Ring Theory

Lecture 18

07/03/2022

Propn. Let $\mathbb{F}$ be a field and $\mathbb{F}[x]$.
Then every ideal in $\mathbb{F}[x]$ is gen by a single elt.

Ex Show that not every ideal in $\mathbb{Z}[x]$ is gen by a single elt.

Show that $(2, x)$ is not gen by a single elt.

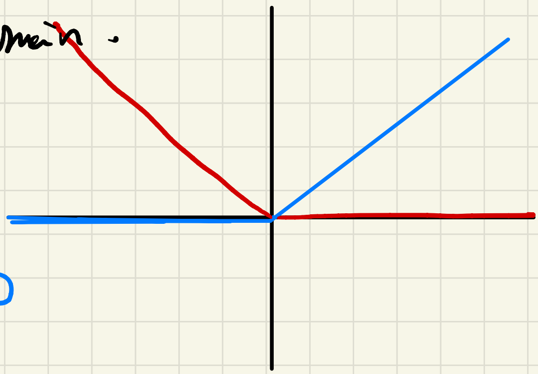
Integral domain: A integral domain R is a non-zero ring having no zero divisor ie if $ab = 0$ then either $a=0$ or $b=0$.

Example (1) Any field is an int domain.

(2) $\mathbb{Z}$, $F[x]$ where F is a field is an integral domain.

(3) $\mathbb{Z}/n\mathbb{Z}$ is an integral domain iff n is a prime number.

(4) $C(\mathbb{R}) = \{ f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ is contn.} \}$ is not an int domain.



Define $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \forall x < 0 \\ x & \forall x \geq 0 \end{cases}$$

and $g : \mathbb{R} \longrightarrow \mathbb{R}$ $\quad g(x) = \begin{cases} x & \forall x \leq 0 \\ 0 & \forall x \geq 0 \end{cases}$

Then $f \cdot g \equiv 0$ but $f \not\equiv 0$, $g \not\equiv 0$.

In $a, b, c \in R$, s.t

$$ab = ac \text{ and } a \neq 0$$

$$b = c$$

Since R is int domain & $a \neq 0$

$$ab - ac = 0$$

$$a(b-c) = 0.$$

$\Rightarrow b - c = 0$

$\therefore b = c.$

Propn. Let R be an integral domain if $ab = ac$ and $a \neq 0$ then $b = c$.

Q Is every integral a field? No.

Propn. A finite integral domain is a field.

Pf: Let R be an int domain.

WTS every $0 \neq x \in R$ has an inverse

Consider the elts $1, x, x^2, x^3, \ldots \in R$

Since R is finite $\exists\ r > s$ s.t

$$x^r = x^s$$

$$\Rightarrow x^r - x^s = 0$$

$$\Rightarrow x^s(x^{r-s} - 1) = 0.$$

$\because$ R is an int domain & $x \neq 0$

so $x^s \neq 0$. $\Rightarrow x^{r-s} - 1 = 0$

$$\Rightarrow x^{r-s} = 1.$$

$$\Rightarrow x \cdot x^{r-s-1} = 1.$$

$\Rightarrow x^{r-s-1}$ is the inverse of x

$\therefore$ R is a field.

Quotient ring: Let I be any ideal of R. Then we have already seen that the set of all left cosets of I forms a gp R/I und addition.

Q. Does R/I has a ring structure?

Want to define multiplication of two left cosets.

$$\boxed{(a+I)\cdot(b+I) = ab+I} \longleftarrow (*)$$

Well defined:

$$a+I = a'+I.$$

$$b+I = b'+I.$$

WTS $\quad ab+I = a'b'+I.$

Since $a+I = a'+I \Rightarrow a-a' \in I$.

$\Rightarrow a-a' = u$

where $u \in I$.

Similarly $b-b' = v$ where $v \in I$.

$a = a'+u$, $b = b'+v$.

$$ab = (a'+u)(b'+v) = a'b' + a'v + ub' + uv$$

Note that $a'v + ub' + uv \in I$.

$\Rightarrow \quad ab - a'b' \in I$.

$\Rightarrow \quad ab + I = a'b' + I$.

Thus R/I has a ring structure with multiplication defined as (*). with multiplicative identity $1+I$.

$$(a+I)\cdot(1+I) = a+I.$$

1st isomorphism Thm: Let $f: R \to S$ be a surjective ring homo then $\bar{f}: R/\ker f \to S$ is an isomorphism

i.e $R/\ker f \cong S$.

Pf: $\bar{f}: R/\ker f \to S$. Let $I = \ker f$.

$$\bar{f}(a+I) = f(a)$$

$$\bar{f}(a+I+b+I) = \bar{f}(a+b+I) = f(a+b)$$

WTS. $\quad \shortparallel$

$$\bar{f}(a+I) + \bar{f}(b+I) = f(a) + f(b) = f(a+b).$$

$$\bar{f}((a+I)\cdot(b+I)) = \bar{f}(ab+I) = f(ab)$$

WTS $\quad \shortparallel$

$$\bar{f}(a+I)\,\bar{f}(b+I) = f(a)\cdot f(b) = f(ab)$$

$\bar{f}(1+I) = f(1) = 1_S.$

$\therefore$ $\bar{f}$ is a ring homo.

Since f is surjective so $\bar{f}$ is surjective.

$$\ker \bar{f} = \{a+I \in R/I \mid \bar{f}(a+I) = 0_S\}$$

$$= \{a+I \in R/I \mid f(a) = 0_S\}.$$

$$= \{a+I \in R/I \mid a \in \ker f = I\},$$

$$= I.$$

$\therefore$ $\bar{f}$ is injective.

$\therefore$ $\bar{f}$ is an isomorphism.

Ex. $\phi: \mathbb{R}[x] \longrightarrow \mathbb{C}.$

$$\phi(f(x)) = f(i).$$

Find $\ker \phi$?.