Lecture 15

Measure in $\mathbb{R}^d$

A (closed) rectangle R in $\mathbb{R}^d$ is given by the product of d one dimensional closed & bounded intervals.

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] \subseteq \mathbb{R}^d$$

where $a_j \le b_j \quad \forall j$

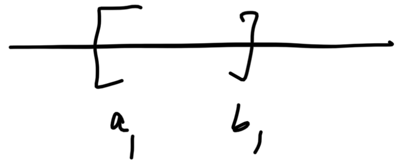
$$= \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \;\middle|\; \begin{array}{l} a_j \le x_j \le b_j \\ \forall j = 1, 2, \ldots, d \end{array} \right\}.$$

Remark:- R is closed & has sides parallel to the co-ordinate axes.

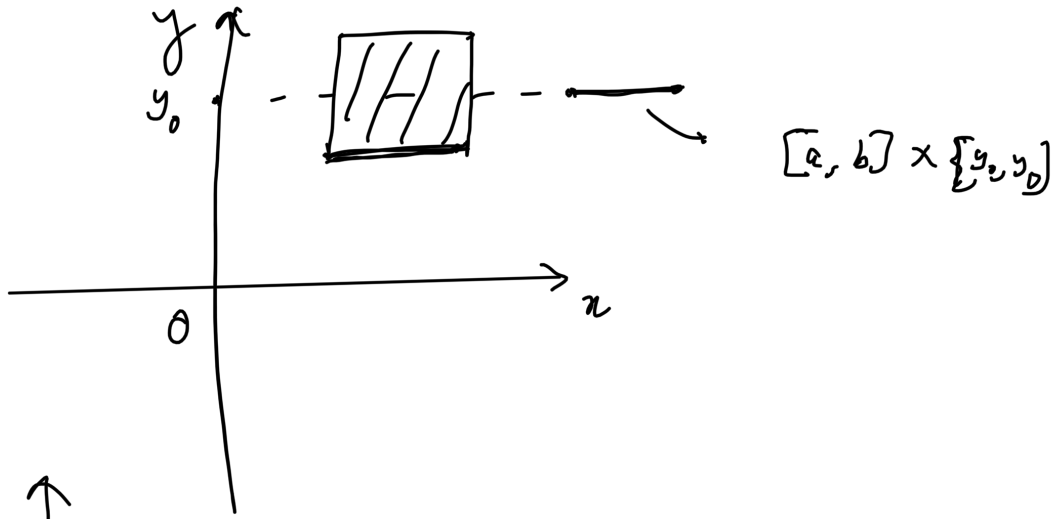
The volume of R is denoted by

$$|R| = (b_1 - a_1)(b_2 - a_2) \cdots (b_d - a_d).$$

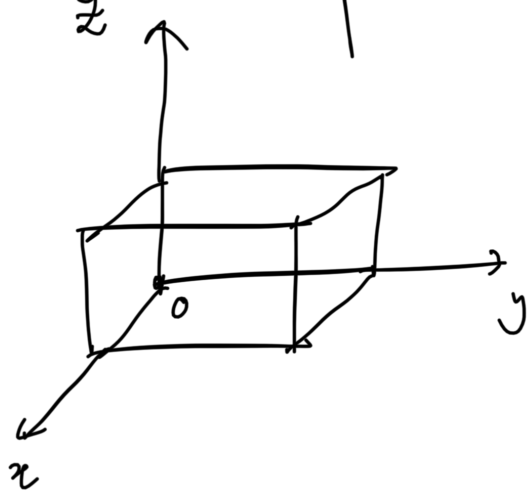
$d=1$: R = closed interval



$d=2$: R = closed rectangle



$d=3$:



<u>Def</u>:- A <u>cube</u> is a rectangle R for which all sides have the same length.

That is, $b_1 - a_1 = b_2 - a_2 = \cdots = b_d - a_d$.

<u>Lemma</u>:- Let $R, R_1, R_2, \ldots, R_n$ be rectangles in $\mathbb{R}^d$ & $R \subseteq \bigcup_{j=1}^{n} R_j$. Then $|R| \leq \sum_{j=1}^{n} |R_j|$

Theorem: Every open set in $\mathbb{R}^d$ can be written as a countable union of almost disjoint closed cubes.

Def: A union of rectangles is said to be <u>almost disjoint</u>, if the interiors of the rectangles are disjoint.

Example:

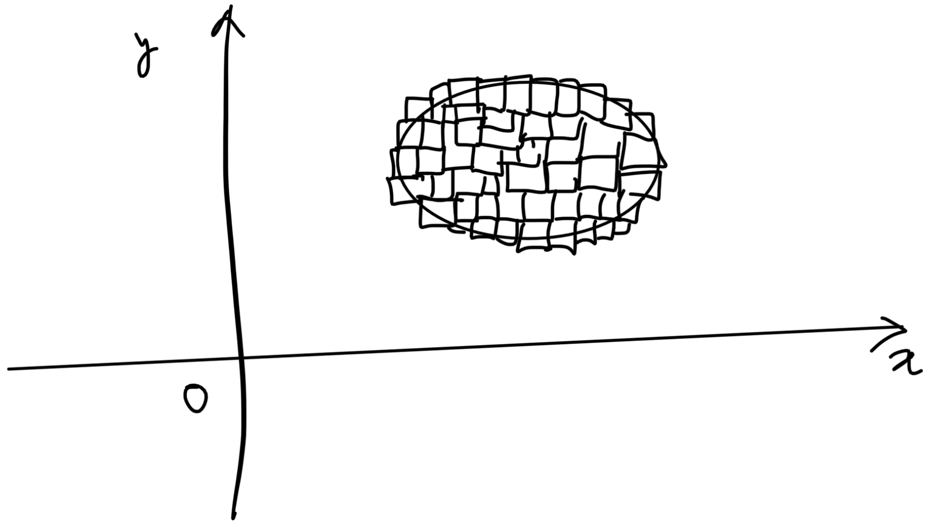
R_1, R_2 are almost disjoint.

R_1^o = interior of R_1

$R_1^o \cap R_2^o = \phi$.

Def: Let $E \subseteq \mathbb{R}^d$. Then the <u>outer measure</u> or <u>Lebesgue outer measure</u> or <u>exterior measure</u> of E is defined as

$$m^*(E) := \inf \left\{ \sum_{j=1}^{\infty} |Q_j| \;\middle|\; E \subseteq \bigcup_{j=1}^{\infty} Q_j \text{ \& } Q_j \text{ are closed cubes} \right\}.$$



Proposition :- In the definition, replace the covering by cubes with covering by rectangles, or with covering by balls, then it yields the same outer measure.

Examples :-

① $\{\underline{a}\} \subseteq \mathbb{R}^d$ $\qquad \underline{a} = (a_1, \dots, a_d)$

any point is a cube with volume zero.

$\therefore m^*(\{\underline{a}\}) = 0.$

② $Q \subseteq \mathbb{R}^d$ be a closed cube. Then

$$m^*(Q) = |Q|.$$

proof of ② :-

Q is covered by $\{Q\}$ itself.

$\therefore \; m^*(Q) \le |Q|$ (by def.)

For reverse inequality, let

$$Q \subseteq \bigcup_{j=1}^{\infty} Q_j \quad , \text{ where } Q_j \text{ cubes.}$$

Let $\varepsilon > 0$. Choose for each j, an open cube S_j such that $Q_j \subseteq S_j$ &

$$|S_j| \le (1+\varepsilon)\,|Q_j| \qquad \forall j$$

$$\therefore \; Q \subseteq \bigcup_{j=1}^{\infty} Q_j \subseteq \bigcup_{j=1}^{\infty} \underbrace{S_j}_{\text{open}}$$

Since Q is compact ($\because$ it is closed & bounded), there exists $n \in \mathbb{N}$ such that

$$Q \subseteq \bigcup_{j=1}^{n} S_j \subseteq \bigcup_{j=1}^{n} \overline{S_j}, \text{ where } \overline{S_j} \text{ closed cube}$$

$$\Rightarrow \; |Q| \le \sum_{j=1}^{n} |\overline{S_j}| = \sum_{j=1}^{n} |S_j| \qquad (\because |A| = |\overline{A}|,\ A \text{ cube}).$$

$$\le \sum_{j=1}^{n} (1+\varepsilon)\,|Q_j|$$

$$\leq (1+\varepsilon)\left(\sum_{j=1}^{\infty} |Q_j|\right)$$

$$\therefore \quad |Q| \leq (1+\varepsilon)\left(\sum_{j=1}^{\infty} |Q_j|\right)$$

for any $\varepsilon > 0$.

$$\Rightarrow \quad |Q| \leq \sum_{j=1}^{\infty} |Q_j|$$

for any $\{Q_j\}$ such that $Q \subseteq \bigcup_j Q_j$

$$\Rightarrow \quad |Q| \leq m^*(Q).$$

$$\therefore \quad m^*(Q) = |Q|.$$

3 If $Q \subseteq \mathbb{R}^d$ is any open cube, then $m^*(Q) = |Q|$

$\left(\because |Q| = |\bar{Q}| = m^*(\bar{Q}),\ Q \subseteq \bar{Q}\right)$

4 If $R \subseteq \mathbb{R}^d$ is a rectangle, then $m^*(R) = |R|$.

5 $m^*(\mathbb{R}^d) = \infty$

Properties of outer measure

① Given $\varepsilon > 0$, there exists a covering $\{Q_j\}$ of $E \subseteq \mathbb{R}^d$ (of closed cubes) such that

$$\sum_{j=1}^{\infty} |Q_j| \leq m^*(E) + \varepsilon.$$

② (Monotonicity) $E_1 \subseteq E_2 \subseteq \mathbb{R}^d \Rightarrow m^*(E_1) \leq m^*(E_2)$.

③ (Countably subadditive) If $E = \bigcup_{j=1}^{\infty} E_j$ in $\mathbb{R}^d$, then $m^*(E) \leq \sum_{j=1}^{\infty} m^*(E_j)$

④ For $E \subseteq \mathbb{R}^d$, $m^*(E) = \inf \left(m^*(U) \right)$, where infimum is taken over all open sets $U \subseteq \mathbb{R}^d$ containing E.

⑤ Let $E = \bigcup_{j=1}^{\infty} Q_j$ almost disjoint closed cubes Q_j, Then $m^*(E) = \sum_{j=1}^{\infty} |Q_j|$.

Def:- A subset $E \subseteq \mathbb{R}^d$ is said to be Lebesgue measurable or measurable, if given $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}^d$ such that $E \subseteq U$ & $m^*(U \setminus E) \leq \varepsilon$.

If E is measurable, then we define the Lebesgue measure or measure $m(E)$ by

$$m(E) := m^*(E).$$
