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1a. $xy'' + y' + y/4 = 0$

$$z = \sqrt{x}, \quad \frac{dz}{dx} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{2\sqrt{x}} \frac{dy}{dz}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dz}\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{1}{x^{3/2}}\right) + \frac{1}{2\sqrt{x}} \frac{d^2y}{dz^2} \cdot \frac{dz}{dx}$$

$$= \frac{1}{2\sqrt{x}} \frac{d^2y}{dz^2} \cdot \frac{dz}{dx} - \frac{1}{4z^{3/2}} \frac{dy}{dz}$$

substituting in given equation, we get

$$x\left(\frac{1}{2\sqrt{x}} \frac{d^2y}{dz^2} \cdot \frac{dz}{dx} - \frac{1}{4x^{3/2}} \frac{dy}{dz}\right) + \frac{dy}{dz} \frac{1}{2\sqrt{x}} + \frac{y}{4} = 0$$

$$\frac{1}{4} \frac{d^2y}{dz^2} + \frac{1}{4\sqrt{x}} \frac{dy}{dz} + \frac{y}{4} = 0$$

$$\Rightarrow \frac{d^2y}{dz^2} + \frac{1}{z} \frac{dy}{dz} + y = 0$$

$$\Rightarrow z^2 \frac{dy}{dz^2} + z \frac{dy}{dz^2} + z^2 y = 0.$$

Above is Bessels function of 'o' order.

Thus, general solution is given by

$y = A J_0(z) + B Y_0(z)$ as '0' is an integer

$\boxed{y = A J_0(\sqrt{x}) + B Y_0(\sqrt{x})}$ where A, B are arbitrary constants.

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16. $\frac{d}{dx}\left(x J_n(x)\, J_{n+1}(x)\right) = x\left\{J_n^2(x) - J_{n+1}^2(x)\right\}$

↳ To prove.

$$\frac{d}{dx}\left[(x)(J_n(x))\, J_{n+1}(x)\right] = J_n(x)\, J_{n+1}(x) + \cancel{x\frac{d}{dx}}$$

$$+ x\frac{d}{dx}\left(J_n(x) J_{n+1}(x)\right)$$

$$= J_n(x) J_{n+1}(x) + x\left(J_{n+1}(x) J_n'(x) + J_n(x) J_{n+1}'(x)\right)$$

$$= J_n(x) J_{n+1}(x) + x\left(J_{n+1}(x)\left[\left(\frac{n}{x}\right) J_n(x) - J_{n+1}(x)\right]\right.$$

$$\left. + J_n(x)\left[J_n(x) - \left(\frac{n+1}{x}\right) J_{n+1}(x)\right]\right)$$

$$= J_n(x) J_{n+1}(x) + x\left(\frac{n}{x} J_n(x) J_{n+1}(x) - J_{n+1}^2(x) + J_n^2(x)\right.$$

$$\left. - \left(\frac{n+1}{x}\right) J_n(x) J_{n+1}(x)\right)$$

$$= J_n(x) J_{n+1}(x) + x\left(J_n^2(x) - J_{n+1}^2(x) - \frac{J_n(x) J_{n+1}(x)}{x}\right)$$

$$= x\left(J_n^2(x) - J_{n+1}^2(x)\right)$$

$$\therefore \frac{d}{dx}\left[x J_n(x)\, J_{n+1}(x)\right] = x\left(J_n^2(x) - J_{n+1}^2(x)\right)$$

hence proved

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2a. We need to show that $J_n'(x)=0$ has no repeated roots except at $x=0$

Assume that $J_n(x)$ has multiple roots when $x\neq 0$.
Then, $J_n(x_0)=0$ & $J_n'(x_0)=0$

As J_n satisfies, $x^2y''+xy'+(x^2-n^2)y=0$,
$J_n(x_0)=0$, $J_n'(x_0)=0 \Rightarrow J_n''(x_0)=0$
as $x_0^2 J_n''(x_0)=0$ & $x_0\neq 0$

This implies $J_n^{(m)}(x_0)=0 \quad \forall m$
But, as J_n is analytic, J_n vanishes identically on a neighbourhood of x_0
i.e $J_n\equiv 0$.

This is a contradiction. Thus $J_n(x)=0$ has no repeated roots when $n\neq 0$

2b. Express $J_4(x)$ in terms of J_0 & J_1

We know by recurrence formula
$$J_n(x)=\frac{x}{2n}(J_{n-1}(x)+J_{n+1}(x))$$

$$\therefore J_{n+1}(x)=\frac{2n}{x}J_n(x)-J_{n-1}(x)$$
for $n=1,2,3$
$$J_2(x)=(2/x)J_1(x)-J_0(x)$$
$$J_3(x)=(4/x)J_2(x)-J_1(x)$$
$$J_4(x)=(6/x)J_3(x)-J_2(x)$$

Thus, $J_3(x)=\frac{4}{x}\left(\frac{2}{x}J_1(x)-J_0(x)\right)-J_1(x))$
$$=\left(\frac{8}{x^2}-1\right)J_1(x)-\frac{4}{x}(J_0(x))$$

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$$J_4(x) = \frac{6}{x}\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{6}{x}\left(\frac{4}{x}\right) J_0(x) - \frac{2}{x} J_1(x) + J_0(x)$$

$$\therefore \boxed{J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)}$$

final Ans

3a. i) $J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots) = 1$

As,

$$J_n'(x) = \left(\frac{n}{x}\right) J_n - J_{n+1} = \left(-\frac{n}{x}\right) J_n + J_{n-1}$$

$$\therefore J'_{(n+1)}(x) = \left(-\frac{(n+1)}{x}\right) J_{n+1} + J_n$$

We know,

$$\frac{d}{dx}(J_n^2 + J_{n+1}^2) = 2J_nJ_n' + 2J_{n+1}J_{n+1}'$$
$$= 2J_n\left(\left(\frac{n}{x}\right)J_n - J_{n+1}\right) + 2J_{n+1}\left(-\frac{(n+1)}{x}J_{n+1} + J_n\right)$$

$$\frac{d}{dx}(J_n^2 + J_{n+1}^2) = 2\left[\left(\frac{n}{x}\right)J_n^2 - \left(\frac{n+1}{x}\right)J_{n+1}^2\right]$$

Applying $\sum_{n=0}^{\infty}$ on both sides, we get

$$\sum_{n=0}^{\infty}\frac{d}{dx}(J_n^2 + J_{n+1}^2) = 2\left(0 - \frac{J_1^2}{x}\right) + 2\left(\frac{1}{x}J_1^2 - \frac{2}{x}J_2^2\right) +$$
$$2\left(\frac{2}{x}J_2^2 - \frac{3}{x}J_3^2\right) + \cdots\cdots$$
$$= \cancel{-\frac{2J_1^2}{x}} + \cancel{\frac{2J_1^2}{x}} - \cancel{\frac{4J_2^2}{x}} + \cancel{\frac{4J_2^2}{x}} - \frac{6J_3^2}{x} + \cdots$$

as consecutive terms cancel out each other

$$\sum_{n=0}^{\infty}\frac{d}{dx}(J_n^2 + J_{n+1}^2) = 0$$

$$\therefore \frac{d}{dx}\left(J_0^2 + 2(J_1^2 + J_2^2 + \cdots)\right) = 0$$

$\rightarrow$ $J_0^2 + 2(J_1^2 + J_2^2 + \cdots\cdots) = c$ $\rightarrow$ arbitrary

We know, $J_0(0) = 1$ & $J_n(0) = 0$, $n \neq 0$

$\therefore$ Using above information,

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$$\left(J_0^2 + 2\left(J_1^2 + J_2^2 + \dots\right)\right)\Big|_{x=0} = 1 = c$$

$\therefore\ c = 1$

Thus, $J_0^2 + 2(J_1^2 + J_2^2 + \dots) = 1$, hence proved

3a. ii) $|J_0(x)| \leq 1$

We have, $J_0^2 + 2(J_1^2 + J_2^2 + \dots) = 1$ &
$x^2 \geq 0 \quad \forall x \in \mathbb{R}$

As each term is non-negative, the individual value of the terms has to be less than or equal to the whole sum, thus
$J_0^2 \leq 1, \quad 2J_1^2 \leq 1, \quad 2J_2^2 \leq 1, \dots, \quad 2J_n^2 \leq 1, \dots$

Thus, $J_0^2 \leq 1 \Rightarrow |J_0(x)| \leq 1$, hence proved

3a. iii) ~~Prove~~ ~~als~~ $|J_n(x)| \leq 2^{-1/2}, \quad n \geq 1$
from above we see that
$J_n^2 \leq 1/2 \quad \forall n \geq 1$
$\Rightarrow |J_n(x)| \leq \frac{1}{\sqrt{2}} \quad \forall n \geq 1$

$\Rightarrow |J_n(x)| \leq 2^{-1/2} \quad \forall n \geq 1$ ~~hence~~ hence proved

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3b. $\lim\limits_{a,b\to\infty} {}_2F_1\left(a,b;\tfrac{1}{2};\dfrac{x^2}{4ab}\right)$

consider

$${}_2F_1\left(a,b;\tfrac{1}{2};\frac{x^2}{4ab}\right) = 1 + \frac{ab}{\frac{1}{2}}\frac{x^2}{4ab} + \frac{a(a+1)b(b+1)}{\frac{1}{2}\cdot\frac{3}{2}\cdot 2!}\frac{x^4}{16a^2b^2} + \cancel{a(a+1)(a+2)}$$

$$+ \frac{a(a+1)(a+2)(b)(b+1)(b+2)}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot 3!}\frac{x^6}{64a^3b^3} + \ldots\ldots$$

$$= 1 + \frac{x^2}{2} + \frac{(a+1)(b+1)x^4}{ab.4.2!.3} + \frac{(a+1)(a+2)(b+1)(b+2)x^6}{a^2b^2.3.5.3!.8} + \ldots$$

applying $\lim\limits_{a,b\to\infty}$ on both sides

$$\cancel{\lim_{a,b\to\infty} {}_2F_1\left(a,b;\tfrac{1}{2};\frac{x^2}{4ab}\right) = \lim_{a,b\to\infty}\left(1 + \frac{x^2}{2} + \frac{a\left(1+\frac{1}{a}\right)b\left(1+\frac{1}{b}\right)x^4}{ab.4.2!.3} + \frac{a^2\left(1+\frac{1}{a}\right)\left(1+\frac{2}{a}\right)(b^2)\left(1+\frac{1}{b}\right)\left(1+\frac{2}{b}\right)}{a^2b^2.3.5.3!.8}\right]}$$

$$\lim_{a,b\to\infty} {}_2F_1\left(a,b;\tfrac{1}{2};\frac{x^2}{4ab}\right) = \lim_{a,b\to\infty}\left[1 + \frac{x^2}{2} + \frac{\cancel{ab}(1+1/a)(1+1/b)x^4}{\cancel{ab}.2!.3.4} + \frac{\cancel{a^2b^2}(1+1/a)(1+2/a)(1+1/b)(1+2/b)x^6}{\cancel{a^2b^2}.3!.3.5.8} + \ldots\right]$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4.3.2!} + \frac{x^6}{3!.8.5.3} + \ldots..$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{3!.5.6.4} + \ldots..$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots..$$

$$\therefore \lim_{a,b\to\infty} {}_2F_1\left(a,b;\tfrac{1}{2};\frac{x^2}{4ab}\right) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots = \cosh(x)$$

4.a. $4x(1-x)y'' + y' + 8y = 0$ → given eq$^{\underline{n}}$

$\Rightarrow x(1-x)y'' + \frac{1}{4}y' + 2y = 0$

The above eq$^{\underline{n}}$ is of the form $x(1-x)y'' + \{\gamma - (\alpha+\beta+1)x\}y' - \alpha\beta y = 0$

i.e hypergeometric eq$^{\underline{n}}$

$\therefore \quad \gamma = 1/4$

$\alpha + \beta + 1 = 0 \Rightarrow \alpha + \beta = -1 \quad \rightarrow (i)$

$-\alpha\beta = 2 \Rightarrow \alpha = -2/\beta \quad \rightarrow (ii)$

substitute (ii) in (i)

$\frac{-2}{\beta} + \beta = -1$

$\Rightarrow -2 + \beta + \beta^2 = 0$

$\Rightarrow \beta^2 + \beta - 2 = 0$

$\Rightarrow \beta = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$

$\Rightarrow \beta = -2, 1$

$\therefore \quad \alpha = 1, \ \beta = -2, \ \gamma = 1/4$

Solution of the given eq$^{\underline{n}}$ is :-

$y = A \, {}_2F_1(\alpha, \beta; \gamma; x) + B x^{1-\gamma} \, {}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x)$

$y = A \, {}_2F_1(1, -2; 1/4; x) + B x^{3/4} \, {}_2F_1(7/4, -5/4; 7/4; x)$

↳ solution

as $\beta < 0$, ${}_2F_1(1, -2; 1/4; x)$ reduces to a polynomial

$${}_2F_1(1, -2; 1/4; x) = 1 + \frac{(1)(-2)}{1/4}x + \frac{(1)(\cancel{2})(-2)(-1)x^2}{\frac{1}{4}\left(\frac{5}{4}\right) \cancel{2!}} + \cancel{\frac{x^3(1)(2)(3)(-2)(-1)0}{\frac{1}{4}\cdot\frac{5}{4}\cdot\frac{9}{4}\, 3!}}$$

0

$${}_2F_1(1, -2; 1/4; x) = 1 - 8x + \frac{32}{5}x^2$$

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$\therefore$ solution of $4x(1-x)y'' + y' + 8y = 0$ is given by

$$y = A\ {}_2F_1(1, -2; 1/4; x) + B x^{3/4}\ {}_2F_1(7/4, -5/4; 7/4; x)$$

$$y = A\left(1 - 8x + \frac{32}{5}x^2\right) + B x^{3/4}\ {}_2F_1(7/4, -5/4; 7/4; x)$$

where A, B are arbitrary constants

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46. $\cos(x\cos\sin\phi) = J_0 + 2\cos(2\phi)J_2 + 2\cos(4\phi)J_4 + \dots$

Generating function of bessel function is $\exp\left(\frac{x}{2}\left(z-\frac{1}{z}\right)\right)$, where coefficient of z^n is $J_n(x)$.

$$\therefore \exp\left(\frac{x}{2}\left(z-\frac{1}{z}\right)\right) = \sum_{m=-\infty}^{\infty} J_m(x) z^m$$

Also, $J_{-n}(x) = (-1)^n J_n(x)$

$$\therefore \exp\left(\frac{x}{2}\left(z-\frac{1}{z}\right)\right) = \sum_{m=0}^{\infty}\left[J_m z^m + J_{-m} z^{-m}\right]$$

$$= \sum_{m=0}^{\infty}\left[J_m z^m + (-1)^m J_m z^{-m}\right]$$

$$= \sum_{m=0}^{\infty} J_m\left(z^m + (-1)^m z^{-m}\right)$$

$\therefore J_m(x)$ is also the coefficient of $\left(z^m + (-1)^m z^{-m}\right)$ when $m \geq 0$

let $z = e^{i\phi}$, $z^m = e^{im\phi}$, $z^{-m} = e^{-im\phi}$

$$\therefore z^m + (-1)^m z^{-m} = \begin{cases} z^m + z^{-m} = e^{i\phi} + e^{-i\phi}, & m \text{ is even} \\ z^m - z^{-m} = e^{i\phi} - e^{-i\phi}, & m \text{ is odd} \end{cases}$$

$$\text{Thus, } z^m + (-1)^m z^{-m} = \begin{cases} 2\cos\phi, & m \text{ is even} \\ 2i\sin\phi, & m \text{ is odd.} \end{cases}$$

$$\therefore \exp\left(\frac{x}{2}\left(e^{i\phi} - e^{-i\phi}\right)\right) = J_0 + J_1\left(e^{i\phi} - e^{-i\phi}\right) + J_2\left(e^{2i\phi} + e^{-2i\phi}\right) + \dots$$

$$\exp(xi\sin\phi) = J_0 + J_1(2i\sin\phi) + J_2(2\cos 2\phi) + J_3(2i\sin 3\phi) + \dots$$

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now, LHS $= e^{i(x \sin\phi)} = \cos(x \sin\phi) + i \sin(x \sin\phi)$

let us separate the imaginary & real parts,
we get RHS =

$J_0 + 2J_2 \cos(2\phi) + 2J_4 \cos(4\phi) + \ldots + i(J_1 2\sin\phi + J_3 2\sin(3\phi) + \ldots)$

Equating the real parts of LHS & RHS

$\cos(x \sin\phi) = J_0 + 2J_2 \cos(2\phi) + 2J_4 \cos(4\phi) + \ldots\ldots$

hence proved