2. Review of Transmission Lines Rev. 9/25/2018

Frequency-domain analysis

2.1 Lossless TL

Loss due to material ($\varepsilon_r=\varepsilon'-j\varepsilon''$, $\mu_r=\mu'-j\mu''$)

Conductor loss (finite σ) copper: $\sigma=5.8\text{x}10^{\wedge}7$ (s/m)

proportional to sqrt(*f*)

Dielectric loss (often specified by loss tangent), non-magnetic case $\mu_r=1$

proportional to frequency

Not discussed: Loss due to mode structures: surface wave loss on PCB

Example of TL loss

coaxial cable: Type 141 (0.141" diameter)

at 1 GHz ~0.4 dB/m

at 10 GHz ~2 dB/m

Loss depends on frequency, TL type, materials, and length

Lossless TL means → Short length

Low freq

Voltage on TL

$$V(z,t)=Ve^{-\gamma z}e^{j\omega t}$$

propagation: $e^{-\gamma z}$, time: $e^{j\omega t}$ usually not included

$$\gamma=\alpha+j\beta$$

$$e^{-\gamma z}=e^{-\alpha z}e^{-j\beta z}$$

attenuation (loss) → phase

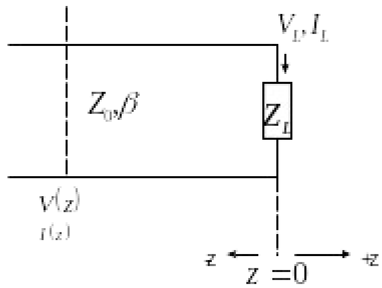
Lossless TL → $\alpha=0$

$$\gamma=\alpha+j\beta=j\beta$$

2-2 Reflection Coefficient

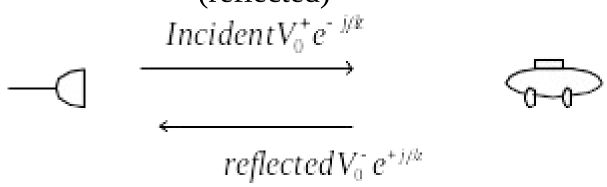
$$V(z) = Ve^{\pm j\beta z}$$

Phasor



$e^{-j\beta z}$: wave (voltage) propagating in +z (incident)

$e^{+j\beta z}$: wave (voltage) propagating in -z (reflected)



Explicit function of time

$$v(z,t) = \text{Re}\left[Ve^{i\omega t}\right] \propto \cos(\omega t \pm \beta z)$$

Measurable quantity: total voltage & current

$$V_{total} = V_{incident} + V_{reflected}$$

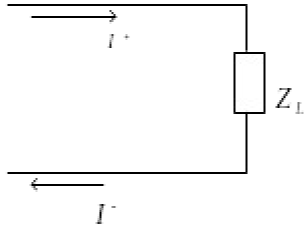
$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}$$

Z_0 **: characteristic impedance.**

$$Z_0 = \frac{V^+}{I^+} = -\frac{V^-}{I^-}$$

Current: reflected current has a negative sign



At $z=0$ (Load position), we get the load voltage and current

$$V(z=0)=V_L=V_0^{+}+V_0^{-}$$

$$I(z=0)=I_L=\frac{V_0^{+}}{Z_0}-\frac{V_0^{-}}{Z_0}$$

Load impedance is given by

Note: When someone measures the impedance (resistance) by DMM, this is the value displayed on DMM.

$$Z_L=\frac{V_L}{I_L}=\left(\frac{Vo^{+}+Vo^{-}}{Vo^{+}-Vo^{-}}\right)Zo$$

Using this we can write, the reflected voltage V_0^{-} as

$$V_0^{-}=\left(\frac{Z_L-Z_o}{Z_L+Z_o}\right)Vo^{+}$$

$$=\Gamma_L Vo^{+}$$

Γ_L : reflection coefficient

$$\Gamma_L=\frac{Vo^{-}}{Vo^{+}}=\frac{Z_L-Z_o}{Z_L+Z_o}$$

$$-1\leq\Gamma_L\leq 1$$

No reflection if $\Gamma_L=0$ → $Z_L=Z_0$

Total reflection if $\Gamma_L=\pm 1$ → $Z_L=0$ (Short) or $Z_L=\infty$ (Open)

In terms of Γ_L

$$V(z)=Vo^{+}\left(e^{-j\beta z}+\Gamma_L e^{j\beta z}\right)$$

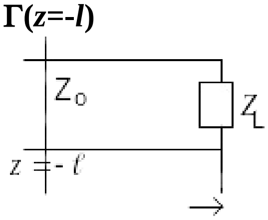
$$I(z)=\frac{Vo^{+}}{Zo}\left(e^{-j\beta z}-\Gamma_L e^{j\beta z}\right)$$

At $z=-\ell$ $z=-\ell$

$$\Gamma(\ell)=\frac{V^{-}(\ell)}{V^{+}(\ell)}=\frac{V_o^{-}e^{-j\beta\ell}}{V_o^{+}e^{+j\beta\ell}}=\Gamma_L e^{-2j\beta\ell}$$

$$\Gamma_L=\Gamma(\ell=0)$$

Γ_L and Γ(*z*=-*l*)



$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}$$

$$\Gamma(\ell) = \Gamma_L e^{-2j\beta\ell}$$

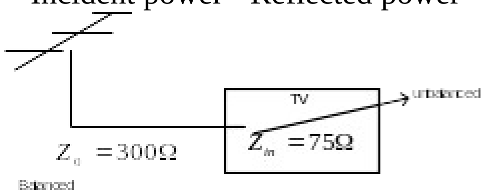
Power delivered to the load

$$P_{ave} = \frac{1}{T}\int P(t)\,dt$$

$$= \frac{1}{2}\mathrm{Re}\left[V(z)I^{*}(z)\right]$$

$$= \frac{1}{2}\frac{|V_0^+|^2}{Z_0}\left(1-|\Gamma_L|^2\right)$$

=Incident power - Reflected power



$$\Gamma_L = \frac{75-300}{75+300} = -0.6$$

$$|\Gamma_L|^2 = 0.36$$

$$1-|\Gamma_L|^2 = 64\,\%$$

is delivered.

Return loss

$$\Gamma_L = \frac{V_o^-}{V_o^+}$$

Return loss $= -20\log_{10}|\Gamma_L|$ dB

20log is used because Γ_L is a voltage ratio.

One port device

$$S_{11} = \frac{b_1}{a_1}\Big|_{a_2=0}$$

$|S_{11}|$ is return loss

Total voltage on TL in terms of Γ_L

$$V(z) = V_o^+\left(e^{-j\beta z} + \Gamma_L e^{j\beta z}\right)$$

$$= V_0^+ e^{-j\beta z}\left(1+\Gamma_L e^{2j\beta z}\right)$$

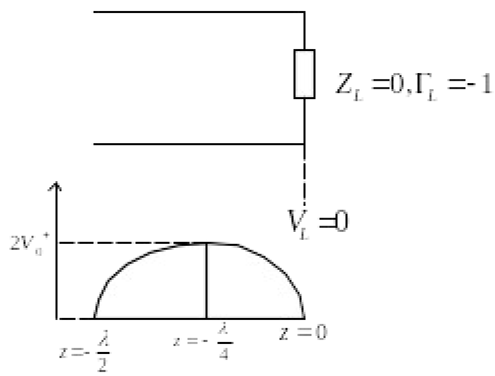
$|Voltage|$ at any point on $TL(z=-\ell)$ is

$$|V(\ell)| = |V_0^+|\,\left|\left(1+|\Gamma_L|e^{j\theta}e^{-2j\beta\ell}\right)\right|$$

$$\Gamma_L = |\Gamma_L| e^{j\theta}$$
$$V_{\max} = V_0^+ (1 + |\Gamma_L|)$$
$$V_{\min} = V_0^+ (1 - |\Gamma_L|)$$

Example $Z_L = 0$ **(short)**

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = -1 = e^{j\pi}$$
$$V\max = 2V_o^+$$
$$V\min = 0$$



$$e^{j\theta} e^{-2j\beta\ell} = e^{-2j\left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{4}\right)} e^{j\theta}$$
$-j\pi$ $+j\pi$

= 1

$V_{\max}$ occurs at $z = -\frac{\lambda}{4}$

Freq 60Hz (power line) $\lambda = 5000$ Km

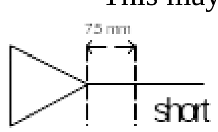
$V_{\max}$ occurs at $\ell = \frac{\lambda}{4} = 1250$ Km

Somewhere in California.

Freq 10GHz $(\mu - wave)$ $\lambda = 30$ mm

$V_{\max}$ occurs at $\ell = \frac{\lambda}{4} = 7.5\,mm$ (amplifier output)

This may damage the amplifier



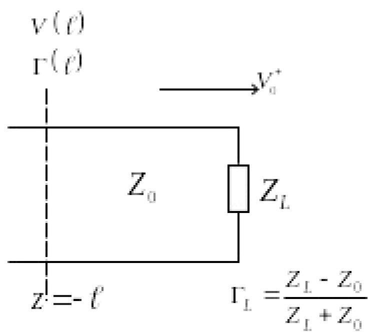
Standing wave ratio

$$SWR=\frac{V_{\max}}{V_{\min}}=\frac{1+|\Gamma_L|}{1-|\Gamma_L|}$$

$$1<SWR<\infty$$

e.g. matched load $(\Gamma_L=0)$ $SWR=$ 1

Short or open $(\Gamma_L=-1)$ $(\Gamma_L=1)$ $SWR=$ ∞



at $z=-\ell$

$$V(\ell)=V_o^+ e^{+j\beta\ell}(1+\Gamma_L e^{-2j\beta\ell})$$

$$\Gamma(z=-\ell)=\frac{V^-(\ell)}{V^+(\ell)}=\frac{V_o^- e^{-j\beta\ell}}{V_o^+ e^{+j\beta\ell}}=\Gamma_L e^{-2j\beta\ell}$$

$$\Gamma_L=|\Gamma_L|e^{j\theta}$$

$$|\Gamma_L|=\frac{SWR-1}{SWR+1}$$

2-3 Characteristic Impedance

Impedance = $\frac{Voltage}{Current}$

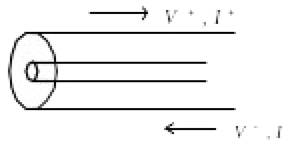
- Characteristic impedance: Z_0
- Input impedance: Z_{in}
- Intrinsic impedance: $\eta_0=120\pi=377\,\Omega=$

- Wave impedance: $\eta=\eta_o\sqrt{\frac{\mu_r}{\varepsilon_r}}=\sqrt{\frac{\mu_o}{\varepsilon_o}}\sqrt{\frac{\mu_r}{\varepsilon_r}}$

- Waveguide impedance: Z_{TE}, Z_{TM}

Characteristic Impedance is defined as

$$Z_o=\frac{V^+}{I^+}=-\frac{V^-}{I^-}$$

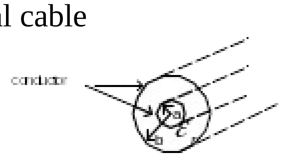


$$\frac{incident \quad voltage}{incident \quad current}$$

Important: 1) Z $_0$ is independent of voltage, current and position.

2) Z $_0$ is a function of materials and geometry.

e.g. coaxial cable



Assume $\mu=\mu_o$: Non magnetic material

$$Z_o=\frac{\eta\ln\frac{b}{a}}{2\pi}$$

$$=\frac{\eta_o}{2\pi\sqrt{\varepsilon_r}}\ln\left(\frac{b}{a}\right)$$

$$\eta=\sqrt{\frac{\mu_r}{\varepsilon_r}}\eta_o=\frac{\eta_o}{\sqrt{\varepsilon_r}}=\frac{120\pi}{\sqrt{\varepsilon_r}}$$

50 Ω semi rigid coaxial cables

Type: 0.087’’ a= 0.02”, b=0.066”, OD = 0.087”

$f_c=60\,GHz$

Type: 0.141” a= 0.036”, b=0.118”, OD = 0.141”

$f_c=34\,GHz$

Type: 0.25” a= 0.074”, b=0.21”, OD = 0.25”

$f_c=21\,GHz$

$\varepsilon_r : PTFE$ Teflon based

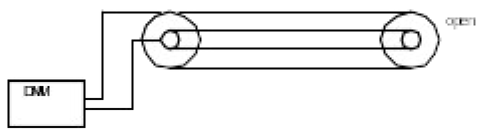
$\varepsilon_r \sim 2.2$

Cable size is determined by

1) Cutoff frequency
2) Power handling capacity (breakdown voltage)

Coaxial cables are designed for TEM wave.
Cut off freq is given by the next TE or TM mode.

Can we measure the characteristic impedance?

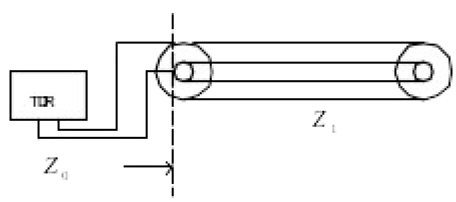


DMM reading will be ∞ (open circuit) because DMM measures

$$R = \frac{voltage}{current} = \frac{finite}{zero} \rightarrow \infty$$

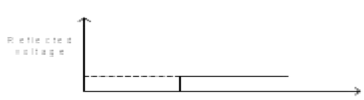
Total values

TDR(Time domain reflectometer) can measure the characteristic impedance.



Reflection

$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_o}$$



TDR can measure the reflected voltage as a function of time.
(TDR can separate the reflected voltage from the total voltage).

2-4 Input Impedance

$$Z_{in}(\ell) = \frac{total \quad voltage}{total \quad current}$$

$$= \frac{V(\ell)}{I(\ell)}$$

$$= \frac{V_0^+\left(e^{j\beta\ell} + \Gamma_L e^{-j\beta\ell}\right)}{\frac{V_0^+}{Z_0}\left(e^{j\beta\ell} - \Gamma_L e^{-j\beta\ell}\right)}$$

$$= Z_0\left[\frac{1+\Gamma_L e^{-2j\beta\ell}}{1-\Gamma_L e^{-2j\beta\ell}}\right]$$

since $\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}$

$$Z_{in}(\ell) = Z_0\left(\frac{Z_L + jZ_0 \tan\beta\ell}{Z_0 + jZ_L \tan\beta\ell}\right)$$

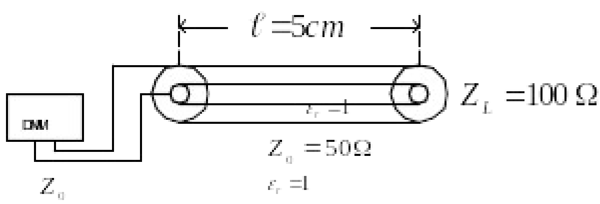
Z_{in} is a function of Z_0 , Z_L , position ℓ and propagation constant β .

Very short TL $\beta\ell = \frac{2\pi}{\lambda}\ell << 1$ [Low freq. [[short ℓ

$\tan\beta\ell \mapsto 0$

then $Z_{in}(\ell) = Z_L$

at DC $(\lambda = \infty)$



DMM reading will be 100Ω because $\beta\ell = 0$ $(\lambda = \infty)$.

How about at 1 GHz? $(\lambda = 0.3m)$

$\beta\ell = \frac{2\pi}{0.3} 0.05 = 1.05$ radian

$$Z_{in}(\ell) = 50\left(\frac{100 + j50 \times \tan(1.05)}{50 + j100 \times \tan(1.05)}\right)$$

$$= 41.9e^{-j30}$$

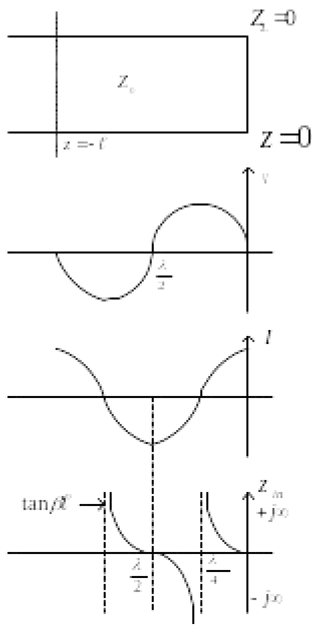
2-5 TL with Open and Short

Input impedance of different cases

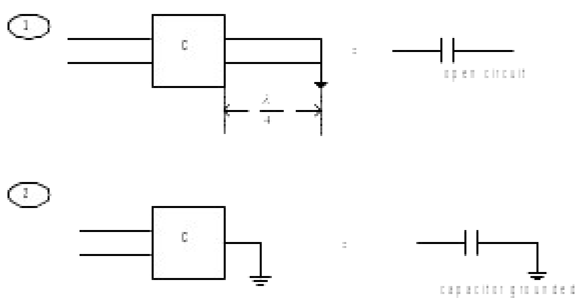
TL with Short circuit

$$Z_{in}(\ell) = jZ_0 \tan \beta\ell$$

A purely imaginary value. It can be negative or positive.



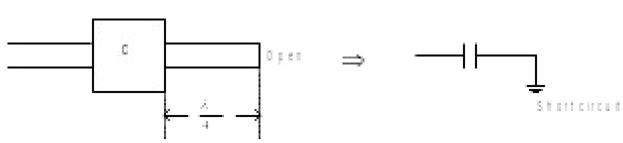
Capacitor



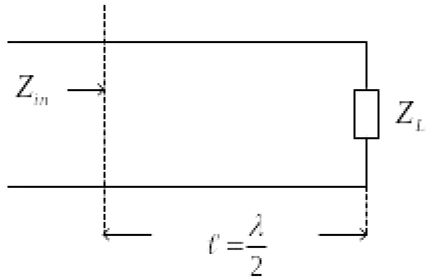
TL with Open $Z_L = \infty$

$$Z_{in}(l) = -jZ_0 \cot \beta l$$

pure imaginary



Load Z_L **and** $\ell = \frac{\lambda}{2}$



$$\beta \ell = \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{2}\right) = \pi$$

$$\tan \pi = 0$$

$$Z_{in}\left(\ell = \frac{\lambda}{2}\right) = Z_L$$

$$Z_{in}\left(\ell = \frac{\lambda}{2} m\right) = Z_L$$

where m: integer

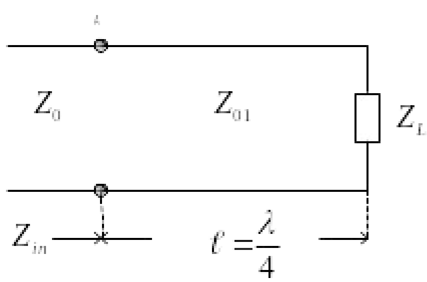
Load Z_L **and** $\ell = \frac{\lambda}{4}$

$$\beta \ell = \left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{4}\right) = \frac{\pi}{2}$$

$$\tan\left(\frac{\pi}{2}\right) = \infty$$

$$Z_{in}\left(\ell = \frac{\lambda}{4}\right) = \frac{Z_0^2}{Z_L}$$

Applications for impedance matching.



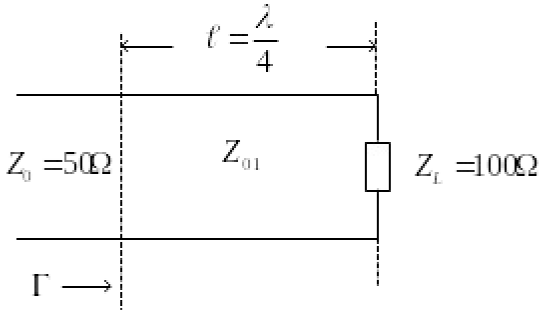
$$Z_{in} = \frac{Z_{01}^2}{Z_L}$$

If $Z_{in}=Z_0$, the impedance is matched at A.

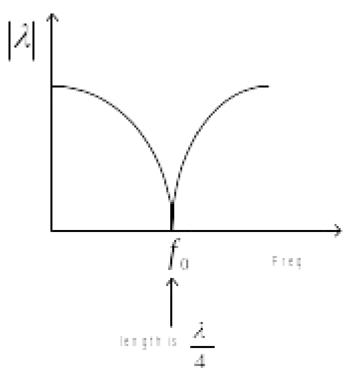
Let $Z_{in}=\frac{Z_{01}^2}{Z_L}=Z_0$

$Z_{01}=\sqrt{Z_0 Z_L}$

If Z_{01} is set to $\sqrt{Z_0 Z_L}$, we can match the impedance at A.



If $Z_{01}=\sqrt{Z_L Z_0}$
$=70.7\Omega$

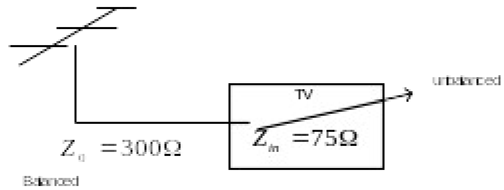


length is $\frac{\lambda}{4}$

Perfectly matched at f_0

This is called " $\frac{\lambda}{4}$ **impedance transformer"**

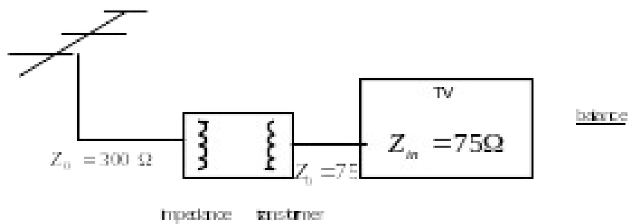
2-6 Impedance Matching Techniques



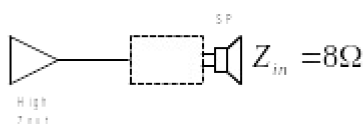
We found 36% of the incoming energy will be reflected.

Solution: Impedance matching

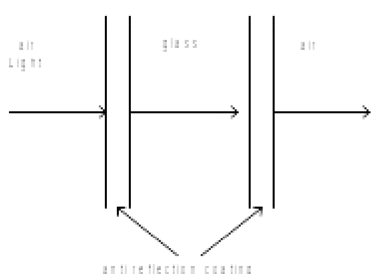
1. Impedance transformer.
   TV



   Audio amp



2. Antireflection coating ($\frac{\lambda}{4}$ impedance transformer)
   Good for microwave and optics

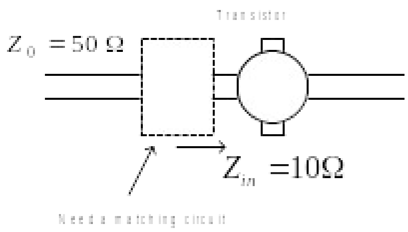


anti-reflection coating

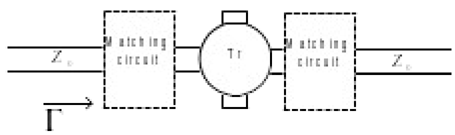
thickness $\frac{\lambda}{4}m$ m: integer

Material $\eta=\sqrt{\eta_0\eta_{glass}}$

3. Stub tuning

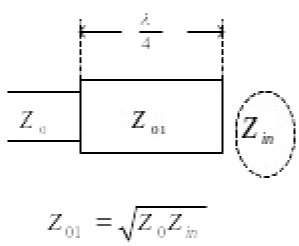


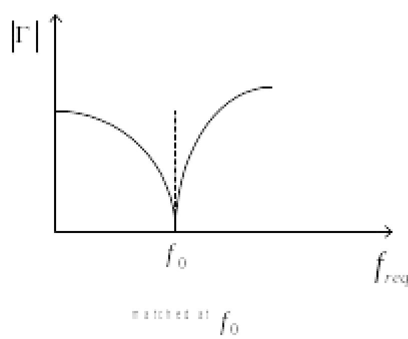
Need a matching circuit



The ideal condition is $Z_{in} = Z_0$ for all frequency.
However, this is impossible to achieve

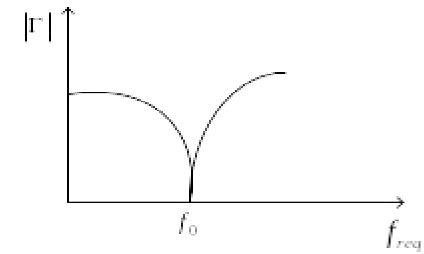
Practical matching circuit

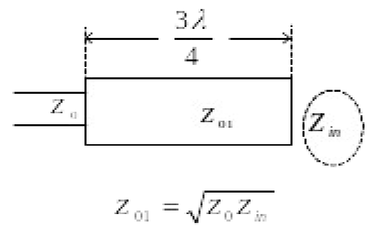


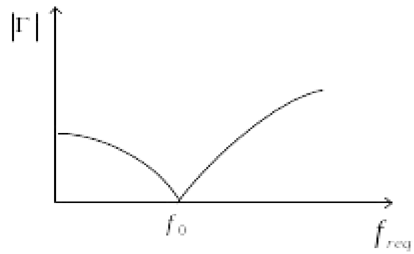


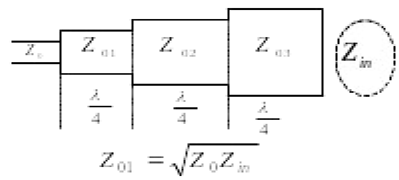
Other cases

Narrow bandwidth



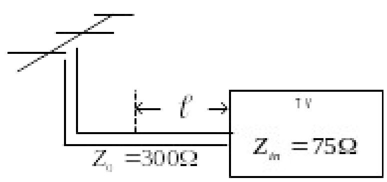






Wide bandwidth

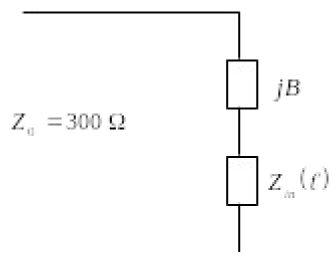
2.7 Series and Parallel (Stub) Matching techniques



$$|\Gamma_L| = \frac{75-300}{75+300} = -0.6$$

$$Z_{in}(l) = Z_0\left(\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right) \quad , \qquad Z_L = Z_{in} = 75\,\Omega$$

If we can get $Z_{in}(l) = 300 + jA$ at $z = -l$, the imaginary part (jA) can be cancelled using a capacitor or inductor.



$Z_{in}(\ell) = 300 + jA$

If $jB = -jA$, then the total load impedance is $300\,\Omega$ "matched".

Normalize the load impedance with Z_0.

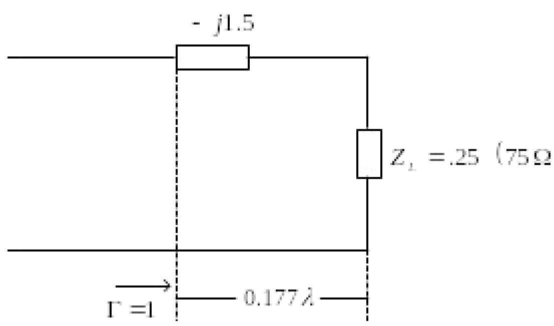
$$Z_{LN} = \frac{75}{300} = 0.25$$

We need to find

$Z_{inn}(\ell) = 300 + jA'$

At $\ell = 0.177\lambda$, we get $Z_{inn}(\ell) = 1 + j1.5$ (Normalized).

Matching circuit (Series matching)



How to get –j1.5?
This is capacitive.

$jX = (-j1.5)(300) = -j450$

Get an actual impedance

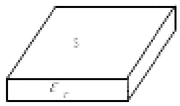
$$= -j\frac{1}{\omega C}$$

at 100MHz

C must be

$$C = \frac{1}{\omega 450} = 3.5\,pF$$

Plate capacitor



$d = 0.1$ mm

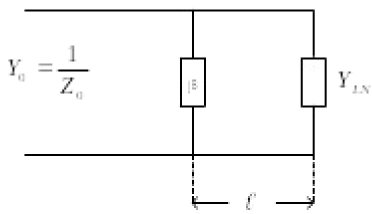
$\varepsilon_r = 2.5$

$$C = \frac{\varepsilon S}{d} = 3.5 \times 10^{-12} \quad \text{(F)}$$

$\therefore$ Size $S = 1.58 \times 10^{-5}$ m^2

~ 4 mm square

Parallel matching:



$$Y_{LN} = \frac{1}{Z_{LN}} = 4$$

We can find

$Y_{inv}(\ell) = 1 - j1.5$

at $\ell = 0.07\lambda$

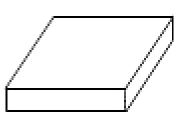
$\therefore jB = +j1.5$

capacitive

$jB = j\omega C$

Matching circuit requires a pure imag. component.

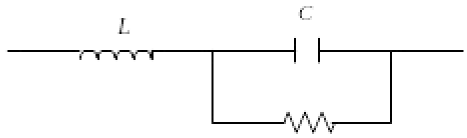
Capacitor



ideal case

$$X = \frac{1}{j\omega C} = -j\frac{1}{\omega C}$$

In reality, the equivalent circuit of C is



$$Z = j\omega L + \frac{1}{j\omega C} \quad \text{if} \quad R = 0$$

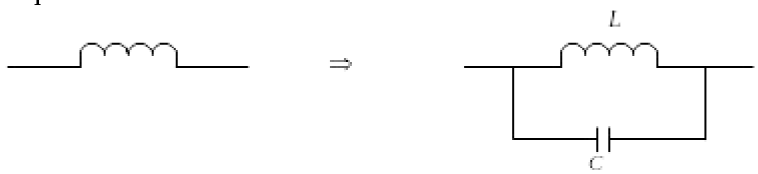
Due to *L*, this has a resonant frequency of f_0

Below f_0 : capacitive

Above f_0 : becomes inductive

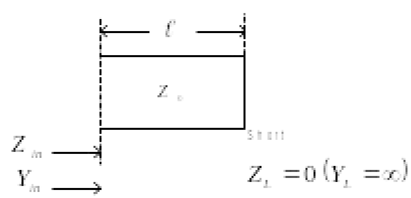
Inductance

Equivalent circuit of an inductance



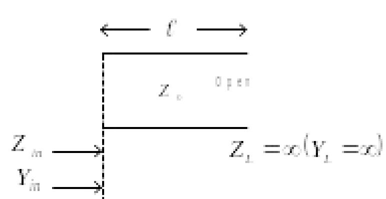
Discrete elements $(C \wedge L)$ may not be suited for high frequency applications.

For microwave circuits, a TL with open or short load is commonly used for matching impedance.



$$Z_{in} = jZ_0 \tan \beta l \ell$$

$$Y_{in} = \frac{1}{Z_{in}} = -jY_0 \cot \beta l \ell$$

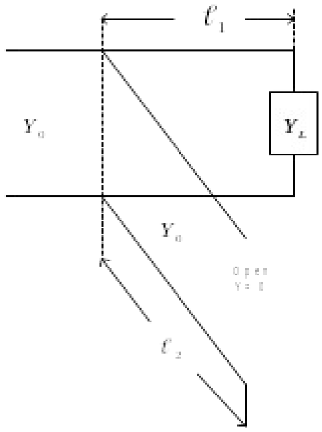


$$Z_{in} = -jZ_0 \cot \beta l \ell$$

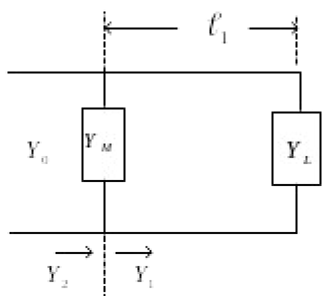
$$Y_{in} = jY_0 \tan \beta l \ell$$

The input impedance is purely imaginary and it can be capacitive or inductive depending on the length ℓ .

Single stub tuning technique



Equivalently



If the stub is open ended, then

$$Y_M = jY_0 \tan \beta \ell_2$$, ℓ_2 is unknown.

without Y_M

$$Y_1(z = -\ell_1) = Y_0 + jB$$, ℓ_1 : unknown

with Y_M

$$Y_2 = Y_1 + Y_M$$
$$= Y_0 + jB + jY_0 \tan \beta \ell_2$$

We want to set

$$Y_2 = Y_0$$
$$\therefore Y_0 \tan \beta \ell_2 = -B$$
$$\tan \beta \ell_2 = -\frac{B}{Y_0}$$
$$\therefore \ell_2 = \frac{1}{\beta} \tan^{-1}\left(-\frac{B}{Y_0}\right)$$

Procedure

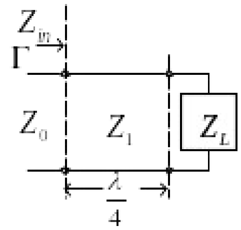
1. Need to find ℓ_1 at which

$$Y_{in}=Y_0+jB$$

2. Need to find ℓ_2 to get

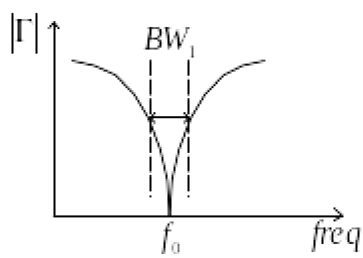
$$Y_M=jY_0\tan\beta\ell_2=-jB$$

2-8 Quarter Wave Transformer



$$Z_L=25\,\Omega \qquad \Gamma_L=0.33$$
$$Z_0=50\,\Omega$$

If $Z_1=\sqrt{Z_0Z_L}=35\,\Omega$, TL is matched.



How to find an approximate bandwidth (BW)?

How to increase BW?

$$Z_{in}=Z_1\frac{Z_L+jZ_1t}{Z_1+jZ_Lt} \qquad \text{where } t=\tan\beta\ell=\tan\theta$$

$$\Gamma=\frac{Z_{in}-Z_0}{Z_{in}+Z_0}$$

$$=\frac{Z_L-Z_0}{Z_L+Z_0+jt\sqrt{Z_0Z_L}}$$

$$|\Gamma|=\frac{1}{\left\{1+\left[4Z_0Z_L/\left(Z_L-Z_0\right)^2\right]+\left[4Z_0Z_Lt^2/\left(Z_L-Z_0\right)^2\right]\right\}^{\frac{1}{2}}}$$

$$|\Gamma| = \frac{1}{\left\{1 + \left[4 Z_0 Z_L / \left(Z_L - Z_0\right)^2\right] \sec^2\theta\right\}^{\frac{1}{2}}}$$

$$1 + t^2 = 1 + \tan^2\theta = \sec^2\theta$$

at $f = f_0$, $\ell = \frac{\lambda}{4}$ and $\theta = \beta l \ell = \frac{\pi}{2}$

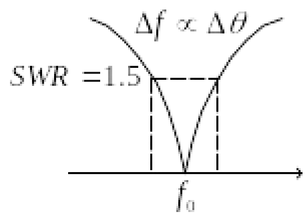
To find a bandwidth for a given SWR

Let the max SWR be 1.5 $(\Gamma = 0.2)$

$$SWR = \frac{1 + |\Gamma_m|}{1 - |\Gamma_m|} = 1.5$$

Find θ_m which gives $SWR = 1.5$

At $f = f_0$, $\theta = \frac{\pi}{2}$ The band width is given by $\Delta\theta$



$$\Delta\theta = 2\left(\frac{\pi}{2} - \theta_m\right)$$

Assume TL has a TEM mode, then $\lambda = \frac{v_p}{f}$: constant

$$\beta\ell = \left(\frac{2\pi f}{v_p}\right)\left(\frac{v_p}{4 f_0}\right)$$, $\frac{\lambda}{4} = \frac{1}{4}\frac{v_p}{f_0}$ at the designed freq.

$$= \frac{\pi f}{2 f_0}$$

$$\text{¿}\theta$$

Bandwidth

$$\Delta f = 2\left(f_0 - f_m\right)$$

$$\frac{\Delta f}{f_0} = 2 - 2\frac{f_m}{f_0}$$

$$= 2 - \frac{4\theta_m}{\pi}$$

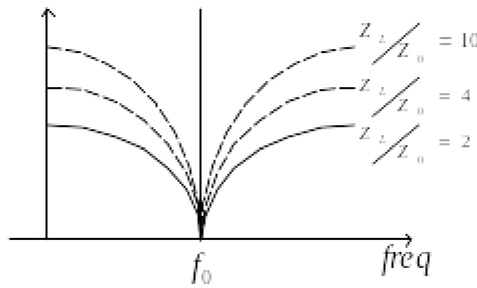
Since $$\cos\theta_m = \left(\frac{\Gamma_m}{\sqrt{1 - \Gamma_m^2}}\right)\frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|}$$

$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi} \cos^{-1}\left[\frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}} \frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|} \right]$$

where Γ_m is given by given the max SWR.

Example

$$\left[\begin{array}{l} f_0 = 3\,GHz \\ SWR = 1.5 \\ Z_L = 10\Omega \\ Z_0 = 50\Omega \end{array} \right.$$



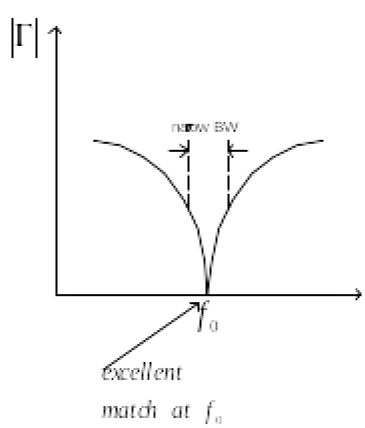
$$\Gamma_m = \frac{SWR-1}{SWR+1} = 0.2$$

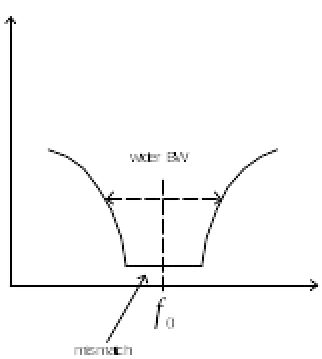
$$\frac{\Delta f}{f} = 0.29$$

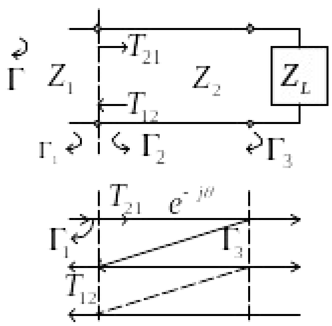
or 29%

Theory of small reflections

From the impedance matching study, we found that BW becomes small if Z_L/Z_0 is large. We want to find a method to increase the BW.







$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$
$$\Gamma_2 = -\Gamma_1$$
$$\Gamma_3 = \frac{Z_L - Z_2}{Z_L + Z_2}$$
$$T_{21} = 1 + \Gamma_1$$
$$T_{12} = 1 + \Gamma_2$$

$$\Gamma = \frac{Z_{in} - Z_1}{Z_{in} + Z_1}$$

$$= \Gamma_1 + T_{12} T_{21} \Gamma_3 e^{-2j\theta} + T_{12} T_{21} \Gamma_3^2 \Gamma_2 e^{-4j\theta} + \ldots..$$

¿ $\Gamma_1 + T_{12} T_{21} \Gamma_3 e^{-2j\theta} \sum_{n=0}^{\infty} \Gamma_2^n \Gamma_3^n e^{-2jn\theta}$

Since

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $|x| < 1$

$(\Gamma_2 = -\Gamma_1$

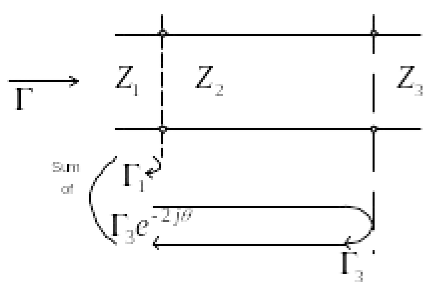
$(T_{21} = 1 + \Gamma_1$ (

$(T_{12} = 1 + \Gamma_2 = 1 - \Gamma_1$

$$\Gamma = \Gamma_1 + \frac{T_{12} T_{21} \Gamma_3 e^{-2j\theta}}{1 - \Gamma_2 \Gamma_3 e^{-2j\theta}}$$
$$= \frac{\Gamma_1 + \Gamma_3 e^{-2j\theta}}{1 + \Gamma_1 \Gamma_3 e^{-2j\theta}}$$

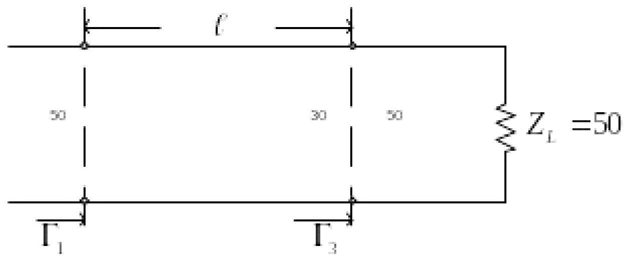
Since $|\Gamma_1 \Gamma_3| << 1$ for most cases

$$\Gamma \sim \Gamma_1 + \Gamma_3 e^{-2j\theta}$$



Multiple reflection is not important if $|\Gamma_1 \Gamma_3| << 1$

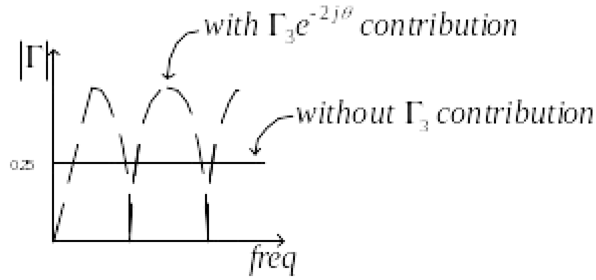
Frequency response of slightly mismatched TL



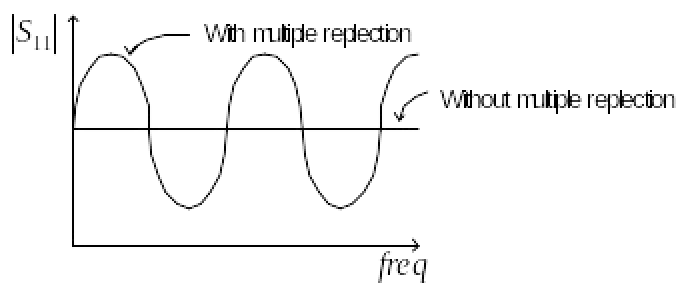
$$\Gamma_1 = \frac{30-50}{30+50} = -0.25$$

$$\Gamma_3 = \frac{50-30}{50+30} = 0.25$$

$$|\Gamma_1 \Gamma_3| = 0.0625 << 1$$

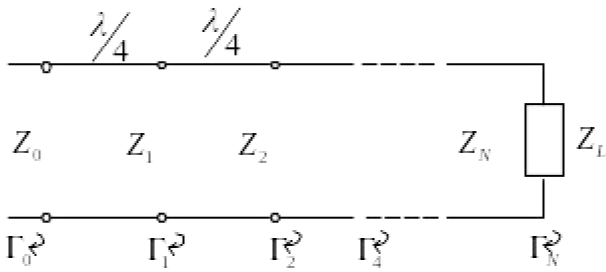


In general



Multi section Transformer

Provide wider bandwidth but a circuit becomes complex.



If each reflection is small, then

$$\Gamma(\theta) \sim \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \ldots\ldots$$

Assume we can satisfy symmetrical condition.

$$\Gamma_0 = \Gamma_N \quad , \quad \Gamma_1 = \Gamma_{N-1} \quad , \quad \Gamma_2 = \Gamma_{N-2}$$

This does not imply $Z_0 = Z_N$, $Z_1 = Z_{N-1}$ ……

For N: even

$$\Gamma(\theta) = 2e^{-jN\theta}\left[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \ldots\ldots \frac{1}{2}\Gamma_{N/2}\right]$$

For N: odd

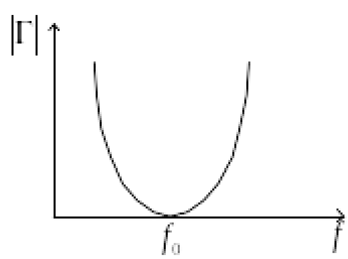
$$\Gamma(\theta) = 2e^{-jN\theta}\left[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \ldots\ldots \Gamma_{(N-1)/2} \cos\theta\right]$$

We need to find a method to calculate Z_n from $\Gamma(\theta)$.

Binomial transformer

(Maximally flat response)

Polynomial of the form $c_0\left(1 + c_1 x^2 + c_2 x^4 + \ldots\ldots\right)$



at f_0 the deviation

$$\frac{d^n|\Gamma(\theta)|}{d\theta^n}\Big|_{f=f_0} = 0 \qquad n = 1\ldots\ldots\ldots N-1$$

Let $\Gamma(\theta) = A\left(1 + e^{-2j\theta}\right)^N$

This gives $\Gamma(\theta)=A\left(1+c_1 e^{-2j\theta}+c_2 e^{-4j\theta}+\ldots\right)$

$$|\Gamma(\theta)|=|A||e^{-j\theta}|^N|e^{j\theta}+e^{-j\theta}|^N$$
$$=2^N|A||\cos\theta|^N$$

We need to relate this to TL impedances

Let $f \to 0$ (DC), then $\beta\ell=0$ $(\theta=0)$

$$|\Gamma(0)|=2^N|A|=\left|\frac{Z_L-Z_0}{Z_L+Z_0}\right|$$

$$A=2^{-N}\left|\frac{Z_L-Z_0}{Z_L+Z_0}\right|$$

Next expand $\Gamma(\theta)$ using binomial coefficient

$$\Gamma(\theta)=A\left(1+e^{-2j\theta}\right)^N=A\sum_{n=0}^{N}C_n^N e^{-2jn\theta}$$

$$C_n^N=\frac{N!}{(N-n)!n!}$$

This must be equal to

$$\Gamma(\theta)=\Gamma_0+\Gamma_1 e^{-2j\theta}+\Gamma_2 e^{-4j\theta}+\ldots\ldots+\Gamma_N e^{-2jN\theta}$$

From these two equations

$$\Gamma_n=AC_n^N$$

Binomial coefficients C_n^N satisfy

$$C_n^N=C_{N-n}^N$$
$$C_0^N=1 \quad \text{and} \quad C_1^N=N=C_{N-1}^N$$

Since

$$\Gamma_n=\frac{Z_{n+1}-Z_n}{Z_{n+1}+Z_n}=AC_n^N$$

We know A and C_n^N then we should be able to find Z_n .

Approximately we can write

here we use

$$\Gamma_n=\frac{Z_{n+1}-Z_n}{Z_{n+1}+Z_n}\sim\frac{1}{2}\ln\frac{Z_{n+1}}{Z_n} \qquad \left[\ln x\sim 2\left[\frac{(x-1)}{x+1}+\frac{(x-1)^3}{3(x+1)^3}+\ldots\ldots\right]\right[\quad [x>0$$

Then

$$\ln\frac{Z_{n+1}}{Z_n}=2\Gamma_n=2AC_n^N=2\left(2^{-N}\right)\left(\frac{Z_L-Z_0}{Z_L+Z_0}\right)C_n^N$$

$$\sim 2^{-N} C_n^N \ln \frac{Z_L}{Z_0}$$

Bandwidth is given by

$$\frac{\Delta f}{f_0} = 2 - \frac{4\theta_m}{\pi}$$

$[T_m : \max \quad value$

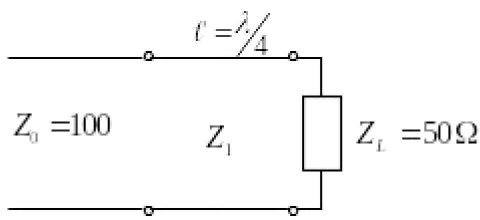
$[that \quad can \quad be \quad tolerated\, [$

$[eg : SWR = 1.5$

Since $\Gamma_m = 2^N |A| \cos^N \theta_m$

$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi} \cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$$

<u>Example</u>

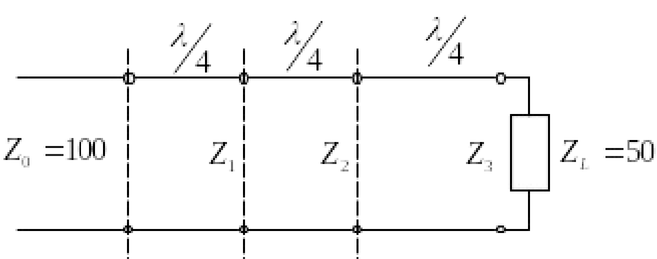


If $Z_1 = \sqrt{Z_0 Z_L} = 70.7$, matched at f_0

Band width $\Delta f / f_0 = 0.18$ for $SWR = 1.1$ $(\Gamma_m = 0.05)$

BW=<u>18%</u>

Multi-section N=3



$$A = 2^{-N} \left|\frac{Z_L - Z_0}{Z_L + Z_0}\right| = 0.0417$$

$\Gamma_m = 0.05$ $(SWR = 1.1)$

then $\frac{\Delta f}{f_0}=0.71$ 71% compared to 18% for a single section

$C_0^3=1$, $C_1^3=3$, $C_2^3=3$

$$\ln Z_1=\ln Z_0+2^{-3}C_0^3\ln\frac{Z_L}{Z_0}$$

Z_1: $Z_1=91.7\Omega$

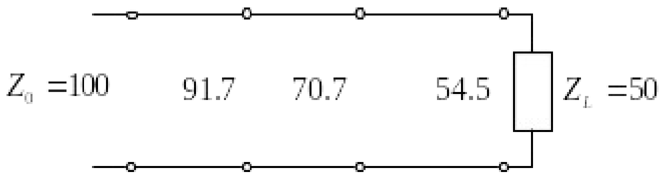
$$\ln Z_2=\ln Z_1+2^{-3}C_1^3\ln\frac{Z_L}{Z_0}$$

Z_2: $Z_1=70.7\Omega$

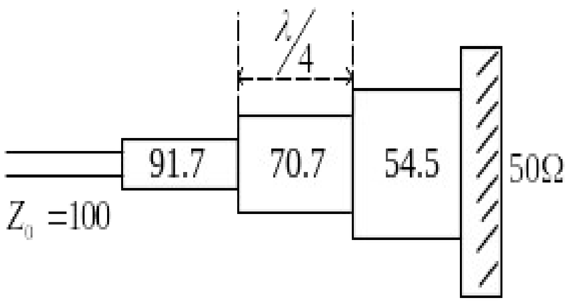
Z_3:

$$\ln Z_3=\ln Z_2+2^{-3}C_2^3\ln\frac{Z_L}{Z_0}$$

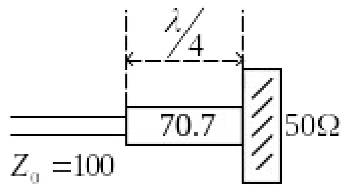
$Z_1=54.5\Omega$

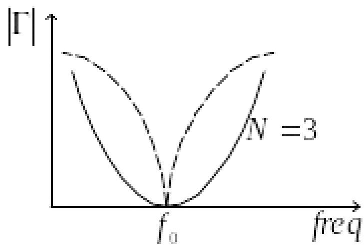


Microstrip circuit (Multi-section)



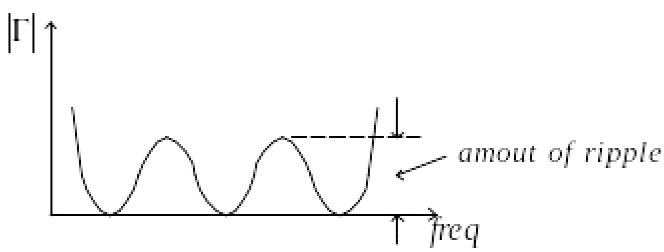
Microstrip circuit (single section)





Chebyshev Multisection Matching Transformers

Use Chebyshev (equal ripple) polynomials to design a matching circuit.



Process

$$\Gamma(\theta)=2e^{-jN\theta}\left[\Gamma_0\cos N\theta+\Gamma_1\cos(N-2)\theta+\ldots\Gamma_n\cos(N-2n)\theta+\ldots\right]$$

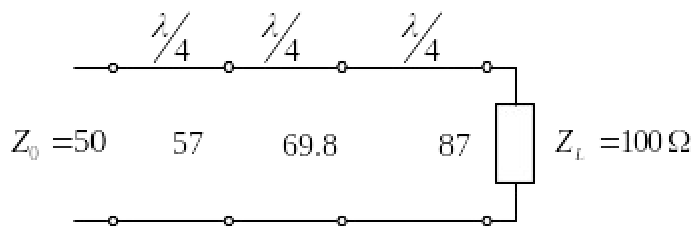
$$=Ae^{-jN\theta}T_N\left(\sec\theta_m\cos\theta\right)$$

$T_N\left(\sec\theta_m\cos\theta\right)$: Chebyshev polynomials.

$$T_1\left(\sec\theta_m\cos\theta\right)=\sec\theta_m\cos\theta$$
$$T_2\left(\sec\theta_m\cos\theta\right)=\sec^2\theta_m(1+\cos 2\theta)-1$$
$$T_3\left(\sec\theta_m\cos\theta\right)=\sec^2\theta_m(\cos 3\theta+3\cos\theta)-3\sec\theta_m\cos\theta$$

<u>Example</u>

3 section transformer $\Gamma_m=0.05$, $N=3$

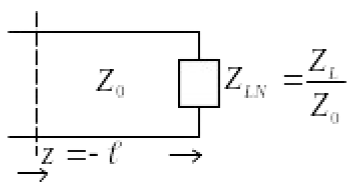


$$\frac{\Delta f}{f_0}=1.02 \qquad \text{better than the binomial method}$$

2-9 Smith Chart

Developed by P. Smith at Bell Lab in 1939.

Graphical method to get $\Gamma(\ell)$ and $Z_{in}(\ell)$ simultaneously.



$\Gamma(\ell)$

$Z_{in}(\ell)$

$$\Gamma_L=\frac{Z_L-Z_0}{Z_L+Z_0}=|\Gamma_L|e^{j\theta}$$

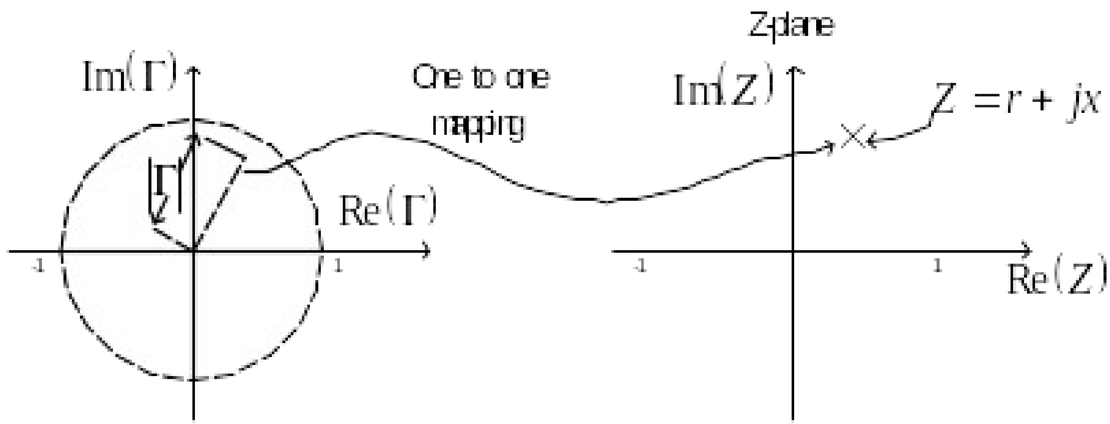
$$Z_{LN}=\frac{1+\Gamma_L}{1-\Gamma_L}$$

$$\Gamma(\ell)=|\Gamma_L|e^{j\theta}e^{-2j\beta\ell}$$

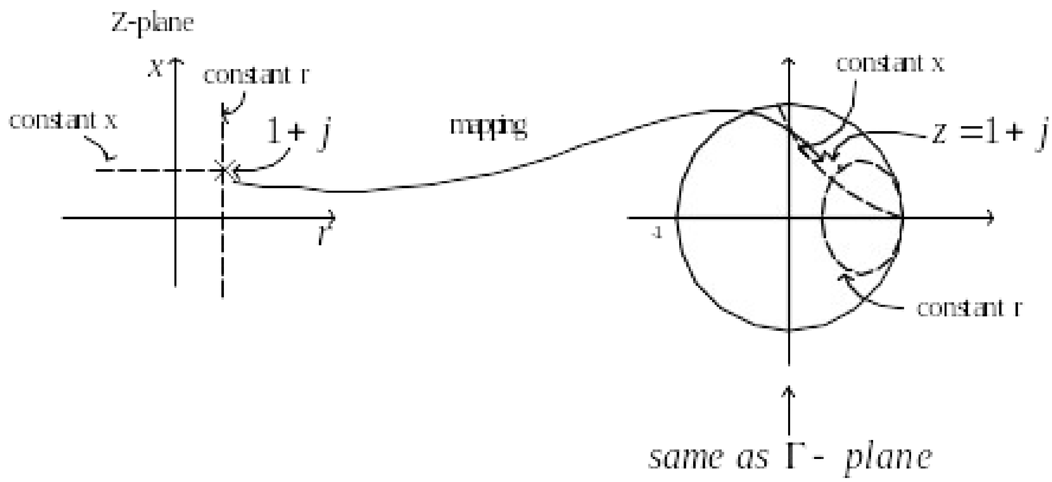
$$Z_{in}(\ell)=Z_0\left(\frac{1+\Gamma_L e^{-2j\beta\ell}}{1-\Gamma_L e^{-2j\beta\ell}}\right)=Z_0\left(\frac{1+|\Gamma_L|e^{j\theta}e^{-2j\beta\ell}}{1-|\Gamma_L|e^{j\theta}e^{-2j\beta\ell}}\right)$$

Range of Γ

$0<|\Gamma|<1$ for $\mathrm{Re}(Z)>0$



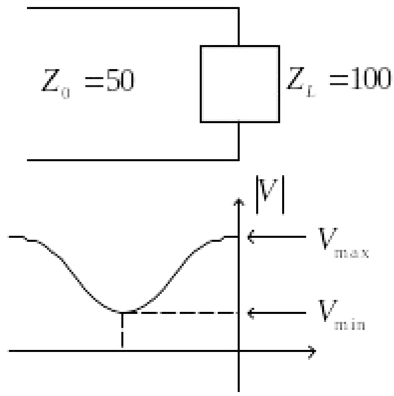
Deformation of Z-plane



1. Relate impedance $Z_{in}(\ell)$ to $\Gamma(\ell)$
2. Can show $Z_{in}(\ell)$ and $\Gamma(\ell)$ on TL
3. Can show SWR and position of $V_{\min}(V_{\max})$ on TL.
4. Can be used for impedance matching.
5. Can show Z_{in} and Γ at different freq.
6. Can be used as impedance (Z) chart or admittance (Y) chart

Standing wave ratio (SWR)

$$\Gamma_L = \frac{100-50}{150} = \frac{1}{3}$$



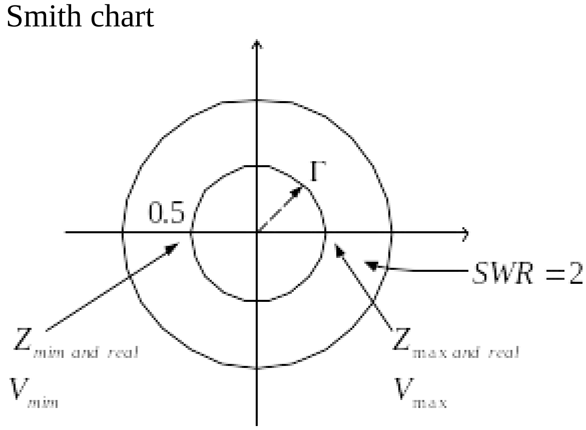
$$V_{\max} = V_0 \left(1 + |\Gamma|\right)$$

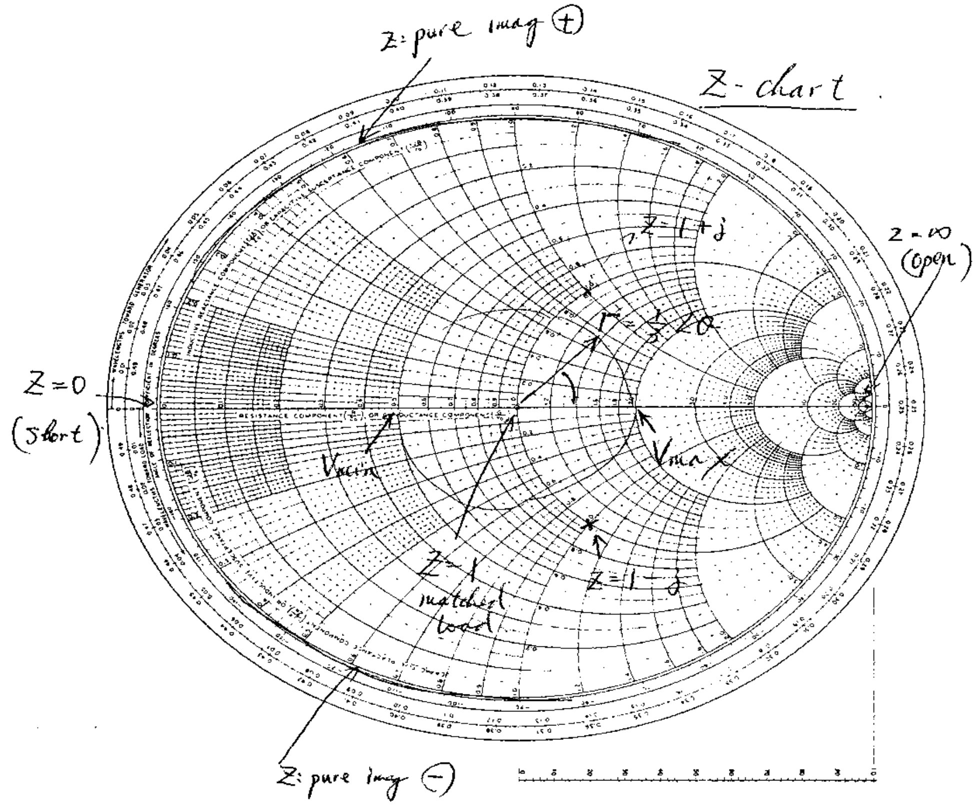
$$\frac{V_{\max}}{I} = Z_0 \left(\frac{|V^+| + |V^-|}{|V^+| - |V^-|} \right) = Z_0 \left(\frac{1 + |\Gamma|}{1 - |\Gamma|} \right) = (Z_0)(SWR)$$

$$= Z_{\text{max and real}}$$

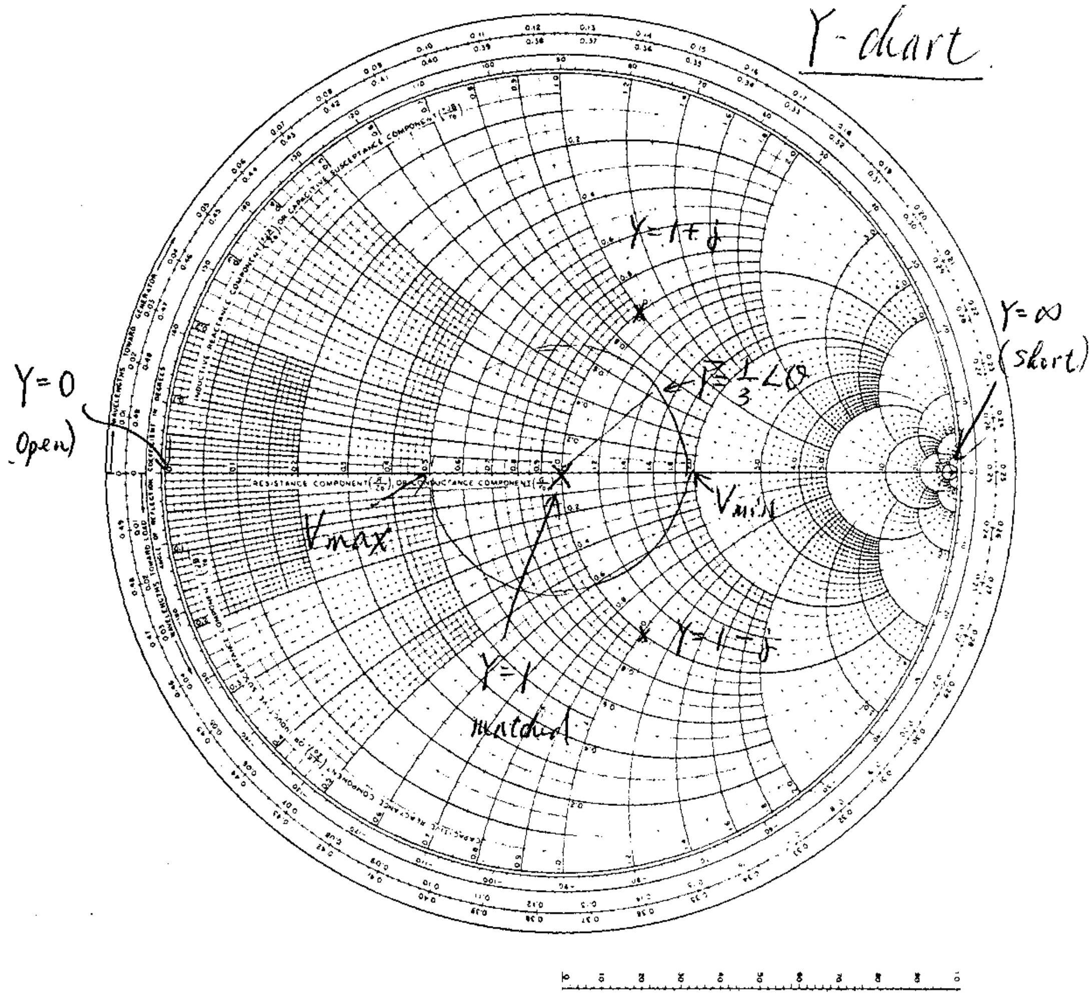
$$\boxed{SWR = \frac{Z_{\text{max and real}}}{Z_0}}$$

Smith chart

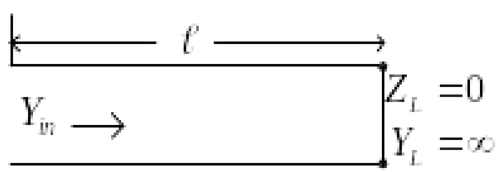




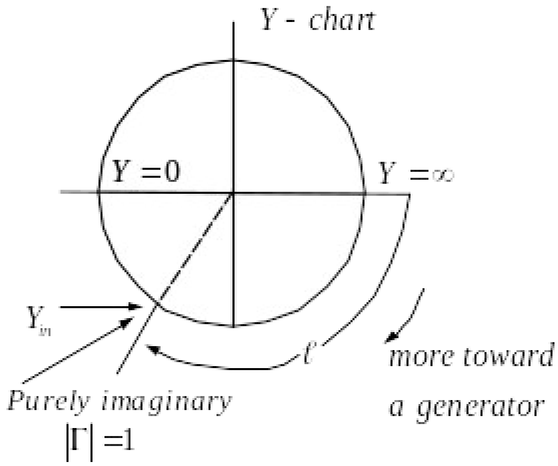
Z: pure imag (+)
Z - chart
Z = 1 + j
Z = ∞
(open)
Γ = 1/3 ∠θ
Z = 0
(short)
Vmin
Vmax
Z = 1
matched
load
Z = 1 − j
Z: pure imag (−)



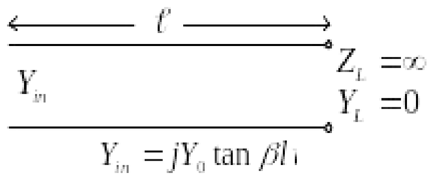
TL with the Short load.

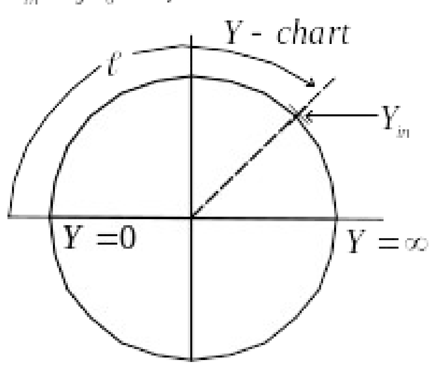


$Y_{in} = -jY_0 \cot \beta l$

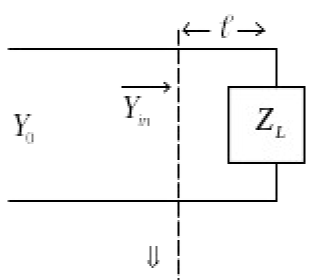


TL with an open load.

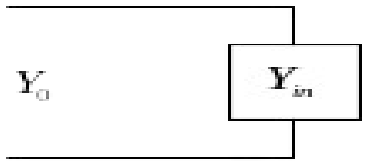




2.10 Single stub tuning using Smith Chart

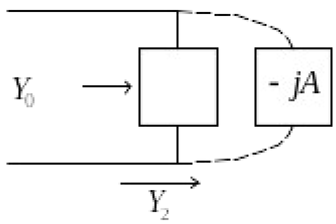


Find ℓ_1 so that Y_{in} becomes $Y_{in} = Y_0 + jA$



$\Downarrow$

Add $-jA$ to cancel the imag. part



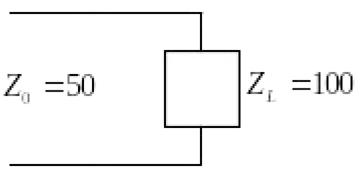
Then $Y_2 = Y_1 - jA$

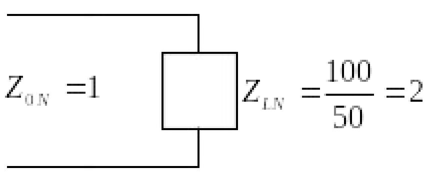
$= Y_0 + jA - jA$

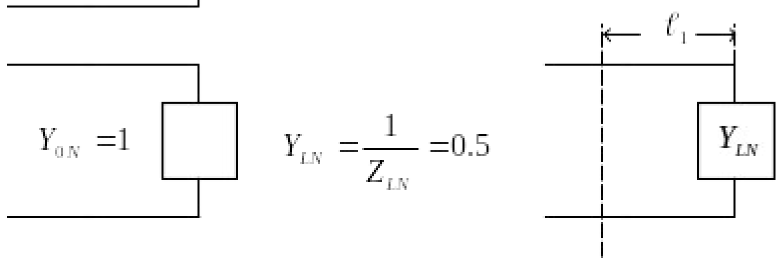
¿Y_0

<u>Matched</u>:

Example:



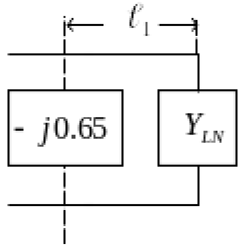




at $l\ell_1 = 0.15\lambda$

$$Y_{inN} = 1 + j0.65$$

Add $b = -j0.65$ at $l\ell$

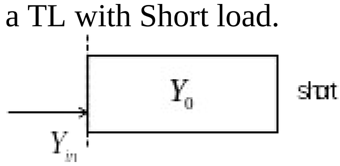


$b = -j0.65$ is a normalized admittance. To un-normalize b

$$Y_M = (-j0.65)Y_0 = -j0.013$$

To get Y_M

Using a TL with Short load.



$$Y_{in} = -jY_0 \cot \beta l\ell = -j0.013$$

$$\cot \beta l\ell = 0.65$$

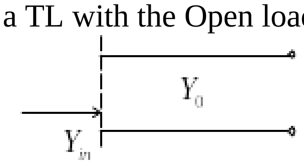
$$\beta\ell = 57^0$$

$$\ell = \left(\frac{\lambda}{2\pi}\right)(1) = 0.16\lambda$$

57^0 in radian

or $\ell = 0.16\lambda + n\dfrac{\lambda}{2}$, $n: 1, 2, \ldots.$

Using a TL with the Open load



$$Y_{in} = jY_0 \tan \beta l\ell = -j0.013$$

$$\tan \beta l\ell = -j0.65$$

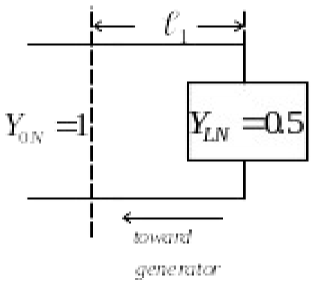
$$\beta l\ell = -33^0$$

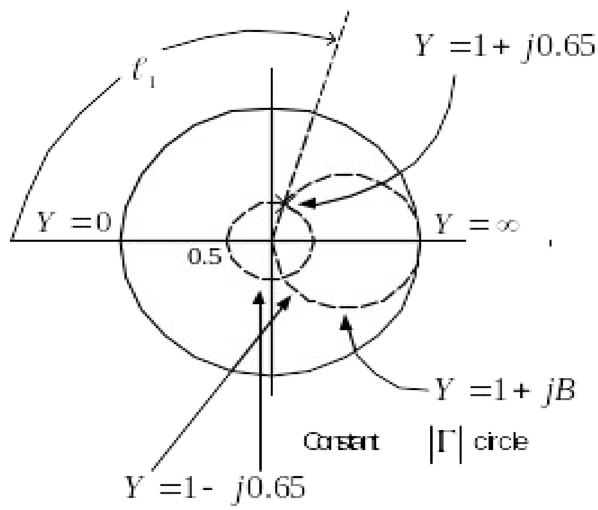
$$\ell' = \left(\frac{\lambda}{2\pi}\right)(-0.58) = -0.09\lambda$$

ℓ must be positive.

$$\therefore \quad \ell' = \frac{\lambda}{2} - 0.09\lambda = 0.41\lambda$$

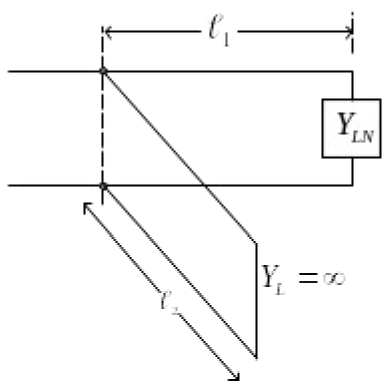
Single stub tuning using Smith Chart

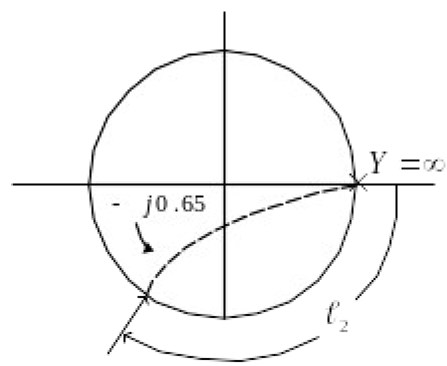




$\ell_1 = 0.15\lambda$

Using a TL with Short load.



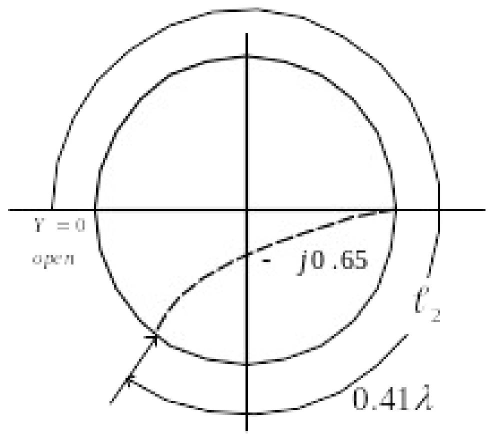


We need to find $l\ell_2$ to get $-j0.65$

$$l\ell_2 = 0.16\lambda$$

or $$l\ell_2 = 0.16\lambda + n\frac{\lambda}{2}, \quad n: 1, 2, \ldots$$

Using a TL with the Open load



$\ell = 0.41\lambda$

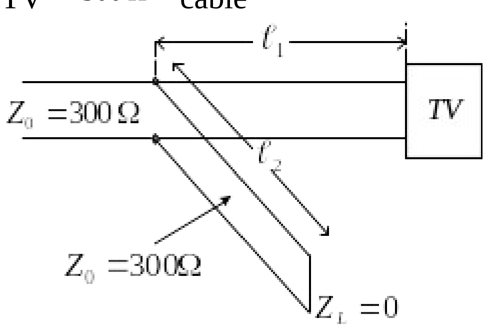
or

$\ell = 0.41\lambda + n\frac{\lambda}{2}$, $n: 1,2,3....$

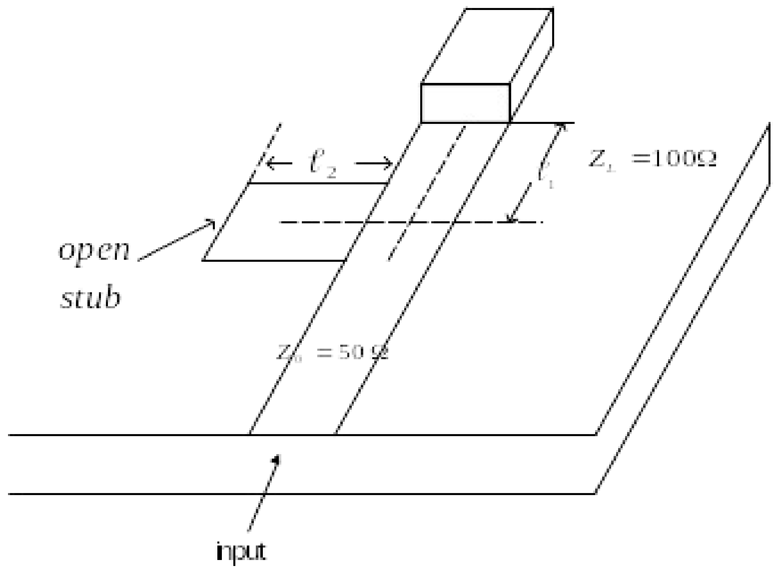
Tuning circuit is designed for the desired wavelength of λ. Therefore, it is a narrow band circuit.

Examples of stub tuning:

1. TV $300\,\Omega$ cable



2. Microstrip line



ℓ_2
ℓ_1
$Z_L = 100\Omega$
open
stub
$Z_0 = 50\,\Omega$
input