Tutorial 4 Discussion

17/02/2022

Tut -4

Q5. N is a normal subgp of G,
$|G|$ is odd and $|N| = 5$.

WTS $N \subseteq Z(G)$

Ans. Define a gp action

$$\phi : G \times N \longrightarrow N$$
$$(g, n) \longmapsto gng^{-1}$$

Permutation representation of the gp action ϕ is

$$\psi : G \longrightarrow S_5$$
$$\psi(g) = \sigma_g$$

$\sigma_g : N \to N$
$\sigma_g(n) = gng^{-1}$.

σ_g is an automorp of N.

$\therefore$ Image $\psi \subseteq \mathrm{Aut}(N)$.

Since $|N| = 5$, therefore N is a cyclic gp of order 5.

hence $\quad \mathrm{Aut}(N) \cong \left(\mathbb{Z}/5\mathbb{Z}\right)^{\times}$

$$\therefore \quad |\mathrm{Aut}(N)| = 4.$$

$$\psi : G \longrightarrow S_5. \qquad \mathrm{Im}\,\psi \leq \mathrm{Aut}(N).$$

$$\ker \psi = \{ g \in G \mid \sigma_g = Id \}$$

$$= \{ g \in G \mid gng^{-1} = n \ \ \forall n \in N \}.$$

By 1st isomorphism Thm.

$$G/\ker\psi \cong \mathrm{Im}\,\psi.$$

Since $|G|$ is odd so $|\mathrm{Im}\,\psi|$ is also odd. But $\mathrm{Im}\,\psi$ is a subgp of $\mathrm{Aut}(N)$ which have even order

$\therefore \ |\mathrm{Im}\,\psi| = 1.$

$\Rightarrow \quad \ker \psi = G.$

$\Rightarrow \quad gng^{-1} = g \quad \forall g \in G$ and $\forall n \in N.$

$\Rightarrow \quad N \subseteq Z(G).$

<u>Lemma</u>: If H is a subgp of S_n, then either all its elts are even permutations or the number of even permutation is same as number of odd permutation.

<u>Pf</u>: Let H is not contained in A_n then $\exists$ at least one odd permutation in H.

Let $\phi : H \longrightarrow \{1, -1\}$.

$$\phi(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

$\ker \phi = H \cap A_n$.

$$H / \ker \phi \cong \mathbb{Z}/2\mathbb{Z}.$$

$$\Rightarrow \quad \frac{|H|}{|H \cap A_n|} = 2.$$

$$\Rightarrow \quad |H| = 2\,|H \cap A_n|$$

$\therefore$ The number of even permutation is same as number of odd permutation.

Q7 $|G| = 2m$ where m is odd.

$\Rightarrow \exists$ an elt of order 2 in G.

i.e $\exists\, x \in G$ s.t $|x| = 2$.

$\phi : G \longrightarrow S_{2m}$.

$\sigma_g : G \to G$.

$\phi(g) = \sigma_g$.

ϕ is an inj gp hm. $\sigma_g(y) = gy$.

$\sigma_x : G \to G$

$\sigma_x(y) = xy$. let $xy = z$.

$\sigma_x(z) = y$. $(y\ z)$.

Since G contains an odd permutation

$\phi(G) \cap A_n$ is a proper subgp of index 2. of $\phi(G)$.

Q10 $20 = 2^2 \cdot 5.$

$n_5 \mid 4$ & $n_5 \equiv 1 \pmod 5$.

$\Downarrow$

$n_5 = 1.$

There is only one Sylow 5-subgp.

There are exactly 4 elts of order 5 as order 5 subgp has to be a cyclic subgp.

Q16 $55 = 5 \cdot 11.$

$n_5 = 1, 11$

$n_{11} = 1.$

There exists only one Sylow 11 subgp.

Say P is a Sylow 11-subgp.

$\therefore$ P is normal and $P = \langle x \rangle$.

Let $\langle y \rangle$ is a Sylow 5-subgp.

$yxy^{-1} \in P$.

$yxy^{-1} = x^r$ for $1 \leq r \leq 10$:

$r^5 \equiv 1 \pmod{11}$.

$r = 3, 4, 5, 9$.

$yxy^{-1} = x^3$. $r = 3$.

$yxy^{-1} = x^4$.

$\Rightarrow$ $\underline{y^2xy^{-2}} = yx^3y^{-1} = x^9$

$\underline{y^3xy^{-3}} = x^5$,