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1) $2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + (1-x^2)y = 0 \longrightarrow (1)$

$\Rightarrow \dfrac{d^2y}{dx^2} - \dfrac{1}{2x}\dfrac{dy}{dx} + \dfrac{1-x^2}{2x^2}y = 0$

$P(x) = -1/2x$ & $Q(x) = (1-x^2)/2x^2$

as $P(x)$ & $Q(x)$ are not defined at $x=0$, they are not analytic about $x=0$

$\therefore$ $x=0$ is not an ordinary point

$xP(x) = -1/2$, $x^2Q(x) = (1-x^2)$ are analytic about $x=0$.

Thus $x=0$ is a regular singular pt.

Let $y = x^k\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{(n+k)}$ be solution of (1)

$\Rightarrow y' = \sum_{n=0}^{\infty}(n+k)\, a_n x^{n+k-1}$ & $y'' = \sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k-2}$

putting y' & y'' in (1)

$2x^2\sum_{n=0}^{\infty}(n+k)(n+k-1)a_n x^{n+k-2} - x\sum_{n=0}^{\infty}(n+k)a_n x^{n+k-1} + (1-x^2)\sum_{n=0}^{\infty} a_n x^{n+k} = 0$

$\Rightarrow \sum_{n=0}^{\infty}\{2(n+k)(n+k-1)-(n+k)\}a_n x^{n+k} + \sum_{n=0}^{\infty} a_n x^{n+k} - \sum_{n=0}^{\infty} a_n x^{n+k+2} = 0$

$\Rightarrow \sum_{n=0}^{\infty} a_n\{(n+k-1)(2n+2k-1)\}x^{n+k} - \sum_{n=0}^{\infty} a_{n-2}x^{n+k} = 0$

$\Rightarrow a_0(k-1)(2k-1)x^k + a_1 k(2k+1)x^{k+1} + \sum_{n=2}^{\infty}(a_n+(n+k-1)(2n+2k-1) - a_{n-2})x^{n+k} = 0$

Equating coefficients of powers of x

$a_0(k-1)(2k-1) = 0 \Rightarrow k = 1$ or $1/2$ for x^k, as $a_0 \neq 0$

$a_1(k)(2k+1) = 0$ but $k = 1$ or $1/2$ $\therefore k \neq 0$ & $2k+1 \neq 0$, $\therefore a_1 = 0$ for x^{k+1}

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$a_n(n+k-1)(2n+2k-1) - a_{n-2} = 0 \Rightarrow a_n = \dfrac{a_{n-2}}{(n+k-1)(2n+2k-1)}$

for x^{n+k} where $n \geq 2$

as $a_1 = 0$, $a_1 = a_3 = a_5 = \dots a_{2m+1} = 0$ for $m \in \mathbb{N}$

Case 1 :- $k = 1$:- $a_n = \dfrac{a_{n-2}}{n(2n+1)} = \dfrac{a_0}{(2.4.\dots n)(5.9.\dots(2n+1))}$

Then, $y_1 = x \sum_{n=0}^{\infty} a_n x^n = a_0 x \left(1 + \dfrac{a_2}{a_0} x^2 + \dfrac{a_4}{a_0} x^4 + \dots\right)$

$\Rightarrow y_1 = a_0 x \left(1 + \dfrac{x^2}{2.5} + \dfrac{x^4}{2.4.5.9} + \dots\right) \rightarrow 2$

Case 2 :- $k = 1/2$:- $a_n = \dfrac{a_{n-2}}{(n-1/2)(2n)} = \dfrac{a_0}{(2.4\dots n)(3.7\dots(2n-1))}$

$y_2 = x^{1/2} \sum_{n=0}^{\infty} a_n x^n = a_0 x^{1/2} \left(1 + \dfrac{a_2}{a_0} x^2 + \dfrac{a_4}{a_0} x^4 + \dots\right)$

$= a_0 x^{1/2} \left(1 + \dfrac{x^2}{2.3} + \dfrac{x^4}{2.4.3.7} + \dots\right) \rightarrow 3$

Gen sol^n of $y \Rightarrow y = Ay_1 + By_2$.
From 2 & 3

$y = Ax\left(1 + \dfrac{x^2}{2.5} + \dfrac{x^4}{2.4.5.9} + \dots\right) + Bx^{1/2}\left(1 + \dfrac{x^2}{2.3} + \dfrac{x^4}{2.4.3.7} + \dots\right)$

2) $\dfrac{d^2y}{dx^2} - y = 0 \longrightarrow 1$

Substitute of $z = 1/x \Rightarrow \dfrac{dz}{dx} = -1/x^2$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx} = \dfrac{-1}{x^2} \cdot \dfrac{dy}{dz}$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{dy}{dz} \times \dfrac{2}{x^3} - \dfrac{1}{x^2} \cdot \dfrac{d}{dx}\left(\dfrac{dy}{dz}\right)$

$= \dfrac{2}{x^3}\dfrac{dy}{dz} + \dfrac{1}{x^4}\dfrac{d^2y}{dz^2} = 2z^3 \dfrac{dy}{dz} + z^4 \dfrac{d^2y}{dz^2}$

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$\therefore$ 1 transforms to, $z^4 \dfrac{d^2y}{dz^2} + 2z^3 \dfrac{dy}{dz} - y = 0 \longrightarrow$ 2

Let $y = z \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^{(n+k)}$ be sol^n of 2

$\Rightarrow y' = \sum_{n=0}^{\infty} (n+k)a_n z^{n+k-1}$ & $y'' = \sum_{n=0}^{\infty} (n+k)(n+k-1)a_n z^{n+k-2}$

putting y' & y'' in 2

$\sum_{n=0}^{\infty} (n+k)(n+k-1)a_n z^{n+k+2} + \sum_{n=0}^{\infty} 2(n+k)a_n z^{n+k+2} - \sum_{n=0}^{\infty} a_n z^{n+k} = 0$

Equating coff. of z^k & z^{k+1} on both sides:

$-a_0 = 0 \Rightarrow a_0 = 0$, $a_1 = 0$

But a_0 has to be $\neq 0$ for sol^n to exist.

Hence, Frobenius method fails here as $z=0$ is not an ordinary / regular singular point of 2

3) i) $P_n(0) = 0$, n is odd

$\Rightarrow (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$

Put $x=0$

$(1+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0) t^n$.

$RHS = 1 - \frac{1}{2}t^2 + \dfrac{\frac{1}{2} \times 3/2}{2!} t^4 - \dfrac{1/2 \times \frac{3}{2} \times \frac{5}{2}}{3!} t^6 + \dots \rightarrow$ 1

Comparing odd powered coefficients from 1

$P_n(0) = 0$ if n is odd

ii) $P_n(0) = -1^{n/2} \dfrac{\times n!}{2^n \times (\frac{n}{2}!)^2}$

$\Rightarrow$ Comparing even powered coefficients from 1

$P_n(0) = (-1)^{n/2} \dfrac{1.3.5.\dots(n-1)}{2^{n/2}.(\frac{n}{2})!} = (-1)^{n/2} \dfrac{(1.2.3.4\dots)(n-1)n}{2^{n/2}.2.4.\dots n.\frac{n}{2}!}$

$= (-1)^{n/2} (n!) / 2^{n/2}.2^{n/2}.1.2.3\dots n/2 . \frac{n}{2}!$

$P_n(0) = (-1) \dfrac{n!}{2^n((\frac{n}{2})!)^2}$ hence proved

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4) $P_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n$

i) $\int_{-1}^{1} P_n(x) \cdot dx = \frac{1}{2^n \cdot n!} \int_{-1}^{1} \frac{d^n}{dx^n} (x^2-1)^n \cdot dx = \frac{1}{2^n \times n!} \left(\frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \right)_{-1}^{1}$

for $n \neq 0$

As, $(x^2-1)^n$ has $(x-1)$ & $(x+1)$ as factors, with multiplicity n, so $(n-1)^{th}$ derivative of $(x^2-1)^n$ will have $(x-1)$ & $(x+1)$ both as factors

So $\left[\frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \right]_{-1}^{1} = 0 \Rightarrow \int_{-1}^{1} P_n(x) dx = 0$ for $n \neq 0$

ii) $\int_{-1}^{1} P_0(x) dx = \int_{-1}^{1} 1 \cdot dx = x \Big|_{-1}^{1} = 2$ (as $P_0(x) = 1$)

$\therefore \int_{-1}^{1} P_0(x) dx = 2$ Hence proved