Inner product Spaces

Let X be a linear space over the field K. An inner product on X is a function $\langle \cdot , \cdot \rangle : X \times X \longrightarrow K$ such that for all $x, y, z \in X$ and $k \in K$, we have

(i) positive-definiteness

$$\langle x, x \rangle \geq 0, \ \forall x \in X$$

and $\langle x, x \rangle = 0 \iff x = 0$.

(ii) Linearity in the first variable.

$$\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

$$\langle kx, y \rangle = k \langle x, y \rangle$$

(iii) Conjugate Symmetry:

$$\langle y, x \rangle = \overline{\langle x, y \rangle}.$$

A linear space X with an inner product so it is called an inner product space (I.P.S) and it is denoted by

$$(X, \langle \cdot , \cdot \rangle).$$

Note :- An inner product is conjugate linear in the second variable

i.e.,

$$\langle x , y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

and

$$\langle x , ky \rangle = \bar{k} \langle x, y \rangle .$$

Ex : let $X = K^n$

For $\left.\begin{array}{l} x = (x(1), x(2), \ldots x(n)) \\ y = (y(1), y(2), \ldots y(n)) \end{array}\right\} \in X,$

define

$$\langle x , y \rangle = \sum_{j=1}^{n} x(j) \overline{y(j)} .$$

Then

$\langle \cdot , \cdot \rangle$ is an inner product on X.

(verify all the properties).

Lemma. Let $\langle \cdot , \cdot \rangle$ be an inner product on a linear space X.

(a) Polarization Identity:

For all $x, y \in X$,

$$4\langle x, y\rangle = \langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle .$$

(b) Let $x \in X$, then $\langle x, y\rangle = 0 \; \forall y \in X \iff x = 0$.

(c) Schwarz-Inequality:

For all $x, y \in X$,

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle,$$

where equality holds iff $\{x, y\}$ is L.D.

Proof: (a) For all $x, y \in X$,

$$\langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle$$
$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle.$$

$\therefore$

$$\langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \quad \text{---(1)}$$

III^{ly}

$$\langle x-y, x-y\rangle = \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \quad \text{---(2)}$$

(1) $-$ (2)

$$\langle x+y, x+y\rangle - \langle x-y, x-y\rangle$$
$$= 2\langle x,y\rangle + 2\langle y,x\rangle \quad \text{---(3)}$$

Replace y by iy and multiply with i to the Eq. (3), we get

$$i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle$$

$$= 2i\langle x, iy\rangle + 2i\langle iy, x\rangle$$

$$= 2i\bar{i}\langle x, y\rangle + 2ii\langle y, x\rangle$$

$$= 2\langle x, y\rangle - 2\langle y, x\rangle \quad \text{--- (4)}$$

Adding (3) & (4) we get

$$\langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle = 4\langle x, y\rangle.$$

b) If $x = 0$, then

$$\langle 0, y\rangle = \langle 0+0, y\rangle = \langle 0, y\rangle + \langle 0, y\rangle.$$

$$\Rightarrow \langle 0, y\rangle = 0, \quad \forall\, y \in X.$$

Conversely, let $\langle x, y\rangle = 0, \ \forall\, y \in X$.

Then in particular for $y = x$, we get

$\langle x, x\rangle = 0 \Rightarrow x = 0$, by positive-definiteness.

(c) Let $x, y \in X$ and consider

$$z = \langle y, y\rangle x - \langle x, y\rangle y.$$

Then
$$0 \le \langle z, z\rangle$$

$$= \big\langle \langle y,y\rangle x - \langle x,y\rangle y \, , \, \langle y,y\rangle x - \langle x,y\rangle y \big\rangle$$

$$= \langle y,y\rangle \big\langle x , \langle y,y\rangle x - \langle x,y\rangle y \big\rangle$$
$$- \langle x,y\rangle \big\langle y , \langle y,y\rangle x - \langle x,y\rangle y \big\rangle$$

$$= \langle y,y\rangle \left[\overline{\langle y,y\rangle} \langle x,x\rangle - \overline{\langle x,y\rangle} \langle x,y\rangle \right]$$
$$- \langle x,y\rangle \left[\overline{\langle y,y\rangle} \langle y,x\rangle - \overline{\langle x,y\rangle} \langle y,y\rangle \right]$$

$$= \langle y,y\rangle \overline{\langle y,y\rangle} \langle x,x\rangle - \langle y,y\rangle \overline{\langle x,y\rangle} \langle x,y\rangle$$
$$- \langle x,y\rangle \cancel{\overline{\langle y,y\rangle}} \langle y,x\rangle + \langle x,y\rangle \cancel{\overline{\langle x,y\rangle}} \langle y,y\rangle$$

$$= \langle y,y\rangle \left[\langle x,x\rangle \langle y,y\rangle - \overline{\langle x,y\rangle} \langle x,y\rangle \right]$$

$$= \langle y,y\rangle \left[\langle x,x\rangle \langle y,y\rangle - |\langle x,y\rangle|^2 \right] \quad \text{--- (1)}$$

So if $\langle y,y\rangle > 0$, then

$$\langle x,x\rangle \langle y,y\rangle - |\langle x,y\rangle|^2 \geq 0$$

$$\Rightarrow |\langle x,y\rangle|^2 \leq \langle x,x\rangle \langle y,y\rangle.$$

If $\langle y,y\rangle = 0 \Rightarrow y = 0$

$$\Rightarrow \langle x,y\rangle = 0$$

$$\therefore |\langle x,y\rangle|^2 = 0 = \langle x,x\rangle \langle y,y\rangle.$$

Now, let $|\langle x,y\rangle|^2 = \langle x,x\rangle \langle y,y\rangle$,

then from (1), $\langle z,z\rangle = 0$

$$\Rightarrow z = 0$$

$$\Rightarrow \langle y,y\rangle x - \langle x,y\rangle y = 0$$

$$\Rightarrow x = \frac{\langle x,y\rangle}{\langle y,y\rangle} y$$

$\therefore \{x,y\}$ is L.D.

Conversely, let $\{x,y\}$ is L.D.

then $y = kx$, $k \in \mathbb{K}$.

Then

$$|\langle x, y\rangle|^2 = \langle x, y\rangle \overline{\langle x, y\rangle}$$

$$= \langle x, kx\rangle \overline{\langle x, kx\rangle}$$

$$= \bar{k} \langle x, x\rangle \cdot \overline{\bar{k} \langle x, x\rangle}$$

$$= |k|^2 |\langle x, x\rangle|^2$$

and

$$\langle x, x\rangle \langle y, y\rangle = \langle x, x\rangle \langle kx, kx\rangle$$

$$= |k|^2 |\langle x, x\rangle|^2.$$

$$\therefore \quad |\langle x, y\rangle|^2 = \langle x, x\rangle \langle y, y\rangle.$$

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Theorem: let $\langle \cdot, \cdot \rangle$ be an inner product on a linear space X. For $x \in X$, define $\|x\| = \langle x, x\rangle^{1/2}$, the nonnegative square root of $\langle x, x\rangle$.

Then

$$|\langle x, y\rangle| \leq \|x\| \|y\|, \quad \forall x, y \in X.$$

The function $\|\cdot\| : X \longrightarrow \mathbb{K}$ is a norm on X, i.e., for all

$x, y \in X$ and $k \in K$, we have

$\|x\| \geq 0$ & $\|x\| = 0 \Leftrightarrow x = 0$

$\|x+y\| \leq \|x\| + \|y\|$

$\|kx\| = |k| \, \|x\|$.

Also the following holds.

(a) if $\|x_n - x\| \to 0$ & $\|y_n - y\| \to 0$

then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

(b) (Parallelogram law) For all $x, y \in X$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\left[\|x\|^2 + \|y\|^2\right]$$

Proof: Let $x, y \in X$.

Then by Schwarz-Inequality, we have

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

$$= \|x\|^2 \|y\|^2$$

$$\Rightarrow |\langle x, y\rangle| \leq \|x\| \|y\|.$$

Also

$$\|x\| = \sqrt{\langle x, x\rangle} \geq 0.$$

$$\|x\| = 0 \iff \sqrt{\langle x, x\rangle} = 0$$

$$\iff \langle x, x\rangle = 0$$

$$\iff x = 0 \text{ (by definition of } \langle \cdot, \cdot \rangle \text{)}.$$

Also

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$

$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$

$$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2$$

$$= \|x\|^2 + 2 \operatorname{Re} \langle x, y\rangle + \|y\|^2$$

$$\leq \|x\|^2 + 2 |\langle x, y\rangle| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

$\Rightarrow$ $\|x+y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$.

Finally for $k \in K$,

$$\|kx\|^2 = \langle kx, kx\rangle = k\bar{k}\langle x, x\rangle$$

$$= |k|^2 \langle x, x\rangle$$

$$= |k|^2 \|x\|^2$$

$\Rightarrow$ $\|kx\| = |k|\,\|x\|$

$\therefore$ $\|\cdot\|$ is a norm on an inner product space X.

(9) let $\|x_n - x\| \to 0$ & $\|y_n - y\| \to 0$

Then

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x_n, y_n\rangle - \langle x_n, y\rangle$$

$$+ \langle x_n, y\rangle - \langle x, y\rangle|$$

$$= |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle|$$

$$\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|$$

$$\leq \underbrace{\|x_n\|}_{<\infty} \underbrace{\|y_n - y\|}_{\to 0} + \underbrace{\|x_n - x\|}_{\to 0} \underbrace{\|y\|}_{<\infty}$$

(by Schwarz inequality).

$\longrightarrow 0$

$[\because \|x_n\| = \|x_n - x + x\|$

$\leq \underbrace{\|x_n - x\|}_{\to 0} + \underbrace{\|x\|}_{<\infty}$

$< \infty$

$\|y_n\| < \infty$

$\therefore \langle x_n, y_n\rangle \to \langle x, y\rangle$.

(b) Let $x, y \in X$, then

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$$

$$= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2$$

$$+ \|x\|^2 - \langle x, y \rangle - \langle y, x \rangle + \|y\|^2$$

$$= 2 \left[\|x\|^2 + \|y\|^2 \right].$$

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Hilbert Space :— An inner product space, which is complete in the norm induced by the inner product is called Hilbert space. We use the letter H to denote the Hilbert space.

Ex: $H = K^n$,

$x = (x(1), x(2), \ldots x(n)) \in H$

$y = (y(1), y(2), \ldots y(n)) \in H$,

Let

$$\langle x, y \rangle = \sum_{i=1}^{n} x(i)\overline{y(i)}.$$

Then $\langle \cdot, \cdot \rangle$ is an inner product on H.

and

$$\|x\|_2 = \sqrt{\langle x, x \rangle}$$

$$= \sqrt{\sum_{i=1}^{n} |x(i)|^2}.$$

Then H is also n.l.s.

Let $\{x_n\}$ be a Cauchy Sequence in H.

Then

$$\|x_n - x_m\|_2 \to 0 \quad \text{as } m, n \to \infty. \quad \text{---(1)}$$

$$\Rightarrow \sqrt{\sum_{i=1}^{n} |x_n(i) - x_m(i)|^2} \to 0$$

$\Rightarrow \{x_n(i)\}_{i=1}^{n}$ is a Cauchy Sequence in $\mathbb{R}$. But $\mathbb{R}$ is Complete.

Let $x_n(i) \to x_0(i)$, $i = 1, 2 \ldots n$

$$\Rightarrow [x_n(i) - x(i)]^2 \to 0 \quad \text{as } n \to \infty$$
$$i = 1, 2 \ldots n.$$

Fixing n and letting $m \to \infty$ in (1) we get

$$\|x_n - x_0\|_2 = \lim_{m \to \infty} \|x_n - x_m\|_2 \to 0.$$

$$= \lim_{m \to \infty} \sqrt{\sum_{i=1}^{n} |x_n(i) - x_m(i)|^2} \to 0$$

$\therefore$ $H = \mathbb{K}^n$ is Complete w.r.t norm $\|\cdot\|_2$. $\therefore$ H is a Hilbert space.

Note:- Among all the norms $\|\cdot\|_p$, $1 \leq p \leq \infty$ on $\mathbb{K}^n$ $(n \geq 2)$, only the norm $\|\cdot\|_2$ is induced by the inner product, because,

if $p \neq 2$, and $x = (1, 0, 0 \cdots 0)$
$y = (0, 1, 0 \cdots 0)$

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 2^{2/p} + 2^{2/p}$$
$$= 2 \cdot 2^{2/p}$$
$$= 2^{1+2/p}.$$

$$\|x\|_p^2 = 1$$

$$\|y\|_p^2 = 1.$$

Now $4 = 2\left(\|x\|_p^2 + \|y\|_p^2\right) \neq 2^{1+\frac{1}{p}}$

$= \|x+y\|_p^2 + \|x-y\|_p^2$

Hence parallelogram does not hold.

Ex: Consider the space ℓ^p, $1 \le p < \infty$ with the norm given by

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{1/p},$$

$x = (x(1), x(2) \ldots) \in \ell^p$.

For $p = 2$, ℓ^2.

$x, y \in \ell^2$, define

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x(i)\overline{y(i)},$$

where $x = (x(1), x(2) \cdots) \in \ell^2$

$y = (y(1), y(2), \cdots) \in \ell^2$.

Then $\langle \cdot, \cdot \rangle$ is an inner product on ℓ^2.

Define $\|x\|_2 = \sqrt{\langle x, x\rangle}$

$$= \sqrt{\sum_{i=1}^{\infty} |x(i)|^2} .$$

Then ℓ^2 is also n.l.s.

Let $\{x_n\}$ be a Cauchy Sequence in ℓ^2

where $x_n = (x_n(1), x_n(2) \ldots \ldots)$.

then given $\epsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ $\ni$

$\|x_n - x_m\|_2 < \epsilon$, $\forall$ $n, m \geq n_0$.

$$\Rightarrow \|x_n - x_m\|_2 = \left(\sum_{i=1}^{\infty} |x_n(i) - x_m(i)|^2\right)^{1/2} < \epsilon \quad \text{(2)}$$

$\Rightarrow \{x_n(i)\}$ is Cauchy Sequen

in $\mathbb{R}$, $i = 1, 2, 3 \ldots \ldots$

$\therefore$ $x_n(i) \longrightarrow x(i)$, as $n \to \infty$.

Fixing n & letting $m \to \infty$, in ⊛, we get

$$\|x_n - x\|_2 = \lim_{m\to\infty} \|x_n - x_m\|_2$$

$$= \lim_{m\to\infty} \sqrt{\sum_{i=1}^{\infty} |x_n(i) - x_m(i)|^2}$$

$$\longrightarrow 0$$

$\therefore$ ℓ^2 is a Hilbert space w.r.t norm $\|\cdot\|_2$ induced by the inner product $\|x\|_2 = \sqrt{\langle x, x\rangle}$.

Note:— For $p \neq 2$, $1 \leq p \leq \infty$, ℓ^p is not a inner product space

$x = (-1, -1, 0, 0, \cdots) \in \ell^p$

$y = (-1, 1, 0, \cdots) \in \ell^p$

$\|x\|_p = 2^{\frac{1}{p}}$, $\|y\|_p = 2^{1/p}$

$\|x+y\|_p = 2 \qquad \|x-y\|_p = 2$

$$\therefore \|x+y\|_p^2 + \|x-y\|_p^2 = 2^2+2^2 = 8$$

$$\& \quad 2\left(\|x\|_p^2 + \|y\|_p^2\right) = 2\left(2^{2/p}+2^{2/p}\right)$$
$$= 4 \cdot 4^{1/p}.$$

Thus in ℓ^p, $p \neq 2$, $\|\cdot\|_p$ does not satisfy parallelogram law.

$\therefore$ ℓ^p, $p \neq 2$ is not an I.P.S and hence it is not an Hilbert space.

Ex: $X = C_{00}$

for $x, y \in X$, define

$$\langle x, y \rangle = \sum_{j=1}^{\infty} x(j)\overline{y(j)}$$

Then $\langle \cdot, \cdot \rangle$ is an I.P on C_{00}.

and the induced norm

$$\|x\|_2 = \langle x, x\rangle^{1/2} = \left(\sum_{j=1}^{\infty} |x(j)|^2\right)^{1/2}$$

Let $x \in l^2$, $x = (x(1), x(2), x(3) \cdots)$

and $x_n = (x(1), x(2), \cdots x(n), 0, 0 \cdots)$
$\in C_{00}$

and

$$\|x - x_n\|_2^2 = \sum_{j=n+1}^{\infty} |x(j)|^2 \longrightarrow 0$$

as $n \to \infty$

$\Rightarrow$ C_{00} is dense in l^2.
i.e., $\overline{C_{00}} = l^2$

However $C_{00} \neq l^2$

$\because (1, \frac{1}{2}, \frac{1}{3} \cdots\cdots) \in l^2$

$\because \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$.

But $(1, \frac{1}{2}, \frac{1}{3}, \cdots) \notin C_{00}$

$\therefore$ $X = c_{00}$ cannot be closed in $H = \ell^2$

Hence c_{00} is incomplete I.P.S.

Ex: Let $X = C[a,b]$.

for $x, y \in X$, define

$$\langle x, y \rangle = \int_a^b x(t)\overline{y(t)}\,dt.$$

Then $\langle \cdot, \cdot \rangle$ is an inner product on $C[a,b]$ and the induced norm is

$$\|x\|_2 = \langle x, x \rangle^{1/2}$$

$$= \left(\int_a^b |x(t)|^2 dt \right)^{1/2}, \quad x \in C[a,b]$$

Orthonormal set :—

Let X be an I.P.S over the field K. For any $x, y \in X$, we say x and y are orthogonal if $\langle x, y \rangle = 0$, $x \neq y$ and we write $x \perp y$.

Let E and F be any two subsets of an I.P.S X. We say E and F are orthogonal if $\langle x, y \rangle = 0$, $\forall\, x \in E$ and $y \in F$.

In this case we write $E \perp F$.

We say a subset E of X is an orthonormal set if $\langle x, y \rangle = 0, x \neq y$

$\forall\ x, y \in E$ and $\|x\| = 1\ \forall x \in E$.

Theorem: Let X be an I.P.S.

(a) (Pythagoras) Let $\{x_1, x_2, \ldots x_n\}$ be an orthogonal set in X.

Then

$$\|x_1 + x_2 + \cdots + x_n\|^2 = \sum_{i=1}^{n} \|x_i\|^2 .$$

(b) Let E be an orthogonal subset of X and $0 \notin E$. Then E is L.I. If, in fact, E is orthonormal, then $\|x-y\| = \sqrt{2}$, $\forall\ x, y \in E,\ x \neq y$.

Proof.

(a) Given that $\langle x_i, x_j \rangle = 0\ \forall\ i \neq j$
$i, j = 1, 2, \ldots n.$

Consider

$$\|x_1+x_2+\cdots+x_n\|^2 = \left\langle \sum_{i=1}^{n} x_i , \sum_{j=1}^{n} x_j \right\rangle$$

$$= \sum_{i=1}^{n} \left\langle x_i , \sum_{j=1}^{n} x_j \right\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x_i , x_j \rangle$$

$$= \sum_{i=1}^{n} \langle x_i , x_i \rangle$$

$$= \sum_{i=1}^{n} \|x_i\|^2 .$$

b) Let $x_1, x_2, \ldots x_n \in E$ and $k_1, k_2 \ldots k_n \in K$ such that

$$k_1x_1 + k_2x_2 + \cdots + k_nx_n = 0$$

Then taking inner product on both side with x_j, $j = 1, 2 \ldots k$, we get

$$\left\langle \sum_{i=1}^{n} k_i x_i , x_j \right\rangle = \langle 0, x_j \rangle$$

$$\Rightarrow \sum_{i=1}^{n} k_i \langle x_i, x_j \rangle = 0$$

$$\Rightarrow \quad k_j \langle x_j, x_j \rangle = 0$$

$$\Rightarrow \quad k_j = 0 \qquad [\because x_j \neq 0 \Rightarrow \langle x_j, x_j \rangle \neq 0]$$

$j = 1, 2 \ldots n.$

$\Rightarrow \{x_1, x_2 \ldots x_n\}$ is L.I

$\Rightarrow E$ is L.I.

If E is orthonormal set, then for any $x, y \in E$, $x \neq y$, we have $\langle x, y \rangle = 0$, and

$$\|x\| = \|y\| = 1.$$

$$\therefore \|x - y\|^2 = \langle x - y, x - y \rangle$$

$$= \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle$$

$$= 1 - 0 - 0 + 1$$

$$= 2.$$

$\therefore \ \|x - y\| = \sqrt{2}, \quad x \neq y.$

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Given a L.I set in an I.P.S X, we can construct an orthonormal set in X.

Theorem:

Gram-Schmidt orthogonalization :—

let $\{x_1, x_2, \ldots\}$ be L.I set in an I.P.S X. Define $y_1 = x_1$,

$u_1 = \frac{y_1}{\|y_1\|}$ and for $n = 2, 3, \ldots$

let, $y_n = x_n - \sum_{i=1}^{n-1} \langle x_n, u_i \rangle u_i$, $u_n = \frac{y_n}{\|y_n\|}$.

Then $\{u_1, u_2, \ldots\}$ is an orthonormal set in X and

$\mathrm{Span}\{x_1, x_2, x_3, \ldots\} = \mathrm{Span}\{u_1, u_2, u_3, \ldots\}$

Proof: Since $\{x_1\}$ is L.I, $y_1 = x_1 \neq 0$,

$\|u_1\| = \left\| \frac{y_1}{\|y_1\|} \right\| = 1$

and $\operatorname{Span}\{u_1\} = \operatorname{Span}\{x_1\}$

Now for $n \geq 1$, assume that we have defined y_n and u_n as stated above, and proved that

$$\operatorname{Span}\{u_1, u_2, \dots u_n\} = \operatorname{Span}\{x_1, x_2, \dots x_n\}.$$

Define $y_{n+1} = x_{n+1} - \sum_{i=1}^{n} \langle x_{n+1}, u_i \rangle u_i$

$\because \{x_1, x_2 \dots x_n, x_{n+1}\}$ is L.I set,

so $x_{n+1} \notin \operatorname{Span}\{x_1, x_2 \dots x_n\} = \operatorname{Span}\{u_1, u_2, \dots u_n\}$.

$\therefore y_{n+1} \neq 0$, Let $u_{n+1} = \dfrac{y_{n+1}}{\|y_{n+1}\|}$.

Then $\|u_{n+1}\| = 1$ and for all $j \leq n$, we have

$$\langle y_{n+1}, u_j \rangle = \left\langle x_{n+1} - \sum_{i=1}^{n} \langle x_{n+1}, u_i \rangle u_i , u_j \right\rangle$$

$$= \langle x_{n+1}, u_j \rangle - \sum_{i=1}^{n} \langle x_{n+1}, u_i \rangle \underbrace{\langle u_i, u_j \rangle}_{\delta_{ij}}$$

$$= \langle x_{n+1}, u_j \rangle - \langle x_{n+1}, u_j \rangle$$

$$= 0.$$

$$\therefore \langle u_{n+1}, u_j \rangle = \left\langle \frac{y_{n+1}}{\|y_{n+1}\|}, u_j \right\rangle$$

$$= \frac{1}{\|y_{n+1}\|} \langle y_{n+1}, u_j \rangle = 0$$

$\Rightarrow \{u_1, u_2, \ldots u_n, u_{n+1}\}$ is an orthonormal set. Also

$$\text{Span}\{u_1, u_2, \ldots u_n, u_{n+1}\} = \text{Span}\{x_1, x_2, \ldots x_n, u_{n+1}\}$$

$$= \text{Span}\{x_1, x_2, \ldots x_n, x_{n+1}\}.$$

Thus by mathematical induction, proof is complete.

Ex: $X = \ell^2$, for $n = 1, 2, 3, \dots$

let $x_n = (\underbrace{1, 1, 1, \dots, 1}_{n \text{ times}}, 0, 0, \dots)$.

Then $\{x_1, x_2, x_3, \dots\}$ is L.I set in ℓ^2.

$x_1 = (1, 0, 0, \dots)$, $\quad y_1 = x_1$, $\|y_1\| = 1$

$$u_1 = \frac{y_1}{\|y_1\|} = (1, 0, 0, \dots)$$

$x_2 = (1, 1, 0, 0, \dots)$

$y_2 = x_2 - \langle x_2, u_1 \rangle u_1$

clearly $\langle x_2, u_1 \rangle = 1$

$$\therefore y_2 = x_2 - \langle x_2, u_1 \rangle u_1 = (1, 1, 0, \dots) - 1 \cdot (1, 0, 0, \dots) = (0, 1, 0, \dots)$$

$\|y_2\| = 1$

$\therefore u_2 = \frac{y_2}{\|y_2\|} = (0,1,0,\ldots)$.

and so on

(Bessel's Inequality):

Theorem: Let $\{u_1, u_2, \ldots\}$ be a countable orthonormal set in an I.P.S X and $x \in X$. Then

$$\sum_n |\langle x, u_n \rangle|^2 \leq \|x\|^2,$$

where the equality holds iff $x = \sum_n \langle x, u_n \rangle u_n$.

Proof: For $m = 1, 2, 3, \ldots$, let

$$x_m = \sum_{n=1}^{m} \langle x, u_n \rangle u_n.$$

Since $\{u_1, u_2, \ldots u_m\}$ is an orthonormal set, we have

$$\langle x, x_m \rangle = \left\langle x, \sum_{n=1}^{m} \langle x, u_n \rangle u_n \right\rangle$$

$$= \sum_{n=1}^{m} |\langle x, u_n \rangle|^2$$

$$= \langle x_M, x \rangle$$
$$= \langle x_M, x_M \rangle$$

Hence

$$0 \leq \|x - x_M\|^2 = \langle x - x_M, x - x_M \rangle$$
$$= \langle x, x \rangle - \langle x, x_M \rangle - \langle x_M, x \rangle + \langle x_M, x_M \rangle$$
$$= \|x\|^2 - \sum_{n=1}^{M} |\langle x, u_n \rangle|^2 \quad \text{---} \ (*)$$

Letting $M \to \infty$, we get

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

This is called Bessel's inequality.

If the equality holds, i.e.,

if $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$, then

from $(*)$

$$0 \le \|x - x_M\|^2 = 0$$

$$\Rightarrow \|x - x_M\|^2 \to 0 \text{ as } M \to \infty.$$

i.e., $\|x - x_M\| \to 0$

$$\Rightarrow x = \lim_{M \to \infty} x_M = \lim_{M \to \infty} \sum_{n=1}^{M} \langle x, u_n \rangle u_n$$

$$= \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n.$$

Conversely, assume

$$x = \sum_n \langle x, u_n \rangle u_n. \text{ Then}$$

$$\|x\|^2 = \langle x, x \rangle$$

$$= \left\langle \sum_n \langle x, u_n \rangle u_n, \sum_p \langle x, u_p \rangle u_p \right\rangle$$

$$= \sum_n |\langle x, u_n \rangle|^2.$$

— // —

Now we discuss the convergence of

a series $\sum_n k_n u_n$ in an I.P.S X where $k_n \in K$ and $\{u_1, u_2, \dots\}$ is an orthonormal set in X.

Theorem: Let X be an IPS, $\{u_1, u_2, u_3, \dots\}$ be a countable orthonormal set in X. Then

(a) If $\sum_n k_n u_n$ converges to some $x \in X$, then $k_n = \langle x, u_n \rangle, \forall n$ and $\sum_n |k_n|^2 < \infty$.

(b) (Riesz – Fischer Theorem):- If X is a Hilbert space and $\sum_n |k_n|^2 < \infty$, then $\sum_n k_n u_n$ converges in X.

Proof: Let $x = \sum_n k_n u_n$ $\left(\because \sum_n k_n u_n \to x\right)$.

Then

$$\langle x, u_n \rangle = \left\langle \sum_j k_j u_j, u_n \right\rangle$$

$$= \sum_j k_j \langle u_j, u_n \rangle$$

$$= k_n .$$

And by Bessel's Inequality we have

$$\sum_n |k_n|^2 = \sum_n |\langle x, u_n \rangle|^2 \leq \|x\|^2 < \infty$$

$$\Rightarrow \sum_n |k_n|^2 < \infty .$$

(b) Now assume that X is a Hilbert space and $\sum_n |k_n|^2 < \infty$.

claim: $\sum_n k_n u_n$ converges in X.

For $m = 1, 2, 3, \ldots$, let $x_m = \sum_{n=1}^{m} k_n u_n$.

Then for $j = 1, 2, 3 \ldots$ and $m > j$, we have

$$x_m - x_j = \sum_{n=j+1}^{m} k_n u_n$$

$\therefore \|x_M - x_j\|^2 = \langle x_M - x_j , x_M - x_j \rangle$

$$= \left\langle \sum_{n=j+1}^{M} k_n e_n , \sum_{p=j+1}^{M} k_p e_p \right\rangle$$

$$= \sum_{n=j+1}^{M} |k_n|^2$$

Hence if $\sum_n |k_n|^2 < \infty$, then it follows that $\sum_{n=j+1}^{M} |k_n|^2 \longrightarrow 0$ as $n, M \longrightarrow \infty$.

$\therefore \|x_M - x_j\|^2 \longrightarrow 0$ as $M, j \longrightarrow \infty$.

$\Rightarrow \{x_m\}$ is a Cauchy sequence in X. But X is a Hilbert space.

$\therefore x_m \longrightarrow x \in X$.

$\therefore \sum_n k_n e_n$ conv in X.

—— // ——

Def (orthonormal basis): —

An orthonormal set $\{u_\alpha\}$ in a Hilbert space H is said to be an orthonormal basis for H if it is maximal in the sense that if $\{u_\alpha\}$ is contained in some orthonormal subset E of H, then

$E = \{u_\alpha\}$.

Let H be a Hilbert space and $H \neq \{0\}$. Let $\mathcal{E}$ be a family of orthonormal sets in H.

Then $\mathcal{E} \neq \phi$

$\because H \neq \{0\}$, $\exists\ 0 \neq x \in H$.

Then $\left\{\frac{x}{\|x\|}\right\}$ is an orthonormal set in H.

Then $\mathcal{C}$ is a POSET with set inclusion.

Let T be any totally ordered subfamily of $\mathcal{C}$.

Then $\bigcup_{A \in T} A$ is an upper bound for T.

$\therefore$ By Zorn's lemma, $\mathcal{C}$ has a maximal element, which is called orthonormal basis.

Theorem: Let $\{u_\alpha\}$ be an orthonormal set in an IPS X and $x \in X$.

Let $E_x = \{u_\alpha \,/\, \langle x, u_\alpha\rangle \neq 0\}$.

Then E_x is a Countable set,

say $E_x = \{u_1, u_2, \dots\}$.

If E_x is denumerable, then

$\langle x, u_n \rangle \longrightarrow 0$ as $n \longrightarrow \infty$.

Further, if X is a Hilbert space, then

$\sum_n \langle x, u_n \rangle u_n$ Converges to some $y \in H$

such that $x - y \perp u_\alpha$, $\forall \alpha$.

Proof: If $x = 0$, there is nothing to prove.

So let $x \neq 0$. For $j = 1, 2, 3, \dots$, let

$$E_j = \{ u_\alpha \mid \|x\| \leq j \, |\langle x, u_\alpha \rangle| \}$$

Fix j, Suppose that E_j Contains

distinct elements $u_{\alpha_1}, u_{\alpha_2}, \ldots u_{\alpha_m}$

Then

$$\|x\| \leq j \, |\langle x, u_{\alpha_1}\rangle|$$

$$\|x\| \leq j \, |\langle x, u_{\alpha_2}\rangle|$$

$$\vdots$$

$$\|x\| \leq j \, |\langle x, u_{\alpha_m}\rangle|$$

$$\Rightarrow 0 \leq m\|x\|^2 \leq j^2 \sum_{n=1}^{m} |\langle x, u_{\alpha_n}\rangle|^2$$

$$\leq j^2 \sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2$$

$$\leq j^2 \|x\|^2 \quad \text{(by Bessel's inequality)}$$

$$\Rightarrow \quad m \leq j^2$$

This shows that each E_j contains atmost j^2 elements.

Also since $E_x = \bigcup_j E_j$, we see that E_x is countable.

Also, if $E_x = \{u_1, u_2, u_3, \dots\}$ is denumerable, then

$$\sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 \leq \|x\|^2 < \infty.$$

Hence the nth term of this convergence series converges to zero

i.e., $\langle x, u_n\rangle \to 0$ as $n \to \infty$.

Further if X is a Hilbert space, then by Riesz - Fisher theorem

$\sum_n \langle x, u_n\rangle u_n$ converges to some $y \in X$.

Then for any α,

$$\langle y, u_\alpha\rangle = \left\langle \sum_n \langle x, u_n\rangle u_n, u_\alpha \right\rangle$$

$$= \sum_n \langle x, u_n \rangle \underbrace{\langle u_n, u_\alpha \rangle}_{\delta_{n\alpha}}$$

$$= \langle x, u_\alpha \rangle$$

$$\Rightarrow \langle x-y, u_\alpha \rangle = 0, \quad \forall \alpha$$

$$\Rightarrow x-y \perp u_\alpha \quad \forall \alpha.$$

Theorem: Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Then the following are equivalent.

(i) $\{u_\alpha\}$ is an orthonormal basis for H.

(ii) (Fourier expansion): For every $x \in H$, we have $x = \sum_n \langle x, u_n \rangle u_n$, where

$$\{u_1, u_2, \dots\} = \{u_\alpha \mid \langle x, u_\alpha \rangle \neq 0\}$$

(iii) (Parseval's formula): for every $x \in H$, we have

$$\|x\|^2 = \sum_n |\langle x, u_n \rangle|^2.$$

when $\{u_1, u_2, \dots\} = \{u_\alpha \mid \langle x, u_\alpha \rangle \neq 0\}$

(iv) $\mathrm{Span}\{u_\alpha\}$ is dense in H

(v) If $x \in H$ and $\langle x, u_\alpha \rangle = 0, \forall \alpha$

..

Then $x = 0$.

Proof: (i) $\Rightarrow$ (ii)

Let $\{u_\alpha\}$ be an orthonormal basis for H. Let $x \in H$.

Then by previous lemma (last class)

$\sum_n \langle x, u_n \rangle u_n$ converges to some

$y \in H$ and $x - y \perp u_\alpha$, $\forall \alpha$.

If $y \neq x$, then $u = \dfrac{y-x}{\|y-x\|}$.

Then $\|u\| = 1$ and $u \perp \{u_\alpha\}$

$\Rightarrow \{u\} \cup \{u_\alpha\}$ is an orthonormal set containing $\{u_\alpha\}$ in H,

Contradicting the maximality of $\{u_\alpha\}$.

$\therefore \; x = y = \sum_n \langle x, u_n \rangle u_n .$

(ii) $\Rightarrow$ (iii)

for any $x \in H$, we have

by (ii) $\quad x = \sum_n \langle x, u_n \rangle u_n$

Then

$$\|x\|^2 = \langle x, x \rangle$$

$$= \left\langle \sum_n \langle x, u_n \rangle u_n , \sum_m \langle x, u_m \rangle u_m \right\rangle$$

$$= \sum_n \langle x, u_n \rangle \sum_m \overline{\langle x, u_m \rangle} \underbrace{\langle u_n , u_m \rangle}_{\delta_{nm}}$$

$$= \sum_n |\langle x, u_n \rangle|^2 .$$

(ii) $\Rightarrow$ (iv)

For any $x \in H$, we have

$x = \sum_n \langle x, u_n \rangle u_n .$

$\because$ For each $m = 1, 2, 3, \cdots$, let

$$x_m = \sum_{n=1}^{m} \langle x, u_n \rangle u_n \in \text{Span}\{u_\alpha\}$$

$$\longrightarrow \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n = x$$

$x_m \in \text{Span}\{u_\alpha\}$ and $x_m \to x$.

$$\therefore \overline{\text{Span}\{u_\alpha\}} = H.$$

(iv) $\Rightarrow$ (v)

given that $\overline{\text{Span}\{u_\alpha\}} = H$.

Let $x \in H$ be such that

$$\langle x, u_\alpha \rangle = 0 \quad \forall \alpha.$$

and let

$$x_m \to x, \quad x_m \in \text{Span}\{u_\alpha\}.$$

$\because \langle x, u_\alpha \rangle = 0 \ \forall \alpha$

$$\Rightarrow \quad \langle x, x_m \rangle = 0, \ x_m \in \text{Span}\{u_\alpha\}$$

$\therefore \quad 0 = \langle x, x_n \rangle \longrightarrow \langle x, x \rangle$

$\Rightarrow \quad \|x\|^2 = 0$

$\Rightarrow \quad x = 0.$

(v) $\Rightarrow$ (i)

Given that $\langle x, u_\alpha \rangle = 0 \ \forall \alpha$

$\Rightarrow x = 0.$

Claim: $\{u_\alpha\}$ is an orthonormal basis for H.

Let E be an orthonormal set in H containing $\{u_\alpha\}$.

If $u \in E$ and $u \neq u_\alpha \ \forall \alpha$,

then $\langle u, u_\alpha \rangle = 0 \ \forall \alpha$

$\Rightarrow \quad u = 0$ (by (v))

But $u \in E \Rightarrow \|u\| = 1$, this

Contradiction shows that

$E = \{u_\alpha\}$.

$\therefore$ $\{u_\alpha\}$ is a maximal orthonormal set in H. That is $\{u_\alpha\}$ is an orthonormal basis for H.

Clearly (iii) $\Rightarrow$ (ii)

See the proof of Bessel's inequality.

$$\therefore\quad \begin{array}{ccccc} (i) & \Rightarrow & (ii) & \Leftarrow & (iii) \\ \Uparrow & & & & \Downarrow \\ (v) & & \Leftarrow & & (iv) \end{array}$$

Projections: —

Let X be a linear space and

X_1 and X_2 be subspaces of X

such that

$$X = X_1 + X_2 \quad , \; X_1 \cap X_2 = \{0\}$$

i.e., $X = X_1 \oplus X_2$;

Then every $x \in X$ can be written uniquely as

$$x = x_1 + x_2 , \quad x_1 \in X_1$$
$$x_2 \in X_2 .$$

Then define $P : X \longrightarrow X_1$ by

$$Px = P(x_1 + x_2) = x_1 .$$

Then P is a linear map.

and for any $u \in X_1$,

we have

$$Pu = P(u+0) = u, \quad \forall\, u \in X_1 = R(P).$$

and for any $v \in X_2$, we have

$$Pv = P(0+v) = 0.$$

$\therefore X_1 = R(P), \quad X_2 = N(P).$

and $P^2 = P$

A linear operator $P : X \to X$ is called projection operator or simply a projection if

$$Pu = u, \quad \forall\, u \in R(P)$$

If $P : X \to X$ is a projection with $R(P) = X_1$ and $N(P) = X_2$, We say P is projection onto X_1 along X_2.

and

$I-P$ is a projection onto X_2 along X_1 with

$$R(I-P) = X_2$$

$$N(I-P) = X_1.$$

Note:

(1) Let X be an I.P.S and $P: X \to X$ be a projection.

We say P is an orthogonal projection if $R(P) \perp N(P)$

[i.e. $X = X_1 \oplus X_2$.

$P: X \to X_1$ is an orthogonal projection and

$$R(P) = X_1 \quad N(P) = X_2$$

$$X_1 \perp X_2]$$

(3) If P is an orthogonal projection, then for any $x \in X$, we have

$$x = \underbrace{Px}_{\in R(P)} + \underbrace{(I-P)x}_{\in N(P)}$$

$$\therefore \|x\|^2 = \|Px\|^2 + \|(I-P)x\|^2$$

(by Pythagoras theory)

$$\geq \|Px\|^2$$

$\therefore \quad \|Px\|^2 \le \|x\|^2$

$\Rightarrow \quad \|Px\| \le \|x\|$

$\Rightarrow \quad \|P\| \le 1 \quad \text{——} (1)$

Also, since

$$P = P^2$$

$\Rightarrow \quad \|P\| = \|P^2\|$

$$= \|P \cdot P\|$$

$$\le \|P\| \, \|P\|$$

$\Rightarrow \quad \|P\| \ge 1 \quad \text{——} (2)$

$\therefore$ from (1) & (2), we get

$$\|P\| = 1.$$

[For any $x \in H$

$$\|ABx\| = \|A(Bx)\|$$

$$\leq \|A\| \, \|Bx\|$$

$$\leq \|A\| \, \|B\| \, \|x\|$$

$$\Rightarrow \|AB\| \leq \|A\| \, \|B\| \,]$$

(Projection theorem) :—

Let H be a Hilbert space and F be a non-empty closed subspace of H. Then $H = F + F^{\perp}$.

Equivalently, there is an orthogonal projection onto F.

Moreover $F^{\perp\perp} = F$

(Projection Theorem) :—

Let H be a Hilbert space and F be a non empty closed subspace of H. Then $H = F + F^{\perp}$.

Equivalently, there is an orthogonal projection onto F. moreover

$F^{\perp\perp} = F$.

Proof: If $F = \{0\}$, then $F^{\perp} = H$

then clearly $H = F + F^{\perp}$.

So let $F \neq \{0\}$. Since F is a closed subspace of H, F itself is a Hilbert space.

Let $\{u_\alpha\}$ be an orthonormal basis for F.

Let $x \in H$. Then $\{u_\alpha \,/\, \langle x, u_\alpha\rangle \neq 0\}$

is a Countable set say $\{u_1, u_2, u_3 \dots\}$ and the series $\sum_n \langle x, u_n \rangle u_n$ Converges to some $y \in H$ such that

$x - y \perp u_\alpha, \ \forall \ \alpha.$

$\because$ each $u_n \in F$ and F is closed,

$$\sum_{n=1}^{m} \langle x, u_n \rangle u_n \longrightarrow y \Rightarrow y \in F$$

Also since $\{u_\alpha\}$ is an orthonormal basis for F, implies $F = \overline{\text{span}\{u_\alpha\}}$.

$\because x - y \perp \{u_\alpha\} \Rightarrow x - y \perp \overline{\text{span}\{u_\alpha\}}$

$$\Rightarrow x - y \perp F$$

$$\Rightarrow x - y \in F^{\perp}$$

Thus every $x \in H$ Can be written

a) $x = y + x-y$, with $y \in F$
$x-y \in F^{\perp}$

$= y + z$

Hence $H = F + F^{\perp}$, $F \cap F^{\perp} = \{0\}$.

Hence there exists an orthogonal projection $P : H \to H$ such that $R(P) = F$ and $N(P) = F^{\perp}$

Claim: $F^{\perp\perp} = F$.

Let $x \in F$. Now for any $z \in F^{\perp}$, we have $\langle x, z \rangle = 0$

$\Rightarrow x \in (F^{\perp})^{\perp} = F^{\perp\perp}$

$\Rightarrow F \subseteq F^{\perp\perp}$ — (1).

Now let $x \in F^{\perp\perp} \Rightarrow x \in H$

Then $x = y + z$, $y \in F$
$z \in F^{\perp}$

$\because y \in F \Rightarrow y \in F^{\perp\perp}$ (by (1))

Thus $z = x - y \in F^{\perp\perp}$ $\left[\begin{array}{l} \because x \in F^{\perp\perp} \\ \quad y \in F^{\perp\perp} \\ \Rightarrow x - y \in F^{\perp\perp} \end{array}\right]$

$\Rightarrow z \in F^{\perp} \cap (F^{\perp})^{\perp} = \{0\}$

$\Rightarrow z = 0$

$\therefore x - y = 0 \Rightarrow x = y \in F$

$\Rightarrow F^{\perp\perp} \subseteq F$ —(2)

$\therefore$ from (1) & (2) we have

$$F^{\perp\perp} = F.$$

—— // ——

The projection theorem show that

every Hilbert space H has complementary subspace property.

That is for every non empty closed subspace F of H, there is a closed subspace G of H such that

$$H = F + G, \quad F \cap G = \{0\}.$$

~~Here~~ $G = F^{\perp}$, it is a closed subspace of H.

* If H is a Hilbert space, every $x \in H$ can be written as

$$x = y + z, \quad y \in F, \quad z \in F^{\perp}$$

Define $P : H \to F$ by

$$P(x) = P(y+z) = y$$

Then P is linear map and

$$P^2 = P.$$

Also $R(P) = F$, $N(P) = F^{\perp}$

$$\therefore \quad R(P) \perp N(P).$$

Thus P is a orthogonal Projection onto $R(P)$ along $N(P)$.

—— // ——

Continuous linear functionals :—

Let X be an I.P.S over K.

Let $f : X \to K$ be a linear

functional on X. let f be continuous on X.

Then f is continuous at zero and $f(0)=0$.

Hence $\exists\ \delta>0 \ni$

$$|f(x)| \leq 1 \quad \forall\ x \in X, \ \|x\| \leq \delta$$

Now for any $x \neq 0$, $y = \frac{\delta x}{\|x\|}$, we see that

$$|f(y)| \leq 1, \quad \|y\| = \delta$$

$$\Rightarrow |f(x)| \leq \alpha \|x\|, \ \alpha = \frac{1}{\delta}.$$

Let $X' = B(X, K)$ be the set of all continuous linear functionals on X.

X' is a linear space. And for $f \in X'$, we let

$$\|f\| = \sup\left\{ |f(x)| \;/\; \begin{array}{c} x \in X \\ \|x\| \le 1 \end{array} \right\}.$$

$$\Rightarrow \quad |f(x)| \le \|f\| \, \|x\|$$

Lemma:- Let X be an I.P.S and $f \in X'$.

(a) Let $\{u_1, u_2, \ldots \ldots\}$ be an orthonormal set in X. Then $\sum_n |f(u_n)|^2 \le \|f\|^2$.

(b) Let $\{u_\alpha\}$ be an orthonormal set in X and

$$E_f = \{u_\alpha \mid f(u_\alpha) \neq 0\}.$$

Then E_f is countable set say $\{u_1, u_2, \dots\}$.

If E_f denumerable, then $f(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof:

(a) For $m = 1, 2, \dots$, let

$$y_m = \sum_{n=1}^{m} \overline{f(u_n)}\, u_n. \text{ Then}$$

$$\|y_m\|^2 = \langle y_m, y_m \rangle$$

$$= \left\langle \sum_{n=1}^{m} \overline{f(u_n)}\, u_n, \sum_{p=1}^{m} \overline{f(u_p)}\, u_p \right\rangle$$

$$= \sum_{n=1}^{M} |f(u_n)|^2 = \beta_M \text{ (say)}$$

— (1)

Also

$$f(y_m) = f\left(\sum_{n=1}^{M} \overline{f(u_n)}\, u_n\right)$$

$$= \sum_{n=1}^{M} \overline{f(u_n)}\, f(u_n)$$

$$= \sum_{n=1}^{M} |f(u_n)|^2 = \beta_M$$

— (2)

Now since

$$|f(y_m)| \leq \|f\|\, \|y_m\|$$

$$= \|f\| \sqrt{\beta_M} \text{ (by (1))}$$

— (*)

Now from (*), (2), we have

$$\beta_M \leq |f(y_m)| \leq \|f\| \sqrt{\beta_M}$$

$$\Rightarrow \beta_M^2 \leq \|f\|^2 \beta_M$$

$\Rightarrow \beta_M \leq \|f\|^2$

$\Rightarrow \sum_{n=1}^{M} |f(u_n)|^2 \leq \|f\|^2$

Letting $M \to \infty$, we get

$$\sum_{n=1}^{\infty} |f(u_n)|^2 \leq \|f\|^2.$$

(b) If $f = 0$,

$E_f = \{u_\alpha \mid f(u_\alpha) \neq 0\} = \varphi$.

So let $f \neq 0$.

For $j = 1, 2, \ldots$, let

$E_j = \{u_\alpha \mid \|f\| \leq j\,|f(u_\alpha)|\}$

Fix j, Suppose E_j contains

distinct elements say $u_{\alpha_1}, u_{\alpha_2}, \ldots, u_{\alpha_M}$

then

$$\|f\| \leq j|f(u_{\alpha_i})|, \quad i=1,2,\ldots,M$$

Squaring and adding

$$\Rightarrow M\|f\|^2 \leq j^2 \sum_{n=1}^{M} |f(u_{\alpha_i})|^2$$

$$\leq j^2 \|f\|^2$$

[by (a)]

$$\Rightarrow M\|f\|^2 \leq j^2 \|f\|^2$$

$$\Rightarrow M \leq j^2$$

Thus E_j contains atmost j^2 elements.

Since $E_f = \bigcup_j E_j$, we see that E_f is countable.

If E_f is denumerable, then

$$\sum_{n=1}^{\infty} |f(u_n)|^2 \leq \|f\|^2 \quad (\text{by } (9))$$

$\sum_{n=1}^{\infty} |f(u_n)|^2$ is cgt series

$\therefore$ nth term $|f(u_n)|^2 \longrightarrow 0$

i.e., $f(u_n) \longrightarrow 0$.

as $n \longrightarrow \infty$.

—— // ——

Let X be an I.P.S over K.

For a fixed $y \in X$, define

$f : X \longrightarrow K$ by

$f(x) = \langle x, y \rangle, \ \forall x \in X$

Then f is linear

$\because f(ax+bz) = \langle ax+bz, y \rangle$

$= a\langle x, y\rangle + b\langle z, y\rangle$

$$= a f(x) + b f(z)$$

Also

$$|f(x)| = |\langle x, y \rangle|$$

$$\leq \|x\| \, \|y\| \quad \text{(Schwarz Inequ)}$$

$$\Rightarrow \|f\| \leq \|y\| \quad \text{---(1)}$$

If $y = 0$, then $f = 0$.

So let $y \neq 0$, let $x = \dfrac{y}{\|y\|}$.

Then $\|x\| = 1$.

and

$$f(x) = \langle x, y \rangle$$

$$= \left\langle \frac{y}{\|y\|}, y \right\rangle$$

$$= \frac{1}{\|y\|} \langle y, y \rangle = \frac{\|y\|^2}{\|y\|} = \|y\|$$

$\therefore \quad \|f\| = \|y\|.$

Riesz Representation Theorem :—

Let H be a Hilbert space and $f \in H'$. Then there is a unique $y \in H$ such that

$$f(x) = \langle x, y \rangle, \ \forall x \in H.$$

In fact, if z is a nonzero element of H such that $z \perp Z(f)$, then $y = \frac{\overline{f(z)}}{\langle z, z \rangle} z$.

Also if $\{u_\alpha\}$ is a orthonormal basis for H and $\{u_\alpha \mid f(u_\alpha) \neq 0\}$

$= \{u_1, u_2, \ldots\}$, then $y = \sum_n \overline{f(u_n)}\, u_n$.

Proof :-

If $f = 0$, then we let $y = 0$

so that $f(x) = 0 = \langle x, y \rangle$

$\forall x \in H$.

So let $f \neq 0$.

Then $Z(f)$ null space of f

is a closed subspace of H.

$\therefore$ By projection theorem, we

have

$$H = Z(f) + Z(f)^{\perp}$$

As $Z(f) \neq H$, so let

$0 \neq z \in Z(f)^{\perp}$

Since $Z(f)$ is a Hyperspace in H, we have

$$H = Z(f) \cup \operatorname{Span}\{z\}.$$

Let $x \in H$. Then

$$x = \omega + kz, \quad \omega \in Z(f)$$
$$kz \in \operatorname{Span}\{z\}.$$

Then

$$\langle x, z\rangle = \langle \omega, z\rangle + \langle kz, z\rangle$$
$$\underset{0}{\parallel} \; + k\langle z, z\rangle \quad \begin{cases} \because \omega \in Z(f) \\ z \in Z(f)^{\perp} \end{cases}$$

$$\therefore k = \frac{\langle x, z\rangle}{\langle z, z\rangle}.$$

$$\therefore x = \omega + \frac{\langle x, z\rangle}{\langle z, z\rangle} \cdot z$$

Applying f on both sides we get

$$f(x) = f(\omega) + \frac{\langle x, z\rangle}{\langle z, z\rangle} \cdot f(z)$$

$$= 0 + \langle x, z\rangle \frac{f(z)}{\langle z, z\rangle}.$$

$$= \left\langle x, \overline{\frac{f(z)}{\langle z, z\rangle}}. z \right\rangle$$

$$= \langle x, y\rangle,$$

where $y = \overline{\frac{f(z)}{\langle z, z\rangle}}. z$

Also $\|f\| = \|y\|$.

Uniqueness of y :—

If $f(x) = \langle x, y_1\rangle, \forall x \in H$

and some $y_1 \in H$ as well.

Then

$f(x) = \langle x, y\rangle = \langle x, y_1\rangle, \forall x \in H$

$\Rightarrow \langle x, y - y_1\rangle = 0, \forall x \in H.$

In Particular letting $x = y - y_1$, we get

$$\langle y-y_1, y-y_1 \rangle = 0$$

$$\Rightarrow \|y-y_1\|^2 = 0$$

$$\Rightarrow y = y_1.$$

Thus for every $f \in H$, $\exists!\, y \in H$ such that $f(x) = \langle x, y \rangle$, $\forall x \in H$.

This y is representer of f and it satisfies $\|y\| = \|f\|$

Alternatively, we proceed as follows:-

Let $\{u_\alpha\}$ be an orthonormal basis for H and $\{u_\alpha \mid f(u_\alpha) \neq 0\}$

$= \{u_1, u_2, \ldots\}$ is

Countable. Then

$$\sum_n |f(u_n)|^2 \leq \|f\|^2 < \infty.$$

Since H is a Hilbert space, then by Riesz-Fisher theorem, we have $\sum_n \overline{f(u_n)}\, u_n$ converges in H.

Let $y = \sum_n \overline{f(u_n)}\, u_n$.

Claim: $f(x) = \langle x, y \rangle, \ \forall x \in H$.

Let $x \in H$ and $\{u_\alpha \mid \langle x, u_\alpha \rangle \neq 0\}$

$= \{v_1, v_2, \ldots \ldots\}$ is

orthonormal basis for H.

Then we have Fourier expansion

$$x = \sum_n \langle x, v_n\rangle v_n.$$

$$\Rightarrow f(x) = \sum_n \langle x, v_n\rangle f(v_n) \quad \text{--- (1)}$$

on the other hand

$$\langle x, y\rangle = \Big\langle \sum_n \langle x, v_n\rangle v_n \, , y\Big\rangle$$

$$= \sum_n \langle x, v_n\rangle \langle v_n, y\rangle \quad \text{--- (2)}$$

We show that from (1) & (2)

$$f(v_n) = \langle v_n, y\rangle$$

Fix n, Then

$$\langle v_m, y \rangle = \langle v_m, \sum_n \overline{f(u_n)}\, u_n \rangle$$

$$= \sum_n f(u_n) \langle v_m, u_n \rangle$$

If $v_m = u_{n_0}$ for some n_0, then

$$\langle v_m, y \rangle = \sum_n f(u_n) \langle u_{n_0}, u_n \rangle$$

$$= f(u_{n_0}).$$

$$= f(v_m)$$

Now let $v_m \neq u_n$ for any n,

Then $f(v_m) = 0$

$$\langle v_m, y \rangle = \langle v_m, \sum_n \overline{f(u_n)}\, u_n \rangle$$

$$= \sum_n f(u_n) \underbrace{\langle v_m, u_n \rangle}_{\substack{0 \\ m \neq n}}$$

$$= 0$$

$\because v_m \neq u_n \; \forall n$

$\therefore$ from (1) & (2) & from above we get

$$f(x) = \langle x, y \rangle, \forall x \in H.$$

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