Lecture 13

Remark:- $E_n = \{x \in E \mid f(x) > \frac{1}{n} + \operatorname{ess\,sup}(f)\}$

$$= f^{-1}\left(\left(\operatorname{ess\,sup}(f) + \tfrac{1}{n}, \infty\right)\right)$$

$$\subseteq f^{-1}\left((\alpha, \infty)\right), \quad \forall \alpha \in U_f$$

where $U_f = \{\alpha \in \mathbb{R} \mid m(f^{-1}((\alpha, \infty))) = 0\}$.

$$\Rightarrow \quad m(E_n) \leq \underset{\displaystyle 0}{\underset{\parallel}{m\left[f^{-1}((\alpha,\infty))\right]}} \quad \forall \alpha \in U_f$$

$\therefore \quad m(E_n) = 0.$

Proposition:- Let $f : E \to \mathbb{R}$ be a measurable function & $B \subseteq \mathbb{R}$ a Borel set. Then $f^{-1}(B)$ is measurable.

proof:- We have for any $\{A_i\}$, $A \subseteq \mathbb{R}$

$$f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i)$$

& $$f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$$

(EXERCISE to verify) (*)

Let $\mathcal{Y} = \{A \subseteq \mathbb{R} \mid f^{-1}(A) \text{ is measurable}\}$.

Since f is a measurable function, we have $f^{-1}[I] \in \mathcal{M}$ for any interval I in $\mathbb{R}$.

$\therefore$ $I \in \mathcal{Y}$, I is any interval.

<u>Claim</u>:- $\mathcal{Y}$ is a σ-algebra.

<u>proof</u> <u>of</u> the <u>claim</u>:- • $\mathbb{R} \in \mathcal{Y}$ $(\because f^{-1}(\mathbb{R}) = E \in \mathcal{M})$.

$\mathcal{Y}$ is closed under countable union & closed under taking complement by $(*)$.

$\therefore$ $\mathcal{Y}$ is a σ-algebra.

Thus $\mathcal{Y}$ is a σ-algebra containing all intervals.

$\Rightarrow$ $\mathcal{Y} \supseteq \mathcal{B}$, where $\mathcal{B}$ is a Borel σ-algebra.

Thus all Borel sets belonging to $\mathcal{Y}$.

$\Rightarrow$ $f^{-1}(B)$ is measurable for any $B \in \mathcal{B}$.

Existance of a non-measurable set in $\mathbb{R}$

ie $\mathcal{M} \neq \mathcal{P}(\mathbb{R})$
$\quad\quad\quad\quad\quad$ ‖ the power set of $\mathbb{R}$.

<u>Theorem</u>:- Let $E \subseteq \mathbb{R}$ be a measurable set. Then for each $y \in \mathbb{R}$, the set

$$E+y = \{ x+y \mid x \in E \}$$ is measurable.

& $m(E) = m(E+y)$.

That is, the Lebesgue measure is translation invariant.

<u>proof</u>:- Given E is measurable.

$\Rightarrow$ Given any $\varepsilon > 0$, there exists an open set $U \supseteq E$ such that $m^*(U \setminus E) \leq \varepsilon$.

Since $E \subseteq U$, we have $E+y \subseteq U+y$

& $U+y$ is open

Also $(U+y) \setminus (E+y) = U \setminus E + y$

$$\therefore m\left(\underline{(U\setminus E)+y}\right) = m^*\left((U\setminus E)+y\right)$$

$$= m^*(U\setminus E)$$

$$= m(U\setminus E) \le \varepsilon$$

$$= m\left(\underline{(U+y)\setminus(E+y)}\right)$$

$A - B$

$A \setminus B$

$m^*(A+x)$
$\|$
$m^*(A)$

$$\therefore \quad m^*\left((U+y)\setminus(\underline{E}+y)\right) \le \varepsilon$$

for some open set $U+y \subseteq \mathbb{R}$.

$\therefore$ $E+y$ is measurable.

Also $m(E+y) = m^*(E+y) = m^*(E)$

$= m(E)$.

Theorem:- There exists a non-measurable set in $\mathbb{R}$.

proof:- Consider $[0,1]$, for any $x,y \in [0,1]$,

define $x \sim y$, if $y-x \in Q_1$

where $Q_1 = Q \cap [-1,1]$

Check that $\sim$ is an equivalence relation on $[0,1]$.

- $x \sim x$ $\quad (\because x - x = 0 \in \mathbb{Q}_1)$
- $x \sim y \Rightarrow y - x \in \mathbb{Q}_1 \Rightarrow x - y \in \mathbb{Q}_1$

$$\Rightarrow y \sim x$$

- $x \sim y$ & $y \sim z \Rightarrow y - x,\ z - y \in \mathbb{Q}_1$

$$\Rightarrow y - x + z - y = z - x \in \mathbb{Q}_1$$

$$\Rightarrow x \sim z.$$

Let $\{E_\alpha\}$ be the set of all equivalence classes & $[0,1] = \bigcup_\alpha^{\cdot} E_\alpha$

Now each E_α is countable

Say $E_\alpha =$ the equivalence class of $x \in [0,1]$

$$= \{y \in [0,1] \,/\, x \sim y\}$$

$$= \{y \in [0,1] \,/\, y - x \in \mathbb{Q}_1\}$$

$$= \{y \in [0,1] \,/\, y \in \mathbb{Q}_1 + x\} \subseteq \underbrace{\mathbb{Q}_1 + x}_{\text{countable set}}$$

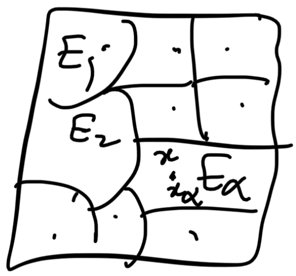
$\therefore E_\alpha$ is countable.

But we know $[0,1]$ is uncountable &

$$[0,1] = \dot{\bigcup_{\alpha}} E_\alpha$$

Then we get that the union is an uncountable union.

$\therefore$ By the axiom of choice, there exists a set V in $[0,1]$ containing just one element x_α from each E_α.



Let $Q_1 = \{r_1, r_2, \ldots\}$

& for each $n \in \mathbb{N}$, write $V_n = V + r_n$

<u>claim</u>:- $V_n \cap V_m = \phi$ if $m \neq n$.

<u>proof of claim</u>:- For $y \in V_n \cap V_m$

$\Rightarrow$ there exist $\underline{x_\alpha, x_\beta} \in V$

such that $y = x_\alpha + r_n = x_\beta + r_m$

$$\Rightarrow \quad x_\alpha - x_\beta = r_m - r_n \in \mathbb{Q}_1$$

$$\Rightarrow \quad x_\alpha = x_\beta \quad (\because \text{ by definition of } V).$$

$$\Rightarrow \quad x_\alpha + r_n = x_\alpha + r_m$$

$$\Rightarrow \quad r_n = r_m$$

$$\Rightarrow \quad n = m.$$

Thus if $m \neq n$, then $V_m \cap V_n = \phi$.

$$V_n = V + r_n, \quad V_m = V + r_m.$$

Now $[0,1] = \bigcup_\alpha E_\alpha \subseteq \bigcup_{n=1}^{\infty} V_n$

For $x \in [0,1]$, then $x \in E_\alpha$ for some α

& then $x = x_\alpha + r_n$ (where $x_\alpha \in V$)

gives that $x \in V_n$. for some n.

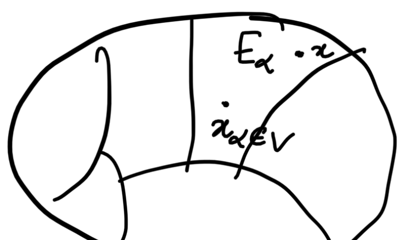
$x \sim x_\alpha \Rightarrow x - x_\alpha \in \mathbb{Q}_1 = \{r_1, r_2, \ldots\}$

$$\underset{E_\alpha}{\underset{\uparrow}{x}} \sim \underset{V.}{\underset{\uparrow}{x_\alpha}} \in E_\alpha$$

$\Rightarrow \quad x - x_\alpha = r_n$ for some n

$\Rightarrow \quad x = x_\alpha + r_n \in V_n = V + r_n$

$\Rightarrow$



& $\bigcup_{n=1}^{\infty} V_n \subseteq [-1, 2]$

$V_n = V + r_n$

$[0,1)$; -1 to 1

Thus

$$\boxed{[0,1] \subseteq \dot{\bigcup_{n=1}^{\infty}} V_n \subseteq [-1, 2]} \longrightarrow \S$$

Claim:- V is not measurable.

proof :- Suppose V is measurable.

Then each $V_n = V + r_n$ is measurable by above thm.

& $m(V) = m(V_n) \quad \forall n.$

$\therefore$ By $\S$, we get

$$m([0,1]) \leq m\left(\dot{\bigcup_{n=1}^{\infty}} V_n\right) \leq m([-1, 2])$$

$$\Rightarrow \quad 1 \leq \sum_{n=1}^{\infty} m(V) \leq 3$$

But $\sum_{n=1}^{\infty} m(V) = 0$ or $+\infty$

$$\| \quad m(V)\left(\sum_{n=1}^{\infty} 1\right)$$

Thus we get a contradiction.

$\therefore$ V is not measurable.