NAME:

# ISyE 601 Midterm

November 3, 2015
8AM-9:15AM

## READ THIS!

1. Please put your name on all the pages of the exam.
2. The exam is open notes.
3. You may use your computers, but *only a PDF reader and GAMS*. You may use your browser only to download the text files containing GAMS "top" matter `midterm1.gms`, `midterm2.gms`, and `midterm3.gms` during the first 5 minutes of the exam. And you may use your web browser to upload your final GAMS models to the `learn@uw` dropbox.
4. be sure to write valid *mathematical* syntax in your written answers. (GAMS syntax in the written portion will not receive full credit). If you need additional space, feel free to write on additional paper, but be sure to write your name on top and staple to exam.
5. The number of points for each problem is displayed at the end of the problem's title. The time required for me to complete each question is also listed. Please use your time wisely. The Dropbox will closely precisely at 9:20, and you will need to have your GAMS files uploaded by that time. Amanda will collect the papers at 9:15.
6. The last problem is the most difficult. Think of it as an extra credit problem. *Do not* work on it unless you have completed the remainder of the exam, as it will take some time, and is not worth too many points.
7. **Good luck!** Don't panic. I'm rooting for you.

Here are the point available for each problem and the time it took me to do it.

| **Problem** | **Points** | **Prof. Linderoth Time** (min.) |
|---|---|---|
| 1 | 14 | 2 |
| 2 | 30 | 7 |
| 3 | 60 | 18 |
| EC | 10 | - |
| **Total:** | | 27 |

# 1 Short Answer

## 1.1 Problem (2 points)

**True or False:** If a linear program has an optimal solution, then it has an optimal solution at an extreme point of the feasible region.

**Answer.** True

## 1.2 Problem (2 points)

**True or False:** The set

$$X = \{x \in \mathbb{R} \mid |x| \geq 2\}$$

is a convex set

**Answer.** False. Note $X' = \{x \in \mathbb{R} \mid |x| \leq 2\}$ is convex.

## 1.3 Problem (2 points)

Give the names of four different linear programming solvers that you can use with GAMS. The answer you give should be valid GAMS code for the line `option lp =` ____________.

**Answer.** N/A.

## 1.4 Problem (2 points)

**True or False:** The simplex method is the theoretically fastest algorithm for solving shortest path problems

**Answer.** False

## 1.5 Problem (2 points)

**True or False:** All linear functions are convex functions

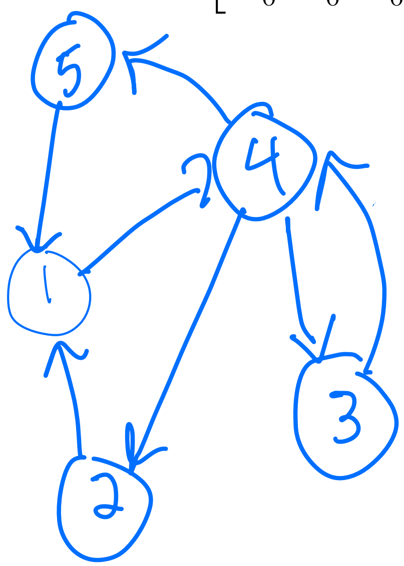
**Answer.** True

### 1.6 Problem (2 points)

Draw the network associated with the following node-arc incidence matrix $A$.

$$A = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

**Answer.**




### 1.7 Problem (2 points)

What is the right hand side ($b$) vector associated with solving a shortest path problem from node 1 to node 5 for the network above?

**Answer.**

$$b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

## 2 The World is Nonlinear!

You are given four linear functions of a vector variable $x = (x_1, x_2, x_3, x_4)$.

$$\ell_1(x) = -x_1 + 2x_2 - 5x_4$$
$$\ell_2(x) = 6x_1 - 8x_2 - 10x_3 - 2x_4$$
$$\ell_3(x) = -8x_1 + 5x_2 + 2x_3$$
$$\ell_4(x) = x_1 + x_2 + x_3 + x_4$$

Let

$$f(x) := \max_{i=1,2,3,4} \{\ell_i(x)\}$$

be the function defined to be the maximum of these four functions, and let

$$g(x) := \frac{\ell_3(x)}{\ell_4(x)}$$

be the ratio of the last two functions.

### 2.1 Problem (15 points)

Write a *linear program* to solve the following optimization problem:

$$\min f(x)$$
$$g(x) \leq 20$$
$$g(x) \geq 10$$
$$\ell_4(x) \geq 1$$
$$x_1, x_2, x_3, x_4 \geq -5$$
$$x_1, x_2, x_3, x_4 \leq 5$$

**Answer.**

$$\min \; z$$

$$z \geq -x_1 + 2x_2 - 5x_4$$
$$z \geq 6x_1 - 8x_2 - 10x_3 - 2x_4$$
$$z \geq -8x_1 + 5x_2 + 2x_3$$
$$y = x_1 + x_2 + x_3 + x_4$$
$$z \geq y$$
$$y \geq 1$$
$$10y \leq -8x_1 + 5x_2 + 2x_3 \leq 20y$$
$$-5 \leq x_i \leq 5 \quad \forall i = 1, 2, 3, 4$$

### 2.2 Problem (15 points)

Implement your linear program from Problem 2.1 in the GAMS modeling language. Top material that may be useful for implementing your model is available in `midterm1.gms`. If you prefer, you may implement your models from scratch. In any event, turn in your GAMS code, in a file named `midterm1.gms` to the learn@uw Dropbox. What is the optimal solution?

$(x_1, x_2, x_3, x_4) =$ NA

Optimal objective value: ______

# 3 Linderoth Brewing Company

Linderoth Brewing Company (LBC) makes a variety of beers $B$, and we are going to help it meet its ever-growing demand for beer on the UW-Madison campus over a planning horizon consisting of a set of months $T$. LBC must produce $d_{bt}$ kegs of beer type $b \in B$ during month $t \in T$. During any month, Linderoth Brewing Company can produce at most $K$ kegs (total) of beer. (This production is split between all of the beer types). It costs LBC $c_{bt}$ to produce one keg of beer type $b \in B$ during month $t \in T$. Beer produced in one month may be held in inventory to meet subsequent demand, but an inventory holding cost of $\alpha_b$ is incurred for each keg of beer type $b \in B$ held in inventory in a month. LBC has $Q_b$ kegs of beer $b \in B$ on hand right now, and it would also like to have $Q_b$ kegs of beer type $b \in B$ on hand at the end of the planning horizon. In this problem, we will write a linear program to minimize the total cost of meeting the next months demands.

## 3.1 Problem (4 points)

What are the decision variables?

**Answer.**

$x_{bt}$ : # kegs of beer of type $b \in B$ LBC produces during month $t \in T$

$h_{bt}$ : # number of kegs of beer type $b \in B$ held in inventory at <u>end</u> of month $t \in T$

## 3.2 Problem (5 points)

As a function of these decision variables, write the objective: to minimize the total production and inventory costs

**Answer.**

$$\min \sum_{b \in B} \sum_{t \in T} \left( c_{bt} x_{bt} + \alpha_b x_{bt} \right)$$

### 3.3 Problem (5 points)

Write the constraints that ensure that no more beer is produced than the maximum production capacity in each time period.

**Answer.**

$$\sum_{b\in B} x_{bt} \le K \qquad \forall t \in T$$

### 3.4 Problem (8 points)

Write all the remaining constraints necessary in the linear program. These constraints ensure that demand is met in each period and production quantities are non-negative.

(assume $T = \{1, 2, \ldots, |T|\}$)

**Answer.**

$$Q_b + x_{b1} = d_{b1} + h_{b1} \qquad \forall b \in B$$

$$h_{b,t-1} + x_{bt} = d_{bt} + h_{bt} \qquad \forall b \in B \quad \forall t \in T \setminus \{1\}$$

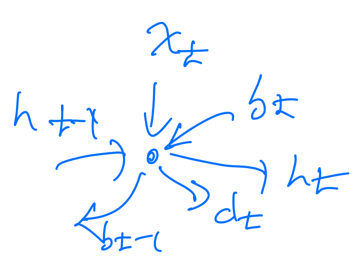
$$h_{b|T|} \ge Q_b$$

$$x_{bt}, h_{bt} \ge 0 \qquad \forall b \in B, \ \forall t \in T$$

### 3.5 Problem (20 points)

Implement the model described in your answers to Problems 3.1— 3.4 in the GAMS modeling language. The top material defining all of the sets and parameters is given in `midterm2.gms`. Turn in your GAMS code (in a file named `midterm2.gms` to the learn@uw Dropbox. What is the optimal solution value?

Optimal objective value: N/A



### 3.6 Problem (8 points)

Now suppose that you have backlogging – Demand for each type of beer must eventually be met, but you can backlog the demand at a cost of \$0.1/keg/month. Write the necessary changes to your linear program linear program for this case, including defining any new decision variables you may need.

**Answer.**

Answer #1:

Let $y_{bt}$ : # of kegs of beer $b \in B$ backlogged to month $t \in T$

Change 1st 2 constraints of 3.4 to

$$Q_b + x_{b1} + y_{b1} = d_{b1} + h_{b1} \quad \forall b \in B$$

$$h_{b,t-1} + x_{bt} + y_{bt} = d_{bt} + h_{bt} + y_{b,t-1} \quad \forall b \in B \ \forall t \in T \setminus \{1\}$$

$$y_{b|T|} = 0 \quad \forall b \in B$$

$$y_{bt} \geq 0 \quad \forall b \in B \ \forall t \in T$$

Answer #2

Let $n_{bt}$ be net inventory position of $b \in B$ at end of $t \in T$. (Also $y_{bt}$ as in Answer 1)

Change 1st 2 constraints of 3.4 to

$$Q_b + x_{b1} = d_{b1} + n_{b1} \quad \forall b \in B$$

$$n_{bt} = h_{bt} - y_{bt} \quad \forall b \in B \ \forall t \in T$$

$$n_{b,t-1} + x_{bt} = d_{bt} + n_{bt} \quad \forall b \in B \ \forall t \in T$$

Obj Function: $\min \sum_{b \in B} \sum_{t \in T} \left( c_{bt} x_{bt} + \alpha_b h_{bt} + 0.1 y_{bt} \right)$

### 3.7 Problem (10 points)

In the same `midterm2.gms` file, create a second model that implements backlogging. What is the optimal objective value in this case?

Optimal objective value: NA

## 4 You Down with LBC, Yeah You Know Me—Extra Credit

**This one is extra credit – only do it if you have time. It isn't worth as many points as the others**

Linderoth Brewing Company has an arbitrary set of locations $L$ that produce a set of beers $B$ to meet demand over time periods $T = \{1, 2, \ldots, |T|\}$ for a set of customers $J$. Each customer $j \in J$ has a demand of $d_{jbt}$ kegs of beer type $b \in B$ during time period $t \in T$. Each location $i \in L$ has a maximum production capacity $K_i$ (in kegs) for total beer produced in any time period. Each location $i \in L$ also has a production cost of $f_{ibt}$ (\$/keg) for producing beer type $b \in B$ during time period $t \in T$. There is a transportation cost of $c_{ij}$ for shipping a keg of (any type) of beer from location $i$ to customer $j$. Beer produced in one period at location $i \in L$ may be held for shipping in a later month at a cost of $\alpha_i$.

### 4.1 Problem (5 points)

Formulate a linear programming model that will determine how to meet each customer's demand each month at a minimum total cost (production + holding + transportation), while not exceeding the supplier production level. Be sure to clearly define all your decision variables

**Answer.** vars: $x_{ibt}$ : Kegs of beer $b \in B$ produced at location $i \in L$ during $t \in T$

$y_{ijbt}$ : Kegs of beer $b \in B$ produced at location $i \in L$, shipped to $j \in J$ during $t \in T$

$h_{ibt}$ : Inventory of beer type $b \in B$ at $i \in L$ at end of $t \in T$

$$\min \sum_{i \in L} \sum_{b \in B} \sum_{t \in T} f_{ibt} x_{ibt} + \sum_{i \in L} \sum_{j \in J} \sum_{b \in B} \sum_{t \in T} c_{ij} y_{ijbt}$$

$$+ \sum_{i \in L} \sum_{b \in B} \sum_{t \in T} \alpha_i h_{ibt}$$

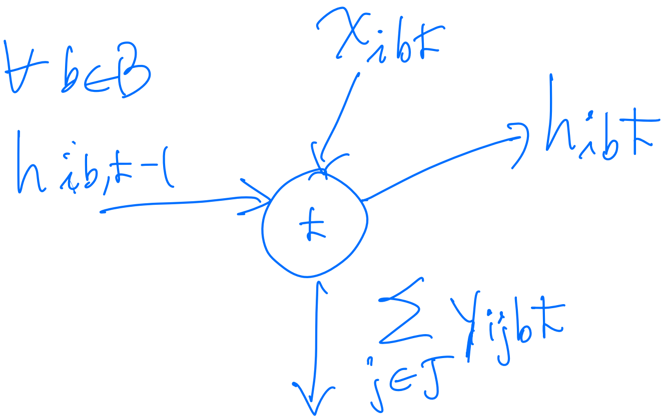
s.t. $$\sum_{b \in B} x_{ibt} \leq K_i \quad \forall i \in L \quad \forall t \in T$$

### 4.2 Problem (5 points)

Implement your model in the GAMS modeling language. Top material defining the sets and parameters is in `midterm3.gms`. What is the optimal solution value?

Optimal objective value: N/A

$\forall i \in L \;\; \forall b \in B$ (Balance)




Inventory Balance:

$$h_{ib,t-1} + x_{ibt} = \sum_{j \in J} y_{ijbt} + h_{ibt} \quad \forall i \in L \;\; \forall b \in B \;\; \forall t \in T$$

$$h_{ib0} = 0 \quad \forall i \in L \;\; \forall b \in B$$

Meet demand:

$$\sum_{i \in L} y_{ijbt} \geq d_{jbt} \quad \forall j \in J \;\; \forall b \in B \;\; \forall t \in T$$

$$h_{ibt},\; x_{ibt} \geq 0 \quad \forall i \in L \;\; \forall b \in B \;\; \forall t \in T$$

$$y_{ijbt} \geq 0 \quad \forall i \in L \;\; \forall j \in J \;\; \forall b \in B \;\; \forall t \in T$$