Assignment-III-Solution

1) If (X, T_1) is a disconnected topological space and (X, T_2) is finer than (X, T_1), then prove that (X, T_2) is dis-connected.

<u>Solution:</u> Given that (X, T_1) is a disconnected topological space. Then $\exists$ two non-empty disjoint open sets U, V in T_1 such that $X = U \cup V$ ——— $(*)$

Now, (X, T_2) is finer than (X, T_1). that is, $T_1 \subset T_2$

Therefore, $U, V \in T_2$ also.

From $(*)$ we can say (X, T_2) is disconnected.

[As $U, V \in T_2$ and U, V are non-empty disjoint open sets and $X = U \cup V$].

2) Let E be a connected subset of a T_1-space with more than one element. Show that E is infinite.

<u>Solution:</u> We know T_1-ness is a hereditary property. so, we consider a connected T_1-space with more than one element.

we have to show: The space is infinite.

let $x \neq y$ be two distinct points in E.

suppose E is not infinite. ie, E is finite.

Then being a T_1-space, every finite subset of E is closed.

Hence $\{x\}$ and $\underbrace{E \setminus \{x\}}_{\text{(a finite set)}}$ are closed sets in E.

Also, $\{x\} \cap (E \setminus \{x\}) = \phi$ and $E = \{x\} \cup (E \setminus \{x\})$

This shows that E is disconnected.

Which is a contradiction.

Hence, E must be infinite.

3) Prove that a discrete space with atleast two elements is disconnected.

Solution: The proof follows from the fact that every every subsets of a discrete space is closed as well as open sets.

4) Is the intersection of a closed set with a compact subspace compact? Justify.

Solution: let (X, T) be a topological space.

let F be a closed sets and C be a compact sets in X.

we have to show $F \cap C$ is compact.

let $\{U_\alpha \mid \alpha \in \Lambda\}$ be an open cover of $F \cap C$.

Now, $\{U_\alpha \mid \alpha \in \Lambda\} \cup \{\underbrace{X \sim F}_{\text{open set}}\}$ is an open cover of C

Since C is compact, it has a finite subcover.

Let $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}, X \sim F$ be such a finite subcover of C.

clearly $F \cap C = \bigcup_{i=1}^{n} U_{\alpha_i}$. This shows that every open cover of $F \cap C$ has a finite subcover and hence $F \cap C$ is compact.

5) Consider the plane $\mathbb{R}^2$. Is it locally compact with respect to

ⓐ Usual topology?

ⓑ Discrete topology?

Justify.

<u>Solution</u>: ⓐ For each point $p = (p_1, p_2) \in \mathbb{R}^2$, the closed ball $[p_1 - \delta, p_1 + \delta] \times [p_2 - \delta, p_2 + \delta]$ is compact. Hence $\mathbb{R}^2$ is locally compact with respect to the usual topology.

ⓑ For every point $p = (p_1, p_2) \in \mathbb{R}^2$, $\{(p_1, p_2)\}$ is a compact neighbourhood of p, as every open cover of $\{(p_1, p_2)\}$ is reducible to a finite cover $\{\{(p_1, p_2)\}\}$. Thus $\mathbb{R}^2$ is locally compact with respect to discrete topology.

6) Let T be the topology on X which consists of ϕ and the complements of countable subsets of X. Show that every infinite subset of X is not sequentially compact.

<u>Solution</u>: We know that a sequence converges in (X, T) iff the sequence is of the form

$$(a_1, a_2, \ldots, a_m, a, a, \ldots), \text{ where } m \in \mathbb{N}.$$

Thus for an infinite subset Y of X, any sequence (b_n) in Y consisting of distinct terms would never have a convergent subsequence.

Hence, any infinite subset of X is not sequentially compact.

7) Show that if X has a countable subbase then there exists a countable base $\mathcal{B}$ for X.

<u>Solution</u>: Let $S = \{S_1, S_2, S_3, \dots\}$ be a countable subbase for X.

Since finite intersection of members of S form a base for X and as we know that number of finite sets in $\mathbb{N}$ is countable, say $\{\lambda_1, \lambda_2, \dots\}$,

$\left\{ \bigcap_{i \in \lambda_1} S_i, \bigcap_{i \in \lambda_2} S_i, \dots \right\}$ forms a countable base for X.

8) Exhibit a countable base for Euclidean m-space.

<u>Solution</u>: $\left\{ (a_{11}, a_{12}) \times (a_{21}, a_{22}) \times \dots \times (a_{n1}, a_{n2}) : a_{11}, a_{12}, a_{21}, a_{22}, \dots, a_{n1}, a_{n2} \in \mathbb{Q} \right\}$

forms a countable base for $\mathbb{R}^n$.

9) Let $\mathcal{A}$ be any collection of disjoint open subsets of a second countable space X. Show that $\mathcal{A}$ is a countable collection.

Solution: Let X be a second countable space.

Since X is countable space. Then there exists a countable base $\mathcal{B}$ for the topology T.

Let $\{B_1, B_2, B_3, \dots\}$ be a countable base for X.

Let $\mathcal{A} = \{U_i, i \in \Lambda\}$ be a collection of disjoint open sets in X.

Now for $U_i, U_j \in \mathcal{A}$, $i \neq j$, $U_i, U_j \neq \phi$,

we have $x_i \in B_i \subset U_i$,

$$x_j \in B_j \subset U_j$$

$$\Rightarrow B_i \cap B_j = \phi$$

$$\therefore |\{U_i\}| \leq |\{B_i\}|$$

$$\Rightarrow \{U_i\} \text{ is countable.}$$

$\therefore$ $\mathcal{A}$ is a countable collection.

10) Show that a continuous image of a Lindelöf space is also a Lindelöf space.

<u>Solution</u>: Let (X, T) be a Lindelöf space and

$f: (X,T) \to (Y, T^*)$ be a continuous map.

<u>claim</u>: $f(X)$ is Lindelof space in (Y, T^*).

Let $\{G_1, G_2, \ldots\}$ be an open cover of $f(X)$.

Then $f(X) \subseteq \bigcup_i G_i$

Then $X = \bigcup_i f^{-1}(G_i)$

since, f is continuous. Then $f^{-1}(G_i)$ is open in X.

so, $\{f^{-1}(G_1), f^{-1}(G_2), \ldots\}$ is an open cover of X.

since X is a Lindelöf space $\{f^{-1}(G_i)\}$ is reducible to a countable cover. say

$$\left\{ f^{-1}(G_{ij}) \;\middle|\; \begin{matrix} i=1,2,\ldots \\ j=1,2,\ldots \end{matrix} \right\}$$

$\therefore X = \bigcup_i \bigcup_j f^{-1}(G_{ij})$

$\Rightarrow f(X) = \bigcup_i \bigcup_j G_{ij}$

Hence any open cover of $f(X)$ reduced to a countable cover of $f(X)$. so, $f(X)$ is Lindelöff space.

11) Show that a discrete space X is separable if and only if X is countable.

Solution: Recall that every subset of a discrete space X is both open and closed. Hence the only dense subset of X is X itself. Hence X contains a countable dense subset iff X is countable, ie, X is separable iff X is countable.

12) Show that a finite subset of a T_1-space X has no accumulation points.

Solution: Suppose $A \subset X$ has n elements, say $A = \{a_1, a_2, \ldots, a_n\}$. Since A is finite it is closed and therefore contains all of its accumulation points. But $\{a_2, \ldots, a_n\}$ is also finite and hence closed. Accordingly, the complement $\{a_2, \ldots, a_n\}^c$ of $\{a_2, \ldots, a_n\}$ is open, contains a_1 and contains no points of A different form a_1. Hence a_1 is not an accumulation point of A. Similarly no other points of A is an accumulation point of A and A has no accumulation points.

13) Let T be the topology on the real line $\mathbb{R}$ generated by the open-closed intervals $(a,b]$. Is $(\mathbb{R}, T)$ Hausdorff?

Solution: Let $a, b \in \mathbb{R}$ with $a \neq b$ say $a<b$.

choose $G = (a-1, a]$ and $H = (a, b]$. Then

$G, H \in T$, $a \in G$, $b \in H$ and $G \cap H = \phi$

Hence (X, T) is Hausdorff.

15) Let $C[a,b]$ denote the collection of all continuous functions on a closed interval $X = [a,b]$, Consider the metrices d and e on $C[a,b]$ defined by

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in X\}$$

$$e(f,g) = \int_a^b |f(x) - g(x)|\,dx$$

show that the topology T_e induced by e is coarser than the topology T_d induced by d. ie, $T_e \subset T_d$.

Solution: Let $S_e(p, \varepsilon)$ be any e-open sphere in $C[a,b]$ with center $p \in C[a,b]$. Let $\delta = \varepsilon/(b-a)$.

[In view that if d and e be metrics on a set X such that for each d-open sphere S_d with center $p \in X$ there exists an e-open sphere S_e with centrer p such that $S_e \subset S_d$. Then the topology T_d induced by d is coarser than the topology T_e induced by e. ie, $T_d \subset T_e$.]

$\therefore$ It is sufficient to show that $S_d(p,\delta)$, the d-open sphere with center p and radius δ, is a subset of $S_e(p,\varepsilon)$ ie, $S_d(p,\delta) \subset S_e(p,\varepsilon)$.

Let $f \in S_d(p,\delta)$, then $\sup\{|p(x)-f(x)|\} < \delta = \frac{\varepsilon}{(b-a)}$

Hence, $e(p,f) = \int_a^b |p(x)-f(x)|\,dx \le \int_a^b \sup\{|p(x)-f(x)|\}\,dx$

$$< \int_a^b \varepsilon/(b-a) = \varepsilon$$

So, $f \in S_e(p,\varepsilon)$ and therefore, $S_d(p,\delta) \subset S_e(p,\varepsilon)$.

16) If $f: X \to Y$ is a continuous function and Y is a Hausdorff space then the graph of f, $G(f) = \{(x, f(x)) : x \in X\}$ is a closed set.

Solution: We consider the function $f \times I_Y : X \times Y \to Y \times Y$ given by $(f \times I_Y)(x,y) = (f(x), y)$. Since f and I_Y are both continuous, $f \times I_Y$ is continuous. Y being Hausdorff, $\Delta = \{(y,y) : y \in Y\}$ is closed set in $Y \times Y$.

Now, $(f \times I_Y)^{-1}(\Delta) = \{(x,y) : (f(x), y) \in \Delta\} = G(f)$ is closed (since f is continuous) in $X \times Y$.

18) Let T be the topology on X which consists of ϕ and the complements of countable subsets of X. Show that every infinite subset of X is not sequentially compact.

<u>Sol^n</u>: Let T be the topology on X which consists of ϕ and the complements of countable subsets of X. We claim that a sequence $\{a_1, a_2, \dots\}$ in X converges to $b \in X$ iff the sequence is also of the form $\{a_1, a_2, \dots, a_{n_0}, b, b, b, \dots\}$. i.e, the set A consisting of the terms $\{a_n\}$ different from b is finite. Now A is countable and so A^c is an open set containing b. Hence if $a_n \to b$ then A^c contains all except a finite number of the terms of the sequence and so A is finite.

So, a sequence in (X, T) converges iff it is of the form $\{a_1, a_2, \dots a_{n_0}, p, p, p, \dots\}$. That is, is constant from some term on. Hence if A is an infinite subset of X, there exists a sequence $\{b_n\}$ in A with distinct terms. Thus $\{b_n\}$ does not contain any convergent subsequence and A is not sequentially compact.