EE 538 Spring 2021
Low-Noise Analog Circuit Design
University of Washington Electrical & Computer Engineering

Instructor: Jason Silver
Assignment #1 (10 points)
Due Sunday, April 11 (Submit on Canvas as a Jupyter Notebook)

Please show your work

Resources:

https://analog.intgckts.com/equivalent-noise-bandwidth/#:~:text=Equivalent%20Noise%20Bandwidth%201%20comment&text=Equivalent%20noise%20bandwidth(ENBW)%20is,ba (https://analog.intgckts.com/equivalent-noise-bandwidth/#:~:text=Equivalent%20Noise%20Bandwidth%201%20comment&text=Equivalent%20noise%20bandwidth(ENBW)%20is,ba

http://www.onmyphd.com/?p=enbw.equivalent.noise.bandwidth (http://www.onmyphd.com/?p=enbw.equivalent.noise.bandwidth)

https://www.k-state.edu/edl/docs/pubs/technical-resources/Technote1.pdf (https://www.k-state.edu/edl/docs/pubs/technical-resources/Technote1.pdf)

http://web.engr.oregonstate.edu/~webbky/ENGR202_files/SECTION%203%20Second%20Order%20Filters.pdf (http://web.engr.oregonstate.edu/~webbky/ENGR202_files/SECTION%203%20Second%20Order%20Filters.pdf)

https://www.ti.com/lit/an/sloa049b/sloa049b.pdf?ts=1617717745044&ref_url=https%253A%252F%252Fwww.google.com%252F (https://www.ti.com/lit/an/sloa049b/sloa049b.pdf?ts=1617717745044&ref_url=https%253A%252F%252Fwww.google.com%252F)

In [1]:

```python
# Imports
import os
import sys
import cmath
import math
import matplotlib.pyplot as plt
import matplotlib
import numpy as np
import pandas as pd
import ltspice
import sympy as sp
from sympy.utilities.lambdify import lambdify
from scipy import signal
%matplotlib inline
from IPython.core.interactiveshell import InteractiveShell
InteractiveShell.ast_node_interactivity = "all"
from matplotlib.ticker import LogLocator
```

In [2]:

```python
def read_ltspice(file_name,ftype='trans',units='db'):
    cols = []
    arrs = []
    with open(file_name, 'r',encoding='utf-8') as data:
        for i,line in enumerate(data):
            if i==0:
                cols = line.split()
                arrs = [[] for _ in cols]
                continue
            parts = line.split()
            for j,part in enumerate(parts):
                arrs[j].append(part)
    df = pd.DataFrame(arrs,dtype='float64')
    df = df.T
    df.columns = cols
    if ftype=='trans':
        return df
    elif ftype=='ac':
        if units=='db':
            for col in cols:
                if df[col].str.contains(',').all():
                    df[f'Mag_{col}'] = df[col].apply(lambda x: x.split(',')[0])
                    df[f'Mag_{col}'] = df[f'Mag_{col}'].apply(lambda x: x[1:-2])
                    df[f'Mag_{col}'] = df[f'Mag_{col}'].astype('float64')
                    df[f'Phase_{col}'] = df[col].apply(lambda x: x.split(',')[1])
                    df[f'Phase_{col}'] = df[f'Phase_{col}'].apply(lambda x: x[0:-2])
                    df[f'Phase_{col}'] = df[f'Phase_{col}'].astype('float64')
        if units=='cartesian':
            for col in cols:
                if df[col].str.contains(',').all():
                    df[f'Re_{col}'] = df[col].apply(lambda x: x.split(',')[0])
                    df[f'Re_{col}'] = df[f'Re_{col}'].astype('float64')
                    df[f'Im_{col}'] = df[col].apply(lambda x: x.split(',')[1])
                    df[f'Im_{col}'] = df[f'Im_{col}'].astype('float64')
        df['Freq.'] = df['Freq.'].astype('float64')
        return df
    else:
        print('invalid ftype')
```

Problem 1: Noise bandwidth

The transfer function of a second-order low-pass filter can be written as

$$H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 \cdot s + \omega_0^2}$$

where ω_0 is the resonant frequency and ζ is the damping factor.

Analysis

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a) Derive a general expression for the noise bandwidth of a second-order low-pass filter.

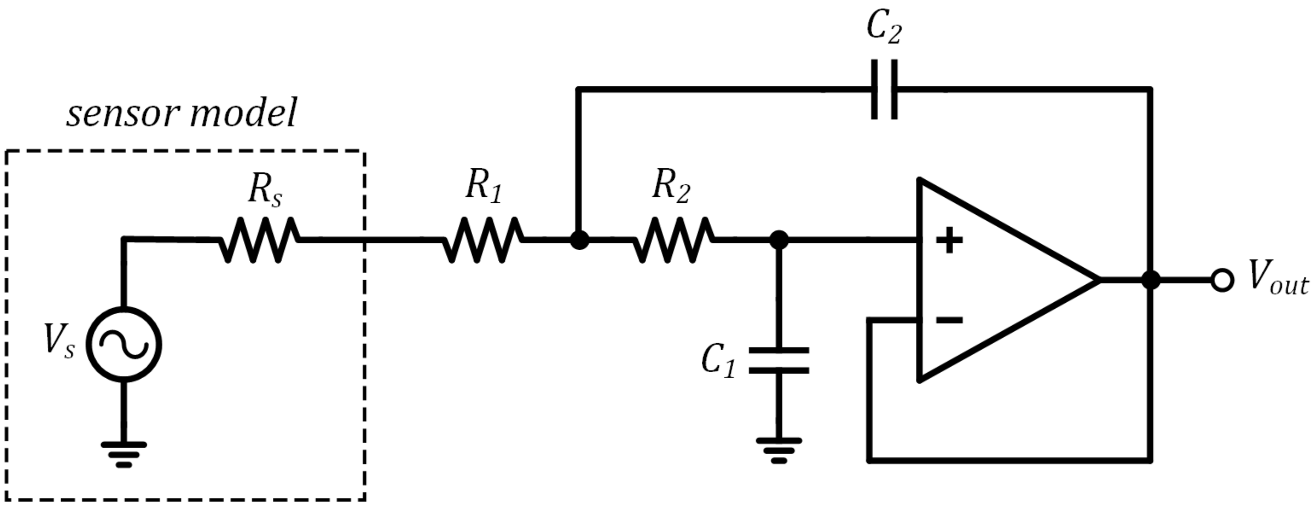
$$
\begin{aligned}
H(s) &= \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 \cdot s + \omega_0^2} \\
&= \frac{\omega_0^2}{s^2 + \frac{s \cdot \omega_0}{Q} + \omega_0^2} \\
&= \frac{1}{\frac{s^2}{\omega_0^2} + \frac{s}{Q\omega_0} + 1} \\
&= \frac{1}{\frac{(j\omega)^2}{\omega_0^2} + \frac{(j\omega)}{Q\omega_0} + 1} \\
&= \frac{1}{\frac{-\omega^2}{\omega_0^2} + \frac{j\omega}{Q\omega_0} + 1} \\
|H(s)| &= \frac{1}{\sqrt{(1 + \frac{-\omega^2}{\omega_0^2})^2 + (\frac{\omega}{Q\omega_0})^2}} \quad \text{and Butterworth } \; Q = \frac{1}{\sqrt{2}} \\
&= \frac{1}{\sqrt{1 - \frac{2\omega^2}{\omega_0^2} + \frac{\omega^4}{\omega_0^4} + \frac{\omega^2}{Q^2\omega_0^2}}} \\
&= \frac{1}{\sqrt{1 - \frac{2\omega^2}{\omega_0^2} + \frac{\omega^4}{\omega_0^4} + \frac{2\omega^2}{\omega_0^2}}} \\
&= \frac{1}{\sqrt{1 + \frac{\omega^4}{\omega_0^4}}} \\
&= \frac{1}{\sqrt{1 + (\frac{\omega}{\omega_0})^4}} \qquad \text{and Butterworth } \; \omega_0 = \omega_c \\
&= \frac{1}{\sqrt{1 + (\frac{\omega}{\omega_c})^4}} \\
\omega_{enb} &= \int_0^\infty |H(s)|^2 \, d\omega \\
&= \int_0^\infty \frac{1}{1 + (\frac{\omega}{\omega_c})^4} \, d\omega \\
f_{enb} &= \frac{1}{2\pi} \int_0^\infty \frac{1}{1 + (\frac{\omega}{\omega_c})^4} \, d\omega
\end{aligned}
$$

---

b) Using your expression from part **a**, determine the relationship between the noise bandwidth and the $-3\mathrm{dB}$ bandwidth of a second-order Butterworth filter.

$$
\begin{aligned}
f_{enb} &= \frac{1}{2\pi}\int_0^\infty \frac{1}{1+(\frac{\omega}{\omega_c})^4}\,d\omega \\
&\quad \text{let } x = \frac{\omega^4}{{\omega_c}^4} \\
&= \frac{1}{2\pi}\int_0^\infty \frac{1}{1+x^4}dx \cdot \omega_c \\
&= \frac{\omega_c}{2\pi}\cdot\frac{\pi}{2\sqrt{2}} \\
&= \frac{2\pi f_c}{2\pi}\cdot\frac{\pi}{2\sqrt{2}} \\
&= \frac{\pi}{2\sqrt{2}} f_c \\
&\approx 1.11 f_c
\end{aligned}
$$

Design



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c) Using the Sallen-Key structure shown above, design an active second-order Butterworth filter (i.e. determine R and C values) to limit the noise bandwidth of a resistive sensor with equivalent resistance $R_s = 1\text{k}\Omega$. Design the filter to achieve an input-referred rms noise of 40nV. Assume that the opamp and filter resistors are noiseless.

$$
\begin{aligned}
T &= 25C = 298K \\
k &= 1.381 \cdot 10^{-23} \\
R_s &= 1000
\end{aligned}
$$

$$v_{n,out(\text{rms})} = \sqrt{e_n^2 \cdot f_{enb}}$$

$$v_{n,in(\text{rms})} = \frac{v_{n,out(\text{rms})}}{A_{v,CL}} \text{ where } A_{v,CL} = 1$$

$$v_{n,in(\text{rms})} = v_{n,out(\text{rms})}$$

$$40\text{nV} \geq \sqrt{e_n^2 \cdot f_{enb}}$$

$$e_n^2 = 4kTR_s$$

$$= 1.6458 \cdot 10^{-17} \frac{V^2}{\text{Hz}}$$

$$40\text{nV} \geq \sqrt{1.6458 \cdot 10^{-17} \cdot f_{enb}}$$

$$f_{enb} \leq \frac{(40\text{nV})^2}{1.6458 \cdot 10^{-17}}$$

$$1.11 f_c \leq \frac{(40\text{nV})^2}{1.6458 \cdot 10^{-17}}$$

$$f_c \leq \frac{(40\text{nV})^2}{1.6458 \cdot 10^{-17} \cdot 1.11}$$

$$f_c \leq 87.6\text{Hz}$$

In [3]:

```
T = 298
k = 1.381 * 1e-23
Rs= 1000
target = 40*1e-9
Vrms = np.sqrt(4*k*T*Rs)
fenb = target**2/Vrms**2
fc = fenb/1.11
print(f'fc: {round(fc,4)}, and f_enb: {round(fenb,4)}')
```

```
fc: 87.5643, and f_enb: 97.1964
```

In [4]:

```
s,m,n,C1,C2,R1,R2,Q,tau,W0,Wc,Cn,K = sp.symbols('s,m,n,C1,C2,R1,R2,Q,tau,omega0,Wc,Cn,K')
```

In [5]:

```
# Low Pass Butterworth
fc1 = 87
systemLP = sp.Matrix([
    [W0 - 1/(tau*sp.sqrt(m*n))],
    [W0 - Cn*Wc],
    [Q - sp.sqrt(m*n)/(1+m)],
    [m - R1/R2],
    [n - C1/C2],
    [tau - R2*C2]
])
myVals = {
    W0:2*np.pi*fc1,  # Cutoff Frequency in radians/second
    Cn:1,            # Table
    Q:1/np.sqrt(2),  # Table
    m:1,             # Chosen Ratio
    C2:1e-9         # Chosen Value
}
systemLP = systemLP.subs(myVals)
eq = sp.solve(systemLP)
eq, myVals
```

Out[5]:

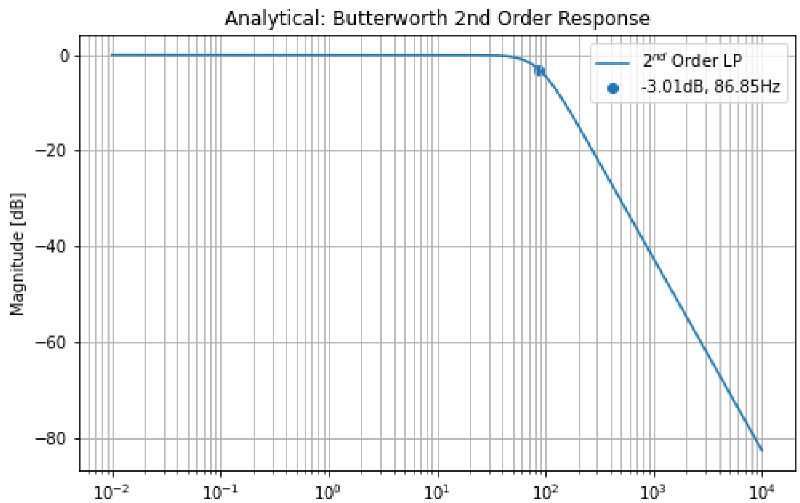
```
({C1: 2.00000000000000e-9,
  R1: 1293557.92551308,
  R2: 1293557.92551308,
  n: 2.00000000000000,
  tau: 0.00129355792551308,
  Wc: 546.637121724624},
 {omega0: 546.637121724624, Cn: 1, Q: 0.7071067811865475, m: 1, C2: 1e-09})
```

In [6]:

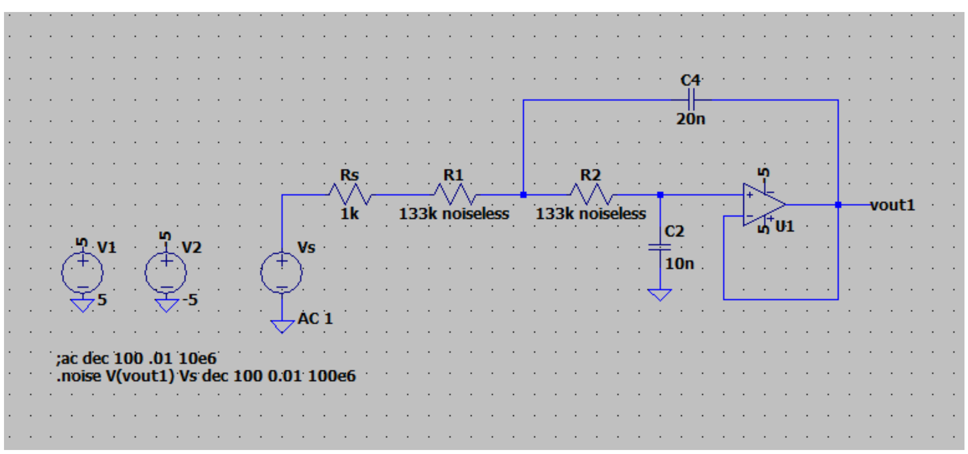
```
f = np.logspace(-2, 4, 10000)
w = 2*np.pi*f
#s = 1j*w

num = W0**2
den = s**2 + s*W0/Q + W0**2

components = {
    #C1:2e-9,
    #C2:1e-9,
    #R1:1293557,
    #R2:1293557,
    W0:546,
    Q :1/np.sqrt(2)
}
H = sp.Matrix([num/den])
H1 = H = H.subs(components)
H = lambdify(s,H,modules='numpy')
H = H(1j*w)
H = H[0][0]
```

In [7]:

```python
fig, ax = plt.subplots(figsize=(8,5))

x1 = np.where(20*np.log10(abs(H))<=-3)[0][0]
label1 = "{:.2f}dB, {:.2f}Hz".format(20*np.log10(abs(H[x1])),f[x1])

ax.set_title('Analytical: Butterworth 2nd Order Response')
ax.semilogx(f, 20*np.log10(abs(H)),label=r'$2^{nd}$ Order LP')
ax.scatter(f[x1],20*np.log10(abs(H[x1])),label=label1,color='tab:blue')
ax.set_ylabel('Magnitude [dB]')
ax.grid(which='both', axis='both')
ax.legend()
plt.show();
```



d) Verify your design in Ltspice using the *UniversalOpamp2* model with a bandwidth of 10MHz. Demonstrate the frequency response and noise performance over a frequency range of 100MHz, and verify that the rms noise meets the specification by integrating over the full 100MHz range.

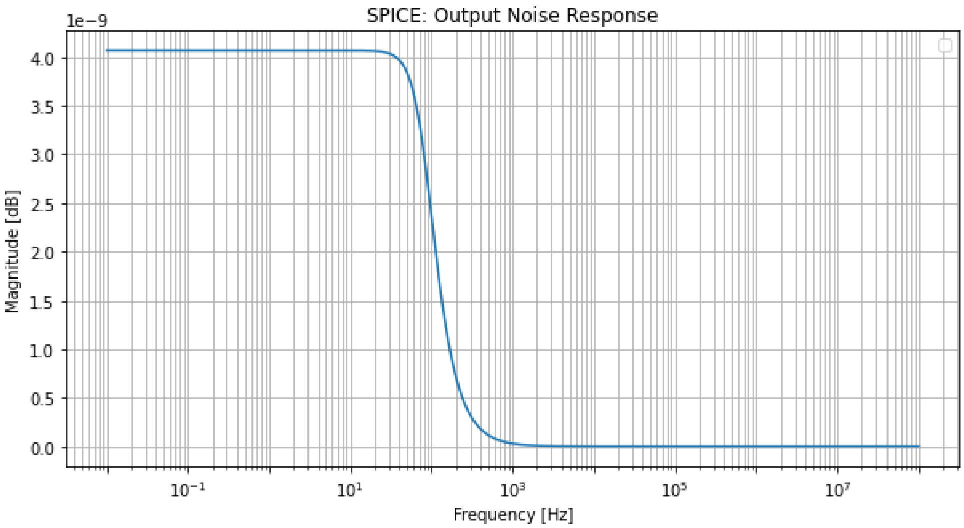


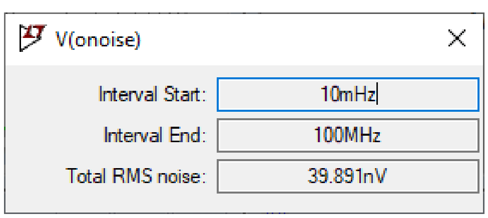
In [8]:

```python
filepath = 'data/butterworth_2nd_order_LP.txt'
df = pd.read_csv(filepath)
freq = df['frequency']
mag = df['V(onoise)']
```

In [9]:
```
fig, ax = plt.subplots(1,figsize=(10,5))

ax.semilogx(freq, mag, color='tab:blue',label='')
ax.grid(True,which='both')
ax.set_xlabel('Frequency [Hz]')
ax.set_ylabel('Magnitude [dB]')
ax.set_title('SPICE: Output Noise Response')

# manipulate x-axis ticks and labels
ax.xaxis.set_major_locator(LogLocator(numticks=15)) #(1)
ax.xaxis.set_minor_locator(LogLocator(numticks=15,subs=np.arange(2,10))) #(2)
for label in ax.xaxis.get_ticklabels()[::2]:
    label.set_visible(False) #(3)

ax.legend()
plt.show();
```

```
No handles with labels found to put in legend.
```





Input noise can be determined from the output noise based on the amplifier gain. Refer to part C. In our case, the input-referred rms noise is equivalent to the output rms noise. The design of the LP Butterworth meets the spec of 40nV.

---

e) If the sensor exhibits $1/f$ noise, the spectral density of the noise can be expressed as

$$\overline{e_n^2} = 4kTR_s \cdot \left(1 + \frac{f_c}{f}\right) \Delta f$$

where f_c is the $1/f$ noise corner frequency. Determine the rms noise (at the output of the filter) between $1\mathrm{Hz}$ and $1\mathrm{MHz}$ if $f_c = 1\mathrm{kHz}$. Verify this in Ltspice. Explain why the concept of noise bandwidth doesn't apply to noise processes that aren't white.

From notes:

- The equivalent noise bandwidth (f_{enb}) of a circuit is the bandwidth of an ideal "brick-wall" filter that would result in the same rms noise as the real filter
- Because white noise has a flat (constant-value) spectrum, multiplication of e_n by the noise bandwidth will conveniently yield the same rms noise value as that obtained via integration of the filter magitude response

My response:

Since the spectral density function is no longer constant, due to 1/f noise, the noise bandwidth cannot simply be multiplied and must be calculated over the integral.

$$\overline{e_n^2} = 4kTR_s \cdot \left(1 + \frac{f_c}{f}\right) \Delta f$$

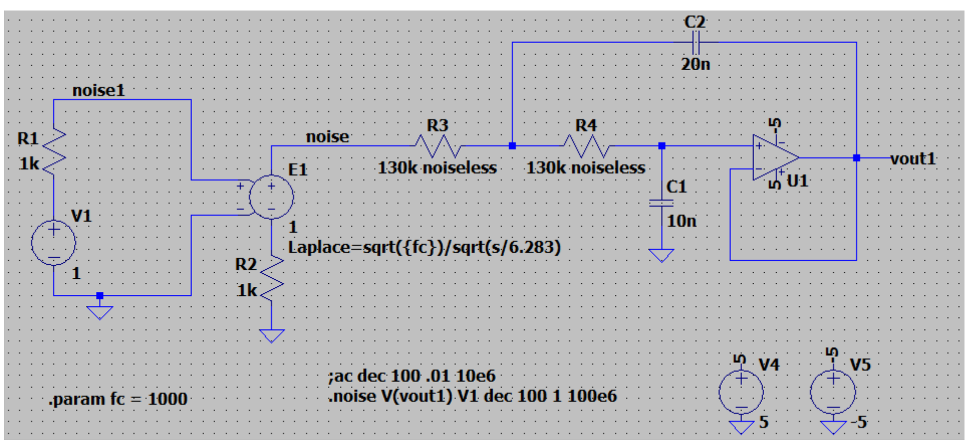
$$v_{n,out(\text{rms})}{}^2 = \int_1^{1MHz} |H(f)|^2 e_n^2 \, df$$

$$v_{n,out(\text{rms})} = \sqrt{\int_1^{1MHz} |H(f)|^2 e_n^2 \, df}$$

$$= \sqrt{\int_1^{1MHz} \frac{1}{1 + (\frac{f}{f_{c_1}})^4} \cdot 4kTR_s \cdot \left(1 + \frac{f_{c_2}}{f}\right) \, df}$$

$$= \sqrt{\int_1^{1MHz} \frac{1}{1 + (\frac{f}{88})^4} \cdot 4kT \cdot 1000 \cdot \left(1 + \frac{1000}{f}\right) \, df}$$

$$= 274.4 \text{ nV}$$



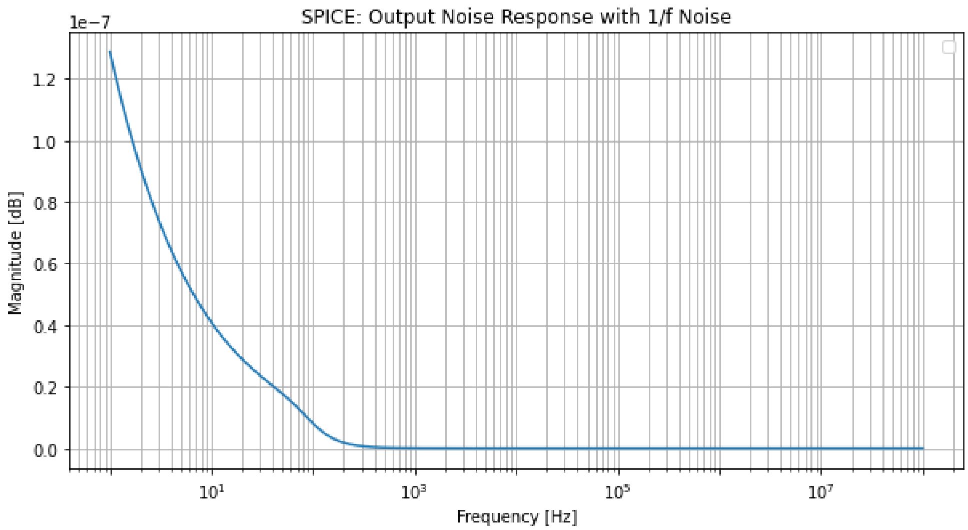
In [10]:

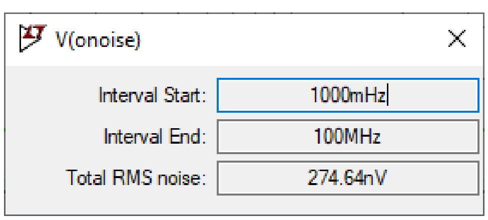
```
filepath = 'data/butterworth_and_1_over_f_noise.txt'
df = pd.read_csv(filepath)
freq = df['frequency']
mag = df['V(onoise)']
```

In [11]:

```python
fig, ax = plt.subplots(1,figsize=(10,5))

ax.semilogx(freq, mag, color='tab:blue',label='')
ax.grid(True,which='both')
ax.set_xlabel('Frequency [Hz]')
ax.set_ylabel('Magnitude [dB]')
ax.set_title('SPICE: Output Noise Response with 1/f Noise')

# manipulate x-axis ticks and labels
ax.xaxis.set_major_locator(LogLocator(numticks=15)) #(1)
ax.xaxis.set_minor_locator(LogLocator(numticks=15,subs=np.arange(2,10))) #(2)
for label in ax.xaxis.get_ticklabels()[::2]:
    label.set_visible(False) #(3)

ax.legend()
plt.show();
```

```
No handles with labels found to put in legend.
```





Use the expression 'noiseless' after the resistance value in Ltspice to prevent specific resistors from generating noise during noise analysis.

To avoid loading between the sensor and filter input, you can use the 'E' component in Ltspice (voltage-controlled voltage source).

In [ ]:

```python

```