Lecture 30

Stage four: the general case

Def:- Let f be a real-valued measurable function on $\mathbb{R}^d$, we say that f is Lebesgue integrable if the non-negative measurable function $|f|$ is Lebesgue integrable in the sense of previous stage.

$$\int |f| = \sup_{0 \le g \le |f|} \left(\int g \right) < \infty$$

measurable, bounded & supported on a set of finite measure.

Define $f^+(x) = \max(f(x), 0)$

& $f^-(x) = \max(-f(x), 0)$.

measurable.

Note that both f^+, f^- are non-negative measurable function &

$$f^+ - f^- = f, \qquad f^+ + f^- = |f|.$$

Also $f^+, f^- \leq |f|$

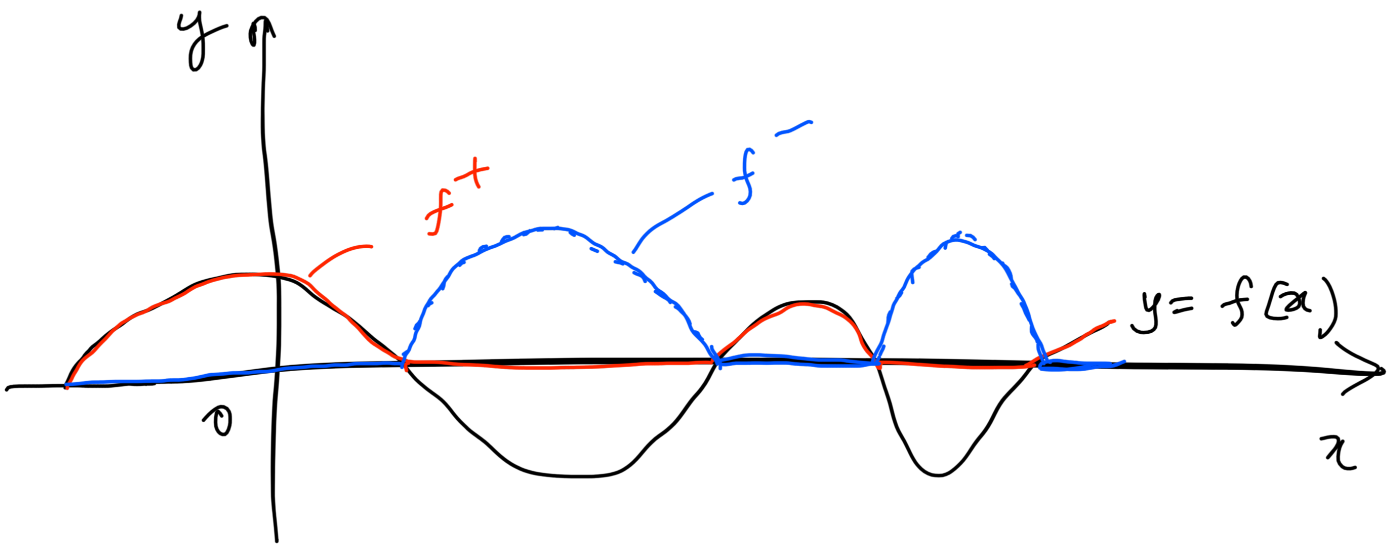
. Both f^+, f^- are integrable if f is integrable

$$\left(\because \infty > \int |f| = \int (f^+ + f^-) \geq \int f^+, \int f^- \right)$$

<u>Definition:-</u>

We define the <u>Lebesgue integral</u> of f by

$$\int f := \int f^+ - \int f^- . \qquad \left(f = f^+ - f^- \right)$$



<u>To show</u>: the definition of the Lebesgue integral of f is well defined.

Suppose $f = f_1 - f_2 = g_1 - g_2$ be two decompositions as difference of two non-negative functions

$$\Rightarrow \quad f_1 + g_2 = g_1 + f_2 \geq 0$$

$$\Rightarrow \quad \int (f_1 + g_2) = \int (g_1 + f_2)$$

$$\Rightarrow \quad \int f_1 + \int g_2 = \int g_1 + \int f_2 .$$

$$\Rightarrow \quad \underline{\int f_1 - \int f_2 = \int g_1 - \int g_2}$$

<u>Example</u>:- $f(x) = x^2 - 2x^4$

$= f_1 - f_2$, $\quad f_1(x) = x^2 \geq 0$

$\quad f_2(x) = 2x^4 . \geq 0$

$= (x^2 + e^x) - (x^4 + e^x)$

$= g_1 - g_2$

where $g_1(x) = x^2 + e^x \geq 0$

$g_2(x) = 2x^4 + e^x \geq 0.$

$$\int f = \int g_1 - \int g_2 = \int f_1 - \int f_2$$

Proposition:— Let f, g be measurable functions.

Then

(i) $\int (af + bg) = a\int f + b\int g$, $\forall\, a, b \in \mathbb{R}$.

(ii) if $f \leq g$, then $\int f \leq \int g$

(iii) $\left|\int f\right| \leq \int |f|$

(iv) If E, F are measurable sets & $E \cap F = \phi$.

Then $\displaystyle\int_{E \cup F} f = \int_E f + \int_F f$.

proof:— EXERCISE.

Recall :-

Let f be a measurable function.

Stages	Def. of Lebesgue integrable
Stage 1 : f is a simple function & $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ where E_i measurable & $m(E_i) < \infty \ \forall i$	$\int f = \sum_{i=1}^{n} a_i m(E_i)$
Stage 2 : f is bounded & supported on a set of finite measure	$\int f = \lim_{n\to\infty}\left(\int \varphi_n\right)$, where $\{\varphi_n\}$ is a sequence of simple functions & bounded & $\varphi_n \to f$ a.e, p.w.
Stage 3 : f is non-negative.	$\int f = \sup_{0 \le g \le f}\left(\int g\right)$ g measurable, bounded, supported on a set of finite measure.
Stage 4 : f any measurable function	$\int f = \int f^+ - \int f^-$ where $f^+ = \max(f,0)$ $f^- = \max(-f,0)$.

Proposition :- Suppose f is Lebesgue integrable on $\mathbb{R}^d$.

Then for every $\varepsilon > 0$:

(i) there exists a set B of finite measure such that $\int_{B^c} |f| < \varepsilon$

(ii) There exists a $\delta > 0$ such that

$$\int_E |f| < \varepsilon, \text{ whenever } m(E) < \delta.$$

(Called "absolute continuity").

proof :- By replace f by $|f|$, we may assume without loss of generality $f \geq 0$.

Let $\varepsilon > 0$.

(i) Let $B_N = B(0, N)$ = the open ball of radius N centered at origin.

Let $f_N(x) = f(x)\,\chi_{B_N}(x)$. $\forall N \geq 1$.

Then $f_N \geq 0$, measurable &

$$f_N(x) \leq f_{N+1}(x) \quad \forall N \geq 1,$$

$$\lim_{N\to\infty} f_N(x) = f(x)$$

$\therefore$ By Montone convergence theorem, we have

$$\lim_{N\to\infty} \int f_N = \int f. \quad \left(\int f_N \nearrow \int f \right.$$

$\left.\right)$ as $N \to \infty$.

$\Rightarrow$ there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, we have

$$0 \leq \left(\int f - \int f_N\right) < \varepsilon$$

In particular,

$$0 \leq \left(\int f - \int f_{N_0}\right) < \varepsilon$$

Now $\displaystyle\int f - \int f_{N_0} = \int (f - f_{N_0})$

$$= \int \left(f - f\chi_{B_{N_0}}\right)$$

$$= \int f\cdot\left(1 - \chi_{B_{N_0}}\right)$$

$$= \int f\chi_{B^c_{N_0}}$$

$$\therefore \quad 0 \le \underset{\displaystyle\int_{B^c_{N_0}} f}{\underset{\|}{\int f\chi_{B^c_{N_0}}}} < \varepsilon$$

$$\Rightarrow \quad \int_{B^c_{N_0}} f < \varepsilon$$

Take $B = B_{N_0}$. & $m(B) = m(B_{N_0}) < \infty$.

(ii). We have $f \geq 0$.

Let $f_N(x) = f(x)\chi_{E_N}(x)$, $\forall N \geq 1$.

$\Rightarrow f_N$

where $E_N = \{x \mid f(x) \leq N\}$ $\forall N \geq 1$.

Then

$f_N \geq 0$, measurable & $f_N \leq f_{N+1}$, $\forall N \geq 1$.

with $\lim_{N \to \infty} f_N(x) = f(x)$

$$\left(f_N \nearrow f\right)_{N \to \infty}.$$

$\therefore$ By Montone Convergence theorem,

$$\int f_N \to \int f \quad \text{as } N \to \infty.$$

There exists $N_0 \in \mathbb{N}$ such that

$$0 \leq \left(\int f - \int f_N\right) < \varepsilon/2, \quad \forall N \geq N_0.$$

In particular, $\int f - \int f_{N_0} < \varepsilon/2$

$$\Rightarrow \int (f - f_{N_0}) < \varepsilon/2$$

Let $\delta > 0$ such that $N_0 \delta < \varepsilon/2$.

For E any measurable set with $m(E) < \delta$,

we have

$$\int_E f = \int_E \left((f - f_{N_0}) + f_{N_0} \right)$$

$$= \int_E (f - f_{N_0}) + \int_E f_{N_0}$$

$$\leq \int_{\mathbb{R}^d} (f - f_{N_0}) + \int_E f_{N_0}$$

$$< \frac{\varepsilon}{2} + \int_E f \chi_{E_{N_0}}$$

$$= \frac{\varepsilon}{2} + \int_{E \cap E_{N_0}} f$$

$\left(\int_E \cdots \leq \int_{\mathbb{R}^d} \cdots \right)$

(A ⊆ B, then $\int_A f \leq \int_B$)

~~$E \subseteq E_{N_0}$~~ ?

$$\Rightarrow \int_E f < \frac{\varepsilon}{2} + \int_{E \cap E_{N_0}} f$$

$$< \frac{\varepsilon}{2} + \int_{E \cap E_{N_0}} N_0 \quad \left(\because \text{ on } E_{N_0},\ f \leq N_0 \right)$$

$E \cap E_{N_0} \subseteq E$

$\subseteq E_{N_0}$

$$< \frac{\varepsilon}{2} + N_0\, m(E \cap E_{N_0})$$

$$< \frac{\varepsilon}{2} + N_0\, m(E)$$

$$< \frac{\varepsilon}{2} + N_0 \cdot \delta$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore \int_E f < \varepsilon.$ whenever $m(E) < \delta.$