CS/ECE/ISYE524: Introduction to Optimization – Linear Optimization Models

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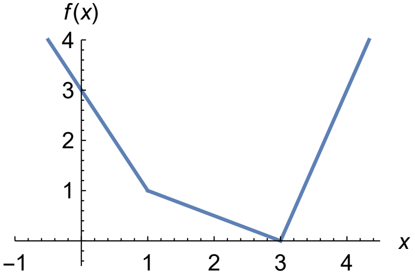
Piecewise linear functions

- Some problems do not appear to be LPs but can be converted to LPs using a suitable transformation.
- An important case: *convex piecewise linear functions*.

Consider the following **nonlinear** optimization:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to:} \quad & x \geq 0 \end{aligned}$$

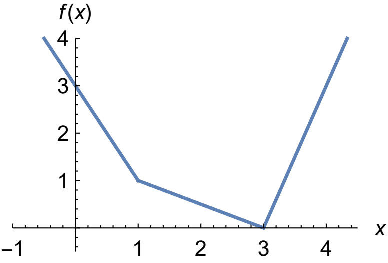
Where $f(x)$ is the function:

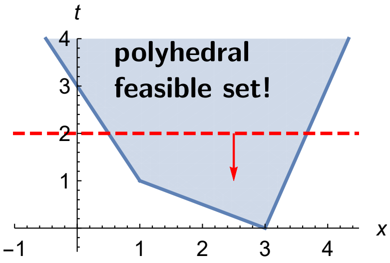


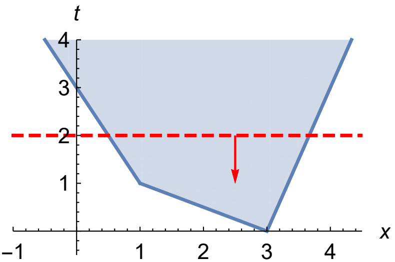
Piecewise Linear Functions

The trick is to convert the problem into **epigraph** form: add an extra decision variable t and turn the cost into a constraint!

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to:} & x \geq 0 \end{array} \quad \Longrightarrow \quad \begin{array}{ll} \underset{x,t}{\text{minimize}} & t \\ \text{subject to:} & t \geq f(x) \\ & x \geq 0 \end{array}$$







$$\begin{array}{ll} \underset{x,t}{\text{minimize}} & t \\ \text{subject to:} & t \geq f(x) \\ & x \geq 0 \end{array}$$

This feasible set is **polyhedral**. It is the set of (x, t) satisfying:

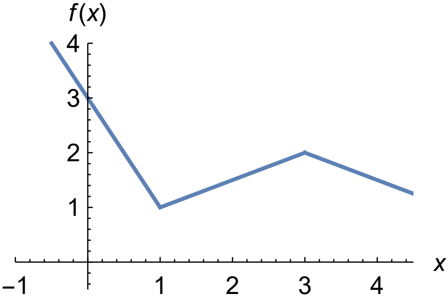
$$\left\{ t \geq -2x + 3, \quad t \geq -\frac{1}{2}x + \frac{3}{2}, \quad t \geq 3x - 9 \right\}$$

Equivalent linear program:

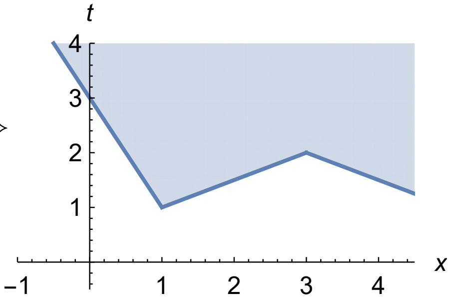
$$\begin{array}{lll} \underset{x,t}{\text{minimize}} & t & \\ \text{subject to:} & t \geq -2x + 3, & t \geq -\frac{1}{2}x + \frac{3}{2} \\ & t \geq 3x - 9, & x \geq 0 \end{array}$$

Piecewise linear functions

Epigraph trick only works if it's a **convex polyhedron**.



$\Longrightarrow$

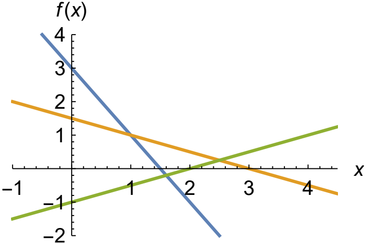


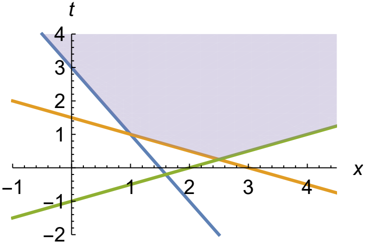
This epigraph is **not a convex polyhedron** so it cannot be the feasible set of a linear program.

Minimax problems

- The maximum of several linear functions is *always* convex. So we can minimize it using the epigraph trick. Example:

$$f(x) = \max_{i=1,\ldots,k} \left\{ a_i^{\mathsf{T}} x + b_i \right\}$$



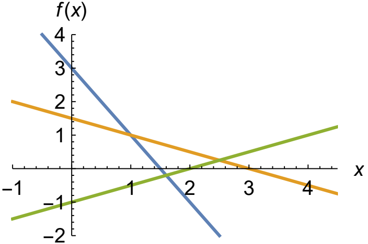


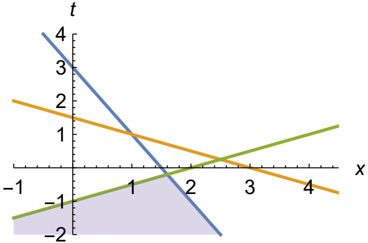
$$\min_{x} \max_{i=1,\ldots,k} \left\{ a_i^{\mathsf{T}} x + b_i \right\} \implies \begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & t \ge a_i^{\mathsf{T}} x + b_i \quad i = 1, 2, \ldots, k. \end{aligned}$$

Maximin problems

- The minimum of several linear functions is *always* concave. So we can maximize it using the epigraph trick. Example:

$$f(x) = \min_{i=1,\ldots,k} \left\{ a_i^\mathsf{T} x + b_i \right\}$$





$$\max_{x} \min_{i=1,\ldots,k} \left\{ a_i^\mathsf{T} x + b_i \right\} \quad \Longrightarrow \quad \begin{aligned} \max_{x,t} \quad & t \\ \text{s.t.} \quad & t \le a_i^\mathsf{T} x + b_i \quad \forall i \end{aligned}$$

Minimax and Maximin problems

- A minimax problem:

$$\min_x \max_{i=1,\dots,k} \{a_i^\mathsf{T} x + b_i\} \quad \Longrightarrow \quad \begin{array}{lll} \min_{x,t} & t & \\ \text{s.t.} & t \geq a_i^\mathsf{T} x + b_i & \forall i \end{array}$$

- A maximin problem:

$$\max_x \min_{i=1,\dots,k} \{a_i^\mathsf{T} x + b_i\} \quad \Longrightarrow \quad \begin{array}{lll} \max_{x,t} & t & \\ \text{s.t.} & t \leq a_i^\mathsf{T} x + b_i & \forall i \end{array}$$

Note: Sometimes called *minmax*, *min-max*, *min/max*.
Of course, minmax $\neq$ maxmin!

Absolute values

- Absolute values are piecewise linear! For $x \in \mathbb{R}$:

$$\begin{aligned} \min_{x} \quad & |x| \\ \text{s.t.} \quad & Ax \le b \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & Ax \le b \\ & t \ge x \\ & t \ge -x \end{aligned}$$

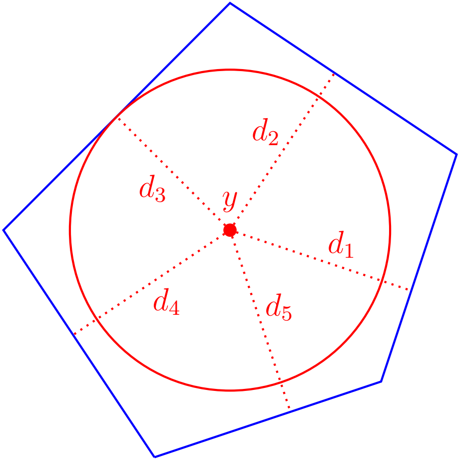
- So are sums of absolute values:

$$\min_{x,y} \quad |x| + |y| \quad \Longrightarrow \quad \begin{aligned} \min_{x,y,t,r} \quad & t + r \\ \text{s.t.} \quad & t \ge x, \quad t \ge -x \\ & r \ge y, \quad r \ge -y \end{aligned}$$

- But not differences! $\min_{x,y} \; |x| - 2|y|$ is not an LP.

Chebyshev center

What is the largest sphere you can fit inside a polyhedron?



If y is the center, then draw perpendicular lines to each face of the polyhedron.

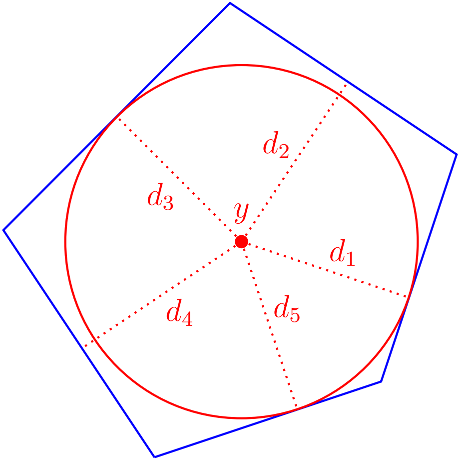
We want to maximize the smallest d_i. In other words,

$$\max_{y} \min_{i=1,\ldots,5} d_i(y)$$

(the y shown here is obviously not optimal!)

Chebyshev center

What is the largest sphere you can fit inside a polyhedron?



If y is the center, then draw perpendicular lines to each face of the polyhedron.

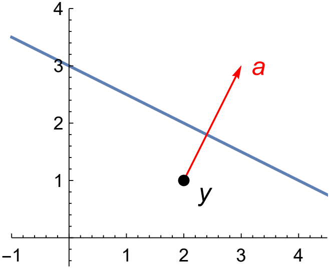
We want to maximize the smallest d_i. In other words,

$$\max_{y} \min_{i=1,\ldots,5} d_i(y)$$

The optimal y is the Chebyshev center

Chebyshev center

Finding the Chebyshev center amounts to solving an LP!



To compute the distance between y and the hyperplane $a^\mathsf{T} x = b$, notice that if the distance is r, then $y + \frac{r}{\|a\|} a$ belongs to the hyperplane:

$$a^\mathsf{T} \left(y + \frac{r}{\|a\|} a \right) = b$$

Simplifying, we obtain: $a^\mathsf{T} y + \|a\| r = b$

"The distance between y and each hyperplane is at least r" is equivalent to saying that $a_i^\mathsf{T} y + \|a_i\| r \le b_i$ for each i.

Chebyshev center

Finding the Chebyshev center amounts to solving an LP!

The transformation to an LP is given by:

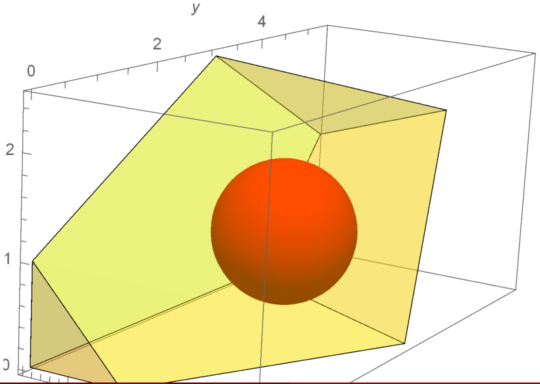
$$\begin{array}{ll} \max\limits_{y} & \min\limits_{i=1,\dots,k} d_i(y) \\ \text{s.t.} & a_i^\mathsf{T} y \le b_i \quad \forall i \end{array} \quad \Longrightarrow \quad \begin{array}{ll} \max\limits_{y,r} & r \\ \text{s.t.} & a_i^\mathsf{T} y + \|a_i\| r \le b_i \quad \forall i \end{array}$$

Chebyshev center

Example: find the Chebyshev center of the polyhedron defined by the following inequalities:

$$2x - y + 2z \leq 2, \ -x + 2y + 4z \leq 16, \ x + 2y - 2z \leq 8,$$
$$x \geq 0, \ y \geq 0, \ z \geq 0$$

Chebyshev.ipynb



Wash and Go With

- Project Scheduling: PERT (Project Evaluation and Review Technique)
- Often used synonomously with CPM: Critical Path Method

PERT

- I: Set of projects
- $P \subset I \times I$: Precedence relationships. ($(i,j) \in P \Rightarrow i$ immediately follows j)
- a_i: Duration of activity $i \in I$

Modeling PERT

Variables

- t_i: Time activity starts

Constraints

- i cannot begin before j finishes:

$$t_i \geq t_j + a_j \qquad \forall (i, j) \in P$$

Objective

- Minimize the latest job completion time (makespan).

$$\min \max\{t_1 + a_1, t_2 + a_2 \ldots, t_{|I|} + a_{|I|}\}.$$

Mini-Max

- Minimax will haunt you

$$T^* = \min z$$

$$\begin{aligned} z &\geq t_i + a_i \ \forall i \in I \\ t_j &\geq t_i + a_i \ \forall (i,j) \in P \\ t_i &\geq 0 \ \forall i \in I \end{aligned}$$

Example: building a house

Several tasks must be completed in order to build a house.

- Each task takes a known amount of time to complete.
- A task may depend on other tasks, and can only be started once those tasks are complete.
- Tasks may be worked on simultaneously as long as they don't depend on one another.
- How fast can the house be built?

Job No.	Description	Immediate predecessors	Normal time (days)
a	Start		0
b	Excavate and pour footers	a	4
c	Pour concrete foundation	b	2
d	Erect wooden frame including rough roof	c	4
e	Lay brickwork	d	6
f	Install basement drains and plumbing	c	1
g	Pour basement floor	f	2
h	Install rough plumbing	f	3
i	Install rough wiring	d	2
j	Install heating and ventilating	d,g	4
k	Fasten plaster board and plaster (including drying)	i,j,h	10
l	Lay finish flooring	k	3
m	Install kitchen fixtures	l	1
n	Install finish plumbing	l	2
o	Finish carpentry	l	3
p	Finish roofing and flashing	e	2
q	Fasten gutters and downspouts	p	1
r	Lay storm drains for rain water	c	1
s	Sand and varnish flooring	o,t	2
t	Paint	m,n	3
u	Finish electrical work	t	1
v	Finish grading	q,r	2
w	Pour walks and complete landscaping	v	5

Source: HBR 1963

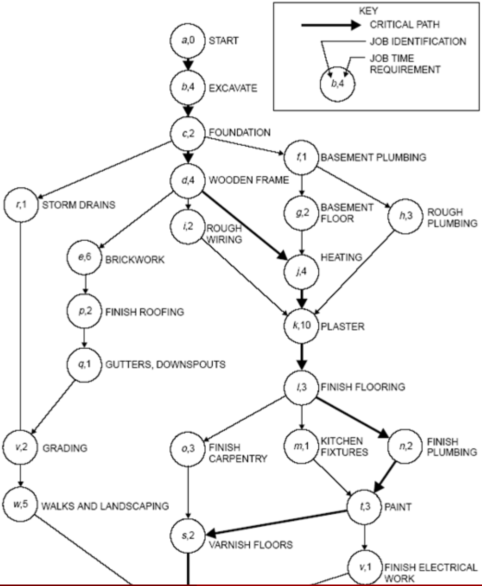
Example: building a house

The data can be visualized using a directed graph.

- Arrows indicate task dependencies.

> What are the decision variables?

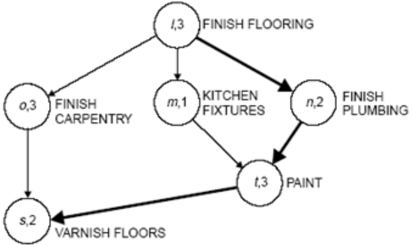
- t_i: *start time* of i^{th} task.
- precedence constraints are expressed in terms of t_i's.
- minimize t_x.



A small sample:

Let t_l, t_o, t_m, t_n, t_t, t_s be start times of the associated tasks.

Now use the graph to write the dependency constraints:



- Tasks o, m, and n can't start until task l is finished, and task l takes 3 days to finish. So the constraints are:
$$t_l + 3 \le t_o, \quad t_l + 3 \le t_m, \quad t_l + 3 \le t_n$$
- Task t can't start until tasks m and n are finished. Therefore:
$$t_m + 1 \le t_t, \quad t_n + 2 \le t_t,$$
- Task s can't start until tasks o and t are finished. Therefore:
$$t_o + 3 \le t_s, \quad t_t + 3 \le t_s$$

Example: building a house

Full implementation in Julia:

House.ipynb

- Follow-up: which tasks in the project are **critical** to finishing on time?
- Which tasks can withstand delays?
- related to notion of *duality* we will see later.

Next...

- more examples of sequential problems
- transportation/shipment problems
- assignment problems
- shortest path problems
- network flows