Mathematical Model of a Balanced Transportation Problem:

$$\min: Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \dots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \dots, n$$

where $x_{ij} \geq 0 \quad \forall\, i, j$

For a Balanced T.P

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Generally Transportation problems are minimization Type. All the constraints are equality Type. It has $m+n$ constraints and mn variables.

Dual of a Transportation Problem:

$$\max: z' = \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j$$

subject to $u_i + v_j \le c_{ij}$

$i = 1, 2, .., m$

$j = 1, 2, .., n$

where u_i and v_j are free variables.

It has $(m+n)$ variables

and $m \times n$ constraints.

All the variables are free.

Example: $m = 3, \ n = 3$

$$\min: Z = \sum_{i=1}^{3} \sum_{j=1}^{3} c_{ij} x_{ij}$$

subject to $\sum_{j=1}^{3} x_{ij} = a_i, \ i = 1, 2, 3$

$\sum_{i=1}^{3} x_{ij} = b_j, \ j = 1, 2, 3$

$x_{ij} \geq 0, \ i = 1, 2, 3$

$j = 1, 2, 3$

where $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$.

Dual of the T.P:

$$\max: z' = a_1u_1 + a_2u_2 + a_3u_3 + b_1v_1 + b_2v_2 + b_3v_3$$

subject to $u_i + v_j \le c_{ij}$ $\quad i = 1,2,3$; $j = 1,2,3$

u_i and v_j are free variables.

	c_{11}	c_{12}	c_{13}	c_{21}	c_{22}	c_{23}	c_{31}	c_{32}	c_{33}	RHS
	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	Var
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	New
y_1 u_1	1	1	1	0	0	0	0	0	0	a_1
y_2 u_2	0	0	0	1	1	1	0	0	0	a_2
y_3 u_3	0	0	0	0	0	0	1	1	1	a_3
y_4 v_1	1	0	0	1	0	0	1	0	0	b_1
y_5 v_2	0	1	0	0	1	0	0	1	0	b_2
y_6 v_3	0	0	1	0	0	1	0	0	1	b_3

$$\left.\begin{aligned} u_1 + v_1 &\le c_{11} \\ u_1 + v_2 &\le c_{12} \\ u_1 + v_3 &\le c_{13} \end{aligned}\right\} \qquad \left.\begin{aligned} u_2 + v_1 &\le c_{21} \\ u_2 + v_2 &\le c_{22} \\ u_2 + v_3 &\le c_{23} \end{aligned}\right\}$$

$$\left.\begin{aligned} u_3 + v_1 &\le c_{31} \\ u_3 + v_2 &\le c_{32} \\ u_3 + v_3 &\le c_{33} \end{aligned}\right\}$$

$u_1, u_2, u_3, v_1, v_2, v_3$ are free

T. P. Example 1

Destination

Source	x_{11} $\boxed{1}$	x_{12} $\boxed{9}$	x_{13} $\boxed{2}$	7 units
	x_{21} $\boxed{3}$	x_{22} $\boxed{8}$	x_{23} $\boxed{4}$	9 units
	x_{31} $\boxed{5}$	x_{32} $\boxed{7}$	x_{33} $\boxed{6}$	14 units
	10	10	10	

units

Primal

$$\min: Z = x_{11} + 9x_{12} + 2x_{13} + 3x_{21} + 8x_{22} + 4x_{23} + 5x_{31} + 7x_{32} + 6x_{33}$$

subject to

$$x_{11} + x_{12} + x_{13} = 7$$

$$x_{21} + x_{22} + x_{23} = 9$$

$$x_{31} + x_{32} + x_{33} = 14$$

$$x_{11} + x_{21} + x_{31} = 10$$

$$x_{12} + x_{22} + x_{32} = 10$$

$$x_{13} + x_{23} + x_{33} = 10$$

$$x_{ij} \geq 0 \quad \forall i,j$$

Dual of T.P:

$$\max: z' = 7u_1 + 9u_2 + 14u_3 + 10v_1 + 10v_2 + 10v_3$$

subject to

$$u_1 + v_1 \le 1$$
$$u_1 + v_2 \le 9$$
$$u_1 + v_3 \le 2$$
$$u_2 + v_1 \le 3$$
$$u_2 + v_2 \le 8$$
$$u_2 + v_3 \le 4$$
$$v_3 + v_1 \le 5$$
$$u_3 + v_2 \le 7$$
$$u_3 + v_3 \le 6$$

where u_i and v_j are free

In general

$$\max: z' \le \min: z$$

For optimal sol^n:

$$\max: z' = \min: z$$

Solution:

⑦ $\boxed{1}$	$\boxed{9}$	$\boxed{2}$	7
③ $\boxed{3}$	⑥ $\boxed{8}$	$\boxed{4}$	~~9~~ 6
$\boxed{5}$	④ $\boxed{7}$	⑩ $\boxed{6}$	14
~~10~~ 3	~~10~~ 4	10	

NWCR:

$x_{11} = 7$

$x_{21} = 3$, $x_{22} = 6$

$x_{32} = 4$ $x_{33} = 10$

Feasible solution

$Z = 7\times1 + 3\times3 + 6\times8 + 4\times7 + 10\times6$

$= 152$

Check for optimality:

$u_1 + v_1 = 1$	$u_1 = 0$	$v_1 = 1$
$u_2 + v_1 = 3$	$u_2 = 3$	$v_2 = 5$
$u_2 + v_2 = 8$	$u_3 = 2$	$v_3 = 3$
$u_3 + v_2 = 7$		
$u_3 + v_3 = 6$		

Check the other constraints:

$u_1 + v_2 = 5 \leq 9$ ✓

$u_1 + v_3 = 3 > 2$ ×

$u_2 + v_3 = 6 > 4$ × → most +ve diff.

$u_3 + v_1 = 3 \leq 5$ ✓

	v_1	v_2	v_3	
u_1	⑦ $\boxed{1}$	$\boxed{9}$	$\boxed{2}$	7
u_2	③ $\boxed{3}$	⑥ − $\boxed{8}$	⊕ $\boxed{4}$	9
u_3	$\boxed{5}$	④ + $\boxed{7}$	⑩ − $\boxed{6}$	14
	10	10	10	

	v_1	v_2	v_3	
u_1	⑦ $\boxed{1}$	$\boxed{9}$	$\boxed{2}$	7
u_2	③ $\boxed{3}$	$\boxed{8}$	⑥ $\boxed{4}$	9
u_3	$\boxed{5}$	⑩ $\boxed{7}$	④ $\boxed{6}$	14
	10	10	10	

$u_1 = 0, v_1 = 1$

$u_2 = 2, v_2 = 3$

$u_3 = 4, v_3 = 2$

$$\left.\begin{array}{l} u_1 + v_2 = 3 \leq 9, \quad u_1 + v_3 = 2 \leq 2 \\ u_2 + v_2 = 5 \leq 8, \quad u_3 + v_1 = 5 \leq 1 \end{array}\right\} \text{optim Soln}$$

$x_{11} = 7, \; x_{21} = 3, \; x_{23} = 6$

$x_{32} = 10, \; x_{33} = 4$

min: $Z = 134$

$u_1 = 0, \; u_2 = 2, \; u_3 = 4$

$v_1 = 1, \; v_2 = 3, \; v_3 = 2$

$$\text{max: } Z' = 0 \times 7 + 9 \times 2 + 4 \times 14 + 1 \times 10 + 3 \times 10 + 2 \times 10$$
$$= 134$$

max: $Z' = 134$: Dual objective function

min: $Z = 134$: Primal objective Function.

This Transportation Problem has an alternate optimal solution.

Example 2:

	v_1	v_2	v_3	
u_1	⑦ 2	9	4	7
u_2	③ 7	− ⑥ 5	⊕ 3	9
u_3	6	④ + 1	⑩ − 8	14
	10	10	10	

$u_1 = 0 \quad v_1 = 2$

$u_2 = 5 \quad v_2 = 0$

$u_3 = 1 \quad v_3 = 7$

Check for optimality:

$u_1 + v_2 = 0 \leq 9$ ✓

$u_1 + v_3 = 7 > 4$ ×

$u_2 + v_3 = 12 > 3$ × $\Longrightarrow$ most +ve diff.

$u_3 + v_1 = 3 \leq 6$ ✓

	v_1	v_2	v_3	
u_1	⑦ 2	9	4	7
u_2	③ − 7	5	⑥ + 3	9
u_3	⊕ 6	⑩ 1	④ − 8	14
	10	10	10	

$u_1 = 0 \qquad v_1 = 2$

$u_2 = 5 \qquad v_2 = -9$

$u_3 = 10 \qquad v_3 = -2$

$u_1 + v_2 = 2 \leq 9$

$u_1 + v_3 = -2 \leq 4$

$u_2 + v_2 = -4 \leq 5$

$u_3 + v_1 = 12 > 6$ × most +ve element

	v_1	v_2	v_3	
u_1	⑦ $\boxed{2}$	$\boxed{9}$	$\boxed{4}$	7
u_2	$\boxed{7}$	$\boxed{5}$	⑨ $\boxed{3}$	9
u_3	③ $\boxed{6}$	⑩ $\boxed{1}$	① $\boxed{8}$	14
	10	10	10	

$u_1 = 0 \qquad v_1 = 2$

$u_2 = -1 \qquad v_2 = -3$

$u_3 = 4 \qquad v_3 = 4$

$u_1 + v_2 = -3 \leq 9$

$u_1 + v_3 = 4 \leq 4$

$u_2 + v_1 = 1 \leq 7$

$u_2 + v_2 = -4 \leq 5$

} optimal sol^n.

$x_{11} = 7$

$x_{23} = 9$

$x_{31} = 3, \ x_{32} = 10, \ x_{33} = 1$

min: $Z = 14 + 27 + 18 + 10 + 8 = 77$

max: $Z' = 7u_1 + 9u_2 + 14u_3$
$+ 10v_1 + 10v_2 + 10v_3$
$= 0 - 9 + 56 + 20 - 30 + 40$
$= 77$

The present solution is optimal. The problem has alternate optimal solution.

Example: 3

$u_1 = 0$
$u_2 = 3$
$u_3 = 6$
$v_1 = 3$
$v_2 = 2$
$v_3 = -5$

	v_1	v_2	v_3	
u_1	⑦ 3	2	7	7
u_2	③ 6	⑥ 5	4	9
u_3	9	④ 8	⑩ 1	14
	10	10	10	

The present solution is optimal. Check it.

⑩(b)

Example 4

	v_1	v_2	v_3	v_4	
u_1	⑧ 2	5	7	3	8
u_2	② 5	⑩ 3	⑧ 2	7	20
u_3	7	5	② 3	⑩ 2	12
	10	10	10	10	

$u_1 = 0$ $\quad v_1 = 2$

$u_2 = 3$ $\quad v_2 = 0$

$u_3 = 4$ $\quad v_3 = -1$

$v_4 = -2$

$$\left.\begin{aligned} u_1+v_2 &= 0 \le 5 \\ u_1+v_3 &= -1 \le 7 \\ u_1+v_4 &= -2 \le 3 \\ u_2+v_4 &= 1 \le 7 \\ u_3+v_1 &= 6 \le 7 \\ u_3+v_2 &= 4 \le 5 \end{aligned}\right\} \text{Present solution is optimal}$$

$\text{min: } Z = 16+10+30+16+6+20 = 98$

$\text{max: } Z' = 0+60+48+(2-1-2)10$

$= 98$

Example 5:

	v_1	v_2	v_3	v_4	
u_1	⑧ 2	5	7	3	8
u_2	② 3	⑦ 2	7	5	9
u_3	5	③ 7	⑧ 2	3	11
u_4	7	⊕ 3	② 5	⑩ 2	12
	10	10	10	10	

$u_1 = 0$

$u_2 = 1$

$u_3 = 6$

$u_4 = 9$

$v_1 = 2$

$v_2 = 1$

$v_3 = -4$

$v_4 = -7$

$u_1 + v_2 = 1 \leq 5$

$u_1 + v_3 = -4 \leq 7$

$u_1 + v_4 = -7 \leq 3$

$u_2 + v_3 = -3 \leq 7$

$u_2 + v_4 = -6 \leq 5$

× $u_3 + v_1 = 8 > 7$

$u_3 + v_4 = -1 \leq 3$

× $u_4 + v_1 = 11 > 7$

× $u_4 + v_2 = 10 > 3$

The present solution is not optimal

min: $Z = 103$ } check it.

max: $Z' = 103$

Example : 6

	D_1	D_2	D_3	D_4	
S_1	4	8	6	8	50
S_2	6	5	8	8	60
S_3	8	8	6	5	50
	40	40	40	40	

(40) 4	− (10) 8	+ 6	8	50
6	+ (30) 5	(30) − 8	8	60
8	8	(10) 6	(40) 5	50
40	40	40	40	

$x_{11} = 40$, $x_{12} = 10$

$x_{22} = 30$, $x_{23} = 30$

$x_{33} = 10$, $x_{34} = 40$,

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$Z = 890$

$u_1 + v_1 = 4$

$u_1 + v_2 = 8$

$u_2 + v_2 = 5$

$u_2 + v_3 = 8$

$u_3 + v_3 = 6$

$u_3 + v_4 = 5$

$u_1 = 0$

$u_2 = -3$

$u_3 = -5$

$v_1 = 4$

$v_2 = 8$

$v_3 = 11$

$v_4 = 10$

Now

$u_1 + v_3 = 11 > 6$ × → most +ve cell

$u_1 + v_4 = 10 > 8$ ×

$u_2 + v_1 = 1 \leq 6$

$u_2 + v_4 = 7 \leq 8$

$u_3 + v_1 = -1 \leq 8$

$u_3 + v_2 = 3 \leq 8$

	v_1	v_2	v_3	v_3	
u_1	㊵ $\boxed{4}$	$\boxed{8}$	⑩ $\boxed{6}$	$\boxed{8}$	50
u_2	$\boxed{6}$	㊵ $\boxed{5}$	⑳ $\boxed{8}$	$\boxed{8}$	60
u_3	$\boxed{8}$	$\boxed{8}$	⑩ $\boxed{6}$	㊵ $\boxed{15}$	50
	40	40	40	40	

$u_1 = 0$ $\quad v_1 = 4$

$u_2 = 2$ $\quad v_2 = 3$

$u_3 = 0$ $\quad v_3 = 6$

$v_4 = 5$

Check for optimality.
Find the improved solution.

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