Lecture 17

Properties of measurable functions

① $f: E \to \mathbb{R}$, $E \subseteq \mathbb{R}^d$ is measurable if and only if $f^{-1}(U)$ is measurable $\forall\, U \subseteq \mathbb{R}^d$ open

② Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set $E \subseteq \mathbb{R}^d$.

Then $\sup(f_n)$, $\inf(f_n)$, $\limsup(f_n)$, $\liminf(f_n)$ are measurable.

③ $f_n: E \to \mathbb{R}$ measurable $\forall\, n \geq 1$. If $f_n \to f$ pointwise, then f is measurable.

④ f, g are measurable $\Rightarrow$ $f \pm g$, fg, f^2 are measurable.
$f, g: E \to \mathbb{R}$

Approximation of measurable funtions by simple funtions or step funtions:

Def:- A step funtion is a finite sum

$$f = \sum_{k=1}^{N} a_k \chi_{R_k}, \quad \text{where } R_k \text{ are rectanges in } \mathbb{R}^d \text{ \& } a_k \in \mathbb{R}.$$

&

$$\chi_{R_k}(x) = \begin{cases} 1 & \text{if } x \in R_k \\ 0 & \text{if } x \notin R_k. \end{cases}$$

Eg:- $f = \chi_{[0,1]} - 2\chi_{[2,3]} + 5\chi_{[3,5]}$ step fun in $\mathbb{R}$.

Def:- A simple function is a finite sum

$$f = \sum_{k=1}^{N} a_k \chi_{E_k},$$ where each E_k is a measurable set of finite measure. & $a_k \in \mathbb{R}$.

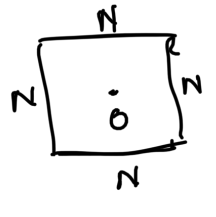
Eg:- $f = 2\chi_P - 3\chi_{[2,5]}$ on $\mathbb{R}$, a simple function, where P is the Cantor set.

Theorem:— Suppose f is a non-negative measurable function on $\mathbb{R}^d$. Then there exists an increasing sequence of non-negative simple functions $\{\varphi_k\}_{k=1}^{\infty}$ that converges pointwise to f, namely, $\varphi_k(x) \leq \varphi_{k+1}(x)$ & $\lim_{k \to \infty} \varphi_k(x) = f(x)$ $\forall x$

proof:—

For $N \geq 1$, let

Q_N = the cube centered at the origin & of side length N.



Define

$$F_N(x) = \begin{cases} f(x) & \text{if } x \in Q_N \text{ \& } f(x) \leq N \\ N & \text{if } x \in Q_N \text{ \& } f(x) > N \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lim_{N \to \infty} F_N(x) = f(x)$ $\forall x$

$\boxed{\text{if } x \in Q_{N_0},\ F_N(x) = f(x)\ \forall N \geq \max(N_0, f(x))}$

Now partition the range of $F_N(x)$, namely $[0, N]$ as follows:

For fixed $N, M \geq 1$, we define

$$E_{l,M} = \left\{ x \in Q_N \;\middle|\; \frac{l}{M} \lneq F_N(x) \leq \frac{l+1}{M} \right\}$$

for $0 \leq l \lneq MN$.

&

$$F_{N,M}(x) = \sum_{l=0}^{NM-1} \frac{l}{M} \chi_{E_{l,M}}$$

$[0, \frac{1}{M}, \frac{2}{M}, \frac{3}{M} \cdots N]$

Each $F_{N,M}(x)$ is a simple function that satisfies

$$0 \leq F_N(x) - F_{N,M}(x) \leq \frac{1}{M} \quad \forall x.$$

For $x \in Q_N$, say $F_N(x) \in \left[\frac{l}{M}, \frac{l+1}{M}\right]$ for some l

$\Downarrow$

$x \in E_{l,M}$

& $F_{N,M}(x) = \frac{l}{M}$

$\therefore$ $F_N(x) \gneq \frac{l}{M} = F_{N,M}(x)$

$\Rightarrow$ $F_N(x) - F_{N,M}(x) > 0.$

$F_N(x) \leq \frac{l+1}{M}$

$F_N(x) - F_{N,M}(x) \leq \frac{l+1}{M} - \frac{l}{M} = \frac{1}{M}.$

Now choose $N = M = 2^k$ with $k \geq 1$ &

let $\varphi_k = F_{2^k, 2^k}$ $\forall k \geq 1$.

Then we have

$$0 \leq F_N(x) - \varphi_k(x) \leq \frac{1}{2^k}, \quad \forall k \geq 1, \longrightarrow (*)$$

$\{\varphi_k\}$ is increasing: if $x \in E_{\ell, 2^k}$ for some $0 \leq \ell < 2^k 2^k$.

then $\varphi_k(x) = \frac{\ell}{2^k}$.

& $\varphi_{k+1}(x) \in \left\{ \frac{\ell}{2^k}, \frac{\ell}{2^k} + \frac{1}{2^{k+1}} \right\}$.

$\Rightarrow \varphi_{k+1}(x) \geq \varphi_k(x)$.

$\varphi_k \to f$ pointwise:

$$|\varphi_k(x) - f(x)| = |\varphi_k(x) - F_{2^k}(x) + F_{2^k}(x) - f(x)|$$

$$\leq \underbrace{|\varphi_k(x) - F_{2^k}(x)|}_{\substack{\downarrow \\ 0 \\ \text{as } k \to \infty \\ (\text{by using } (*))}} + \underbrace{|F_{2^k}(x) - f(x)|}_{\substack{\downarrow \\ 0 \text{ as} \\ k \to \infty}}$$

$F_N(x) \to f(x)$

$\therefore |\varphi_k(x) - f(x)| \to 0$ as $k \to \infty$

That is $\varphi_k(x) \to f(x)$ as $k \to \infty$.

Thus: $\{\varphi_k\}$ seq. of simple functions non-negative, increasing & converges pointwise to f.

<u>Remark</u>:- Let $f : E \to \mathbb{R}$ be any function, $E \subseteq \mathbb{R}^d$.

Define $f^+ : E \to \mathbb{R}$, $f^+(x) := \max\{f(x), 0\} \geq 0$

$$= \frac{f(x) + |f(x)|}{2} \quad (\text{Ex})$$

& $f^- : E \to \mathbb{R}$, $f^-(x) := -\min\{f(x), 0\}$

$$= \max\{-f(x), 0\} \geq 0$$

$$= \frac{|f(x)| - f(x)}{2} \quad (\text{Ex}).$$

Then

(i) f^+, f^- are non-negative functions.

(ii) $f = f^+ - f^-$

(iii) $|f| = f^+ + f^-$.

Theorem:- Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of simple funtions $\{\varphi_k\}_{k=1}^{\infty}$ that satisfies

$$|\varphi_k(x)| \leq |\varphi_{k+1}(x)| \quad \forall k \geq 1 \quad \&$$

$$\lim_{k \to \infty} \varphi_k(x) = f(x) \quad \forall x.$$

proof:- We have $f(x) = f^+(x) - f^-(x)$

where f^+, f^- are non-negative measurable funtions.

$\therefore$ By above theorem, there exist sequence of simple funtions $\{\varphi_k^{(1)}(x)\}$ & $\{\varphi_k^{(2)}(x)\}$

such that $\varphi_k^{(1)}(x) \to f^+(x)$ &

$\varphi_k^{(2)}(x) \to f^-(x)$ as $k \to \infty$.

Let $\underline{\varphi_k(x) := \varphi_k^{(1)}(x) - \varphi_k^{(2)}(x)}$ $\forall x$, $\forall k$.

Then $\lim_{k\to\infty} \varphi_k(x) = f^+(x) - f^-(x)$

$= f(x) \quad \forall x.$

Now $|\varphi_k| = |\varphi_k^{(1)} - \varphi_k^{(2)}|$.

$= \varphi_k^{(1)} + \varphi_k^{(2)}$.

(Fact: If $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$, then $|\varphi| = \sum_{k=1}^{N} |a_k| \chi_{E_k}$)

Suppose $\varphi_k^{(1)} = \sum_{j=1}^{N_1} a_{k,j} \chi_{E_{k,j}^{(1)}}$

$\varphi_k^{(2)} = \sum_{j=1}^{N_2} b_{k,j} \chi_{E_{k,j}^{(2)}}$

$\varphi_k^{(1)} - \varphi_k^{(2)} = \sum_{j=1}^{N_1} a_{k,j} \chi_{E_{k,j}^{(1)}} + \sum_{j=1}^{N_2} (-b_{k,j}) \chi_{E_{k,j}^{(2)}}$

a simple function

$\Rightarrow |\varphi_k^{(1)} - \varphi_k^{(2)}| = \sum_{j=1}^{N_1} |a_{k,j}| \chi_{E_{k,j}^{(1)}} + \sum_{j=1}^{N_2} \underbrace{|-b_{k,j}|}_{= |b_{k,j}|} \chi_{E_{k,j}^{(2)}}$

$= |\varphi_k^{(1)}| + |\varphi_k^{(2)}|$

Since $\varphi_k^{(1)}, \varphi_k^{(2)}$ are increasing, therefore

$$|\varphi_k| = \varphi_k^{(1)} + \varphi_k^{(2)}$$ is also increasing.