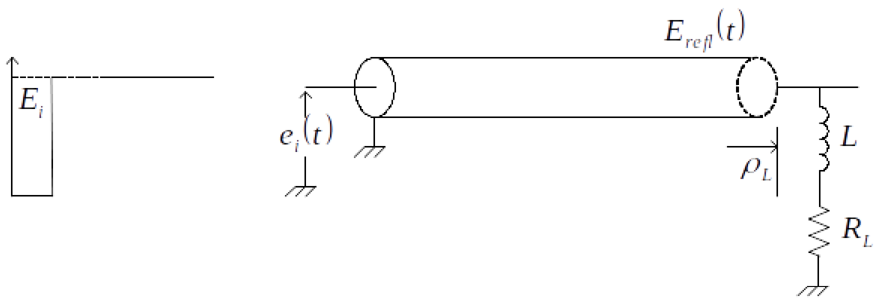
5. Laplace transform analysis of a TL terminated with a complex load impedance **Revised 10/18/18**

In the previous section, we studied the reflection of a unit step function from the unmatched load which is purely resistive. The rise-time (waveform) of the reflected signal does not change if the unmatched load is resistive. In reality, however, the input/load impedances of the digital circuit contain a reactive element such as a parasitic capacitance and inductance. In this section, therefore, we will study the time-domain analysis of reflected signal from a complex load impedance. The analysis is based on Laplace transform . We express the input waveform and complex load impedance using Laplace transform and calculate the reflected voltage. By taking the inverse Laplace transform, we can derive the time-domain response. The waveform of the reflected signal shows the distinctive characteristics depending on the load type. This can be used for inferring the load type. In addition, we will show how to obtain the values of each element using different techniques.

5-1. Unit step function as an input (Reflection case)

We have a L-R series load attached to a lossless TL which does not distort the waveform. The input signal is a unit step function. To simplify the analysis, we will neglect the delay due to a TL.



Using the Laplace transform, the load impedance is given by $Z_L = R_L + sL$.

The reflection coefficient at the load can be expressed as

$$\rho_L(s) = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{s + \dfrac{R_L - Z_0}{L}}{s + \dfrac{R_L + Z_0}{L}}$$

The input voltage is the unit step function and is given by

$$e_i(t) = u(t)$$

The Laplace transform of this is

$$E_i(s) = \frac{1}{s}$$

where Laplace transform of $[u(t)] = \frac{1}{s}$

The reflected voltage from the load is given by

$$E_{refl}(s) = E_i(s)\,\rho_L(s) = \left(\frac{1}{s}\right)\left(\frac{s + \frac{R_L - Z_0}{L}}{s + \frac{R_L + Z_0}{L}}\right)$$

Now we use the inverse Laplace transform to get the time-domain response $E_{refl}(t)$

We need to separate the above equation into simple terms which represent each response such as a unit step function u(t). We can write the term as

$$\frac{s+A}{s(s+B)} = \frac{C_1}{s} + \frac{C_2}{(s+B)} = \frac{C_1(s+B) + C_2 s}{s\,(s+B)}$$

Therefore

$$C_1 + C_2 = 1$$
$$C_1 B = A$$

$$C_1 = \frac{A}{B} = \frac{R_L - Z_0}{R_L + Z_0}$$

$$C_2 = 1 - C_1 = \frac{2Z_0}{R_L + Z_0}$$

Now we take the inverse Laplace transform and obtain

$$\left[\frac{R_L - Z_0}{(R_L + Z_0)} + \frac{2Z_0}{(R_L + Z_0)} e^{-\left(\frac{R_L + Z_0}{L}\right)t}\right] u(t)$$

where $\frac{1}{S+B} \rightarrow e^{-Bt}$

The final expression of the reflected signal is

$$E_{refl}(t) = \left[\frac{R_L - Z_0}{(R_L + Z_0)} + \frac{2Z_0}{(R_L + Z_0)} e^{-\left(\frac{R_L + Z_0}{L}\right)t}\right] u(t)$$

The total voltage is the sum of incident and reflected at the load. Therefore, we have the load voltage of

$$E_{total}(t) = E_{inc} + E_{ref} = \left[1 + \frac{R_L - Z_0}{(R_L + Z_0)} + \frac{2Z_0}{(R_L + Z_0)} e^{-\left(\frac{R_L + Z_0}{L}\right)t}\right] u(t)$$

It is easy to see the peak voltage occurs at t=0 and the value is given by

$$E_{total}(t=0) = \left[\frac{2R_L + 2Z_0}{(R_L + Z_0)} \right] = 2$$

This is expected because the inductance is an open circuit at t=0 and the total voltage becomes twice the incident voltage.

The final voltage is given by setting $t \to \infty$. The value is

$$E_{total}(t=\infty) = \left[\frac{2R_L}{(R_L + Z_0)} \right]$$

The transient region is given by the exponential decay $e^{-\left(\frac{R_L+Z_0}{L}\right)t}$.
If we take a natural log of this response, we get a line as a function of time.

$$\log[e^{-\left(\frac{R_L+Z_0}{L}\right)t}] = -\left(\frac{R_L+Z_0}{L}\right)t = -mt$$

The above expression shows the slope of this line *m* is proportional to 1/L. Because we can find R_L from the $t=\infty$ data, we should be able to obtain the value of L from the slope *m*. This process can be used with the TDR response to obtain the values of L and R_L.

If the load is given by the other combinations of L, C and R, we need to replace Z_L by

Parallel L and R

$$Z_L = R_L // sL = sR_L L/(R_L + sL)$$

Series C and R

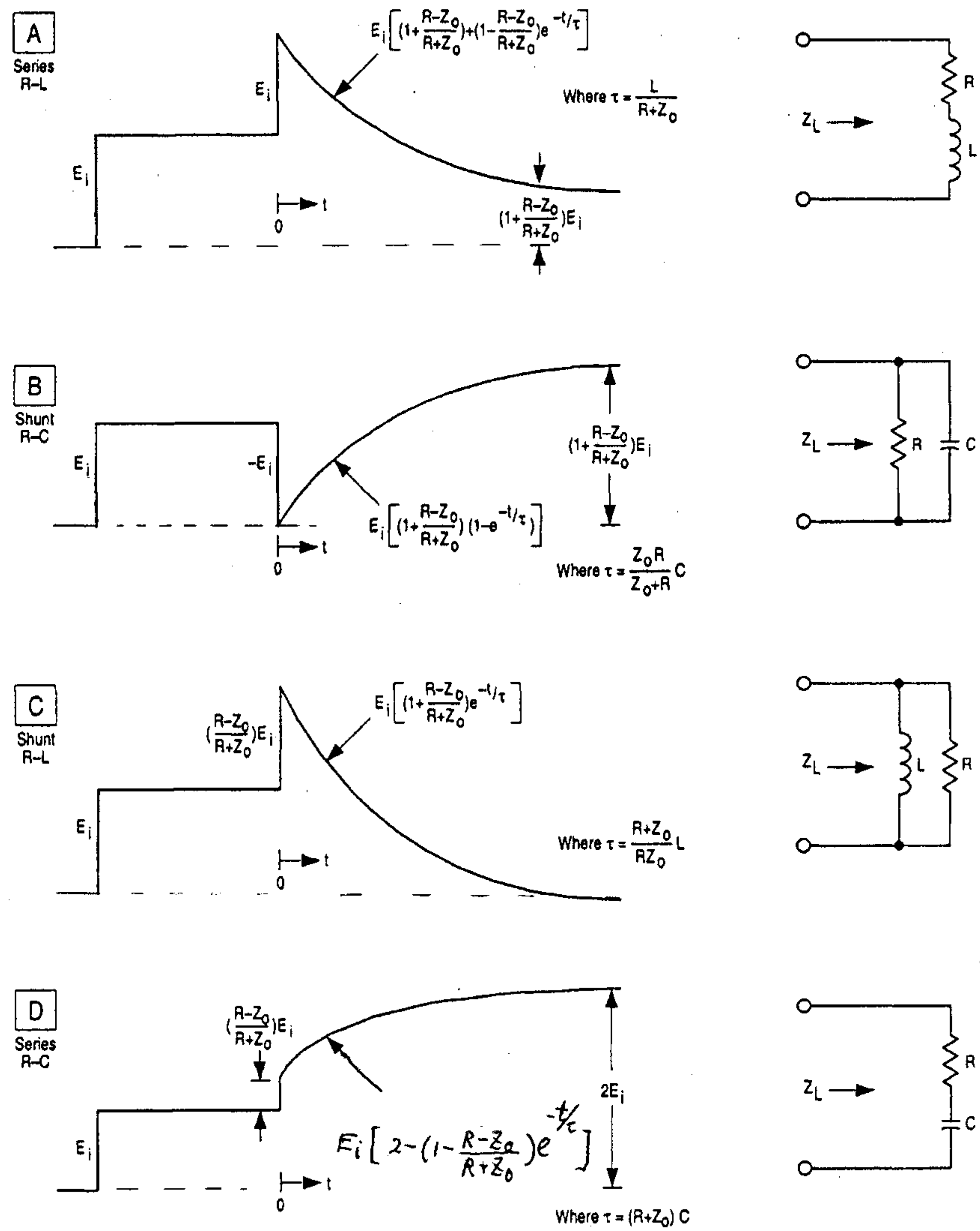
$$Z_L = R_L + (1/sC)$$

Parallel C and R

$$Z_L = R_L // sC = R_L/(1 + sCR_L)$$

The expected waveforms and responses are shown below. In all cases, the values of L and C can be estimated by taking a natural log of the time-domain responses. However, it is also clear that neither the initial peak voltage nor the final voltage can be used for estimating L and C. This limitation is due to the use of an ideal unit step function.
In the next section, we will study the finite rise-time case and show that the peak voltage value can be used for estimating the value of L when the inductance is present.

Expected TDR responses from different complex loads



Estimation of complex load values for the unit step function excitation

Assume we have a series L and R circuit and we get

$$E_{total}(t) = C_1\left[1 + \frac{R_L - Z_0}{(R_L + Z_0)} + \frac{2Z_0}{(R_L + Z_0)} e^{-\left(\frac{R_L + Z_0}{L}\right)t}\right] u(t)$$

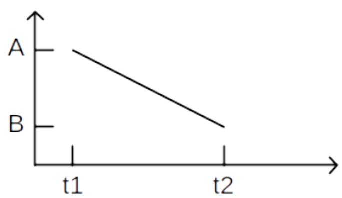
where C_1 is a constant related to the initial peak voltage of TDR.

We can assume E_{total} is also our measured data. We can find R_L from either the initial or final condition. To find L, we set

$$e^{-\left(\frac{R_L + Z_0}{L}\right)t} = [(E_{total}(t)/C_1) - (1 + \frac{R_L - Z_0}{(R_L + Z_0)})]\left[\frac{(R_L + Z_0)}{2Z_0}\right] = C_2$$

$$\ln(e^{-\left(\frac{R_L + Z_0}{L}\right)t}) = -\left(\frac{R_L + Z_0}{L}\right)t = -mt = \ln(C_2)$$

where m is a slope of y=-mt line of the experimental data. C_2 must be positive. C_2 has a linear section and you can find the slope as m=(A-B)/(t2-t1). Although E_{total} is your measured data, the constant term in C_2 is not important. You can use your experimental data as C_2 in the analysis as shown below.



Linear section of C_2 is shown.

When we have R and C, we have (1-exp(-mt)) response. In this case you cannot take ln(experimental data) to get a slope. You need to change the data to get the form exp(-mt)=C_x equation.

Assume we have a parallel R and C load.

$$E_{total}(t) = C_1\left[(1 + \frac{R_L - Z_0}{(R_L + Z_0)})(1 - e^{-mt})\right] u(t)$$

$$e^{-mt} = 1 - E_{total}(t) / [C_1\left[(1 + \frac{R_L - Z_0}{(R_L + Z_0)})\right]] = C_3$$

where C_1 is a constant related to the initial peak voltage of TDR. In this expression E_{total} is your measured data. Also C_3 must be positive to use ln() function.

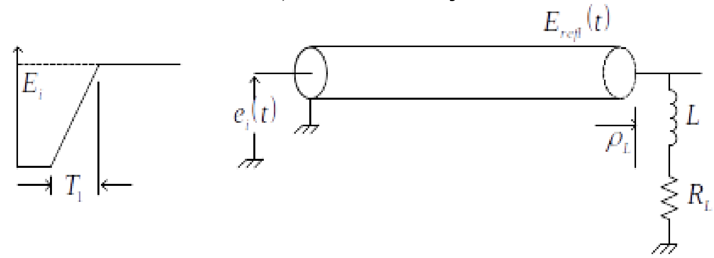
Another approach is the "trial and error" or "supervised parameter estimation". Assume you already know C_1 and R_L in the following equation and the only unknown is L.

$$E_{total}(t) = C_1\left[1 + \frac{R_L - Z_0}{(R_L + Z_0)} + \frac{2Z_0}{(R_L + Z_0)} e^{-\left(\frac{R_L + Z_0}{L}\right)t}\right] u(t)$$

Set L to be a certain value then calculate E_{total}() and compare it with experimental data. Time t=0 corresponds to the start of the reflected voltage. The initial voltage exists for t<0, shown as E_i. If the tail is too long, L is too large. If the tail is too short, L is too small. You may use the "binary search" method to find the optimum value of L.

5-2. Finite rise-time input (Reflection case) (This is more realistic model.)

In 5-1, we studied the simple unit step function responses. The practical digital circuits, however, have a finite rise-time signal and complex load. In this section, we will use the Laplace transform technique to analyze both finite rise-time signal and reflection from a complex load.   We assume the TL is lossless. The lossy TL case which will create dispersion (distortion of the waveform) will be analyzed later.



Let assume the input pulse rise-time (10-90%) is specified as t_r =80ps. Using the straight line approximation, the total transient time is given by

$$T_1 \approx \frac{t_r}{0.8} = 100\,ps$$

The slope of the input signal (rise) is then $m = \dfrac{E_i}{T_1}$.

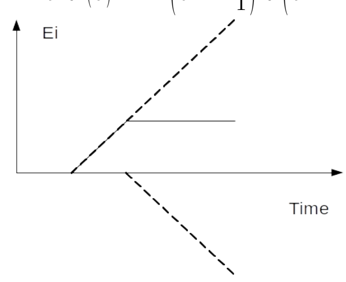
We use the same complex load given by $Z_L = R_L + sL$

The reflection coefficient at the load can be expressed as

$$\rho_L(s) = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{s + \dfrac{R_L - Z_0}{L}}{s + \dfrac{R_L + Z_0}{L}}$$

The input voltage can be decomposed into two terms and it is given by

$$e_i(t) = m\,t\;u(t) - m\,(t - T_1)\,u(t - T_1)$$



We take Laplace transform of $e_i(t)$ and obtain

$$E_i(s) = \frac{m}{s^2}\left(1 - e^{-T_1 s}\right)$$

$e^{-T_1 s}$: delay

where $u(t)=\frac{1}{s}$

$tu(t)=\frac{1}{s^2}$

The reflected voltage from the load is given by

$$E_{refl}(s)=E_i(s)\,\rho_L(s)=\left(\frac{m}{s^2}\right)\left(1-e^{-T_1 s}\right)\left(\frac{s+\frac{R_L-Z_0}{L}}{s+\frac{R_L+Z_0}{L}}\right)$$

Now we use an inverse Laplace transform to get the time-domain response $E_{refl}(t)$

$$E_{refl}(s)= \left(\frac{m}{s^2}\right)\left(\frac{s+\frac{R_L-Z_0}{L}}{s+\frac{R_L+Z_0}{L}}\right) - \left(\frac{m}{s^2}\right) e^{-T_1 s} \left(\frac{s+\frac{R_L-Z_0}{L}}{s+\frac{R_L+Z_0}{L}}\right)$$

We need to separate the above equation into simple terms which represent each response such as a unit step function u(t). We can write the first term as

$$\frac{s+A}{s^2(s+B)}=\frac{C_1}{s}+\frac{C_2}{s^2}+\frac{C_3}{(s+B)}=\frac{C_1 s(s+B)+C_2(s+B)+C_3 s^2}{s^2(s+B)}$$

Therefore

$$C_1+C_3=0$$
$$C_1 B+C_2=1$$
$$C_2 B=A$$

$$C_2=\frac{A}{B}=\frac{R_L-Z_0}{R_L+Z_0}$$
$$C_1=\frac{1-C_2}{B}=\frac{2Z_0 L}{(R_L+Z_0)^2}$$
$$C_3=-C_1$$

Take an inverse Laplace transform and obtain

1st term

$$\left[\frac{2Z_0 L}{(R_L+Z_0)^2}+\left(\frac{R_L-Z_0}{R_L+Z_0}\right)t-\frac{2Z_0 L}{(R_L+Z_0)^2}e^{-\left(\frac{R_L+Z_0}{L}\right)t}\right]u(t)$$

$$\frac{1}{S+B}\rightarrow e^{-Bt}$$

We can obtain the 2nd term using the same method.

Then the final expression of the reflected voltage is

$$E_{refl}(t)=\left[\frac{2Z_0L}{(R_L+Z_0)^2}+\left(\frac{R_L-Z_0}{R_L+Z_0}\right)t-\frac{2Z_0L}{(R_L+Z_0)^2}e^{-\left(\frac{R_L+Z_0}{L}\right)t}\right]m\ u(t)$$
$$-\left[\frac{2Z_0L}{(R_L+Z_0)^2}+\left(\frac{R_L-Z_0}{R_L+Z_0}\right)(t-T_1)-\frac{2Z_0L}{(R_L+Z_0)^2}e^{-\left(\frac{R_L+Z_0}{L}\right)(t-T_1)}\right]m\ u(t-T_1)$$

The total voltage is the sum of incident and reflected at the load. Therefore, we have the load voltage of

$$E_{total}(t)=E_{inc}+E_{ref}$$

This should provide the complete voltage response.

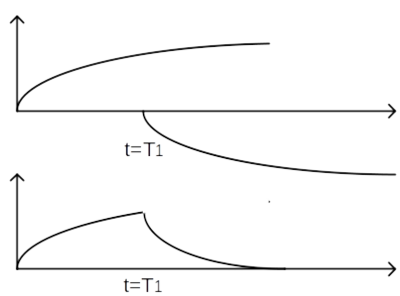
It is often important to estimate the maximum reflected voltage. As we found in 5-1, the maximum value is $2E_{inc}$ if the input is a unit step function.

**** This is not correct

To obtain the maximum reflected voltage, we take a derivative of $E_{refl}(t)$ with respect to *t* and set it to zero. This gives us the condition that the maximum voltage occurs at $t=T_1$.

*****

To simplify, assume $R_L=Z_0$ and plot $E_{refl}(t)$. We can show the peak occurs at $t=T_1$.



Therefore

$$E_{refl\max}=E_{refl}(t=T_1)$$

For a special case of $R_L=Z_0$, the maximum voltage becomes

$$E_{refl}(t=T_1)=\frac{mL}{2Z_0}\left[1-e^{-\left(\frac{2Z_0}{L}\right)T_1}\right]$$

Because *L* is contained at two places, there is no simple formula to estimate the inductance *L*.

If we approximate exp() as $e^x\sim1+x+x^2/2$ ($e^x\sim1+x$ does not work in this case), we get

$$E_{refl}(t=T_1) \approx \frac{mL}{2Z_0}\left[1-[1-\left(\frac{2Z_0}{L}\right)T_1+[\left(\frac{2Z_0}{L}\right)T_1]^2/2\right] = mT_1\left[1-\left(\frac{Z_0}{L}\right)T_1\right]$$

Then the approximate value of L is

$$L \approx Z_0 T_1 /[1 - E_{refl}(t=T_1)/mT_1]$$

Another approach is to calculate the area of the reflected voltage which is the same as integration over the reflected voltage. If $R_L = Z_0$, the final value is the same as the incident voltage and the effect of the incident voltage can be neglected. However, if $R_L \neq Z_0$, the integration over the initial and final values must be carefully done.

Assume $R_L = Z_0$. The integration of the reflected voltage is given by

$$\int_0^\infty E_{refl}(t)dt = \int_0^\infty \frac{mL}{(2Z_0)}\left[1-e^{-\left(\frac{2Z_0}{L}\right)t}\right]dt - \int_{T_1}^\infty \frac{mL}{(2Z_0)}\left[1-e^{-\left(\frac{2Z_0}{L}\right)(t-T_1)}\right]dt = \frac{mLT_1}{2Z_0}$$

This should be set equal to the measured area under the reflected voltage. Then L can be estimated as

$$L = \frac{2Z_0}{mT_1}\int_0^\infty E_{refl\_measured}(t)dt$$

5.3 Estimation of the inductance *L* (Examples using the next figure)

(a) Based on the peak value

$T_1 \approx \frac{t_r}{0.8} = 100\,ps$ t_r : 10 to 90%, T_1: total risetime

We want to find the parasitic inductance of the load.

The peak reflected voltage is given by = 0.44V

$m = \frac{0.94}{100\,ps}$ 0.94V: initial voltage

From $0.44 = \frac{mL}{2Z_0}\left[1-e^{-\left(\frac{2Z_0}{L}\right)T_1}\right]$, $Z_0 = 50\,\Omega$

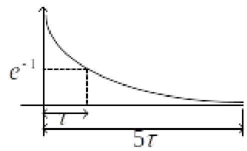
We can find L to be

$$L = 6\,nH$$

If we use the approximation $L \approx Z_0 T_1 /[1 - E_{refl}(t=T_1)/mT_1]$, we get $L \approx 10nH$ which is substantially different from the previous estimate. In this case the argument of the exp() is close to -1 and the approximation based on the 3 terms is not sufficient.

(b) Based on e^{-1} value. If the time-domain response is close to the exponential function and the inductance can also be estimated from the decay time as shown below.

$$\tau = \frac{L}{2Z_0} \quad \text{where} \quad e^{-\frac{t}{\tau}}$$



$$L = (\tau)(2Z_0)$$

If the decay time is $5\tau \sim 0.3ns$, then

$$L \approx (60\,ps)(100) \approx 6\,nH$$

This value is close to an estimate based on the peak value.

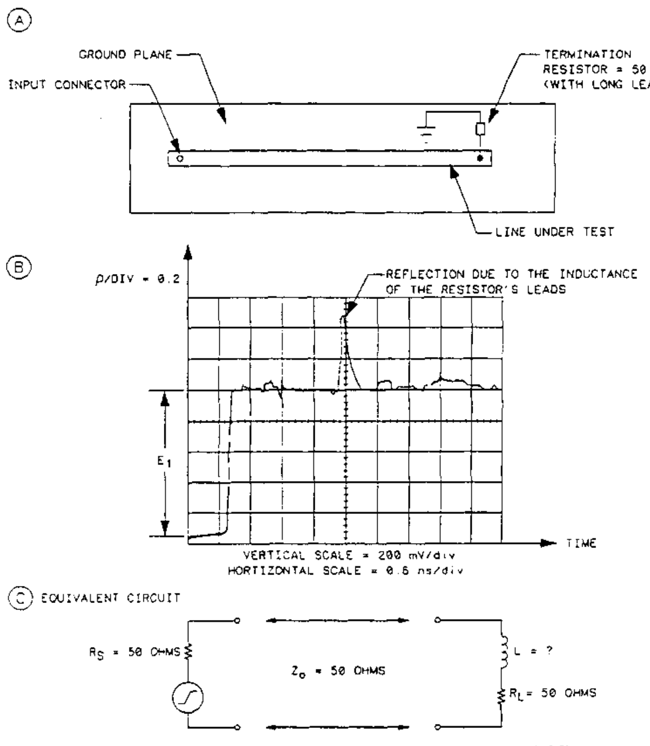


Figure E-10. Effects Due to Termination Resistor Leads

These two methods to estimate L have serious problems. For example, the max value of E_{refl} depends on the rise-time t_r in the first approach. If the rise time is close to 1ns, then the peak voltage will be reduced to

$$E_{refl_{\max}} \sim 56\,mV$$

This is much less than 0.44V for $t_r = 80\,ps$ and a small amount of noise will introduce an error in estimation. Similarly, it can be shown that the voltage waveform becomes far from an ideal exponential decay when the rise-time is increased.

5.4 Estimation of *L* based on the area (Transmission case)

Unlike a small capacitance, the small inductance measurement is much more difficult. The time-domain approach shown here can be used. In many cases it is difficult to find the accurate exp(-1) time or slope *m* from the measured data due to a slow rise-time. Also these techniques are sensitive to measurement errors. In the following section, we will show the area under the reflected voltage can be used for estimating L.

The right figure shows the voltage observed at the unknown inductor. If there is no resistor, the final voltage is 0 (or the final current is ΔV /Rs). The initial current *I*(t=0) = 0. We can express the area under the voltage waveform is

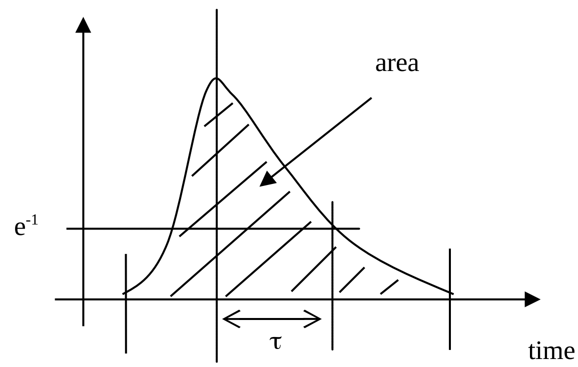
$$V_{inductor} = L\frac{dI_{inductor}}{dt}$$

$$\int_0^{\infty} V_{inductor}(t)\,dt = L\int_0^{\infty}\frac{dI_{inductor}(t)}{dt}dt$$

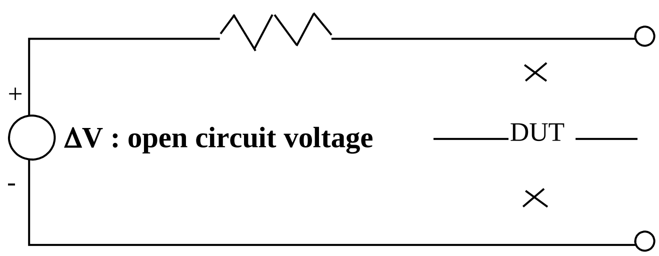
$$\int_0^{\infty} V_{inductor}(t)\,dt = L[I(\infty) - I(0)]$$

area = L[I(∞) - I(0)]

$$L = \frac{(area)}{DI} = \frac{(area)(R_s)}{DV}$$



R_s : Source impedance



R_s=equivalent source impedance due to source
ΔV is the open circuit voltage.

Example of Small Inductance Measurement.

The following figures show how to find the unknown inductance from the measured time-domain data. Unlike a small capacitor, measuring a small inductor with TDR is a challenging task. The time constant of series or parallel RL circuits has exp(-tL/R). If the value of L is small, R cannot to too large to observe a time-domain response within a reasonable time.

$$\tau = \frac{L}{R}$$

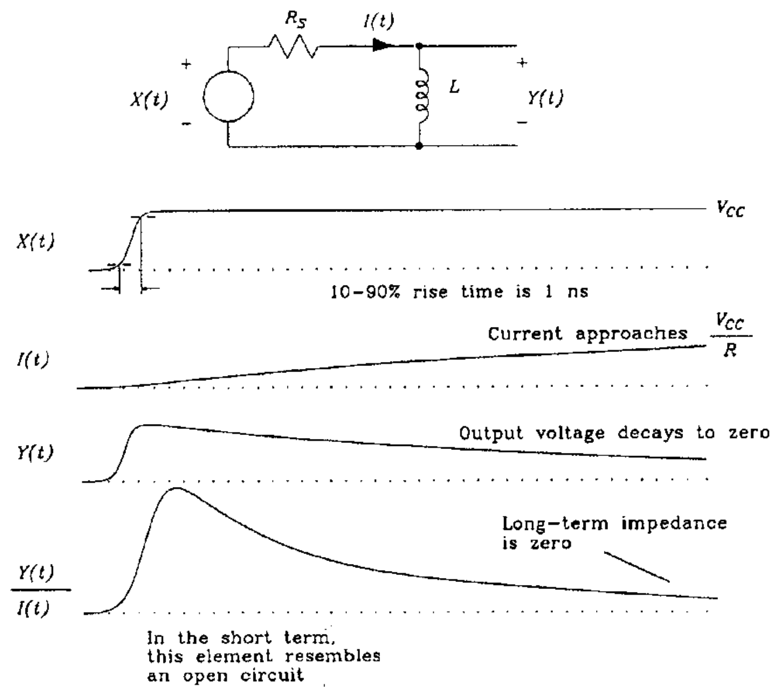


Figure 1.10 Instantaneous resistance of a perfect inductor.

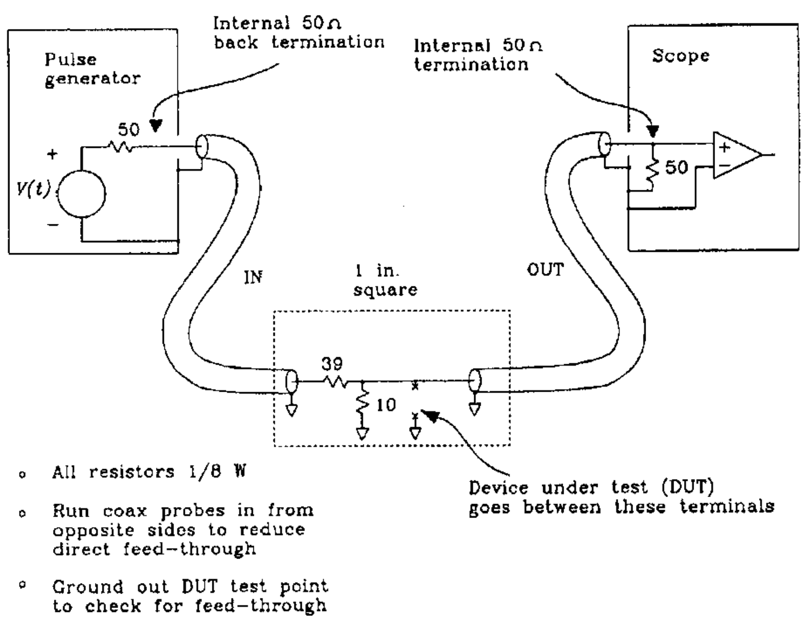
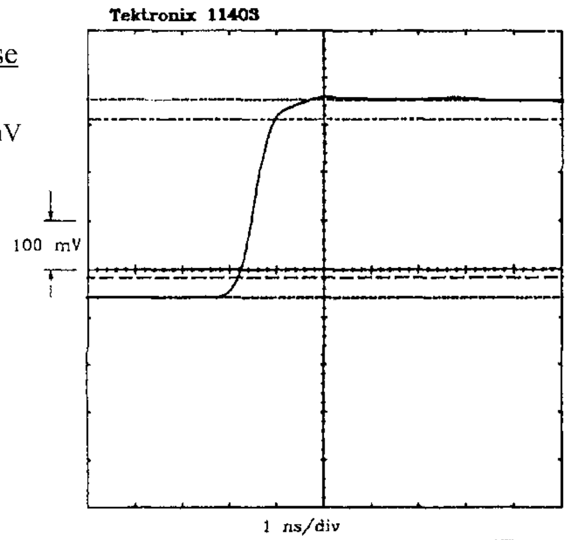


Figure 1.11 A 7.6-Ω lab setup for measuring inductance.

System response

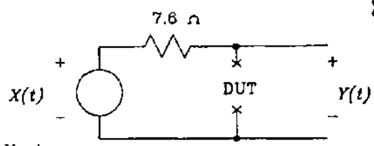
Open circuit voltage ~ 418mV



Peak to peak, 417.6 mV	Rise, 788.1 ps			

Figure 1.12 *Open-circuit response of a 7.6-Ω inductance test setup.*

$$89//10//50 = 7.6\Omega$$



417-mV step

788-ps 10-90% rise time

Figure 1.13 Thevenin equivalent of a 7.6-Ω inductance test setup.

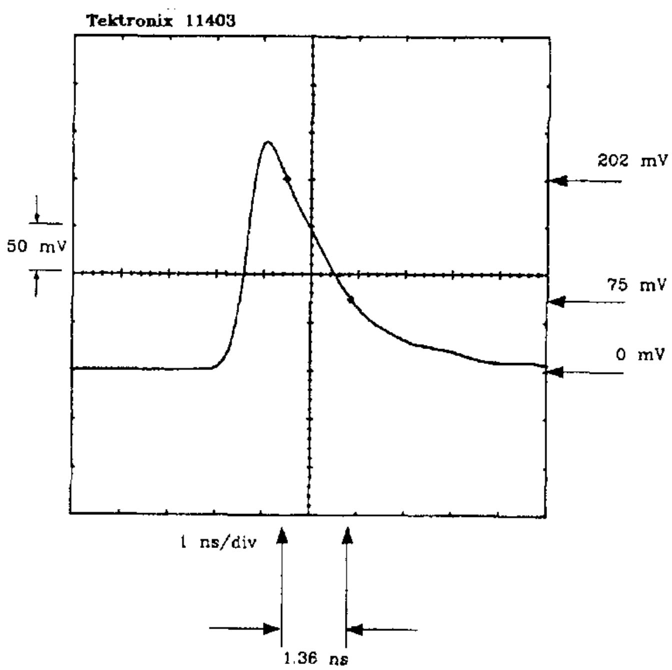


Figure 1.14 Decaying exponential response of a 7.6-Ω inductance test setup.

Based on the time constant

Find e^{-1} time

$$L = R\tau = (7.6)(1.36\times10^{-9}) = 10.3\,nH$$

R: must be small to get a large value of τ=L/R.

System response without *L:* 0.8ns (rise time of the system)
System response with *L*: 1.36ns (inductor decay time)

***L* estimation from the area**

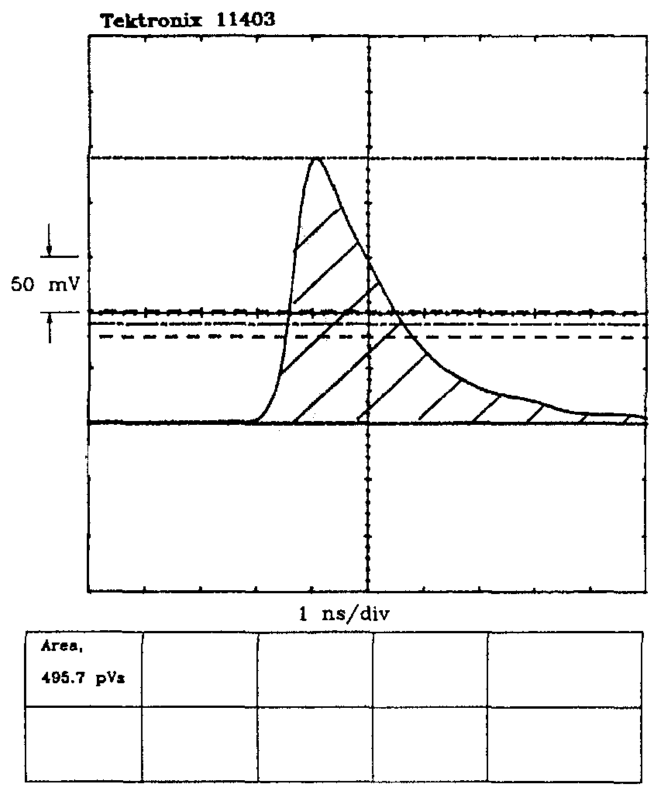


Figure 1.15 Area under the response of a 7.6-Ω inductance test setup.

$$= L[I(\infty) - I(0)]$$
$$= L\Delta I$$

$$\int_0^\infty V_{inductor}(t)\,dt = Area$$

$$L = \frac{Area}{\Delta I} = \frac{Area\cdot R_s}{\Delta V}$$

ΔV : open circuit voltage

R_s = 7.6Ω

$$L = \frac{(495)(7.6)}{0.418} = 9\,nH$$

Rs=(50+39)//10//50=7.6 ohm

50+39:	toward source (pulse generator and series resistor)
10:	to ground
50:	toward scope (input impedance of the scope)

5.5 Small Capacitance Measurements

Unlike a small inductor, measuring a small capacitor with TDR is relatively easy. The time constant of series or parallel RC circuits has exp(-t/RC) which can be significantly increased by choosing a large R value.

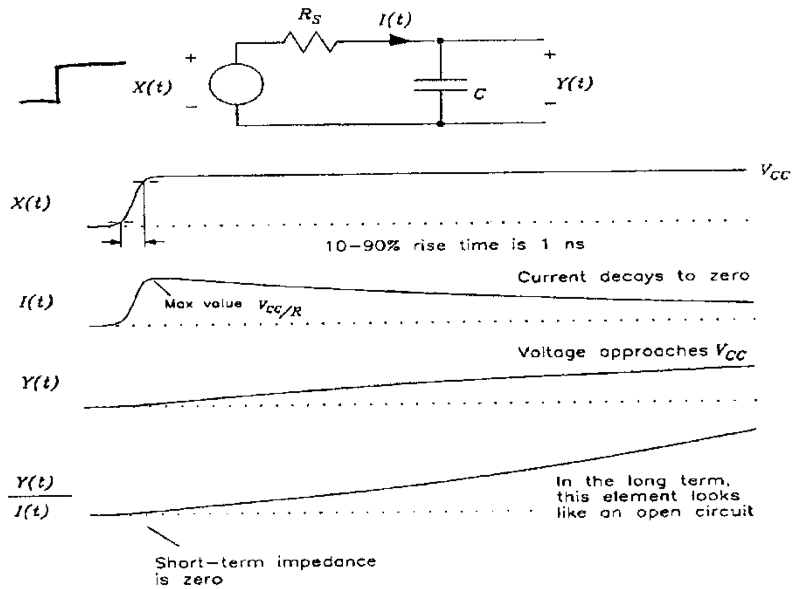


Figure 1.5 Step response of a perfect capacitor.

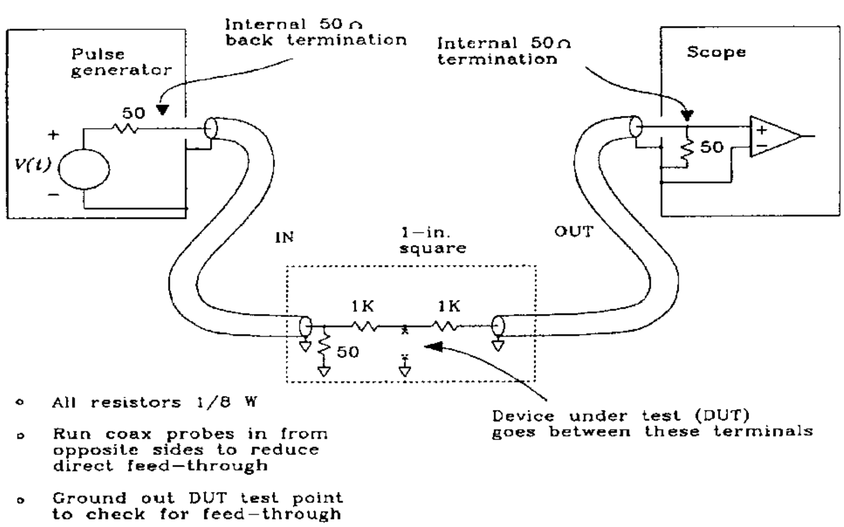
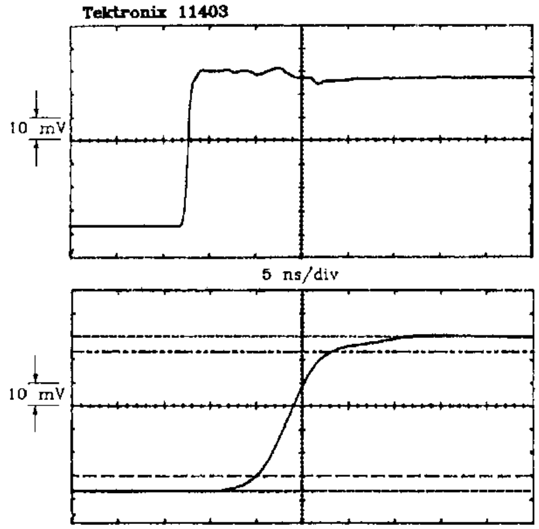


Figure 1.6 A 500-Ω lab setup for measuring capacitance.



Peak to peak, 67.16 mV	Rise, 816.9 ps			

Figure 1.7 Open-circuit response of a 500-Ω capacitance test setup.

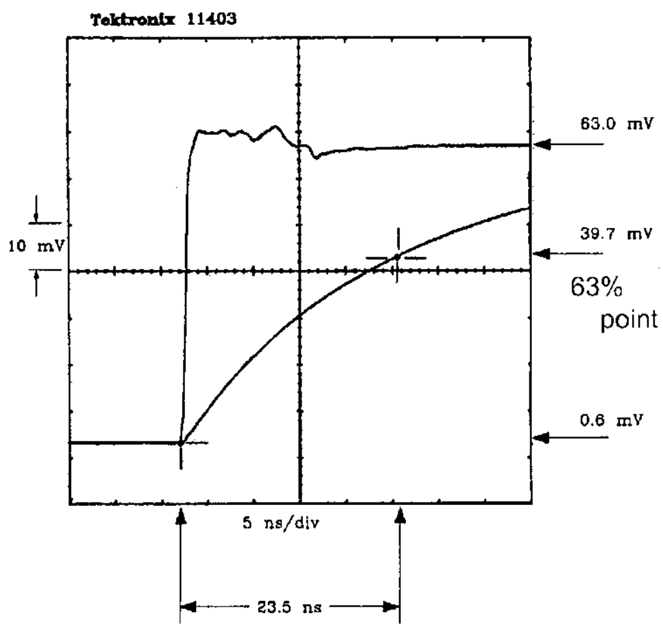
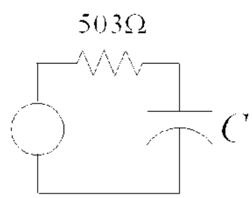


Figure 1.9 Finding a time constant using the 63% method.

$$1025 // 1050 = 503\Omega$$



$$C = \frac{23.5 * 10^{-9}}{503} = 467\,pF$$

Laplace Transform Theorem

$$X(s)=\int_0^{\infty} x(t)e^{-st}\,dt$$

$$x(t)=\frac{1}{2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

	Name	Transform Pair
1.	Linearity	$ax(t)+by(t) \leftrightarrow aX(s)+bY(s)$
2.	Scale change	$x(at) \leftrightarrow \frac{1}{a}X(\frac{s}{a}) \qquad a>0$
3.	Time delay	$x(t-t_0) \leftrightarrow X(s)e^{-st_0} \qquad t_0>0$
4.	s-Shift	$e^{-at}x(t) \leftrightarrow X(s+a)$
5.	Multiplication by t^n	$t^n x(t) \leftrightarrow (-1)^n \frac{d^n X(s)}{ds^n} \qquad n=1,2,\ldots$
6.	Time differentiation	$\frac{d^n x(t)}{dt^n} \leftrightarrow s^n X(s) - \sum_{i=0}^{n-1} s^{n-1-i}x^{(i)}(0^-)$ where $x^{(i)}(t)=\frac{d^i x(t)}{dt^i}$
7.	Time integration	$y(t)=\int_{0^-}^{t} x(\lambda)\,d\lambda + y(0^-) \leftrightarrow \frac{X(s)}{s}+\frac{y(0^-)}{s}$
8.	Convolution	$x(t)*y(t) \leftrightarrow X(s)Y(s)$

The final two theorems present equal expressions rather than transform pairs. Also, these two theorems are valid only if the conditions stated in Chapter 7 are satisfied.

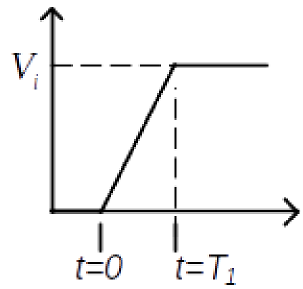
9.	Final value	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$
10.	Initial value	$\lim_{t\to 0^+} x(t) = \lim_{s\to\infty} sX(s)$

Table C.4 Laplace Transform Pairs

	$x(t)$	$X(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$\frac{t^n}{n!}u(t)$	$\frac{1}{s^{n+1}}$
4.	$e^{-at}u(t)$	$\frac{1}{s+a}$
5.	$\frac{t^n e^{-at}}{n!}u(t)$	$\frac{1}{(s+a)^{n+1}}$
6.	$\sin(\omega_0 t)\,u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
7.	$\cos(\omega_0 t)\,u(t)$	$\frac{s}{s^2+\omega_0^2}$
8.	$t\sin(\omega_0 t)\,u(t)$	$\frac{2\omega_0 s}{\left(s^2+\omega_0^2\right)^2}$
9.	$t\cos(\omega_0 t)\,u(t)$	$\frac{\left(s^2-\omega_0^2\right)}{\left(s^2+\omega_0^2\right)^2}$
10.	$e^{-at}\sin(\omega_0 t)\,u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$
11.	$e^{-at}\cos(\omega_0 t)\,u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
12.	$2Ce^{-at}\cos(\omega_0 t)\,u(t)$ $+\,2De^{-at}\sin(\omega_0 t)\,u(t)$	$(C+jD)\big/(s+a+j\omega_0)$ $+\,(C-jD)\big/(s+a-j\omega_0)$
13.	$Ee^{-at}\cos(\omega_0 t)\,u(t)$ $+\,\left((F-Ea)\big/\omega_0\right)e^{-at}\sin(\omega_0 t)u(t)$	$\frac{Es+F}{s^2+es+f}$ $a=e/2$, $\omega_0=\sqrt{f-a^2}$

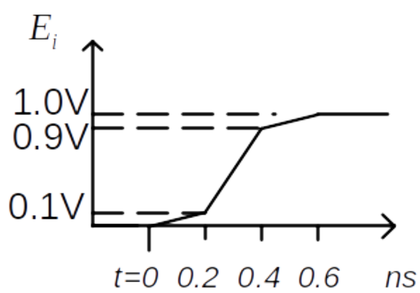
Shapes of rise-time and function to represent them

(1) Linear slope



dV_{in}/dt= a rectangular pulse.

(2) Piece-wise linear



dV_{in}/dt= a series of rectangular pulses.

(3) Square function (Non-linear)

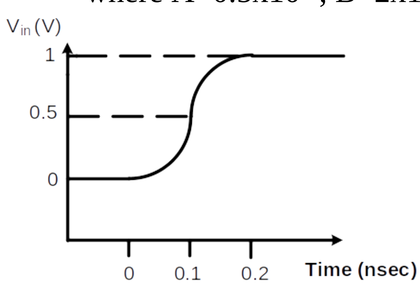
The input voltage can be expressed as

$V_{in} = At^2$ for $0 =< t =< 0.1$ns

$V_{in} = -At^2 + Bt + C$ for 0.1ns $=< t =< 0.2$ns

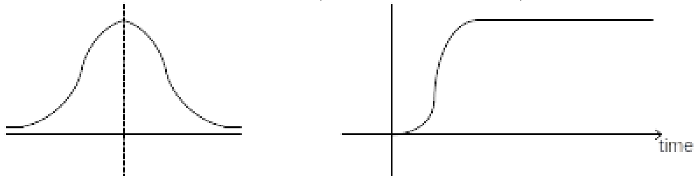
$V_{in} = 1$V for $t >= 0.2$ns

where $A=0.5\times10^{20}$, $B=2\times10^{10}$, $C=-1$



dV_{in}/dt= a triangular pulse.

(4) Gaussian function (Non-linear)



$$g_{step}(t)=\frac{1}{2}\left[1+erf\left(\frac{t}{2\tau_3}\right)\right]$$

$$erf(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-\lambda^2}d\lambda$$

$dV_{in}/dt=$

(5) Exponential function (Non-linear)

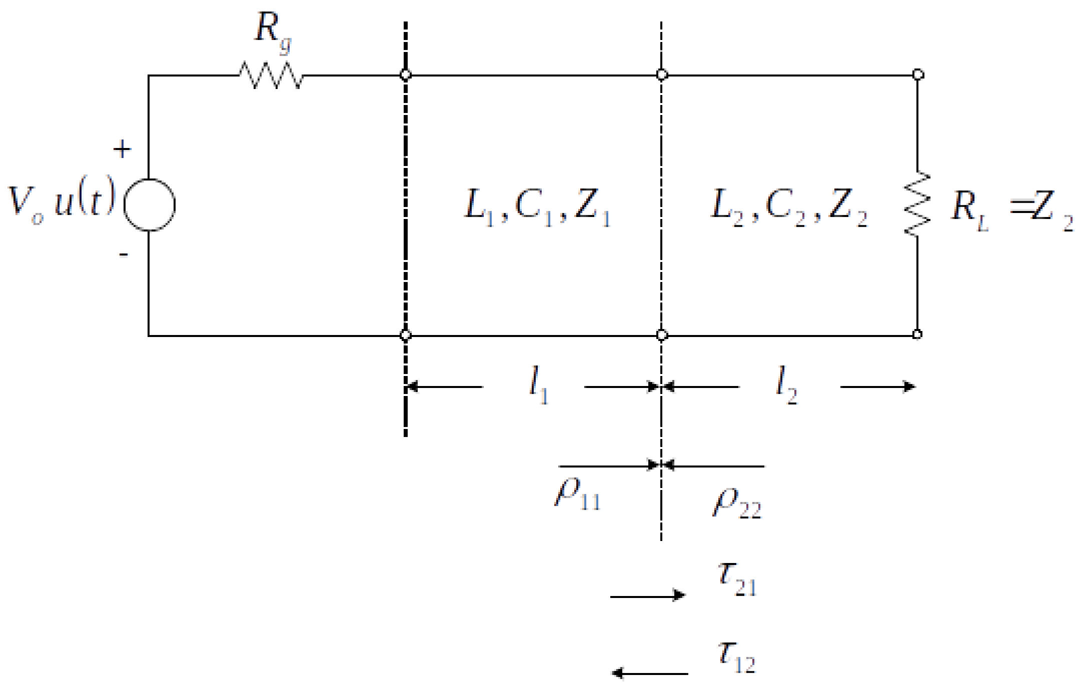
Single pole exponential

$$h_{step}(t)=u(t)\left(1-e^{-\left(\frac{t}{\tau_1}\right)}\right)$$

$dV_{in}/dt=$

6. Reflection on lines with discontinuities and branches

The voltage reflection/transmission on cascaded TLs can be analyzed using the space-time diagram as shown below. Assume we have two lossless TLs given by Z1 and Z2. To make the problem simple, we also assume the load impedance is matched to Z2.



$$Z_1 = \sqrt{\frac{L_1}{C_1}}$$

$$Z_2 = \sqrt{\frac{L_2}{C_2}}$$

$$v_1 = \frac{1}{\sqrt{L_1 C_1}}$$

$$v_2 = \frac{1}{\sqrt{L_2 C_2}}$$

$$\rho_G = \frac{R_g - Z_1}{R_g + Z_1}$$

$$\rho_{11} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$\rho_{22} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

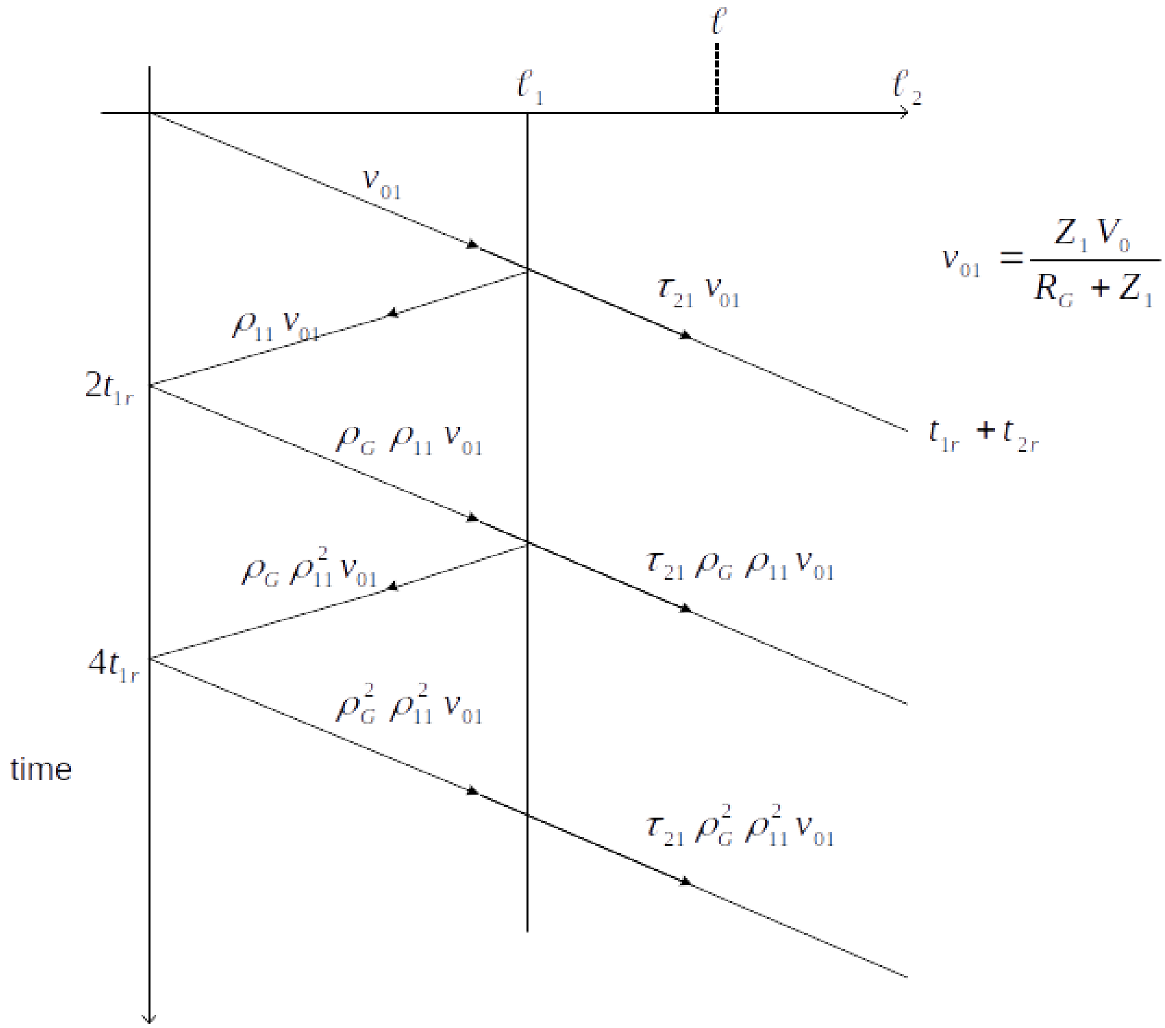
$$\tau_{21} = 1 + \rho_{11} = \frac{2Z_2}{Z_2 + Z_1}$$

$$\tau_{12} = 1 + \rho_{22} = \frac{2Z_1}{Z_1 + Z_2}$$

$$t_{1r}=\frac{l_1}{v_1}$$

$$t_{2r}=\frac{l_2}{v_2}$$ travel time

The space-time diagram is given by

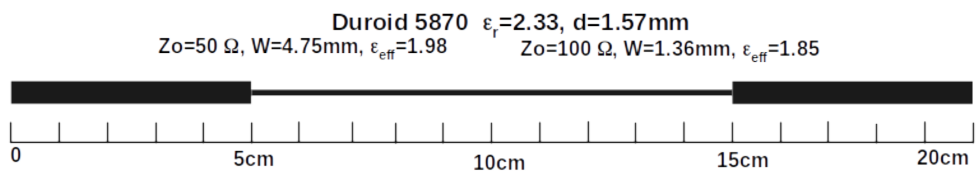


The transmission coefficient τ_{21} can be obtained from the reflection coefficient ρ_{11}. If we want to find the total voltage at the position l on TL2, we can write it as

$$v(l,t)=\left(\frac{V_0 Z_1}{R_G+Z_1}\right)\left[\tau_{21}u\left(t-t_{1r}-\frac{l}{v_2}\right)+\tau_{21}\rho_{11}\rho_G u\left(t-3t_{1r}-\frac{l}{v_2}\right)+\right.$$
$$\left.+\tau_{21}\rho_{11}^2\rho_G^2 u\left(t-5t_{1r}-\frac{l}{v_2}\right)..\right]$$

If the load impedance R_L is not matched to Z_2, we have reflected wave in TL_2 which will be transmitted into TL_1 with the transmission coefficient τ_{12}. This can be analyzed using the same space-time diagram with added reflections at load.

Problem: Obtain the space-time diagram of the following microstrip circuit and show the reflected and transmitted waveforms as a function of time. Assume both input and output are connected to the matched impedance (Z_o=50 Ω). The signal velocity on the microstrip TL is given by $v = c_o / \sqrt{\varepsilon_{eff}}$.

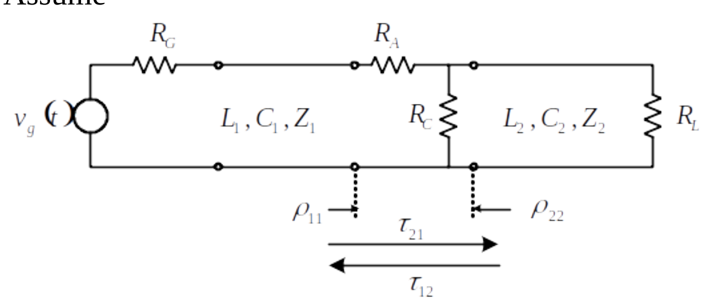


Resistive network can be used to minimize the reflection

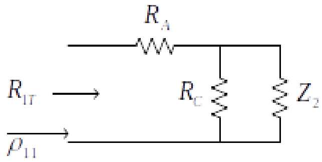
In the TL section, we study the different impedance matching techniques including the λ/4 transformer and open/short-stub tuning. These methods utilize a specific TL length to minimize the reflection at the designed frequency. It is effective only within a limited frequency range and it is difficult to use at low frequency.

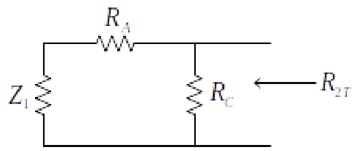
The matching conditions required for the digital circuits are different. The impedance matching must be effective at low frequency and it should not depend on the frequency. (This applies to digital circuits).
In this case, it will be shown that a simple resistive network can be applied to match the two TL with different characteristic impedance. The matching circuit consists of two resistors, R_A and R_C, as shown below. This configuration is valid if Z_2<Z_1. The required circuit for Z_2>Z_1 will be shown later. The values of R_A and R_C must be determined to satisfy the no reflection condition at the junction.

Assume $Z_2 < Z_1$



Equivalent circuits from Z1 looking into Z2 and fromZ2 looking into Z1 are





$$R_{1T}=R_A+\frac{R_C Z_2}{R_C+Z}$$

$$\rho_{11}=\frac{R_{1T}-Z_1}{R_{1T}+Z_1}$$

$$R_{2T}=\frac{R_C\left(R_A+Z_1\right.}{R_C+R_A+Z}$$

$$\rho_{22}=\frac{R_{2T}-Z_2}{R\quad +Z}$$

$$\tau_{21}=(1+\rho_{11})\left(\frac{R_c//Z_2}{R_A+R_c//Z_2}\right)=(1+\rho_{11})\left(\frac{R_c Z_2}{R_c+Z_2}\right)\left(\frac{1}{R_{1T}}\right)$$

$$\tau_{12}=(1+\rho_{22})\left(\frac{Z_1}{Z_1+R_A}\right)$$

To minimize the reflection, we want to set

$$\rho_{11}=\frac{R_{1T}-Z_1}{R_{1T}+Z_1}=0 \quad \text{and} \quad \rho_{22}=\frac{R_{2T}-Z_2}{R_{2T}+Z_2}=0$$

These give us the conditions

$$Z_1=R_A+\frac{R_c Z_2}{R_c+Z_2}$$

$$Z_2=\frac{R_c\left(R_A+Z_1\right)}{R_c+R_A+Z_1}$$

By solving for R_A and R_c, we can obtain

$$R_A=\sqrt{Z_1\left(Z_1-Z_2\right)}$$

$$R_c=Z_2\sqrt{\frac{Z_1}{Z_1-Z_2}} \qquad Z_1>Z_2$$

Also we can find the transmission as

$$\tau_{12}=\frac{Z_1}{Z_1+R_A}$$

$$\tau_{21}=\frac{R_C Z_2}{(R_C+Z_2)Z_1}$$

Example:

Let Z_1=200 Ω, Z_2=100 Ω.

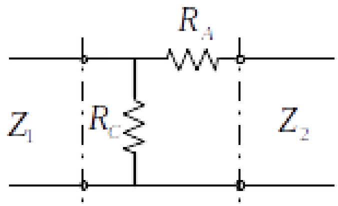
Then we get

R_A=141 Ω and R_C=141 Ω.

R_{1T}=200 Ω which creates ρ_{11}=0.
R_{2T}=100 Ω which creates ρ_{22}=0.
τ_{21}=0.586

For the matched impedance case (Z_1=Z_2), we know τ_{21}=1. The above example shows the resistive matching circuit introduces the signal loss due to resistors. The signal loss may not be a major concern for digital circuits. However, it is a very important issue for the microwave circuits and the resistive matching technique is usually not used in microwave circuits for this reason.

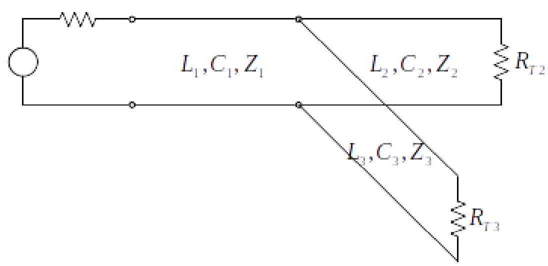
If $Z_1 < Z_2$, the resistive network must be changed to the following configuration and we need to solve for R_A and R_C.



Problem: Obtain R_A and R_C when Z_1=100 Ω and Z_2=200 Ω.

Reflection and transmission at the multiple-line junction

If more than one TL are connected in parallel, we can use the equivalent impedance and solve for the transmission coefficients. This example shows two TLs in parallel. The transmission coefficients of TL2 (τ_{21}) and TL3 (τ_{31}) from TL1are the same because the same voltage is entering into TL2 and TL3 at the junction. Similarly if the signal is coming from TL2, we have τ_{12}=τ_{32}



$$R_{1T} = \frac{Z_2 Z_3}{Z_2 + Z_3}$$

$$R_{2T} = \frac{Z_1 Z_3}{Z_1 + Z_3}$$

$$\rho_{11} = \frac{R_{1T} - Z_1}{R_{1T} + Z_1}$$

$$\tau_{31} = \tau_{21} = 1 + \rho_{11} = \frac{2R_{1T}}{R_{1T} + Z_1}$$

$$\tau_{12} = \tau_{32} = \frac{2R_{2T}}{R_{2T} + Z_2}$$

7. Pulse on lossy transmission lines and dispersion

If the phase and group velocities vary as a function of frequency, the TL is called dispersive. One example is the rectangular waveguide in which the phase velocity approaches to infinite and group velocity approaches to 0 at the cutoff frequency. When a pulse is transmitted through a dispersive TL, the output pulse shape will be distorted because the different frequency components will propagate at different speeds through a TL. The dispersion will be caused by several mechanisms. The wave guiding structure will create dispersion. The waveguide dispersion and multimode dispersion in an optical fiber are examples. If the material characteristics such as ε_r depend on the frequency as $\varepsilon_r(f)$, then the wave velocity becomes a function of frequency and the dispersion occurs. This is called "material dispersion". Another case is the dispersion caused by lossy TL. For the very high-speed digital circuits, dispersion due to lossy TL will be a major problem. In this section we will analyze the dispersion due to a lossy TL using the Laplace transform technique.

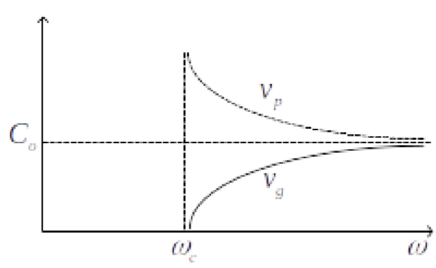
Before discussing the dispersion due to a lossy TL, we will show two examples of dispersive TLs.

<u>Waveguide dispersion</u>

A TE (Transverse Electric) or TM (Transverse Magnetic) wave inside a guiding structure has a cutoff frequency given by ω_c. Below ω_c, the wave becomes evanescent wave (non-propagating wave). Both phase and group velocities depend on ω_c as shown below.

$$v_p = \frac{c_0}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$$ Phase velocity

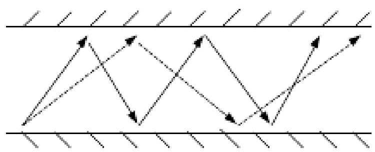
$$v_g = c_0\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}$$ Group velocity.



If we transmit a wideband signal through a waveguide, the signal propagates at different speed and the waveform of the transmitted signal will be distorted. This is called the waveguide dispersion.

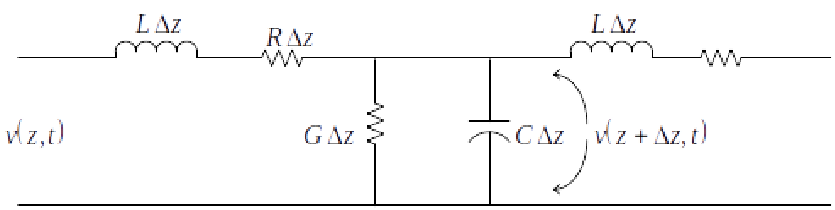
Mode dispersion

When we transmit light through an optical fiber, one way to visualize is that light is bouncing between top and bottom layers as shown below. This figure shows two signals with different angle of propagation. The velocity of propagation is determined by taking the horizontal component of the bouncing light. Then it is clear that these two signals are propagating at different velocities. Each light wave is called mode. The number of modes that can be transmitted through an optical fiber depends on the core size. If the core size is very small, it can support only one mode and it is called the single-mode fiber. If the core size is increased, many modes can be transmitted through it and it is called a multi-mode optical fiber. The figure shown below is the multi-mode case. When an optical pulse is sent through the multi-mode optical fiber, the different modes will propagate with different speed resulting in the pulse broadening. This is called the multi-mode dispersion.



TL equation for a lossy TL

To obtain the signal velocity, first we need to derive the TL equations for a lossy TL. In addition to LC for ideal TL, we need to add the series R to represent conductor loss and parallel G to represent material loss. The lumped element circuit of a lossy TL is given by



The TL equations are

$$\begin{cases} \dfrac{\partial v(z,t)}{\partial z} = -Ri(z,t) - L\dfrac{\partial}{\partial t}i(z,t) \\ \dfrac{\partial}{\partial z}i(z,t) = -Gv(z,t) - C\dfrac{\partial}{\partial t}v(z,t) \end{cases}$$

$$\begin{cases} \dfrac{\partial^2}{\partial z^2}v(z,t) = RGv(z,t) + (RC+LG)\dfrac{\partial}{\partial t}v(z,t) + LC\dfrac{\partial^2}{\partial t^2}v(z,t) \\ \dfrac{\partial^2}{\partial z^2}i(z,t) = RGi(z,t) + (RC+LG)\dfrac{\partial}{\partial t}i(z,t) + LC\dfrac{\partial^2}{\partial t^2}i(z,t) \end{cases}$$

For the lossless TL case, we can set R=0 and $G = \infty$, and obtain the time-harmonic solutions as

$$v_1(z,t) = v^+ \operatorname{Re}\left[e^{j\omega t} e^{-j\beta z}\right] = v^+ \cos(\omega t - \beta z)$$
$$v_2(z,t) = v^- \operatorname{Re}\left[e^{j\omega t} e^{+j\beta z}\right] = v^- \cos(\omega t + \beta z)$$
$$\beta = \omega \sqrt{LC}$$

For the lossy TL, the form of the solution should be similar to the lossless TL case. We can set the expected solution to be

$$v = v^+ e^{-\gamma z} = v^+ e^{-\alpha z} e^{-j\beta z}$$

where γ is modified to include loss.

$$\gamma = \alpha + j\beta$$

By inserting it to the TL equation, we can obtain γ as.

$$\gamma = \alpha + j\beta$$
$$= \sqrt{RG + (RC + LG) j\omega + LC (j\omega)^2}$$
$$¿ \sqrt{(R + j\omega L)(G + j\omega C)}$$

Now the time-harmonic solutions for the lossy TL solution are

$$v_1(z,t) = v^+ e^{-\alpha x} \cos(\omega t - \beta z)$$
$$and$$
$$v_2(z,t) = v^- e^{\alpha x} \cos(\omega t + \beta z)$$

The addition of $e^{-\alpha z}$ term shows that the wave envelope is exponentially decreasing as it propagates.

Next we will consider the phase velocity.

For the lossless TL case, we have $\beta = \omega \sqrt{LC}$ and we get

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$
.

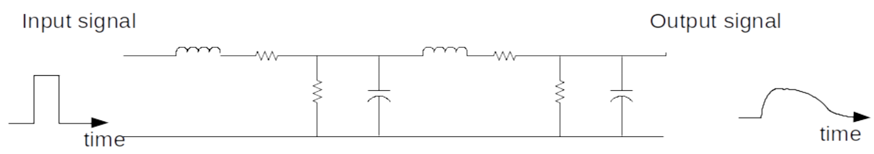
v_p becomes independent of frequency.

For the lossy TL case, β must be modified and we have

$$\beta = \operatorname{Imag}\left[\sqrt{(R + j\omega L)(G + j\omega C)}\right]$$

$$v_p = \frac{\omega}{\beta}$$

Unlike the lossless TL case, ω cannot be eliminated in v_p and the phase velocity still contain ω. This shows the lossy TL is dispersive (velocity depends on ω). A pulse signal contains many frequency components. Therefore, we should expect the pulse shape will be distorted on a lossy TL as shown below. This is called **"waveform distortion due to attenuation".** One example of a lossy TL is a microstrip/stripline TL fabricated on a low cost FR4 PCB which is commonly used for digital circuits.



Characteristic impedance of lossy TL

For a lossless TL, we know the characteristic impedance is given by

$$Z_0 = \sqrt{\frac{L}{C}}$$

and Z_o is real.

The current i_1 on a lossless TL is given by

$$i_1(z,t) = \frac{v_1(z,t)}{Z_0} = \frac{v^+}{Z_0}\cos(\omega t - \beta z)$$

It is clear i_1 and v_1 are in-phase (no phase difference).

For a lossy TL, we have

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}} = |Z_0| e^{j\theta}$$

and Z_o is complex.

The current i_1 on a lossy TL is given by

$$i_1(z,t) = \frac{v_1(z,t)}{Z_0} = \frac{v^+}{|Z_0|}\cos(\omega t - \beta z - \theta)$$

Now i_1 and v_1 are no longer in-phase.

Propagation constant of lossy TL

In the previous section, we obtain the propagation constant on a lossy TL as

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

We consider the special case so that we can write $\gamma = \alpha + j\beta$.

Let us assume R and G are small compare to ωL and ωC. This is called "High frequency limit".

ωL¿ R¿ωC¿¿G¿¿¿
¿

Then we can write

$$\gamma = \sqrt{j\omega L\left(1+\frac{R}{j\omega L}\right) j\omega C\left(1+\frac{G}{j\omega C}\right)}$$

$\sqrt{1 \pm u}$ ~ $1 \pm \frac{u}{2}$ $|u|$ ¿¿

$$\gamma \approx j\omega\sqrt{LC}\left[1+\frac{1}{2}\left(\frac{R}{j\omega L}+\frac{G}{j\omega C}\right)\right]$$

$$¿\, j\omega\sqrt{LC}+\frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}}$$

The real and imaginary parts of γ are α and β, respectively.
Then we have

$$\beta = \omega\sqrt{LC}$$

The phase velocity is $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$ which is the same as the lossless case.
The attenuation can be written as

$\alpha \approx \frac{R}{2}\sqrt{\frac{C}{L}}$ because in general $\frac{G}{C}$ ¿¿

$$\alpha = \frac{R}{2z_o'}$$

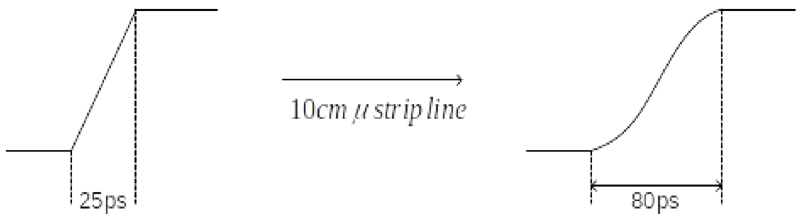
where $z_0' = \sqrt{\frac{L}{C}}$:Characteristic impedance of a lossless TL

Suppose we have Z_0 =50 Ω and Z_0 =100 Ω TLs. The above α shows that Z_0 =100 Ω TL has 1/2 loss compared to Z_0 =50 Ω.

Lossy TL: time-domain analysis

The previous analysis is useful for estimating the loss and phase velocity. However, it does not show the dynamic characteristics such as a change of the rise-time on a lossy TL. In this section, to understand the pulse broadening due to attenuation, we will obtain the time-domain response using the Laplace transform technique.

Suppose we have a fast rise-time input signal as an input (left). After going through a 10cm of a lossy TL, the output may look like the one shown in right. We may see the substantial increase in the rise-time as well as the distortion in the waveform.



The rise-time increase if TL is lossy

TL equation using Laplace transform

In the previous section, we obtain the TL equation for the time-harmonic (jω) case. To obtain the time domain response, we will derive the TL equation using Laplace transform.

We can write the Laplace transform definition as

$$V(z,s)=\int_0^\infty v(z,t)e^{-st}dt=\pounds\left[v(z,t)\right]$$

$$\pounds\left[\frac{\partial}{\partial t}v(z,t)\right]=sV(z,s)-v(z,0)$$

where *v(z,0)* is the initial value.

Laplace transform is an operation in time. It does not do anything with respect to space. Therefore,

$$\pounds\left[\frac{\partial}{\partial z}v(z,t)\right]=\frac{\partial}{\partial z}v(z,t)$$

Using these, the lossless TL equations become

$$\frac{\partial V(z,s)}{\partial z}=-sLI(\mathrm{z,s})$$

$$\frac{\partial I(z,s)}{\partial z}=-sCV(\mathrm{z,s})$$

If TL is lossy, L becomes Z and C becomes Y.

$$Z = R + sL$$
$$Y = G + sC$$

Then we have

$$\frac{\partial V(z,s)}{\partial z} = -Z(s)\,I(z,s)$$
$$\frac{\partial I(z,s)}{\partial z} = -Y(s)\,V(z,s)$$

By combining these equations, we can obtain the TL equation for *V*(z,s).

$$\frac{\partial^2 V(z,s)}{\partial z^2} = Z(s)\,Y(s)\,V(z,s)$$
$$= +\gamma^2(s)\,V(z,s)$$

where $\gamma(s) = \sqrt{Z(s)Y(s)}$

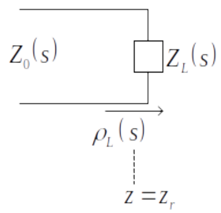
The solutions for the above equation are

$$V(z,s) = V_A(s)\,e^{-\gamma(s)z} + V_B(s)\,e^{+\gamma(s)z}$$
$$I(z,s) = \frac{1}{Z_o(s)}\left[V_A(s)\,e^{-\gamma(s)z} - V_B(s)\,e^{+\gamma(s)z}\right]$$
$$Z_o(s) = \sqrt{\frac{Z(s)}{Y(s)}}$$

Boundary conditions at load and source

At load $Z_L(s)$, we have the total voltage $V(z_r,s)$ and total current $I(z_r,s)$.

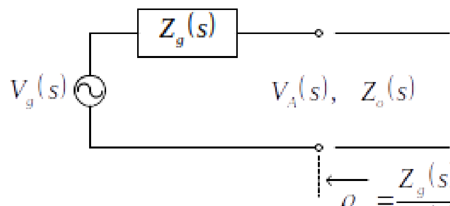
$$V(z_r,s) = Z_L(s)\,I(z_r,s)$$



The reflected voltage is given by the load reflection coefficient.

$$\rho_L(s) = \frac{Z_L(s) - Z_o(s)}{Z_L(s) + Z_o(s)}$$

At source, we have a source impedance and corresponding source reflection coefficient.



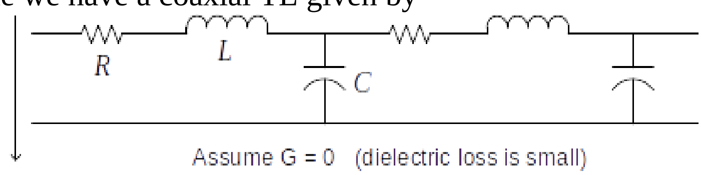
$$\rho_g = \frac{Z_g(s) - Z_o(s)}{Z_g(s) + Z_o(s)}$$

Because $Z_o(s)$ is complex for a lossy TL, both ρ_g and ρ_L can be complex even if Z_g and Z_L are real.

Example:
To obtain the solution, we need to specify the complex propagation constant $\gamma(s)$ for a given TL. We will use the coaxial TL as an example. We assume the dielectric material is almost lossless (G=0) and the conductor loss is small.

REF: Approximation of the Skin-Effect using complex Impedance
(Magusson,Alexander,Tripathi,1992)

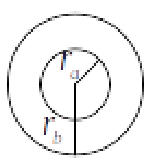
Assume we have a coaxial TL given by



$$L = \frac{\mu_o}{2\pi} \ln\left(\frac{r_b}{r_a}\right)$$

$$C = \frac{2\pi\varepsilon_o\varepsilon_r'}{\ln\left(\frac{r_b}{r_a}\right)}$$

$$R = \frac{R_s}{2\pi}\left(\frac{1}{r_a} + \frac{1}{r_b}\right)$$



$$G = \frac{2\pi\omega\varepsilon_o\varepsilon_r''}{\ln\left(\frac{r_b}{r_a}\right)}$$

$$z_s = \frac{1+j}{\sigma\delta_s} = R_s + j\omega L_s$$

$$R_s = \frac{1}{\sigma\delta_s} = \sqrt{\frac{\pi f \mu_o}{\sigma}}$$

$$L_s = \frac{1}{2}\sqrt{\frac{\mu_o}{\pi f \sigma}}$$

$$\sqrt{j} = (1+j)/\sqrt{2}$$

This is valid for waves in good conductor (plate).

Z_s: surface impedance. δ_s Skin depth

$\varepsilon = \varepsilon_o(\varepsilon_r' - j\varepsilon_r'')$

For a lossless TL, the characteristic impedance is given by

$$Z_o = \sqrt{\frac{L}{C}} = \frac{1}{2\pi}\ln\left(\frac{r_b}{r_a}\right)\sqrt{\frac{\mu_o}{\varepsilon_o\varepsilon_r'}}$$

If the coaxial cable is 0.141" type, then we may have

r_b = 2.997mm (0.118"), r_a = 0.9144mm (0.036")

ε_r'=2.03, σ = 5.8x10^7 S/m (copper), Z_o=50 ohm

Now we can write

Replace R_s in R by Z_s. Use $\sqrt{j}=(1+j)/\sqrt{2}$ and get $\sqrt{j\omega}$ in R.

$$Z = R + j\omega L \approx j\omega\left[\underbrace{\frac{\mu_o}{2\pi}\ln\left(\frac{r_b}{r_a}\right)}_{L}\right] + \underbrace{\sqrt{j\omega}\left[\frac{1}{2\pi}\sqrt{\frac{\mu_o}{\sigma}}\left(\frac{1}{r_a}+\frac{1}{r_b}\right)\right]}_{\text{due to surface impedance}}$$

Now we change $j\omega$ to s so that the equation can be put into the Laplace transform.

$$Z(s) = sL + \xi\sqrt{s}\ , \qquad \xi = \frac{1}{2\pi}\left[\sqrt{\frac{\mu_o}{\sigma}}\left(\frac{1}{r_a}+\frac{1}{r_b}\right)\right]$$

$$\gamma(s) = \sqrt{Z(s)Y(s)} = \sqrt{(sL+\xi\sqrt{s})sC}$$

$$\sim s\sqrt{LC} + \frac{\xi}{2}\sqrt{\frac{Cs}{L}} \qquad \text{here} \quad \sqrt{Z(s)} \sim \sqrt{Ls} + \frac{\xi}{2\sqrt{L}}$$

$$= s\sqrt{LC} + a\sqrt{s} \qquad \text{where} \quad a = \frac{\xi}{2}\sqrt{\frac{C}{L}}$$

By setting $s=j\omega$, we get the complex propagation constant

$$\gamma(j\omega) = j\omega\sqrt{LC} + \frac{a}{\sqrt{2}}(1+j)\sqrt{\omega}$$

$\gamma = \alpha + j\beta$ where α is attenuation constant and β is the propagation constant.

$$\alpha(\omega) = \frac{a}{\sqrt{2}}\sqrt{\omega}$$

$$\beta(\omega) = j[\omega\sqrt{LC} + \frac{a}{\sqrt{2}}\sqrt{\omega}]$$

The complex characteristic Impedance is

$$Z_o(s) = \sqrt{\frac{Z(s)}{Y(s)}}$$

$$¿\sqrt{\frac{L}{C}}\left(1+\frac{\xi}{2L\sqrt{s}}\right)$$

Let

$$Z_{o1}=\sqrt{\frac{L}{C}}$$

$$b=\frac{\xi}{2L}=\frac{a}{\sqrt{LC}}$$

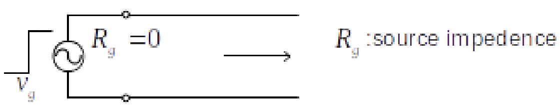
then we get

$$Z_o(s)=Z_{o1}\left(1+\frac{b}{\sqrt{s}}\right)$$

Now we are ready to calculate *v(t)* and *i(t)* using the inverse Laplace transform.

Case A: Unit step function response on semi-Infinite TL with $R_g=0$

First, we will take a look at a simple case. Assume we have a semi-infinite TL (no reflected voltage and current) and R_g=0. Also the input is a unit step function.



The source voltage in the Laplace transform is

$$V_g(s)=\frac{V_o}{s}$$

The propagating voltage and current on TL are given by

$$V(z,s)=\frac{V_o}{s}e^{-z\gamma(s)}$$

$$I(z,s)=\frac{V_o}{sZ_o(s)}e^{-z\gamma(s)}$$

By changing $Z_o(s)$ and $\gamma(s)$, we get

$$V(z,s)=\frac{V_o}{s}\exp\left[-sz\sqrt{LC}-az\sqrt{s}\right]$$

$$I(z,s)=\frac{V_o}{Z_{o1}}\frac{1}{\sqrt{s}\left(\sqrt{s}+b\right)}\exp\left[-sz\sqrt{LC}-az\sqrt{s}\right]$$

where

$e^{zs\sqrt{LC}}$ → delayed response at $t_z=z\sqrt{LC}$ (shifting theorem)

The inverse Laplace transform of this can be expressed using the complementary error function which is defined here.

$$\pounds^{-1}\left[\frac{e^{-az\sqrt{s}}}{s}\right]=erfc\left(\frac{az}{2\sqrt{t}}\right)u(t)$$

$$\pounds^{-1}\left[\frac{e^{-az\sqrt{s}}}{\sqrt{s}(\sqrt{s}+b)}\right]=\exp\left(b^2 t+abz\right)erfc\left(b\sqrt{t}+\frac{az}{2\sqrt{t}}\right)u(t)$$

The complementary error function and error function are defined as
Complementary error function (CEF)

$$erfc(p)=\frac{2}{\sqrt{\pi}}\int_p^{\infty}\exp\left(-u^2\right)du$$

P	erfc(P)
0	1.0
0.0889	0.9
0.41769	0.5
1.1631	0.1
∞	0

Error function (EF)

$$erf(p)=\frac{2}{\sqrt{\pi}}\int_0^{p}\exp\left(-u^2\right)du$$

$$erfc(p)=1-erf(p)$$

Both complementary error function and error function are frequently used in communications. They appear when a Gaussian function is integrated over a finite interval.

Useful formula is the asymptotic approximation of CEF.

$p>>1$

$$erfc(p)\sim\frac{\exp\left(-p^2\right)}{p\sqrt{\pi}}\left[1-\frac{1}{2p^2}+\frac{3}{\left(2p^2\right)^3}-..........\right]$$

$p<<1$

$$erfc(p)\approx 1-\frac{2}{\sqrt{\pi}}\left(p-\frac{p^3}{3}+..........\right)$$

Using the complementary error function, the solutions are given by

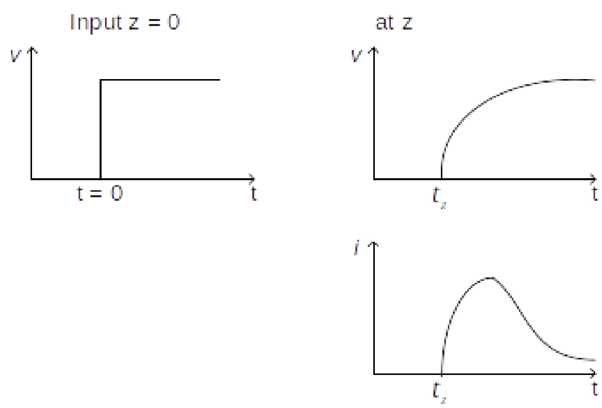
$$v(z,t)=V_o\, erfc\left(\frac{az}{2\sqrt{t-t_z}}\right)u\left(t-t_z\right)$$

$$i(z,t)=\frac{V_o}{Z_{o1}}\exp\left[b^2(t-t_z)+abz\right] \qquad erfc\left[b\sqrt{t-t_z}+\frac{az}{2\sqrt{t-t_z}}\right]u(t-t_z)$$

$$t_z=z\sqrt{LC}$$

at $t=t_z$

$\frac{az}{2\sqrt{t-t_z}}$¿¿1¿ $\qquad erfc()$ → 0



At $t\rightarrow\infty$

$$\frac{az}{2\sqrt{t-t_z}}\rightarrow 0 \qquad erfc()\rightarrow 1$$

Using the asymptotic approximation $p<<1, \quad erfc(p)\approx 1-\frac{2}{\sqrt{\pi}}(p)$

$$v(z,T)\approx V_o\left[1-\frac{T_z}{\sqrt{\pi(T-T_z)}}\right] \qquad T_z=b^2t_z$$

$$T=b^2t$$

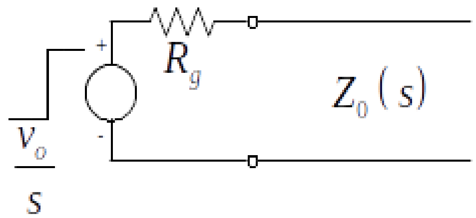
$$i(z,T)\approx\frac{V_o}{Z_{o1}}\frac{1}{\sqrt{\pi(T-T_z)}}$$

where $b=\frac{a}{\sqrt{LC}}$ and $t_z=z\sqrt{LC}$ are used.

Case B: Unit step function response on the semi-Infinite TL with non-zero R_g

If the source impedance R_g is added, we can obtain the following case.

We are still assuming a semi-infinite TL (no reflected voltage and current) and the input is a unit step function.



At $z=0$

$$I(z=0,s)=\frac{V_o}{s}\left[\frac{1}{R_g+Z_o(s)}\right]$$

$$=\frac{V_0}{s\left[R_g+Z_{01}+\frac{bZ_{01}}{\sqrt{s}}\right]}$$

$$=\frac{V_o}{\sqrt{s}\left[(R_g+Z_{o1})(\sqrt{s}+bk)\right]}$$

$$k=\frac{Z_{o1}}{R_g+Z_{o1}}$$

For $z\neq 0$

$$I(z,s)=\frac{V_o}{\sqrt{s}(R_g+Z_{o1})(\sqrt{s}+bk)}\exp\left(-st_z-az\sqrt{s}\right)$$

at $z=0$

$$V(z=0,s)=\frac{V_o}{s}-R_g I(z=0,s)$$

at $z\neq 0$

$$R_g I(z,s)=R_g I(z=0,s)\exp[-\gamma(s)z]$$

$$V(z,s)=\frac{V_o}{s}\exp\left(-st_z-az\sqrt{s}\right)-R_g I(z,s)$$

By taking the inverse Laplace transform, we get

$$v(z,t)=V_o\,erfc\left(\frac{az}{2\sqrt{t-t_z}}\right)u(t-t_z)-R_g i(z,t)$$

$$i(z,t)=\frac{kV_o}{Z_{o1}}\exp\left[b^2k^2(t-t_z)+abkz\right]\quad erfc\left(bk\sqrt{t-tz}+\frac{az}{2\sqrt{t-t_z}}\right)u(t-t_z)$$

$$k=\frac{Z_{o1}}{R_g+Z_{o1}}$$

at $t>t_z$

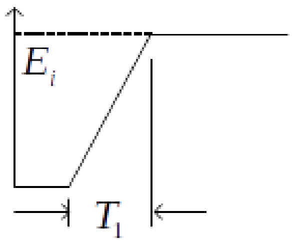
For $t \to \infty$

$v(z,t): voltage$ divider due to R_g

$v(z,t) \to V_0$ because $i(z,t) \to 0$

Case C: Finite rise-time input
As we discussed in Section 5, the input voltage is given by

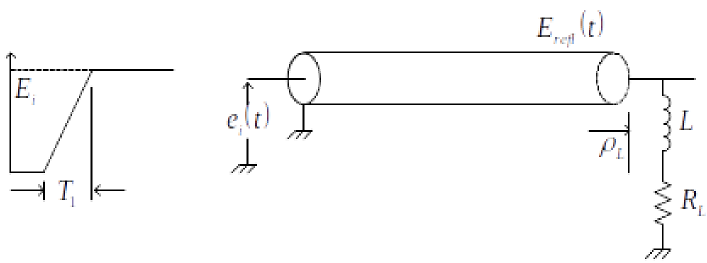
$$e_i(t) = m\,t\,u(t) - m\,(t - T_1)\,u(t - T_1)$$
$$E_i(s) = \frac{m}{s^2}\left(1 - e^{-T_1 s}\right)$$



The unit step function voltage $V_g(s) = \frac{V_o}{s}$ must be replaced by the above expression.

Case D: Finite rise-time input and reflection at load impedance

If the load impedance is present as shown below and we want to find the reflected voltage at the source, we need to analyze it in 3 steps.

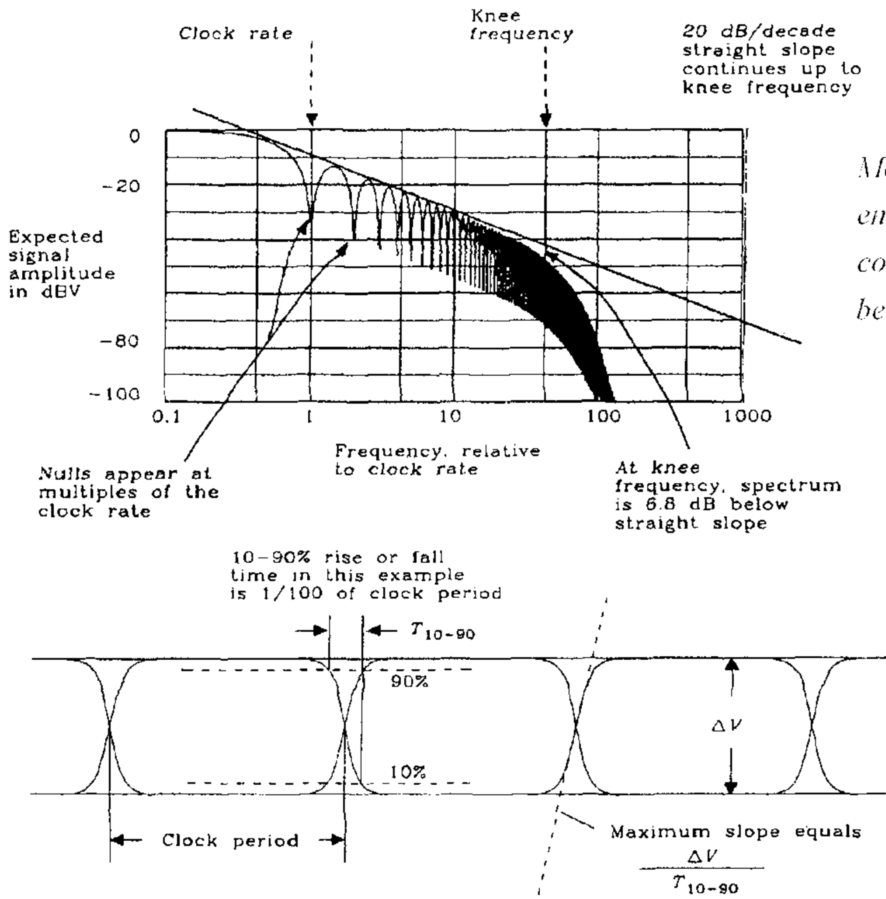


Step 1: Obtain the incident voltage at the load
    The waveform will not be the same as the input voltage.
Step 2: Obtain the reflected voltage from the load
    The Laplace transform technique can be used if the load is complex.
Step 3: Obtain the reflected voltage traveled from the load to source
    Additional waveform distortion will occur.

In principle, this case can be solved using the Laplace transform method. In practice, however, the final results become very complicated.

8. High Speed Digital Circuits and Effects

Rise-time and frequency spectrum F_{knee}, F_{3dB} and F_{rms}



Most energy is contained below F_{knee}

Figure 1.1 Expected spectral power density of a random digital waveform.

$$F_{kner} \approx \frac{0.5}{T_r}$$

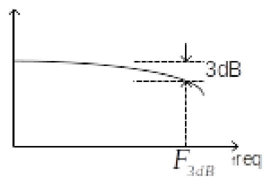
T_r : Pulse rise-time (10-90%)

<u>Risetime</u>

100ns	slow	No problem
10ns	medium	Still OK
1ns	Problem with EMI (electromagnetic Interference)	
0.2ns	Ground noise Coupling noise Reflection	

$\underline{F_{3dB}}$

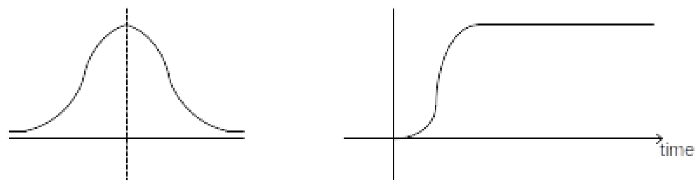
Assume we have a pulse shape (eg Gaussian pulse) with a rise-time. Take a FT and define -3dB frequency.



$$F_{3dB} \sim \frac{1}{T_r}$$

T_r : 10~90% rise-time

<u>Gaussian pulse</u>



$$F_{3dB} = \frac{0.338}{T_r}$$

$$g_{step}(t) = \frac{1}{2}\left[1 + erf\left(\frac{t}{2\tau_3}\right)\right]$$

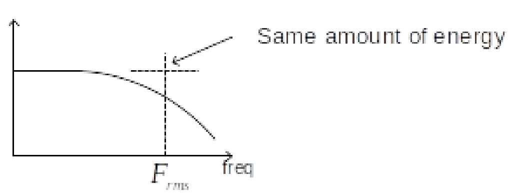
$$erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\lambda^2} d\lambda$$

<u>Single pole exponential</u>

$$h_{step}(t) = u(t)\left(1 - e^{-\left(\frac{t}{\tau_1}\right)}\right)$$

$$F_{3d} = \frac{0.35}{T_r}$$

$\underline{F_{rms}}$



$$F_{rms} = \frac{0.361}{T_r} \qquad \text{Gaussian}$$

$$F_{rms} = \frac{0.549}{T_r} \qquad \text{Single pole}$$

Lumped vs Distributed Circuits

When can we neglect the TL effects?

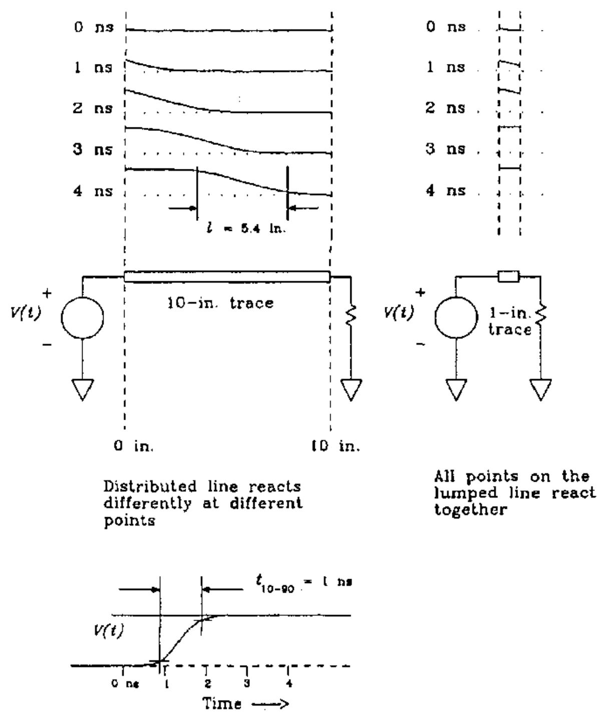


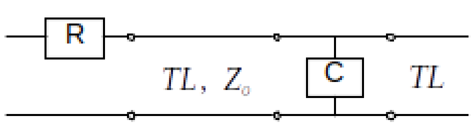
Figure 1.3 Snapshots in time of the electric potential on distributed and lumped transmission lines.

POINTS TO REMEMBER:

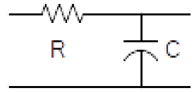
$$\text{Length of rising edge } l = \frac{\text{Rise time (ps)}}{\text{Delay (ps/in.)}} \qquad [1.4]$$

Circuits smaller than $l/6$ are lumped circuits.

Distributed model: use TL between lumped elements



Lumped without TL (low speed)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