Lecture 19

Recall:- If $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ be a simple function, canonical representation,

then $$\int \varphi = \sum_{k=1}^{N} a_k \, m(E_k)$$

Eg:- $\varphi = \chi_{[0,1]} - 2\chi_{[2,5]} + 3\chi_{[-2,-1]}$.

$$\int \varphi = m([0,1]) - 2\, m([2,5]) + 3\, m([-2,-1])$$
$$= 1 - 2(3) + 3(1)$$
$$= -2.$$

Definition:- If $E \subseteq \mathbb{R}^d$ is a measurable set with finite measure, then $\varphi(x)\chi_E(x)$ is a simple function & we define

$$\int_E \varphi(x)\, dx := \int \varphi(x)\chi_E(x)\, dx$$

Other notation: $\int_E \varphi$ or $\int_E \varphi(x)\, dm(x)$

Proposition:—

The integral of simple functions defined above satisfies the following properties.

(i) [Independent of the representation]

If $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ is any representation of φ,

then $\int \varphi = \sum_{k=1}^{N} a_k \, m(E_k)$.

(ii) (Linearity) If φ, ψ are simple functions & $a, b \in \mathbb{R}$,

then $\int (a\varphi + b\psi) = a\int \varphi + b\int \psi$.

(iii) (Additive) If E, F are disjoint subsets of $\mathbb{R}^d$ with finite measure, then

$$\int_{E \cup F} \varphi = \int_E \varphi + \int_F \varphi .$$

(iv) (Monotonicity) If $\varphi \leq \psi$ are simple functions, then $\int \varphi \leq \int \psi$.

(v) (Triangular inequality) If φ is a simple function, then $|\varphi|$ is also simple & $\left|\int \varphi\right| \leq \int |\varphi|$.

proof:- case-1

(i) Suppose $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$, where E_k are disjoint but a_k's are not distinct & non-zero.

For each distinct non-zero value a, among the $\{a_k\}$, we define $E'_a = \bigcup E_k$, where union is taken over those indices k such that $a_k = a$.

i.e. $E'_a = \varphi^{-1}(\{a\})$.

Note that E'_a are disjoint. &

$m(E'_a) = m(\bigcup E_k) = \sum m(E_k)$, where the sum is over those k such that $a_k = a$.

Then $\varphi = \sum a \chi_{E'_a}$, where — in canonical form

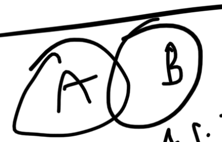
the sum is over the distinct non-zero values of $\{a_k\}$.

Thus $\int \varphi = \sum a \, m(E'_a)$ " "

$$= \sum_{k=1}^{N} a_k \, m(E_k)$$

case-2:

Suppose $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$, where a_k's are distinct & E_k's are not disjoint.



$A \cup B = A \cup (B \setminus A)$

Then we can refine the decomposition $\bigcup_{k=1}^{N} E_k$ by finding the sets $E_1^*, \dots, E_n^*$ with the property that $\bigcup_{k=1}^{N} E_k = \bigcup_{j=1}^{n} E_j^*$ & the sets E_j^* are disjoint.

For each k, $E_k = \bigcup E_j^*$, where the union is taken over those E_j^* that are contained in E_k.

For each j, let now $a_j^* = \sum a_k$, where the summation is taken over all k such that $E_k \supseteq E_j^*$.

Then $\varphi = \sum_{j=1}^{n} a_j^* \chi_{\underline{E_j^*}}$, this representation may have a_j^* need not be distinct. but E_j^* are disjoint.

Then by case 1, we have

$$\int \varphi = \sum_{j=1}^{N} a_j^* \, m(E_j^*)$$

$$= \sum_{k=1}^{N}{}' \sum_{E_k \supseteq E_j^*} a_k \, m(E_j^*)$$

$$= \sum_{k=1}^{N}{}' a_k \, m(E_k) .$$

Thus $\int \varphi$ is independent of representation of φ.

(ii) Follows from (i) by taking any representation of φ & ψ:

Let $\varphi = \sum_{k=1}^{N}{}' a_k \chi_{E_k}$, $\psi = \sum_{k=1}^{M}{}' b_k \chi_{E_k'}$ any repres.

Then $$a\varphi + b\psi = \sum_{k=1}^{N}{}' a a_k \chi_{E_k} + \sum_{k=1}^{M}{}' b b_k \chi_{E_k'}$$

a simple funtion.

$\therefore$ By (i) $$\int (a\varphi + b\psi) = \sum_{k=1}^{N}{}' a a_k \, m(E_k) + \sum_{k=1}^{M}{}' b b_k \, m(E_k') .$$

$$= a \left(\sum_{k=1}^{N}{}' a_k \, m(E_k) \right) + b \left(\sum_{k=1}^{M}{}' b_k \, m(E_k') \right)$$

$$= a \int \varphi + b \int \psi \quad \text{(by (i))}$$

(iii) Suppose E, F are disjoint.

Note that $\chi_{E \cup F} = \chi_E + \chi_F$.

(For x, $\chi_{E \cup F}(x) = \begin{cases} 1 & \text{if } x \in E \cup F \\ 0 & \text{if } x \notin F \cup F \end{cases}$

$$(\chi_E + \chi_F)(x) = \chi_E(x) + \chi_F(x)$$

$$= \begin{cases} 1 & \text{if } x \in E \cup F \\ 0 & \text{if } x \notin F \cup F \end{cases}$$

$$= \chi_{E \cup F}(x).\ )$$

Let $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ be any simple function.

Then
$$\int_{E \cup F} \varphi = \int \varphi \cdot \chi_{E \cup F}$$

$$= \int \varphi \cdot (\chi_E + \chi_F)$$

$$= \int \varphi \cdot \chi_E + \int \varphi \cdot \chi_F$$

$$= \int_E \varphi + \int_F \varphi.$$

[(iv) Suppose $\varphi \leq \psi$. simple functions.

$\Rightarrow \underbrace{\psi - \varphi}_{\text{a simple function}} \geq 0.$

$$\psi - \varphi = \sum_{k=1}^{N} a_k \chi_{E_k}, \text{ where } a_k \geq 0 \ \forall k$$

$$\Rightarrow \int (\psi - \varphi) = \sum_{k=1}^{N} a_k \, m(E_k) \geq 0$$

$$\Rightarrow \int \psi - \int \varphi \geq 0$$

$$\Rightarrow \int \psi \geq \int \varphi.$$

(v) Let $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ canonical form.

then $|\varphi| = \sum_{k=1}^{N} |a_k| \chi_{E_k}$

Now
$$\left|\int \varphi\right| = \left| \sum_{k=1}^{N} a_k \, m(E_k) \right|$$
$$\leq \sum_{k=1}^{N} |a_k \, m(E_k)|$$
$$= \sum_{k=1}^{N} |a_k| \, m(E_k)$$
$$= \int |\varphi|.$$

$\therefore \left|\int \varphi\right| \leq \int |\varphi|.$

Proposition:— Let $A, B \subseteq \mathbb{R}^d$. Then.

① $\chi_{A\cap B} = \chi_A \cdot \chi_B$

② $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$.

proof:— (1) $\chi_{A\cap B}(x) = \begin{cases} 1 & \text{if } x \in A\cap B \\ 0 & \text{otherwise} \end{cases}$.

$(\chi_A \cdot \chi_B)(x) = \chi_A(x) \cdot \chi_B(x)$

$= 1$ if both $\chi_A(x)$ & $\chi_B(x)$ equal to 1.

& 0 otherwise

$= \begin{cases} 1 & \text{if } x \in A\cap B \\ 0 & \text{otherwise} \end{cases}$

$= \chi_{A\cap B}(x)$.

② For $x \in A\cup B$, say $x \in A$ — $x \in B$ / $x \notin B$.

$\chi_A(x) = 1$, if $x \in B$, then $\chi_B(x) = 1$

& $\chi_{A\cap B}(x) = 1$

& if $x \notin B$, then $\chi_B(x) = 0 = \chi_{A\cap B}(x)$

Thus $(\chi_A + \chi_B - \chi_{A\cap B})(x) = \chi_A(x) + \chi_B(x)$
$- \chi_{A\cap B}(x)$

$$= \begin{cases} 1+1-1 & \text{if } x \in B \\ 1+0-0 & \text{if } x \notin B \end{cases}$$

$$= \begin{cases} 1 & \text{if } x \in A \end{cases}$$

|||ly $(\chi_A + \chi_B - \chi_{A\cap B})(x) = \begin{cases} 1 & \text{if } x \in B \end{cases}$

Thus $(\chi_A + \chi_B - \chi_{A\cap B})(x) = 1$ if $x \in A\cup B$.

<u>To show</u>: if $x \notin A\cup B$, then

$(\chi_A + \chi_B - \chi_{A\cap B})(x) = 0$.

$x \notin A\cup B \Rightarrow x \notin A$ & $x \notin B$

$\Rightarrow \chi_A(x) = 0$, $\chi_B(x) = 0$

& $\chi_{A\cap B}(x) = 0$.

$\therefore$ $\underline{(\chi_A + \chi_B - \chi_{A\cap B})(x)} = 0 + 0 - 0 = \underline{0}$.

Thus $\quad x_A + x_B - x_{A \cap B} = x_{A \cup B}$.