CS/ECE/ISYE524: Introduction to Optimization – Linear Optimization Models

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Exam #1 Preparation

Today's Docket

- Duality Review and Examples
- Review Midterm #1 Details
- Quick reminder of topics (few details)
- Questions from you
- Practice problems

First Midterm—Wednesday February 28

- In class—4pm-5:15pm
- Topics—Anything covered in class through this lecture
- Midterm has Four Problems
  1. True/False
  2. Short Answer
  3. Modeling
  4. Modeling

First Midterm

- **Note Sheet:** A single sheet of 8.5" x 11" paper is allowed
  - It will be *much better* to know the material than have to try and find and decipher it on your note sheet
- No calculators or other electronic devices
- Bring your ID to the exam and place it out. We will be checking IDs during the exam.
- Spread out in the room! (We may ask you to move)

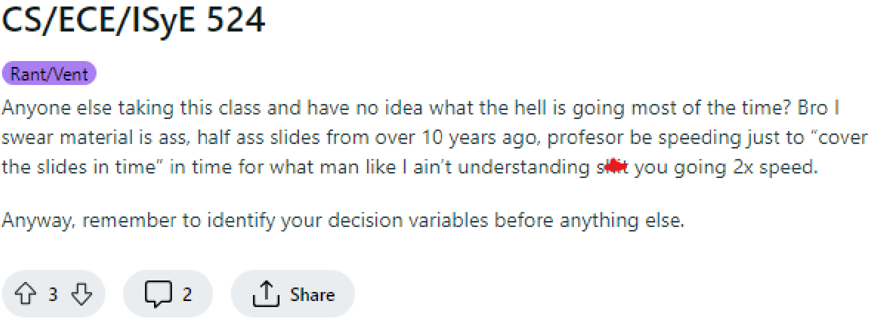
Who Says There's No Such Thing?

Free Lunch!

Eric and Sanjai[a] will take the top three scorers out to lunch!

[a] Prof. Linderoth will also join and pay

Happy Customers!



CS/ECE/ISyE 524

Rant/Vent

Anyone else taking this class and have no idea what the hell is going most of the time? Bro I swear material is ass, half ass slides from over 10 years ago, profesor be speeding just to "cover the slides in time" in time for what man like I ain't understanding [redacted] you going 2x speed.

Anyway, remember to identify your decision variables before anything else.

3 2 Share

- I apologize for covering duality material so quickly. This was necessary so that the material students needed to know in order to do the homework problems was covered before the due date. Lectures are available to watch again—even at 0.5x speed. :-)
- We will review duality today

Dual of Dual is Primal

To find the dual of an LP that is **not** in standard form:

1. convert the LP to standard form
2. write the dual
3. make simplifications

Example: What is the dual of the dual? *the primal!*

$$\begin{aligned} \min_{\lambda} \quad & b^{\mathsf{T}}\lambda \\ \text{s.t.} \quad & A^{\mathsf{T}}\lambda \geq c \\ & \lambda \geq 0 \end{aligned}$$

equiv

$$\begin{aligned} -\max_{\lambda} \quad & (-b)^{\mathsf{T}}\lambda \\ \text{s.t.} \quad & (-A^{\mathsf{T}})\lambda \leq (-c) \\ & \lambda \geq 0 \end{aligned}$$

dual

$$\begin{aligned} -\min_{z} \quad & (-c)^{\mathsf{T}}z \\ \text{s.t.} \quad & (-A)z \geq (-b) \\ & z \geq 0 \end{aligned}$$

equiv

$$\begin{aligned} \max_{z} \quad & c^{\mathsf{T}}z \\ \text{s.t.} \quad & Az \leq b \\ & z \geq 0 \end{aligned}$$

More duals

Standard form:

$$\begin{aligned} \max_x \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned} \quad \overset{\text{dual}}{\Longleftrightarrow} \quad \begin{aligned} \min_\lambda \quad & b^\mathsf{T} \lambda \\ \text{s.t.} \quad & \lambda \ge 0 \\ & A^\mathsf{T} \lambda \ge c \end{aligned}$$

Free form:

$$\begin{aligned} \max_x \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax \le b \\ & x \text{ free} \end{aligned} \quad \overset{\text{dual}}{\Longleftrightarrow} \quad \begin{aligned} \min_\lambda \quad & b^\mathsf{T} \lambda \\ \text{s.t.} \quad & \lambda \ge 0 \\ & A^\mathsf{T} \lambda = c \end{aligned}$$

Mixed constraints:

$$\begin{aligned} \max_x \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax \le b \\ & Fx = g \\ & x \text{ free} \end{aligned} \quad \overset{\text{dual}}{\Longleftrightarrow} \quad \begin{aligned} \min_{\lambda,\mu} \quad & b^\mathsf{T} \lambda + g^\mathsf{T} \mu \\ \text{s.t.} \quad & \lambda \ge 0 \\ & \mu \text{ free} \\ & A^\mathsf{T} \lambda + F^\mathsf{T} \mu = c \end{aligned}$$

More duals

Equivalences between primal and dual problems

Minimization	Maximization
Nonnegative variable $\geq$	Inequality constraint $\leq$
Nonpositive variable $\leq$	Inequality constraint $\geq$
Free variable	Equality constraint $=$
Inequality constraint $\geq$	Nonnegative variable $\geq$
Inequality constraint $\leq$	Nonpositive variable $\leq$
Equality constraint $=$	Free Variable

Example

$$\begin{aligned} \min \quad & 3x_1 - x_2 \\ & x_1 + 2x_2 \leq 6 \\ x_1 \geq 0, & x_2 \text{free} \end{aligned}$$

Weak Duality

Primal problem (P)

$$\begin{array}{ll} \underset{x}{\text{maximize}} & c^{\mathsf{T}}x \\ \text{subject to:} & Ax \leq b \\ & x \geq 0 \end{array}$$

Dual problem (D)

$$\begin{array}{ll} \underset{\lambda}{\text{minimize}} & b^{\mathsf{T}}\lambda \\ \text{subject to:} & A^{\mathsf{T}}\lambda \geq c \\ & \lambda \geq 0 \end{array}$$

If x and λ are feasible points of (P) and (D) respectively:

$$c^{\mathsf{T}}x \leq p^{\star} \leq d^{\star} \leq b^{\mathsf{T}}\lambda$$

Weak Duality: The value of every feasible dual solution provides an (upper) bound on the value of every feasible primal solution.

Strong Duality

Primal problem (P)

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & c^{\mathsf{T}} x \\ \text{subject to:} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual problem (D)

$$\begin{aligned} \underset{\lambda}{\text{minimize}} \quad & b^{\mathsf{T}} \lambda \\ \text{subject to:} \quad & A^{\mathsf{T}} \lambda \geq c \\ & \lambda \geq 0 \end{aligned}$$

If x and λ are feasible points of (P) and (D) respectively:

$$c^{\mathsf{T}} x \leq p^{\star} \leq d^{\star} \leq b^{\mathsf{T}} \lambda$$

Strong Duality: if $p^{\star}$ and $d^{\star}$ exist and are finite, then $p^{\star} = d^{\star}$.

Primal/Dual Cases

Primal problem (P)

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & c^{\mathsf{T}} x \\ \text{subject to:} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual problem (D)

$$\begin{aligned} \underset{\lambda}{\text{minimize}} \quad & b^{\mathsf{T}} \lambda \\ \text{subject to:} \quad & A^{\mathsf{T}} \lambda \geq c \\ & \lambda \geq 0 \end{aligned}$$

There are **exactly four** possibilities:

1. (P) and (D) are both feasible and bounded, and $p^\star = d^\star$.
2. $p^\star = +\infty$ (unbounded primal) and $d^\star = +\infty$ (infeasible dual).
3. $p^\star = -\infty$ (infeasible primal) and $d^\star = -\infty$ (unbounded dual).
4. $p^\star = -\infty$ (infeasible primal) and $d^\star = +\infty$ (infeasible dual).

Complementary slackness

- Every primal solution x has (at least one) "complementary solution" λ that satisfies the *complementary slackness* conditions:

Either a primal constraint is tight **or** its dual variable is zero.

Either a dual constraint is tight **or** its primal variable is zero.

Complementary Slackness Theorem

If x is feasible for the primal, and λ is feasible for the dual, then (x, λ) are optimal solutions to their respective problems **if and only if** complementary slackness holds.

Example

- In the following linear program with unknown parameter c_3, give a value c_3 such that the optimal primal solution is $x_1^* = 3, x_2^* = 2,$ and $x_3^* = 1$ and the optimal dual solution is $\lambda_1^* = 2, \lambda_2^* = 0$. If this is not possible, say "Not Possible," and justify your answer.

$$\begin{aligned} \max 2x_1 + 2x_2 + c_3 x_3 & \\ x_1 + 3x_2 - x_3 &\leq 9 \\ 3x_1 - 2x_3 &\leq 7 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution

- If $x^* = (3, 2, 1)$ is optimal, the C.S. conditions must hold:

$$x_1 + 3x_2 - x_3 = 3 + 3(2) - 1 = 8 < 9 \Rightarrow \lambda_1^* = 0$$
$$3x_1 - 2x_3 = 3(3) - 2(1) = 7 \leq 7$$

- An optimal solution must have $\lambda_1^* = 0$, so no value of c_3 will make $x^* = (3, 2, 1)$, $\lambda^* = (2, 0)$ optimal. NOT POSSIBLE.

Example

- In the following linear program with unknown parameter c_3, give a value c_3 such that the optimal primal solution is $x_1^* = 3, x_2^* = 0,$ and $x_3^* = 1$ and the optimal dual solution is $\lambda_1^* = 0, \lambda_2^* = 1$. If this is not possible, say "Not Possible," and justify your answer.

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Solution

- If $x^* = (3, 0, 1)$ is optimal, the C.S. conditions must hold:

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 3 + 3(0) - 1 = 4 < 9 \\ 3x_1 - 2x_3 &= 3(3) - 2(1) = 7 \leq 7 \end{aligned}$$

- If $\lambda^* = (0, 1)$ is optimal, it must be feasible to the dual constraints:

$$\begin{aligned} \lambda_1 + 3\lambda_2 &\geq 2 \\ 3\lambda_1 &\geq 2 \\ -\lambda_1 - 2\lambda_2 &\geq c_3 \\ \lambda_1, \lambda_2, \lambda_3 &\geq 0 \end{aligned}$$

- So this is not possible.

Example

- In the following linear program with unknown parameters c_2, c_3, give values of c_2, c_3 such that the optimal primal solution is $x_1^* = 3, x_2^* = 0,$ and $x_3^* = 1$. If this is not possible, say "Not Possible," and justify your answer.

$$\begin{aligned} \max 2x_1 + c_2x_2 + c_3x_3 & \\ x_1 + 3x_2 - x_3 &\leq 9 \\ 3x_1 - 2x_3 &\leq 7 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution

- If $x^* = (3, 0, 1)$ is optimal, the C.S. conditions must hold, so like last time $\lambda_1^* = 0$.
- λ^* must be feasible, so it should satisfy the dual constraints:

$$\begin{aligned} \lambda_1^* + 3\lambda_2^* &\geq 2 \\ 3\lambda_1^* &\geq c_2 \\ -\lambda_1^* - 2\lambda_2^* &\geq c_3 \\ \lambda_1^*, \lambda_2^*, \lambda_3^* &\geq 0 \end{aligned}$$

- With $\lambda_1^* = 0$, these are equivalent to $c_2 \leq 0$, and

$$-\frac{c_3}{2} \geq \lambda_2 \geq \frac{2}{3}$$

  so setting $c_3 = -4/3$ should allow for $\lambda^* = (0, 2/3)$ to be a feasible dual solution.
- One answer is $c_2 = 0, c_3 = -4/3$. (Can check objective values match.)

Sensitivity in general

Primal problem (P)

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & c^{\mathsf{T}} x \\ \text{subject to:} \quad & Ax \leq b + e \\ & x \geq 0 \end{aligned}$$

Dual problem (D)

$$\begin{aligned} \underset{\lambda}{\text{minimize}} \quad & (b + e)^{\mathsf{T}} \lambda \\ \text{subject to:} \quad & A^{\mathsf{T}} \lambda \geq c \\ & \lambda \geq 0 \end{aligned}$$

Suppose we add a small e to the constraint vector b.

Sensitivity in general

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- The optimal $x^\star$ (and therefore $p^\star$) may change, since we are changing the feasible set of (P). Call new values $\hat{x}^\star$ and $\hat{p}^\star$.

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- The optimal $x^\star$ (and therefore $p^\star$) may change, since we are changing the feasible set of (P). Call new values $\hat{x}^\star$ and $\hat{p}^\star$.
- As long as e is small enough, the optimal λ will not change, since the feasible set of (D) is the same.

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- As long as e is small enough, the optimal λ will not change, since the feasible set of (D) is the same.
- Before: $p^\star = b^{\mathsf{T}} \lambda^\star$. After: $\hat{p}^\star = b^{\mathsf{T}} \lambda^\star + e^{\mathsf{T}} \lambda^\star$

Sensitivity in general

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Dual problem (D)

$$\begin{aligned} \underset{\lambda}{\text{minimize}} \quad & (b+e)^{\mathsf{T}}\lambda \\ \text{subject to:} \quad & A^{\mathsf{T}}\lambda \ge c \\ & \lambda \ge 0 \end{aligned}$$

Suppose we add a small e to the constraint vector b.

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- As long as e is small enough, the optimal λ will not change, since the feasible set of (D) is the same.
- Before: $p^\star = b^{\mathsf{T}}\lambda^\star$. After: $\hat{p}^\star = b^{\mathsf{T}}\lambda^\star + e^{\mathsf{T}}\lambda^\star$
- Therefore: $(\hat{p}^\star - p^\star) = e^{\mathsf{T}}\lambda^\star$.

Sensitivity in general

Primal problem (P)

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Suppose we add a small e to the constraint vector b.

- The optimal $x^\star$ (and therefore $p^\star$) may change, since we are changing the feasible set of (P). Call new values $\hat{x}^\star$ and $\hat{p}^\star$.
- As long as e is small enough, the optimal λ will not change, since the feasible set of (D) is the same.
- Before: $p^\star = b^\mathsf{T} \lambda^\star$. After: $\hat{p}^\star = b^\mathsf{T} \lambda^\star + e^\mathsf{T} \lambda^\star$
- Therefore: $(\hat{p}^\star - p^\star) = e^\mathsf{T} \lambda^\star$. Letting $e \to 0$, $\nabla_b (p^\star) = \lambda^\star$.

Midterm 1 What to know—LP Stuff

For 2-Var LP

- Plot/shade feasible region and isoprofit line(s)
- Determine if feasible region is unbounded
- Find an optimal solution or set of all optimal solutions, determine if LP is infeasible or unbounded
- For multiple optimal solutions: Be able to mathematically write the set of all optimal solutions (convex combinations!)

Definitions, LP properties

Definitions

- Feasible point
- Feasible region
- Optimal solution
- Convex set

LP Properties

- Convex feasible region (a polyhedron)
- Finitely many extreme points
- **If an LP has an optimal solution**, it has an extreme point optimal solution

LP—Possible Outcomes

Possible Outcomes When Solving a Linear Program

- A unique optimal solution
- Infinitely many optimal solutions
- Infeasible
- Unbounded

LP Modeling

Things to Remember

- We will give you *all the decision variables* you need
- Re-read the problem to make sure you didn't miss anything
- Remember units check!
- Try to put some numbers in for the decision variables – do constraints make sense?

Lots of Example Structures

- Diet (Input/Output) problem
  - General modeling and mathematical notation: $\sum, \forall, \in$
- Blending
  - When can one apply the blending trick?
- Multiperiod modeling
  - Including Backlog
- Piecewise linear convex functions & Epigraph Trick
  - Modeling $\min |\cdot|$ in objective
  - $f(x) = \max_{i=1,\ldots,k} \left\{a_i^\intercal x + b_i\right\}$ is convex function.
  - $f(x) = \min_{i=1,\ldots,k} \left\{a_i^\intercal x + b_i\right\}$ is concave function.
  - Minimizing convex function is "easy": You can model it is an LP
  - Maximizing concave function is "easy": You can model it is an LP
- CPM, Precedence constraints, and minimizing makespan

Duality to Know

- How to take a dual.
- Duality properties
  - Weak duality, Strong duality, Complementary slackness
  - Relationships between primal and dual problems
- Economic interpretation

Duality Properties

Weak Duality Property

If x is a feasible solution for the primal (max) problem and λ is a feasible solution for the dual (min) problem, then

$$c^{\mathsf{T}}x \leq b^{\mathsf{T}}\lambda.$$

Duality Properties

Weak Duality Property

If x is a feasible solution for the primal (max) problem and λ is a feasible solution for the dual (min) problem, then

$$c^{\mathsf{T}}x \leq b^{\mathsf{T}}\lambda.$$

Strong Duality Property

If x^* is an optimal solution for the primal problem and λ^* is an optimal solution for the dual problem, then

$$c^{\mathsf{T}}x^* = b^{\mathsf{T}}\lambda^*$$

Infeasible and Unbounded Problems

Duality Theorem

1. If (P) is *feasible* and *bounded*, then so is (D). Both weak and strong duality apply.
2. If (P) is *feasible* and *unbounded*, then (D) is *infeasible*.
3. If (P) is *infeasible*, then (D) is either *infeasible* or *unbounded*.

Infeasible and Unbounded Problems

Duality Theorem

1. If (P) is *feasible* and *bounded*, then so is (D). Both weak and strong duality apply.
2. If (P) is *feasible* and *unbounded*, then (D) is *infeasible*.
3. If (P) is *infeasible*, then (D) is either *infeasible* or *unbounded*.

- And: If you switch (P) and (D) in the Duality Theorem, all three results still hold.

Complementary Slackness

Complementary Slackness Conditions (CSCs)

If x^* is an optimal primal solution and λ^* is an optimal dual solution:

- For every $j = 1, \ldots, n$,
  - either $x_j^* = 0$ (jth primal variable is 0)
  - or $\sum_{i=1}^{m} \lambda_i^* a_{ij} = c_j$ (no slack in jth dual constraint).
- For every $i = 1, \ldots, m$,
  - either $\lambda_i^* = 0$ (ith dual variable is 0)
  - or $\sum_{j=1}^{n} a_{ij} x_j^* = b_i$ (no slack in ith primal constraint).

Complementary Slackness Theorem

If x is feasible for the primal, and λ is feasible for the dual, then (x, λ) are optimal solutions to their respective problems if and only if complementary slackness holds.

Network Things You Should Know

- Graph/Network Terminology
- Min Cost Network Flow Problem
- Transportation/Assignment Problems
- Shortest Path Problem
- Maximum Flow Problem

- Know how to recognize all of these and write them as MCNF
- Balanced/unbalanced MCNF
- Integrality properties of solutions.
- Node-Arc Incidence Matrix: Totally Unimodular Matrix

MCNF: Mathematical Formulation

- Given directed graph $G = (N, A), c, p, q \in \mathbb{R}^A, b \in \mathbb{R}^N$:

$$\min \sum_{(i,j) \in A} c_{ij} x_{ij}$$

$$\begin{aligned} \sum_{j \,:\, (i,j) \in A} x_{ij} - \sum_{j \,:\, (j,i) \in A} x_{ji} &= b_i \quad \forall i \in N \\ p_{ij} \leq x_{ij} &\leq q_{ij} \quad \forall (i,j) \in A \end{aligned}$$

- First set of constraints are "flow balance"
- "Out" - "In" $=$ "Net supply"

Special Cases

- Transportation Problem: G is *bipartite*
- Assignment Problem: G is *bipartite*, $b_i \in \{-1, 1\} \forall i \in N$
- Shortest Path Problem. Arc distances c. $s, t \in N$.
  $b_s = 1, b_t = -1, b_i = 0, i \in N \setminus s \setminus t$
  - Formulation also works for longest path $(\max)$ if there are no directed cycles.
- Max Flow Problem. $A \leftarrow A \sup\{(t, s)\}$. $b_i = 0, i \in N$,
  $c_{ts} = -1$, $c_{ij} = 0 \forall (i, j) \in A \setminus (t, s)$

General Modeling Practice

PaperCo:

- Paperco owns a set I of paper mills in northern Wisconsin
- These paper mills provide paper to a set of customers J different customers
- Producing a roll of paper at mill $i \in I$ costs p_i
- Shipping a roll of paper from mill $i \in I$ to customer $j \in J$ costs c_{ij}
- Each customer $j \in J$ requires at least d_j rolls of paper per week
- Each paper mill $i \in I$ can produce at most s_i rolls per week

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- Formulate a general linear programming model that will help PaperCo meet customer demands at minimum cost.

Sparse Solutions

- Given $A \in \mathbb{R}^{m \times n}$ with $m < n$ and $b \in \mathbb{R}^m$, an important problem in statistics is to find the "sparsest" solution $\hat{x}$ such that $A\hat{x} = b$.
- Finding the "shortest" vector $\hat{x}$, where "short" is measure in $\|\cdot\|_1$ can often given sparse solutions.
- Formulate the following as a linear program:

$$\min_x \sum_{j=1}^n |x_j|$$

$$Ax = b$$