Lecture 35

① Monotone Convergence theorem $\Leftrightarrow$ Fatou's lemma.

proof:- We already prove that Fatou's lemma implies the Monotone convergence theorem.

Conversely assume monotone convergence.

To show: Fatou's lemma.

Let $\{f_n\}$ be a sequence of non-negative measurable funtions & $f_n \to f$ a.e. p.w.

To show: $\int f \leq \liminf_{n \to \infty} \int f_n$.

Let $g_k = \inf\{f_k, f_{k+1}, \cdots\cdots\}$, $\forall k \geq 1$.

Then g_k is non-negative, measurable, $\forall k \geq 1$.

Also $\{g_k\}$ is increasing $(g_k \leq g_{k+1}, \ \forall k \geq 1)$.

$\therefore$ By monotone Convergence theorem,

$$\int \left(\lim_{k \to \infty} g_k \right) = \lim_{k \to \infty} \int g_k .$$

$$\Rightarrow \int \lim_{k \to \infty} \left(\inf_{n \geq k} (f_n) \right) = \lim_{k \to \infty} \int g_k$$

$$\Rightarrow \int \liminf_{k \to \infty} (f_k) = \underline{\lim_{k \to \infty} \int g_k} .$$

Now, We have $\underset{\underset{\inf\{f_k, f_{k+1}, \dots, f_n, \dots\}}{\shortparallel}}{g_k} \leq f_n \quad \forall n \geq k$

$$\Rightarrow \int g_k \leq \int f_n \quad \forall n \geq k$$

$$\Rightarrow \int g_k \leq \inf_{n \geq k} \left(\int f_n \right)$$

$$\Rightarrow \lim_{k \to \infty} \int g_k \leq \lim_{k \to \infty} \left(\inf_{n \geq k} \int f_n \right) = \liminf_{k \to \infty} \int f_k$$

$$\Rightarrow \int \liminf_{k\to\infty} (f_k) \leq \liminf_{k\to\infty} \int f_k$$

$$\Rightarrow \int f \leq \liminf_{k\to\infty} \int f_k .$$

This proves Fatou's lemma.

if $f_k \to f$ as $n \to \infty$

then $\lim_{k\to\infty} f_k = \liminf_{k\to\infty} (f_k)$

$= \limsup_{k\to\infty} (f_k)$

$= f$

Remark: Fatou's lemma is true for non-negative measurable functions only.

We can not drop the non-negative assumption in the Fatou's lemma.

Example:-

Let $f_n(x) = \begin{cases} -n , & \frac{1}{n} \leq x \leq 2/n \\ 0 & \text{otherwise.} \end{cases}$

$$\lim_{n\to\infty} f_n(x) = 0 \quad \forall\, x \in [0,1]$$
$$= f \ (\text{say})$$

$$\int_{[0,1]} f_n = \int_0^1 f_n = \int_{1/n}^{2/n} (-n)\, dx$$

$$= -n x \Big|_{x=1/n}^{2/n}$$

$$= (-n)\left(\frac{2}{n} - \frac{1}{n}\right)$$

$$= -1 \quad \text{for any } n \in \mathbb{N}.$$

$$\therefore \liminf_{n\to\infty} \int_{[0,1]} f_n = \lim_{n\to\infty} \int_{[0,1]} f_n = -1$$

$$\& \int_{[0,1]} f = \int_{[0,1]} 0 = 0 \not\leq -1 = \liminf_{n\to\infty} \int_{[0,1]} f_n$$

$\therefore$

Take $g_k = \inf(\{f_k, f_{k+1}, \dots\}) \nearrow f = 0.$

(2) Fatou's lemma $\Rightarrow$ Dominated Convergence theorem

$\Downarrow$

Bounded Convergence theorem.

Fatou's $\Rightarrow$ DCT :

proof:- Assume $\{f_n\}$ is a sequence of measurable functions such that $f_n \to f$ a.e as $n \to \infty$

& $|f_n| \leq g$, where $g \in L^1(\mathbb{R}^d)$.

(i.e, g is L-integrable) $g \geq 0$.

To show: $\int f_n \longrightarrow \int f$ as $n \to \infty$.

We have, $|f| \leq g$ &

$g - f_n \longrightarrow g - f$ as $n \to \infty$ a.e

$g + f_n \longrightarrow g + f$ as $n \to \infty$. a.e

& $g - f_n$, $g + f_n$ are non-negative functions $\forall\, n \geq 1$.

$\therefore$ By Fatou's lemma,

$$\left.\begin{aligned}
&\liminf_{n\to\infty}\left(\int (g-f_n)\right) \geq \int (g-f) \\
\& \quad &\liminf_{n\to\infty}\left(\int (g+f_n)\right) \geq \int g+f.
\end{aligned}\right\} \rightarrow (*)$$

But

$$\liminf_{n\to\infty}\left(\int (g-f_n)\right) = \liminf_{n\to\infty}\int g + \liminf_{n\to\infty}\left(-\left(\int f_n\right)\right)$$

$$= \int g - \limsup_{n\to\infty}\left(\int f_n\right)$$

&

$$\liminf_{n\to\infty}\left(\int (g+f_n)\right) = \liminf_{n\to\infty}\int g + \liminf_{n\to\infty}\left(\int f_n\right)$$

$$= \int g + \liminf_{n\to\infty}\left(\int f_n\right).$$

$\therefore$ From $(*)$, we have

$$\cancel{\int g} - \limsup_{n\to\infty}\left(\int f_n\right) \geq \cancel{\int g} - \int f$$

&

$$\cancel{\int g} + \liminf_{n\to\infty}\left(\int f_n\right) \geq \cancel{\int g} + \int f$$

$$\Rightarrow \limsup_{n\to\infty}\left(\int f_n\right) \le \int f \le \liminf_{n\to\infty}\left(\int f_n\right)$$

$$\Rightarrow \limsup_{n\to\infty}\left(\int f_n\right) = \int f = \liminf_{n\to\infty}\left(\int f_n\right)$$

$$\Rightarrow \lim_{n\to\infty}\left(\int f_n\right) = \int f.$$

DCT $\Rightarrow$ BCT:

Let $|f_n| \le M$ $\forall n \ge 1$, supported on a set E of finite measure.

$f_n \to f$ a.e. as $n \to \infty$.

To show: $\int |f_n - f| \to 0$ as $n \to \infty$.

$$\boxed{\begin{array}{c}\text{DCT}\\ |f_n| \le g \in L^1(\mathbb{R}^d)\\ \Downarrow\\ \int |f_n - f| \to 0.\end{array}}$$

Let $g \equiv M$ constant function M.

& $\int_E g = \int_E M = M\, m(E) < \infty$

(that is g is L-integrable).

$\therefore$ By DCT, $\int |f_n - f| \to 0$ as $n \to \infty$.

L^p-space

We have seen L^1-space

$$L^1(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{C} \;\middle|\; \int_{\mathbb{R}^d} |f(x)|\, dx < \infty \right\}.$$

& L^1-norm $\|f\|_{L^1} = \int_{\mathbb{R}^d} |f(x)|\, dx$.

Let $p > 0$ be any integer. Define

$$L^p(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{C} \;\middle|\; \int_{\mathbb{R}^d} |f(x)|^p\, dx < \infty \right\}$$

& $\|f\|_{L^p} \doteq \left(\int_{\mathbb{R}^d} |f(x)|^p\, dx \right)^{1/p}$. called $\underline{L^p\text{-norm}}$.

- $\left(L^p(\mathbb{R}^d), \|-\|_{L^p} \right)$ is a normed linear space.

Theorem:- $\left(L^p(\mathbb{R}^d), \|-\|_{L^p}\right)$ is Complete.

($p > 0$ integer)

Called "L^p-Space."

$\mathbb{R}^d$ may have other type of measures other than Lebesgue measure.

$(\mathbb{R}^d, \mu)$ measure space, where μ is a measure on $\mathbb{R}^d$.
