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Functional Analysis

(1) Functional Analysis by B. V. LIMAYE, New AGE International publications

(2) Functional Analysis, A first course, by M. T. NAIR, PHI publication

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Let $K = \mathbb{R}$ or $\mathbb{C}$ be field.

Let X be a linear space over the field K. A norm on X is a function $\|\cdot\| : X \longrightarrow \mathbb{R}^+$ such that for all $x, y \in X$, $\alpha \in K$,

(i) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$.

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$

(iii) $\|x+y\| \leq \|x\| + \|y\|$.

A normed linear space X is a linear space with a norm $\|\cdot\|$ on it. It is denoted by $(X, \|\cdot\|)$

Ex: let $K^n = \{ (x_1, x_2, \ldots x_n) \mid x_i \in K, \ i = 1, \ldots n \}$ of all n-tuples of numbers in K such that

$x = (x_1, x_2, \ldots x_n) \in K^n$

$y = (y_1, y_2 \ldots y_n) \in K^n$

$$x+y = (x_1+y_1, x_2+y_2, \dots x_n+y_n)$$

$$\alpha x = (\alpha x_1, \alpha x_2, \dots \alpha x_n),$$
$\alpha \in K$.

K^n is a linear space.

[$x = (x_1, x_2 \dots x_n)$
or $(x(1), x(2), \dots x(n))$]

Define
$$\|x\|_1 = \sum_{i=1}^{n} |x(i)|$$

$\because |x(i)| \geq 0 \quad \forall\, i = 1, 2 \dots n$

$$\Rightarrow \sum_{i=1}^{n} |x(i)| \geq 0$$

$$\Rightarrow \|x\|_1 \geq 0$$

let

$$\|x\|_1 = 0$$

$$\Rightarrow \sum_{i=1}^{n} |x(i)| = 0$$

$$\Rightarrow |x(i)| = 0 \ \forall \ i = 1 \ldots n$$

$$\Rightarrow x(i) = 0, \ \forall \ i = 1 \ldots n$$

$$\Rightarrow x = (0, 0 \ldots 0).$$

We know that $\forall \ i = 1, \ldots n$

$$|x(i) + y(i)| \leq |x(i)| + |y(i)|$$

$$\Rightarrow \sum_{i=1}^{n} |x(i) + y(i)| \leq \sum_{i=1}^{n} |x(i)| + \sum_{i=1}^{n} |y(i)|$$

$$\Rightarrow \|x+y\|_1 \leq \|x\|_1 + \|y\|_1$$

where $y = (y(1), y(2) \ldots y(n))$

Now for any $\alpha \in K$,

$$\alpha x = (\alpha x(1), \alpha x(2) \ldots \alpha x(n))$$

$$\|\alpha x\|_1 = \sum_{i=1}^{n} |\alpha x(i)|$$

$$= \sum_{i=1}^{n} |\alpha| \, |x(i)|$$

$$= |\alpha| \sum_{i=1}^{n} |x(i)|$$

$$= |\alpha| \, \|x\|_1$$

$\therefore$ $\|\cdot\|_1$ is a norm on K^n.

Ex: $X = K^n$, for any $x = (x_1, x_2, \dots x_n)$

$$\|x\|_\infty = \max_{i=1,\dots n} |x(i)|$$

Show that $(K^n, \|\cdot\|_\infty)$ is a normed linear space (n.l.s).

Ex: $X = C[0\ 1]$ be a linear space of all continuous functions defined on $[0\ 1]$.

For $f, g \in C[0\ 1]$, $\alpha \in \mathbb{R}$

$$(f+g)(t) = f(t) + g(t) \quad \forall t \in [0\ 1]$$

$$(\alpha f)(t) = \alpha f(t).$$

For any $f \in C[0\ 1]$,

$$\|f\|_1 = \int_0^1 |f(t)|\, dt$$

$$\|f\|_\infty = \max_{t \in [0\ 1]} |f(t)|.$$

$$\left[\ \forall t \quad |(f+g)(t)| \le |f(t)| + |g(t)| \right.$$

$$\le \max_t |f(t)| + \max_t |g(t)|$$

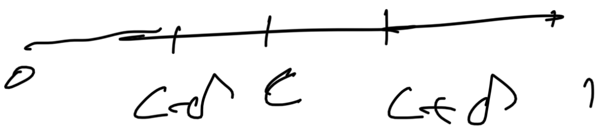
$$= \|f\|_1 + \|g\|_1$$

$$\Rightarrow \max_t |(f+g)(t)| \le \|f\|_1 + \|g\|_1$$

$$\Rightarrow \|f+g\|_1 \le \|f\|_1 + \|g\|_1$$

* $\int_0^1 |f(t)|\,dt = 0 \overset{?}{\Longrightarrow} f = 0$

if $f(t) \neq 0$ at $c \in [0,1]$



$\because f \in C[0,1]$, we can find $\delta > 0$

$\ni \; f(t) > 0, \; \forall t \in (c-\delta, c+\delta)$

$$\int_0^1 |f(t)|\,dt \geq \int_{c-\delta}^{c+\delta} |f(t)| \geq \beta \int_{c-\delta}^{c+\delta} dt$$

$$= \beta \cdot 2\delta > 0$$

when $f(t) \geq \beta > 0$ on $(c-\delta, c+\delta)$

$$0 < \int_{c-\delta}^{c+\delta} \beta \leq \int_{c-\delta}^{c+\delta} |f(t)|\,dt$$

which is contradiction to

$$\int_0^1 |f(t)|\,dt = 0.$$

$\therefore \; f \equiv 0$ on $C[0,1]$.

$$\|\alpha f\|_1 = \int_0^1 |\alpha f(t)|\, dt$$

$$= \int_0^1 |\alpha|\, |f(t)|\, dt$$

$$= |\alpha| \int_0^1 |f(t)|\, dt$$

$$= |\alpha|\, \|f\|_1 .$$

$\therefore$ $(C[0,1], \|\cdot\|_1)$ is a n.l.s.

Illy on $X = C[0,1]$,

define

$$\|f\|_\infty = \max_{t \in [0,1]} |f(t)| .$$

Then $(C[0,1], \|\cdot\|_\infty)$ is also a normed linear space.

Theorem: (i) For any $x = (x_1, x_2, \ldots x_n)$ and $y = (y_1, y_2, \ldots y_n)$ in K^n,

$$\sum_{i=1}^{n} |x(i)y(i)| \leq \left(\sum_{i=1}^{n} |x(i)|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |y(i)|^2\right)^{\frac{1}{2}} \quad \text{--- } (*)$$

(ii) For any $x, y \in C[a,b]$,

$$\int_a^b |x(t)y(t)| \, dt \leq \left(\int_a^b |x(t)|^2\right)^{\frac{1}{2}} \left(\int_a^b |y(t)|^2\right)^{\frac{1}{2}}.$$

Proof:

(i) For $x = (x_1, x_2, \ldots x_n) \in K^n$,

define $\|x\|_2 = \left(\sum_{i=1}^{n} |x(i)|^2\right)^{\frac{1}{2}}$.

For $x = (x_1, x_2, \ldots x_n) = (0, 0 \ldots 0)$, the inequality $(*)$ is true.

So let $x \neq 0$, $y \neq 0$ in $\mathbb{R}^n$.

Then
$$\|x\|_2 = \left(\sum_{i=1}^{n} |x(i)|^2\right)^{1/2} \neq 0$$

&
$$\|y\|_2 = \left(\sum_{i=1}^{n} |y(i)|^2\right)^{1/2} \neq 0$$

Also for any $a, b \in \mathbb{R}$,

$$ab \leq \frac{a^2 + b^2}{2}$$

Letting $a = \dfrac{|x(i)|}{\|x\|_2}$, $b = \dfrac{|y(i)|}{\|y\|_2}$,

we have

$$\frac{|x(i)|}{\|x\|_2} \cdot \frac{|y(i)|}{\|y\|_2} \leq \frac{1}{2}\left[\frac{|x(i)|^2}{\|x\|_2^2} + \frac{|y(i)|^2}{\|y\|_2^2}\right]$$

Now taking summation for $i = 1, 2 \ldots n$,

$$\sum_{i=1}^{n} \frac{|x(i)\, y(i)|}{\|x\|_2 \|y\|_2} \leq \frac{1}{2}\left\{ \frac{\cancel{\sum_{i=1}^{n} |x(i)|^2}}{\cancel{\|x\|_2^2}} + \frac{\cancel{\sum_{i=1}^{n} |y(i)|^2}}{\cancel{\|y\|_2^2}} \right\}$$

$$= \frac{1}{2}[1 + 1]$$

$$= 1$$

$$\Rightarrow \sum_{i=1}^{n} |x(i)\, y(i)| \leq \|x\|_2 \|y\|_2 .$$

(ii) Define for any $x \in C[a, b]$,

$$\|x\|_2 = \left(\int_a^b |x(t)|^2 dt \right)^{1/2}$$

$$\Rightarrow \|x\|_2^2 = \int_a^b |x(t)|^2 dt .$$

Assume $x \neq 0$, $y \neq 0$ in $C[a, b]$.

let $a = \dfrac{|x(t)|}{\|x\|_2}$, $b = \dfrac{|y(t)|}{\|y\|_2}$

$\forall\, t \in [a, b]$.

Using

$$ab \leq \frac{1}{2}\left[a^2 + b^2\right]$$

for any $a, b \in \mathbb{R}$,

we have

$$\frac{|x(t)|\,|y(t)|}{\|x\|_2\,\|y\|_2} \leq \frac{1}{2}\left[\frac{|x(t)|^2}{\|x\|_2^2} + \frac{|y(t)|^2}{\|y\|_2^2}\right]$$

Integrate on both side from a to b,

$$\int_a^b \frac{|x(t)\,y(t)|}{\|x\|_2\,\|y\|_2} \leq \frac{1}{2}\left[\frac{\int_a^b |x(t)|^2\,dt}{\|x\|_2^2} + \frac{\int_a^b |y(t)|^2}{\|y\|_2^2}\right]$$

$$= \frac{1}{2}\left[1 + 1\right]$$

$$= 1$$

$$\Rightarrow \int_a^b |x(t)\, y(t)|\, dt \leq \|x\|_2 \|y\|_2 .$$

Problem: let $X = K^n$,
for any $x = (x_1, x_2, \dots x_n) \in X$,
define $\|x\|_2 = \left(\sum_{i=1}^{n} |x(i)|^2\right)^{1/2}$.

Then $\|\cdot\|_2$ is a norm on K^n.

Sol

(i) $\because |x(i)|^2 \geq 0 \quad \forall\, i = 1, \dots n$

$$\Rightarrow \sum_{i=1}^{n} |x(i)|^2 \geq 0$$

$$\Rightarrow \|x\|_2 \geq 0 .$$

And $\|x\|_2 = 0 \Rightarrow \sum_{i=1}^{n} |x(i)|^2 = 0$

$$\Rightarrow |x(i)|^2 = 0, \ i = 1, \dots n$$

$\Rightarrow x(i) = 0 \quad \forall i = 1, \ldots n.$

$\therefore x = (0, 0, \ldots 0).$

For any $\alpha \in K,$

$$\|\alpha x\|_2 = \left(\sum_{i=1}^{n} |\alpha x(i)|^2 \right)^{1/2}$$

$$= \left(\sum_{i=1}^{n} \left(|\alpha| \, |x(i)| \right)^2 \right)^{1/2}$$

$$= |\alpha| \left(\sum_{i=1}^{n} |x(i)|^2 \right)^{1/2}$$

$$= |\alpha| \, \|x\|_2 .$$

Now for any $x = (x_1, x_2, \ldots x_n)$
$y = (y_1, y_2, \ldots y_n)$

we have

$$\|x+y\|_2^2 = \sum_{i=1}^{n} |x(i)+y(i)|^2$$

$$\leq \sum_{i=1}^{n} \left[|x(i)|+|y(i)|\right]^2$$

$$= \sum_{i=1}^{n} \left[|x(i)|^2 + |y(i)|^2 + 2|x(i)|\,|y(i)|\right]$$

$$= \sum_{i=1}^{n} |x(i)|^2 + \sum_{i=1}^{n} |y(i)|^2 + 2\sum_{i=1}^{n} |x(i)y(i)|$$

$$\leq \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2 .$$ (by previous theorem)

$$= \left[\|x\|_2 + \|y\|_2\right]^2$$

$\therefore$ $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$

$\therefore$ $(K^n, \|\cdot\|_2)$ is a n.l.s.

Illy we can prove that $(C[a,b], \|\cdot\|_2)$ is also a normed linear space, where

$$\|x\|_2 = \left(\int_a^b |x(t)|^2\right)^{1/2}, \quad x \in C[a,b].$$

Lemma: Let p and q be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for every positive real numbers a and b, we have

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof: Note that a function φ is a convex function on a interval J, if for any $\alpha, \beta \in J$, and $\lambda, \mu \in [0, 1]$, $\lambda + \mu = 1$,

$$\varphi(\lambda\alpha + \mu\beta) \leq \lambda\varphi(\alpha) + \mu\varphi(\beta).$$

letting $\varphi(t) = e^t$, $t \geq 0$, which is a convex function, we have

$$e^{\lambda\alpha + \mu\beta} \leq \lambda e^{\alpha} + \mu e^{\beta}$$

$$\Rightarrow e^{\lambda\alpha} \cdot e^{\mu\beta} \leq \lambda e^{\alpha} + \mu e^{\beta} \longrightarrow (*)$$

let $\lambda = \frac{1}{p}$, $\mu = \frac{1}{q}$

and α and β be such that

$$a = e^{\alpha/p}, \quad b = e^{\beta/q}$$

$$\Rightarrow a^p = e^{\alpha}, \quad b^q = e^{\beta}$$

Using these in (*), we get

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

when $\frac{1}{p} + \frac{1}{q} = 1$.

$[\quad 1 \le p < \infty$

$$\|x\|_p = \left(\sum_{i=1}^{n} |x(i)|^p \right)^{1/p}$$

on $\mathbb{K}^n$