Lecture 8 (January 27, 2022)

HW 2 due now

HW3 will be uploaded today - due coming Thursday. (2/3)

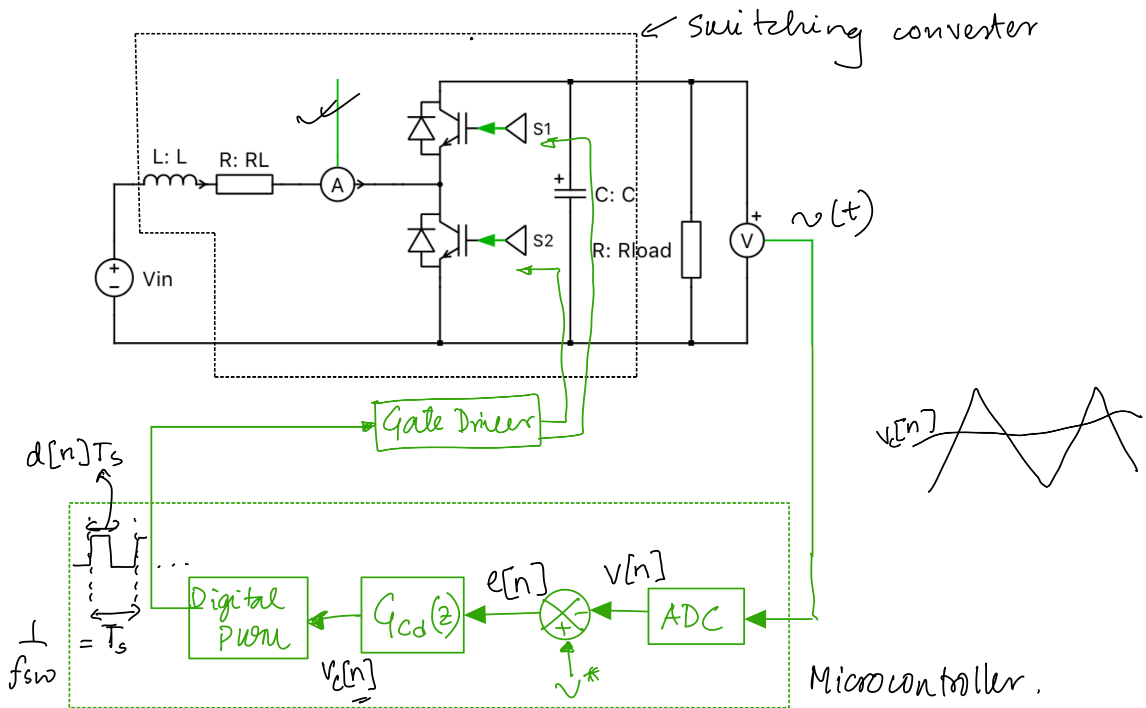
Midterm (take home assignment) Friday (2/4)

Submit by. End of Sunday

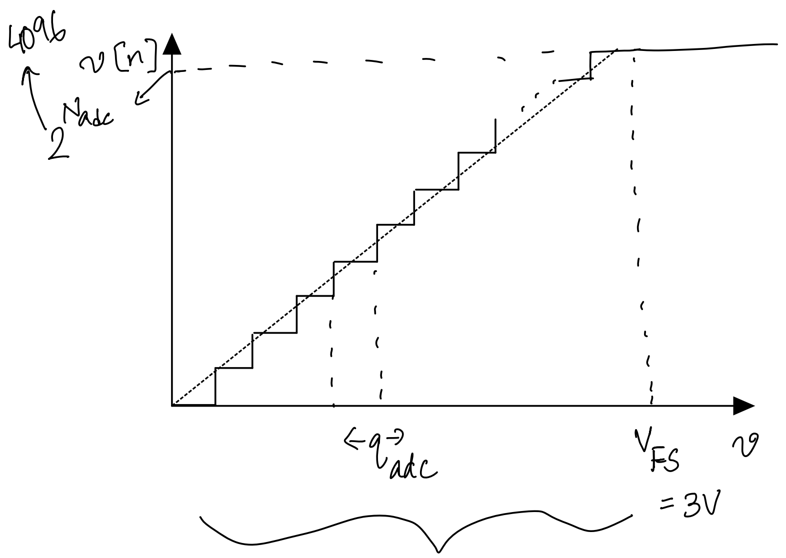
- Friday 3-4 pm (Come with clarificaticati on questions on finals)

→ Linearization / Modeling
→ Control Design
→ ADC / PLOW related ques.

Open book / internet mid term (But cannot discuss with any students).



Quantization characteristics of ADC.

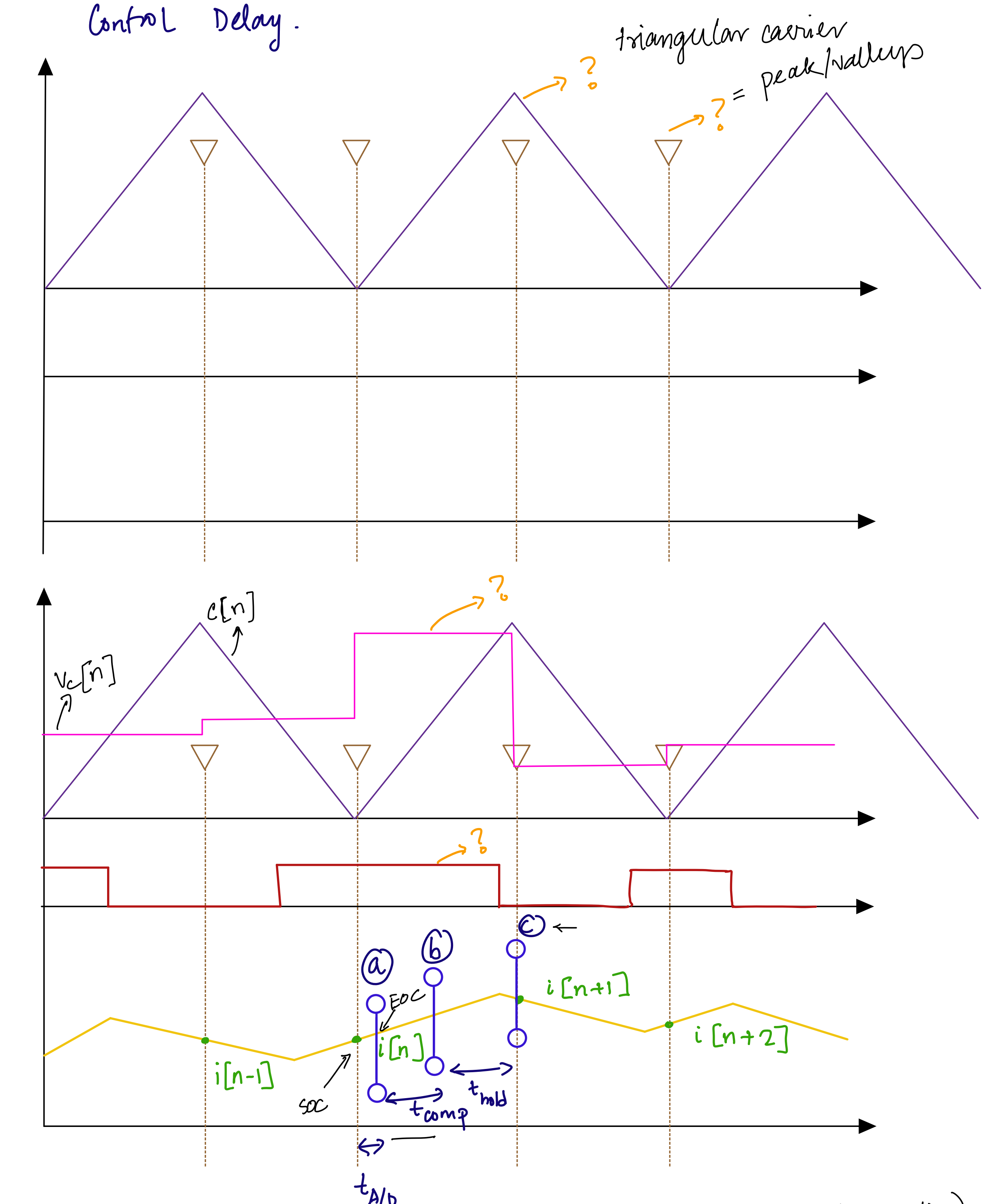


$$q_{adc} = \frac{V_{FS}}{2^{N_{adc}}}.$$

3V i/p in analog

= 4096 O/P in digital.

Control Delay.



$$t_{ctrl,delay} = t_{A/D} + t_{comp} + t_{hold} = \begin{cases} T_{sw}/2 \text{ or} & \text{(peak valley sampling)} \\ T_{sw} \text{ (only peak / only valley)} & \end{cases}$$

(a) EOC is reached by ADC.

Processor now starts the controller computation

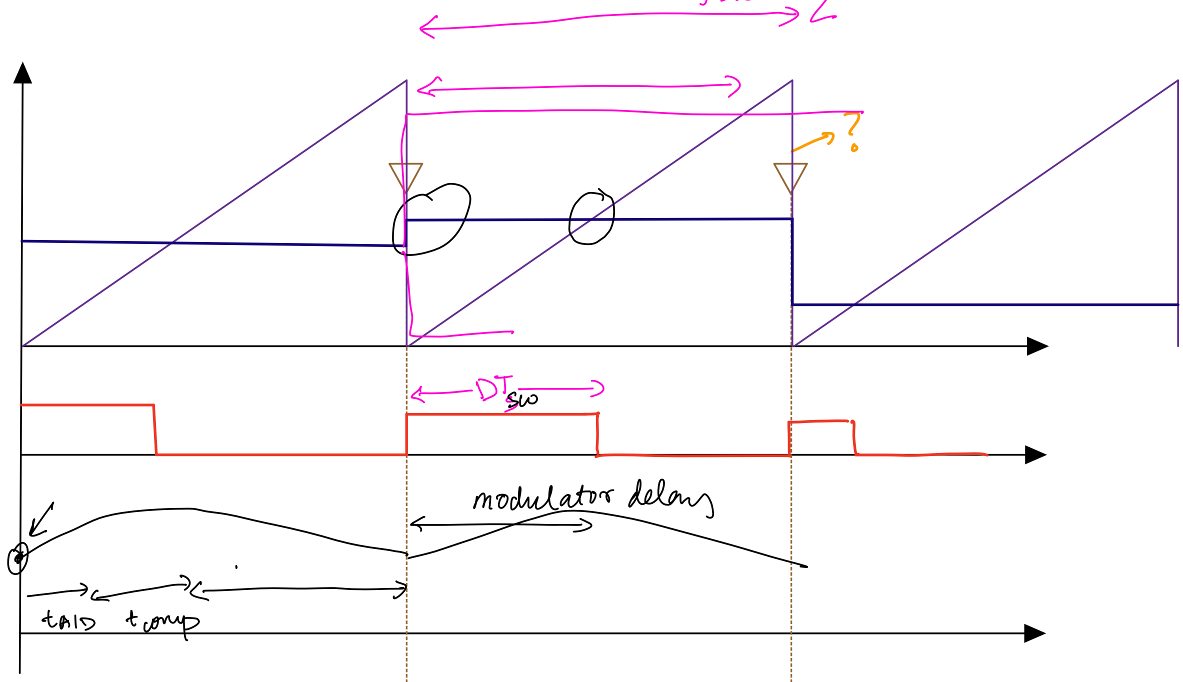
⇒ Here is where your math goes in.

(b) Processor finishes computation & new controller output which is the duty ratio is available.

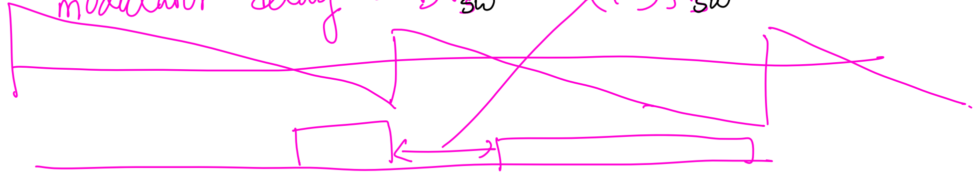
This new d[n] is stored in a "shadow register".

(c) This value uploaded to main register

$T_{sw} = \frac{1}{f_{sw}} = \frac{1}{20\,kHz}$.



modulator delay $= DT_{sw}$ $(1-D)T_{sw}$



Total delay

$= \underbrace{\text{Control Delay}} + \text{Modulator delay}$

$$t_{d\,(secs)} = \underbrace{\overbrace{t_{A/D} + t_{computation} + t_{hold}} + \text{Modulator delay}}_{T_{sw}/2}$$

$$\mathcal{L}[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = X(s)$$

$$\mathcal{L}[x(t-t_d)] = \int_{-\infty}^{\infty} \underbrace{x(t-t_d)}\, e^{-st}\, \underline{dt}$$

$t - t_d = \tau \longrightarrow dt - \cancel{dt_d}^{\,0} = d\tau$

as $t \to \infty$ $\quad \tau \to \infty$

$$\int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau - st_d}\, d\tau$$

$$\int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau}\, \boxed{e^{-st_d}}\, d\tau$$

$$e^{-st_d} \underbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau}\, d\tau}_{X(s)}$$

$$\therefore \mathcal{L}[x(t-t_d)] = \underbrace{e^{-st_d}}_{\text{delay function.}} X(s)$$

Triangular Carrier (peak valley sampling)

$$t_d = \underbrace{t_{A/D} + t_{comp} + t_{hold}}_{T_{sw}/2} + \underbrace{t_{mod}}_{T_{sw}/2}$$

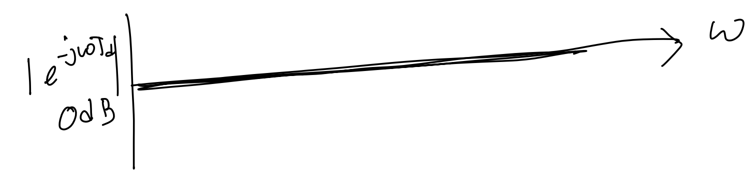
$t_d = T_{sw}$ (T_{sw} = inverse of sw. frequency)

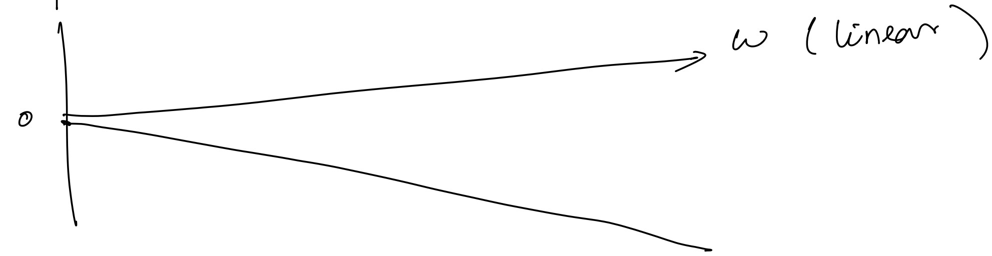
Delay T.F = $e^{-sT_{sw}}$

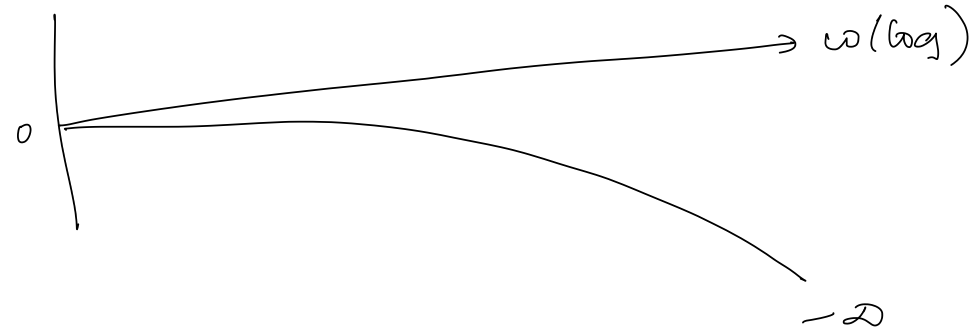
$$e^{-j\omega T_d} = \cos(-\omega T_d) + j\sin(-\omega T_d)$$
$$= \cos(\omega T_d) - j\sin(\omega T_d)$$

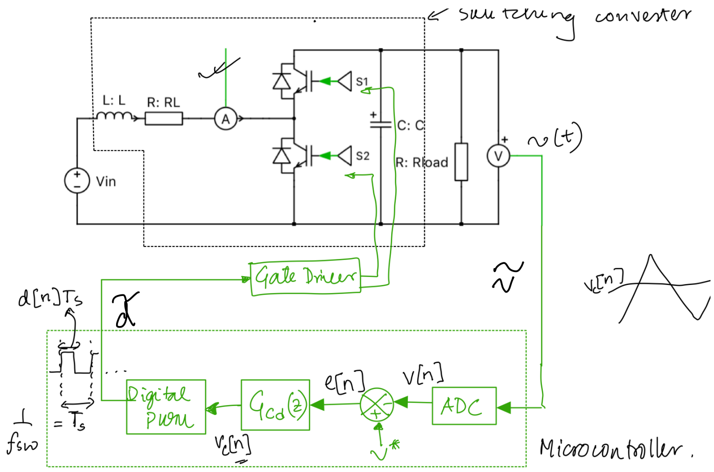
$$|e^{-j\omega T_d}| = \sqrt{\cos^2(\omega T_d) + \sin^2(\omega T_d)} = 1.$$

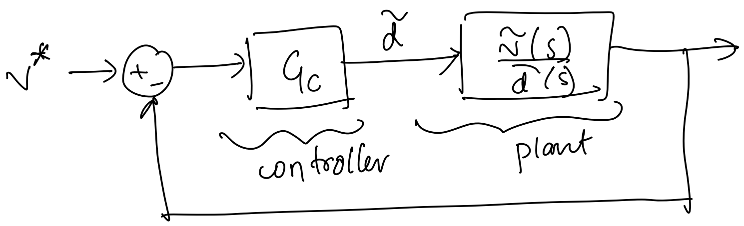
$$\angle e^{-j\omega T_d} = \tan^{-1}\left[-\frac{\sin(\omega T_d)}{\cos(\omega T_d)}\right] = -\tan^{-1}\tan(\omega T_d)$$
$$= \underline{-\omega T_d}.$$

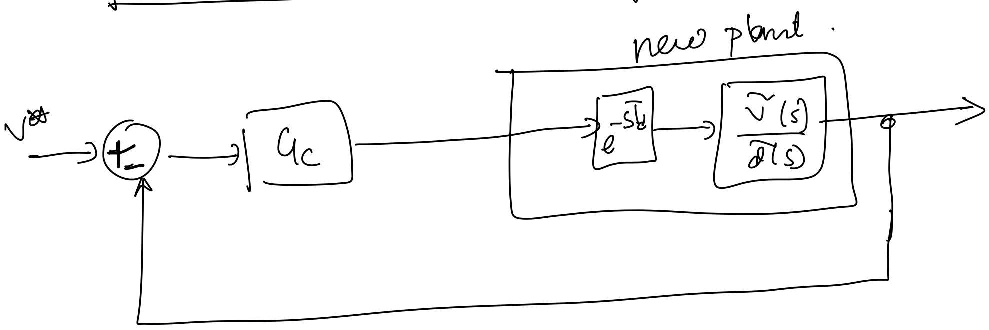


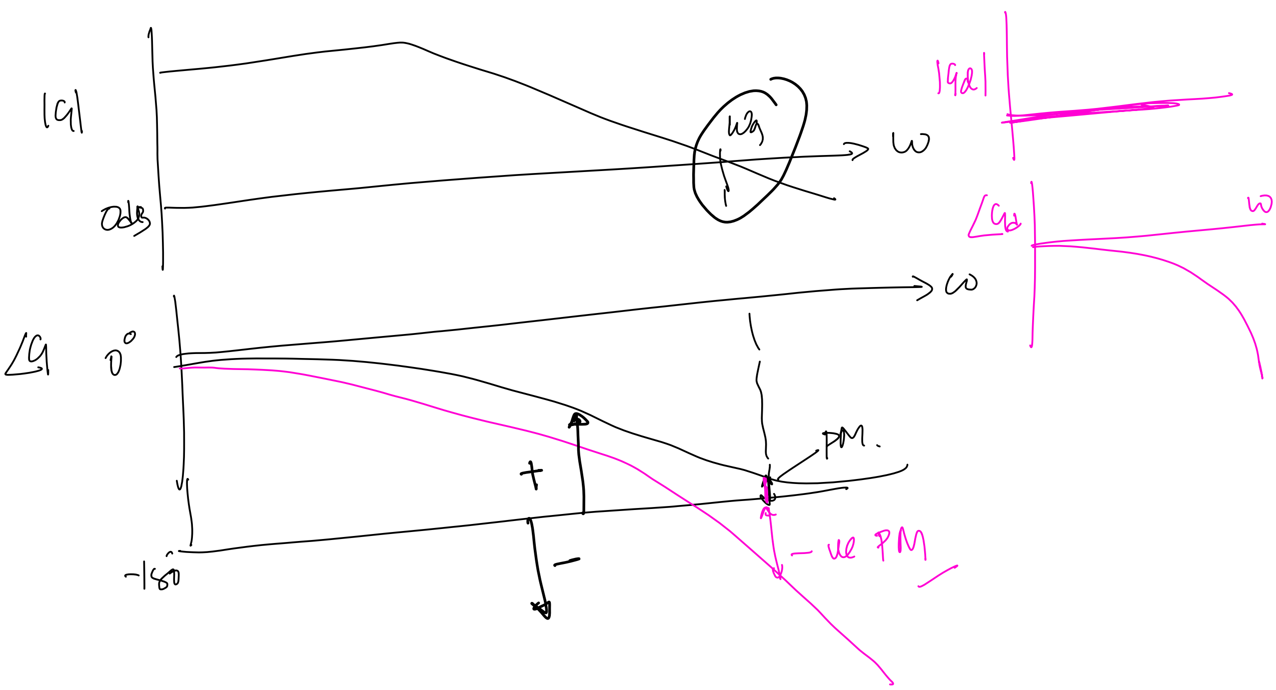




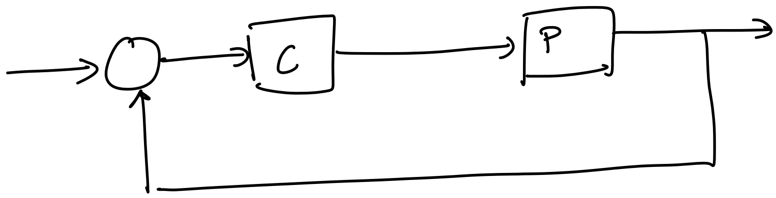

switching converter
L: L
R: RL
Vin
S1
S2
C: C
R: Rload
v(t)
Gate Driver
d[n]Ts
Ts
fsw
Digital PWM
Gcd(z)
e[n]
v[n]
ADC
vc[n]
v*
Microcontroller.


v*
Gc
d̃
ṽ(s) / d̃(s)
controller
plant


new plant.
v*
Gc
e^{-sTd}
ṽ(s) / d̃(s)


|G|
0dB
ωg
ω
|Gd|
∠Gd
ω
∠G
0°
-180
PM.
+
-
-ve PM

Terminologies



P : plant transfer function.

C : Controller transfer function.

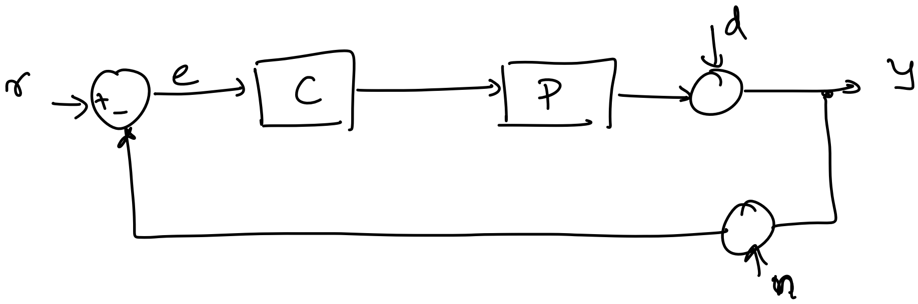
C.P = loop gain or, open loop gain.
or, compensated loop gain. = l

P = uncompensated loop gain

l = loop gain or, open loop transfer function

$\frac{l}{1+l}$ = closed loop transfer function
or complementary sensitivity transfer function.

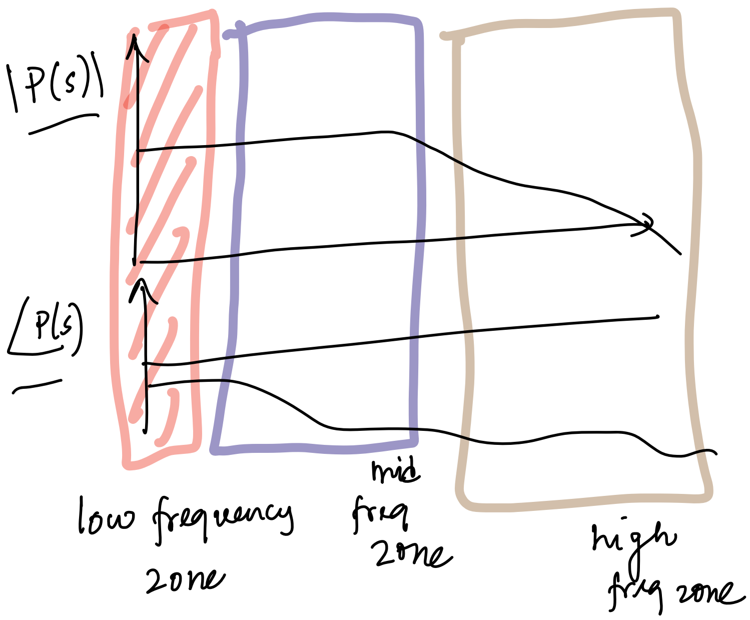
$\frac{1}{1+l}$ = sensitivity transfer function.

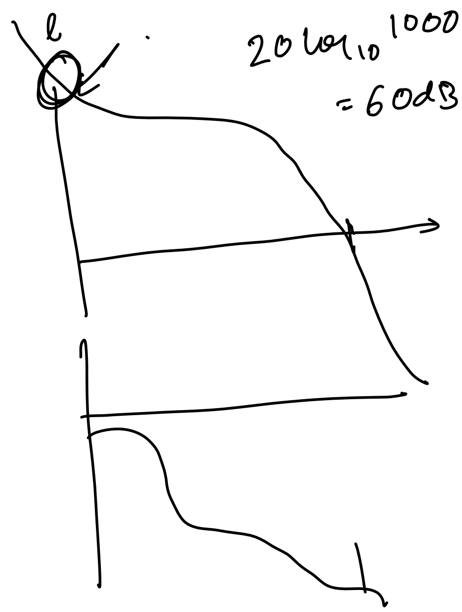


$$y = T\, r + S\, d + T\, n$$

$$= \frac{l}{1+l}\, r + \frac{1}{1+l}\, d + \frac{l}{1+l}\cdot n.$$

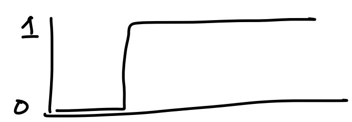
where $l = C \cdot P$.





$$y = \frac{l}{1+l}\, r.$$

$r = 1$



track ref with $\underbrace{\text{max } 0.1\% \text{ error}}$

$$|y| = \left|\frac{l}{1+l}\right| \cdot |r|$$

$y = r$

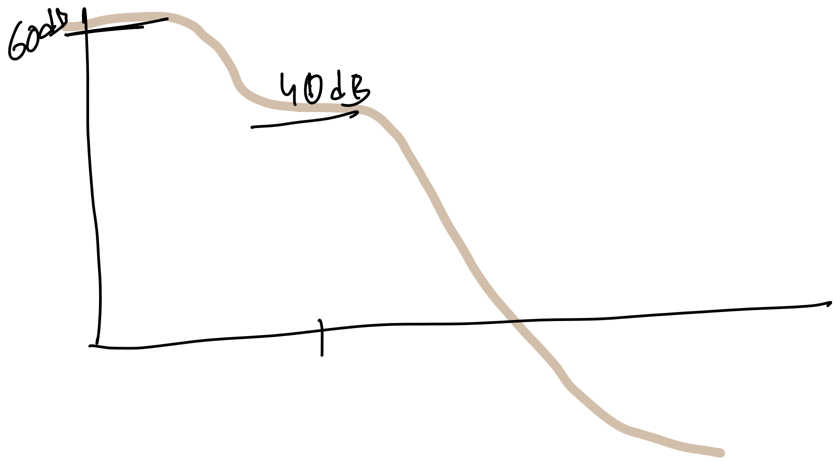
$$\left|\frac{y}{r}\right| = 0.999 = \left|\frac{l}{1+l}\right| \Rightarrow \frac{l}{1+l} = 0.999$$

$$l = 0.999 + 0.999l$$

or, $0.001l = 0.999$

$$\therefore l = 999 \sim \underline{1000}$$

(e)



$$V_0 = M(D) \cdot V_{in}$$

$$\Delta V_o = \underbrace{\left[M(D) - M(D \pm \overbrace{\Delta D}^{\text{quantization}})\right]} V_{in}$$

$$V_0 = M(D)\, V_{in}$$

$$\frac{\partial V_o}{\partial D} = V_{in}\left[\underbrace{\frac{\partial M(D)}{\partial D}}\right]$$

$$\partial V_o = q \cdot V_{in}\left[\frac{\partial M(D)}{\partial D}\right]\Bigg|_{D = D_{nominal}}$$

IN 227 Control Systems Design

Lectures 7 and 8

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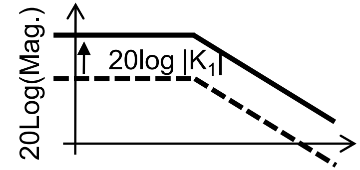
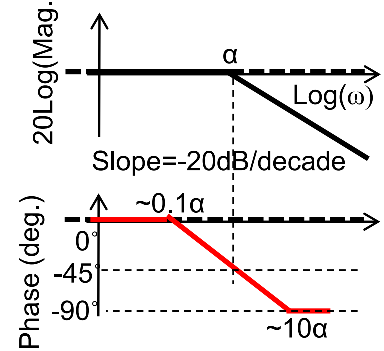
Bode Plots

- If $C(j\omega)P(j\omega)$ is the open-loop transfer function, Bode plots jointly refer to two graphs, one depicting $20\log|C(j\omega)P(j\omega)|$ versus $\log(\omega)$ (*the magnitude plot*) and the other depicting the phase of $C(j\omega)P(j\omega)$ versus $\log(\omega)$ (*the phase plot*).
- Since Bode Plots would be used to visualize the effect of the design, we briefly recount the nature of the changes introduced by specific kinds of transfer functions on the Bode plot of a system.
- Summation Property: The open-loop transfer function of a plant commanded by a controller is $C(j\omega)P(j\omega)$. Thus, the Bode plot of the open-loop transfer function is the sum of the individual Bode plots of $C(j\omega)$ and $P(j\omega)$. This is one key reason that has made Bode Plots such popular tools. It indicates that the bode-plots of the controller can be separately designed with the full awareness of its consequence on the overall Bode plot.
- Effect of a Scaling Operation: Multiplying $P(j\omega)$ by a factor K_1 (>0) amounts to vertically displacing the entire Magnitude plot by an amount $20\log|K_1|$ while leaving the phase plot untouched.
- Effect of "Adding a Pole": To "add a pole" at a frequency 'α' is to multiply the existing transfer function $P(j\omega)$ by $C_1(j\omega)=1/(j\omega/\alpha+1)$. The effect of doing this is to attenuate the magnitude plot at a rate 20dB/10 units of frequency=20dB/decade starting from $\omega=\alpha$. Likewise, in the phase plot, the function $C_1(j\omega)$ adds -90˚ over a wide frequency range. For $\omega<<\alpha$ the phase is close to 0˚. For $\omega=\alpha$, the phase is -45˚ and for $\omega>>\alpha$, the phase is -90˚. It is evident from the summation property that adding multiple poles has the same effect as the sum the effect of each pole.
- Effect of "Adding a Zero": To "add a zero" at a frequency 'α' is to multiply the existing transfer function $P(j\omega)$ by $C_1(j\omega)=(j\omega/\alpha+1)$. The effect of doing this is to amplify the magnitude plot at a rate 20dB/10 units of frequency=+20dB/decade starting from $\omega=\alpha$. Likewise, in the phase plot, the function $C_1(j\omega)$ adds +90˚ over a wide frequency range. for $\omega<<\alpha$ the phase is close to 0˚. For $\omega=\alpha$, the phase is +45˚ and for $\omega>>\alpha$, the phase is +90˚. It is evident from the summation property that adding multiple zeros has the same effect as the sum of the effect of each zero.

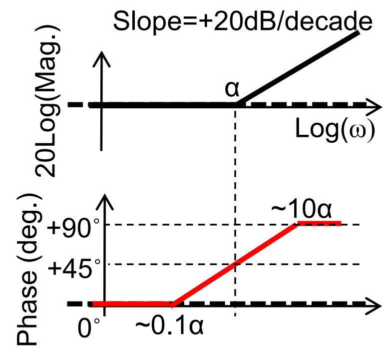
Effect of scaling



Effect of adding a pole

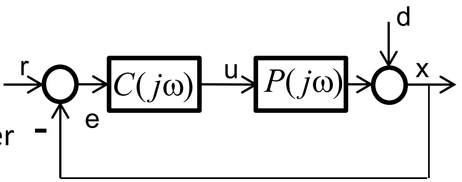


Effect of adding a zero



Design of One Degree of Freedom Control Systems

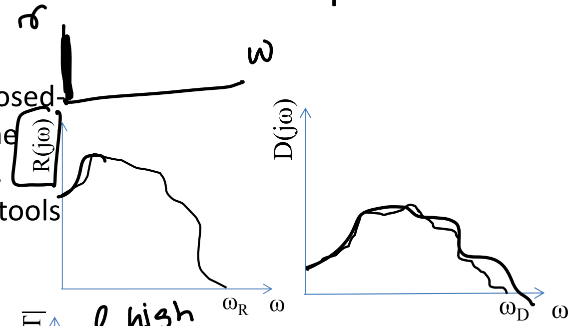
- We have so far familiarized ourselves with the general constraints associated with the design of linear control systems due to the possibility of instability. The most important tool to study stability in the frequency domain is the Nyquist plot. We used it to learn that stability constraints imposed limits on the bandwidth of the control systems, demanded the use of dynamic controllers and necessitated performance specifications for the accuracy with which the control objective x(t)=r(t) is achieved (constraints that we, control engineers, would like to wish away if we could!). The most important tool to design a controller that achieves the desired performance based on frequency domain specifications is the Bode Plot.
- General Design Philosophy: The goal of this and the next lecture is to outline the process of control design, and to provide an understanding of what each of the control techniques do, i.e., explain the usefulness of specific tools such as Proportional/ Integral/ Derivative/ Lead/Lag control techniques. How these tools are to be used in practice, and in what combination, is a judgment that has to be made by the design engineer once he/she ***understands*** what each of these tools can accomplish. Formulae/recipe oriented approach to design inhibits understanding and thus, is generally avoided in the lecture. In building a controller, we seek *simplicity* and *understanding*, and refrain from introducing *any more* complexity to the controller than the situation demands. So, in introducing each specific tool in this lecture, we proceed from the simplest to the more complicated, and at each stage justify *why* we are forced to choose a more complicated design over a simpler one.

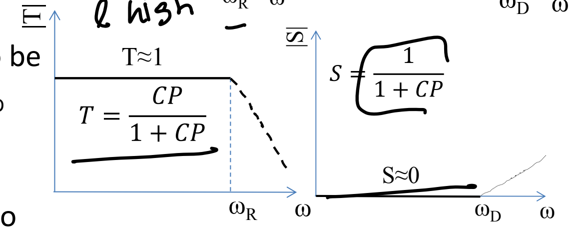


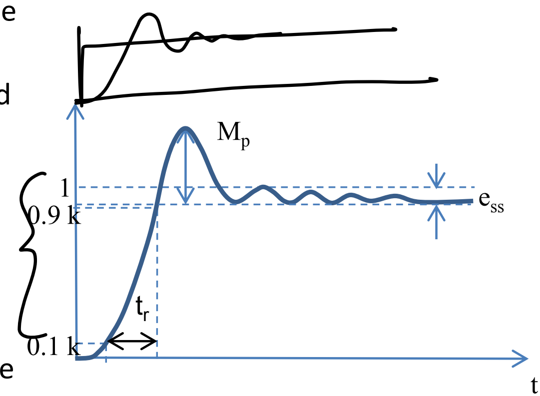
- **One Degree of Freedom Control System** is one that possesses just a single controller C(jω) to design. Once we have designed C(jω) to address a few constraints, we have no more freedom left to independently address any remaining constraints, especially those that could not be addressed by C(jω).
- There are two transfer functions of interest: the transmission function T=X(s)/R(s) $= \dfrac{CP}{1+CP}$ and the Sensitivity function S=X(s)/D(s) $= \dfrac{1}{1+CP}$

One Degree of Freedom Control Design: Determination of closed-loop specifications

- Control design begins with gathering together, all the requirements of the closed-loop system: the desired maximum tracking errors for different inputs and the desired maximum suppression of typical disturbances that affect the system. These are typically time domain specifications. However, since all the useful tools of analysis are in the frequency domain, it is necessary to translate the time-domain specs into frequency domain specs. For example, if the frequency spectrum of the reference inputs or the output disturbances extend up to frequencies ω_R and ω_D respectively, it is necessary to achieve T=CP/(1+CP) to be close to 1 and S=1/(1+CP) to be close to zero for frequencies up to ω_R and ω_D respectively. Their exact values depend on the permissible tracking error or desired rejection of disturbance.
- However, since the exact spectrum of the inputs or disturbance are specific to particular applications, it is desirable to choose a **standard input** whose response can be employed to quantify the performance of a general control system.
- A little thought reveals that a **step input** is an ideal candidate for such a standard input: since it's magnitude does not change beyond the instant of application, it clearly showcases the transient response of the plant itself. Likewise, since it possesses a non-zero steady-state value, it showcases steady-state errors in the response, if any.
- While a lot of features can be defined on the step response of the closed-loop system, and guidelines can be derived to relate them to closed-loop frequency domain requirements on the transmission function T=CP/(1+CP), we shall choose to restrict ourselves to defining three features. They are:





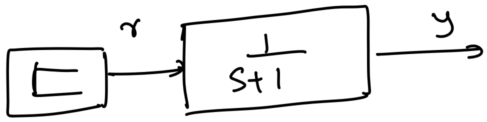


Rise time (t_r): Time taken for the response to rise from 10% to 90% of its steady-state value

Peak overshoot (M_p): largest overshoot of the response beyond its steady-state value.
Steady state error (e_{ss}): The difference between the reference(1) and output (k) after the transients have disappeared, i.e., e_{ss}=1-k.

→ go to simulations

$P \quad \boxed{\frac{1}{s+1}}$



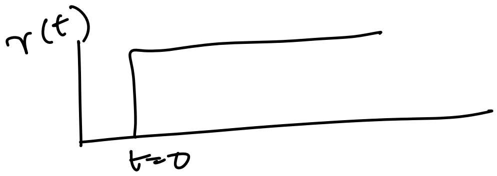
$\frac{0.9}{s+1}$

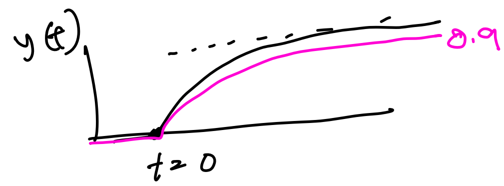
$$y(s) = \frac{1}{s+1} \cdot r(s)$$

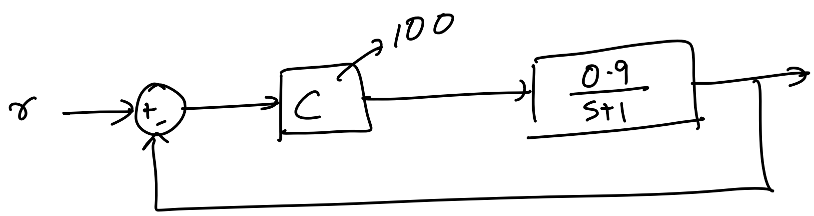
$$= \frac{1}{s+1} \cdot \frac{1}{s}$$

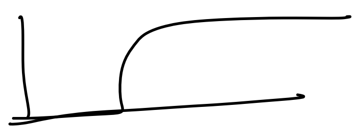
$$= \frac{s+1-s}{(s+1) \cdot s} = \frac{1}{s} - \frac{1}{s+1}$$

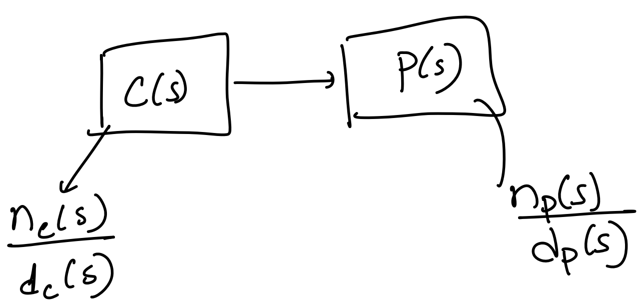
$$\therefore \; y(t) = 1 - e^{-t} =$$









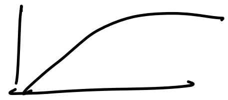


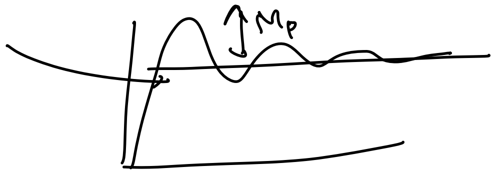
$$l = \frac{n_c(s) \cdot n_p(s)}{\underbrace{d_c(s)}\;\underbrace{d_p(s)}}$$

$\frac{1}{s}$ $\qquad s^1 \ldots$

$l = \frac{}{s^2 + \dots\dots}$

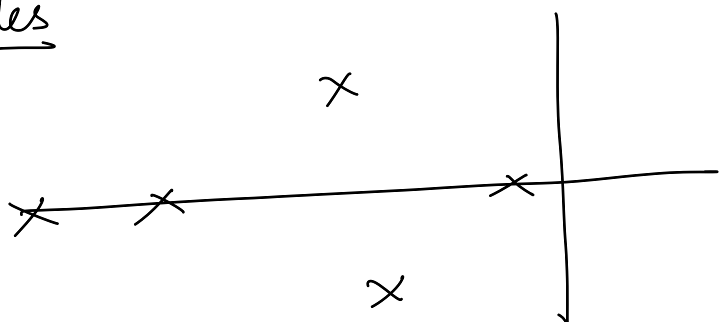
$\frac{1}{s+a}$

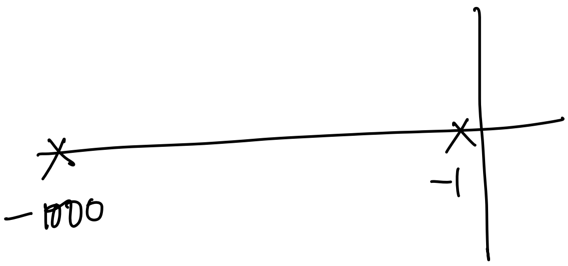




$$l = \frac{1}{\underbrace{s^5 + 1s^4 + 2s^3 + 99s^2 + s + 3}_{\text{characteristic polynomial}}}$$

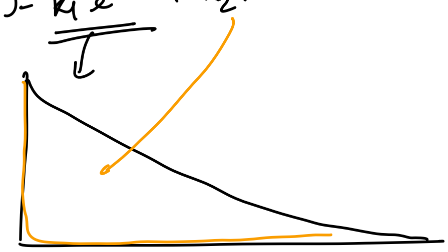
Dominant poles



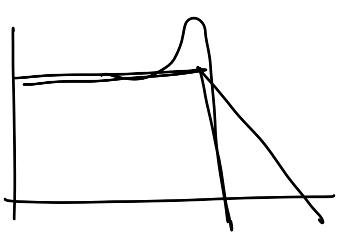


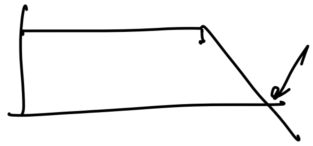
$g(s) = \frac{1}{(s+1)(s+1000)}$

$g(t) = \underline{k_1 e^{-t}} + k_2 e^{-1000t}$



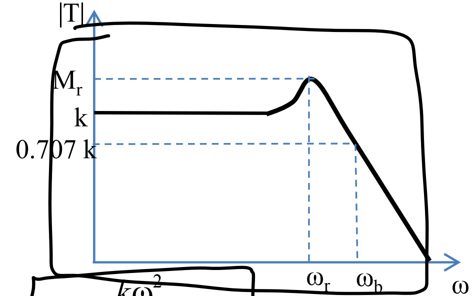
dynamics of response is dominated by the pole closest to the imaginary axis.





$\omega_g = BW$

One Degree of Freedom Control Design: Determination of open-loop specifications



- Since we have not yet designed the controller C, it is not possible to exactly relate the time-domain closed-loop specifications on the step response described earlier to the closed-loop transfer function T(jω). However, reasonable approximations can be made by employing the notion of dominant poles: in general the closed-loop system is assumed to have at least one pair of complex conjugate poles, and that this pair is assumed to be closer to the imaginary axis than all the others. Thus, the closed-loop system is assumed to "look very much like" a second order system: $\boxed{T(s) = \dfrac{k\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}}$
- The steady-state error is related to the DC value of the transmission function, i.e., $T(0)=k=1-e_{ss}$.
- For a second order system, the rise time is indicative of the speed of response and is therefore related also to the closed-loop bandwidth ω_b. It is given by $\boxed{t_r \approx \dfrac{0.4}{f_b} = \dfrac{2.5}{\omega_b}}$

$\boxed{BW = \omega_g}$

$\xi = \dfrac{PM}{100}$

- The peak overshoot M_p is related to the amount of damping in the closed-loop system. In the absence of any damping the overshoot is 100%. For more reasonable cases, the relationship between M_p and the damping factor ξ is given by $\boxed{M_p \approx \exp\left(-\dfrac{\xi}{\sqrt{1-\xi^2}}\pi\right)}$
- In an underdamped system (ξ<1), the response oscillates before it dies down. The oscillation frequency is close to the resonant frequency ω_r of the system. The resonant frequency itself is close to the closed-loop bandwidth and is given by $\omega_r \approx \dfrac{\omega_b}{1.5}$
- Since $\omega_r = \omega_n\sqrt{1-2\xi^2}$ for a second order system, all parameters of T(s), viz., K, ξ and ω_n can therefore be determined.
- Once we have gathered the closed-loop frequency domain requirements, we derive the corresponding open-loop specifications. The flow chart for the entire design process is given on the right.
- From the equation T=CP/(1+CP), it is seen that the steady-state error is related to the loop gain L(j0)=CP(j0) as $CP(j0) = \dfrac{e_{ss}-1}{e_{ss}}$

Convert time domain closed-loop requirements to closed loop frequency domain requirements

↓

Convert closed loop frequency domain requirements to open-loop frequency domain specs

↓

Perform design using Bode plots and study the response to a standard (step) input

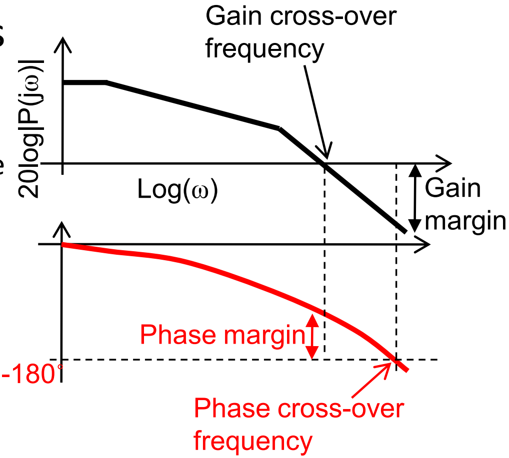
↓

Study the response to the actual inputs/ disturbances

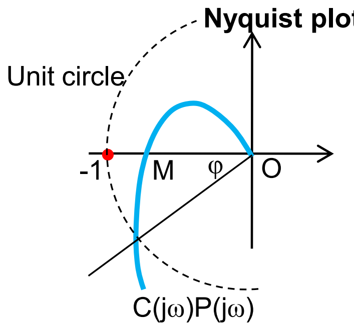
↓

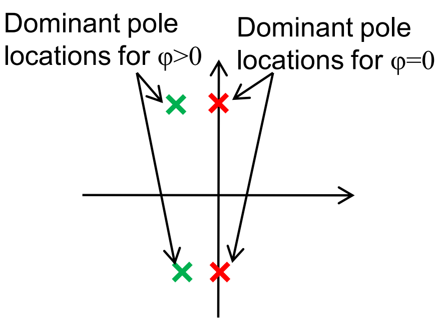
Stop

Determination of open-loop specifications



- From the point of view of stability, there are four variables of interest in the Bode plot of a plant (or more generally, the open-loop transfer function C(jω)P(jω)). The ***Gain Cross-Over Frequency*** is the frequency at which the magnitude graph of 20log|C(jω)P(jω)| crosses the 0dB line. The gain cross-over frequency is related to the closed-loop resonant frequency as $\omega_{gc} \approx \omega_{r.}$ The ***Phase Cross-Over Frequency*** is the frequency at which the phase of C(jω)P(jω) crosses -180˚.
- Their significance becomes evident in the Nyquist plot of the open-loop system. The Nyquist plot of C(jω)P(jω) is plotted on the right. Since the Nyquist plot does not encircle -1, we know that the system is stable. The effect of C(jω) is to alter the magnitude of P(jω) and/or its phase. It is seen that if C(jω) scales up the magnitude by a value more than 1/|OM| then the system will encircle -1 and thus, become unstable. Likewise, if C(jω) adds a phase less than $-\phi$, again the system becomes unstable. OM is equal to|C(jω)P(jω)| of the open-loop system at the phase cross-over frequency. $-180+\phi$ is the phase of the system at the gain cross-over frequency. Thus, for the sake of stability, OM<1 and $\phi>0$. -20log|OM| is defined as the ***gain margin*** and ϕ is defined as the ***phase margin***. If the gain margin and the phase margin are both greater than zero, then the closed loop system is stable. If one of them is less than 0, the closed loop system is unstable.





Determination of open-loop specifications

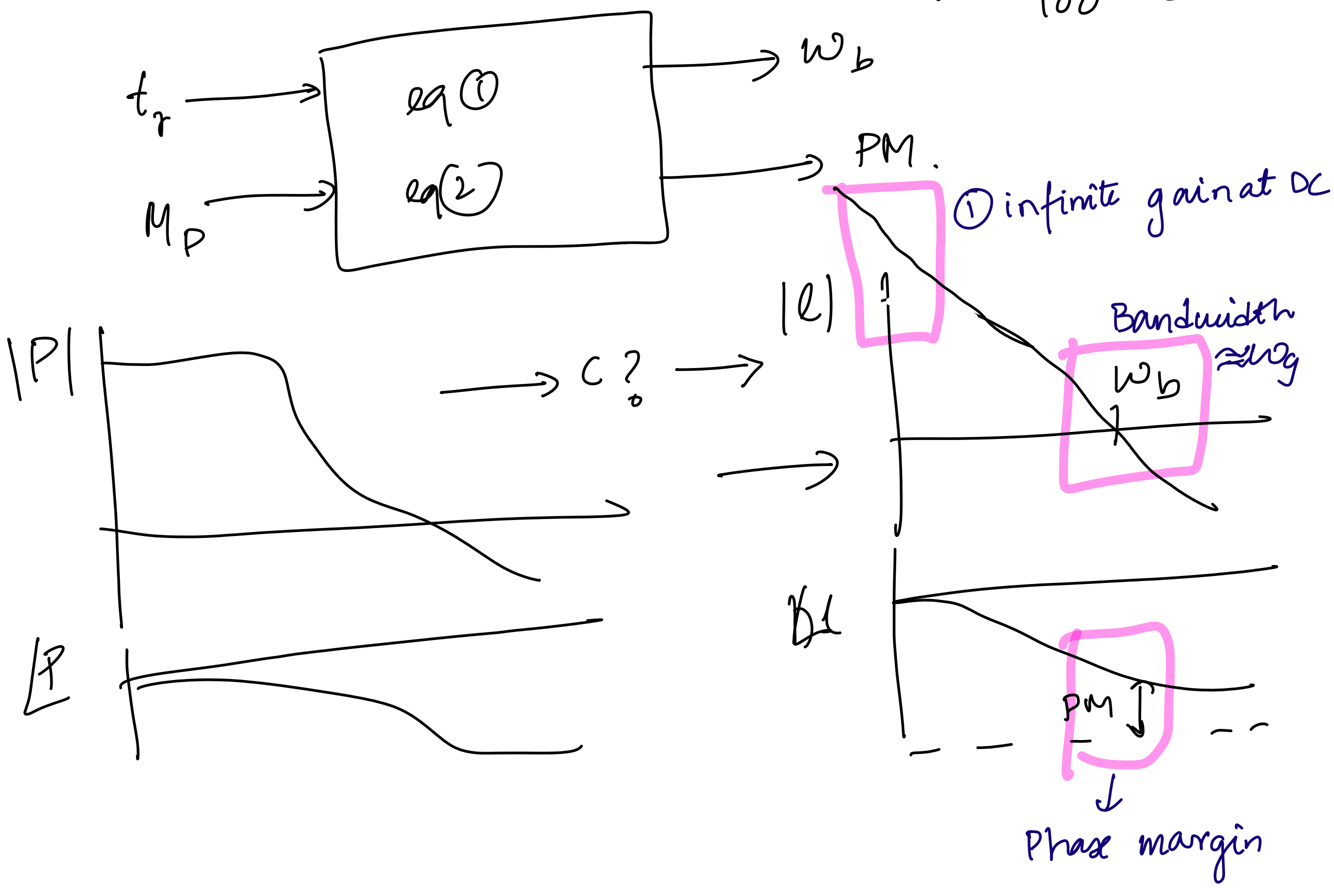
- Significance of Phase Margin: Let the approximate model for closed loop system described in the previous slide be further approximated by setting k≈1. For this case, the open-loop system L can be seen to be given by $L(s)=\omega_n^2/[s(s+2\xi\omega_n)]$
- The gain cross-over frequency ω_{gc} for $L(j\omega)=\omega_n^2/[j\omega(j\omega+2\xi\omega_n)]$ can be obtained by setting |L|=1. This is found to be $\omega_{gc}=\omega_n\sqrt{1-2\xi^2}$. The phase of L at this frequency is $\angle L(j\omega)=-\pi/2-\tan^{-1}(\omega_{gc}/2\xi\omega_n)=-\pi+\tan^{-1}(2\xi\omega_n/\omega_{gc})$
- Thus the phase margin is found to be ϕ= $\tan^{-1}(2\xi\omega_n/\omega_{gc})=\tan^{-1}(2\xi/\sqrt{1-2\xi^2})$
- For closed-loop damping factors ξ <0.6, $\tan^{-1}(2\xi/\sqrt{1-2\xi^2})\approx 2\xi$ **.** Thus, there exists a proportional relationship between the *open-loop* phase margin ϕ and the *closed-loop* damping ratio ξ. If ϕ is expressed in degrees, the proportionality constant is given by π/360≈1/100. Thus, the relationship between the two is given by **ξ =ϕ/100**. Thus, specifying the phase margin ϕ in the frequency domain is equivalent to specifying a particular damping ratio ξ for the closed-loop system. Note that this proportional relationship holds for second order systems. For higher order systems, the relationship holds to the extent that a pair of complex conjugate poles are the dominant closed-loop poles of the system.

① Rise time (t_r)

$$\boxed{\omega_b} = \frac{2.5}{\boxed{t_r}} \quad (1)$$

② M_p (Maximum overshoot)

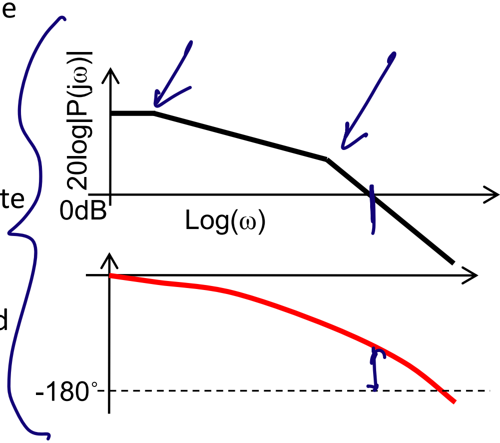
$$\left.\begin{aligned} \boxed{M_p} &= e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \\ \zeta &= \frac{\boxed{PM}}{100} \end{aligned}\right\} (2)$$



The Plant

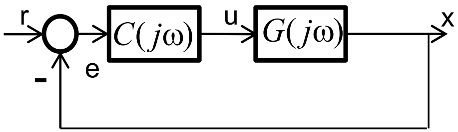
- A plant, in general, refers to the system that we want to control. A well designed plant would have been constructed so as to minimize the amount of feedback control necessary. Here we assume that the plant cannot be altered any further to improve the feedback control performance and the controller has to make do with what has been given.
- Modeling a plant: Mathematically, a linear plant is represented by the transfer function P(jω). P(jω) can be obtained in two ways **(a)** apply a known input u(t), such as a sinusoid, or a step, obtain the response x(t) and evaluate X(jω)/U(jω). **(b)** Understand the physics of each of the subsystems that constitute the plant and use engineering intuition to pick those subsystems that are relevant to the overall dynamics (slowest dynamics are generally the most important because they limit the overall speed of the system) and mathematically model their resultant response. In order to gain an understanding of the overall system, it is recommended to adopt **(b)** to the extent possible before turning to a more accurate model using **(a)**.
- General plant: While illustrating the design procedures, we assume that a general plant can be modeled as $P(j\omega)=K_0\frac{(j\omega/z_1+1)(j\omega/z_2+1)..(j\omega/z_m+1)}{(j\omega/p_1+1)(j\omega/p_1+1)..(j\omega/p_n+1)}$ where, n>m, and $z_1,..\ z_m,\ p_1,..\ p_m,K_0>0$.

  The Bode plot of a general plant looks as shown on the right.



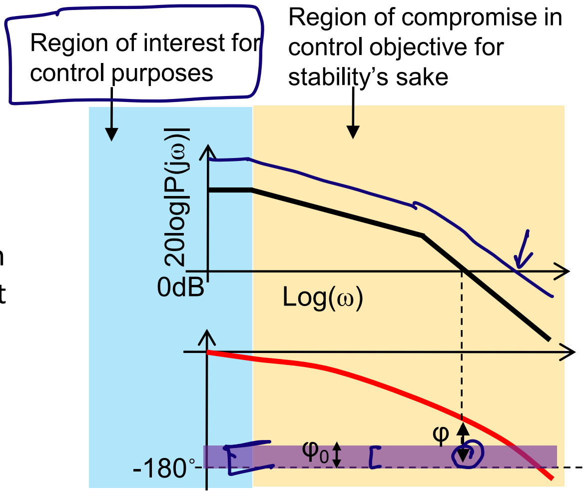
- It is desired that the output of the plant track a specified set of reference inputs or be insensitive to certain disturbances by the specified amount. Our job as control engineers is to design a feedback network around the plant that ensures that these specifications are met. The important steps in the design process are outlined first.

Goals of Design of the Control Systems

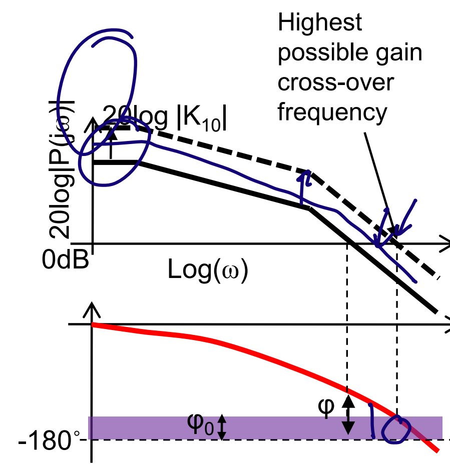


$$X = \frac{1}{1+\frac{1}{C(j\omega)G(j\omega)}} R$$

- Achieving the control objective: Before we plunge into investigation of the specific design tools, it is good to remind ourselves what we expect from the design. The goal of a control engineer is to ensure x=r. The block diagram and the associated input-output relationship on the right indicates that this is achieved when $|C(j\omega)P(j\omega)|>>1$, i.e., when loop gain is high. Thus, if the plant does not already possess sufficiently high gain, our controller should provide it. However, we learnt from Nyquist analysis that we are limited in our ability to supply arbitrarily high gain when the phase of the plant is close to -180°. Thus, our best bet to achieve high loop gain, given the plant on the right (below), is at low frequencies when the plant gain itself is maximum and the phase is close to zero. At higher frequencies, the attenuating plant gain and the specter of instability force us to be more conservative in increasing loop gain. This makes us less of *control engineers* and more of *stability engineers* at these frequencies. As stability engineers, we want to make sure that the antics we play out to achieve high gain at low frequencies do not spill over too much into the high frequency region and destabilize the overall system.
- Phase margin specification: It was evident from the discussion on phase-margins that ensuring sufficient phase-margin for the open-loop system is equivalent to ensuring sufficient damping for the closed-loop system, which is clearly a desirable trait. Thus, we decide that our open-loop systems must have a predetermined satisfactory phase margin at least of ϕ_0.
- The plant: In order to throw full spotlight on the controllers and not on the plant, we assume that our plant has no specific characteristic that demands special attention…i.e., it is very 'normal'. Its parameters are assumed to be fixed and its phase margin ϕ is greater than ϕ_0. (Fig. on the right).



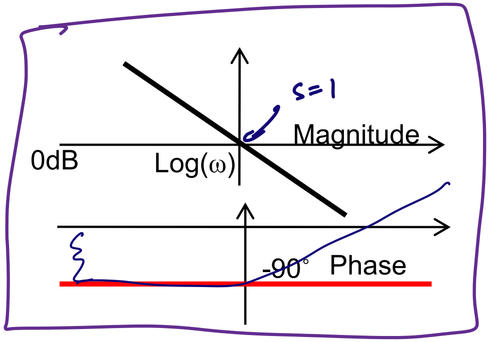
Tools for Design of the Control Systems

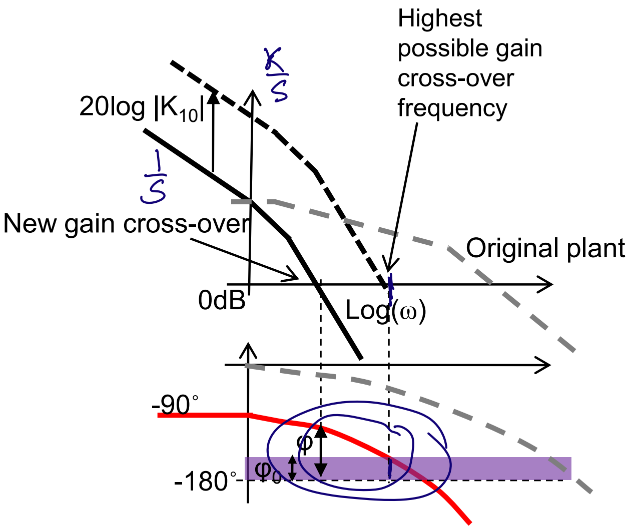


- Tool#1 The Proportional Controller: A proportional controller is a simple gain K_1. It pushes up the magnitude plot but leaves the phase plot untouched.
- It is evident from the figure on the right that the best we can do to increase the loop gain using this strategy is by the amount K_{10} for which the system phase margin ϕ at the gain-cross-over frequency is ϕ_0. Any larger gain and the system phase margin would be less than ϕ_0, thereby leading to excessive transients.
- Thus, the improvement in loop gain is rather modest over all frequencies and thus, at no frequency can the controller engineer achieve adequately high gain. For this reason, a plain proportional controller is not an attractive design choice.
- There is one case where proportional controllers can be used: if the plant is of first order, then it's phase does not dip below -90° and thus never crosses ϕ_0. For this case alone proportional controllers work very well. Of course, no plant is strictly first order. However, many systems, particularly thermal systems, possess their higher poles much farther away than the first pole and therefore resemble first order systems.

Tools for Design of the Control Systems

- Tool#2 The Integral Controller: A integral controller has a transfer function K_1/s. The figure on the right is a Bode plot of an integral controller.
- It is evident from the bode plot that the really useful range for this type of controller is at low frequencies, when its gain is large. Indeed, its gain tends to ∞ as $\omega \to 0$. Thus, we can achieve close-to-perfect control around $\omega=0$.
- On the other end, it is not at all useful at high frequencies. As seen from the figure below, it drastically reduces the gain cross-over frequency. Furthermore it adds a constant phase of -90˚ at all frequencies thereby reducing the phase cross-over frequency as well.
- If the $\phi > \phi_0$ at the new gain cross-over frequency, then we have excessive phase margin, which may be unnecessary and too conservative. Thus, the gain of integral control can be scaled up at every frequency by an amount K_{10} by using a proportional controller, such that the final phase-margin just satisfies the minimum specifications. If $\phi < \phi_0$ then the proportional controller should scale down the gain until the desired phase-margin is achieved.
- An integrator's performance can be safely declared to be adequate only around $\omega=0$. At other low frequencies, it continues to be valuable in terms of increasing the gain, but if a *specified* minimum gain needs to achieved at a particular frequency that is higher than what the integrator can supply, the demand cannot be met. This is because we have only one parameter to tune, viz., K_1, and this gets fixed by the need to maintain sufficient phase margin.

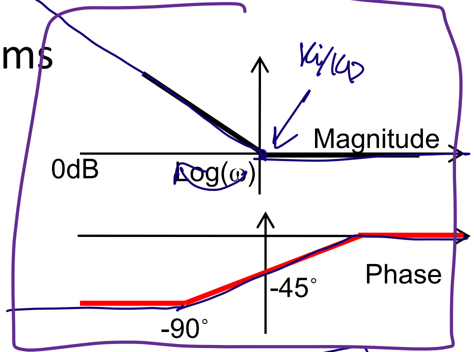




Tools for Design of the Control Systems

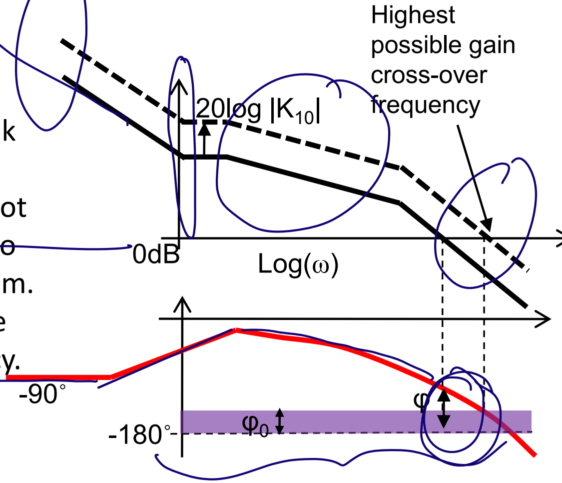
- Tool#3 The Proportional-Integral Controller: It is evident from the previous slide that the use of an integral controller at high frequencies runs against the credo of control engineers since this results in deliberate attenuation of the loop gain.
- The frequency at which this tendency begins is when the controller crosses the 0dB line (if $K_1=1$, this happens at $\omega=1$rad/s). Therefore, it would be wiser to curb the tendency beyond this frequency by adding a zero at the frequency where integral control is no longer useful for increasing the loop gain. The resulting Bode plot is shown on the right. In general it has a transfer function $C(s)=K_1\dfrac{s/\alpha+1}{s}$
- The Bode plot reveals another unintended attraction of cutting of integrator-characteristics at high frequencies. The phase of the controller returns to 0. This tends to restore the phase-cross-over frequency to that of the original plant. Indeed, since the controller also has unity gain at high frequencies, it is as though there is no specific form of control for the plant at frequencies $\omega>\alpha$.
- Having recovered the original plant's dynamics at mid-and high-frequencies, we can inspect to see if the phase margin $\phi>\phi_0$. If yes, we can crank up the gain (K_1) to a value at which the phase margin just equals ϕ_0.
- However, if $\phi<\phi_0$, we have two options: the first option is, as before, to crank down the gain until the phase margin increases to ϕ_0. However, there is a possibility that the poor phase-margin is because the controller phase was not restored to 0° early enough and the residual negative phase has "leaked" into the high frequency region and reduced the phase margin of the overall system. Thus, the second option is to shift the zero 'α' to the left so that the negative phase introduced by the controller returns to zero at a much lesser frequency.

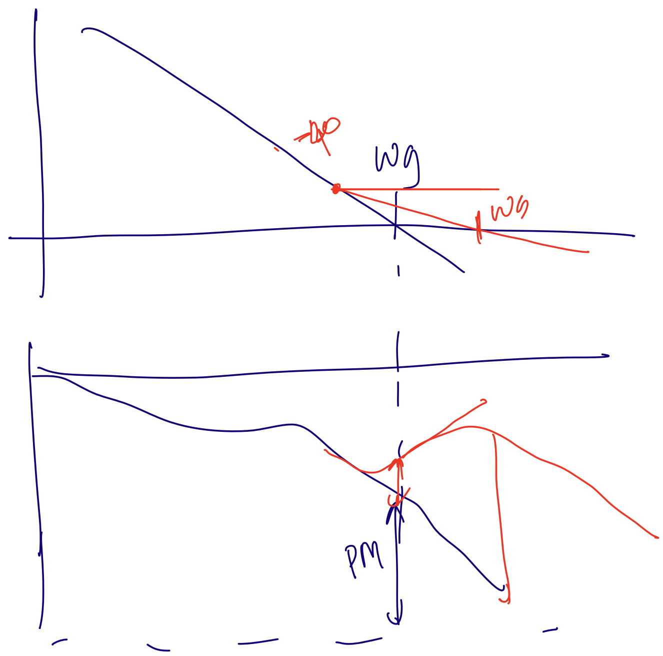
$\omega=0$



$K_p + \dfrac{ki}{S}$ PI.

$\dfrac{sK_p+ki}{s}$

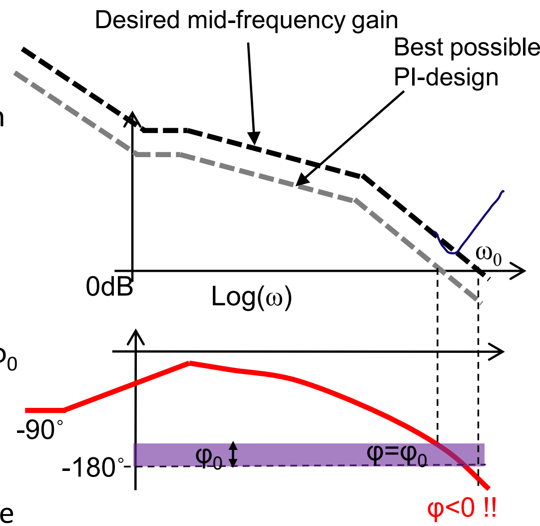




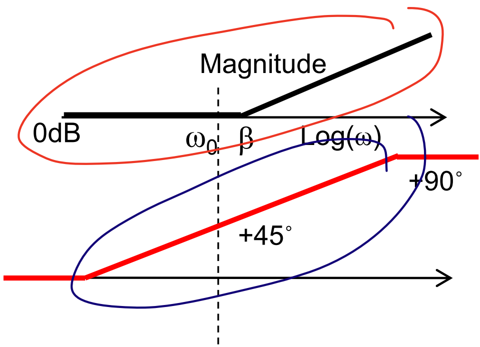
$$K_p + \frac{K_i}{s} + \frac{K_d s}{s+1}$$

$$\underbrace{\left(K_p + \frac{K_i}{s}\right)}\left(\frac{s/\alpha + 1}{s/\gamma + 1}\right)$$

Tools for Design of the Control Systems

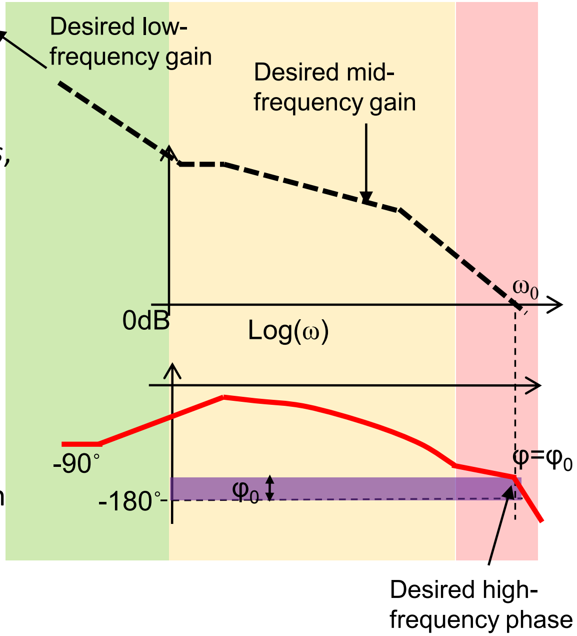


- Tool#4 The Proportional-Integral-Derivative Controller: The best loop gain that we can achieve with a PI controller is shown on the right. Suppose we desire an *even* higher gain in the mid frequency region than what is supplied by the best PI-design, our only option is to increase the proportional gain K_1 even further until the desired loop gain is achieved.
- However, in increasing the gain, we pay a heavy (and possibly fatal) price in terms of the phase margin. We see from the schematic that our phase margin (the new gain-cross-over frequency is named ω_0) has not only reduced below ϕ_0 but could have possibly become negative. This implies that for this gain, the closed-loop system is unstable.
- In this situation, we desire a controller that adds positive phase so that we can pull the phase curve back up and achieve a phase margin $\geq \phi_0$. Furthermore, we desire that the controller does not add any gain since any additional gain will push the gain cross-over frequency even more to the right and make the phase margin even lesser.
- We notice that if we add a zero at β to the controller situated slightly to the *right* of ω_0, then we can add a phase of approximately +45° at ω_0 but no gain (Fig. on the right). This is exactly what we want!



- If we include this new transfer function to the original PI-controller design, the overall transfer function looks as follows: $C(s) = K_1 \dfrac{(s/\alpha+1)(s/\beta+1)}{s}$
- If this is expanded out we see that $C(s) = \dfrac{K_1}{\alpha\beta} s + K_1\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}\right) + K_1 \dfrac{1}{s}$. Since it can be expressed as a summation of proportional, integral and derivative controllers this is called Proportional-Integral-Derivative (PID) control.
- Of course we cannot add a stand-alone zero at β because we would have a non-causal transfer function. So, we add a far-off pole at $\gamma \gg \omega_0$. Thus, in practice $C(s) = K_1 \dfrac{(s/\alpha+1)(s/\beta+1)}{s(s/\gamma+1)}$

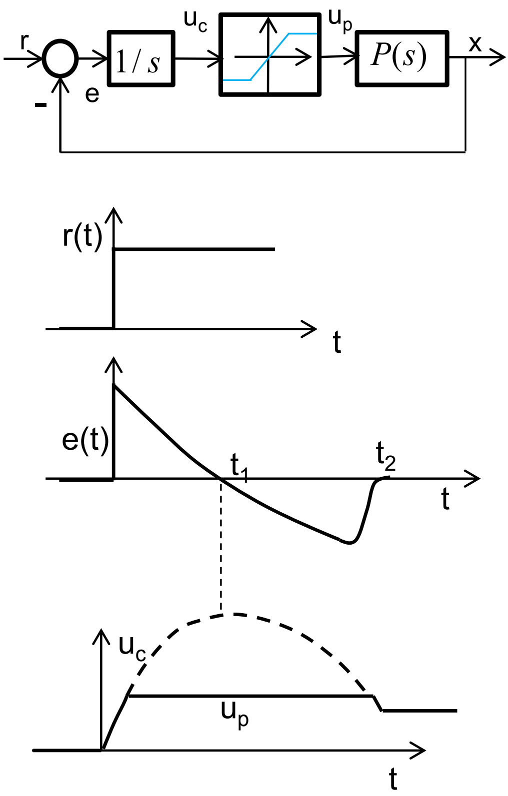
Some notes on PID Controllers



- The overall Bode Plot of the Loop Gain shaped by means of a PID controller looks as shown on the right.
- The role of each control is specific and independent of the others. The Integral-controller is meant to achieve very high gain at *very low frequencies*, thereby ensuring perfect input-tracking and/or disturbance rejection at these frequencies. Its negative effects on high-frequency range is minimized by adding a zero around the frequency at which its gain dips below 0dB. At the mid-frequencies, compromise in the achievable loop gain is inevitable and the loop gain is in general much less than that at low frequencies. Nevertheless, we try to maximize the *mid-frequency gain* to the extent possible by using Proportional-control and thereby also extend the control bandwidth. Finally comes the derivative part of the controller, which *does not* worry about the gain (its low/mid-frequency gain is 1). Its prime concern is to ensure stability. Its role is to add phase alone near the cross-over frequency and thus improve the phase margin.
- Conventionally, PID controllers are introduced as though they are designed using the Zeigler-Nichols tuning rules. However, these rules are meant for a specific type of plant model $P(s)=K_0e^{-sT}/(s+a)$, which happens to be a suitable approximation for most plants found in industries. For other plants their performance is sub-optimal.
- PID controllers are today the most prevalent type controllers. However, this is not because of the enduring relevance of Zeigler-Nichols (Z-N) tuning rules. Rather, it is because PID represents a *family* of controllers (and not a specific set of control gains according to Z-N prescription) whose general characteristics enable achieving high loop gain in every relevant part of a plant's frequency response while simultaneously addressing the stability constraint.

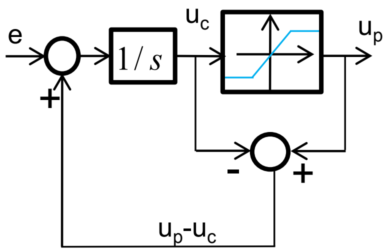
Issue with integral control: Integrator windup

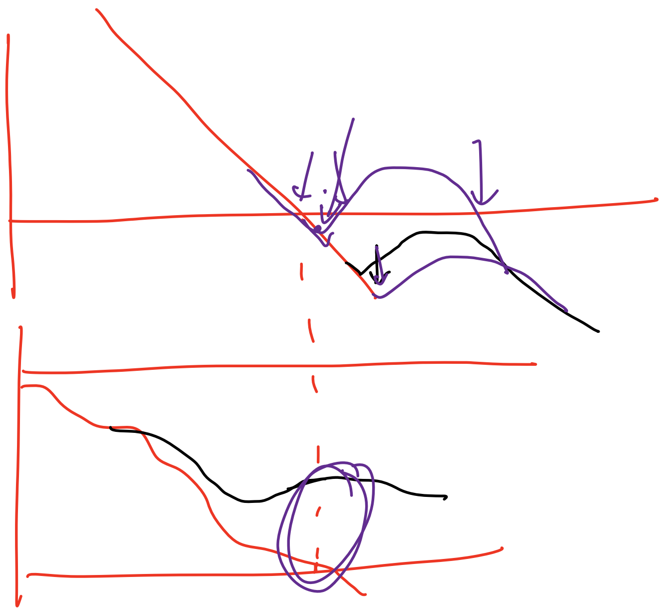
- An important practical problem associated with the integral controller is the integrator "wind up": this occurs in the presence of a nonlinear element such as saturation. Saturation is a common nonlinearity associated with the finite magnitude of control voltages or actuator ranges. The corresponding control block-diagram is shown on the right. We shall label u_c as the output of the integrator and u_p as the control effort applied to the plant. Within the linear range of the actuator/electronics, $u_c = u_p$. However, once either of them saturates, u_p =constant=u_{p0}.
- Let us suppose the reference r(t) was being tracked with negligible error to start with. However, at t=0, there is a sudden large change in the reference r(t) as shown on the right. Consequently, the error e(t) will suddenly assume a large value and cause the integrator's output u_c to increase. However, when $u_c > u_{p0}$, u_p saturates at u_{p0}. Under the action of u_{p0}, the error gradually reduces and reaches zero at t_1. However, in the mean time u_c has continued to integrate the positive error and grow. Thus, even after the error reaches zero at t_1, the output of the saturation block remains saturated at u_{p0}. This causes the error to dip below zero and keep reducing. This in turn causes the output of the integrator to reduce and ultimately re-enter the linear regime. Once it is back in the linear regime, the error is finally restored back to zero at a time t_2.
- Thus, it is seen that it takes altogether t_2 units of time to reach the set-point though in principle, it could have been reached in just t_1 units of time. This was because the integral action did not stop once the nonlinear element saturated.

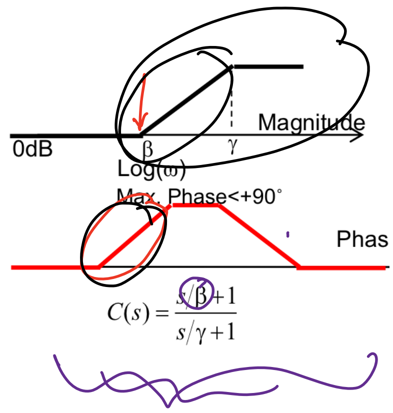


The integrator anti-wind-up circuit

- To address the problem described on the previous slide, it is necessary to stop integrating once the output of the integrator u_c exceeds the saturation u_{p0}. This can be accomplished by an "anti-windup circuit" shown on the right. The circuit is inserted in the feedback loop between the output of the actuator and the error e. It is seen that the difference $u_p – u_c$ is fed back and added to the error e. In the linear regime of operation, $u_p – u_c = 0$, and thus the circuit does nothing. Once $u_c > u_p$, a negative signal is fed to the integrator so that the output u_c is never allowed to exceed u_{p0}.
- It is worth noting that the anti-windup circuit requires the signal u_c to be accessible. If the integrator is implemented using an opamp, and the saturation voltage of the opamp is less than that of the actuator, u_c will not be available over the entire linear range of the actuator. This issue however does not exist if the integrator is implemented by a computer.







1. Try a proportional controller
   - problem is finite dc gain, hence poor tracking
2. Try integral + proportional controller
   - integral control automatically ensures dc tracking
   - PI control ensures higher PM at higher freq.
   - vary K_p, K_i gain to adjust bandwidth
3. Make sure dc gain $\underbrace{\text{is infinite}}_{\text{integral cont}}$ bandwidth requirement is met.
   - check if P.M is met;
   - if not, add a lead compensator ($\beta > \omega_g$)

$C(s) \cdot \left(\frac{s}{a}+1\right)$ ← Adding a zero to your controller

$C(s) \left[\frac{\left(\frac{s}{a}+1\right)}{\left(\frac{s}{b}+1\right)}\right]$

$C(s) \frac{\left(\frac{s}{a}+1\right)}{s}$ ← PI. controller.

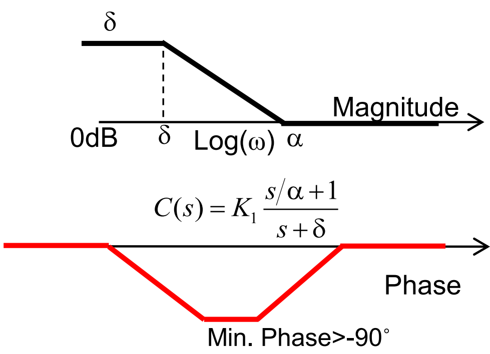
$= \frac{s/a}{s} + \frac{1}{s} = \frac{1}{a} + \frac{1}{s}$

$\boxed{k_p + \frac{k_i}{s}}$

$k_p = \frac{1}{a}$; $k_i = 1$

Tools for Design of the Control Systems

- Having briefly paused to highlight the significance of PID control, we now cover two more control schemes that derive inspiration from PID based control.
- Tool#5 Lag Compensator: While anti-wind up strategies exist, it can be altogether avoided if the output of the controller, to a constant DC input, does not build up. Thus, in order to curb the characteristics of 1/s at DC, we replace it with a very low bandwidth low-pass filter $1/(s+\delta)$. For $\omega>\delta$, this function behaves exactly as an integrator. However, for $\omega<<\delta$, it outputs a constant gain, independent of frequency. This is called a lag compensator. Its transfer function is $C(s)=K_1\dfrac{s/\alpha+1}{s+\delta}$
- The bode plot of a lag compensator is shown on the right. It is intended to mimic PI-control. Thus, $\alpha>>\delta$. However, having invented it, we see that a lag compensator can also be used for other reasons than just as replacements for PI control. In particular, we can use it to *add gain* below any specified frequency α. The amount of gain we add is determined δ (and if the freedom exists, by K_1).
- Thus, as an example of the use of a lag compensator, if we cannot achieve sufficiently high gain with an integrator at a particular low frequency $\omega(\neq 0)$, we can introduce a lag compensator whose corner α sufficiently to the right of ω so that the extra gain necessary is added. Likewise, if we do not desire perfect tracking at DC, we can altogether replace PI control by a lag compensator.
- The "lag" compensator gets its name from the fact that its phase is always negative.

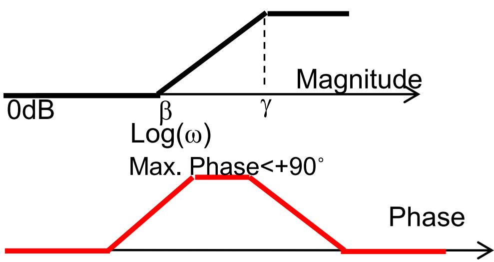


Tools for Design of the Control Systems

- Tool#6 Lead Compensator: In the case of PID control, we added a zero at β in order to add phase. To ensure causal implementation, we added a pole at γ>>β. Between frequencies β and γ, the controller behaves as a differentiator. Thus, if γ is excessively large, our controller would be rendered extremely susceptible to high frequency noise. Instead, if we include γ also as a design parameter and limit its value in order to cutoff the derivative action early enough, we get a lead compensator. The general transfer function of a lead compensator is $C(s) = \dfrac{s/\beta + 1}{s/\gamma + 1}$

- The bode plot of a lead compensator is shown on the right. The purpose of a lead compensator is exactly the same as that of its equivalent in PID controller: it is to *add phase* near the gain cross-over frequency. It is called a "lead compensator" because its phase is always positive.

- As with PID control, lead lag controllers operate at two different frequency regimes and achieve two different goals. For the most versatile design (i.e., one that achieves the maximum gain bandwidth), it is suggested that both lag as well as lead compensators be used.



$$C(s) = \frac{s/\beta + 1}{s/\gamma + 1}$$