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March 7th, 2018
Probability and Random Processes
HW2

HW2: Coupon Collecting

Given:
There is an urn containing n balls numbered from 1 -> n.
Sampling with replacement: A ball is drawn, its number is recorded and then the ball is thrown back in the urn.

1: How many draws does it take on average to record k different numbers?
When (k<n)

Solution:

1) K= balls drawn with replacement

$\therefore$ n^K = total possible ways of drawing

so when we want the different numbered balls, the total possibilities are given by,

$$k!\binom{n}{k} = \frac{n!}{(n-k)!}$$

now applying this the probability formula

$$p = \frac{|A|}{|S|} = \frac{\left[\frac{n!}{(n-k)!}\right]}{n^k} = \frac{n!}{(n-k)!\,n^k}$$

but since we want the average we have to put the probability of the number of draws:

$$\frac{k}{\frac{n!}{(n-k)!\,n^k}} \rightarrow \boxed{\frac{k(n-k)!\,n^k}{n!} \text{ for } k<n}$$

Explanation:

To calculate this we have to start off by first calculating the total number of ways to draw k balls from n balls in the urn. We first do this by setting n balls in the urn to the power of k balls being drawn because the k balls are being drawn with replacement and we don't care about repetition of the samples. Then we have to move to calculating the number of ways of drawing k balls with different results (without repetition) rather than drawing the same ball over and over again, because again it is with replacement we get, the factorial of n balls divided by the the factorial of the difference of n-k. We then apply this to the probability formula using the non-repetition as our desired set and the repetition as our sample set. This yields the probability though because we want the average amount of draws to record k different numbers we must take k and divide it by the probability yielding the result for k<n.

2: How many draws does it take on average to record numbers 1 through k ?
Solution:

2)

the number of draws expected to get
the first ball from k balls is given by,

$p=\frac{k}{n}$ ∴ $\to \frac{1}{\frac{k}{n}} \to \frac{n}{k}$

then when trying to get a second ball,

$= \frac{n}{k-1}$

while trying to get a third,

$= \frac{n}{k-2}$

so for k^{th} ball

we get

$p=\frac{1}{n} \to n \to n$

∴ expected # of draws from $1 \to k$ is given by,

$$k = \frac{n}{k} + \frac{n}{k-1} + \frac{n}{k-2} \dots \to n\left(\frac{1}{k}+\frac{1}{k-1}+\frac{1}{k-2} \dots +1\right)$$

Explanation:
This is a geometric probability. Therefore the first ball being drawn's probability is given as k/n which we then flip. Then as we draw more balls with replacement we take this fraction and subtract the denominator by 1 less than the draw number. We then see that the result average is the expected values of k's summation.