week-7 class-2

PDE :

$u(x,y)$ PDE $\Rightarrow$ $F(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$

2nd order PDE

$F(u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0$, $u_x = \frac{\partial u}{\partial x}$

$u_{xx} = \frac{\partial^2 u}{\partial x^2}$

Linear Principal part $\rightarrow$ the part involving order 2 ie. u_{xx}, u_{yy}, u_{xy}

$A(x,y)\, u_{xx} + 2B(x,y)\, u_{xy} + C(x,y)\, u_{yy} + f(u, u_x, u_y, x, y) = 0$

The $f(u, u_x, u_y, x, y)$ can be linear / non-linear but the principal part must be linear.

The transformation $(x,y) \rightarrow (\xi, \eta)$ is cannonical transformation.

classification $\rightarrow$

① Hyperbolic PDE $\rightarrow$ $\boxed{B^2 - AC > 0}$

then its cannonical form $(x,y) \rightarrow (\xi, \eta)$

$u_{\xi\eta} + G(u, u_\xi, u_\eta, \xi, \eta) = 0$

eg: wave eqn $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

$\Rightarrow \frac{\partial^2 u}{\partial t^2} - \frac{c^2 \partial^2 u}{\partial x^2} = 0$

So, $B(x,y) = 0$, $A(x,y) = 1$, $C(x,y) = -c^2$

So,

$0 - (1)(-c^2) \Rightarrow c^2 > 0 \quad \forall\, c \in \mathbb{R}$

we get its cannonical form as

$$\frac{\partial^2 \mu}{\partial \xi \partial \eta} = 0 \quad , \quad \xi = x + ct, \quad \eta = x - ct$$

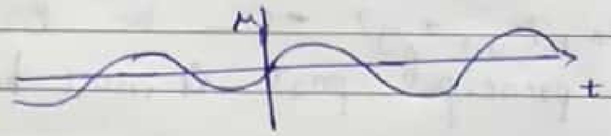
$$\Rightarrow \frac{\partial}{\partial \xi}\left(\frac{\partial \mu}{\partial \eta}\right) = 0$$

$$\Rightarrow \frac{\partial \mu}{\partial \eta} = g(\eta)$$

$$\Rightarrow \mu = F(\eta) + g(\xi) = F(x - ct) + g(x + ct)$$

$$\Rightarrow \mu(x,t) = F(x - ct) + g(x + ct) \quad , \quad t > 0$$

If $\mu(x,0) = G(x)$, $\frac{\partial \mu}{\partial t}(x,0) = F_1(x)$



② Parabolic PDE → $\boxed{B^2 - AC = 0}$

Cannonical form $\quad \mu_{\xi\xi} = F(\;--\;)$

eg: $\frac{\partial \mu}{\partial t} = c \frac{\partial^2 \mu}{\partial x^2}$ heat conduction eqn. by diffusion

$\Rightarrow B(x,y) = 0$, $C(x,y) = 0$, $A(x,y) = c$

so, $B^2 - AC = 0$

③ Elliptic PDE - $\quad \frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} = f(\mu, \mu_x, \mu_y, x, y)$

$\boxed{B^2 - AC < 0}$

$\nabla^2 \mu = 0$, Laplace eqn

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad , \quad \nabla^2 \mu = g(x,y)$$

poission eqn/ Laplace eqn

$B(x,y) = 0$, $A(x,y) = 1$, $C(x,y) = 1$

$B^2 - AC = 0 - (1)(1) = -1 < 0$

Numerical soln for Parabolic PDE :

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2} , \quad t > 0, \quad \underbrace{0 < x < a}_{\text{spatial domain}}$$

we need 2 B.C. for x

① $u(0,t) = u_0 \quad \forall\, t > 0$

② $u(a,t) = u_a \quad \forall\, t > 0$

or sometimes derivates are given at the end points

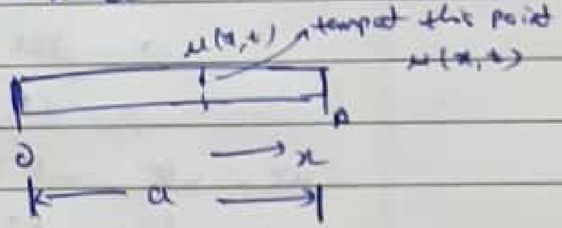
$\frac{\partial u}{\partial x}(0,t) = \alpha$, $\frac{\partial u}{\partial x}(a,t) = \beta$

and,

Initial condition at $t = 0$

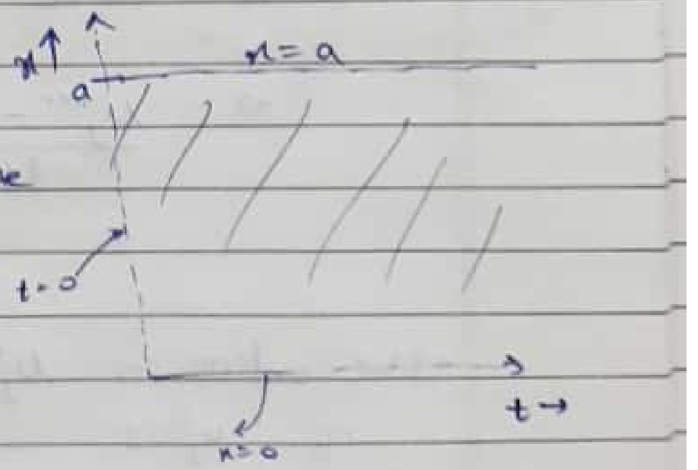
$u(x,0) = f(x)$, $0 < x < a$

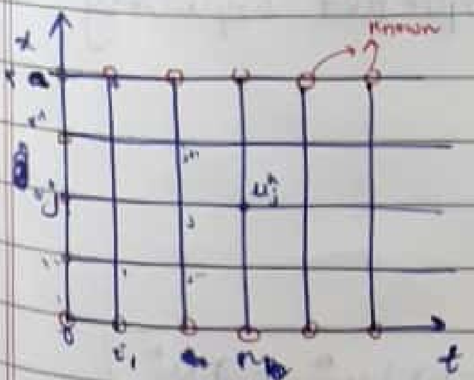
~~somethin~~



domain $[0, a]$ & $t > 0$

we need to discretize this semi-infinite domain into grid points.





$u_j^n = u(x_j, t_n)$

$t_n = n\,\delta t$

$x_j = j\,\delta x$

$j = 0, 1, 2, \ldots, N$

n can theoritically go upto ∞.

find u_j^n

given: IC ① $u(x,0) = f(x)$

$\Rightarrow u_j^0 = f_j$, $j = 1, 2, \ldots, N$

BC & ② $u(0,t) = u_0$ & $u(a,t) = u_a$

B.C. $\Rightarrow$ $u_0^n = u_0$ & $u_N^n = u_N$

find u_j^n, $j = 1, 2, 3, \ldots, N-1$
when $n > 0$

we adopt a forward marching in time procedure.
At any stage u_j^n is known & task is to find u_j^{n+1}
$j = 1, 2, 3, \ldots, N-1$
$u_j^n \rightarrow u_j^{n+1}$ when $n \geq 0$
satisfy the PDE at (x_j, t_n)

$$\left.\frac{\partial u}{\partial t}\right|_{(x_j, t_n)} = c \left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_j, t_n)}$$

$$\Rightarrow \left.\frac{\partial u}{\partial t}\right|_j^n = c \left.\frac{\partial^2 u}{\partial x^2}\right|_j^n$$

we use FTCS $\rightarrow$ forward time central space
to discretize.

$u_j^n \rightarrow u_j^{n+1}$

$$\Rightarrow \frac{u_j^{n+1} - u_j^n}{\delta t} = c \left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2} \right)$$

$n \geq 0$, $j = 1, 2, \ldots, N-1$

The unknown u_j^{n+1} can be expressed explicitly in terms of u_j^n s

$\Rightarrow$ let $\quad r = \dfrac{c(\delta t)}{(\delta x)^2}$

$$\Rightarrow \boxed{u_j^{n+1} = r u_{j+1}^n + (1 - 2r) u_j^n + r u_{j-1}^n}$$

$j = 1, 2, \ldots, N-1$
when $n \geq 0$

Scanned with CamScanner

This is called explicit discritization/scheme (EDM)

Truncation Error → $O(\delta t, (\delta x)^2)$

T.E. → is the residue by which the exact soln of the PDE fails to satisfy the difference scheme

$u(x_j, t_n) = u_j^n$ → exact soln

$$T.E. = \left(\frac{u_j^{n+1} - u_j^n}{\delta t}\right) - c\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2}\right)$$

Expand by Taylor series about (x_j, t_n)

$$\left(\frac{\partial u}{\partial t} - c\frac{\partial^2 u}{\partial x^2}\right)\Big|_j^n = 0$$

$$T.E. = \frac{1}{(\delta t)}\left[\delta t \frac{\partial u}{\partial t}\Big|_j^n + \frac{(\delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}\Big|_j^n + \text{---}\right]$$

$$- c\left[\delta x \frac{\partial u}{\partial x}\Big|_j^n + \frac{\delta x^2}{2!}\frac{\partial^2 u}{\partial x^2}\Big|_j^n + \frac{(\delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_j^n + \frac{(\delta x)^4}{4!}\frac{\partial^4 u}{\partial x^4}\Big|_j^n + \text{---}\right.$$

$$= \left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right)\Big|_j^n + \frac{\delta t}{2} u_{tt}\Big|_j^n + c^2\frac{(\delta x)^2}{4!}$$

$$= O(\delta t, (\delta x)^2)$$

Consistency → if T.E. → 0 as $\delta x, \delta t \to 0$

The explicit scheme is stable for $\boxed{r \le \frac{1}{2}}$

$\Rightarrow c(\delta t) \le 0.5\,(\delta x)^2$

i.e. $(\delta t) \ll 1$

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Implicit scheme

Satisfy the PDE at unknown time (x_j, t_{n+1})

$$\frac{\partial u}{\partial t}\Big|_j^{n+1} = \nu \frac{\partial^2 u}{\partial x^2}\Big|_j^{n+1}$$

use backwood time & central space

$$\Rightarrow \frac{(u_j^{n+1} - u_j^n)}{\delta t} = \nu\left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\delta x)^2}\right)$$

$$-r\, u_{j-1}^{n+1} + (1+2r)\, u_j^{n+1} - r(u_{j+1}^{n+1}) = u_j^n$$

$j = 1, 2, \_\_\_, N-1$

$n \geq 0$

which are $(N-1)$ linear algebric eq^n involving unknowns $\rightarrow$ $u_1^{n+1}, u_2^{n+1}, \_\_\_, u_{N-1}^{n+1}$ i.e $(N-1)$ unknown

$$X^T = [u_1^{n+1}, \_\_\_\_, u_{N-1}^{n+1}]$$

$AX = d$, $A \rightarrow$ tri-diagonal system

with $a_j = -r$, $b_j = 1+2r$, $c_j = -r$, $d_j = u_j^n$

$$d_1 = u_1^n + r u_0$$
$$d_{N-1} = u_{N-1}^n + r u_N$$

Solve the compact tri-diagonal system.

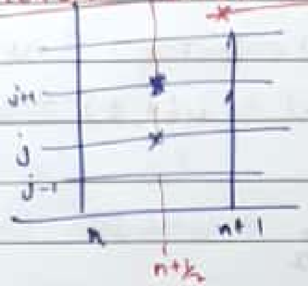
H.T.

Truncation error $= O(\delta t, (\delta x)^2)$ consistent

Stable for any choice of r.

Boundary condition have direct influence on the solution at every time step.

Crank-Nicolson scheme :



$t_n \to t_{n+1}$

$$\boxed{u_t = c^2 u_{xx}}$$

$$\frac{\partial u}{\partial t}\Big|_j^{n+1/2} = \vartheta \frac{\partial^2 u}{\partial x^2}\Big|_j^{n+1/2}$$

at $x = x_j$ integrate both sides of the PDE w.r.t. t b/w t_n & t_{n+1}

$$\int_{t_n}^{t_{n+1}} \frac{\partial u}{\partial t}\Big|_{x=x_j} dt = \vartheta \int_{t_n}^{t_{n+1}} \frac{\partial^2 u}{\partial x^2}\Big|_{x_j} dt$$

use Trapezoidal rule

$$\Rightarrow \quad u_j^{n+1} - u_j^n = \frac{\vartheta\,\delta t}{2}\left[\frac{\partial^2 u}{\partial x^2}\Big|_j^{n+1} + \frac{\partial^2 u}{\partial x^2}\Big|_j^{n}\right] + O(\delta t^3)$$

now use central diff scheme for space derivatives

$$\Rightarrow \quad \frac{(u_j^{n+1} - u_j^n)}{\delta t} = \frac{\vartheta}{2}\left[\frac{(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})}{(\delta x)^2} + \frac{(u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n})}{(\delta x)^2}\right]$$

let $r = \frac{\vartheta\,\delta t}{(\delta x)^2}$

$$+ O((\delta t)^2, (\delta x)^2)$$

$$-\frac{r}{2}u_{j+1}^{n+1} + (1+r)u_j^{n+1} - r u_{j-1}^{n+1} = d_j^n$$

$j = 1, 2, \text{——}, N-1$

which is a Compact & tri-diagonal system.

$$Ax = d$$

Stable for any choice of r.

Advantage - 2nd order in both time & space

H.T. Show T.E. $O(\delta t^2, \delta x^2)$.

Hint: Expand about $(n+\frac{1}{2}, j)$ consistent

H.T. Q1 $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(x,0) = \sin\pi x$, $0 < x < 1$ IC

$u(0,t) = u(1,t) = 0$, $t > 0$ BC

Let $\delta x = \frac{1}{4}$, $r = \frac{1}{2}$

Solve by all schemes.

Lab → $\delta x = 0.25$, 0.125, 0.0625

$r = \frac{1}{2}$

H.T. Q2 $u_t = u_{xx}$, $u(x,0) = \cos\frac{\pi x}{2}$ $-1 \le x \le 1$

$u(-1,t) = u(1,t) = 0$, $t > 0$

$r = \frac{1}{3}$, $\delta x = \frac{1}{4}$

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* we call a method consistent if $T.E \to 0$ when $h \to 0$ & $\delta t \to 0$

Let $F_{ij}(u) = 0$ is the Finite Difference equation (FDE) for the PDE $L(u) = 0$ for a given linear initial BVP.

$U \to$ numerical soln

$u \to$ exact solution of PDE

T.E. = the residue by which the exact soln of the PDE fails to satisfy the PDE.

$$T.E. = F_{ij}(u)$$

Explicit scheme for $L(u) = u_t - c\,u_{xx} = 0$

$$\Rightarrow \frac{(u_j^{n+1} - u_j^n)}{\delta t} - c\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2}\right) = 0 \quad \to F_{jn}(u)$$

$$T.E. = \frac{(u_j^{n+1} - u_j^n)}{\delta t} - c\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2}\right) \quad \text{———} (*)$$

u_j^n is the exact solution of PDE $L(u) = 0$

Expand $(*)$ by Taylor Series about (x_j, t_n)

$$T_j^n = \frac{1}{(\delta t)}\left[u_j^n + \delta t \frac{\partial u}{\partial t}\Big|_j^n + \frac{(\delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}\Big|_j^n + \text{———} + u_j^n\right]$$

$$-\frac{c}{(\delta x)^2}\left[u_j^n + \delta x \frac{\partial u}{\partial x}\Big|_j^n + \frac{(\delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}\Big|_j^n + \frac{(\delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_j^n\right.$$

$$\left. + \frac{\delta x^4}{4!}\frac{\partial^4 x}{\partial x^4}\Big|_j^n + \text{———}\right.$$

$T_j^n = O(\delta t, \delta x^2)$ and $T_j^n \to 0$ as $\delta t, \delta x \to 0$

Thus, the numerical scheme is consistent.

H.T Find T.E. and check for consistency of
i) Implicit scheme
(ii) Crank-Nicolson scheme

Lax-Equivalence Theorem → $U_j^n \to u_j^n$
i.e. the numerical soln should converge to exact solution.

For a linear Initial boundary-value problem, the stability and consistency of the numerical scheme leads to converged soln.

Stability of numerical scheme :-
$u_j^{n+1} \to u_j^n$, $n >> 1$

u_j^n be the exact solution of the numerical scheme
$\bar{u}_j^n$ be the solution obtained.

$u_j^n = \bar{u}_j^n + \xi_j^n$, ξ_j^n is the error, mostly the round-off error.

Because of the linearity of the problem, ε_j^n satisfy PDE

A discrete distribution of ϵ_j^n is considered ϵ_j^n means bounded within the Von-neuman stability Analysis.

we approximate the error distribution by a finite Fourier Series.

$\epsilon(x,t) = \sum_{m=0}^{M} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$

L is the interval over which x varies.

$\epsilon(x,t) = \sum_{m=0}^{M} A_m(t)\, e^{im\pi x/L}$, A_m's are arbitrarily complex coefficient which is the amplitude for the wave $A_m e^{\frac{im\pi x}{L}}$

$\epsilon(x,t)$ satisfy the linear FDE.

let at $t=t_n$ $\bar{A}_m^n$ be the amplitude such that $\max_m \{|A_m(t_n)|\}$

and denote the term at $t=t_n$ be

$\epsilon_m(x,t_n) = A_m(t_n)\, e^{im\pi x/L}$

at $x = x_j$, $\epsilon_j^n = A_m^n\, e^{im\pi x_j/L}$

$x_j = j(\delta x)$

$\epsilon_j^n = \bar{A}_m^n\, e^{\frac{im\pi \delta x j}{L}}$, $\theta = \frac{m\pi \delta x}{L}$, is the phase angle

$\epsilon_j^n = \bar{A}^n e^{i\theta j}$, is a wave propagating the x-direction

$|\epsilon_j^n| = |\bar{A}^n|$

we define the amplification factor

$\xi = \frac{\bar{A}_{n+1}}{\bar{A}_n}$, for stability $|\xi| \le 1$

$\Rightarrow |\bar{A}^{n+1}| \le |\bar{A}^n| \Rightarrow |\xi| \le 1 \rightarrow$ stability

$|\xi| > 1 \Rightarrow |\bar{A}^{n+1}| > |\bar{A}^n|$ unstable

consider the term $\epsilon_j^n = \bar{A}^n e^{i\theta j}$ which satisfy the FDE then find ξ and check $|\xi| \le 1$ for stability

* Explicit scheme

$$u_j^{n+1} = u_j^n + r(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

$$\epsilon_j^{n+1} = \epsilon_j^n + r(\epsilon_{j+1}^n - 2\epsilon_j^n + \epsilon_{j-1}^n)$$

$$\epsilon_j^n = \bar{A}^n e^{i\theta j}$$

$$\Rightarrow \bar{A}^{n+1} e^{i\theta j} = \bar{A}^n e^{i\theta j} + r\left[\bar{A}^n e^{i\theta(j+1)} - 2\bar{A}^n e^{i\theta j} + \bar{A}^n e^{i\theta(j-1)}\right]$$

$$\xi = \frac{\bar{A}^{n+1}}{\bar{A}^n} = 1 + r\left[e^{i\theta} - 2 + e^{-i\theta}\right]$$

$$= 1 + 2r(\cos\theta - 1) \qquad \theta = \frac{m\pi(\delta x)}{L}$$

$$|\xi| \le 1$$

$$\Rightarrow -1 \le 1 + 2r(\cos\theta - 1) \le 1$$

$$-1 \le 1 - 4r\sin^2\theta/2 \le 1$$

$$\Rightarrow 1 - 4r\sin^2\theta/2 \ge -1$$

$$\Rightarrow 2 \ge 4r\sin^2\theta/2$$

$$\Rightarrow \frac{1}{2} \ge r\sin^2\theta/2$$

So, $\boxed{r \le 1/2}$ as $\sin^2\theta/2$ is max $= 1$

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Stability Analysis (Recap):

$$\epsilon(x, t_n) = \sum_m A_m^n e^{im\pi x/L}$$

$$\epsilon_j^n = \bar{A}^n e^{im\pi x_j/L} = \bar{A}^n e^{i\theta j} \quad , \quad \theta = \frac{m\pi \Delta x}{L}$$

Amplification factor $\quad \xi = \dfrac{\bar{A}^{n+1}}{\bar{A}^n}$,

stability $|\xi| \leq 1$ & unstable $|\xi| > 1$

Von Neuman Stability Analysis.

Explicit scheme, $|\xi| \leq 1$ for $\boxed{r \leq 1/2}$

* Implicit scheme *

$$u_j^{n+1} - u_j^n = r\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right)$$

$$\epsilon_j^{n+1} - \epsilon_j^n = r\left(\epsilon_{j+1}^{n+1} - 2\epsilon_j^{n+1} + \epsilon_{j-1}^{n+1}\right)$$

$$\epsilon_j^n = \bar{A}^n e^{i\theta j}$$

$$\bar{A}^{n+1} - \bar{A}^n = r A^{n+1}\left(e^{i\theta} - 2 + e^{-i\theta}\right)$$

$$\xi = \frac{1}{1 - 2r(\cos\theta - 1)} = \frac{1}{1 + 4r\sin^2(\theta/2)}$$

$\Rightarrow |\xi| < 1$ & $r > 0$ $\quad \forall \; r \geq 0$

So,

Implicit scheme is unconditionally stable for any choice of r.

Check for the stability of Crank-Nicolson Method.

CTCS Leapfrog scheme :

3 - time level

$$u_t = c\, u_{xx}$$

$$\Rightarrow \frac{u_j^{n+1} - u_j^{n-1}}{2(\delta t)} = c\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2}\right), \quad n \geq 1$$

$$+ O(\delta t^2, \delta x^2)$$

- HT → check for TE & consistency of this scheme

$$\Rightarrow u_j^{n+1} = u_j^{n-1} + 2r\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

stability analysis, $\xi = \frac{\bar{A}^{n+1}}{\bar{A}^n} \simeq \frac{\bar{A}^n}{\bar{A}^{n-1}}$

$$\bar{A}^{n+1} = \bar{A}^{n-1} + 2r\bar{A}^n\left(e^{i\theta} - 2 + e^{-i\theta}\right)$$

$$\xi = \frac{1}{\xi} + 4r(\cos\theta - 1)$$

$$\Rightarrow \xi - \frac{1}{\xi} = 4r(\cos\theta - 1)$$

$$\Rightarrow \xi^2 - 4r(\cos\theta - 1)\xi - 1 = 0$$

$$\xi = \frac{4r(\cos\theta - 1) \pm \sqrt{16r^2(1-\cos\theta)^2 + 4}}{2}$$

θ ~~$4r(\cos$~~ $\xi = \frac{-b \pm \sqrt{b^2+4}}{2}$

$|\xi| \leq 1$, $b = -4r(\cos\theta - 1) = 8r\sin^2\theta/2$

$\xi = -b/2 \pm \frac{(\sqrt{b^2+4})}{2}$, $|\xi| = 1$ if $b = 0$

$b \neq 0$

$|\xi| = 1$ if $b=0$, $b \neq 0$

$|\xi| > 1 \quad \forall\, b \neq 0$

$$-1 \le -\frac{b}{2} \pm \frac{\left(\sqrt{b^2+4}\right)}{2} \le 1$$

$$\frac{\sqrt{b^2+4}}{2} < \frac{b}{2}$$

$\boxed{|\xi| > 1} \quad \forall \quad b \neq 0$

Leap-frog scheme is unconditionally unstable.

The numerical solution will oscillate/wiggle around the solution and not converge to it.

Du Fort Frankel scheme:

Consider the scheme

$$\frac{u_j^{n+1} - u_j^{n-1}}{2k} - \frac{1}{h^2}\left\{ u_{j+1}^n - 2\left\{ \theta u_j^{n+1} + (1-\theta) u_j^{n-1} \right\} + u_{j-1}^n \right\} = 0$$

$0 < \theta < 1$ is a parameter, $k = \delta t$, $h = \delta x$

Show that (i) if $k = rh$ the scheme is not consistent to the given PDE.

(ii) if $k = rh^2$, scheme is not consistent for $\theta \neq \frac{1}{2}$

$\Rightarrow$ ~~$(\bar{A}^{n+1} - \bar{A}^{n-1})$ $- \frac{2k}{h^2}$~~

$$T.E. = \frac{\partial u}{\partial t}\Big|_j^n + \frac{k^2}{6} u_{ttt}\Big|_j^n + (1-2\theta)\frac{(2k)}{h^2}\frac{\partial u}{\partial t}\Big|_j^n - \frac{\partial u}{\partial x^2}\Big|_j^n$$

$$+ \frac{k^2}{h^2} u_{tt}\Big|_j^n - \frac{h^2}{12} u_{xxxx}\Big|_j^n + O\left(\frac{k^3}{h^2}, h^4, k^4\right)$$

$$= \left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right)\Big|_j^n + (1-2\theta)\frac{2k}{h^2} u_t\Big|_j^n + \frac{k^2}{h^2} u_{tt}\Big|_j^n - \frac{h^2}{12} u_{xxxx}\Big|_j^n$$

$$+ O\left(\frac{k^3}{h^2}, h^4, k^4\right)$$

(i) If $h, k \to 0$ with $k = rh$, $\frac{k}{h} = r$, ~~then~~ if $\theta \neq \frac{1}{2}$ then $T.E. \to \infty$

$$T.E. = \left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right)\Big|_j^n + r^2 u_{tt}\Big|_j^n$$

if $\theta = \frac{1}{2}$

The FDE is consistent with PDE

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + r^2\frac{\partial^2 u}{\partial t^2} = 0$$

(ii) $k = rh^2$, $k, h \to 0$

$$T.E. \to (u_t - u_{xx})\Big|_j^n + 2r(1-2\theta)\, u_t\Big|_j^n$$

if $\theta = \frac{1}{2}$, the numerical scheme is consistent if $k = rh^2$

* Dufort Frankel scheme

$$u_j^n = \frac{1}{2}\left(u_j^{n+1} + u_j^{n-1}\right)$$

$$\frac{\left(u_j^{n+1} - u_j^{n-1}\right)}{2(\delta t)} = c\left(\frac{u_{j+1}^n - (u_j^{n+1} + u_j^{n-1}) + u_{j-1}^n}{(\delta x)^2}\right)$$

is stable and consistent

$$\boxed{\delta t = r(\delta x)^2} = c\frac{\delta t}{\delta x}(\delta x)^2, \quad \boxed{\delta x = \frac{1}{c}}$$

H.T. check its consistency $|\xi| \le 1$

stable and consistent $\Rightarrow$ convergence

Burgers Equation :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2}$$

LAB

Q1 $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $t>0$, $0<x<1$ $\quad u(x,0) = 1 \quad$ IC

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= u \quad \text{at } x=0 \\ \frac{\partial u}{\partial x} &= -u \quad \text{at } x=1 \end{aligned}\right\} t \ge 0$$

solve by Crank-Nicolson scheme with $r=1$, $\delta t = 0.01$

(Fictitious or first order Forward/Backward)

Q2 $u_t = u_{xx}$ $\quad u(x,0) = \frac{\cos \pi x}{2}$, $-1 \le x \le 1$

$u(-1,t) = u(1,t) = 0$, $\delta x = \frac{1}{3}$ $\quad r = \frac{1}{3}$

Explicit , implicit & Crank-Nicolson scheme.

H.T. LAB

Q $u_t = 4u_{xx}$ $\quad u(0,t) = u(8,t) = 0$

$u(x,t) = 4x - \frac{x^2}{2}$

$\delta x = 1$, $\delta t = 1/8$ H.T. by Crank-Nicolson scheme.

Non-linear transport Eqⁿ / Burgers equation

$$u_t + u u_x = \nu u_{xx} \qquad \nu \sim O(1)$$

I.C. $\quad u(x,0) = f(x), \quad 0 < x < l$

B.C.

$$u(0,t) = U_0, \quad u(l,t) = u_l, \quad t > 0$$

use Crank-Nicolson scheme followed by newton's linearization technique

$$\frac{(u_j^{n+1} - u_j^n)}{(\delta t)} + \frac{1}{2}\left(u \frac{\partial u}{\partial x}\Big|_j^n + u \frac{\partial u}{\partial x}\Big|_j^{n+1} \right) = \frac{\nu}{2}\left(\frac{\partial^2 u}{\partial x^2}\Big|_j^n + \frac{\partial^2 u}{\partial x^2}\Big|_j^{n+1} \right)$$

use central difference scheme.
then use newton's linearization technique.

$$(u_j^{n+1})^{(k+1)} = (u_j^n)^{(k)} + \Delta u_j, \quad \forall j$$

get the reduced tri-diagonal system $Ax = b$ at every iteration.

(Q) $\quad u_t + c u_{xx} = \nu u_{xx}$

Q. I.C. $u(x,0) = \sin x$, $0 < x < 1$ $\vartheta = 1$

B.C. $u(0,t) = u(1,t) = 0$, $t > 0$

Q. $u(x,0) = 4x(1-x)$, $0 < x < 1$

$u(0,t) = u(1,t) = 0$, $t > 0$

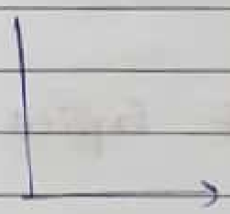
Hypeobolic

$u_t + c u_x = 0 \longrightarrow$ Hypeobolic eq^n.

$u = f(x+ct)$, $t > 0$

$u(x,t) = f(x - ct)$

$u(x,0) = f(x) = \begin{cases} 1 & , x < 0 \\ 0 & , x > 0 \end{cases}$



Q. $u_t + c u_x = \vartheta u_{xx}$ LAB problem

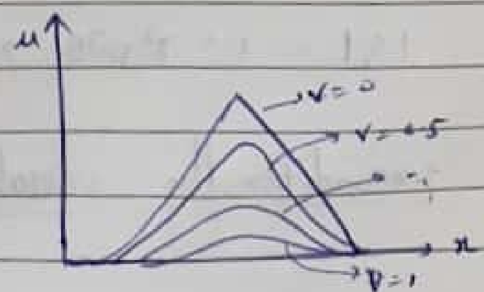
$u(x,0) = \begin{cases} x & ; 0 < x < 1 \\ 2-x & ; 1 < x < 2 \end{cases}$

$u(0,t) = 0$

$u(10,t) = 0$

B.C

Vary $\vartheta = 0.01, 0.05, 0.1, 0.5, 1$



non-linear Burgers eq^n

$$u_t + u u_x = \nu u_{xx}$$

$u(0,t) = u_0, \quad u(L,t) = u_1$

$u(x,0) = f(x)$

Discretize the above eq^n by crank-nicolson scheme and solve iteratively through newton's linearization tech.

Determine the tri-diagonal system at each iteration.

Explicit scheme for linear Burgers eq^n (or advection-diff eq^n)

$$u_t + c u_x = \nu u_{xx}$$

FTCS

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c\left(\frac{u_{j+1}^n - u_{j-1}^n}{2(\delta x)}\right) = \nu\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2}\right)$$

if $\nu << 1$, let $\nu = 0$

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\delta x}\right) = 0$$

$\xi = 1 - i r \sin\theta$ $\qquad r = \frac{c\,\delta t}{\delta x}$

$|\xi| = 1 + r^2 \sin^2\theta > 1 \quad \forall\, \theta$

Unconditionally unstable in this case.

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N.T. now let $\nu \sim O(1)$ value i.e. constant

Modified eq^n i.e. the FDE expand all variable about (x_j, t_n) and remove the 2nd and higher order of derivative w.r.t t by x using ~~$u_t + cu_x$~~ series expansion.

$u_t + cu_x = \text{---}$ at a grid point (x_j, t_n)

$$\left(\frac{u_j^{n+1} - u_j^n}{\delta t}\right) + c\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\delta x}\right) = \nu\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2}\right)$$

(x_j, t_n)

$$u_t + cu_x = (\text{---})u_{tt} + (\text{---})u_{xx} + \text{---}$$

$$r = \frac{\nu c\,\delta t}{(\delta x)^2} \leq \frac{1}{2}$$

Find the range of $\nu = \frac{c\delta t}{\delta x}$ for stability

Q. lab task $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0$

I.C. $u(x,0) = \sin \pi x \qquad 0 < x < 1$

B.C. $u(0,t) = u(1,t) = 0 \qquad , t > 0$

$\delta x = 0.05 \qquad , \qquad r = \frac{\delta t}{(\delta x)^2} = 1$

N.T. get the ensuing tri-diagonal system at every iteration using crank-nicolson scheme for discretization.

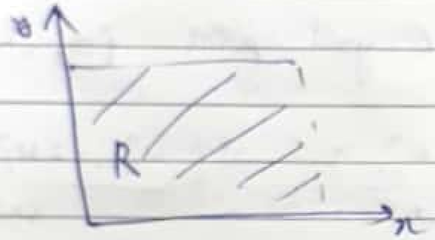
** Eliptic eqn

$$\frac{\partial u}{\partial t} = c\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = c^2 \nabla^2 u \quad , t > 0$$

$x, y \in R$

$\nabla^2 \rightarrow$ Laplacian.

$u(x, y, t)$



$R: 0 < x < a, \quad 0 < y < b$

$R: x^2 + y^2 \leq a^2$

$\partial R \rightarrow$ is the boundry of R

Conditions : I.C. $u(x, y, 0) = f(x, y)$ in R

B.C. $u(x, y, t)$ is prescribed on ∂R

Grids are (x_i, y_j, t_n) , $u(x_i, y_j, t_n) = u_{ij}^n$

$t_n \rightarrow t_{n+1}$ Through the knowledge of u_{ij}^n obtain

N.T. The drawback of using Crank-Nicolson here or any implicit scheme

$u_t = c\nabla^2 u$, $u_{ij}^0 = f_{ij}$

B.C. $u_{ij}^n \ \forall \ i = 0, - N$

$j = 0, - M$

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$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u$$

Forward marching in time

$$u_{ij}^{n} \rightarrow u_{ij}^{n+1}, \quad \forall\, i,j$$

Crank-Nicolson scheme

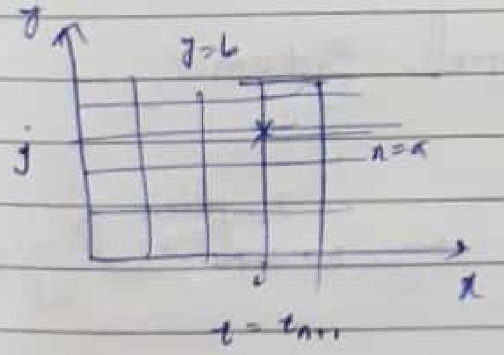
$$\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\delta t} = \frac{1}{2}\left[\frac{u_{i+1,j}^{n+1} - 2u_{ij}^{n+1} + u_{i-1,j}^{n+1}}{\delta x^2} + \frac{u_{i+1,j}^{n} - 2u_{ij}^{n} + u_{i-1,j}^{n}}{\delta x^2}\right]$$

$$+ \frac{1}{2}\left[\frac{u_{i,j+1}^{n+1} - 2u_{ij}^{n+1} + u_{i,j-1}^{n+1}}{\delta y^2} + \frac{u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n}}{\delta y^2}\right]$$

$$+\, O(\delta t^2, \delta x^2, \delta y^2)$$

$$\left.\begin{array}{l} i = 1, 2, \ldots, N-1 \\ j = 1, 2, \ldots, M-1 \end{array}\right\} \text{— } (*)$$

$$\text{B.C.} \left\{\begin{array}{l} u_{0j}^{n+1},\ u_{Nj}^{n+1}, \quad \forall\, j \\ u_{i0}^{n+1},\ u_{iM}^{n+1} \quad \forall\, i \end{array}\right.$$



To get u_{ij}^{n+1} for $i = 1, 2, \ldots, N-1$

$j = 1, 2, \ldots, M-1$

i.e. $(N-1) \times (M-1) \rightarrow$ unknowns.

& involves $(N-1) \times (M-1)$ number of equations — (*)

But, not a tri-diagonal system as every eq^n involves 5 unknowns i.e. u_{ij}^{n+1}, $u_{i+1,j}^{n+1}$, $u_{i-1,j}^{n+1}$, $u_{i,j+1}^{n+1}$, $u_{i,j-1}^{n+1}$

ADI scheme

This is not trivial to solve. So we do this discretization

$$t_n \xrightarrow{\text{step-1}} t_{n+1/2} \xrightarrow{\text{step-2}} t_{n+1}$$

Alternating ~~Discretization~~ Direction Implicit (ADI) scheme

To obtain solution at time step t_{n+1} using the solution at time step t_n, we go by two steps -

1) In step-1, an advancing from t_n to $t_{n+1/2}$ is done in which an implicit (explicit discritization) of derivaties w.r.t x (or y) is considered and an explicit (or implicit) discretization w.r.t y (or x) is made to obtain $u_{ij}^{n+1/2}$ $\forall\, i,j$

2) In Step-2, The solution is advanced from $t_{n+1/2}$ to t_{n+1} by reverse procedure i.e an explicit (or implicit) in x and implicit (or explicit) in y are made $\rightarrow u_{i,j}^{n+1}$, $\forall\, i,j$

we obtain a tri-diagonal System.

Now do the discretization by taking x - ~~explicit~~ Implicit & y explicit

$$\frac{\partial u}{\partial t} = \vartheta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$t_n \rightarrow t_{n+1/2}$ at (i,j) let $u_{ij}^{n+1/2}$

$$\frac{u_{ij}^{n+1/2} - u_{ij}^{n}}{\delta t} = \vartheta\left[\frac{\partial^2 u}{\partial x^2}\Big|_{i,j}^{n+1/2} + \frac{\partial^2 u}{\partial y^2}\Big|_{ij}^{n}\right]$$

$i = 1, 2, \_\_\_\_, N-1$

$j = 1, 2 \_\_\_\_ M-1$

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$$\left(\frac{\nu}{2\delta x^2}\right) u_{i-1,j}^{n+1/2} + \left(-\frac{\nu}{\delta x^2} - \frac{1}{\delta t}\right) u_{ij}^{n+1/2} + \left(\frac{\nu}{2\delta x^2}\right) u_{i+1,j}^{n+1/2} = -\frac{\nu}{2}\left(\frac{u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n}}{\delta y^2}\right) - \frac{u_{ij}^{n}}{\delta t}$$

At any fixed j

$$a_i u_{i-1,j}^{n+1/2} + b_i u_{ij}^{n+1/2} + c_i u_{i+1,j}^{n+1/2} = d_i$$

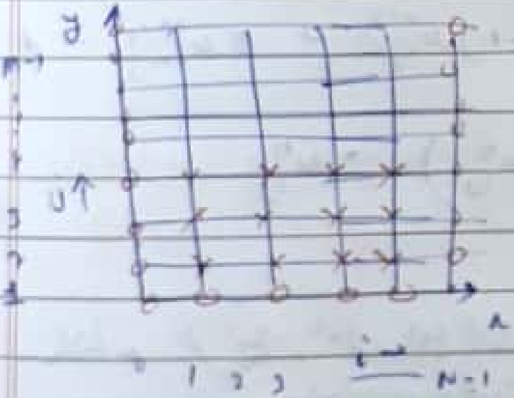
$$\Rightarrow \quad a_i = \frac{\nu}{2(\delta x)^2}, \quad b_i = \frac{-1}{\delta t} - \frac{\nu}{(\delta x)^2}$$

$$c_i = \frac{\nu}{2(\delta x)^2}, \quad d_i = -\left(\frac{\nu}{2}\right)\frac{\partial^2 u}{\partial y^2}\bigg|_{ij}^{n} - \frac{u_{ij}^{n}}{\delta t}$$

$$i = 1, 2, \_\_\_, N-1$$

$$A^j U_j^{n+1/2} = D^j, \quad U_j^{n+1/2} = \begin{bmatrix} u_{1j}^{n+1/2} \\ u_{2,j}^{n+1/2} \\ \vdots \\ u_{N-1,j}^{n+1/2} \end{bmatrix}$$

$$j = 1, 2, \_\_\_, M-1$$



For $j = 1, \_\_\_, M-1$

solve $A^j U_j^{n+1/2} = D^j$

we obtain $U_{ij}^{n+1/2}, \ \forall i, j$

Step-II $\quad t_{n+1/2} \rightarrow t_{n+1}$ $\quad$ This time x is explicit & y is implicit (reverse of step-1).

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+1/2}}{\delta t / 2} = \frac{\nu}{2}\left[\frac{\partial^2 u}{\partial x^2}\bigg|_{ij}^{n+1/2} + \frac{\partial^2 u}{\partial y^2}\bigg|_{ij}^{n+1}\right]$$

At a fixed i ($i = 1, 2, \_\_\_, N-1$)

$$A^i U_i^{n+1} = D^i$$

$$\frac{\vartheta}{2(\delta y)^2} u_{i,j-1}^{n+1} + \left(-\frac{\vartheta}{\delta y^2} - \frac{1}{\delta t}\right) u_{i,j}^{n+1} + \frac{\vartheta}{2(\delta y)^2} u_{i,j+1}^{n+1} = -\frac{\vartheta}{2}\left[\frac{u_{i,j+1}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i,j-1}^{n+1/2}}{\delta x^2}\right] - \frac{u_{i,j}^{n+1/2}}{\delta t}$$

$A^i \to \cancel{(M-1)\times(M-1)}$ $(M-1)\times(M-1)$ tri-diagonal system.

$$O(\delta t^2, \delta x^2, \delta y^2)$$

Q. $\frac{\partial u}{\partial t} = \nabla^2 u$, $-1 < x, y < 1$, $t > 0$

$u(x,y,0) = \cos\frac{\pi x}{2} \cos\frac{\pi y}{2}$

$u = 0$ on $x = \pm 1$, $y = \pm 1$

$\delta x = \delta y = 1/2$ & $r = \frac{1}{6}$ $\quad r = \frac{\delta t\,\vartheta}{\delta x^2}$

use ADI scheme & get tri-diagonal system at each step.

Sol

$$\frac{u_{i,j}^{n+1/2} - u_{i,j}^{n}}{\delta t} = \vartheta\left[\frac{u_{i+1,j}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i-1,j}^{n+1/2}}{\delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\delta x^2}\right]$$

$$\Rightarrow u_{i,j}^{n+1/2} - u_{i,j}^{n} = r(u \_\_\_\_)$$

$$\Rightarrow r u_{i-1,j}^{n+1/2} + (-2r-1) u_{i,j}^{n+1/2} + r u_{i+1,j}^{n+1/2} = d_i$$

$$d_i = -r(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}) - u_{i,j}^{n}$$

For the ADI scheme is $O(\delta t^2, \delta x^2, \delta y^2)$ which is same as the crank-nicolson scheme.
The von-neuman stability.

Poission's equation : CDS

$$\frac{\partial u}{\partial t} + c_1 \frac{\partial u}{\partial x} + c_2 \frac{\partial u}{\partial y} = \nu \nabla^2 u$$ Poission eqⁿ.

$\nabla^2 u = 0$ Laplace eqⁿ $\qquad \frac{\partial u}{\partial t} \to 0$

$\nabla^2 u = f(x,y) \to$ Poisson eqⁿ.

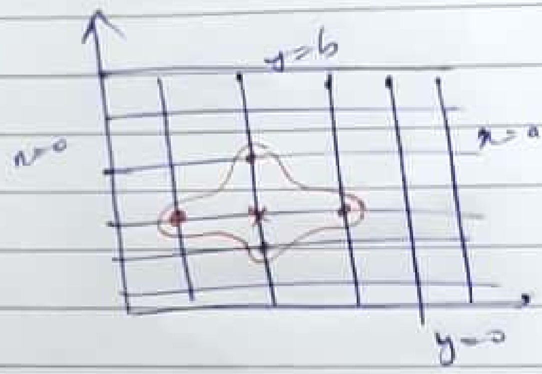
Elliptic PDE , a Boundary value problem.

$x, y \in R$, u is prescribed on ∂R.

$\nabla^2 T = 0$ $\to$ T on surface $\qquad x^2 + y^2 + z^2 \le a^2$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \qquad 0 < x < a$$
$$0 < y < b$$

u is prescribed at $x = 0, a$ & $y = 0, b$



$u_{ij} = u(x_i, y_j)$

$x_i = i\delta x$

$y_j = j\delta y$

use central diff scheme

$$\frac{u^{(k)}_{i+1,j} - 2u^{(k+1)}_{i,j} + u^{(k+1)}_{i-1,j}}{\delta x^2} + \frac{\left(u^{(k)}_{i,j+1} - 2u^{(k+1)}_{i,j} + u^{(k+1)}_{i,j-1}\right)}{\delta y^2} = f_{ij} \qquad \text{— } (*)$$

$i = 1, 2, \_\_\_, N-1$
$j = 1, \_\_\_, M-1$

B.C

u_{0j}, u_{Nj} for $j = 1, 2, \_\_\_, M-1$ are prescribed
$u_{i,0}, u_{iM}$ for $i = 1, 2, \_\_\_, N-1$ prescribed

$u_{i,j}$ for $i = 1, \_\_\_, N-1$ &
$j = 1, \_\_\_, M-1$

The system of eq^n $(*)$ are $(N-1)\times(M-1)$ numbers of equation & variable i.e. compact system.

$O(\delta x^2, \delta y^2)$

solve using gauss-seidel iteration. ~~as gaussian~~ ~~diminat~~

Gauss - Seidel iterative method to solve

Thus, at any iteration level $(k+1)$, $k \geq 0$

$$u_{ij}^{(k+1)} = \left[u_{i+1,j}^{(k)} + u_{i-1,j}^{(k+1)} + \beta^2 \left(u_{i,j+1}^{(k)} + u_{i,j-1}^{(k+1)} \right) - \delta x^2 f_{ij} \right] \times \frac{1}{2(1+\beta^2)}$$

$\boxed{\beta = \delta x / \delta y}$, $k \geq 0$ $\quad$ $i = 1, \ldots, N-1$

$j = 1, \ldots, M-1$

To start the iteration we guess u_{ij} $\forall\, i,j$

Iteration converges

$$\max_{i,j} \left| u_{ij}^{(k+1)} - u_{ij}^{(k)} \right| < \xi \quad , \forall\, k \geq K$$

$$|a_{ii}| \geq \sum_{j=1}^{n-1} |a_{ij}| \quad \forall\, i$$

$j \neq i \rightarrow$ sufficient condition.

Convergence may occur without being diagonally dominant.

But, very slow converges.

HT

Q1 $\quad \nabla^2 u = -10(x^2 + y^2 + 10)$ $\quad$ $0 < x < 3$, $0 < y < 3$

$u = 0$ on the boundary

$\delta x = 1$, $\delta y = 1$ $\quad$ u_{ij}

Q2 $\quad -\nabla^2 u + 0.1\, u = 1$, $\quad$ $0 \leq x, y \leq 1$

$u = 0$ on $x = 0$, $y = 0$

$\frac{\partial u}{\partial n} = 0$ on $x = 1$ & $y = 1$

n is the unit normal

$\delta x = \delta y = 0.5$

Rep $\rightarrow$ $\delta x = \delta y = 0.1, 0.05, \ldots,$

Q $\nabla^2 u = 2\frac{\partial u}{\partial x} - 2$, $0 \le x, y \le 1$

$u = 0$ on the boundaries.

$\delta x = \delta y = 1/3$

Get the discretized eqn.

Lab → $\delta x = \delta y = 0.1, 0.05$

Problem $\nabla^2 u = 0$ $0 \le x, y \le 1$

B.C.

$u = 2x$, $y = 0$; $u = 2x - 1$, $y = 1$ } $0 < x < 1$

$u_x + u = 2 - y$, $x = 0$; $u = 2 - y$, $x = 1$ } $0 \le y \le 1$

$h = k = 1/3$

Exact soln

$u(x, y) = 2x - y$

$u_1 = -1/3$, $u_2 = 1/3$

$u_3 = 1$, $u_4 = -2/3$

$u_5 = 0$

$u_6 = 2/3$

$h = \delta x$

$k = \delta y$

Sol $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

let $u_{ij} = u(x_i, y_j)$

$\Rightarrow \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\delta y)^2} = 0$

B.C.

$u_{i0} = 2x_i$

$u_{iM} = 2x_i - 1$

$\frac{\partial u}{\partial x}\Big|_{0,j} + u_{0j} = 2 - y_j$

$u_{Nj} = 2 - y_j$

u_{ij} , $i = 0, 1, \ldots, N-1$

$j = 0, 1, \ldots, M-1$

Fictitious point $u_{-1,j}$

$\frac{u_{0j} - u_{-1,j}}{\delta x} + u_{0j} = 2 - y_j$

$\Rightarrow -u_{-1,j} + (\delta x + 1)u_{0j} = 2 - y_j$

$\Rightarrow u_{-1,j} = \cancel{(u_{0j} - 2 + y_j)\delta x} + (\delta x + 1)u_{0j} - 2 + y_j$

$(N-1) \times (M-1)$ compact system $\quad N = \frac{1}{1/3} = 3 \quad M = 3$

2×2

$\cancel{u_{0,0}} - 2u_{1,0} + u_{2,0} + u_{0,1} - 2u_{1,1} + u_{2,1}$

six ~~convergence acceleration.~~

convergence acceleration.

$$u_{ij}^{(k+1)} = \frac{1}{2(1+\beta^2)}\left[u_{i-1,j}^{k} + u_{i+1,j}^{(k+1)} + \beta^2\left(u_{i,j+1}^{k+1} + u_{i,j-1}^{k+1}\right)\right]$$

$k \geq 0$

$$\left|\bar{u}_{ij} - u_{ij}^{(k)}\right| > \alpha \left|\bar{u}_{ij} - \bar{u}_{ij}^{(k+1)}\right| \qquad \alpha < 1$$

$k \quad k+1 \quad u_{ij}$

Successive – Over – Relaxation Technique (SOR)

Let $u_{ij}^{(k+1)}$ is the gauss-seidel iteration at $(k+1)^{th}$ iteration. Then to accelerate the convergence of the iteration procedure, we consider

$$\bar{u}_{ij}^{(k+1)} = \bar{u}_{ij}^{k} + \omega\left(u_{ij}^{(k+1)} - \bar{u}_{ij}^{(k)}\right)$$

$\bar{u}_{ij}^{(k)}$ = is the modified value at the i^{th} iteration

ω = is the relaxation parameter

over-relaxation if $1 < \omega < 2$

No correct procedure to choose ω. (Hit & trial)

if $\omega = 1$ then it is our normal gauss-seidel iteration method only.

under-relaxation, $0 < \omega < 1$

$k \quad (\overline{k+1}) \quad (k+1)$

H.T

eg: $\nabla^2 T = -100$, $0 < x < 3$, $0 < y < 6$

B.C.

$T = 0$, $x = 0$; $T = 200$, $x = 3$

$\frac{\partial T}{\partial y} = 100$, $y = 0$; $\frac{\partial T}{\partial y} = -100$, $y = 6$

solve using SOR Technique.

we also have multi-grid method and SSOR technique.

$u_{ij}^{(k+1)} \rightarrow \bar{u}_{ij}^{(k+1)}$

H.T

eg2 $-\nabla^2 u + 0.1 u = 1$, $0 \le x, y \le 1$

$u = 0$ on $x = 0$ & $y = 0$

$\frac{\partial u}{\partial x} = 0$, $x = 1$ & $y = 1$

$\delta x = \delta y = 0.5$ → LOB

eg B3 $\nabla^2 u = 2\frac{\partial u}{\partial x} - 2$, $0 \le x$, $y \le 1$

$u = 0$ on the boundary condition

$R : \begin{cases} 0 \le x \le 1, & 0 \le y \le 1 \\ \delta x = \delta y = 1/3 \end{cases}$

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H.T Q $\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = 0$

$$\left(\frac{0.5\,T_j^{n-1} - 2T_j^n + 1.5\,T_j^{n+1}}{\delta t}\right) - \alpha\left[(1+d)\frac{\left(T_{j-1}^n - 2T_j^n + T_{j+1}^n\right)}{(\delta x)^2} - d\frac{\left(T_{j-1}^{n-1} - 2T_j^{n-1} + T_{j+1}^{n-1}\right)}{(\delta x)^2}\right] = 0$$

3 - level explicit discretization

1) T.E. of any arbitary value of d
check for consistency.

2) Choose d as a function of $\frac{\alpha\,\delta t}{(\delta x)^2}$ so that the scheme is 4^{th}-order accurate in δx.

H.T Q $\frac{\partial \theta}{\partial t} = \alpha \nabla^2 \theta + \frac{q}{\rho C_p}$, $\theta(r, \phi)$

$$\frac{\partial \theta}{\partial t} = \alpha\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \theta}{\partial \phi^2}\right) + \frac{q}{\rho C_p}$$

θ is prescribed at $r=1$, $\frac{\partial \theta}{\partial r} = 0$ at $r=0$

$\theta(r, 0) = \theta(r, 2\pi)$

$\theta(0, r, \phi) = f(r, \phi) \rightarrow$ I.C.

We will use implicit ADI scheme to discretize.

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Hyperbolic PDE

wave eq^n $\quad \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

$(x,t) \longrightarrow (\xi, \eta)$ $\quad \xi = x - at$

$\eta = x + at$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\xi_t + \frac{\partial u}{\partial \eta}\eta_t$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\quad \right)$$

Reduced equation is $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$

$$\Rightarrow \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = 0 \Rightarrow \frac{\partial u}{\partial \eta} = h(\eta)$$

$$u = f(\xi) + g(\eta) = f(x+at) + g(x-at)$$

f, g are arbitary functions

$$u(x,0) = F(0) = f(x) + g(x)$$

I.C. $\quad u(x,0) = F(x) \quad , \quad \frac{\partial u}{\partial t}(x,0) = G(x)$

$$u(x,t) = f(x+at) + g(x-at) \quad \text{f, g arbitary}$$

$$u(x,0) = F(x) = f(x) + g(x) \quad -(i)$$

$$\frac{\partial u}{\partial t} = f'(x+at)\cdot a - a\, g'(x-at)$$

$$\frac{\partial u}{\partial t}(x,0) = af' - ag' = G(x) \quad -(ii)$$

After integrating

$$f(x) - g(x) = \frac{1}{a}\int_0^x G(\tau)\, d\tau + C_1$$

& put $g(x) = F(x)$

So, we get

$$f(x) = \frac{1}{2}F(x) + \frac{1}{2a}\int_0^x G(\tau)d\tau + c_1$$

$$g(x) = \frac{1}{2}F(x) - \frac{1}{2a}\int_0^x G(\tau)d\tau - c_1$$

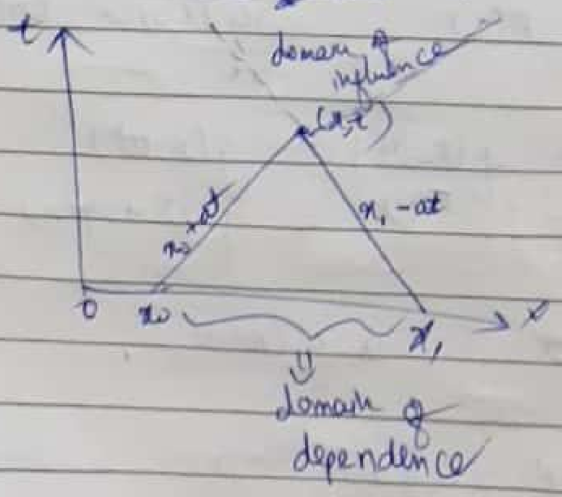
$$u(x,t) = f(x+at) + g(x-at)$$

$$= \frac{1}{2}\left[F(x+at) + F(x-at)\right]$$

$$+ \frac{1}{2a}\int_0^{x+at} G(\tau)d\tau - \frac{1}{2a}\int_0^{x-at} G(\tau)d\tau$$

$$u(x,t) = \frac{1}{2}\left[F(x+at) + F(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} G(\tau)d\tau, \quad t>0$$

It is called D'Alemberts solution of for PDE



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Numerical domain of dependence of $u(x_j, t_{n+1})$ should contain the analytic domain of dependence $\rightarrow$ Courant's condition.

3 - time step leap-frog scheme

Q
$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 , \quad t>0 , \quad 0<x<a$$

I.C. $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$

$u(a,t)$ & $u(b,t)$ are prescribed.

Sol central diff scheme

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\delta t^2} - c^2 \left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\delta x^2} \right) = 0$$

I.C. $u_j^0 = f_j$, $\frac{\partial u}{\partial t}(x,0) = g(x)$

$\Rightarrow \frac{u_j^1 - u_j^{-1}}{2\delta t} = g_j$, $u_j^{-1} = u_j^{1} - 2(\delta t) g_j$

$\vartheta = \frac{c\,\delta t}{\delta x}$

$n=0$

$$u_j^1 - 2u_j^0 + u_j^{-1} - \vartheta^2 (u_{j+1}^0 - 2u_j^0 + u_{j-1}^0) = 0$$

$$\Rightarrow u_j^1 = \vartheta^2 (u_{j+1}^0 + u_{j-1}^0) + 2(1-\vartheta^2) u_j^0 - (u_j^{1} - 2\delta t\, g_j)$$

$$\Rightarrow u_j^1 = \frac{\vartheta^2}{2} (u_{j+1}^0 + u_{j-1}^0) + (1-\vartheta^2) u_j^0 + \delta t\, g_j$$

$$u_j^1 = \frac{\vartheta^2}{2} (f_{j+1} + f_{j-1}) + (1-\vartheta^2) f_j + \delta t\, g_j$$

$j = 1, 2, \text{—} , N-1$

$u_j^0 , u_j^1 \rightarrow u_j^2$ (time marching Algo)

Q: $u_{tt} = c^2 u_{xx}$ $\qquad u(0,t) = u(1,t) = 0$

$u(x,0) = \sin\pi x \;,\; \frac{\partial u}{\partial t}(x,0) = 0$

$0 \le x \le 1$

$c = 1, \quad \delta x = 1/5 \;,\; r = \frac{\delta t}{\delta x} = 0.5$

Find $u_j^1 \;,\; u_j^2$

Sol:
$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\delta t^2} = c^2\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\delta x^2}\right)$$

$\Rightarrow$
$$u_j^{n+1} - 2u_j^n + u_j^{n-1} = r^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$\Rightarrow u_j^{n+1} = r^2\left(u_{j+1}^n\right) + r^2\left(u_{j-1}^n\right) + (2-2r^2)u_j^n - u_j^{n-1}$$

n = 0

$$u_j^1 = r^2\left(u_{j+1}^0 + u_{j-1}^0\right) + (2-2r^2)u_j^0 - u_j^{-1}$$

we have

$$\frac{\partial u}{\partial t}(x,0) = 0$$

$$\Rightarrow \frac{u_j^1 - u_j^{-1}}{2\delta t} = 0$$

$$\Rightarrow \boxed{u_j^{-1} = u_j^1}$$

$x_0 = 0$
$x_1 = \frac{1}{5}$
$x_2 = \frac{2}{5}$
$x_3 = \frac{3}{5}$
$x_4 = \frac{4}{5}$
$x_5 = 1$

$\Rightarrow$
$$u_j^1 = \frac{r^2}{2}\left(u_{j+1}^0 + u_{j-1}^0\right) + (1-2r^2)u_j^0$$

$u_j^0 = \sin\pi x_j \qquad j = 1, 2, 3, 4$

$N = \frac{1}{1/5} = 5$

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$$u_1^1 = \frac{\nu^2}{2}(u_2^0 + u_0^0) + (2 - 2\nu^2)u_1^0$$

$$= \frac{(0.5)^2}{2}\left(\sin\frac{2\pi}{5} + 0\right) + \left[2 - 2\times(0.5)^2\right]\left(\sin\frac{\pi}{5}\right)$$

$$= 0.01919034$$

$u_1^1 = 0.05597$
$u_2^1 = 0.9057$

$$u_2^1 = \frac{\nu^2}{2}(u_3^0 + u_1^0) + (2 - 2\nu^2)u_2^0$$

$$= \frac{(0.5)^2}{2}\left(\sin\frac{3\pi}{5} + \sin\frac{\pi}{5}\right) + \left(2 - 2(0.5)^2\right)\sin\left(\frac{2\pi}{5}\right)$$

$$= 0.03837888$$

$n-1, n \rightarrow n+1$

Similarly others

Q2 $u_{tt} = c^2 u_{xx}$, $u(x,0) = 0, \quad 0 \le x \le 100$

$= 100\sin\left(\frac{\pi(x-100)}{120}\right), \quad 100 < x \le 220$

$= 0, \quad 220 \le x \le 300$

$\frac{\partial u}{\partial t}(x,0) = 0$, BC $u(t,0) = 0$
$u(t,300) = 0$

$\delta x = 0.2$

$\nu = c\frac{\delta t}{\delta x} = 1$, $\delta x = 1$

$\bar{x} = \frac{x}{100}$

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial t} - \frac{c\,\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial t} + \frac{c\,\partial u}{\partial x}\right) = 0$$

$$dt = \frac{dx}{c} = \frac{du}{0}, \quad u = f(x - ct) \quad f = \text{arbitrary}$$

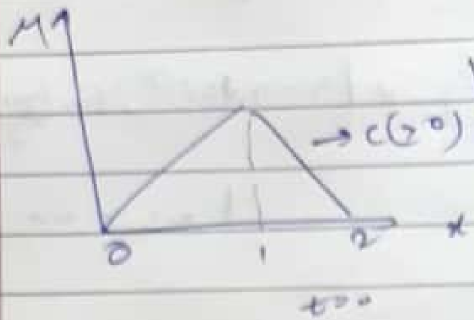
$u(x,t) = f(x - ct)$ $\quad u_t + c u_x = 0$

$u(x,0) = g(x)$

$u(0,t) = u_0, \quad t \geq 0$

$u(x,t) = f(x-ct)$

$$u(x,0) = \begin{cases} 0 & , \quad x < 0 \\ x & , \quad 0 < x < 1 \\ 2-x & , \quad 1 \le x \le 2 \\ 0 & , \quad x > 2 \end{cases}$$



$$u(x,t) = f(x-ct) = \begin{cases} 0 & , \quad x - ct < 0 \\ x - ct & , \quad t \le x, \ t \le 1 \ldots \\ 2 - x + ct & , \quad 1 + ct \le x \le 2 + c\ldots \\ 0 & , \quad x > 2 + ct \end{cases}$$

at $t = t_0 = 1$

$u(x, 1) =$

\# $u_t + c u_x = 0$, $c = c(u)$

↳ linear if c is a constant or $c = c(x,t)$
else non-linear

let us consider $c(u)$ is constant

If $u_t + c u_x = 0$, $u(x,0) = f(x)$, $x > 0$
$u(0,t) = u_0$, $t > 0$

$t_n \to t_{n+1}$ use FTCS
Forward time Central Scheme

can not use Implicit method here so, we try with some explicit method

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\,\delta x}\right) = 0$$

$$\Rightarrow u_j^{n+1} = u_j^n - \frac{\nu}{2}\left(u_{j+1}^n - u_{j-1}^n\right)$$

$\nu \rightarrow$ Courant number

This scheme comes out to be unconditionally unstable.

$\xi \rightarrow$ amplication factor. $\quad \nu = c\frac{\delta t}{\delta x}$

$$\cancel{\xi^2 - \xi\left(1 + \frac{\nu}{2}\right) + \frac{\nu}{2}}$$

$$\xi = 1 - \frac{\nu}{2}\left(e^{i\theta} - e^{-i\theta}\right) = 1 - \nu i \sin\theta$$

$$|\xi|^2 = 1 + \nu^2 \sin^2\theta > 1 \quad \forall\ \theta$$

FTCS fails

so, now, we try FTBS, forward time backward space.

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c\left(\frac{u_j^n - u_{j-1}^n}{\delta x}\right) = 0$$

$$\Rightarrow u_j^{n+1} = (1-\nu)u_j^n + \nu u_{j-1}^n, \quad n \geq 0, \quad j = 1, 2, 3 \ldots$$

$$\epsilon_j^n = A^n e^{i\theta j}, \quad \xi = \frac{A^{n+1}}{A^n}$$

$$\xi = \frac{A^{n+1}}{A^n} = (1-\nu) + \nu(\cos\theta - i\sin\theta)$$

$$|\xi|^2 = (1 - \nu + \nu\cos\theta)^2 + \nu^2 \sin^2\theta$$

$$\vartheta \le 1 \quad , \quad |\xi| \le 1 \ \forall \ \theta$$

$$\frac{c\,\delta t}{\delta x} \le 1$$

H.W

Q) $u_t + u_x = 0$, $u(x,0) = \begin{cases} x^2 & , \ 0 \le x \le 1 \\ 0 & , \ x > 1 \end{cases}$

$\delta x = 1/4$, $\vartheta = \delta t/\delta x = 1/2$

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I) $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$ FTCS is unconditionally unstable

II) FTBS $\frac{u_j^{n+1} - u_j^n}{\delta t} + c\frac{(u_j^n - u_{j-1}^n)}{\delta t} = 0$

T.E. $\sim O(\delta t, \delta x)$

$e_j^n = A^n e^{ij\theta}$, $\xi = \frac{A^{n+1}}{A^n}$, $|\xi| \le 1 \ \forall \theta$

$u_j^{n+1} = (1-\nu) u_j^n + \nu u_{j-1}^n$, $\nu = c\,\delta t/\delta x$ Courant number

$n \ge 0$, $j = 1, 2, \_\_\_\_,$ $|\xi| \le 1$

$\xi = (1-\nu) + \nu(\cos\theta - i\sin\theta)$

$|\xi|^2 = \cancel{(1-\nu)^2 + \nu^2(\cos^2\theta}$

$= (1-\nu+\nu\cos\theta)^2 + \nu^2\sin^2\theta$

$\Rightarrow |\xi|^2 = (1-\nu)^2 + \nu^2 + 2\nu\cos\theta(1-\nu)$

$= 1 - 2\nu(1-\nu)(1-\cos\theta)$

$\Rightarrow 2\nu(1-\nu)(1-\cos\theta) \ge 0$

$\Rightarrow 0 \le \nu \le 1$ | $\Rightarrow 1-\nu \ge 0$

$\Rightarrow \boxed{1 \ge \nu}$

$\Rightarrow \boxed{\nu \le 1}$

The scheme FTBS is stable provided $0 < \nu \le 1 \Rightarrow c\,\delta t/\delta x \le 1$, where $c > 0$

when $\underline{c < 0}$, $c(x,t)$

$\frac{\partial u}{\partial t} + c(x,t)\frac{\partial u}{\partial x} = 0$

$c < 0$, Apply FTFS

when $c > 0$, FTBS i.e. $c \frac{\partial u}{\partial x}\Big|_j^n = c_j^n \frac{u_j^n - u_{j-1}^n}{\delta x}$

when $c < 0$, FTFS i.e

$$c \frac{\partial u}{\partial x}\Big|_j^n = c_j^n \frac{u_{j+1}^n - u_j^n}{\delta x}$$

This is called upwind scheme

$\xrightarrow{c>0} \quad \xleftarrow{c<0}$

$j-1 \quad j \quad j+1$

M.T.

use upwind scheme to

$u_t + c u_x = 0$, $c = c(x,t)$

Find the condition for stability.

M.T. Q2

$u_t + u_x = 0$, $x > 0$, $t < 0$

$$u(x,0) = \begin{cases} x^2, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

$u(0,t) = 0$ $\delta x = 1/4$, $\vartheta = 1/2$

$u_j^{n+1} = (1-\vartheta) u_j^n + \vartheta u_{j-1}^n$ | $u_j^0 = x_j^2$, $u_0^n = 0$

n=0

$u_j^1 = (1-\vartheta) u_j^0 + \vartheta u_{j-1}^0$

$j = 1, 2, \ldots$

$x = t_1$

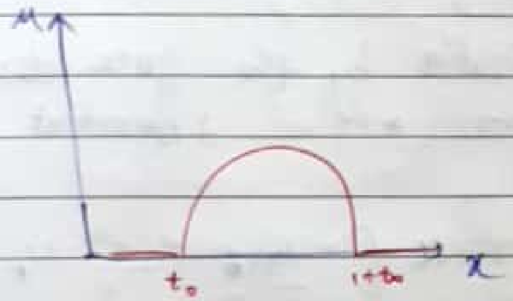
0.0156 0.0937 0.2812 0.5937

$$\frac{du}{1} = \frac{dt}{1} = \frac{du}{0}, \qquad u = f(x-t)$$

exact solution $\quad u = \begin{cases} (x-t)^2, & t \le x \le 1+t \\ 0, & x > 1+t \end{cases}$

$$\left(x - \frac{1}{4}\right)^2, \quad \frac{1}{4} \le x \le 1 + \frac{1}{4}$$

at $t = 2\delta t = \frac{1}{4}$ $\qquad u = \begin{cases} \left(x - \frac{1}{4}\right)^2, & \frac{1}{4} \le x \le \frac{5}{4} \\ 0, & x > \frac{5}{4} \end{cases}$



$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = 0 \qquad u = 0.1$$

advection speed

$$T(x,0) = \begin{cases} 200x, & 0 \le x \le 0.5 \\ 200(1-x), & 0.5 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$\delta x = 0.05 \qquad \vartheta = 0.1$

Solve & compare with exact solution.

$$\frac{u_i^{n+1} - u_i^n}{\delta t} + c\,\frac{u_i^n - u_{i-1}^n}{\delta x} = 0, \quad c > 0$$

FTBS

Expand by Taylor series abt (x_i, t_n)

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$$\frac{1}{\delta t}\Big[\Big\{u_j^n + \delta t\, u_t\big|_j^n + \frac{\delta t^2}{2} u_{tt}\big|_j^n + \frac{(\delta t)^3}{2} u_{tt}\big|_j^n + \frac{(\delta t)^3}{3!} u_{ttt}\big|_j^n + \text{——}\Big\} + \frac{c}{\delta x}\Big[u_j^n - \Big\{u_j^n - \delta x \ldots + \frac{\delta x^2}{2} u_{xx}\big|_j^n - \frac{\delta x^3}{3!} u_{xxx}\big|_j^n + \text{——}\Big\}\Big] = 0$$

$$u_t + c u_x = -\frac{\delta t}{2} u_{tt} + \frac{c\,\delta x}{2} u_{xx} - \frac{\delta t^2}{6} u_{ttt} - c\,\delta x^2 u_{xxx} + \text{——}$$

Replace time derivaties by the space derivaties in RHS
differentiate w.r.t $(*)$ — differentiate w.r.t (x)

$$u_{tt} = c^2 u_{xx} + \delta t\Big[-\frac{u_{ttt}}{2} + \frac{c^2}{2} u_{txx} + \text{——}$$

in a similar manner get u_{ttt} & substitute in $(*)$ to get

$$u_t + c u_x = c\frac{\delta x}{2}(1-\vartheta)u_{xx} - c\frac{\delta x^2}{6}(2\vartheta^2 - 3\vartheta + 1)u_{xx} + O\big(\delta x^3, \delta t\,\delta x^2, \text{——}\big]$$

n.t. find this modified equation

The least order term of the modified equation is
$\frac{c}{2}(1-\vartheta)\,\delta x\, u_{xx}$

Let $\mu = \frac{c}{2}(1-\vartheta)\,\delta x$

If $\vartheta \le 1$ then $\mu \ge 0$

then $\boxed{u_t + c u_x - \mu u_{xx} = 0}$ is called numerical viscosity

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The least order term i.e. $\mu\, \mu_{xx}$ involves 2nd-order space derivatives, which refers as the dissipative error involves even-order derivatives than the numerical scheme is referred as dissipative numerical scheme

If the error terms in the least-order involve odd order spatial derivatives. Then the numerical scheme is called dispersive scheme.

In hyperbolic type of PDE dissipative scheme is preferred as due to this dissipative term (-ve n-order spatial derivative) it spreads the sharp discontinuity in the variable whereas the dispersive error (odd order spatial derivative in T.E) leads to the formation of wiggles or oscillation near the discontinuity.

FTCS: Forward time central scheme --- Time- forward diff, space central diff (yt+c yx=0)

Lax scheme

In FTCS scheme ~~replace~~ (Euler scheme) replace μ_j^n with the average of two-time step solution as

$$\mu_j^n = \frac{1}{2}\left(\mu_{j+1}^n + \mu_{j-1}^n\right)$$

$$\Rightarrow \frac{\mu_j^{n+1} - \frac{1}{2}\left(\mu_{j+1}^n + \mu_{j-1}^n\right)}{\delta t} + c\left(\frac{\mu_{j+1}^n - \mu_{j-1}^n}{2\,\delta x}\right) = 0$$

This gives us a stable numerical scheme.

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K.T. modified eqⁿ

$$u_t + c u_x = \frac{c}{2}\delta x\left(\frac{1}{\nu} - \nu\right) u_{xx} + \frac{c(\delta x)^2}{3}(1-\nu^2) u_{xxx}$$

T.E. $\to 0$ as $\delta x, \delta t \to 0$

K.T. Stable if $|\nu| = |c|\frac{\delta t}{\delta x} \le 1$

T.E. $= O(\delta t, \delta x^2/\delta t)$

$$\mu = \frac{c}{2}\delta x\left(\frac{1}{\nu} - \nu\right)$$

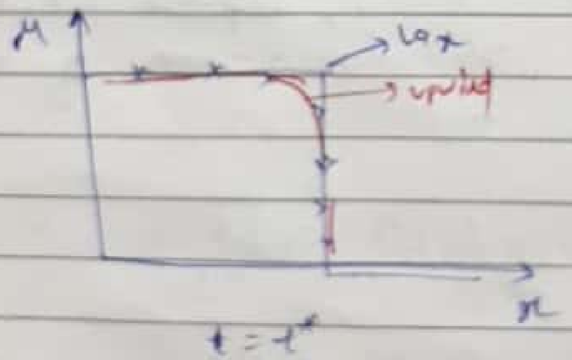
upwind we had

$$\mu = \frac{c}{2}\delta x(1-\nu)$$

Large dissipative error as compared to upwind scheme in Lax scheme

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{\nu}{2}\left(u_{j+1}^n + u_{j-1}^n\right)$$

$n \ge 0$



K.T. Solve by upwind scheme & Lax scheme

$$T_t + \mu T_x = 0$$

$T(x,0) = 200x, \quad 0 \le x \le 0.5$

$= 200(1-x), \quad 0.5 < x \le 1$

$= 0, \quad$ otherwise

$\mu = 0.1$

$\delta x = 0.05, \quad \nu = 0.1 \quad$ v=|c|(dt/dx)

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Lax - Wendroff scheme :

expand $u_j^{n+1} = u(x_j, t_{n+1})$ by Taylor series and replace the time derivatives by space derivatives using the PDE $u_t = -c u_x$

$$u_j^{n+1} = u_j^n + \delta t \left.\frac{\partial u}{\partial t}\right|_j^n + \frac{\delta t^2}{2} \left.\frac{\partial^2 u}{\partial t^2}\right|_j^n + O(\delta t^3)$$

$$u_t = -c u_x \quad , \quad u_{tt} = c^2 u_{xx}$$

$$u_{tt} = -c u_{xt} \quad ,$$
$$u_{xt} = -c u_{xx}$$

$$\frac{\partial^2 u}{\partial t^2} = -c \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = c^2 \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = c^2 u_{xx}$$

So,

$$u_j^{n+1} = u_j^n - c(\delta t) \left.\frac{\partial u}{\partial x}\right|_j^n + \frac{\delta t^2}{2} c^2 \left.\frac{\partial^2 u}{\partial x^2}\right|_j^n + O(\delta t^3)$$

Approximate the space derivatives by central diff scheme.
i.e.

$$u_j^{n+1} = u_j^n - \frac{c}{2}\frac{\delta t}{\delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$
$$+ \frac{c^2}{2}\frac{\delta t^2}{\delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) + O(\delta t^3)$$

T.E. is $O(\delta t^2, \delta x^2)$ and this explicit scheme is stable if $|\nu| \le 1$

Thus, the explicit one-step Lax - Wendroff Scheme is

$$u_j^{n+1} = u_j^n - \frac{c}{2}\frac{\delta t}{\delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{c^2 \delta t^2}{2\delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

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The amplication factor

H.T. $\xi = \left[(1-\nu^2) + \nu^2 \cos\theta\right] - i\nu \sin\theta$

~~Stability~~
Stability $|\xi| \le 1 \Rightarrow |\nu| \le 1$

$\cdot\,(j, n+1)$

$(j-1, n) \quad (j, n) \quad (j+1, n)$

Modified eqn for Lax - wendroff scheme

$$u_t + c u_x = -c \frac{(\delta x^2)}{6}(1-\nu^2) u_{xxx}$$

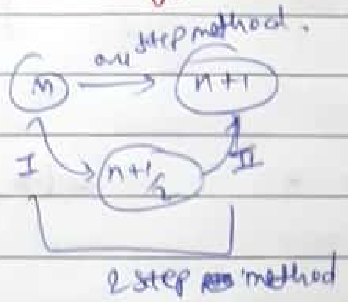
$$- c\frac{(\delta x^3)}{8} \nu (1-\nu^2) u_{xxxx} + \cdots$$

T. E. $\to 0$ as $\delta x\ \delta t \to 0$
T. E. is $O(\delta t^2, \delta x^2)$

The lowest order term in the modified eqn involves 3rd-order derivatives u_{xxx}, which produces a dispersion error $\to$ which leads to the formation of wiggles near a sharpness in the variable.

H.T. apply Lax - wendroff scheme to the previous problem

Two-step Lax-Wendroff scheme



Step-I Apply Lax method at the mid-point $(j+1/2)$, with half time step.

$$\frac{u_{j+1/2}^{n+1/2} - \frac{1}{2}\left(u_{j+1}^{n} + u_{j}^{n}\right)}{\delta t/2} + c\left(\frac{u_{j+1}^{n} - u_{j}^{n}}{2(\delta x/2)}\right) = 0$$

$$u_{j+1}^{n+1/2} = \frac{1}{2}\left(u_{j+1}^{n} + u_{j}^{n}\right) - \frac{\nu}{2}\left(u_{j+1}^{n} - u_{j}^{n}\right) \quad — (1)$$

correction(j+1/2)

which is the predictor step

Step-II (corrector-step)

central diff approximation at $(j, n+1/2)$ with half space and time step

$$\frac{u_{j}^{n+1} - u_{j}^{n}}{2(\delta t/2)} + c\left(\frac{u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}}{2(\delta x/2)}\right) = 0$$

$$u_{j}^{n+1} = u_{j}^{n} - \nu\left(u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}\right) \quad — (2)$$

if $u_{j+1/2}^{n+1/2}$ & $u_{j-1/2}^{n+1/2}$ are eliminated by using (I) we get the single step Lax-Wendroff scheme

T.E. $\sim O(\delta t^2, \delta x^2)$ dissipative and stable for $|\nu| \le 1$

~~HW~~ Q (a) $u_t + c\,u_x = 0$, $u(x,0) = \exp(-20(x-2)^2) + \exp(-(x-5)^2)$

Solve upto $t = 17$, $0 \le x \le 25$

$h = \delta x = 0.05$, $\nu = 0.8$

(b) $u(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$

(c) $u(x,0) = 4x(1-x)$, $0 < x < 1$

$= 0$, $x \ge 1$

Non-linear PDE

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0$$

$$F = F(u) = \frac{1}{2}u^2$$

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0, \quad F = F(u)$$

~~HW~~ Derive single step Lax-Wendroff scheme for (A)

i.e.

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} = -\frac{\partial F}{\partial x} = -\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} = -A\frac{\partial u}{\partial x}$$

$$A = \frac{\partial F}{\partial u}, \quad \text{Jacobian} =$$

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 F}{\partial t\,\partial x} = -\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial t}\right), \quad \left|\ \frac{\partial F}{\partial t} = \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial t} = A\frac{\partial u}{\partial t}\right.$$

$$= -\frac{\partial}{\partial x}\left(A\frac{\partial u}{\partial t}\right)$$

i.e.

$$\frac{\partial^2 u}{\partial t^2} = \frac{-\partial}{\partial x}\left(A\frac{-\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(A\frac{\partial F}{\partial x}\right)$$

If u is a vector then A is a Jacobian.

H.T → Get the single-step lax-wendroff scheme from here.

Q Use this lax-wendroff scheme $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$

$$u(x,0) = \sqrt{x}, \quad 0 \le x \le 1$$
$$= 0, \quad \text{otherwise}$$

$\delta x = 0.2, \quad \nu = 0.5$

Q2 $u_t + u u_x = 0, \quad u(x,0) = 1, \quad 0 \le x \le 2$

$= 0, \quad 2 \le x \le 4$

Q) $\mu_t + \cancel{\ } \overset{\mu}{} \mu_x = 0,$

$$\mu(x,0) = \begin{cases} 20x, & 0 < x \le 0.05 \\ 20(1-x), & 0.05 \le x < 0.1 \\ 0, & 0.1 < x \le 1 \end{cases}$$

$$\mu(0,t) = 0, \quad t \ge 0$$

$$\vartheta = 0.8 \qquad \delta x = 0.05$$

$$\frac{\partial \mu}{\partial t} + \frac{\partial F}{\partial x} = 0$$

$$\mu_j^{n+1} = \mu_j^n - \delta t \left.\frac{\partial F}{\partial x}\right|_j^n + \frac{\delta t^2}{2} \frac{\partial}{\partial x}\left(A \frac{\partial F}{\partial x}\right)\Big|_j^n + O(\delta t^3)$$

use central diff scheme

$$\mu_j^{n+1} = \mu_j^n - \frac{\delta t}{\delta x}\left(\frac{F_{j+1}^n - F_{j-1}^n}{2}\right) + \frac{\delta t^2}{2\delta x^2}\Big[A_{j+\frac{1}{2}}^n \left(F_{j+1}^n - F_j^n\right) - A_{j-\frac{1}{2}}^n \left(F_j^n - F_{j-1}^n\right)$$

$$\left.\begin{aligned} A_{j+\frac{1}{2}}^n &= A\left(\frac{\mu_j^n + \mu_{j+1}^n}{2}\right) \\ A_{j-\frac{1}{2}}^n &= A\left(\frac{\mu_j^n + \mu_{j-1}^n}{2}\right) \end{aligned}\right\} \text{Jacobian}$$

Dt - 4/4/22

MacCormack Scheme : Similar to Lax-Wendroff scheme, we expand u_j^{n+1} by Taylor series

$$u_j^{n+1} = u_j^n + \frac{\partial u}{\partial t}\Big|_j^n \delta t + \frac{\partial^2 u}{\partial t^2}\Big|_j^n \frac{\delta t^2}{2} + O(\delta t^3) \quad \text{--- (1)}$$

Unlike Lax-Wendroff scheme we donot replace u_{tt} by the PDE. Instead we consider first order expansion of $u_t|_j^n$.

$$u_t\big|_j^{n+1} = u_t\big|_j^n + \delta t\, u_{tt}\big|_j^n + O(\delta t^2)$$

$$u_{tt}\big|_j^n = \frac{\left(u_t\big|_j^{n+1} - u_t\big|_j^n\right)}{\delta t} + O(\delta t)$$

Substitute in (1)

$$u_j^{n+1} = u_j^n + u_t\big|_j^n \delta t + \frac{\delta t}{2}\left[u_t\big|_j^{n+1} - u_t\big|_j^n\right] + O(\delta t^3)$$

$$= u_j^n + \frac{\delta t}{2}\left[u_t\big|_j^n + u_t\big|_j^{n+1}\right] + O(\delta t^3)$$

now,

$$u_t = c u_x \quad , \quad \text{replace } u_t\big|_j^n$$

so,

$$u_j^{n+1} = u_j^n - \frac{c\,\delta t}{2}\left[u_x\big|_j^n + u_x\big|_j^{n+1}\right] + O(\delta t^3)$$

replace u_x by central diff scheme it will be an implicit scheme of T.E. $O(\delta x^2, \delta t^3)$ which is difficult to solve

MacCormack proposed a predicted u_j^{n+1}, denoted by $\bar{u}_j^{n+1}$ through a predictor step obtained by FTFS

predictor step for $u_t + c u_x \rightarrow$ FTFS

$$\bar{u}_j^{n+1} = u_j^n - \vartheta\left(u_{j+1}^n - u_j^n\right) \quad — (1)$$

$$\vartheta = \frac{c\,\delta t}{\delta x}$$

Using $\bar{u}_j^{n+1}$ evaluate $u_x|_j^n$ by first order forward scheme and $u_x|_j^{n+1}$ by first-order backward difference

$$u_j^{n+1} = u_j^n - \frac{c\,\delta t}{2}\left[u_x|_j^{n+1} + u_x|_j^n\right] + O(\delta t^3)$$

$$u_j^{n+1} = u_j^n - \frac{c\,\delta t}{2}\left[\frac{\left(u_{j+1}^n - u_j^n\right)}{\delta x} + \frac{\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}}{\delta x}\right]$$

$$= \frac{1}{2}u_j^n + \frac{1}{2}\left[u_j^n - \vartheta\left(u_{j+1}^n - u_j^n\right)\right] - \frac{\vartheta}{2}\left[\left(\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}\right)\right]$$

$$= \frac{1}{2}\left(u_j^n + \bar{u}_j^{n+1}\right) - \frac{\vartheta}{2}\left(\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}\right)$$

Corrector step

$$u_j^{n+1} = \frac{1}{2}\left(u_j^n - \bar{u}_j^{n+1}\right) - \frac{\vartheta}{2}\left(\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}\right) \quad — (II)$$

with predictor step

$$\bar{u}_j^{n+1} = u_j^n - \vartheta\left(u_{j+1}^n - u_j^n\right) \quad — (I)$$

For a linear PDE, the MacCormack P.C method and Lax - Wendroff method are identical.

N.I.

$T.E \sim O(\delta x^2, \delta t^2)$

stability $\nu = |c| \frac{\delta t}{\delta x} \leq 1$

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0$$ Euler equation, $F(u)$ is flux

$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0$, $F = F(u)$

MacCormack PC method

predictor step

$$\bar{u}_j^{n+1} = u_j^n - \frac{\delta t}{\delta x}\left(F_{j+1}^n - F_j^n\right)$$

N.I.

Corrector step

$$u_j^{n+1} = \frac{1}{2}\left[\bar{u}_j^n + u_j^n\right] - \frac{\delta t}{2\delta x}\left[\bar{F}_j^{n+1} - \bar{F}_{j-1}^{n+1}\right]$$

$u_t + c u_x = 0$

$$u_j^{n+1} = \frac{1}{2}\left[u_j^n + \bar{u}_j^{n+1} - \frac{c\,\delta t}{\delta x}\left(\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}\right)\right]$$

here $\bar{F}_j^{n+1} = F(\bar{u}_j^{n+1})$

& $F_j^n = F(u_j^n)$

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classmate

Date

Page

1. $u_t + u u_x = 0$, $u(x,0) = 1$; $0 \le x \le 2$

$u(x,0) = 0$; $2 \le x \le 4$

Solve by both Lax - wendroff & MacCromack method

2. $u_t + u u_x = 0$, $u(x,0) = x$, $x \ge 0$

$= 0$, $x < 0$