Lecture-23 (08-04-2024)

Separated Sets:

Two subsets A and B of a topological space $(X, \mathcal{T})$ are said to be separated sets if

(i) A and B are disjoint, i.e., $A \cap B = \phi$

(ii) neither containing the accumulation points of the other

i.e., $A \cap \bar{B} = \phi$ and $\bar{A} \cap B = \phi$.

[That is A and B are separated sets if $A \cap \bar{B} = \phi$ and $\bar{A} \cap B = \phi$].

Ex: $(\mathbb{R}, \mathcal{U})$

$A = (0, 1)$, $B = (1, 2)$, $C = [2, 3)$

Since $A \cap B = \phi$

and $\bar{A} \cap B = [0, 1] \cap (1, 2) = \phi$

$A \cap \bar{B} = (0, 1) \cap [1, 2] = \phi$.

$\therefore$ A and B are separated sets.

But B and C are not separated sets, since

$\because \bar{B} \cap C = [1,2] \cap [2,3) = \{2\} \neq \varphi$.

Connected set :—

A subset A of a topological space (X, T) is said to be disconnected set if there exist open subsets G and H of X such that $A \cap G$ and $A \cap H$ are disjoint non-empty sets whose union is A.

That is $G, H \in T \ni$

$A \cap G \neq \varphi$, $A \cap H \neq \varphi$, $(A \cap G) \cap (A \cap H) = \varphi$

and $A = (A \cap G) \cup (A \cap H)$.

In this case $G \cup H$ is called a disconnection of the set A.

A set 'A' is said to be a connected set if it not a disconnected set.

* Note that

$$A = (A \cap G) \cup (A \cap H)$$

$$= A \cap (G \cup H)$$

$$\iff A \subseteq G \cup H.$$

Also

$$\phi = (A \cap G) \cap (A \cap H)$$

$$= A \cap (G \cap H)$$

$$\iff G \cap H \subset A^c$$

$\therefore$ $G \cup H$ is disconnection of the set A iff $A \cap G \neq \phi$, $A \cap H \neq \phi$, $A \subseteq G \cup H$ and $G \cap H \subset A^c$, where G and H are open sets.

* ϕ and $\{p\}$ are always Connected sets.

Ex: $X = \{a, b, c, d, e\}$

$T = \{X, \phi, \{a, b, c\}, \{c, d, e\}, \{c\}\}$

Then (X, T) is a topological space.

Let $A = \{a, d, e\} \subset X$.

Let $G = \{a, b, c\}$, $H = \{c, d, e\}$

Now $A \cap G = \{a\}$, $A \cap H = \{d, e\}$.

$\therefore A = \{a, d, e\} = (A \cap G) \cup (A \cap H)$

$= \{a\} \cup \{d, e\}$

$(A \cap G) \cap (A \cap H) = \{a\} \cap \{d, e\} = \phi$.

Thus A can be written as disjoint union of non-empty sets $A \cap G$, $A \cap H$, where $G, H \in T$.

$\therefore$ A is disconnected.

Note that the open sets G and H are not disjoint.

Note :- Note that a topological space $(X, \mathcal{T})$ is disconnected if there exist open sets G and H such that

$$X = (X \cap G) \cup (X \cap H)$$
$$= G \cup H$$

Illy

$$\varphi = (X \cap G) \cap (X \cap H)$$
$$= G \cap H.$$

$$X \cap G \neq \varphi \implies G \neq \varphi$$
$$X \cap H \neq \varphi \implies H \neq \varphi \quad]$$

Problem: If A and B are nonempty separated sets in a topological space $(X, \mathcal{T})$, then $A \cup B$ is disconnected.

Sol: Since A and B are separated sets, we have

$$A \cap \overline{B} = \varphi \quad \text{and} \quad \overline{A} \cap B = \varphi .$$

Let $G = \overline{B}^c$ and $H = \overline{A}^c$

Then G and H are open sets.

Also

$$(A \cup B) \cap G = A \cup (B \cap G)$$
$$= A \cup (B \cap \overline{B}^c)$$
$$= A \cup \varphi$$
$$= A \neq \varphi$$

Illy

$$(A \cup B) \cap H = H \cap (A \cup B)$$
$$= (H \cap A) \cup (H \cap B)$$
$$= (\overline{A}^c \cap A) \cup B \neq \varphi$$
$$= B \neq \varphi .$$

$$A \cup B = \left[((A \cup B) \cap G) \cup ((A \cup B) \cap H) \right]$$

and

$$[(A \cup B) \cap G] \cap [(A \cup B) \cap H]$$

$$= A \cap B$$

$$= \phi \qquad [\because A \text{ and } B \text{ are separated sets}].$$

Thus $G \cup H$ form the disconnection of $A \cup B$.

$\therefore$ $A \cup B$ is disconnected

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Problem: let $G \cup H$ be disconnection of the set A in a topological space (X, T). Then $A \cap G$ and $A \cap H$ are separated sets.

Sol: Since $G \cup H$ is disconnection of the set A, we have

$$(A \cap G) \cap (A \cap H) = \phi.$$

Hence we only need to show that each of the sets $A \cap G$ and $A \cap H$ contains no accumulation points of the other.

So let $P \in (A \cap G)'$ — $(*)$

Claim: $P \notin A \cap H$.

Suppose $P \in A \cap H$

$\Rightarrow P \in A$ and $P \in H$.

Since $P \in H$ and H is an open set containing P, by $(*)$ it follows that H must contain at least one point of $A \cap G$ (by definition of a limit point)

$\therefore (A \cap G) \cap H \neq \varphi$ — $(**)$

But

$\varphi = (A \cap G) \cap (A \cap H)$

$= (A \cap G) \cap H \neq \varphi$ which is not true.

$\therefore \; p \notin A \cap H$.

Illy we can prove that

if $p \in (A \cap H)'$, then $p \notin (A \cap G)$.

$\therefore$ $A \cap G$ and $A \cap H$ are separated sets

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Attendce

[65, 11, 63, 16, 27, 32, 06]

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Theorem: A set S in a topological space (X, T) is connected if and only if S is not union of two nonempty separated sets.

Proof: The above statement is equivalent to say that " A set S is disconnected if and only if S is the union of two nonempty separated sets ".

Suppose the set S is disconnected, and let $G \cup H$ be disconnection of S.

Then by the above problem, $S \cap G$ and $S \cap H$ are separated sets.

Also

$$S = (S \cap G) \cup (S \cap H), \quad \begin{matrix} S \cap G \neq \varphi \\ S \cap H \neq \varphi \end{matrix}$$

$$\varphi = (S \cap G) \cap (S \cap H)$$

Conversely if S is the union of two nonempty separated sets say A and B,

$S = A \cup B$, $\quad$ A and B are separated sets.

$\therefore$ S is disconnected by one of the previous problem.

Problem: Let $G \cup H$ be disconnection of the set A in a topological space (X, T). Let B be a connected subset of A. Then either $B \cap H = \varphi$ or

$B \cap G = \phi$, so either $B \subset G$ or $B \subset H$.

Sol: $\because$ $G \cup H$ is disconnection of A, we have

$$A = (A \cap G) \cup (A \cap H),$$

$$(A \cap G) \cap (A \cap H) = \phi, \quad A \cap G \neq \phi$$
$$A \cap H \neq \phi,$$
$$G, H \in T.$$

Given that B is connected and

$$B \subset A = (A \cap G) \cup (A \cap H)$$
$$= A \cap (G \cup H)$$

$$\Rightarrow \quad B \subset G \cup H \quad \text{—— (1)}$$

Also $(A \cap G) \cap (A \cap H) = \phi$

$$\Rightarrow \quad A \cap (G \cap H) = \phi$$
$$\Rightarrow \quad G \cap H \subset A^c \quad \text{—— (2)}$$
$$\Rightarrow \quad G \cap H \subset A^c \subset B^c \quad [\because A \subset B \Rightarrow B^c \subset A^c]$$

$\Rightarrow G \cap H \subset B^c$

$\Rightarrow (G \cap H) \cap B = \phi$

$\Rightarrow (B \cap G) \cap (B \cap H) = \phi$. — (3)

Now by (1) & (3) we have

$B \subset G \cup H$ and $(B \cap G) \cap (B \cap H) = \phi$

If $B \cap G \neq \phi$ and $B \cap H \neq \phi$, then B becomes a disconnected set, which is contradiction to B is a connected set

$\therefore$ either $B \cap G = \phi$ or $B \cap H = \phi$

$\Rightarrow$ either $B \subset H$ or $B \subset G$.

Theorem: If A and B are connected sets in a topological space (X, T), which are not separated, then $A \cup B$ is connected set.

Proof: Suppose $A \cup B$ is disconnected set.

and $G \cup H$ be disconnection of $A \cup B$.

i.e.,

$$A \cup B = [(A \cup B) \cap G] \cup [(A \cup B) \cap H]$$

$$(A \cup B) \cap G \neq \varphi, \quad (A \cup B) \cap H \neq \varphi,$$

$$\varphi = [(A \cup B) \cap G] \cap [(A \cup B) \cap H], \quad G, H \in T$$

Since A and B are connected sets

and $A, B \subset A \cup B$, implies

either $A \subset G$ or $A \subset H$

and $B \subset G$ or $B \subset H$ [by previous problem]

So if $A \subset G$ and $B \subset H$

$\Rightarrow A \cap G \neq \varphi$ & $A \cap H = \varphi$ $\quad$ $B \cap H \neq \varphi$, $B \cap G = \varphi$.

[by previous problem]

$$\begin{aligned}(A \cup B) \cap G &= (A \cap G) \cup (B \cap G) \\ &= A \cup \varphi \\ &= A \quad \text{—— (1)}\end{aligned}$$

Illy $(A \cup B) \cap H = (A \cap H) \cup (B \cap H)$

$= \phi \cup B$

$= B, \quad \text{———} (2)$

But $(A \cup B) \cap G$ and $(A \cup B) \cap H$ are separated sets by one of the previous problem. This implies by (1) and (2) that A and B are separated sets, which is contradiction to A and B are not separated sets.

Illy we can prove for other case $A \subset H$ and $B \subset G$.

$\therefore$ We have either $(A \cup B) \subset G$ or $A \cup B \subset H$, so that $G \cup H$ is not disconnection of $A \cup B$.

$\therefore$ $A \cup B$ is connected.

H.W

Problem: Let $\mathcal{A} = \{A_i\}$ be a class of connected subsets of a topological space (X, T), such that no two members of $\mathcal{A} = \{A_i\}$ are separated sets. Then $\bigcup_i A_i$ is a connected set.

[5 pm
11, 27, 06, 60]

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