Calculus

Limit

"If the successive values attributed to the same variable approach to a fixed value, such that they finally differ from it by as little as one wishes, the latter is called the limit of others" (Def. by Cauchy)

$\lim_{x\to a} f(x) = l$ (Notation)

Mathematical Expression

ε - Epsilon (used for small no.'s)
$\exists$ - there exists

Given, $\varepsilon > 0$, if $\exists$ a $\delta(\varepsilon)$, such that whenever $|x-a| < \delta$, then $|f(x) - l| < \varepsilon$

Eg:-

$\lim_{x\to 2} (2x+3) = 7$

Proof :-

Given $\varepsilon > 0$

$|x-2| < \delta$

$|2x+3-7| = |2x-4| = 2|x-2|$
$< 2\delta (= \varepsilon)$
$< \varepsilon$

$\therefore 2\delta = \varepsilon$
$\delta = \varepsilon/2$
$\delta(\varepsilon) = \varepsilon/2$

There is no unique choice of δ it can be $2\varepsilon/3$ or $\varepsilon/2$ . . .

8) $\lim_{x\to x_0} \sin x = \sin x_0$

Proof :-

Given $\varepsilon > 0$

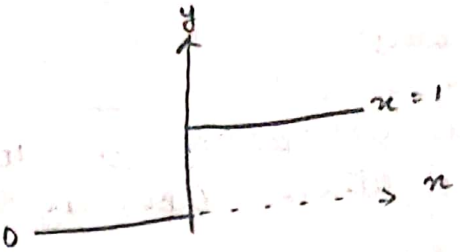
$|x - x_0| < \delta$, $|\sin x - \sin x_0|$
$= 2\sin\left(\frac{x-x_0}{2}\right)\cos\left(\frac{x+x_0}{2}\right)$
$= \delta \cos\left(\frac{x+x_0}{2}\right)$

$\sin\theta \to \theta$ when θ is very small

$\therefore$ Maxm case possible is $\delta = \varepsilon$.

Limit doesn't exist, if for every δ there exists ε_0 such that $|f(x) - l| > \varepsilon_0$

Consider $f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$



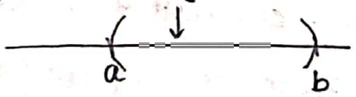
?? The $\lim_{x \to 0} f(x)$ doesn't exist for the reason that the l is not same on left and right.
The limits are one distance apart and ε should

Cluster point : (Limit point)

Let there be a subset $S \subseteq R$ and there exists an 'a' $\in R$
'a' is said to be a cluster point (limit point) of S if for every $\delta > 0$, $(a-\delta, a+\delta) \cap S$ contains at least one point other than 'a'.

Here 'a' need not be a part of S
$S \neq \{x \in R \mid a-\delta < x < a+\delta\}$
δ- Neighbourhood of the point 'a'

Consider the intervals
(i) (a, b)



Any pt. between (a, b) are limit points whereas a, b are not in set so they aren't limit pts.

(ii) $[a, b]$ Here a, b are limit points.

Closed set :-
A set $A \subseteq R$ is closed if it contains all of it's limit points

Open set :-
A set O is said to be open, if every point of O is an interior point.
Bounded interval is a set where the end pt. isn't ∞

Interior pt.

$S \subseteq R$, $a \in R$ is said to be interior pt. if $\exists\ \epsilon > 0$, such that $(a-\epsilon, a+\epsilon) \subset S$

Cluster point characterizes closed set
Interior point characterizes open set.

An empty is both open and closed

Rational no.'s are dancing part of real no's
single no. is a closed set

Continuous Functions :-

Real valued fn. $\Rightarrow f : R \rightarrow R$

→ Continuity is defined in the neighbourhood of the point

→ A fn. $f(x)$ is said to continuous at a point 'a' if

$$\lim_{x \to a} f(x) = f(a)$$

Mathematical proof

Given $\epsilon > 0$, if $\exists$ a $\delta(\epsilon)$
such that whenever $|x-a| < \delta$, then $|f(x) - f(a)| < \epsilon$

Eg of a fn. which is not continuous at any pt.

$$f(x) = \begin{cases} 1, & x \in Q \\ 0, & x \in Q^c \end{cases}$$

It's discont. at every point of R
← Dirichlet fn.

For all sequences
$x_0 \rightarrow a$
If $f(x_0) \rightarrow f(a)$

Then f will be cont. at $x = a$

Didnt understand

$x_n \rightarrow x_0$

For every $\epsilon > 0$, if $\exists$ $n \in N$ such that whenever, $n \geq n_0$, $|x_n - x_0| < \epsilon$

Let x_0 is irrational no.'s such that $x_n \rightarrow x_0$

$x_0 \in Q$ $f(x_0) = 1$

$f(x_n) \longrightarrow f(x_0) = 1$

$\downarrow$

0

$0 \neq 1$ Hence it's discont.

unction which is cont exactly at one point

$$f(x) = \begin{cases} x, & x \text{ is rational} \\ -x, & x \text{ is irrational} \end{cases}$$

The fn. is cont. at $x = 0$ and at other pts. x is discont.

Properities of continuous fn:

→ IMVT :-

Let $f(x)$ be a cont. fn. such that $f : [a, b] \xrightarrow{\text{cont}} \mathbb{R}$

and $f(a) < \alpha < f(b)$ then $\exists$ an 'c' such that $a < c < b$

and $f(c) = \alpha$

→ Min. Max Property :-

$[a, b] \leftarrow$ compact set (closed + bounded)

Let f be cont. in $[a, b]$

Then $\exists$ at least one $x_0 \in [a, b]$ such that $f(x_0) = \underset{x \in [a,b]}{\text{Max}} f(x)$ and is attainable

and $\exists$ at least one $y_0 \in [a, b]$ such that $f(y_0) = \underset{y \in [a,b]}{\text{Min}} f(y)$ and is attainable

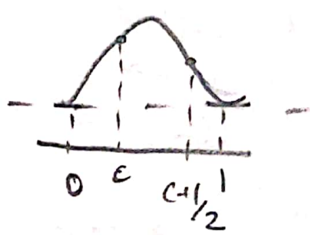
1) f is cont. on $[0, 1]$ such that $f(0) = f(1)$. Prove that there exists a $c \in [0, \frac{1}{2}]$ s.t $f(c) = f(c + \frac{1}{2})$

$g(x) = f(x) - f(x + \frac{1}{2})$

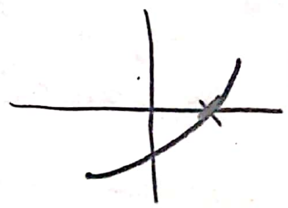
$g(0) = f(0) - f(\frac{1}{2})$

$g(\frac{1}{2}) = f(\frac{1}{2}) - f(1)$

$- g(0) = g(\frac{1}{2})$



Means $g(x)$ cuts x-axis



$g(c) = 0$ in $[0, \frac{1}{2}]$

$\Rightarrow f(c) = f(c + \frac{1}{2})$

⇒ Let $f: [0,1] \to R$ is cont and takes only rational values. P.T f is a constant fn.

A) Suppose $f(0) \neq f(1)$

Means f has atleast two distinct values

consider $f(0) < f(1)$

In the interval $[f(0), f(1)]$ f must take all the values in this means, it takes irrational values also which is a contradiction

$\Rightarrow f(0) = f(1)$

Differentiability

A cont. fn. $f : (a, b) \to R$ is said to be diff. on (a, b) of $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ exists for all $x \in (a, b)$

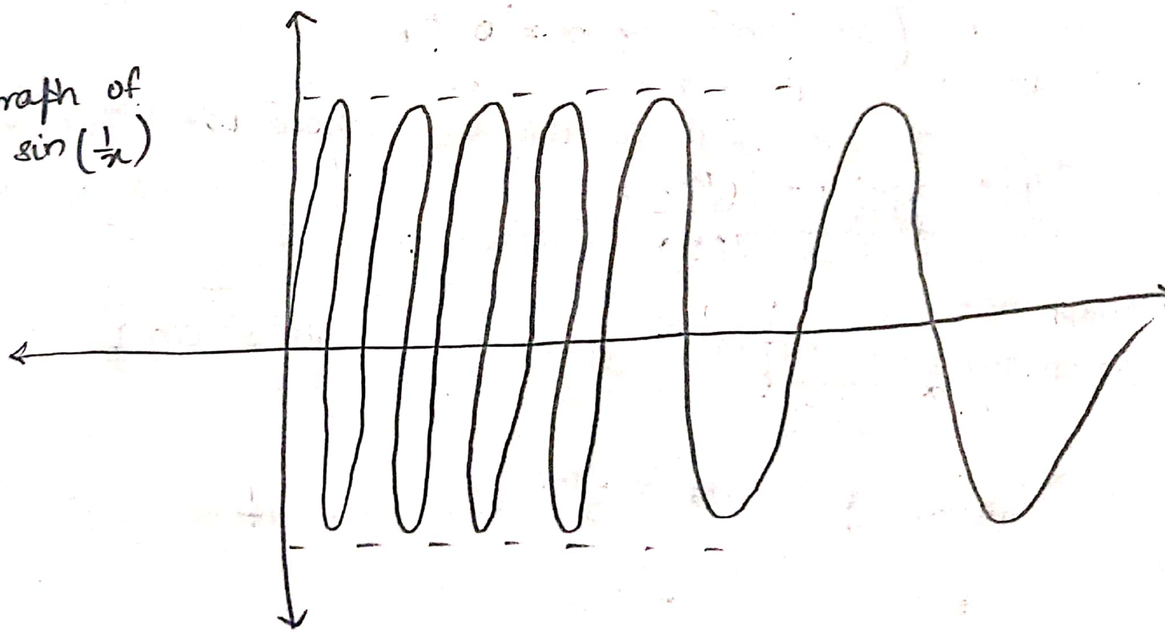
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

All diff. fn are cont. but not vice versa

$\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ doesn't exist

$\sin\left(\frac{1}{x}\right) = f(x)$ is cont. on $R - \{0\}$

Graph of $\sin\left(\frac{1}{x}\right)$



$x_n = \frac{1}{n\pi}$

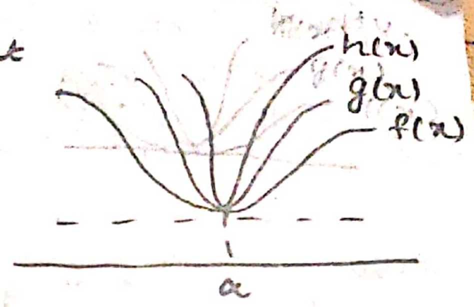
$f\left(\frac{1}{n\pi}\right) = \sin n\pi = 0$

$f\left(\frac{1}{(n+\frac{1}{2})\pi}\right) = \sin\frac{n\pi}{2} + n\pi = 1$

× contradiction

Sandwich Theorm

Consider three fn's $f(x)$, $g(x)$, $h(x)$ such that
$f(x) \le g(x) \le h(x)$



Then

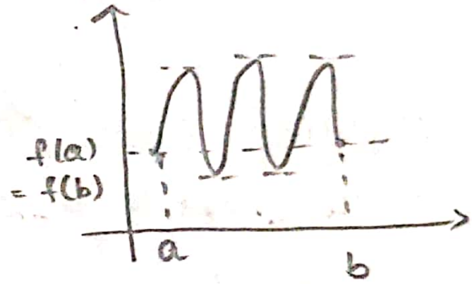
$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x) \le \lim_{x\to a} h(x)$$

Mean Value Theorm

→ Rolle's Theorm

Suppose f is a cont. function $[a,b]$, differentiable on (a,b) and $f(a) = f(b)$. Then there exists at least one c between a and b such that $f'(c) = 0$

There is atleast one point in this case



Proof

Let m is minimum of $f(x)$ in $[a,b]$
M is maximum of $f(x)$ in $[a,b]$

Case-1
$m = M$,
Then f must be constant $\Rightarrow f'(c) = 0 \ \forall x \in [a,b]$

Case-2
$m \neq M$
$\exists\ a < c < b$ s.t $f(c) = M$

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

$$\lim_{h\to 0^+} \frac{\overbrace{f(c+h)-f(c)}^{<0}}{\underbrace{h}_{>0}} \le 0$$

$$\lim_{h\to 0^-} \frac{\overbrace{f(c+h)-f(c)}^{<0}}{\underbrace{h}_{<0}} \ge 0$$

But $f'(c)$ exists $\Rightarrow f'(c) = 0$

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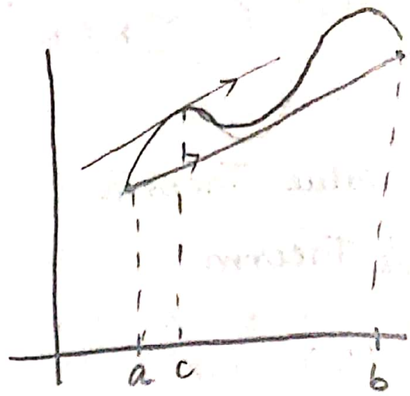
Lagrange's Mean Value Theorm

Suppose f is a cont. fn in $[a, b]$, differentiable in (a, b)
Then there exists atleast one c ~~such~~ between (a, b)
such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$



Proof

$g(x) = f(x) + \lambda x$

$g(a) = g(b)$ (consider)

$\Rightarrow f(a) + a\lambda = f(b) + \lambda b$

$-\lambda(b - a) = f(b) - f(a)$

$\Rightarrow \lambda = \dfrac{f(b) - f(a)}{a - b}$

Applying Rolle's Theorm to $g(x)$

$\Rightarrow$ There exists a c in b/w (a, b) such that $g'(c) = 0$

$\Rightarrow f'(c) + \lambda = 0$

$\Rightarrow f'(c) = -\lambda$

$\therefore f'(c) = \dfrac{f(b) - f(a)}{b - a}$

→ Cauchy's Mean Value Theorm

Suppose $f(x)$ and $g(x)$ are diff. on (a, b). Let $g'(x) \neq 0 \ \forall x \in (a, b)$
Then there exists a 'c' b/w (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof

$h(x) = f(x) + \lambda g(x)$

$h(a) = h(b)$ (consider)

$\Rightarrow h(a) = f(a) + \lambda g(a)$

||ly

$h(b) = f(b) + \lambda g(b)$

$\Rightarrow \dfrac{f(a) - f(b)}{g(b) - g(a)} = \lambda$

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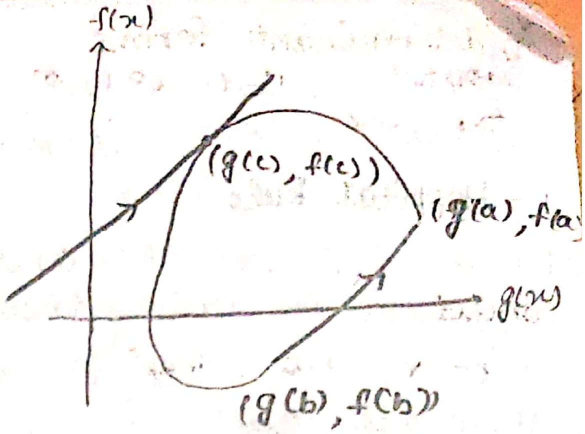
$$f'(c) = -\lambda g'(c)$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = -\lambda$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$



Also
for Rolle's

$$F(x) = f(x) - f(a) - \left[\frac{f(b) - f(a)}{b - a}(x - a)\right]$$

for cauchy

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a))$$

Indeterminant forms

$\frac{0}{0}, \frac{\infty}{\infty}, 0.\infty, 0^0, \infty^0, 1^\infty, \infty - \infty$

But $\infty + \infty, \infty^\infty, \infty \cdot \infty, 0^\infty$

L- Hospital Rule

Let the function $f(x)$ and $u(x)$ in $[a,b]$ satisfy the conditions of Cauchy theorm and vanish at $x=a$ i.e $f(a) = u(a) = 0$. Then, if the ratio $\frac{f'(x)}{u'(x)}$ has a limit as $x \to a$, there also exists $\lim_{x\to a} \frac{f(x)}{u(x)}$ and

$$\lim_{x\to a} \frac{f(x)}{u(x)} = \lim_{x\to a} \frac{f'(x)}{u'(x)}$$

Proof:-

Let $x \in [a,b]$ and $x \neq a$

$$\frac{f(x) - f(a)}{u(x) - u(a)} = \frac{f'(\varepsilon)}{u'(\varepsilon)} \quad \varepsilon \in (a,x)$$

$$u(a) = f(a) = 0 \Rightarrow \frac{f(x)}{u(x)} = \frac{f'(\varepsilon)}{a'(\varepsilon)}$$

$x \to a \Rightarrow \varepsilon \to a$ since $\varepsilon \in (a,x)$

Then

$$\lim_{x\to a} \frac{f(x)}{u(x)} = \lim_{x\to a} \frac{f'(\varepsilon)}{u'(\varepsilon)} = \lim_{x\to a} \frac{f'(x)}{u'(x)}$$

Cauchy sequence:-

Sequence x_n is said to be 'Cauchy' if for every $\varepsilon > 0$, $\exists$ $n_0 \in N$ such that whenever $m > n \geq n_0$, then $|x_n - x_0| < \varepsilon$

A sequence in $\mathbb{R}$ converges $\Rightarrow$ it must be Cauchy

Let f be cont. on $[a,b]$, $a > 0$ and diff. on (a,b). P.T there exists c between (a,b) such that $\frac{bf(a) - af(b)}{b-a} = f(c) - cf'(c)$

$g(x) = \frac{f(x)}{x}$ ✓ $\quad h(x) = \frac{1}{x} \quad g'(x) = \frac{xf'(x) - f(x)}{x^2}$

$$\frac{\frac{f(b)}{b} - \frac{f(a)}{a}}{b-a} = f'(c) \Rightarrow \frac{af(b) - bf(a)}{ab(b-a)} = f'(c)$$

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$$\frac{\frac{f(b)}{b} - \frac{f(a)}{a}}{\frac{1}{b} - \frac{1}{a}} = \frac{g'(c)}{h'(c)}$$

$$\Rightarrow \frac{af(b) - bf(a)}{a-b} = \frac{cf'(c) - f(c)}{-\cancel{c}}$$

$$\Rightarrow \frac{bf(a) - af(b)}{b-a} = f(c) - cf'(c)$$

② using CMVT
P.T $1 - \frac{x^2}{2} < \cos x \quad \forall x \neq 0$

A) $f(x) = 1 - \cos x$ in $[0, x]$
$g(x) = \frac{x^2}{2}$

using CMVT,

$$\frac{\sin c}{c} = \frac{1-\cos x}{x^2/2} < 1$$

$$\Rightarrow 1 - \cos x < \frac{x^2}{2}$$

→ set of all real no-s aren't countable

Indeterminant forms

If $\lim_{x\to a} f(x) = l$, $\lim_{x\to a} g(x) = m$

If $m \neq 0$, then, $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{l}{m}$

If $m = 0$, $(l \neq 0)$ then the $\lim_{x\to a} \frac{f(x)}{g(x)}$ doesn't exist

If $l = m = 0$

Then $\lim_{x\to a} \frac{f(x)}{g(x)}$
1) may exist as a finite value
2) may not exist
3) may exist as an infinite value

Let $l \neq 0$

$$l = \lim_{x\to a} f(x) \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} \cdot \lim_{x\to a} g(x)$$
$$= l \times m$$
$$= 0 \neq l]$$

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L-Hospital Rule :-

Suppose f and g are diff. at $x=a$ and $g'(x) \neq 0$. Then

Let $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Proof :-

$f(a) = g(a) = 0$

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \frac{f'(a)}{g'(a)}$$

$\frac{\infty}{\infty}$ form :-

Let $f(x)$ and $g(x)$ be cont. and diff. $\forall x \neq a$ in the neighbourhood of a

Let $g'(x) \neq 0$, $(x \neq a)$

let $\lim_{x \to a} f(x) = \infty = \lim_{x \to a} g(x)$, then

If $\lim_{x \to a} \frac{f'(x)}{g'(x)} = l$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = l$

8) $\lim_{x \to \infty} \left[\sqrt[n]{(x+a_1)(x+a_2)\cdots(x+a_n)} - x \right]$

A $\Rightarrow \lim_{x \to \infty} \left[\sqrt[n]{x\left(1+\frac{a_1}{x}\right)\left(1+\frac{a_2}{x}\right)\cdots\left(1+\frac{a_n}{x}\right)} - x \right]$

$$\Rightarrow \lim_{x \to \infty} \left[\left(1+\frac{a_1}{nx}\right)\left(1+\frac{a_2}{nx}\right)\cdots\left(1+\frac{a_n}{x}\right) - 1\right]$$

$$\Rightarrow \lim_{x \to \infty} \frac{x \sum_{i=1}^{n} a_i}{nx} \Rightarrow \frac{\sum_{i=1}^{n} a_i}{n}$$

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Q) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = \lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x}\right) = 0$

Q) $\lim_{x\to 0} x\ln x = \lim_{x\to 0}\frac{\ln x}{1/x} = \lim_{x\to 0}\frac{1/x}{-1/x^2}$

$= -\lim_{x\to 0} x = 0$

P^Q

$1^\infty \rightarrow e^{\lim_{x\to a} Q(P-1)}$

Q) $\lim_{x\to 0} x^x = e^{\lim_{x\to 0} x\ln x}$

$= 1$

Q) $\lim_{x\to \infty}\left(1+\frac{1}{x}\right)^x = e^{\lim_{x\to 0} x\times\frac{1}{x}} = e$

Note:-

$\lim_{x\to a}\ln y = \ln\left(\lim_{x\to a} y\right)$

It is because log is cont. function

Taylor's formula:-

consider f is diff. $(n+1)$ times

$P_n(x)$ is a n^{th} degree polynomial such that

$f(a) = P_n(a)$, $f'(a) = P_n'(a)$, ... $f^n(a) = P_n^n(a)$

Suppose

$P_n(x) = c_0 + c_1(x-a) + \dots + c_n(x-a)^n$

where $c_0, c_1 \dots c_n$ needs to be determined

$P_n(a) = c_0 = f(a)$

$P_n'(a) = c_1 = \frac{f'(a)}{1!}$

$P_n''(a) = 2c_2 = f''(a) \Rightarrow \frac{f''(a)}{2!} = c_2$

$\vdots$

$P_n^n(a) = c_n = \frac{f^n(a)}{n!}$

$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots \frac{f^n(x)(x-a)^n}{n!}$

$f(x) = P_n(x) + R_n(x)$

↳ is error or remainder

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form of remainder

Let $R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} Q(x)$ $\quad x \in [a,b]$

$Q(x)$ is a new function that has to be determined

$$f(x) = f(a) + (x-a)f'(x) + \dots + \frac{f^n(a)}{n!}(x-a)^n + \frac{(x-a)^{n+1}}{(n+1)!}Q(x)$$

$$F(t) = f(x) - f(t) - (x-t)f'(t) + \dots - \frac{(x-t)^n}{n!}f^n(t) - \frac{(x-t)^{n+1}}{(n+1)}Q$$

(Q is the value that is found by fixing a and x in f(x))

$$F'(t) = -\cancel{f'(t)} + \cancel{f'(t)} \dots - \frac{(x-t)^n}{n!}f^n(t) + \frac{(n+1)(x-t)^n}{(n+1)!}Q$$

$F(x) = F(a) = 0$

By Rolle's Theorm

$F'(c) = 0$

$\Rightarrow Q = f^{n+1}(c)$

$a < c < x$

$\Rightarrow c - a < x - a$

$\Rightarrow c - a = \theta(x-a) \Rightarrow c = a + \theta(x-a)$

$\Rightarrow R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^n(a + \theta(x-a))$ $\quad 0 < \theta < 1$

Langrange's form of remainder

If $a = 0$, we call the expansion as Machlauran's expansion
i.e

$$f(x) = f(0) + xf'(0) + \dots + \frac{x^n f^n(x)}{n!} + R_n(x)$$

where $\boxed{R_n = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(a + \theta(x-a))}$

$0 < \theta < 1$

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Cauchy's form of remainder

$R_n(x) = f(x) - f(a) - (x-a)f'(a) - \dots - \frac{(x-a)^n}{n!} f^n(a)$

Let $F(t) = f(x) - f(t) - (x-t)f'(t) - \dots - \frac{(x-t)^n}{n!} f^n(t)$, $t \in [a, x]$

$F(a) = R_n(x)$, $F(x) = 0$

$F'(t) = -\frac{(x-t)^n}{n!} f^{n+1}(t)$,

Applying LMVT to $F(x)$ on $[0, x]$ $\exists\ a < c < x$ such that $\frac{F(x) - F(a)}{x-a} = F'(c)$

$$\Rightarrow \frac{0 - R_n(x)}{x-a} = -\frac{(x-c)^n}{n!} f^{n+1}(c)$$

$$\Rightarrow R_n(x) = \frac{(x-c)^n (x-a) f^{n+1}(c)}{n!}$$

↑ Cauchy's form of remainder

8) $f(x) = \begin{cases} e^{-1/x^2} , & x \neq 0 \\ 0 , & x = 0 \end{cases}$ check the diff. at $x = 0$

$f'(x) = \frac{e^{-1/x^2}}{x^2}$

$\lim_{x \to 0} \frac{e^{-1/x^2}}{x^2} = L$ Let $u = \frac{1}{x}$

$\Rightarrow L = \lim_{x \to 0} u e^{-u^2}$

$\Rightarrow L = \lim_{x \to 0} \frac{u}{e^{u^2}}$

$= 0$

It happens that $f'(0) = f''(0) = f'''(0) = f^{iv}(0) = \dots = 0$ — ①, it happens so in taylor series if we use that ①, $f(x) = 0$ so it's a contradiction

↑
Non-zero

$\Rightarrow$ Even the fn. is diff., there may not exist taylor series

$\Rightarrow$ So the $f(x)$ doesn't have taylor series in the neighbourhood of $x = 0$ even if fn. is infinitely diff.

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Radius of convergence

If $f(x) = \frac{1}{1-x}$

and $g(x) = 1 + x + x^2 + \dots x^n$

$f(x) = g(x)$ Not for all x, only for $|x| < 1$

In this range the series of $f(x)$ is $g(x)$, and $f(x)$ is converging with $g(x)$ except at $x = 1$

Q) Find $R_n(x)$ for $f(x) = e^x$

(i) L.F.O.R

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} e^{a + \theta(x-a)}$$

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} (e^{\theta x}) \quad (0 < \theta < 1)$$

Taylor's formula

x is fixed, find $\lim\limits_{n \to \infty} \frac{x^{n+1}}{(n+1)!} = ?$

Ans) $\lim\limits_{n \to \infty} \frac{x^{n+1}}{(n+1)!} = 0$

$|x| < N$

$\Rightarrow \left|\frac{x}{N}\right| = q < 1$

$$\left|\frac{x^{n+1}}{(n+1)!}\right| = \left|\frac{x}{1}\right| \left|\frac{x}{2}\right| \cdots \left|\frac{x}{N}\right| \left|\frac{x}{N+1}\right|$$

$$\leq \frac{x^{n-1}}{N-1} q^{n-N+2}$$

As $n \to \infty$ $\longrightarrow 0$

since $q < 1$

and $n - N + 2 \to \infty$

$\Rightarrow q^{n-N+2} \to \infty$

$$\therefore \lim_{n \to \infty} R_n(x) = \lim_{n \to \infty} \frac{x^{n+1}}{(n+1)!} e^{\theta x} = 0$$

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Functions of several real variables
(two or more)

If to each point (x,y) of a certain part of x-y plane, there corresponds a real value z according to some given rule $f(x,y)$, then $f(x,y)$ is called real valued fn. of two variables

Represented as $z = f(x,y)$; $x,y \in R^2$, $z \in R$

x, y - Independent variable
z - Dependent variable

A real valued fn. of n-variables is defined as
$$z = f(x_1, x_2 \dots x_n) ; (x_1, x_2 \dots x_n) \in R^n, z \in R$$

Limit (Two variables).

Let $z = f(x,y)$ be a fn. of two variables defined in a domain

Let $P(x_0, y_0)$ be a point in D

Given $\varepsilon > 0$, if there exists a $\delta > 0$ such that whenever $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$, then $|f(x,y) - l| < \varepsilon$

where $l = \lim_{(x,y \to x_0, y_0)} f(x,y)$

We have infinitely many directions to approach a point

If $\lim_{(x,y) \to (x_0,y_0)} f(x,y) = f(x_0, y_0)$

Then the fn. is said to be cont.

Q) $f(x,y) = \begin{cases} \dfrac{x^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

check continuity of the fn.

A) We can approach from any direction, and if we consider any two directions and find both the values are not equal it is suff. to say Limit doesn't exist
AS

"LIMIT IS UNIQUE"

So, lets consider $y = 2x$ and $y = 3x$

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$\lim_{(x,y)\to(0,0)} \frac{x^2}{5x^2} = \frac{1}{5}$ (Approaching from $y = 2x$) $\Big\} \Rightarrow$ Limit DE

$\lim_{(x,y)\to(0,0)} \frac{x^2}{10x^2} = \frac{1}{10}$ (Approaching from $y = 3x$)

If we use $y = mx$

$\lim_{(x,y)\to(0,0)} \frac{x^2}{(1+m^2)x^2} = \frac{1}{1+m^2}$ (So we are finding infinitely many paths to prove limit DE)

Q) $f(x,y) = y + x \ln \frac{1}{y}$. Find $\lim_{(x,y)\to(0,0)} f(x,y)$

A) $f(x,y) = y + x \ln \frac{1}{y}$

$|f(x,y) - 0| = |y + x \ln \frac{1}{y}|$

$|y + x \ln \frac{1}{y}| < |y| + |x| |\ln \frac{1}{y}|$

$\Rightarrow |y + x \ln \frac{1}{y}| \leq |y| + |x|$

$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\therefore \sqrt{x^2 + y^2} = \sqrt{\frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4}} = \frac{\varepsilon}{2} = \delta$

Note:-

→ To prove LE don't use 2-path test, use only ε-method

→ To prove LDE we can use 2-path test

Q) $f(x,y) = \begin{cases} \frac{x(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

A) $\lim_{(x,y)\to(0,0)} \frac{x^3(1-m^2)}{x^2(1+m^2)} = 0$

$0 \leq |f(x,y) - 0| = \left| \frac{x(x^2-y^2)}{x^2+y^2} \right| \leq |x|$

$\therefore$ limit exists and is equal to 0

Q) $f(x,y) = \begin{cases} \dfrac{x^4y^4}{(x^2-y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ (use 2-path test)

"Baboi Champayandi"

A) $|f(x,y) - 0| = \left|\dfrac{2x^4y^4}{2(x^2+y^2)^3} - 0\right| < \left|\dfrac{x^8+y^8}{2(x^2+y^2)^3}\right|$

Q) $f(x,y) = \begin{cases} \dfrac{2x^2+y^2}{3+\sin x}, & (x,y) \neq (0,0) \\ (0), & (x,y) = (0,0) \end{cases}$

Check if $f(x,y)$ is cont. at $(0,0)$

Q_2) $f(x,y) = \begin{cases} \dfrac{xy(x-y)}{x^2y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Check if $f(x,y)$ is cont. at $(0,0)$

A) $|f(x,y) - f(0,0)| = \left|\dfrac{2x^2+y^2}{3+\sin x} - 0\right| \leq \dfrac{1}{2}|2x^2+y^2| \leq x^2+y^2 \leq \varepsilon$

$[|3+\sin x| \not\geq 2]$

$\Rightarrow \varepsilon = x^2+y^2$

$\sqrt{x^2+y^2} < \delta \Rightarrow \delta = \sqrt{\varepsilon}$

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Partial Differentiation

$$\frac{\partial f(x,y)}{\partial x} = f(x,y_0) = \lim_{h\to 0} \frac{f(x+h, y_0) - f(x, y_0)}{h}$$ (Partial derivative w.r. Here y is fixed

$$\frac{\partial f(x,y)}{\partial y} = f(x_0,y_0) = \lim_{k\to 0} \frac{f(x_0, y_0+k) - f(x_0,y_0)}{k}$$ (Partial derivative

Conditions for ~~P.D~~ Differentiability to exist

Derivative

1) $f_x(x_0,y_0)$ & $f_y(x_0,y_0)$ exists
2) Either $f_x(x_0,y_0)$ or $f_y(x_0,y_0)$ is cont. at (x_0,y_0)

Relationship between Continuity and the existance of partial derivative

A fn: can have partial derivative w.r.t both x and y at a point without being cont. there. On the other hand a cont. fn: may not have partial derivatives

Eg :-

$$f(x,y) = \begin{cases} (x+y)\sin\left(\frac{1}{(x+y)}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

M1

$$|f(x,y) - f(0,0)| = \left|(x+y)\sin\left(\frac{1}{x+y}\right)\right| \le |x+y|$$
$$\le \sqrt{2}\,|x| + |y|$$
$$\Rightarrow \delta \le \frac{\varepsilon}{\sqrt{2}}$$

M2

Let $x+y = t$

$$\Rightarrow \lim_{x+y\to 0} f(x+y) = \lim_{x,y\to 0} t\sin\frac{1}{t} \to 0 \quad \Rightarrow \text{It is cont.}$$

Now consider $f(x+\Delta x, y)$

$$\Rightarrow \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta x \sin\left(\frac{1}{\Delta x}\right)}{\Delta x}$$

$\Rightarrow$ Limit doesn't exist

So fn: is cont. but not diff.

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Eg :-

$$f(x,y) = \begin{cases} \frac{xy}{x^2+2y^2} , & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$$

Let $y = mx$

$$\lim_{x,y \to 0} f(x,y) = \lim_{x,y \to 0} \frac{my^2}{(m^2+2)y^2} = \frac{m}{m^2+2}$$

$\Rightarrow$ Diff. for diff 'm'
so fn isn't cont.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = f_x(0,0)$$

The partial derivatives of f_x, f_y exists at $(0,0)$

Theorm : (Suff. condition for continuity at (0,0))

One of the first order partial derivative exists and is bounded in neighbourhood of (x_0, y_0) and the other exists at (x_0, y_0)

Note :-

$$\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)}_{f_{xx}} , \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)}_{f_{yx}} , \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)}_{f_{xy}}$$ are called 2° p.d.

f_{yx}, f_{xy} : Mixed derivatives

If f_{yx}, f_{xy} are cont. in an open domain D, then at any pt $(x,y) \in D$

$$f_{xy} = f_{yx}$$

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...entiability

$$\lim_{(h,k)\to(0,0)} \frac{f(x+h, y+k) - f(x,y) - hf_x(x,y) - kf_y(x,y)}{\sqrt{h^2+k^2}} = 0$$

f is diff at (x,y) if

$\Rightarrow \lim_{\Delta P\to 0} \frac{\Delta f - df}{\Delta P}$

$\Delta f = f(x+h, y+k) - f(x,y)$ ← Total increment

$df = hf_x + kf_y$ ← Total differential

$h = \Delta x \approx dx$, $k = \Delta y \approx dy$

$$\boxed{df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.}$$

If there is fn. $f: R^n \to R^m$

$f(x_1, x_2 \dots x_n) = (f_1(x_1,\dots,x_n), f_2(x_1,\dots x_n) \dots f_m(x_1, \dots x_n))$

derivative of $f(x_1, \dots x_n)$ (df) $=$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m\times n} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix}$$

$m\times n$ matrix

Eg:-

$f: R^2 \to R^1 \Rightarrow [f_x(x,y), f_y(x,y)]_{1\times 2}$

↑ is the derivative

$f: R^2 \to R^2$

$f(x,y) = (\overset{f_1}{u(x,y)}, \overset{f_2}{v(x,y)})$

$D_f = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. $h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - D_f \times h}{|h|}$$

→ vector in R^n

↙ vector in R^m

but length is R

so $\lim_{|h|\to 0} \frac{|f(x+h) - f(x) - D_f \times h|}{|h|}$

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Sufficient condition for differentiability

* If $f_x(x,y)$ & $f_y(x,y)$ exist
* One of em is cont at (x,y)

Then f is diff. at (x,y)

Proof :-

$\Delta f = f(x+h, y+k) - f(x,y)$

$= [f(x+h, y+k) - f(x, y+k)] + [f(x, y+k) - f(x,y)]$

$= h f_x(\bar{x}, y+k) + [f(x, y+k) - f(x,y)] \quad x < \bar{x} < x+h$ — ①

Since f_x is cont. at (x,y)

$$\lim_{(h,k)\to(0,0)} f_x(\bar{x}, y+k) = f_x(x,y)$$

$\Leftrightarrow f_x(\bar{x}, y+k) = f_x(x,y) + \eta_1$ — ②

where $\eta_1 \to 0$ as $(h,k) \to (0,0)$

$f_y(x,y)$ exist

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x, y+k) - f(x,y)}{k}$$

$\Rightarrow \frac{f(x,y+k) - f(x,y)}{k} = f_y(x,y) + \eta_2$

$f(x, y+k) - f(x,y) = k f_y(x,y) + \eta_2 k$

where $\eta_2 \to 0$ as $(h,k) \to 0$ — ③

$\Delta f = h f_x(x,y) + \eta_1 h + k f_y(x,y) + \eta_2 k$ where $\eta_1, \eta_2 \to 0$ as $(h,k) \to (0,0)$

$$\frac{\Delta f - df}{\Delta \rho} = \frac{\eta_1 h}{\Delta \rho} + \frac{\eta_2 k}{\Delta \rho} \qquad \left|\frac{h}{\Delta\rho}\right| \le 1 \qquad \left|\frac{k}{\Delta\rho}\right| \le 2$$

Then $\lim$ as $\Delta\rho \to 0$

$$\lim_{\Delta\rho\to 0} \frac{\Delta f - df}{\Delta\rho} = 0$$

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$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

$\rightarrow$ fn: is cont. at $(0,0)$ partial derivatives exists

$$f_x(x,y) = \frac{y\sqrt{x^2+y^2} - \dfrac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2}$$

$x = r\sin\theta$
$y = r\cos\theta$

$$= \frac{y(x^2+y^2) - 2x}{(x^2+y^2)^{3/2}}$$

$$f(x,y) = \begin{cases} \dfrac{r\sin 2\theta}{2} & (\theta) \neq 0 \\ 0 & \theta = 0 \end{cases}$$

$f_x(0,0) = 0$

$f_y(0,0) = 0$

$$f_x = \begin{cases} \dfrac{y^3}{(x^2+y^2)^{3/2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$\Rightarrow$ fn isn't cont. at origin (As $x = r\cos\theta$, $y = r\sin\theta$)

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$$f(x,y) = \begin{cases} \frac{x^6 - 2y^4}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Test diff. at $(0,0)$?

$$\Delta f = \frac{h^6 - 2k^4}{h^2+k^2}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = 0$$

$$\lim_{\Delta\rho\to 0} \frac{\Delta f - df}{\Delta \rho} = \lim_{\Delta\rho\to 0} \frac{h^6 - 2k^4}{(h^2+k^2)^{3/2}}$$

$$= \lim_{r\to 0} \frac{r^6\cos^6\theta - 2r^4\sin^4\theta}{r^3}$$

$$= 0$$

Derivative in several variables at a point means a tangent plane containting all the tangents drawn to that point. The plane is approaching $f(x,y)$ in that neighbourhood.

Chain rule :-

Consider $z = F(u,v)$ where $u = \phi(x,y)$, $v = \psi(x,y)$

$$\Rightarrow z = F[\phi(x,y), \psi(x,y)]$$ ← we don't know what the fn: is explicitly.

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial u}{\partial x} \\ \frac{\partial z}{\partial y} &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} \end{aligned}}$$

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struct info *next;
}
TOOK

$x \to x + \Delta x$, $\Rightarrow$ y is held constant

$\Delta_x z = z(x+\Delta x, y) - z(x,y)$

Suppose u receives an increment of the amount $\Delta_x u$

$\Delta_x u = u(x+\Delta x, y) - u(x,y)$

& v recieves an increment of the amount $\Delta_x v$

$\Rightarrow \Delta_x z = \frac{\partial F}{\partial u} \Delta_x u + \frac{\partial F}{\partial v} \Delta_x v + \gamma_1 \Delta_x u + \gamma_2 \Delta_x v$

where γ_1 & $\gamma_2 \to 0$ as $\Delta_x u, \Delta_x v \to 0$

If $\Delta x \to 0$, then $\gamma_1, \gamma_2 \to 0$

$\frac{\Delta_x z}{\Delta x} = \frac{\partial F}{\partial u} + \frac{\Delta_x u}{\Delta x} + \frac{\partial F}{\partial v} * \frac{\Delta_x v}{\Delta x} + \gamma_1 \frac{\Delta_x u}{\Delta x} + \gamma_2 \frac{\Delta_x v}{\Delta x}$

(As $\Delta x \to 0$)

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial x}}$$

Eg:-

1) $z = f(u, v)$

$u = e^x \cos y$, $v = e^x \sin y$

Show That $y \frac{\partial z}{\partial u} + x \frac{\partial z}{\partial v} = e^{2u} \frac{\partial z}{\partial y}$

A) $\frac{\partial z}{\partial x} =$

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2) Verify the chain rule

$u = x e^{yz}$, $(x, y, z) \to (e^t, t, \sin t)$

A) $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$

$= e^{yz} e^t + zx e^{zt} + xy e^{yz} \cos t$

$\frac{du}{dt} = \frac{d(e^{t \sin t} \cdot e^t)}{dt}$

$= \frac{d e^{(t \sin t + 1)}}{dt} = [e^{t \sin t} (t \cos t + \sin t)] e^t$

$= [e^{t \sin t} (t \cos t + \sin t)] e^t + e^{t \sin t + 1}$

Implicit Differentiation

$y = f(x)$, $F(x, y) = 0$

$F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = 0$

$\left.\frac{dy}{dx}\right|_{(x_0, y_0)} = -\frac{F_x(x_0, y_0)}{F_y(x_0, y_0)}$ provided $F_y(x_0, y_0) \neq 0$

$z = f(x, y)$, $F(x, y, z) = 0$

$z_x?$, $z_y?$

$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\underbrace{\frac{dy}{dx}}_{0} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$

↳ as x, y are independent

$\Rightarrow \boxed{z_x = \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}}$

lly $\boxed{z_y = \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$

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Q) Find z_x, z_y ?

(i) $xy^2 + z^3 + \sin(xyz) = 0$

$$\frac{\partial F}{\partial z} = 3z^2 + xy\cos(xyz)$$

$$\frac{\partial F}{\partial x} = y^2 + yz\cos(xyz)$$

$$z_x = -\left[\frac{y^2 + yz\cos(xyz)}{3z^2 + xy\cos(xyz)}\right]$$

$x^y + y^x = c$

$[y\log x][x\log y] = \log c$

$\Rightarrow xy\log(x+y) = \log c$

$xy \times \frac{1}{x+y}\left[1 + \frac{dy}{dx}\right] + \log(x+y)\left[y + \right.$

$= 0$

$\Rightarrow \frac{xy}{x+y} + \left(\frac{xy}{x+y}\right)\frac{dy}{dx} + y\log(x+y) + x\log(x+y)$

$=$

$$\Rightarrow \frac{dy}{dx} = -\frac{\left[\frac{xy}{x+y} + y\log(x+y)\right]}{\left[\frac{xy}{x+y} + x\log(x+y)\right]}$$

(ii) $x - yz + \cos(xyz) - x^2z^2 = 0$

$$\frac{\partial F}{\partial x} = 1 - yz\sin(xyz) - 2xz^2$$

$$\frac{\partial F}{\partial z} = -y - xy\sin(xyz) - 2zx^2$$

$$z_x = \frac{1 - yz\sin(xyz) - 2xz^2}{y + xy\sin(xyz) + 2zx^2}$$

Higher order partial derivative

$$z = f(x,y)$$

$f_x(x,y)$, $f_y(x,y) \leftarrow$ 1st order partial derivative

$$f_{xx}(x,y) = \frac{\partial (f_x(x,y))}{\partial x} = \lim_{h\to 0} \frac{f_x(x+h,y) - f_x(x,y)}{h}$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}(f_x(x,y)) = \lim_{k\to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k}$$

$$f_{yx}(x,y) = \frac{\partial}{\partial x}(f_y(x,y)) = \lim_{h\to 0} \frac{f_y(x+h,y) - f_y(x,y)}{h}$$

$$f_{yy}(x,y) = \frac{\partial}{\partial y}(f_y(x,y)) = \lim_{k\to 0} \frac{f_y(x,y+k) - f_y(x,y)}{h}$$

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When mixed derivatives are equal

Suppose f_x, f_y, f_{xy}, f_{yx} are all cont. at (x,y)

Then $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

or $f_{yx} = f_{xy}$

Proof :-

consider $A = [f(x+h, y+k) - f(x+h, y)] - [f(x, y+k) - f(x,y)]$

$f(x,y)$ is cont. and diff., so we can apply LMVT

$A = k f_y(x+h, \bar{y}) - k f_y(x, \bar{\bar{y}})$ $\quad y < \bar{y} < y+k$

$\quad y < \bar{\bar{y}} < y+k$

$A = k[f_y(x+h, \bar{y}) - f_y(x, \bar{\bar{y}})]$

$= hk\ f_{yx}(\bar{x}, \bar{y})$ $\quad x < \bar{x} < x+h$

$\quad y < \bar{y} < y+h$

$A = [f(x+h, y+k) - f(x, y+k)] - [f(x+h, y) - f(x,y)]$

$= h\ f_x(\bar{x}, y+k) - h\ f_x(\bar{x}, y)$

$= hk\ f_{xy}(\bar{x}, \bar{y})$; $x < \bar{x} < x+h$

$y < \bar{y} < y+k$

$f_{yx}(\bar{x}, \bar{y}) = f_{xy}(\bar{x}, \bar{\bar{y}})$

$\lim_{(h,k)\to(0,0)} f_{yx}(\bar{x}, \bar{y}) = \lim_{(h,k)\to(0,0)} f_{xy}(\bar{x}, \bar{y})$

$\Rightarrow f_{yx}(x,y) = f_{xy}(x,y)$

Proof :-

$A = [f(x+h, y+k) - f(x+h, y)] - [f(x, y+k) - f(x,y)]$

$= \phi(x+h) - \phi(x)$ $\quad$ when $\phi(x) = f(x, y+k) - f(x,y)$

$= h\ \phi_x(\bar{x})$, $x < \bar{x} < x+h$ $\quad$ $\phi_x(\bar{x}) = f_x(\bar{x}, y+k) - f_x(\bar{x}, y)$

$= hk\ f_{xy}(\bar{x}, \bar{y})$

$A = [f(x+h, y+k) - f(x, y+k)] - \underbrace{[f(x+h, y) - f(x,y)]}_{\psi(y)}$

$A = \psi(y+k)$

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(i)

$$f(x,y) = \begin{cases} \dfrac{x^2 y(x-y)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$f(x,y) = \begin{cases} \dfrac{x^3 y - y^2 x^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\bigg|_{(0,0)} = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

(ii)

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{h^2 k(h-k)}{h^2+k^2} - 0}{k}$$

$$= \lim_{k\to 0} \frac{h^2(h-k)}{h^2+k^2}$$

$$= \frac{\cancel{h^2}(h)}{\cancel{h^2}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\bigg|_{(0,0)} = \lim_{h\to 0} \frac{h-0}{h} = 1$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\bigg|_{(0,0)} = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h^2 k(h-k)}{h^2+k^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{h^{\cancel{2}} k(h-k)}{(h^2+k^2)\cancel{h}}$$

$$= 0$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\bigg|_{(0,0)} = 0$$

$\Rightarrow$ Atleast one of f_{xy}, f_{yx} aren't cont. at $(0,0)$

4

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omogeneous Function

onsider a fn: $z = f(x,y)$ is said to homogeneous fn: of certain egree 'n' if

$$\boxed{f(tx, ty) = t^n f(x,y) \quad \forall\, t \in R}$$

Euler's Theorm on Homogeneous fn:

Suppose f has partial derivative of first order and f is a homogeneous function of degree n, then

$$\boxed{x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf}$$

Proof

$f(tx, ty) = t^n f(x,y), \forall t \in R$ — ①

let $x' = tx$, $y' = ty$

① $\Rightarrow \frac{\partial f}{\partial x'} \frac{\partial x'}{\partial t} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial t} = nt^{n-1} f(x,y)$ (Differentiate w.r.t t)

$\Rightarrow x \frac{\partial f}{\partial x'} + y \frac{\partial f}{\partial y'} = nt^{n-1} f(x,y)$

Let $t = 1$

$\Rightarrow x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf$

Suppose f is a homogeneous fn: of nth degree and has pd 1st and 2nd

Then

$$\boxed{x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n-1)f}$$

Proof :-

$\left(x \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x} + y \frac{\partial^2 f}{\partial y \partial x} = n \frac{\partial f}{\partial x}\right) x \Rightarrow x^2 \frac{\partial^2 f}{\partial x^2} + x \frac{\partial f}{\partial x} + xy \frac{\partial^2 f}{\partial y \partial x} = nx \frac{\partial f}{\partial x}$ — ①

$\Rightarrow \left(x \frac{\partial^2 f}{\partial y \partial x} + y \frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y} = ny \frac{\partial f}{\partial y}\right) y$

$\Rightarrow xy \frac{\partial^2 f}{\partial y \partial x} + y^2 \frac{\partial^2 f}{\partial y^2} + y \frac{\partial f}{\partial y} = ny \frac{\partial f}{\partial y}$ — ②

Adding ① & ②

$2xy \frac{\partial^2 f}{\partial x \partial y} + x^2 \frac{\partial f^2}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2} = (n^2 - n) f$

$\Rightarrow x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = (n-1)nf$

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8) If

$$u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

Using Euiler's theorm S.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

A) $\sin u = \left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

Taylor's Theorm for functions of two variables

$D \subseteq R^2 \rightarrow R$

Consider a fn: $f(x,y)$ such that $f: R^2 \rightarrow R$

$$f(x,y) = f(x_0,y_0) + (x-x_0)f_x(x_0,y_0) + (y-y_0)f(x_0,y_0) + \frac{1}{2!}\Big[(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\Big]$$

+ Higher order term *

$$f(x,y) = f(x,y_0) + (y-y_0)f_y(x,y_0) + \frac{(y-y_0)^2}{2!}f_{yy}(x,y_0) + \frac{(y-y_0)^3}{3!}f_{yyy}(\eta_1,y_1)$$

$y_0 < \eta_1 < y$ — ①

$$f(x,y_0) = f(x_0,y_0) + (x-x_0)f_x(x_0,y_0) + \frac{(x-x_0)^2}{2!}f_{xx}(x_0,y_0) + \frac{(x-x_0)^3}{3!}f_{xxx}(\eta_2,y_0)$$

$x_0 < \eta_2 < x$ — ②

$$f_y(x,y_0) = f_y(x_0,y_0) + (x-x_0)f_{yx}(x_0,y_0) + \frac{(x-x_0)^2}{2!}f_{yxx}(\eta_3,y_0)$$

$x_0 < \eta_3 < x$ — ③

$$f_{yy}(x,y_0) = f_{yy}(x_0,y_0) + (x-x_0)f_{yyx}(\eta_4,y_0), \quad x_0 < \eta_4 < x$$ — ④

*

$$= \frac{1}{3!}\Big[(x-x_0)^3 f_{xxx}(\,) + 3(x-x_0)^2(y-y_0)f_{xxy}(\,) + 3(x-x_0)(y-y_0)^2 f_{xyy}(\,) + (y-y_0)^3 f_{yyy}(\,)\Big]$$

✓→ All these are bounded

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$$R_2 = \frac{(\Delta\rho)^3}{3!}\left[\left(\frac{x-x_0}{\Delta\rho}\right)^3 f_{xxx}(\,) + 3\left(\frac{x-x_0}{\Delta\rho}\right)^2\left(\frac{y-y_0}{\Delta\rho}\right) f_{xxx}(\,) + 3\,\frac{x-x_0}{\Delta\rho}\left(\frac{y-y_0}{\Delta\rho}\right)^2 f_{xyy}(\,) + \left(\frac{y-y_0}{\Delta\rho}\right)^3 f_{yyy}(\,)\right]$$

$$= \alpha \times (\Delta\rho)^3 \qquad \Delta\rho = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

↳ It is bounded

$R_2 \to 0$ as $\Delta\rho \to 0$ as α is bounded.

$f(x,y) = \ln(x^2+y^2-1)$ at $(x_0, y_0) \to (1,1)$

$$f_x(1,1) = \lim_{h\to 0} \frac{f(h+1,1) - f(1,1)}{h} = \lim_{h\to 0} \frac{\ln((h+1)^2)}{h} = 2\lim_{h\to 0}\frac{\ln(h+1)}{h} = 2$$

$f(x,y) = (x-1) f_x(1,1) + (y-1) f(1,1) + 0$

$= (x-1)(2) + (y-1)\,2$

$= 2x + 2y - 4$

8) $f(x,y) = \frac{x}{y} + \frac{y}{x}$

what is $x f_x + y f_y = ?$

A) $f_x = \frac{1}{y} - \frac{y}{x^2}$

$f_y = \frac{1}{x} - \frac{x}{y^2}$

$x f_x + y f_y = \cancel{\frac{x}{y}} - \cancel{\frac{y}{x}} + \cancel{\frac{y}{x}} - \cancel{\frac{x}{y}}$

$= 0$

8) If $u = \underbrace{\frac{(ax^3+by^3)^n}{3n(n-1)}}_{f_1} + \underbrace{x f\left(\frac{y}{x}\right)}_{f_2}$, Then P.T

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = (ax^3+by^3)^n$$

$u = u_1 + u_2$

Using Euler's Theorm,

$$x^2 (u_1)_{xx} + 2xy(u_1)_{xy} + y^2 (u_1)_{yy} = 3n(n-1)\frac{(ax^3+by^3)^n}{(3n)(3n-1)}$$

$$\Rightarrow x^2 (u_1)_{xx} + 2xy(u_1)_{xy} + y^2(u_{1yy}) = (ax^3+by^3)^n$$

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Minima & Maxima

Consider a fn: $f : [a,b] \to R$

Let $x = x_0$

$f(x_0+h) < f(x_0)$ $\forall$ small h

Then x_0 is called local max^m

$f(x_0+h) > f(x_0)$ $\forall$ small h

x_0 is called local min.

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = 0$$

$$\Rightarrow f'(x_0) = 0.$$

$$\Delta f = f(x_0+h) - f(x_0) \simeq h\cancel{f'(x_0)}^{0} + \frac{h^2}{2!} f''(x_0)$$

If $f''(x) > 0 \Rightarrow \Delta f > 0$

$\Rightarrow f(x_0+h) > f(x_0)$

$\Rightarrow x_0$ is local min.

If $f''(x) < 0 \Rightarrow \Delta f < 0$

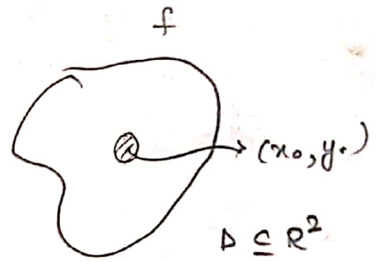
$\Rightarrow f(x_0+h) < f(x_0)$

$\Rightarrow x_0$ is local max^m.

Local extremum

In Two Variables

Suppose (x_0, y_0) is a pt. of extremum for f



$\Delta f = f(x_0+h, y_0+k) - f(x_0, y_0) > 0$

(x_0, y_0) will be a pt. of local min^m.

If $\Delta f < 0$

(x_0, y_0) will be a pt. of local ~~min~~ max^m.

Consider

$F(x, y = y_0)$ is a fn: of single variable

$$\Rightarrow \frac{\partial f(x_0, y_0)}{\partial x} = 0$$

$F(x = x_0, y)$ is also a fn: of a single variable

$$\Rightarrow \frac{\partial f(x_0, y_0)}{\partial y} = 0$$

Only necessary

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Consider $f(x,y) = x^2 - y^2)$

$f_x = 2x$ $\quad$ $(0,0)$ is critical pt

$f_y = -2y$ $\quad$ Here for some (h,k) f is +ve while for some (h,k) f is -ve

Saddle pt. is the point where f has neither maxm. nor min m.

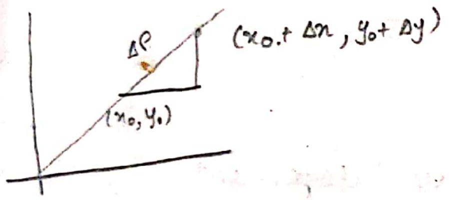
Suppose (x_0, y_0) is critical pt. for $f(x,y)$

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 = \frac{\partial f}{\partial y}(x_0, y_0)$$

$$f(x_0 + \Delta x,\ y_0 + \Delta y) = f(x_0, y_0) + \Delta x\, f_x(x_0, y_0) + \Delta y\, f_y(x_0, y_0) + \frac{1}{2!}\Big[\underbrace{f_{xx}(x_0; y_0)}_{A}\Delta x^2 + 2\Delta x\,\Delta y\,\underbrace{f_{xy}(x_0, y_0)}_{B} + \overbrace{f_{yy}}^{C}(\Delta y)^2\Big] + \alpha_0 [\Delta \rho]^3$$

$$\Delta f = f(x_0 + \Delta x,\ y_0 + \Delta y) - f(x_0, y_0) = \frac{1}{2!}\left[A(\Delta x)^2 + 2\Delta x\,\Delta y\, B + C(\Delta y)^2\right] + \alpha_0 (\Delta \rho)^3$$

$$\Delta \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2} \qquad \alpha_0 = \text{bounded quantity}$$



$$\Delta x = \Delta \rho \cos\phi$$

$$\Delta y = \Delta \rho \sin\phi$$

$$\Delta f = \frac{1}{2}\left[A(\Delta\rho)^2\cos^2\phi + 2B(\Delta\rho)^2\cos\phi\sin\phi + C(\Delta\rho)^2\sin^2\phi\right] + \alpha_0(\Delta\rho)^3$$

$$\Delta f = \frac{(\Delta\rho)^2}{2}\left[\frac{A}{A}(A\cos^2\phi + 2B\cos\phi\sin\phi) + C\sin^2\phi + 2\alpha_0(\Delta\rho)\right]$$

$$\Delta f = \frac{(\Delta\rho)^2}{2}\left[\frac{(A\cos\phi + B\sin\phi)^2 + (AC - B^2)\sin^2\phi}{A} + 2\alpha_0\Delta\rho\right] \quad (A \neq 0)$$

Case - I: $AC - B^2 > 0$; $A > 0$

$\Delta f > 0 \Rightarrow (x_0, y_0)$ is a pt. of local min.

$$AC - B^2 = f_{xx}(x_0, y_0) \times f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

Case 2: $AC - B^2 > 0$, $A < 0$

$\Delta f < 0 \Rightarrow (x_0, y_0)$ is pt. of local maxm.

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Case - 3: $AC-B^2<0$, $A \neq 0$

1) $A>0$

If $\phi=0 \Rightarrow \Delta f>0$

If ϕ is such that $A\cos\phi + B\sin\phi = 0 \Rightarrow \Delta f<0$

In this situation (x_0, y_0) is a "Saddle pt."

2) $A<0$

Similarly as the above case

$\therefore$ $AC-B^2<0 \Rightarrow$ saddle pt.

3) $A=0$

$$\Delta f = \frac{(\Delta\rho)^2}{2}\left[\sin\phi\left(\underbrace{2B\cos\phi + C\sin\phi}\right) + 2\alpha_0\Delta\rho\right]$$

when we consider small ϕ's $\Rightarrow$ The braces break down to $2B$ as $\cos\phi\to 1$, $\sin\phi \to 0$

Let's take $\phi>0$

$\Rightarrow$ Δf will be +ve

If $\phi<0 \Rightarrow \Delta f <$ -ve

$\therefore$ It's a saddle pt.

Case 4: $AC-B^2=0$

$\Rightarrow$ Further investigation requried

Coz the value of Δf becomes completely dependent on α which is bounded.

Q) Find local maxm. and min:

A) $f(x,y) = x\sin y$

$f_x = \sin y$ $\quad f_{xx}=0$ $\quad f_{xy} = \cos y$.

$f_y = x\cos y$ $\quad f_{yy} = -x\sin y$

$A=0$

$B = 2\Delta x\Delta y\cos y$

$C = \cos y$

$AC-B^2<0 \Rightarrow$ Saddle pt.

$(0, n\pi) \rightarrow$ critical saddle pt.

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(vi) $f(x,y) = (x^2 - y^2)\, e^{-\frac{x^2 - y^2}{2}}$

Let f_x = [illegible]

$f_y = -2y\, e^{-\frac{x^2-y^2}{2}}$ [illegible]

[illegible]

$\Rightarrow y\,[\,2 - (x^2 - y^2)\,] = 0$ [illegible]

$x^2, y^2 = 2$

$\therefore$ critical pt's are
$(0,0)$ $\quad (\pm\sqrt{2}, 0)$
$(0, \pm\sqrt{2})$

Method of Langrange Multiplier

Let $f(x,y)$ $\longleftarrow$ To find : Maxm. and Min ①

Subject to $\phi(x,y) = 0 \rightarrow$ side cond. - ②

$$\frac{du}{dx} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad - ③$$ (True at all extremum points)

From ② $\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0$ - ④ (True at all the pts. satisfying ②)

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right) + \lambda\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}\right) = 0$$

(λ is arbitrary parameter)

(True at all extremum pts. as well as all pt. that satisfy ②)

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right) + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial f}{\partial y}\right)\frac{dy}{dx} = 0 \quad - ⑤$$

Choose λ such that at the extremum pts.

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial f}{\partial y} = 0 \quad - ⑥$$, provided $\frac{\partial f}{\partial y} \neq 0$

$$\Rightarrow \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \quad - ⑦$$

$F(x,y,\lambda) = f(x,y) + \lambda\,\phi(x,y) \rightarrow$ Parital derivative with λ, y, x Then we get eqn. ②, ⑥, ⑦ and λ is called Langrange's Multiplier

$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial x} =$

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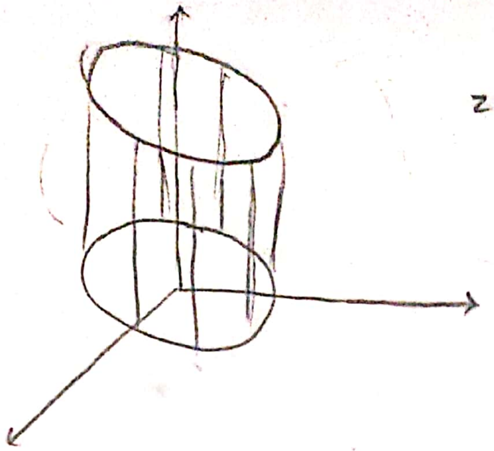


$z = f(x,y)$ Min/Max

Show that $x^2+y^2=1$, $\phi(x,y)=0$

$\phi(x,y) = x^2+y^2-1$

$F(x,y,\lambda) = f(x,y) + \lambda\,\phi(x,y)$

Solve for (x,y,λ)

$\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$

$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$

$\frac{\partial F}{\partial \lambda} = 0 \rightarrow \phi(x,y) = 0$

langranges Multiplier can only give the points and not their nature

8) Find maxm./minm. for $f(x,y) = x^2-y^2$, show that $g(x,y)=0$ where $g(x,y) = x^2+y^2-1$

A) $F(x,y,\lambda) = x^2-y^2+\lambda(x^2+y^2-1)$

$\frac{\partial F}{\partial x} = 2x + 2\lambda x = 0 \Rightarrow \lambda = -1,\ x=0$

$\frac{\partial F}{\partial y} = -2y + 2\lambda y = 0 \Rightarrow \lambda = 1,\ y=0$

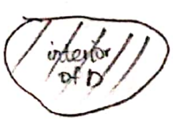
$\frac{\partial F}{\partial \lambda} = x^2+y^2-1=0$

$\therefore$ Points are $(\pm1, 0)$, $(0, \pm1)$

Note :-

→ D is domain whose boundary is C (smooth closed curve).

To Find Max/Min for $z = f(x,y)$ on D



Step - 1

$\left.\begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix}\right\}$ Find critical pts. in the interior of D

Step - 2

Find extremum on C using langranges Multiplier method

Step - 3

compare all those values to find abesoulte (global) maxm./minm.

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8) Find ab. maxm / minm for $x^2+y^2 \le 1$ $x = y$

(i) $f(x,y) = xy$

in $D = \{(x,y) : x^2+y^2 \le 1\}$

$F(x,y,\lambda) = xy + \lambda(x^2+y^2-1)$

$\frac{\partial F}{\partial x} = y + 2\lambda x = 0$ $y + 2\lambda x = 0$

$\lambda = +\frac{y}{2x} = +\frac{x}{2y}$ $y^2 = x^2$

$\frac{\partial F}{\partial y} = x + 2\lambda y = 0$ $(0,0)$ will be a saddle pt.

$\frac{\partial F}{\partial \lambda} = x^2+y^2 = 1$ $x = \frac{1}{\sqrt{2}}$ $\Rightarrow \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

Global maxm → $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ & $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $= 1/2$

Global minm → $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ & $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ $= -1/2$

(ii) $f(x,y) = \frac{x^2+y^2}{2}$

s.t $\frac{x^2}{2} + y^2 \le 1$

$F(x,y,\lambda) = \frac{x^2+y^2}{2} + \lambda\left(\frac{x^2}{2} + y^2\right)$

$\frac{\partial F}{\partial x} = \frac{2x}{2} + \lambda x = 0$ $x = 0, \lambda = -1$

$\frac{\partial F}{\partial y} = \frac{2y}{2} + 2y\lambda = 0$ $y = 0, \lambda = -\frac{1}{2}$

$\frac{\partial F}{\partial \lambda} = \frac{x^2}{2} + y = 1$

8) Suppose f is twice diff and $f(x) \le 0$, $f'(x) \ge 0$, $f''(x) \ge 0$

P.T f is constant

A) $f'(x)$ is increasing

let $f(x)$ be not const., then $\exists x_0$ such that $f'(x_0) > 0$

Consider an interval $[x_0, P]$

Applying LMVT

$f(P) - f(x_0) = (P - x_0) f'(c)$ $x_0 < c < P$

$\Rightarrow f'(x_0) \le f'(c)$ As f' is ↑

$\underset{-ve}{f(P)} \ge \underset{-ve}{f(x_0)} + \underset{+ve}{(P-x_0)} \underset{+ve}{f'(x_0)}$

$\Rightarrow$ Only possibility is $P = x_0$

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Differential Equations

↳ Easyily scorable
↳ No silly mistake.

Ordinary Differential Eqn:

A diff eqn: which involves single independent variable

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = f(x)$$

$$\frac{dy}{dx} = f(x,y)$$

Order of a Differential Eqn:

It is the order of the highest order derivative invowed in the DE

Degree of a Differential Eqn:

It is exponent of the highest order differential eqn: which from radicals and ractions

(i) $\left(\frac{d^3y}{dx^3}\right)^{2/3} = \left(1 + 2\frac{dy}{dx}\right)$ Order - 3, Degree - 2

(ii) $y\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 + 1 = 0$ Order - 1, Degree - 2

~~Note~~: Picard's Exsistance and uniqueness Theorm

For $\frac{dy}{dx} = f(x,y)$

If the fn: $f(x,y)$ is cont. in the neighbour hood of (x_0, y_0) and satisfies

$$|f(x,y_1) - f(x,y_2)| < M|y_1 - y_2| \text{ for some } M > 0$$

The diff eqn: has unique soln:

Lipsitz cont w.r.t y variable is req. for uniqness

How to solve a Diff. eqn.?

Variable Seperable

If the eqn: is of the form $\frac{dy}{dx} = f_1(x) f_2(y)$

Then $\frac{dy}{dx} = f_1(x) f_2(y)$

$$\Rightarrow \int \frac{dy}{f_2(y)} = \int f_1(x)\,dx$$

$g_2(y) = g_1(x) + C$ is the soln:

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Homogeneous ~~Eq~~ Diff. Eqn.

If $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$, and $f_1(x,y)$ and $f_2(x,y)$ are homogeneous fn. of same degree then the eqn. is called homogeneous diff. eqn.

To solve such eqn., we use $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + \frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{f_1(v)}{f_2(v)} = g(v)$$ and now it can solved by variable seperable.

Q) $x^2 dy + y(x+y)dx = 0$

A) $-\frac{y(x+y)}{x^2} = \frac{dy}{dx}$ $\qquad y = vx$

$$v + \frac{dv}{dx} = -\frac{\cancel{x^2}(1+v)v}{\cancel{x^2}}$$

$$\frac{dv}{dx} \; \frac{dv}{dx} = v^2 + 2v$$

$$\int \frac{dv \; dv}{v(2+v) \; 2v} = -\int \frac{dx}{x}$$

$\frac{a}{v} + \frac{b}{(2+v)}$

$a(2+v) + bv = \frac{1}{v(2+v)}$

$2a + v(a+b) = 1$

$a = -1/2$

$b = 1/2$

$$-\int \frac{dv}{2v} + \frac{1}{2}\int \frac{dv}{2+v} = -\int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2}\ln v + \frac{1}{2}\ln(2+v) = -\ln x + c$$

If it is of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$$ a, b, c, a', b', c' are consts.

(Assume $\frac{a}{a'} \neq \frac{b}{b'}$)

Making the substitution $X + h = x$, $Y + k = y$ $\Rightarrow$ $dx = dX$, $dy = dY$

h, k are constants which are to be found

$$\frac{dY}{dX} = \frac{a(X+h) + b(Y+k) + c}{a'(X+h) + b'(Y+k) + c'} : \frac{aX + bY + (ah+bk+c)}{a'X + b'Y + (a'h+b'k+c')}$$

Choose (h,k) such that $\left.\begin{array}{l} ah+bk+c=0 \\ a'h+b'k+c'=0 \end{array}\right\}$ Solve for (h,k)

$$\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y} \leftarrow \text{Homogeneous D.E}$$

$Y = f(X)$

Now $X + h = x$

$Y + k = y$

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Q) $\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$

A) $h+2k+3=0 \Rightarrow 2h+4k+6=0$ $\qquad h-4+3=0$

$2h+3k+4=0 \quad (-)\ 2h+3k+4=0$ $\qquad h=1$

$k+2=0$

$k=-2$

$O'(1,-2)$ $\qquad X+1=x$

$y=Y-2$

$\frac{dY}{dX} = \frac{X+2Y}{2X+3Y}$

$\frac{2+3v}{1-3v^2} = \frac{A}{(1+\sqrt{3}v)} + \frac{B}{(1-\sqrt{3}v)}$

$v+\frac{dv}{dx}(x) = \left(\frac{1+2v}{2+3v}\right)$

$A-\sqrt{3}Av+B+\sqrt{3}Bv$

$(x)\frac{dv}{dx} = \frac{1+\cancel{2v}-\cancel{2v}-3v^2}{2+3v}$

$A+B=2$

$\sqrt{3}(B-A)=3\sqrt{3}$

$\frac{(2+3v)}{(1-3v^2)}dv = \frac{dx}{x}$ $\qquad \frac{1}{2a}\ln\left|\frac{1-\sqrt{3}v}{1+\sqrt{3}v}\right|$

$2B = 2+\sqrt{3}$

$B = \frac{2+\sqrt{3}}{2}$

$1+\frac{\sqrt{3}}{2}$

$\left(1\mp\frac{\sqrt{3}}{2}\right)\ln(1+\sqrt{3}v) + \left(1+\frac{\sqrt{3}}{2}\right)\ln(1\mp\sqrt{3}v)$ //

$= \ln x + c$

$v^2=t$

$3v^2=t$

$6v\,dv = dt$

$2\int\frac{\cancel{1}\,dv}{1-3v^2} + \int\frac{3v\,dv}{1-3v^2} = \ln x + c$

$\cancel{2}\ln\left|\frac{1-\sqrt{3}v}{1+\sqrt{3}v}\right| + \frac{1}{2}\int\frac{2v\,dv}{1-3v^2} = \ln x + c$

$\Rightarrow \ln\left|\frac{1-\sqrt{3}v}{1+\sqrt{3}v}\right| + \frac{1}{2}\ln|1-\sqrt{3}v| = \ln x + c$ //

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Linear First order Diff. Eqn :-

$$\frac{dy}{dx} + p(x)\,y = q(x)$$

consider a fn.

$$f(x) = e^{\int p\,dx}$$

Multiplying both sides with $f(x)$

$$\Rightarrow e^{\int p\,dx}\frac{dy}{dx} + e^{\int p\,dx}p(x)\,y = q(x)\,e^{\int p\,dx}$$

$$\frac{d}{dx}\left[y\,e^{\int p\,dx}\right] = q\,e^{\int p\,dx}$$

Integrating on both sides

$$y\,e^{\int p\,dx} = \int\left[Q\,e^{\int p\,dx}\right]dx + c$$

$\therefore$ Soln. is

$$y = e^{-\int p\,dx}\left[\int Q\,e^{\int p\,dx}\,dx + c\right]$$

c - arbitrary const

Q) $x^2(x^2-1)\frac{dy}{dx} + x(x^2+1)\,y = (x^2-1)$

A) $\frac{dy}{dx} + \frac{(x^2+1)}{(x^2-1)x}\,y = \frac{1}{x^2}$

$$\frac{x^2+1}{(x^2-1)x} = \frac{A}{x+1} + \frac{B}{x}$$

$$I.F = e^{\int p\,dx}$$

$$e^{\int \frac{x^2+1}{(x^2-1)x}dx} = e^{\log\left(\frac{x^2-1}{x}\right)} = \frac{x^2-1}{x}$$

$$\frac{x^2-1}{x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3}$$

$Ax^2 + Bx + C$

$B = 0$

$A = 1$

$C = -1$

$$\Rightarrow y\left(\frac{x^2-1}{x}\right) = \int \frac{x^2-1}{x^3}\,dx$$

$$= \int \frac{1}{x}\,dx + \int -\frac{1}{x^3}\,dx$$

$$\Rightarrow y\left(\frac{x^2-1}{x}\right) = \ln x + \frac{1}{2x^2} + c$$

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$$\frac{dy}{dx} + Py = qy^n$$

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = q \quad -①$$

$$y^{1-n} = z$$

$$(1-n)y^{-n}\frac{dy}{dz} = \frac{dz}{dx} \Rightarrow y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx} \quad -②$$

From ① & ②

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = q$$

$$\Rightarrow \frac{dz}{dx} + (1-n)Pz = (1-n)q$$

Q) Solve $\frac{dy}{dx} - xy = x^3y^2$

$y^{1-2} = z \Rightarrow z = \frac{1}{y}$

A) $\frac{dz}{dx} + (1-2)xz = (-1)x^3$

$$\Rightarrow \frac{dz}{dx} - xz = -x^3$$

$$I \cdot F = e^{\int -x\,dx} = e^{-\frac{x^2}{2}dx}$$

$$\Rightarrow \frac{d}{}ze^{-x^2/2} = -\int x^3e^{-x^2/2}dx$$

$$\frac{e^{-x^2/2}}{y} = x^2 + 2 + ce^{-1/2x^2}$$

$x^3 \quad e^{-x^2/2}$

$3x^2 \quad -\frac{e^{-x^2/2}}{2}(2x)$

$6x \quad +\frac{e^{-x^2/2}}{4}4x^2$

$6 \quad -\frac{e^{-x^2/2}}{8}8x^3$

$+\frac{e^{-x^2/2}}{16}16x^4$

Exact Differential Eqn:

$$M(x,y)dx + N(x,y)dy = 0 \quad -①$$

A differential form $Mdx + Ndy$ is called "exact" if there exists a diffable fn $U(x,y)$ such that

$$du = Mdx + Ndy$$

Necessary and sufficient condition for exact diff.

① is exact DE iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

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Proof :-

Suppose ① is exact $\exists\, U$ such that
$$dU = M\,dx + N\,dy$$

$$\Rightarrow \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = M\,dx + N\,dy$$

$$M = \frac{\partial U}{\partial x} \quad \& \quad N = \frac{\partial U}{\partial y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial x \partial y} \quad \& \quad \frac{\partial N}{\partial x} = \frac{\partial^2 U}{\partial y \partial x}$$

Assuming U_{xy}, U_{yx} are cont

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Let $U = \int M\,dx$

Then $\frac{\partial U}{\partial x} = M$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)$$

$$N = \frac{\partial U}{\partial y} + f(y) \quad -②$$

$$M\,dx + N\,dy = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + f(y)\,dy$$

$$= d\left[U + \int f(y)\,dy\right]$$

$\Rightarrow$ D.E ① is exact.

$$U + \int f(y)\,dy = c$$

$$\Rightarrow \int M\,dx + \int f(y)\,dy = c$$

$$\Rightarrow \int M\,dx + \int \{\text{Those terms inde. of } x\}\,dy = c$$

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R

8) $(x^3 + 3xy^2)\,dx + (3x^2y + y^3)\,dy = 0$

$3x^2y$

4) $\frac{\partial M}{\partial x} = 3x^2 + 3y^2$

$\frac{\partial N}{\partial y} = 3x^2 + 3y^2$

$\int M\,dx + \int (\text{terms ind. of } x)\,dy = 0$

$\int (x^3 + 3xy^2)\,dx + \int y^3\,dy = 0$

$\Rightarrow \frac{1}{4}x^4 + \frac{6x^2y^2}{4} + \frac{y^4}{4} = c$

$\Rightarrow \frac{1}{4}(x^4 + 6x^2y^2 + y^4) = c$

Non-Exact Differential Eqn:

$M\,dx + N\,dy = 0 \quad - (1)$

Rule-1 where $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ → Eqn (1) isn't exact

Multiplying by $\mu(x)$ [Fn. of x ony] → Integrating Factor

$M\mu\,dx + N\mu\,dy = 0$

$\frac{\partial(M\mu)}{\partial y} = \frac{\partial(N\mu)}{\partial x}$

$M_y\,\mu + M\cancel{\frac{d\mu}{dy}}^{0} = \mu N_x + N\frac{d\mu}{dx}$

$\Rightarrow M_y\,\mu = \mu N_x + N\frac{d\mu}{dx}$

$\Rightarrow N\frac{d\mu}{dx} = \mu\,(M_y - N_x)$

$\int \frac{d\mu}{\mu} = \int \left(\frac{M_y - N_x}{N}\right) dx$

Rule-2

$\mu(x) = \exp\left(\int \left(\frac{\frac{\partial N}{\partial y} - \frac{\partial U}{\partial x}}{N}\right) dx\right)$ → If this is a fn. of x only then μ is an integrating factor

$\mu(y) = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\,dy\right)$ → If this a fn. of y only Then this is int. factor.

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Rule - 3

$M\,dx + N\,dy = 0$ is of the form

$f(xy)\,y\,dx + \phi(xy)\,x\,dy = 0$

Then $\frac{1}{Mx - Ny}$ is an integrating factor

provided $Mx - Ny \neq 0$

Rule - 4

$x^a y^b [my\,dx + nx\,dy] + x^{a'} y^{b'} (m'y + n'x\,dy) = 0$

Then $I \cdot F = x^h y^k$ where h, k are integers such that

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}$$

&

$$\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

Rul - 5

$M\,dx + N\,dy = 0$

M, N are homogeneous eqn. of same degree

and $Mx + Ny \neq 0$, then $I \cdot F = \frac{1}{Mx + Ny}$

First order D. E

Recall

$\frac{dy}{dx} = f(x, y)$

$y = \phi(x) + c \rightarrow$ General soln.

$M\,dx + N\,dy = 0$

Exact diff $\rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$du = M\,dx + N\,dy = 0$

$\Rightarrow u = c \rightarrow$ soln.

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \simeq \varphi(x)$ $\quad I \cdot F \rightarrow e^{\int \varphi(x)\,dx}$

$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \simeq \phi(y)$ $\quad I \cdot F \rightarrow e^{\int \phi(y)\,dy}$

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$P = P(x)$
$Q = Q(x)$

$$\frac{dy}{dx} + Py = Q \Rightarrow \underbrace{\left(\frac{d}{dx} + P\cdot I\right)}_{D}(y_1 + y_2) = Dy_1 + Dy_2$$

Homogeneous Linear Differential equations of second order

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

where $P \equiv p(x)$
$Q \equiv Q(x)$

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}$$

$$\Rightarrow (D^2 + PD + QI)\,y = 0$$

If y_1, y_2 are the soln of the above eqn.
Then the general soln is

$$y = c_1 y_1 + c_2 y_2$$

where c_1, c_2 are constants.

(y_1, y_2) are called linearly independent if one isn't a scalar multiple of other

i.e $\quad y_1 \neq k y_2$

If this is the case, then we call y_1 & y_2 are linearly dependent.

→ Degree of polynomial of atmost n form the vector space (i.e $1, x, x^2 \dots$) all are independent
Dimension of these is $(n+1)$

If $\{y_1, y_2 \dots y_n\}$ are linearly independent

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$$

iff, $c_1 = c_2 = c_3 \dots c_n = 0$

Then $\{y_1, y_2 \dots y_n\}$ is linearly independent

If one or more $c_i \neq 0$ then $\{y_1, \dots y_n\}$ is linearly dependent.

If one of $y_i = 0$, the set becomes linearly dependent as

let $y_1 = 0$, $\quad \underset{\neq 0}{c_1} \cdot \underset{0}{y_1} + 0 \cdot y_2 + 0 \cdot y_3 + \dots = 0$

$\Rightarrow$ linearly dependent

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Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} \neq 0 \Rightarrow \text{linearly independent}$$

If wronskian is zero ⇒ linearly dependent

Basis of a soln:

For ①, if y_1 & y_2 are linearly independent they will form basis of soln: space. $\{y_1, y_2\}$

$$y = c_1 y_1 + c_2 y_2$$

where c_1, c_2 are arbitrary const.

If y_1 is soln: and (constant) y_1 is also a soln: , then the D. E is called homogeneous D. E

→ 0 is never included in the basis of the soln:

How to find the basis?

Reduction of order method

Suppose y_1 is a soln: of ① (By inspection)

$$y_1'' + py_1' + qy_1 = 0 \quad -(*)$$

Let y_1, y_2 be the soln: of the D. E

$$y_2'' + py_2' + qy_2 = 0$$

$p \equiv p(x)$

$q \equiv q(x)$

let $u \equiv u(x)$, then

$$y_2 = u(x) y_1$$

$$\Rightarrow y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

$$\Rightarrow u''y_1 + 2u'y_1' + \cancel{uy_1''} + pu'y_1 + \cancel{puy_1'} + \cancel{quy_1} = 0 \rightarrow (\text{From } *)$$

$$\Rightarrow * \; u''y_1 + (2y_1' + py_1)u' = 0 \qquad u(y_1'' +$$

let $U(x) = u'$

$$\Rightarrow * \; U'(x) y_1 + (2y_1' + py_1) U = 0 \leftarrow \text{First order diff. eqn.}$$

solve for $U(x)$

and $* \; u(x) = \int U(x)\,dx$

and $* \; y_2 = u(x) y_1$

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~~er~~ linear homogeneous eqn: with constant co-eff

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

Aim to find several soln:

Let $y = e^{\lambda x}$ → Derivate becomes easy

where ↳ Only such a fn: can form the above D.E

λ is any scalar

$$\Rightarrow \frac{dy}{dx} = \lambda e^{\lambda x}$$

$$\frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}$$

$$\Rightarrow (\lambda^2 + a\lambda + b)\, e^{\lambda x} = 0$$

↳ never zero

$\Rightarrow \lambda^2 + a\lambda + b = 0$ → This eqn: has two values

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Case 1 :- λ_1, λ_2 are distinct real variables

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

Case 2 :- $a^2 - 4b = 0$

$\lambda, \lambda \Rightarrow \{ e^{\lambda x}, e^{\lambda x} \}$

$$\Rightarrow y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

Case 3 :- $a^2 - 4b < 0$, $i = \sqrt{-1}$

$\lambda = \lambda_1 + i\lambda_2$, $\lambda - i\lambda_2$

$\{ e^{(\lambda_1 + i\lambda_2)x}, e^{(\lambda_1 - i\lambda_2)x} \}$

$$y = c_1 e^{(\lambda_1 + i\lambda_2)x} + c_2 e^{(\lambda_1 - i\lambda_2)x}$$

$$= c_1 e^{\lambda_1 x}[\cos\lambda_2 x + i\sin\lambda_2 x] + c_2 e^{\lambda_1 x}[\cos\lambda_2 x - i\sin\lambda_2 x]$$

$$= e^{\lambda_1 x}(A\cos\lambda_2 x + B\sin\lambda_2 x)$$

A, B are arbitrary const (can be real / imaginary) next ;

b ;

A & B are arbitrary co

$e^{\lambda_1 x} \cos \lambda_2 x$ if add and sub by 2

$e^{\lambda_1 x} \sin \lambda_2 x$ " " ... by $2i$

Q) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$, $y(0) = 3$, $y'(0) = 1$

A) $\{e^{2x}, x e^{2x}\}$

$y = c_1 e^{2x} + c_2 x e^{2x}$

$y \Rightarrow 3 = c_1$

$y = 3e^{2x} + c_2 x e^{2x}$

$y' = 6e^{2x} + e^{2x}(c_2 x + c_2)$

$1 = 6 + c_2 \Rightarrow c_2 = -5$

$y = 3e^{2x} - 5e^{2x}(x)$

Q) $16\frac{d^2y}{dx^2} - 8y' + 5y = 0$

A) $\Rightarrow \frac{d^2y}{dx^2} - \frac{y'}{2} + \frac{5y}{16} = 0$

$$\lambda = \frac{+\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{5}{4}}}{2}$$

$$\Rightarrow \lambda = \frac{+\frac{1}{2} \pm \sqrt{-1}}{2} = \frac{1}{4} \pm \frac{i}{2}$$

$y = c_1 e^{\frac{x}{2}} + c_2 e^{-ix}$

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Note :-

→ $a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$

↳ $\{y_1, y_2 \cdots y_n\}$ are linearly independent soln. of the equation

Then they form the basis for the soln.

And any soln. of the equation can be written as

$y = c_1 y_1 + c_2 y_2 + \cdots c_n y_n$, where c_i are arbitrary constants and $i \in [1, n]$

↳ There are n arbitrary constants

The diff. eqn. is a homogeneous diff. eqn.

As $y = 0$ is always a soln. of the above eqn.

If n intial conditions are given, then we can find the n arbitrary constants.

$$\left.\begin{aligned} y(0) &= y_0 \\ y'(0) &= y_1 \\ &\vdots \\ y^{n-1}(0) &= y'_{n-1} \end{aligned}\right\} \text{n - intial value conditions}$$

$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = 0$

can be written as $(a_n D^n + a_{n-1} D^{n-1} + \cdots a_1 D + a_0 I) y = 0$

Where $D^n = \frac{d^n}{dx^n}$

→ $W\{y_1, \cdots y_n\} = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \vdots \\ y_1^n & y_2^n & \cdots & y_n^n \end{vmatrix}$ $\neq 0$ ⇒ Independent

$= 0$ ⇒ Dependent

If W is 0 even for one pt. in $I(-\alpha, \alpha)$, then it is zero in $(-\alpha, \alpha)$ i.e for all points $\in (-\alpha, \alpha)$ $W = 0$

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→ $ay'' + by' + cy = 0$, $a, b, c \in$ constants

Let $y = e^{mx}$ be a trial soln.

The Auxilary eqn. is $am^2 + bm + c = 0$

(i) $m_1 \neq m_2$, $y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ $c_1, c_2 \in$ constants

(ii) $m_1 = m_2 = m$, $y_h = (c_1 + c_2 x) e^{mx}$

(iii) $m_i = \alpha \pm i\beta$, $y_h = e^{\alpha x} (c_1 \cos\beta x + c_2 \sin\beta x]$

→ If the above eqn. is of higher order i.e uptil n^{th} derivative,

If the auxillary eqn. has

(i) n distinct roots Soln. is

$(m_1 \neq m_2 \neq \cdots \neq m_n)$

$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots c_n e^{m_n x}$

(ii) If r roots are repeated and $r < m$

Soln. is

$$y_h = \underbrace{(c_0 + c_1 x + \cdots + c_{r-1} x^{r-1})}_{r \text{ const's}} e^{mx} + \underbrace{c_r e^{m_1 x} + \cdots + c_{n-r} e^{m_{n-r} x}}_{(n-r) \text{ consts.}}$$

(iii) If r roots are repeated and all are complex

$$y_h = e^{\alpha x} \left[(c_0 + c_1 x + \cdots c_{r-1} x^{r-1}) \cos\beta x + (d_0 + d_1 x + \cdots + d_{r-1} x^{r-1}) \sin\beta x \right]$$

Example :-

1) $y''' - y'' + 100y' - 100y = 0$

A) $\lambda^3 - \lambda^2 + 100\lambda - 100 = 0$ is the auxilary eqn.

$\Rightarrow \lambda = 1$ ✓

$$\Rightarrow \begin{array}{r|l} \lambda - 1 & \lambda^3 - \lambda^2 + 100\lambda - 100 \; (\lambda^2 + 100 \\ & \underline{(-)\lambda^3 \overset{(+)}{-} \lambda^2} \\ & 100\lambda - 100 \\ & \underline{(-) 100\lambda \overset{(+)}{-} 100} \\ & 0 \end{array}$$

$\lambda = 10i$
$\lambda = -10i$

$f(\lambda) = (\lambda - 1)(\lambda^2 + 100)$

$= (\lambda - 1)(\lambda - 10i)(\lambda + 10i)$

$\Rightarrow y = c_0 e^x + (c_1 \cos 10x + c_2 \sin 10x) e^0$

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Q) $y^{v} - 3y^{iv} + 3y''' - y'' = 0$

A) The auxiliary eqn is

$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$

$\lambda^2(\lambda^3 - 3\lambda^2 + 3\lambda - 1) = 0$, $\lambda = 0, \lambda = 0$

$\Rightarrow \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$, $\lambda = 1$

$$\begin{array}{r|l} \lambda - 1 & \lambda^3 - 3\lambda^2 + 3\lambda - 1 \;(\; \lambda^2 - 2\lambda + 1 \\ & \underline{(-)\lambda^3 \; (+)\lambda^2} \\ & -2\lambda^2 + 3\lambda \\ & \underline{(+)2\lambda^2 \; (-)2\lambda} \\ & \lambda - 1 \\ & \underline{(-)\lambda \; (+)1} \\ & 0 \end{array}$$

$f(\lambda) = (\lambda - 1)(\lambda - 1)^2$

$= (\lambda - 1)^3$

$y = (c_1 + c_2 x + c_3 x^2)e^x + (c_4 + c_5 x)$

$\uparrow$ $\lambda = 0$ twice

Cauchy - Euler Eqn:

$$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 0$$

How to solve?

A) Substitute $\boxed{z = \log x}$

$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}$

$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$

$\Rightarrow \boxed{x\frac{dy}{dx} = \frac{dy}{dz}, \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}}$

$\Rightarrow \frac{d^2y}{dz^2} - \frac{dy}{dz} - 3\frac{dy}{dz} + 4y = 0$

$\Rightarrow \frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = 0$ → Now it is sowalble

$(D^2 - 4D + 4)y = 0 \Rightarrow m_1 = 2$, $m_2 = 2$

soln is $y = (c_1 + c_2 z)e^{2z} \Rightarrow y = (c_1 + c_2 \log x)x^2$

Vardha

ifo stack;

Note:-

We can generalise the above method, as follows

$D \equiv \frac{d}{dz}$

$x\frac{dy}{dx} = Dy$

$x^2 \frac{d^2y}{dx^2} = (D^2 - D)y = D(D-1)y$

$\vdots$

$x^n \frac{d^n x}{dx} = D(D-1)(D-2)\cdots(D-n+1)y$

Coz if y is soln cy isn't a soln

$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = r(x)$ — ① (Non - Homogeneous)

$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = 0$ - ② (Homogeneous)

→ If y_1, y_2 is a general soln of ①, Then $y = y_1 - y_2$ is for sure a general soln of ②

(∵ r(x), r(x) gets cancelled when we subract both)

→ If y_h is general soln of ②, if y_p is a particular soln of ①, then $y = y_h + y_p$ is a general soln of ①

To solve the above NHD·E

Step - ① : Find general soln of Eqn ②

Step - ② : Find "1" soln (particular soln) of the Eqn ①
→ No arbitrary consts.

Step - ③ : Add both.

How to find a particular soln ?

Method of D- operator

1) The eqn can be written in the following form

$(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_n I)y = r(x)$, $D^n = \frac{d^n}{dx^n}$

consider this as a polynomial in D

$\Rightarrow f(D)y_p = r(x)$ → This is like $AX = D$, $A_{m\times n}[\ ]_{n\times 1} = [\ ]_{m\times 1}$

where $f(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots a_1 D + a_0 I$

Operator eqn — Treat similarly

fn of operators → acting on a vector

$AX = D$ → giving another vector
↳ Operator

$\Rightarrow \boxed{y_p = \frac{1}{f(D)} r(x)}$

↳ Inverse of f(D)

For eg :- f(D) is D

Then $\frac{1}{f(D)}$ is integration of r(x)

Scanned with CamScanner

$\Rightarrow \frac{1}{D} X = \int X$ w.r.t. x upto ...

$X = X(x)$

$\Rightarrow D(\int X) = X$ Inverse of differentiation ~~of~~ is integration

↳ We are getting back the X, then this is the inverse of D

→ What do we mean by $\frac{1}{D-\alpha} X = y$

$\Leftrightarrow (D-\alpha)y = X$

$\Rightarrow \frac{dy}{dx} - \alpha y = X$

$\Rightarrow y = e^{\alpha x}\left[\int X e^{-\alpha x}\right]$

$y = e^{\alpha x}\int X e^{-\alpha x} dx = \frac{1}{D-\alpha}(X)$

Eg :-

1) $(D-\alpha)^2 y = 0$

$\Rightarrow (D-\alpha)(D-\alpha)y = 0$

Let $(D-\alpha)y = z$

$(D-\alpha)z = 0 \Rightarrow z = c_1 e^{\alpha x}$

$(D-\alpha)y = c_1 e^{\alpha x}$

$\Rightarrow y = e^{\alpha x}[c_1 x + c_2]$

$\Rightarrow$

$$\underbrace{(a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 I)}_{f(D)} y = r(x).$$

$f(D) = (D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)$

$$\Rightarrow \boxed{y_p = \frac{1}{(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)} r(x)}$$

$$\frac{1}{(D-a)(D-b)} f(x) = \frac{1}{(D-a)}\left[\frac{1}{(D-b)} f(x)\right] = \frac{1}{(D-b)}\left[\frac{1}{(D-a)} f(x)\right]$$

We can also solve the above eqn: by using partial fractions

i.e

$$\boxed{\frac{1}{f(D)} X = \frac{1}{(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)}(X) = \left[\frac{a_1}{D-\alpha_1} + \frac{a_2}{D-\alpha_2} + \cdots \frac{a_n}{D-a_n}\right] X}$$

Vardhan

ruct

Rules for finding particular integral

(i) $X = e^{mx}$

$y_p = \frac{1}{f(D)} e^{mx}$

(ii) $X = \sin(mx)$ or $(\cos mx)$

$f(D)\, e^{mx} = f(n$

?? Didn't understand

$f(D) = (D-m)^r\, \phi(D), \quad \phi(m) \neq 0$

$$\frac{1}{f(D)} e^{mx} = \frac{1}{(D-m)^r}\left[\frac{1}{\phi(D)} e^{mx}\right] = \frac{1}{(D-m)^r} \frac{1}{\phi(m)} e^{mx}$$

$$= \frac{1}{(D-m)^r} \frac{1}{\phi(m)} e^{mx}$$

$$f(D) = (D-m)^r\, \phi(D) = \frac{1}{\phi(x)} \frac{x^r e^{mx}}{r!}$$

Eg:-

1) $(D^2+6D+9)y = 2e^{-3x}$, $D \equiv \frac{d}{dx}$

A) $m^2 + 6m + 9 = 0$

$(m+3)^2 = 0$

$\Rightarrow m = -3, -3$

$y_c = (c_1 + c_2 x)e^{-3x}$ $\quad c_1, c_2$ are constants

$y_p = \frac{1}{D^2+6D+9} 2e^{-3x}$

$= 2\left(\frac{1}{(D+3)^2}\right) e^{-3x} = 2 \times \frac{x^2}{2!} e^{-3x} = x^2 e^{-3x}$

$\Rightarrow$ The general soln. of the D.E is

$y = y_p + y_c$

$= x^2 e^{-3x} + (c_1 + c_2 x) e^{-3x}$ $\quad$ where c_1, c_2 are arbitrary constants.

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Rule 2 :- $X = \sin mx$ or $X = \cos mx$

$\frac{1}{f(D)} X$

Case 1 :-
Suppose $f(D)$ contains even powers of D only

$f(D) = \phi(D^2) = (D^2)^n + a_1(D^2)^{n-1} + \dots a_{n-1}D^2 + a_n$

$D(\sin mx) = m\cos mx$

$D^2(\sin mx) = -m^2 \sin mx$

$D^4(\sin mx) = (-m^2)^2 \sin mx$

$\vdots$

$D^6(\sin mx) = (-m^2)^3 \sin mx$

$f(D)\sin mx = \phi(D^2)\sin mx = \phi(-m^2)\sin mx$

$\Rightarrow \sin mx = \frac{1}{f(D)} \phi(-m^2)\sin mx$

$$\Rightarrow \boxed{\frac{1}{f(D)} \sin mx = \frac{\sin mx}{\phi(-m^2)}, \quad \phi(-m^2) \neq 0}$$

$f(D)$

$D^2 = -m^2$

$D =$

$\phi(D^2)$

$D^2 = -m^2$

$\phi(D^2)$ $f(D)$

$\boxed{D^2 = -m^2}$

$\phi(-m^2)$

$\frac{\sin mx}{\phi(-m^2)}$

Case 2 :-
Suppose $f(D)$ contains odd powers of D also

$\frac{1}{f(D)} \sin mx$

$= \frac{1}{\phi_1(D^2) + \phi_2(D^2) \times D} \sin mx$

$= \frac{1}{\underbrace{\phi_1(-m^2)}_{P} + D\underbrace{\phi_2(-m^2)}_{q}} \sin mx$

$D^2 = -m^2$

$= \frac{P - Dq}{P^2 + D^2q^2} \sin mx$

$= \frac{(P - Dq)}{P^2 + m^2q^2} \sin mx$

$$\therefore \boxed{\frac{1}{f(D)} \sin mx = \frac{P\sin mx - qm\cos mx}{P^2 + m^2q^2}}$$

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Eg :-

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin 3x$$

4) $(D^2 - 5D + 6)y = \sin 3x$

$$\lambda^2 - 5\lambda + 6 = 0$$
$$\lambda^2 - 3\lambda - 2\lambda + 6 = 0$$
$$= \lambda(\lambda - 3) - 2(\lambda - 3) = 0$$
$$= (\lambda - 2)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 2, 3$$

$$y_c = c_1 e^{2x} + c_2 e^{3x}$$

$$y_p = \frac{1}{D^2 - 5D + 6} \sin 3x$$

$$D^2 = -m^2 \Rightarrow -m^2 = -9$$

$$y_p = \frac{1}{-9 - 5D + 6} \sin 3x = \frac{1}{-3 - 5D} \sin 3x$$

$$y_p = -\left(\frac{3 - 5D}{9 + 25D^2}\right) \sin 3x = -\left(\frac{3 - 5D}{234}\right) \sin 3x$$

$$y_p = \frac{-3 \sin 3x}{234} + \frac{5}{234} \frac{d(\sin 3x)}{dx}$$

$$\Rightarrow y_p = -\frac{3 \sin 3x}{234} + \frac{15}{234} \cos 3x$$

$$\therefore y = y_p + y_c$$
$$= -\frac{3 \sin 3x}{234} + \frac{15}{234} \cos 3x + c_1 e^{2x} + c_2 e^{3x}$$

Alternate method:-

$$y_p = \frac{1}{D^2 - 5D + 6} \sin 3x$$

$$= \left[\frac{A}{(D-3)} + \frac{B}{(D-2)}\right] \sin 3x$$

$$AD - 2A + BD - 3B$$

$$2A + 3B = -1 \qquad B = -1$$
$$A + B = 0 \qquad A = 1$$
$$\Rightarrow A = -B$$

$$= \left[\frac{1}{D-3} - \frac{1}{(D-2)}\right] \sin 3x$$

$$= e^{3x} \int \sin 3x \, e^{-3x} dx$$

Rule 3:-

$$X = x^m$$

$$\frac{1}{f(D)} X = \frac{1}{f(D)} x^m$$

$$= \frac{1}{(D - \alpha_1) \cdots (D - \alpha_n)} x^m \quad \Big/ \left(\frac{a_1}{D - \alpha_1} + \frac{a_2}{D - \alpha_2} + \cdots \frac{a_n}{D - \alpha_n}\right) x^m$$

$$\frac{1}{D - \alpha_1} x^m = \frac{1}{-\alpha_1\left(1 - \frac{D}{\alpha_1}\right)} x^m = -\frac{1}{\alpha_1}\left(1 - \frac{D}{\alpha_1}\right)^{-1} x^m$$

$$= -\frac{1}{\alpha_1}\left(1 + \frac{D}{\alpha_1} + \frac{D}{\alpha_1^2} + \cdots\right) x^m$$

$$\Rightarrow \boxed{x^m = \frac{1}{\alpha_1}(D - \alpha_1)\left(1 + \frac{D}{\alpha_1} + \frac{D^2}{\alpha_1^2} + \cdots\right) x^m}$$

Vardhan

1) $(D^3+2D^2+4D+8)y = x^2$

A) $\lambda^3+2\lambda^2+4\lambda+8=0$, $\lambda=-2$

$$\begin{array}{r|l} \lambda+2 & \lambda^3+2\lambda^2+4\lambda+8 \;(\lambda^2+4 \\ & (-)\lambda^3+2\lambda^2 \\ \hline & 4\lambda+8 \\ & (-)4\lambda+8 \\ \hline & 0 \end{array}$$

$f(\lambda) = (\lambda+2)(\lambda^2+4)$

$y_c = c_1 e^{-2x} + (c_2\cos 2x + c_3 \sin 2x)$

$y_p = \dfrac{1}{(D+2)(D^2+4)} x^2 = \dfrac{1}{8}\left(\dfrac{1}{1+\underbrace{\left(\frac{D^3+2D^2+4D}{8}\right)}_{\alpha}}\right)x^2$

$= \dfrac{1}{8}\,\dfrac{1}{1+\alpha}\,x^2$

$= \dfrac{1}{8}\,(1-\alpha+\alpha^2 - \cdots)x^2$

$y_p = \frac{1}{8}\left(1-(\;)+(\;)^2\right)$

The whole cube term will "kill" x^2

$y_p = \frac{1}{8}\left[x^2 - x - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}}\right]$

$= \frac{1}{8}(x^2-x)$

$\Rightarrow y = y_p + y_c = c_1 e^{-2x} + c_2\cos 2x + c_3 \sin 2x + \frac{1}{8}(x^2-x)$

) $(D^4-a^4)y = x^4 + \sin bx$

A) $f(\lambda) = \lambda^4 - a^4 = 0$

$\Rightarrow \lambda^4 - a^4 = 0$

$(\lambda^2-a^2)(\lambda^2+a^2)=0$

$(\lambda+a)(\lambda-a)(\lambda-ia)(\lambda+ia)=0$

$y_c = c_1 e^{ax} + c_2 e^{-ax} + c_3 \sin ax + c_4 \cos ax$

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$$y_P = \frac{1}{D^4-a^4}(x^4+\sin bx)$$

$$= \frac{1}{D^4-a^4}x^4 + \frac{1}{D^4-a^4}\sin bx \qquad 4\times3\times2$$

$$= \frac{1}{a^4\left(1-\frac{D^4}{a^4}\right)}x^4 + \frac{D^4+a^4}{(D^4-a^4)(D^4+a^4)}\sin bx$$

$$= a^4\left(1+\frac{D^4}{a^4}\right)x^4 + \frac{D^4+a^4}{D^8-a^8}\sin bx$$

$$= a^4x^4 + 24 + \frac{(-b)^4\sin bx + a^4\sin bx}{b^8-a^8}$$

$$y = c_1e^{ax} + c_2e^{-ax} + c_3\sin ax + c_4\cos ax - (a^4x^4+24) + \frac{b^4\sin bx + a^4\sin bx}{b^8-a^8}$$

Rule 4 :-

$f(D)y = e^{ax}(v)$, v is a fn. of x

$$\boxed{\frac{1}{f(D)}(e^{ax}(v)) = e^{ax}\left(\frac{1}{f(D+a)}v\right)}$$

Eg :-

$(D^2-2D+1)y = x^2e^{3x}$

$f(D) = D^2-2D+1$

$f(a) = (a-1)^2$

$y_c = (c_1+c_2x)e^x$

$$y_P = \frac{1}{(D-1)^2}x^2e^{3x} = e^{3x}\frac{1}{(D+3-1)^2}x^2$$

$$= e^{3x}\frac{1}{(D+2)^2}x^2$$

$$= \frac{e^{3x}}{4}$$

$$y = (c_1+c_2x)e^x + \frac{e^{3x}}{4}$$

Note :-

$\frac{1}{f(D^2)}\sin ax$, $f(-a^2) = 0$

$e^{iax} = \cos ax + i\sin ax$

$$\frac{1}{D^2+a^2}e^{iax} = \frac{1}{(D+ai)}\times\frac{1}{(D-ai)}e^{ai} = \frac{1}{2ai}\times\frac{1}{D-ai}e^{iax}$$

$$= \frac{e^{iax}}{2ai}\times\frac{1}{D+ia-ia}$$

$$= \frac{e^{iax}}{2ai}\times\frac{1}{D} = \frac{xe^{iax}}{2ai}$$

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$\frac{1}{D^2+a^2}\sin ax$

$= \frac{x}{2a}(-i)(\cos ax + i\sin ax)$

$= \frac{x}{2a}(\sin ax - i\cos ax)$

$$\boxed{\begin{aligned}\frac{1}{D^2+a^2}\cos ax &= \frac{x}{2a}\sin ax \\ \frac{1}{D^2+a^2}\sin ax &= -\frac{x}{2a}\cos ax\end{aligned}}$$

$\cos ax = \frac{}{2a}$

$\frac{1}{D^2+a^2}\cos ax = -\frac{x}{2a}\cos ax$

$\frac{1}{D^2+a^2}$

$\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax$

$\frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax$

$\frac{A}{D}$

8) Solve

1) $(D^3+a^2D)y = \sin ax$

A) $f(\lambda) = \lambda^3+a^2\lambda = 0$

$\lambda(\lambda^2+a) = 0$

$\lambda = 0,\ \lambda = \pm ai$

$y_c = c_1 + c_2\sin ax + c_3\cos ax$

$AD^2 + aA + BD^2 + CD = 1$

$A + B = 0 \quad C = 0$

$aA = 1$

$A = 1/a$

$B = -1/a$

$y_p = \frac{1}{(D^3+a^2D)}\sin ax$

$\Rightarrow y_p = \frac{1}{D(D^2+a)}\sin ax$

$y_p = \left(\frac{A}{D} + \frac{BD+C}{D^2+a^2}\right)\sin ax$

$y_p = \frac{1}{a}\left(\frac{1}{D} - \frac{1}{D^2+a^2}\right)\sin ax$

$y_p = -\frac{\cos ax}{a \times a} - \frac{x}{2a}\sin ax$

$y = c_1 + c_2\sin ax + c_3\cos ax - \frac{\cos ax}{a^2} - \frac{x}{2a}\sin ax$

Rule 5

$$\boxed{\frac{1}{f(D)}xV = \left[x - \frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}} \checkmark$$

$\left(x - \frac{f'(D)}{f(D)}\right)\frac{1}{f(D)} \checkmark$

1 Solve $(D^2+4)y = x\sin x$

A) $y_c =$

$\lambda^2+4 = 0$

$\lambda = \pm 2i$

$y_c = c_1\sin 2x + c_2\cos 2x$

$y_p = \frac{1}{D^2+4}x\sin x = \left[x - \frac{2D}{D^2+4}\right] \times \frac{1}{D^2+4}\sin x$

$= \left(x - \frac{1}{D^2+4}2D\right)\frac{\sin x}{3}$

$= +\frac{x\sin x}{3} - \frac{2}{9}\cos x$

Vardhan

1) $(x+2)^2 \frac{d^2y}{dx^2} - 4(x+2)\frac{dy}{dx} + 6y = x$

A) $z = \log(x+2)$

$(x+2)\frac{dy}{dx} = \frac{dy}{dz}$, $(x+2)^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$

$\Rightarrow \frac{d^2y}{dz^2} - 5\frac{dy}{dz} + 6y = e^{z} - 2$

Auxiliary eqn is $\lambda^2 - 5\lambda + 6 = 0$

$\lambda = 3, \lambda = 2$

$-\frac{y_2 r}{W}$ $\quad \frac{y_1 r}{W}$

$y_c = c_1 e^{3z} + c_2 e^{2z}$

$y_P = \frac{1}{D^2 - 5D + 6}(e^z - 2)$

$A + B = 0$

$2A + 3B = 1$

$A = 1$

$B = -1$

Method of variation parameter

$y'' + p(x)y' + q(x)y = r(x)$ — ①

When ② = 0 we get c_1, c_2

But ① = 0 we get c_1 as a fn of x and c_2 as fn of x

Consider $y'' + py' + qy = 0$ — ②

Suppose $\{y_1, y_2\}$ is basis for soln space of ②

Then $y_h = c_1 y_1 + c_2 y_2$ is general soln of ②

where c_1, c_2 are arbitrary constants.

Let $y_P = u(x)y_1 + v(x)y_2$ where u, v are arbitrary fn of x and both needs to be determined.

$y_P' = uy_1' + u'y_1 + v'y_2 + vy_2'$

Impose :- $u'y_1 + v'y_2 = 0$ — ③

$u''y_1 + v''y_2 + u'y_1' + v'y_2' = 0$

$u''y_1 + v''y_2 = -r(x)$

$$\left.\begin{aligned} y_P' &= uy_1' + vy_2' \\ y_P'' &= u'y_1' + uy_1'' + v'y_2' + vy_2'' \end{aligned}\right\} \text{Ⓐ}$$

From ① & Ⓐ

$u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + u'y_1' + v'y_2' = r(x)$

$u' = -\frac{y_2 r(x)}{W}$

$\Rightarrow u'y_1' + v'y_2' = r(x)$ — ④

From ③ & ④ $\quad u' = \frac{-y_2 r}{y_1 y_2' - y_1' y_2}$ $\quad v' = \frac{y_1 r}{y_1 y_2' - y_1' y_2}$

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↳ Wronskian of y_1, y_2 which can't be 0 as they are L.I

$$\therefore \boxed{u' = \frac{-y_2 r}{W(y_1, y_2)} \qquad v' = \frac{y_1 r}{W(y_1, y_2)}}$$

$$u = \int \frac{-y_2 r}{W}\,dx \quad , \quad v = \int \frac{y_1 r}{W}\,dx$$

$$\boxed{y_p = -\int \frac{y_2 r}{W}\,dx \; y_1 + \int \frac{y_1 r}{W}\,dx \; y_2}$$

Eg :-

1) $y'' + y = \sec x$ $\qquad 0 \pm i$ $\qquad y = e^{0x}(c_1 \cos(x) + c_2 \sin(-x))$

A) $y'' + y = 0$ $\quad (y = e^{\lambda x})$

$\lambda^2 + 1 = 0$ $\qquad W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$\Rightarrow \lambda \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$\Rightarrow u' = \frac{-\cos x \sec x}{1}$ $\qquad v' = \frac{\sin x \sec x}{1}$

$u = \int -1\,dx = -x$ $\qquad v = \int \tan x\,dx = \ln|\sec x| + c$

$y_p = -x \sin x + \ln|\sec x| \cos x$

$y = y_c + y_p$

$= c_1 \cos x + c_2 \sin x - x \cos x + \sin x \ln|\sec x|$

2) $y'' - 4y' + 4y = \frac{e^{2x}}{x}$

A) $\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$ $\qquad y_1 = e^{2x}$

$\lambda = 2, 2$ $\qquad y_2 = x e^{2x}$

$y_c = e^{2x}(c_1 + c_2 x)$ $\qquad W = \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & y_2' \end{vmatrix}$

$u' = -\frac{x e^{2x} \times \frac{e^{2x}}{x}}{e^{4x}}$ $\qquad v' = \frac{e^{2x} \times \frac{e^{2x}}{x}}{e^{4x}}$ $= e^{4x}$

$u = -x$ $\qquad v = \int \frac{1}{x}\,dx = \ln x$

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next;

$$y_p = -xe^{2x} + x\ln x\, e^{2x}$$

$$y_c = e^{2x}(c_1 + c_2 x)$$

$$y = y_c + y_p = e^{2x}(c_1 + c_2 x) + (-xe^{2x}) + x\ln x\, e^{2x}$$

3) $y'' + 2y' + y = e^{-x}\cos x$

A) $\lambda^2 + 2\lambda + 1 = 0$

$(\lambda+1)^2 = 0$

$\lambda = -1, -1$

$y_1 = e^{-x}$

$y_2 = xe^{-x}$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}$$

$$= e^{-2x} \cancel{- xe^{-2x}} + \cancel{xe^{-2x}}$$

$$= e^{-2x}$$

$$y_c = e^{-x}(c_1 + c_2 x)$$

$$u' = \frac{-x\cancel{e^{-x}} \times \cancel{e^{-x}}\cos x}{\cancel{e^{-2x}}} \qquad v' = \frac{\cancel{e^{-x}} \times \cancel{e^{-x}}\cos x}{\cancel{e^{-2x}}}$$

$$u' = -x\cos x \qquad v' = \cos x$$

$x \quad \cos x$

$1 \quad -\sin x$

$0 \quad -\cos x$

$$u = -\int x\cos x\, dx \qquad v = \int \cos x\, dx$$

$$= -(-x\sin x + \cos x) \qquad = -\sin x$$

$$y_p = e^{-x}(x\sin x - \cos x) - xe^{-x}\sin x = -e^{-x}\cos x$$

$$y_c = e^{-x}(c_1 + c_2 x)$$

$$y = -e^{-x}\cos x + e^{-x}(c_1 + c_2 x)$$

System of Linear diff. eqn:

$$\frac{dy_1}{dx} = f_1(x, y_1, \cdots y_n) \quad - (1)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, \cdots y_n) \quad - (2)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, \cdots y_n) \quad - (n)$$

Diff. eqn: (1) w.r.t x

$0 \rightarrow$ All these are known $f_2, \cdots f_n$

$$\frac{d^2y_1}{dx^2} = \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y_1}\frac{\partial y_1}{\partial x} + \frac{\partial f_1}{\partial y_2} \times \frac{\partial y_2}{\partial x} + \cdots \frac{\partial f_1}{\partial y_n}\frac{\partial y_n}{\partial x}$$

$$\frac{d^2y_1}{dx^2} = F(x, y_1, y_2 \cdots y_n) \quad - (*)$$

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Diff. (*) with x

$$\frac{d^n y_1}{dx^n} = F_3(x, y_1, y_2 \cdots y_n)$$

$\vdots$

$$\frac{d^n y_1}{dx^n} = F_n(x, y_1 \cdots y_n)$$

So the new system of eqn:s are

$$\left.\begin{aligned} \frac{dy_1}{dx} &= f_1(x, y_1 \cdots y_n) \\ \frac{d^{n-1} y_1}{dx^{n-1}} &= F_{n-1}(x, y_1 \cdots y_n) \\ \frac{d^n y_1}{dx^n} &= F_n(x, y_1 \cdots y_n) \end{aligned}\right\} \text{(**)}$$

Considering these as an algebraic sys. of eqn: Find $y_2, y_3 \cdots y_n$ in terms of $x, y_1, y_1' \cdots y^{n-1}$

Take the first $(n-1)$ eqn:s and solve $y_2, y_3 \cdots y_n$ interms of $x, y_1, y_1' \cdots y_1^{n-1}$ (soln: of an algebraic system)

Now, substitute these obtained values in the last eqn: (**)

Thus we get,

$$\frac{d^n y_1}{dx^n} = \phi(x, y_1, y_1', y_1'' \cdots y_1^{n-1})$$

and this is an n^{th} order diff. eqn: and WK how to solve it

From this eqn: we find $y_1 = \psi(x, \underbrace{c_1, c_2, \cdots, c_n}_{n \text{ arbitrary constants}})$

Calculate $\frac{dy_1}{dx}, \frac{d^2 y_1}{dx^2}, \cdots \frac{d^{n-1} y_1}{dx^{n-1}}$ and find $y_2, y_3 \cdots y_n$

as they are fn: of $x, y_1, y_1', y_1'' \cdots y_1^{n-1}$

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Eg :-

1) $\frac{dy}{dx} = x + y + z$ $\qquad$ x is independent variable.

$\frac{dz}{dx} = -4y - 3z + 2x$

$y|_{x=0} = 1, \quad z|_{x=0} = 0$

A) $\frac{d^2y}{dx^2} = 1 + \frac{\partial(x+y+z)}{\partial y}\frac{dy}{dx} + \frac{\partial(x+y+z)}{\partial z}\frac{\partial z}{\partial x}$

$= 1 + \frac{dy}{dx} + \frac{\partial z}{\partial x}$

$\frac{d^2y}{dx^2} = 1 + x + y + z - 4y - 3z + 2x = 1 + 3x - 3y - 2z$

$= 1 + \cancel{3}x - y - 2y - 2z - 2x$

$= 1 + 5x - y - 2y'$

$\frac{d^2y}{dx^2} = 1 + 5x - y - 2y'$

$$\boxed{\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = (1 + 5x)}$$

$\lambda^2 + 2\lambda + 1 = 0$

$(\lambda + 1)^2 = 0$

$\lambda = -1, -1$

$y_c = (c_1 + c_2 x)e^{-x}$

$y_P = \frac{1}{D^2 + 2D + 1}(1 + 5x) = \frac{1}{(D+1)^2}(1 + 5x)$

$= (1 + D)^{-2}(1 + 5x)$

$= (1 - 2D + 3D^2)(1 + 5x)$

$= 1 + 5x - 10$

$y_P = 5x - 9$

$y = (c_1 + c_2 x)e^{-x} + 5x - 9$

$z = 5 + c_1 e^{-x} + c_2(-x e^{-x} + e^{-x})$

$y = 1$ at $x = 0$

$c_1 = 10$

$z = 0$ at $x = 0$

$0 = 5 + 10 + c_2(1)$

$c_2 = -15$

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Scanned with CamScanner

Complex Analysis

Complex Numbers

Famous eqn: $e^{i\pi} = -1$

$z = x + iy \rightarrow$ Rep. of complex no.'s z is "reserved" for $\mathbb{C}$

$Re(z) = x \quad Im(z) = y$

$\mathbb{C}$ = set of all complex numbers
↳ complex plane

There is no order relation (Total ordering) in $\mathbb{C}$

$z_1 > z_2$ (It's a crime) × Not allowed

Consider $1, i$, $\quad i > 1$

$i^2 > 1 \Rightarrow -1 > 1$ (contradiction)

$i < 1$

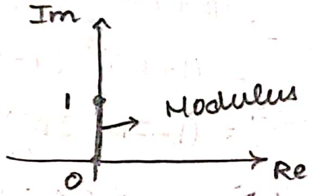
$-i > -1 \Rightarrow i^2 > (-1)^2 \Rightarrow -1 > 1$ ×

Every non-zero no. has an inverse, so a $\mathbb{C}N$ also has an inverse and is a complex no.

It all started here $x^2 + 1 = 0$, n-th degree polynomial should have n roots

$\Rightarrow x^2 = -1$

$x = \pm\sqrt{-1} \Rightarrow x = \pm i$



Modulus of i is this distance = 1

Field is a structure or set where every point has a multiplicative inverse.

$\mathbb{R}^2 \qquad \mathbb{C}$
$\qquad\qquad x+iy$

If we make this as multiplication rule then $\mathbb{R}^2$ becomes a field and

$\mathbb{R}^2 \equiv \mathbb{C}$

$(x_1, y_1) \times (x_2, y_2)$

$= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$

$\underset{i}{(0,1)} \times \underset{i}{(0,1)}$

$= (-1, 0)$

$(x+iy)^{-1} = \dfrac{x - iy}{x^2+y^2}$

$(x, y) \times \left(\dfrac{x}{x^2+y^2}, \dfrac{-y}{x^2+y^2}\right)$

$= (1, 0)$

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** top)

Properities of complex Numbers

→ If $|z| = x + iy$ then $|z| \geq 0$ as $|z| = \sqrt{x^2+y^2}$

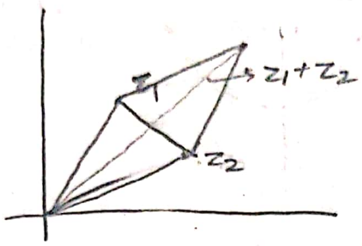
→ conjugate of a complex no. is again a complex no.

If $z = x + iy$, then $\bar{z} = x - iy$ $(\bar{\bar{z}} = z)$

→ If $z = z_1 + z_2$

Then $\bar{z} = \bar{z}_1 + \bar{z}_2$

→ $\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|$



$|z_1 + z_2|^2$

$= (z_1+z_2).(\bar{z}_1 + \bar{z}_2)$

$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1, \bar{z}_2)$

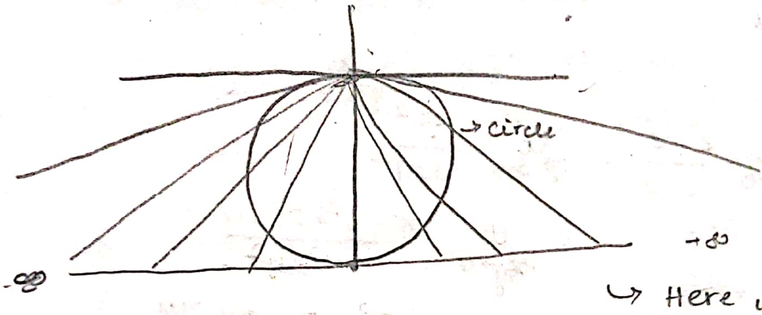
$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\alpha > 0$

$\leq (|z_1| + |z_2|)^2$

Equality holds when z_1, z_2 are co-linear

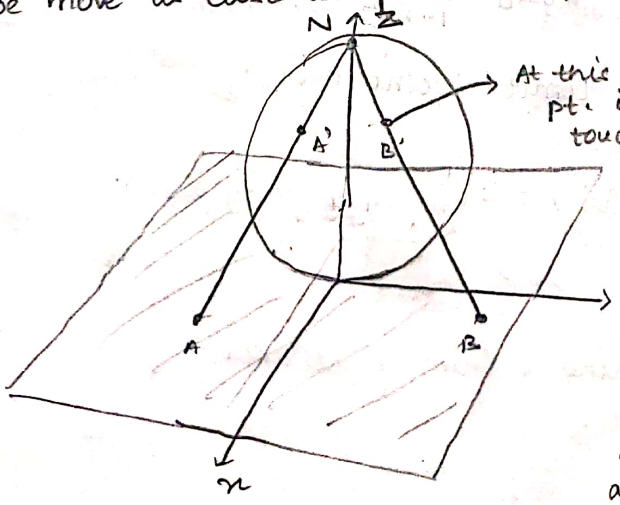
complex infinity



↳ Here when we draw from P we can reach all points except $+\infty, -\infty$ and the point P itself

A line is a circle of infinite radius

So we move as close as possible to p



→ At this pt. it is touching the sphere

Distance b/w A and B is A'B'

We can reach all pts. except the point N

Now this sphere is the complex set for us, we are able to reach all pts. except point N, so N is our infinity

$N \equiv \infty$

sphere

$\mathbb{C} \cup \{\infty\}$ is compact set

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n^{th} roots of unity

$z^n = 1$

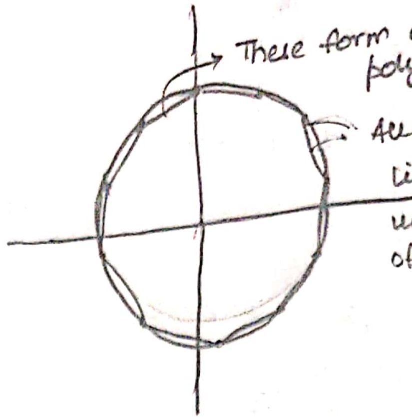
$e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow z^n = e^{2k\pi i}$

$e^{2k\pi i} = 1$

$\therefore \boxed{z = e^{\frac{2k\pi i}{n}} \quad k = 0, 1, 2, \ldots n-1}$

$|z| = 1$



These form a regular polygon

All these roots lie on a unit circle of rad 1

Limits, Continuity & Differentiability :-

Same def. as in the case of two variable calculus

$f(z) : \mathbb{C} \xrightarrow{=\mathbb{R}^2} \mathbb{C} = \mathbb{R}^2$

$\lim_{z \to z_0} f(z) = l = f(z_0)$ and it is defined

$\Rightarrow$ Fn. is cont.

Eg :-

$$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{z\to 0} \frac{\bar{z}\cdot\bar{z}}{z\cdot\bar{z}} = \lim_{z\to 0} \frac{\bar{z}^2}{|z|^2}$$

$$= \lim_{x,y\to 0} \frac{x^2 - y^2 + 2ixy}{x^2+y^2}$$

$$= \lim_{x,y\to 0} \frac{x^2-y^2}{x^2+y^2} + i\left(\frac{xy}{x^2+y^2}\right)$$

$$= \lim_{x,y\to 0} \frac{1-m^2}{1+m^2} + i\left(\frac{m^2}{1+m^2}\right)$$

Limit doesn't exist.

or

$$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{z\to 0} \frac{-iy}{iy}$$ (Approaching 0 from imaginary axis)

$$= -1$$

$$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{z\to 0} \frac{x}{x}$$ (Approaching 0 from real axis)

$$= 1$$

$\Rightarrow$ Limit doesn't exist.

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$z = x + iy$

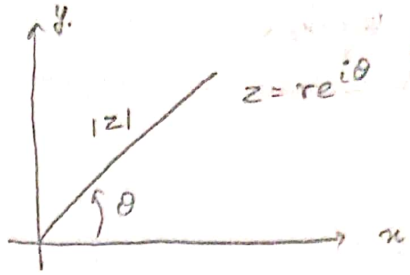
$z = re^{i\theta} = r\cos\theta + i\, r\sin\theta$

$x = r\cos\theta$ $\quad |z| = \sqrt{x^2+y^2} = r \rightarrow$ Modulus of z

$y = r\sin\theta$ $\quad \theta \rightarrow$ called argument of z.

For $z = 0$, arg is not defined

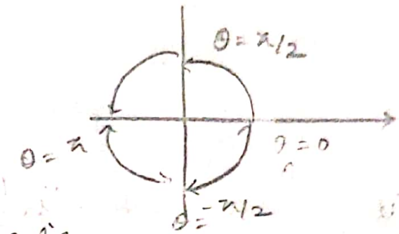
However modulus is zero for $z = 0$



$z = re^{i\theta} = re^{i(2k\pi+\theta)}$ (As $e^{i2k\pi} = 1$)

Principal Arg :- $-\pi < \theta \leq \pi$

As $\arg z = \theta + 2k\pi$



Let $f(z) = \operatorname{Arg} z$

Domain - $\mathbb{C} - \{0\} \longrightarrow (-\pi, \pi]$

The fn. is discont. on -ve real axis

$\mathbb{C} \rightarrow \left\{ x+iy \;;\; \begin{matrix} x \leq 0 \\ y = 0 \end{matrix} \right\}$

↑ Arg is cont. in this

Differentiability

A fn. $f : \mathbb{C} \rightarrow \mathbb{C}$ is said to be complex diff ($\mathbb{C}$ - diff.) at a point z_0 iff

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$ exists and denote the derivative as $f'(z_0)$

(h is a complex variable)

$z_0 = x_0 + iy_0 \equiv (x_0, y_0)$

$h = h_1 + ih_2 \equiv (h_1, h_2)$

$z_0 + h = (x_0 + h_1) + i(y_0 + h_2)$

Analytic fn. :- A fn. f is said to be analytic on an open set $\Omega \subseteq$ if it is differentiable at every pt. of Ω

↑ In a set ↑ At a pt.

f is analytic at $z = z_0$ means that there is a small neighbourhood around $z = z_0$ where the fn. is differentiable.

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Necessary condition for $\mathbb{C}$-differentiability

Suppose f is $\mathbb{C}$-diff. at a pt. $z_0 = (x_0, y_0)$. Let $f(z) = u(x,y) + i v(x,y)$

Then,

$$\boxed{\begin{aligned} u_x(x_0,y_0) &= v_y(x_0,y_0) \\ \& \quad u_y(x_0,y_0) &= -v_x(x_0,y_0) \end{aligned}} \rightarrow \boxed{\text{Cauchy - Riemann eqn}}$$ CR eqns

Most preferable

Eg :-

$f(z) = \bar{z}$

$= x - iy$ $\quad u_x = 1$

$u(x,y) = x$ $\quad v_y = -1$

$v(x,y) = -y$

The fn is cont. everywhere, but nowhere diff.

Proof :-

$$f'(z_0) = \lim_{(h,k)\to(0,0)} \frac{u(x_0+h, y_0+k) - u(x_0,y_0) + i\,(v(x_0+h, y_0+k) - v(x_0,y_0))}{h+ik}$$

Take $k = 0$

$$\Rightarrow \lim_{h\to 0} \frac{u(x_0+h, y_0) - u(x_0,y_0)}{h} + i \lim_{h\to 0} \frac{v(x_0+h,y_0) - v(x_0,y_0)}{h}$$

$$\Rightarrow u_x(x_0,y_0) + i\, v_x(x_0,y_0) \quad \text{— ①}$$

Take $h = 0$

$$f'(z_0) = \lim_{k\to 0} \frac{u(x_0, y_0+k) - u(x_0,y_0) + i\,(v(x_0,y_0+k) - v(x_0,y_0))}{ik}$$

$$\rightarrow f'(z_0) = \frac{1}{i} \lim_{k\to 0} \frac{u(x_0,y_0+k) - u(x_0,y_0)}{k} + \lim_{k\to 0} \frac{v(x_0,y_0+k) - v(x_0,y_0)}{k}$$

$$= -i\, u_y(x_0,y_0) + v_y(x_0,y_0) \quad \text{— ②}$$

since fn is diff. ① = ②

$$\Rightarrow \boxed{\begin{aligned} u_x(x_0,y_0) &= v_y(x_0,y_0) \\ u_y(x_0,y_0) &= -v_x(x_0,y_0) \end{aligned}}$$ CR eqns

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track;

Sufficient Condition for complex - differentiability

(i) C-R eqn's are true at $z \equiv (x_0, y_0)$

(ii) f_x, f_y and u_x, u_y, v_x, v_y are cont. in a neighbourhood of (x_0, y_0)

Proof :-

$f(x,y) = u + iv$

$f_x = u_x + iv_x$, $f_y = u_y + iv_y$

$f_y = i f_x \leftarrow$ C-R eqn.

$u_y + iv_y = i(u_x + iv_x)$

$= iu_x - v_x$

$$f(x+h, y+k) - f(x,y) = hf_x + kf_y + \varepsilon_1 h + \varepsilon_2 k$$

$$\lim_{(h,k)\to 0} \frac{f(x+h, y+k) - f(x,y)}{h+ik} = \lim_{(h,k)\to(0,0)} \frac{hf_x + kf_y + \varepsilon_1 h + \varepsilon_2 k}{h+ik}$$

$$= \lim_{(h,k)\to(0,0)} \frac{hf_x + kf_y + \varepsilon_1 h + \varepsilon_2 k}{h+ik}$$

$$= f_x + \lim_{h,k\to(0,0)} \varepsilon_1 \underbrace{\left(\frac{h}{h+ik}\right)}_{\text{Bounded}} + \varepsilon_2 \underbrace{\left(\frac{k}{h+k}\right)}_{\text{Bounded}}$$

$$= f_x = -if_y$$

$f(z) \equiv z = re^{i\theta}$

$f(r,\theta) = u(r,\theta) + iv(r,\theta)$

conditions for diff. $\boxed{\begin{aligned} ru_r &= v_\theta \\ rv_r &= -u_\theta \end{aligned}}$ $(r \neq 0)$

Consider $f(z) = f(x,y)$

$z = x + iy \Rightarrow \bar{z} = x - iy$

$x = \frac{z+\bar{z}}{2}$ $y = \frac{z-\bar{z}}{2}$

Substituting x, y

$f(x,y) = f(z, \bar{z})$

$\Rightarrow \boxed{\frac{\partial f}{\partial \bar{z}} = 0} \leftarrow$ complex form of CR eqn.

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Harmonic fn:

A real valued fn: $u(x,y)$ defined on an open set $\Omega \subseteq \mathbb{C}$ is said to be harmonic if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ i.e $u_{xx} + u_{yy} = 0$

(Laplace eqn.)

→ It is defined on an open set

Eg:-

$u(x,y) = x^2+y^2$ ✗ $\quad u_{xx} + u_{yy} \neq 0$

$x^2 - y^2$ ✓ $\quad u_{xx} + u_{yy} = 0$

→ "Real & imaginary part of analytic fn's are harmonic"

$f(z) = u(x,y) + i\,v(x,y)$

use CR eqn: $u_x = v_y \rightarrow u_{xx} = v_{yx}$

$u_y = -v_x \rightarrow u_{yy} = -v_{xy}$

(x_0,y_0)

$f(z)$ → Taylor series can be applied in a radius r for $z = z_0$

$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

and the series must converge at this pt.

⇒ then r is called radius of convergence

→ Analytical fn's are locally power series

Eg:- $\sum_{r=0}^{\infty} z^n = \frac{1}{1-z}$

↑ Power series

Q) Given a harmonic fn 'u'. Does there exist an analytic fn f whose real part is u

A) Given a harmonic fn u, v is said to be harmonic conjugate of u, if $f = u + iv$ is analytic

$u(x,y) = x^2 - y^2$

$u(x,y) = 4xy - x^3 + 3xy^2$

} Find harmonic conjugate (v)

$u_y = v_x$ ⇒ $u_y = v_x$ ⇒ $u_y = 2y$ ⇒ $v_x = 2y$

$v_y = -u_x$ $\quad v_y = -u_x$ ⇒ $v_y = -2x$ ⇒ $v_y = +2x$

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Domain → open + connected

Simply connected Domain → A domain which has no hole

Now in the domain if we draw the closed loop

All points in the loop must be in domain

Not SCD

SCD ×

drawn curve

open boundary

$D - \{z_0\}$

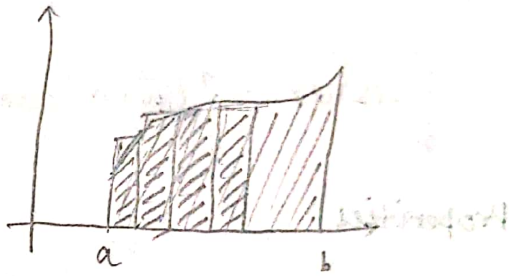
curve will not satisfy the req. question

choosing this curve will show that the domain isn't SCD

Complex Integration

$$\int_a^b f(x)\,dx :$$

(Riemann Integration)



$$m_i = \min_{[a_{i-1},\, a_i]} f(x)$$

$$M_i = \max_{[a_{i-1},\, a_i]} f(x)$$

$L(f)$ = Lower sum = $\sum_i m_i (a_i - a_{i-1})$

$U(f)$ = Upper sum = $\sum_i M_i (a_i - a_{i-1})$

we have considered only one partition, now considering all the partitions, taking max of $L(f)$, min of $U(f)$ when both are equal to R, R is called integral.

$$L(f) = \max_P \left(L(P, f) \right)$$

$$U(f) = \min_P \left(U(P, f) \right)$$

" If $L(f) = U(f) = R$, then the fn f is said to be Riemann integrable and $\int_a^b f(x)\,dx = R$

$$f(x) = \begin{cases} 1 & x \in Q \\ 0 & x \in I \end{cases}$$

(a, b) Lower limit = 0, upper limit = $b-a$

of $b-a$ Hence $f(x)$ isn't diff.

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$$\int_\gamma f(z)\,dz = \int_{t=a}^{b} f(\gamma(t_1))$$

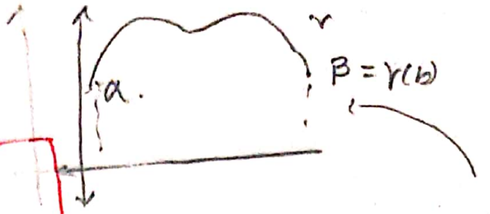
$z = x + iy$

$dz = dx + i\,dy$

$f(z) = u(x,y) + i\,v(x,y)$

$z = \gamma(t)$

$dz = \gamma'(t)\,dt$



$\gamma(t)$

$a \le t \le b$

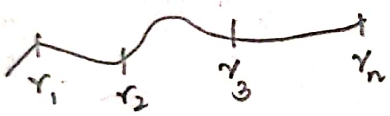
$$\boxed{\int_\gamma f(z)\,dz = \int_{t=a}^{b} f(\gamma(t))\,\gamma'(t)\,dt}$$

← Working rule / defn: for path integral

$$\int_a^b f(z)\,dz = \int_a^b (u+iv)\,(dx + i\,dy)$$

Properities

→ $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cdots \cup \gamma_n$



$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \cdots \int_{\gamma_n} f(z)\,dz$$

→ $\displaystyle\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$ If γ is from (a,b) then $-\gamma$ is from (b,a)

→ $$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|d(z)|$$

$$\le M \int_\gamma |d(z)|$$

M·L · inequality

Eg :-

$z = \gamma(t) = x(t) + i y(t)$

$dz = (x'(t) + i y'(t))\,dt$

$|dz| = \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \displaystyle\int_{t=a}^{b} \sqrt{x'(t)^2 + y'(t)^2}\,dt$

$= l(\gamma)$

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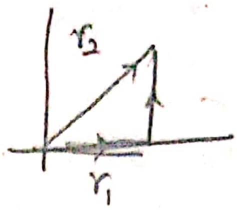
Eg:-

1) $f(z) = \int_{\gamma_1} \bar{z}\, dz$



$= \int_{\gamma_2} \bar{z}\, dz$

$$\boxed{\begin{aligned} \gamma(t) &= a + (b-a)t \\ &= a(1-t) + bt \end{aligned}}$$

(ii) $\displaystyle\int_{t=0}^{1} f(\gamma_2(t))\, \gamma_2'(t)\, dt = \int_{t=0}^{1} (2-i)t(2+i)\, dt$

$= 5/2$

(i) $\displaystyle\int_{t=0}^{1} (2t)\cdot 2\, dt + \int_{0}^{1} (2-it)\, i\, dt$

$= \left[\frac{4}{2} + 2it + \frac{t^2}{2}\right]_{t=0}^{1} = \frac{5}{2} + 2i$

$\rightarrow \displaystyle\int f(z)\, dz = \int_{t=a}^{b} f(\gamma(t))\cdot\gamma'(t)\, dt \qquad f = F'$

$\displaystyle = \int_{t=a}^{b} F'(\gamma(t))\, \gamma'(t)\, dt = \int_{a}^{b} \frac{d}{dt}\left(F(\gamma(t))\right)$

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Cauchy's theorm

Let f be an analytic fn. on a S.C.D Ω. Let γ be a simple closed curve in Ω. Then

$$\int_\gamma f(z)\,dz = 0$$

$$\int_{\partial\Omega} p\,dx + q\,dy = \iint_\Omega \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy$$

$f(z) = u + iv$

$dz = dx + i\,dy$

$$\int_\gamma (u+iv)(dx + i\,dy) = \int_{\Omega} u\,dx - v\,dy + i \cdots$$

$$= \iint_\Omega (-v_x - u_y)\,dy\,dx + i(\quad)$$

Eg:-

1) $\int \frac{dz}{z}$ ← can't use cauchy's theorm, as the fn isn't scb and the fn isn't analytic

$z = e^{it}, \quad 0 \le t \le 2\pi$

$dz = ie^{it}dt$

$$\int \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}dt}{e^{it}} = i\int_0^{2\pi} dt = 2\pi i$$

Cauchy's Integral Formula

Suppose f is analytic on a set Ω. Let γ be a simple closed curve inside Ω, a is a point, that is closed by the curve γ

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz$$

$$\int_\gamma \frac{f(z)}{z-a}\,dz = \int \frac{f(z)}{z-a}\,dz$$

$|z-a| = \varepsilon$

$$= \int \frac{f(z) - f(a) + f(a)}{z-a}\,dz$$

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$$\int \frac{f(z)}{z-a}\,dz = \int \frac{f(z)-f(a)}{z-a}\,dz + f(a)\int \frac{dz}{z-a}$$

$|z-a| = \varepsilon$

$$= I_1 + 2\pi i f(a)$$

$$|I_1| = \left| \int \frac{f(z)-f(a)}{z-a}\,dz \right|$$

$|z-\alpha| = \varepsilon$

$$\leq \int \frac{|f(z)-f(a)|}{|z-a|}\,|dz|$$

$|z-\alpha| = \varepsilon$

Recall :-

Line Integration

$\int_\gamma f(z)\,dz \qquad \gamma(t) : [a,b] \xrightarrow[\text{cont.}]{} \mathbb{C}$

$\gamma'(t) \in C'$

$-\gamma = z(-t)$

$-b \leq -t \leq -a$

$\hookrightarrow z = \gamma = \gamma(t) \qquad a \leq t \leq b$

$$\int_{-c} f(z)\,dz = -\int_c f(z)\,dz$$

$$\int_{t=a}^{b} f(\gamma(t))\,\gamma'(t)\,dt = \left[\int_{t=a}^{b} \mathrm{Re}(\ )\right] + i\left[\int \mathrm{Im}(\ )\right]$$

$\left| \int_\gamma f(z)\,dz \right| \quad$ if $|f(z)| < M$ for $z \in \gamma$

$$\leq \int_\gamma |f(z)|\,|dz|$$

$$\leq M \int_\gamma |dz| = M\ \text{length}(\gamma)$$

$$= ML$$

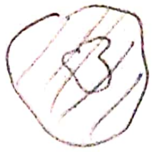
$z = x + iy \qquad |dz| = \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$ Vardhan

$dz = (x'(t) + i\,y'(t))\,dt \qquad \text{length}(\gamma) = \int_{t=a}^{b} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

Cauchy's Theorem

$\Omega \to$ SCD (no hole)

γ SCC $C\ \Omega$

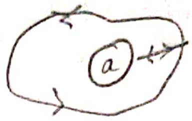


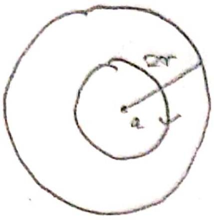
$\int_\gamma f\, dz = 0$

f is analytic in Ω

Cauchy's Integral formula

$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}\, dz$$





$|h| < r$

$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)}\, dz$$

$|z-a| = 2i$

$$f(a+h) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a-h}\, dz$$

$$\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\frac{1}{h}\left[\int \frac{f(z)\,[z-a-z+a+h]}{(z-a-h)(z-a)}\, dz\right]$$

$$= \frac{1}{2\pi i}\int \frac{f(z)\, dz}{(z-a-h)(z-a)}$$

$$= \frac{1}{2\pi i}\int \frac{((z-a-h)+h)\, f(z)}{(z-a-h)(z-a)^2}\, dz$$

$|z-a| = 2r$

$$\left|\frac{f(a+h)-f(a)}{h} - \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^2}\, dz\right|$$

$$\le \frac{|h|}{2\pi}\int \frac{|f(z)|\,|dz|}{|z-a-h|\,|z-a|^2}$$

$$\le \frac{1}{2\pi}\,\frac{M}{2\times 4r^2}\times\frac{1}{\cancel{r}}\,(2\pi\times 2\cancel{r})$$

$$= \frac{M\,|h|}{2r^2}$$

Vardhan

$$\boxed{f^n(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)\,dz}{(z-a)^{n+1}}}$$

a must be inside γ
f(z) must be analytic fn.

$\gamma = |z-a| = R$

$$|f^n(a)| \le \frac{M n!}{2\pi} \times \frac{2\pi R}{R^{n+1}}$$

$$= \frac{M n!}{R^n} \quad n \geq 1$$

Liouville's Theorm :- An entire (diff on $\mathbb{C}$) and bounded fn: is constant

$$|f'(a)| \le \frac{M}{R}$$

As $R \to \infty$ $|f'(a)| = 0$
$\Rightarrow f'(a) = 0$

$f'(z) = 0 \quad \forall z \in \mathbb{C}$

Proof that n^{th} degree polynomial has n roots

$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$

suppose p has no root in $\mathbb{C}$

then $\frac{1}{p(z)}$ is entire

$\lim_{|z| \to \infty} p(z) = 0$

$|z| \le R$, $|z| > R$

Take that root out, now n-1 deg
apply again $\to$ n roots

Vardhan

1) $\int \frac{ze^z}{z^2-4}\,dz$

$|z| = 3$

$= \int \frac{ze^z}{(z-2)(z+2)}\,dz$

$f(z) = \frac{1}{2}\frac{e^z}{(z-2)} + \frac{1}{2}\frac{e^z}{(z+2)}$

$\int f(z)\,dz = \frac{1}{2}\int \frac{e^z}{(z-2)}\,dz + \frac{1}{2}\int \frac{e^z}{(z+2)}\,dz$

$= \frac{1}{2} \times 2\pi i\, e^2 + \frac{1}{2} \times 2\pi i\, e^{-2}$

$|z| = 3$ → It is a circle in which there are both $2, -2$

↑

It is given to show that both $2, -2$ will contribute to the result

If $|z-2| = \frac{1}{2}$ is only given, then only 2 will contribute to the result

2) $\int \frac{ze^z}{(z^2-4)^2}\,dz = \int \frac{ze^z}{(z-2)^2(z+2)^2}\,dz$

$|z| = 3$

$= \frac{1}{8}\left((z+2)^2 - (z-2)^2\right)$

$f(z) = \sum_{n \geq 0} \frac{f^n(z_0)}{n!}(z-z_0)^n, \quad |z-z_0| < \rho$

where $f^n(z_0) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}}\,dz$

$|z-z_0| = r < \rho = |\zeta - z_0|$

$f(z) = \frac{1}{2\pi i}\int \frac{f(\zeta)\,d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int \frac{f(\zeta)\,d\zeta}{\zeta - z_0 + z_0 - z}$

$= \frac{1}{2\pi i}\int \frac{f(\zeta)\,d\zeta}{(\zeta-z_0)-(z-z_0)}$

$= \frac{1}{2\pi i}\int \frac{f(\zeta)\,d\zeta}{\zeta - z_0}$

$+ \left(\frac{1}{2\pi i}\int \frac{f(\zeta)\,d\zeta}{(\zeta-z_0)\left(1-\frac{z-z_0}{\zeta-z_0}\right)}\right)$

Vardha

$$|\zeta - z_0| = r \qquad \left|\frac{z-z_0}{\zeta - z_0}\right| \leq \frac{r}{\rho} < 1$$

$$= \frac{1}{2\pi i}\int \frac{f(\zeta)}{\zeta - z_0}d\zeta \left[1 + \left(\frac{z-z_0}{\zeta-z_0}\right) + \left(\frac{z-z_0}{\zeta-z_0}\right)^2 + \cdots + \frac{\left(\frac{z-z_0}{\zeta-z_0}\right)^n}{1 - \frac{z-z_0}{\zeta-z_0}}\right]$$

$$|\zeta - z_0| = r$$

$$= \frac{1}{2\pi i}\int \frac{f(\zeta)}{\zeta - z_0}d\zeta + \left(\frac{1}{2\pi i}\int \frac{f(\zeta)d\zeta}{(\zeta-z_0)^2}\right)(z-z_0)$$

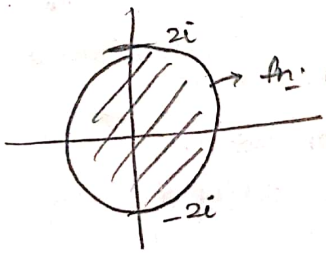
$$+ \cdots \frac{1}{2\pi i}\int\left(\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^n}\right)(z-z_0)^{n-1}$$

$$+ R_n$$

where $R_n =$

$$\boxed{f(z) = f(z_0) + f'(z_0)(z-z_0) + \cdots + \frac{f^{n-1}(z_0)}{(n-1)!}(z-z_0)^{n-1} + R_n}$$

$\frac{1}{z^2+4}$ $\quad$ $z = 0$



fn is analytic $(|z| < 2)$

$$\frac{1}{1-z} = \sum_{n\geq 0} z^n, \quad |z| < 1$$

$$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)} = \frac{1}{4i}\left[\frac{1}{z-2i} - \frac{1}{z+2i}\right]$$

→ Every fn which is anlaytic in a domain has a Taylor series in any disc.

Vardhan

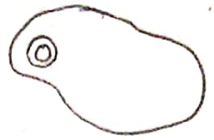
Laurant series

$f \quad \Omega$

$z_0 \in \Omega$

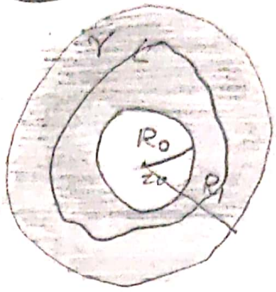
$d = \text{dist}\,(z_0, \partial\Omega)$

$f(z) = \sum_{n \geq 0} a_n (z-z_0)^n, \quad |z-z_0| < d$



$a_n = \frac{f^n(z_0)}{n!}$

$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} dz$



Suppose f is analytic

$|z - z_0| > R$

$\Rightarrow \frac{1}{|z-z_0|} < \frac{1}{R} = R_1$

If I consider this as my variable and use Taylor series

$R_1 > R_0$

$f(z) = \sum_{n \geq 0} b_n \left(\frac{1}{z-z_0}\right)^n$ ①

$+ \sum_{n \geq} b_n \left(\frac{1}{z-z_0}\right)^n$ ②

$|z-z_0| > R_1$
$|z-z_0| > R_0$ } Interection is the annular ring.

This expansion is true only in the annular ring

$= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ ✓

$R_0 < |z-z_0| < R_1$

(Two tailed series)

First observe the fn is analytic or not

If it is anllytic then it is a series of $\frac{1}{z-z_0}$ and it has negative powers of z

In the annular region both the expansion ① and ② are valid

$$\lim_{\substack{M\to\infty \\ N\to\infty}} \sum_{n=-M}^{N} a_n (z-z_0)^n = l$$

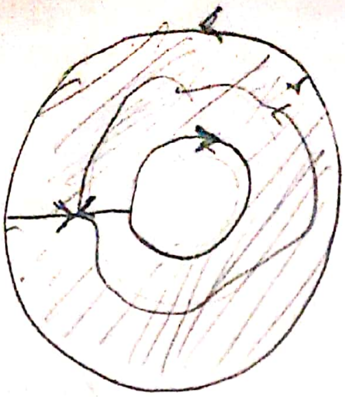
M, N are independent of each other

M isn't a fn of N

M and N go to ∞ independently.

$$a_n = \frac{1}{2\pi i} \int_{\substack{\gamma \\ G}} \frac{f(z)}{(z-z_0)^{n+1}} dz \checkmark$$

Vardhan



$$a_n = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$n = 0, \pm 1, \pm 2 \cdots$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \underbrace{\cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0}}_{\text{principal part}} + \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots}_{\text{Analytic part}}$$

Singular pt

Isolated

Eg :- $\frac{1}{z}$, $\frac{\sin z}{z}$

$z=0$ is an isolated singular pt.

Non - isolated

$\frac{1}{\sin(1/z)}$ $\quad \frac{1}{n\pi} \cup \{0\}$

isolated $\quad$ non-isolated

→ Removable singular pt. ($a_n = 0 \; \forall \; n = 1,2,3 \ldots$

Eq. $\lim_{z\to z_0} f(z) = a_0 =$ finite)

singular pt. can be removed by doing so.

→ Pole (Finitely many terms are present in principal part)

→ Essential singular pt

Pole

$\lim_{z\to z_0} f(z) = \infty$

$\lim_{z\to z_0} (z-z_0)^k f(z) = a_{-k} \neq 0, \infty$

$\frac{1}{z^2}$, order of pole is order of zeroes of polynomial

$\frac{1}{z^2 Sz}$ $\quad z^2 Sz = z^2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)$

$0, n\pi$ $\quad = z^3\left(1 - \frac{z^2}{3!} + \cdots\right)$ $\quad \therefore \frac{1}{z^3}(\quad)$

Vardhan

Essential singularity.

$a_{-n} \neq 0$ for infinitely many n

$\lim_{z \to z_0} f(z)$ DNE

$0 < |z| < \infty$

$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{z^2} \times \frac{1}{2!} + \frac{1}{z^3} \times \frac{1}{3!}$

$a_n = \frac{1}{2\pi i} \int \frac{f(z)\, dz}{(z-z_0)^{n+1}}$

what is a_{-1}?

$z = 1/n \to \infty$, $\to 0$

co-eff of

$z = -1/n \to 0$, $\to -\infty$

$\frac{1}{z-z_0} := a_{-1} = \frac{1}{2\pi i} \int_c f(z)\, dz$

$f(z) = e^{1/z}$ $\quad e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \frac{1}{3! z^3}$

$\frac{1}{z}$ co-eff $= 1$ so $a_{-1} = 1$

for $\int_{|z|=1} z^2 e^{1/z}\, dz$

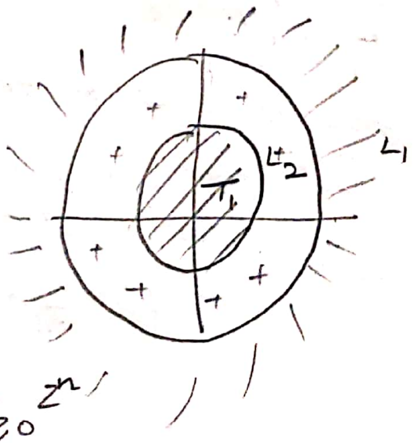
A) $z^2 e^{1/z} = z^2 + z + \frac{1}{2!} + \frac{1}{3!} \times \frac{1}{z}$

co-eff of $1/z = 1/3!$

$\Rightarrow a_{-1} = \frac{1}{3!} \times 2\pi i$

Q) $f(z) = \frac{1}{z^2 - 3z + 2}$ Laurent abt $z = 0$

A) problem at 1, 2



$|z| < 1$:

$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$

$= \frac{1}{-2(1 - z/2)} + \frac{1}{1-z}$

$= -\frac{1}{2} \sum_{n \geq 0} \left(\frac{z}{2}\right)^n + 1 \sum_{n \geq 0} z^n$

Vardhan

$1 < |z| < 2 \rightarrow$ Laurent

$$\frac{1}{z-2} - \frac{1}{(z-1)} = \frac{1}{-2(1-z/2)} - \frac{1}{z(1-\frac{1}{z})} = -\frac{1}{2}\sum_{n\geq 0}\left(z/2\right)^n - \frac{1}{z}\sum_{n\geq 0}\left(\frac{1}{z}\right)^n$$

Singularity

Isolated

$\tan z = \frac{\sin z}{\cos z}$

$\hookrightarrow (2n+1)\frac{\pi}{2} = z$

$\frac{1}{z \sin z}$ $\quad z = n\pi \cup \{0\}$

$1/e^z \rightarrow$ No singular points

$e^{1/z}$ $\quad z = 0$

It's isolated

$\hookrightarrow$ Removable

Consider the fn: $\frac{\sin z}{z}$

It is not existing at $z = 0$

So $z = 0$ is a singular isolated point

By redefining the fn: we can make the fn: ~~cont.~~ ~~and~~ analytic

$$g(z) = \begin{cases} f = \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

$\lim_{z \to 0} f(z)$, exists as a finite value.

Non-isolated

$\frac{1}{\sin(1/z)}$, $z = 0$, $\boxed{1/n\pi}$ $n = \pm 1, \pm 2 \dots$

Isolated

Non-isolated

Vardhan

→ Pole

$\frac{1}{z^2}$, $\frac{1}{z^2 \sin z}$

$\lim_{z \to z_0} f(z) = \infty$
$\Rightarrow$ pole

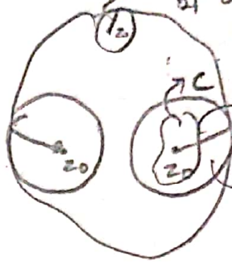
$\lim_{z \to z_0} (z-z_0)^k f(z) =$ finite

~~then~~ "the pole of order k"

Laurant expansion

A fn is analytic in Ω

Consider $z_0 \in \Omega$



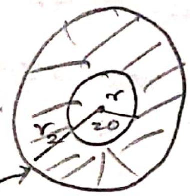
$r = \text{dist}(z_0, \partial\Omega)$

$f(z) = \sum_{n \geq 0} (z-z_0)^n a_n \qquad |z-z_0| < r$

$$a_n = \frac{f^n(z_0)}{n!} = \frac{1}{n!} \times \frac{n!}{2\pi i} \int \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

If z_0 is non-analytic at z_0. where there is no z_0
So we have to consider an anulus,

So we can expand in the anulus



$r_1 < |z-z_0| < r_2$

Two sided series

May
$\boxed{\begin{matrix} r_1 \to 0 \\ r_2 \to \infty \end{matrix}}$

we have both +ve and -ve powers of $z-z_0$

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$$

The series is valid only in the annulus region

The total series is converging to z_0
↳ The value of series is going to a finite value

$\lim_{n \to \infty} \sum_{-n}^{n} a_n (z-z_0) = L$ ← convergent
↑
finite value

Vardhan

Laurant series → $-$ve powers of $z-z_0$ → Principal part
→ $+$ve powers of $z-z_0$ → Analytic part
(Literally forget about it as our Taylor series is always convergent in this part)

Principal part → No term available (Removable) As the limit exists
→ n - number of terms is available (The order is n) → ∞
→ Infinitely many terms also

Residue

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$a_{-1} = \frac{1}{2\pi i} \int f(z)\, dz$$

→ co-eff of $(z-z_0)$ in the expansion of f

→ Residue of $f(z)$ at $z = z_0$

Suppose z_0 is a $\boxed{\text{simple pole}}$ (pole of order 1)

So laurant expansion starts from $n = -1$

$$f(z) = \sum_{n=-1}^{+\infty} a_n (z-z_0)^n$$

$$f(z) = \frac{a_{-1}}{(z-z_0)} + a_1 (z-z_0) + \cdots$$

$$(z-z_0) f(z) = a_{-1} + a_{-1} (z-z_0)^2 + \cdots$$

$$\lim_{z\to z_0} (z-z_0) f(z) = \lim_{z\to z_0} a_{-1}(\cancel{z-z_0})^2 + \cdots \quad + a_{-1}$$

$$\therefore \boxed{a_{-1} = \lim_{z\to z_0} (z-z_0) f(z)}$$

Vardhan

If it is a $\boxed{\text{pole of order } k}$

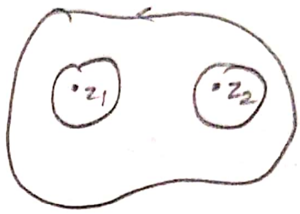
$$f(k) = \frac{a_{-k}}{(z-z_0)^k} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$$

$$(z-z_0)^k f(z) : a_{-k} + \cdots + a_{-1}(z-z_0)^{k-1} + a_0(z-z_0)^k + \cdots$$

$$\boxed{a_{-1} = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} \{(z-z_0)^k f(z)\}}$$

Cauchy's Residue Theorm

Let $z_1, z_2 \ldots z_n$ are isolated singular pts. of f and γ be a curve which encloses all these singular pts.



Then $\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Residue}\,(f(z), z = z_k)$

Radius of convergence

Check nearest singular pt., find distance, that is

$$\boxed{\frac{1}{R} = \lim_{n \to \infty} \sqrt[n]{|a_n|}}$$

$\frac{1}{\sin z}$ about i

$\Rightarrow \boxed{r = 1}$

Vardhan

Q) Find using Taylor's thm $\lim_{x \to 0} \frac{(\sin x - x)^3}{x^9}$

A) $\lim_{x \to 0} \frac{\left(\cancel{x} - \frac{x^3}{3!} - \cancel{h}\right)^3}{x^9} = -\frac{1}{(3!)^3} = -\frac{1}{216}$ //

2) $f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0 \\ 0 & z = 0 \end{cases}$

Verify CR eqn at $z=0$. Is fn diff. at $z=0$

A)

A) $\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{h^5}{|h|^4} - 0}{h} = \lim_{h \to 0} \frac{h^4}{|h|^4}$

$= \left(\lim_{h \to 0} \frac{h}{|h|}\right)^4$

$\lim_{h \to 0} e^{i4\theta}$ ⇒ limit is depending on θ

⇒ Fn is non-diff. at origin

If this limit exists along x-direction and y-direction and are equal (in this case $=1$) ⇒ CR eqn are verified

3) Classify all singular points for $f(z) = \frac{\sin z}{e^z - 1}$ $z \in \mathbb{C}$

A) $z = 0$ $\frac{1}{e^z}$

$\lim_{z \to 0} \frac{\sin z}{e^z - 1} = \lim_{z \to 0} \left(\frac{\sin z}{z}\right) \left(\frac{z}{e^z - 1}\right)$

$= 1$ // Removable

$z = 0$

$z = 2k\pi i$, $k \in \mathbb{Z} - \{0\}$

↑

Simple Poles (order 1)

$\lim_{z \to 0} \frac{\sin z}{z^3(e^z - 1)}$

↑

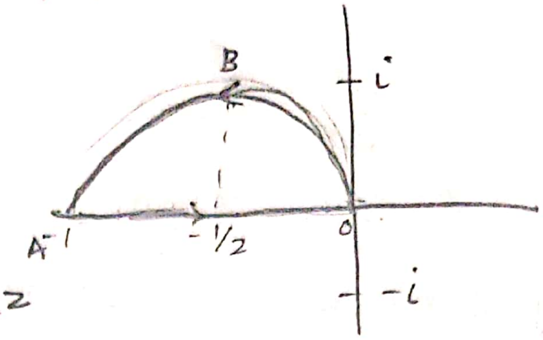
$z=0$ is Pole of order 3

Vardhan

4) $\int_C \frac{z\,dz}{z^2+1}$, where C is upper semi-circle of circle $\left|z+\frac{1}{2}\right| = \frac{1}{2}$ in counter clockwise direction

A) $f(z) = \frac{z}{z^2+1}$ Problem at $z = \pm i$

$$\int_{OBA} f(z)\,dz + \int_{OA} f(z)\,dz = 0$$



$$\Rightarrow \int_{OBA} f(z)\,dz = -\int_{OA} f(z)\,dz$$

$$-\int_{-1}^{0} \frac{x\,dx}{x^2+1} = -\frac{1}{2}\int \frac{dt}{t+1}$$

$x^2 = t$

$2x\,dx = dt$

$$= -\frac{1}{2}\int_{1}^{0} \frac{dt}{t+1}$$

$$= -\frac{1}{2}\ln(t+1)$$

$$= +\frac{1}{2}\ln 2 = \frac{1}{2}\ln 2$$

5) Find the taylor series of the following fn about $z_0 = 1$

$$f(z) = \frac{2z^2+9z+5}{z^3+z^2-8z-12}$$

$$\frac{A}{(z-2)} + \frac{B}{(z-2)^2} + \frac{C}{(z-3)}$$

A) $f(z) = \frac{1}{(z+2)^2} + \frac{2}{(z-3)}$

$$= \frac{2}{-2\left(1-\left(\frac{z-1}{2}\right)\right)} + \frac{1}{9^2\left(1+\frac{z-1}{3}\right)^2}$$

$|z-1| < 3$

$|z-1| < 2$

$\hookrightarrow$ Both the series is valid in $|z-1| < 2$

$$= -\frac{2}{2}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n + \frac{1}{9}\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{z-1}{3}\right)^n$$

$\downarrow$

Valid in $|z-1| < 2$

$$= -\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n + \frac{1}{9}\sum \frac{(-1)^n(n+1)}{3^{n+2}}(z-1)^n$$

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6) Find all Taylor and Laurent series along $z=0$ for

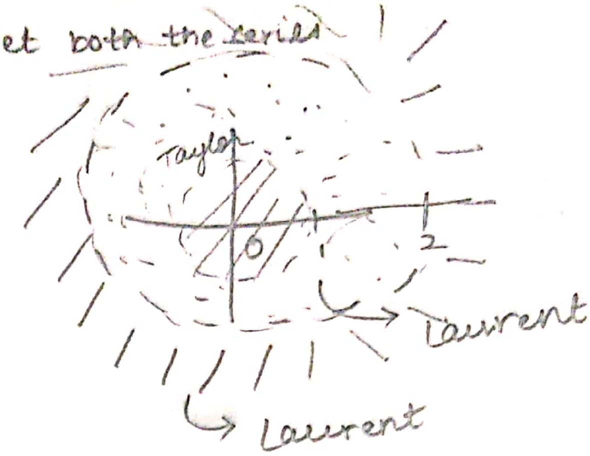
$$f(z) = \frac{-2z+3}{z^2-3z+2}$$

A) If the center about which we expand the series is problematic then we only get Laurent series

If it is not problematic, we get both the series

$$f(z) = \frac{-2z+3}{z^2-3z+2}$$

$$f(z) = -\frac{1}{z-1} - \frac{1}{z-2}$$



Taylor $|z| < 1$

No -ve powers

Laurent $1 < |z| < 2$

Both +ve & -ve powers

Laurent $\infty > |z| > 2$

Both +ve & -ve powers

For $|z| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \sum_{n=0}^{\infty} z^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \leftarrow \text{combinedly } |z| < 1$$

$$= \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^{n+1}}\right) z^n$$

$$f(z) = \frac{1}{1-z} - \frac{1}{z-2}$$

$$= -\frac{1}{z}\left(1 - \frac{1}{z}\right) + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= -\frac{1}{z} \sum \left(\frac{1}{z}\right)^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$|z| > 1$ $\qquad |z| < 2$

$\Rightarrow 1 < |z| < 2$

Vardhan

$2 < |z| < \infty$

$$f(z) = -\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n + \frac{1}{z} \left(\frac{1}{1-\frac{2}{z}}\right)$$

$$= -\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n + -\frac{1}{z} \sum_{n=0}^{\infty} \left(+\frac{2}{z}\right)^n$$

Vardhan