MA 41017/MA 60067

Stochastic Processes / Stochastic Process and Simulation

| | Marks | |
|---|---|---|
| CT-1 | 8 | (Fri) 3 Feb. 11:10 – 12:10 |
| Mid | 30 | |
| CT-2 | 8 | (Fri) 31 March 11:10 – 12:10 |
| End | 50 | |
| Project / Attendance / Tutorial | 4 | |
| | 100 | |

Books

1) Intro. to prob. models by S.M. Ross
2) Stochastic processes by S.M. Ross
3) An intro. to stochastic modeling by MA Pinsky, S. Karlin

N(t) indep increment $t_2 > t_1$

$$P(N(t_2)-N(t_1) = x_2 - x_1,\ N(t_1) = x_1)$$

$$= P(N(t_2) - N(t_1) = x_2 - x_1)\ P(N(t_1) = x_1)$$

$P(A,B) = P(A|B)P(B)$

$$= P(N(t_2 - t_1) = x_2 - x_1)\ P(N(t_1) = x_1)$$

N(t) stationary increment

$$\boxed{E(X) = E(E(X|Y))}$$

$$E(X|Y=y) = \sum_x x\, p_{X|Y=y}(x) = \phi(y)$$

$$E(X|Y) = \phi(Y)$$

$$E(E(X|Y)) = E(\phi(Y)) = \sum_y \phi(y)\, p_Y(y)$$

$$= \sum_y \sum_x x\, p_{X|Y=y}(x)\, p_Y(y)$$

$$\frac{p(x,y)}{p_Y(y)}$$

$$E(X) = \sum_x x\, p_x$$

$$E(g(X)) = \sum_x g(x)\, p_x$$

$$= \sum_y \sum_x x \, p(x,y)$$

$$= \sum_x x \sum_y p(x,y) = \sum_x x \, p_X(x) = E(X)$$

$\sum_y p(x,y) \to p_X(x)$

—x—

$X_i \sim Poiss(\lambda_i)$ , $i=1,2$

$\underbrace{X_1, X_2}_{\text{indep}}$ $\quad m_{X_i}(t) = e^{\lambda_i(e^t-1)}$ , $i=1,2$

$S_2 = X_1 + X_2 \sim Poiss(\lambda_1 + \lambda_2)$

$$m_{S_2}(t) = m_{X_1}(t) \, m_{X_2}(t)$$

$$= e^{(\lambda_1+\lambda_2)(e^t-1)}$$

—x—

Syllabus : DTMC, Poiss. Process and related distributions, CTMC, queueing theory, renewal process, martingales, Brownian Motion, simulation.

—x—

Stochastic Process (S.P.) is a family of random variables (rv's) $\{X(t), t \in T\}$, defined on a given probability space, indexed by the parameter $t$, $t \in T$ index set

values assumed by $X(t) \in S$ is called state $\quad$ $S$ ← state space

$T$ parameter space or time space

(1) discrete state , discrete parameter SP

(2) " " , continuous " "

(3) continuous " , " " "

(4) , " , discrete " "

<u>Example</u> Consider a queueing system with jobs arriving at random point in time, queueing for service and departing from the system after service completion.

a) $X(t)$ # of jobs in the system at time $t$

$\{X(t), t \in T\}$

$X(t) \in \{0, 1, 2, \ldots\} = S$ discrete state,

$T = [0, \infty)$ continuous parameter S.P.

b) $W_k$ time that the $k^{th}$ customer has to wait in the system before receiving service.

$\{W_k, k \in T\}$

$W_k \in [0, \infty) = S$ continuous state,

$T = \{1, 2, \ldots\}$ discrete parameter SP

c) $Y(t)$ : cumulative service requirement (experience) of all jobs in the system at time $t$.

$Y(t) \in [0, \infty) = S$ $T = [0, \infty)$

$\{Y(t)\}$ is continuous state, continuous parameter S.P.

d) $N_k$ # of jobs in the system at the time of departure of the $k^{th}$ customer (after service completion).

$N_k \in \{0, 1, 2, \ldots\} = S$

$T = \{1, 2, \ldots\}$

$\{N_k, k \in T\}$ is discrete state, discrete parameter S.P.

— x —

## Discrete time Markov Chain: (DTMC)

S.P. $\{X_n, n = 0,1,2,\ldots\}$ that takes values on a finite or countable number of values

$\{0,1,2,\ldots\} = S \Rightarrow$ discrete state space

discrete parameter S.P. $\{X_n\}$

$i, j, i_0, i_1, \ldots \in S$

$$P(X_{n+1}=j \mid X_n = i, \cancel{X_{n-1}=i_{n-1}, \ldots, X_0 = i_0})$$

$$= P(X_{n+1}=j \mid X_n = i)$$

$$= P_{ij}(n, n+1)$$

$$= P_{ij}^{(1)}$$ stationary transition probability or Homogeneous M.C.

($i$: initial state, $j$: final state)

$$= P_{ij}$$

$P^{(1)} = P$ transition probability matrix (tpm)

$$= \begin{array}{c|c} & j \rightarrow \;\; 0 \quad\quad 1 \quad\quad 2 \quad \ldots \\ i \downarrow \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} & \begin{bmatrix} \boxed{P_{00} \quad P_{01} \quad P_{02} \quad \cdots} \\ P_{10} \quad P_{11} \quad P_{12} \quad \cdots \\ P_{20} \quad P_{21} \quad P_{22} \quad \cdots \\ \cdots \\ \cdots \end{bmatrix} \end{array}$$

$\Sigma = 1$

$\pi = [\frac{1}{n}, \frac{1}{n}, \quad ]$

DSP

$$0 \le P_{ij} \le 1, \quad \forall i, \forall j$$

$$\sum_j P_{ij} = 1 \quad \text{for fixed } i$$

—x—

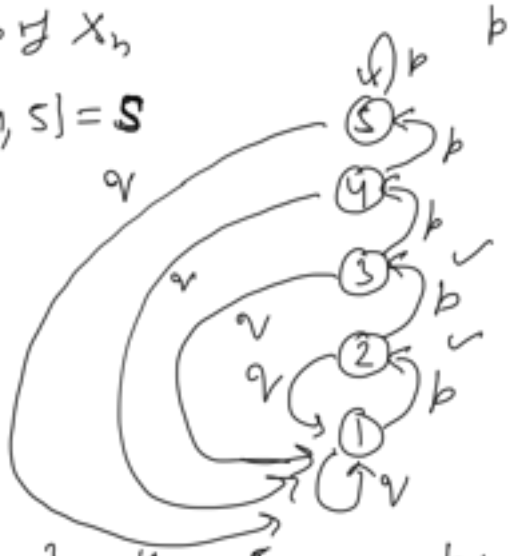
**Example 1** Consider a game of ladder climbing.

There are 5 levels in the game, level 1 is lowest (bottom) and level 5 is the highest (top). A player starts at the bottom. Each time, a fair coin is tossed. If it turns up heads, the player moves up one rung. If tails, the player moves down to the very bottom. Once at the top level, the player moves to the very bottom if tails turns up and stays at the top if head turns up.

Let $X_n$ be the level of the game in the $n$th step / transition. tpm of $X_n$ $\quad p=q=\frac{1}{2}$

$$X_n \in \{1,2,3,4,5\} = S$$

$X_n$ DTMC



$$P_{ij} = P(X_{n+1}=j \mid X_n=i)$$
$$= P(X_1=j \mid X_0=i)$$

tpm

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} q & p & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ q & 0 & 0 & p & 0 \\ q & 0 & 0 & 0 & p \\ q & 0 & 0 & 0 & p \end{bmatrix} \end{array}$$

$p+q=1$

$p=q=\frac{1}{2}$

$$\left\{ P(\underline{X_{n+1}=j} \mid \underline{X_n=i}, \cancel{X_{n-1}=i_{n-1}, \dots, X_0=i_0}) = P(\underline{X_{n+1}=j} \mid \underline{X_n=i}) \right.$$

— x —

<u>Example</u>: Let $\{X_n\}_{n=0,1,2,\dots}$ be a sequence of i.i.d. (independently & identically distributed)

discrete r.v. with $p_j = P(X_1=j) = \left(\frac{1}{2}\right)^{j+1}$, $\forall j = 0,1,2,3,-$

Determine whether each of the following chain is Markovian or not. If so find its corresponding state space (S) and tpm

(i) $\{S_n\}_{n=0,1,2,--}$ where $S_n = \sum_{i=1}^{n} X_i$

(ii) $\{M_n\}_{n=0,1,2,—}$ where $M_n = \max\{X_1, X_2, --, X_n\}$

Sol (i) $S_n \in \{0,1,2,---\}$ $\quad S_{n+1} = S_n + X_{n+1}$

$P_{ij} = P(S_{n+1}=j \mid S_n = i)$

tpm

$$P = \begin{array}{c|c} & S_{n+1}=j \rightarrow \\ S_n=i & \begin{array}{ccccc} 0 & 1 & 2 & 3 & ---- \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} & \begin{bmatrix} p_0 & p_1 & p_2 & p_3 & ---- \\ 0 & p_0 & p_1 & p_2 & -- \\ 0 & 0 & p_0 & p_1 & -- \\ - & - & — & & \\ — & — & & & \end{bmatrix} \end{array} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & -- \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & -- \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & -- \\ - & - & — & & \end{bmatrix}$$

Example (Transformation of a process into M.C.)

Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Suppose that if it has rained for the past two days, then it will rain tomorrow with prob. (w.p) 0.7; if it has rained today but not yesterday, then it will rain tomorrow w.p 0.5; if it has rained yesterday but not today, then it will rain tomorrow w.p 0.4; if it has not rained in the past two days, then it will rain tomorrow w.p. 0.2.

Sol: $X_n$ state at any time is determined by the weather condition during both that day and the previous day

| State $X_n$ | Rained yesterday | Rained today |
|---|---|---|
| 0 | ✓ | ✓ |
| 1 | X | ✓ |
| 2 | ✓ | X |
| 3 | X | X |

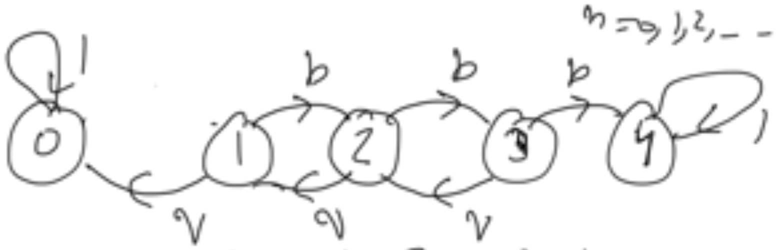
$S = \{0, 1, 2, 3\}$

$$P = \begin{array}{ccc|cccc} & & \text{yesterday today} & \text{today tomorrow} & & X_{n+1}=j & \\ & & & 0 & 1 & 2 & 3 \\ & & & \checkmark\checkmark & \times\checkmark & \checkmark\times & \times\times \\ & 0 & \checkmark\checkmark & 0.7 & 0 & 0.3 & 0 \\ X_n=i & 1 & \times\checkmark & 0.5 & 0 & 0.5 & 0 \\ & 2 & \checkmark\times & 0 & 0.4 & 0 & 0.6 \\ & 3 & \times\times & 0 & 0.2 & 0 & 0.8 \end{array}$$

<u>Example 1</u> Particle performs a random walk in states $\{0, 1, 2, 3, 4\}$. It remains in state 0 and 4 with probability 1. It moves from state $r$ $(0 < r < 4)$ to $r+1$ with prob $p$; and from state $r$ to $r-1$ with prob. $q = 1 - p$. $(0 < r < 4)$

Let $X_n$ : position of particle at time/step $n$.

$X_n \in \{0, 1, 2, 3, 4\} = S$ $\quad \{X_n\}$ M.C. $n = 0, 1, 2, \ldots$




0, 4 absorbing states

0 1 2 3 4

P 0 [ 1 0 0 0

tpm $\quad P =$

$$
\begin{array}{c|ccccc}
 & & & & & \\
1 & q & 0 & p & 0 & 0 \\
2 & 0 & q & 0 & p & 0 \\
3 & 0 & 0 & q & 0 & p \\
4 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

n-step transition probability:

$i,j \in S \qquad \{X_n\}$ M.C. $\qquad S=\{0,1,2,\dots\}$

$$P_{ij}^{(n)} = P(X_{m+n}=j \mid X_m=i) = P(X_n=j \mid X_0=i)$$ homogeneous

n step tpm $\rightarrow$ $P^{(n)} = \left(\left(P_{ij}^{(n)}\right)\right)$

$A = [a_{ij}]$

$$
P^{(n)} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\
\begin{bmatrix}
P_{00}^{(n)} & P_{01}^{(n)} & P_{02}^{(n)} & \cdots \\
P_{10}^{(n)} & P_{11}^{(n)} & P_{12}^{(n)} & \cdots \\
- & - & - & \\
 & - & - & \\
 & & &
\end{bmatrix}
\end{array}
$$

$$\sum_j P_{ij}^{(n)} = 1 \quad \text{for fixed } i$$

$$0 \le P_{ij}^{(n)} \le 1 \quad \forall i, \forall j$$

Chapman Kolmogorov equation $\qquad i,j,k \in S$

$$\boxed{P_{ij}^{(m+n)} = \sum_k P_{ik}^{(m)} P_{kj}^{(n)} = \sum_k P_{ik}^{(n)} P_{kj}^{(m)}}$$

(i,j)th cell of $P^{(m+n)}$ Sol

$$P_{ij}^{(m+n)} = P(X_{m+n}=j \mid X_0=i)$$

$$= \sum_k P(X_{m+n}=j, X_n=k \mid X_0=i)$$

$$P(A) = P\left(\bigcup_i (A \cap E_i)\right)$$
$$= \sum_i P(A \cap E_i)$$

$$= \sum_k P(X_{m+n}=j \mid X_n=k, \cancel{X_0=i})\, P(X_n=k \mid X_0=i)$$

$$\boxed{P(AB \mid C) = P(A \mid BC)\, P(B \mid C)}$$

$$= \sum_k P(X_{\underset{m}{\cancel{m+n}}}=j \mid X_{\underset{0}{\cancel{n}}}=k)\, P(X_n=k \mid X_0=i)$$

$$= \sum_k p^{(m)}_{kj}\, p^{(n)}_{ik}$$

$$\underset{P^{(n)}}{\begin{pmatrix} - & - & \\ p^{(n)}_{i0} & p^{(n)}_{i1} & \cdots \\ - & - & \\ - & - & \end{pmatrix}} \underset{P^{(m)}}{\begin{pmatrix} - & p^{(m)}_{0j} & - \\ - & p^{(m)}_{1j} & - \\ \cdots & \vdots & - \\ - & | & - \end{pmatrix}}$$

$$= P^{(m+n)}$$

$$\underline{P^{(m+n)}} = P^{(n)}P^{(m)} = \underline{P^{(m)}P^{(n)}}$$ matrix form

$$P^{(1)} = P$$

$$P^{(2)} = P^{(1)}P^{(1)} = P.P = P^2$$

$$\boxed{P^{(n)} = P^n}\ \checkmark$$

$$P^{(n)} = \left\{ p^{(n)}_{00} \right.$$

$$\cancel{P^n = \{ p^n_{00}}$$

—x—

<u>pmf of step $X_n$</u>

$i \in S = [0, 1, 2, \ldots]$

$$P(X_n = i) = p_i^{(n)}$$

pmf of $X_n$

→ prob

$$\underset{\sim}{p}^{(n)} = (P(X_n=0), P(X_n=1), \ldots) = (p_0^{(n)}, p_1^{(n)}, \ldots)$$

pmf of $X_0$

$$\underset{\sim}{p}^{(0)} = (p_0^{(0)}, p_1^{(0)}, \ldots)$$

$P \leftarrow$ tpm

$\underset{\sim}{p}^{(n-2)} P P = \underset{\sim}{p}^{(n-2)} P^2$

<u>Claim</u> $\underset{\sim}{p}^{(n)} = \underset{\sim}{p}^{(n-1)} P = \cdots = \underset{\sim}{p}^{(0)} P^n$

$$p_i^{(n)} = p_0^{(n-1)} P_{0i} + p_1^{(n-1)} P_{1i} + \cdots$$
$$= \sum_k p_k^{(n-1)} P_{ki}$$

Sol $p_i^{(n)} = P(X_n = i)$

$$= \sum_k P(X_n = i, X_{n-1} = k)$$
$$= \sum_k P(X_n = i \mid X_{n-1} = k) P(X_{n-1} = k)$$
$$= \sum_k P_{ki}\, p_k^{(n-1)}$$

$$(p_0^{(n)}, p_1^{(n)}, \ldots p_i^{(n)}) = (p_0^{(n-1)}, p_1^{(n-1)}, \ldots) \begin{pmatrix} P_{00} & P_{01} & P_{0i} \\ \cdot & \text{---} & P_{1i} \end{pmatrix}$$

$$\boxed{\underset{\sim}{p}^{(n)} = \underset{\sim}{p}^{(n-1)} P}$$

pmf of $X_n$

$[P(X^n = 1)\ P(X^n = 2)\ P(X^n = 3)]$

$[P(X^{n-1} = 3)\ P(X^{n-2} = 2)]$

—x—

<u>Example</u>: $\{X_n\}$ M.C. $S = \{1, 2, 3\}$

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.1 & 0.5 & 0.4 \\ 2 & 0.6 & 0.2 & 0.2 \\ 3 & 0.3 & 0.4 & 0.3 \end{array}$$

$P(X_0 = 1) = 0.7$, $P(X_0 = 2) = 0.2$, $P(X_0 = 3) = 0.1$

(a) $P(X_0 = 2, X_2 = 3, X_1 = 3, X_3 = 2)$

$P(X_3, X_2, X_1, X_0)$
$= P(X_3 \mid X_2, \cancel{X_1, X_0})$
$P(X_2 \mid X_1, \cancel{X_0}) P(X_1 \mid X_0) P(X_0)$

$= P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$

$= P_{32} P_{33} P_{23} P(X_0 = 1)$

$= P(X_3 = 2 \mid X_2 = 3, X_1 = 3, X_0 = 2)\, P(X_2 = 3 \mid X_1 = 3, X_0 = 2)$

$$\cdot\, P(X_1=3 \mid X_0=2)\, P(X_0=2)$$

$$= P(X_3=2 \mid X_2=3)\, P(X_2=3 \mid X_1=3)\, P(X_1=3 \mid X_0=2)\, P(X_0=2)$$

$$= P_{32}\, P_{33}\, P_{23}\, P(X_0=2) = 0.4 \times 0.3 \times 0.2 \times 0.2$$
$$= 0.0048$$

(b) $P(X_2=3, X_1=3 \mid X_0=2)$

$$= P(X_2=3 \mid X_1=3, \cancel{X_0=2})\, P(X_1=3 \mid X_0=2)$$

$$= P(X_2=3 \mid X_1=3)\, P(X_1=3 \mid X_0=2)$$

$$= P_{33}\, P_{23} = 0.3 \times 0.2$$

(c) $P(X_3=2, X_0=2, X_1=3)$ $\quad P^{(2)}_{32}\, P_{23}\, P(X_0=2)$

$$= P(X_3=2, X_1=3, X_0=2)$$

$$= P(X_3=2 \mid X_1=3, \cancel{X_0=2})\, P(X_1=3 \mid X_0=2)\, P(X_0=2)$$

$$= P(X_3=2 \mid X_1=3)\, P(X_1=3 \mid X_0=2)\, P(X_0=2)$$

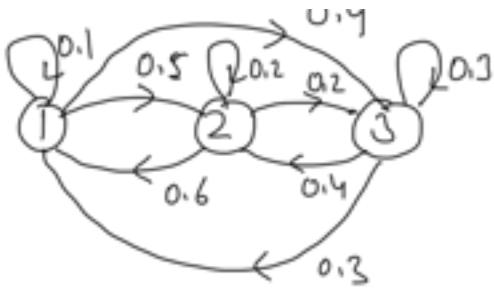
$$= \underline{P^{(2)}_{32}}\, P_{23}\, P(X_0=2) = 0.35 \times 0.2 \times 0.2$$
$$= 0.014$$

$$\underline{P^{(2)} = P^2 = P.P} \qquad k \in S = \{1,2,3\}$$

$$P^{(2)}_{32} = \sum_k P_{3k}\, P_{k2}$$

$$= P_{31}P_{12} + P_{32}P_{22} + P_{33}P_{32}$$

$$= 0.3 \times 0.5 + 0.4 \times 0.2 + 0.3 \times 0.4$$
$$= 0.35$$




(d) $P(X_2 = 3) = p_3^{(2)}$

$\underset{\sim}{p}^{(2)}$ needed

$P \leftarrow$ tpm

$[P(X_2 = 0)\ P(X_2 = 1) \uparrow ]$

$\underset{\sim}{p}^{(2)} = \underset{\sim}{p}^{(1)} P$

$\underset{\sim}{p}^{(1)} = \underset{\sim}{p}^{(0)} P \qquad \underset{\sim}{p}^{(0)} = (0.7, 0.2, 0.1)$

$$\underset{\sim}{p}^{(1)} = (0.7, 0.2, 0.1)\begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$= (0.22, 0.43, 0.35)$$

$\underset{\sim}{p}^{(2)} = \underset{\sim}{p}^{(1)} P$

$$p_3^{(2)} = 0.22 \times 0.4 + 0.43 \times 0.2 + 0.35 \times 0.3$$
$$= 0.279$$

**Example** Consider a two state M.C. $(X_n)$ having state space $S = \{0, 1\}$ with tpm

$$Q = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{array}$$

Whether $Z_n = (X_{n-1}, X_n)$ is a M.C.? If so determine state space and tpm of $\{Z_n\}$.

$$\begin{array}{c|cccc} & X_n, X_{n+1} & & & \\ X_{n-1}, X_n & 00 & 01 & 10 & 11 \\ \hline 00 & 1/2 & 1/2 & 0 & 0 \\ 01 & 0 & 0 & 1/3 & 2/3 \\ 10 & 1/2 & 1/2 & 0 & 0 \\ 11 & 0 & 0 & 1/3 & 2/3 \end{array}$$

Sol. $\{Z_n\}$ is a M.C.

$S = \{(0,0), (0,1), (1,0), (1,1)\}$

| | (0,0) | (0,1) | (1,0) | (1,1) |
|---|---|---|---|---|
| (0,0) | 1/2 | 1/2 | 0 | 0 |

| | | | | |
|---|---|---|---|---|
| (0,1) | 0 | 0 | 1/3 | 2/3 |
| (1,0) | 1/2 | 1/2 | 0 | 0 |
| (1,1) | 0 | 0 | 1/3 | 2/3 |

$$P_{(i,j),(k,l)} = P(\underset{(X_n, X_{n+1})}{Z_{n+1}} = (k,l) \mid \underset{(X_{n-1}, X_n)}{Z_n} = (i,j))$$

$i,j,k,l \in \{0,1\}$

$$P(Z_{n+1}=(1,1) \mid Z_n = (0,1)) =$$

$$= P(\underbrace{X_n=1}_{B}, \underbrace{X_{n+1}=1}_{A} \mid \underbrace{X_{n-1}=0}_{C}, \underbrace{X_n=1}_{B})$$

$$= P(X_{n+1}=1 \mid X_n=1, \cancel{X_{n-1}=0})$$

$$= P(X_{n+1}=1 \mid X_n=1) = Q_{11} = \frac{2}{3}$$

$$P(AB \mid BC) = \frac{P(ABBC)}{P(BC)} = \frac{P(ABC)}{P(BC)} = P(A \mid BC)$$

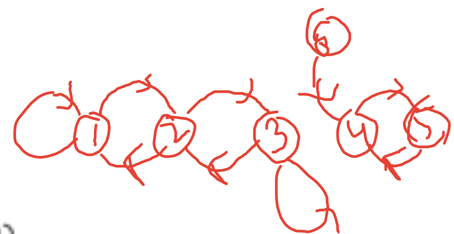
—x—

## Classification of states:

$\{X_n\}$ m.c. $S=\{0,1,2,3,--\}$

$i,j,k \in S$

Def$^n$ $i \to j$, state $j$ is accessible from state $i$ if $P_{ij}^{(n)} > 0$ for some $n$.




Def$^n$ $i \leftrightarrow j$, state $i$ and $j$ communicate with each other if $i \to j$ and $j \to i$

Result $i \leftrightarrow j$, $j \leftrightarrow k \Rightarrow i \leftrightarrow k$

Sol $\exists\ n, m$ s.t. $P_{ij}^{(n)} > 0$, $P_{jk}^{(m)} > 0$

$$P_{ik}^{(m+n)} = \sum_{l} P_{il}^{(n)} P_{lk}^{(m)} \quad \text{(CK equations)}$$

$$\geq P_{ij}^{(n)} P_{jk}^{(m)}$$

$$> 0$$

$i \to k$ . Similarly $k \to i$ $\therefore$ $i \leftrightarrow k$

<u>Defn</u> M.C $\{X_n\}$ is <u>irreducible</u> or (connected) <u>if every</u> <u>state communicate with every other state</u> otherwise reducible.

<u>Defn</u> <u>period of state $i$</u>

$$\begin{cases} d(i) = \gcd\ \mathbb{I}^+ = \{1,2,\ldots\}\ n \text{ st. } P_{ii}^{(n)} > 0 \\ (\text{If } P_{ii}^{(n)} = 0\ \forall n \geq 1, \text{ define } d(i) = 0) \end{cases}$$

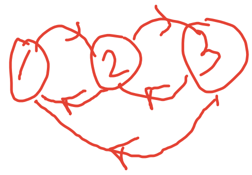
gcd $(1,2,3,4,5\ )$

$d(1) = 1$

$d(1) = 0$

<u>Example</u>: ① $\{X_n\}$ $S = \{0,1\}$

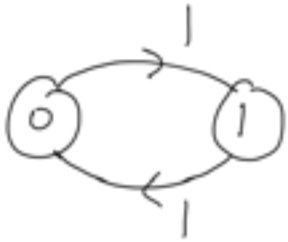
$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$



Irreducible M.C

gcd $\{2,3,4,5\ \}$

$d(1) = 1$




$0 \leftrightarrow 1$

Class $\{0,1\}$

same

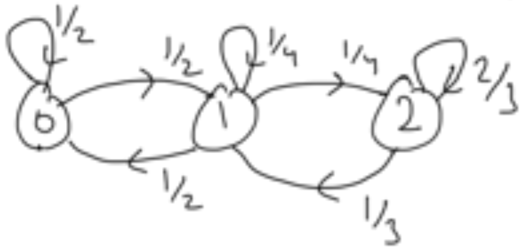
$d(0) = \gcd\{2,4,6,\ldots\}$ $P_{00}^{(m)} > 0$

$= 2 = d(1)$

② M.C $\{X_n\}$ $S = \{0,1,2\}$ having tpm

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1/2 & 1/2 & 0 \\ 1 & 1/2 & 1/4 & 1/4 \end{array}$$

$$2 \begin{vmatrix} 0 & 1/3 & 2/3 \end{vmatrix}$$




Classes $\{0,1\}$ $\{2\}$

$0 \leftrightarrow 1 \leftrightarrow 2$ Class $= \{0,1,2\}$

Irreducible M.C.

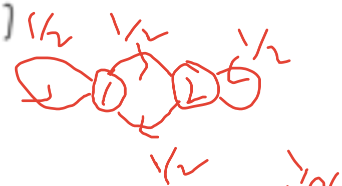
$d(0) = \gcd\{1,2,3,\dots\} = 1$ $P_{00}^{(4)} > 0$

$= d(1) = d(2)$

—x—

$\{X_n\}$ M.C. $S = \{0,1,2,\dots\}$

For state $i \in S$




$f_{11}^{(1)} = 1/2$

$f_{11}^{(2)} = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}$

$f_{11}^{3} = 3$ terms

$$f_{ii}^{(n)} = P(X_n = i,\ X_k \neq i,\ k = 1,2,\dots,n-1 \mid X_0 = i)$$

recurrence time prob

probability of first visit to state $i$ in $n$ transitions/steps, starting from state $i$

$$\boxed{f_{ii}^{(0)} = 1}$$

$$f_{ii} = f_{ii}^{(1)} + f_{ii}^{(2)} + f_{ii}^{(3)} + \cdots$$

probability of ever visiting state $i$, starting from state $i$

→ $f_{ii} = 1$, i.e., return to state $i$ is certain, starting from state $i$

$i$ recurrent state

→ $f_{ii} < 1$, i.e., return to state $i$ is uncertain

$i$ transient state

—x—

$$P_{ij}^{(n)} = P(X_{m+n} = j \mid X_m = i)$$

Let $\quad I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & , X_n \neq i \end{cases}$

$\sum_{n=1}^{\infty} I_n$ : # of time periods, the process is in state $i$

$$E\left(\sum_{n=1}^{\infty} I_n \,\middle|\, X_0 = i\right) = \sum_{n=1}^{\infty} E(I_n | X_0 = i)$$

$$= \sum_{n=1}^{\infty} \left[1 . P(X_n = i | X_0 = i) + 0 . P(X_n \neq i | X_0 = i)\right]$$

$$= \sum_{n=1}^{\infty} P_{ii}^{(n)}$$

$$\boxed{i \text{ recurrent} \Leftrightarrow f_{ii} = 1 \Leftrightarrow \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty}$$

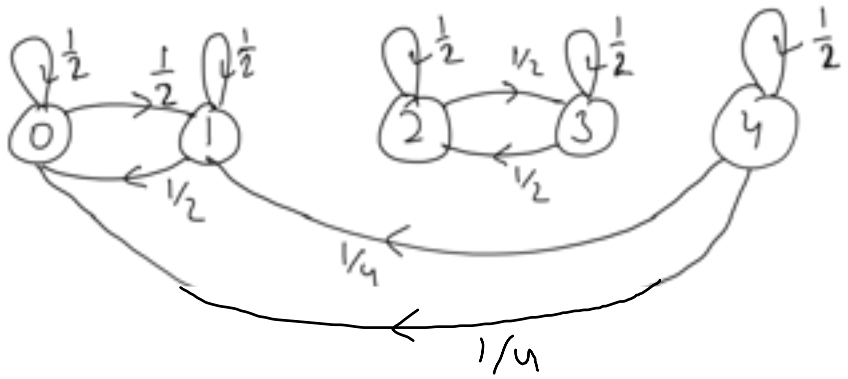
$$i \text{ transient} \Leftrightarrow f_{ii} < 1 \Leftrightarrow \sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$$

**Example** Consider a M.C having states 0, 1, 2, 3, 4 and tpm

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

$\pi \frac{1}{2} = \pi$

$\rightarrow \pi = 0$ for 4




$\{0, 1\}$

$\{2, 3\}$

$\{4\}$

$f_{22} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

$= \dfrac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ recurrent

$f_{44} = \frac{1}{2} < 1$ transient

$1/4$

$0 \leftrightarrow 1, \; 2 \leftrightarrow 3, \; 4$ Reducible M.C.

Class $\{0,1\}, \{2,3\}, \{4\}$

$\{0,1\}, \{2,3\}$ → recurrent; $\{4\}$ → transient

$$f_{00} = f_{00}^{(1)} + f_{00}^{(2)} + f_{00}^{(3)} + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)$$

$$= \frac{1}{2} \times \frac{1}{1-\frac{1}{2}} = 1$$

0 recurrent.

$$f_{44} = f_{44}^{(1)} + f_{44}^{(2)} + \cdots = \frac{1}{2} + 0 + 0 + \cdots < 1$$

4 transient.

—x—

P1. $i \leftrightarrow j$, $i$ recurrent $\Rightarrow$ $j$ recurrent & $|||^{ly}$ for transient

Sol

Given $\begin{cases} i \leftrightarrow j \Rightarrow \exists\, n, m \text{ s.t. } P_{ij}^{(n)} > 0, \; P_{ji}^{(m)} > 0 \\ i \text{ recurrent} \Leftrightarrow \sum_{\nu} P_{ii}^{(\nu)} = \infty \end{cases}$

$$P_{jj}^{(m+n+\nu)} \geq P_{ji}^{(m)} P_{ii}^{(\nu)} P_{ij}^{(n)} \quad \text{[Using CK eqs}$$

$$\sum_{\nu} P_{jj}^{(m+n+\nu)} \geq P_{ji}^{(m)} P_{ij}^{(n)} \underbrace{\sum_{\nu} P_{ii}^{(\nu)}}_{=\infty}$$

$$= \infty$$

$\Rightarrow$ $j$ recurrent.

P2 $\quad i \leftrightarrow j$, $i$ transient $\Rightarrow$ $j$ transient

P3 In a finite state M.C all states can not be transient.

P4 In a finite state, irreducible M.C. all states are +ve recurrent.

all states communicate

Hence, if any 1 state transient $\Rightarrow$ Reducible MC

sol/ using P1 and P3.

$f_{11}$: $<1$ transient; $=1$ recurrent

recurrent: $m_{11}<\infty$ positive recurrent (finite); $m_{11}=\infty$ null recurrent

Def: Let $i$ recurrent

$$\boxed{m_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)}}$$ mean recurrence time

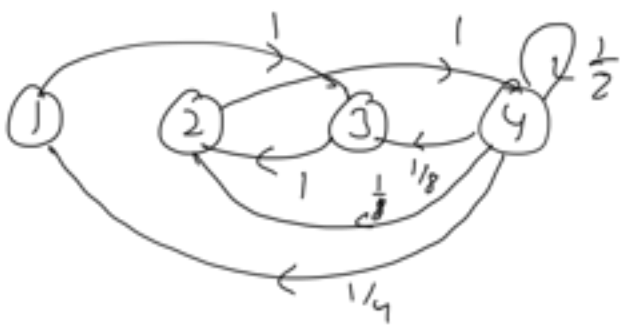
If $m_{ii} = \infty$, $i$ null recurrent

If $m_{ii} < \infty$, $i$ non-null recurrent/positive recurrent

Example: $(X_n)$ M.C. $S = \{1,2,3,4\}$

tpm

$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 0 \\ 4 & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{2} \end{array}$$




$1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$ Irreducible M.C

Class $\{1,2,3,4\}$ finite state

all states are positive recurrent.

$f_{44} = f_{44}^{(1)} + f_{44}^{(2)} + f_{44}^{(3)} + f_{44}^{(4)} + f_{44}^{(5)} \ldots$

$f_{44} = f_{44}^{(1)} + f_{44}^{(2)} + f_{44}^{(3)} + f_{44}^{(4)} + f_{44}^{(5)} + \cdots$

$$= \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{4} + 0 + 0 + \cdots$$

$$= 1 \qquad 4 \text{ recurrent}$$

$$m_{44} = \sum_{n=1}^{\infty} n f_{44}^{(n)} = 1 \times \frac{1}{2} + 2 \times \frac{1}{8} + 3 \times \frac{1}{8} + 4 \times \frac{1}{4} + 0 + 0 + \cdots$$

$$= \frac{17}{8} < \infty$$

→ Finite state, irreducible M.C. all states are +ve recurrent

recurrent — +ve → if finite; null

→ Irreducible M.C., all states are either +ve recurrent or null recurrent or transient.

transient

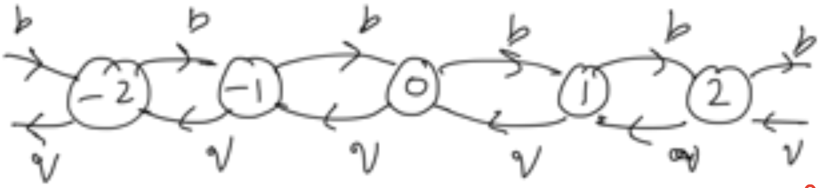
## Example

One-dimensional random walk

$S = \{ \cdots, -2, -1, 0, 1, 2, \cdots \}$

$X_n$ position of particle at $n$th step

$P_{i,i+1} = p \; ; \; P_{i,i-1} = q = 1-p, \; P_{i,j} = 0, \; j \neq i-1, i+1$



$$P_{ii}^{(n)} = \begin{cases} \binom{2m}{m} \boxed{p^m (1-p)^m} & n = 2m \\ 0 & n = 2m+1 \end{cases} \quad , m = 1, 2, 3, \cdots$$

no of paths

single path probability of returning

$$= \begin{cases} a_m & , n = 2m \\ 0 & , n = 2m+1 \end{cases}$$

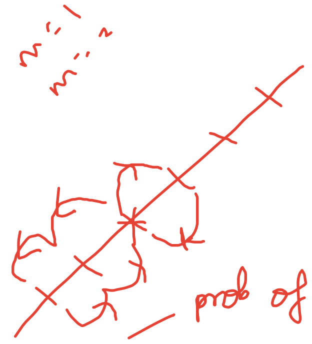
Ratio test

$$\lim_{m \to \infty} \frac{a_{m+1}}{a_m} = \begin{cases} < 1, \sum_m a_m \text{ converge} \end{cases}$$

$\lim_{m\to\infty} a_m$ $\left\{ >1, \sum_m a_m \text{ diverges} \right.$

$$\frac{a_{m+1}}{a_m} = \frac{\binom{2m+2}{m+1} p^{m+1} (1-p)^{m+1}}{\binom{2m}{m} p^m (1-p)^m}$$

$$= \frac{(2m+2)(2m+1)}{(m+1)(m+1)}\, p\,(1-p)$$



$$\lim_{m\to\infty} \frac{a_{m+1}}{a_m} = 4p(1-p) = \begin{cases} 1 & ; \quad p=\frac{1}{2} \\ <1 & \quad p\neq \frac{1}{2} \end{cases}$$

prob of return

$P(m=1) = P(m=2)$

$P(m=1) > P(m=2$

$p \neq \frac{1}{2} \Rightarrow \sum_n P_{ii}^{(n)} < \infty$, i.e, $i$ transient (return gets difficult)

$p \neq \frac{1}{2}$ irreducible M.C all states are transient

irreducible

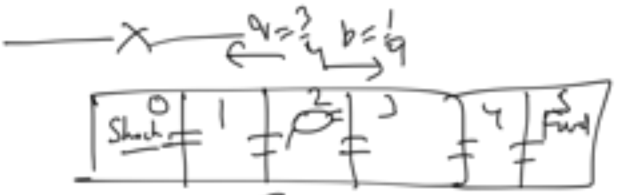
show $p = \frac{1}{2}$ $i$ recurrent .. — — — recurrent

Food

$$P_2 = \frac{1-(3)^2}{1-3^5} =$$

$\frac{q}{p} = 3$

$i=2, N=5$



Gambler's ruin problem:

Shock $\quad 1-P_2$

initial capital Rs $i$ $\qquad$ aim Rs $N$

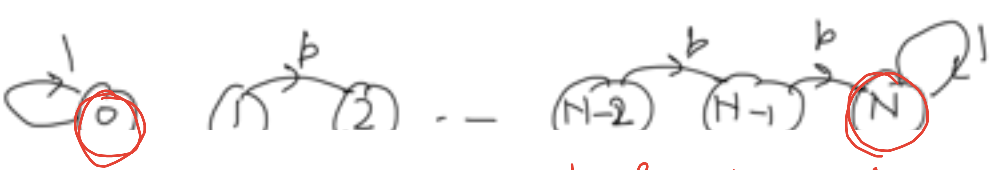
$i = 0, 1, 2, \ldots, N$

$P(Z_i = +1) = p$ ; $P(Z_i = -1) = q = 1-p$

$Z_i$ $i^{th}$ bet / step / transition.

$X_n$ : player's fortune after $n^{th}$ bet/steps

$\in \{0, 1, 2, \ldots, N\}$ $\qquad$ M.C



absorbing/recurrent | others transient


0, N recurrent/absorbing
1, 2, --, N-1 transient

$P_{00} = 1$ , $P_{NN} = 1$

tpm

$$P = \begin{array}{c|c} & \begin{array}{cccccc} 0 & 1 & 2 & - & - & N \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ \\ N \end{array} & \begin{bmatrix} 1 & 0 & 0 & - & - & 0 \\ q & 0 & p & 0 & - & 0 \\ 0 & q & 0 & p & - & 0 \\ - & - & - & - & - & - \\ & - & - & - & - & \\ 0 & 0 & 0 & - & - & 1 \end{bmatrix} \end{array}$$

$T_0$ : time he broke

$T_0 = \inf\{n : X_n = 0\}$

$T_N$ : time he has Rs. N

$T_N = \inf\{n : X_n = N\}$

$P_i = P(T_N < T_0)$ : Prob. that starting with $i$, the gambler's fortune will reach N before reaching 0.

$$= P(T_N < T_0 \mid Z_1 = -1)\, \underline{P(Z_1 = -1)} + \underline{P(T_N < T_0 \mid Z_1 = 1)}\, \underline{P(Z_1 = 1)}$$

$$= q P_{i-1} + p P_{i+1}$$


$$\boxed{P_i = p P_{i+1} + q P_{i-1}}$$

$$\Rightarrow q P_i + p P_i = q P_{i-1} + p P_{i+1}$$

$$\Rightarrow P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1})$$

$i=1$ $\quad P_2 - P_1 = \frac{q}{p} P_1$ $\qquad P_0 = 0, P_N = 1$

$i=2$ $\quad P_3 - P_2 = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1$

$\vdots$

$$P_i - P_{i-1} = \left(\frac{q}{p}\right)^{i-1} P_1$$

$$P_i - P_1 = P_1\left(\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right)$$

$$\Rightarrow P_i = P_1\left(1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right)$$

$$P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}}\, P_1 & \text{if } \frac{q}{p} \neq 1 \\[2ex] i\, P_1 & \text{if } \frac{q}{p} = 1 \end{cases}$$

$$\because P_N = 1 \Rightarrow P_1 = \begin{cases} \dfrac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^N} & \text{if } \frac{q}{p} \neq 1 \\[2ex] \dfrac{1}{N} & \text{if } \frac{q}{p} = 1 \end{cases}$$

$$\boxed{P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & \text{if } \frac{q}{p} \neq 1 \Leftrightarrow \underline{p \neq \frac{1}{2}} \\[2ex] \dfrac{i}{N} & \text{if } \frac{q}{p} = 1 \Leftrightarrow \underline{p = \frac{1}{2}} \end{cases}}$$

$P(T_N < T_0)$ starting at $i$

$\underline{N \to \infty}$

$$P_i = \begin{cases} \underline{0}, \ \cancel{\frac{i}{N}} \to 0 & \underline{p = \frac{1}{2}} \\[1ex] \underline{0}, \ \cancel{\left(\frac{q}{p}\right)^{N}} \to \infty & p < \frac{1}{2} \Leftrightarrow \underline{\frac{q}{p} > 1} \\[1ex] \underline{1-\left(\frac{q}{p}\right)^i}, \ \cancel{\left(\frac{q}{p}\right)^{N}} \to 0 & p > \frac{1}{2} \Leftrightarrow \underline{\frac{q}{p} < 1} \end{cases}$$

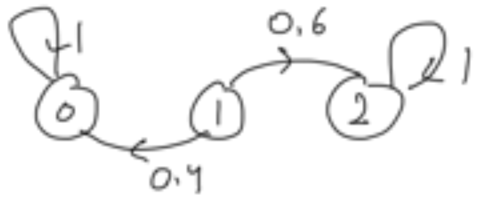
—x—

<u>Example 1</u> ① tpm

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0 & 0.6 \end{bmatrix} \end{array}$$

$$2\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Starting in 1, determine the prob. that M.C ends in state 0.




$p = 0.6,\ q = 0.4$ $\qquad p \neq \frac{1}{2}$

$i = 1,\ N = 2$ $\qquad \frac{q}{p} = \frac{2}{3}$

$$1 - P_1 = 1 - \frac{1 - \left(\frac{2}{3}\right)}{1 - \left(\frac{2}{3}\right)^2} = 0.4$$

(2) The probability of the thrower winning in the dice game called "Craps" is $p = 0.49$. Suppose Player A is the thrower and begins the game with \$5, and Player B, his opponent, begins with \$10. What is the probability that player A goes bankrupt before player B? Assume that the bet is \$1 per round.

$i = 5,\ N = 15$

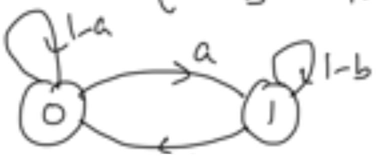
$p = 0.49,\ q = 1 - p = 0.51$

$$1 - P_5 = 1 - \frac{1 - \left(\frac{0.51}{0.49}\right)^5}{1 - \left(\frac{0.51}{0.49}\right)^{15}}$$

—x—

Limiting prob.:

$$\text{Ex}^n \quad P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-a & a \\ 1 & b & 1-b \end{array}, \qquad 0 \le a, b \le 1$$




then $|1 - a - b| < 1$

$(a = b = 0 \text{ or } a = b = 1)$

b

we solve separately

$$P^n = \begin{bmatrix} \dfrac{b + a(1-a-b)^n}{a+b} & \dfrac{a - a(1-a-b)^n}{a+b} \\[2ex] \dfrac{b - b(1-a-b)^n}{a+b} & \dfrac{a + b(1-a-b)^n}{a+b} \end{bmatrix}$$

Sol $\quad P_{00} = 1-a \;,\; P_{01} = a,\; P_{10} = b,\; P_{11} = 1-b$

$$P_{00}^{(n)} = P_{00}^{(n-1)} P_{00} + P_{01}^{(n-1)} P_{10}$$

$$= (1-a)\, P_{00}^{(n-1)} + b\, P_{01}^{(n-1)} \qquad \because P_{00}^{(n-1)} + P_{01}^{(n-1)} = 1$$

$$= (1-a)\, P_{00}^{(n-1)} + b\left(1 - P_{00}^{(n-1)}\right)$$

$$= b + (1-a-b)\, \underbrace{P_{00}^{(n-1)}} \qquad , n \geq 1$$

$$= b + b(1-a-b) + b(1-a-b)^2 + \cdots + b(1-a-b)^{n-2} + P_{00}(1-a-b)^{n-1}$$

$$P_{00} \to (1-a)$$

$$= b \sum_{k=0}^{n-2} (1-a-b)^k + \underline{(1-a)}\,(1-a-b)^{n-1}$$

$$\frac{1-(1-a-b)^{n-1}}{1-(1-a-b)} = \frac{1-(1-a-b)^{n-1}}{a+b}$$

$$= \frac{b}{a+b} + \frac{a(1-a-b)^n}{a+b}$$

$$(1-a-b)^{n-1}\left[1-a-\frac{b}{a+b}\right]$$

$$\frac{a+b-a^2-ab-b}{a+b}$$

$$= \frac{a(1-a-b)}{a+b}$$

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \dfrac{b}{a+b} & \dfrac{a}{a+b} \\ \vdots & \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 \\ \pi_0 & \pi_1 \end{pmatrix}$$

$$\left(\frac{b}{a+b} \quad \frac{a}{a+b}\right)$$

$$\lim_{n\to\infty} p_i^{(n)} = \lim_{n\to\infty} P(X_n = i)$$

$\lim_{n\to\infty} \underset{\sim}{p}^{(n)} = \underset{\sim}{p}^{(0)} P^n$

$$\underline{\underline{\left(\frac{1}{3}, \frac{2}{3}\right)}} \left( \quad \right) = \left(\frac{b}{a+b}, \frac{a}{a+b}\right)$$

$$= (\pi_0, \pi_1)$$

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

$a = b = 1$

periodic with period $= 2$

$d(0) = 2$

$$P^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P^3 = P \; ; \; P^4 = P^2$$

$\underset{\sim}{p}^{(0)}$

$(\alpha, 1-\alpha)$

$$\underset{\sim}{p}^{(1)} = \underset{\sim}{p}^{(0)} P = (1-\alpha, \alpha)$$

$\underset{\sim}{\pi} = \underset{\sim}{\pi} P$

$\sum \pi_i = 1$

$\pi_0 = \pi_1$

$\pi_0 + \pi_1 = 1$

$$\underset{\sim}{p}^{(2)} = \underset{\sim}{p}^{(1)} P = (\alpha, 1-\alpha)$$

$\underset{\sim}{\pi} = (\pi_0, \pi_1) \Rightarrow \pi_0 = \pi_1 = \frac{1}{2}$

limiting prob DNE.

$a = b = 0$

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$P^n = I$$

Regular & Recurrent

Finite state, irreducible, aperiodic (period $= 1$),

limiting prob. exist

$\pi P = \pi$ unique

$\sum \pi_i = 1$

$$\pi_j = \lim_{n\to\infty} p_j^{(n)}$$

$p_j^{(n)} = P(X_n = j)$

$\underset{\sim}{p}^{(n)} = \underset{\sim}{p}^{(n-1)} P$

$$\underline{\underline{p_j^{(n)}}} = \sum_i p_i^{(n-1)} P_{ij}$$

Take limit

$$\left.\begin{aligned}\pi_j &= \sum_i \pi_i P_{ij} \\ \sum_i \pi_i &= 1\end{aligned}\right\} \rightarrow \begin{aligned}\underset{\sim}{\pi} &= \underset{\sim}{\pi} P \\ \sum_i \pi_i &= 1\end{aligned}$$

$\underset{\sim}{\pi} = (\pi_0, \pi_1, \dots)$

<u>Regular tpm:</u>

Given tpm $P$ is regular if $P^k$ has all element $>0$ for some $k$.

$$P = \begin{pmatrix} + & + & 0 \\ + & 0 & + \\ + & 0 & + \end{pmatrix} \qquad P^2 = \begin{pmatrix} + & + & 0 \\ + & 0 & + \\ + & 0 & + \end{pmatrix}\begin{pmatrix} + & + & 0 \\ + & 0 & + \\ + & 0 & + \end{pmatrix}$$

regular

$N=3$ $\quad P^9$

$$= \begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix}$$

states $N$

$P \qquad \underline{P^{N^2}}$

$N=2 \quad P^4$

$$P = \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix}; \quad P^2 = \begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix}$$

not regular

$P^4 = P^2 \cdot P^2$

★ Show that finite state aperiodic irreducible M.C. is regular and recurrent

<u>Thm</u> (✱) Let $P$ regular tpm $S = \{0, 1, \dots, N\}$. Then limiting pmf $\underset{\sim}{\pi} = (\pi_0, \pi_1, \dots, \pi_N)$ is <u>unique</u>

Solⁿ to equations

$$\underset{\sim}{\pi} = \underset{\sim}{\pi} P \qquad \pi_j = \sum_{k=0}^{N} \pi_k P_{kj}, \quad j = 0, 1, \dots, N$$

$$\sum_{k=0}^{N} \pi_k = 1 \qquad \sum_{k=0}^{N} \pi_k = 1$$

Sol M.C. is regular, we have limiting prob

$P^{(n)} \qquad N$

$$\lim_{n\to\infty} P_{ij} = \pi_j \quad ; \quad \sum_{j=0} \pi_j = 1$$

$$P_{ij}^{(n)} = \sum_{k=0}^{N} P_{ik}^{(n-1)} P_{kj}$$

Take Limit as $n \to \infty$ $\quad P_{ij}^{(n)} \to \pi_j \; ; \; P_{ik}^{(n-1)} \to \pi_k$

$$\pi_j = \sum_{k=0}^{N} \pi_k P_{kj}$$

$$\sum_{j=0}^{N} \pi_j = 1$$

T.S. sol$^n$ is unique

$\exists \; x_0, \ldots, x_N$ s.t.

$$x_j = \sum_{k=0}^{N} x_k P_{kj} \;, \; j = 0, 1, \ldots, N \qquad \text{—— (1)}$$

$$\sum_{k=0}^{N} x_k = 1$$

$P_{jl}$

$$\underbrace{\sum_{j=0}^{N} x_j P_{jl}}_{x_l \text{ using (1)}} = \sum_{j=0}^{N} \sum_{k=0}^{N} x_k P_{kj} P_{jl}$$

$$= \sum_{k=0}^{N} x_k \underbrace{\sum_{j=0}^{N} P_{kj} P_{jl}}_{P_{kl}^{(2)}}$$

$$\Rightarrow x_l = \sum_{k=0}^{N} x_k P_{kl}^{(2)}$$

$\vdots$

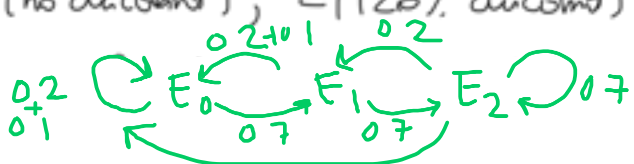
$$x_l = \sum_{k=0}^{N} x_k P_{kl}^{(n)} \quad , \; l = 0, 1, \ldots, N$$

as $n \to \infty$ $\quad P_{kl}^{(n)} \to \pi_l$

$$x_l = \sum_{k=0}^{N} x_k \pi_l = \pi_l \underbrace{\sum_{k=0}^{N} x_k}_{1} = \pi_l$$

— x —

<u>Example</u> An NCD system has discount class $E_0$ (no discount), $E_1$ (20% discount) and $E_2$ (40% discount)



0 1

Movement in the system is determined by the rule whereby one steps back one discount level (or stays in $E_0$) with one claim in a year, and returns to a level of no discount if more than one claim is made. A claim free year results in a step up to a higher discount level (or one remains in class $E_2$ if already there).

| NCD class | $E_0$ | $E_1$ | $E_2$ |
|---|---|---|---|
| % discount | 0 | 20 | 40 |
| annual premium | 100 | 80 | 60 |

If we suppose that for someone in this scheme the prob. of one claim in a year is 0.2 while the prob of two or more claims is 0.1 (i) Find tpm

(ii) In long run what proportion of time is the process in each of the discount classes $\pi$

(iii) Find the av annual premium paid.

$E_i \equiv i$

$i = 0,1,2$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0.3 & 0 & 0.7 \\ 0.1 & 0.2 & 0.7 \end{pmatrix} \end{array}$$

Class = {0,1,2} irreducible, aperiodic finite state

limiting prob. exist and same as stationary state probs.

$\pi P = \pi$

$$\underset{\sim}{\Pi} P = \underset{\sim}{\Pi} \qquad \underset{\sim}{\Pi} = (\Pi_0, \Pi_1, \Pi_2)$$

$$\rightarrow \sum_{i=0}^{2} \Pi_i = 1$$

$$\Rightarrow \begin{cases} \Pi_0 = 0.3\Pi_0 + 0.3\Pi_1 + 0.1\Pi_2 \\ \Pi_1 = 0.7\Pi_0 + 0.2\Pi_2 \\ \Pi_0 + \Pi_1 + \Pi_2 = 1 \end{cases}$$

$$\Rightarrow \Pi_0 = 0.1860, \ \Pi_1 = 0.2442, \ \Pi_2 = 0.5698$$

av. annual premium paid

$$= 0.1860 \times 100 + 80 \times 0.2442 + 60 \times 0.5698$$

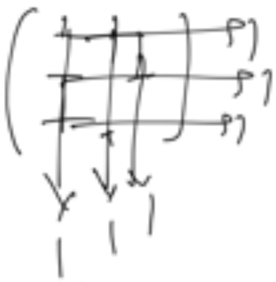
$$= 72.324$$

—x—

Doubly stochastic matrix

tpm $P$

$$\sum_k P_{ik} = \sum_i P_{ik} = 1$$



eg $\quad P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

—x—

let $P$ doubly stochastic, regular

$S = \{0, 1, \rightarrow N-1\}$

limiting prob. $\underset{\sim}{\Pi} = \left(\frac{1}{N}, \frac{1}{N}, \rightarrow \frac{1}{N}\right)$

$$\begin{cases} \Pi_j = \sum_k \Pi_k P_{kj} \\ \sum_k \Pi_k = 1 \end{cases}$$

$$\frac{1}{N} = \sum_k \frac{1}{N} P_{kj} = \frac{1}{N} \sum_k P_{kj}$$

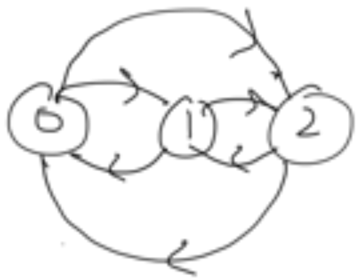
$\therefore \ \underset{\sim}{\pi} = \left(\frac{1}{N}, \dots, \frac{1}{N}\right)$ is unique → since
soln using thm (*) doubly stochastic tpm

—x—

$S = \{0, 1, 2\}$

$N = 3$

tpm
$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1/2 & 1/2 \\ 1 & 1/2 & 0 & 1/2 \\ 2 & 1/2 & 1/2 & 0 \end{array}$$



irreducible, aperiodic, finite state M.C, doubly stochastic

$$\underset{\sim}{\pi} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

eg doubly stochastic tpm may or may not be symmetric

not symmetric

$$P = \begin{pmatrix} 7/12 & 0 & 5/12 \\ 2/12 & 6/12 & 4/12 \\ 3/12 & 6/12 & 3/12 \end{pmatrix}$$

regular

$$, P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Doubly stoch

not regular

$\pi$ DNE

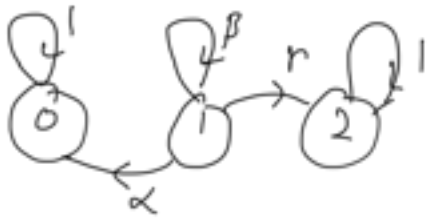
—x—

## Simple first step analysis

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & 0 & 0 \\ 1 & \alpha & \beta & \gamma \\ 2 & 0 & 0 & 1 \end{array}$$

$\alpha + \beta + \gamma = 1$

$0 < \alpha, \beta, \gamma < 1$



0,2 absorbing

1 transient

$T = \min\{n : X_n = 0 \text{ or } X_n = 2\}$

↳ time of absorption of the process

mean time to absorption, start 1 $v_1 = E(T \mid X_0 = 1)$

starting from

$u_1 = P(X_T = 0 \mid X_0 = 1)$ $\quad S = \{0,1,2\}$

↳ Prob. of ultimate absorption into state 0 given that starting from state 1.

$$u_1 = P(X_T = 0 \mid X_0 = 1)$$

$$u_1 = \sum_{k \in S} P(X_T = 0, X_1 = k \mid X_0 = 1)$$

$$u_1 = \sum_{k \in S} P(X_T = 0 \mid X_1 = k, \cancel{X_0 = 1})\, P(X_1 = k \mid X_0 = 1)$$

$$u_1 = \sum_{k \in S} P(X_T = 0 \mid X_1 = k)\, P_{1k}$$

$$u_1 = \underbrace{P(X_T=0 \mid X_1=0)}_{1}\underbrace{P_{10}}_{\alpha} + \underbrace{P(X_T=0 \mid X_1=1)}_{u_1}\underbrace{P_{11}}_{\beta} + \underbrace{P(X_T=0 \mid X_1=2)}_{0}\underbrace{P_{12}}_{r}$$

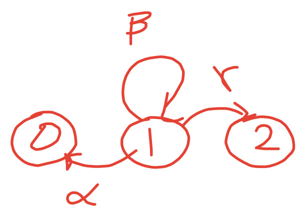
$$u_1 = \alpha + \beta u_1 \Rightarrow (1-\beta)u_1 = \alpha \Rightarrow u_1 = \frac{\alpha}{1-\beta}$$

$p_{10}$

$p_{11}$

$$\nu = E(T \mid X_0 = 1)$$

$$= \underset{\text{min step}}{1} + \overbrace{\underbrace{\alpha \times 0 + r \times 0}_{X_1 = 0 \text{ or } X_1 = 2} + \underbrace{\beta \times \nu}_{X_1 = 1}}^{\text{excess time}}$$




$$\Rightarrow \nu = 1 + \beta\nu \Rightarrow (1-\beta)\nu = 1 \Rightarrow \nu = \frac{1}{1-\beta}$$

eg

$$P = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

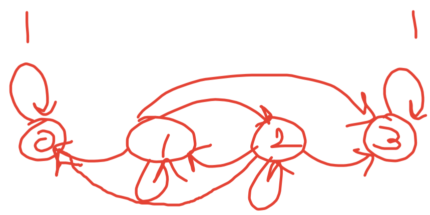
$u_1 = P(X_T = 0 \mid X_0 = 1) \rightarrow 1$

$u_2 = P(X_T = 0 \mid X_0 = 2) \rightarrow 2$

0, 3 absorbing

1, 2 transient

$0 < p_{ij} < 1$



u10+u13 =1

$1-u_1 = P(X_T=3 \mid X_0=1)$

$u_1 = P(X_T=0 \mid X_0=1)$

$$\begin{cases} u_1 = p_{10} + p_{11}u_1 + p_{12}u_2 \\ u_2 = p_{20} + p_{21}u_1 + p_{22}u_2 \end{cases}$$

$= \sum_k P(X_T=0 \mid X_1=k)\, P_{1k}$

$= P_{10} + P_{11}u_1 + P_{12}u_2$

$u_2 = P(X_T=0 \mid X_0=2)$

$= \sum_k P(X_T=0 \mid X_1=k)\, P_{2k}$

$= P_{20} + P_{21}u_1 + P_{22}u_2$

$v_i = E(X_T \mid X_0=i)$, $i=1,2$

$v_1 = 1 + p_{11}v_1 + p_{12}v_2$

$v_2 = 1 + p_{21}v_1 + p_{22}v_2$

$v_1 = 1 + (p_{10}\times 0 + P_{13}\times 0) + (p_{11}v_1 + p_{12}v_2)$

$v_2 = 1 + p_{21}v_1 + p_{22}v_2$

Finite state m.c

$S = \{0, 1, \ldots, N\}$

$0, 1, \ldots, r-1 \rightarrow$ transient

$r, \ldots, N \rightarrow$ absorbing

tpm $P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$

$$\begin{array}{c|cc} & r & N-r+1 \\ \hline r & Q_{r\times r} & R_{r\times N-r+1} \\ N-r+1 & O_{N-r+1\times r} & I_{N-r+1\times N-r+1} \end{array}$$

($N+1$ rows, $N+1$ columns)

$k \in \{r, \ldots, N\}$, $i \in \{0, 1, \ldots, r-1\}$

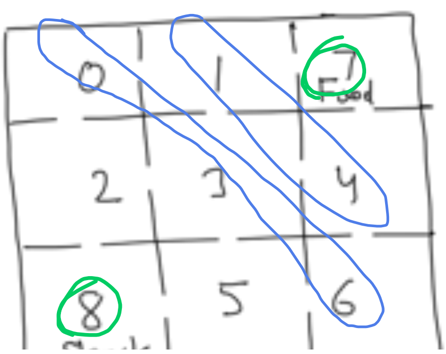
$u_{ik} = u_i = P(\text{absorbed in } k \mid X_0 = i)$

$$= P_{ik}\times 1 + \sum_{\substack{j=r \\ j\neq k}}^{N} P_{ij}\times 0 + \sum_{j=0}^{r-1} P_{ij}u_j$$

$$= P_{ik} + \sum_{j=0}^{r-1} P_{ij}u_j \quad , i = 0, 1, \ldots, r-1.$$

$$v_i = 1 + \sum_{j=0}^{r-1} P_{ij}v_j \quad \text{for } i = 0, 1, \ldots, r-1$$

Example




$u_0 = u_6$

$u_2 = u_5$

$u_1 = u_4$

$u_3 = \frac{1}{2}$ symmetric

shock

| tpm | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | 1/2 | 1/2 | | | | | | |
| 1 | 1/3 | | | 1/3 | | | | 1/3 | |
| 2 | 1/3 | | | 1/3 | | | | | 1/3 |
| 3 | | 1/4 | 1/4 | | 1/4 | 1/4 | | | |
| 4 | | | | 1/3 | | | 1/3 | 1/3 | |
| 5 | | | | 1/3 | | | 1/3 | | 1/3 |
| 6 | | | | | 1/2 | 1/2 | | | |
| 7 | | | | | | | | 1 | |
| 8 | | | | | | | | | 1 |

absorbing: 7, 8

$i = 0, 1, \ldots, 6$

$u_{i7} = u_i$

$$\left.\begin{aligned} u_0 &= \tfrac{1}{2}u_1 + \tfrac{1}{2}u_2 \\ u_1 &= \tfrac{1}{3}u_0 + \tfrac{1}{3}u_3 + \tfrac{1}{3}\times 1 \\ u_2 &= \tfrac{1}{3}u_0 + \tfrac{1}{3}u_3 \end{aligned}\right] \rightarrow \quad u_0 = \tfrac{1}{2},\ u_1 = \tfrac{2}{3},\ u_2 = \tfrac{1}{3},\ u_3 = \tfrac{1}{2}.$$

**Mean time spent in transient state:**

Finite state M.C.

$T = \{1, 2, \ldots, t\}$ set of transient states

$$P_T = \begin{bmatrix} P_{11} & \cdots & P_{1t} \\ \vdots & & \vdots \\ P_{t1} & \cdots & P_{tt} \end{bmatrix}$$

$i, j \in T$

Mean time spent in transient state

$s_{ij}$ = expected # of time period the M.C. is in state $j$ given that it starts in state $i$

$$I_{n,j} = \begin{cases} 1 & \text{if } X_n = j \\ 0 & \text{o.w.} \end{cases} \qquad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{o.w.} \end{cases}$$

$\ldots \in T$

$i, j \in$

$$s_{ij} = \delta_{ij} + E\left(\sum_{n=1}^{\infty} I_{n,j} \,\middle|\, X_0 = i\right)$$

$$= \delta_{ij} + \sum_{n=1}^{\infty} E\left(I_{n,j} \mid X_0 = i\right)$$

$$1 \times P(X_n = j \mid X_0 = i) + 0 \times P(X_n \neq j \mid X_0 = i)$$

$$P_{ij}^{(n)}$$

$$= \delta_{ij} + \sum_{n=1}^{\infty} P_{ij}^{(n)} \quad \text{——— } (**)$$

$$= \delta_{ij} + \sum_{n=1}^{\infty} \sum_{k} P_{ik} P_{kj}^{(n-1)} \qquad , k \in S, \; i, j \in T$$

$$= \delta_{ij} + \sum_{k} P_{ik} \sum_{n=1}^{\infty} P_{kj}^{(n-1)}$$

$$\delta_{kj} + \sum_{n=2}^{\infty} P_{kj}^{(n-1)}$$

$$\underbrace{\delta_{kj} + \sum_{n=1}^{\infty} P_{kj}^{(n)}}$$

$$s_{kj} \quad \text{using } (**)$$

$$= \delta_{ij} + \sum_{k} P_{ik} s_{kj}$$

$$s_{ij} = \delta_{ij} + \sum_{k=1}^{t} P_{ik} s_{kj}, \quad \longleftarrow$$

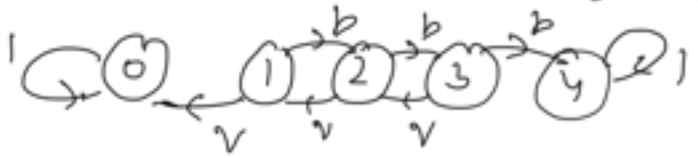
since it is impossible to go from a recurrent to a transient state $\Rightarrow$ $s_{kj} = 0$, when $k$ is recurrent state

$S = ((s_{ij}))$

$$S = I + P_T S$$

$$\Rightarrow (I - P_T) S = I \Rightarrow \boxed{S = (I - P_T)^{-1}}$$

Example Gambler ruin problem $p=0.4$, $N=4$




transient 1,2,3

absorbing 0,4

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \rightarrow P_T$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad P_T = \begin{bmatrix} 0 & 0.4 & 0 \\ 0.6 & 0 & 0.4 \\ 0 & 0.6 & 0 \end{bmatrix}$$

$$(I - P_T) = \begin{bmatrix} 1 & -0.4 & 0 \\ -0.6 & 1 & 0 \\ 0 & -0.6 & 1 \end{bmatrix}$$

$$((s_{ij})) = S = (I-P_T)^{-1} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.46 & 0.76 & 0.31 \\ 1.15 & 1.92 & 0.76 \\ 0.69 & 1.15 & 1.46 \end{bmatrix} \end{array}$$

$$s_{2,3} = 0.76 ; \quad s_{2,1} = 1.15$$

$f_{ij}$ : prob. that M.C ever makes a transition into state $j$ given that it starts in state $i$

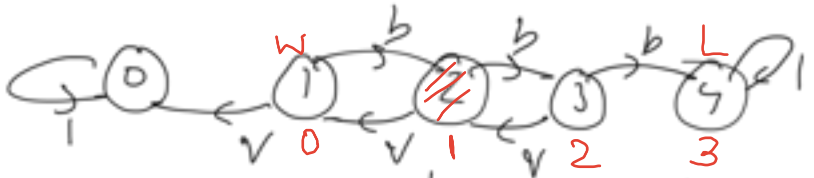
way-1 $f_{2,1} = f_{2,1}^{(1)} + f_{2,1}^{(2)} + f_{2,1}^{(3)} + \cdots$

$$= q + pq^2 + p^2q^3 + p^3q^4 + \cdots$$

$$= q + pq\left[q + pq^2 + p^2q^3 + \cdots\right]$$

$= q + pq\, f_{2,1}$

$$\Rightarrow f_{2,1} = \frac{q}{1-pq} = \frac{0.6}{1-0.4\times 0.6} = 0.78$$




way-2 $$f_{2,1} = 1 - \frac{1-\left(\frac{q}{p}\right)^1}{1-\left(\frac{q}{p}\right)^3} = 1 - \frac{1-\left(\frac{0.6}{0.4}\right)}{1-\left(\frac{0.6}{0.4}\right)^3} = 0.78$$

$$s_{ij} = E(\text{time in } j \mid \text{start } i)$$

$$= \underbrace{E(\text{time in } j \mid \text{start } i,\ \text{ever transit to } j)}_{(\delta_{ij}+s_{jj})} f_{ij} + \underbrace{E(\text{time in } j \mid \text{start } i,\ \text{never transit to } j)}_{\delta_{ij}} (1-f_{ij})$$

$$= (\delta_{ij}+s_{jj})\cdot f_{ij} + \delta_{ij}(1-f_{ij})$$

$$= \delta_{ij} + s_{jj} f_{ij}$$

way-3 $$\Rightarrow \boxed{f_{ij} = \frac{s_{ij}-\delta_{ij}}{s_{jj}}}$$

$s_{2,1} = 1.15$, $\delta_{2,1} = 0$

$s_{1,1} = 1.46$

$$f_{2,1} = \frac{1.15-0}{1.46} = 0.78$$

Particular case:

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$$

$0, 1, \ldots, n-1$ transient

$n, \ldots, N$ absorbing

$$S = I + QS \Rightarrow \boxed{S = (I-Q)^{-1}}$$

$$s_{ij} = \delta_{ij} + \sum_{l=0}^{r-1} P_{il}\, s_{lj} \qquad i, j = 0, 1, \ldots, r-1$$

$T \to$ time of absorption

$$T = \min\{n ; r \le X_n \le N\}$$

$$s_{ij} = E\left( \sum_{n=0}^{T-1} 1(X_n = j) \,\middle|\, X_0 = i \right)$$

$$v_i = E(T \mid X_0 = i)$$

$$\sum_{j=0}^{r-1} \sum_{n=0}^{T-1} 1(X_n = j) = \sum_{n=0}^{T-1} \underbrace{\sum_{j=0}^{r-1} 1(X_n = j)}_{1} = T$$

$$\sum_{j=0}^{r-1} s_{ij} = E\left( \sum_{j=0}^{r-1} \sum_{n=0}^{T-1} 1(X_n = j) \,\middle|\, X_0 = i \right)$$

$$= E(T \mid X_0 = i)$$

$$= v_i$$

$$\boxed{\sum_{j=0}^{r-1} s_{ij} = v_i}$$

$s_{ij} \to$ start from $i$, time in $j$

$v_i \to$ start from $i$, total transient time

$v_i = \sum_k s_{ik}$ $\quad k =$ trans

## Reducible M.C. :

(1) tpm

$\pi = (\pi_0, \pi_1)$

$\pi = \pi P$

$\pi_0 + \pi_1 = 1$

$\pi_0 = \frac{1}{2}\pi_0 + \frac{1}{4}\pi_1$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 2 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{array}$$


Class $\{0,1\}$ $\{2,3\}$

$$\pi_0+\pi_1=1 \Rightarrow \pi_0=\frac{1}{3}, \quad \pi_1=\frac{2}{3}$$

$$3 \left( 0 \quad 0 \quad \frac{2}{3} \quad \frac{1}{3} \right)$$

stochastic regular — recurrent aperiodic — recurrent aperiodic

$$\pi_0^{(2)}=\frac{1}{2}, \quad \pi_1^{(2)}=\frac{1}{2}$$

$$P=\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \quad P^2=\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}=\begin{pmatrix} P_1^2 & 0 \\ 0 & P_2^2 \end{pmatrix}$$

$$P^n=\begin{pmatrix} P_1^n & 0 \\ 0 & P_2^n \end{pmatrix}$$

$$\lim_{n\to\infty} P^n=\begin{pmatrix} \lim_{n\to\infty} P_1^n & 0 \\ 0 & \lim_{n\to\infty} P_2^n \end{pmatrix}=\begin{pmatrix} \pi_0^{(1)} \; \pi_1^{(1)} & 0 \\ 0 & \pi_0^{(2)} \; \pi_1^{(2)} \end{pmatrix}$$

$$\lim_{n\to\infty} P^n = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

$$\pi_0=\frac{1}{2}\pi_0+\frac{1}{4}\pi_1, \quad \pi_0+\pi_1=1$$

$$\Rightarrow \pi_0^{(1)}=1/3, \quad \pi_1^{(1)}=2/3$$

(2)

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Classes $C_1=\{0,1\}$, $C_2=\{2\}$, $C_3=\{3\}$

recurrent aperiodic — transient — absorbing

$C_1$

$$u_2=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}u_2 \quad P(0 \text{ or } 1 \mid X_0=2)$$

$$\Rightarrow u_2=\frac{2}{3}$$

$$1-u_2=\frac{1}{3}=P(3 \mid X_0=2)$$

$$\lim_{n\to\infty} P^n = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \\ \frac{2}{3}\times\frac{1}{3} & \frac{2}{3}\times\frac{2}{3} & 0 & \frac{1}{3}\times 1 \end{bmatrix} \end{array}$$

$$\begin{array}{c|c} 3 & \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(3)

$u_2$

$$\begin{array}{c|c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 0 & 1/3 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$u_2 = \frac{1}{3} + \frac{1}{3}u_3$

$u_3 = \frac{2}{6} + \frac{1}{6}u_2$

$\Downarrow$

$u_2 = \frac{8}{17}$ $\quad u_3 = \frac{7}{17}$

$\pi_0 = \frac{2}{5}, \quad \pi_1 = \frac{3}{5}$

$C_1 = \{0,1\}$, $C_2 = \{2,3\}$, $C_3 = \{4,5\}$

recurrent, aperiodic — transient — recurrent, period = 2

$$\lim_{n\to\infty} P^n = \begin{array}{c|c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 \\ \frac{8}{17}\times\frac{2}{5} & \frac{8}{17}\times\frac{3}{5} & 0 & 0 & X & X \\ \frac{7}{17}\times\frac{2}{5} & \frac{7}{17}\times\frac{3}{5} & 0 & 0 & X & X \\ 0 & 0 & 0 & 0 & X & X \\ 0 & 0 & 0 & 0 & X & X \end{bmatrix} \end{array}$$

transient calculated via this

X DNE

For time av

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} P^m = \begin{array}{c|c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 2/5 & 3/5 & 0 & 0 & 0 & 0 \\ 2/5 & 3/5 & 0 & 0 & 0 & 0 \\ \frac{8}{17}\times\frac{2}{5} & \frac{8}{17}\times\frac{3}{5} & 0 & 0 & \frac{9}{17}\times\frac{1}{2} & \frac{9}{17}\times\frac{1}{2} \\ \frac{7}{17}\times\frac{2}{5} & \frac{7}{17}\times\frac{3}{5} & 0 & 0 & \frac{10}{17}\times\frac{1}{2} & \frac{10}{17}\times\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix} \end{array}$$

$$\lim_{n\to\infty} \frac{P^0 + P^1 + \quad + P^{n-1}}{n}$$

$$5 \mid 0 \quad 0 \quad 0 \quad 0 \quad 1/2 \quad 1/2$$

Ergodic → irreducible, aperiodic, +ve recurrent

## Branching process:

$X_0$ size of zeroth generation

0 j

offsprings with probability $P_j$, independent of number produced by other individual

$X_1$ size of first generation, i.e., all offsprings of zeroth generation

$X_n \in \{0, 1, 2, \dots\}$

$p_0 > 0$

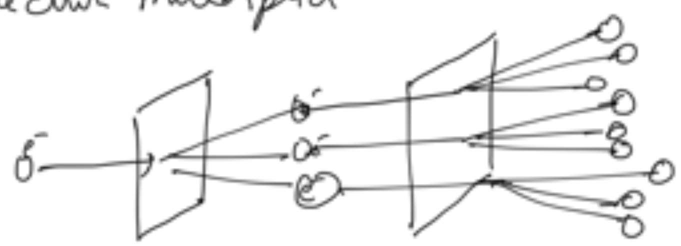
$X_n$ Size of nth generation, i.e., all offsprings of $(n-1)^{th}$ generation.

$E(X_n)$ ?

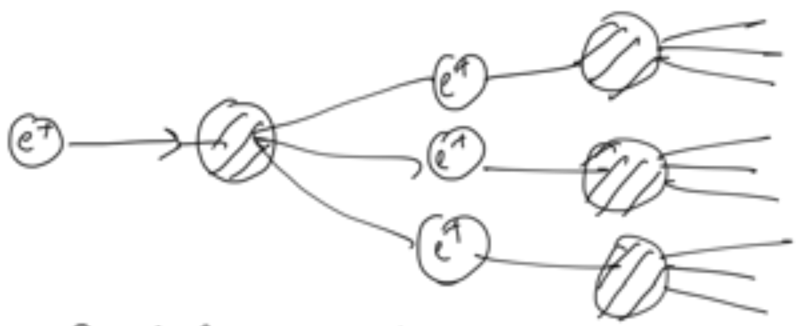
$V(X_n)$ ?

Application

Electron multiplier



Neutron chain reaction



Suppose $X_0 = 1$ Survival of family name:

mean # of offsprings of a single individual is $\mu = \sum_{j=0}^{\infty} j p_j = E(Z_1)$

var. " " " " " " $\sigma^2 = \sum_{j=0}^{\infty} (j-\mu)^2 p_j = V(Z_1)$

formula $V(X) = E[(X-\mu)^2] = E[X^2] - \mu^2$

(n-1) generation: $1 \quad 2 \quad \cdots \quad X_{n-1}$

Size of nth generation: $X_n = Z_1 + Z_2 + \cdots + Z_{X_{n-1}} = \sum_{i=1}^{X_{n-1}} Z_i$

$Z_i$: # of offsprings of ith individual of $(n-1)^{st}$ generation. $E(Z_i)=\mu$, $V(Z_i)=\sigma^2$

$$E(X_n) = E\left[E(X_n \mid X_{n-1})\right]$$

$$= E\left(E\left(\sum_{i=1}^{X_{n-1}} Z_i \,\middle|\, X_{n-1}\right)\right)$$

$$E\left(\sum_{i=1}^{X_{n-1}} Z_i \,\middle|\, X_{n-1}=x\right) = E\left(\sum_{i=1}^{x} Z_i\right) = \sum_{i=1}^{x} \underbrace{E(Z_i)}_{\mu} = x\mu \checkmark$$

$$= E(X_{n-1}\mu)$$

$$= \mu E(X_{n-1})$$

$$= \mu^2 E(X_{n-2})$$

$$\boxed{E(X_n) = \mu^n E(X_0)}$$

$$= \mu^n \quad \text{if } X_0 = 1$$

$$V(X_n) = E(\underbrace{V(X_n \mid X_{n-1})}_{X_{n-1}\sigma^2}) + V(\underbrace{E(X_n \mid X_{n-1})}_{X_{n-1}\mu})$$

$$V\left(\sum_{i=1}^{X_{n-1}} Z_i \,\middle|\, X_{n-1}=x\right) = V\left(\sum_{i=1}^{x} Z_i\right) = \sum_{i=1}^{x} \underbrace{V(Z_i)}_{\sigma^2} = x\sigma^2$$

$$= E(X_{n-1}\sigma^2) + V(X_{n-1}\mu)$$

$$= \sigma^2 E(X_{n-1}) + \mu^2 V(X_{n-1})$$

$$= \sigma^2 \mu^{n-1} + \mu^2 V(X_{n-1})$$

$$= \sigma^2\mu^{n-1} + \mu^2 V(X_{n-1})$$

$$= \sigma^2\mu^{n-1} + \mu^2\left[\sigma^2\mu^{n-2} + \mu^2 V(X_{n-2})\right]$$

$$= \sigma^2\left[\mu^{n-1} + \mu^{n}\right] + \mu^4 V(X_{n-2})$$

$$= \sigma^2\left[\mu^{n-1} + \mu^{n}\right] + \mu^4\left[\sigma^2\mu^{n-3} + \mu^2 V(X_{n-3})\right]$$

$$= \sigma^2\left[\mu^{n-1} + \mu^{n} + \mu^{n+1}\right] + \mu^6 V(X_{n-3})$$

$$\vdots$$

$$= \sigma^2\left[\mu^{n-1} + \mu^{n} + \cdots + \mu^{2n-2}\right] + \mu^{2n} \underbrace{V(X_0)}_{0}$$

$$= \sigma^2\mu^{n-1}\left[1 + \mu + \cdots + \mu^{n-1}\right]$$

$$\boxed{V(X_n) = \begin{cases} \sigma^2\mu^{n-1}\left(\dfrac{1-\mu^n}{1-\mu}\right) & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}}$$

—x—

$$u_{n+1} = P(X_{n+1}=0) = \sum_{j=0}^{\infty} \underbrace{P(X_{n+1}=0 \mid X_1=j)}_{[P(X_n=0)]^j}\, p_j$$

$$P(X_{n+1}=0 \mid X_0=1)$$
$$= P(X_{n+1}=0 \mid X_1=j)\, p_j$$
$$= \underbrace{P(X_n=0 \mid X_0=j)}_{[P(X_n=0)]^j}\, p_j$$

$$\boxed{u_{n+1} = \sum_{j=0}^{\infty} u_n^j\, p_j}$$

—x—

$\pi_0$ prob. of ultimate extinction, i.e., prob. that the popl. will eventually die out (under the assumption that $X_0=1$)

$$\pi_0 = \lim_{n\to\infty} P(X_n=0 \mid X_0=1)$$ (prob of ultimate extinction)

$\mu \le 1 \quad \pi_0 = 1$

$\mu > 1 \quad \pi_0 = \sum_{j=0}^{\infty} \pi_0^j p_j$

$\rightarrow \Pi_0 = 1$ if $\underline{\mu < 1}$

$$\mu^n = E(X_n) = \sum_{j=1}^{\infty} j\, P(X_n = j)$$

$$\geq \sum_{j=1}^{\infty} 1 \cdot P(X_n = j)$$

$$= P(X_n \geq 1)$$

$$\lim_{n\to\infty} P(X_n \geq 1) = 0 \Rightarrow \lim_{n\to\infty} P(X_n = 0) = 1$$

$$\Rightarrow \Pi_0 = 1$$

$\rightarrow \Pi_0 = 1$ if $\mu = 1$

$\rightarrow$ When $\mu > 1$

$$\Pi_0 = P(\text{popl}^n \text{ die out})$$

$$= \sum_{j=0}^{\infty} \underbrace{P(\text{popl}^n \text{ die out} \mid X_1 = j)}_{\Pi_0^j} p_j$$

$$\Rightarrow \boxed{\Pi_0 = \sum_{j=0}^{\infty} \Pi_0^j p_j} \rightarrow$$

$\Pi_0$ is smallest +ve number which is the solution of this equation

Example: $X_0 = 1$

(1) $p_0 = \frac{1}{2}, \; p_1 = \frac{1}{4}, \; p_2 = \frac{1}{4}$

$\Pi_0 = 1$

$$\mu = 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 2 \times \frac{1}{4} = \frac{3}{4} < 1$$

$X_0 = n$

$\Pi_0^n = 1^n = 1$

(2) $p_0 = \frac{1}{4}, \; p_1 = \frac{1}{4}, \; p_2 = \frac{1}{2} \qquad \Pi_0$

Pg 40 onwards