INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR

Date: 25.04.2018, FN, Time: 3 Hrs, Full Marks: 50, Deptt: MA/ ME/PH
No. of Students: 85, End-Spring Semester Examination, 2018
Sub. No. MA40002/ MA51004/ MA61052,
Sub. Name Integral Equations and Variational methods

Instruction: *Answer all the questions. Notations have their usual meaning. Calculators are not allowed.*

1. Let $\lambda = \lambda_0$ be an eigenvalue of $u(x) = \lambda \int_{-1}^{1} (x \cosh t - t \sinh x)\, u(t)dt$ ... (1)
(a) Find λ_0 and the corresponding eigenfunction for the integral equation (1).
(b) If $\lambda \neq \lambda_0$, does solution to the integral equation

$$u(x) = \sinh x + \lambda \int_{-1}^{1} (x \cosh t - t \sinh x)\, u(t)dt$$

exist? Justify. Is the solution unique? If solution(s) exist, find it/them.
(c) Can $\lambda = 2 + \mathrm{i}$ be an eigenvalue of the integral equation

$$u(x) = \lambda \int_0^x \sinh t \sinh(x-1)\, u(t)dt + \lambda \int_x^1 \sinh x \sinh(t-1)\, u(t)dt\, ?$$

Give reason in support of your answer. [4+3+1=8]

2. (a) Derive the solution to the integral equation $f(x) = \int_0^x \frac{u(t)dt}{\{g(x)-g(t)\}^{\alpha}}$; $0 < \alpha < 1$
where $f(x)$, $g(x)$ are known continuous functions in $[0, a]$ $(a > 0)$.
(b) Solve $u(x) = 2x^2 - 4\int_0^x (x-t)\, u(t)dt$ by the method of successive approximation taking $u_0 = 0$.
You must compute u_1, u_2, u_3 to find the solution. [4+4=8]

3. Using Green's function technique solve the boundary value problem

$$\frac{d^2u}{dx^2} - u = 4e^x;\ \ 0 \le x \le 2;\ \ u(0) = u'(0),\ u(2) + u'(2) = 0.$$ [7]

4. (a) For what value of α and β there exists a solution to the variational problem

$$I[y(x)] = \int_0^1 (xy + y^2 - 2y^2 y')\, dx;\ \ y(0) = \alpha,\ y(1) = \beta.$$

(b) (i) Write down the problem of finding the shortest path between two points in xy -plane as a variational problem. (ii) Change the governing functional $I[y(x)]$ to a functional of the form $J[r(\theta)]$ by changing (x, y) coordinates to (r, θ) coordinates taking $x = r\cos\theta$, $y = r\sin\theta$. (iii) If the extremal is of the form $r = f(\theta)$, find $f(\theta)$. [3+5=8]

5. (a) Derive Euler-Poisson equation.
(b) Find the curve that extremizes the functional $I[y(x)] = \int_0^1 \{x^4 (y'')^2 + 4x^2 (y')^2\}\, dx$, given that y is not singular at $x = 0$ and that $y(1) = y'(1) = 1$. [6+4=10]

6. Find the shortest distance between the circle $x^2 + y^2 = 1$ and the straight line $x + y = 4$, using the concept of calculus of variations. [9]

****