Directional Couples 2/18/04

(1) Waveguide two-hole couplers

Analysis
Constructive and destructive interference effects

Co-directional coupler
Coupled signal will travel in the same direction as the incident wave.

(2) Microstrip or stripline couplers

Analysis
Even-and odd-mode
Set unwanted output to zero

Counter-directional coupler
Coupled signal will travel in the opposite direction as the incident wave.

(3) Multi-section coupled line couplers

Single section: Narrow bandwidth

Multi-section: Wider bandwidth
Multi-octave, Example: 2- 18 GHz
2- 8 GHz

Common type: Odd numbers of sections and each one is $\lambda/4$.
Analysis: Small coupling (C: small)
Does not satisfy the energy conservation

Two- hole Couplers (waveguide)

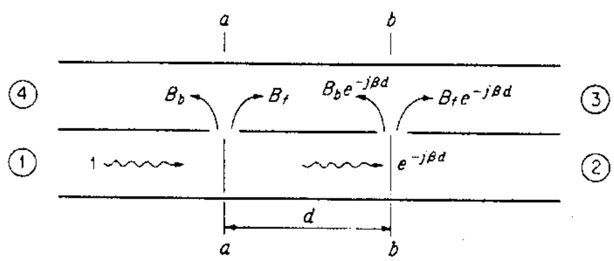


FIGURE 6.21
Two-hole directional coupler.

Forward direction is always in phase.
Path length is the same.

Backward direction may get no output if $d=\frac{\lambda_g}{4}$
Due to phase cancellation of two signals.

Coupled output at $|3|$

$$C = \underbrace{B_f e^{-j\beta d}}_{\text{through (a)}} + \underbrace{B_f e^{-j\beta d}}_{\text{through (b)}} = 2\,B_f e^{-j\beta d}$$

Coupled output at (4)

$$D = B_b + B_b\, e^{-2j\beta d}$$
$$D = B_b\, 2e^{-j\beta d}\cos\beta d$$
$$|D| = 2B_b\,|\cos\beta d|$$

$$D=0 \quad at \quad \beta d=\frac{\pi}{2}$$
$$or \quad d=\frac{\lambda_g}{4}$$

Waveguide multi hole couplers

(1) Co directional coupler
Group velocity is in the same direction.
Phase velocity is in the same direction

(2) Port 4. (Undesired output port)
is zero if $d=\frac{\lambda_g}{4}$.
If the operating freq is changed, the output is non-zero.

(3) Fairly narrow band device

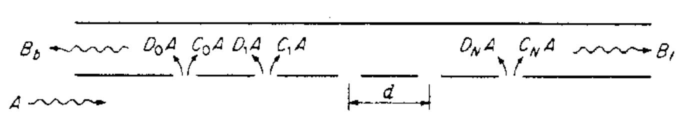


FIGURE 6.23
A multielement directional coupler.

Coupled-line Directional Couplers

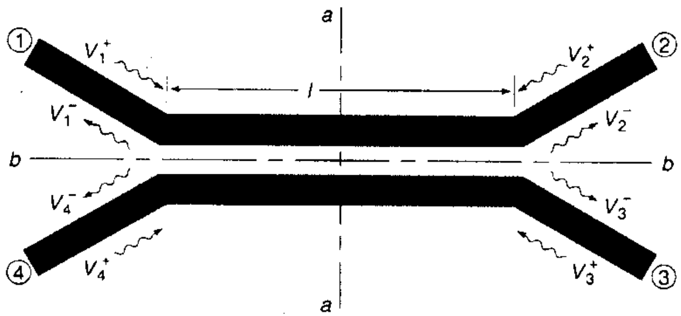
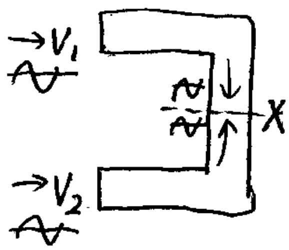


FIGURE 6.25
A microstrip coupled-line directional coupler.

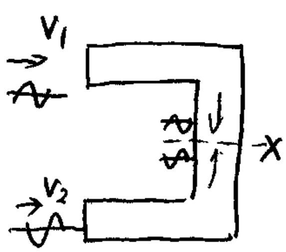
Odd-even mode analysis to get S-parameters.

Even mode



$V_1 = V_2 = +V_0$
effectively X is <u>open</u>
or $\Gamma = 1$ at X

Odd mode



$V_1 = +V_0$, $V_2 = -V_0$
effectively X is <u>short</u>
or $\Gamma = -1$ at X

Directional coupler has 2 symmetry planes: a-a and b-b
a-a can be OPEN or SHORT. Similarity, b-b can be OPEN or SHORT. Therefore, we have 4 combinations.

a/ a-a: OPEN

b-b: OPEN $\rightarrow$ *Even* Z_e, β_e

$$V_1^+ = V_2^+ = V_3^+ = V_4^+ = V^+$$

b/

a-a: SHORT

b-b: OPEN $\rightarrow$ *Even* Z_e, β_e

$$V_1^+ = V_4^+ = V^+$$
$$V_2^+ = V_3^+ = -V^+$$

c/

a-a: OPEN

b-b: SHORT $\rightarrow$ *Odd* Z_0, β_0

$$V_1^+ = -V_4^+ = V^+$$
$$V_2^+ = -V_3^+ = V^+$$

d/

a-a: SHORT

b-b: SHORT $\rightarrow$ *Odd* Z_0, β_0

$$V_1^+ = -V_4^+ = V^+$$
$$V_2^+ = -V_3^+ = -V^+$$

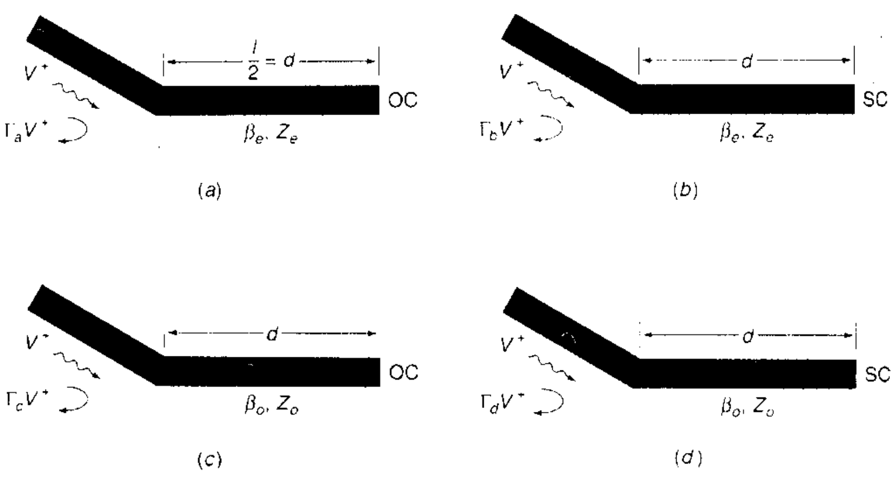
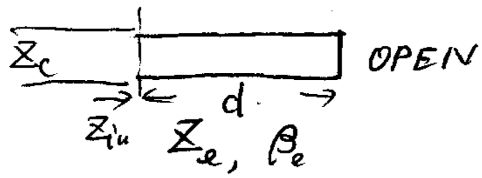


FIGURE 6.26
Equivalent circuit for one-quarter of the coupled-line directional coupler when (*a*) the planes *aa* and *bb* are magnetic walls, (*b*) *aa* is an electric wall and *bb* is a magnetic wall, (*c*) *aa* is a magnetic wall and *bb* is an electric wall, (*d*) *aa* and *bb* are both electric walls.

Input impedance & the reflection coefficient of a TL with OPEN

a/

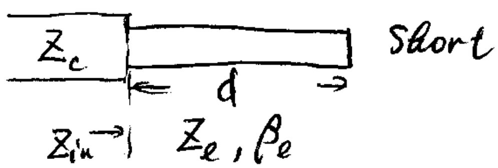


$$Z_{in} = -jZ_e \cot \beta_e d$$
$$\Gamma_a = \frac{Z_{in} - Z_c}{Z_{in} + Z_c} = \frac{-jZ_e \cot \beta_e d - Z_c}{-jZ_e \cot \beta_e d + Z_c}$$

Z_c : characteristic impedance of the line w/o coupled section

$$V_1^- = \Gamma_a V_1^+$$
$$V_2^- = \Gamma_a V_2^+$$
$$V_3^- = \Gamma_a V_3^+$$
$$V_4^- = \Gamma_a V_4^+$$

b/



$$Z_{in} = jZ_e \tan \beta_e d$$
$$\Gamma_b = \frac{jZ_e \tan \beta_e d - Z_c}{jZ_e \tan \beta_e d + Z_c}$$

c/

$$Z_{in} = -jZ_0 \cot \beta_0 d \quad , \quad \Gamma_c = \frac{-jZ_0 \cot \beta_0 d - Z_c}{-jZ_0 \cot \beta_0 d + Z_c}$$

d/

$$Z_{in} = jZ_0 \tan \beta_0 d \quad , \quad \Gamma_d = \frac{jZ_0 \tan \beta_0 d - Z_c}{jZ_0 \tan \beta_0 d + Z_c}$$

If we add (a), (b), (c) & (d), we have

$[V_1^+ = 4\,V^+$
$[V_2^+ = 0$ $\quad [$
$[V_3^+ = 0$
$[V_4^+ = 0$ only part 1 is excited

V_1^- is due to V_1^+ , V_2^+ , V_3^+ , V_4^+ . We can write

$$V_1^- = \Gamma_a V_1^+ + \Gamma_b V_2^+ + \Gamma_c V_3^+ + \Gamma_d V_4^+ = \frac{1}{4}\left(\Gamma_a + \Gamma_b + \Gamma_c + \Gamma_d\right) V_1^+$$

Similarly

$$V_2^- = \frac{1}{4}\left(\Gamma_a - \Gamma_b + \Gamma_c - \Gamma_d\right) V_1^+$$

$$V_3^- = \frac{1}{4}\left(\Gamma_a - \Gamma_b - \Gamma_c + \Gamma_d\right) V_1^+$$

$$V_4^- = \frac{1}{4}\left(\Gamma_a + \Gamma_b - \Gamma_c - \Gamma_d\right) V_1^+$$

Desired characteristics

$V_1^- = 0$ No reflection (matched)

$V_3^- = 0$ matched port

$V_4^- \ or\ S_{41}$ Coupled output

Or we can see

$$[\Gamma_a + \Gamma_d = 0$$
$$)[\Gamma_b + \Gamma_c = 0 \quad [\,) \rightarrow \ then \quad V_1^- = 0 \wedge V_3^- = 0$$
¿

$$\Gamma_a + \Gamma_d = \frac{2\left(Z_c^2 t_e - Z_e Z_0 t_e\right)}{jZ_c\left(Z_c - Z_0 t_e t_0\right) - Z_c^2 t_e - Z_e Z_0 t_0}$$

$$\Gamma_b + \Gamma_c = \frac{2\left(Z_c^2 t_0 - Z_e Z_0 t_e\right)}{jZ_c\left(Z_0 - Z_e t_e t_0\right) - Z_c^2 t_0 - Z_e Z_0 t_e}$$

$$t_0 = \tan\beta_0 d \quad , \quad t_e = \tan\beta_e d$$

The ideal directional coupler has $\beta=\beta_0=\beta_e \; (t_e=t_0)$

If we choose $Z_c^2=Z_e\,Z_0$, then

$$\begin{cases} \Gamma_a+\Gamma_d=0 \\ \Gamma_b+\Gamma_c=0 \end{cases}$$

Therefore, if we set $Z_c^2=Z_e\,Z_0$, we get

$$V_1^-=0$$

$$V_3^-=0$$

Using $t=t_e=t_0$ and $Z_c^2=Z_e\,Z_0$, we can obtain

$$\Gamma_a+\Gamma_b-\Gamma_c-\Gamma_d=2(\Gamma_a+\Gamma_b)=4\left[\frac{(Z_c^2-Z_e^2)\;t}{jZ_cZ_e(1-t^2)-(Z_c^2+Z_e^2)\;t}\right]$$

or

$$\Gamma_a+\Gamma_b=\frac{-2(Z_e-Z_0)\sin 2\beta d}{j2\,Z_c\cos 2\beta d-(Z_e+Z_0)\sin 2\beta d}$$

Finally we get

$$S_{41}=\frac{1}{2}(\Gamma_a+\Gamma_b)=\frac{j\left(\dfrac{Z_e-Z_0}{Z_e+Z_0}\right)\sin 2\beta d}{\left(\dfrac{2Z_c}{Z_e+Z_0}\right)\cos 2\beta d+j\sin 2\beta d}$$

If we use

$$c=\frac{Z_e-Z_0}{Z_e+Z_0} \quad \text{and} \quad 2d=l$$

We get $|S_{41}|$

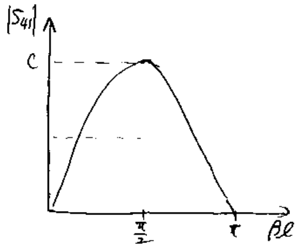
$$|S_{41}|=\frac{c\sin\beta l}{\sqrt{(1-c^2)\cos^2\beta l+\sin^2\beta l}}=\frac{c\sin\beta l}{\sqrt{1-c^2\cos^2\beta l}}$$

$|S_{41}|$ is max at $\beta l=\dfrac{\pi}{2}$

$$|S_{41}|=c$$

$|S_{41}|$ is min at $\beta l=\pi$

$$|S_{41}|=0$$



$$S_{21} = \frac{\sqrt{1-c^2}}{\sqrt{1-c^2}\cos\beta l + j\sin\beta l}$$

$|S_{21}| = 1$ at $\beta l = \pi$

$|S_{21}| = \sqrt{1-c^2}$ at $\beta l = \frac{\pi}{2}$

Coupled-line directional coupler (Microstrip Type)

1/ Contra-directional

- Group velocity V_g is in the opposite direction.
- Phase velocity is in the same direction.

2/ Port 1 & 3 are matched at all freq

$$S_{11} = 0 \wedge S_{31} = 0$$

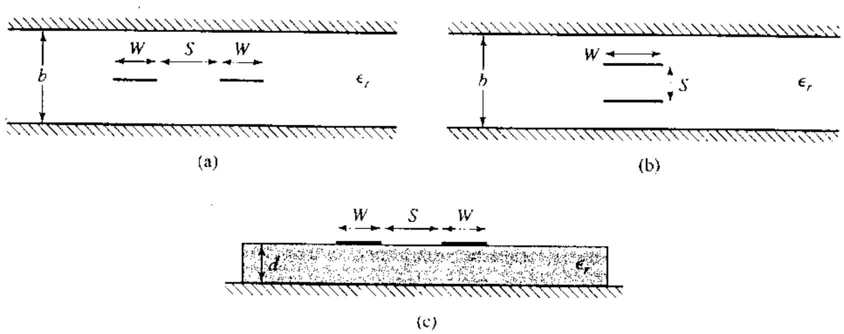
3/ Fairly narrow band
Wideband response can be obtained using multi-section coupled lines

4/ Coupled length $l = \frac{\lambda}{4}$

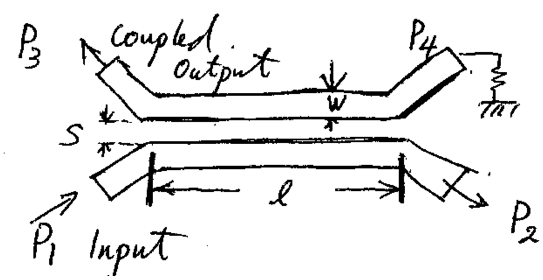
5/ Control $Z_0 \wedge Z_e$ to get a desired coupling coefficient.

Another method to analyze coupled line

Coupled Line Directional Couplers



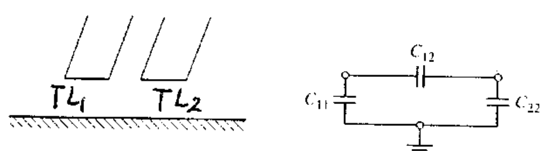
Various coupled transmission line geometries. (a) Coupled stripline (planar, or edge-coupled). (b) Coupled stripline (stacked, or broadside-coupled). (c) Coupled microstrip.



Unlike a waveguide coupler, the coupled output appears at P_3 (back direction)
We want to determine

S: gap width
W: line width
L: TL length

Analysis

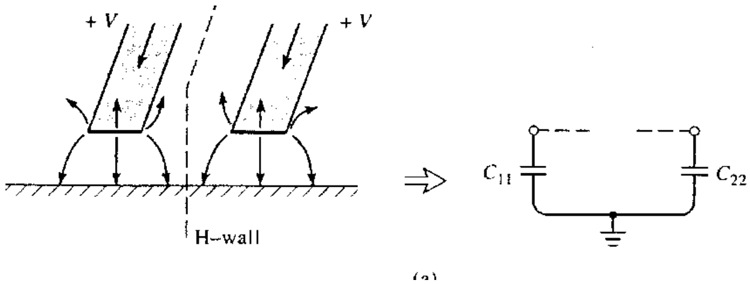


A three-wire coupled transmission line and its equivalent capacitance network.

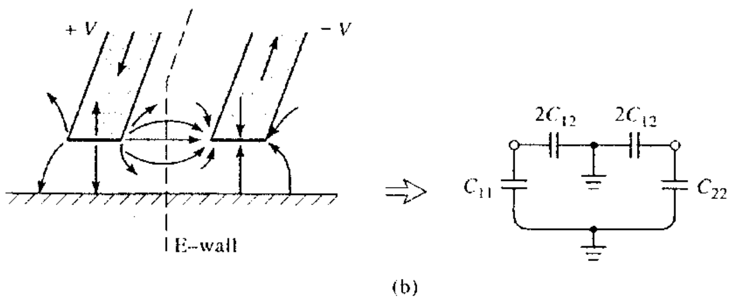
C_{11} due to TL_1 without TL_2
C_{22} due to TL_2 without TL_1
C_{12} mutual coupling

Even & Odd mode analysis

Even
Mode



Odd
Mode



Even- and odd-mode excitations for a coupled line, and the resulting equivalent capacitance networks. (a) Even-mode excitation. (b) Odd-mode excitation.

Even mode

Let $C_e = C_{11} = C_{22}$

$Z_{0e} = \sqrt{\dfrac{L}{C_e}} = \dfrac{1}{vC_e}$ where $v = \dfrac{1}{\sqrt{LC_e}}$ phase velocity

$$Z_{0e} = \frac{V^{+}_{even}}{I^{+}_{even}}$$

Odd mode

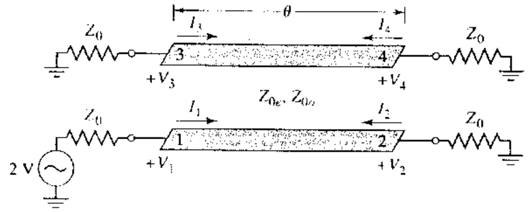
$C_0 = C_{11} + 2C_{12} = C_{22} + 2C_{12}$

$Z_{00} = \sqrt{\dfrac{L}{C_0}} = \dfrac{1}{vC_0}$ where $v = \dfrac{1}{\sqrt{LC_0}}$

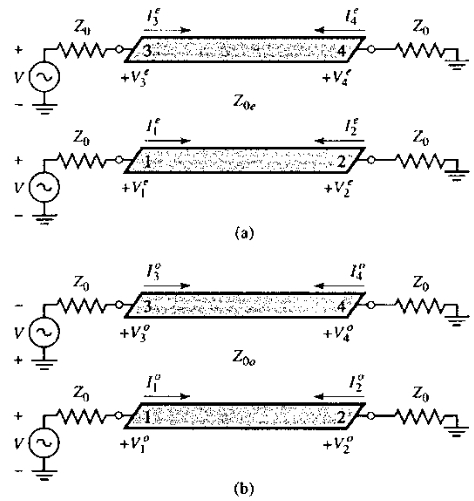
We want to obtain

$$\frac{P_3}{P_1},\ \frac{P_2}{P_1},\ \frac{P_4}{P_1}$$

We want to set $S_{11}=S_{22}=S_{33}=S_{44}=0$



Even case



$V_{1e} = V_{3e}$
$V_{4e} = V_{2e}$
$I_{1e} = I_{3e}$
$I_{4e} = I_{2e}$

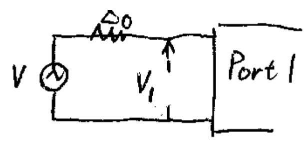
Odd case

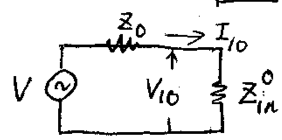
$V_{1o} = -V_{3o}$
$V_{4o} = -V_{2o}$
$I_{1o} = -I_{3o}$
$I_{4o} = -I_{2o}$

Decomposition of the coupled line coupler circuit of Figure 7.31 into even- and odd-mode excitations. (a) Even mode. (b) Odd mode.

Input matching (Port 1)

$$Z_{in}=\frac{V_1}{I_1}=\frac{V_{1e}+V_{10}}{I_{1e}+I_{10}}$$





$$V_{10}=V\left(\frac{Z_{in}^0}{Z_{in}^0+Z_0}\right)$$

$$V_{1e}=V\left(\frac{Z_{in}^e}{Z_{in}^e+Z_0}\right)$$

$$Z_{in}^0=Z_{00}\left(\frac{Z_0+jZ_{00}\tan\beta l}{Z_{00}+jZ_0\tan\beta l}\right)$$

$$Z_{in}^{e} = Z_{0e}\left(\frac{Z_0 + jZ_{0e}\tan\beta l}{Z_{0e} + jZ_0\tan\beta l}\right)$$

$$I_{10} = \frac{V_{10}}{Z_{in}^{0}} = \frac{V}{Z_{in}^{0} + Z_0}$$

$$I_{1e} = \frac{V_{1e}}{Z_{in}^{e}} = \frac{V}{Z_{in}^{e} + Z_0}$$

Then

$$Z_{in} = \frac{V_1}{I_1} = Z_0 + \frac{2\left(Z_{in}^{0} Z_{in}^{e} - Z_0^2\right)}{Z_{in}^{e} + Z_{in}^{0} + 2Z_0}$$

We need to set $Z_{in} = Z_0$. This can be done by setting $\underline{Z_0 = \sqrt{Z_{0e} Z_{00}}}$ because

$$Z_{in}^{0} Z_{in}^{e} = Z_{0e} Z_{00} = Z_0^2$$

This gives us

$Z_{in} = Z_0$ Port 1 matched.

Port 2, 3, 4 are the same

Coupled voltage at Port 3

$$V_3 = V_{3e} + V_{30} = V_{1e} - V_{10} = V\left[\frac{Z_{in}^{e}}{Z_{in}^{e} + Z_0} - \frac{Z_{in}^{0}}{Z_{in}^{0} + Z_0}\right]$$

$$V_3 = V\left[\frac{j\left(Z_{0e} - Z_{00}\right)\tan\theta}{2Z_0 + j\left(Z_{0e} + Z_{00}\right)\tan\theta}\right] \qquad \theta = \beta l$$

$$= V\,\frac{jC\tan\theta}{\sqrt{1 - C^2} + j\tan\theta}$$

Where

$$C = \frac{Z_{0e} - Z_{00}}{Z_{0e} + Z_{00}}$$

Coupling coefficient

Special cases

At DC (Low frequency)

$\theta \rightarrow 0$

$V_3 = 0$ No coupling

at $\theta = \frac{\pi}{2} \quad \left(l = \frac{\lambda}{4}\right) \qquad \tan\theta \rightarrow \infty$

$V_3 = V\,C$

at $\theta=\pi \quad \left(l=\frac{\lambda}{4}\right) \qquad \tan\theta=0$

$V_3=0$

Periodic function

Let $V_{10}=A_0+B_0 \quad \wedge \quad V_{1e}=A_e+B_e$

A_0 , A_e : Incident

B_0 , B_e : Reflected

$$V_{20}=A_0 e^{-jx}+B_0\, e^{jx} \qquad x=\frac{\pi}{2}$$

$$V_{2e}=A_e e^{-jx}+B_e\, e^{jx}$$

$$V_{2e}-V_{20}=A_e e^{-jx}+B_e\, e^{jx}-A_0 e^{-jx}-B_0\, e^{jx}$$

$$=-jA_e+j\,B_e+jA_0-jB_0$$

¿$-j\left(A_e-B_e\right)+j\left(A_0-B_0\right)$

¿$-j\left[\left(A_e-B_e\right)-\left(A_0-B_0\right)\right]$

Current

$$I_{10}=\frac{V_{10}}{Z_{in}^0}=\frac{1}{Z_{00}}\left(A_0-B_0\right)=\frac{V}{Z_{in}^0+Z_0}$$

$$I_{1e}=\frac{V_{1e}}{Z_{in}^e}=\frac{1}{Z_{0e}}\left(A_e-B_e\right)=\frac{V}{Z_{in}^e+Z_0}$$

$$V_{2e}-V_{20}=-j\left[\frac{VZ_{0e}}{Z_{in}^e+Z_0}-\frac{VZ_{00}}{Z_{in}^0+Z_0}\right]$$

If $\beta\, l=\frac{\pi}{2}$, $Z_{in}^0=\frac{Z_{00}^2}{Z_0}$, $Z_{in}^0\ Z_{in}^0=Z_{0e}Z_{00}=Z_0^2$

Then

$V_{2e}-V_{20}=0$ No output from port 4

Coupled voltage at Port 4

$$V_4=V_{4e}+V_{40}=V_{2e}-V_{20}=0$$

Port 4 isolated

Coupled voltage at Port 2

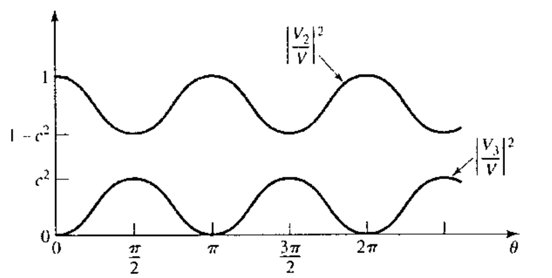
$$V_2 = V_{2e} + V_{20} = V\frac{\sqrt{1-c^2}}{\sqrt{1-c^2}\cos\theta + j\sin\theta}$$

at $\theta = \frac{\pi}{2}$

$V_2 = V\left(-j\sqrt{1-c^2}\right)$ min

at $\theta = \pi$

$V_2 = V$ max



Coupled and through port voltages (squared) versus frequency for the coupled line coupler of Figure 7.31.

at $\theta = \frac{\pi}{2}$

$\frac{V_3}{V} = c$

$)\frac{V_2}{V} = -j\sqrt{1-c^2})$

¿ 90^0 phase shift

Coupler Design Process

1/ Desired c eg 20dB coupler $c = 0.1$

2/ Obtain $Z_{0e} \wedge Z_{00}$

$$Z_{0e} = Z_0\sqrt{\frac{1+c}{1-c}}$$

$$Z_{00} = Z_0\sqrt{\frac{1-c}{1+c}}$$

3/ Find $Z_{00} \wedge Z_{0e}$ in terms of $C_{11}, C_{12} \wedge C_{22}$

$$Z_{00}=\frac{1}{vC_0}=\frac{1}{v}\frac{1}{(C_{11}+2C_{12})} \quad , \; C_{11}=C_{22}$$

$$Z_{0e}=\frac{1}{vC_e}=\frac{1}{v}\frac{1}{C_{11}}$$

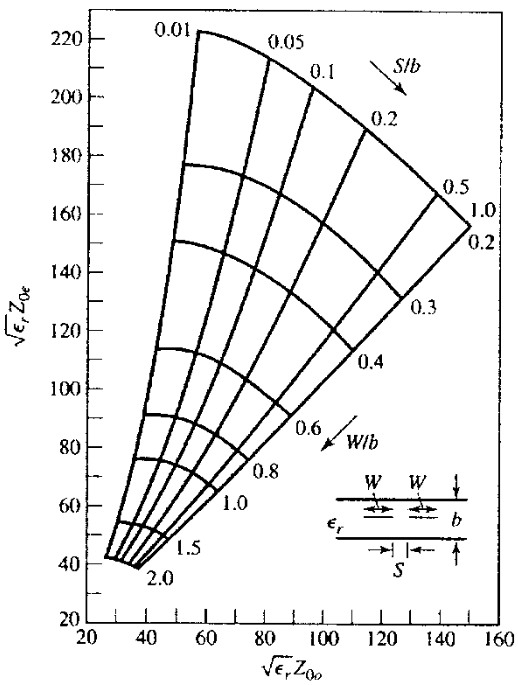
4/ Calculate $C_{11}, C_{12} \wedge C_{33}$ in terms of the gap width S, line width W, PCB characteristic. (geometry)

Example 20 dB stripline coupler

$$C=0.1$$

$$Z_{0e}=50\sqrt{\frac{1+0.1}{1-0.1}}=55.28\ \Omega$$

$$Z_{00}=50\sqrt{\frac{1-0.1}{1+0.1}}=45.23\ \Omega$$



Normalized even- and odd-mode characteristic impedance design data for edge-coupled striplines.

Let $\varepsilon_r=2.56$, $b=1.59\ mm$

$$\sqrt{\varepsilon_r}Z_{0e}=88.4 \quad , \quad \sqrt{\varepsilon_r}Z_{00}=72.4$$

Then

$$\frac{w}{b} = 0.72 \qquad w = 1.14\ mm$$

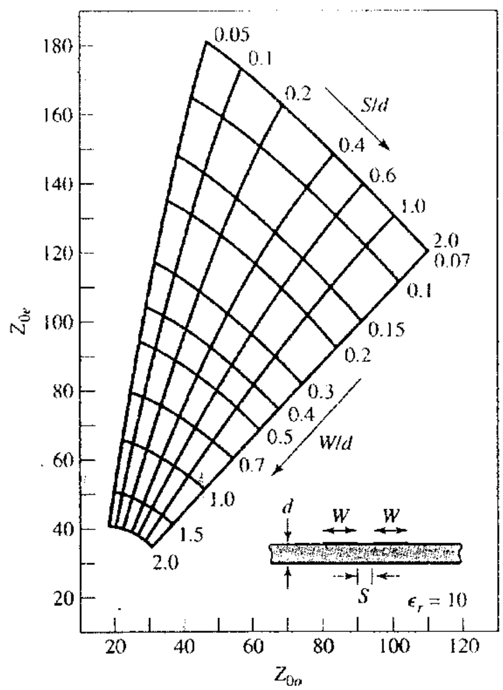
$$\frac{S}{b} = 0.34 \qquad S = 0.54\ mm \quad \leftarrow Small$$

Example $20\ dB$ $\mu - strip$ **coupler**

$\varepsilon_r = 10$, $d = 1.59\ mm$

$$Z_{0e} = 50\sqrt{\frac{1+0.1}{1-0.1}} = 55.28\ \Omega$$

$$Z_{00} = 50\sqrt{\frac{1-0.1}{1+0.1}} = 45.23\ \Omega$$



Even- and odd-mode characteristic impedance design data for coupled microstrip lines.

$$\frac{w}{d} = 0.9 \qquad w = 1.43\ mm$$

$$\frac{S}{d} = 1.5 \qquad S = 2.385\ mm$$

Coupled Mode Theory Based on Small Coupling

Wave propagation on a TL is given by

$$a_1 = e^{-j\beta z}$$

$$or$$

$$\frac{da_1}{dz} = -j\beta\, a_1$$

Suppose if we have two coupled lines ; we get

$$\frac{da_1}{dz} = -j\beta_{10}\, a_1 - jc_{12} a_2$$

$$\frac{da_2}{dz} = -j\beta_{20}\, a_2 - jc_{21} a_1$$

β_{10} and β_{20} are the propagation constant of lines 1 & 2

c_{12} and c_{21} are coupling coefficients.

Assumptions: weak coupling

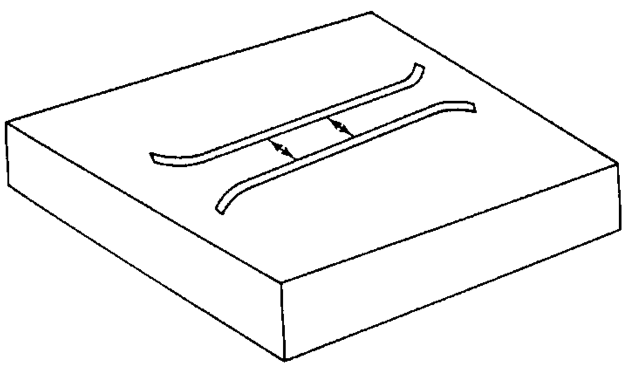


Figure 7-15 Coupling between two strip lines.

<u>Lossless case</u>

$$P_1 = a_1\, a_1^{*} \qquad unit\ \ of\ \ a_1 \quad \frac{V}{\sqrt{Z_0}}$$

Power

$$\frac{dP_1}{dz} = \frac{d}{dz}\left(a_1\, a_1^{*}\right) = a_1 \frac{da_1^{*}}{dz} + a_1^{*}\,\frac{da_1}{dz}$$

$$\frac{dP_1}{dz} = a_1\left[\, j\beta_{10}^{*}\, a_1^{*} + jc_{12}^{*}\, a_2^{*}\right] + a_1^{*}\left[-j\beta_{10}\, a_1 - jc_{12}\, a_2\right]$$

$$= -ja_1\left[\beta_{10} - \beta_{10}^{*}\right] a_1^{*} + ja_1\, c_{12}^{*}\, a_2^{*} - j\, a_1^{*}\, c_{12}\, a_2$$

$$\text{¿} -ja_1\left[\beta_{10} - \beta_{10}^{*}\right] a_1^{*} + 2R_e\left[ja_1\, c_{12}^{*}\, a_2^{*}\right]$$

$$\frac{dP_2}{dz} = -ja_2\left[\beta_{20} - \beta_{20}^{¿}\right] a_2^{¿} + 2R_e\left[-ja_1\ c_{21}\ a_2^{¿}\right]$$

Lossless $\beta_{10} \wedge \beta_{20}$ are real

Or $\beta_{10} - \beta_{10}^{¿} = 0$

$\beta_{20} - \beta_{20}^{¿} = 0$

Also energy must be conserved

$[If\ P_1\ and\ P_2\ are\ in\ the\ same\ direction\ (same\ group\ velocity),\ then\ \frac{d}{dz}(P_1 + P_2) = 0\ [$

$[\quad Then\ c_{12} = c_{21}^{¿}$

$[If\ P_1\ and\ P_2\ are\ in\ the\ opposite\ direction\ (group\ velocity\ is\ opposite),\ then\ \frac{d}{dz}(P_1 - P_2) = 0\ [$

$[\quad or\ c_{12} = -c_{21}^{¿}$

<u>Co-directional Coupler</u> ($c_{12} = c_{21}^{¿}$)

$$\frac{da_1}{dz} = -j\beta_{10}\ a_1 - jc_{12} a_2$$

$$\frac{da_2}{dz} = -j\ c_{21} a_1 - j\beta_{20} a_2$$

Let $a_1 = A_1 e^{-j\beta z}$ Coupled line : periodic behavior

$\beta:\ eigen\ value$

$)A_1:\ eigen\ vector)$

$)A_2$

$a_2 = A_2 e^{-j\beta z}$ ¿ determine the characteristics of this wave

We get

$$\begin{bmatrix} \beta_{10} & c_{12} \\ c_{21} & \beta_{20} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \beta \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$\begin{bmatrix} \beta_{10} - \beta & c_{12} \\ c_{21} & \beta_{20} - \beta \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

The determinant must be 0

$$(\beta_{10} - \beta)(\beta_{20} - \beta) - c_{21}\ c_{12} = 0$$

Solutions

$$\beta_1 = \beta_a + \beta_b$$
$$\beta_2 = \beta_a - \beta_b$$

where

$$\beta_a = \frac{1}{2}\left(\beta_{10} + \beta_{20}\right)$$
$$\beta_b = \sqrt{\beta_d^2 + c_{12}\, c_{21}} = \sqrt{\beta_d^2 + |c_{12}|^2}$$
$$\beta_d = \frac{1}{2}\left(\beta_{10} - \beta_{20}\right)$$

Eigen vectors

$$\frac{A_2}{A_1} = \frac{\beta - \beta_{10}}{c_{12}} = \frac{c_{21}}{\beta - \beta_{20}}$$

Solutions

$$a_1 = c_1\, e^{-j\beta_1 z} + c_2\, e^{-j\beta_2 z}$$
$$a_2 = c_1 \left(\frac{\beta_1 - \beta_{10}}{c_{12}}\right) e^{-j\beta_1 z} + c_2 \left(\frac{\beta_2 - \beta_{10}}{c_{12}}\right) e^{-j\beta_2 z}$$

If the input is given by

$$a_1(z=0) = a_0 \quad and \quad a_2(z=0) = 0$$

Then

$$a_1(z) = a_0 \left(\cos\beta_b z - j\frac{\beta_d}{\beta_b}\sin\beta_b z \right) \exp(-j\beta_a z)$$
$$a_2(z) = a_0 \frac{c_{21}}{\beta_b}\sin\beta_b z \;\; \exp(-j\beta_a z)$$

$$P_1 = |a_1(z)|^2$$
$$P_2 = |a_2(z)|^2 \qquad\qquad P_1 + P_2 = \text{ constant if lossless}$$

$P_{2\ max}$ is given by

$$P_{2\ max} = \frac{|c_{12}|^2}{|\beta_d|^2 + |c_{12}|^2}\; |a_0|^2$$

If $\beta_d = 0$ $(or \;\; \beta_{10} = \beta_{20})$, then $P_{2\ max} = |a_0|^2$

100% power transfer

<u>Contra direction Coupler</u>

$\beta_{10} > 0$

$\beta_{20} > 0$ but V_{g1} *and* V_{g2} are in the opposite direction

$c_{12} = -c_{21}^{*}$

Similar to the co-directional case, we write

$$a_1 = A_1 \exp(-j\beta\ z)$$
$$a_2 = A_2 \exp(-j\beta\ z)$$

Two solutions are

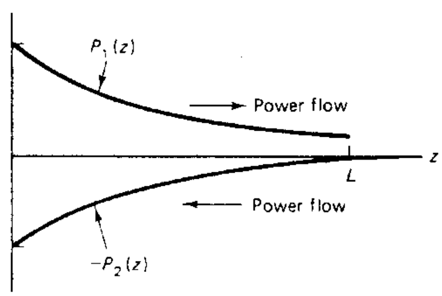
$$\beta_1 = \beta_a + \beta_b$$
$$\beta_2 = \beta_a - \beta_b$$

where

$$\beta_a = \frac{1}{2}(\beta_{10} + \beta_{20})$$
$$\beta_b = \sqrt{(\beta_d)^2 - |c_{12}|^2}$$
$$\beta_d = \frac{1}{2}(\beta_{10} - \beta_{20})$$

β_b will be imaginary if $\beta_d^2 < |c_{12}|^2$

Then $|a_1|$ decreases exponentially & $|a_2|$ increases exponentially. (assume a_1 is input)



$P_1 - P_2$: constant

Figure 7-17 Contradirectional coupler.

a_1 *and* a_2 are given by

$$a_1 = c_1\, e^{-j\beta_1 z} + c_2\, e^{-j\beta_2 z}$$
$$a_2 = c_1 \left(\frac{\beta_1 - \beta_{10}}{c_{12}}\right) e^{-j\beta_1 z} + c_2 \left(\frac{\beta_2 - \beta_{10}}{c_{12}}\right) e^{-j\beta_2 z}$$

Boundary conditions

$$a_1(z=0) = a_0$$
$$a_2(z=L) = 0$$

$$c_1 + c_2 = a_0$$

$$X = \frac{\beta_1 - \beta_{10}}{c_{12}}$$
$$Y = \frac{\beta_2 - \beta_{10}}{c_{12}}$$

$$c_1 X e^{-j\beta_1 L} + c_2 Y e^{-j\beta_2 L} = 0$$

$$\beta_1 = \beta_a + \beta_b$$
$$\beta_2 = \beta_a - \beta_b$$

$\beta_b :$ Imaginary

$$c_1 X e^{-j\beta_a L} e^{-\beta_b L} + c_2 Y e^{-j\beta_a L} e^{+\beta_b L} = 0$$

or

$$c_1 X e^{-\beta_b L} + c_2 Y e^{+\beta_b L} = 0$$

From this we get

$$c_1 = \frac{-a_0 Y e^{+2\beta_b L}}{X - Y e^{+2\beta_b L}}$$
$$c_2 = \frac{a_0 X}{X - Y e^{+2\beta_b L}}$$

$$a_1 = \left(c_1 e^{-\beta_b z} + c_2 e^{+\beta_b z}\right) e^{-j\beta_a z}$$
$$a_2 = \left(c_1 X e^{-\beta_b z} + c_2 Y e^{+\beta_b z}\right) e^{-j\beta_a z}$$