9th Sept 2021

Matrix of a Linear Map :-

Let U be n-dimensional vector space.
with ordered basis. $B = \{u_1\ u_2 \ldots u_n\}$
and let V be m-dimensional vector space.

$$B' = \{v_1\ v_2 \ldots v_n\}$$

let $T : U \to V$ be a linear map.

Then $\{T(u_1), T(u_2), \ldots T(u_n)\}$ is subset of V.

<u>But</u> $V = [B'] = [v_1\ v_2 \ldots v_n]$

$\therefore$ For each $T(u_j) \in V$ we have $T(u_j) = \sum_{i=1}^{n} a_{ij} v_i$

The scalars. $a_{1j}\ a_{2j} \ldots a_{nj}$ $\qquad j = 1, 2, 3, \ldots, m.$
are coordinates of $T(u_j)$ wrt basis B'

$$[T(u_j)]_B = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

write the co-ordinate vectors of $T(u_1)\ T(u_2) \ldots T(u_n)$ succesfully as column vectors in the form of rectangular array as.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & & a_{in} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} = (a_{ij})_{m \times n}.$$

This is called matrix of T wrt basis B and B'.
And it is denoted by $[T : B, B'] = (a_{ij})_{m \times n}$

* If $U = V$ $B = B'$

$$[T : B\, B'] = [T : B]$$
$$[T(u_j)]_B = (a_{ij})_{n \times n}$$

<u>Ex</u> let $T : V_2 \to V_3$ be defined by
$$T(x_1 x_2) = (x_1 + x_2,\ 2x_1 - x_2,\ 7x_2).$$

<u>Sol^n</u> $B_1 = \{e_1\ e_2\}$ be standard basis for V_2.
$B_2 = \{f_1\ f_2\ f_3\}$ be standard basis for V_3.

$$T(e_1) = (1, 2, 0) = 1 f_1 + 2 f_2 + 0 . f_3$$
$$T(e_2) = (1, -1, 7) = 1 f_1 - 1 f_2 + 7 f_3.$$

$$[T : B_1, B_2] = [\, T(e_1) \quad T(e_2) \,]$$
$$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}_{3 \times 2}$$

<u>Ex</u>. $T : U_3 \longrightarrow V_3$ be defined by
$$T(x_1\ x_2\ x_3) = (x_1 - x_2 + x_3,\ 2x_1 + 3x_2 - \tfrac{1}{2}x_3,\ x_1 + x_2 - 2x_3)$$
let $B_1 = \{e_1\ e_2\ e_3\}$ be standard basis for V_3.
~~Then~~ $B_2 = \{(1, 1, 0)\ (1, 2, 3)\ (-1, 0, 1)\}$
Then find $[T : B_1, B_2]$.

<u>Sol^n</u>
$$T(e_1) = (1, 2, 1) = a(1,1,0) + b(1,2,3) + c(-1,0,1)$$
$$T(e_2) = (-1, 3, 1) = a_1(1,1,0) + b_1(1,2,3) + c_1(-1,0,1)$$
$$T(e_3) = (1, -\tfrac{1}{2}, -2) = a_2(1,1,0) + b_2(1,2,3) + c_2(-1,0,1)$$

$a = 2$ $b = 0$ $c = 1$
$a_1 = \cancel{6}$ $b_1 = -3/2$ $c_1 = 11/2$
$a_2 = 0$ $b_2 = -1/4$ $c_2 = -5/4$

$$[T : B_1\ B_2] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3/2 & -1/4 \\ 1 & 11/2 & -5/4 \end{bmatrix}$$

Ex Let $I : U \to V$ be identity linear map.
$\dim(U) = n$ Let B be standard Basis for U.

$$[I]_B = I_{n \times n}$$

Ex $O : U_n \to V$

$$[O : B] = O_{n \times n}$$

Linear map associated with a given matrix

let ~~U and V A = adj~~
$A = (a_{ij})_{m \times n}$ be a given matrix.
let U and V be the n and m dimensional vector spaces with ~~bas~~ ordered basis

$$B_1 = \{u_1, u_2, \dots u_n\}$$
and $$B_2 = \{v_1\ v_2 \dots v_n\}.$$

Now consider ~~the scalars~~

$$\sum_{i=1}^{m} a_{i1} v_i, \quad \sum_{i=1}^{m} a_{i2} v_i, \quad \dots \quad \sum_{i=1}^{m} a_{in} v_i$$

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \dots \begin{bmatrix} a_{1n} \\ a_{mn} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Now consider n vectors in V
then there exists a unique linear map $T : U \to V$
st $\boxed{T(u_1) = \sum_{i=1}^{m} a_{i1} v_i}$ $\boxed{T(u_2) = \sum_{i=1}^{m} a_{i2} v_i}$ ... $\boxed{T(u_n) = \sum_{i=1}^{m} a_{in} v_i}$

The T is given at $u_1, u_2, \ldots u_n$
so we can extend T linearly to whole Space U.
for any $u \in U = [u_1\ u_2 \cdots u_n]$

$$u = \sum_{i=1}^{n} \alpha_i u_i$$

$$T(u) = \sum_{i=1}^{n} \alpha_i T(u_i)$$

Thus we have T is the linear map associated with $(a_{ij})_{m\times n}$

$$\therefore [T : B_1, B_2] = (a_{ij})_{m\times n}.$$

* If again $U = V$ $B_1 = B_2 = B$.
Then $[T : B_1, B_2] = [T]_B = (a_{ij})_{n\times n}$.

Ex: consider the following matrix $\begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & 2 & -2 \end{bmatrix} = (a_{ij})_{4\times 3}$

Solⁿ choose V_3 and V_4
be vector spaces with standard basis
$B_1 = \{u_1, u_2, u_3\}$
and $B_2 = \{v_1\ v_2\ v_3\ v_4\}$ respectively

$$\begin{bmatrix} T(u_1) = 2v_1 + 1v_2 - 2v_3 + v_4 \\ T(u_2) = -3v_1 + 0.v_2 + 1v_3 + 2v_4 \\ T(u_3) = 4v_1 + -1.v_2 + 0.v_3 + -2v_4 \end{bmatrix}$$

Now let $u = (x_1, x_2\, x_3) \in V_3$
Then. $u = x_1 u_1 + x_2 u_2 + x_3 u_3$.
Then $T(u) = T(x_1, x_2, x_3) = x_1(2, 1, -2, 1) + x_2(-3, 0, 1, 2)$
$+ x_3(4, -1, 0, -2)$

Linear-Map $\boxed{T(x_1\, x_2\, x_3) = (2x_1 - 3x_2 + 4x_3,\ x_1 - x_3,\ -2x_1 + x_2,\ x_1 + 2x_2 - 2x_3)}$

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* let $\tau : L(U,V) \longrightarrow M_{m\times n}$

$\tau(T) = [T : B_1, B_2]$

is 1-1 & onto

$$T \in L(U,V) \overset{\text{1-1 \& onto}}{\Longleftrightarrow} [T : B_1, B_2] \in M_{m\times n}$$

$$T \in L(U,V) \Longleftarrow (a_{ij}) \in M_{m\times n}.$$