7 to 1, one millihenry transformer. Two calculated values of rise time can be obtained for checking purposes. The first value is for a no-load condition ($G_L=0$). By interpolation from Fig. 4, a 7 to 1, one millihenry transformer corresponds to an x of about 1.8. This corresponds to a rise time of 0.13 microsecond which gives the point labeled (1) on Fig. 11(b). The second value of rise time is for a loaded condition. Knowing the turns ratio and the transistor parameters, (17) can be solved for the value of x. This value of x substituted into (16) gives a value of G_L. The value of x can be used to solve for the rise time with this particular load. Thus, a second point can be calculated and it is labeled (2) on Fig. 11(b).

Previously it was shown that a 5 to 1 turns ratio resulted in the fastest rise time for the unloaded case. If the circuit is loaded as shown in Fig. 11(a), the rise time will become slower because the load robs some of the feedback current. However, by increasing the turns ratio the rise time may be shortened by restoring some of the feedback current. From (16) and (17) it is possible to calculate either the optimum turns ratio for a given load or the optimum load for a given turns ratio. Fig. 11(b) shows that the calculated and experimental value of optimum load for a given turns ratio are about the same.

Eq. (20) shows how the pulse width changes with the load conductance. Experimental measurements confirm this variation.

Theory of Shot Noise in Junction Diodes and Junction Transistors*

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Summary—A theory of shot noise in junction diodes and transistors is presented, based upon a transmission line analogy. The noise is caused by the randomness in the diffusion of the minority carriers and the randomness in the recombination of minority and majority carriers. In transmission line analogy, first process corresponds to distributed noise emf's in series with transmission line, and second process corresponds to distributed noise current generators in parallel to transmission line. After magnitude of these equivalent distributed noise sources has been determined, noise problem is solved by standard methods. It is found that theoretical results for a junction diode agree with experimental work published earlier.

An equivalent circuit describing the noise in transistors is presented. It contains two partially correlated noise current generators i_{p1} and i_{p2}, i_{p1} connected across the emitter junction and i_{p2} connected across the collector junction. $\langle i_{p1}^2\rangle_{av}$, $\langle i_{p2}^2\rangle_{av}$ and $\langle i_{p1}^* i_{p2}\rangle_{av}$ are calculated. At low frequencies emitter and collector current show full shot effect and i_{p1} and i_{p2} are almost completely correlated; at higher frequencies the correlation becomes less complete.

The noise can also be described by an emf e_s in series with the emitter and a current generator i_p in parallel to the collector junction. $\langle e_s^2\rangle_{av}$, $\langle i_p^2\rangle_{av}$ and $\langle e_s^* i_p\rangle_{av}$ are calculated. At low frequencies e_s and i_p are uncorrelated, but some correlation is expected at higher frequencies. Low-frequency results resemble earlier equivalent circuits of Montgomery, Clark, and van der Ziel, and of Giacoletto.

Introduction

In THE past, diffusion problems have been solved occasionally with the help of a transmission line analogy. Recently Dr. D. O. North, RCA Laboratories, showed the author that the same method could be extended to noise problems in diodes and transistors.[1]

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[1] D. O. North, 1955 Conference on Semiconductor Device Research, Philadelphia, Pa.; June, 1955; and earlier discussions.

This idea, for which Dr. North deserves the credit, is applied in this paper. To put the paper in a self-consistent form and to help readers that are unfamiliar with the method, we first develop the analogy, then apply it to derive known diode and transistor formulas, and finally extend it to noise problems.

The noise mechanisms discussed in this paper do not refer to the IF noise mechanism which predominates at low frequencies. For good transistors the IF noise is important below 1,000 cycles, for poorer transistors its influence extends even to higher frequencies. The noise mechanisms discussed here are thus those that predominate at higher frequencies, at which the influence of the IF noise is negligible.

I. The Transmission Line Analogy of Diffusion Problems in Semiconductors

The differential equations for a distributed line with negligible distributed self-inductance are

$$\partial E/\partial x = -RI,$$
$$\partial I/\partial x = -GE - C\partial E/\partial t, \tag{1}$$

where E is the voltage, I the current and R, G, and C the resistance, conductance and capacitance per unit length. We shall show that the diffusion equations for current carriers in a semiconductor can be written in a form that is identical with (1).

Consider the one-dimensional diffusion of minority carriers injected into semiconducting material, such as occurs in junction diodes and in the base region of junction transistors. Let "drift" of the minority carriers be negligible in comparison with "diffusion." Further let the material be n-type, so that the minority carriers are holes. Let the material have a cross-sectional area A and a length w. If we integrate over the cross-sectional area A, we have the diffusion equations

$$\partial p/\partial t = -(p - p_n)/\tau_p - (1/e)\partial i_p/\partial x$$
$$i_p = -eD_p \partial p/\partial x, \tag{2}$$

where D_p is the hole diffusion coefficient, τ_p the hole lifetime, e is the electronic charge, i_p is the hole current and p_n is the equilibrium hole concentration, whereas p is the actual hole concentration (both per unit length, of course). Introducing the excess hole concentration $p' = (p - p_n)$, we rewrite (2) as

$$\partial p'/\partial x = -(1/eD_p)i_p$$
$$\partial i_p/\partial x = -(e/\tau_p)p' - e\partial p'/\partial t, \tag{2b}$$

which shows a complete analogy with (1). We thus have the following correspondences: E corresponds to the excess hole concentration p', I corresponds to the hole current i_p, R corresponds to $(1/eD_p)$ G corresponds to (e/τ_p) and C corresponds to e. To solve the problem of hole flow in n-type material, we solve the corresponding transmission line problem by standard methods and then translate back to the original problem.

Two important quantities in transmission line theory are the characteristic impedance $Z_0 = R^{1/2}/(G + j\omega C)^{1/2}$ and the propagation constant $\gamma = R^{1/2}(G + j\omega C)^{1/2}$. In the diffusion problem the corresponding quantities are

$$Z_0 = [e^2 D_p(1 + j\omega\tau_p)/\tau_p]^{-1/2}:$$
$$\gamma = a + jb = [(1 + j\omega\tau_p)/D_p\tau_p]^{1/2}. \tag{2c}$$

We shall call these quantities the *characteristic impedance* and the *propagation constant* of the semiconductor. For low frequencies these quantities reduce to:

$$Z_0 = Z_{00} = [e^2 D_p/\tau_p]^{-1/2} \quad \gamma = \gamma_0 = (D_p\tau_p)^{-1/2}, \tag{2d}$$

so that (2*c*) may be written

$$Z_0 = Z_{00}/(1 + j\omega\tau_p)^{1/2}: \quad \gamma = \gamma_0(1 + j\omega\tau_p)^{1/2}. \tag{2e}$$

II. Application to p-n Junctions

Consider a p-n junction in which the current is all carried by holes and in which the n-region is infinitely long for all practical purposes (that is, the length w of the n-region is such that practically none of the holes injected at one end reach the electrode at the other end). We put the origin of our co-ordinate system at the point at which the holes are injected into the n-region. If no voltage is applied to the junction, the concentration at $x = 0$ is equal to the equilibrium concentration p_n. If a voltage V is applied to the diode, the hole concentration at $x = 0$ becomes:

$$p = p_n \exp(eV/kT). \tag{3a}$$

The hole concentration follows the applied voltage almost instantaneously. If in particular a dc voltage V_0 and a small ac voltage $V_1 \exp(j\omega t)$, $(eV_1/kT \ll 1)$ are applied, we may write

$$p = p_n + p_0' + p_1' \exp(j\omega t), \tag{3b}$$

where

$$p_0' = p_n[\exp(eV_0/kT) - 1]:$$
$$p_1' = p_n(eV_1/kT)\exp(eV_0/kT) \tag{3c}$$

For an infinite transmission line of characteristic impedance Z_0 an applied voltage E gives rise to an input current E/Z_0. Applying this directly to our p-n junction, we have for the dc hole current at $x = 0$,

$$I_p = p_0'/Z_{00} = I_{p0}[\exp(eV_0/kT) - 1]:$$
$$I_{p0} = ep_n(D_p/\tau_p)^{1/2}. \tag{4}$$

In the same way we have for the ac hole current at $x = 0$,

$$i_p = I_1 \exp(j\omega t) = p_1' \exp(j\omega t)/Z_0$$
$$= G_0(1 + j\omega\tau_p)^{1/2} V_1 \exp(j\omega t). \tag{5}$$

Consequently, ac admittance of junction becomes

$$Y = G + jB = I_1/V_1 = G_0(1 + j\omega\tau_p)^{1/2}, \tag{6a}$$

where G_0 is the dc conductance of the junction

$$G_0 = (e/kT)e(D_p/\tau_p)^{1/2} p_n \exp(eV_0/kT)$$
$$= (e/kT)(I_p + I_{p0}). \tag{6b}$$

Eq. (6a) may be rewritten as

$$G = G_0[\tfrac{1}{2}(1 + \omega^2\tau_p^2)^{1/2} + \tfrac{1}{2}]^{1/2}$$
$$B = G_0[\tfrac{1}{2}(1 + \omega^2\tau_p^2)^{1/2} - \tfrac{1}{2}]^{1/2}. \tag{6c}$$

Finally, we obtain for the dc hole concentration in the n-region,

$$p'(x) = p_0' \exp(-\gamma_0 x) = p_0' \exp[-x/(D_p\tau_p)^{1/2}]. \tag{7}$$

We have thus derived all well-known formulas of junction diode theory.

III. Application to Junction Diode Noise Problems

Let the junction be short-circuited as far as ac is concerned, then, according to (3c), the ac hole concentration at $x = 0$ is zero. The fluctuating hole *current* at $x = 0$ is not zero, however, and the noise current in the external short-circuit is equal to it. We thus have to calculate the mean square value of this fluctuating hole current at $x = 0$.

The noise in the p-n junction is caused by two mechanisms: fluctuations in the hole concentration at all points of the n-region due to diffusion, and to recombination. Both mechanisms are independent; their effects should, therefore, be added quadratically. For that reason, both effects are discussed separately. We assume again that all current is carried by holes and the n-region is infinitely long for all practical purposes.

We first turn to the random processes of hole-electron pair creation and recombination. We split the semiconductor into sections of length Δx. According to (2b), the above process corresponds to a dc hole current ($\partial p'/\partial t = 0$ for stationary current flow):

$$e(p' + p_n)\Delta x/\tau_p - ep_n\Delta x/\tau_p,$$

disappearing between x and $(x+\Delta x)$. The first term is due to hole-electron pair recombination and the second term is due to hole-electron pair creation. Since pair recombination and pair creation are independent random processes, the two currents fluctuate independently: because the individual events are independent and random, and because the chance of recombination within the interval Δx is very small, one would expect full shot effect for these currents. The mean square value of the Fourier component Δi_{px} is, therefore,

$$\langle \Delta i_{px}^2 \rangle_{av} = 2edf[e(p' + p_n)\Delta x/\tau_p + ep_n\Delta x/\tau_p] \quad (8)$$
$$= 2e^2 df(p' + 2p_n)\Delta x/\tau_p.$$

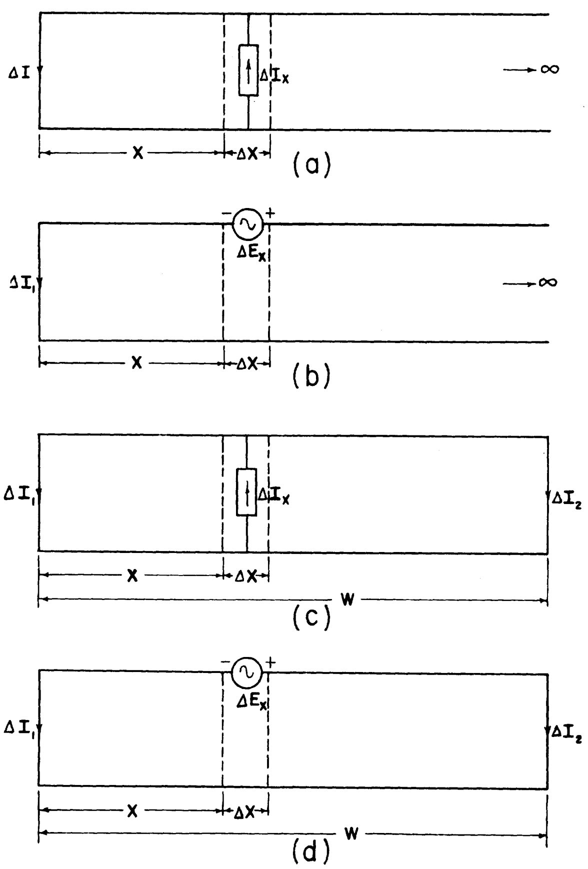

Fig. 1—Transmission line equivalent circuits of semi-conductor noise. (a) Equivalent circuit of recombination fluctuations in a junction diode. (b) Equivalent circuit of diffusion fluctuations in a junction diode. (c) Equivalent circuit of recombination fluctuations in a transistor. (d) Equivalent circuit of diffusion fluctuations in a transistor.

In our transmission line analogy, a current generator ΔI_x connected across the line at a distance x from the input gives rise to a current ΔI in the lead short-circuiting the input (Fig. 1(a)):

$$\Delta I = \Delta I_x \exp(-\gamma x).$$

Applying this to our $p-n$ junction, we find for the mean square value of the hole current in the lead short-circuiting the input, due to the hole current generator Δi_{px},

$$\langle \Delta i_p^2 \rangle_{av} = \langle \Delta i_{px}^2 \rangle_{av} \exp(-2ax)$$
$$= [2e^2 df(p' + 2p_n)\Delta x/\tau_p] \exp(-2ax), \quad (9a)$$

where a is the real part of γ:

$$a = [(1 + \omega^2\tau_p^2)^{1/2} + 1]^{1/2}/(2D_p\tau_p)^{1/2} = \gamma_0(G/G_0). \quad (9b)$$

We now turn to the effect of the random diffusion processes. We split the semi-conductor into sections of length Δx and calculate the density fluctuations δp in these sections. Making a Fourier analysis of these fluctuations, we find for the mean square value of the Fourier components Δp_x of δp:

$$\langle \Delta p_x^2 \rangle_{av} = 4(p' + p_n)\Delta x df/D_p. \quad (10)$$

An indirect proof of this formula is given at the end of this section.

In our transmission line analogy a fluctuating emf ΔE in series with line between x and $(x+\Delta x)$ produces a current ΔI in lead short-circuiting input [Fig. 1(b)]:

$$\Delta I = (\Delta E/Z_0) \exp(-\gamma x). \quad (11)$$

Therefore, the mean square value of the Fourier component of the fluctuating hole current in the lead short-circuiting the input of the junction is

$$\langle \Delta i_p^2 \rangle_{av} = [(\Delta p_x^2)_{av}/|Z_0|^2] \exp(-2ax)$$
$$= [4e^2 df(p' + p_n)\Delta x/\tau_p](1 + \omega^2\tau_p^2)^{1/2}$$
$$\cdot \exp(-2ax). \quad (12)$$

We now add (9a) and (12) and integrate with respect to x. This yields for the mean square value of the Fourier component i_p of the hole current fluctuation at $x=0$ (equal to the current fluctuation in the external circuit):

$$\langle i_p^2 \rangle_{av} = (2e^2 df/\tau_p) \left[\int_0^\infty (p'+2p_n) \exp(-2ax)dx \right.$$
$$\left. + 2(1+\omega^2\tau_p^2)^{1/2} \int_0^\infty (p'+p_n) \exp(-2ax)dx \right]$$
$$= 2edf[I_p(-1+2G/G_0) + I_{p0}(2G/G_0)]$$
$$= 4kTGdf - 2eI_p df. \quad (13a)$$

The noise is thus represented by a current generator i_p in parallel to the junction. The final result in (13a) follows from substituting (7), putting

$$(1+\omega^2\tau_p^2)^{1/2} = -1 + 2(G/G_0)^2; \quad \gamma_0 + 2a = \gamma_0(1+2G/G_0);$$
$$Z_{00}/(\gamma_0\tau_p) = 1/e \quad (13b)$$

and finally substituting (4).

Equating

$$\langle i_p^2 \rangle_{av} = 2eI_{eq}df. \quad (14a)$$

we find for the equivalent saturated diode current I_{eq} of the junction,

$$I_{eq} = I_p(-1 + 2G_1G_0) + I_{p0}(2G/G_0). \quad (14b)$$

For low frequencies $G \simeq G_0$, and (14b) reduces to

$$I_{eq} = (I_p + 2I_{p0}). \quad (14c)$$

This allows a very simple interpretation; the noise is just as if two currents (I_p+I_{p0}) and $-I_{p0}$ were flowing across the p-n barrier, each showing full shot effect.[2] That this interpretation gives the correct result, however, is accidental; it does not describe the physical mechanism and it would give erroneous results at higher frequencies. Equating

$$\langle i_p{}^2\rangle_{av} = n\cdot(4kTGdf), \tag{15a}$$

we find, for the noise ratio of the junction,

$$n = 1 - (1/2)(G_0/G)I_p/(I_p + I_{p0}). \tag{15b}$$

For low frequencies $G\simeq G_0$, and (15b) reduces to the well-known formula

$$n = n_0 = (1/2)(I_p + 2I_{p0})/(I_p + I_{p0}), \tag{15c}$$

which follows immediately from combining (14c) and (6b). At very high frequencies the noise ratio rises from the low-frequency value n_0 to the limiting value $n_\infty=1$. This agrees very well with the experimental results.[3] The calculations thus give a simple explanation of the fact that n is relatively independent of frequency and also explain the finer details of the experiments.

For $I_p=0$ we find $n=1$ at all frequencies, as would be expected from Nyquist's theorem. Note that this result is only achieved because of (10). Any other expression would have resulted in a violation of Nyquist's theorem for $I_p=0$. Our final result thus gives an indirect verification of (10) for $p'=0$. Since the random diffusion processes affect the equilibrium hole concentration p_n and the excess hole concentration p' in the same manner, p' and p_n have to occur in the formula in the same way. We therefore conclude, even without a detailed proof, that (10) must be correct. Up to now we have not yet been able to give a *direct* proof of (10).

It is sometimes convenient to know the noise resistance R_n of the junction. Since R_n is a measure for the noise emf in series with the junction, we find its value by putting

$$4kTR_ndf = \langle i_p{}^2\rangle_{av}/\,|\,Y\,|^2, \tag{16a}$$

where Y is the junction admittance. This yields

$$R_n = (1/2G_0)[2(G/G_0)(I_p + I_{p0}) - I_p] \cdot\{(I_p + I_{p0})[2(G/G_0)^2 - 1]\}^{-1}: \tag{16b}$$

for low frequencies $(G\simeq G_0)$ this reduces to the well-known formula,

$$R_n = R_{n0} = (1/2G_0)(I_p + 2I_{p0})/(I_p + I_{p0}). \tag{16c}$$

We thus see that R_n decreases with increasing frequency.

[2] The result obtained at low frequencies is, therefore, identical with what would be expected from the emission theory (compare: A. van der Ziel, "Noise," Prentice-Hall, New York, 1954: p. 216). An earlier treatment appears in V. F. Weisskopf, "On the Theory of Noise in Conductors, Semi-conductors, and Crystal Rectifiers." NDRC 14-33, May 15, 1943.

[3] R. L. Anderson and A. van der Ziel. "On the shot effect of p-n junctions," Trans. IRE, vol. EO-1, pp. 20–24; November, 1952.

IV. Dc and Ac Operation of Transistors

We now apply our theory to p-n-p transistors and assume again that all current is carried by holes. The difference with the junction diode case is that the n-region is now much shorter. It corresponds to a transmission line of finite length. Let w be the width of the n-region.

In our calculations we often have to apply emf's to the emitter junction and (or) to the collector junction. Since the base layer (n-layer) has a true base resistance r_b', this is not quite equivalent to applying these emf's between the emitter and base electrodes and (or) between the collector and base electrodes, respectively. For the same reason, short-circuiting the emitter and collector junction is not quite equivalent to putting short-circuiting connections between the emitter and base electrodes and between the collector and base electrodes, respectively. This is important for the equivalent circuit that is going to be developed.

If we have a transmission line of length w, with a voltage E_s at the sending end and a voltage E_R at the receiving end, then the voltage distribution along the line is

$$E(x) = E_s[\sinh\gamma(w - x)/\sinh\gamma w] + E_R(\sinh\gamma x/\sinh\gamma w), \tag{17}$$

and the input and output currents are

$$I_s = E_s/(Z_0\tanh\gamma w) - E_R/(Z_0\sinh\gamma w)$$
$$I_R = E_s/(Z_0\sinh\gamma w) - E_R/(Z_0\tanh\gamma w). \tag{18}$$

We can apply this directly to the n-region of our p-n-p transistor. Let V_e be the dc signal applied across the emitter junction and V_c the dc signal applied across the collector junction. If p_n is the equilibrium hole concentration in the n-region, then the excess hole concentration p_{e0}' at the emitter junction $(x=0)$ and the excess hole concentration p_{c0}' at the collector junction $(x=w)$ are given by the equations

$$p_{e0}' = p_n[\exp(eV_e/kT) - 1];$$
$$p_{c0}' = p_n[\exp(eV_c/kT) - 1]. \tag{19}$$

$p_{c0}'\simeq -p_n$ if the collector is properly biased.

If small ac signals

$$v_e = V_{e1}\exp(j\omega t) \quad \text{and} \quad v_c = V_{c1}\exp(j\omega t)$$

are applied across the emitter and the collector junction, respectively, the ac hole densities at the emitter junction $(x=0)$ and at the collector junction $(x=w)$ are

$$p_{e1}'\exp(j\omega t) = (e/kT)V_{e1}\exp(j\omega t)\cdot p_n\exp(eV_e/kT),$$
$$p_{c1}'\exp(j\omega t) = (e/kT)V_{c1}\exp(j\omega t)\cdot p_n\exp(eV_c/kT),$$

because the hole densities at $x=0$ and at $x=$w readjust themselves almost instantaneously to the applied voltages. $p_{c1}'\simeq 0$ if the collector is properly biased. This is usually the case and we shall assume that it is true here.

We thus have for the dc excess hole density distribution in the n-region, according to (17),

$$p'(x) = p_{e0}'[\sinh \gamma_0(w - x)/\sinh \gamma_0 w] + p_{c0}'(\sinh \gamma_0 x/\sinh \gamma_0 w); \quad (20)$$

whereas the dc hole currents at $x=0$ and $x=w$ are

$$I_e = p_{e0}'/(Z_{00} \tanh \gamma_0 w) - p_{c0}'/(Z_{00} \sinh \gamma_0 w).$$

$$I_c = p_{e0}'/(Z_{00} \sinh \gamma_0 w) - p_{c0}'/(Z_{00} \tanh \gamma_0 w). \quad (21a)$$

If $V_c=0$, then $p_{c0}'=0$ and, consequently,

$$I_e = I_{ee}[\exp(eV_e/kT)-1]; \; I_{ee}=p_n/(Z_{00} \tanh \gamma_0 w). \quad (21b)$$

If $V_e=0$, then $p_{e0}'=0$ and, consequently,

$$I_c = -I_{cc}[\exp(eV_c/kT)-1]; \; I_{cc}=p_n/(Z_{00} \tanh \gamma_0 w). \quad (21c)$$

In that case $I_c \simeq I_{cc}$ if the collector is properly biased. $I_{cc}=I_{ee}$ if p_n is uniform in the n-region; we tacitly assumed this to be the case. The base current I_b is, if $p_{c0}' \simeq -p_n$,

$$I_b = (I_e - I_c) = (1-\alpha_0)[I_e - I_{cc}(1+\alpha_0)]; \quad (21d)$$

where

$$\alpha_0 = 1/(\cosh \gamma_0 w) \quad (21e)$$

is a quantity that is close to unity if the width w of the n-region is small. Hence,

$$I_e = I_{cc}(1+\alpha_0) \quad (21f)$$

for the case of zero base current (floating base). This does not seem to agree with experiments which seems to require a much larger emitter current for floating base. Leakage currents are partly responsible for this discrepancy.[4]

We now apply small ac voltages across the two junctions and neglect effects due to the change in width w of the junction; we shall take that into account later. We also assume that the collector is biased such that p_{c1}' is negligible. We then have at the emitter side

$$i_e = p_{e1}' \exp(j\omega t)/(Z_0 \tanh \gamma w) = V_{e1} \exp(j\omega t) Y_e, \quad (22)$$

and at the collector side,

$$i_c = p_{e1}' \exp(j\omega t)/(Z_0 \sinh \gamma w) = \alpha V_{e1} \exp(j\omega t) Y_e = \alpha i_e; \quad (23a)$$

where

$$\alpha = 1/(\cosh \gamma w) = 1/\cosh[\gamma_0 w(1+j\omega\tau_p)^{1/2}]. \quad (23b)$$

α is the current amplification factor: its value α_0 for low frequencies was already introduced by (21e). The emitter junction admittance Y_e is

$$Y_e = (p_{e1}/V_{e1})/(Z_0 \tanh \gamma w) = (e/kT)p_n \exp(eV_e/kT)/(Z_0 \tanh \gamma w). \quad (24a)$$

For low frequencies this expression reduces to

$$Y_e = g_{e0} = (e/kT)p_n \exp(eV_e/kT)/(Z_{00} \tanh \gamma_0 w) = (e/kT)[I_e + (1-\alpha_0)I_{cc})], \quad (24b)$$

so that the expression for high frequencies may be written

$$Y_e = g_{e0}(1+j\omega\tau_p)^{1/2} \tanh \gamma_0 w/\tanh[\gamma_0 w(1+j\omega\tau_p)^{1/2}]. \quad (24c)$$

We have thus verified the well-known transistor formulas by this method. We note that our problem reduces to the junction diode problem mentioned previously if $w \to \infty$, this corresponds to $\alpha_0 \to 0$.

We now turn to the effects caused by the change in width of the n-region due to the ac voltages V_e and V_c applied across the emitter and the collector junction, respectively. These effects occur because the width of the transition region of a p-n junction depends upon the voltage applied across the junction. Usually the effects of the fluctuating boundary at the emitter side can be neglected but the effects of the fluctuating boundary at the collector side have to be taken into account. This has three effects:[5]

1. It modulates the true resistance r_b' of the base region; described by an emf $\mu_{bc}V_c$ in series with r_b'.
2. It modulates the current amplification factor α, this gives rise to a collector junction admittance Y_c'.
3. It modulates the current flow across the emitter junction; this can be described by a fluctuating emf $\mu_{ec}V_c$ in series with Y_e.

In addition true capacitance C_c of collector junction is in parallel to Y_c'. Equivalent circuit is shown in Fig. 2. At high frequencies μ_{ec} and Y_c' may be complex.

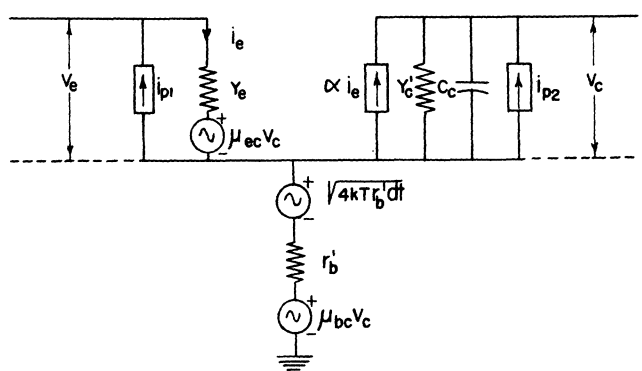

Fig. 2—Equivalent circuit of noise in junction transistors. The noise is described by two partially correlated current generators i_{p1} and i_{p2}, the feedback properties by two emf's $\mu_{ec}v_c$ and $\mu_{bc}v_c$.

V. Application to Noise Problems in Transistors

We now want to calculate the noise currents flowing across the emitter and collector junctions when both junctions are short-circuited. Our diffusion problem

[4] L. J. Giacoletto, private communication.

[5] J. M. Early, "Effects of space-charge layer widening in junction transistors," Proc. IRE, vol. 40, pp. 1401–1406; November, 1952.
L. J. Giacoletto, "Study of p-n-p alloy junction transistor from dc through medium frequencies," *RCA Rev.*, vol. 15, pp. 506–562; December, 1954.

thus corresponds to a transmission line with input and output short-circuited.

We divide the n-region of the transistor into small sections of length Δx. The noise due to hole-electron pair creation and recombination in the region between x and $(x+\Delta x)$ may then again be represented by a parallel current generator Δi_{px} and the noise due to random diffusion processes may be represented by a fluctuating hole density Δp_x between x and $(x+\Delta x)$. According to Section III,

$$\langle \Delta i_{px}^2 \rangle_{av} = 2e^2 df(p' + 2p_n)\Delta x/\tau_p;$$
$$\langle \Delta p_x^2 \rangle_{av} = 4(p' + p_n) df \Delta x/D_p. \tag{25}$$

We now apply the same reasoning as in Section III; the only difference is that we now have an equivalent transmission line of finite length w, short-circuited at both ends, and that we have to know the short-circuit currents at both the input and the output.

In our transmission line case, a parallel-connected current generator ΔI_x between x and $(x+\Delta x)$ produces currents ΔI_1 and ΔI_2 in input and output [Fig. 1(c)]:

$$\Delta I_1 = \Delta I_x \sinh \gamma(w - x)/\sinh \gamma\, w;$$
$$\Delta I_2 = \Delta I_x \sinh \gamma x/\sinh \gamma w|; \tag{26}$$

and an emf ΔE_x connected between x and $(x+\Delta x)$ produces currents $\Delta I_1'$ and $\Delta I_2'$ in input and output [Fig. 1(d)]:

$$\Delta I_1' = -(\Delta E_x/Z_0) \cosh \gamma(w - x)/\sinh \gamma w;$$
$$\Delta I_2' = +(\Delta E_x/Z_0) \cosh \gamma x/\sinh \gamma w. \tag{27}$$

We thus have for our diffusion problem,

$$\Delta i_{p1} = \Delta i_{px} \sinh \gamma(w - x)/\sinh \gamma w$$
$$\qquad - (\Delta p_x/Z_0) \cosh \gamma(w - x)/\sinh \gamma w$$
$$\Delta i_{p2} = \Delta i_{px} \sinh \gamma x/\sinh \gamma w$$
$$\qquad + (\Delta p_x/Z_0) \cosh \gamma x/\sinh \gamma w. \tag{28}$$

Since Δi_{px} and Δ_{px} are uncorrelated, Δi_{p1} and Δi_{p2} will be only *partially* correlated.

We find, therefore, that the noise of the transistors can be represented by two partially correlated current generators i_{p1} and i_{p2} across the emitter and collector junctions, respectively. These quantities are obtained by summing the effects of the individual sections Δx. Since fluctuations in different intervals are independent and since Δi_{px} and Δp_x are independent also, the fluctuations have to be added quadratically. Replacing the summation by an integration, we obtain

$$\langle i_{p1}^2 \rangle_{av} = \sum \langle \Delta i_{p1}^2 \rangle_{av}$$
$$= \int_0^w \langle \Delta i_{px}^2 \rangle_{av} |\sinh \gamma(w-x)/\sinh \gamma w|^2$$
$$+ \int_0^w \langle \Delta p_x^2 \rangle_{av} |\cos \gamma(w-x)/\sinh \gamma w|^2/Z_0|^2 \tag{29}$$

$$\langle i_{p2}^2 \rangle_{av} = \sum \langle \Delta i_{p2}^2 \rangle_{av}$$
$$= \int_0^w \langle \Delta i_{px}^2 \rangle_{av} |\sinh \gamma x/\sinh \gamma w|^2$$
$$+ \int_0^w \langle \Delta p_x^2 \rangle_{av} |\cosh \gamma x/\sinh \gamma w|^2/|Z_0|^2 \tag{30}$$

$$\langle i_{p1}^* i_{p2} \rangle_{av} = \sum \langle \Delta i_{p1}^* \Delta i_{p2} \rangle_{av}$$
$$= \int_0^w \langle \Delta i_{px}^2 \rangle_{av} [\sinh \gamma^*(w-x) \sinh \gamma x/|\sinh^2 \gamma w|]$$
$$- \int_0^w \langle \Delta p_x^2 \rangle_{av} [\cosh \gamma^*(w-x)$$
$$\cdot \cosh \gamma x/|\sinh^2 \gamma w|]/|Z_0|^2. \tag{31}$$

The problem of noise in transistors is hereby solved in principle for all frequencies. For small w the term with $\langle \Delta p_x^2 \rangle_{av}$ predominates; that is, diffusion is the most important effect. In that case, the correlation coefficient for i_1 and i_2 is almost equal to -1, except at the highest frequencies (See Section VI).

The quantities i_{p1} and i_{p2} represent the effect of shot noise in transistors. Besides these two current generators, one has to introduce the thermal noise emf $(4kTr_b'df)^{1/2}$ of the true base resistance r_b'. The full equivalent circuit, as far as noise is concerned, is, therefore, as given in Fig. 2.

VI. Further Discussion of Transistor Noise

For relatively low frequencies we may put $\gamma = \gamma_0$ and $Z_0 = Z_{00}$, this simplifies the integrations in (29)–(31). Introducing (25) for $\langle \Delta i_{px}^2 \rangle_{av}$ and $\langle \Delta p_x^2 \rangle_{av}$, substituting (20) for $p'(x)$, carrying out the integrations in (29)–(31), substituting the expressions for I_e, I_c and I_{cc} and bearing in mind that $p'_{c0} \simeq -p_n$ in practice, we obtain:

$$\langle i_{p1}^2 \rangle_{av} = 2e[I_e + 2I_{cc}(1 - \alpha_0)]df \tag{32}$$
$$\langle i_{p2}^2 \rangle_{av} = 2eI_c df \tag{33}$$
$$\langle i_{p1}^* i_{p2} \rangle_{av} = -2e[I_c - I_{cc}(1 - \alpha_0)]df. \tag{34}$$

The emitter and collector current thus show full shot effect; the correlation coefficient c of the emitter current and the collector current fluctuations is

$$c = \langle i_{p1}^* i_{p2} \rangle_{av} / [\langle i_{p1}^2 \rangle_{av} \cdot \langle i_{p2}^2 \rangle_{av}]^{1/2}$$
$$\simeq -(I_c/I_e)^{1/2} \simeq -\alpha_0^{1/2}; \tag{34a}$$

the latter part of the equation holds if we neglect a few small terms of the order $I_{cc}(1-\alpha_0)$. The correlation coefficient is thus closely equal to -1 at low frequencies.

Except perhaps for very small currents, the terms containing the quantity I_{cc} may be neglected at low frequencies. We would have obtained that approximation also, if we had neglected the terms p_{c0}' and p_n in $\langle \Delta i_{px}^2 \rangle_{av}$ and $\langle \Delta p_x^2 \rangle_{av}$. This should still be a reasonable approximation at very high frequencies, even though it may not be as accurate as at low frequencies. By substituting

$$p'(x) = p_{e0}' \sinh \gamma_0(w - x)/\sinh \gamma_0 w; \qquad \gamma = a + jb;$$
$$a = \gamma_0\left[\tfrac{1}{2} + \tfrac{1}{2}(1 + \omega^2\tau_p^2)^{1/2}\right]^{1/2};$$
$$b = \gamma_0\left[-\tfrac{1}{2} + \tfrac{1}{2}(1 + \omega^2\tau_p^2)^{1/2}\right]^{1/2} \tag{35}$$

into (29)–(31) and carrying out the integrations, we obtain

$$\langle i_{p1}^2\rangle_{av} = 2eI_e df[-1 + \tanh \gamma_0 w\,(a \sinh 2aw + b \sin 2bw)/(\gamma_0 \mid \sinh^2 \gamma w \mid)] \tag{36}$$
$$\langle i_{p2}^2\rangle_{av} = 2eI_c df \tag{37}$$
$$\langle i_{p1}^* i_{p2}\rangle_{av} = -2eI_c df[\gamma \sinh \gamma_0 w/(\gamma_0 \sinh \gamma w)]. \tag{38}$$

Introducing the emitter junction admittance $Y_e = g_e + jb_e$ [(24)], and bearing in mind that $I_c = \alpha_0 I_e$ and that $Y_e = g_{e0} = eI_e/kT$ at low frequencies, these equations may be written:

$$\langle i_{p1}^2\rangle_{av} = 2eI_e df(-1 + 2g_e/g_{e0}) = 2kTdf\cdot(2g_e - g_{e0}) \tag{36a}$$
$$\langle i_{p2}^2\rangle_{av} = 2eI_c df = 2kTdf\cdot\alpha_0 g_{e0} \tag{37a}$$
$$\langle i_{p1}^* i_{p2}\rangle_{av} = -2kTdf\cdot\alpha Y_e \tag{38a}$$

Eq. (36a) is equivalent to (13a) of the junction diode.

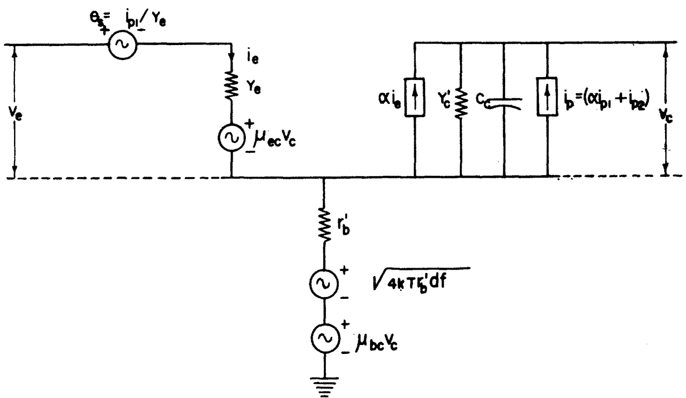

Fig. 3—Other form of equivalent circuit. The noise is now described by an emf $e_s = i_{p1}/Y_e$ in series with the input and a current generator $i_p = (i_{p2} + \alpha i_{p1})$ in parallel to the collector junction. This circuit is completely equivalent to the one shown in Fig. 2.

Our next step is to modify the equivalent circuit of Fig. 2. We observe that it is formally equivalent to circuit shown in Fig. 3 (above), where noise is represented by a noise emf $e_s = i_{p1}/Y_e$ in series with the input and a noise current generator $i_p = (i_{p2} + \alpha i_{p1})$ in parallel to the collector admittance. Applying (32)–(34) and neglecting a few small terms of the order $I_{cc}(1-\alpha_0)$, we obtain at low frequencies:

$$\langle e_s^2\rangle_{av} = \langle(i_{p1}/Y_e)^2\rangle_{av} = 2kTr_{e0}df \tag{39}$$
$$\langle i_p^2\rangle_{av} = \langle(i_{p2} + \alpha_0 i_{p1})^2\rangle_{av} = 2e\alpha_0(1 - \alpha_0)I_e df \tag{40}$$
$$\langle e_s^* i_p\rangle_{av} = \langle i_{p1}^*(i_{p2} + \alpha_0 i_{p1})\rangle_{av} r_{e0} = 0, \tag{41}$$

where r_{e0} is the emitter junction resistance at low frequencies. Hence e_s and i_p are practically uncorrelated. Except for the term due to the shot effect of the collector saturated current I_{c0}, the equivalent circuit of Fig. 3 shows resemblance with a phenomenological equivalent circuit published earlier.[6]

At higher frequencies this is no longer the case; e_s and i_p have a complex correlation coefficient $c \neq 0$ and their mean square values differ from the corresponding low-frequency values. Applying (36)–(38), we obtain:

$$\langle e_s^2\rangle_{av} = \langle i_{pi}^2\rangle_{av}/\mid Y_e\mid^2 = 2kTdf\cdot(2g_e - g_{e0})/(g_e^2 + b_e^2) \tag{42}$$
$$\langle i_p^2\rangle_{av} = 2eI_c(1 - \mid\alpha\mid^2/\alpha_0)df = 2kTdf\cdot\alpha_0 g_{e0}(1 - \mid\alpha\mid^2/\alpha_0) \tag{43}$$
$$\langle e_s^* i_p\rangle_{av} = 2kTdf\cdot\alpha(1 - g_{e0}/Y_e^*). \tag{44}$$

At low frequencies (42) reduces to (39), (43) reduces to (40), and (44) reduces to (41).

In the phenomenological equivalent circuit the emf e_s was ascribed to "shot noise" and the current generator i_p was ascribed to "partition noise"; the latter meant that the noise is caused by fluctuations in the recombination processes. To show that at low frequencies the recombination fluctuations predominate over diffusion fluctuations in i_p, we apply (28) and write

$$i_p = \sum(\Delta i_{p2} + \alpha\Delta i_{p1}) = \alpha \sum(\Delta i_{px} \cosh \gamma x + \Delta p_x \sinh \gamma x/Z_0),$$

where the summation extends over all sections Δx of the base region. Consequently,

$$\langle i_p^2\rangle_{av} = \mid\alpha^2\mid\left[\int_0^w \langle\Delta i_{px}^2\rangle_{av}\mid\cosh^2\gamma x\mid + \int_0^w \langle(\Delta p_x/Z_0)^2\rangle_{av}\mid\sinh^2\gamma x\mid\right]. \tag{45}$$

The first term in this equation is due to recombination fluctuations and the second is due to diffusion fluctuations. For good junction transistors α_0 is close to unity and $|\cosh^2 \gamma x| \gg |\sinh^2 \gamma x|$ at low frequencies; therefore the recombination fluctuations predominate in i_p at low frequencies. In the same way it may be shown that diffusion fluctuations predominate in e_s at low frequencies.

The present calculation does not verify the existence of a noise term due to the shot effect of the collector saturated current. Further experimental and theoretical work is needed to clarify that situation.

The equivalent circuit also contains two feedback emf's $\mu_{ec}v_c$ and $\mu_{bc}v_c$. A calculation shows that these emf's, which are important for the amplification properties of the transistor, hardly alter the noise figure; they may therefore be omitted as far as noise considerations are concerned. This indicates that the "base resistance r_b" in the earlier phenomenological equivalent circuit actually should be the "true" base resistance r_b' of the new equivalent circuit. The numerical value for r_b that

[6] H. C. Montgomery and M. A. Clark, "Shot noise in junction transistors," *Jour. Appl. Phys.*, vol. 24, pp. 1397–1398; November, 1953.

A. van der Ziel, "Note on shot and partition noise in junction transistors," *Jour. Appl. Phys.*, vol. 25, pp. 815–816; June, 1954.

was used by Montgomery and Clark in their calculation of the noise figure actually represented the true base resistance of their transistor.[7]

It is also possible to change the equivalent circuit of Fig. 2 by replacing the emf $\mu_{ec}v_c$ by a current generator $Y_{ec}v_c$, where $Y_{ec}=\mu_{ec}Y_e$, the current generator αi_e by a current generator $Y_{ce}v_e$, where $Y_{ce}=\alpha Y_e$, and the admittance Y_c' by an admittance:

$$Y_c'' = Y_c' + \alpha\mu_{ec}Y_e \tag{46}$$

(Y_c' is the collector junction admittance for open emitter junction and Y_c'' the collector junction admittance for short-circuited emitter junction). The circuit thus obtained closely resembles a phenomenological circuit proposed by Giacoletto.[8]

VII. Noise Figure

Let a signal source of internal impedance $Z_s=(r_s+jX_s)$ be connected to the input and let the thermal noise of the signal source be represented by an emf $(4kTr_sdf)^{1/2}$ in series with Z_s. Let the output terminals be open and let $\langle v_{ct}^2\rangle_{av}$ and $\langle v_{cs}^2\rangle_{av}$ represent the mean square value of the noise voltage across the collector junction due to transistor noise and the thermal noise of the signal source, respectively. Since $(Y_c'+j\omega C_c)$ is a small admittance, we may in first approximation assume that the noise voltages developed in the base resistance are small in comparison with the noise voltage across $(Y_c'+j\omega C_c)$. The noise figure F may then be defined as:

$$F = 1 + \langle v_{ct}^2\rangle_{av}/\langle v_{cs}^2\rangle_{av} \tag{47}$$

Calculating v_{ct} and v_{cs}, introducing the emitter junction impedance $Z_e=1/Y_e=(r_e+jX_e)$ and putting $Z_{tot}=(Z_e+Z_s+r_b')$, we obtain:

$$F = 1 + r_b'/r_s + \langle(- e_s + i_pZ_{tot}/\alpha]\cdot(- e_s^* + i_p^*Z_{tot}^*/\alpha^*)\rangle_{av}/(4kTr_sdf). \tag{48}$$

This does not contain any reference to the feedback emf's, indicating that feedback does not affect the noise figure in this approximation.

It may be shown that F is a minimum if:

$$X_s + X_e = - X_e/(- 1 + \alpha_0/|\alpha|^2) \tag{49}$$

$$r_s = \{[(- 1 + \alpha_0/|\alpha|^2)(r_e + r_b') + r_e]^2 - (\alpha_0/|\alpha|^2)(r_e^2 + X_e^2)\}^{1/2}/(- 1 + \alpha_0/|\alpha|^2)^{1/2} \tag{50}$$

in which case:

$$F = F_{\min} = [(- 1 + \alpha_0/|\alpha|^2)(r_e + r_b') + r_e]/r_{e0} + \{[(- 1 + \alpha_0/|\alpha|^2)(r_e + r_b') + r_e]^2/r_{e0}^2 - (\alpha_0/|\alpha|^2)(r_e^2 + X_e)^2/r_{e0}^2\}^{1/2} \tag{51}$$

For low frequencies $\alpha=\alpha_0$, $r_e=r_{e0}$ and $X_e=0$, so that:

$$F_{\min} = \{[1 + (1 - \alpha_0)r_b'/r_{e0}] + \sqrt{[1 + (1 - \alpha_0)r_b'/r_{e0}]^2 - \alpha_0}\}/\alpha_0 \tag{52}$$

The minimum noise figure at low frequencies is thus mainly determined by the quantity $(1-\alpha_0)r_b'/r_{e0}$; for low noise figure this quantity should be small. At higher frequencies the noise figure increases because the current amplification factor α goes to zero and $\langle i_p^2\rangle_{av}$ increases from the low-frequency value $2eI_c(1-\alpha_0)df$ to the high-frequency value $2eI_cdf$.

Conclusion

The above developments show that the transmission line analogy can be successfully applied to the solution of noise problems in junction diodes and transistors. It is interesting to note that at low frequencies the new theory shows resemblance with older phenomenological equivalent circuits. Besides, the theory also gives the hf noise behavior of the devices, a result that could not be obtained by the old method. The earlier experimental results on noise in junction diodes find hereby an adequate theoretical explanation.

The theory is a *one-dimensional* theory and the recombination is assumed to be volume recombination. Actually most recombination occurs at the surface and the problem is multi-dimensional. Deviations between theory and experiment might thus be expected in some cases. A research program is under way at the University of Minnesota to check this theory with experiment.

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[7] H. C. Montgomery, private communication.
[8] L. J. Giacoletto, Semiconductor Devices Conference. Minneapolis, Minn., June, 1954.