After Midterm

Date 03 10 2023

How to solve primal using dual or vice versa?

Q- Apply simplex method to solve the following LPP-

$$\text{Max } Z = 30x_1 + 23x_2 + 29x_3$$
$$\text{s.t. } 6x_1 + 5x_2 + 3x_3 \leq 26$$
$$4x_1 + 2x_2 + 5x_3 \leq 7$$
$$x_1, x_2, x_3 \geq 0$$

From the final table find the optimal solution of the dual problem.

Dual →

$$\text{min } w = 26v_1 + 7v_2$$
$$\text{s.t. } 6v_1 + 4v_2 \geq 30$$
$$5v_1 + 2v_2 \geq 23$$
$$3v_1 + 5v_2 \geq 29$$
$$v_1, v_2 \geq 0$$

Std. form- (Primal)

$$\text{Max } Z = 30x_1 + 23x_2 + 29x_3 + 0.x_4 + 0.x_5$$
$$\text{s.t. } 6x_1 + 5x_2 + 3x_3 + x_4 = 26$$
$$4x_1 + 2x_2 + 5x_3 + x_5 = 7$$
$$x_j \geq 0, \ j = 1, 2, 3, 4, 5$$

			c_j	30	23	29	0	0	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	
0	a_4	x_4	26	6	5	3	1	0	52/12
0	a_5	x_5	7	$\boxed{4}$	2	5	0	1	21/12 →
			$z_j - c_j$	−30 ↑	−23	−29	0	0	

0	a_4	x_4	31/2	0	2	−3/2	1	−3/2	31/4
✓ 30	a_1	x_1	7/4	1	$\boxed{1/2}$	5/4	0	1/4	14/4 →
			$z_j - c_j$	0	−8 ↑	17/2	0	1/2	

final Table →.

0	a_4	x_4	17/2	−4	0	−19/2	1	−5/2
23	a_2	x_2	7/2	2	1	5/2	0	1/2
			$z_j - c_j$	16	0	57/2	$\underbrace{0}$	$\underbrace{23/2}$

Optimal soln for primal –

$x_1 = 0$, $x_2 = 7/2$, $x_3 = 0$

$Z_{max} = \frac{161}{2}$

Optimal soln for dual –

$v_1 = 0$, $v_2 = \frac{23}{2}$

$Z_{max} = \frac{161}{2}$

c_j, $C_B B^{-1} a_j$

Q- By solving the dual of the following problem, show that the given problem has no feasible solution-

min. $z = x_1 - x_2$

s.t. $2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$

$x_1, x_2 \geq 0$

Dual→ max $z = 2v_1 + 2v_2$

s.t. $2v_1 - v_2 \leq 1$

$-v_1 + v_2 \geq 1$

$v_1, v_2 \geq 0$

Std. form-

max $w = 2v_1 + 2v_2 + 0\cdot v_3 + 0\cdot v_4 - M v_5$

s.t. $2v_1 - v_2 + v_3 = 1$

$-v_1 + v_2 - v_4 + v_5 = 1$

$v_1, v_2, v_3, v_4, v_5 \geq 0$

			c_j	2	2	0	0	-M	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	Min. Ratio
0	a_3	v_3	1	2	-1	1	0	0	
-M	a_5	v_5	1	-1	1	0	-1	1	
		$z_j - c_j$ →		M-2	-M-2	0	M ↑	0	

~~Cannot find departing vector. dual is unbounded.~~

Final→

2	a_1	v_1	2	1	0	1	-1	1
1	a_2	v_2	3	0	1	1	-2	2
		$z_j - c_j$		0	0	3	-4	4+M

Unable to decide leaving variable so this problem has unbounded solution so primal has infeasible solution

Primal	Dual	Conclusion
f.s	f.s	both have optimal f.s.
No f.s	f.s.	dual obj. func. is unbounded
f.s	No. f.s.	primal obj. func. is unbounded
No. f.s.	No. f.s.	func. is unbounded No. soln exists

Duality Theorem

Theorem If any of the constraints in the primal problem is a perfect equality, then the corresponding dual variable is unrestricted in sign.

Theorem If any variable of the primal problem is unrestricted in sign then the corresponding constraint of the dual is an equality.

Theorem Dual of the dual is the primal

Theorem If x is any f.s. to the primal problem and v is any f.s. to the dual problem then

$$cx \leq b^T v$$

$$\begin{array}{ll} \max\ z = cx & \min\ \omega = b^T v \\ Ax \leq b & \text{s.t. } A^T v \geq c^T \\ x \geq 0 & v \geq 0 \end{array}$$

Theorem If x^* is a f.s. of the primal problem and v^* is the f.s. to the dual problem such that

$$cx^* = b^T v^*$$

then both x^* and v^* are optimal solution to the respective problem.

Theorem: Fundamental Theorem of Duality — A f.s. x^* to the primal problem is optimal iff there exists a f.s v^*

to the dual problem such that

$$c x^* = b^T v^*$$

If a finite opt. soln exists for primal then there exists a finite opt soln for the dual and conversely.

Revised Simplex Method

Solving LPP with less labour-

Example → Use the revised simplex method to solve the LPP -

$$\text{Max } Z = x_1 + x_2$$

$$\text{s.t. } 3x_1 + 2x_2 \le 6$$

$$x_1 + 4x_2 \le 4$$

$$x_1, x_2 \ge 0$$

$$y_j = B^{-1} a_j, \quad x_B = B^{-1} b, \quad z_j - c_j = c_B y_j - c_j$$

$$= c_B B^{-1} a_j - c_j$$

→ only transformed quantities are needed

If a_k is the entering vector.

y_k not all y_j, $j = 1, 2, \ldots, n$

only X_B, z, $c_B B^{-1}$, B^{-1}

→ are transformed not all y_j

$$\text{Max } z = x_1 + x_2 + 0 \cdot x_3 + 0 \cdot x_4$$

$$\text{s.t. } 3x_1 + 2x_2 + x_3 = 6$$

$$x_1 + 4x_2 + x_4 = 4 \quad \longrightarrow \quad \begin{array}{l} \text{max } z = cx \\ Ax \le b \\ x \ge 0 \end{array}$$

$$x_1, x_2, x_3, x_4 \ge 0$$

Rewrite

$$\begin{aligned} z - x_1 - x_2 \quad &= 0 \\ 3x_1 + 2x_2 + x_3 \quad &= 6 \\ x_1 + 4x_2 \quad + x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

$\rightarrow A^* x^* = b^*$

$x^* \geq 0$

$A^* = \left[\begin{array}{c|c} 1 & -c \\ \hline 0 & B \end{array}\right]$

$a_0^* = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad a_1^* = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ $\quad a_2^* = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$

$a_3^* = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad a_4^* = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad b^* = \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix}$

Initial basis :-

$B^* = (a_0^*, a_3^*, a_4^*)$ → ~~$(\beta_0^*, \beta_1^*, \beta_2^*)$~~

$(B^*)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\beta_0^*, \beta_1^*, \beta_2^*)$.

$x_B^* = (B^*)^{-1} b^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix}$

Determining the entering vector

(Compute $z_j - c_j$ correspond to non-basic vectors)

$z_1 - c_1 = (1 \; 0 \; 0) \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = -1$

↓ first row of $(B^*)^{-1} \times a_j$

$z_2 - c_2 = (1 \; 0 \; 0) \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = -1$

Min $\{-1, -1\} \xRightarrow{\text{tie}}$ Take $k = 1$ $\quad a_k^*$ entering

- Determining departing vector

Compute $y_1^* = (B^*)^{-1} a_1^* = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

2nd & 3rd col. of $(B^*)^{-1}$

y_k column

Table 1

Basis (B^*)	β_1^*	β_2^*	x_B^*	y_1^*	Min Ratio
a_0^*	0	0	0	-1	
a_3^*	1	0	6	3	$6/3 = 2 \rightarrow$
a_4^*	0	1	4	1	$4/1 = 4$
				$\uparrow$	

a_3^* is the leaving vector.

$$\bar{\beta}_1^* = \begin{pmatrix} 1/3 \\ 1/3 \\ -1/3 \end{pmatrix} \qquad \bar{\beta}_2^* = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \qquad x_B^* = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$B^* = (a_0^*, a_1^*, a_4^*)$$

$$(B^*)^{-1} = (\bar{\beta}_0^*, \bar{\beta}_1^*, \bar{\beta}_2^*)$$

Find entering vector →

$z_2 - c_2 = (\text{first row } (B^*)^{-1}) \times a_2^* = \begin{pmatrix} 1 & 1/3 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = -1/3$

~~$z_3 - c_3 =$~~

$z_3 - c_3 = \begin{pmatrix} 1 & 1/3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1/3$

a_2^* is entering vector.

(See case for artificial variable)

-determining departing vector

mpute. $y_2^* = (B^*)^{-1} a_3^* = \begin{pmatrix} 1 & 1/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & -1/3 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \\ 10/3 \end{pmatrix}$

Table 2

→ entering vector

Basis (B^*)	β_1^*	β_2^*	x_B^*	y_2^*	Min Ratio
a_0^*	$1/3$	0	2	$-1/3$	
a_1^*	$1/3$	0	2	$2/3$	3
a_4^*	$-1/3$	1	2	$10/3$	$3/5$ →
				↑	

a_4^* is the leaving vector

$\bar{\beta}_1^* = \begin{pmatrix} 3/10 \\ 2/5 \\ -1/10 \end{pmatrix}$ $\bar{\beta}_2^* = \begin{pmatrix} 1/10 \\ -1/5 \\ 3/10 \end{pmatrix}$ $\bar{x}_B^* = \begin{pmatrix} 11/5 \\ 8/5 \\ 3/5 \end{pmatrix}$

$B^* = (a_0^*, a_1^*, a_2^*)$

$(B^*)^{-1} = (\bar{\beta}_0^*, \bar{\beta}_1^*, \bar{\beta}_2^*)$

$z_3 - c_3 = (\text{first row of } (B^*)^{-1}) \times a_3^* = \begin{pmatrix} 1 & 3/10 & 1/10 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{3}{10} >$

$z_4 - c_4 = \begin{pmatrix} 1 & \frac{3}{10} & \frac{1}{10} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{10} > 0$

Optimal condition reached. Optimal solution ↓

Table 3 -

Basis B^*	$\hat{\beta}_1^*$	$\hat{\beta}_2^*$	$\hat{x}_B^*$	y_k^*
a_0^*	$3/10$	$1/10$	$11/5$	
a_1^*	$2/5$	$-1/5$	$8/5$	
a_2^*	$-1/10$	$3/10$	$3/5$	

$x_1^* = \frac{8}{5}$ $x_2^* = \frac{3}{5}$ $Z^*_{max} = \frac{11}{5}$

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Revised Simplex Method (Artificial Variable)

Computation of Inverse by Partitioning

$M_{n\times n} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad a_{l\times l}\ ,\ b_{l\times m}$

$\quad c_{m\times l}\ ,\ d_{m\times m}$

$n = l+m$

$M^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

$MM^{-1} = I_n$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_l & 0 \\ 0 & I_m \end{bmatrix}$$

$aA + bC = I_l \quad , \quad aB + bD = 0$

$cA + dC = 0 \quad\quad cB + dD = I_m$

If d has inverse, we get

$A = (a - bd^{-1}c)^{-1}$ $\quad \boxed{B = -Abd^{-1}}$

$C = -d^{-1}cA$ $\quad D = d^{-1} - d^{-1}cB$

$M = \begin{bmatrix} I & \Phi \\ O & R \end{bmatrix}$ & R^{-1} exists

then $M^{-1} = \begin{bmatrix} I & -\Phi R^{-1} \\ O & R^{-1} \end{bmatrix}$

Max $Z = CX$

$AX = b$ $\qquad A_{m \times n}$

$x \geq 0$

Rewrite

$$\left.\begin{aligned} Z - CX &= 0 \\ AX &= b \end{aligned}\right]$$

$$X \geq 0$$

$\boxed{A^* x^* = b^*}$

$$\boxed{\begin{aligned} x^* &= \begin{bmatrix} Z \\ \cancel{x} \end{bmatrix} \\ b^* &= \begin{bmatrix} 0 \\ b \end{bmatrix} \end{aligned}}$$

A

$$B^*_{(m+1)\times(m+1)} = \left[\begin{array}{c|c} 1 & -C_B \\ \hline 0 & B \end{array}\right]$$

$$(B^*)^{-1} = \begin{bmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{bmatrix}$$

$$x^*_B = (B^*)^{-1} b^* = \begin{bmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{pmatrix} C_B B^{-1} b \\ B^{-1} b \end{pmatrix} = \begin{pmatrix} C_B x_B \\ x_B \end{pmatrix}$$

$$= \begin{pmatrix} Z \\ x_B \end{pmatrix}$$

$j \neq 0$

$$\begin{aligned} y^*_j &= (B^*)^{-1} a^*_j \\ &= \begin{bmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} -c_j \\ a_j \end{bmatrix} \\ &= \begin{bmatrix} -c_j + C_B B^{-1} a_j \\ B^{-1} a_j \end{bmatrix} \\ &= \begin{bmatrix} z_j - c_j \\ y_j \end{bmatrix} \end{aligned}$$

Q- Solve the following LPP by revised simplex method

$$\text{Min } z = x_1 + 2x_2$$
$$\text{s.t. } 2x_1 + 5x_2 \geq 6$$
$$x_1 + x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

$$\text{Max } z' = -z = -x_1 - 2x_2$$

Phase I

$$\text{Max } Z_a = -x_{a_1} - x_{a_2}$$
$$\text{s.t. } 2x_1 + 5x_2 - x_3 + x_{a_1} = 6$$
$$x_1 + x_2 - x_4 + x_{a_2} = 2$$
$$x_1, x_2, x_3, x_4 \geq 0$$
$$x_{a_1}, x_{a_2} \geq 0$$

$\alpha_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ Z'

$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ x1

$\alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 5 \\ 1 \end{pmatrix}$ x2

$\alpha_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ s1

$\alpha_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ s2

$\alpha_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ Za

$\alpha_6 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ a1

$\alpha_7 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ a2

$d = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 2 \end{pmatrix}$

Initial basis .

$$S = \begin{array}{cccc} \alpha_{\textcircled{1}} & \alpha_5 & \alpha_6 & \alpha_7 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \qquad S^{-1} = \begin{array}{cccc} g_0 & g_1 & g_2 & g_3 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}$$

$$x_s = S^{-1} d = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -8 \\ 6 \\ 2 \end{pmatrix}$$

Initial Table

Basis (S)	g_1	g_2	g_3	x_s ✓	η_x ✓	min ratio
$\alpha_{\textcircled{4}}$	0	0	0	0	2	
α_5	1	-1	-1	8	-6	↙ pivot ~~table~~ element.
α_6	0	1	0	6	$\boxed{5}$	6/5 ⟶
α_7	0	0	1	2	1	2/1

Determining entering vector   Compute $z_j - c_j$ for all non-basic vectors $j = 1, 2, 3, 4$

$$z_1 - c_1 = \underline{\text{2nd row of } S^{-1}} \times \alpha_1 = (0 \ 1 \ -1 \ -1) \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} = -3$$

$\underline{z_2 - c_2 = -6}$

$z_3 - c_3 = 1$

$z_4 - c_4 = 1$

$$\text{Compute } \eta_2 = S^{-1} \alpha_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 5 \\ 1 \end{pmatrix}$$

Modified Table

Basis (S)	$\bar{g}_1$	$\bar{g}_2$	$\bar{g}_3$	x_s	η_k
α_0	0	$-2/5$	0	$-12/5$	
α_5	1	$1/5$	-1	$-9/5$	
α_2	0	$1/5$	0	$6/5$	
α_7	0	$-1/5$	1	$4/5$	

↙ artificial variable

Determining the entering vector

$z_j - c_j \quad j = 1, 3, 4, 6$

$z_1 - c_1 = 2^{nd}$ row of $(S)^{-1} \times \alpha_1$

$$= \left(0, 1, \tfrac{1}{5}, -1\right) \times \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} = \boxed{-\frac{3}{5}}$$

$z_3 - c_3 = -1/5$

$z_4 - c_4 = 1$

$z_6 - c_6 = 6/5$

Determining the leaving vector

$$\eta_1 = S^{-1}\alpha_1 = \begin{pmatrix} 1 & 0 & -2/5 & 0 \\ 0 & 1 & 1/5 & -1 \\ 0 & 0 & 1/5 & 0 \\ 0 & 0 & -1/5 & 1 \end{pmatrix} = \begin{pmatrix} 1/5 \\ -3/5 \\ 2/5 \\ 3/5 \end{pmatrix}$$

Table

Basis (S)	$\hat{g}_1$	$\hat{g}_2$	$\hat{g}_3$	x_0	η_k
α_0	0	$-1/3$	$-1/3$	$-8/3$	
α_5	1	0	0	0	
α_2	0	$1/3$	$-2/3$	$2/3$	
α_1	0	$-1/3$	$5/3$	$4/3$	

Non-basic variables x_3, x_4, x_6, x_7

$$z_3 - c_3 = (2^{nd} \text{ row of } S^{-1}) \times \alpha_3$$

$$= (0, 1, 0, 0) \times \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = 0$$

$z_4 - c_4 = 0$

$z_6 - c_6 = 1$

$z_7 - c_7 = 1$

all $z_j - c_j \geq 0$

10/10/23

Revised simplex method (Artifical variables)

Max $z = cx$

$Ax = b$

$x \geq 0$

$$Z_a = -\sum_{i=1}^{m} x_{ai}$$

$x_{ai} \rightarrow$ artificial variable

horizontal order -
Z' and normal variables first
slack/surplus next
Za and artificial variables next

$$\begin{array}{cccccccc} \alpha_0 & \alpha_1 & \alpha_2 & & \alpha_n & \alpha_{n+1} & \cdots & \alpha_{n+m+1} \\ Z & -c_1x_1 & -c_2x_2 & - \cdots & -c_nx_n & & & = 0 \end{array}$$

$$Z_a + x_{a_1} + x_{a_2} + \cdots + x_{a_m} = 0$$

vertical order -
z'
za
constraints

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \cdots + a_{1n}x_n + x_{a_1} = b_1$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 \cdots + x_{am} = b_m$$

$\alpha_j = [-c_j, 0, a_j]$ $\quad j = 1, 2, 3 \ldots n$

$\alpha_j = [0, 1, e_i]$ $\quad j = n+1+i$ $\quad i = 1, 2, 3 \ldots m$

$$S = (\overset{\nearrow z}{e_1}\ \overset{\nearrow z_a}{s_1}\ \overset{\nearrow x_{a_1}}{s_2} \cdots \overset{\nearrow x_{am}}{s_{m+1}}) = \begin{pmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & & 1 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & 1 \end{pmatrix}$$

$m+2 \times$
$m+2$

$$= \left(\begin{array}{cc|c} 1 & 0 & -c_B \\ 0 & 1 & -c_a \\ \hline 0 & 0 & B \end{array}\right)$$

$d = [0, 0, b]$

$Cl = [Cl_1, Cl_2, Cl_3 \ldots Cl_n]$

$$\text{where } Cl_i = \begin{cases} 0 & \text{corresponding to } \alpha_j \quad j \leq n \\ -1 & \text{corresponding to } \alpha_j \quad j \geq n+1 \end{cases}$$

$$S^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$A = (a - bd^{-1}c)^{-1} = a^{-1} = I_2$$

$$B = -Abd^{-1} = -I_2 bd^{-1} = \begin{bmatrix} c_B B^{-1} \\ c_a B^{-1} \end{bmatrix}$$

$$C = d^{-1}cA = 0$$

$$D = d^{-1} - d^{-1}cA = B^{-1}$$

$$S^{-1} = \begin{bmatrix} 1 & 0 & c_B B^{-1} \\ 0 & 1 & c_a B^{-1} \\ 0 & 0 & B^{-1} \end{bmatrix}$$

$$x_r = S^{-1} d = \begin{pmatrix} 1 & 0 & c_B B^{-1} \\ 0 & 1 & c_a B^{-1} \\ 0 & 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$$

$$= \begin{pmatrix} c_B B^{-1} b \\ c_a B^{-1} b \\ B^{-1} b \end{pmatrix} = \begin{pmatrix} Z \\ Z_a \\ x_B \end{pmatrix}$$

$j = 1, 2, 3 \ldots n$

$$\eta_j = S^{-1} d_j$$

$$= \begin{pmatrix} 1 & 0 & c_B B^{-1} \\ 0 & 1 & c_a B^{-1} \\ 0 & 0 & B^{-1} \end{pmatrix} \begin{pmatrix} -c_j \\ 0 \\ a_j \end{pmatrix}$$

$$= \begin{pmatrix} -c_j + c_B B^{-1} a_j \\ c_a B^{-1} a_j \\ B^{-1} a_j \end{pmatrix} = \begin{pmatrix} z_j - c_j \\ (z_j - c_j)_a \\ y_j \end{pmatrix}$$

- If all artificial variables are removed in phase I and optimality is reached → done.
- artifical variables in phase I is at 0 level in the final table where optimality condition is reach, we go for phase-II

?

i) remove z_a column

ii) 1st colum corresponding to z never leaves the basis

iii) $z_j - c_j$ → multiply 1st row of s^{-1} with α_j . α_j non basic vector

Dual simplex method.

Example:

use dual simplex method to solve the LPP

$$\text{Max } z = -2x_1 - 3x_2 - x_3$$

$$\text{st. } 2x_1 + x_2 + 2x_3 \geq 3$$

$$3x_1 + 2x_2 + x_3 \geq 4$$

$$x_1, x_2, x_3 \geq 0.$$

solution:

std form

$$\text{Max } z = -2x_1 - 3x_2 - x_3 + 0.x_4 + 0.x_5$$

$$\text{st. } 2x_1 + x_2 + 2x_3 - x_4 = 3$$

$$3x_1 + 2x_2 + x_3 - x_5 = 4$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

select the Basis

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2$$

$$B^{-1} = -I_2$$

$$x_B = B^{-1}b = -I_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

$$c_B = 0$$

$$z_j - c_j = c_B y_j - c_j = -c_j \geq 0 \ \forall j.$$

- ✓ Max
- ✓ Initial basis yeilds optimality condition, $z_j - c_j \geq 0 \ \forall j$ and atleast one x_B component is -ve.

Max $z = -2x_1 - 3x_2 - x_3 + 0.x_4 + 0.x_5$

s.t. $-2x_1 - x_2 - 2x_3 + x_4 = -3$

$-3x_1 - 2x_2 - x_3 + x_5 = -4$

$x_j \geq 0 \ \forall \ j = 1,2,3,4,5.$

			c_j	-2	-3	-1	0	0	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	
0	a_4	x_4	-3	-2	-1	-2	1	0	
0	a_5	x_5	-4	$\boxed{-3}$	-2	-1	0	1	$\frac{2}{-3}$ $\frac{3}{-2}$ -1
				2	3	1	0	0	
				$\uparrow$					

- leaving variable is determined first by choosing the most -ve component of b $\rightarrow$ a_r
- Max ratio

max $\left\{ \frac{z_j - c_j}{y_{rj}} \quad y_{rj} < 0 \right\}$

C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5
0	a_4	x_4	$-1/3$	0	$1/3$	$-4/3$	1	$-2/3$
-2	a_1	x_1	$4/3$	1	$2/3$	$1/3$	0	$-1/3$
		$z_j - c_j$		0	$5/3$	$1/3$	0	$2/3$

C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5
-1	a_3	x_3	$1/4$	0	$-1/4$	1	$-3/4$	$1/2$
-2	a_1	x_1	$5/4$	1	$3/4$	0	$1/4$	$-1/2$
		$z_j - c_j$		0	$7/4$	0	$3/4$	$1/2$

final table

all x_B are non negative and

$$z_j - c_j \geq 0 \ \forall j$$

The optimal soln is

$$x_1 = 5/4 \cdot x_2 = 0, x_3 = 1/4$$

$$Z_{max} = -11/4$$

Example:

Using dual simplex method, prove that the following problem has no feasible solution.

$$\text{Min } Z = 7x_1 + 3x_2$$

$$\text{st} \quad x_1 - 3x_2 \geq 1$$

$$x_1 + x_2 \geq 2$$

$$-2x_1 + 2x_2 \geq 1$$

$$x_1, x_2 \geq 0.$$

std form

$$\text{Max } z^* = -7x_1 - 3x_2 + 0.x_3 + 0.x_4 + 0.x_5$$

$$\text{st.} \quad x_1 - 3x_2 - x_3 = 1 \quad \rightarrow b.$$

$$x_1 + x_2 - x_4 = 2$$

$$-2x_1 + 2x_2 - x_5 = 1$$

Initial basis

$$B = [a_3 \ a_4 \ a_5]$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -I_3$$

$$B^{-1} = -I_3$$

$$x_B = B^{-1}b = -I_3\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\\-1\end{pmatrix} \rightarrow -ve$$

$$c_B = (c_3\ c_4\ c_5) = 0$$

$$z_j - c_j = c_B y_j - c_j = -c_j \geq 0$$

→ We can apply the dual simplex method

			c_j	-7	-3	0	0	0
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5
0	a_3	x_3	-1	-1	3	1	0	0
0	a_4	x_4	-2	-1	$\boxed{-1}$	0	1	0
0	a_5	x_5	-1	2	-2	0	0	1
			$z_j - c_j$	7	3 ↑	0	0	0

-ve most x_4

a_4 leaving

$\text{Max}\left\{\frac{7}{-1}, \frac{3}{-1}\right\}$

c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5
0	a_3	x_3	-7	$\boxed{-4}$	0	1	2	0
-3	a_2	x_2	2	1	1	0	-1	0
0	a_5	x_5	3	4	0	0	-2	1
			$z_j - c_j$	4 ↑	0	0	3	0

$\text{Max}\left\{\frac{-4}{4}\right\}$

c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5
-7	a_1	x_1	7/4	1	0	-1/4	-2/4	0
-3	a_2	x_2	1/4	0	1	1/4	1/4	0
0	a_5	x_5	-4	0	0	1	1	1
			$z_j - c_j$	0	0	1	6	0

∴ Max { } not computable

No feasible soln.

Example: (Artificial constraint method)

Add constraint

$$\sum x_j \le M$$

summation is over all the j's for which $z_j - c_j < 0$ and M sufficiently large +ve number

$$\sum x_j + x_M = M$$

$j = p \rightarrow z_p - c_p$ -ve most

$$x_p = M - \left(\sum_{j \ne p} x_j + x_M\right)$$

use the artificial constraint method to find the initial basis solution of the following problem and then apply the dual simplex algm to solve it.

$\text{Max } Z = 2x_1 - 3x_2 - 2x_3$

s.t. $x_1 - 2x_2 - 3x_3 = 8$

$2x_2 + x_3 \le 10$

$x_2 + 2x_3 \ge 4$

$x_1, x_2, x_3 \ge 0.$

std form $\quad \text{Max } Z = 2x_1 - 3x_2 - 2x_3 + 0.x_4 + 0.x_5$

s.t. $x_1 - 2x_2 - 3x_3 = 8$

$2x_2 + x_3 + x_4 = 10$

$-x_2 - 2x_3 + x_5 = -4$

$x_1, x_2, x_3, x_4, x_5 \ge 0.$

Basis

$B = [a_1, a_4, a_5] = I_3$

$B^{-1} = I_3$

$x_B = B^{-1} b = \begin{pmatrix} 8 \\ 10 \\ -4 \end{pmatrix}$

$Z_j - C_j = C_B y_j - C_j$

$C_B = [2, 0, 0]$

			C_j	2	-3	-2	0	0
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5
2	a_1	x_1	8	1	-2	-3	0	0
0	a_4	x_4	10	0	2	1	1	0
0	a_5	x_5	-4	0	-1	-2	0	1
	$Z_j - C_j$			0	-1	-4	0	0

Add new constraint

$x_2 + x_3 \leq M$

Add artificial variable x_M

$x_2 + x_3 + x_M = M$

$\Rightarrow x_3 = M - x_2 - x_M$

The augmented problem then becomes

$\text{Max } Z = 2x_1 - 3x_2 - 2(M - x_2 - x_M)$

$= 2x_M + 2x_1 - x_2 - 2M$

st $x_1 - 2x_2 - 3(M - x_2 - x_M) = 8$

$2x_2 + (M - x_2 - x_M) + x_4 = 10$

$-x_2 - 2(M - x_2 - x_M) + x_5 = 4$

Max $Z = 2x_M + 2x_1 - x_2 - 2M$

$$
\begin{aligned}
3x_M + x_1 + x_2 &= 8 + 3M \\
-x_M + x_2 + x_4 &= 10 - M \\
2x_M + x_2 + x_5 &= -4 + 2M \\
x_M + x_2 + x_3 &= M
\end{aligned}
$$

			c_j	2	2	-1	0	0	0
c_B	B	x_B	b	a_M	a_1	a_2	a_3	a_4	a_5
2	a_1	x_1	$8+3M$	3	1	1	0	0	0
0	a_4	x_4	$10-M$	-1	0	1	0	1	0
0	a_5	x_5	$-4+2M$	-2	0	-3	0	0	1
0	a_3	x_3	M	1	0	1	1	0	0
		$z_j - c_j$		4	0	3	0	0	0

B	x_B	b						
a_1	x_1	$94/5$						
a_M	x_M	$5M-16$						
a_2	x_2	$24/5$						
a_3	x_3	$2/5$						
	$z_j - c_j$		0	0	0	0	$9/5$	$7/5$

N

Eliminate the extra row and extra column corresponding to a_M and put the original costs.

			c_j	2	-3	-2	0	0
C_B	B	X_B	b	a_1	a_2	a_3	a_4	a_5
2	a_1	94/5		1	0	0	-1/5	4/5
-3	a_2	24/5		0	1	0	2/5	-1/5
-2	a_3	2/5		0	0	1	1/5	7/5
		$z_j - c_j$		0	0	0	6/5	7/5

Date 16/10/2023

Duality Theorem

Theorem - The dual of the dual is the primal.

Proof -

Primal
$$\left.\begin{aligned} \text{Max } & z = cx \\ \text{s.t. } & Ax \le b \\ & x \ge 0 \end{aligned}\right\} - ①$$

Dual
$$\left.\begin{aligned} \text{Min } & w = b^T v \\ \text{s.t. } & A^T v \ge c^T \\ & v \ge 0 \end{aligned}\right\} - ②$$

$$\downarrow$$

Rewrite.
$$\left.\begin{aligned} \text{let } & w_1 = -w \\ \text{Max } & w_1 = -b^T v \\ \text{s.t. } & -A^T v \le -c^T \\ & v \ge 0 \end{aligned}\right\} - ③$$

Dual
$$\begin{aligned} \text{min } & Z_1 = -(c^T)^T x \\ \text{s.t. } & -(A^T)^T x \ge -(b^T)^T \\ & x \ge 0 \end{aligned}$$

$$\left.\begin{aligned} \text{max } & Z_1 = cx \\ \text{s.t. } & A \cdot x \le b \\ & x \ge 0 \end{aligned}\right\} \text{Same as } ①$$

Theorem (Weak Duality Theorem)

If x is any f.s. to the primal problem ① and v is any f.s. to the associated dual problem ②.

then $cx \le b^T v$

$z \le w$

Proof

$$Ax \le b$$
$$v^T(Ax) \le v^T b$$
$$(v^T A)x \le v^T b \quad — ③$$
$$(A^T v)^T x \le (b^T v)^T \quad \leftarrow$$

$$A^T v \geq c^T$$
$$x^T (A^T v) \geq x^T c^T$$
$$x^T (v^T A)^T \geq (cx)^T$$
$$(v^T A x)^T \geq c x^T$$
$$v^T A x \geq cx \quad — ④$$

③, ④ →

$$cx \leq v^T A x \leq v^T b \leq b^T v$$

Theorem (Strong Duality Theorem)

If x^* is a f.s. to the primal ① & v^* is the f.s. to the associated dual ② s.t.

$$cx^* = b^T v^*$$

then both x^* and v^* are optimal solution to the respective problems.

Proof - Given $cx^* = b^T v^*$

Now $cx \leq \underset{= cx^*}{b^T v^*}$. for any f.s. x of ① and any f.s. v such as v^* of ②

$cx \leq cx^*$ for any f.s. x of ①

$\Rightarrow$ x^* is optimal f.s. to ①.

Other part.. → Same way.

Theorem (Fundamental Theorem of Duality)

If a finite optimal f.s. exists for the primal then ∃ a finite optimal f.s. for the dual & conversely.

Restate - A f.s. x^* to the primal is optimal iff ∃ a f.s. v^* to the associated dual s.t.

$$cx^* = b^T v^*$$

Proof -

① ⇒

$$\text{Max } z = cx + 0 \cdot x_s$$
$$Ax + I x_s = b$$
$$x, \quad x_s \geq 0$$

x_s → vector of slack variables

let x_B^* → optimal bfs. of ③
B → corresponding basis
C_B → associated cost vector

x_B^* optimal f.s. ⇒ $z_j - c_j \geq 0 \quad \forall j$

$$z_j = \sum_{i=1}^{m} c_{Bi} y_{ij}$$
$$= C_B y_j$$
$$= C_B B^{-1} a_j$$

let $v^{*T} = C_B B^{-1}$

Along with slack variables we get

$$C_B B^{-1} (A, I) \geq (C, 0)$$
$$(v^*)^T A \geq C \implies A^T v^* \geq C^T$$
$$(v^*)^T \geq 0$$

v^* satisfies the constraints of the dual ②

Then v^* is a f.s to the dual ②

Claim - $(v^*)^T = C_B B^{-1}$ is an optimal soln of ②.

Proof $\quad z_{max} = C_B x_B^* = C_B B^{-1} b = (v^*)^T b$

$$= b^T v^*$$

$$= W_{min}$$

x^*, v^* → f.s of the primal & the associated dual respectively with

$$cx^* = b^T v^*$$

⇒ x^* is a finite optimal f.s. of the primal & v^* is a finite optimal f.s. of the dual by the strong duality theorem.

Theorem -

If the primal an unbounded objective function then the dual has no f.s.

Proof - Suppose (Use contradiction)

primal → dual has ⟶ dual of dual → primal

u.o.f. $\qquad$ finite $\qquad$ f.o.s.

optimal soln

f.o.s.

Theorem -

If the dual has no f.s. and a primal has a f.s then primal obj. func is unbounded

Proof - Suppose dual has no f.s.

But primal a f.s. ⟶ x^0

value of obj. func $= cx^0$

Claim - Obj. func of the primal is unbounded

x^0 cannot be optimal soln of the primal as by the Fundamental Th. of Duality the dual will have a f.s.. The primal has no optimal soln

⇒ obj func of primal is unbounded.

2nd classtest & assignment → 13th Nov.

Date 17/10/2023

Complementary Slackness Theorem

for any part of optimal solution to a LPP and its associated dual

(a) the product of the jth variable of the primal and the jth surplus variable of the dual is zero,
for each $j = 1, 2, \dots, n$

(b) the product of the ith variable of the dual and the ith slack variable of the primal is zero,
for each $i = 1, 2, \dots, n$.

$x_j (v_s)_j = 0$ $\qquad$ $v_i (x_s)_i = 0$

Proof-

Primal ① $\begin{cases} \text{Max } z = cx \\ \text{s.t. } Ax \le b \\ x \ge 0 \end{cases}$ $\qquad$ ③ $\begin{cases} \text{max } z = cx \\ v^T X \quad \text{s.t. } \underset{m\times n}{A} x + \underset{n\times 1}{x_s} = \underset{m\times 1}{b} \end{cases}$

$x, x_s \ge 0$.
x_s - vector of slack varia

Dual ② $\begin{cases} \text{Min } w = b^T v \\ \text{s.t. } A^T v \ge c^T \\ v \ge 0 \end{cases}$ $\qquad$ ④ $\begin{cases} \text{min } w = b^T v \\ x^T X \quad \text{s.t. } A^T v - v_s = c^T \\ v, v_s \ge 0, \ v_s = \text{vector of surplus variable} \end{cases}$

$v^T A x + v^T x_s = v^T b$

$x^T A^T v + v^T x_s = b^T v$ —⑤

$x^T A^T v - x^T v_s = x^T c^T$

$x^T A^T v - x^T v_s = cx$ —⑥

(x°, x_s°) , (v°, v_s°)

↓ feasible optimal solution of the primal ③ $\qquad$ ↓ feasible optimal solution of the dual ④

$(v^\circ)^T x_s^\circ = 0$

$(x^\circ)^T v_s^\circ = 0$

$b^T v = cx$.

$x^T \cancel{A^T} v + v^T x_s = x^T \cancel{A^T} v - x^T v_s$

$x^T v_s + v^T x_s = 0$

Theorem - If (x, x_s), (v, v_s) are feasible solution of the primal ① and the associated dual ② under conditions where complementary slackness holds then (x, x_s) and (v, v_s) are also their respective optimal solution.

Proof - Complementary slackness holds

$$\Rightarrow v^T x_s + x^T v_s = 0$$

$$v^T x_s = -x^T v_s = -v_s^T x$$

Add $v^T A x \Rightarrow$

$$v^T A x + v^T x_s = v^T A x - v_s^T x$$

$$v^T (Ax + x_s) = x(A^T v - v_s)$$

$$v^T b = x^T c^T = (cx)^T$$

$$cx = b^T v$$

Result follows by the Fundamental Theorem of Duality

Assignment Problem (Hungarian Method)

↓

Based on the work of two Hungarian mathematician Konig and Egervary

Q- Find the optimal assignment for a problem with the following cost matrix -

$$\begin{array}{c|ccccc} & M_1 & M_2 & M_3 & M_4 & M_5 \\ \hline J_1 & 8 & 4 & 2 & 6 & 1 \\ J_2 & 0 & 9 & 5 & 5 & 4 \\ J_3 & 3 & 8 & 9 & 2 & 6 \\ J_4 & 4 & 3 & 1 & 0 & 3 \\ J_5 & 9 & 5 & 8 & 9 & 5 \end{array} \longrightarrow \begin{array}{c|ccccc} & M_1 & M_2 & M_3 & M_4 & M_5 \\ \hline J_1 & 7 & 3 & 1 & 5 & 0 \\ J_2 & 0 & 9 & 5 & 5 & 4 \\ J_3 & 1 & 6 & 7 & 0 & 4 \\ J_4 & 4 & 3 & 1 & 0 & 3 \\ J_5 & 4 & 0 & 3 & 4 & 0 \end{array}$$

$$\begin{array}{c|ccccc} & M_1 & M_2 & M_3 & M_4 & M_5 \\ \hline J_1 & 7 & 3 & 0 & 5 & 0 \\ J_2 & 0 & 9 & 4 & 5 & 4 \\ J_3 & 1 & 6 & 6 & 0 & 4 \\ J_4 & 4 & 3 & 0 & 0 & 3 \\ J_5 & 4 & 0 & 2 & 4 & 0 \end{array}$$

$$\begin{array}{cc|ccccc} & & M_1 & M_2 & M_3 & M_4 & M_5 \\ \hline \times & J_1 & 7 & 3 & 0 & 5 & \boxed{0} \\ & J_2 & \boxed{0} & 9 & 4 & 5 & 4 \\ & J_3 & 1 & 6 & 6 & \boxed{0} & 4 \\ & J_4 & 4 & 3 & \boxed{0} & 0 & 3 \\ \times & J_5 & 4 & \boxed{0} & 2 & 4 & 0 \\ & & \times & & \times & \times & \end{array}$$

First check rowwise for single 0 & then col wise

$J_1 \rightarrow M_5$

$J_2 \rightarrow M_1$

$J_3 \rightarrow M_4$

$J_4 \rightarrow M_3$

$J_5 \rightarrow M_2$

Min Cost $= 1 + 0 + 2 + 1 + 5$

$= 9$

Q- The head of the department has five jobs A, B, C, D, E and five sub ordinates V, W, X, Y, Z. The no. of hours each men could take to perform each job is as follows-

	V	W	X	Y	Z
A	3	5	10	15	8
B	4	7	15	18	8
C	8	12	20	20	12
D	5	5	8	10	6
E	10	10	15	25	10

→

	V	W	X	Y	Z
A	0	2	7	12	5
B	0	3	11	14	4
C	0	4	12	12	4
D	0	0	3	5	1
E	0	0	5	15	0

↓

	V	W	X	Y	Z
A	0	2	4	7	5
B	0	3	3	8	9
C	0	4	9	7	4
D	0	0	0	0	1
E	0	0	2	18	0

Draw min no. of lines (horizontal & vertical) to cover all zeros ←

min among uncovered = 2

Add 2 at the intersection

Subtract 2 from all uncovered costs.

↓

	V	W	X	Y	Z
A	0	0	2	5	3
B	0	1	6	7	2
C	0	2	7	5	2
D	2	0	0	0	1
E	2	0	2	10	0

min no of lines = 4 < m

min among uncovered cost = 2.

←

	V	W	X	Y	Z
A	0	0	0	3	3
B	0	1	4	5	2
C	0	2	5	3	2
D	4	2	0	0	3
E	2	0	0	8	0

4 < m

$$\begin{array}{c|ccccc} & V & W & X & Y & Z \\ \hline A & 1 & 0 & 0 & 3 & 3 \\ \rightarrow B & 0 & 0 & 3 & 4 & 1 \\ C & 0 & 1 & 4 & 2 & 1 \\ D & 5 & 2 & 0 & 0 & 3 \\ E & 3 & 0 & 0 & 8 & 0 \end{array} \longrightarrow \begin{array}{c|ccccc} & V & W & X & Y & Z \\ \hline A & 1 & 0 & \boxed{0} & 3 & 3 \\ B & 0 & \boxed{0} & 3 & 4 & 7 \\ C & \boxed{0} & 1 & 4 & 2 & 1 \\ D & 5 & 2 & 0 & \boxed{0} & 3 \\ E & 3 & 0 & 0 & 8 & \boxed{0} \end{array}$$

Variation of Assignment Problems

(i) Max problem ⟶ min problem

$$\begin{array}{c|c} 3 & 9 \\ \hline 6 & 4 \end{array} \longrightarrow \begin{array}{c|c} -3 & -9 \\ \hline -6 & -4 \end{array} \quad \text{or} \quad \begin{array}{c|c} 6 & 0 \\ \hline 3 & 5 \end{array}$$

(ii) Unbalanced → Dummy row / column with 0 cost

(iii) Impossible Assignment → Put a large +ve M for min problem

Transportation Problem

Q- For the following problem ~~Attained~~ obtain the different starting solution by adapting -

(i) the North-West Corner Method
(ii) the Vogel's approximation method
(iii) the Matrix minima method (least cost entry method)

	D1	D2	D3	ai
O1	5	1	8	12
O2	2	4	0	14
O3	3	6	7	4
bj	9	10	11	

North-West Corner Method -

	D1	D2	D3	ai
O1	$\boxed{9}$ 5	$\boxed{3}$ 1	8	~~12~~ ~~3~~
O2	2	$\boxed{7}$ 4	$\boxed{7}$ 0	~~14~~ ~~7~~
O3	3	6	$\boxed{4}$ 7	~~4~~
bj	~~9~~	~~10~~ ~~7~~	~~11~~	

Initial b.f.s.

$x_{11}=9$, $x_{12}=3$, $x_{22}=7$,
$x_{23}=7$, $x_{33}=4$

$$\text{Total Cost} = \sum_i \sum_j c_{ij} x_{ij}$$
$$= 5\times9 + 1\times3 + 4\times7 + 0\times7 + 7\times4$$
$$= 104$$

VAM -

diff (smallest, second smallest)

	D1	D2	D3	ai	
O1	5	1	8	12	(4)
O2	2	4	$\boxed{11}$ 0	~~14~~ 3	(2)
O3	3	6	7	4	(3)
bj	9	10	~~11~~		
	(1)	(3)	(7)		

max diff -> min cost

	D1	D2	ai	
O1	$\boxed{2}$ 5	$\boxed{10}$ 1	~~12~~ 2	(4) ✓
O2	$\boxed{3}$ 2	4	3	(2)
O3	$\boxed{4}$ 3	6	4	(3)
bj	9	~~10~~		
	(1)	(3)		

	D_1	D_2	D_3	a_i
O_1	$\boxed{2}$ 5	$\boxed{10}$ 1	8	12
O_2	$\boxed{3}$ 2	4	$\boxed{11}$ 0	14
O_3	$\boxed{4}$ 3	6	7	4
b_j	9	10	11	

$x_{11} = 2$, $x_{12} = 10$, $x_{21} = 3$, $x_{23} = 11$

$x_{31} = 4$

Total cost $= 10+10+6+0+12 = 38$

Matrix Minima Method.

	D_1	D_2	D_3	D_4	a_i
O_1	2	$\boxed{30}$ 1	3	4	~~30~~
O_2	3	2	1	4	50
O_3	5	2	3	8	20
b_j	20	~~40~~ 10	30	10	

$\longrightarrow$

	D_1	D_2	D_3	D_4	a_i
O_2	3	2	$\boxed{30}$ 1	4	~~50~~ 20
O_3	5	2	3	8	20
b_j	20	10	~~30~~	10	

$\downarrow$

	D_1	D_2	D_4	a_i
O_2	3	$\boxed{10}$ 2	4	~~20~~ 10
O_3	5	2	8	20
b_j	20	~~10~~	10	

$\longleftarrow$

	D_1	D_4	a_i
O_2	3	$\boxed{10}$ 4	~~10~~
O_3	5	8	20
b_j	20	~~10~~	

$\downarrow$

	D_1	a_i
O_3	$\boxed{20}$ 5	~~20~~
b_j	~~20~~	

	D_1	D_2	D_3	D_4	a_i
O_1	2	$\boxed{30}$ 1	3	4	30
O_2	3	$\boxed{10}$ 2	$\boxed{30}$ 1	$\boxed{10}$ 4	50
O_3	$\boxed{20}$ 5	2	3	8	20
b_j	20	40	30	10	

Total Cost $= 30+20+30+40+100$

$= 220$

F.S. cannot have more than $(m+n-1)$ quantities

30/10/23

Travelling - salsemen problem

Example:

solve the TSP with following cost matrix $(C_{ij})_{4\times4}$.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & 15 & 30 & 4 \\ 2 & 6 & \infty & 4 & 1 \\ 3 & 10 & 15 & \infty & 16 \\ 4 & 7 & 18 & 13 & \infty \end{array}$$

when C_{ij} is cost of travelling from city i to city j.

NP - hard problem

$n=2 \longrightarrow 1!$

$n=3 \longrightarrow 2!$

$n=4 \longrightarrow 3!$

$\vdots$

$n \longrightarrow (n-1)!$

Hungarian method

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & 11 & 26 & 0 \\ 2 & 5 & \infty & 3 & 0 \\ 3 & 0 & 5 & \infty & 6 \\ 4 & 0 & 11 & 6 & \infty \end{array}$$

$\Downarrow$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & 6 & 23 & 0 \\ 2 & 5 & \infty & 0 & 0 \\ 3 & 0 & 0 & \infty & 6 \\ 4 & 0 & 6 & 3 & \infty \end{array}$$

4 = order of the matrix = min. no. of hor. and vertical lines to cover all the 0's.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & 6 & 23 & \boxed{0} \\ 2 & 5 & \infty & \boxed{0} & 0 \\ 3 & 0 & \boxed{0} & \infty & 6 \\ 4 & \boxed{0} & 6 & 3 & \infty \end{array}$$

$1 \to 4 \to 1$

$2 \to 3 \to 2$

cost = 4+4+15+7 = 30.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & 6 & 23 & \boxed{0} \\ 2 & \boxed{5} & \infty & 0 & 0 \\ 3 & 0 & \boxed{0} & \infty & 6 \\ 4 & 0 & 6 & \boxed{3} & \infty \end{array}$$

min cost → 3

assign 3 in cell (4,3)

next min cost → 5

assign 5 in cell (2,1)

$1 \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 1$

cost = 36.

Example: (production scheduling problem)

$$\begin{array}{cc|ccccc} & & & & \text{TO} & & \\ & & A & B & C & D & E \\ \hline & A & \infty & 2 & 5 & 7 & 1 \\ & B & 6 & \infty & 3 & 8 & 2 \\ & C & 8 & 7 & \infty & 4 & 7 \\ \text{FROM} & D & 12 & 4 & 6 & \infty & 5 \\ & E & 1 & 3 & 2 & 8 & \infty \end{array}$$

Given the matrix of the set-up cost of machine when there is a change of item for processing on the machine.

Show how to sequence production of five items as to minimize set up cost per cycle.

C_{ij} = set up. cost of the processing machine when item a_i is followed by item a_j.

$$Z = \sum_j \sum_i C_{ij} x_{ij}$$

$$x_{ij} = \begin{cases} 1 & \text{if} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{c|ccccc|} & A & B & C & D & E \\ \hline A & \infty & 1 & 3 & 6 & \boxed{0} \\ B & 4 & \infty & \boxed{0} & 6 & 0 \\ C & 4 & 3 & \infty & \boxed{0} & 3 \\ D & 8 & \boxed{0} & 1 & \infty & 1 \\ E & \boxed{0} & 2 & 0 & 7 & 7 \\ \hline \end{array}$$

Starting matrix

order of matrix = the min no. of horizontal and vertical lines.

$$\left.\begin{array}{l} A \to E \to A \\ B \to C \to D \to B \end{array}\right\} \text{cost} = 13 \text{ (lower bound)}$$

Does not satisfy the condition of a TSP

↓

so go for enumeration to solve the problem.

Min cost → 1

assign 1 in cell (A, B)

$$\begin{array}{c|ccccc|} & A & B & C & D & E \\ \hline A & \infty & \boxed{1} & 3 & 6 & 0 \\ B & 4 & \infty & \boxed{0} & 6 & 0 \\ C & 4 & 3 & \infty & \boxed{0} & 3 \\ D & 8 & 0 & 1 & \infty & \boxed{1} \\ E & \boxed{0} & 2 & 0 & 7 & 7 \\ \hline \end{array}$$

A → B → C → D → E → A

cost = 15

no feasible solution.

$$\begin{array}{c|ccccc|} & A & B & C & D & E \\ \hline A & \infty & 1 & 3 & 6 & 0 \\ B & 4 & \infty & 0 & 6 & 0 \\ C & 4 & 3 & \infty & 0 & 3 \\ D & 8 & 0 & \boxed{1} & \infty & 1 \\ E & 0 & 2 & 0 & 7 & 7 \\ \hline \end{array}$$

Min cost → 1

assign 1 in cell (D, C)

D → C also B → C

2nd min cost → 1

assign 1 in cell (A, B)

A → B → E

Transporation problem

Finding initial solution

- → Vogel's approx method
- → Northwest corner method
- → Matrix minima method
- → Row minima method
- → column minima method

	W_1	W_2	W_3	W_4	a_i
P_1	$\boxed{5}$ 19	80	50	$\boxed{2}$ 10	~~7~~ ~~2~~
P_2	70	30	$\boxed{7}$ 40	$\boxed{2}$ 60	~~9~~ ~~2~~
P_3	40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	~~18~~ ~~10~~
b_j	~~5~~	~~8~~	~~7~~	~~14~~	

x_{ij} (min)

other x_{ij}

$x_{11} = 5$

$x_{14} = 2$

$x_{23} = 7$

$x_{24} = 2$

$x_{32} = 8$

$x_{34} = 10$

$$\sum_i a_i = \sum_j b_j$$

x_{ij}

$$\min Z = \sum_i \sum_j c_{ij} x_{ij}$$

\# of allocation $= 6 = \underset{3}{m} + \underset{4}{n} - 1$

Example (Test for optimality) VAM

	D1	D2	D3	D4	ai			
O1	20 1	2	1	4	30 (0)	30(0) 10 ~~30(0)~~	10(0) ~~30(0)~~	10(0
O2	3	3	2	10 1	40 ~~50~~(1)	40(1)	40(1)	40(1)
O3	4	20 2	5	3	20 (2)	~~20(2)~~		
bj	20 (2)	40 (0)	30 (1)	~~10~~ (3)				
	20(2)	~~40(0)~~ 20	30(1)					
	~~20(2)~~	~~20(0)~~	30(1)					

Example: (Test for optimality)

Ch

A → D.

B → A

C → B

D → A

31/10/23

	D1	D2	D3	D4	a_i	u_i
O_1	1	2	1	4	30	-1
O_2	3	3	2	1	50	0
O_3	4	2	5	9	20	-1
b_j	20	40	30	10		
v_j	~~3~~ 2	3	2	1		

VAM

Initial b/s.

How to check optimality of this?

allocted cells

$c_{ij} = u_i + v_j$.

$x_{11} = 20$ $\quad x_{13} = 10$

$x_{22} = 20$ $\quad x_{23} = 20$ $\quad x_{24} = 10$

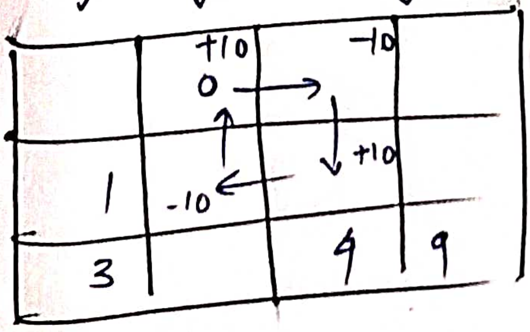
$x_{32} = 20$.

\# of allocations $= 6$
$= m+n-1$
$= 3+4-1$

Non degenerate solution.

cell evaluation : for non allocated cells.

$\Delta_{ij} = c_{ij} - u_i - v_j$.



all $\Delta_{ij} \geq 0$

$\Rightarrow$ ① is optimal

$\Delta_{12} = 0 \Rightarrow$

An alternative soln exists.

Δij

	D1	D2	D3	D4	ai	ui
O1	20 $\boxed{1}$	10 $\boxed{2}$	⓪ $\boxed{2}$	④ $\boxed{1}$	30	-1
O2	⑦ $\boxed{3}$	10 $\boxed{3}$	30 $\boxed{2}$	10 $\boxed{1}$	50	0
O3	③ $\boxed{4}$	20 $\boxed{12}$	④ $\boxed{5}$	⑨ $\boxed{9}$	20	-1
bj	20	40	30	10		
vj	2	3	2	1		

$x_{11} = 20, x_{12} = 10$

$x_{22} = 10, x_{23} = 30, x_{24} = 10$

$x_{32} = 20$

Cost = $20 \times 1 + 10 \times 2 + 3 \times 10 + 2 \times 30 + 1 \times 10 + 2 \times 20$

= 180.

Example:

(cell evalution -ve)

	D1	D2	D3	ai
O1	2	7	4	5
O2	3	3	1	8
O3	5	4	7	7
O4	1	6	2	14
bj	7	9	18	

$$\min z = \sum_i \sum_j c_{ij} x_{ij}.$$

$\sum a_j = \sum b_j.$ ✓

such that $\sum_j x_{ij} = \cancel{a_j}\ a_i$

$\sum_i x_{ij} = \cancel{b_j}\ b_j$

$x_{ij} \geq 0.$

Subset of LPP

Min $Z = Cx$

such that $Ax = b.$

$x \geq 0.$

$$① \Rightarrow \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

$$② \Rightarrow \sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j$$

$$\sum_{i=1}^{m} \left\{ \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} x_{ij} \right\} = b_n$$

$$\sum_{i=1}^{m} x_{in} = b_n$$

precisely the last eqn.

$$x_{11} + x_{12} + x_{13} + \dots + x_{1n} = a_1$$

$$x_{21} + x_{22} + \dots + x_{2n} = a_2$$

$$x_{31} + x_{32} \dots + x_{3n} = a_3$$

$$\vdots$$

$$x_{m1} + x_{m2} + \dots x_{mn} = a_n$$

$$x_{11} + x_{21} + x_{m1} = b_1$$

$$x_{12} + \dots + x_{m2} = b_2$$

$$x_{1n} + x_{2n} \dots + x_{mn} = b_n$$

$$A = (a_{ij})_{m \times n}$$

$$\cancel{a_{ij} = e_i + e_{j+b}}.$$

$$a_{ij} = e_i + e_{j+m}$$

- ✓ TP always has a feasible soln
- ✓ TP solution never unbounded.

?

$$T = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j .$$

claim:

$x_{ij} = \frac{a_i b_j}{T}$ is a feasible soln of ①.

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{m} \frac{a_i b_j}{T} = \frac{a_i}{T} \sum_{j=1}^{m} b_j = a_1$$

$$0 \leq x_{ij} \leq \overset{\text{Min}}{\cancel{\text{Max}}}(a_i, b_j)$$

↳ cannot be artitrarly large.

	D_1	D_2	D_3	a_i	u_i
O_1	5 2	⓪ 7	① 4	5	1
O_2	③ 3	-2 +2 3	8 -2 1	8	-1
O_3	⑥ 5	7 4	⑦ 7	7	-2
O_4	2 1	2 -2 6	10 +2 2	14	0
b_i	7	9	18		
v_i	1	6	2		

	D1	D2	D3	a_i	u_i
O1	5 2	⑤ 7	③ 4	5	1
O2	⓪ 3	2 3	6 1	8	7
O3	⑦ 5	7 4	② 7	7	0
O4	2 1	~~4~~ ④ 6	12 2	14	0
b_j	7	9	18		
v_j	1	4	2		

all $\Delta_{ij} > 0$ -

solution is optimal

min cost = 78.

Degeneracy

VAM

				a_i	u_i
		ε $-\varepsilon$ 7	60 $+\varepsilon$ 3	60	0
	50 3	$+\varepsilon$ 8	20 $-\varepsilon$ 9	70	6
	11	80 3	5	80	−4
b_j	50	80	80		
v_j	−3	7	3		

Note: row 1, column 1 cost is 8.

$m+n-1 = 3+3-1 = 5$.

\# of allocations $= 4 < m+n-1$

This has degenerate solution.

$\varepsilon > 0$.

cell evaluation Δ_{ij}

11		
	−5	
18		6

			a_i	u_i
⑪ 8	④ 7	60 3	60	-6
50 3	ε 8	20 9	70	0
⑬ 11	80 3	① 5	80	-5
b_j 50	80	80		
v_j 3	8	9		

$\Delta_{ij} > 0$ for non allocated cells.

Variations in TP.

a) unbalanced

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

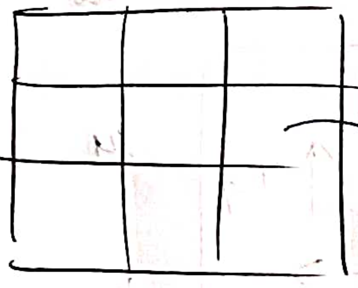
Make it balanced with a dummy row/column with zero costs.

b) max .

$M-c_{11}$	$M-c_{12}$	. .
	M	

→ min

c) No allocation in a particular cell.



no allocation in cell i, j .

$\rightarrow c_{ij} = M > 0.$ $x_{ij} = 0.$

$\Uparrow$

min

or

$c_{ij} = M < 0$

max

d) Some +ve allocation in a particular cell

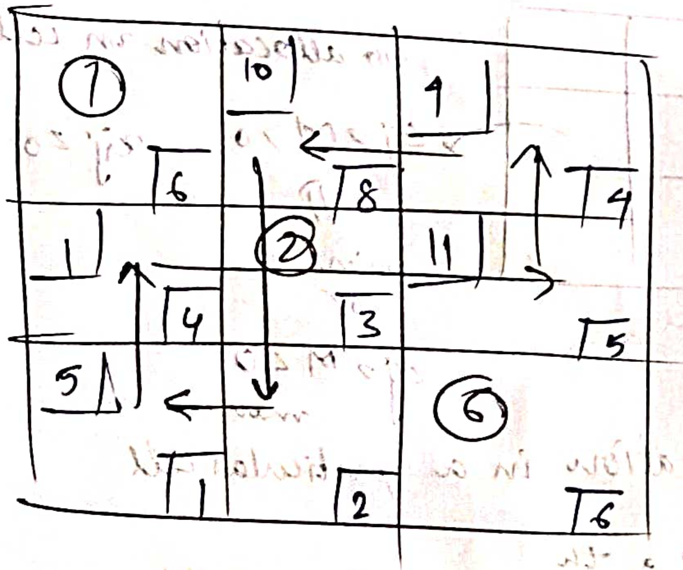
x_{ij} $(i,j)^{th}$ $\longrightarrow$ p quantity

$a_i = p$ $b_j = p.$

	A	B	C	a_i
I	1 6	10 8	9 4	14
II	4	3	12 5	12
III	5 1	2	6	5
b_j	6	10	15	

non degenerate bfs.

⇓



				a_i	u_i
	⑦ 6	10 8	9 4	14	0
	1 4	② 3	11 5	12	1
	5 1	2	⑥ 6	5	−4
b_j	6	10	15		
v_j	5	−8	4		

ILP

Integer linear programming

optimize

$$Z = \sum_{j=1}^{n} c_j x_j$$

such that

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1, 2, 3 \dots m$$

$x_j \ge 0$ and some or all x_j are integer.

Branch and Bound

(Lang and Doig)

Example:

Max $Z = 2x_1 + 3x_2$

st $6x_1 + 5x_2 \le 25$

$x_1 + 3x_2 \le 10$

$x_1, x_2 \ge 0$ and integer.

A

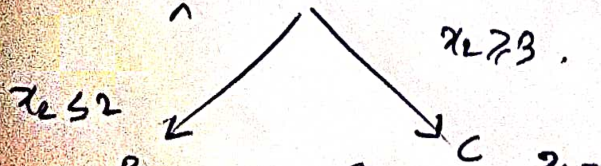
$x_1 = 1.92$ $x_2 = 2.69$

$Z = 11.92$

Lower bound

$Z = 11$ $x_1 = 1$ $x_2 = 3$. because maximization problem.

x_2 has more fractional part



$x_2 \ge 3$. add these contrains to the problem.

$x_2 \le 2$

B

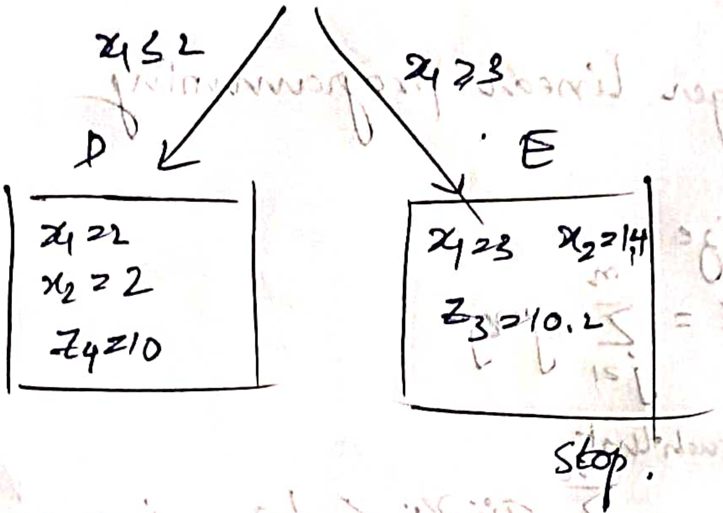
$x_1 = 2.5$

~~$x_1 = 1$, $x_2 = 0$~~ $x_2 = 2$

$Z_3 = 11$ $Z_2 = 11$

C $x_1 = 1$ $x_2 = 3$

$Z_3 = 11$


$x_1 \le 2$
$x_1 \ge 3$
D
E
$x_1 = 2$
$x_2 = 2$
$Z_4 = 10$
$x_1 = 3$ $x_2 = 1.4$
$Z_3 = 10.2$
Stop.