Tutorial 6 Discussion

Tut - 6

Q2. Let R be a ring and $a \in R$ be nilpotent i.e $a^n = 0$ for some $n > 0$.

WTS $(1+a)$ is a unit.

$$(1+a)\left(1 - a + a^2 - a^3 + \cdots + (-1)^{n-1} a^{n-1}\right)$$

$$= 1 + (-1)^{n-1} a^{n-1} \cdot a$$

$$= 1.$$

If u is a unit and a is nilpotent then $u^{-1}a$ is nilpotent.

$1 + u^{-1}a$ is an unit by previous argument

$u(1 + u^{-1}a) = u + a$ is an unit.

Q4 $\quad f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + \cdots$

Suppose $f(t)$ is an unit

$\Rightarrow \exists\ g(t) \in R[[t]]$ s.t

$$f(t) \cdot g(t) = 1.$$

$$\Rightarrow \boxed{(a_0 + a_1 t + a_2 t^2 + \cdots)(b_0 + b_1 t + b_2 t^2 + \cdots) = 1.} \quad (*)$$

$$\Rightarrow a_0 b_0 + (a_1 b_0 + a_0 b_1) t + (\cdots) = 1.$$

$$\Rightarrow a_0 b_0 = 1.$$

$\Rightarrow a_0$ is an unit.

Suppose a_0 is an unit a_0^{-1} exist

$$f(t) = a_0 (1 + a_1 a_0^{-1} t + a_2 a_0^{-1} t^2 + \cdots)$$

$$g(t) = b_0 (1 - a_1 a_0^{-1} t + (a_1^2 a_0^{-2}) t^2 + \cdots)$$

If $f(t)$ an unit then $\exists\, g(t)$

s.t $f(t)\, g(t) = 1$.

From eq (i) we have

$$a_0 b_0 = 1.$$

$$\Rightarrow \quad b_0 = a_0^{-1}.$$

we have $\quad a_0 b_1 + b_0 a_1 = 0$

$$\Rightarrow \quad a_0 b_1 = -b_0 a_1$$

$$\Rightarrow \quad b_1 = -b_0 a_1 a_0^{-1}.$$

$$a_2 b_0 + a_1 b_1 + b_2 a_0 = 0.$$

<u>WTS</u> If $f(t)$ is nilpotent then all a_i's are nilpotent.

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots$$

If $f(t)$ is nilpotent then

$$f(t)^n = 0.$$

$$\Rightarrow \left(a_0 + a_1 t + \cdots\right)^n = 0.$$

$$\Rightarrow a_0^n = 0.$$

$\Rightarrow$ a_0 is nilpotent.

$$\boxed{a_1^n + a_0 t_0}$$

$\rightarrow$ coeff of t^n in $f(t)^n$ is of the form $\boxed{a_1^n + a_0 t_0} = 0.$

$\rightarrow$ nilpotent.

$\Rightarrow$ a_1 is nilpotent.

The converse is not true.

$$B = C[y_1, y_2, \dots].$$

where C is an int domain.

and y_i's are inderminates.

$$I = (y_1, y_2^2, y_3^3, \dots, y_n^n \dots).$$

Consider the ring $A = B/I$.

Let z_n denote the image of y_n in A. i.e $z_n = y_n + I$.

Then in A $z_n^n = 0$ and $z_n^{n-1} \neq 0$.

Let $f = \sum_{n \geq 0} z_n x^n \in A[|x|]$. then each coeff of f is nilpotent but $f^n \neq 0 \; \forall n$, Q5. $\mathbb{Q}[\sqrt{2}] = \{ a + \sqrt{2}\, b. \mid a, b \in \mathbb{Q} \}$

WTS $\underline{\mathbb{Q}[\sqrt{2}+\sqrt{3}]} = \underline{\mathbb{Q}[\sqrt{2}, \sqrt{3}]}$.

$\subseteq$

Easy to see. $\underline{\sqrt{2}+\sqrt{3} \in \mathbb{Q}[\sqrt{2}, \sqrt{3}]}$.

WTS. $\mathbb{Q}[\sqrt{2}, \sqrt{3}] \subseteq \mathbb{Q}[\sqrt{2}+\sqrt{3}] = K$.

$\sqrt{2}+\sqrt{3} \in K$.

$\Rightarrow (\sqrt{2}+\sqrt{3})^2 \in K$.

$\Rightarrow 5 + 2\sqrt{6} \in K$

$\Rightarrow \underline{\sqrt{6}} \in K$.

$\sqrt{6}(\sqrt{2}+\sqrt{3}) \in K$.

$= 2\sqrt{3} + 3\sqrt{2} \in K$.

$2(\sqrt{2}+\sqrt{3}) \in K$

$\sqrt{2} \in K$.

Q9.

$$\boxed{\mathbb{R}[[x]]} = R.$$

$$I \subseteq \mathbb{R}[x].$$

$$I = \langle f \rangle$$

g.

$$f = \sum a_i x^i$$

$$\text{ord}(f) = \min\{i \mid a_i \neq 0\}.$$

Let I be any I in R.
and $f(x) \in R$ be the power series with smallest order present in I.

WTS $I = (f(x))$.

Q8 $a, b \in I$ = Set of nilpotent elts of R.

$(a+b) \in I$.

and $ra \in I \quad \forall r \in R$.

$a^n = 0, \quad b^m = 0$.

Then $(a+b)^{m+n} = 0$.

$\mathbb{Z}/12\mathbb{Z}$

$12 = 0$.

$\Rightarrow 2^2 \cdot 3 = 0$.

$2 \cdot 3 = 6$.

$\mathrm{nil}(\mathbb{Z}/12\mathbb{Z}) = 6\mathbb{Z}$.

$\boxed{\mathrm{nil}(R) = \bigcap P. \quad P \text{ is prime ideal.}}$