Lecture - 15 (05-03-2024)

Def: A topological space (X, T) is said to be Lindelöf space if every open cover of X is reducible to a countable cover.

Hence every second countable topological space is a Lindelöf space.

Separable space:—

A topological space (X, T) is said to be a separable space if it containing a countable dense subset.

That is (X, T) is separable if there exists a finite or denumerable subset A of X such that $\overline{A} = X$.

Ex: Let (R, U) be a usual topological space.

$\because \overline{Q} = R$, Q is countable

$\therefore$ (R, U) is separable space.

Ex 2: (R, D) discrete topological space.

Then every subset of R is both open and closed, so the dense subset of R is R itself, which is uncountable.

$\bar{A} = R$

$A = \bar{A}$

$\bar{R} = R$

$\therefore$ (R, D) is not separable.

Theorem: Every second countable topological space is separable.

Proof: Let (X, T) be a second countable topological space. Then X has a countable base say

$$\mathcal{B} = \{B_n / n \in I\},$$ where I is index set.

Now for each $n \in I$, choose a point $a_n \in B_n$

Let $A = \{ a_n \mid n \in I \}$.

Then A is a countable set.

Claim: $\overline{A} = X$.

$[\because \quad A \cup A' = \overline{A} \overset{?}{=} X = A \cup A^c]$

So it is enough to prove that every point $P \in A^c$ is an accumulation point of A.

So let $P \in A^c$ and G be an open set containing P.

$[P \in G = \bigcup_n B_n \Rightarrow \exists\, n_0 \in I \ni P \in B_{n_0} \subset G]$

Then there exists $B_{n_0} \in \mathcal{B}$ such that

$P \in B_{n_0} \subset G$.

But $a_{n_0} \in B_{n_0}$ and $B_{n_0} \subset G \Rightarrow a_{n_0} \in G$.

Then $(G-\{p\}) \cap A \neq \varphi$, $\because a_{n_0} \neq p$

$\Rightarrow$ p is the limit point of A.

$\Rightarrow$ $p \in A'$.

$\Rightarrow$ $A^c \subseteq A'$.

$\therefore$ $\overline{A} = X$.

$\because$ A is Countable and $\overline{A} = X$

$\Rightarrow$ (X, T) is separable.

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H.w

Problem: A separable space need not be a second countable space.

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Theorem: let (X, d) be a separable metric space. Then X is a second countable topological space.

Proof: let (X, d) be a separable metric space. Then X has a

Countable dense subset say A.

i.e., $\bar{A} = X$ and A is countable.

Let

$$B = \{ S(a, \delta) \mid a \in A,\ \delta \in \mathbb{Q} \},$$

where

$$S(a, \delta) = \{ x \in X \mid d(x, a) < \delta \}.$$

Then B is countable set.

Claim: B is a base for X.

Let G be an open set containing a point $p \in X$.

Since G is an open set containing $p \in X$, so there exists an open sphere $S(p, \epsilon)$ with center p and radius ϵ such that

$$p \in S(p, \epsilon) \subset G.$$

Since $\overline{A} = X$, and $P \in X$, so there exists $a_0 \in A$ such that $d(P, a_0) < \epsilon/3$. ——(1)

Now choose δ_0 be a rational number such that $\epsilon/3 < \delta_0 < \frac{2\epsilon}{3}$.

$\therefore$ From (1), we have

$$d(P, a_0) < \epsilon/3 < \delta_0$$

$$\Rightarrow P \in S(a_0, \delta_0).$$

Hence it is enough to prove that

$$S(a_0, \delta_0) \subset S(P, \epsilon)$$

[If this is true, then

$$P \in S(a_0, \delta_0) \subset S(P, \epsilon) \subset G$$

$$\Rightarrow P \in S(a_0, \delta_0) \subset G]$$

So let $x \in S(a_0, \delta_0)$

$$\Rightarrow d(a_0, x) < \delta_0$$

Now using Δle inequality, we have

$$d(p, x) \leq d(p, a_0) + d(a_0, x)$$

$$< \epsilon/3 + \delta_0$$

$$< \epsilon/3 + 2\epsilon/3 \qquad [\because \delta_0 < 2\epsilon/3]$$

$$= \epsilon$$

$\Rightarrow d(p, x) < \epsilon \Rightarrow x \in S(p, \epsilon)$.

$\therefore S(a_0, \delta_0) \subset S(p, \epsilon)$.

But $S(a_0, \delta_0) \in \mathcal{B}$

$\therefore \mathcal{B}$ is a countable base for X.

$\Rightarrow X$ is second countable.

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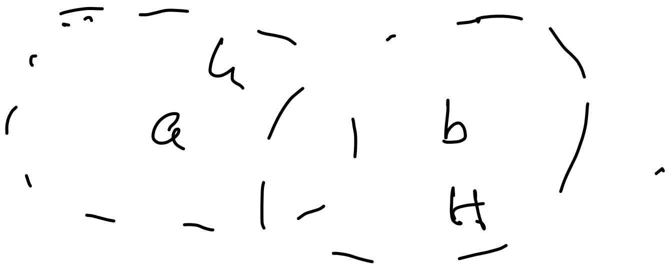
Separation Axiom

T_1 — Space :—

Let (X, T) be a topological space. We say X is a T_1-space if given any pair of distinct points $a, b \in X$, there exists open sets G and H such that $a \in G$, $a \notin H$ and $b \in H$, $b \notin G$.



The open sets need not be disjoint sets.

Ex: $X = \{a, b\}$, $T = \{X, \varphi, \{a\}\}$

$a \neq b$, $a \in \{a\}$, $b \in X$, but $a \in X$.

$\Rightarrow (X, T)$ is not a T_1-Space.

Theorem: A topological space (X, T) is a T_1-space iff every singleton subset of X is a closed set.

Proof: Suppose (X, T) is a T_1-space and $p \in X$.

Claim: $\{p\}$ is a closed set.

i.e., we prove $\{p\}^c$ is an open set.

Let $x \in \{p\}^c \Rightarrow x \notin \{p\}$

$\Rightarrow x \neq p$

$\because$ (X, T) is a T_1-space and $x, p \in X$ with $x \neq p$, so there exists an open set G_x such that

$x \in G_x$ and $p \notin G_x$.

$\Rightarrow \{p\} \not\subset G_x$.

$\Rightarrow G_x \subset \{p\}^c$

Thus $x \in G_x \subset \{p\}^c \Rightarrow \{p\}^c = \cup\{G_x \mid x \in \{p\}^c\}$

$\Rightarrow \{p\}^c$ is an open set.

$\Rightarrow \{p\}$ is a closed set.

Conversely suppose that $\{p\}$ is a closed set for every $p \in X$.

Claim: (X, τ) is a T_1-space.

So let $a, b \in X$, with $a \neq b$.

$\Rightarrow \{a\}$ and $\{b\}$ are closed sets.

Then $a \in \{b\}^c$ and $b \in \{a\}^c$

$\Rightarrow \{b\}^c$ is an open set containing a and $\{a\}^c$ is an open set containing b.

$\therefore (X, T)$ is a T_1-space.

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Note :— (1) Since finite union of closed sets are closed, by above Theorem, it follows that (X, T) is a T_1-Space iff T contains a cofinite topological space.

(2) All finite sets are closed in T_1-Space.

Ex: (1) Let (X, d) be a metric space. Then X is a T_1-Space, since we know that finite subsets of X are closed.

(2) $X = \{a, b\}$, $T = \{X, \phi, \{a\}\}$

closed sets: $X, \phi, \{b\}$.

$\because \{a\}$ is not closed, so X is not a T_1-Space.

Attendance

[36, 11, 19, 62, 42, 35, 58, 40, 51, 60, 06]