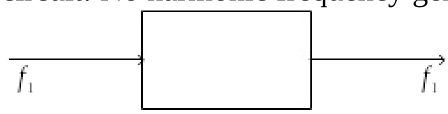
1. Linear, Lossless and Reciprocal Network 1/20/16

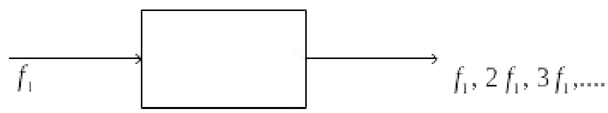
1-1. Linear and non-linear circuits

If a circuit does not change the frequency spectrum and the transfer function does not depend on the input level, the circuit is known as a linear circuit. If a circuit contains the non-linear element such as a saturated amplifier, the output spectrum will contain higher order harmonics. This is called a non-linear circuit. In addition to higher-order harmonics, the non-linear circuit will generate intermodulation if 2 or more signals are entered.

Linear circuit: No harmonic frequency generation



Non-linear circuit



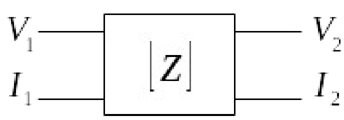
1-2. Reciprocal network

S_{21} is known as the forward gain whereas S_{12} is known as the reverse gain. If a DUT shows $S_{21}=S_{12}$, the DUT is reciprocal. One example is a TL. Whether we measure it from left-to-right or right-to-left, we get the same response for S_{21} and S_{12}.

We will derive the reciprocity in terms of Z-parameters.
The reciprocity condition of a 2-port network is

$$V_{12}I_1 = V_{21}I_2$$

$$(Voltage\,1\,due\,to\,I_2)(I_1\,source\,at\,1) = (Voltage\,2\,due\,to\,I_1)(I_2\,source\,at\,2)$$



We start with

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

In V_1, we set I_1=0 and define V_{12} to be due to current I_2. Similarly, V_{21} is due to current I_1 as shown below.

$$Let\ I_1 = 0 \quad \rightarrow \quad V_{12} = Z_{12}I_2$$
$$Let\ I_2 = 0 \quad \rightarrow \quad V_{21} = Z_{21}I_1$$

By multiplying with I_1 and I_2, we get

$$\begin{cases} V_{12}I_1 = Z_{12}I_2I_1 \\ V_{21}I_2 = Z_{21}I_1I_2 \end{cases}$$

Since $V_{12}I_1 = V_{21}I_2$, we obtain the reciprocity condition

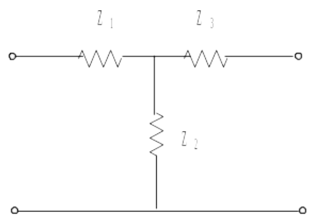
$$Z_{12} = Z_{21}$$

We don't have any condition for Z_{11} and Z_{22}. Therefore, the reciprocity condition can be written as

$$[Z] = [Z]^t \quad (transpose)$$

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \end{bmatrix}$$

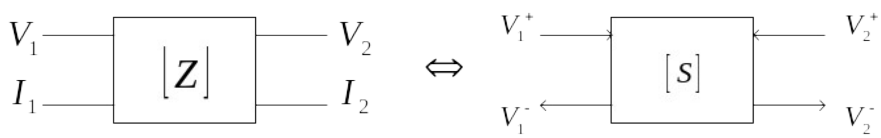
Example: T network



$$Z_{11} = Z_1 + Z_2 \ , \quad Z_{22} = Z_3 + Z_2 \ , \quad Z_{12} = Z_{21} = Z_2$$

Reciprocity using S-parameters

As we show when $[Z]$ satisfies $[Z] = [Z]^t$, the network is reciprocal. Now we consider this in terms of S-parameters.



First, we write

$$[V] = [V^+] + [V^-]$$

$$[I] = [I^+] - [I^-]$$

$$Assume\ Z_0 = 1$$

$$[I] = \frac{1}{Z_0}\left([V^+] - [V^-]\right) = [V^+] - [V^-]$$

$$[V] = [V^+] + [V^-] = [Z][I] = [Z][V^+] - [Z][V^-]$$

Then we can find [S] in terms of [Z].

$$([Z] + [U])[V^-] = ([Z] - [U])[V^+]$$

$$[V^-] = \underbrace{([Z] + [U])^{-1}([Z] - [U])}_{[S]}[V^+]$$

$$[S] = ([Z] + [U])^{-1}([Z] - [U])$$

Also we can write.

$$2[V^+] = [V] + [I] = ([Z] + [U])[I]$$

$$2[V^-] = [V] - [I] = ([Z] - [U])[I]$$

$$[V^-] = ([Z] - [U])([Z] + [U])^{-1}[V^+]$$

Then [S] can be expressed as

$$[S] = ([Z] - [U])([Z] + [U])^{-1}$$

By taking a transpose

$$[S]^t = \left[([Z] - [U])([Z] + [U])^{-1}\right]^t$$

$$= \left[([Z] + [U])^{-1}\right]^t ([Z] - [U])^t$$

Since $[Z] = [Z]^t$, we get

$$[S]^t = ([Z] + [U])^{-1}([Z] - [U])$$

This is the same as the one we got

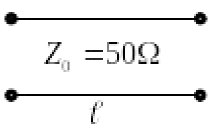
$$[S] = ([Z] + [U])^{-1}([Z] - [U])$$

Therefore

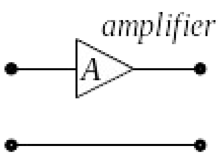
$$[S] = [S]^t \quad \textit{If a network is reciprocal}$$

This shows $S_{12}=S_{21}$ if a circuit is reciprocal as we expected.

Examples:



$$[S] = \begin{bmatrix} 0 & e^{-j\beta\ell} \\ e^{-j\beta\ell} & 0 \end{bmatrix} \qquad \textit{reciprocal}$$



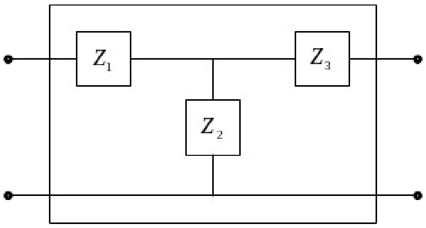
$$S_{21} \neq S_{12} \qquad \textit{non-reciprocal}$$

$$[S] = \begin{bmatrix} 0.1 & 0.15 \\ 10 & 0.2 \end{bmatrix}$$

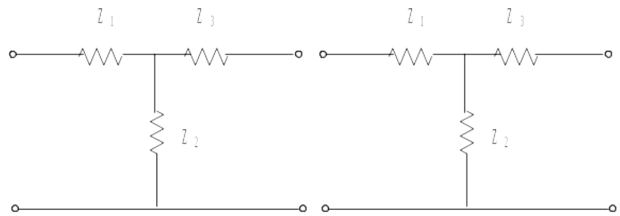
Problem:
What is the reciprocity condition in terms of T-parameters?

1-3. Lossless circuit

If the circuit contains a resistive element (resistor) or lossy TL, the circuit is lossy (dissipative). The portion of the input energy will be converted into heat. However, if it contains only reactive elements (C and L) and lossless TL, the circuit is lossless.



$[Z]$



If $Z_1, Z_2, and\ Z_3$ contain real (resistive) impedance, the network is lossy.
If $Z_1, Z_2, and\ Z_3$ are purely imaginary, the circuit is lossless.

Now we derive the lossless condition in terms of S-parameters.

Lossless means there is no power loss. We can state this as

$$P_{ave} = \frac{1}{2}\mathrm{Re}\left\{[V]^t [I]^*\right\} = \frac{1}{2}\mathrm{Re}\left\{[I]^t [Z]^t [I]^*\right\} = 0$$

here $[V]^t = [ZI]^t = [I]^t [Z]^t$.

This can be satisfied if $[Z]$ is purely imaginary.
In terms of S-parameters (assume Zo=1).

$$P_{ave} = \frac{1}{2}\mathrm{Re}\left\{[V]^t [I]^*\right\} = \frac{1}{2}[V^+]^t [V^+]^* - \frac{1}{2}[V^-]^t [V^-]^* = 0$$

$$[V^+]^t [V^+]^* = [V^-]^t [V^-]^*$$

$$= \left\{[S][V^+]\right\}^t \left\{[S][V^+]\right\}^*$$

$$= [V^+]^t \underbrace{[S]^t [S]^*}_{[U]} [V^+]^*$$

Note: The top equation has both real and imaginary parts. Only the real part is kept.

Therefore, we find the lossless condition as

$$[S]^t [S]^* = [U]$$

In terms of elements of S-parameter

$$\sum_{k=1}^{N} S_{ki} S_{kj}^* = 1 \qquad i = j$$

$$\sum_{k=1}^{N} S_{ki} S_{kj}^* = 0 \qquad i \neq j$$

Example:

$$[S]=¿[0.3+j0.7 \quad ¿ \quad j0.6¿]¿¿¿$$

This is reciprocal. $S_{12}=S_{21}$

Lossless?

$$|S_{11}|^2+|S_{12}|^2=0.09+0.49+0.36=0.94\neq 1$$

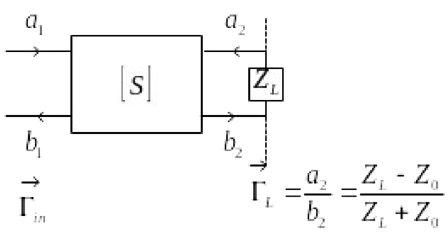
$$|S_{21}|^2+|S_{22}|^2=0.94\neq 1$$

This is not lossless.

$$S_{11}S_{12}^{¿}+S_{22}S_{21}^{¿}=(0.3+j0.7)(-j0.6)+(0.3-j0.7)(-j0.6)$$
$$=-j0.6$$
$$\neq 0$$

2. Two-Port Network with an Unmatched Impedance

The input reflection coefficient Γ_{in} is equal to S_{11} if there is no signal coming into the output port (a2=0). However, in many cases, we have reflected signal from the load impedance Z_L which becomes a_2 as shown below. In this case Γ_{in} is no longer equal to S_{11}.



We want to find Γ_{in}

If $Z_L=Z_0$ (matched load), $\Gamma_{in}=\frac{b_1}{a_1}=S_{11}$

If $Z_L\neq Z_0$, then

$$a_2=\Gamma_L b_2$$
$$(b_1=S_{11}a_1+S_{12}(\Gamma_L b_2)($$
$$(b_2=S_{21}a_1+S_{22}(\Gamma_L b_2)$$
$$b_2=\left(\frac{S_{21}}{1-\Gamma_L S_{22}}\right)a_1$$
$$b_1=S_{11}a_1+\Gamma_L S_{12}\left(\frac{S_{21}}{1-\Gamma_L S_{22}}\right)a_1$$
$$\Gamma_{in}=\frac{b_1}{a_1}=S_{11}+\left(\frac{S_{12}\ S_{21}\ \Gamma_L}{1-\Gamma_L S_{22}}\right)$$

Example:

Assume S is given by

$$[s] = \begin{bmatrix} 0.1 & j0.8 \\ j0.8 & 0.2 \end{bmatrix}$$

If $Z_L = Z_o$, $\Gamma_{in} = S_{11} = 0.1$.

Now we consider two extreme cases.

$$if\ \Gamma_L = -1 \quad (Z_L = 0, \text{output short})$$

$$\Gamma_{in} = 0.1 - \frac{(j0.8)(j0.8)}{1 + 0.2} = 0.633$$

$$if\ \Gamma_L = 1 \quad (Z_L = \infty, output\ open)$$

$$\Gamma_{in} = 0.1 + \frac{(j0.8)(j0.8)}{1 - 0.2} = -0.7$$

much larger than S_{11}

3. Useful methods and formula for network analysis

3-1. Shift in Reference Planes

The modern microwave network analyzer (NWA) such as Agilent 8720 has the PORT EXTENSION (one-way phase shift) and ELECTRICAL DELAY (two-way phase shift) capabilities. These are signal processing functions performed on the measured data. NWA must be calibrated at a certain location (usually the end of the test cable) to obtain the accurate data. This location, however, is often different from the measurement plane at which the device must be characterized. The process of moving the calibration position to the desired position is called port extension. Figure 4 shows one example of the port extension applied for the desired impedance measurement technique. If the TL is lossless and the distance *l* is known, the port extension is essentially the phase shift given by $\theta=\beta l$ where β is the propagation constant. In Fig. 3-1, the input impedance Z_{in} can be expressed by Z_L, Z_o, β, and *l*. Rather than calculating Z_L from Z_{in}, if we can remove the phase shift caused by βl, we should be able to get Z_L directly from the instrument. The port extension on NWA does this function.

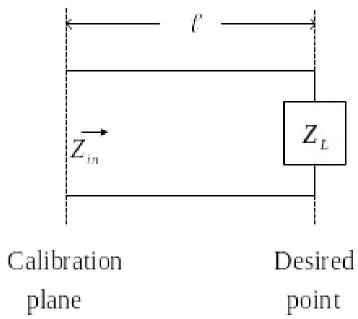
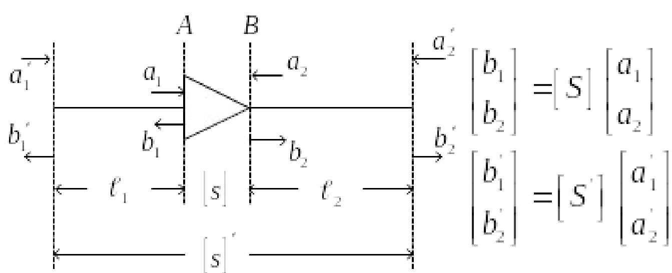


Fig. 3-1: NWA uses the port extension to convert Z_{in} to Z_L.

In this section, we will derive the formula used for the port extension. Assume NWA is calibrated at the plane specified by [S]'. The desired device is given by [S]. We want to relate the measured [S]' to the desired [S]. We assume the TL connecting between [S] and [S]' are lossless.



Measured S-parameter: [S]’
Desired S-parameter: [S]
Assume A and B are reference planes.

Fig. 3-2: Relationship between [S]' and [S].

We can relate [S]' to [S] using

$$a_1' = a_1 e^{+j\beta\ell_1} = a_1 e^{j\theta_1}$$
$$a_2' = a_2 e^{+j\beta\ell_2} = a_2 e^{j\theta_2}$$

$$b_1' = b_1 e^{-j\theta_1}$$
$$b_2' = b_2 e^{-j\theta_2}$$

Then we have

$$\begin{bmatrix} b_1' e^{j\theta_1} \\ b_2' e^{j\theta_2} \end{bmatrix} = [S] \begin{bmatrix} a_1' e^{-j\theta_1} \\ a_2' e^{-j\theta_2} \end{bmatrix}$$

$$\begin{bmatrix} e^{j\theta_1} & 0 \\ 0 & e^{j\theta_2} \end{bmatrix} \begin{bmatrix} b_1' \\ b_2' \end{bmatrix} = [S] \begin{bmatrix} e^{-j\theta_1} & 0 \\ 0 & e^{-j\theta_2} \end{bmatrix} \begin{bmatrix} a_1' \\ a_2' \end{bmatrix}$$

$$\begin{bmatrix} b_1' \\ b_2' \end{bmatrix} = \underbrace{\begin{bmatrix} e^{-j\theta_1} & 0 \\ 0 & e^{-j\theta_2} \end{bmatrix} [S] \begin{bmatrix} e^{-j\theta_1} & 0 \\ 0 & e^{-j\theta_2} \end{bmatrix}}_{[S]'} \begin{bmatrix} a_1' \\ a_2' \end{bmatrix}$$

Finally, we obtain

$$\therefore \quad [S] = \begin{bmatrix} e^{j\theta_1} & 0 \\ 0 & e^{j\theta_2} \end{bmatrix} [S'] \begin{bmatrix} e^{j\theta_1} & 0 \\ 0 & e^{j\theta_2} \end{bmatrix}$$

3-2. Cascaded Network

It is common to find a system consisting of many devices in series connection. This is called the cascaded network and one example is shown in Fig. 3-3. Assume we know the S-parameters of all devices. What we need to find is the S-parameter of the cascaded network given by [S_{total}] in Fig. 3-3.

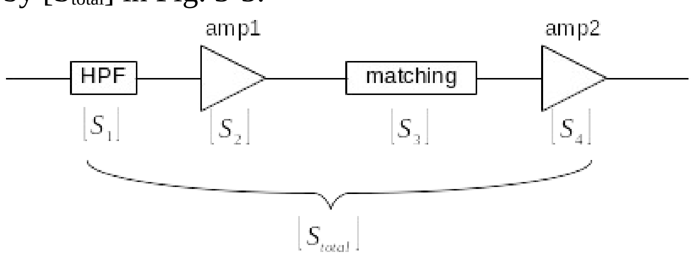


Fig. 3-3: Cascaded network

What is $[S_{total}]$? W cannot obtain the total S-parameter by taking a matrix multiplication of cascaded [S].

$$[S_{total}] \neq [S_1][S_2][S_3][S_4]$$

This is because the definition of S-parameter contains both input and output on the left side as shown below. $[S]$ is defined as

Input side

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = [S] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

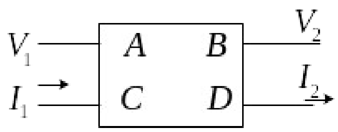
Output side

To analyze the cascaded network, we usually use ABCD or T-parameters. ABCD parameter is defined in terms of total voltage and current and it is similar to Z- and Y-parameters.

We write the two-port network using ABCD-parameter as

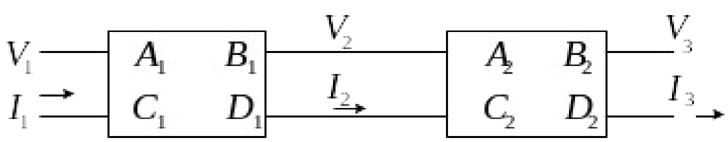
$[V_1$ ¿ $]$ ¿ ¿ ¿
¿

Note: direction of I2



It is important to remind that V_1 and V_2 are total voltages and contain both incident and reflected voltages. Similarly, I_1 and I_2 are total currents.

When two devices given by ABCD parameters are cascaded, we have



$[V_1$ ¿ $]$ ¿ ¿ ¿
¿

$[A \quad B$ ¿ $]$ ¿ ¿ ¿
¿

ABCD parameters of common impedance network are shown in the following table.

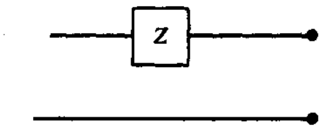
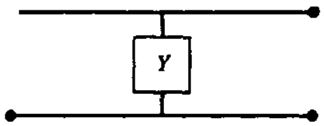
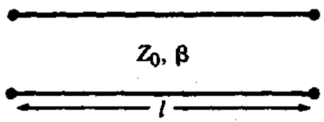
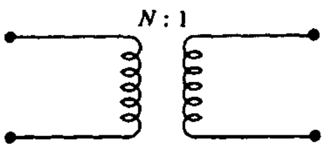
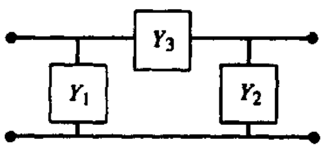
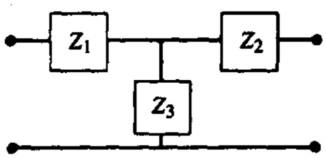
Since we are describing the same device using different parameters, we should be able to relate ABCD to any other parameters including S-parameter. The conversion table for S, Z, Y, and ABCD is shown in the following page.

For example, S-parameters are given by

$$S_{11} = \frac{A + B/Z_o - CZ_o - D}{A + B/Z_o + CZ_o + D}, \quad S_{12} = \frac{2(AD - BC)}{A + B/Z_o + CZ_o + D}$$

$$S_{21} = \frac{2}{A + B/Z_o + CZ_o + D}, \quad S_{22} = \frac{-A + B/Z_o - CZ_o + D}{A + B/Z_o + CZ_o + D}$$

TABLE 5.1 The *ABCD* Parameters of Some Useful Two-Port Circuits

Circuit	*ABCD* Parameters	
	$A = 1$ $C = 0$	$B = Z$ $D = 1$
	$A = 1$ $C = Y$	$B = 0$ $D = 1$
	$A = \cos \beta\ell$, $C = jY_0 \sin \beta\ell$	$B = jZ_0 \sin \beta\ell$ $D = \cos \beta\ell$
	$A = N$ $C = 0$	$B = 0$ $D = \frac{1}{N}$
	$A = 1 + \frac{Y_2}{Y_3}$ $C = Y_1 + Y_2 + \frac{Y_1 Y_2}{Y_3}$	$B = \frac{1}{Y_3}$ $D = 1 + \frac{Y_1}{Y_3}$
	$A = 1 + \frac{Z_1}{Z_3}$ $C = \frac{1}{Z_3}$	$B = Z_1 + Z_2 + \frac{Z_1 Z_2}{Z_3}$ $D = 1 + \frac{Z_2}{Z_3}$

which indicates that A is found by applying a voltage V_1 at port 1, and measuring the open-circuit voltage V_2 at port 2. Thus, $A = 1$. Similarly,

$$B = \left.\frac{V_1}{I_2}\right|_{V_2=0} = \frac{V_1}{V_1/Z} = Z,$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = 0,$$

$$D = \left.\frac{I_1}{I_2}\right|_{V_2=0} = \frac{I_1}{I_1} = 1.$$

Table 1:

Table 2:

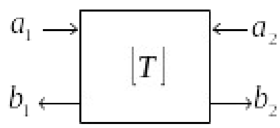
	S	Z	Y	$ABCD$
S_{11}	S_{11}	$\frac{(Z_{11}-Z_0)(Z_{22}+Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0-Y_{11})(Y_0+Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{A+B/Z_0-CZ_0-D}{A+B/Z_0+CZ_0+D}$
S_{12}	S_{12}	$\frac{2Z_{12}Z_0}{\Delta Z}$	$\frac{-2Y_{12}Y_0}{\Delta Y}$	$\frac{2(AD-BC)}{A+B/Z_0+CZ_0+D}$
S_{21}	S_{21}	$\frac{2Z_{21}Z_0}{\Delta Z}$	$\frac{-2Y_{21}Y_0}{\Delta Y}$	$\frac{2}{A+B/Z_0+CZ_0+D}$
S_{22}	S_{22}	$\frac{(Z_{11}+Z_0)(Z_{22}-Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0+Y_{11})(Y_0-Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{-A+B/Z_0-CZ_0+D}{A+B/Z_0+CZ_0+D}$
Z_{11}	$Z_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{11}	$\frac{Y_{22}}{\lvert Y\rvert}$	$\frac{A}{C}$
Z_{12}	$Z_0\frac{2S_{12}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{12}	$\frac{-Y_{12}}{\lvert Y\rvert}$	$\frac{AD-BC}{C}$
Z_{21}	$Z_0\frac{2S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{21}	$\frac{-Y_{21}}{\lvert Y\rvert}$	$\frac{1}{C}$
Z_{22}	$Z_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{22}	$\frac{Y_{11}}{\lvert Y\rvert}$	$\frac{D}{C}$
Y_{11}	$Y_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{22}}{\lvert Z\rvert}$	Y_{11}	$\frac{D}{B}$
Y_{12}	$Y_0\frac{-2S_{12}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{12}}{\lvert Z\rvert}$	Y_{12}	$\frac{BC-AD}{B}$
Y_{21}	$Y_0\frac{-2S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{21}}{\lvert Z\rvert}$	Y_{21}	$\frac{-1}{B}$
Y_{22}	$Y_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{11}}{\lvert Z\rvert}$	Y_{22}	$\frac{A}{B}$
A	$\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{11}}{Z_{21}}$	$\frac{-Y_{22}}{Y_{21}}$	A
B	$Z_0\frac{(1+S_{11})(1+S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{\lvert Z\rvert}{Z_{21}}$	$\frac{-1}{Y_{21}}$	B
C	$\frac{1}{Z_0}\frac{(1-S_{11})(1-S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{1}{Z_{21}}$	$\frac{-\lvert Y\rvert}{Y_{21}}$	C
D	$\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{22}}{Z_{21}}$	$\frac{-Y_{11}}{Y_{21}}$	D

$\lvert Z\rvert = Z_{11}Z_{22}-Z_{12}Z_{21}$; $\lvert Y\rvert = Y_{11}Y_{22}-Y_{12}Y_{21}$; $\Delta Y = (Y_{11}+Y_0)(Y_{22}+Y_0)-Y_{12}Y_{21}$; $\Delta Z = (Z_{11}+Z_0)(Z_{22}+Z_0)-Z_{12}Z_{21}$; $Y_0 = 1/Z_0$

3-3. T-Parameters

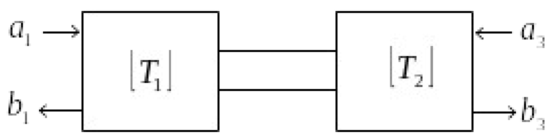
ABCD parameter is useful for many applications but we often need to express the device in terms of incident and reflected signal as we do with S-parameters. If we want to express the cascaded network using the incident and reflected signal, the device description based on T-parameter is better suited. We will show that T-parameters can be cascaded.

The two-port network is given by the incident and reflected signals. Similar to the S-parameter case, a_1 and a_2 are incident and b_1 and b_2 are reflected signals. To express T-parameter, we put the input signals on the left side and they are given by the output signals and T-parameters as shown below.



¿ $[b_1$ ¿ $]$ ¿ ¿ ¿
¿
¿

When two devices are cascaded, the total T-parameter can be expressed as



$[b_1$ ¿ $]$ ¿ ¿ ¿
¿

$$[T]_{total} = [T_1][T_2]$$

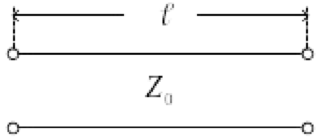
The conversions between $[T]$ and $[S]$ are given by

$$[T] = \begin{bmatrix} -\left(\frac{S_{11}S_{22} - S_{12}S_{21}}{S_{21}}\right) & \frac{S_{11}}{S_{21}} \\ -\frac{S_{22}}{S_{21}} & \frac{1}{S_{21}} \end{bmatrix}$$

$$[S] = \begin{bmatrix} \frac{T_{12}}{T_{22}} & \frac{T_{11}T_{22} - T_{12}T_{21}}{T_{22}} \\ \frac{1}{T_{22}} & -\frac{T_{21}}{T_{22}} \end{bmatrix}$$

Notice that [T] and [S] cannot be obtained if $S_{21}=0$ or $T_{22}=0$, respectively.

As an example of [S] and [T] parameters, a matched TL with the length ℓ is shown below.



$$\theta = \beta \ell$$

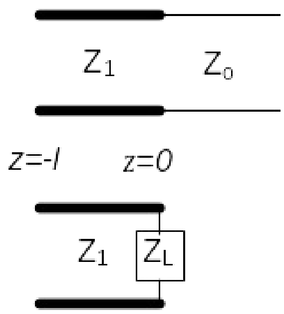
$$[S] = \begin{bmatrix} 0 & e^{-j\theta} \\ e^{-j\theta} & 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} e^{-j\theta} & 0 \\ 0 & e^{j\theta} \end{bmatrix}$$

If ℓ =0 in the above example, [T] becomes a unit matrix. [T] also becomes a unit matrix when $\theta = m\pi$ where *m* is an integer.

ABCD parameter of an ideal TL

The ABCD parameters of an ideal TL is shown in Table 1. We will show the method to get those from the TL model. Assume we have a TL with 3 sections as shown below.



Top: Two TL sections, Bottom: equivalent circuit.

In Z_1 section, we get

$$V(z=-l) = V_0^+(e^{+j\beta l} + \Gamma_L e^{-j\beta l})$$

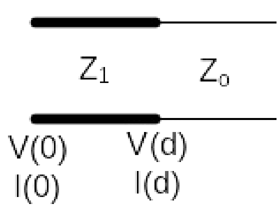
$$I(z=-l) = \frac{V_0^+}{Z_0}(e^{+j\beta l} - \Gamma_L e^{-j\beta l})$$

$$V_L = V_0^+ + V_0^- = V_0^+(1+\Gamma_L)$$

$$V_0^+ = V_L / (1+\Gamma_L)$$

$$\Gamma_L = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_1}{Z_L + Z_1}$$

Consider the same case with ABCD. From Table 1, this should be



$$\begin{bmatrix} V(0) \\ I(0) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V(d) \\ I(d) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & jZ_1 \sin\theta \\ jY_1 \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} V(d) \\ I(d) \end{bmatrix}$$

$$\theta = \beta d, \quad Y_1 = \frac{1}{Z_1}$$

To show these two cases are the same, we write the TL equations as

$$V(z=-l)=\frac{V_L}{1+\Gamma_L}(e^{+j\theta}+\Gamma_L e^{-j\theta})$$

$$I(z=-l)=\frac{1}{Z_1}\frac{V_L}{1+\Gamma_L}(e^{+j\theta}-\Gamma_L e^{-j\theta})$$

Rearranging the voltage equation, we get

$$\begin{aligned}V(z=-l)&=\frac{Z_0+Z_1}{2Z_0}(e^{+j\theta}+\frac{Z_0-Z_1}{Z_0+Z_1}e^{-j\theta})V_L\\&=\frac{1}{2Z_0}(2Z_0\cos\theta+j2Z_1\sin\theta)V_L\\&=\cos\theta V_L+jZ_1\sin\theta I_L\end{aligned}$$

where $I_L=\frac{V_L}{Z_L}=\frac{V_L}{Z_0}$

Similarly for the current, we get

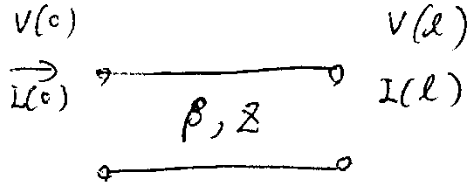
$$\begin{aligned}I(z=-l)&=\frac{Z_0+Z_1}{2Z_0}(e^{+j\theta}-\frac{Z_0-Z_1}{Z_0+Z_1}e^{-j\theta})\frac{V_L}{Z_1}\\&=\frac{1}{2Z_0}(Z_0(e^{+j\theta}-e^{-j\theta})+Z_1(e^{+j\theta}+e^{-j\theta}))\frac{V_L}{Z_1}\\&=(j\sin\theta+\frac{Z_1}{Z_0}\cos\theta)\frac{V_L}{Z_1}\\&=jY_1\sin\theta V_L+\cos\theta I_L\end{aligned}$$

Therefore, the voltage and current equations can be expressed as

$$\begin{bmatrix}V(z=-l)\\I(z=-l)\end{bmatrix}=\begin{bmatrix}\cos\theta & jZ_1\sin\theta\\ jY_1\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}V_L\\I_L\end{bmatrix}$$

This is the same as

$$\begin{bmatrix}V(0)\\I(0)\end{bmatrix}=\begin{bmatrix}\cos\theta & jZ_1\sin\theta\\ jY_1\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}V(d)\\I(d)\end{bmatrix}$$



$$\begin{bmatrix} V(0) \\ I(0) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V(l) \\ I(l) \end{bmatrix}$$

$$\begin{cases} A = \cos\beta l \\ B = jZ\sin\beta l \\ C = \frac{j}{Z}\sin\beta l \\ D = \cos\beta l \end{cases}$$

$$V(z) = V^{+}e^{-j\beta z} + V^{-}e^{+j\beta z}$$

$$I(z) = \frac{V^{+}}{Z}e^{-j\beta z} - \frac{V^{-}}{Z}e^{+j\beta z}$$

$$\begin{bmatrix} V(z) \\ I(z) \end{bmatrix} = \begin{bmatrix} e^{-j\beta z} & e^{+j\beta z} \\ \frac{1}{Z}e^{-j\beta z} & -\frac{1}{Z}e^{+j\beta z} \end{bmatrix} \begin{bmatrix} V^{+} \\ V^{-} \end{bmatrix}$$

at $z = 0$

$$\begin{bmatrix} V(0) \\ I(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{Z} & -\frac{1}{Z} \end{bmatrix} \begin{bmatrix} V^{+} \\ V^{-} \end{bmatrix}$$

$$\begin{bmatrix} V(0) \\ I(0) \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{Z} & -\frac{1}{Z} \end{pmatrix} \begin{bmatrix} & \\ & \end{bmatrix}^{-1} \begin{bmatrix} V(l) \\ I(l) \end{bmatrix}$$

$$= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V(l) \\ I(l) \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{Z} & -\frac{1}{Z} \end{bmatrix} \left(\frac{-2}{Z}\right) \begin{bmatrix} -\frac{1}{Z}e^{+j\beta l} & -e^{j\beta l} \\ -\frac{1}{Z}e^{-j\beta l} & e^{-j\beta l} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta l & jZ\sin\beta l \\ \frac{j}{Z}\sin\beta l & \cos\beta l \end{bmatrix} \qquad AD - BC = 1$$

Discontinuities and equivalent circuits

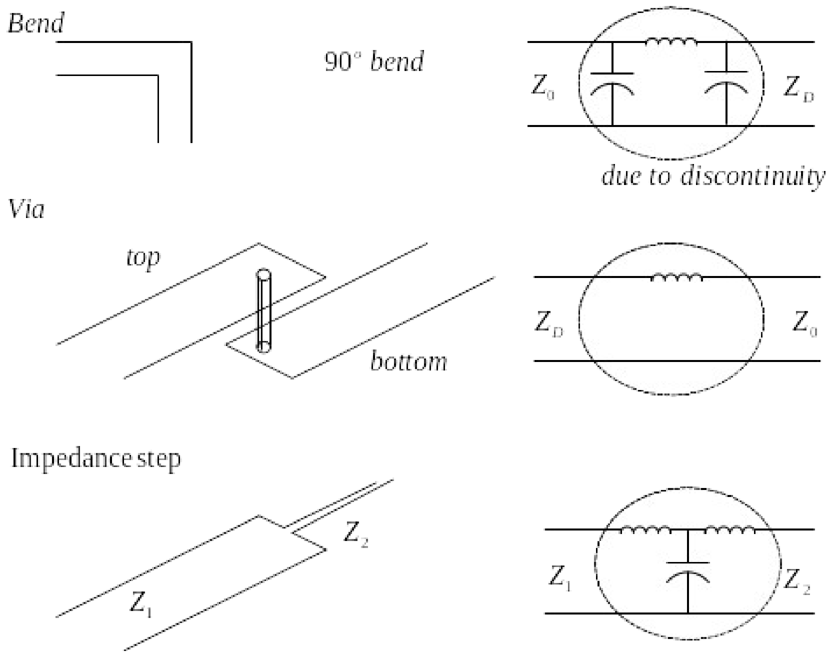


Fig.: Examples of discontinuities

Comparison of ideal TL model and EM simulations of test PCB

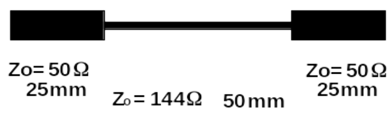


Fig. 4-1: Non-uniform TL test PCB.

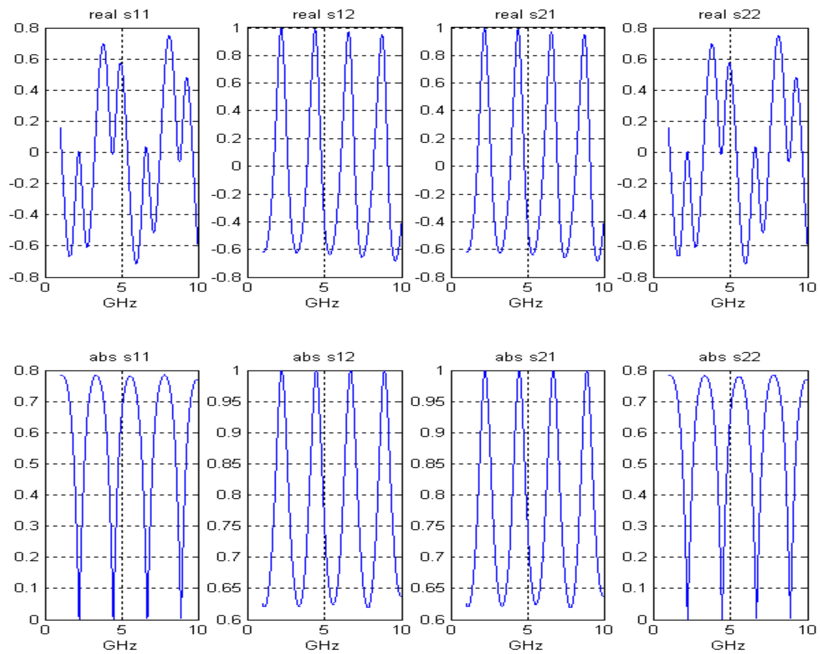


Fig. 4-2: S-parameters based on the ideal transmission line model (50-144-50)

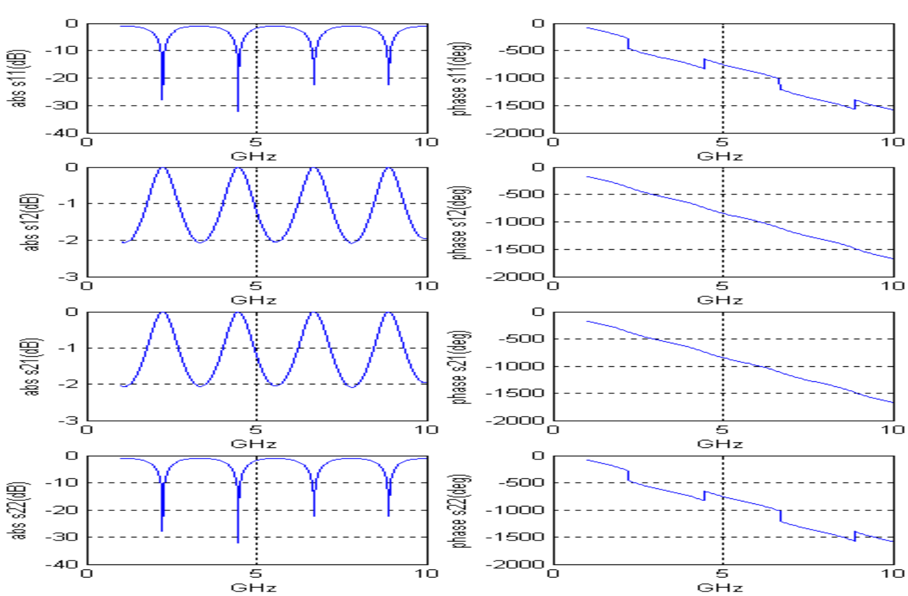


Fig 4-3: S-parameters of ideal transmission line (Mag in dB and phase)

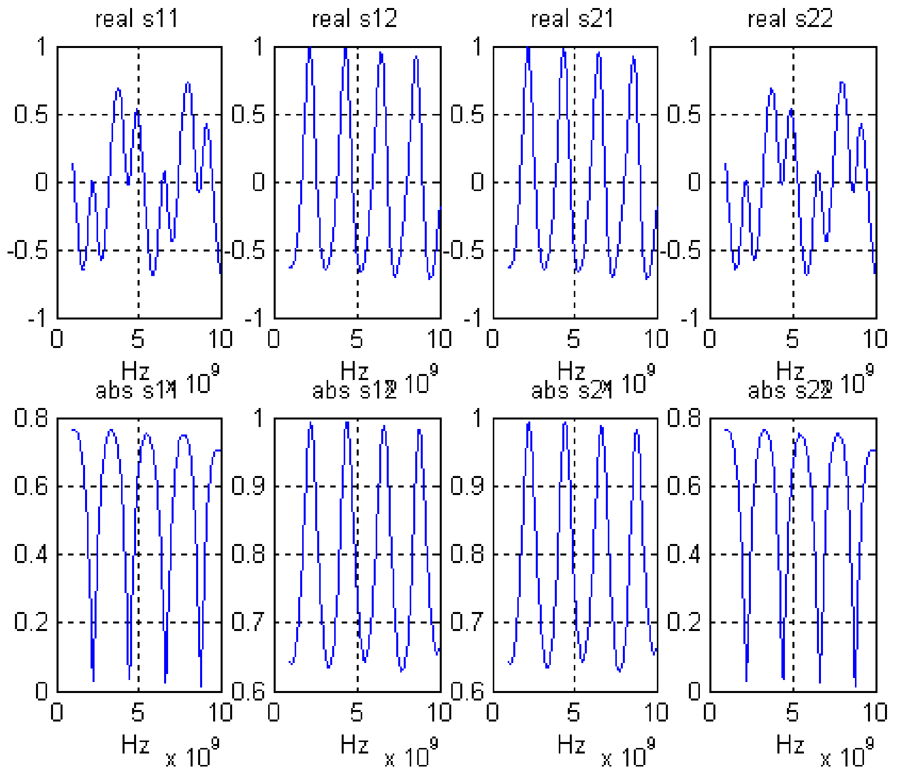


Fig 4-4: S-parameters of test PCB with HSFF

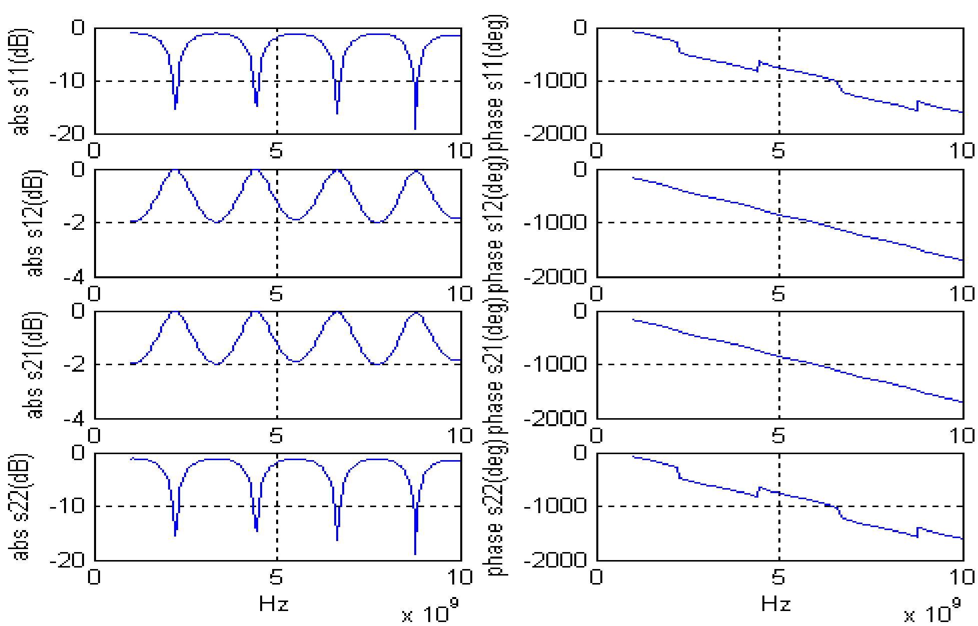


Fig 4-5: S-parameters of the test PCB with HFSS (Mag in dB and phase)

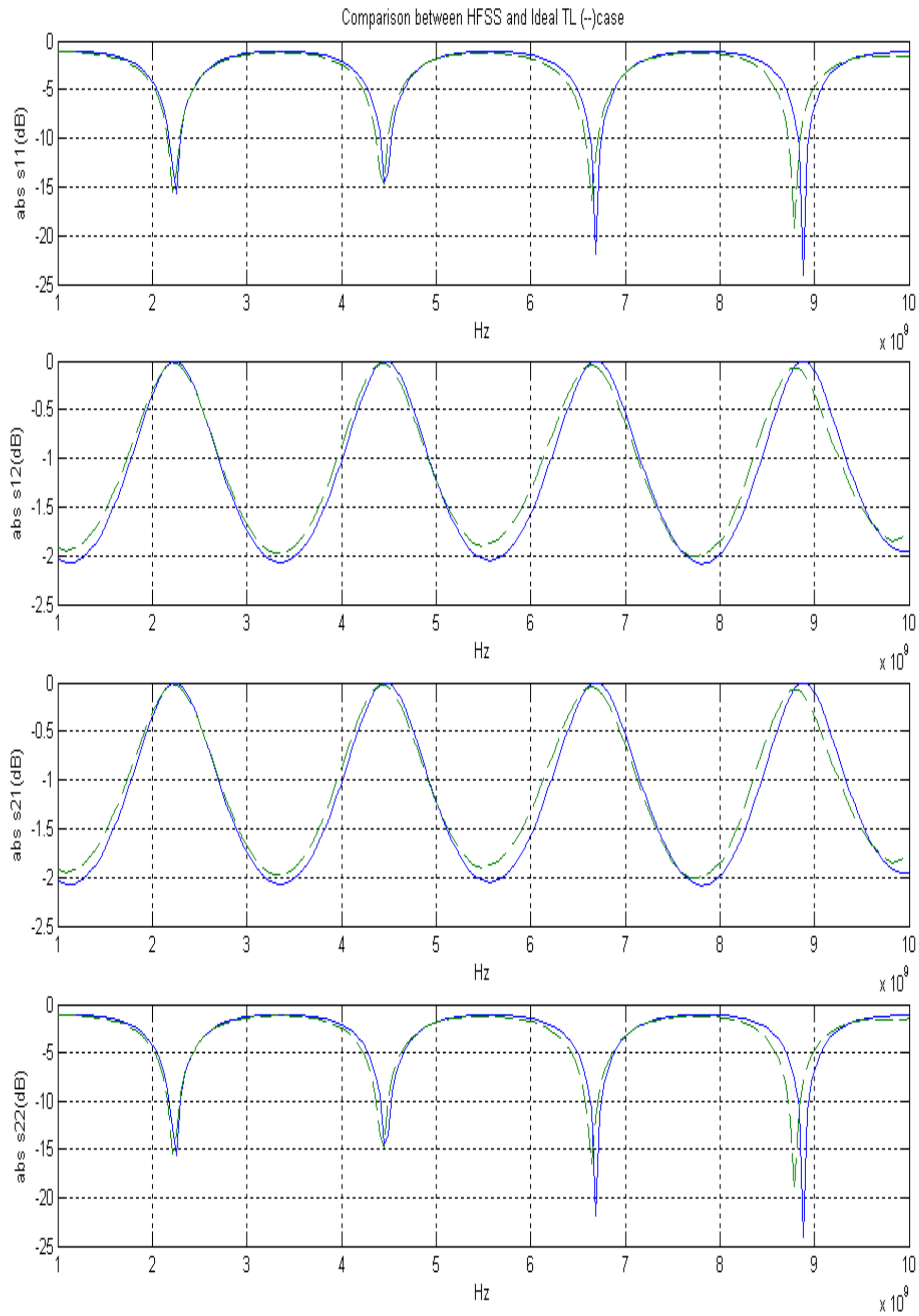


Fig. 4-6: Comparison between HFSS (with discontinuity effect) and Ideal TL

The difference between the ideal TL model and HFSS simulations is small at low frequency but increases at high frequency as shown in Fig. 4-6.