Instructor: Jason Silver
Midterm (100 points)
Due Sunday, May 16 (Submit on Canvas as a Jupyter Notebook or PDF)

Show all work

In [1]:
```
import numpy as np
from scipy import integrate
k = 1.38e-23
T = 300
q = 1.602e-19
V_T = k*T/q
```

Problem 1:Noise figure (25 points)

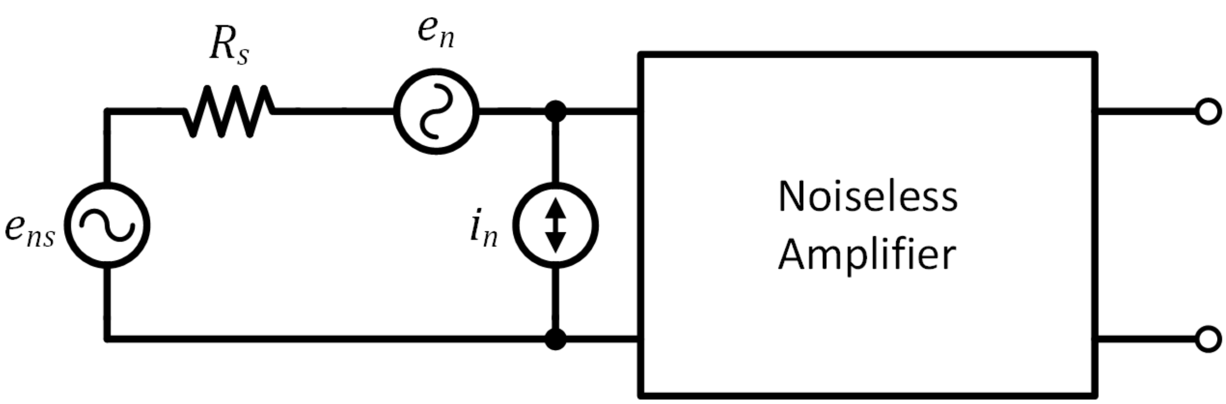


Figure 1. Amplifier noise model

An amplifier datasheet provides noise measurements that show it has a noise figure of $3dB$ for $R_s = 100\Omega$ and $R_s = 10k\Omega$ (same noise figure for both values of R_s) at $T = 300K$ and $f = 10kHz$.

a) Assuming all noise sources are white and e_n and i_n are uncorrelated, determine e_n and i_n at $10kHz$ from the noise figure data. (10 points)

$$\begin{aligned}
\text{NF} &= 10\log F = 10\log\left[1 + \frac{e_n^2 + R_s^2 i_n^2}{e_{ns}^2}\right] \\
\text{NF} &= 3\ \text{dB} \rightarrow F \approx 2 \\
2 &= 1 + \frac{e_n^2 + R_s^2 i_n^2}{e_{ns}^2} \\
1 &= \left.\frac{e_n^2 + R_s^2 i_n^2}{4kTR_s\Delta f}\right|_{R_s = 100\Omega,\ 10k\Omega}
\end{aligned}$$

$$
\begin{aligned}
4kT \cdot 100 = e_n^2 + 100^2 i_n^2 \quad &\text{and} \quad 4kT \cdot 10000 = e_n^2 + 10000^2 i_n^2 \\
4kT = \frac{e_n^2}{100} + 100 i_n^2 \quad &\text{and} \quad 4kT = \frac{e_n^2}{10000} + 10000 i_n^2 \\
\frac{e_n^2}{100} + 100 i_n^2 \quad &= \quad \frac{e_n^2}{10000} + 10000 i_n^2 \\
e_n \quad &= \quad 1000 i_n
\end{aligned}
$$

$$
\begin{aligned}
e_n &= 1.28\text{nV} \\
i_n &= 1.28\text{pV}
\end{aligned}
$$

b) What are the optimum source resistance R_{opt} of the amplifier and the corresponding minimum noise figure NF_{min}? (5 points)

From Notes

- To minimize the noise factor/figure, we can take the first derivative of F with respect to the source impedance and set it equal to zero. Assuming $Z_s = R_s$ (no reactive components), this yields

$$
-\frac{e_n^2}{R_{opt}^2} + i_n^2 = 0
$$

- From which we find the optimum source resistance to be

$$
R_{opt} = \frac{e_n}{i_n}
$$

- The noise factor corresponding to R_{opt} is thus

$$
F_{opt} = 1 + \frac{e_n^2 + R_{opt}^2 i_n^2}{4kTR_{opt}\Delta f} = 1 + \frac{2e_n^2}{4kT\Delta f e_n / i_n} = 1 + \frac{e_n \cdot i_n}{2kT\Delta f}
$$

In [2]:

```python
e_n = 1.28*1e-9
i_n = 1.28*1e-12
```

In [3]:

```python
Rs = e_n/i_n
F = 1 + (e_n*i_n)/(2*k*T)
NF = 10*np.log10(F)
print(f'Rs = {round(Rs)}, F = {round(F,2)}, Noise Figure = {round(NF,2)} dB')
```

```
Rs = 1000, F = 1.2, Noise Figure = 0.78 dB
```

$$
\begin{aligned}
R_{opt} &= \frac{e_n}{i_n} = 1000\Omega \\
F_{opt} &= 1 + \frac{e_n \cdot i_n}{2kT\Delta f} \\
&= 1 + \frac{(1.28 \cdot 10^{-9}) \cdot (1.28 \cdot 10^{-12})}{2kT} = 1.2 \\
NF_{min} &= 10 \log(1.2) = 0.78\text{dB}
\end{aligned}
$$

c) Upon measuring the output noise of the amplifier with $R_s = 0$, you find that it has a $1/f$ noise corner of $1kHz$. What is the noise figure at $100Hz$ for $R_s = R_{opt}$? (10 points)

In [4]:

```
e_n = 1.28*1e-9
i_n = 1.28*1e-12
```

In [5]:

```
Rs = e_n/i_n
F = (1 + (e_n*i_n)/(2*k*T))*(1+1000/100)
NF = 10*np.log10(F)
print(f'Rs = {round(Rs)}, F = {round(F,2)}, Noise Figure = {round(NF,2)} dB')
```

```
Rs = 1000, F = 13.18, Noise Figure = 11.2 dB
```

$$
\begin{aligned}
\text{NF} &= 10\log\left[F_{opt}(1+\frac{f_c}{\Delta f})\right] \text{ where } \Delta f = 100\text{Hz} \\
&= 10\log\left[(1+\frac{e_n \cdot i_n}{2kT})(1+\frac{f_c}{\Delta f})\right] \\
&= 10\log\left[(1+\frac{0.12810^{-6} \cdot 0.12810^{-9}}{2kT})(1+\frac{1000}{100})\right] \\
&= 10\log[13.18] \\
&= 11.2 \text{ dB}
\end{aligned}
$$

Problem 2: Noise in opamp circuits (25 points)

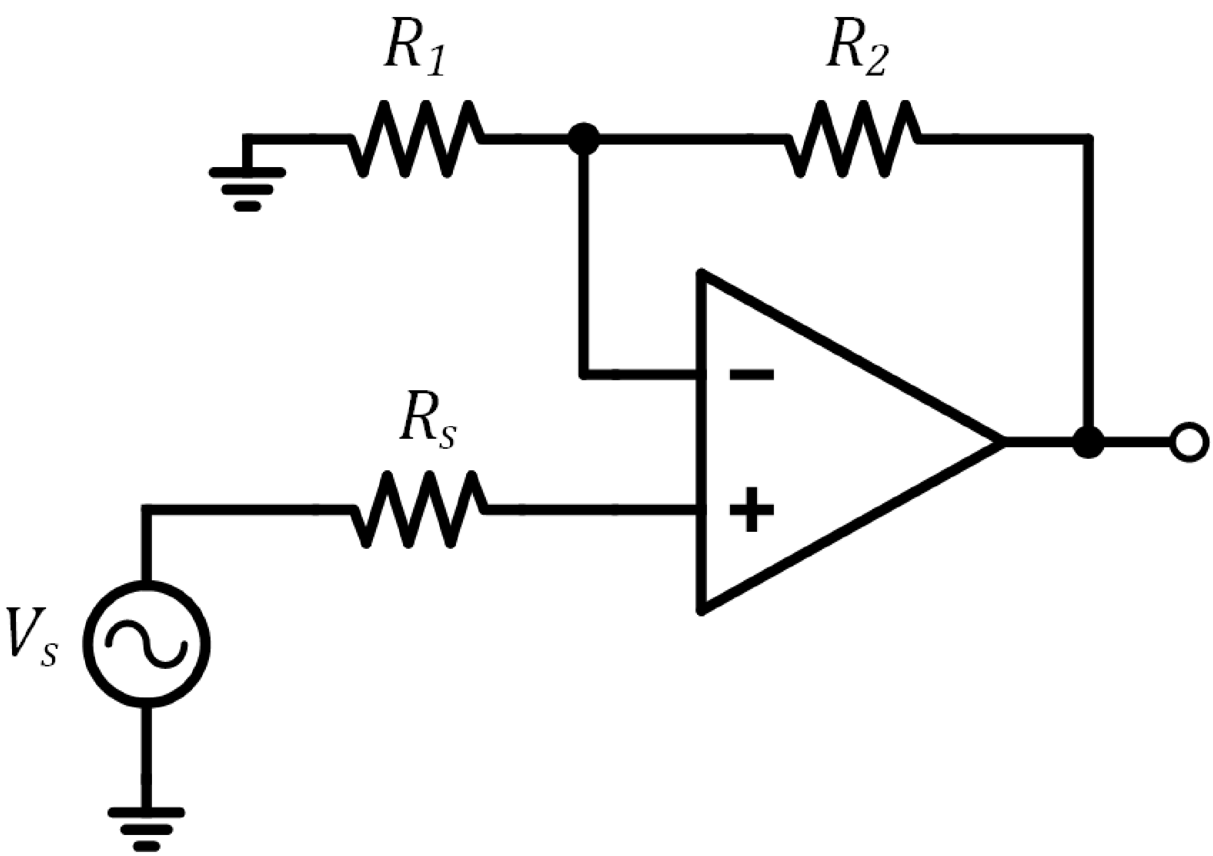


Figure 2. Non-inverting amplifier with source resistance

For the non-inverting amplifier in Fig. 2, suppose you have a choice between two opamps. Opamp A has $e_{na} = 10nV/\sqrt{Hz}$ and $i_{na} = 10pA/\sqrt{Hz}$, while opamp B has $e_{na} = 5nV/\sqrt{Hz}$ and $i_{na} = 0.5pA/\sqrt{Hz}$.

a) What is the optimum source resistance for each opamp and its corresponding minimum noise figure? If $R_s = 1k\Omega$, which opamp should you use to minimize noise figure? (10 points)

$$
\begin{aligned}
G &= 1 + \frac{R_2}{R_1} \\
e_{n,out}^2 &= 4kTR_sG^2 + e_n^2G^2 + 4kTR_2 + 4kTR_1(\frac{{R_2}^2}{{R_1}^2}) + i_n^2R_s^2G^2 + i_n^2R_2^2 \\
e_{ns}^2 &= 4kTR_s \\
F &= \frac{4kTR_sG^2 + e_n^2G^2 + 4kTR_2 + 4kTR_1(\frac{{R_2}^2}{{R_1}^2}) + i_n^2R_s^2G^2 + i_n^2R_2^2}{4kTR_s} \\
\frac{\partial F}{\partial R_s} &= -\frac{e_n^2G^2}{R_{opt}^2} - \frac{R_2}{R_{opt}^2} - \frac{(\frac{{R_2}^2}{R_1})}{R_{opt}^2} + i_n^2G^2 - \frac{i_n^2R_2^2}{R_{opt}^2} = 0 \\
R_{opt} &= \frac{e_n^2G^2 + R_2 + (\frac{{R_2}^2}{R_1}) + i_n^2R_2^2}{i_n^2G^2} \\
R_{opt}\Big|_{R_1,R_2=noiseless} &= \frac{e_n^2G^2}{i_n^2G^2} = \frac{e_n}{i_n} \\
F_{opt} &= 1 + \frac{e_n^2 + R_{opt}^2 i_n^2}{4kTR_{opt}} = 1 + \frac{e_n \cdot i_n}{2kT}
\end{aligned}
$$

In [6]:

```python
e_n = 10*1e-9
i_n = 10*1e-12
```

In [7]:

```python
Rs = e_n/i_n
F = 1 + (e_n*i_n)/(2*k*T)
NF = 10*np.log10(F)
print(f'Rs = {round(Rs)}, F = {round(F,2)}, Noise Figure = {round(NF,2)} dB')
```

```
Rs = 1000, F = 13.08, Noise Figure = 11.17 dB
```

$$
\begin{aligned}
R_{optA} &= \frac{10 \cdot 10^{-9}}{10 \cdot 10^{-12}} = 1K\Omega \\
F_{optA} &= 1 + \frac{(10 \cdot 10^{-9}) \cdot (10 \cdot 10^{-12})}{2kT} = 13.08 \\
NF_{minA} &= 10 \log{[13.08]} = 11.17 \text{ dB} \\
NF\Big|_{R_s=1000} &= NF_{minA}
\end{aligned}
$$

In [8]:

```python
e_n = 5*1e-9
i_n = 0.5*1e-12
```

In [9]:

```
Rs = e_n/i_n
F = 1 + (e_n*i_n)/(2*k*T)
NF = 10*np.log10(F)
print(f'Rs = {round(Rs)}, F = {round(F,2)}, Noise Figure = {round(NF,2)} dB')
```

```
Rs = 10000, F = 1.3, Noise Figure = 1.15 dB
```

In [10]:

```
Rs = 1000
F = 1 + ( e_n**2 + (Rs**2) * (i_n**2) )/(4*k*T*Rs)
NF = 10*np.log10(F)
print(f'Rs = {round(Rs)}, F = {round(F,2)}, Noise Figure = {round(NF,2)} dB')
```

```
Rs = 1000, F = 2.52, Noise Figure = 4.02 dB
```

$$
\begin{aligned}
R_{opt\,B} &= \frac{5 \cdot 10^{-9}}{0.5 \cdot 10^{-12}} = 10K\Omega \\
F_{opt\,B} &= 1 + \frac{(5 \cdot 10^{-9}) \cdot (0.5 \cdot 10^{-12})}{2kT} = 1.30 \\
NF_{min\,B} &= 10 \log [1.30] = 1.15 \text{ dB} \\
NF\Big|_{R_s=1000} &= 10 \log \left[1 + \frac{e_n^2 + R_s^2 i_n^2}{4kTR_s} \right] \\
&= 10 \log \left[1 + \frac{(5 \cdot 10^{-9})^2 + 1000^2 (0.5 \cdot 10^{-12})^2}{4kT \cdot 1000} \right] \\
&= 4.02 \text{dB}
\end{aligned}
$$

Response: I would choose opamp B, because it still has a lower noise figure even though it's not at optimal source impedance.

---

b) You found an even better opamp with $e_{na} = 2nV/\sqrt{Hz}$ and $i_{na} = 1pA/\sqrt{Hz}$ and decided to use it instead of the first two. What is the noise figure of the amplifier if $R_1 = 500\Omega$ and $R_2 = 4.5k\Omega$? (7.5 points)

In [11]:

```
e_n = 2*1e-9
i_n = 1*1e-12
Rs = 1000 # e_n/i_n
R1 = 500
R2 = 4500
G = 1 + R2/R1
```

In [12]:

```
F = ( 4*k*T*Rs*G**2 + (e_n**2)*(G**2) + 4*k*T*R2 + \
      4*k*T*R1*(R2**2)/(R1**2) + (i_n**2)*(Rs**2)*(G**2) + \
      (i_n**2)*(R2**2) )/(4*k*T*Rs)
NF = 10*np.log10(F)
print(f'Rs = {round(Rs)}, F = {round(F,2)}, Noise Figure = {round(NF,2)} dB')
```

```
Rs = 1000, F = 176.42, Noise Figure = 22.47 dB
```

Assume $R_s = R_{opt}$

$$G = 1 + \frac{R_2}{R_1} = 1 + \frac{4500}{500} = 10$$

$$F = \frac{4kTR_sG^2 + e_n^2G^2 + 4kTR_2 + 4kTR_1(\frac{R_2{}^2}{R_1{}^2}) + i_n^2R_s^2G^2 + i_n^2R_2^2}{4kTR_s}$$

$$F = \frac{(4kT \cdot 1000 \cdot 10^2) + (2 \cdot 10^{-9})^2 10^2 + (4kT \cdot 4500) + 4kT \cdot 500(\frac{4500^2}{500^2}) + (1 \cdot 10^{-12})^2 1000^2 10^2 + (1 \cdot 10^{-}}{4kT(1000)}$$

$$F = 176.42$$

$$NF = 10 \log [147.27] = 22.47 \text{ dB}$$

c) If the transit frequency of the opamp is $10MHz$ and its $1/f$ corner is $10kHz$, what is the signal-to-noise ratio for an input signal given by $v_{in} = v_a \cdot \sin(\omega_0 t)$, where $v_a = 1mV$ and $\omega_0 = 2\pi \cdot 1kHz$? (7.5 points)

In [13]:

```
e_nout_sq = 4*k*T*Rs*G**2 + (e_n**2)*(G**2) + 4*k*T*R2 + \
      4*k*T*R1*(R2**2)/(R1**2) + (i_n**2)*(Rs**2)*(G**2) + \
      (i_n**2)*(R2**2)
print(f'Power spectral density = {round(e_nout_sq,18)} V^2/Hz')
```

```
Power spectral density = 2.921e-15 V^2/Hz
```

In [14]:

```
beta = 1/10
f_enb = (np.pi/2)*(beta)*(10*1e6)
print(f'Noise bandwidth = {round(f_enb)} Hz')
```

```
Noise bandwidth = 1570796 Hz
```

In [15]:

```
v_srms = 0.707*1e-3
v_nrms = 7.07*1e-6
SNR = 20*np.log10(v_srms/v_nrms)
print(f'Signal-to-noise ratio = {round(SNR,2)} dB')
```

```
Signal-to-noise ratio = 40.0 dB
```

$$
\begin{aligned}
SNR &= 20\log\frac{v_{s(rms)}}{v_{n,in(rms)}} \\
v_{s(rms)} &= 0.707mV \\
v_{n,in(rms)} &= \frac{\sqrt{e_n^2\cdot f_{enb}}}{1/\beta} \\
1/\beta &= G = 10 \\
{e_n}^2 &= 4kTR_sG^2 + e_n^2G^2 + 4kTR_2 + 4kTR_1(\frac{{R_2}^2}{{R_1}^2}) + i_n^2R_s^2G^2 + i_n^2R_2^2 \\
{e_n}^2\Big|_{(1/f)noise} &= e_n^2(1+\frac{f_c}{\Delta f}) \\
f_{enb} &= \frac{\pi}{2}f_{3dB} \approx \frac{\pi}{2}\beta f_T \\
&= 1.57\text{ MHz} \\
v_{n,in(rms)} &= \frac{\sqrt{\int_1^{1.57MHz}(2.92\cdot 10^{-15})(1+\frac{f_c}{f})df}}{10} \text{ where } f_c = 10\text{ kHz} \\
&= 7.07\mu\text{V} \\
SNR &= 20\log\left[\frac{0.707\cdot 10^{-3}}{7.07\cdot 10^{-6}}\right] = 20\log[100] \\
&= 40\text{ dB}
\end{aligned}
$$

Problem 3: Low-noise amplifier design (25 points)

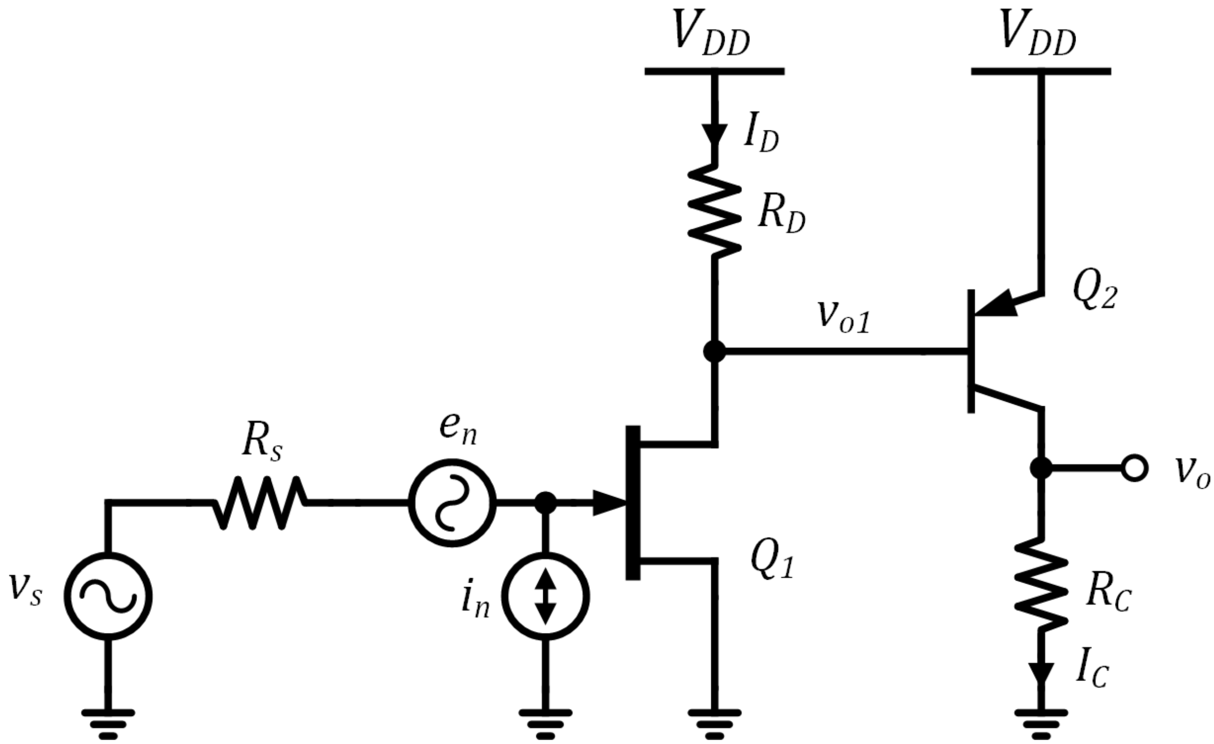


Figure 3. CS-CE Amplifier

As shown in Fig. 3, a common-source amplifier can be combined with a common-emitter amplifier for high input impedance and high gain (biasing circuitry not shown).

Unless otherwise specified, assume only drain current thermal noise for Q_1 and base/collector shot noise for Q_2 (i.e. ignore $1/f$ noise and assume $i_n = 0$ for Q_1 and $r_b = 0$ for Q_2). Ignore all capacitances except C_L.

For both transistors, assume $r_o \to \infty$.

Use $\gamma = 2/3$, $T = 300K$, and $\beta = I_C/I_B = 200$ for Q_2 for your calculations.

a) Noise measurements of Q_1 reveal that $e_n = 3nV/\sqrt{Hz}$ for $I_D = 100\mu A$. What is the corresponding transconductance efficiency, g_m/I_D? (5 points)

In [16]:

```python
I_D = 100 * 1e-6
e_n = 3*1e-9
gamma = 2/3
gm = 4*k*T*gamma/(e_n**2)
print(f'Transconductance gm = {round(gm,4)}')
print(f'Transconductance efficiency gm/I_D = {round(gm/I_D,2)}')
```

```
Transconductance gm = 0.0012
Transconductance efficiency gm/I_D = 12.27
```

$$\begin{aligned}
I_D &= 100 \cdot 10^{-6} \\
{e_n}^2 &= \frac{{i_{nd}}^2}{{g_m}^2} = \frac{4kT\gamma g_m}{{g_m}^2} = \frac{4kT\gamma}{g_m} \\
g_m &= \frac{4kT\gamma}{{e_n}^2} = \frac{4kT(2/3)}{(3 \cdot 10^{-9})^2} = 0.0012 \\
\frac{g_m}{I_D} &= \frac{0.0012}{100 \cdot 10^{-6}} = 12.27
\end{aligned}$$

b) Assuming the g_m/I_D value determined in part **a)**, determine values for I_D and R_D that give an input-referred voltage noise density of the common-source stage to be $e_{n1} = 1nV/\sqrt{Hz}$ and a voltage gain of $20dB$. (10 points)

In [17]:

```python
R_D = 10/gm
print(f'R_D = {round(R_D,2)} Ohms')
```

```
R_D = 8152.17 Ohms
```

In [18]:

```python
I_D = (4*k*T*(2/3))/(12.27 * (1*1e-9)**2)
print(f'I_D = {round(I_D,6)} A')
```

```
I_D = 0.0009 A
```

$$\begin{aligned}
|A_v| &= g_m(r_0 \parallel R_D) \approx g_m R_D \\
10 &= g_m R_D \\
R_D &= \frac{10}{g_m} = \frac{10}{0.0012} = 8152.17\Omega
\end{aligned}$$

Assume only drain current thermal noise

$$
\begin{aligned}
e_n^2 &= \frac{i_{nd}^2}{g_m^2} = \frac{4kT\gamma}{g_m} \\
e_n^2 &= \frac{4kT\gamma}{(g_m/I_D)\cdot I_D} \\
(1nV/\sqrt{Hz})^2 &= \frac{4kT\gamma}{(g_m/I_D)\cdot I_D} \\
(1nV/\sqrt{Hz})^2 &= \frac{4kT(2/3)}{(12.27)\cdot I_D} \\
I_D &= \frac{4kT(2/3)}{(12.27)\cdot(1nV/\sqrt{Hz})^2} \\
I_D &= 0.9\ \text{mA}
\end{aligned}
$$

---

c) Design the common-emitter stage (i.e. determine I_C and R_C) for a gain of $20dB$ and such that with the addition of the second stage the total input-referred *voltage* noise is only 1% higher than the $1nV/\sqrt{Hz}$ target. (10 points)

In [19]:
```python
e_n2 = np.sqrt(10**2 * ((1.01*1e-9)**2-(1*1e-9)**2))
print(f'e_n2 = {round(e_n2,12)} V/sqrt(Hz)')
```

```
e_n2 = 1.418e-09 V/sqrt(Hz)
```

In [20]:
```python
beta = 200
I_C = (4*k*T)/((1.42*1e-9)**2) * (V_T/2 + V_T/(2*beta))
print(f'I_C = {round(I_C,5)} A')
```

```
I_C = 0.00011 A
```

In [21]:
```python
R_C = 10*V_T/I_C
print(f'R_C = {round(R_C,2)} Ohms')
```

```
R_C = 2423.15 Ohms
```

$$
\begin{aligned}
r_e &= \frac{1}{g_m} = \frac{V_T}{I_C} \\
V_T &= 26\ \text{mV}
\end{aligned}
$$

$$
\begin{aligned}
e_{n,in1} &= 1nV/\sqrt{Hz} \\
\sqrt{e_{n,in1}^2 + (\frac{e_{n,in2}}{A_{c1}})^2} &= 1.01nV/\sqrt{Hz} \\
e_{n,in1}^2 + (\frac{e_{n,in2}}{A_{c1}})^2 &= (1.01nV/\sqrt{Hz})^2 \\
{e_{n,in2}}^2 &= {A_{c1}}^2 \cdot [(1.01nV/\sqrt{Hz})^2 - (1nV/\sqrt{Hz})^2] \\
e_{n,in2} &= \sqrt{10^2 \cdot (2.01 \cdot 10^{-20})} V/\sqrt{Hz} \\
e_{n,in2} &= 1.42nV/\sqrt{Hz}
\end{aligned}
$$

Assume only base/collector shot noise

$$
\begin{aligned}
e_n^2 &= i_{nc}^2 + i_{nb}^2 \\
e_n^2 &= 2qI_C r_e^2 + \frac{2qI_C}{\beta} r_e^2 \\
e_n^2 &= 4kT\left(\frac{r_e}{2} + \frac{r_e}{2\beta}\right) = 4kT\left(\frac{1}{2g_m} + \frac{1}{2\beta g_m}\right) \\
(1.42nV/\sqrt{Hz})^2 &= 4kT\left(\frac{1}{2g_m} + \frac{1}{2\beta g_m}\right) \\
(1.42nV/\sqrt{Hz})^2 &= 4kT\left(\frac{V_T}{2I_C} + \frac{V_T}{2\beta I_C}\right) \\
(1.42nV/\sqrt{Hz})^2 &= \frac{4kT}{I_C}\left(\frac{V_T}{2} + \frac{V_T}{2(200)}\right) \\
I_C &= \frac{4kT}{(1.42nV/\sqrt{Hz})^2}\left(\frac{26\cdot 10^{-3}}{2} + \frac{26\cdot 10^{-3}}{2(200)}\right) \\
I_C &= 0.11\cdot 10^{-3} = 0.11\text{ mA}
\end{aligned}
$$

$$
\begin{aligned}
|A_v| &= g_m(r_0 \parallel R_C) \approx g_m R_C \\
10 &= g_m R_C \\
10 &= \frac{I_C}{V_T} R_C = \frac{I_C}{26\cdot 10^{-3}} R_C \\
R_C &= \frac{26\cdot 10^{-2}}{I_C}\Omega \\
R_C &= \frac{26\cdot 10^{-2}}{0.11\cdot 10^{-3}}\Omega \\
R_C &= 2423\Omega
\end{aligned}
$$

Problem 4: DC-coupled differential amplifier (25 points)

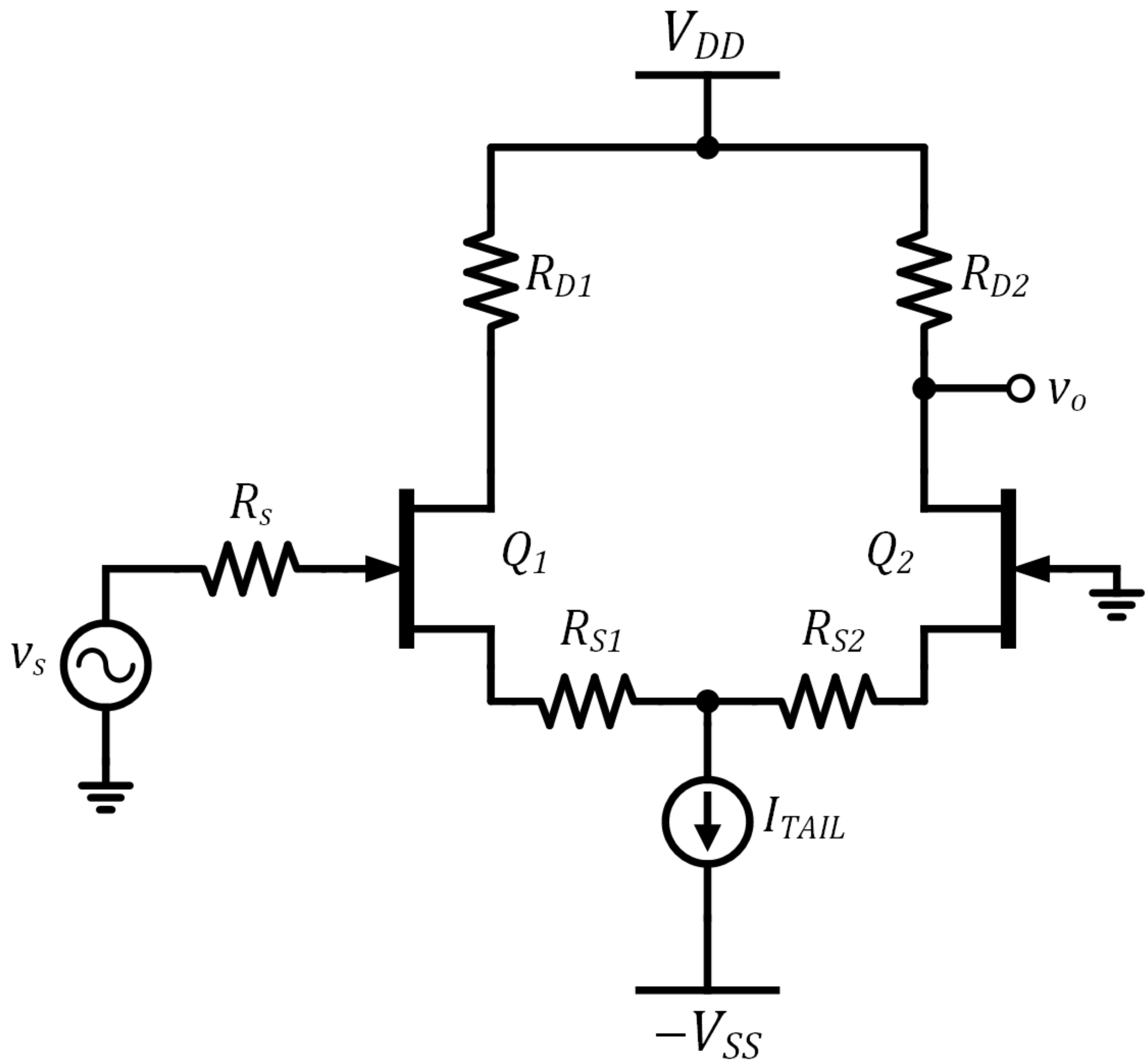


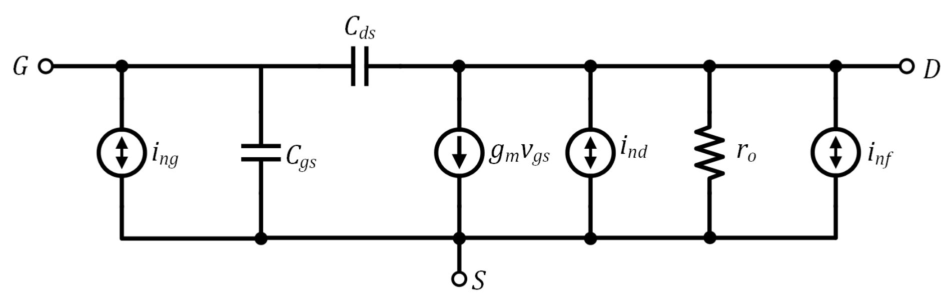
Figure 4. DC-coupled differential amplifier

The JFET differential pair in Fig. 4 is to be used for a DC-coupled sensor application.

I_{TAIL} is an ideal current source with $R_{out} \to \infty$.

Unless otherwise specified, assume only drain current thermal noise for Q_1 and Q_2 (i.e. ignore $1/f$ noise and assume $i_{ng} = 0$ for Q_1 and Q_2). Ignore all capacitances.

For both transistors, assume $r_o \to \infty$, $\gamma = 2/3$, and $g_m/I_D = 10S/A$. $T = 300K$.



a) Determine an expression for the input-referred voltage noise density of the amplifier, e_n, in terms of R_D, R_{S1}, R_{S2}, I_{TAIL}, γ, and kT. Do NOT assume balanced operation for noise analysis. (15 points)

Assume $i_{ng} = 0$

$$\begin{aligned}
e_{n,in} &= \frac{e_{n,out}}{|A_v|} \\
|A_v| &\approx \frac{R_D}{2R_S} \\
e_{n,out} &= \sqrt{R_D^2 \cdot (i_{nd_1}^2 + i_{nRD_2}^2)} \\
e_{n,out} &= \sqrt{R_D^2 \cdot (4kT\gamma g_{m1} + \frac{4kT}{R_{D_2}})} \\
e_{n,in} &= \frac{\sqrt{R_D^2 \cdot (4kT\gamma g_{m1} + \frac{4kT}{R_{D_2}})}}{|A_v|}
\end{aligned}$$

b) Calculate the input-referred noise density if $I_{tail} = 1mA$, $R_D = 10k\Omega$, and $R_{S1} = R_{S2} = 1k\Omega$. (10 points)

$$\begin{aligned}
I_D &\approx I_{tail} \\
|A_v| &\approx \frac{R_D}{2R_S} = 5 \\
g_m &= \frac{g_m}{I_D} I_D \\
&= 10 \cdot 1\text{mA} \\
&= 0.01S
\end{aligned}$$

In [22]:
```
gamma = (2/3)
Itail = 1*1e-3
R_D = 10*1e3
Rs1 = 1000
Rs2 = 1000
G = R_D/Rs1/2
gm = 0.01
```

In [23]:
```
e_nout = np.sqrt(R_D**2 * (4*k*T*gamma*gm + 4*k*T/R_D))
e_nin = e_nout/G
print(f'Input-referred noise density = {round(e_nin,12)} V/sqrt(Hz)')
```

```
Input-referred noise density = 2.1171e-08 V/sqrt(Hz)
```

In [ ]:
```

```