CS/ECE/ISYE524: Introduction to Optimization – Linear Optimization Models

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You Deserve a Break Today

- We're hungry! —Let's determine how many of the following items to eat in order to meet our daily nutritional requirements.

Mmmmmmmmmmm.

- QP: Quarter Pounder
- MD: McLean Deluxe
- BM: Big Mac
- FF: Filet-O-Fish
- MC: McGrilled Chicken
- FR: Small Fries
- SM: Sausage McMuffin
- 1M: 1% Milk
- OJ: Orange Juice

Nutrients

- Prot: Protein
- VitA: Vitamin A
- VitC: Vitamin C
- Calc: Calcium
- Iron: Iron
- Cals: Calories
- Carb: Carbohydrates

Data

	QP	MD	BM	FF	MC	FR	SM	1M	OJ	Req'd
Cost	1.84	2.19	1.84	1.44	2.29	0.77	1.29	0.6	0.72	
Prot	28	24	25	14	31	3	15	9	1	55
VitA	15	15	6	2	8	0	4	10	2	100
VitC	6	10	2	0	15	15	0	4	120	100
Calc	30	20	25	15	15	0	20	30	2	100
Iron	20	20	20	10	8	2	15	0	2	100
Cals	510	370	500	370	400	220	345	110	80	2000
Carb	34	33	42	38	42	26	27	12	20	350

Elements of an Optimization

Variables

- What are we trying to decide?
- How many of each item to eat.
- Let x_j: Be the number of item j to eat.
    - (e.g. x_{QP}: Number of quarter pounders).

Objective

- Let's minimize our cost
- But how much does a daily menu cost?

Costing

- So if I bought my regular lunch: 3 quarter pounders, 2 small fries, and a 1% milk, my cost would be

$$3(1.84) + 2(1.44) + 1(0.6) = \$9.00$$

- A general expression for my cost as a function of my decision on what to buy is

$$\begin{aligned}1.84x_{QP} + 2.19x_{MD} + 1.84x_{BM} + 1.44x_{FF} + 2.29x_{MC}\\ + 0.77x_{FR} + 1.29x_{SM} + 0.6x_{1M} + 0.72x_{OJ}\end{aligned}$$

- This is our linear objective function

Nag, Nag, Nag :-)

- My wife tells me that I need to get 100% of my daily nutritional requirements from eating at McGreasy's
- A general expression for the daily amount of Vitamin A that I get by eating at McGreasy's is[2]

$$15x_{QP} + 15x_{MD} + 6x_{BM} + 2x_{FF} + 8x_{MC} + 4x_{SM} + 10x_{1M} + 2x_{OJ}$$

- In general I need that

$$15x_{QP} + 15x_{MD} + 6x_{BM} + 2x_{FF} + 8x_{MC} + 4x_{SM} + 10x_{1M} + 2x_{OJ} \geq 100$$

- You can write similar constraints for each nutrient:

[2] I could eat 50 Filet-O-Fish to get my Vitamin A requirements!

The Final Model (1 of 3)

minimize

$$1.84x_{QP} + 2.19x_{MD} + 1.84x_{BM} + 1.44x_{FF} + 2.29x_{MC} + 0.77x_{FR} + 1.29x_{SM} + 0.6x_{1M} + 0.72x_{OJ}$$

subject to

Protein: $$28x_{QP} + 24x_{MD} + 25x_{BM} + 14x_{FF} + 31x_{MC} + 3x_{FR} + 15x_{SM} + 9x_{1M} + x_{OJ} \geq 55$$

Vitamin A: $$15x_{QP} + 15x_{MD} + 6x_{BM} + 2x_{FF} + 8x_{MC} + 4x_{SM} + 10x_{1M} + 2x_{OJ} \geq 100$$

Final McGreasy's Model (2 of 3)

Vitamin C: $$6x_{QP} + 10x_{MD} + 2x_{BM} + 15x_{MC} + 15x_{FR} + 4x_{1M} + 120x_{OJ} \geq 100$$

Calcium: $$30x_{QP} + 20x_{MD} + 25x_{BM} + 15x_{FF} + 15x_{MC} + 20x_{SM} + 30x_{1M} + 2x_{OJ} \geq 100$$

Iron: $$20x_{QP} + 20x_{MD} + 20x_{BM} + 10x_{FF} + 8x_{MC} + 2x_{FR} + 15x_{SM} + 2x_{OJ} \geq 100$$

Final McGreasy's Model (3 of 3)

Calories: $$510x_{QP} + 370x_{MD} + 500x_{BM} + 370x_{FF} + 400x_{MC} + 220x_{FR} + 345x_{SM} + 110x_{1M} + 80x_{OJ} \geq 2000$$

Carbs: $$34x_{QP} + 35x_{MD} + 42x_{BM} + 38x_{FF} + 42x_{MC} + 26x_{FR} + 27x_{SM} + 12x_{1M} + 20x_{OJ} \geq 350$$

$$x_{QP}, x_{MD}, x_{BM}, x_{FF}, x_{MC}, x_{FR}, x_{SM}, x_{1M}, x_{OJ} \geq 0$$

Check Out The Notebook

McDonaldsDiet.ipynb

- Use of `Dict(zip(indexList,ValuesList))` to create indexed parameters
- Use of `NamedArrays` package to allow array to be indexed by element names, not by number
- `(m, [i in nutrients], sum(A_NA[i,j]*x[j] for j in foods) >= required[i])` creates one constraint for every element in `nutrients`

The Sets View—A General Model

Sets

- F: Set of possible foods
- N: Set of nutrional requirements

Parameters

- c_j: Per unit cost of item $j \in F$
- ℓ_i: Lower Bound on amount of nutrient $i \in N$
- u_i: Upper Bound on amount of nutrient $i \in N$
- a_{ij}: Amount of nutrient $i \in N$ in food $j \in F$

The Diet Problem

$$\min \sum_{j \in F} c_j x_j$$

$$\begin{aligned} \ell_i \leq \sum_{j \in F} a_{ij} x_j \quad & \leq \quad u_i \qquad \forall i \in N \\ x_j \quad & \geq \quad 0 \qquad \forall j \in F \end{aligned}$$

$$\min_{x \in \mathbb{R}_+^{|F|}} \{c^T x \mid \ell \leq Ax \leq u\}$$

Check Out The Notebook

McDonaldsDiet-CSV.ipynb

- Uses julia `DataFrames`, like Pandas in python, or R Data Data Frames
- Uses julia `CSV` to read the CSV file into a dataframe
- Extract sets and parameters from dataframe, put into Dictionaries
- Extract 'A' matrix from dataframe, then put it into a `NamedArray`
- Code solving model is exact same: If `mcdonalds.csv` had 10,000 rows and columns, it would just solve a bigger problem!

Recall the Simplex Algorithm

The Simplex Method

0. Start from an extreme point.
1. Find an improving direction d. If none exists, STOP. The extreme point is an optimal solution.
2. Move along d until you hit a new extreme point. Go to 1.

Simplex Method – What can go wrong?

Simplex Method: Step 2

Move along d until you hit a new extreme point.

- What if we don't hit an extreme point?

$$\max x_1 + x_2$$

$$\text{s.t. } \begin{array}{rcl} x_1 + 2x_2 & \geq & 1 \\ x_1, x_2 & \geq & 0 \end{array}$$

- Usually this means you forgot some constraints. Maybe your variable bounds?
- N.B.: Just because the region is unbounded doesn't mean that the LP is unbounded.

I Will Glady Pay You Tuesday...

- I really like hamburgers.
- Let's suppose in the diet problem, I decide to maximize the number of hamburgers I eat
- Let $B \subset F$

$$B = \{\text{QP, MD, BM}\}$$

- My new objective is to

$$\max \sum_{j \in B} x_b$$

- McDonaldsDiet-LPCases.ipynb

Mmmmmmmmmm. Beef

- Always check the Model status in the solution report

```
Model   status        : Unbounded
Simplex   iterations: 3
Objective value       :  5.0000000000e+01
HiGHS run time        :           0.00
Maximum Number Hamburgers 0.50:
Eat 0.50 of menu item :BM
```

- The Model status is unbounded!

```
stat = termination_status(m)
if stat != MOI.OPTIMAL
println("Solver did not find optimal solution, status:
", stat)
end
```

Simplex Method – What can go wrong?

Simplex Method: Step 0

Start from an extreme point

- What if there *are* no extreme points?
  - This (usually) means that the feasible region is empty.
  - The instance is infeasible.
  - $P = \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 1, x_1 + x_2 \geq 2\}$
- How will we know if an instance is infeasible?
  - "Big-M", "Two-Phase"?
  - The solver will tell us!

Warning!

- It may be hard to "blame" one constraint for being infeasible.
- When building models for the real world determining what is "causing" the infeasibility may be tough.
- Whose "fault" is this?

$$x_1 - x_2 \geq 1, x_2 - x_3 \geq 1, -x_1 + x_3 \geq 1$$

My Wife Loves Me!

- In the interest of extending my life, Helen has requested that I obey the following constraints:

1. Don't eat more than 3 sandwiches per day

$$x_{QP} + x_{MD} + x_{BM} + x_{FF} + x_{MC} + x_{SM} \leq 3$$

2. Don't drink too much: $x_{1M} + x_{OJ} \leq 3$
3. Only two french fries per day: $x_{FF} \leq 2$

- But with these constraints, the problem is infeasible!

```
{Model   status       : Infeasible
Simplex   iterations: 5
Objective value      :  2.4751250000e+01}
```

Handling Infeasibility

Our First Trick

- Introduce slack/surplus variables and try to minimize the slack/surplus.

- Suppose I think that the "too much drinking" constraint is the one causing the problem to be infeasible
- New decision variable s: Number of extra drinks (over three) that I must drink in order to get a feasible solution

$$x_{1M} + x_{OJ} - s \leq 3, s \geq 0$$

- New Objective: $\min s$

- Be sure to go through McDonaldsDiet-LPCases.ipynb

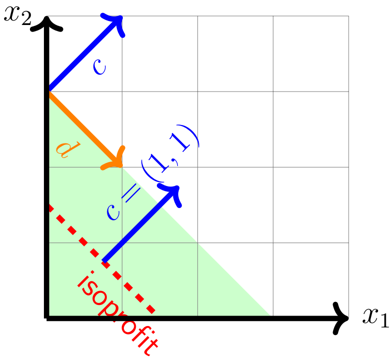
Multiple Optimal Solutions

- What if c is orthogonal to an "improving" direction d? (Rate of change 0)

maximize

$$x_1 + x_2$$

$$\begin{aligned} x_1 + x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

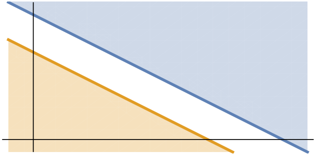


- We get an infinite number of optimal solutions.
- Every point that is a convex combination of the extreme points of the optimal face is also optimal

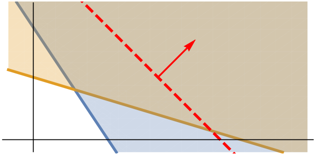
Solutions of an LP

There are exactly three possible cases:

1. Model is *infeasible*: there is no x that satisfies all the constraints.
(is the model correct?)
2. Model is feasible, but *unbounded*: the cost function can be arbitrarily improved. (forgot a constraint?)
3. Model has a solution which occurs *on the boundary* of the set.
(there may be many solutions!)



infeasible



unbounded