EE P 538
Analog Circuits for Sensor Systems

Spring 2020

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ELECTRICAL ENGINEERING
UNIVERSITY of WASHINGTON

Announcements

- Midterm due Sunday, May 17 at midnight
- Assignment 6 due Saturday, May 23 at midnight

Week 7

- AoE Chapter 6 – Filters
  - Sections 6.1, 6.2
- James Karkin – Active Low-Pass Filter Design
  - TI Application Note SLOA049B
- James Karkin – Sallen–Key Filter Analysis
  - TI Application Note SLOA024B

Overview

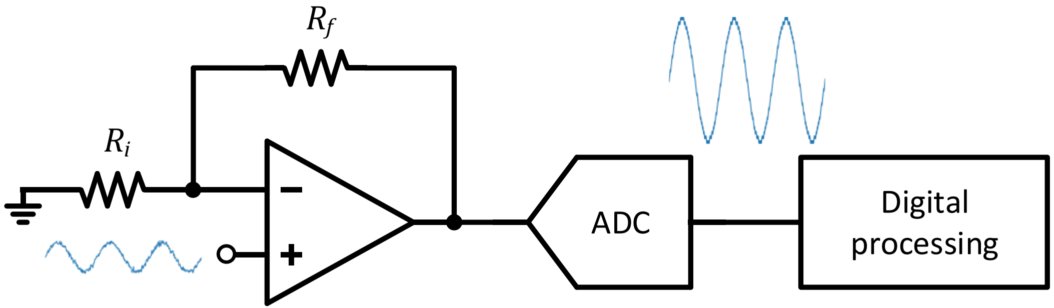
- Last time…
  - Thermal, flicker, and shot noise
  - Analyzing circuits with noise
  - Noise bandwidth
  - Opamp noise
- Today…
  - Noise filtering
  - Butterworth, Chebyshev, and Bessel filters
  - Sallen–Key filter architecture

Lecture 7 – Filters 1

“An ideal electrical filter should not only completely reject the unwanted frequencies but should also have uniform sensitivity for the wanted frequencies.”

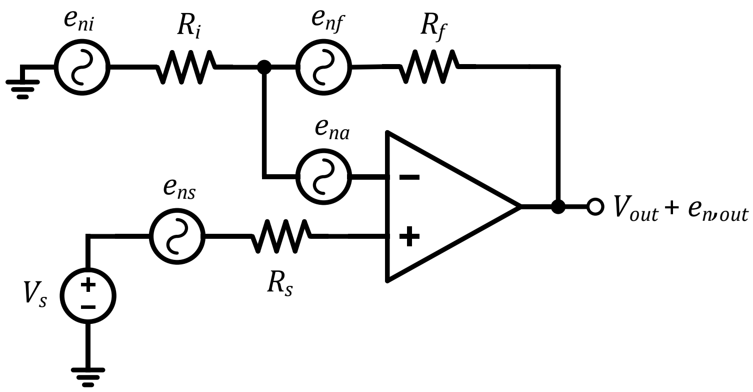
– *Stephen Butterworth*

Front-End Gain



- The first circuit a sensor or transducer signal sees typically has a substantial amount of gain (20dB or higher)
- Gain is used to desensitize signals both to downstream noise and the finite resolution of ADCs
- An ideal amplifier merely increases the amplitude of a signal without adding noise

Noise Bandwidth



$$V_{out(rms)} = V_{out,pp}/\sqrt{2}$$

$$v_{n,out(rms)} \approx \sqrt{e_{n,out}^2 f_{ENB}}$$

$$SNR = V_{out(rms)}/v_{n,out(rms)}$$

- Unfortunately, all electronic components used to process signals contribute some amount of additional noise
- That is, a circuit can only *increase*, never reduce, the RMS noise in a given bandwidth
- In order to build precision circuits, we need to limit bandwidth to what is required for a given application

Signal-to-Noise Ratio (SNR)

- Signal-to-noise ratio is the ratio of RMS signal amplitude the RMS noise
- For example, suppose we want to amplify and digitize a sensor signal with a peak-to-peak amplitude of 5mV and achieve an SNR of ≥80dB
- The signal-to-noise ratio is given by

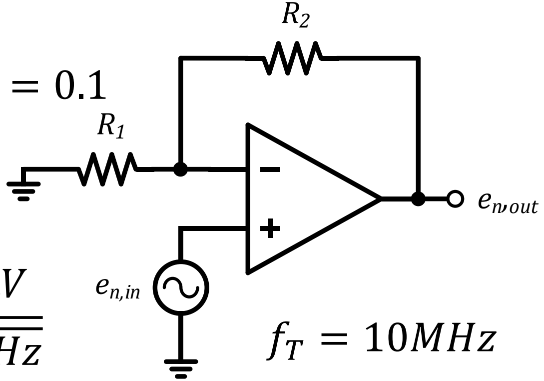
$$SNR = 20\log\frac{v_{s(rms)}}{v_{n(rms)}} = 20\log\frac{1.77mV}{v_{n(rms)}} = 80dB$$

- To meet this specification, the noise should be

$$v_{n(rms)} \leq \frac{1.77mV}{10^{\frac{80dB}{20}}} = 177nV$$

Gain Stage Noise

$$\beta = \frac{R_1}{R_1 + R_2} = 0.1$$



$$e_{n,in} = 10\frac{nV}{\sqrt{Hz}}$$

$$f_T = 10MHz$$

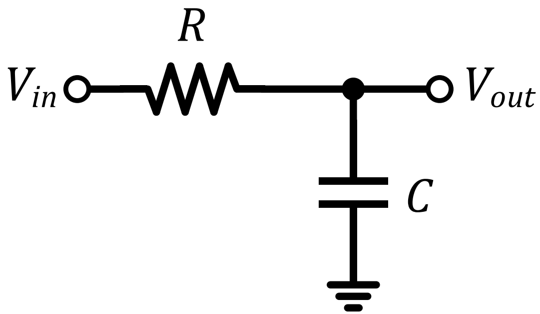
$$f_{3dB,CL} = \beta f_T = 1MHz$$

$$f_{ENB,amp} = \frac{\pi}{2}\beta f_T \approx 1.57MHz$$

$$v_{n,in(rms)} \approx \sqrt{e_{n,in}^2 f_{ENB}} \approx 12.5\mu V$$

- For high precision, opamp bandwidth may be much wider than that required for the signal
- This results in a wide noise bandwidth, and consequently a large amount of noise and degraded SNR
- In order to limit the RMS noise and improve the SNR, we need to add filtering

RC Lowpass Filter



$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1 + (f/f_{3dB})^2}} \qquad f_{3dB} = 1/RC$$

$$f_{ENB} = \frac{\pi}{2} f_{3dB} \approx 1.57 f_{3dB}$$

- What about adding a single-pole passive RC filter?
- The noise bandwidth is approximately ~1.57f_{3dB}, which can be used to limit the total noise from the amplifier
- However, we need to ensure that the filter's "gain" doesn't vary significantly over the bandwidth of the signal

Signal vs Noise Bandwidth

- Suppose we want to pass a given signal with less than 5% attenuation across its bandwidth
- To do this, we need a filter that satisfies

$$\frac{1}{\sqrt{1+(f_{SBW}/f_{3dB})^2}} = 0.95 \longrightarrow f_{SBW} \approx 0.33 f_{3dB}$$

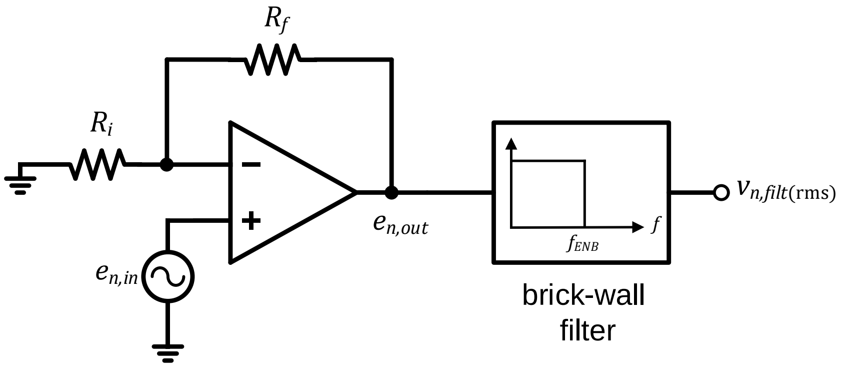
- We know the equivalent noise bandwidth (f_{ENB}) is ~1.57f_{3dB}, so the ratio between the noise and signal bandwidths is

$$f_{ENB}/f_{SBW} \approx 4.8$$

- This means that the SNR performance is 4.8× worse than that of a brick wall filter with a bandwidth of f_{SBW}! (Ltspice: RC filter)

Ideal Filter

- The bandwidth of the signal is fixed by the application, and sets the minimum limit on the circuit bandwidth
- For example for audio applications, the range of audible frequencies is from about 20Hz to 20kHz
- To maximize SNR, we should ensure that the noise bandwidth is as narrow as possible, ideally equal to the signal bandwidth

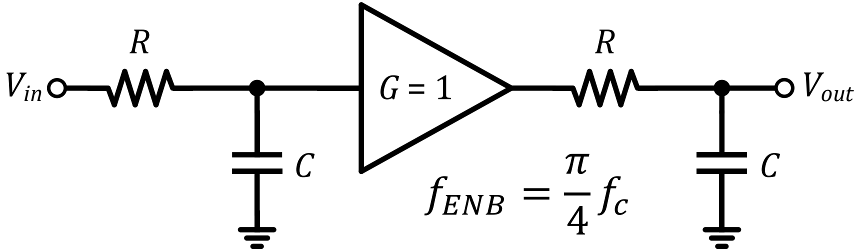


$$v_{n,filt(rms)} = \sqrt{e_{n,in}^2 A_{CL}^2 f_{ENB}}\, V/\sqrt{Hz}$$

$$f_{ENB,ideal} = f_{SBW}$$

Filter Cascading

- If we cascade multiple passive RC filters, can we do better?
- The equivalent noise bandwidth, which is determined by the total noise power across the frequency spectrum, is reduced by half
- However, the “passband” is also reduced, resulting in only a modest improvement
- To improve upon this, we need less attenuation in the pass band and steeper roll-off



$$f_{ENB} = \frac{\pi}{4} f_c$$

5% attenuation: $$\frac{1}{1 + (f_{SBW}/f_0)^2} = 0.95$$

$$f_{SBW} = 0.23 f_0$$

$$f_{ENB}/f_{SBW} \approx 3.4$$

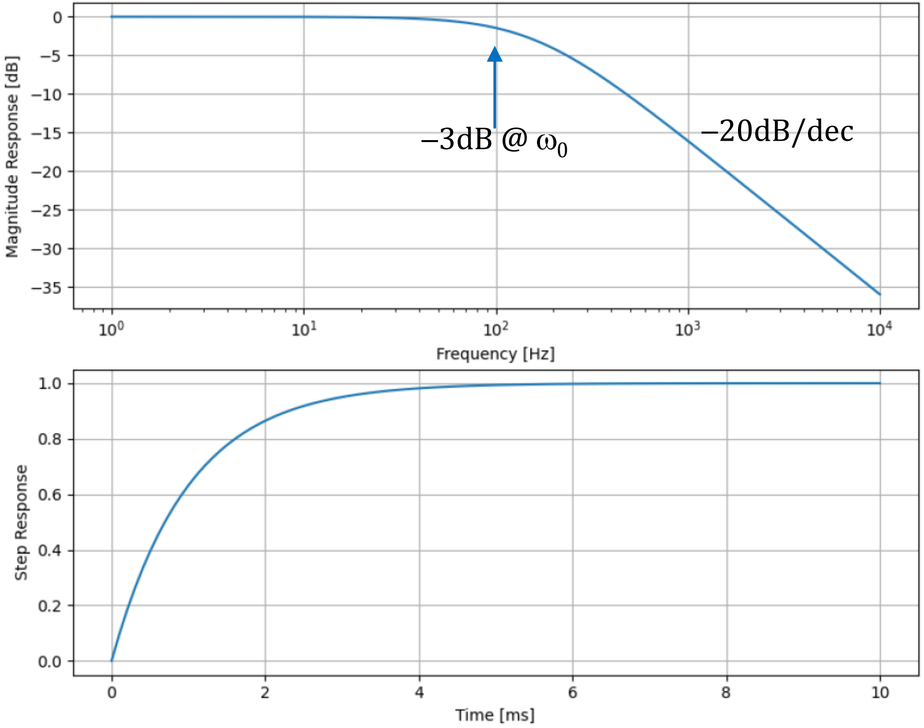
First-Order Filter

- The dynamic behavior of a 1st order filter is fully described by a single parameter, ω_0:

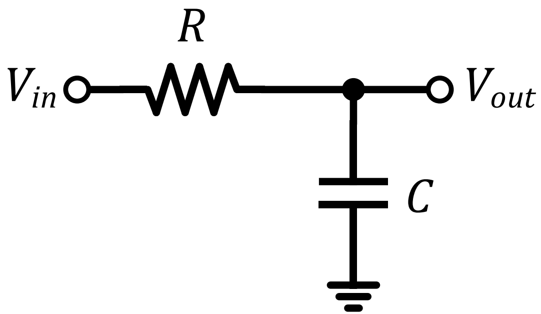
$$H(s) = \frac{\omega_0}{s + \omega_0}$$

- To determine ω_0, we solve for the pole frequency by setting the denominator of the transfer function equal to 0:

$$s_0 = -\omega_0 \qquad\qquad \omega_0 = |s_0|$$

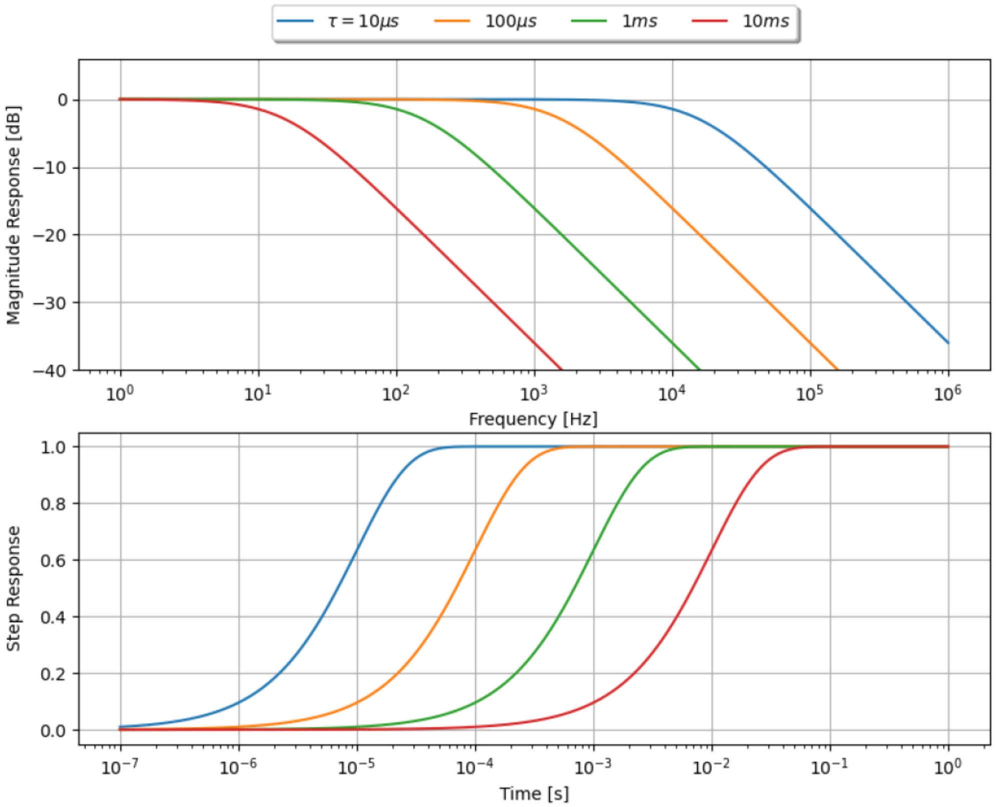


1st Order RC Filter



$$\frac{V_{out}}{V_{in}} = \frac{1/RC}{s + 1/RC} = \frac{\omega_0}{s + \omega_0}$$

$$\omega_0 = \frac{1}{RC} = \frac{1}{\tau} \qquad f_{3dB} = \frac{\omega_0}{2\pi}$$



Second Order Filter

- The “order” of a filter is determined the number of poles in its transfer function, which for an analog filter depends on the number of reactive components (i.e. inductors and capacitors)
- For a second-order filter, rather than depending on only a single parameter ω_0, the response is governed by two parameters, ω_0 and Q
- The transfer function of a second order filter is given by

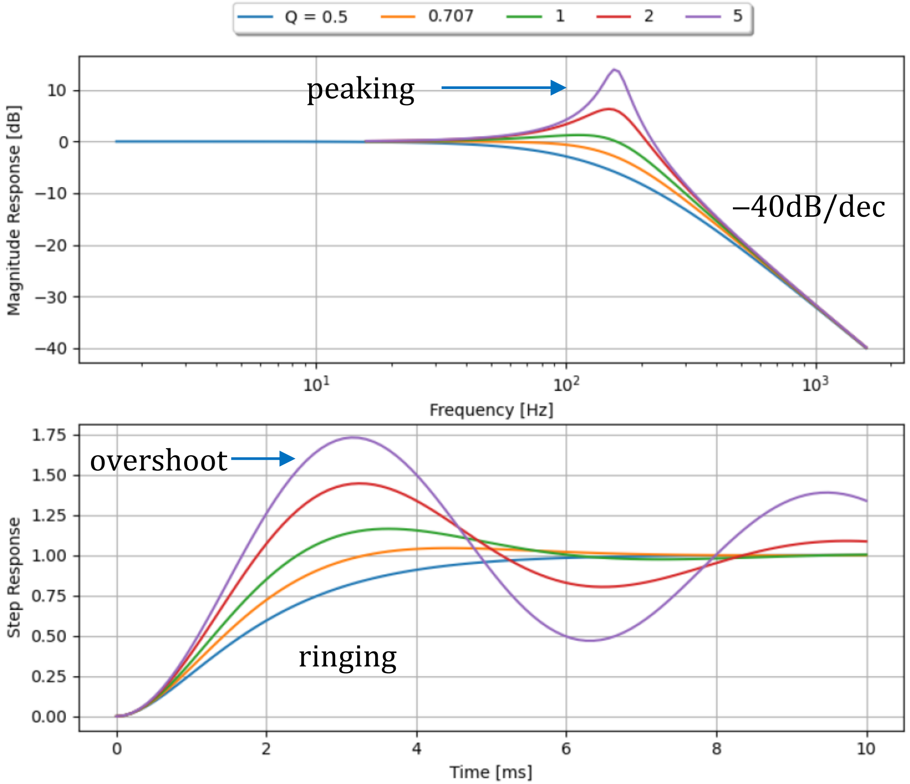
$$H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

where

$$Q = \frac{\omega_0}{2\zeta\omega_0}$$

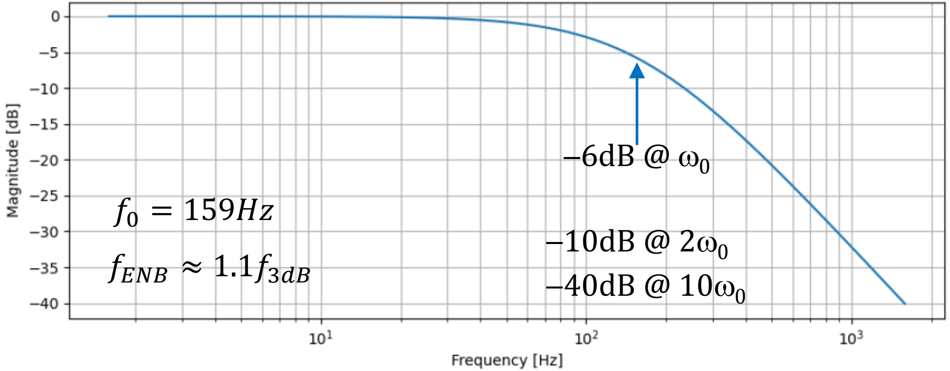
Quality Factor (Q)

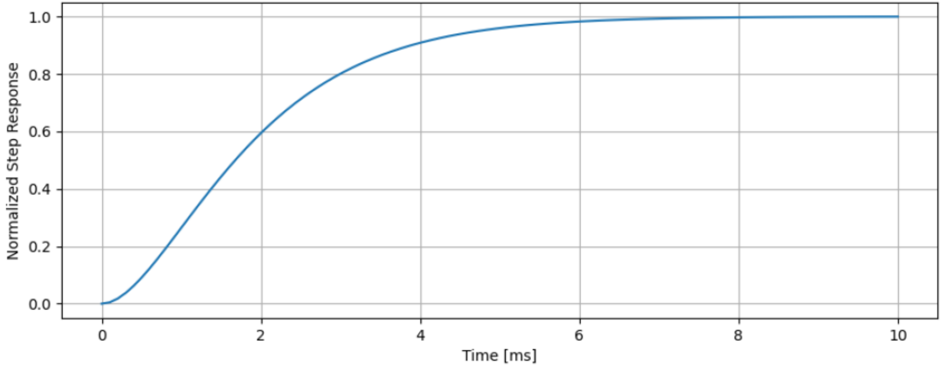
- While ω_0 still somewhat controls the settling of the transient response and the bandwidth of the frequency response, the *quality* of the response is determined by Q
- A higher value of Q results in more *peaking* in the frequency response and more *ringing* in the step response
- Depending on the application, both peaking and ringing (or neither) may be acceptable



2^{nd} Order RC Filter Response

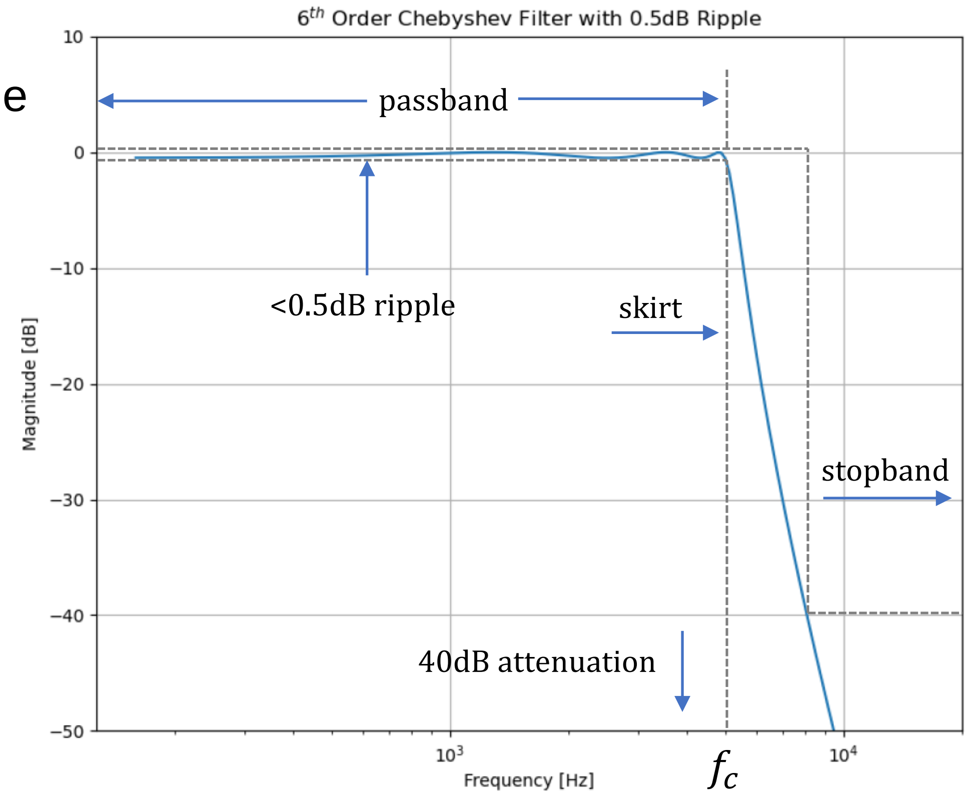
- How does the RC filter cascade with a buffer in between stages perform?
- The Q factor is low (0.5), resulting in an *overdamped* response in the time domain
- The roll-off eventually reaches –40dB/dec, but it does so over a wide frequency range
- Attenuation at the "cutoff" frequency is severe at 6dB, so the useful signal bandwidth low





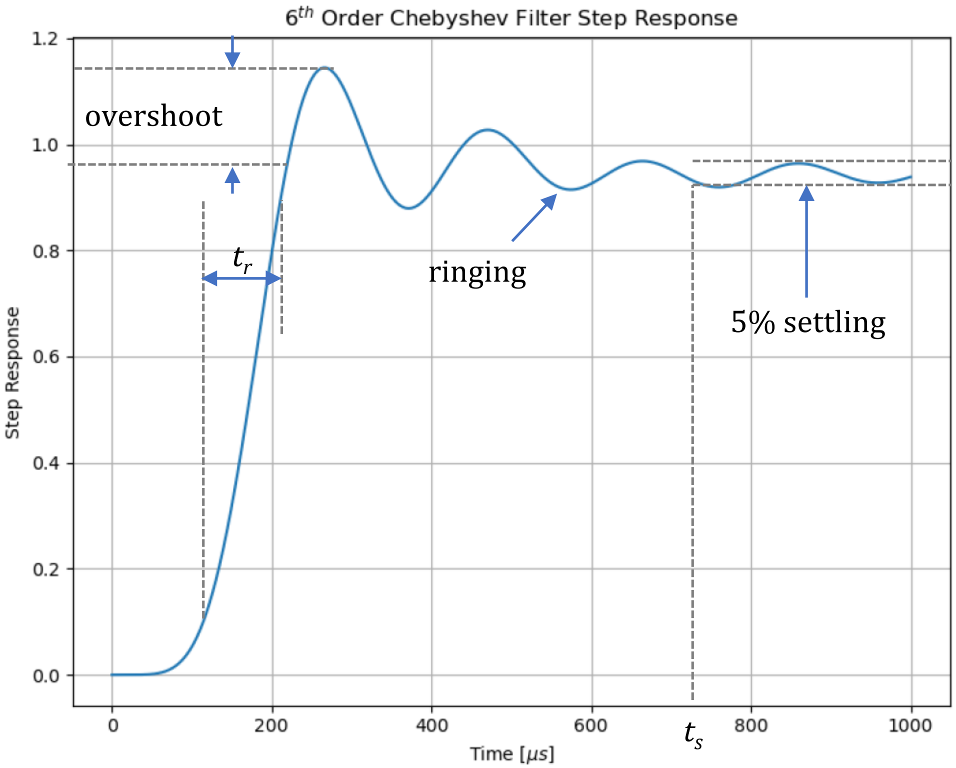
Frequency Domain Specifications

- *Passband*
  - Region of frequencies with little attenuation
- *Cutoff frequency (f_c)*
  - End of the passband
- *Transition region*
  - Transition from passband to stopband
- *Stopband*
  - Defined by a minimum required attenuation above a given frequency



Time-Domain Specifications

- *Rise time* is the time required to go from 10% to 90%
- *Settling time* is the time required to arrive within some specified range of the final value and remain there
- *Overshoot* and *ringing* describe (typically) undesirable characteristics that affect settling
- The step response of a filter may be of particular interest if the signal contains steps/pulses



Butterworth Filter

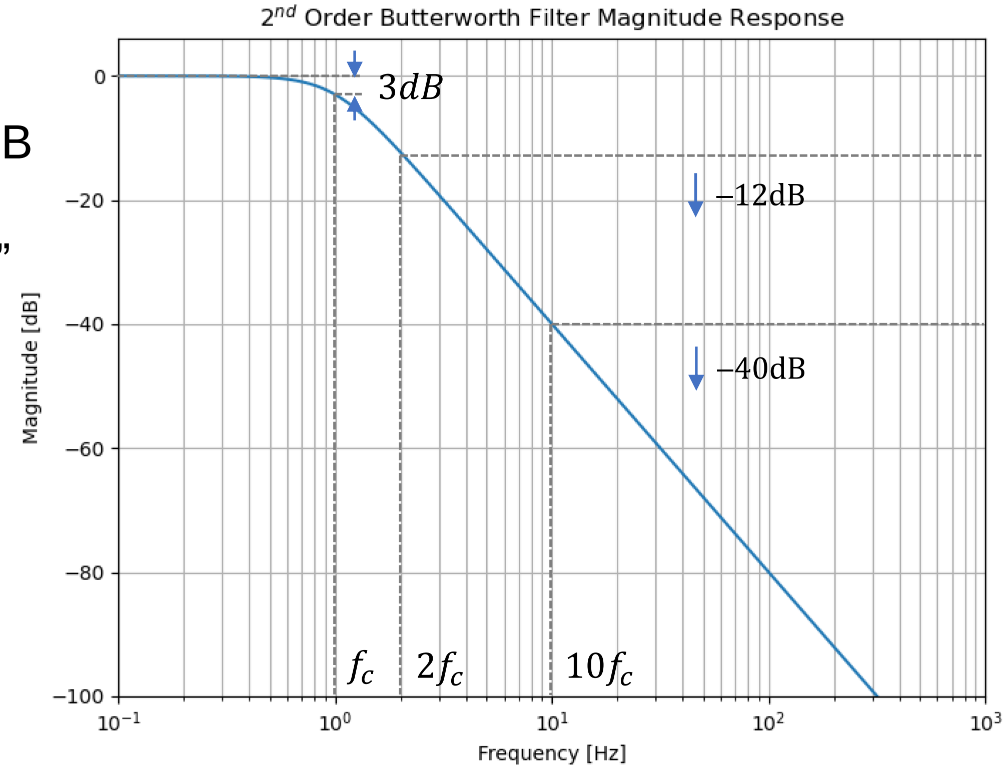
- Designed to ensure minimal variation in the filter gain over the passband
- The magnitude response of a Butterworth filter is given by

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\left(1 + (f/f_c)^{2n}\right)^{\frac{1}{2}}}$$

- n is the order of the filter and f_c is the cutoff frequency, the frequency at which the magnitude has decreased by 3dB
- The Q factor of a second order Butterworth filter is $1/\sqrt{2}$, giving an underdamped response with some peaking in the frequency domain and minor ringing in the time domain

Butterworth Magnitude Response

- No ripple in the passband, and minimal attenuation at low frequencies (approximately 0.1dB up to $f_c/2$)
- However, the passband "droops" early, resulting in $-3dB$ attenuation at f_c
- The stopband begins somewhat late, due to the slow transition from the passband



Chebyshev Filters

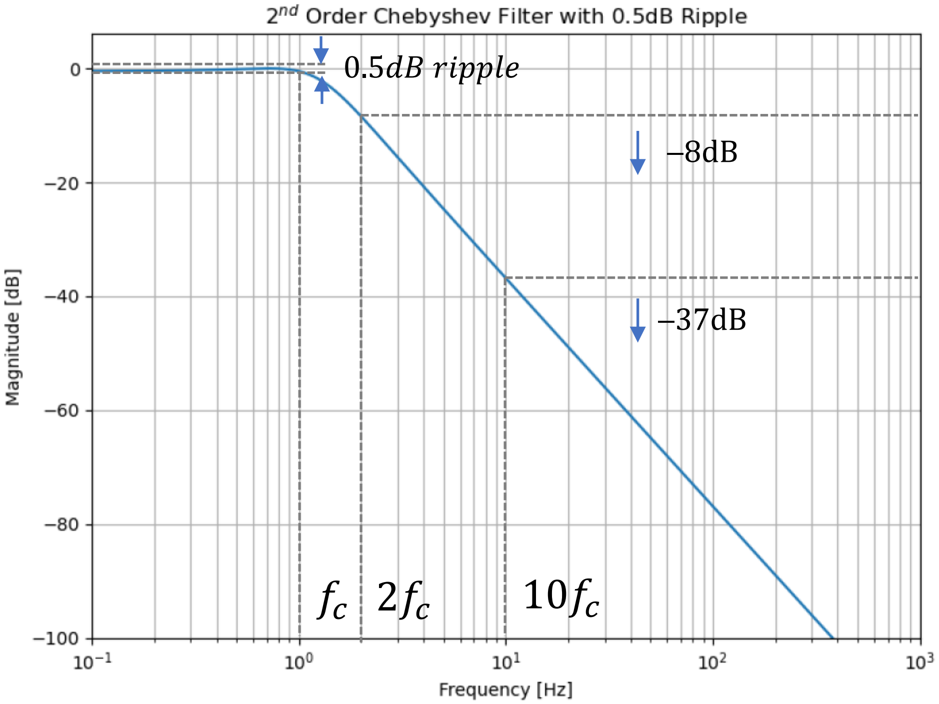
- Maximum flatness in the passband may not be critical, as long as ripple is kept below some critical value (e.g. 0.1 or 0.5 dB)
- Chebyshev filters achieve much steeper roll-off in the transition band at the expense of increased ripple in the passband
- This tradeoff often proves beneficial, as design specifications can be achieved with fewer poles (relative to a Butterworth design)
- The magnitude response of a Chebyshev filter is given by

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\left(1+\varepsilon^2 C_n^2(f/f_c)\right)^{\frac{1}{2}}}$$

where C_n is an n^{th} order Chebyshev polynomial and ε is a constant that sets the passband ripple (2^{nd} order Q factor depends on ε)

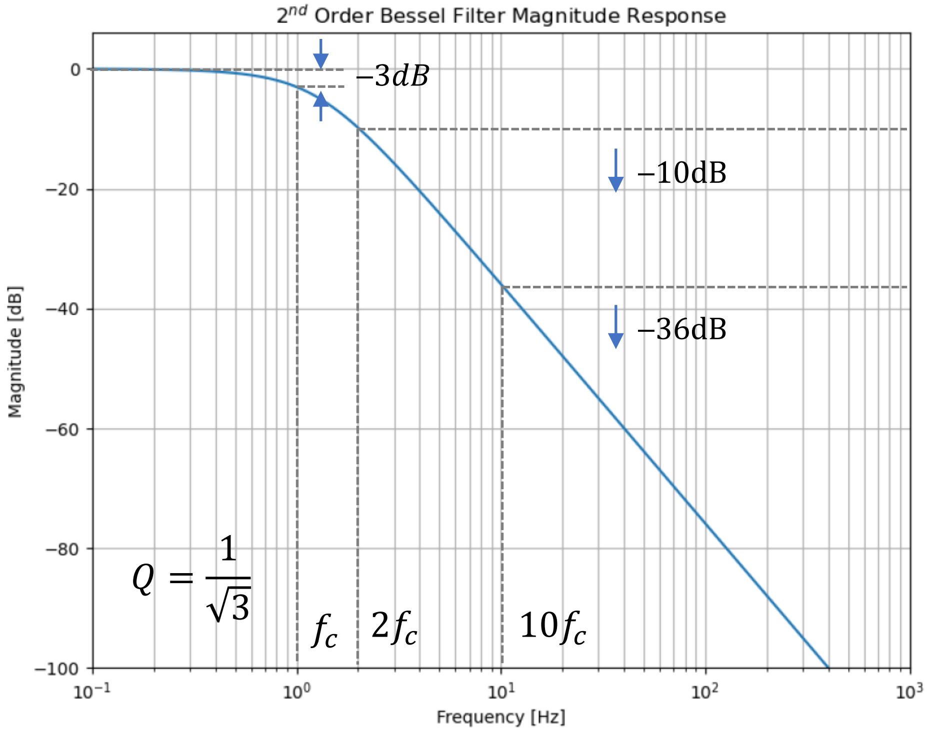
Chebyshev Magnitude Response

- Designed for a sharp transition region, which is a trade-off for ripple in the passband
- For a Chebyshev design, f_c is defined as the frequency at which the response exits the ripple band
- The narrower filter bandwidth can reduce noise and improve SNR (depending on ripple spec and filter order)
- For the 2nd order Chebyshev, the stopband begins later than for the Butterworth design



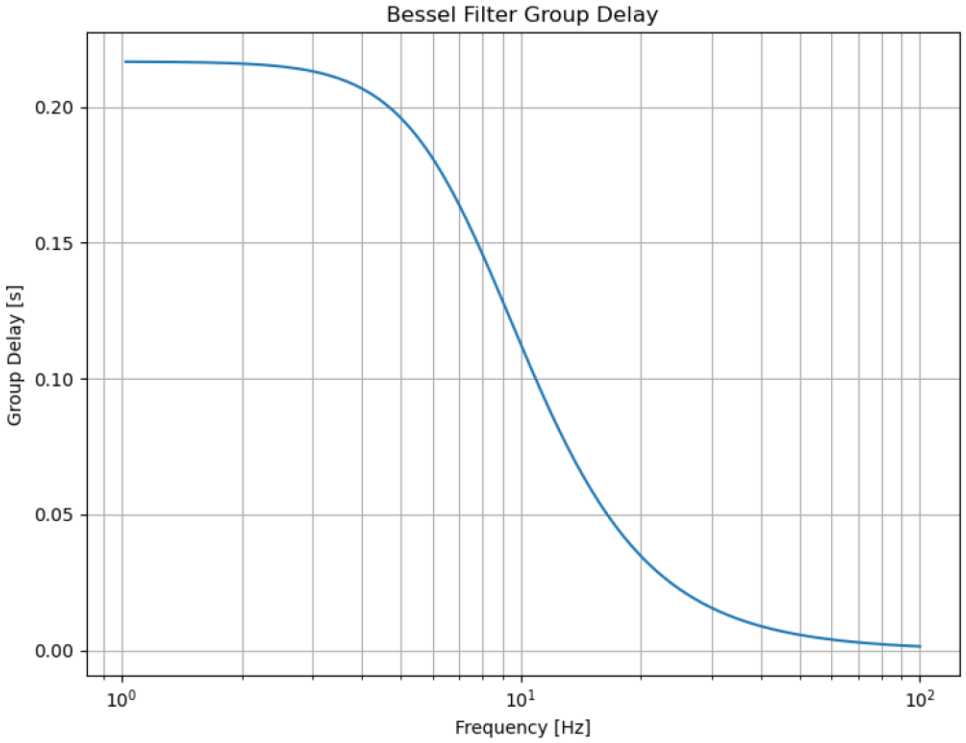
Bessel Magnitude Response

- The magnitude response of the Bessel filter exhibits even more “passband droop” than the Butterworth filter
- f_c for a 2nd order Bessel filter is 1.272 times lower than f_0 for –3dB attenuation at f_c
- The primary advantage of the Bessel filter lies in the phase domain, with a parameter called “group delay”, which ensures equal time delays for different frequencies



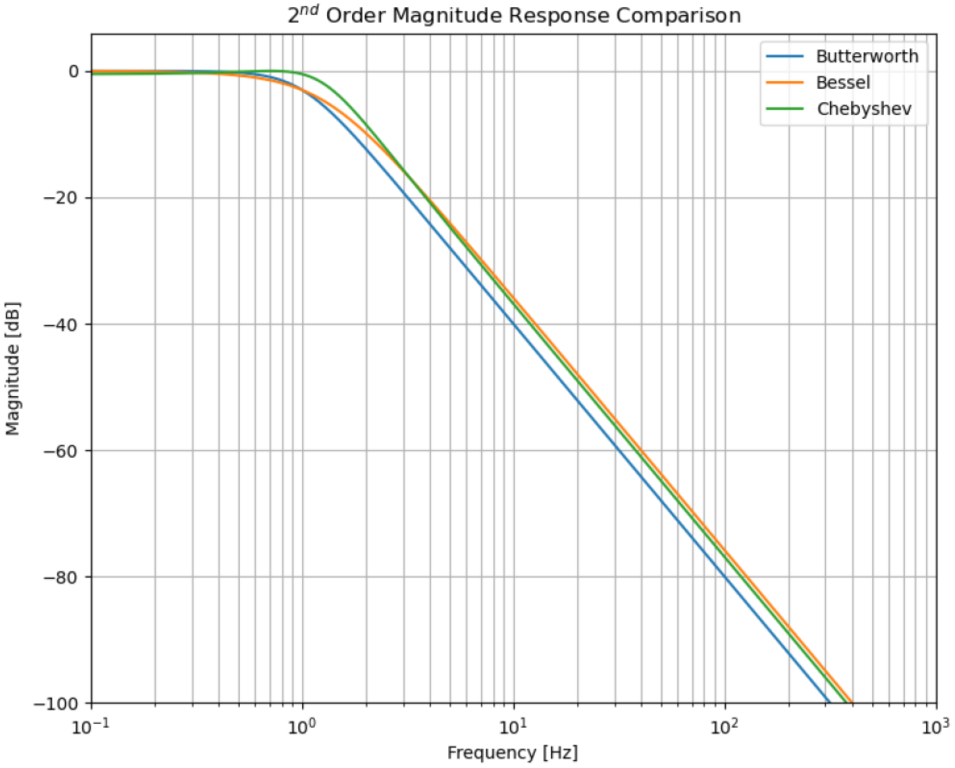
Bessel Group Delay

- Group delay is the time delay for different frequencies through a filter or system
- It is defined as the derivative of phase with respect to frequency
- A constant group delay means that different frequency sinusoids pass through with identical time delays
- Bessel filters exhibit maximally flat group delay, similar to the magnitude of a Butterworth filter



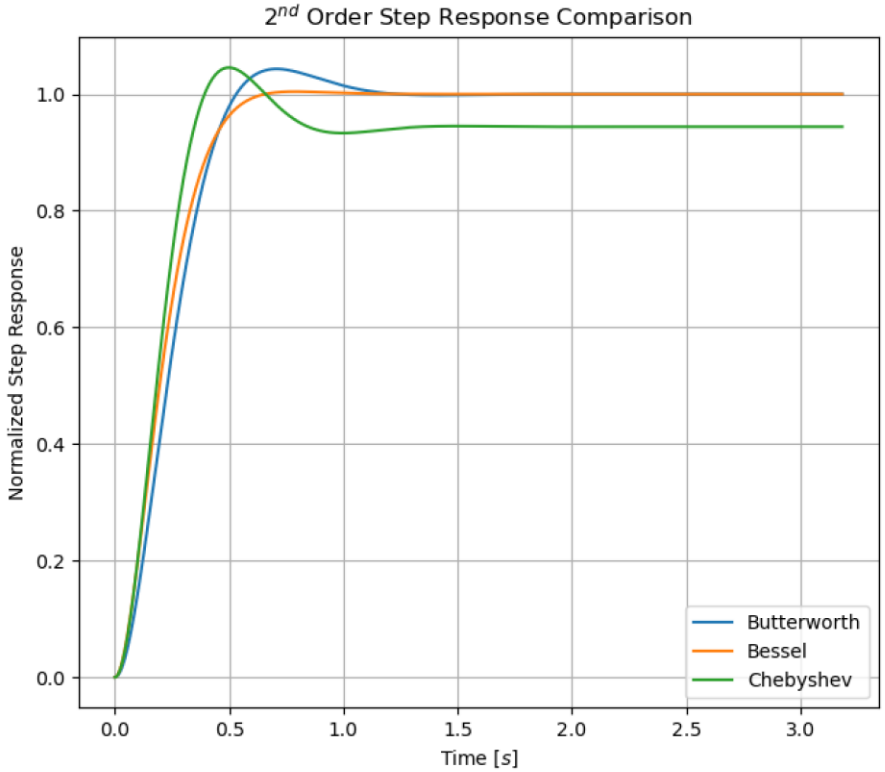
Magnitude Response Comparison

- In the frequency domain, the salient characteristics of the three filters types can be summarized as follows:
  - *Butterworth*: maximally flat, minor peaking
  - *Chebyshev*: widest passband, with ripple as a design parameter
  - *Bessel*: Almost no peaking, slower transition to maximum roll-off



Step Response Comparison

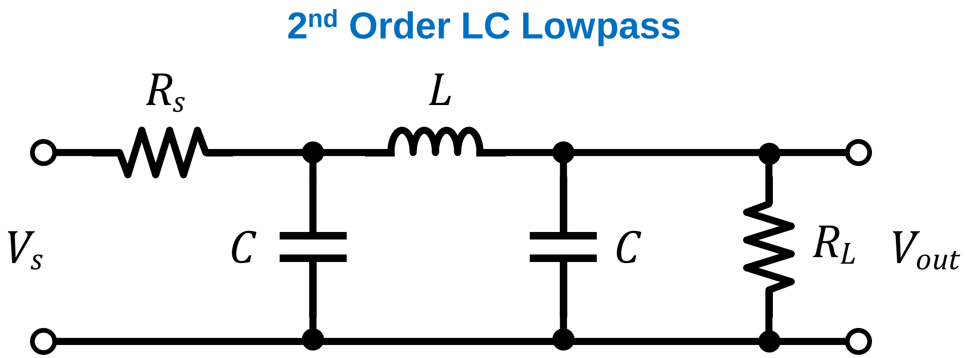
- The Chebyshev trades its sharp transition and wide passband in the for significant ringing in the time domain
- The Bessel filter exhibits almost no overshoot or ringing, and has a slower rise time (t_r)
- The Butterworth filter constitutes the middle ground of the three in both the frequency and the time domain, with moderate peaking (frequency domain) and ringing (time domain)



Filter Performance Comparison

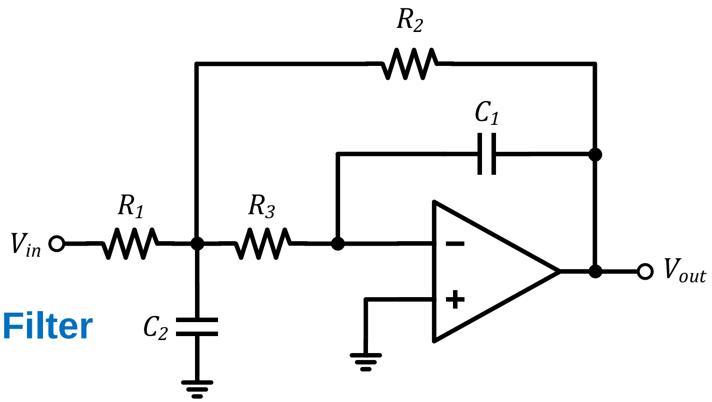
Type	f_{3dB}/f_c	Poles	Rise Time (0 – 90%)	Overshoot (%)	0.1% Settling (s)	Stopband attenuation 2fc	10f_c
Bessel (–3dB at f_c)	1.0	2	$0.4/f_c$	0.4	$1.1/f_c$	10	36
	1.0	4	$0.5/f_c$	0.8	$1.2/f_c$	13	66
	1.0	6	$0.6/f_c$	0.6	$1.2/f_c$	14	92
Butterworth (–3dB at f_c)	1.0	2	$0.4/f_c$	4	$1.7/f_c$	12	40
	1.0	4	$0.6/f_c$	11	$2.8/f_c$	24	80
	1.0	6	$0.9/f_c$	14	$3.9/f_c$	36	120
Chebyshev 0.5dB ripple (–0.5dB at f_c)	1.39	2	$0.4/f_c$	1.1	$1.6/f_c$	8	37
	1.09	4	$0.7/f_c$	3.0	$5.4/f_c$	31	89
	1.04	6	$1.1/f_c$	5.9	$10.4/f_c$	54	141
Chebyshev 2dB ripple (–2dB at f_c)	1.07	2	$0.4/f_c$	21	$2.7/f_c$	15	44
	1.02	4	$0.7/f_c$	28	$8.4/f_c$	37	96
	1.01	6	$1.1/f_c$	32	$16.3/f_c$	60	148

Passive LC Filters



- LC filters are bulky and expensive, and aren't electrically tunable
- Further, they aren't able to achieve gain, but rather exhibit attenuation
- However, they tend to be the only viable option at high frequencies (i.e. above ~100kHz), due to the limited bandwidth of opamps
- For low-frequency applications, active filters are the preferred approach

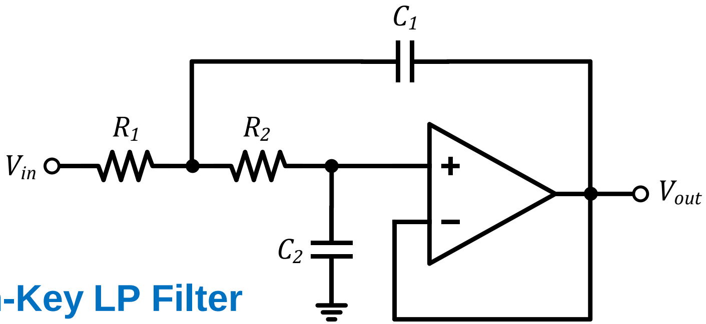
Active Filters



Multi-Feedback Active Filter

- We can synthesize RLC filter characteristics without inductors by using opamps
- Such filters are known as "active" filters, due to the presence of active devices (i.e. opamps)
- Active filters can be used to make lowpass, highpass, bandpass, and band-reject filters, with Butterworth, Bessel, and Chebyshev responses

Sallen–Key Filter

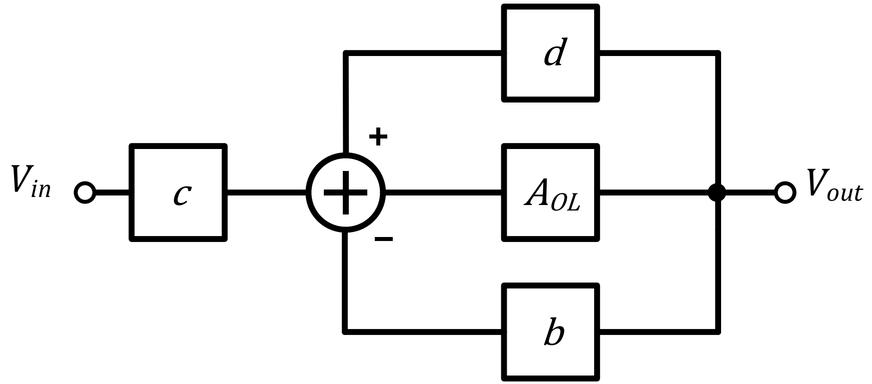


Sallen-Key LP Filter

$$\frac{V_{out}}{V_{in}} = \frac{\frac{1}{R_1 R_2 C_1 C_2}}{s^2 + s\frac{1}{C_1}\left(\frac{R_1 + R_2}{R_1 R_2}\right) + \frac{1}{R_1 R_2 C_1 C_2}}$$

- Similar to the cascade of two RC stages, with the bottom of C_1 “bootstrapped” by the active follower
- The energy loss in the RC cascade is compensated by the opamp, raising the Q factor
- Similar configuration can be used to realize highpass and bandpass filters

Sallen–Key Transfer Function



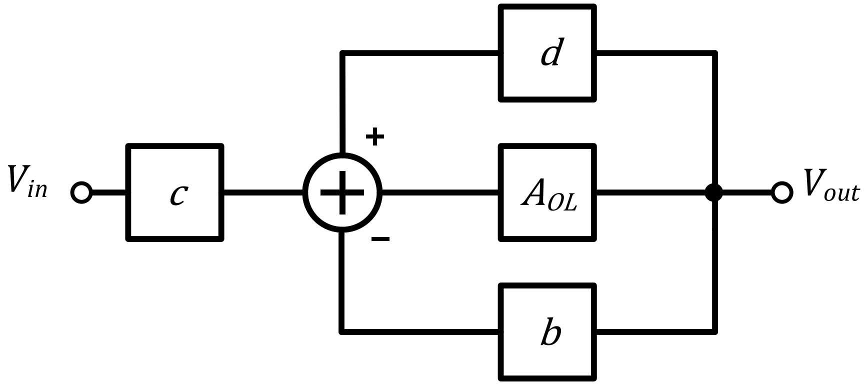
$$V_{out} = A_{OL}[cV_{in} + V_{out}(d - b)]$$

$$A_{OL} \to \infty$$

$$\frac{V_{out}}{V_{in}} = \frac{c}{b - d}$$

- Positive feedback allows the realization of higher Q factors than were possible with a passive filter, and since b > d the *net* feedback is still negative
- c and d can be determined via circuit analysis, using the same techniques used for standard feedback structures (i.e. inverting, non-inverting, etc)

Sallen–Key Transfer Function



$$V_{out} = A_{OL}[cV_{in} + V_{out}(d - b)]$$

$$A_{OL} \to \infty$$

$$\frac{V_{out}}{V_{in}} = \frac{c}{b - d}$$

$$b = 1$$

$$c = \frac{1}{s^2 C_1 C_2 R_1 R_2 + s(R_1 C_2 + R_2 C_1 + R_1 C_1) + 1}$$

$$d = \frac{s C_1 R_1}{s^2 C_1 C_2 R_1 R_2 + s(R_1 C_2 + R_2 C_1 + R_1 C_1) + 1}$$

$$\rightarrow \boxed{\frac{V_{out}}{V_{in}} = \frac{\frac{1}{R_1 R_2 C_1 C_2}}{s^2 + s\frac{1}{C_1}\left(\frac{R_1 + R_2}{R_1 R_2}\right) + \frac{1}{R_1 R_2 C_1 C_2}}}$$

Sallen–Key Transfer Function (2)

- The transfer function as derived using circuit analysis can be cast in the standard form to determine ω_0 and Q
- Component values R_1, R_2, C_1, and C_2 are selected to achieve a desired response (e.g. Butterworth, Chebyshev, etc.)
- Analysis of Sallen–Key highpass and bandpass structures is the same, with the exception that the impedances are different

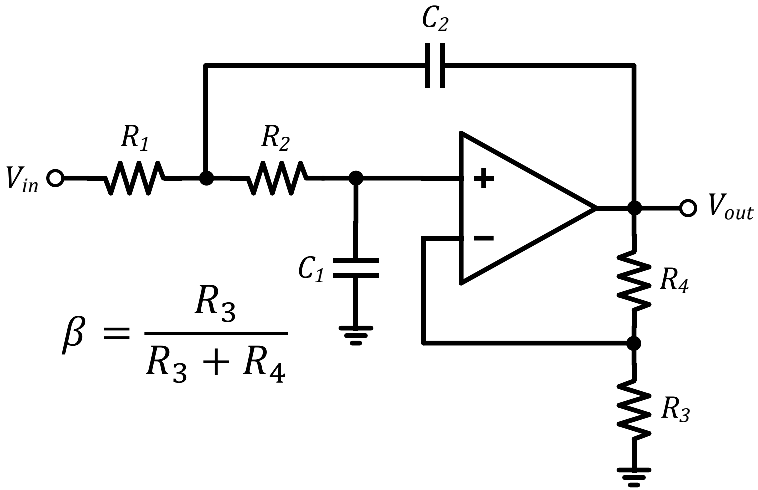
$$\frac{V_{out}}{V_{in}} = \frac{\frac{1}{R_1 R_2 C_1 C_2}}{s^2 + s\frac{1}{C_1}\left(\frac{R_1 + R_2}{R_1 R_2}\right) + \frac{1}{R_1 R_2 C_1 C_2}}$$

$$= \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$\boxed{\begin{gathered} \omega_0 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}} \\ Q = \frac{\omega_0}{2\zeta\omega_0} = \frac{1}{C_1}\frac{1}{\sqrt{R_1 R_2 C_1 C_2}}\left(\frac{R_1 + R_2}{R_1 R_2}\right) \end{gathered}}$$

Sallen–Key Filter Design

- The design process for active filters can be challenging to carry out manually, but second-order structures can be constructed easily if we make certain assumptions
- Component values are determined based on the desired response
- *Caveat*: Here the DC gain is coupled to its response type, which may be unacceptable if a particular DC gain is needed (e.g. unity)



$$\beta = \frac{R_3}{R_3 + R_4}$$

$$C_1 = C_2 = C_{nom}$$

$$R_1 = R_2$$

Sallen–Key Design Process

- To simplify the design procedure, let $C_1 = C_2$ and $R_1 = R_2$
- Solving for the transfer function gives

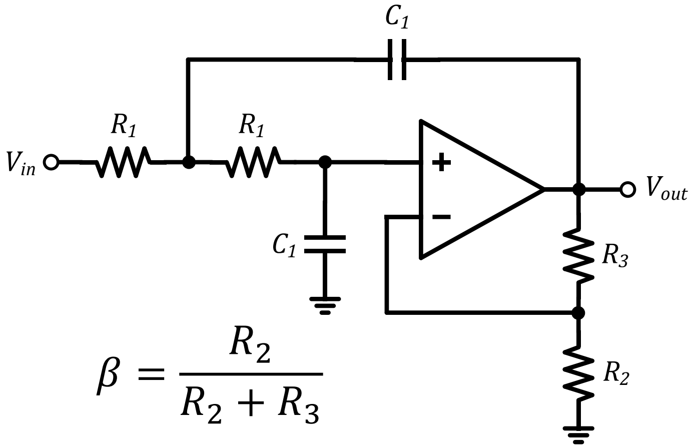
$$\frac{V_{out}}{V_{in}} = \frac{1}{\beta} \frac{1/R_1^2 C_1^2}{s^2 + \frac{s}{R_1 C_1}\left(3 - \frac{1}{\beta}\right) + \frac{1}{R_1^2 C_1^2}} = K \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

- Which results in expressions for ω_0 and Q:

$$\omega_0 = \frac{1}{R_1 C_1} \qquad Q = 1/(3 - 1/\beta)$$

- To achieve a certain response (i.e. Butterworth, Chebyshev, etc.), we can select values for ω_0 and Q and solve for the component values

2^{nd} Order Butterworth Design



$$\beta = \frac{R_2}{R_2 + R_3}$$

$$\omega_0 = 2\pi \cdot 5kHz$$

$$\omega_0 = \frac{1}{R_1 C_1} \qquad Q = 1/(3 - 1/\beta)$$

- Choose C_1 = 10nF and solve for R_1:

$$R_1 = \frac{1}{\omega_0 C_1} \approx 15.9k\Omega$$

- For a Butterworth response, we need $Q = 1/\sqrt{2}$, which leads to

$$\beta = \frac{1}{3 - \sqrt{2}} \approx 0.63$$

- Setting $R_2 = 10k\Omega$ results in $R_3 \approx 5.86k\Omega$, which gives us a gain of ~4dB

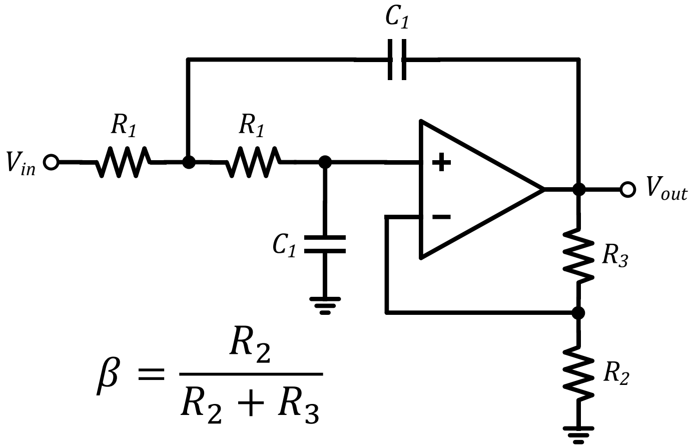
Frequency Scaling Factor

- To achieve other types of responses (i.e. Bessel or Chebyshev), we need to scale ω_0 so the filter provides a given attenuation at ω_c
- A Butterworth filter exhibits 3dB attenuation at ω_0, so $\omega_0 = \omega_c$
- For a Bessel Filter, the 3dB frequency occurs at a frequency *lower than* ω_0, so we need to scale ω_0 in our design equation(s)
- Similarly, for a Chebyshev filter, a *frequency scaling factor* needs to be applied to ω_0 to obtain the appropriate attenuation (e.g. 0.5dB) at ω_c
- To do this, we merely replace ω_0 in our design equations with the relation

$$\omega_0 = c_n \omega_c$$

- The frequency scaling factor c_n is always greater than 1 and depends on the filter order

2^{nd} Order Bessel Design



$$\beta = \frac{R_2}{R_2 + R_3}$$

$$\omega_c = 2\pi \cdot 5kHz = \omega_0/c_n$$

$$\omega_c = \frac{1}{c_n R_1 C_1} \qquad Q = 1/(3 - 1/\beta)$$

- For a second order Bessel, $\omega_c \approx 1.272$
- Choose C_1 = 10nF and solve for R_1:

$$R_1 = \frac{1}{\omega_c C_1} \approx 20.2k\Omega$$

- For a Bessel response, we need $Q = 1/\sqrt{3}$, which leads to

$$\beta = \frac{1}{3 - \sqrt{3}} \approx 0.789$$

- Setting $R_2 = 10k\Omega$ results in $R_3 \approx 2.68k\Omega$, giving a gain of ~2.06dB

Generalized Design Approach

- Depending on the application, the dependence of the filter response on closed-loop gain may be unacceptable
- So, although the previous design procedure is intuitive and simple, it may not be ideal
- A generalized design approach for low-pass filters is presented by James Karki of Texas Instruments here: Active Low-Pass Filter Design
- This approach utilizes ratios of component values to achieve a desired response with arbitrary gain, including a gain of 1
- The basic approach is
    1. Choose ratios m and n to achieve the desired value of Q, and
    2. Select component values to achieve the target value of f_c

Source: SLOA049B: Active Low-Pass Filter Design. Texas Instruments, September 2002. http://www.ti.com/lit/an/sloa049b/sloa049b.pdf?&ts=1589465107385. Accessed 14 May 2020.

2^{nd} Order Butterworth

- Defining parameters m, n, and τ as

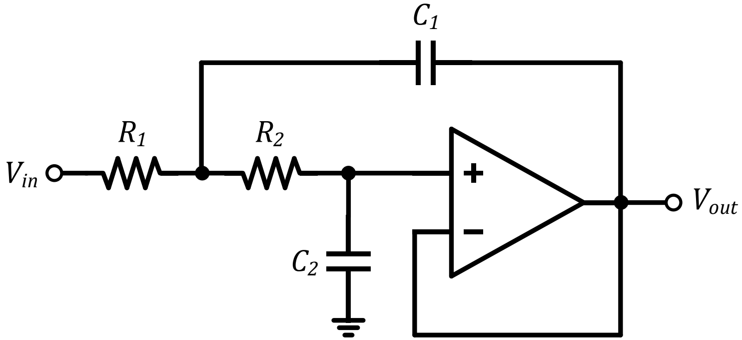
$$m = \frac{R_1}{R_2} \qquad n = \frac{C_1}{C_2} \qquad \tau = R_2 C_2$$

- The 2-pole filter section has a cutoff frequency

$$f_c = \frac{1}{2\pi\tau\sqrt{mn}}$$

- And a Q given by

$$Q = \frac{\sqrt{mn}}{1 + m + mn(1 - K)}$$

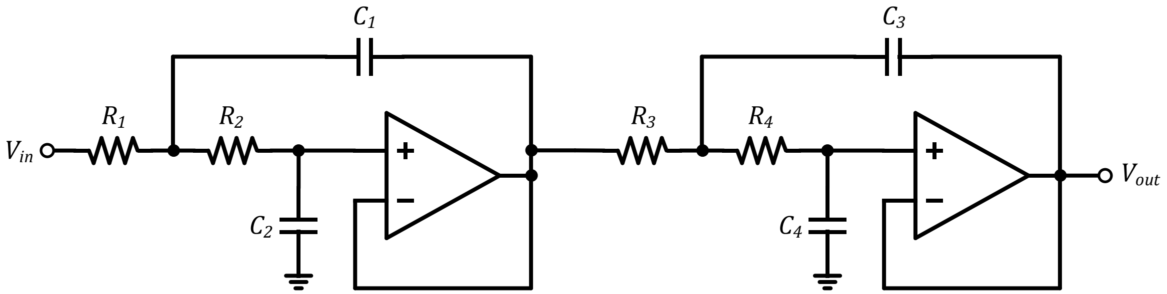


$$Q = \frac{1}{\sqrt{2}} \qquad f_c = 1kHz$$

n	m	C_2	C_1	R_1	R_2
3.3	0.229	0.01uF	0.033uF	4.22k	18.2k

Higher Order Filters

4th Order Sallen–Key



- Higher order and bandpass filters are typically constructed as cascades of 1st and 2nd order stages
- To realize the desired response, ω_c and Q for each stage need to be modified from the values used in a second-order design
- For higher order filters, it is generally *much* easier to use a table-based approach or a filter design tool (TI Filter Design Tool)