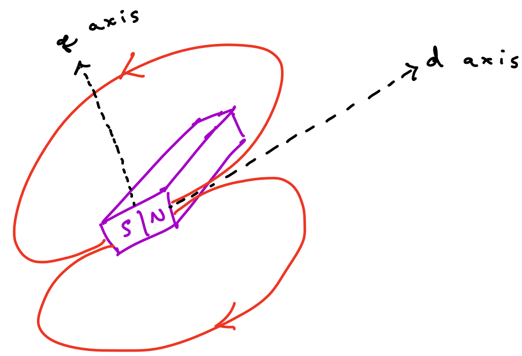
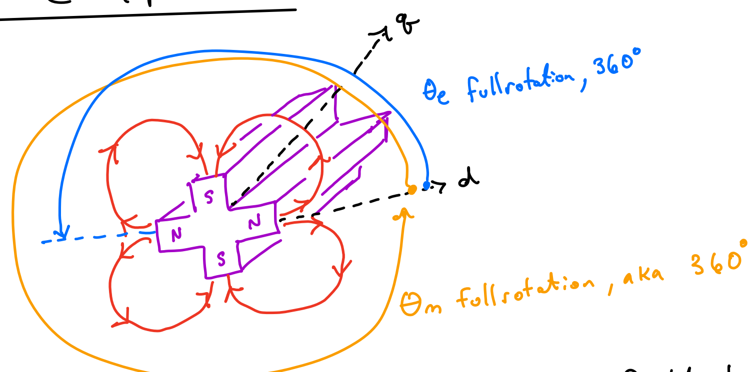
— Consider a 2 pole magnet rotor



- Magnetic $\vec{B}$ field lines exit N & return to S.
- So-called "d-axis" is aligned with $\vec{B}$ field exiting N pole
- The "q-axis" is orthognonal to d axis, based on right-hand rule.

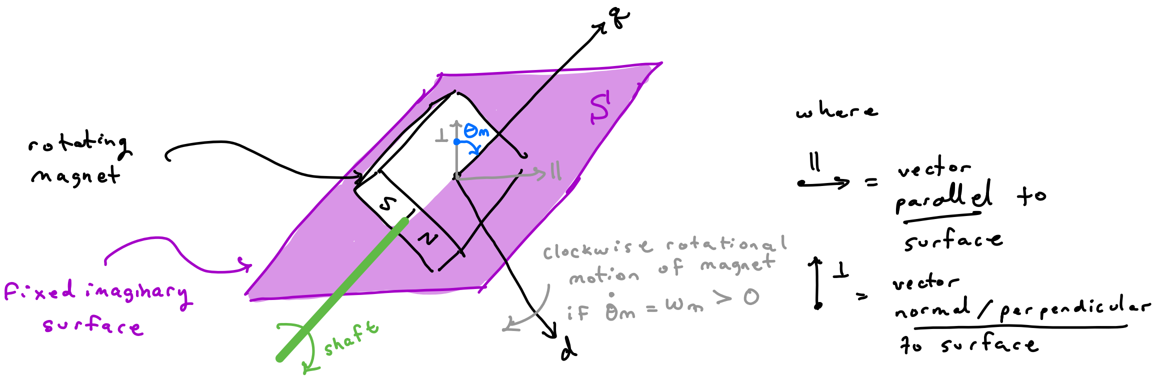
— Look @ 4 pole rotor



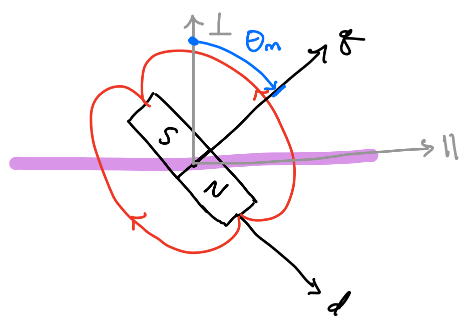
- Use simple N → S rule to understand field lines
- Easy to get higher pole counts & visualize them

– Flux versus angle

- Look at 2 pole magnet which rotates w/ angle θ_m.
- Consider a stationary surface area S that magnet is halfway submerged in.



Looking at 2D side view & angles more closely for clarity



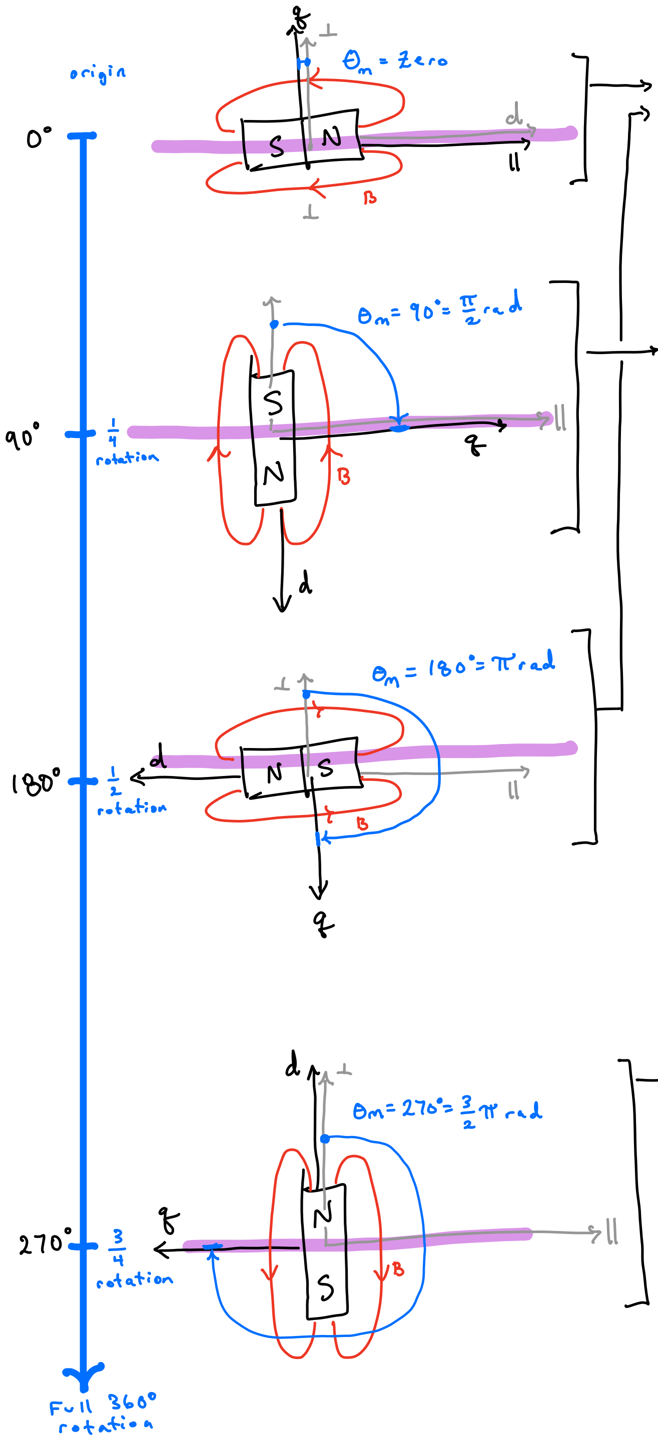
- The "Flux" Φ through the surface is defined by the surface integral

$$\Phi = \int_S \vec{B} \cdot d\perp$$

where $d\perp$ is the unit length normal vector for the surface.

Intuitively Φ totals up all the $\vec{B}$ field which "cuts" through area S. Only fields normal to surface contribute to Φ. The dot product, $\vec{B} \cdot \vec{d\perp} = \underbrace{\|B\|}_{\text{magnitude of } \vec{B} \text{ at one point on } S} \cos(\text{angle between } \vec{B} \text{ \& } \perp \text{ vectors})$, is responsible for capturing the orthogonal components. Note that this integral is generally not practical to compute -- just know its meaning.

- We can actually deduce the form of Φ without computing the integral.
- Look @ 4 positions to see how flux changes



* For both cases:
Notice all $\vec{B}$ field lines are parallel to S'. Hence

$$\Phi(0) = \Phi(\pi) = 0$$

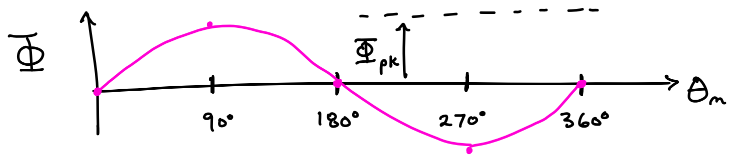
* $\vec{B}$ is perpendicular to surface S'.
Since the amount of field cutting S' is maximized and $\vec{B}$ is parallel to $\bar{I}$, we say

$$\Phi\left(\frac{\pi}{2}\right) = \text{large } \oplus \text{ value}$$

* $\vec{B}$ is perpendicular to surface S'.
Since the amount of field cutting S' is maximized and $\vec{B}$ is anti-parallel to $\bar{I}$, we say

$$\Phi\left(\frac{3\pi}{2}\right) = \text{large } \ominus \text{ value}$$

- Extrapolating from these points, $\Phi(\theta_m)$ looks like a sine wave as it is rotated one revolution.



Hence Φ takes the form

$$\Phi(\theta_m) = \Phi_{pk} \sin \theta_m \qquad (1)$$

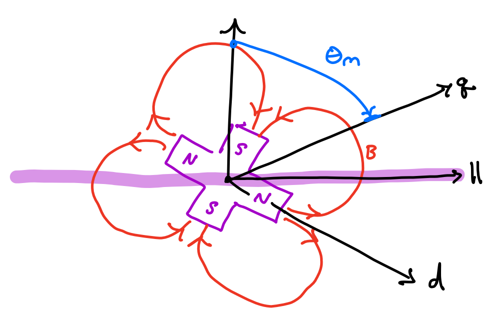
where Φ_{pk} is the magnitude. Note how we computed $\Phi(\theta_m)$ while avoiding the nasty integral. All you need is intuition & understanding!

- More than one pole pair

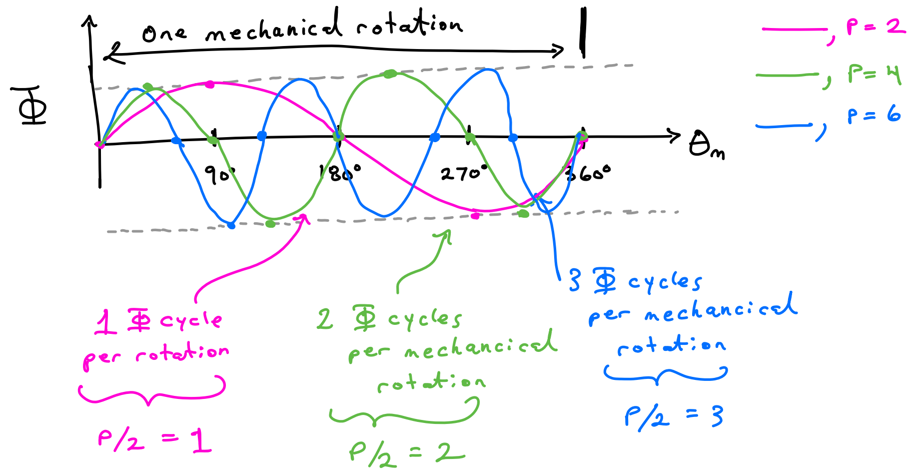
  - We could do a similar excercise and show that for P total poles ($P/2$ pairs), we get the following form for the magnet and one surface S with flux Φ:

$$\Phi(\theta_m) = \Phi_{pk} \sin \Big(\underbrace{\tfrac{P}{2} \theta_m}_{\theta_e} \Big) \qquad (2)$$

  - Here is a visualization with $P = 4$ poles



- For higher pole counts, Φ vs θ_m becomes

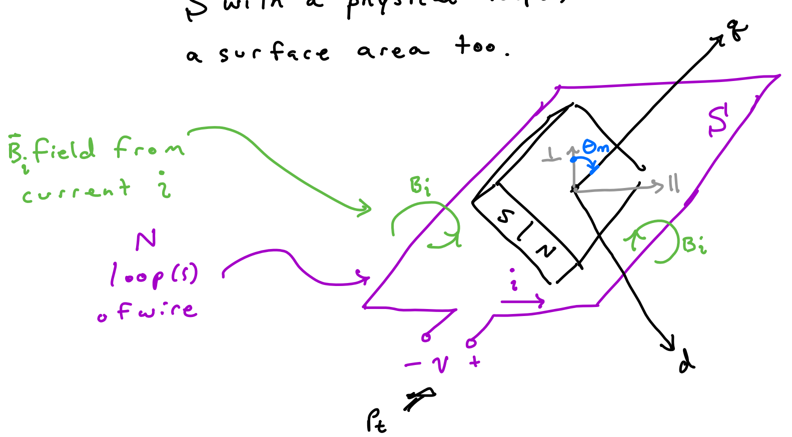


We will eventually show that electrical waveforms have the same periodicity / frequency as flux. Hence, lets define the electrical angle, θ_e, as

$$\theta_e = \frac{P}{2}\theta_m \qquad (3)$$

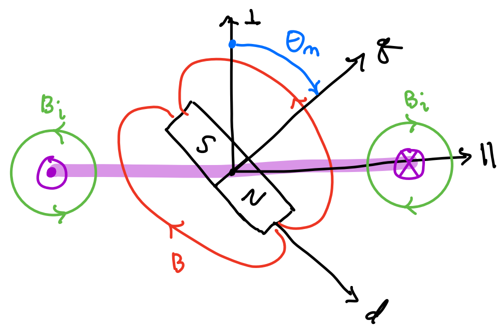
$$\Rightarrow \text{electrical angle} = \frac{P}{2} \times \text{mechanical angle}$$

- Consider a 2-pole magnet & 1 coil w/ N turns (AKA: 2 pole, single phase machine)
  - Basically, all we do is replace an imaginary surface S with a physical loop(s) of wire. The loop does enclose a surface area too.

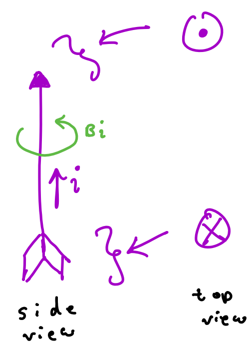


- Note use of motor/load convention where i enters + terminal. Implies coil absorbs power when power, $P_t = v i > 0$, is ⊕.
- P_t is power into terminals.

• Look at side view:



where



• Where the green $\vec{B}_i$ field is associated with the current in the coil. Use right hand rule to obtain direction. Remember any current produces a $\vec{B}$ field.

• Notice $\vec{B}_i$ produces $\oplus$ flux since $\vec{B}_i$ cutting surface points down.

▫ Define "Flux Linkage"

• B/c of magnet & current in wire, there are now 2 contributions to flux in area S.

• "Flux" is just for imaginary surface S

• "Flux linkage" is a new quantity which accounts for total flux through wire loops w/ N turns.

$$\lambda(\theta_m) := \text{"Flux Linkage"}$$

$$\left.\begin{aligned} &= (\#\text{ turns}) \times (\text{magnet Flux through surface } S) \\ &\quad + (\text{flux produced by current } i) \end{aligned}\right\} \text{2 contributions}$$

$$= \underbrace{N\,\Phi(\theta_m)}_{} \qquad + \qquad \underbrace{L\,i}_{}$$

* from magnet

* use (2), (3)

* from current i

* Note role of inductance L

* L = "inductance" of N wire loops

$$= \lambda_m \sin\left(\frac{P}{2}\theta_m\right) + L\,i \qquad (4)$$

where $\lambda_m = N\,\Phi_{pk}$

- Use Faraday to get voltage

Recall

$\mathcal{E} :=$ induced voltage

$$= -\frac{d\lambda(\theta_m)}{dt} \qquad (5)$$

Where this is what we see looking into coil

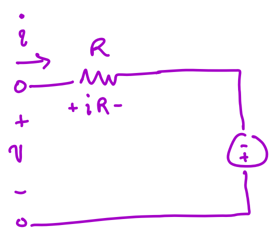
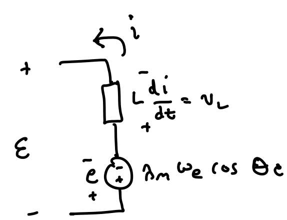


Fig 1

KVL gives:

$$\Rightarrow \quad v - iR + \mathcal{E} = 0 \qquad (6)$$



Plug (4) → (5) & expand

$$\mathcal{E} = -\frac{d}{dt}\left(\lambda_m \sin\left(\frac{P}{2}\theta_m\right) + L\,i\right)$$

$$= -\lambda_m \overbrace{\frac{P}{2}\underbrace{\left(\frac{d\theta_m}{dt}\right)}}^{\omega_e} \cos\left(\frac{P}{2}\theta_m\right) - L\frac{di}{dt}$$

↳ * $\frac{d\theta_m}{dt} = \omega_m$ & recall $\omega_e = \frac{P}{2}\omega_m$

$$= \underbrace{-\lambda_m \omega_e \cos(\theta_e)} - L\frac{di}{dt}$$

*denote as e

*called "Back EMF"

we will stick to using electrical angle/speed from here forward.

$$= -e - L\frac{di}{dt} \qquad (7)$$

where

$$e = \lambda_m \omega_e \cos(\theta_e) \qquad (8)$$

Plug (7)-(8) into (6) to get updated KVL eqn.

$$v = iR - \mathcal{E}$$
$$= iR + \underbrace{L\frac{di}{dt}}_{v_L} + e$$

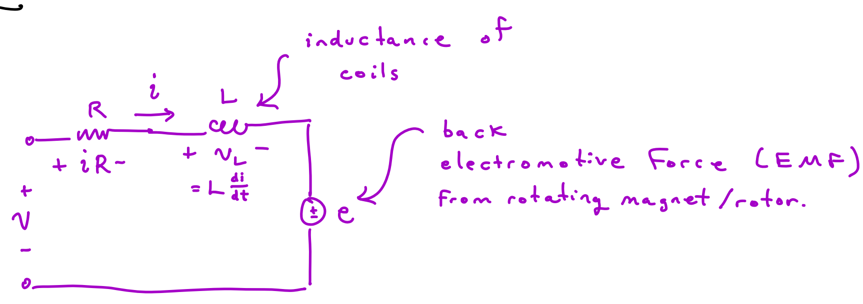


Fig 2: ckt w/ back EMF

□ Torque & Electromechanical Energy Transfer

Q: How to link electrical & mechanical?

A: Use energy to relate electrical & mechanical variables. Energy is universal!

- Glance at physical system below. P_t is electrical power going into stator coil terminal. P_m is mechanical power extracted from shaft.

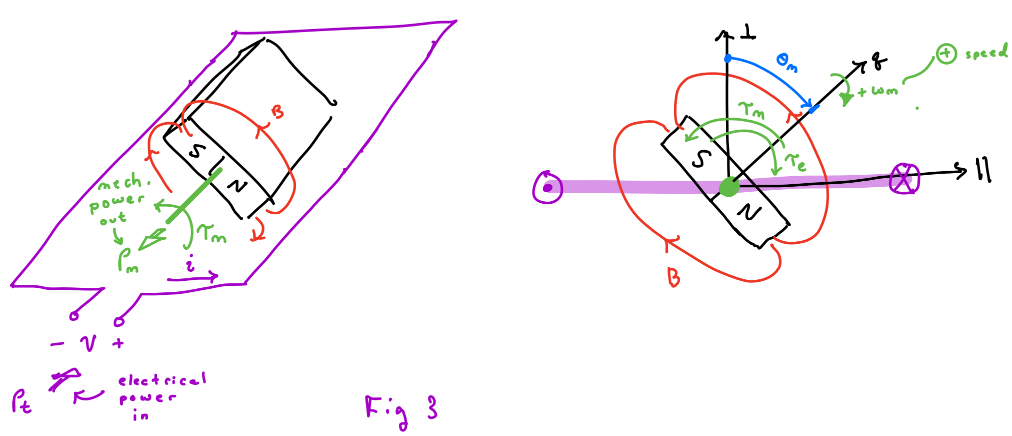


Fig 3

- Look @ ckt model to show more clearly where power is transfered:

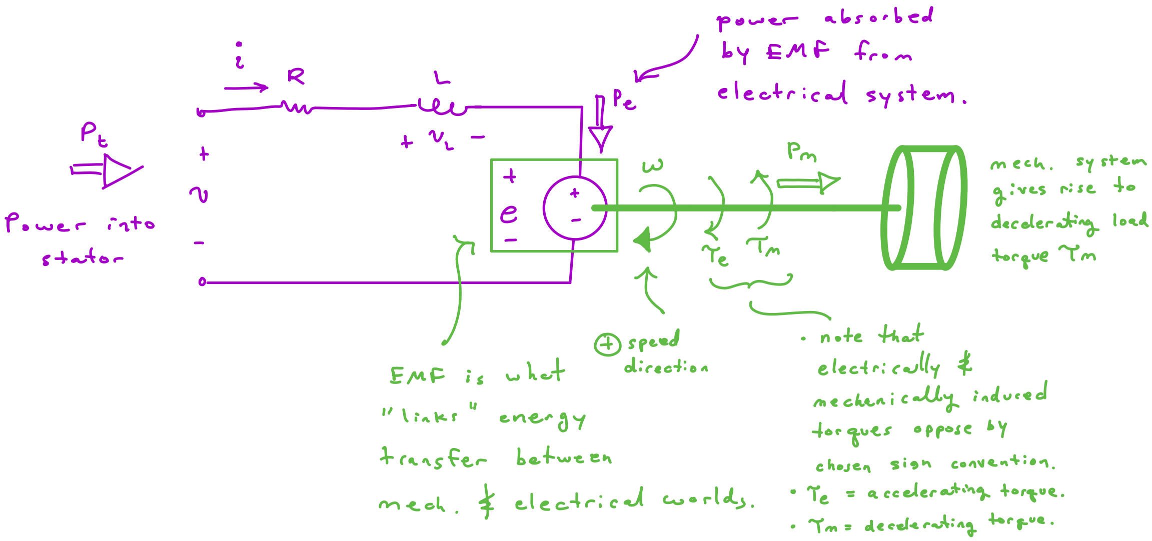


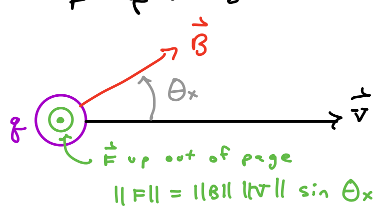
Fig 4: EMF is energy link.

- EMF is where "magic" happens. Electrical & mechanical are linked & energy is transfered/converted

↳ Q: What is fundamental reason why?

A: B/c we know that any moving charged particle w/ velocity $\vec{v}$ & charge q in a field $\vec{B}$ experiences the force

$$\vec{F} = q\vec{v} \times \vec{B}$$



The electromagnetic force is fundamental to the universe & exists between moving charged particles in coil w/ current i (recall current is moving electrons w/ charge!) & electrons in magnet...

to be discussed more later.

- Energy is common language. Write power on elec. & mech. sides

  - Electrical Power

$$P_t = iv$$

Total power injected into terminals by power electronics

$$= i\left(iR + L\frac{di}{dt} + e\right)$$

$$= i^2R + Li\frac{di}{dt} + \boxed{ei} \quad (9)$$

P_e = power transfered from elec. side.

power dissipated in winding resistance

power flowing into coil inductance to be stored.

power going into EMF and becoming T_e torque

Power absorbed by EMF from electrical side is P_e. Can write P_e as

$$P_e := \boxed{\begin{aligned} &e\,i \\ =\ &T_e\,\omega_m \end{aligned}}$$

(10 a) — in elec. variables

(10 b) — in mech. variables

$$\Rightarrow T_e = \frac{ei}{\omega_m} \quad (11) \text{ — Electrically induced torque}$$

  - Mechanical Power

Recall that (mech power) = (torque) × (speed)

$$P_m := T_m\,\omega_m \quad (12)$$

P_m comes from mechanical load (e.g., mass, gravity, friction)

□ Newton's 2nd Law & Net Power Transfer.

- Signs picked such that net torque accellerates rotational mass, Newton's 2nd Law gives

$$J\dot{\omega}_m = T_e - T_m$$

- "moment of inertia"
- Quantifies how "heavy" mass is.

- accelerate
- From drive

- decelerate
- From mech load

$*$ use (11)

$$= \frac{\overbrace{ei}}{\omega_m} - T_m$$

$*$ use (7) For "e"

$$* \; e = \lambda_m \underbrace{\frac{P}{2}\omega_m}_{\omega e} \cos(\theta e)$$

$$= \frac{\left(\lambda_m \frac{P}{2} \cancel{\omega_m} \cos\left(\frac{P}{2}\theta_m\right)\right) i}{\cancel{\omega_m}} - T_m$$

$$= \underbrace{i(t)}\,\lambda_m \frac{P}{2} \cos(\theta_e(t)) - T_m(t) \qquad (13)$$

- Our power electronics will control $i(t)$ to follow a command.
- Shape $i(t)$ to manipulate acceleration $\dot{\omega}_m$.

□ Ideal Sinusoidal Current Control.

- Assume controller can perfectly manipulate $i(t)$ to follow command

$$\underbrace{i(t)}_{\text{actual}} \longrightarrow \underbrace{i^*(t) = I_{pk}\cos(\theta_e)}_{\text{command}} \qquad (14)$$

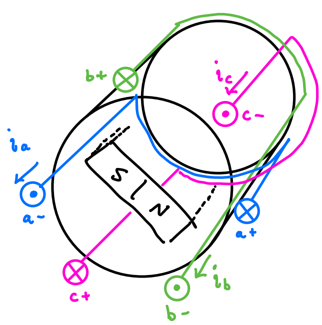
- If $i(t) \approx i^*(t)$, then

$$J\dot{\omega} = \underbrace{I_{pk}\lambda_m \frac{P}{2}\cos^2(\theta_e)} - T_m \qquad (15)$$

↳ • The single phase machine has an undesirable $\cos^2(\cdot)$ term, which gives rise to pulsating/"jerky" mechanical torque from drive.

• 3-phase will fix this & give smooth torque!

- Three-phase machine w/ 2 pole magnet
  - Add 2 more coils for 3 total. Label them a, b, & c.



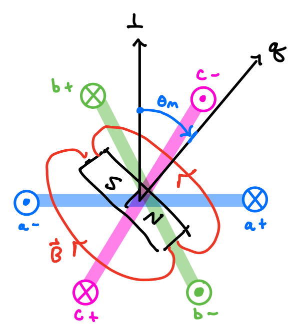


Fig 5: 3 phase Machine

↳ Note that there are 6 terminals total (⊕ & ⊖ for a, b, & c). But a typical machine, like our E-bike motor, only has 3 terminals. What's the deal?

↳ A: The three ⊖ terminals typically shorted together to form a "neutral" terminal. Only see ⊕ a, b, c terminals & neutral on a real machine.

- Analyze 3 flux linkages (one for each coil).

$$
\left.
\begin{aligned}
\lambda_a(\theta_m) &= \lambda_m \sin(\theta_e) + L i_a \\
\lambda_b(\theta_m) &= \lambda_m \sin\left(\theta_e - \frac{2\pi}{3}\right) + L i_b \\
\lambda_c(\theta_m) &= \lambda_m \sin\left(\theta_e + \frac{2\pi}{3}\right) + L i_c
\end{aligned}
\right\} \quad (16)
$$

assume same λ_{pk}, $\frac{P}{2}$, L for all coils

angle shifts due to coil angular displacements

Apply Faraday's Law again:

$$
\left.
\begin{aligned}
\mathcal{E}_a &= \frac{d\lambda_a}{dt} = \lambda_m \omega_e \cos(\theta_e) + L\frac{di_a}{dt} \\
\mathcal{E}_b &= \frac{d\lambda_b}{dt} = \lambda_m \omega_e \cos\left(\theta_e - \frac{2\pi}{3}\right) + L\frac{di_b}{dt} \\
\mathcal{E}_c &= \frac{d\lambda_c}{dt} = \lambda_m \omega_e \cos\left(\theta_e + \frac{2\pi}{3}\right) + L\frac{di_c}{dt}
\end{aligned}
\right\} \quad (17)
$$

where we now have 3 coils w/ KVL expressions:

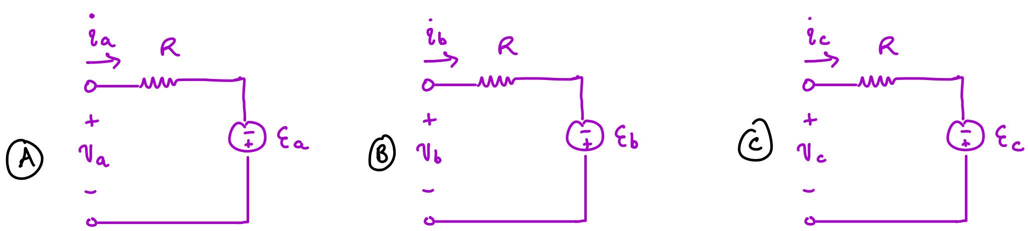


Fig 6: 3 KVL loops for 3 phase machine

And the KVL expressions are:

$$
\left.
\begin{aligned}
v_a &= i_a R - \mathcal{E}_a \\
v_b &= i_b R - \mathcal{E}_b \\
v_c &= i_c R - \mathcal{E}_c
\end{aligned}
\right\} \quad (18)
$$

(17) $\rightarrow$ (18) gives

$$
\begin{aligned}
v_a &= i_a R + L\frac{di_a}{dt} + \underbrace{\lambda_m \omega_e \cos(\theta_e)}_{e_a} \\
v_b &= i_b R + L\frac{di_b}{dt} + \underbrace{\lambda_m \omega_e \cos\left(\theta_e - \frac{2\pi}{3}\right)}_{e_b} \\
v_c &= i_c R + L\frac{di_c}{dt} + \underbrace{\lambda_m \omega_e \cos\left(\theta_e + \frac{2\pi}{3}\right)}_{e_c}
\end{aligned}
\qquad (19)
$$

We now have 3 back EMF's that link electrical & mechanical energy transfer. How to visualize?

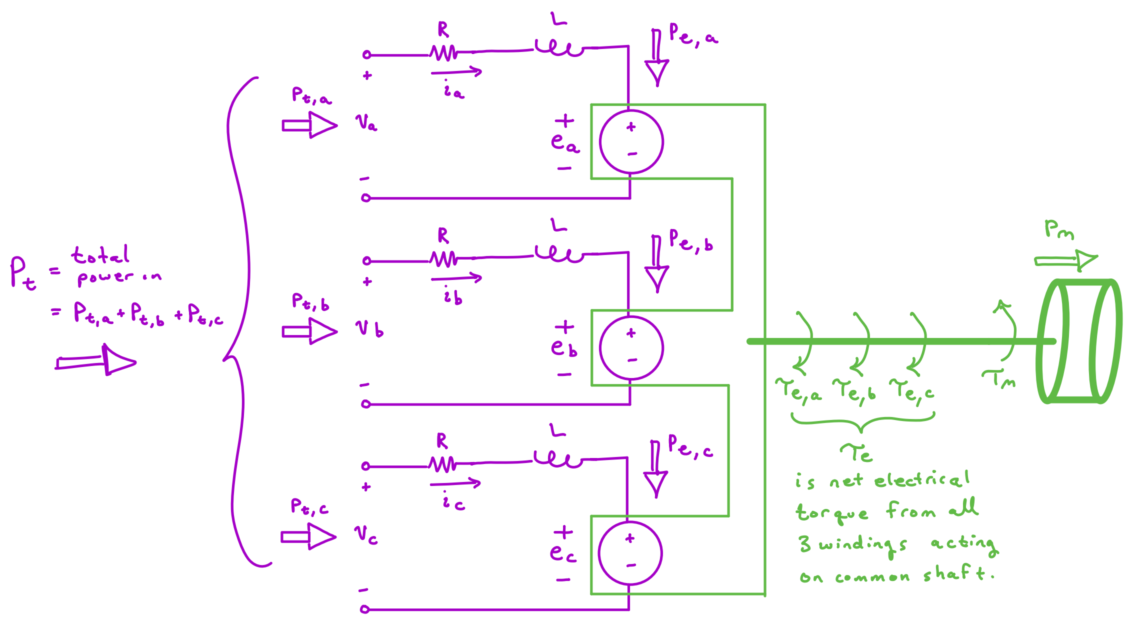


Fig 7 : 3 phase Machine

- Looking @ Fig. 7, the main idea is that 3 back EMFs can now absorb electric power. The power all 3 absorb translate into 3 torques on the common mechanical shaft.

- Analyze EMF absorbed power

P_e = total power absorbed in 3 EMFs

$$= P_{e,a} + P_{e,b} + P_{e,c}$$

$$= e_a i_a + e_b i_b + e_c i_c$$

$$\begin{aligned} = & \, i_a \lambda_m \omega_e \cos(\theta_e) \\ & + i_b \lambda_m \omega_e \cos\left(\theta_e - \tfrac{2\pi}{3}\right) \\ & + i_c \lambda_m \omega_e \cos\left(\theta_e + \tfrac{2\pi}{3}\right) \end{aligned} \qquad (20)$$

Assume we have perfect control of i_a, i_b, i_c such that they take the form

$$\begin{aligned} i_a &\rightarrow I_{pk} \cos(\theta_e) \\ i_b &\rightarrow I_{pk} \cos(\theta_e - 2\pi/3) \\ i_c &\rightarrow I_{pk} \cos(\theta_e + 2\pi/3) \end{aligned} \qquad (21)$$

(21) → (20) gives:

$$P_e = I_{pk} \lambda_m \omega_e \underbrace{\left(\cos^2(\theta_e) + \cos^2\left(\theta_e - \tfrac{2\pi}{3}\right) + \cos^2\left(\theta_e + \tfrac{2\pi}{3}\right) \right)}_{\text{* after some trig, this boils down to } \frac{3}{2}\text{!}}$$

$$\left.\begin{aligned} &= \tfrac{3}{2} I_{pk} \lambda_m \omega_e \\ &= P_e \end{aligned}\right\} \qquad (22)$$

(22) can be written in mech. variables too

$$P_e = \frac{3}{2} I_{pk} \lambda_m \underbrace{\left(\frac{P}{2}\omega_m\right)}^{\omega_e} \leftarrow \text{elec}$$

$$= T_e \omega_m \qquad \leftarrow \text{mech} \qquad \Bigg\} \; (23)$$

Where T_e is the torque experienced on shaft.

- Apply Newton's 2nd Law:

$$J\dot{\omega}_m = \underbrace{T_e}_{\text{*use (23)}} - T_m$$

$$= \frac{3}{2} \frac{I_{pk} \lambda_m \frac{P}{2} \cancel{\omega_m}}{\cancel{\omega_m}} - T_m$$

$$\boxed{\begin{aligned} &= \lambda_m \frac{P}{2} \frac{3}{2} I_{pk} - T_m \\ &= J\dot{\omega}_m \end{aligned}} \qquad (25)$$

where

$$T_e = \text{electrically induced torque}$$
$$= \lambda_m \frac{P}{2} \frac{3}{2} I_{pk}$$

- (25) is a beautiful equation to behold. It says that the amplitude of the injected current sinusiods can manipulate mech. acceleration $\dot{\omega}_m$.

— Steady State Torque

- When constant speed occurs, $\dot{\omega}_m = 0$ & system is said to be in steady state. This occurs when electrical & mech torques are balanced:

$$\cancel{J \dot{\omega}_m}^{\to 0} = \lambda_m \frac{P}{2} \frac{3}{2} I_{pk} - \tau_m$$

$$\Rightarrow \quad \tau_m = \lambda_m \frac{P}{2} \frac{3}{2} I_{pk}$$