# CTMC

state $i, j$

$v_i$ rate at which the process makes a transition when in state $i$

$P_{ij}$ prob. that this transition is into state $j$

failure rate $\lambda(t) = -\frac{d}{dt} \ln \bar{F}(t)$

$$= \frac{f(t)}{\bar{F}(t)} = \frac{\lambda \cancel{e^{-\lambda t}}}{\cancel{e^{-\lambda t}}}$$

$q_{ij}$ rate, when in state $i$, at which the process makes a transition into state $j$

For $i \neq j$

$$q_{ij} = v_i P_{ij}$$

$$\sum_j q_{ij} = v_i \underbrace{\sum_j P_{ij}}_{=1} = v_i, \ i \neq j$$

$P_{ij}(t) = P(X(t)=j \mid X(0)=i)$

$\underset{\sim}{P}(t) = ((P_{ij}(t)))$

$$\lim_{h \to 0} \frac{\underset{\sim}{P}(h) - I}{h} = Q$$

**Lemma**

(a) $$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i$$

(b) $$\lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \quad \text{when } i \neq j$$

Sol. (a) $1 - P_{ii}(h) = v_i h + o(h)$

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i$$

(b) $P_{ij}(h) = q_{ij} h + o(h)$

$$\lim_{h\to 0} \frac{P_{ij}(h)}{h} = q_{ij}$$

— x —

$$P_{ij}(t+s) = P(X(t+s)=j \mid X(0)=i)$$

$$= \sum_k P(X(t+s)=j, X(t)=k \mid X(0)=i)$$

$$= \sum_k P(X(t+s)=j \mid X(t)=k, \cancel{X(0)=i})\, P(X(t)=k \mid X(0)=i)$$

$$= \sum_k P_{kj}(s)\, P_{ik}(t)$$

$$\boxed{P_{ij}(t+s) = \sum_k P_{ik}(t)\, P_{kj}(s)}$$

Chapman Kolmogorov equation.

$$\underset{\sim}{P}(t+s) = \underset{\sim}{P}(t)\, \underset{\sim}{P}(s)$$

— x —

$$P_{ij}(h+t) - P_{ij}(t) = \sum_k P_{ik}(h)\, P_{kj}(t) - P_{ij}(t)$$

$$= \sum_{k\neq i} P_{ik}(h)\, P_{kj}(t) - (1 - P_{ii}(h))\, P_{ij}(t)$$

$$\lim_{h\to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \sum_{k\neq i} \left(\lim_{h\to 0} \frac{P_{ik}(h)}{h}\right) P_{kj}(t) - \left(\lim_{h\to 0} \frac{1-P_{ii}(h)}{h}\right) P_{ij}(t)$$

$$\boxed{P'_{ij}(t) = \sum_{k\neq i} q_{ik}\, P_{kj}(t) - v_i\, P_{ij}(t)} \quad \text{——} \; \text{(I)}$$

## Kolomogorov Backward equations

Using lemma $\underset{\sim}{P}'(t) = Q\,\underset{\sim}{P}(t)$

### Example (1) Backward equations for B & D process

$$\nu_0 = \lambda_0, \quad \nu_i = \lambda_i + \mu_i, \quad P_{01} = 1, \quad P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i},$$

$$P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

$$q_{i,i+1} = \nu_i P_{i,i+1} = \cancel{(\lambda_i + \mu_i)} \times \frac{\lambda_i}{\cancel{\lambda_i + \mu_i}} = \lambda_i$$

$$q_{i,i-1} = \mu_i$$

From (1)

$$P'_{ij}(t) = \mu_i P_{i-1,j}(t) + \lambda_i P_{i+1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t)$$

$$P'_{0j}(t) = \lambda_0 P_{1j}(t) - \lambda_0 P_{0j}(t)$$

(2) Backward equations for Pure Birth process

B & D process $\lambda_i$, $\mu_i = 0$

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) - \lambda_i P_{ij}(t)$$

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$$P_{ij}(t+h) - P_{ij}(t) = \sum_k P_{ik}(t) P_{kj}(h) - P_{ij}(t)$$

$$= \sum_{k \neq j} P_{ik}(t) P_{kj}(h) - (1 - P_{jj}(h)) P_{ij}(t)$$

$$\lim \frac{P_{ij}(t+h) - P_{ij}(t)}{} = \sum \lim \frac{P_{kj}(h)}{} P_{ik}(t)$$

$h \to 0 \quad h \qquad k \neq j \; h \to 0 \quad h$ ...

$$- \left( \lim_{h \to 0} \frac{1 - P_{jj}(h)}{h} \right) P_{ij}(t)$$

$$\boxed{P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)} \quad \text{——(2)}$$

Kolmogorov forward equation $\Rightarrow \underset{\sim}{P}'(t) = \underset{\sim}{P}(t) Q$

Example B & D process (forward equations)

$$v_0 = \lambda_0, \; v_i = \lambda_i + \mu_i, \; q_{i,i+1} = \lambda_i, \; q_{i,i-1} = \mu_i$$

$$(2) \Rightarrow P'_{ij}(t) = \lambda_{j-1} P_{i,j-1}(t) + \mu_{j+1} P_{i,j+1}(t) - (\lambda_j + \mu_j) P_{ij}(t)$$

$$P'_{i0}(t) = \mu_1 P_{i1}(t) - \lambda_0 P_{i0}(t)$$

— x —

$$\underset{\sim}{P}(t) = ((P_{ij}(t)))$$

$$Q = ((q_{ij}))$$

$q_{ii} = -v_i = -q_i$

$$\begin{cases} \underset{\sim}{P}'(t) = \underset{\sim}{P}(t) Q = Q \underset{\sim}{P}(t) \\ \underset{\sim}{P}(0) = I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \end{cases}$$

$$f'(t) = c f(t)$$

$$\Rightarrow f(t) = f(0) e^{ct}$$

$$\Rightarrow \underset{\sim}{P}(t) = \underset{\sim}{P}(0) e^{Qt} = e^{Qt}$$

$$= \sum_{n \geq 0} \frac{Q^n t^n}{n!}$$

$$\left. \frac{d^n}{dt^n} \underset{\sim}{P}(t) \right|_{t=0} = Q^n$$

Example (Two state M.C.) $\quad X(t) \quad S = \{0, 1\}$

$$Q = 0 \begin{pmatrix} -\alpha & \alpha \end{pmatrix}$$

$$Q = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

$$\nu_0 = \alpha, \quad q_{01} = \alpha, \quad q_{10} = \beta, \quad \nu_1 = \beta$$

$$P_{01} = \frac{q_{01}}{\nu_0} = 1 \qquad\qquad q_{01} = \nu_0 P_{01}$$

$$P_{10} = \frac{q_{10}}{\nu_1} = 1$$

$$\det(Q - kI) = 0 \quad \Rightarrow \quad k_1 = 0, \quad k_2 = -(\alpha+\beta)$$

$$Q = U D U^{-1}, \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & -\alpha-\beta \end{pmatrix}$$

$$P(t) = e^{Qt} = \sum_{k \geq 0} \frac{t^k}{k!} Q^k$$

$$= \sum_{k \geq 0} \frac{t^k}{k!} U D^k U^{-1}$$

$$= U \Big( \underline{\sum_{k \geq 0} \frac{t^k}{k!} D^k} \Big) U^{-1} = U e^{tD} U^{-1}$$

$$= U \begin{pmatrix} 1 & 0 \\ 0 & e^{-t(\alpha+\beta)} \end{pmatrix} U^{-1}$$

$$P_{ij}(t) = A_{ij} + B_{ij} e^{-(\alpha+\beta)t}$$

$$P_{ij}(0) = A_{ij} + B_{ij} = \delta_{ij}$$

$$\left. \frac{dP_{ij}(t)}{dt} \right|_{t=0} = -(\alpha+\beta) B_{ij} = q_{ij}$$

$$P_{11}(t) = A_{11} + B_{11} e^{-(\alpha+\beta)t} = \frac{\alpha}{\phantom{x}} + \frac{\beta}{\phantom{x}} e^{-(\alpha+\beta)t}$$

$$A_{11} + B_{11} = 1$$

$$-(\alpha+\beta) B_{11} = -\beta \Rightarrow B_{11} = \frac{\beta}{\alpha+\beta} \; ; \; A_{11} = \frac{\alpha}{\alpha+\beta}$$

—x—

$$Q = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

$\det(Q - kI) = 0$

$k = 0, -2, -3, -3$

$$P_{ij}(t) = A_{ij} + B_{ij} e^{-2t} + (C_{ij} + D_{ij} t) e^{-3t}$$

—x—

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -1 & 1/2 & 1/2 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \end{array}$$

$P_{ij}(t)$ ?

$$\det(Q - kI) = 0 \Rightarrow k\left(-k^2 - 4k - \frac{7}{2}\right) = 0$$

$$\Rightarrow k_1 = 0 \, , \, k_2 = -2 + \frac{1}{\sqrt{2}} \, , \, k_3 = -2 - \frac{1}{\sqrt{2}}$$

$$P_{ij}(t) = A_{ij} + B_{ij} e^{\left(-2 + \frac{1}{\sqrt{2}}\right)t} + C_{ij} e^{\left(-2 - \frac{1}{\sqrt{2}}\right)t} \quad , \; i,j = 1,2,3$$

Ex. $P_{11}(t)$

$$A_{11} = \frac{2}{7} \, , \, B_{11} = \frac{5 + 3\sqrt{2}}{14} \, , \, C_{11} = \frac{5 - 3\sqrt{2}}{14}$$

$$P_{11}(t) = A_{11} + B_{11} e^{\left(-2 + \frac{1}{\sqrt{2}}\right)t} + C_{11} e^{\left(-2 - \frac{1}{\sqrt{2}}\right)t}$$

as $t \to \infty$ , $P_{11}(t) = \frac{2}{7}$

—x—

Limiting prob. or steady state sol^n

CTMC

$$\pi_j = \lim_{t\to\infty} P_{ij}(t)$$

Using forward equations and taking Lim as $t \to \infty$

$$0 = \sum_{k\neq j} q_{kj}\pi_k - v_j\pi_j$$

$$\sum_{k\neq j} q_{kj}\pi_k = v_j\pi_j \qquad \left(\underset{\sim}{\pi} Q = 0\right)$$

$$\sum_j \pi_j = 1 \qquad \sum \pi_i = 1$$

$$\sum_k q_{kk}\pi_k = 0 \qquad q_{jj} = -v_j$$

$$\pi Q = 0$$

Example (Shoe shine shop) (contd)

$$v_0 = \lambda, \quad v_1 = \mu_1, \quad v_2 = \mu_2$$

$$P_{01} = P_{12} = P_{20} = 1$$

$$q_{01} = v_0 P_{01} = \lambda, \quad q_{12} = v_1 P_{12} = \mu_1, \quad q_{20} = \mu_2$$

generator

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} -\lambda & \lambda & 0 \\ 0 & -\mu_1 & \mu_1 \\ \mu_2 & 0 & -\mu_2 \end{bmatrix} \end{array}$$

$(\pi_0, \pi_1, \pi_2)$

$$\pi Q = 0$$

$$\Pi_0 + \Pi_1 + \Pi_2 = 1$$

$$\left.\begin{aligned} -\lambda \Pi_0 + \mu_2 \Pi_2 &= 0 \\ \lambda \Pi_0 - \mu_1 \Pi_1 &= 0 \\ \Pi_0 + \Pi_1 + \Pi_2 &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} \Pi_0 &= 1/\left(1 + \frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2}\right) \\ \Pi_1 &= \frac{\lambda}{\mu_1}\Pi_0 \\ \Pi_2 &= \frac{\lambda}{\mu_2}\Pi_0 \end{aligned}$$

—x—

<u>Example</u> (B & D process)

$$\nu_n \Pi_n = \sum_{k \neq n} q_{kn} \Pi_k$$

$$\Rightarrow \quad (\lambda_n + \mu_n) P_n = \mu_{n+1} P_{n+1} + \lambda_{n-1} P_{n-1}$$

$$\lambda_0 P_0 = \mu_1 P_1$$

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 \quad , \quad P_2 = \frac{\lambda_1 P_1}{\mu_2} = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0$$

$$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0 = C_n P_0$$

where $C_n = \dfrac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$

$$\sum_i P_i = 1$$

$$\Rightarrow \quad P_0 (1 + C_1 + C_2 + \cdots) = 1$$

$SP_0 = 1$ , where $S = 1 + C_1 + C_2 + \cdots$

limiting prob exist iff $S < \infty$

<u>Example</u> ① Multiserver exponential queueing system (M/M/s)

$$0 \xrightarrow{\lambda_0 = \lambda} 1 \xrightarrow{\lambda_1 = \lambda} 2 \;\cdots\; s-1 \xrightarrow{\lambda_{s-1} = \lambda} s \xrightarrow{\lambda_s = \lambda} s+1 \;\cdots$$

$\mu_1 = \mu \quad \mu_2 = 2\mu \qquad\qquad \mu_s = s\mu \quad \mu_{s+1} = s\mu$

$$\mu_n = \begin{cases} n\mu, & n = 1, 2, \ldots, s \\ s\mu, & n = s+1, s+2, \ldots \end{cases}$$

B & D process

$$\lambda_n = \lambda, \quad n = 0, 1, 2, \ldots$$

$$S < \infty \equiv \sum_{n=1}^{\infty} \left(\frac{\lambda}{s\mu}\right)^n < \infty \equiv \frac{\lambda}{s\mu} < 1$$

(2) Linear growth model with immigration

B&D process $\mu_n = n\mu, \quad n = 1, 2, \ldots$

$$\lambda_n = n\lambda + \theta, \quad n = 0, 1, 2, \ldots$$

$$S < \infty \equiv \sum_{n=1}^{\infty} \underbrace{\frac{\theta(\theta+\lambda)\cdots(\theta+(n-1)\lambda)}{\mu \cdot 2\mu \cdots n\mu}}_{T_n} < \infty$$

ratio test

$$\lim_{n\to\infty} \frac{T_{n+1}}{T_n} < 1$$

$$\Leftrightarrow \lim_{n\to\infty} \frac{\theta(\theta+\lambda)\cdots(\theta+n\lambda)}{(n+1)!\,\mu^{n+1}} \times \frac{n!\,\mu^n}{\theta(\theta+\lambda)\cdots(\theta+(n-1)\lambda)} < 1$$

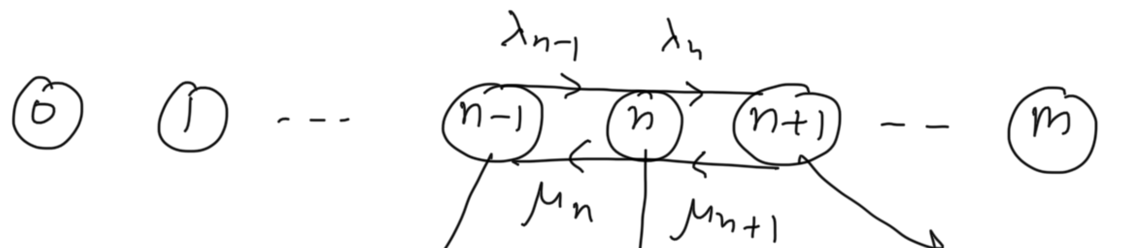
$$\Leftrightarrow \lim_{n\to\infty} \frac{(\theta+n\lambda)}{(n+1)\mu} < 1 \Leftrightarrow \frac{\lambda}{\mu} < 1$$

<u>Example</u>: Consider a job shop that consists of $M$ machines and 1 repairman. Suppose that the

amount of time each machine runs before breaking down is exponentially distributed with mean $\frac{1}{\lambda}$, and suppose that the amt of time it takes for the serviceman to fix a machine is expo. distn with mean $\frac{1}{\mu}$ (i) What is the av. number of machines not in use?

(ii) What proportion of times is each machine in use?

Sol $X(t)$ # of machines not in use at time $t$.
$\in \{0, \ldots, m\}$




$\mu_n = \mu$, $n = 1, 2, \ldots, m$

$m-n+1$ working

$n$ machines not in use, i.e. $m-n$ machines working

$m-n-1$ working

$$\lambda_n = (m-n)\lambda, \quad n = 0, 1, \ldots, m-1$$

$P_n$ : prob that $n$ machines not in use

$$P_n = C_n P_0$$

BD Process

$$P_0 = \frac{1}{S} = \frac{1}{1 + C_1 + C_2 + \cdots}$$

$$= \frac{1}{1 + \sum_{n=1}^{m} \frac{m\lambda (m-1)\lambda \cdots (m-n+1)\lambda}{\mu^n}}$$

$$= \frac{1}{1 + \sum_{n=1}^{m} \frac{m!}{(m-n)!} \left(\frac{\lambda}{\mu}\right)^n}$$

$$P_n = C_n P_0 = \frac{\frac{m!}{(m-n)!} \left(\frac{\lambda}{\mu}\right)^n}{1 + \sum_{n=1}^{m} \frac{m!}{(m-n)!} \left(\frac{\lambda}{\mu}\right)^n}, \quad n = 0, 1, \ldots m$$

(i) av. # of machines not in use $= \sum_{n=0}^{m} n P_n$

(ii) $P(\text{machine is working}) = E\left(P(\text{machine is working} \mid X(t))\right)$

$$= \sum_{n=0}^{m} \underbrace{P(\text{machine is working} \mid X(t)=n)}_{\frac{m-n}{m}} \; \underbrace{P(X(t)=n)}_{P_n}$$

$$= \sum_{n=0}^{m} \frac{m-n}{m} P_n = 1 - \frac{1}{m} \sum_{n=0}^{m} n P_n$$

—x—