CS/ECE/ISYE524: Introduction to Optimization – Linear Optimization Models

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The Top Brass example revisited

$$\begin{aligned}
\underset{f,s}{\text{maximize}} \quad & 12f + 9s \\
\text{subject to:} \quad & 4f + 2s \leq 4800, \quad f + s \leq 1750 \\
& 0 \leq f \leq 1000, \quad 0 \leq s \leq 1500
\end{aligned}$$

Suppose the maximum profit is $p^\star$. How can we *bound* $p^\star$?

- Finding a *lower* bound is easy... pick any feasible point!
  - $\{f = 0, s = 0\}$ is feasible. So $p^\star \geq 0$ (we can do better...)
  - $\{f = 500, s = 1000\}$ is feasible. So $p^\star \geq 15000$.
  - $\{f = 1000, s = 400\}$ is feasible. So $p^\star \geq 15600$.
- Each feasible point of the LP yields a lower bound for $p^\star$.
- Finding the largest lower bound $=$ solving the LP!

Estimating upper bounds

$$\begin{aligned} \underset{f,s}{\text{maximize}} \quad & 12f + 9s \\ \text{subject to:} \quad & 4f + 2s \le 4800, \quad f + s \le 1750 \\ & 0 \le f \le 1000, \quad 0 \le s \le 1500 \end{aligned}$$

Suppose the maximum profit is $p^\star$. How can we *bound* $p^\star$?

- Finding an *upper* bound is harder... (use the constraints!)
  - $12f + 9s \le 12 \cdot 1000 + 9 \cdot 1500 = 25500$. So $p^\star \le 25500$.
  - $12f + 9s \le f + (4f + 2s) + 7(f + s)$
    $\le 1000 + 4800 + 7 \cdot 1750 = 18050$. So $p^\star \le 18050$.
- Combining the constraints in different ways yields different upper bounds on the optimal profit $p^\star$.

Estimating upper bounds

$$\begin{aligned}
\underset{f,s}{\text{maximize}} \quad & 12f + 9s \\
\text{subject to:} \quad & 4f + 2s \le 4800, \quad f + s \le 1750 \\
& 0 \le f \le 1000, \quad 0 \le s \le 1500
\end{aligned}$$

Suppose the maximum profit is $p^\star$. How can we *bound* $p^\star$?

What is the **best** upper bound we can find by combining constraints in this manner?

Estimating upper bounds

$$\begin{aligned} \underset{f,s}{\text{maximize}} \quad & 12f + 9s \\ \text{subject to:} \quad & 4f + 2s \le 4800, \quad f + s \le 1750 \\ & 0 \le f \le 1000, \quad 0 \le s \le 1500 \end{aligned}$$

- Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$ be the multipliers. If we can choose them such that for *any* feasible f and s, we have:

$$12f + 9s \le \lambda_1(4f + 2s) + \lambda_2(f + s) + \lambda_3 f + \lambda_4 s \qquad (1)$$

Then, using the constraints, we will have the following upper bound on the optimal profit:

$$12f + 9s \le 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4$$

Estimating upper bounds

$$\begin{array}{ll} \underset{f,s}{\text{maximize}} & 12f + 9s \\ \text{subject to:} & 4f + 2s \leq 4800, \quad f + s \leq 1750 \\ & 0 \leq f \leq 1000, \quad 0 \leq s \leq 1500 \end{array}$$

- Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ be the multipliers. If we can choose them such that for *any* feasible f and s, we have:

$$12f + 9s \leq \lambda_1(4f + 2s) + \lambda_2(f + s) + \lambda_3 f + \lambda_4 s \tag{1}$$

  Rearranging (1), we get:

$$0 \leq (4\lambda_1 + \lambda_2 + \lambda_3 - 12)f + (2\lambda_1 + \lambda_2 + \lambda_4 - 9)s$$

  We can ensure this always holds by choosing $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ to make the bracketed terms nonnegative.

Estimating upper bounds

$$\begin{array}{ll} \underset{f,s}{\text{maximize}} & 12f + 9s \\ \text{subject to:} & 4f + 2s \leq 4800, \quad f + s \leq 1750 \\ & 0 \leq f \leq 1000, \quad 0 \leq s \leq 1500 \end{array}$$

- **Recap**: If we choose $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ such that:

$$4\lambda_1 + \lambda_2 + \lambda_3 \geq 12 \quad \text{and} \quad 2\lambda_1 + \lambda_2 + \lambda_4 \geq 9$$

Then we have a *upper* bound on the optimal profit:

$$p^\star \leq 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4$$

Finding the best (smallest) upper bound is... an LP!

The dual of Top Brass

$$
\begin{aligned}
\max_{f,s} \quad & 12f + 9s \\
\text{subject to:} \quad & 4f + 2s \le 4800, \quad f + s \le 1750 \\
& 0 \le f \le 1000, \quad 0 \le s \le 1500
\end{aligned}
$$

To find the best upper bound, solve the **dual** problem:

$$
\begin{aligned}
\min_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \quad & 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4 \\
\text{subject to:} \quad & 4\lambda_1 + \lambda_2 + \lambda_3 \ge 12 \\
& 2\lambda_1 + \lambda_2 + \lambda_4 \ge 9 \\
& \lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0
\end{aligned}
$$

The dual of Top Brass

Primal problem:

$$\begin{aligned} \underset{f,s}{\text{maximize}} \quad & 12f + 9s \\ \text{subject to:} \quad & 4f + 2s \le 4800 \\ & f + s \le 1750 \\ & f \le 1000 \\ & s \le 1500 \\ & f, s \ge 0 \end{aligned}$$

Solution is $p^\star$.

Dual problem:

$$\begin{aligned} \underset{\lambda_1,\dots,\lambda_4}{\text{minimize}} \quad & 4800\lambda_1 + 1750\lambda_2 \\ & \quad + 1000\lambda_3 + 1500\lambda_4 \\ \text{subject to:} \quad & 4\lambda_1 + \lambda_2 + \lambda_3 \ge 12 \\ & 2\lambda_1 + \lambda_2 + \lambda_4 \ge 9 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0 \end{aligned}$$

Solution is $d^\star$.

- Primal is a maximization, dual is a minimization.
- There is a dual variable for each primal constraint.
- There is a dual constraint for each primal variable.
- (any feasible primal point) $\le p^\star \le d^\star \le$ (any feasible dual point)

The dual of Top Brass

Primal problem:

$$\max_{f,s} \quad \begin{bmatrix} 12 \\ 9 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} f \\ s \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} 4 & 2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f \\ s \end{bmatrix} \le \begin{bmatrix} 4800 \\ 1750 \\ 1000 \\ 1500 \end{bmatrix}$$

$$f, s \ge 0$$

Dual problem:

$$\min_{\lambda_1,\dots,\lambda_4} \quad \begin{bmatrix} 4800 \\ 1750 \\ 1000 \\ 1500 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} 4 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \ge \begin{bmatrix} 12 \\ 9 \end{bmatrix}$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$$

Using matrix notation...

Code: Top Brass dual.ipynb

Danger!

Warning

JuMP's definition of duality is independent of the objective sense. That is, the sign of feasible duals associated with a constraint depends on the direction of the constraint and not whether the problem is maximization or minimization. **This is a different convention from linear programming duality in some common textbooks.** If you have a linear program, and you want the textbook definition, you probably want to use `shadow_price` and `reduced_cost` instead.

Weak Duality

Primal problem (P)

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & c^\mathsf{T} x \\ \text{subject to:} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual problem (D)

$$\begin{aligned} \underset{\lambda}{\text{minimize}} \quad & b^\mathsf{T} \lambda \\ \text{subject to:} \quad & A^\mathsf{T}\lambda \geq c \\ & \lambda \geq 0 \end{aligned}$$

If x and λ are feasible points of (P) and (D) respectively:

$$c^\mathsf{T} x \leq p^\star \leq d^\star \leq b^\mathsf{T} \lambda$$

Weak Duality: The value of every feasible dual solution provides an (upper) bound on the value of every feasible primal solution.

Weak Duality

Primal problem (P)

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & c^{\mathsf{T}} x \\ \text{subject to:} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual problem (D)

$$\begin{aligned} \underset{\lambda}{\text{minimize}} \quad & b^{\mathsf{T}} \lambda \\ \text{subject to:} \quad & A^{\mathsf{T}} \lambda \geq c \\ & \lambda \geq 0 \end{aligned}$$

If x and λ are feasible points of (P) and (D) respectively:

$$c^{\mathsf{T}} x \leq p^\star \leq d^\star \leq b^{\mathsf{T}} \lambda$$

Strong Duality: if $p^\star$ and $d^\star$ exist and are finite, then $p^\star = d^\star$. This is a powerful and amazing fact.

General LP duality

Primal problem (P)

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & c^{\mathsf{T}}x \\ \text{subject to:} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

1. optimal $p^\star$ is attained
2. unbounded: $p^\star = +\infty$
3. infeasible: $p^\star = -\infty$

Dual problem (D)

$$\begin{aligned} \underset{\lambda}{\text{minimize}} \quad & b^{\mathsf{T}}\lambda \\ \text{subject to:} \quad & A^{\mathsf{T}}\lambda \geq c \\ & \lambda \geq 0 \end{aligned}$$

1. optimal $d^\star$ is attained
2. unbounded: $d^\star = -\infty$
3. infeasible: $d^\star = +\infty$

Which combinations are possible? Remember: $p^\star \leq d^\star$.

General LP duality

Primal problem (P)

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & c^{\mathsf{T}} x \\ \text{subject to:} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

Dual problem (D)

$$\begin{aligned} \underset{\lambda}{\text{minimize}} \quad & b^{\mathsf{T}} \lambda \\ \text{subject to:} \quad & A^{\mathsf{T}} \lambda \ge c \\ & \lambda \ge 0 \end{aligned}$$

There are **exactly four** possibilities:

1. (P) and (D) are both feasible and bounded, and $p^\star = d^\star$.
2. $p^\star = +\infty$ (unbounded primal) and $d^\star = +\infty$ (infeasible dual).
3. $p^\star = -\infty$ (infeasible primal) and $d^\star = -\infty$ (unbounded dual).
4. $p^\star = -\infty$ (infeasible primal) and $d^\star = +\infty$ (infeasible dual).

More properties of the dual

To find the dual of an LP that is **not** in standard form:

1. convert the LP to standard form
2. write the dual
3. make simplifications

Example: What is the dual of the dual? *the primal!*

$$\begin{array}{rl} \min\limits_{\lambda} & b^{\mathsf{T}}\lambda \\ \text{s.t.} & A^{\mathsf{T}}\lambda \geq c \\ & \lambda \geq 0 \end{array}$$

equiv

$$\begin{array}{rl} -\max\limits_{\lambda} & (-b)^{\mathsf{T}}\lambda \\ \text{s.t.} & (-A^{\mathsf{T}})\lambda \leq (-c) \\ & \lambda \geq 0 \end{array}$$

dual

$$\begin{array}{rl} -\min\limits_{z} & (-c)^{\mathsf{T}}z \\ \text{s.t.} & (-A)z \geq (-b) \\ & z \geq 0 \end{array}$$

equiv

$$\begin{array}{rl} \max\limits_{z} & c^{\mathsf{T}}z \\ \text{s.t.} & Az \leq b \\ & z \geq 0 \end{array}$$

More duals

Standard form:

$$\begin{array}{ll} \max_x & c^{\mathsf{T}} x \\ \text{s.t.} & Ax \le b \\ & x \ge 0 \end{array} \quad \xleftrightarrow{\text{dual}} \quad \begin{array}{ll} \min_\lambda & b^{\mathsf{T}} \lambda \\ \text{s.t.} & \lambda \ge 0 \\ & A^{\mathsf{T}} \lambda \ge c \end{array}$$

Free form:

$$\begin{array}{ll} \max_x & c^{\mathsf{T}} x \\ \text{s.t.} & Ax \le b \\ & x \text{ free} \end{array} \quad \xleftrightarrow{\text{dual}} \quad \begin{array}{ll} \min_\lambda & b^{\mathsf{T}} \lambda \\ \text{s.t.} & \lambda \ge 0 \\ & A^{\mathsf{T}} \lambda = c \end{array}$$

Mixed constraints:

$$\begin{array}{ll} \max_x & c^{\mathsf{T}} x \\ \text{s.t.} & Ax \le b \\ & Fx = g \\ & x \text{ free} \end{array} \quad \xleftrightarrow{\text{dual}} \quad \begin{array}{ll} \min_{\lambda,\mu} & b^{\mathsf{T}} \lambda + g^{\mathsf{T}} \mu \\ \text{s.t.} & \lambda \ge 0 \\ & \mu \text{ free} \\ & A^{\mathsf{T}} \lambda + F^{\mathsf{T}} \mu = c \end{array}$$

More duals

Equivalences between primal and dual problems

Minimization	Maximization
Nonnegative variable $\geq$	Inequality constraint $\leq$
Nonpositive variable $\leq$	Inequality constraint $\geq$
Free variable	Equality constraint $=$
Inequality constraint $\geq$	Nonnegative variable $\geq$
Inequality constraint $\leq$	Nonpositive variable $\leq$
Equality constraint $=$	Free Variable

Simple example

Why should we care about the dual?

1. It can sometimes make a problem easier to solve

$$\begin{aligned} \max_{x,y,z} \quad & 3x + y + 2z \\ \text{s.t.} \quad & x + 2y + z \leq 2 \\ & x, y, z \geq 0 \end{aligned} \quad \overset{\text{dual}}{\Longleftrightarrow} \quad \begin{aligned} \min_{\lambda} \quad & 2\lambda \\ \text{s.t.} \quad & \lambda \geq 3 \\ & 2\lambda \geq 1 \\ & \lambda \geq 2 \\ & \lambda \geq 0 \end{aligned}$$

   - Dual is much easier in this case!
   - Many solvers take advantage of duality.

2. Duality is related to the idea of sensitivity: how much do each of your constraints affect the optimal cost?

Sensitivity

Primal problem:

$$
\begin{aligned}
\max_{f,s} \quad & 12f + 9s \\
\text{subject to:} \quad & 4f + 2s \le 4800 \\
& f + s \le 1750 \\
& f \le 1000 \\
& s \le 1500 \\
& f, s \ge 0
\end{aligned}
$$

Solution is $p^\star$.

Dual problem:

$$
\begin{aligned}
\min_{\lambda_1,\dots,\lambda_4} \quad & 4800\lambda_1 + 1750\lambda_2 \\
& + 1000\lambda_3 + 1500\lambda_4 \\
\text{subject to:} \quad & 4\lambda_1 + \lambda_2 + \lambda_3 \ge 12 \\
& 2\lambda_1 + \lambda_2 + \lambda_4 \ge 9 \\
& \lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0
\end{aligned}
$$

Solution is $d^\star$.

If Millco offers to sell me more wood at a price of \$1 per board foot, should I accept the offer?

Sensitivity

Primal problem:

$$\begin{aligned} \underset{f,s}{\text{maximize}} \quad & 12f + 9s \\ \text{subject to:} \quad & 4f + 2s \le 4800 \\ & f + s \le 1750 \\ & f \le 1000 \\ & s \le 1500 \\ & f, s \ge 0 \end{aligned}$$

Solution is $p^\star$.

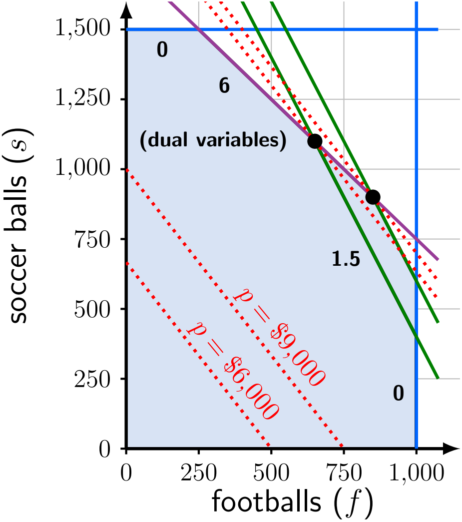
Dual problem:

$$\begin{aligned} \underset{\lambda_1,\dots,\lambda_4}{\text{minimize}} \quad & 4800\lambda_1 + 1750\lambda_2 \\ & + 1000\lambda_3 + 1500\lambda_4 \\ \text{subject to:} \quad & 4\lambda_1 + \lambda_2 + \lambda_3 \ge 12 \\ & 2\lambda_1 + \lambda_2 + \lambda_4 \ge 9 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0 \end{aligned}$$

Solution is $d^\star$.

- changes in primal *constraints* are changes in the dual *cost*.
- a small change to the feasible set of the primal problem can change the optimal f and s, but $\lambda_1, \dots, \lambda_4$ will not change!
- if we increase 4800 by 1, then $p^\star = d^\star$ increases by λ_1.

Sensitivity of Top Brass



$$\max_{f,s} \quad 12f + 9s$$
$$\text{s.t.} \quad 4f + 2s \le 5200$$
$$f + s \le 1750$$
$$0 \le f \le 1000$$
$$0 \le s \le 1500$$

What happens if we add 400 wood?

Profit goes up by \$600!

shadow price is \$1.50

Units

- In Top Brass, the primal variables f and s are the number of football and soccer trophies. The total profit is:

$$\text{(profit in \$)} = \left(12 \, \frac{\$}{\text{football trophy}}\right)(f \text{ football trophies}) + \left(9 \, \frac{\$}{\text{soccer trophy}}\right)(s \text{ soccer trophies})$$

- The dual variables also have units. To find them, look at the cost function for the dual problem:

$$\text{(profit in \$)} = (4800 \text{ board feet of wood})\left(\lambda_1 \, \frac{\$}{\text{board feet of wood}}\right) + (1750 \text{ plaques})\left(\lambda_2 \, \frac{\$}{\text{plaque}}\right) + \cdots$$

λ_i is the price that item i is worth to us.

Sensitivity in general

Primal problem (P)

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & c^{\mathsf{T}} x \\ \text{subject to:} \quad & Ax \le b + e \\ & x \ge 0 \end{aligned}$$

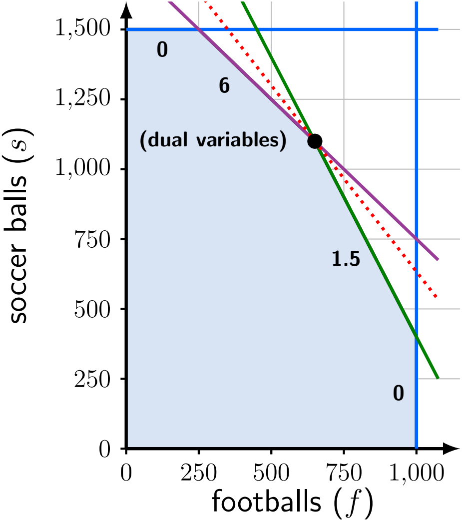
Dual problem (D)

$$\begin{aligned} \underset{\lambda}{\text{minimize}} \quad & (b + e)^{\mathsf{T}} \lambda \\ \text{subject to:} \quad & A^{\mathsf{T}} \lambda \ge c \\ & \lambda \ge 0 \end{aligned}$$

Suppose we add a small e to the constraint vector b.

- The optimal $x^\star$ (and therefore $p^\star$) may change, since we are changing the feasible set of (P). Call new values $\hat{x}^\star$ and $\hat{p}^\star$.
- As long as e is small enough, the optimal λ will not change, since the feasible set of (D) is the same.
- Before: $p^\star = b^{\mathsf{T}} \lambda^\star$. After: $\hat{p}^\star = b^{\mathsf{T}} \lambda^\star + e^{\mathsf{T}} \lambda^\star$
- Therefore: $(\hat{p}^\star - p^\star) = e^{\mathsf{T}} \lambda^\star$. Letting $e \to 0$, $\nabla_b(p^\star) = \lambda^\star$.

Sensitivity of Top Brass



$$\max_{f,\,s} \quad 12f + 9s$$

$$\text{s.t.} \quad 4f + 2s \leq 4800$$

$$f + s \leq 1750$$

$$0 \leq f \leq 1000$$

$$0 \leq s \leq 1500$$

Constraints that are loose at optimality have corresponding dual variables that are zero; those items aren't *worth* anything.

Complementary slackness

- At the optimal point, some inequality constraints become *tight*. Ex: wood and plaque constraints in Top Brass.
- Some inequality constraints may remain loose, even at optimality. Ex: brass football/soccer ball constraints. These constraints have *slack*.

Either a primal constraint is tight **or** its dual variable is zero.

The same thing happens when we solve the dual problem. Some dual constraints may have slack and others may not.

Either a dual constraint is tight **or** its primal variable is zero.

These properties are called *complementary slackness*.

Proof of complementary slackness

- **Primal**: $\max_x \; c^\top x$ s.t. $Ax \le b, \; x \ge 0$
- **Dual**: $\min_\lambda \; b^\top \lambda$ s.t. $A^\top \lambda \ge c, \; \lambda \ge 0$

Suppose (x, λ) is feasible for the primal and the dual.

- Because $Ax \le b$ and $\lambda \ge 0$, we have: $\lambda^\top Ax \le b^\top \lambda$.
- Because $c \le A^\top \lambda$ and $x \ge 0$, we have: $c^\top x \le \lambda^\top Ax$.

Combining both inequalities: $c^\top x \le \lambda^\top Ax \le b^\top \lambda$.

By strong duality, $c^\top x^\star = {\lambda^\star}^\top A x^\star = b^\top \lambda^\star$

Proof of complementary slackness

$$\boxed{c^{\mathsf{T}} x^{\star} = {\lambda^{\star}}^{\mathsf{T}} A x^{\star} = b^{\mathsf{T}} \lambda^{\star}}$$

$u_i v_i = 0$ means that: $u_i = 0$, or $v_i = 0$, or *both*.

The first equation says: ${x^{\star}}^{\mathsf{T}}(A^{\mathsf{T}}\lambda^{\star} - c) = 0$.
But $x^{\star} \geq 0$ and $A^{\mathsf{T}}\lambda^{\star} \geq c$, therefore:

$$\sum_{i=1}^{n} x_i^{\star}(A^{\mathsf{T}}\lambda^{\star} - c)_i = 0 \quad \Longrightarrow \quad \boxed{x_i^{\star}(A^{\mathsf{T}}\lambda^{\star} - c)_i = 0 \quad \forall i}$$

Similarly, the second equation says: ${\lambda^{\star}}^{\mathsf{T}}(Ax^{\star} - b) = 0$.
But $\lambda^{\star} \geq 0$ and $Ax^{\star} \leq b$, therefore:

$$\sum_{j=1}^{m} \lambda_j^{\star}(Ax^{\star} - b)_j = 0 \quad \Longrightarrow \quad \boxed{\lambda_j^{\star}(Ax^{\star} - b)_j = 0 \quad \forall j}$$

Another simple example

Primal problem:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & x_1 + x_2 \\ \text{subject to:} & 2x_1 + x_2 \geq 5 \\ & x_1 + 4x_2 \geq 6 \\ & x_1 \geq 1 \end{array}$$

Dual problem:

$$\begin{array}{ll} \underset{\lambda}{\text{maximize}} & 5\lambda_1 + 6\lambda_2 + \lambda_3 \\ \text{subject to:} & 2\lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1 + 4\lambda_2 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array}$$

Question: Is the feasible point $(x_1, x_2) = (1, 3)$ optimal?

- Second primal constraint is slack, therefore $\lambda_2 = 0$.
- Solving dual equations gives $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -1$
- The only dual solution satisfying complementary slackness is not feasible: This does not satisfy $\lambda_i \geq 0$

$(1, 3)$ is **not optimal** for the primal.

Another simple example

Primal problem:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & x_1 + x_2 \\ \text{subject to:} \quad & 2x_1 + x_2 \geq 5 \\ & x_1 + 4x_2 \geq 6 \\ & x_1 \geq 1 \end{aligned}$$

Dual problem:

$$\begin{aligned} \underset{\lambda}{\text{maximize}} \quad & 5\lambda_1 + 6\lambda_2 + \lambda_3 \\ \text{subject to:} \quad & 2\lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1 + 4\lambda_2 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{aligned}$$

Another question: Is $(x_1, x_2) = (2, 1)$ optimal?

- Third primal constraint is slack, therefore $\lambda_3 = 0$.
- Costs should match, so $5\lambda_1 + 6\lambda_2 = 3$.
- Solving dual constraints gives: $\lambda_1 = \frac{3}{7}$, $\lambda_2 = \frac{1}{7}$, $\lambda_3 = 0$, which is dual feasible!

$(2, 1)$ is **optimal** for the primal. (Objective values are =!)