Assignment 4 - MM

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1) $x^2 \frac{d^2y}{dx^2} + (2x^3+1)\frac{dy}{dx} + y = 0$

Adjoint is given by

$My(x) \equiv \frac{d^2}{dx^2}(x^2 \cdot y) - \frac{d}{dx}((2x^3+1)\cdot y) + y$

$= \frac{d}{dx}(2xy + x^2 \cdot y') - (6x^2y) - (2x^3+1)y' + y$

$= 2y + 2xy' + 2xy' + x^2y'' - 6x^2y - (2x^3+1)y' + y$

$= x^2y'' - (2x^3 - 4x + 1)y' - 3(2x^2-1)y$

Adjoint $eq^n \Rightarrow My(x) = 0 \Rightarrow$

$x^2y'' - (2x^3-4x+1)y' - 3(2x^2-1)y = 0$

2) $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

~~Adjoint is given by~~

For the given $eq^n$ to be self adjoint

$\frac{d}{dx}(x^2)$ should be equal to $-2x$

but, $\frac{d}{dx}(x^2) = 2x \neq -2x$

Thus,

$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$ is not a self adjoint $eq^n$

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3) $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ ; $-L \leq x \leq L$

Let $\{f_n\}_{n=0}^{\infty} = \left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$

We need to prove that $\{f_n\}_{n=0}^{\infty}$ is a set of mutually orthogonal functions on $-L \leq x \leq L$

Proof :- for $m \neq n$, $\int_{-L}^{L} f_m(x) \cdot f_n(x)\, dx$

$$= \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_{-L}^{L} 2\cos\left(\frac{m\pi x}{L}\right) \cdot \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_{-L}^{L} \left[\cos\left(\frac{(m+n)\pi x}{L}\right) + \cos\left(\frac{(m-n)\pi x}{L}\right)\right] dx$$

$$= \frac{1}{2} \frac{\sin\left((m+n)\pi x/L\right)}{((m+n)\pi)/L}\Bigg|_{-L}^{L} + \frac{1}{2} \frac{\sin\left((m-n)\pi x/L\right)}{((m-n)\pi)/L}\Bigg|_{-L}^{L}$$

$$= \frac{1}{2} \times 0 + \frac{1}{2} \times 0 = 0$$

Thus $\left\{\cos \frac{n\pi x}{L}\right\}_{n=1}^{\infty}$ is mutually orthogonal on $-L \leq x \leq L$

4) $\left\{\sin \frac{n\pi x}{L}\right\}_{n=1}^{\infty}$ ; $-L \leq x \leq L$

Let $\{f_n\}_{n=1}^{\infty} = \left\{\sin \frac{n\pi x}{L}\right\}_{n=0}^{\infty}$

for $m \neq n$, $\int_{-L}^{L} f_m(x) \cdot f_n(x)\, dx = \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) . \sin\left(\frac{n\pi x}{L}\right) dx$

$$= \frac{1}{2}\int_{-L}^{L} \left[\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right] dx$$

$$= \frac{1}{2} \frac{\sin\left((m-n)\pi x/L\right)}{((m-n)\pi)/L}\Bigg|_{-L}^{L} - \frac{1}{2} \frac{\sin\left((m+n)\pi x/L\right)}{((m+n)\pi)/L}\Bigg|_{-L}^{L}$$

$$= 0 - 0 = 0$$

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Hence, $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ is mutually orthogonal on $-L \leq x \leq L$

5) $-y''(x) = \lambda y(x)$, $y(0) = 0 = y(L)$

Case 1:- $\lambda = 0$

Gen. Sol$^{\underline{n}}$:- $y(x) = Ax + B$

Using BC's:- $A = B = 0$

Only trivial solution, $y \equiv 0$ is possible in this case

Case 2:- $\lambda < 0$. Let $\lambda = -\mu^2$, $\mu \neq 0$

Homogeneous part:- $-y'' = -\mu^2 y \Rightarrow y'' - \mu^2 y = 0$

Gen. Sol$^{\underline{n}}$:- $Ae^{\mu x} + Be^{-\mu x} = y(x)$

Using BC's:- $A = 0$, $B = 0$

Only trivial solution, $y \equiv 0$ is possible in this case

Case 3:- $\lambda > 0$, Let $\lambda = \mu^2$, $\mu \neq 0$

Homogeneous part:- $-y'' = \mu^2 y \Rightarrow y'' + \mu^2 y = 0$

Gen sol$^{\underline{n}}$:- $y(x) = A\cos(\mu x) + B\sin(\mu x)$

Using BC's:- $A = 0$ (from $y(0) = 0$)

$$y(L) = \cancel{A\cos(\mu L)}^{0} + B\sin(\mu L) = 0 \quad \text{as } (A = 0)$$

$$B\sin(\mu L) = 0$$

For non-trivial solution to exist, $B \neq 0$,

$\therefore \sin(\mu L) = 0$

$\mu L = n\pi$, $n = \cancel{0}, 1, 2, \ldots$ as $\mu \neq 0$ & $L \neq 0$

$\mu = \frac{n\pi}{L}$, $n = 1, 2, \ldots$

$\mu^2 = (n\pi)^2 / L^2$

Then, $\lambda_n = \mu_n^2 = \frac{n^2\pi^2}{L^2}$ are the eigenvalues of the BVP, i.e. values where non-zero solutions to the BVP exist.

The corresponding solutions are:-

$y_n(x) = B\sin(\mu x) = B\sin\left(\frac{n\pi x}{L}\right)$

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6) $y''(x) + \lambda(y(x)) = 0, \quad y'(0) = 0 = y'(L)$

Case 1:- $\lambda = 0$

Gen Sol$^n$:- $y(x) = Ax + B$

Using BC's :- $A = 0$, $B$ is arbitrary

$\therefore y(x) = B$, $B \neq 0$ has non-zero solutions to the given BVP

Thus, $\lambda = 0$ is an eigenvalue with eigen function, $y_0(x) = B$

Case 2:- $\lambda < 0$, Let $\lambda = -\mu^2$, $\mu \neq 0$

Homogeneous part:- $y'' - \mu^2 y = 0$

Gen Sol$^n$ :- $y = Ae^{\mu x} + Be^{-\mu x} \Rightarrow y' = \mu A e^{\mu x} - \mu B e^{-\mu x}$

Using BC's :- $\mu A - \mu B = 0 \Rightarrow A - B = 0 \Rightarrow A = B$ (using $y'(0)=0$)

$\mu A e^{\mu L} - \mu B e^{-\mu L} = 0 \Rightarrow Ae^{2\mu L} - B = 0 \Rightarrow A = B = 0$

Only trivial sol$^n$ $y(x) = 0$ in this case, so $\lambda < 0$ is not an eigenvalue for BVP given.

Case 3:- $\lambda > 0$, Let $\lambda = \mu^2$, $\mu = 0$

Homogeneous part :- $y'' + \mu^2 y = 0$

Gen sol$^n$ :- $y(x) = A\cos(\mu x) + B\sin(\mu x)$

$\Rightarrow y' = -A\mu \sin(\mu x) + B\mu \cos(\mu x)$

Using BC's:- $y'(0) = B\mu = 0 \Rightarrow B = 0$

$y'(L) = -A\mu \sin(\mu L) = 0$; now $\mu \neq 0$ & $A \neq 0$ for non-trivial sol$^n$

$\sin(\mu L) = 0 \Rightarrow \mu L = n\pi$, $n = 1, 2, \ldots$

$\Rightarrow \mu = \frac{n\pi}{L}$

$\Rightarrow \lambda = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, \ldots$

$\lambda_n = \mu_n^2 = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, \ldots$ are also eigen values of this BVP with eigenfunctions $y_n(x) = A\cos\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, 3, 4 \ldots$

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Combining all three cases,
~~eigenvect~~ eigenvalues :- $\lambda_n = \frac{n^2\pi^2}{L^2}$, $n = 0, 1, 2 \ldots\ldots$

Corresponding eigenfunctions :- $y_n(x) = A\cos\left(\frac{n\pi x}{L}\right)$, $n = 0, 1, 2, \ldots$

7. $y''(x) + \lambda y(x) = 0$, $y(0) = 0$, $y(1) + y'(1) = 0$

Case 1 :- $\lambda = 0$
Gen Sol$^n$ :- $y(x) = Ax + B$
Using BC's :- $B = 0$, $A = 0$
Only trivial sol$^n$, $y \equiv 0$ is possible. // $y \not\equiv 0$
$\therefore \lambda = 0$ is NOT an eigenvalue

Case 2 :- $\lambda < 0$, Let $\lambda = -\mu^2$, $\mu \neq 0$
Gen sol$^n$ :- $y = Ae^{\mu x} + Be^{-\mu x}$, $y' = \mu Ae^{\mu x} - \mu Be^{-\mu x}$
Using BC's :- $A + B = 0$, $Ae^{\mu} + Be^{-\mu} + \mu Ae^{\mu} - \mu Be^{-\mu} = 0$
$\Rightarrow A \times 2\cosh(\mu) \times [\tanh(\mu) + \mu] = 0$
~~As $\mu \neq 0$, $A \neq 0$ (for non-trivial sol cosh($\mu$) ≠ 0~~,
As $\mu \neq 0$, $\cosh(\mu) \neq 0$, $\tanh(\mu) \neq -\mu$
Thus $A = 0$
Only trivial sol$^n$, $y \equiv 0$ is possible. $\therefore \lambda < 0$ is not eigenvalue.

Case 3 :- $\lambda > 0$, Let $\lambda = \mu^2$; $\mu \neq 0$
Gen sol$^n$ :- $y = A\cos(\mu x) + B\sin(\mu x)$, $y' = -A\mu \sin(\mu x) + B\mu\cos(\mu x)$
Using BC's :- $A = 0$, $B\mu(\cos\mu) + B\sin\mu = 0 \Rightarrow B(\mu\cos\mu + \sin\mu) = 0$
For non-trivial sol$^n$, $B \neq 0$, $\therefore$ $\mu\cos\mu + \sin\mu = 0 \Rightarrow \tan\mu = -\mu$
(as $\cos\mu \neq 0$)

$\therefore$ Eigenvalue :- $\lambda = \mu^2$ where $\mu$ satisfies $\tan\mu = -\mu$, $\mu \neq 0$
Eigenfunction :- $y = B\sin(\mu x)$