CS/ECE/ISYE524: Introduction to Optimization – Integer Optimization Models

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Example: ClothCo

ClothCo is capable of manufacturing three types of clothing: shirts, shorts, and pants. Each type of clothing requires that ClothCo have the appropriate type of machine available. The machines can be rented at a fixed weekly cost. The manufacture of each type of clothing also requires some amount of cloth and labor, and returns some profit, indicated below. Each week, 150 hours of labor and 160 sq yd of cloth are available. How should ClothCo tune its production to maximize profits? Note:
If we don't produce a particular item, we don't pay the rental cost!

Clothing item	**Labor per item**	**Cloth per item**	**Profit per item**	**Machine rental**
Shirt	3 hours	4	\$6	\$200/wk.
Shorts	2 hours	3	\$4	\$150/wk.
Pants	6 hours	4	\$7	\$100/wk.

Example: ClothCo

Obvious decision variables:

- $x_1 \geq 0$: number of shirts produced each week
- $x_2 \geq 0$: number of shorts produced each week
- $x_3 \geq 0$: number of pants produced each week

- Constraints:

$$\begin{aligned} 3x_1 + 2x_2 + 6x_3 &\leq 150 \qquad \text{(labor budget)} \\ 4x_1 + 3x_2 + 4x_3 &\leq 160 \qquad \text{(cloth budget)} \end{aligned}$$

- Maximize weekly profit:

$$6x_1 + 4x_2 + 7x_3$$

- Still need to account for machine rental costs...

Example: ClothCo

Binary variables:

- $z_1 = \begin{cases} 1 & \text{if any shirts are manufactured} \\ 0 & \text{otherwise} \end{cases}$
- $z_2 = \begin{cases} 1 & \text{if any shorts are manufactured} \\ 0 & \text{otherwise} \end{cases}$
- $z_3 = \begin{cases} 1 & \text{if any pants are manufactured} \\ 0 & \text{otherwise} \end{cases}$

- Maximize net weekly profit:

$$6x_1 + 4x_2 + 7x_3 - 200z_1 - 150z_2 - 100z_3$$

Example: ClothCo

Optimization model:

$$
\begin{aligned}
\underset{x,z}{\text{maximize}} \quad & 6x_1 + 4x_2 + 7x_3 - 200z_1 - 150z_2 - 100z_3 \\
\text{subject to:} \quad & 3x_1 + 2x_2 + 6x_3 \le 150 \quad \text{(labor budget)} \\
& 4x_1 + 3x_2 + 4x_3 \le 160 \quad \text{(cloth budget)} \\
& x_i \ge 0, \quad z_i \in \{0, 1\} \\
& \text{if } x_i > 0 \text{ then } z_i = 1
\end{aligned}
$$

- We need to find an algebraic representation for the relationship between x_i and z_i.

Detour: Logic!

How do we represent: "if $x > 0$ then $z = 1$"?

- Statements of the form "if P then Q" are written as:

$$P \implies Q$$

- This is equivalent to the *contrapositive*:

$$\neg Q \implies \neg P$$

- But this is **not** equivalent to the *converse*:

$$Q \implies P$$

Detour: Logic!

P: I am drunk.
Q: I have been drinking.

- Basic statement $(P \implies Q)$ true
  "if I am drunk, then I have been drinking"
- Contrapositive $(\neg Q \implies \neg P)$ also true
  "if I have not been drinking, then I'm not on drunk"
- Converse $(Q \implies P)$ not true
  "if I have been drinking, then I am drunk"

Detour: Logic!

How do we represent: "if $x > 0$ then $z = 1$"?

- Contrapositive: "if $z = 0$ then $x \leq 0$"
- Since $x \geq 0$, this is the same as: "if $z = 0$ then $x = 0$"
- Model logical condition "if $x > 0$ then $z = 1$" as:

$$\boxed{x \leq Mz}$$

  where M is *any* upper bound on the value x can take at the optimal solution: $x_{\text{opt}} \leq M$.
- This is called the "Big M method"

Example: ClothCo

Optimization model:

$$\begin{array}{rll} \underset{x,z}{\text{maximize}} & 6x_1 + 4x_2 + 7x_3 - 200z_1 - 150z_2 - 100z_3 & \\ \text{subject to:} & 3x_1 + 2x_2 + 6x_3 \leq 150 & \text{(labor budget)} \\ & 4x_1 + 3x_2 + 4x_3 \leq 160 & \text{(cloth budget)} \\ & x_i \geq 0, \quad z_i \in \{0,1\} & \\ & x_i \leq M_i z_i & \end{array}$$

- Where M_i is an upper bound on x_i.
- IJulia notebook: ClothCo.ipynb

Example: ClothCo

We can choose very large bounds, e.g. $M_i = 10^5$...

...or we can choose M_i using constraints!

- $3x_1 + 2x_2 + 6x_3 \leq 150$ (labor budget)
  Since we have $x_i \geq 0$, we have the obvious bounds:
  $x_1 \leq 50$, $x_2 \leq 75$, $x_3 \leq 25$
- $4x_1 + 3x_2 + 4x_3 \leq 160$ (cloth budget)
  Using a similar argument, we conclude that:
  $x_1 \leq 40$, $x_2 \leq 54$, $x_3 \leq 40$
- Combining these bounds, we obtain:
  $x_1 \leq 40$, $x_2 \leq 54$, $x_3 \leq 25$

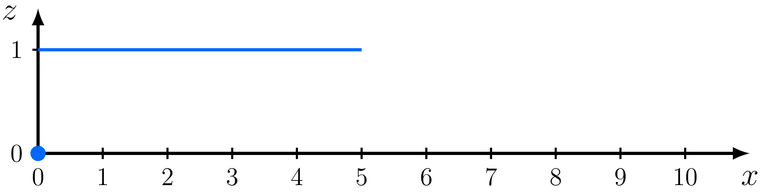
Choosing an upper bound

It's generally desirable to pick the smallest possible M.

(The MIP solver can run much faster that way.)

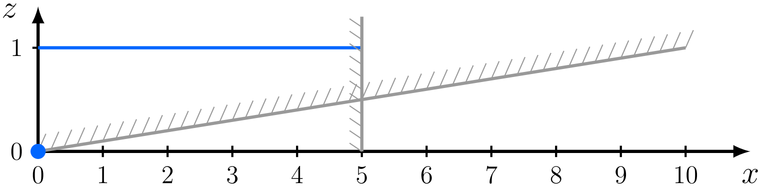
Simple example:

$$P = \left\{ 0 \le x \le 5,\ z \in \{0,1\} \;\middle|\; \text{if } x > 0 \text{ then } z = 1 \right\}$$

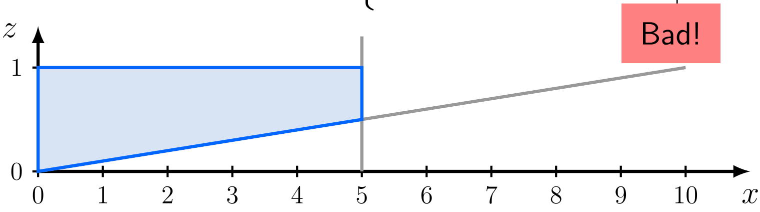


Choosing an upper bound

upper bounding: $P_1 = \left\{ 0 \le x \le 5,\ z \in \{0,1\} \;\middle|\; x \le 10z \right\}$



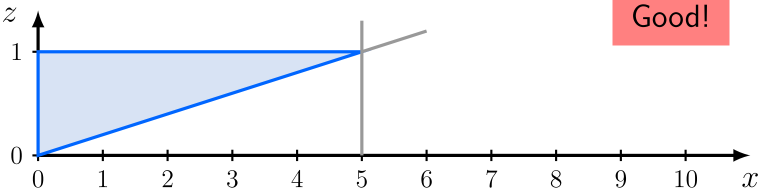
LP relaxation: $P_2 = \left\{ 0 \le x \le 5,\ 0 \le z \le 1 \;\middle|\; x \le 10z \right\}$



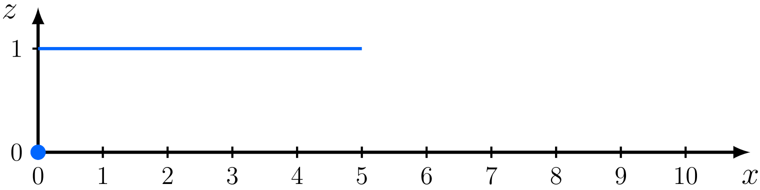
Choosing an upper bound

tightest bound: $P_3 = \left\{ 0 \le x \le 5,\ 0 \le z \le 1 \;\middle|\; x \le 5z \right\}$

Good!

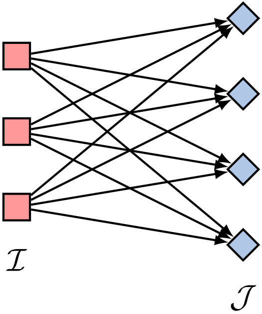


Same as the convex hull of the original set!



Simple facility location problem

- Facilities ■ : $\mathcal{I} = \{1, 2, \ldots, I\}$
- Customers ◇ : $\mathcal{J} = \{1, 2, \ldots, J\}$
- c_{ij} is the cost for facility i to serve customer j. (e.g. transit cost)
- Each customer must be served and there is **no limit** on how many customers each facility can serve.



Even easier than an assignment problem! Simply assign each customer to the cheapest facility for them:

$$\text{minimum cost} = \sum_{j \in \mathcal{J}} \left(\min_{i \in \mathcal{I}} c_{ij} \right)$$

Simple facility location problem

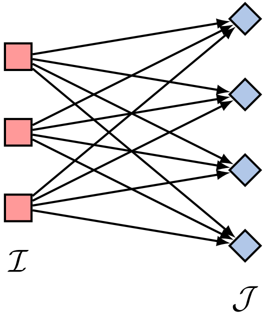
LP formulation

$$
\begin{aligned}
\underset{y}{\text{minimize}} \quad & \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} c_{ij} y_{ij} \\
\text{subject to:} \quad & \sum_{i \in \mathcal{I}} y_{ij} = 1 \quad \text{for all } j \in \mathcal{J} \\
& y_{ij} \geq 0 \quad \text{for all } i \in \mathcal{I} \text{ and } j \in \mathcal{J}
\end{aligned}
$$

- no reason to use the LP formulation for this problem, but we'll use this formulation as a starting point for a modified version of the problem.

Uncapacitated facility location

- Facilities $\square$: $\mathcal{I} = \{1, 2, \ldots, I\}$
- Customers $\diamond$: $\mathcal{J} = \{1, 2, \ldots, J\}$
- c_{ij} is the cost for facility i to serve customer j. (e.g. transit cost)
- f_i is the cost to build facility i. We can choose which ones to build.
- No limit on how many customers each facility can serve.



Let $\mathcal{S} \subseteq \mathcal{I}$ be the subset of the facilities we choose to build. This is a much more difficult problem.

$$\text{minimum cost} = \min_{\mathcal{S}} \left(\sum_{i \in \mathcal{S}} f_i + \sum_{j \in \mathcal{J}} \left(\min_{i \in \mathcal{S}} c_{ij} \right) \right)$$

Uncapacitated facility location

MIP formulation

$$
\begin{aligned}
\underset{y,z}{\text{minimize}} \quad & \sum_{i \in \mathcal{I}} f_i z_i + \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} c_{ij} y_{ij} \\
\text{subject to:} \quad & \sum_{i \in \mathcal{I}} y_{ij} = 1 \quad \text{for all } j \in \mathcal{J} \\
& y_{ij} \in \{0,1\} \text{ for all } i \in \mathcal{I} \text{ and } j \in \mathcal{J} \\
& z_i \in \{0,1\} \text{ for all } i \in \mathcal{I} \\
& \text{if } z_i = 0 \text{ then } y_{ij} = 0 \text{ for all } j \in \mathcal{J}
\end{aligned}
$$

- need to find an upper bound on $y_{ij} \leq M$ so we can write the logical constraint as: $y_{ij} \leq M z_i$.

Uncapacitated facility location

MIP formulation

$$\begin{aligned}
\underset{y,z}{\text{minimize}} \quad & \sum_{i\in\mathcal{I}} f_i z_i + \sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}} c_{ij} y_{ij} \\
\text{subject to:} \quad & \sum_{i\in\mathcal{I}} y_{ij} = 1 \quad \text{for all } j \in \mathcal{J} \\
& y_{ij} \in \{0,1\} \text{ for all } i \in \mathcal{I} \text{ and } j \in \mathcal{J} \\
& z_i \in \{0,1\} \text{ for all } i \in \mathcal{I} \\
& \text{if } z_i = 0 \text{ then } y_{ij} = 0 \text{ for all } j \in \mathcal{J}
\end{aligned}$$

- First option: $y_{ij} \le z_i$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$.
- Clever simplification: $\sum_{j\in\mathcal{J}} y_{ij} \le J z_i$ for all $i \in \mathcal{I}$.
- (or is it?) Julia notebook: UFL.ipynb

Uncapacitated facility location

Random instance of the problem with $I = J = 100$, and f_i, c_{ij} uniform in $[0, 1]$. Solved using JuMP+Cbc.

- clever constraint: $\sum_{j \in \mathcal{J}} y_{ij} \leq J z_i$ for all $i \in \mathcal{I}$.
    - Optimal solution found (all variables binary) in 4.2 sec.
    - Same solution found if we let $0 \leq y_{ij} \leq 1$. Now 3.7 sec.
- tighter constraint: $y_{ij} \leq z_i$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$.
    - Optimal solution found (all variables binary) in 0.65 sec.
    - Same solution found if we let $0 \leq y_{ij} \leq 1$. Now 0.45 sec.
    - Same solution if we also let $0 \leq z_i \leq 1$. Now 0.02 sec.
- about 15 facilities selected

Uncapacitated facility location

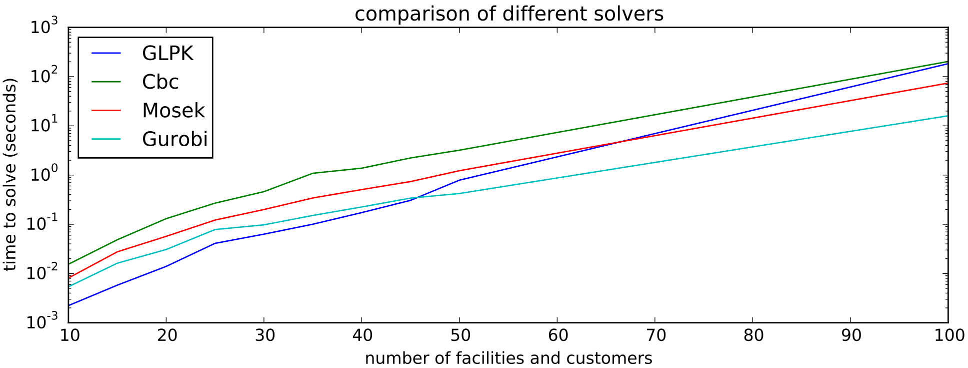
Random instance of the problem with $I = J = 100$, and $f_i = 0.5$, c_{ij} uniform in $[0, 1]$. Solved using JuMP+Cbc.

- clever constraint: $\sum_{j \in \mathcal{J}} y_{ij} \leq J z_i$ for all $i \in \mathcal{I}$.
  - Optimal solution found (all variables binary) in 32 min.
  - Same solution found if we let $0 \leq y_{ij} \leq 1$. Now 15 min.
- tighter constraint: $y_{ij} \leq z_i$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$.
  - Optimal solution found (all variables binary) in 3.3 min.
  - Same solution found if we let $0 \leq y_{ij} \leq 1$. Now 3.8 min.
  - Non-integer if we also let $0 \leq z_i \leq 1$. Now 0.025 sec.
- about 10 facilities selected

Be careful with integer programs!

Solver comparison

- $f_i = 0.5$ and c_{ij} uniform in $[0, 1]$.
- the z_i are binary and $0 \leq y_{ij} \leq 1$.
- disaggregated constraint $y_{ij} \leq z_i$.



- Most solvers are substantially slower if we use the aggregated

Recap: Fixed Costs

- Producing x has a fixed cost if the cost has the form:

$$\text{cost} = \begin{cases} f + cx & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- Define a binary variable $z \in \{0, 1\}$ where:

$$z = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

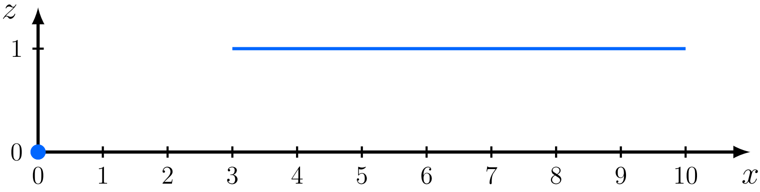
- The constraint becomes: $x \leq Mz$
  where M is any upper bound of x.
- The cost becomes: $fz + cx$
- Small M's are usually better!

Variable lower bounds

(lower bounds that vary, not lower bounds on variables!)

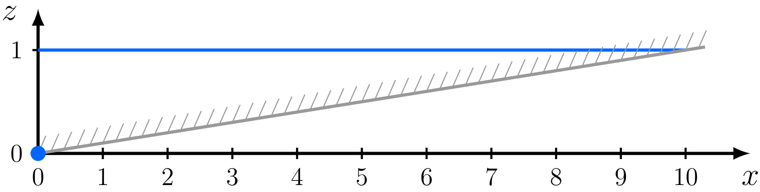
We have a variable $x \geq 0$, but we want to prevent solutions where x is small but not zero, for example $x = 0.001$.

- Model the constraint: "either $x = 0$ or $3 \leq x \leq 10$".
- Define a binary variable $z \in \{0, 1\}$ that characterizes whether we are dealing with the case $x = 0$ or the case $3 \leq x \leq 10$. The set we'd like to model:

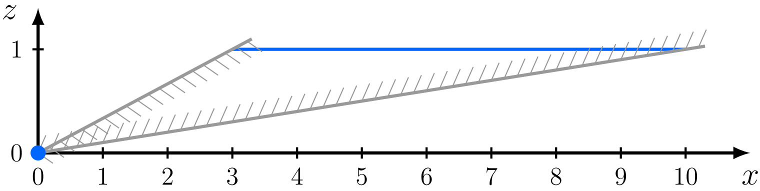


Variable lower bounds

upper bounding: $\left\{ 0 \leq x \leq 10,\ z \in \{0,1\} \;\middle|\; x \leq 10z \right\}$

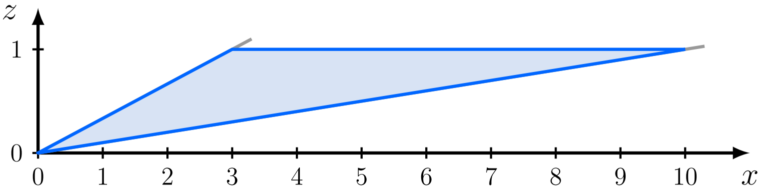


lower bounding: $\left\{ 0 \leq x \leq 10,\ z \in \{0,1\} \;\middle|\; 3z \leq x \leq 10z \right\}$

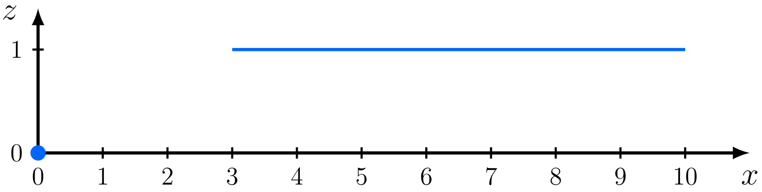


Variable lower bounds

LP relaxation: $\left\{ 0 \le x \le 10,\ 0 \le z \le 1 \;\middle|\; 3z \le x \le 10z \right\}$



Same as the convex hull of the original set!



Variable lower bounds

- The MIP is exact (can serve as a substitute to the original set).
- The LP relaxation may not be exact if there are other constraints:

Original problem

$$\begin{aligned} \max_{x,y} \quad & x + y \\ \text{s.t.} \quad & 3 \le y \le 4 \\ & x + y \le 5 \\ & x = 0 \text{ or } 3 \le x \le 4 \end{aligned}$$

$x = 0,\ y = 4$
obj $= 4$

MIP formulation

$$\begin{aligned} \max_{x,y,z} \quad & x + y \\ \text{s.t.} \quad & 3 \le y \le 4 \\ & x + y \le 5 \\ & 3z \le x \le 4z \\ & z \in \{0, 1\} \end{aligned}$$

$x = 0,\ y = 4,\ z = 0$
obj $= 4$

LP relaxation

$$\begin{aligned} \max_{x,y,z} \quad & x + y \\ \text{s.t.} \quad & 3 \le y \le 4 \\ & x + y \le 5 \\ & 3z \le x \le 4z \\ & 0 \le z \le 1 \end{aligned}$$

$x = 1,\ y = 4,\ z = 0.25$
obj $= 5$