Geometry of invariant norm for linear dynamical systems

Asiiat Musaeva
Scientific advisor: Prof. V. Yu. Protasov

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The discrete-time system

Let us consider a discrete-time system

$$\begin{cases} x(k+1) = A(k)x(k), \quad k > 1 \\ x(0) = x_0, \end{cases} \tag{1}$$

when $A(k) \in \{A_1, .., A_m\}$.

Definition

A family of linear operators $A = \{A_1, ...A_m\}$ acting in $\mathbb{R}^d$ is given. Their *joint spectral radius* is a value

$$\rho(A) = \lim_{k \to \infty} \max_{d_1, \ldots, d_k \in \{1, \ldots, m\}} \|A_{d_1} \cdots A_{d_k}\|^{\frac{1}{k}},$$

where the norm $\| \cdot \|\|$ is operator Euclidean.

The discrete-time system

Theorem (N.Barabanov)

*For an irreducible family $\{A_1, A_2\}$ in $\mathbb{R}^d$, there exists $\lambda > 0$ and there exists a norm $f(x)$ (**Barabanov norm**) in $\mathbb{R}^d$ such that*

$$max\{f(A_1x), f(A_2x)\} = \lambda f(x) \qquad \forall x \in \mathbb{R}^d.$$

Moreover, for any such norm $\lambda = \rho(A_1, A_2)$.

Theorem (A.Dranishnikov-S.Konyagin)

For an irreducible family $\{A_1, A_2\}$ in $\mathbb{R}^d$, there exists $\lambda > 0$ and a symmetric convex body $G \subset \mathbb{R}^d$(invariant body) such that

$$co\{A_1G, A_2G\} = \lambda G$$

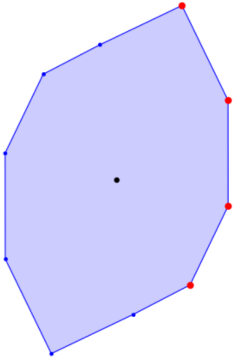
Moreover, for any such body $\lambda = \rho(A_1, A_2)$.

IPA

For constructing that norm they put to good use the *invariant polytope algorithm*. The algorithm finds a *dominant product* $\Pi = A_{s_n} \cdots A_{s_1}$ such that $|\lambda|^{1/n} = \rho(\mathcal{A})$, where λ is the *leading*.
The algorithm halts within finite time if and only if the product Π is dominant and its leading eigenvalue λ is unique and simple.
In this case the obtained invariant body G is either a polytope (if $\lambda \in \mathbb{R}$) or a convex hull of several ellipses (if $\lambda \notin \mathbb{R}$).

Example 1. For the family $\mathcal{A} = \{A_1, A_2\}$, where

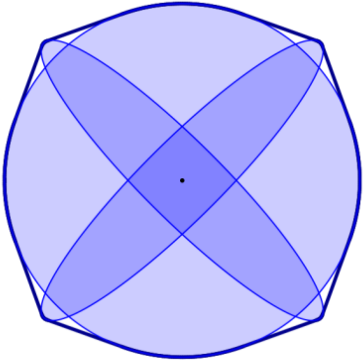
$$A_1 = \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} \ ; \quad A_2 = \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}.$$



Example

Example 2 The family

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; ; \quad A_2 = \begin{pmatrix} 0.890 & 0.646 \\ -0.129 & -0.178 \end{pmatrix}.$$



The discrete-time system

Theorem (N.Barabanov)

For an irreducible family $\{A_1, A_2\}$ *in* $\mathbb{R}^d$, *there exists* $\lambda > 0$ *and there exists a norm* $f(x)$ *(**Barabanov norm**) in* $\mathbb{R}^d$ *such that*

$$max\{f(A_1x), f(A_2x)\} = \lambda f(x) \qquad \forall x \in \mathbb{R}^d.$$

Moreover, for any such norm $\lambda = \rho(A_1, A_2)$.

Theorem (A.Dranishnikov-S.Konyagin)

For an irreducible family $\{A_1, A_2\}$ *in* $\mathbb{R}^d$, *there exists* $\lambda > 0$ *and a symmetric convex body* $G \subset \mathbb{R}^d$ *(invariant body) such that*

$$co\{A_1G, A_2G\} = \lambda G$$

Moreover, for any such body $\lambda = \rho(A_1, A_2)$.

Duality

Theorem (F.Wirth, E.Plischke)

Let B be the unit ball of the Barabanov norm for the operators A_1 and A_2. Then B^* is an invariant convex body for the operators A_1^*, A_2^* in the Dranishnikov-Konyagin theorem.

Definition

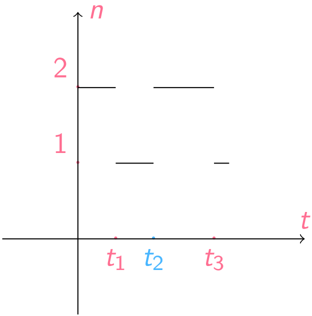
Continuous linear switching system is a system of ODE

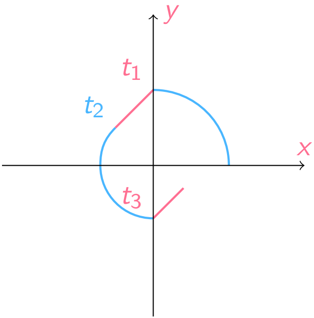
$$\begin{cases} \dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t), \quad t \geq 0 \\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases}$$

where the matrix $A(t) : \mathbb{R}^+ \to \mathcal{A}$ is a measurable function.
$\mathcal{A}$ is a set of $d \times d$-matrices (*control set*).

Example: The case of two matrices: $U = \{A_1, A_2\}$

The matrix $A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the matrix $A_2 = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}$





Lyapunov exponent

The following value is called Lyapunov exponent

$$\sigma(\mathcal{A}) = \inf\{\alpha \mid \exists C \; \|\boldsymbol{x}(t)\| \leq C \cdot e^{\alpha t}\}$$

when inf is over the set of all trajectories $\boldsymbol{x}(t)$ and by the set of all values of t .

The properties of a Lyapunov exponent

- $\sigma(\mathcal{A}) = \sigma(co(\mathcal{A}))$
- $\sigma(\mathcal{A} + sI) = \sigma(\mathcal{A}) + s$

Barabanov theorem

Theorem (Barabanov,1989)

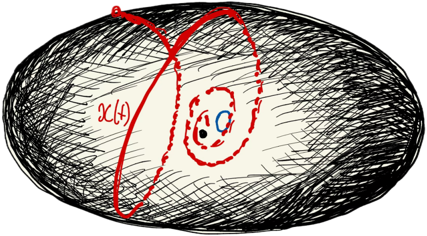
If $\sigma(\mathcal{A}) = 0$ *and matrix from* $\mathcal{A}$ *do not have common proper invariant subspaces, then there is a norm* f *such that :*
1. for all $\bar{\boldsymbol{x}}(t)$ *function* $f(\bar{\boldsymbol{x}}(t))$ *is non-increasing*
2. for every $\boldsymbol{x}_0$ *there exists* $\bar{\boldsymbol{x}}(t)$ *:*

$$f(\bar{\boldsymbol{x}}(t)) = constant, \;\; for\ all\ t > 0$$

$$\bar{\boldsymbol{x}}(0) = \boldsymbol{x}_0$$

Any such norm f is called an *invariant norm*.
There may be several such norms.
The invariant norm is a Lyapunov function of the system.

The geometric sense



Main result

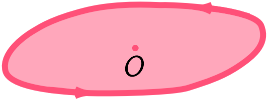
Theorem

Suppose that the linear operators A_1,A_2 *have an invariant norm* f *with a unit ball* B. *Then the Minkowski's functional of the* B^* *is an invariant norm for adjoint operators* A_1^*, A_2^*.

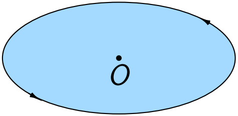
Main result

In $\mathbb{R}^2$ there are 3 cases of the Barabanov norm (supposed $\sigma(\mathcal{A}) = 0$) :

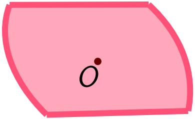
1. There are no dominant matrices.



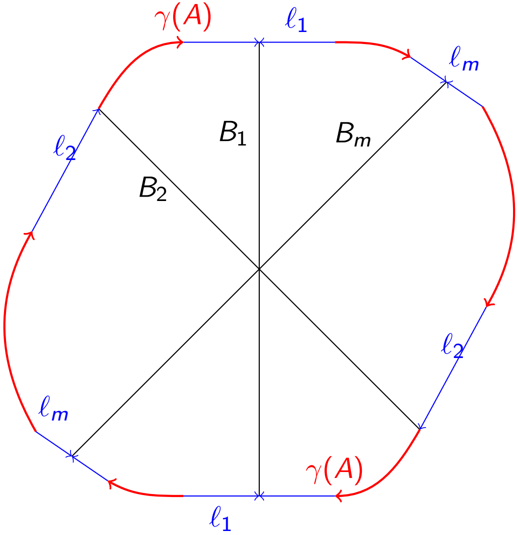
2. There is a matrix with complex dominance.
   In this case $\boldsymbol{x}(t)$ is an ellipse, $f(\boldsymbol{x}) = \sqrt{x^T M x}$ – elliptic norm.



3. The real dominance (the most interesting case).



Example



Open problem

1. Every convex curve symmetric about the origin can be the unit sphere of the Barabanov norm of a suitable system with real dominance.
2. For any polytope in $\mathbb{R}^d$ there exists a suitable system.
3. For any system in $\mathbb{R}^2$ there is an exact criterion that determines whether the invariant norm can be a polygon.
4. **For $\mathbb{R}^d$ this question remains open.**

Thank you for your attention!