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DM Assignment 4

2) $a^2 = e$

$a * b * a = b * a * a = b$ (abelian group)

Given:- $a * b^7 * a = b^7$

$\therefore b^7 = b$

$b^6 = e$

$\therefore b^{48} = (b^6)^8 = e^8 = e$

$\boxed{\therefore b^{48} = e}$

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4) $G = \langle g \rangle \longrightarrow$ cyclic group of order 30

a) $o(g^m) = \dfrac{o(g)}{\gcd(m, o(g))} = 5$

$= \dfrac{30}{\gcd(m, o(g))} = 5$

$\gcd(m, o(g)) = 6$

$m = 6, 12, 18, 24$

b) $o(g^m) = \dfrac{\cancel{o(g)}^{30}}{\gcd(m, o(g))} = 6$

$\gcd(30, m) = 5 \longrightarrow m = 5, 25$

5) $o(a) = 3 \longrightarrow a^3 = e$

$a * b * a^{-1} = b^2$

$(a * b * a^{-1}) * (a * b * a^{-1}) = b^2 * b^2 \Rightarrow a * b^2 * a^{-1} = b^4$

$a * (a * b * a^{-1}) * a^{-1} = b^4 \Rightarrow a^2 * b * a^{-2} = b^4$

$(a^2 * b * a^{-2}) * (a^2 * b * a^{-2}) = b^4 * b^4$

$a^2 * b^2 * a^{-2} = b^8$

$a^3 * b * a^{-3} = b^8$

$b = b^8$

$b^7 = 1$

$\boxed{o(b) = 7}$

1)
a) semigroup
b) semigroup
c) groupoid
d) semigroup
e) semigroup
f) groupoid

9) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{pmatrix}$

$= \underbrace{(1\ 4\ 5)}_{3\text{ cycle}}\underbrace{(2\ 6)}_{2\text{ cycle}}$

order of permutation = LCM(2, 3) = 6

now

$(145)(26) = (14)(15)(26)$

= 3 transposition

$\Rightarrow$ ODD permutation

10) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 5 & 6 & 1 \end{pmatrix}$

$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 4 & 5 & 3 & 2 \end{pmatrix}$

$fg = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 6 & 3 & 4 \end{pmatrix}$

$gf = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$

$f = (1\ 2\ 4\ 5\ 6) \Rightarrow f^{-1} = (6\ 5\ 4\ 2\ 1)$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 2 & 4 & 5 \end{pmatrix}$

$g = (26)(3\ 4\ 5) \Rightarrow g^{-1} = (6\ 2)(5\ 4\ 3)$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 3 & 4 & 2 \end{pmatrix}$

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8) $G = (\mathbb{Z}, +)$ $H = (3\mathbb{Z}, +)$

$H + 0 = \{3n \mid n \in \mathbb{Z}\}$
$H + 1 = \{3n+1 \mid n \in \mathbb{Z}\}$
$H + 2 = \{3n+2 \mid n \in \mathbb{Z}\}$ → Distinct Right cosets of 1

~~7) $S : \{x^5 = 1 \mid x \in \mathbb{C}\} = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4\}$~~
~~$\alpha = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = e^{i\frac{2\pi}{5}}$ $x \in \mathbb{C}$ ∴ $x = e^{i\theta}$~~
~~now, $\forall\ a, b \in S$ $a \cdot b = b \cdot a \in S$~~
~~So, S is commutative wrt multiplication~~

~~6)~~

7) $S : \{x^5 = 1 \mid x \in \mathbb{C}\} = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4\}$

we know that $\alpha^5 = 1$

Also, $\forall\ a, b \in S$, $a \cdot b \in S$

As complex multiplication is commutative,
$\forall\ a, b \in S$, $a \cdot b = b \cdot a$.

hence S is commutative wrt multiplication

6) $\mathbb{Q}^+$ forms abelian group wrt $*$ defined by
$a * b = \frac{1}{2}ab\ \forall\ a, b \in \mathbb{Q}^+$

Proof

i) checking for closure property
$\forall\ a, b \in \mathbb{Q}^+$, $a * b = \frac{1}{2}ab \in \mathbb{Q}^+$
↳ closure property satisfied

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ii) Checkig for associative property
$\forall\ a,b,c \in Q^+$
$a*(b*c) = \frac{1}{4}abc = (a*b)*c$
↳ Associative property satisfied

iii) checking for commutative property
$\forall\ a,b \in Q^+$
$a*b = \frac{1}{2}ab = b*a$
↳ commutative property satisfied

iv) Checking for existance of identity element

$e*a = a*e = a \Rightarrow \frac{1}{2}ea = a \Rightarrow e = 2 \in Q^+$
↳ Identity element exists

v) Checking for inverse element

$a*b = b*a = e$
$\frac{1}{2}ab = 2 \longrightarrow b = \frac{4}{a} \in Q^+$
↳ Hence inverse of every element a is $4/a \in Q^+$

Thus Q^+ forms abelian group wrt $*$ defiend by $a*b = \frac{1}{2}ab$.

3) $O(a) = 4$, $a^2 = b^2$, $ba = a^3b$, $a^4 = e$

$a^2 = b^2$ & $ba = a^3b$
$\Rightarrow aba = a^4b = b$

also $ba = a(a^2)b = ab^3$
$\Rightarrow aba = a^2b^3 = b^5$
$b^5 = b$
$b^4 = e$
$\boxed{O(b) = 4}$

now $|G| = ?$ 19MA20059

Generators $\rightarrow$

element

$\downarrow$

a	b	
a	b	
a^2	b^2	$\Rightarrow$ same
a^3	b^3	
a^4	b^4	$\Rightarrow$ same

a, a^2, a^3, a^4, b, b^3 — already in group

few elements can be formed by composite usage of generators.
but we ignore a^2, b^2, a^4, b^4 terms for composite elements formation as they are equal.

$a\ b$
$a^3\ b^3$ $\rightarrow$ $ab, ab^3, ba, b^3a, a^3b, a^3b^3, ba^3, b^3a^3$

as $ba = a^3b \Rightarrow aba = b$
$ba = a(b^2)b = ab^3 \Rightarrow bab = a$

$aba = b$ $\quad$ ~~bab = a~~ $\quad$ $bab = a$
$\Rightarrow aba^4 = ba^3,\ a^4ba = a^3b \Rightarrow bab^4 = ab^3,\ b^4ab = b^3a$
$\Rightarrow ab = ba^3,\ ba = a^3b \Rightarrow ba = ab^3,\ ab = b^3a$

$ba^3 = b^3a$
$a^3b = ab^3$ $\rightarrow$ ab, ba are the composite elements of the group

triple composite elements:- $aba = b$, $bab = a$
(repeated) $\quad aba^2 = ba,\ bab^2 = ab$
$a^2ba = ab,\ b^2ab = ba$

quadra. composite elements:- $abab = b^2$, $baba = a^2$
(repeating) $\rightarrow$ Higher composite elements would repeat

$\therefore G = \{e, a, a^2, a^3, b, b^3, ab, ba\} \Rightarrow \boxed{|G| = 8}$