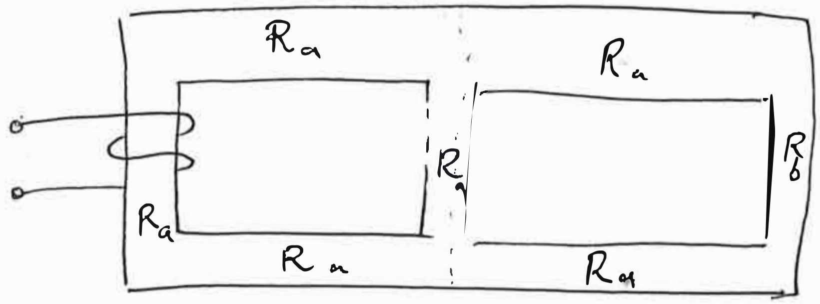
Homework Solution

(a) Draw magnetic ckt:



- All sections labeled R_a have identical reluctances

$$R_a = \frac{\text{length of leg}}{\mu \text{ area}}$$

$$= \frac{l}{\mu_0 \mu_r A_a}$$

* where $l = 3cm = \boxed{0.03\,m = l}$
* $A_a = 1cm \times 1cm$

$$= (0.01m)^2 = (10^{-2})^2 m^2$$

$$= \boxed{10^{-4} m^2 = A_a}$$

$$= \frac{(0.03\,m)}{(4\pi \times 10^{-7}\,H/m)\,1000\,(10^{-4}\,m^2)}$$

$$\underbrace{\approx 238.7 \times 10^3 \; H^{-1} = R_a}$$

- The "narrow" leg has reluctance

$$\mathcal{R}_b = \frac{l}{\mu_0 \mu_r A_b}$$

* where $A_b = 0.5\text{cm} \times 1\text{cm}$

$$= (0.5\times10^{-2}\text{m})(1\times10^{-2}\text{m})$$

$$= 0.5\times10^{-4}\text{ m}^2$$

$$= 5\times10^{-5}\text{ m}^2$$

$$= \frac{0.03\text{ m}}{(4\pi 10^{-7}\text{ H/m})\,1000\,(5\times10^{-5}\text{m}^2)}$$

$$\approx \underbrace{477.5\times10^3\ H^{-1} = \mathcal{R}_b}$$

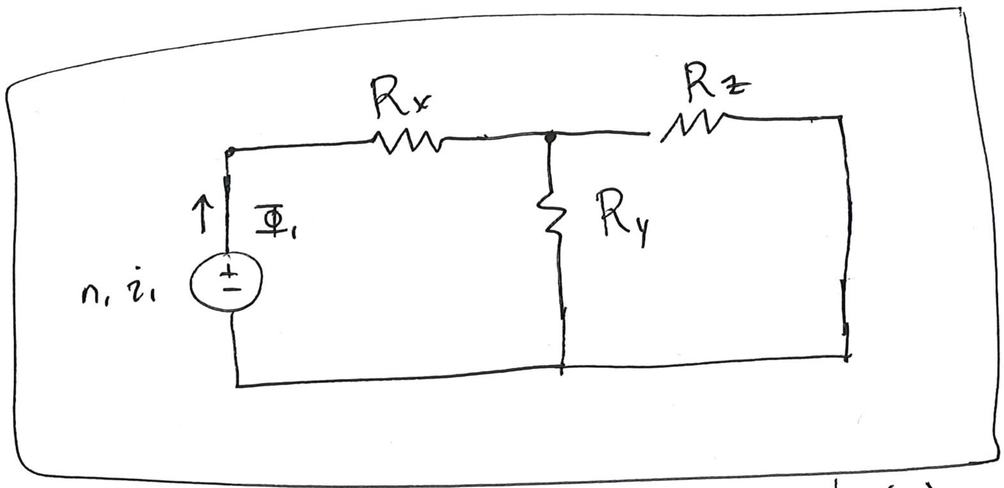
- The core has 3 branches

Left branch total reluctance $= \mathcal{R}_x = 3\,\mathcal{R}_a = 716.2\times10^3\ H^{-1}$

Center branch total reluctance $= \mathcal{R}_y = \mathcal{R}_a = 238.7\times10^3\ H^{-1}$

Right branch total reluctance $= \mathcal{R}_z = 2\,\mathcal{R}_a + \mathcal{R}_b = 954.9\times10^3\ H^{-1}$

We can now redraw the circuit as



(a)

b) Determine the inductance

Compute "effective" reluctance from coil #1

$$\mathcal{R}_{eff} = \mathcal{R}_x + \mathcal{R}_y \,||\, \mathcal{R}_z$$

$$= \mathcal{R}_x + \frac{\mathcal{R}_y \mathcal{R}_z}{\mathcal{R}_y + \mathcal{R}_z}$$

$$\cong 907.2 \ H^{-1}$$

$$\rightarrow \quad \Phi_1 = \frac{n_1 i_1}{\mathcal{R}_{eff}}$$

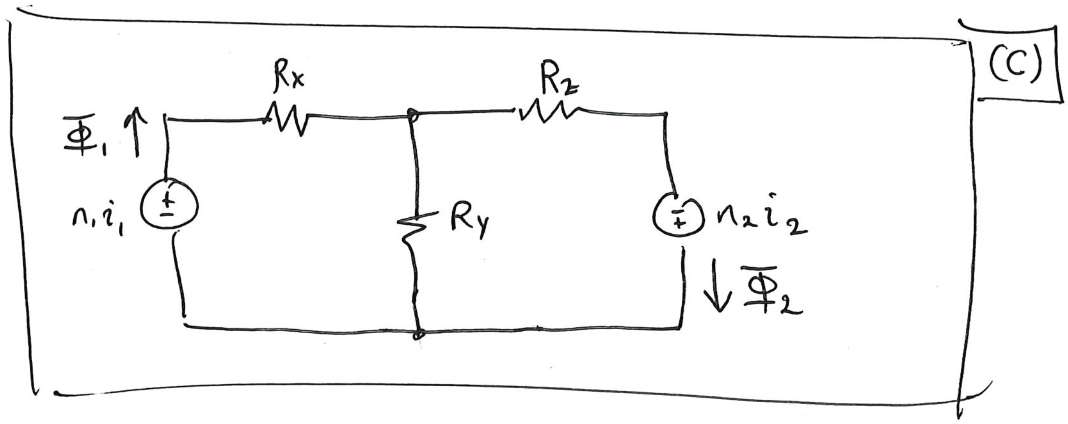
$$\text{\&} \quad v_1(t) = n_1 \frac{d\Phi_1}{dt} = \overbrace{\frac{n_1^2}{\mathcal{R}_{eff}}}^{L} \frac{di_1(t)}{dt}$$

$$\Rightarrow L = \frac{n_1^2}{\mathcal{R}_{eff}} \approx \boxed{110.2 \ \mu H = L}$$

$n_1 = 10$

(b)

c) Modify ckt model w/ 2nd winding



d) Solve via superposition to get

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}$$

- Main Strategy:
  - We need to solve for Φ_1 & Φ_2 in the diagram above. Once we obtain Φ_1 & Φ_2, then Faraday's law gives

$$\left. v_1(t) = n_1 \frac{d\Phi_1}{dt} \quad \& \quad v_2(t) = n_2 \frac{d\Phi_2}{dt} \right\} \text{gives us}$$

  which we can put into the form needed.

  - The main idea of superposition is that we will write the circuit above as the sum of two circuits:

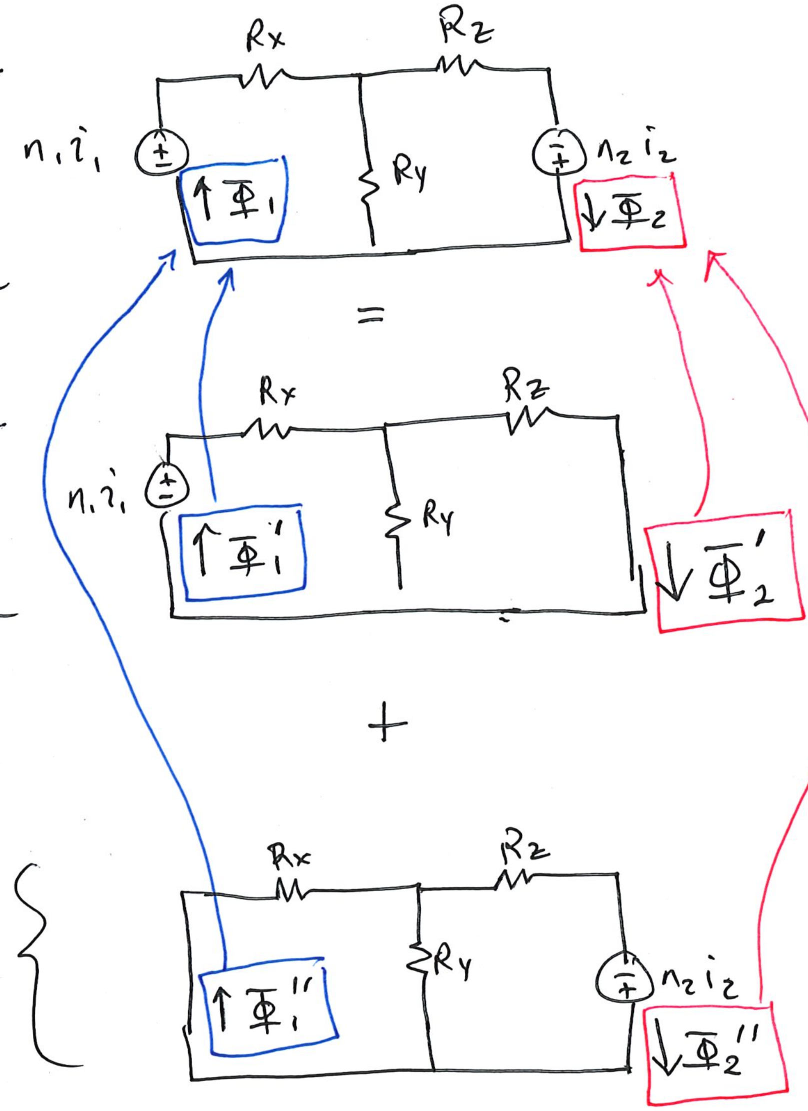
actual ckt

ckt'
w/ $n_2 i_2 = 0$

solve for Φ_1', Φ_2'

ckt''
w/ $n_1 i_1 = 0$

solve for Φ_1'', Φ_2''



where $\Phi_1 = \Phi_1' + \Phi_1''$ & $\Phi_2 = \Phi_2' + \Phi_2''$

- First set $n_2 i_2 = 0$ to get ckt'... solve for Φ_1' & Φ_2'.

Looking above, KVL for ckt' gives

$$n_1 i_1 = \Phi_1' \left(R_x + \underbrace{\frac{R_y R_z}{R_y + R_z}}_{R_y \| R_z} \right)$$

$$\Rightarrow \boxed{\Phi_1' = \frac{n_1 i_1}{R_x + \frac{R_y R_z}{R_y + R_z}}} \quad (1)$$

Next solve for Φ_2'

KVL also says:

$$R_z \Phi_2' = n_1 i_1 - \Phi_1' R_x$$

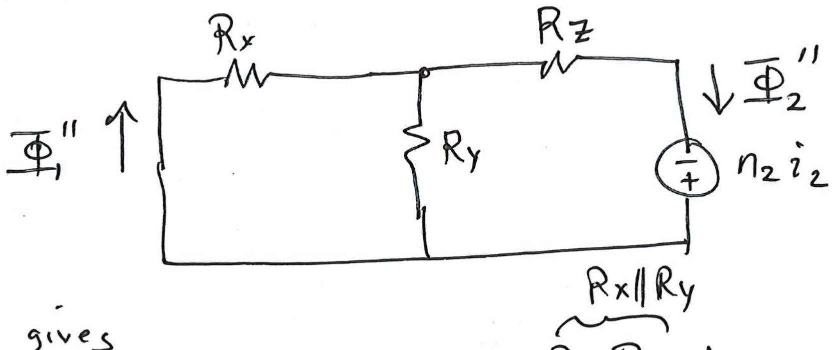
$$\Rightarrow \Phi_2' = \frac{n_1 i_1 - \Phi_1' R_x}{R_z}$$

* use (1) for Φ_1'

$$= \frac{n_1 i_1}{R_z} - \frac{R_x}{R_z} \frac{n_1 i_1}{R_x + \frac{R_y R_z}{R_y + R_z}}$$

$$\boxed{= n_1 i_1 \left(\frac{1}{R_z} - \frac{R_x}{R_z} \frac{1}{\left(R_x + \frac{R_y R_z}{R_y + R_z}\right)} \right) = \Phi_2'} \quad (2)$$

- Next set $n_1 i_1 = 0$ & solve again



need to solve for Φ_1'' & Φ_2''

KVL gives

$$n_2 i_2 = \Phi_2'' \Big(R_z + \underbrace{\frac{R_x R_y}{R_x + R_y}}_{R_x \| R_y} \Big)$$

$$\Rightarrow \boxed{\Phi_2'' = \frac{n_2 i_2}{R_z + \frac{R_x R_y}{R_x + R_y}}} \quad (3)$$

- Do KVL again but include $\Phi_1'' R_x$ branch

$$0 = n_2 i_2 - \Phi_1'' R_x - \Phi_2'' R_z$$

→ solve for Φ_1''

$$\Rightarrow \Phi_1'' = \frac{n_2 i_2 - \Phi_2'' R_z}{R_x}$$

*substitute (3)

$$= \frac{n_2 i_2}{R_x} - \frac{R_z}{R_x}\left(\frac{n_2 i_2}{R_z + \frac{R_x R_y}{R_x + R_y}}\right)$$

$$= n_2 i_2 \left(\frac{1}{R_x} - \frac{R_z}{R_x} \frac{1}{R_z + \frac{R_x R_y}{R_x + R_y}}\right) = \Phi_1'' \quad (4)$$

- Now combine & apply superposition:

$$\Phi_1 = \underbrace{\Phi_1'}_{(1)} + \underbrace{\Phi_1''}_{(4)}$$

$$= n_1 i_1 \frac{1}{R_x + \frac{R_y R_z}{R_y + R_z}} + n_2 i_2 \left(\frac{1}{R_x} - \frac{R_z}{R_x} \frac{1}{R_z + \frac{R_x R_y}{R_x + R_y}}\right) \quad (5)$$

and voltage is

$$v_1(t) = n_1 \frac{d\Phi_1}{dt}$$

$$= \underbrace{\boxed{n_1^2 \frac{1}{R_x + \frac{R_y R_z}{R_y + R_z}}}}_{L_{11}} \frac{di_1}{dt} + \underbrace{\boxed{n_1 n_2 \left(\frac{1}{R_x} - \frac{R_z}{R_x}\left(\frac{1}{R_z + \frac{R_x R_y}{R_x + R_y}}\right)\right)}}_{L_{12}} \frac{di_2}{dt} \quad (6)$$

and

$$\Phi_2 = \overbrace{\Phi_2'}^{(2)} + \overbrace{\Phi_2''}^{(3)}$$

$$= n_1 i_1 \left(\frac{1}{R_z} - \frac{R_x}{R_z} \frac{1}{\left(R_x + \frac{R_y R_z}{R_y + R_z}\right)} \right) + n_2 i_2 \frac{1}{R_z + \frac{R_x R_y}{R_x + R_y}} \qquad (4)$$

where v_2 is

$$v_2(t) = n_2 \frac{d\Phi_2}{dt}$$

$$= \underbrace{\boxed{n_1 n_2 \left(\frac{1}{R_z} - \frac{R_x}{R_z} \frac{1}{\left(R_x + \frac{R_y R_z}{R_y + R_z}\right)} \right)}}_{L_{12}} \frac{di_1}{dt} \underbrace{\boxed{+\, n_2^2 \left(\frac{1}{R_z + \frac{R_x R_y}{R_x + R_y}} \right)}}_{L_{22}} \frac{di_2}{dt}$$

- L_{12} in (8) & (6) "look" different but are in fact equal once you plug in the numbers. You can also show they are equivalent by doing a bunch of algebra. Plugging in numbers on a computer we get.

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 110.2\,\mu H & 44.1\,\mu H \\ 44.1\,\mu H & 352.7\,\mu H \end{bmatrix}}_{\begin{bmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{bmatrix}} \frac{d}{dt} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}$$

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