524 Midterm Exam (S'23) – SOLUTIONS

Name: ________________________________

Student number: ________________________________

Instructions:

- You have 2 hours to complete this exam. I suggest you skim through the entire exam before starting, so that you can pace yourself better.
- Blank pages appear at the end of this exam package. You may tear these off and use them for scratch and/or for additional pages containing solutions.
  - Do not submit scratch paper!
  - Reattach any pages containing solutions using the stapler at the end of the exam — do not leave them loose!
- This is a *closed-book* exam. The only exception is that you may bring one (1) double-sided hand-written aid sheet. You are not permitted to bring into the exam any other notes, aides, calculators, the internet, etc. No electronic devices except a watch.
- Please write neatly. Your exam may be scanned and graded online.

1. **[15 pts total]** Multiple choice questions. For each question, you are asked to circle **all that apply**. This means a "correct answer" might require you to circle more than one statement. You will get full credit only if you circle all the true statements (and nothing else).

Do not provide additional explanation or justification. Just circle the correct answer(s).

a) **[2 pts]** After modeling your LP that you know should have a solution, the solver unexpectedly returns "INFEASIBLE". Which of the following modeling errors could have caused this? (circle all that apply)

i. incorrect objective function
ii. forgot to include a constraint
iii. added an extra constraint by mistake.

EXPLANATION: Infeasible means that it's impossible to satisfy all the constraints. This could mean one of the constraints doesn't belong. Infeasibility has nothing to do with the objective.

RUBRIC: -1 point for each wrong answer, with a minimum of 0 points

b) **[2 pts]** After modeling your LP that you know should have a solution, the solver unexpectedly returns "UNBOUNDED". Which of the following modeling errors could have caused this? (circle all that apply)

i. incorrect objective function
ii. forgot to include a constraint
iii. added an extra constraint by mistake.

EXPLANATION: Unbounded means the feasible set is too big in the wrong direction. This could be caused by a missing constraint that would have prevented this or an incorrect objective.

RUBRIC: -1 point for each wrong answer, with a minimum of 0 points

c) **[2 pts]** Suppose you have a constrained minimization problem. You then add an extra linear constraint to the problem. What things might happen to the optimal objective value? (circle all that apply)

i. stay the same
ii. increase
iii. decrease

EXPLANATION: Adding an extra constraint will either make the feasible set smaller or leave it unchanged. This means we are optimizing over a set of points that is smaller (or the same), so our optimal objective can't improve (get smaller). Therefore it can stay the same or get larger.

RUBRIC: -1 point for each wrong answer, with a minimum of 0 points

d) **[3 pts]** Suppose we have a minimum-cost network flow problem of the form min $c^T x$ subject to $Ax = b$ and $0 \le x \le q$, where A is the incidence matrix for the network, x are the flows on the edges, q are the capacity constraints, and b is the vector indicating how much material is produced or consumed at each node. Which of the following combination of properties are sufficient to guarantee that any solution x of the problem with have *integer* flows? (circle all that apply)

i. b contains all integer values
ii. b and q contain all integer values
iii. c and b contain all integer values
iv. c and q contain all integer values

EXPLANATION: it is sufficient for b and q to contain all integer values. Then any basic feasible point can be represented as the inverse of a submatrix of A times a subvector of (b, q), which will have all integer values.

RUBRIC: -1 point for each wrong answer, with a minimum of 0 points

e) **[3 pts]** The strong duality theorem for linear programming indicates that for any linear program (the "primal") and its dual, only certain combinations of outcomes are possible. For example, the combination of primal infeasible / dual unbounded can occur. Indicate which combinations of outcomes for the primal and dual linear programs **are not** possible. (Circle all that apply.)

i. primal has a solution but the dual is infeasible
ii. primal and dual are both unbounded
iii. primal and dual are both infeasible
iv. primal and dual both have a solution, but the optimal objective values are different.

EXPLANATION: These are immediate consequences of strong duality for LP

RUBRIC: -1 point for each wrong answer, with a minimum of 0 points

f) **[3 pts]** Indicate which of the following problems can be reformulated as a linear program. (Circle all that apply.)

i. $\min_{x_1,x_2} \max\{2x_1 + 3x_2 - 5, 4x_2 - 6, -x_1 + 5x_2 - 3\}$
ii. $\min_{x_1,x_2} \min\{2x_1 + 3x_2 - 5, 4x_2 - 6, -x_1 + 5x_2 - 3\}$
iii. $\min_{x_1,x_2} 2x_1 + 5 - |x_2|$

EXPLANATION: Max of linear function is convex, and can be solved by introducing one variable t that is $\geq$ each of the pieces. The other two examples are not convex.

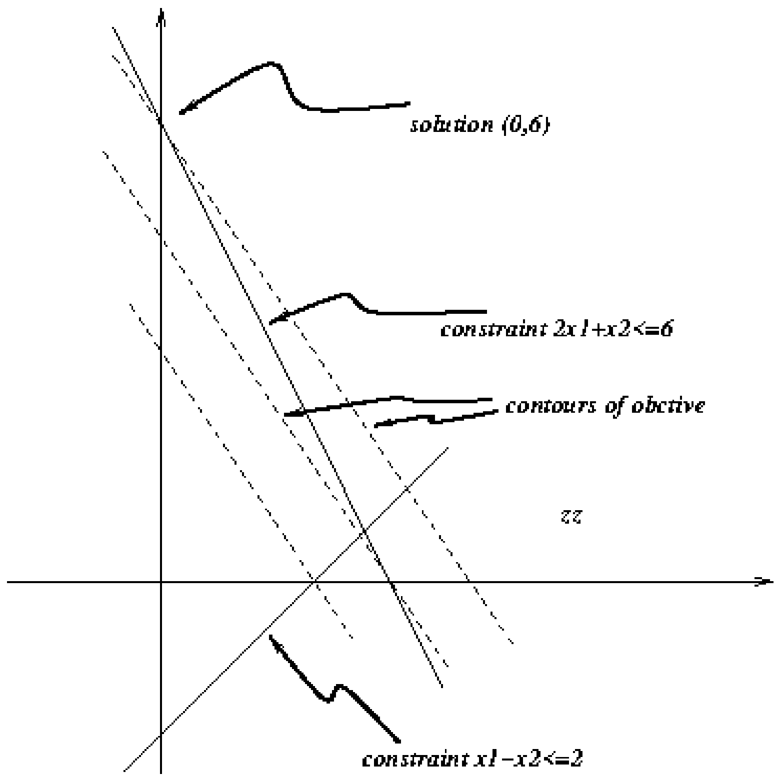
RUBRIC: -1 point for each wrong answer, with a minimum of 0 points

2. [14 pts total] Consider the following linear program:

$$\begin{aligned} \max_{x_1,x_2}\ & 3x_1 + 2x_2 \\ \text{subject to}\quad & 2x_1 + x_2 \le 6, \\ & x_1 - x_2 \le 2, \\ & x_1, x_2 \ge 0. \end{aligned}$$

(a) **[8 pts]** Sketch the constraints and objective contours on a graph (with horizontal axis x_1 and vertical axis x_2). Mark the solution point on the graph, and indicate the values of x_1 and x_2 at the solution.

SOLUTION: The solution is at $x^* = (0, 6)^T$.



RUBRIC:

- -2 points for incorrect solution (or solution not marked)
- -2 points for incorrect objective contours
- -2 points for each incorrect constraint

(b) **[6 pts]** Write down the dual of this linear program. (Use λ_1, λ_2 to denote the variables in the dual.)

SOLUTION:

$$\begin{aligned} & \min 6\lambda_1 + 2\lambda_2 \\ \text{subject to} \quad & 2\lambda_1 + \lambda_2 \geq 3, \\ & \lambda_1 - \lambda_2 \geq 2, \\ & \lambda_1, \lambda_2 \geq 0. \end{aligned}$$

RUBRIC:

- -2 for incorrect objective, or for "max" instead of "min"
- -2 for incorrect constraints
- -2 for not putting lower bounds on the variables.

3. [24 pts total] Consider the following modeling problems.

a) [8 pts] At Amazon's warehouse, robots are deployed to move packages to K stations with coordinates $(a_1, b_1), \dots, (a_K, b_K)$ (measured in meters). Assuming that robots can only travel along North-South and East-West directions, one can define the distance between any two locations in the warehouse with coordinates (p_1, q_1) and (p_2, q_2) as $|p_1 - p_2| + |q_1 - q_2|$, which is commonly known as the *Manhattan distance*. Your task is to figure out where to place a hub in the warehouse so that it best serves the stations. That is, the sum of the (Manhattan) distances between the hub and each of the K stations should be as small as possible. Model this problem as a **linear program**. Clearly identify the variables, the constraints, and the objective function and include a short description of what each part of the model does.

SOLUTION: Let's call (x, y) the location of the hub. This is what we are looking to optimize so x, y are decision variables. The sum of the Manhattan distances to the stations is:

$$\sum_{i=1}^{K} (|x - a_i| + |y - b_i|)$$

This is our objective. As it stands, this isn't a linear program, but we can convert it to one by using the epigraph trick. Introduce extra variables $t_1, \dots, t_K$ and $s_1, \dots, s_K$ and the optimization problem becomes:

$$\begin{aligned} \underset{x,y,\{t_i\},\{s_i\}}{\text{minimize}} \quad & \sum_{i=1}^{K} t_i + \sum_{i=1}^{K} s_i \\ \text{subject to} \quad & |x - a_i| \le t_i \qquad i = 1, \dots, K \\ & |y - b_i| \le s_i \qquad i = 1, \dots, K \end{aligned}$$

Now use the fact that $|z| \le t$ is equivalent to $-t \le z \le t$. The model becomes:

$$\boxed{\begin{aligned} \underset{x,y,\{t_i\},\{s_i\}}{\text{minimize}} \quad & \sum_{i=1}^{K} t_i + \sum_{i=1}^{K} s_i \\ \text{subject to} \quad & -t_i \le x - a_i \le t_i \qquad i = 1, \dots, K \\ & -s_i \le y - b_i \le s_i \qquad i = 1, \dots, K \end{aligned}}$$

The first line is the objective, which is are upper bounds on the Manhattan distances to each of the stations. The next two lines are constraints, which spell out those bounds. The variables in our problem are: x, y (the coordinates of the hub) and $\{t_i\}$ and $\{s_i\}$, the variables introduced when we used the epigraph trick.

RUBRIC:

- -2 if location of hub (x, y) is not among the variables.
- -4 if wrong formulation.
- -2 if formulation is correct but not written as a LP (e.g. if there is only the "sum of absolute values" form)

b) [**8 points**] Suppose that we modify the problem in part **a)** as follows. There is a charging station located at coordinates (a_0, b_0), where the robots go when there is a need for recharging or repair. In addition to serving the K stations optimally, the hub should be no more than 100 meters away (Manhattan distance) from the charging station. By modifying your model from part **a)**, model this problem as a **linear program**. Clearly identify the variables, constraints, and objective.

SOLUTION: If the hub must be a distance of 100 from the charging station at (a_0, b_0), we can write this as a constraint:

$$|x - a_0| + |y - b_0| \leq 100$$

We must convert this constraint into a linear constraint, and to this effect we do something similar in spirit to the epigraph trick. Introduce variables t_0 and s_0. Then, the constraint above is equivalent to:

$$|x - a_0| \leq t_0 \qquad \text{and} \qquad |y - b_0| \leq s_0 \qquad \text{and} \qquad t_0 + s_0 \leq 100$$

Including this into our previous model, we obtain:

$$\boxed{\begin{aligned} \underset{x,y,\{t_i\},\{s_i\}}{\text{minimize}} \quad & \sum_{i=1}^{K} t_i + \sum_{i=1}^{K} s_i \\ \text{subject to} \quad & -t_i \leq x - a_i \leq t_i \qquad i = 0, \dots, K \\ & -s_i \leq y - b_i \leq s_i \qquad i = 0, \dots, K \\ & s_0 + t_0 \leq 100 \end{aligned}}$$

Note that the list of constraints now starts from $i = 0$ to include the new constraint for the charging station. The sums in the objective function do not include s_0 and t_0, of course, since it's the same as it was before.

RUBRIC:

- -3 if the additional constraint involving (a_0, b_0) is not included
- -1 if the new constraint is not added in LP form i.e. if there are still absolute values in the formulation.
- -4 if other aspects of the model are wrong (carried over from part (a))

c) [**8 points**] Suppose that the stations are reorganized so that the robots can move in a straight line (in any direction) from the hub to any station. To obtain the optimal location of the hub, we modify part **a)** by using Euclidean distance instead of Manhattan distance. That is, the distance between any two locations in the warehouse with coordinates (p_1, q_1) and (p_2, q_2) is now taken to be $\sqrt{(p_1 - p_2)^2 + (q_1 - q_2)^2}$. Write down a **second-order cone program** whose solution gives the location of the hub that minimizes the sum of (Euclidean) distances between the hub and each of the K stations.

SOLUTION: The natural formulation would be

$$\min_{x,y} \sum_{i=1}^{K} \sqrt{(x - a_i)^2 + (y - b_i)^2}.$$

We can write this in second-order cone form by introducing variable t_i to represent the distance of the hub from station i. We then obtain

$$\boxed{\begin{array}{lll} \underset{x,y,\{t_i\}}{\text{minimize}} & \displaystyle\sum_{i=1}^{K} t_i & \\ \text{subject to} & \left\| \begin{bmatrix} x - a_i \\ y - b_i \end{bmatrix} \right\| \le t_i & i = 1, \dots, K \end{array}}$$

or alternatively

$$\boxed{\begin{array}{lll} \underset{x,y,\{t_i\}}{\text{minimize}} & \displaystyle\sum_{i=1}^{K} t_i & \\ \text{subject to} & \sqrt{(x - a_i)^2 + (y - b_i)^2} \le t_i & i = 1, \dots, K \end{array}}$$

RUBRIC:

- -6 if model is incorrect
- -2 is model is not expressed in SOCP form, i.e. with variables t_i added to capture the norm of the distance.

4. [9 pts total] We are interested in the set of points (x, y) that satisfy the inequality:

$$73x^2 - 72xy + 52y^2 \leq 625. \tag{1}$$

To aid in our analysis, we can divide (1) by 625 and rewrite it as:

$$\begin{bmatrix} x \\ y \end{bmatrix}^\mathsf{T} \begin{bmatrix} \frac{73}{625} & -\frac{36}{625} \\ -\frac{36}{625} & \frac{52}{625} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq 1.$$

Then, we can find the eigenvalue decomposition of the symmetric matrix, which turns out to be:

$$\begin{bmatrix} \frac{73}{625} & -\frac{36}{625} \\ -\frac{36}{625} & \frac{52}{625} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{25} & 0 \\ 0 & \frac{4}{25} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}^\mathsf{T}.$$

Answer the following questions.

a) [3 pts] What is the name given to shape formed by the points satisfying the inequality (1) in $\mathbb{R}^2$?

SOLUTION: Since the eigenvalues of the matrix are positive, this set is an ellipsoid. ("Ellipse" is also OK as an answer.)

RUBRIC:

- 3 points for correct answer
- 1 point if they say "convex"

b) [3 pts] Find a matrix A and scalar b such that (1) can be written in the equivalent form $\left\| A \begin{bmatrix} x \\ y \end{bmatrix} \right\| \leq b$.

SOLUTION:

$$A = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \frac{3}{25} & \frac{4}{25} \\ -\frac{8}{25} & \frac{6}{25} \end{bmatrix}, \quad b = 1.$$

In fact any orthogonal multiple of A would be a valid answer, but this particular A is the obvious one, given the eigenvalue decomposition.

RUBRIC:

- -2 points for incorrect or missing A.
- -1 point for incorrect or missing b.

c) **[3 pts]** The set of points satisfying (1) is centered at the origin $(0,0)^T$. Consider how far we can move (backward and forward) along each of the two directions $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ and $\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$ while staying inside the set. Which of the following options is true? Circle the correct one (no need for explanation).

i. The set is longer in the direction $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ than in the direction $\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$

ii. The set is shorter in the direction $\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ than in the direction $\begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$

iii. The set is equally long in both directions.

EXPLANATION: Consider points along the direction $t(\frac{3}{5}, \frac{4}{5})^T$ for any $t \in \mathbb{R}$. By applying the constraint, we have

$$\frac{1}{25}(\frac{9}{25} + \frac{16}{255})^2 t^2 \le 1 \;\Leftrightarrow\; t^2 \le 25,$$

so in this direction the ellipsoid runs from $(-3,-4)^T$ to $(3,4)^T$. For points along the other direction $t(-\frac{4}{5}, \frac{3}{5})^T$, the constraint is

$$\frac{4}{25}(\frac{16}{25} + \frac{9}{25})^2 t^2 \le 1 \;\Leftrightarrow\; t^2 \le 25/4 \;\Leftrightarrow\; -5/2 \le t \le 5/2,$$

so the ellipsoid runs from $(2, -3/2)^T$ to $(-2, 3/2)^T$. Hence the ellipse is longer in the direction $(\frac{3}{5}, \frac{4}{5})^T$

RUBRIC: 0 points for incorrect answer.

Extra paper

Extra paper

Extra paper

Extra paper

Extra paper