Assignment-1

SPRING 2022

MEASURE THEORY & INTEGRATION (MA51002)

1. Show that the set of real numbers in $[0,1]$ which possess decimal expansion not containing the digit 5 has measure zero.

2. (a) Find two subsets E_1, E_2 in $I\!R$ such that $E_1 \cap E_1 = \emptyset$ and $m^*(E_1 \cup E_1) \neq m^*(E_1) + m^*(E_2)$? On what conditions on E_1, E_2 will give that the equality holds above ?

   (b) Does the Lebesgue outer measure countably additive ?

3. Let $\mathcal{F} = \{A \subseteq I\!R \mid A \text{ or } A^c \text{ is a finite set}\}$. Check that whether $\mathcal{F}$ is a σ-algebra or not. If $\mathcal{F}$ is not a σ-algebra, then find a necessary and sufficient conditions for $\mathcal{F}$ to form a σ-algebra ?

4. Let $\{E_k\}$ be a sequence of measurable sets in $I\!R$.

   (a) Suppose $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$ and $m(E_1) < \infty$. Then show that
$$\lim_{k\to\infty} m(E_k) = m(\cap_{k=1}^{\infty} E_k).$$

   (b) Suppose $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$. Then show that
$$\lim_{k\to\infty} m(E_k) = m(\cup_{k=1}^{\infty} E_k).$$

5. Show that every monotonically increasing (and decreasing) function defined on a measurable set in $I\!R$ is Borel measurable.

6. Let $f : E \to I\!R$ be a function and $E \subseteq I\!R$ is a measurable set. Define a function $g : I\!R \to I\!R$ as $g(x) = f(x)$ if $x \in E$ and $g(x) = 0$ if $x \notin E$. Then show that f is measurable if and only if g is measurable.

7. Let $f : I\!R \to I\!R$ be a measurable function and $g : I\!R \to I\!R$ be a continuous function. Then show that $g \circ f$ is measurable. What can you say about the measurability of $f \circ g$?

8. Let $A \subseteq I\!R$. Show that there exists a Borel set U such that $U \supseteq A$ and $m^*(A) = m^*(U)$.

9. Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E in $I\!R$. Then show that $limsup(f_n), liminf(f_n)$ are measurable.

10. (a) What are Littlewood's 3 principles ?

    (b) Show that if f is a measurable function which is almost everywhere differentiable, then its derivative $\frac{df}{dx}$ is also measurable.