Lecture – 1 (08-01-2024)

(1) Topology, by James R. Munkres,
PHI Publication
Second Edition

(2) Theory and Problems of
General topology, Schaum's outline series

—— // ——

Topology :—

Let X be a nonempty set. A class T of subsets of X is said to be a topology on X if T satisfies the following properties:

(1) $X, \phi \in T$

(2) Union of any number of sets in T also belongs to T.

(3) The intersection of any two sets in T is also in T.

That is

(1) $X, \phi \in T$

(2) $\bigcup_{i \in I} A_i \in T, \quad \forall A_1, A_2, \ldots \in T$

(3) $A_i \cap A_j \in T, \quad \forall A_i, A_j \in T$.

If T is a topology on X, then the members of T are called T-open sets or open sets.

The pair (X, T) is called a topological space.

Ex: $X = \{a, b, c, d, e\}$

$T_1 = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

Clearly (i) $X, \phi \in T_1$

(ii) $\bigcup_i A_i \in T_1$, $\forall A_i \in T_1$

(iii) $A_i \cap A_j \in T_1$, $\forall A_i, A_j \in T_1$

$\therefore$ T_1 is a topology on X.

Now let

$T_2 = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\},$
$\{b, c, d\}\}$

be a class of sub-sets of $X = \{a, b, c, d, e\}$

$\because \{a, c, d\} \cup \{b, c, d\} = \{a, b, c, d\} \notin T_2$

$\therefore$ T_2 is not a topology on X.

let $T_3 = \{X, \phi, \{a\}, \{c, d\},$
$\{a, c, d\}, \{a, b, d, e\}\}$.

$\because \{a, c, d\} \cap \{a, b, d, e\}$

$= \{a, d\} \notin T_3$

$\therefore T_3$ is not a topology on X.

Ex: $X = \{a, b\}$

$T_1 = \{X, \phi, \{a\}\}$

$T_2 = \{X, \phi, \{b\}\}$

$T_3 = \{X, \phi, \{a\}, \{b\}\}$

$T_4 = \{X, \phi\}$.

Clearly T_1, T_2, T_3, T_4 are topologies on X.

Discrete topology :—

Let X be a nonempty set and $\mathcal{D}$ be the class of all subsets of X. Then $\mathcal{D}$ is a topology on X. It is called discrete topology on X and $(X, \mathcal{D})$ is called discrete topological space.

Indiscrete topology :—

Let X be a nonempty set and $\mathcal{I} = \{X, \phi\}$. Then $\mathcal{I}$ is also a topology on X, it is called indiscrete topology on X.

$(X, \mathcal{I})$ is called Indiscrete topological space.

Cofinite topology :

Let T be the class of all subsets of a nonempty set X whose complements are finite together with the empty set φ.

That is

$$T = \{ \varphi, \{A_i \subseteq X \mid A_i^c \text{ is finite}\} \}$$

Let $A_1, A_2, \ldots \in T$

$\Rightarrow A_1^c, A_2^c, \ldots$ are finite

$\Rightarrow A_1^c \cap A_2^c \ldots$ are finite

$\Rightarrow \left(\bigcup_i A_i\right)^c$ is finite

$\Rightarrow \bigcup_i A_i \in T$

Now let $A_1, A_2 \in T$

$\Rightarrow A_1^c, A_2^c$ are finite

$\Rightarrow \quad A_1^c \cup A_2^c$ is also finite
$\quad\quad\quad\quad \parallel$
$\Rightarrow \quad (A_1 \cap A_2)^c$ is finite

$\Rightarrow \quad A_1 \cap A_2 \in T, \ \forall A_1, A_2 \in T,$

Clear $\phi \in T$

$X^c = \phi$ is finite

$\Rightarrow X \in T$

$\therefore (X, T)$ is a topological space.

This topology is called Cofinite topology.

Cocountable topological space :-

Let X be a nonempty set and T be a class of subsets of X whose complements are countable along with the empty set ϕ. Then T is a topology on X. It is called cocountable topology on X.

Ex: Let $X = \mathbb{R}$, the set of real numbers, and let U be the class of all open subsets of X. Then U is a topology on $\mathbb{R}$. This topology is called usual topology or standard topology on $\mathbb{R}$.

$\because$ $U = \{G \subseteq \mathbb{R} \mid G \text{ is open set}\}$.

$\because$ $\mathbb{R}, \phi$ are both open, implies
$\mathbb{R}, \phi \in U$.

$\because$ Arbitary union of open sets is also an open set in $\mathbb{R}$, it follows that

$\bigcup_i A_i \in U, \ \forall \ A_i \in U$.

$\because$ finite intersection of open sets is also an open set, we have

$A_i \cap A_j \in U, \ \forall \ A_i, A_j \in U$.

Note :- If $\{T_i \mid i \in I\}$ is a class of topologies on a nonempty set X, then $\bigcap_{i \in I} T_i$ is also a topology on X.

$\because$ $X, \phi \in T_i \;\; \forall i$

$\Rightarrow X, \phi \in \bigcap_i T_i$

let $A_1, A_2, A_3 \ldots \in \bigcap_i T_i$

$\Rightarrow A_1, A_2, A_3 \ldots \in T_i \;\; \forall i$

$\Rightarrow \bigcup_j A_j \in T_i \;\; \forall i$ [$\because$ each T_i is a topology]

$\Rightarrow \bigcup_j A_j \in \bigcap_i T_i$

llly if $A_1, A_2 \in \bigcap_i T_i$

$\Rightarrow A_1 \cap A_2 \in \bigcap_i T_i$.

[46, 36, 44, 61, 55, 42, 43, 59, 27, 32, 25, 35, 06, 57, 11, 60, 13, 26, 23, 41] .