} c_2\right]$$

where, $c_1 = \int_1^2 t\, g(t)\, dt$ & $c_2 = \int_1^2 \frac{1}{t} g(t)\, dt$

$$\therefore c_1 = \int_1^2 t\, \lambda\left(t c_1 + \frac{1}{t} c_2\right) dt$$

$$= \lambda c_1 \left[\frac{t^3}{3}\right]_1^2 + \lambda c_2 \left[t\right]_1^2$$

$$= \lambda c_1 \left(\frac{7}{3}\right) + \lambda c_2$$

$$\Rightarrow \left(1 - \frac{7}{3}\lambda\right) c_1 - \lambda c_2 = 0 \quad \text{——— (A)}$$

Now,

$$c_2 = \int_1^2 \frac{1}{t}\, \lambda\left(t c_1 + \frac{1}{t} c_2\right) dt$$

$$= \lambda c_1 \left[t\right]_1^2 + \lambda c_2 \left[\frac{t^{-2+1}}{-2+1}\right]_1^2$$

$$c_2 = \lambda c_1 + \frac{\lambda c_2}{2}$$

$$\Rightarrow -\lambda c_1 + (1 - \lambda/2) c_2 = 0 \quad \text{——— (B)}$$

$\therefore$ from Ⓐ & Ⓑ

$$\left(1-\frac{7}{3}\lambda\right)c_1 - \lambda c_2 = 0$$

$$-\lambda c_1 + \left(1-\frac{\lambda}{2}\right)c_2 = 0$$

$$D(\lambda) = \begin{vmatrix} 1-\frac{7}{3}\lambda & -\lambda \\ -\lambda & 1-\lambda/2 \end{vmatrix} = 1 - \frac{7}{3}\lambda - \lambda/2 + \frac{7\lambda^2}{6} - \lambda^2$$

$$= 1 - \frac{17\lambda}{6} + \frac{\lambda^2}{6}$$

$$D(\lambda) = 0 \Rightarrow 1 - \frac{17\lambda}{6} + \frac{\lambda^2}{6} = 0$$

$$\Rightarrow \lambda = \frac{17 \pm \sqrt{17^2 - 24}}{2} = 16.64,\ 0.36$$

For $\underline{\lambda = 16.64}$: $c_2 = \dfrac{1-7/3\,\lambda}{\lambda} c_1 = -2.27\, c_1$

$$g(s) = 16.64\, c_1 \left[s - \frac{2.27}{s}\right] \quad \text{——— ①}$$

For $\underline{\lambda = 0.36}$,

$$c_2' \simeq 0.44\, c_1'$$

$$\therefore g(s) = 0.36\, c_1' \left[s + \frac{0.44}{s}\right] \quad \text{——— ②}$$

① & ② are eigen function corresponding to eigen values 16.64 & 0.36

③ $g(s) = e^s - s - \int_0^1 s(e^{st}-1)\, g(t)\, dt$

Reduce, $s(e^{st}-1) = s^2t + 1/2\, s^3t^2 + 1/6\, s^4t^3$

$$g(s) = e^s - s - \int_0^1 \left(s^2t + \frac{1}{2}s^3t^2 + \frac{1}{6}s^4t^3\right) g(t)\, dt$$

$$= e^s - s - \left[s^2 c_1 + \frac{s^3}{2}c_2 + \frac{s^4}{6}c_3\right]$$

$$c_1 = \int_0^1 t\, g(t)\, dt$$

$$= \int_0^1 t\left[e^t - t - c_1 t^2 - \frac{c_2}{2}t^3 - \frac{c_3}{6}t^4\right] dt$$

$$= \int_0^1 t e^t\, dt - \frac{1}{3} - \frac{c_1}{4} - \frac{c_2}{10} - \frac{c_3}{36}$$

$$= \left[t e^t - e^t\right]_0^1 - \frac{1}{3} - \frac{c_1}{4} - \frac{c_2}{10} - \frac{c_3}{36}$$

$$= \frac{2}{3} - \frac{c_1}{4} - \frac{c_2}{10} - \frac{c_3}{36}$$

$$\Rightarrow \frac{5c_1}{4} + \frac{c_2}{10} + \frac{c_3}{36} = 2/3 \quad \text{——— (A)}$$

$$c_2 = \int_0^1 t^2\, g(t)\, dt$$

$$= \int_0^1 t^2\left[e^t - t - c_1 t^2 - \frac{c_2}{2}t^3 - \frac{c_3}{6}t^4\right] dt$$

$$= \int_0^1 t^2 e^t\, dt - \frac{1}{4} - \frac{c_1}{5} - \frac{c_2}{12} - \frac{c_3}{42}$$

$$= \left[t^2 e^t - \int 2t e^t\, dt\right]_0^1 - \frac{1}{4} - \frac{c_1}{5} - \frac{c_2}{12} - \frac{c_3}{42}$$

$$= \left[t^2 e^t - 2\{t e^t - e^t\}\right]_0^1 - \frac{1}{4} - \frac{c_1}{5} - \frac{c_2}{12} - \frac{c_3}{42}$$

$$= \left[e - 2e + 2e - 0 + 0 - 2\right] - \frac{1}{4} - \frac{c_1}{5} - \frac{c_2}{12} - \frac{c_3}{42}$$

$$= e - \frac{9}{4} - \frac{c_1}{5} - \frac{c_2}{12} - \frac{c_3}{42}$$

$$\Rightarrow \frac{13c_2}{12} + \frac{c_1}{5} + \frac{c_3}{42} = e - \frac{9}{4} \quad \text{——— (B)}$$

IIly,

$$c_3 = \int_0^1 t^3\, g(t)\, dt$$

$$= \int_0^1 t^3\left[e^t - t - c_1 t^2 - \frac{c_2}{2}t^3 - \frac{c_3}{6}t^4\right] dt$$

$$= \int_0^1 t^3 e^t\, dt - \frac{1}{5} - \frac{c_1}{6} - \frac{c_2}{14} - \frac{c_3}{48}$$

$$c_3 = \left[t^3 e^t - \int 3t^2 e^t\right]_0^1 - \frac{1}{5} - \frac{c_1}{6} - \frac{c_2}{14} - \frac{c_3}{48}$$

$$= e - 3\left[t^2 e^t - 2te^t + 2e^t\right]_0^1 - \frac{1}{5} - \frac{c_1}{6} - \frac{c_2}{14} - \frac{c_3}{48}$$

$$= e - 3\left[e - \cancel{2e} + \cancel{2}e - 2\right] - \frac{1}{5} - \frac{c_1}{6} - \frac{c_2}{14} - \frac{c_3}{48}$$

$$= -2e + \frac{29}{5} - \frac{c_1}{6} - \frac{c_2}{14} - \frac{c_3}{48}$$

$$\Rightarrow \frac{c_1}{6} + \frac{c_2}{14} + \frac{49c_3}{48} = -2e + \frac{29}{5} \quad \text{------} \quad (c)$$

$$D(\lambda) = \begin{vmatrix} \frac{5}{4} & \frac{1}{10} & \frac{1}{36} \\ \frac{1}{5} & \frac{13}{12} & \frac{1}{42} \\ \frac{1}{6} & \frac{1}{14} & \frac{49}{48} \end{vmatrix}$$

Method of Successive Approximation :-

① Solve the integral equation,

$$g(s) = f(s) + \lambda \int_0^1 e^{s-t} g(t)\, dt. \quad \text{①}$$

Solⁿ:

here, $K(s,t) = e^{s-t}$

$\therefore K_1(s,t) = K(s,t) = e^{s-t}$

$$K_2(s,t) = \int_0^1 K_1(s,x) K(x,t)\, dx = \int_0^1 e^{s-x} e^{x-t}\, dx = e^{s-t}$$

Thus, $K_m(s,t) = K_1(s,t) = e^{s-t} \quad \forall\, m \in \mathbb{N}$

Now, $g_0(s) = f(s)$

$$g_1(s) = f(s) + \lambda \int_0^1 e^{s-t} g_0(s)\, dt$$

$$\vdots$$

$$g_n(s) = f(s) + \lambda \int_0^1 e^{s-t} g_{n-1}(s)\, dt$$

On simplification we get,

$$g_n(s) = f(s) + \sum_{m=1}^{n} \lambda^m \int_0^1 K_m(s,t) f(t)\, dt$$

and, $g(s) = \lim_{n\to\infty} g_n(s) = f(s) + \sum_{m=1}^{\infty} \lambda^m \int_0^1 K_m(s,t) f(t)\, dt \quad \text{②}$

where, $K_m(s,t) = \int_0^1 K_{m-1}(s,x) K(x,t)\, dx$

Now, Equⁿ ② can be written as,

$$g(s) = f(s) + \lambda \int_0^1 \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t) f(t)\, dt :$$

take, $\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t)$

$$= \sum_{m=1}^{\infty} \lambda^{m-1} e^{s-t} = \frac{e^{s-t}}{1-\lambda}, \quad \text{for } \lambda \neq 1,\ |\lambda|<1$$

The series $\sum_{m=1}^{\infty} \lambda^{m-1}$ converges only for $|\lambda| < 1$

Hence, the solⁿ of equⁿ (1) is,

$$g(s) = f(s) - \frac{\lambda}{(\lambda-1)} \int_0^1 e^{s-t} f(t)\, dt.$$

(2) Solve the Fredholm Integral Equation,

$$g(s) = 1 + \lambda \int_0^1 (1-3st)\, g(t)\, dt$$

and evaluate the resolvant Kernel.

Solⁿ:- $g_0(s) = f(s) = 1$, $K_1(s,t) = K(s,t) = 1-3st$

$$g_1(s) = f(s) + \lambda \int_0^1 K(s,t)\, g_0(t)\, dt$$

$$= 1 + \lambda \int_0^1 (1-3st)\, dt = 1 + \lambda\left(1 - \frac{3s}{2}\right)$$

$$g_2(s) = 1 + \lambda \int_0^1 (1-3st)\left[1 + \lambda\left(1 - \frac{3t}{2}\right)\right] dt$$

$$= 1 + \lambda\left(1 - \frac{3s}{2}\right) + \lambda^2\left[1 - \frac{3}{4}\right]$$

$$= 1 + \lambda\left(1 - \frac{3s}{2}\right) + \frac{\lambda^2}{4}$$

continuing this,

$$g(s) = 1 + \lambda\left(1 - \frac{3s}{2}\right) + \frac{\lambda^2}{4} + \frac{\lambda^3}{4}\left(1 - \frac{3s}{2}\right) + \frac{\lambda^4}{16} + \frac{\lambda^5}{16}\left(1 - \frac{3}{2}s\right) + \cdots$$

$$= \left(1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{16} + \cdots\right)\left[1 + \lambda\left(1 - \frac{3s}{2}\right)\right]$$

$$= \sum_{n=0}^{\infty} \left(\frac{\lambda}{2}\right)^{2n} \left[1 + \lambda\left(1 - \frac{3s}{2}\right)\right]$$

$$= \frac{1 + \lambda\left(1 - \frac{3s}{2}\right)}{1 - \frac{\lambda^2}{4}}$$

[The series $\sum_{n=0}^{\infty} \left(\frac{\lambda}{2}\right)^{2n}$ is conv. for $|\lambda| < 2$]

$$= \frac{4 + 2\lambda(2-3s)}{4 - \lambda^2}$$

(3) Solve the integral Equ^n,

$$g(s) = 1 + \lambda \int_0^{\pi} \sin(s+t)\, g(t)\, dt$$

Sol^n:- $K(s,t) = \sin(s+t)$, $f(s) = 1$

$$K_1(s,t) = K(s,t) = \sin(s+t)$$

$$K_2(s,t) = \int_0^{\pi} K_1(s,x)\, K(x,t)\, dx$$

$$= \int_0^{\pi} \sin(s+x) \sin(x+t)\, dx$$

$$= \int_0^{\pi} (\sin s \cos x + \cos s \sin x)(\sin x \cos t + \cos x \sin t)$$

$$= \int_0^{\pi} \left[\frac{\sin s \cos t}{2} \sin 2x + \frac{\cos s \cos t}{2}(1 - \cos 2x) + \frac{\cos s \sin t}{2} \sin 2x + \frac{\sin s \sin t}{2}(1 + \cos 2x) \right] dx$$

$$= \int_0^{\pi} \left[\frac{\sin(s+t)}{2} \sin 2x + \frac{\cos(s-t)}{2} - \frac{\cos(s-t)}{2} \cos 2x \right] dx$$

$$= \frac{\sin(s+t)}{2} \times 0 + \frac{\pi}{2} \cos(s-t) - \frac{\cos(s-t)}{2} \times 0$$

$$= \frac{\pi}{2} \cos(s-t)$$

$$K_3(s,t) = \int_0^{\pi} K_2(s,x)\, K(x,t)\, dx$$

$$= \int_0^{\pi} \frac{\pi}{2} \cos(s-x) \sin(x+t)\, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} (\cos s \cos x + \sin s \sin x)(\sin x \cos t + \cos x \sin t)\, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} \left[\frac{\cos s \cos t}{2} \sin 2x + \frac{\sin s \cos t}{2}(1 - \cos 2x) + \frac{\cos s \sin t}{2}(1 + \cos 2x) + \frac{\sin s \sin t}{2} \sin 2x \right]$$

$$K_3(s,t) = \frac{\pi}{2}\int_0^{\pi}\left[\frac{\cos(s+t)}{2}\sin 2x + \frac{\sin(s-t)}{2} - \frac{\sin(s+t)}{2}\cos 2x\right] dx$$

$$= \frac{\pi}{2}\left[\frac{\cos(s+1)}{2}\times 0 + \frac{\pi}{2}\sin(s+t) - \frac{\sin(s+1)}{2}\times 0\right]$$

$$= \left(\frac{\pi}{2}\right)^2 \sin(s+t)$$

$$K_4(s,t) = \int_0^{\pi} K_3(s,x)\, K(x,t)\, dx = \left(\frac{\pi}{2}\right)^2 \int_0^{\pi} \sin(s+x)\sin(x+t)\, dx$$

$$= \left(\frac{\pi}{2}\right)^3 \cos(s-t)$$

$$K_6(s,t) = \left(\frac{\pi}{2}\right)^5 \cos(s-t) \dots$$

$$\text{lly}, \quad K_5(s,t) = \left(\frac{\pi}{2}\right)^4 \sin(s+t);$$

this gives, $K_{2n} = \left(\frac{\pi}{2}\right)^{2n-2} K_2$, $\quad K_{2n+1} = \left(\frac{\pi}{2}\right)^{2n} K_1$

Now,

$$g(s) = f(s) + \lambda \int_0^{\pi} \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t)\, dt$$

$$= 1 + \lambda \int_0^{\pi} [K_1(s,t) + \lambda K_2(s,t) + \lambda^2 K_3(s,t) + \dots]\, dt$$

$$= 1 + \int_0^{\pi} \lambda [K_1 + \lambda K_2 + \lambda^2 K_3 + \dots]$$

$$= 1 + \int_0^{\pi} (\lambda^2 K_2 + \lambda^4 K_4 + \lambda^6 K_6 + \dots) + (\lambda K_1 + \lambda^3 K_3 + \lambda^5 K_5 + \dots)\, dt$$

$$= 1 + \int_0^{\pi} \left[\lambda^2 + \lambda^4 \left(\frac{\pi}{2}\right)^2 + \lambda^6 \left(\frac{\pi}{2}\right)^4 + \dots\right] \frac{\pi}{2}\cos(s-t) + \left[\lambda + \lambda^3 \left(\frac{\pi}{2}\right)^2 + \lambda^5 \left(\frac{\pi}{2}\right)^4 + \dots\right] \sin(s-t)\, dt$$

$$= 1 + \int_0^{\pi} \lambda^2 \frac{\pi}{2}\cos(s-t) + \lambda^2 \left(\frac{\pi}{2}\right)\cos(s-t)\left[\left(\frac{\pi\lambda}{2}\right)^2 + \left(\frac{\pi\lambda}{2}\right)^4 + \dots\right] + \lambda\left[1 + \left(\frac{\pi\lambda}{2}\right)^2 + \left(\frac{\pi\lambda}{2}\right)^4 + \dots\right] \sin(s-t)\, dt$$

$$= \int_0^{\pi} \left[1 + \left(\frac{\pi\lambda}{2}\right)^2 + \left(\frac{\pi\lambda}{2}\right)^4 + \dots\right]\left[\lambda^2 \frac{\pi}{2}\cos(s-t) + \lambda \sin(s-t)\right] dt \quad \cancel{+ \frac{\pi}{2}\cos(s-t)}$$

$$+ 1$$

$$= 1 + \int_0 \frac{\frac{\pi\lambda^2}{2}\cos(s-t) + \lambda\sin(s+t)}{\left(1 - \frac{\pi^2\lambda^2}{4}\right)}\,dt \qquad \text{for } |\lambda| <$$

$$= 1 + \frac{1}{1 - \frac{\pi^2\lambda^2}{4}}\left[\frac{\pi\lambda^2}{2}\frac{\sin(s-t)}{-1} + \lambda\frac{\cos(s+t)}{-1}\right]_0^{\pi}$$

$$= 1 + \frac{1}{\left(1 - \frac{\pi^2\lambda^2}{4}\right)}\left[\frac{\pi\lambda^2}{2}\sin s + \lambda\cos s + \frac{\pi\lambda^2}{2}\sin s + \lambda\cos s\right]$$

$$= 1 + \left[\left(2\lambda\cos s + \lambda^2\pi\sin s\right)\Big/\left(1 - \frac{\lambda^2\pi^2}{4}\right)\right]$$

Ex ③ (i) $g(s) = e^s - \frac{1}{2}e + \frac{1}{2} + \frac{1}{2}\int_0^1 g(t)\, dt.$

Sol^n:- $f(s) = e^s + \frac{(1-e)}{2}$, $K(s,t) = 1$, $\lambda = 1/2$

$K_1(s,t) = K(s,t) = 1$

$K_2(s,t) = \int_0^1 K_1(s,x)\, K(x,t)\, dx = 1$

$\therefore K_m(s,t) = 1, \quad \forall\, m \in \mathbb{N}$

Now, $g(s) = f(s) + \lambda \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t)$

$= f(s) + \lambda \sum_{m=1}^{\infty} \lambda^{m-1}\cdot 1 = f(s) + \frac{\lambda}{\lambda - 1}$, $|\lambda| < 1$.

$\therefore g(s) = e^s + \frac{(1-e)}{2} + \frac{\lambda}{\lambda-1} = e^s + \frac{(1-e)}{2} + \frac{1/2}{-1/2} = e^s - \frac{(e+1)}{2}$

Required.

Iter^n

$g_0(s) = f(s) = e^s + \frac{(1-e)}{2}$

$g_1(s) = f(s) + \frac{1}{2}\int_0^1 e^t + \frac{(1-e)}{2}\, dt$

$= f(s) + \frac{1}{2}\frac{(1-e)}{2} + \frac{1}{2}(e-1)$ $\cancel{= f(s) = e^s + \frac{(1-e)}{2}}$

$= e^s + \frac{1}{4}(1-e)$

$\therefore \cancel{g_n(s) = e^s + \frac{(1-e)}{2} \quad \forall\, n \in \mathbb{N}}$

$\therefore \cancel{g_1(s) = f(s) - \frac{1}{4}(1-e) = e^s + \frac{1}{4}}$

$\cancel{g_2(s) = f(s) + \frac{1}{2}\int_0^1 \left(f(t) - \frac{1}{4}(1-e)\right)} dt$

$= \cancel{f(s) + \frac{1}{2}\int_0^1 e^t + \frac{3}{4}(1-e)}\, dt$

$= \cancel{f(s) + \frac{2}{4}(e-1) + \frac{3}{4}(1-e)}$

$= \cancel{f(s) + \frac{1}{4}(1-e)}$

$g_2(s) = e^s + \frac{1}{8}(1-e)$

$g(s) = g_1(s) + g_2(s) + \cdots$

$=$

(ii) $g(s) = \sin s - \frac{s}{4} + \frac{1}{4}\int_0^{\pi/2} st\, g(t)\, dt$

$f(s) = \sin s - \frac{s}{4}$, $\lambda = 1/4$

$K(s,t) = st = K_1(s,t)$

$$K_2(s,t) = \int_0^{\pi/2} K_1(s,x)\, K(x,t)\, dx$$

$$= \int_0^{\pi/2} sx \cdot xt\, dx = \frac{st}{3} \cdot \left(\frac{\pi}{2}\right)^3$$

$$K_3(s,t) = \int_0^{\pi/2} K_2(s,x)\, K(x,t)\, dx$$

$$= \int_0^{\pi/2} \left(\frac{\pi}{2}\right)^3 \cdot \frac{sx}{3} \cdot xt\, dx$$

$$= \left(\left(\frac{\pi}{2}\right)^3 \cdot \frac{1}{3}\right)^2 st.$$

$$\therefore K_m(s,t) = \left(\frac{\pi^3}{24}\right)^{m-1} st$$

Now,

$$g(s) = f(s) + \lambda \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t)$$

$$= f(s) + \frac{1}{4} \sum_{m=1}^{\infty} \left(\frac{1}{4} \cdot \frac{\pi^3}{24}\right)^{m-1} st$$

$$= f(s) + \frac{st}{4} \cdot \frac{1}{1 - \frac{\pi^3}{96}} \qquad \because \frac{\pi^3}{96} < 1$$

$$= f(s) + \frac{st}{4} \cdot \frac{96}{96 - \pi^3}$$

$$g(s) = \sin s - \frac{s}{4} + \frac{24\, st}{96 - \pi^3}$$

— Required.

2. $g(s) = 1 + \lambda \int_0^1 st\, g(t)\, dt$

(a) make use of the relation $|\lambda| < B^{-1}$ to show that the iterative procedure is valid for $|\lambda| < 3$.

$K(s,t) = st$.

$$B^2 = \int_0^1 \int_0^1 |K(x,t)|^2 \, dx\, dt = \int_0^1 \int_0^1 x^2 t^2 \, dx\, dt$$

$$= \int_0^1 t^2 \frac{1}{3}\, dt = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

$$B^2 = \frac{1}{9} \Rightarrow B = \frac{1}{3}$$

The series converges for $|\lambda| < B^{-1}$ i.e. $|\lambda| < 3$

(b) show that the iterative procedure leads formally to the solution,

$$g(s) = 1 + s\left[\lambda/2 + (\lambda^2/6) + (\lambda^3/18) + \cdots\right]$$

$$g_0(s) = f(s) = 1, \quad K(s,t) = st$$

$$g_1(s) = f(s) + \lambda \int_0^1 st.\, dt = f(s) + \lambda \frac{s}{2} = 1 + s\frac{\lambda}{2}$$

$$g_2(s) = f(s) + \lambda \int_0^1 st\left(1 + \frac{\lambda}{2} t\right) dt = 1 + \frac{\lambda s}{2} + \frac{\lambda^2 s}{6}$$

$\vdots$

$$g_n(s) = 1 + \frac{\lambda s}{2} + \frac{\lambda^2 s}{6} + \frac{\lambda^3 s}{18} + \cdots + \frac{\lambda^n s}{2 \times 3^{n-1}}$$

$\vdots$

$$\therefore g(s) = 1 + s\left(\frac{\lambda}{2} + \frac{\lambda^2}{6} + \frac{\lambda^3}{18} + \cdots\right)$$

— Required.

(c) use the method of the previous chapter to obtain the exact solution, $g(s) = 1 + \left[\frac{3\lambda s}{2(3-\lambda)}\right]$, $\lambda \neq 3$

Sol^n:

$$g(s) = 1 + \lambda \int_0^1 st\, g(t)\, dt$$

$$\Rightarrow g(s) = 1 + \lambda s \int_0^1 t\, g(t)\, dt = 1 + \lambda s\, q, \quad q = \int_0^1 t\, g(t)\, dt$$

$$\therefore g(s) = 1 + \lambda s q$$

$$\Rightarrow q = \int_0^1 t\, g(t)\, dt$$

$$\Rightarrow q = \int_0^1 t\left[1 + \lambda t q\right] dt$$

$$\Rightarrow q = \frac{1}{2} + \lambda q \cdot \frac{1}{3}$$

$$\Rightarrow q\left(1 - \frac{\lambda}{3}\right) = 1/2$$

$$\Rightarrow q = 1/2 \cdot \left(1 - \frac{\lambda}{3}\right)^{-1} = \frac{3}{2} \cdot \frac{1}{(3-\lambda)}$$

Hence, $g(s) = 1 + \lambda s q = 1 + \frac{3\lambda s}{2(3-\lambda)}$

Required.

(3) $g(s) = 1 + \lambda \int_0^1 (s+t)\, g(t)\, dt$ solve by method of successive app.

show that the estimate afforded by the relation $|\lambda| < B^{-1}$ is conservative in this case.

Sol^n: $K(s,t) = s + t = K_1(s,t)$

$$K_2(s,t) = \int_0^1 K_1(s,x)\, K(x,t)\, dx = \int_0^1 (s+x)(x+t)\, dx$$

$$= \int_0^1 \left(x(s+t) + x^2 + st\right) dx$$

$$= \frac{(s+t)}{2} + \frac{1}{3} + st$$

$g_0(s) = 1 = f(s)$

$g_1(s) = 1 + \lambda \int_0^1 (s+t)\,dt = 1 + \lambda\left(s + 1/2\right)$

$g_2(s) = 1 + \lambda \int_0^1 (s+t)\left[1 + \lambda\left(t + 1/2\right)\right] dt$

$= 1 + \lambda\left(s + 1/2\right) + \lambda^2 \int_0^1 \left(t\left(s + 1/2\right) + t^2 + s/2\right) dt$

$= 1 + \lambda\left(s + 1/2\right) + \lambda^2 \left[\left(s + 1/2\right) \cdot \frac{1}{2} + 1/3 + s/2\right]$

$\cancel{= 1 + \lambda\left(s + 1/2\right) + \lambda^2\left[s + \frac{7}{12}\right]}$

$\cancel{g_3(s) = 1 + \lambda \int_0^1 (s+t)\left[1 + \lambda\left(t + 1/2\right) + \lambda^2\left(t + 7/12\right)\right] dt}$

$\cancel{g_3(s) = 1 + \lambda\left(s + 1/2\right) + \lambda^2\left(s + \frac{7}{12}\right) + \lambda^3 \int_0^1 \left(s + \frac{7}{12}\right) t + t^2 + \frac{7s}{12}\, dt}$

$\cancel{= 1 + \lambda\left(s + \frac{1}{2}\right) + \lambda^2\left(s + \frac{7}{12}\right) + \lambda^3\left[\left(s + \frac{7}{12}\right)\frac{1}{2} + \frac{1}{3} + \frac{7s}{12}\right]}$

$g_2(s) = 1 + \lambda\left(s + 1/2\right) + \lambda^2\left(s + 7/12\right)$

$g_3(s) = 1 + \lambda \int_0^1 (s+t)\left[1 + \lambda\left(t + \frac{1}{2}\right) + \lambda^2\left(t + \frac{7}{12}\right)\right] dt$

$g_3(s) = 1 + \lambda\left(s + 1/2\right) + \lambda^2\left(s + 7/12\right) + \lambda^3 \int_0^1 \left[\left(s + \frac{7}{12}\right)t + t^2 + \frac{7s}{12}\right] dt$

$g_3(s) = 1 + \lambda\left(s + \frac{1}{2}\right) + \lambda^2\left(s + \frac{7}{12}\right) + \lambda^3\left[\left(s + \frac{7}{12}\right)\frac{1}{2} + \frac{1}{3} + \frac{7s}{12}\right]$

$= 1 + \lambda\left(s + \frac{1}{2}\right) + \lambda^2\left(s + \frac{7}{12}\right) + \lambda^3\left[\frac{13s}{12} + \frac{7}{24} + \frac{8}{24}\right]$

$= 1 + \lambda\left(s + \frac{1}{2}\right) + \lambda^2\left(s + \frac{7}{12}\right) + \lambda^3\left[\frac{26s}{24} + \frac{15}{24}\right]$

$= 1 + \lambda\left(s + \frac{1}{2}\right) + \lambda^2\left(s + \frac{7}{12}\right) + \lambda^3\left(\frac{26s + 15}{24}\right)$

4. Find the resolvant kernel associated with the following kernels: (i) $k(s,t) = |s-t|$ in the interval $(0,1)$

(ii) $k(s,t) = \exp(-|s-t|)$ in $(0,1)$

(iii) $k(s,t) = \cos(s+t)$ in $(0,2\pi)$

(i) $\cancel{k(s,t) = |s-t| = k_1(s,t)}$

$\cancel{k_2(s,t) = \int_0^1 k_1(s,x)\, k(x,t)\, dx}$

$\cancel{= \int_0^1 |s-x||x-t|\, dx}$

(iii) $k(s,t) = \cos(s+t) = k_1(s,t)$

$$k_2(s,t) = \int_0^{2\pi} k_1(s,x)\, k(x,t)\, dx$$

$$= \int_0^{2\pi} \cos(s+x)\cos(x+t)\, dx$$

$$= \int_0^{2\pi} (\cos s \cos x - \sin s \sin x)(\cos x \cos t - \sin x \sin t)\, dx$$

$$= \int_0^{2\pi} \Big[\frac{\cos s \cos t}{2}(1+\cos 2x) - \frac{\sin s \cos t}{2}\sin 2x - \frac{\cos s \sin t}{2}\sin 2x + \frac{\sin s \sin t}{2}(1-\cos 2x)\Big] dx$$

$$= \int_0^{2\pi} \Big[\frac{1}{2}\cos(s-t) + \frac{1}{2}\cos(s+t)\cos 2x - \frac{1}{2}\sin(s+t)\sin 2x\Big] dx$$

$$= \pi \cos(s-t)$$

$$k_3(s,t) = \int_0^{2\pi} k_2(s,x)\, k(x,t)\, dx$$

$$= \int_0^{2\pi} \pi \cos(s-x)\cos(x+t)\, dx$$

$$= \pi \int_0^{2\pi} (\cos s \cos x + \sin s \sin x)(\cos x \cos t - \sin x \sin t)\, dx$$

$$\pi \int_0^{2\pi} \Big[\frac{\cos s \cos t}{2}(1+\cos 2x) + \frac{\sin s \cos t}{2}\sin 2x - \frac{\cos s \sin t}{2}\sin 2x - \frac{\sin s \sin t}{2}(1-\cos 2x)\Big] dx$$

$$= \pi \int_0^{2\pi} \left[\frac{1}{2}\cos(s+t) + \frac{1}{2}\cos(s-t)\cos 2x + \frac{1}{2}\sin(s-t)\sin 2x \right] dx$$

$$= \pi \cdot \pi \cos(s+t)$$

$$= \pi^2 \cos(s+t).$$

$$\therefore K_1(s,t) = \cos(s+t)$$

$$K_2(s,t) = \pi \cos(s-t)$$

$$K_3(s,t) = \pi^2 \cos(s+t)$$

$$\text{III}^{ly},\ K_4(s,t) = \pi^3 \cos(s-t)$$

hence,
$$K_n(s,t) = \begin{cases} \pi^{n-1}\cos(s+t), & n = \text{odd} \\ \pi^{n-1}\cos(s-t), & n = \text{even} \end{cases}$$

i.e.
$$K_n(s,t) = \begin{cases} \pi^{n-1} K_1, & n = \text{odd} \\ \pi^{n-2} \cdot K_2, & n = \text{even} \end{cases}$$

Now,
$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t)$$

$$\Gamma(s,t;\lambda) = K_1 + \lambda K_2 + \lambda^2 K_3 + \cdots \cdots$$

$$= (K_1 + \lambda^2 K_3 + \lambda^4 K_5 + \cdots) + (\lambda K_2 + \lambda^3 K_4 + \lambda^5 K_6 + \cdots)$$

$$= K_1 (1 + \lambda^2\pi^2 + \lambda^4\pi^4 + \cdots) + \lambda K_2 (1 + \lambda^2\pi^2 + \lambda^4\pi^4 + \cdots)$$

$$= (1 + \lambda^2\pi^2 + \lambda^4\pi^4 + \cdots)(K_1 + \lambda K_2)$$

$$= \frac{1}{1 - \lambda^2\pi^2}\left[\cos(s+t) + \lambda\pi\cos(s-t)\right], \quad |\lambda\pi| < 1$$

① Find the Neumann series for the solⁿ of the I.E

$$g(s) = (1+s) + \lambda \int_0^s (s-t)\, g(t)\, dt.$$

$$K_1(s,t) = s-t$$

$$K_2(s,t) = \int_t^s (s-x)(x-t)\, dx = \int_t^s \left[x(s+t) - x^2 - st\right] dx$$

$$= (s+t)\frac{(s^2-t^2)}{2} - \frac{1}{3}(s^3-t^3) - st(s-t)$$

$$= \frac{3(s^3 - st^2 + s^2t - t^3) - 2(s^3-t^3) - 6(s^2t - st^2)}{\lfloor 3}$$

$$= \frac{(s-t)^3}{\lfloor 3}$$

$$K_3(s,t) = \int_t^s (s-x)\frac{(x-t)^3}{\lfloor 3}\, dx$$

$$= \int_t^s s\frac{(x-t)^3}{\lfloor 3}\, dx - \int_t^s (x-t+t)\frac{(x-t)^3}{\lfloor 3}\, dx$$

$$= \int_t^s (s-t)\frac{(x-t)^3}{\lfloor 3}\, dx - \int_t^s \frac{(x-t)^4}{\lfloor 3}\, dx$$

$$= (s-t)\left[\frac{(x-t)^4}{\lfloor 4}\right]_t^s - \left[\frac{(x-t)^5}{5\lfloor 3}\right]_t^s$$

$$= \frac{(s-t)^5}{\lfloor 4} - \frac{1}{5}\frac{(s-t)^5}{\lfloor 3} = \frac{5(s-t)^5}{\lfloor 5} - \frac{4(s-t)^5}{\lfloor 5}$$

$$= \frac{(s-t)^5}{\lfloor 5}$$

Now,

$$g(s) = f(s) + \lambda \int_0^s \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t)\, f(t)\, dt$$

$$\therefore g(s) = (1+s) + \lambda \int_0^s \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t)\,(1+s)\, dt$$

$$\therefore g(s) = 1+s+\lambda\left[\int_0^s K_1(1+t)\,dt + \int_0^s \lambda K_2(1+t)\,dt + \dots\right]$$

Now, $$\int_0^s K_1(1+t)\,dt = \int_0^s (s-t)(1+t)\,dt = \left[\frac{(s-t)^2}{-2} + \frac{st^2}{2} - \frac{t^3}{3}\right]$$

$$= 0 + \frac{s^3}{2} - \frac{s^3}{3} + \frac{s^2}{2}$$

$$= \frac{s^2}{2} + \frac{s^3}{6} = \frac{s^2}{\lfloor 2} + \frac{s^3}{\lfloor 6}$$

$$\int_0^s K_2(1+t)\,dt = \int_0^s \frac{(s-t)^3}{\lfloor 3}(1+t)\,dt$$

$$= \cancel{\frac{1}{\lfloor 3}\left[\frac{(s-t)^4}{4}\right]_0^s - \frac{1}{\lfloor 3}\int_0^s (t-s+s)}(t-s)^3\,dt$$

$$= \cancel{\frac{1}{\lfloor 4}(+s^4)} + \frac{s^5}{5\lfloor 3} - \frac{s}{\lfloor 3}\left(-\frac{s^4}{4}\right)$$

$$= \cancel{+\frac{s^4}{\lfloor 4} + \frac{4s^5}{\lfloor 5}} + \frac{5s^5}{\lfloor 5}$$

$$= \int_0^s \frac{(s-t)^3}{\lfloor 3} - (t-s+s)\frac{(t-s)^3}{\lfloor 3}\,dt$$

$$= \int_0^s \frac{(s-t)^3}{\lfloor 3} - \frac{(t-s)^4}{\lfloor 3} - s\frac{(t-s)^3}{\lfloor 3}\,dt$$

$$= \left[\frac{(s-t)^4}{-\lfloor 4} - \frac{(t-s)^5}{5\lfloor 3} - \frac{s(t-s)^4}{\lfloor 4}\right]_0^s$$

$$= \left[0 - 0 - 0 + \frac{s^4}{\lfloor 4} - \frac{s^5}{5\lfloor 3} + \frac{s^5}{\lfloor 4}\right.$$

Alt

$$\int_0^s \left(\frac{(s-t)^3}{\lfloor 3} - (s-t-s)\frac{(s-t)^3}{\lfloor 3}\right)dt$$

$$= \int_0^s \frac{(s-t)^3}{\lfloor 3} - \frac{(s-t)^4}{\lfloor 3} + s\frac{(s-t)^3}{\lfloor 3}\,dt$$

$$= \left[\frac{(s-t)^4}{\dots}\right.$$

$$\int_0^s k_2(1+t)\,dt = \int_0^s \frac{(s-t)^3}{\lfloor 3}(1+t)\,dt$$

$$= \int_0^s \left[\frac{(s-t)^3}{\lfloor 3} + t\,\frac{(s-t)^3}{\lfloor 3}\right] dt$$

$$= \int_0^s \left[\frac{(s-t)^3}{\lfloor 3} - (s-t-s)\,\frac{(s-t)^3}{\lfloor 3}\right] dt$$

$$= \int_0^s \left[\frac{(s-t)^3}{\lfloor 3} - \frac{(s-t)^4}{\lfloor 3} + s\,\frac{(s-t)^3}{\lfloor 3}\right] dt$$

$$= \left[\frac{(s-t)^4}{-\lfloor 4} + \frac{(s-t)^5}{5\lfloor 3} - \frac{s(s-t)^4}{\lfloor 4}\right]_0^s$$

$$= \frac{s^4}{\lfloor 4} - \frac{4s^5}{\lfloor 5} + \frac{s^5}{\lfloor 4} = \frac{s^4}{\lfloor 4} - \frac{4s^5}{\lfloor 5} + \frac{5s^5}{\lfloor 5}$$

$$= \frac{s^4}{\lfloor 4} + \frac{s^5}{\lfloor 5}$$

Thus,

$$q(s) = 1 + s + \lambda\left(\frac{s^2}{\lfloor 2} + \frac{s^3}{\lfloor 3}\right) + \lambda^2\left(\frac{s^4}{\lfloor 4} + \frac{s^5}{\lfloor 5}\right) + \cdots$$

For $\lambda = 1$,

$$q(s) = 1 + s + \frac{s^2}{\lfloor 2} + \frac{s^3}{\lfloor 3} + \cdots$$

$$= e^s.$$

Ex :- (5) (i) $g(s) = 1 + \int_0^s (s-t)\, g(t)\, dt$.

$$K(s,t) = (s-t) = K_1(s,t)$$

$$K_2(s,t) = \int_t^s K(s,x)\, K_1(x,t)\, dx = \int_t^s (s-x)(x-t)\, dx$$

$$= \int_t^s \left[x(s+t) - x^2 - st\right] dx$$

$$= \left[\frac{(s+t)}{2}x^2 - \frac{1}{3}x^3 - st\,x\right]_t^s$$

$$= \frac{(s+t)}{2}(s^2-t^2) - \frac{1}{3}(s^3-t^3) - st(s-t)$$

$$= \frac{3s^3 + 3s^2t - 3st^2 - 3t^3 - 2s^3 + 2t^3 - 6s^2t + 6st^2}{6}$$

$$= \frac{s^3 + 3st^2 - 3s^2t - t^3}{6} = \frac{(s-t)^3}{\lfloor 3}$$

$$K_3(s,t) = \int_t^s K(s,x)\, K_2(x,t)\, dx$$

$$= \int_t^s (s-x)\frac{(x-t)^3}{\lfloor 3}\, dx$$

$$= \frac{1}{\lfloor 3}\int_t^s s(x-t)^3 - (x-t+t)(x-t)^3\, dx$$

$$= \frac{1}{\lfloor 3}\int_t^s \left[(s-t)(x-t)^3 - (x-t)^4\right] dx$$

$$= \frac{1}{\lfloor 3}\left[(s-t)\frac{(x-t)^4}{4} - \frac{(x-t)^5}{5}\right]_t^s$$

$$= \frac{1}{\lfloor 3}\left[\frac{(s-t)^5}{4} - \frac{(s-t)^5}{5}\right] = \frac{(s-t)^5}{\lfloor 5}$$

Thus, $K_m(s,t) = \dfrac{(s-t)^{2m-1}}{\lfloor 2m-1}$

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t)$$

$$= \sum_{m=1}^{\infty} \lambda^{m-1} \frac{(s-t)^{2m-1}}{\lfloor 2m-1 \rfloor}$$

6. Find an approximate solution of the I.E
$g(s) = (\sinh s) + \int_0^s e^{t-s} g(t)\,dt$ by method of iteration

Sol^n: here, $f(s) = \sinh s = \frac{e^s - e^{-s}}{2} = s + \frac{s^3}{\lfloor 3} + \frac{s^5}{\lfloor 5} + \cdots$

$g_0(s) = f(s)$

$$g_1(s) = \sinh s + \int_0^s e^{t-s}\left(\frac{e^t - e^{-t}}{2}\right) dt$$

$$= \sinh s + \int_0^s \frac{e^{2t-s} - e^{-s}}{2}\,dt$$

$$= \sinh s + \frac{1}{2}\left[\frac{e^{2t-s}}{2} - e^{-s}t\right]_0^s$$

$$= \frac{e^s - e^{-s}}{2} + \frac{1}{2}\left[\frac{e^s}{2} - s e^{-s} - \frac{e^{-s}}{2}\right]$$

$$= \sinh s + \frac{1}{2}\left[\sinh s - s e^{-s}\right]$$

$$= \frac{3}{2}\sinh s - \frac{s}{2}e^{-s} = \frac{3}{2}\left(\frac{e^s - e^{-s}}{2}\right) - \frac{s}{2}e^{-s}$$

$$g_2(s) = \sinh s + \int_0^s e^{t-s}\left[\frac{3}{2}\left(\frac{e^t - e^{-t}}{2}\right) - \frac{t}{2}e^{-t}\right] dt$$

$$= \sinh s + \frac{3}{2}\left[\frac{1}{2}\sinh s - \frac{s}{2}e^{-s}\right] - \frac{1}{2}\int_0^s e^{-s}t\,dt$$

$$= \sinh s + \frac{3}{4}\sinh s - \frac{3}{4}s e^{-s} - \frac{e^{-s}}{4}s^2$$

$$= \sinh s + \frac{3}{4}\sinh s - \frac{3}{4}s e^{-s} - \frac{e^{-s}s^2}{4}$$

$$g_3(s) = \sinh s + \int_0^s e^{t-s}\left[\frac{7}{4}\left(\frac{e^t - e^{-t}}{2}\right) - \frac{3}{4}t e^{-t} - \frac{t^2 e^{-t}}{4}\right] dt$$

$$= \sinh s + \frac{7}{4}\left[\frac{1}{2}\sinh s - s e^{-s}\right] - \frac{3}{4}e^{-s}\frac{s^2}{2} - \frac{1}{4}e^{-s}\cdot\frac{s^3}{3}$$

$$= \sinh s + \frac{7}{8}\sinh s - \frac{7}{4}s e^{-s} - \frac{3}{8}s^2 e^{-s} - \frac{1}{12}s^3 e^{-s}$$

Ex-2 $g(s) = 1 + \lambda \int_0^{\pi} \sin(s+t)\, g(t)\, dt.$

$$\Gamma(s,t;\lambda) = \frac{D(s,t;\lambda)}{D(\lambda)}, \quad D(s,t;\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{L^n} \lambda^n B_n(s,t)$$

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{L^n} C_n \lambda^n$$

$$B_0(s,t) = K(s,t)$$
$$= \sin(s+t)$$

$$C_0 = 1, \quad C_n = \int_0^{\pi} B_{n-1}(s,s)\, ds$$

$$B_n(s,t) = C_n K(s,t) - n \int_0^{\pi} K(s,x) B_{n-1}(x,t)\, dx$$

$$C_1 = \int_0^{\pi} B_0(s,s)\, ds = \int_0^{\pi} \sin 2s\, ds = \left[-\frac{\cos 2s}{2}\right]_0^{\pi}$$
$$= \frac{1}{2} - \frac{1}{2} = 0$$

$$B_1(s,t) = -\int_0^{\pi} \sin(s+x)\, \sin(x+t)\, dx$$

$$= -\int_0^{\pi} (\sin s \cos x + \cos s \sin x)(\sin x \cos t + \cos x \sin t)\, dx$$

$$= -\int_0^{\pi} \left[\frac{\sin s \cos t}{2} \sin 2x + \frac{\cos s \cos t}{2}(1 - \cos 2x) + \frac{\sin s \sin t}{2}(1 + \cos 2x) + \frac{\cos s \sin t}{2} \sin 2x\right] dx$$

$$= -\int_0^{\pi} \left[\frac{\sin(s+t)}{2} \sin 2x + \frac{\cos(s-t)}{2} - \frac{\cos(s+t)}{2} \cos 2x\right] dx$$

$$= -\frac{\cos(s-t)}{2} \cdot \pi = -\frac{\pi}{2} \cos(s-t)$$

$$C_2 = \int_0^{\pi} -\frac{\pi}{2} \cdot ds = -\frac{\pi^2}{2}$$

$$B_2 = -\frac{\pi^2}{2}\sin(s+t) - 2\int_0^{\pi} \sin(s+x)\left(-\frac{\pi}{2}\cos(x-t)\right) dx$$

$$= -\frac{\pi^2}{2}\sin(s+t) + \pi\int_0^{\pi} \sin(s+x)\cos(x-t)\, dx$$

$$= -\frac{\pi^2}{2}\sin(s+t) + \pi\int_0^{\pi} (\sin s\cos x + \cos s\sin x)(\cos x\cos t + \sin x\sin t)\, dx$$

$$= -\frac{\pi^2}{2}\sin(s+t) + \pi\int_0^{\pi}\left[\frac{\sin s\cos t}{2}(1+\cos 2x) + \frac{\cos s\cos t}{2}\sin 2x + \frac{\sin s\sin t}{2}\sin 2x + \frac{\cos s\sin t}{2}(1-\cos 2x)\right] dx$$

$$= -\frac{\pi^2}{2}\sin(s+t) + \pi\int_0^{\pi}\left[\frac{\cos(s-t)}{2}\sin 2x + \frac{\sin(s+t)}{2} + \frac{\sin(s-t)}{2}\cos 2x\right] dx$$

$$= -\frac{\pi^2}{2}\sin(s+t) + \pi\left[\frac{\cos(s-t)}{2}\cdot\frac{-\cos 2x}{2} + \frac{\sin(s+t)}{2}x + \frac{\sin(s-t)}{2}\cdot\frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= -\frac{\pi^2}{2}\sin(s+t) + \pi\cdot\frac{\pi}{2}\sin(s+t) = 0$$

$\therefore\ C_n = 0 \quad \forall\ n \geq 3$

$B_n = 0 \quad \forall\ n \geq 2$

$$\therefore\ D(s,t;\lambda) = \sin(s+t) + \frac{\lambda\pi}{2}\cos(s-t)$$

$$D(\lambda) = 1 + \frac{\lambda^2}{2}\cdot\left(-\frac{\pi^2}{2}\right) = 1 - \frac{\lambda^2\pi^2}{4}$$

$\therefore$ Resolvant kernel,

$$\Gamma(s,t;\lambda) = \frac{\sin(s+t) + \frac{\pi}{2}\lambda\cos(s-t)}{1 - \frac{\lambda^2\pi^2}{4}}$$