2.8.3: Introduction to Bode plots

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Overview

Plotting a systems' frequency response on a linear scale, as done in chapters 2.8.1 and 2.8.2, has a number of drawbacks. An alternate format for plotting frequency responses, called a *Bode plot*, is therefore commonly used. On Bode plots, the amplitude response is essentially presented as a log-log plot, while the phase response is a semi-log plot. Some reasons for this are

- Use of logarithms converts the operation of multiplication and division to addition and subtraction. This can simplify the creation of frequency response plots for higher order systems.
- Frequencies and amplitudes of interest commonly span many orders of magnitude. Logarithmic scales improve the presentation of this type of data.
- Human senses are fundamentally logarithmic. The use of logarithmic scales is therefore more "natural". (This is the reason for use of the Richter scale in measuring earthquake intensity, and the decibel scale in measuring sound levels. It is also the reason that increasing a musical tone by one octave corresponds to doubling its frequency.)

Before beginning this module, you should be able to:

- Calculate and plot magnitude and phase responses on a linear scale. (Chapters 2.8.0 and 2.8.1)
- State from memory the definition of high-pass and low-pass filters (Chapter 2.8.2)
- Calculate the cutoff frequency for high-pass and low-pass filters (Chapter 2.8.2)
- Determine the DC gain of filters (Chapter 2.8.2)

After completing this module, you should be able to:

- Write, from memory, the equation used to convert gains to decibel form
- Sketch straight-line amplitude approximations to Bode plots
- Sketch straight-line phase approximations to Bode plots

This module requires:

- N/A

Properties of logarithms:

Since Bode plots employ logarithms extensively, we will briefly review some of the basic properties of logarithms before proceeding further. Bode plots rely upon base-10 logarithms ($\log_{10}$), so we will restrict our attention to base-10 logarithms.

A plot of $\log_{10}(x)$ vs. x is shown in Figure 1 below. A few important features to note are:

- $\log_{10}(x)$ is a real number only for positive values of x.
- $\log_{10}(x)$ asymptotically approaches $-\infty$ as $x \to 0$. The slope of $\log_{10}(x)$ becomes very large as $x \to 0$.
- The slope of $\log_{10}(x)$ becomes small as $x \to \infty$.
- From the comments above relative to the slope of $\log_{10}(x)$, it can be seen that the sensitivity of $\log_{10}(x)$ to variations in x decreases as x increases (this is the reason why logarithmic scales are used when large variations in x are encountered – as in Richter scales and musical scales).
- $\log_{10}(1) = 0$

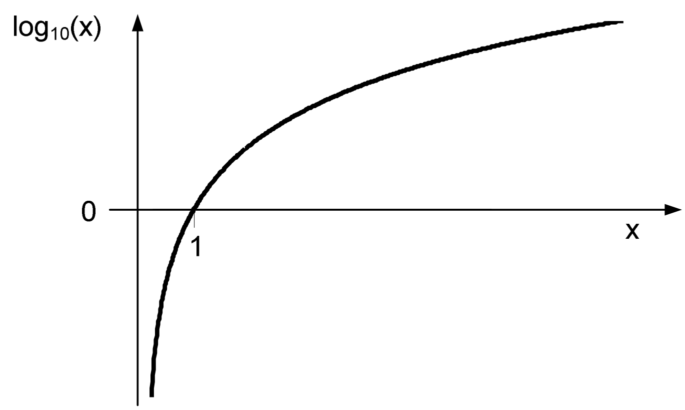


Figure 1. Plot of $\log_{10}(x)$ vs. x

The basic defining property of a base-10 logarithm is that $x = 10^y$, then $y = log_{10} x$. This property leads to the following rules governing logarithmic operations:

1. Logarithms convert multiplication and division to addition and subtraction, respectively. Thus,

$$log_{10}(xy) = log_{10} x + log_{10} y$$

and

$$log_{10}\left(\frac{x}{y}\right) = log_{10} x - log_{10} y$$

This property is especially useful for us, since determining the spectrum of an output signal results from the product of an input signal's spectrum with the frequency response. Thus, the output spectrum on a logarithmic scale can be obtained from a simple addition.

2. Logarithms convert exponentiation to multiplication by the exponent, so that

$$log_{10}\left(x^n\right) = n\,log_{10}\,x$$

Decibel Scales:

Magnitude responses are often presented in terms of decibels (abbreviated dB). Decibels are a logarithmic scale. A magnitude response is presented in units of decibels according to the following conversion:

$$\left|H(j\omega)\right|_{dB} = 20\,log_{10}\left(\left|H(j\omega)\right|\right) \tag{1}$$

Strictly speaking, magnitudes in decibels are only appropriate if the amplitude response is unitless (e.g. the units of the input and output must be the same in order for the logarithm to be a mathematically appropriate operation). However, in practice, magnitude responses are often presented in decibels regardless of the relative units of the input and output – thus, magnitude responses are provided in decibels even if the input is voltage and the output is current or vice-versa.

Brief historical note:

Decibel units are related to the unit "bel", which are named after Alexander Graham Bell. Units of bels are, strictly speaking, applicable only to power. Power in bels is expressed as $log_{10}(P/P_{ref})$, where P_{ref} is a "reference" power. Bels are an inconveniently large unit, so these were converted to decibels, or tenths of a bel. Thus, power in decibels is $10\,log_{10}(P/P_{ref})$. Since the units of interest to electrical engineers are generally voltages or currents, which must be squared to obtain power, we obtain $20\,log_{10}\left(\left|H(j\omega)\right|\right)$. The significant aspect of the decibel unit for us is not, however, the multiplicative factor of "20), but the fact that the unit is logarithmic.

We conclude this subsection with a table of common values for $\left|H(j\omega)\right|$ and their associated decibel values.

$\left\|H(j\omega)\right\|$	$\left\|H(j\omega)\right\|$ in decibels
10	20
1	0
0.1	-20
$1/\sqrt{2}$	-3
$1/2$	-6

Bode Plots:

Bode plots are simply plots of the magnitude and phase response of a system using a particular set of axes. For Bode plots,

- Units of frequency are on a base-10 logarithm scale.
- Amplitudes (or magnitudes) are in decibels (dB)
- Phases are presented on a linear scale

Notes:

- Since frequencies are on a logarithmic scale, frequencies separated by the same multiplicative factor are evenly separated on a logarithmic scale. Some of these multiplicative factors have special names. For example, frequencies separated by a factor of two are said to be separated by *octaves* on a logarithmic scale and frequencies separated by a factor of 10 are said to be separated by *decades*.
- Since decibels are intrinsically a logarithmic scale, magnitudes which are separated by the same multiplicative factor are evenly separated on a decibel scale. For example, magnitudes which are separated by a factor of 10 are separated by 20dB on a decibel scale.

One convenient aspect of the presentation of frequency responses in terms of Bode plots is the ability to generate a reasonable sketch of a frequency response very easily. In general, this approach consists of approximating the Bode plot of a system by its asymptotic behavior as a set of straight lines. This is called a “straight line approximation” of the Bode plot; the approach is illustrated for a typical low-pass filter in the following subsection.

Bode plots for first order low-pass filters:

The frequency response of a general first order low-pass filter is provided in chapter 2.8.2 as:

$$H(j\omega) = \frac{K}{j\omega + \omega_c} \tag{1}$$

Thus, the magnitude response of the circuit is:

$$\left|H(j\omega)\right| = \frac{K}{\sqrt{\omega^2 + \omega_c^2}} \tag{2}$$

and the phase response of the circuit is

$$\angle H(j\omega) = -tan^{-1}\left(\frac{\omega}{\omega_c}\right) \tag{3}$$

To estimate the asymptotic behavior of the frequency response, we consider the behavior of equations (2) and (3) for the low frequency and high frequency cases. In general, we consider "low" frequencies to be frequencies which are less than a factor of 10 below the cutoff frequency (i.e. $\omega < \omega_c/10$, or frequencies more than a decade below the cutoff frequency). High frequencies are typically assumed to be frequencies which are more than a factor of 10 above the cutoff frequency (i.e. $\omega > 10\omega_c$, or more than a decade above the cutoff frequency). We consider the high and low frequency cases separately below.

- Low frequencies:

  The magnitude response given by equation (2) is $\left|H(j\omega)\right| = \dfrac{K}{\sqrt{\omega^2 + \omega_c^2}}$. If $\omega \ll \omega_c$, the denominator is approximately $\sqrt{\omega_c^2} = \omega_c$ and the magnitude response $\left|H(j\omega)\right| \approx \dfrac{K}{\omega_c}$.

  If $\omega \ll \omega_c$, $\omega/\omega_c \approx 0$ and the phase response is approximately $\angle H(j\omega) \approx -tan^{-1}(0) = 0°$.

- High frequencies:

  If $\omega \gg \omega_c$, the denominator of the amplitude response is $\sqrt{\omega^2 + \omega_c^2} \approx \sqrt{\omega^2} = \omega$. Therefore, for high frequencies, the magnitude response $\left|H(j\omega)\right| \approx \dfrac{K}{\omega}$. If, for high frequencies, we increase the frequency by a factor of 10, we reduce the magnitude response by 20dB (since $\left|H(j \cdot 10\omega)\right| \approx \dfrac{K}{10\omega} = 0.1\dfrac{K}{\omega}$ and the multiplicative factor of 0.1 corresponds to -20dB). Thus, for frequencies well above the cutoff frequency the magnitude response, presented in Bode plot form, decreases by 20dB/decade. When $\omega \gg \omega_c$, the phase response is given by

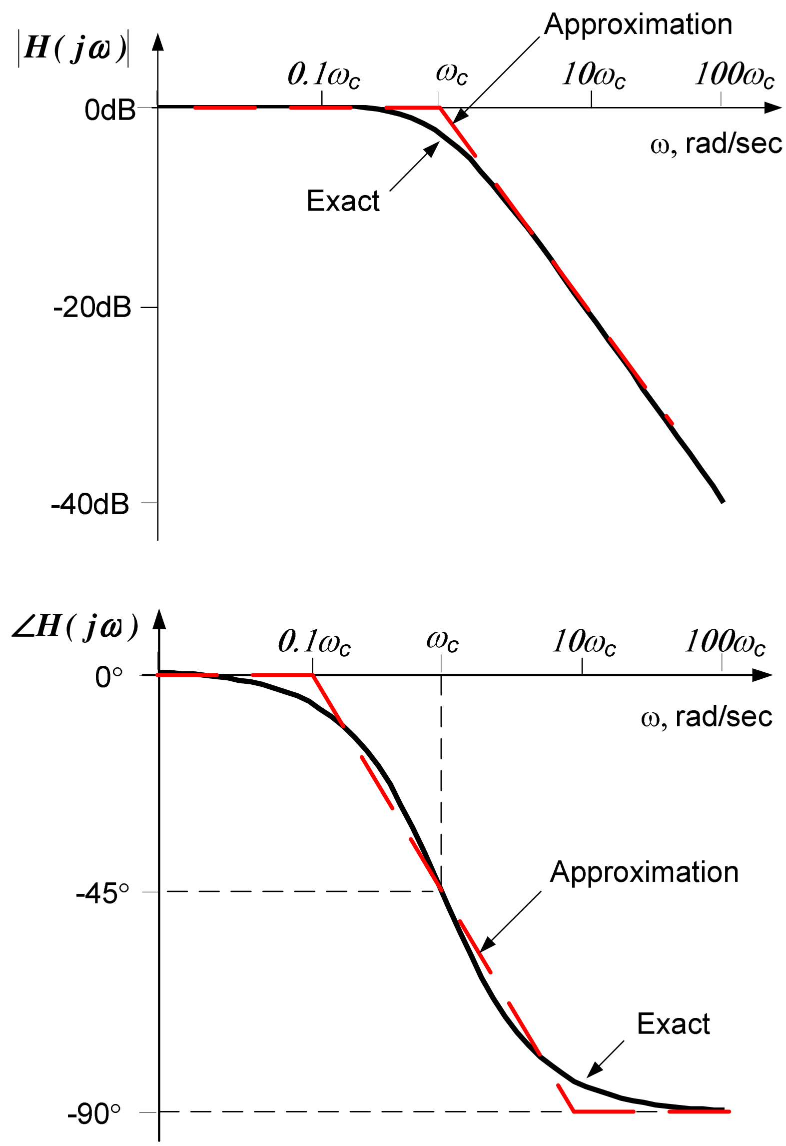
  $\angle H(j\omega) = -tan^{-1}\left(\dfrac{\omega}{\omega_c}\right) \approx -tan^{-1}(\infty) = -90°$.

Summary – Low-pass filter straight-line Bode plot approximations:

The straight line approximation to the magnitude response is constant below the cutoff frequency, with a value (in decibels) of $20log_{10}\left(K/\omega_c\right)$. Above the cutoff frequency, the Bode plot straight-line approximation has a constant slope of -20 dB/decade.

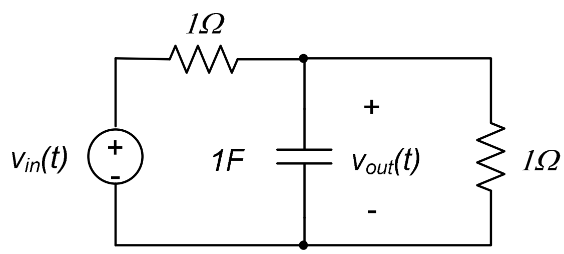
The straight line approximation to the phase response is zero degrees up to a frequency of $\omega_c/10$ and is -90° above a frequency of $10\omega_c$. A straight line is used to connect the $\omega_c/10$ and $10\omega_c$ frequencies.

A straight-line approximation to the Bode plot for a typical low-pass circuit, with $K = \omega_c$, (so that the frequency response is $H(j\omega) = \frac{\omega_c}{j\omega + \omega_c}$ and the DC gain is 1, or 0dB) along with an exact curve is provided below in Figure 2.



We conclude this section with a numerical example of the straight-line approximation to a Bode plot for a specific circuit.

Example 1: Sketch a straight-line approximation to the Bode plot for the circuit below. The input is $v_{in}(t)$ and the output is $v_{out}(t)$.



The frequency response for this circuit is $H(j\omega) = \frac{1}{j\omega + 2}$. Therefore, the cutoff frequency is $\omega_c = 2$ rad/sec and the gain in decibels at low frequencies is $|H(j0)|_{dB} = 20 log_{10}\left(\frac{1}{2}\right) \approx -6dB$. Thus, the straight-line magnitude response is -6dB below the cutoff frequency and decreases by 20dB/decade above the cutoff frequency. The straight-line phase response is 0° below 0.2rad/sec, -90° above 20 rad/sec and a straight line between these frequencies. The associated plots are shown below.