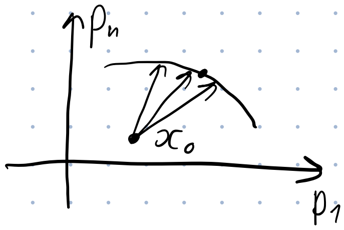
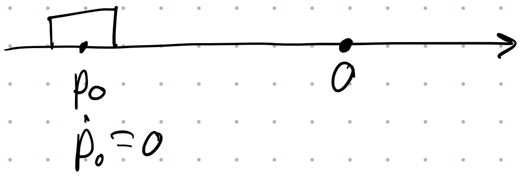
08.10.25 Model Predictive Control

$$x_{k+1} = f(x_k, u_k)$$

$$u_0^* = \underset{u_0 \in U}{\arg\min}\ G(x_1, u_0)$$



$$u_0^*, u_1^* = \underset{u_0, u_1 \in U}{\arg\min}\ G(x_1, x_2, u_0, u_1)$$

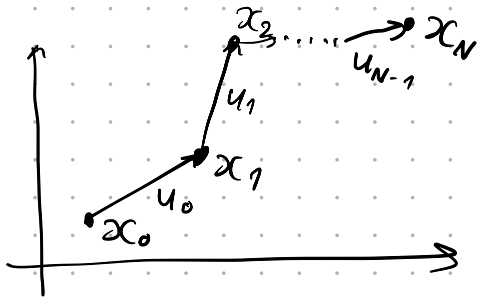


ЗАДАЧА В ОБЩ. ВИДЕ:

$$\begin{cases} \{u_i^*\}_{i=0}^{N-1} = \underset{u_i \in U}{\arg\min} \sum_{i=0}^{N-1} G(x_{i+1}, u_i)\gamma^i \\ \text{s.t. } x_{i+1} = f(x_i, u_i) \end{cases}$$

$$x_i \in X$$

robust MPC
$x_{i+1} = f^M(x_i, u_i)$
$M_i \in [\ldots, \ldots]$



ПОСЛЕ РЕШ. ЗАД. ВЫПОЛН. ТОЛЬКО u_0

$$x_0$$

$$x_1 = f(x_0, u_0)$$

$$x_2 = f(x_1, u_1) = f(f(x_0, u_0), u_1)$$

$$x_N = f(f(f(\ldots), u_{N-2}), u_{N-1})$$

ВЫВОД ЛИН. MPC (ДИСКР. LQR С КОН. ГОР.)

$$x_{k+1} = Ax_k + Bu_k$$

$$J = \sum_{i=0}^{N-1} x_{i+1}^T Q x_{i+1} + u_i^T R u_i$$

$$\{u_i\}_{i=0}^{N-1} - ?$$

$$x_1 = Ax_0 + Bu_0$$

$$x_2 = A x_1 + B u_1 = A^2 x_0 + A B u_0 + B u_1$$

$$x_k = A^k x_0 + A^{k-1} B u_0 + \dots + A B u_{k-2} + B u_{k-1}$$

$$\overbrace{\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_N \end{pmatrix}}^{X \in \mathbb{R}^{N \cdot n}} = \begin{pmatrix} A x_0 \\ A^2 x_0 \\ \cdots \\ A^N x_0 \end{pmatrix} + \begin{pmatrix} B u_0 \\ AB u_0 + B u_1 \\ \cdots \\ A^{N-1} B u_0 + A^{N-2} B u_1 + \dots + A B u_{N-2} + B u_{N-1} \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} A \\ A^2 \\ \cdots \\ A^N \end{pmatrix}}_{\hat{A} \in \mathbb{R}^{N \cdot n \times n}} x_0 + \underbrace{\begin{pmatrix} B & 0_{n \times m} & \cdots & 0_{n \times m} \\ AB & B & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A^{N-1} B & A^{N-2} B & \cdots & B \end{pmatrix}}_{\hat{B} \in \mathbb{R}^{N \cdot n \times N \cdot m}} \underbrace{\begin{pmatrix} u_0 \\ u_1 \\ \cdots \\ u_{N-1} \end{pmatrix}}_{U \in \mathbb{R}^{N \cdot m}}$$

$$X = \hat{A} x_0 + B U$$

$$\underset{\in \mathbb{R}^{N \cdot n \times N \cdot n}}{\hat{Q}} = \begin{pmatrix} Q & & & 0 \\ & Q & & \\ & & \ddots & \\ 0 & & & Q \end{pmatrix} ; \quad \underset{\in \mathbb{R}^{Nm \times Nm}}{\hat{R}} = \begin{pmatrix} R & & \\ & \ddots & \\ & & R \end{pmatrix}$$

$$J = \sum_{i=0}^{N-1} x_{i+1}^T Q x_{i+1} + u_i^T R u_i = X^T \hat{Q} X + U^T \hat{R} U =$$

$$= (\hat{A} x_0 + \hat{B} U)^T \hat{Q} (\hat{A} x_0 + \hat{B} U) + U^T \hat{R} U =$$

$$= x_0^T \hat{A}^T \hat{Q} \hat{A} x_0 + 2 x_0^T \hat{A}^T \hat{Q} \hat{B} U + U^T \hat{B}^T \hat{Q} \hat{B} U + U^T \hat{R} U =$$

$$= x_0^T \hat{A}^T \hat{Q} \hat{A} x_0 + 2 x_0^T \hat{A}^T \hat{Q} \hat{B} U + U^T (\hat{B}^T \hat{Q} \hat{B} + \hat{R}) U$$

$$\frac{\partial J}{\partial U} = 2 \hat{B}^T \hat{Q} \hat{A} x_0 + 2 (\hat{B}^T \hat{Q} \hat{B} + \hat{R}) U = 0$$

$$(\hat{B}^T \hat{Q} \hat{B} + \hat{R}) U = -\hat{B}^T \hat{Q} \hat{A} x_0 \quad \Big| \, (\hat{B}^T \hat{Q} \hat{B} + \hat{R})^{-1} \cdot$$

$$U^* = -(\hat{B}^T\hat{Q}\hat{B}+\hat{R})^{-1}\hat{B}^T\hat{Q}\hat{A}x_0$$

$$(\hat{B}^T\hat{Q}\hat{B})^{-1}$$

$$y = Ax$$

$$A^Ty = A^TAx \quad | \ (A^TA)^{-1}\cdot$$

$$x = (A^TA)^{-1}A^Ty$$

$$U^* = \left((\hat{B}^T\hat{Q}\hat{B}+\hat{R})^T(\hat{B}^T\hat{Q}\hat{B}+\hat{R})\right)^{-1}(\hat{B}^T\hat{Q}B+R)^T(-\hat{B}^T\hat{Q}\hat{A}x_0) =$$

$$= -(B^TQBB^TQB + 2B^TQBR + R^T\cancel{B^T}QB + R^TR)^{-1}(B^TQB+R)B^TQAx_0 =$$

$$\lrcorner \ R = \varepsilon\mathbb{I}$$

$$= -(B^TQB(B^TQB + 2\varepsilon\mathbb{I} + \cancel{(B^TQB)^{-1}}\varepsilon^2\mathbb{I}))^{-1}(B^TQB+\varepsilon\mathbb{I})B^TQAx_0 =$$

$$= -(B^TQB)^{-1}(B^TQB+2\varepsilon\mathbb{I})^{-1}(B^TQB+\varepsilon\mathbb{I})B^TQAx_0$$

$$(A+\varepsilon\mathbb{I})^{-1} = (A(\mathbb{I}+\varepsilon A^{-1}))^{-1} =$$

$$= A^{-1}(\mathbb{I}+\varepsilon A^{-1})^{-1}$$

$$x_2 = A^2x_0 + ABu_0 + Bu_1 =$$

$$= \begin{pmatrix} A \\ A^2 \end{pmatrix} x_0 + \begin{pmatrix} B & 0 \\ AB & B \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

$$\begin{pmatrix} B & 0 \\ AB & B \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = x_2 - \begin{pmatrix} A \\ A^2 \end{pmatrix} x_0$$

$$\hat{B}(u_0) = x_1 - Ax_0$$

$$U^* = -(\hat{B}^T\hat{Q}\hat{B}+\hat{R})^{-1}\hat{B}^T\hat{Q}\hat{A}x_0 =$$

при $R = 0$

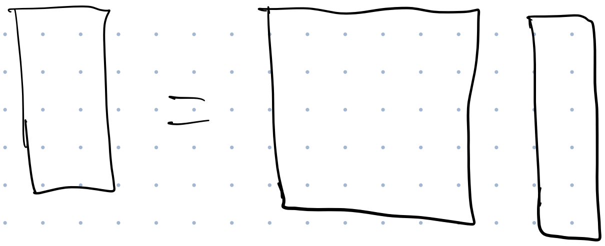
$$\hat{A} = BC$$

$$= -\cancel{(\hat{B}^T\hat{Q}\hat{B})^{-1}\hat{B}^T\hat{Q}\hat{B}}Cx_0 = -(\hat{B}^T\hat{B})^{-1}\hat{B}^TAx_0$$

$$\underset{\in \mathbb{R}^{N\cdot n\times n}}{\hat{A}} = \underset{\in \mathbb{R}^{Nn\times Nm}}{\hat{B}} \cdot \underset{\in \mathbb{R}^{N\cdot m\times n}}{C}$$

$$\hat{B}^T\hat{A} = \hat{B}^T\hat{B}C \quad \Big| (\hat{B}^T\hat{B})^{-1}\cdot$$

$$C = (\hat{B}^T\hat{B})^{-1}\hat{B}^T A$$



$$X = \hat{A}x_0 + \hat{B}V = \hat{A}x_0 - \hat{B}(\hat{B}^T\hat{B})^{-1}\hat{B}^T A x_0 =$$

$$= \hat{A}x_0 - \hat{A}x_0 = 0$$