ELECTRICAL ENGINEERING

UNIVERSITY *of* WASHINGTON

Lecture 15: Multipliers

Based on material prepared by prof. Visvesh S. Sathe

ELECTRICAL & COMPUTER ENGINEERING

Acknowledgements

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UW (2013-2022)
GaTech (2022-present)

Roadmap for today

- Unsigned multiplication
- Signed multiplication (Baugh-Wooley)
- Booth encoding to improve performance
- Prefix computation…. **again?!!**

Multipliers

- How do you multiply binary numbers?

- Multiplication
  - Generate partial products
  - Sum partial products together
  - M X N multiplication → M+N bit result

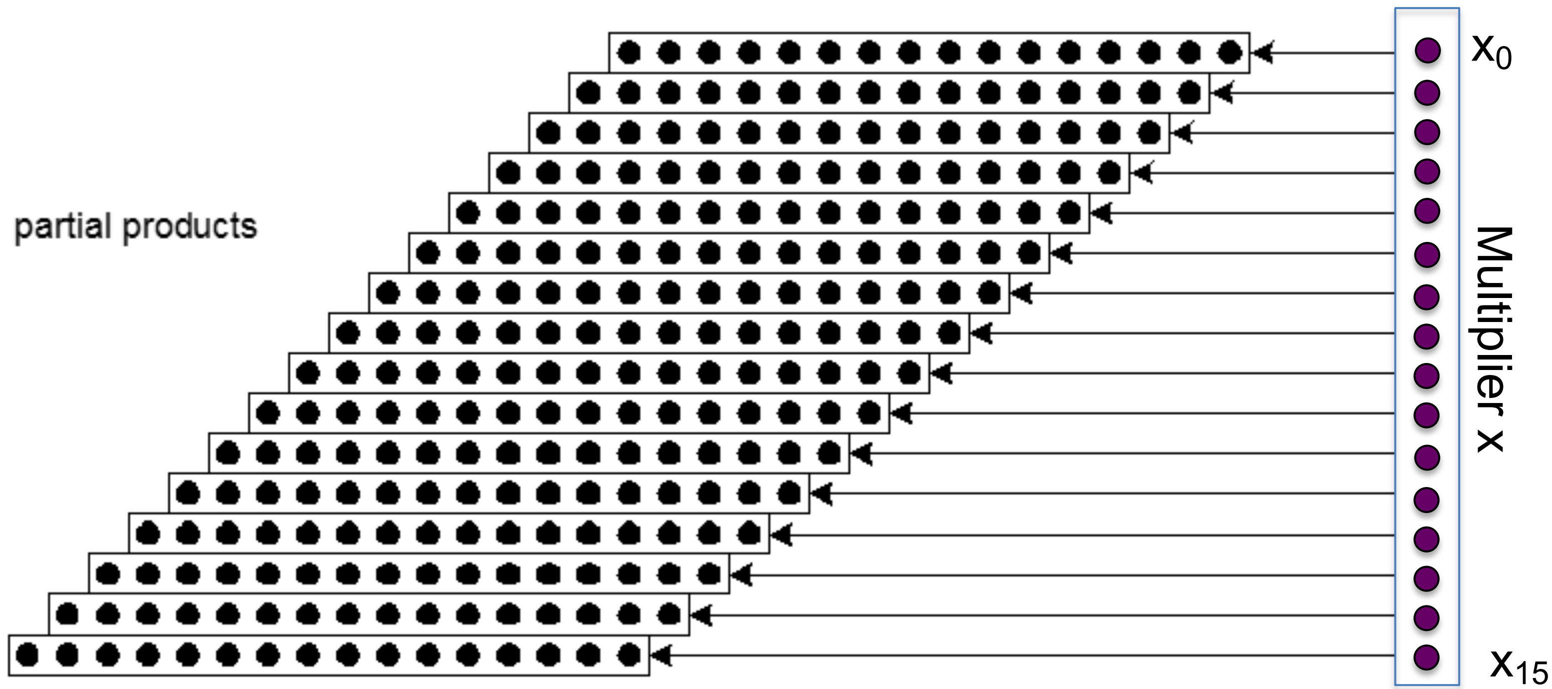
Multiplication

- Given multiplicand Y = $y_{n-1}\ y_{n-2}\ y_{n-3}\ \ldots\ y_1\ y_0$
- Given multiplicand X = $x_{m-1}\ x_{n-2}\ x_{n-3}\ \ldots\ x_1\ x_0$
- $X \times Y = \left(\sum_{i=0}^{m-1} x_i 2^i\right)\left(\sum_{j=0}^{n-1} y_j 2^j\right) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} x_i y_j 2^{i+j}$
- In the case m=5, n=5

						y_5	y_4	y_3	y_2	y_1	y_0	Multiplicand
						x_5	x_4	x_3	x_2	x_1	x_0	Multiplier
						x_0y_5	x_0y_4	x_0y_3	x_0y_2	x_0y_1	x_0y_0	
					x_1y_5	x_1y_4	x_1y_3	x_1y_2	x_1y_1	x_1y_0		
				x_2y_5	x_2y_4	x_2y_3	x_2y_2	x_2y_1	x_2y_0			Partial Products
			x_3y_5	x_3y_4	x_3y_3	x_3y_2	x_3y_1	x_3y_0				
		x_4y_5	x_4y_4	x_4y_3	x_4y_2	x_4y_1	x_4y_0					
	x_5y_5	x_5y_4	x_5y_3	x_5y_2	x_5y_1	x_5y_0						
p_{11}	p_{10}	p_9	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0	Products

Source: Weste-Harris

Dot Diagram



Source: Weste-Harris

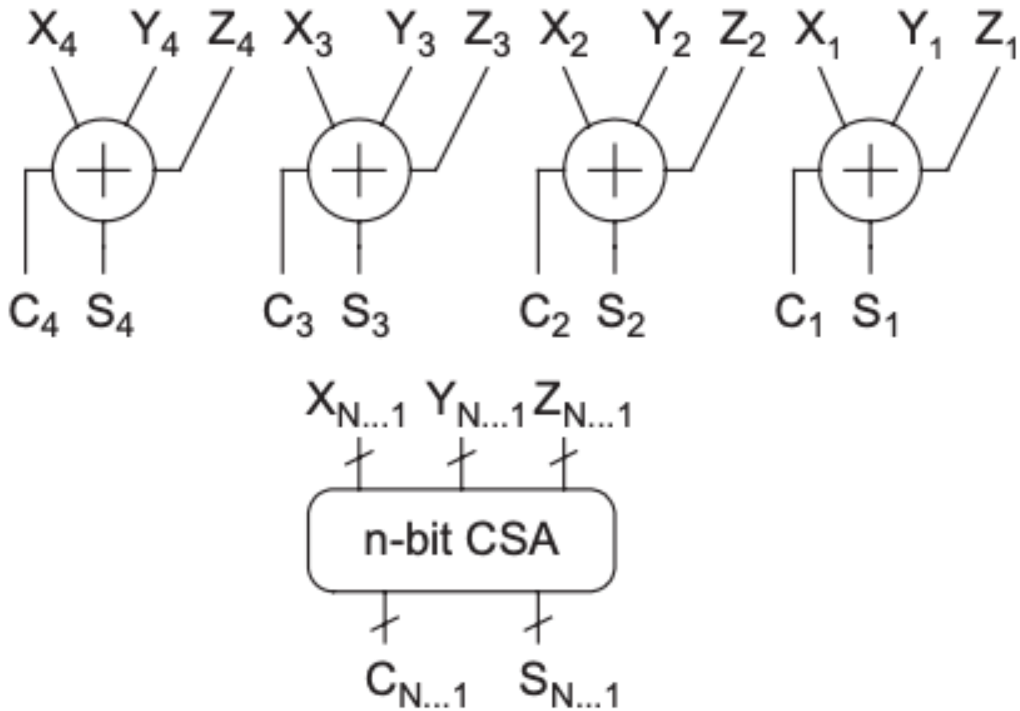
- Each “●“ represents a bit-product

How to sum more than 2 numbers?

How to sum more than 2 numbers?

- Remember full-adders?

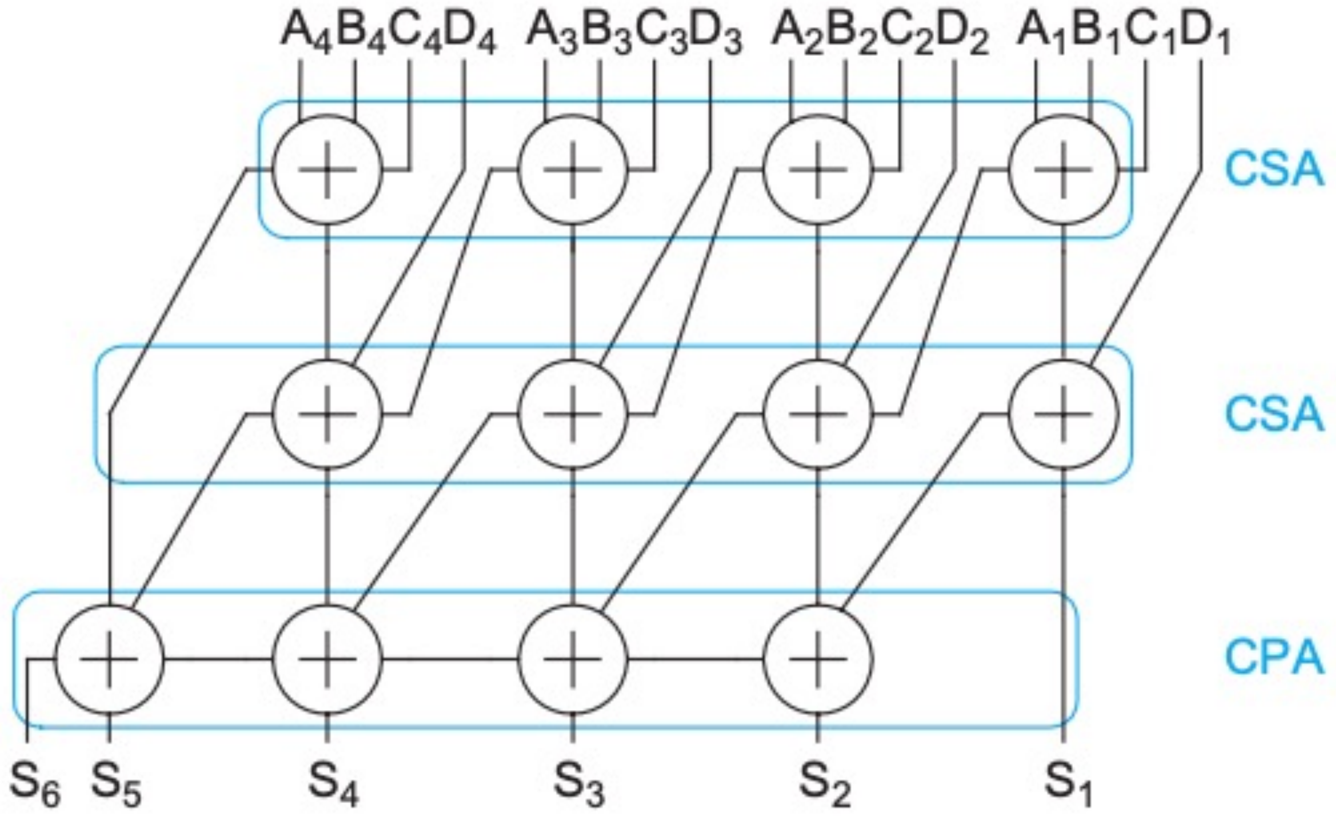
Carry Save Adder (CSA)



$$X + Y + Z = S + 2C$$

Source: Weste-Harris

CSAs to add four numbers



Source: Weste-Harris

Array Multiplier

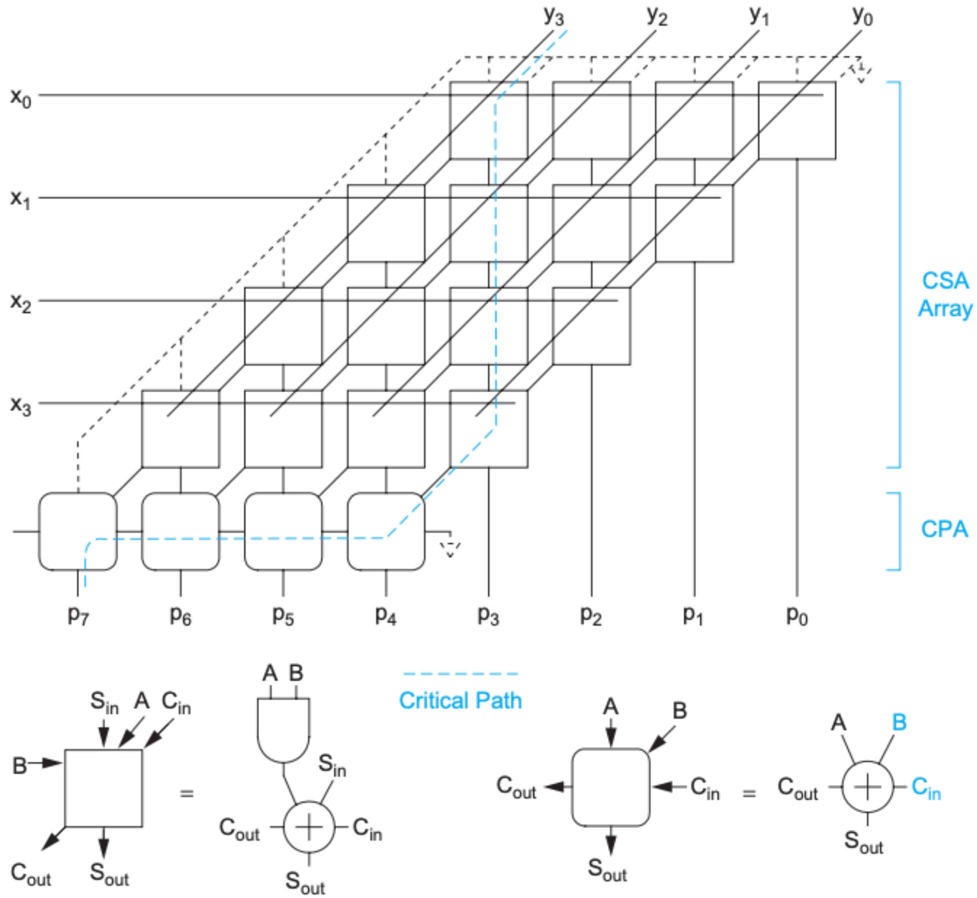


FIGURE 11.74 Array multiplier

Source: Weste-Harris

- Sequential addition of partial products (Analogous to Ripple-Carry Addition)
- Final addition of partial products is an Adder operation
  - All inputs do not arrive at the same time

Array Multiplier

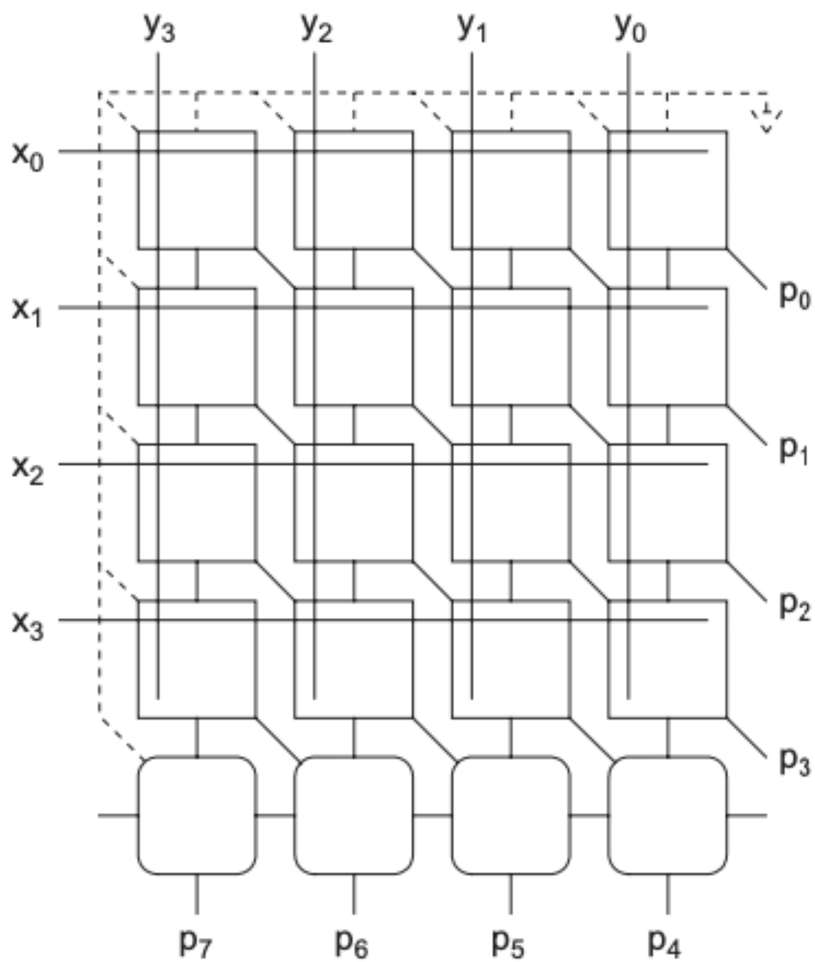


FIGURE 11.75 Rectangular array multiplier

Source: Weste-Harris

- “Straightening it” to be rectangular

Do you remember how to compute -x?

Signed Multiplication (Baugh-Wooley)

- Multiplier topology thus far has been signed multiplication
- $P_{signed} = X \times Y$

$$\left(-x_{m-1}2^{m-1} + \sum_{i=0}^{m-2} x_i 2^i\right)\left(-y_{n-1}2^{n-1} + \sum_{j=0}^{n-2} y_j 2^j\right)$$

$$= y_{n-1}\, x_{m-1} 2^{m+n-2}$$

$$- 2^{n-1} \sum_{i=0}^{m-2} y_{n-1} x_i 2^i \; - \; 2^{m-1} \sum_{j=0}^{n-1} x_{m-1} y_i 2^i$$

$$+ \sum_{i=0}^{m-2} \sum_{j=0}^{n-2} x_i y_j 2^{i+j}$$

Signed Array Multiplier (Baugh-Wooley)

$\sum_{i=0}^{m-2} \sum_{j=0}^{n-2} x_i y_j 2^{i+j}$

				x_0y_4	x_0y_3	x_0y_2	x_0y_1	x_0y_0
			x_1y_4	x_1y_3	x_1y_2	x_1y_1	x_1y_0	
		x_2y_4	x_2y_3	x_2y_2	x_2y_1	x_2y_0		
	x_3y_4	x_3y_3	x_3y_2	x_3y_1	x_3y_0			
x_4y_4	x_4y_3	x_4y_2	x_4y_1	x_4y_0				

Signed Array Multiplier (Baugh-Wooley)

$\sum_{i=0}^{m-2}\sum_{j=0}^{n-2} x_i y_j 2^{i+j}$

				x_0y_4	x_0y_3	x_0y_2	x_0y_1	x_0y_0
			x_1y_4	x_1y_3	x_1y_2	x_1y_1	x_1y_0	
		x_2y_4	x_2y_3	x_2y_2	x_2y_1	x_2y_0		
	x_3y_4	x_3y_3	x_3y_2	x_3y_1	x_3y_0			
x_4y_4	x_4y_3	x_4y_2	x_4y_1	x_4y_0				

$+\, y_{n-1}\, x_{m-1} 2^{m+n-2}$ x_5y_5

Signed Array Multiplier (Baugh-Wooley)

								x_0y_4	x_0y_3	x_0y_2	x_0y_1	x_0y_0
							x_1y_4	x_1y_3	x_1y_2	x_1y_1	x_1y_0	
$\sum_{i=0}^{m-2}\sum_{j=0}^{n-2} x_i y_j 2^{i+j}$						x_2y_4	x_2y_3	x_2y_2	x_2y_1	x_2y_0		
					x_3y_4	x_3y_3	x_3y_2	x_3y_1	x_3y_0			
				x_4y_4	x_4y_3	x_4y_2	x_4y_1	x_4y_0				
$+\,y_{n-1}\,x_{m-1}2^{m+n-2}$		x_5y_5										
												1
$-\,2^{m-1}\sum_{j=0}^{n-1} x_{m-1}y_i 2^i$	$\overline{0}$	$\overline{0}$	$\overline{x_5y_4}$	$\overline{x_5y_3}$	$\overline{x_5y_2}$	$\overline{x_5y_1}$	$\overline{x_5y_0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$

Signed Array Multiplier (Baugh-Wooley)

									x_0y_4	x_0y_3	x_0y_2	x_0y_1	x_0y_0
								x_1y_4	x_1y_3	x_1y_2	x_1y_1	x_1y_0	
$\sum_{i=0}^{m-2}\sum_{j=0}^{n-2} x_i y_j 2^{i+j}$						x_2y_4	x_2y_3	x_2y_2	x_2y_1	x_2y_0		
					x_3y_4	x_3y_3	x_3y_2	x_3y_1	x_3y_0			
				x_4y_4	x_4y_3	x_4y_2	x_4y_1	x_4y_0				
$+\, y_{n-1}\, x_{m-1} 2^{m+n-2}$		x_5y_5										
												1
$-\, 2^{m-1}\sum_{j=0}^{n-1} x_{m-1} y_i 2^i$	$\overline{0}$	$\overline{0}$	$\overline{x_5y_4}$	$\overline{x_5y_3}$	$\overline{x_5y_2}$	$\overline{x_5y_1}$	$\overline{x_5y_0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$-\, 2^{n-1}\sum_{i=0}^{m-2} y_{n-1} x_i 2^i$	$\overline{0}$	$\overline{0}$	$\overline{x_4y_5}$	$\overline{x_3y_5}$	$\overline{x_2y_5}$	$\overline{x_1y_5}$	$\overline{x_0y_5}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
												1

Signed Array Multiplier (Baugh-Wooley)

								x_0y_4	x_0y_3	x_0y_2	x_0y_1	x_0y_0
							x_1y_4	x_1y_3	x_1y_2	x_1y_1	x_1y_0	
$\sum_{i=0}^{m-2}\sum_{j=0}^{n-2} x_i y_j 2^{i+j}$						x_2y_4	x_2y_3	x_2y_2	x_2y_1	x_2y_0		
					x_3y_4	x_3y_3	x_3y_2	x_3y_1	x_3y_0			
				x_4y_4	x_4y_3	x_4y_2	x_4y_1	x_4y_0				
$+\, y_{n-1}\, x_{m-1} 2^{m+n-2}$		x_5y_5										1
$-\, 2^{m-1}\sum_{j=0}^{n-1} x_{m-1} y_i 2^i$	1	1	$\overline{x_5y_4}$	$\overline{x_5y_3}$	$\overline{x_5y_2}$	$\overline{x_5y_1}$	$\overline{x_5y_0}$	1	1	1	1	1
$-\, 2^{n-1}\sum_{i=0}^{m-2} y_{n-1} x_i 2^i$	1	1	$\overline{x_4y_5}$	$\overline{x_3y_5}$	$\overline{x_2y_5}$	$\overline{x_1y_5}$	$\overline{x_0y_5}$	1	1	1	1	1
												1

Signed Array Multiplier (Baugh-Wooley)

							x_0y_4	x_0y_3	x_0y_2	x_0y_1	x_0y_0	
						x_1y_4	x_1y_3	x_1y_2	x_1y_1	x_1y_0		
$\sum_{i=0}^{m-2}\sum_{j=0}^{n-2} x_i y_j 2^{i+j}$					x_2y_4	x_2y_3	x_2y_2	x_2y_1	x_2y_0			
				x_3y_4	x_3y_3	x_3y_2	x_3y_1	x_3y_0				
			x_4y_4	x_4y_3	x_4y_2	x_4y_1	x_4y_0					
$+\, y_{n-1}\, x_{m-1} 2^{m+n-2}$		x_5y_5										1
$-\, 2^{m-1}\sum_{j=0}^{n-1} x_{m-1} y_i 2^i$	1	1	$\overline{x_5y_4}$	$\overline{x_5y_3}$	$\overline{x_5y_2}$	$\overline{x_5y_1}$	$\overline{x_5y_0}$	1	1	1	1	1
$-\, 2^{n-1}\sum_{i=0}^{m-2} y_{n-1} x_i 2^i$	1	1	$\overline{x_4y_5}$	$\overline{x_3y_5}$	$\overline{x_2y_5}$	$\overline{x_1y_5}$	$\overline{x_0y_5}$	1	1	1	1	1
												1

				x_0y_4	x_0y_3	x_0y_2	x_0y_1	x_0y_0
			x_1y_4	x_1y_3	x_1y_2	x_1y_1	x_1y_0	
		x_2y_4	x_2y_3	x_2y_2	x_2y_1	x_2y_0		
	x_3y_4	x_3y_3	x_3y_2	x_3y_1	x_3y_0			
x_4y_4	x_4y_3	x_4y_2	x_4y_1	x_4y_0				

Removing leading 1's from signed numbers?

Signed Array Multiplier (Baugh-Wooley)

$$\sum_{i=0}^{m-2} \sum_{j=0}^{n-2} x_i y_j 2^{i+j}$$

$$+\, y_{n-1}\, x_{m-1} 2^{m+n-2}$$

$$-\, 2^{m-1} \sum_{j=0}^{n-1} x_{m-1} y_i 2^i$$

$$-\, 2^{n-1} \sum_{i=0}^{m-2} y_{n-1} x_i 2^i$$

					1	$\overline{x_0y_5}$	x_0y_4	x_0y_3	x_0y_2	x_0y_1	x_0y_0
					$\overline{x_1y_5}$	x_1y_4	x_1y_3	x_1y_2	x_1y_1	x_1y_0	
				$\overline{x_2y_5}$	x_2y_4	x_2y_3	x_2y_2	x_2y_1	x_2y_0		
			$\overline{x_3y_5}$	x_3y_4	x_3y_3	x_3y_2	x_3y_1	x_3y_0			
		$\overline{x_4y_5}$	x_4y_4	x_4y_3	x_4y_2	x_4y_1	x_4y_0				
1	x_5y_5	$\overline{x_5y_4}$	$\overline{x_5y_3}$	$\overline{x_5y_2}$	$\overline{x_5y_1}$	$\overline{x_5y_0}$					

Signed Array Multiplier (Baugh-Wooley)

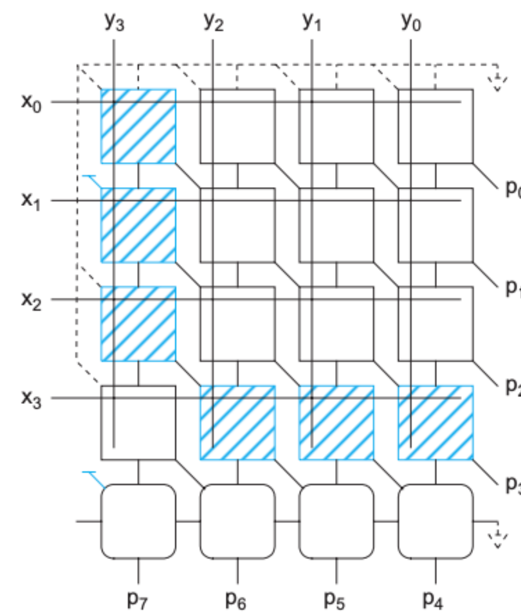
					1	$\overline{x_0y_5}$	x_0y_4	x_0y_3	x_0y_2	x_0y_1	x_0y_0
					$\overline{x_1y_5}$	x_1y_4	x_1y_3	x_1y_2	x_1y_1	x_1y_0	
				$\overline{x_2y_5}$	x_2y_4	x_2y_3	x_2y_2	x_2y_1	x_2y_0		
			$\overline{x_3y_5}$	x_3y_4	x_3y_3	x_3y_2	x_3y_1	x_3y_0			
		$\overline{x_4y_5}$	x_4y_4	x_4y_3	x_4y_2	x_4y_1	x_4y_0				
1	x_5y_5	$\overline{x_5y_4}$	$\overline{x_5y_3}$	$\overline{x_5y_2}$	$\overline{x_5y_1}$	$\overline{x_5y_0}$					

$$\sum_{i=0}^{m-2}\sum_{j=0}^{n-2} x_i y_j 2^{i+j}$$

$$+\, y_{n-1}\, x_{m-1} 2^{m+n-2}$$

$$-\, 2^{m-1}\sum_{j=0}^{n-1} x_{m-1} y_i 2^i$$

$$-\, 2^{n-1}\sum_{i=0}^{m-2} y_{n-1} x_i 2^i$$



Source: Weste-Harris

FIGURE 11.78 Modified Baugh-Wooley two's complement multiplier

Reducing the Number of Partial Products

- Array multipliers require merging M partial products
  - We're adding too many partial products!

- Choose r for "radix-2^r" Booth Encoding
- → reduction to M/r partial products

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  - Referred to as radix-2^r encoding
  - Traditional radix-2 encoding: PP is selected as either 0 or Y (AND Array)
  - Radix-4 encoding: PP is selected as either 0, Y, 2Y or 3Y (Selection logic)

Similar ideas are relatively common in Digital VLSI. See for example:

Jeong, J., Collins, N., & Flynn, M. P. (2016). *A 260 MHz IF Sampling Bit-Stream Processing Digital Beamformer With an Integrated Array of Continuous-Time Band-Pass ΔΣ Modulators.* IEEE Journal of Solid-State Circuits, 51(5), 1168-1176.

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- Radix-4 and radix-8 encoding used more often
  - Increased complexity in generating partial products with ↑r
  - Use of even higher radix justified in rare cases

Booth Encoding (Radix-4)

- Idea: Scale partial product by 0, 1, 2 or 3
  - 2Y generated from a shift <<1
  - 3Y is a problem.

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- But.. what if previous group is 11 and the current group is 10?

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- Ugh… Prefixes again!! ☹

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- WAIT! There's an easy fix with a clever idea ☺

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Booth Encoding (Radix-4)

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- Idea: Scale partial product by 0, 1, 2 or 3
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  - 2Y implemented as
    - sometimes 2Y
    - sometimes 4Y-2Y

Booth Encoding (Radix-4)

$$\boxed{\mathbf{2 = 4 - 2}}$$

- Idea: Scale partial product by 0, 1, 2 or 3
  - 2Y generated from a shift <<1
  - 3Y is a problem. Try 4-1 instead of 2+1
  - 2Y implemented as
    - sometimes 2Y
    - sometimes 4Y-2Y

- Why is this a good idea?
  - We only need to look at the MSB bit of the previous 2-bit group! (No prefix ☺)

Booth-Encoding: rules and how it works

- Add extra group with leading 0s if needed
- Form partial products according to the following rules:
  - Add 4Y to next group based on MSB of current group

Inputs (numbers to multiply)			**Outputs**
Current Group		**MSB from previous group**	**Partial product**
x_{2i+1}	x_{2i}	x_{2i-1}	PP_i
0	0	0	0
0	1	0	Y
1	0	0	-2Y
1	1	0	-Y
0	0	1	Y
0	1	1	2Y
1	0	1	-Y
1	1	1	0

Tree Multipliers

- Array multipliers inherently process PPs serially
- This addition can also be performed in parallel

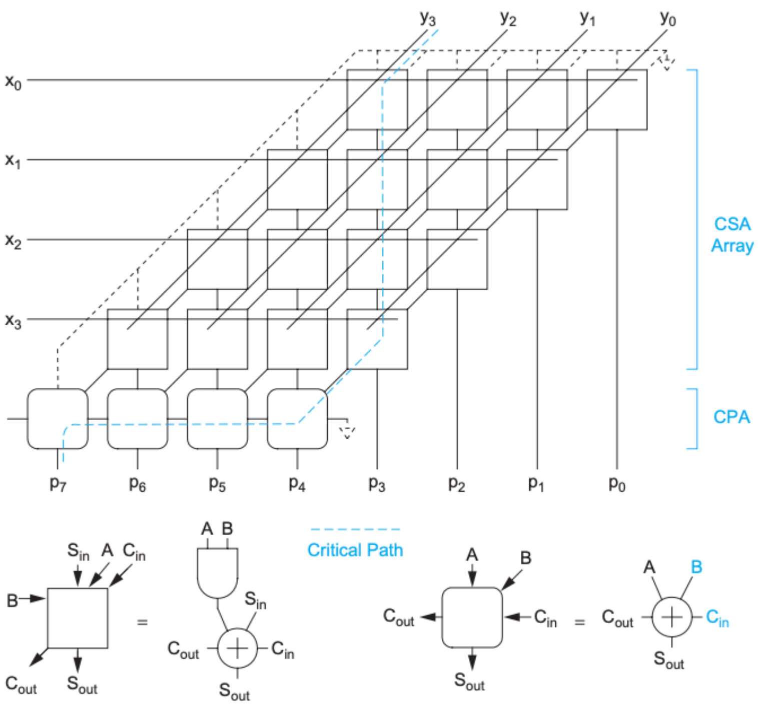

FIGURE 11.74 Array multiplier

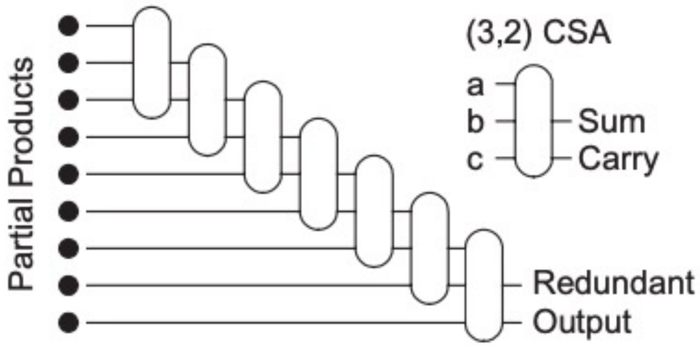

FIGURE 11.84 Dot diagram for array multiplier

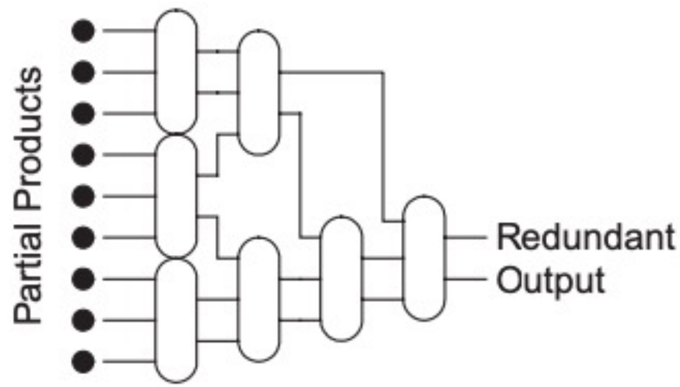

FIGURE 11.85 Dot diagram for Wallace tree multiplier

Source: Weste-Harris

Tree Multipliers

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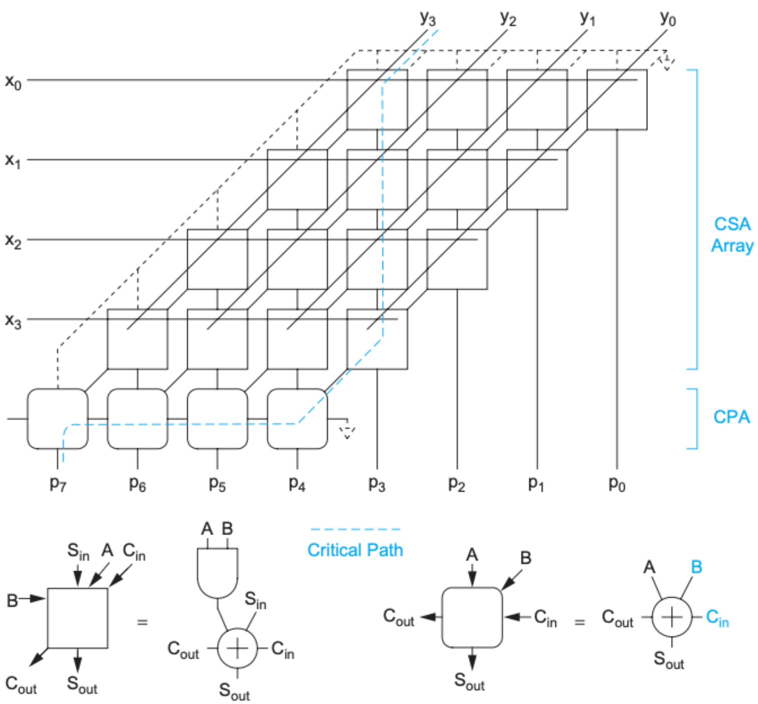

FIGURE 11.74 Array multiplier

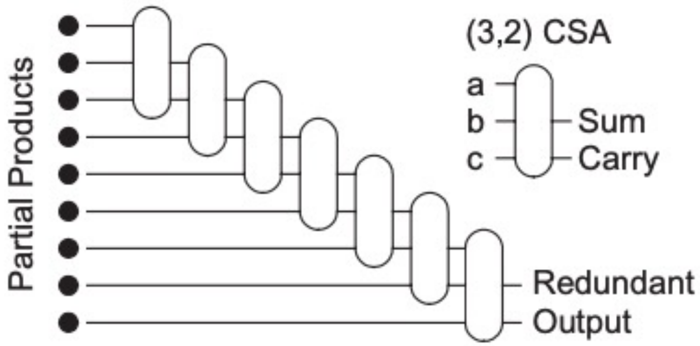

FIGURE 11.84 Dot diagram for array multiplier

Source: Weste-Harris

Questions/Assignments

- Signed booth multiplier?
- Non-equal width of the carry-save segment