# CS/ECE/ISYE524: Introduction to Optimization – Linear Optimization Models

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# Blending Constraints

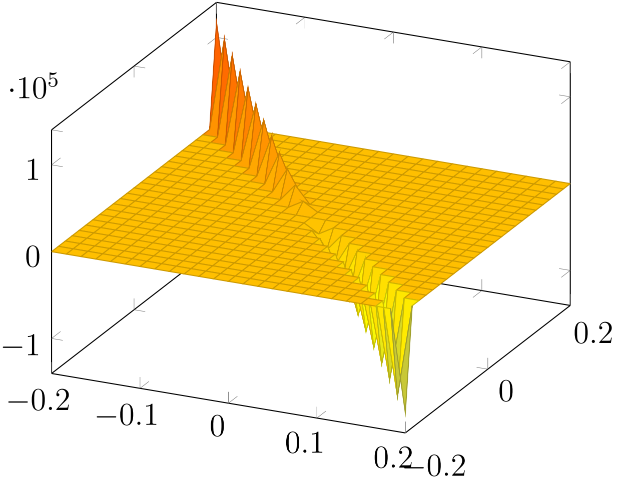
- Back to McGreasy's — Imagine that Helen has "relaxed" her constraint on my hamburger intake.
- Now, I can eat as many hambugers as I want, with two new requirements:
  1. We now have some maximum amount of every nutrient (say three times the minimum requirement)
  2. Keep my calories to a specified percentage of my vitamin intake:

$$\frac{\sum_{j \in F} a_{Cals,f} x_f}{\sum_{j \in F} a_{VitC,f} x_f} \leq \rho$$

$$\frac{\sum_{j \in F} a_{Cals,f} x_f}{\sum_{j \in F} a_{VitA,f} x_f} \leq \rho$$

- Is this a linear constraint?

NO! : $\frac{2x_1+x_2}{x_1+x_2}$



# Solving with HiGHS

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Constraints of type
MathOptInterface.ScalarNonlinearFunction-in-MathOptInterface.I
are not supported by the solver.

If you expected the solver to support your problem, you may
have an error in your formulation. Otherwise,
consider using a different solver.

The list of available solvers, along with the problem
types they support, is available at
https://jump.dev/JuMP.jl/stable/installation/#Supported-solver
```

# Making the Nonlinear Into Linear

- By doing some algebra, we can write the set of points satisfying this (nonlinear) inequality as a linear inequality...
- Multiply both sides of the inequality by $\sum_{j \in F} a_{VitC,f} x_f$
- What (very important) assumption did I just make?
- $\sum_{j \in F} a_{VitC,f} x_f > 0$ in any feasible solution!
- The moral of the story...
  - Not everything that looks nonlinear is nonlinear
- This is called a "blending" constraint.

# Making Alloy

- We would like to make an amount $d$ of a specific alloy
- There is a set $E$ of elements
- For each element $e \in E$, there is both a minimum (%) grade $\ell_e$ and a maximum (%) grade (%) $u_e$ that the alloy must have.
- Alloy is made from a set $R$ of raw materials, each costing $c_r$ per unit and having a maximum amount $K_r$ available $(\forall r \in R)$
- Raw material $r$ is made up up $\alpha_{re}$ percent of element $e \in E$

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**Assumption: Production is "linear'**

- All raw materials converted into alloy
- Final alloy element percentages is weighted average of element composition of input raw materials

# Math Model

- $x_r$ : Amount of raw $r$ to produce

$$\min \sum_{r \in R} c_r x_r$$

$$\sum_{r \in R} (\alpha_{re} - \ell_e) x_r \geq 0 \quad \forall e \in E$$

$$\sum_{r \in R} (\alpha_{re} - u_e) x_r \leq 0 \quad \forall e \in E$$

$$\sum_{r \in R} x_r \geq d$$

$$0 \leq x_r \leq K_r \qquad \forall r \in R$$

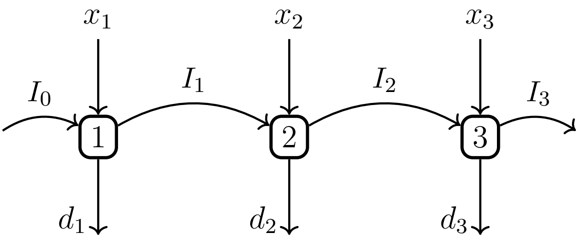
**Check the Notebook**

- Alloy.ipynb

# Modeling Multi-Period Problems

- One of the most important uses of optimization is in multi-period planning.
- Partition time into a number of periods.
- Usually distinguished by Inventory or Carry-Over variables.
- Suppose there is a "planning horizon" $T = \{1, 2, \ldots, |T|\}$.
- Also suppose there is a known demand $d_t$ for each $t \in T$
- Define...
  - $x_t$ : Production level during period $t, \forall t \in T$
  - $I_t$ : Inventory level at end of period $t, \forall t \in T$

# Modeling Multi-Period Problems




$$I_0 + x_1 = d_1 + I_1$$

$$I_1 + x_2 = d_2 + I_2$$

$$I_{t-1} + x_t = x_t + I_t$$

- To model "losses or gains", just put appropriate multipliers (not 1) on the arcs

# Another Story: Aggregate Planning

- Complex production process involving many pieces
    - Demands
    - Variable workforce size
    - Overtime possibilities
    - Inventory requirements

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### We're Making Shoes: ShoeCo

- Plan production of shoes for next several months
- Meet forecast demands on time
- Hire and/or lay off workers
- Make overtime decisions
- Objective: minimize total cost

# ShoeCo: It's All Greek To Me

- Planning horizon $T = \{1, 2, \dots |T|\}$. ($|T| = 4$).
- Meet demand $d_t$ for shoes in period $t \in T$.
  $d = (3000, 5000, 2000, 1000)$
- Initial Shoe Inventory: $\mathcal{I}_0 = 500$
- Have $\mathcal{W}_0 = 100$ workers currently employed
- Workers paid $\$\alpha = 1500$/month for working $H = 160$ hours
- They can work overtime (max of $O = 20$ hours/worker) and get paid $\$\beta = 13$/hour.

# ShoeCo: Greek Letter Zoo

- It take $a = 4$ hours of labor and $\delta = \$15$ in raw material costs to produce a shoe
- Hire-Fire costs: $\eta = 1600$ to hire a worker, $\zeta = \$2000$ to fire a worker.
- Running out of greek letters, $\iota = \$3$ holding cost incurred for each pair of shoes held at the end of the month.
    - Inventory costs are sometime compuer as cost of capital—You could better invest your money rather than having that investment tied up in produced inventory

# Your Mission

- Minimize all costs: labor (regular + overtime), production, inventory, hiring and firing
- What decision variables do we need?
  - HINT: If you're having trouble getting the decision variables, try and write the objective

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## Decision Variables

- $x_t$: # of shoes to produce during month $t$
- $I_t$: Ending inventory in month $t$, $t \in T \cup \{0\}$
- $w_t$: # of workers available in month $t$, $t \in T \cup \{0\}$.
- $o_t$: # of overtime hours used in month $t$
- $h_t$: # workers hired at the beginning of month $t$
- $f_t$: # workers fired at the beginning of month $t$

# Objective, Minimize Total Costs

1. Raw Material Costs: $\sum_{t \in T} \delta x_t$
2. Regular Labor Costs: $\sum_{t \in T} \alpha w_t$
3. Overtime Labor Costs: $\sum_{t \in T} \beta o_t$
4. Hiring Costs: $\sum_{t \in T} \eta h_t$
5. Firing Costs: $\sum_{t \in T} \zeta f_t$
6. Inventory Costs: $\sum_{t \in T} \iota I_t$

# Constraints

## Limit on Monthly Production

- Not given explicitly
- Determined by number of workers available and overtime decisions
- Math-speak: $ax_t \leq Hw_t + o_t \quad \forall t \in T$

## Upper limit on overtime hours/month

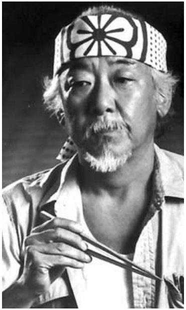
- Depends on how many workers you have
- Aggregate planning: Don't worry about individual workers
- Math-speak: $o_t \leq Ow_t \quad \forall t \in T$

# Constraints

## Demand must be met on time

- Equivalent to having nonnegative ending inventory each month (no backlogging)
- Math-speak: $I_t \geq 0 \quad \forall t \in T$
- This assumes we have balance between production, demand, and inventory
- We'll see backlogging later

# Balance, Daniel-Son



## Shoes

- Draw Picture, Math Speak:

$$I_{t-1} + x_t = d_t + I_t \quad \forall t \in T$$

- Boundary: $I_0 = \mathcal{I}_0$ (Maybe $I_{|T|} \geq \mathcal{I}_0$).

## People

- Hiring/Firing Affects worker levels. Math speak:

$$w_t = w_{t-1} + h_t - f_t \quad \forall t \in T$$

- Boundary: $w_0 = \mathcal{W}_0$

# Full Model

$$\min \sum_{t \in T} (\delta x_t + \alpha w_t + \beta o_t + \eta h_t + \zeta f_t + \iota I_t)$$

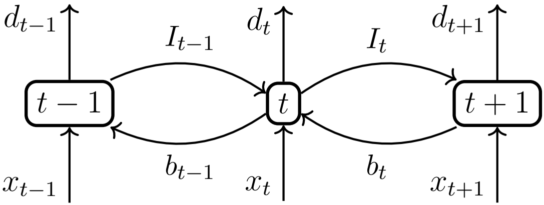
$$\begin{array}{rcl} \text{s.t. } a x_t & \leq & H w_t + o_t \quad \forall t \in T \\ o_t & \leq & O w_t \quad \forall t \in T \\ I_{t-1} + x_t & = & d_t + I_t \quad \forall t \in T \\ I_0 & = & \mathcal{I}_0 \\ w_t & = & w_{t-1} + h_t - f_t \quad \forall t \in T \\ w_0 & = & \mathcal{W}_0 \\ x_t, I_t, w_t, h_t, f_t & \geq & 0 \quad \forall t \in T \end{array}$$

- Check out the notebook ShoeCo.ipynb

# Stuff Happens

- Suppose you don't have to meet forecast demands in every period.
- Meeting demand is often too stringent a requirement for the real-world
- Demand does not have to be met on time, but it must be met *eventually*
- There is a shortage cost $\theta = \$20$ per unit per month backlogged
- \$1 Question: How should the minimum cost compare with cost of earlier model?

# Backlog model: Revised inventory balance




- Interpretation: $b_t$ represents a flow from the future to the current period
- New inventory balance constraints, for $t = 1, \ldots, T$

$$I_{t-1} + b_t + x_t = d_t + I_t + b_{t-1}$$

- Backlog variables also have the sign restriction:

$$b_t \geq 0, \quad t = 1, \ldots, T$$

# Problem with model?

In our model, it is feasible to have both $b_t > 0$ and $I_t > 0$

- In period $t$, we hold inventory *and* have backlogged demand
- This doesn't make sense! Should use inventory to satisfy the unmet demand

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- It's OK: Won't happen in an optimal solution
  - Both $b_t > 0$ and $I_t > 0$ incur *costs* in objective
  - $b_t$ and $I_t$ *always appear together* in constraints

$$I_{t-1} + b_t + x_t + = d_t + I_t + b_{t-1}$$
$$\Leftrightarrow (I_{t-1} - b_{t-1}) + x_t = d_t + (I_t - b_t)$$

  - Can decrease both by the same amount and still be feasible, until one becomes zero

# Inventory position

- The quantity $I_t - b_t$ is sometimes called the inventory position.
  - It represents a *net* inventory level
  - Can be positive or negative (i.e., it is *unrestricted in sign*)
  - Positive $\Rightarrow I_t > 0$ and $b_t = 0$, we are holding inventory
  - Negative $\Rightarrow I_t = 0$ and $b_t > 0$, we have a backlog

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- We need separate decision variables for (positive) inventory level and backlog, to account for the costs of those
- There is another way to think about backlogging

# How to Model Backlogging

- Think of inventory being allowed to go negative, and let $n_t$ be this "net inventory position"
- Picture still makes sense, since if inventory is negative, you need to "make up" for it during one of the next periods
- You can set last period demand $n_{|T|} \geq 0$ to ensure that all demand is *eventually* met.
- Cost function $F(n_t)$:

$$F(n_t) = \begin{cases} \iota n_t & \text{if } n_t \geq 0 \\ -\theta n_t & \text{if } n_t < 0 \end{cases}$$

- Is $F(n_t)$ a linear function of $n_t$? no!

# Another Nonlinear/Linear Trick

- To model the case where we are minimizing a convex piecewise linear function (like $F(\cdot)$ or $|\cdot|$), we can introduce a variable for each piece
- Write constraints $n_t = I_t - b_t \quad \forall t \in T$
  - Think of this as (Leftover - Shortage)
- Objective gets terms:

$$\sum_{t \in T} (\iota I_t + \theta b_t)$$

- This trick only works if we are *minimizing* costs. Then at most one of $I_t$ and $b_t$ will ever be positive in an optimal solution.
- We will learn more about modeling piecewise linear functions next time