Lecture 3 (11th January 2022)

Linear operator

input : x
output : y

$$x \longrightarrow \boxed{\mathcal{L}(\cdot)} \longrightarrow y$$

$$y = \mathcal{L}(x)$$

Q: When is '$\mathcal{L}$' a linear operator.

A: $\mathcal{L}$ is a linear operator iff it satisfies the following property :—

(a) Scaling : If input is scaled by α (α is a scalar) the o/p is also scaled by α.

$$x \longrightarrow \boxed{\mathcal{L}(\cdot)} \longrightarrow y \quad \Rightarrow \quad \mathcal{L}(x) = y$$

$$\alpha x \longrightarrow \boxed{\mathcal{L}(\cdot)} \longrightarrow \alpha y$$

$$\Rightarrow \quad \mathcal{L}(\underline{\alpha x}) = \underline{\alpha . y}$$

But we know $\mathcal{L}(x) = y$

$$\boxed{\therefore \; \mathcal{L}(\alpha x) = \alpha \mathcal{L}(x)}$$

(b) Superposition

$$x \longrightarrow \boxed{\mathcal{L}(\cdot)} \longrightarrow y$$

$$p \longrightarrow \boxed{\mathcal{L}(\cdot)} \longrightarrow q$$

$$x+p \longrightarrow \boxed{L(\cdot)} \longrightarrow y+q$$

$$L(x+p) = y+q$$

$$\because \quad y = L(x) \ \& \ q = L(p)$$

$$\boxed{\therefore \ L(x+p) = L(x) + L(p)}$$

Q: what is linear system?

A: Any system in which input : x and output : y are related by :

$L_1(x) = L_2(y)$; where L_1 & L_2 are linear operators. then we call the s/s to be linear.

Linear operators :—

1. k = gain
2. $\frac{d}{dt}, \frac{d^2}{dt^2} \ \ldots$
3. $\int (\ ) \, dt$

Linear s/s

1. $y = k \cdot x$
2. $\frac{d}{dt} y = kx$
3. $\frac{d^2}{dt^2} y + \frac{dy}{dt} + y = \frac{d}{dt} u + u$

(4) $t^2 \frac{d}{dt} y = x \leftarrow$ linear s/s.

Q Is $x \frac{dx}{dt} + y = 2 \rightarrow$ linear?

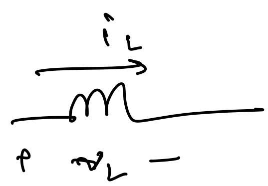
$$\underbrace{\left[\frac{d}{dt} \cdot x\right]}_{\text{linear}} + y$$

$\mathcal{L}(\cdot x)$

$$\underbrace{x \cdot \frac{d}{dt} x}_{\text{non-linear}}$$

Q $L \frac{di_L}{dt} = v_L$

linear



$$\underbrace{L(i_L) \frac{di_L}{dt}}_{\text{non-linear}} = v_L$$

Q $$\underbrace{\dot{x} = ax + bu}_{\text{linear s/s}}$$

$\dot{x} = \frac{dx}{dt}$

$\dot{\underline{x}} = A \underline{x} + B \underline{u}$ → vector } Linear s/s

vector

Matrices

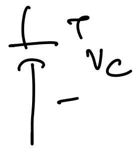
Differential Equations

State variables: Any variable that has a time derivative.

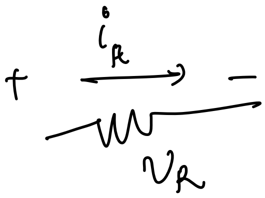
i_L, v_C are state variables

$$v_L = L\frac{di_L}{dt}$$

$$i_C = C\frac{dv_C}{dt}$$

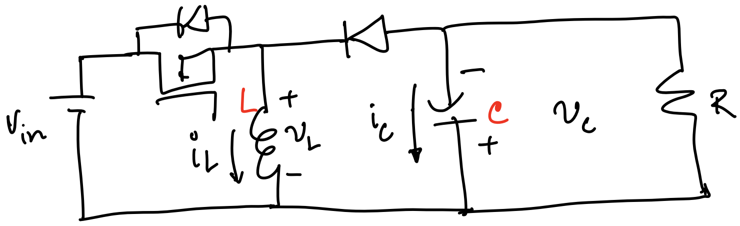


v_L, i_C are not state variables.

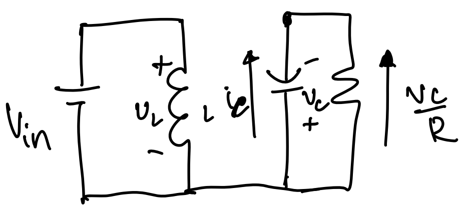


v_R, i_R are not state " .

Buck Boost Converter



Mode 1



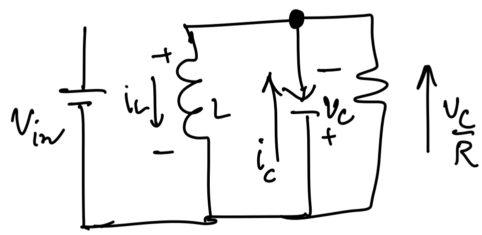
$$v_L = L\frac{di_L}{dt} = V_{in}$$

$$i_C + \frac{v_C}{R} = 0$$

$$C\frac{dv_C}{dt} + \frac{v_C}{R} = 0$$

$$\therefore \underbrace{C\frac{dv_C}{dt} = -\frac{v_C}{R}.}$$

Mode 2



$$v_L = -v_C = L\frac{di}{dt}$$

$$i_C + \frac{v_C}{R} = i_L$$

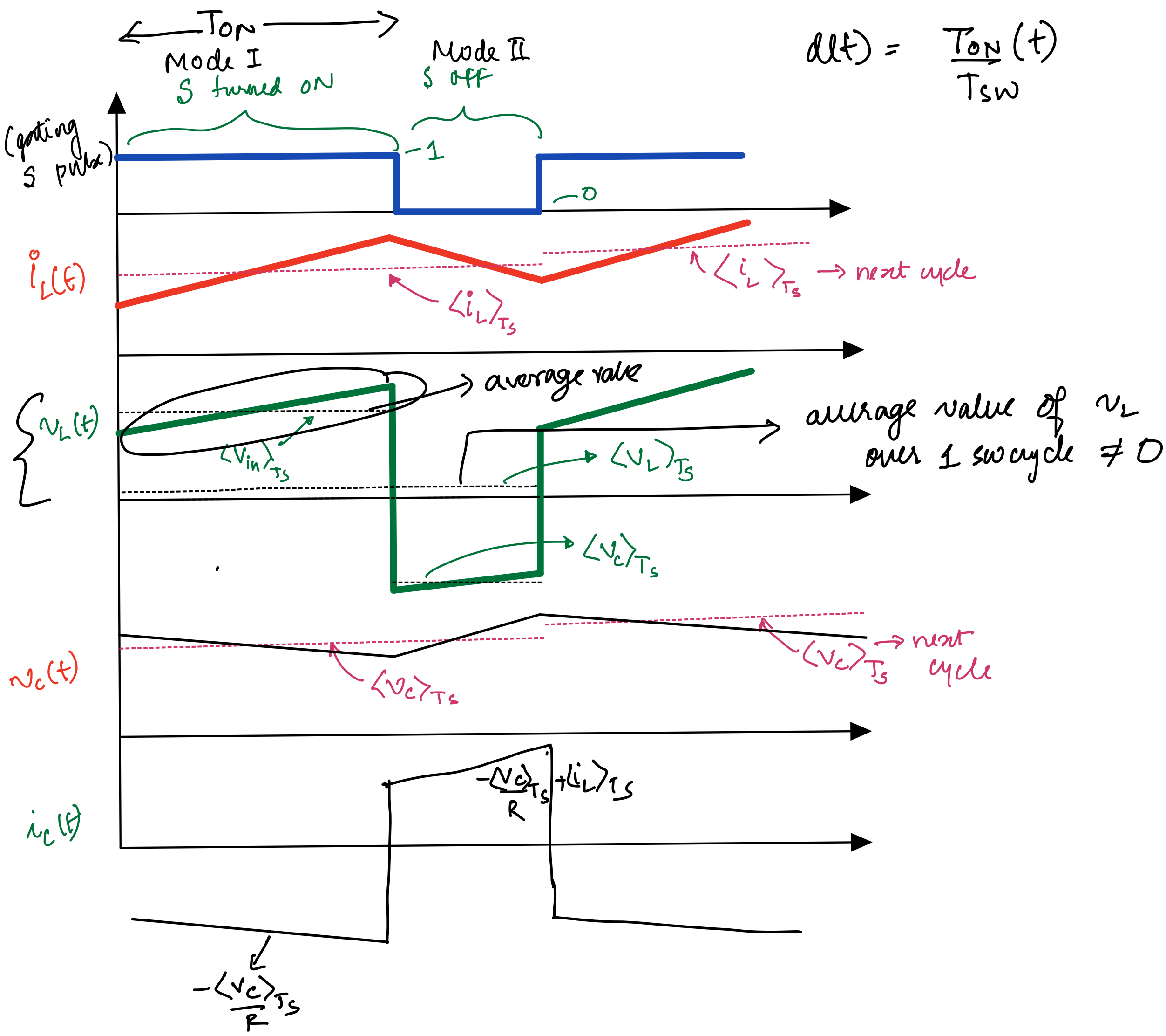
$$C\frac{dv_C}{dt} = \underbrace{-\frac{v_C}{R} + i_L}$$

EE452 : $(v_{in})D + (-v_C)D' = 0$

$$\langle x(t)\rangle_{T_s} = \int_t^{t+T_s} x(\tau)\,d\tau$$

EE452 : $\left(-\frac{v_C}{R}\right)D + \left(\overset{I_o}{-\frac{v_C}{R}} + I_L\right)D' = 0$

$$d(t) = \frac{T_{ON}(t)}{T_{sw}}$$



$$\langle v_L(t) \rangle_{T_s} = [v_L \text{ during mode 1}]\, d(t) + [v_L \text{ during mode 2}] \cdot d'(t)$$

$$= \langle v_{in}(t) \rangle_{T_s} d(t) + \langle -v_c(t) \rangle_{T_s} d'(t)$$

$$= \langle v_{in}(t) \rangle_{T_s} d(t) - \langle v_c(t) \rangle_{T_s} d'(t)$$

$$\langle i_c(t) \rangle_{T_s} = [\, i_c \text{ during mode 1}]\, d(t) + [i_c \text{ during mode 2}]\, d'(t)$$

$$= \frac{-\langle v_c(t)\rangle_{T_s}}{R} \cdot d(t) + \left(\frac{-\langle v_c(t)\rangle_{T_s}}{R} + \langle i_L(t)\rangle_{T_s} \right) d'(t)$$

$$\langle v_L(t)\rangle_{T_s} = L \frac{d\langle i_L(t)\rangle_{T_s}}{dt} = \langle v_{in}(t)\rangle_{T_s} d(t) - \langle v_c(t)\rangle_{T_s} d'(t)$$

$$\langle i_c(t)\rangle_{T_s} = C \frac{d}{dt}\langle v_c(t)\rangle_{T_s} =$$

$$\frac{-\langle v_c(t)\rangle_{T_s}}{R} \cdot d(t) + \left(\frac{-\langle v_c(t)\rangle_{T_s}}{R} + \langle i_L(t)\rangle_{T_s} \right) d'(t)$$

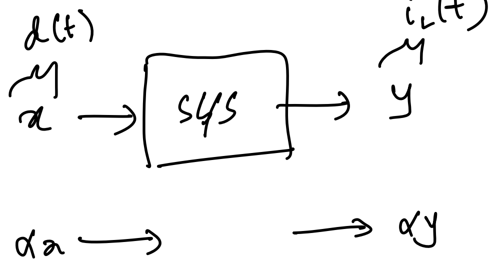
Q: Is this a linear s/s?

States :- $\underbrace{\langle i_L(t)\rangle_{T_s}, \quad \langle v_c(t)\rangle_{T_s}}_{\frac{d}{dt}x}$ } which have derivative.

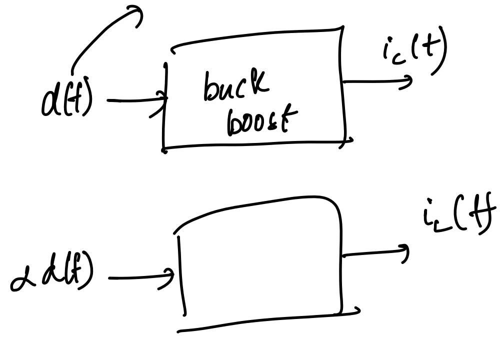
inputs : $d(t)$, $\langle v_{in}(t)\rangle_{T_s}$

things you can control.

No, its not a linear s/s.



Mode-I

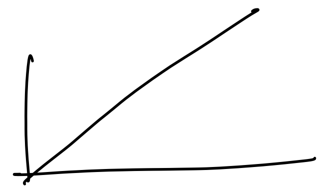


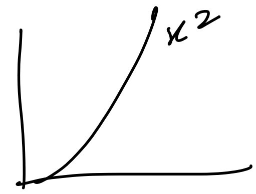
Mode II

$$\frac{dx}{dt} = y \rightarrow L$$

$$x\frac{dx}{dt} = y \rightarrow NL$$

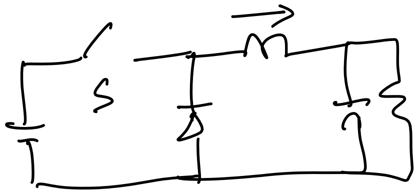
$$\frac{dx}{dt} = x \cdot y \rightarrow NL$$





D: constant

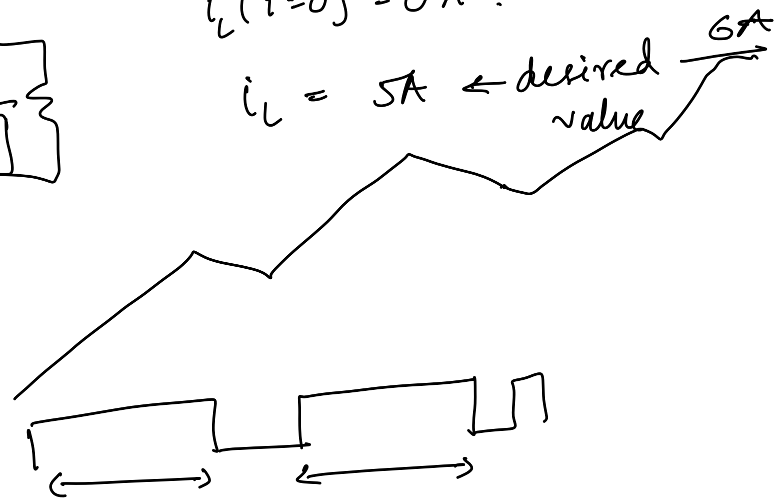
$d(t)$ = time varying signal.



$i_L(t=0) = 0\,A$.

$i_L = 5A$ ← desired value

turn ON S



$d(t)$

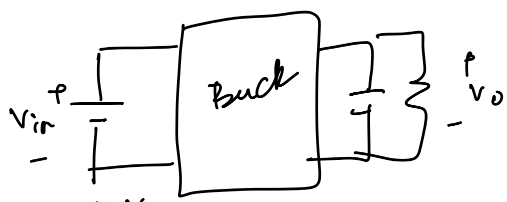
Q $\langle i_L(t)\rangle_{T_s}$ can be influenced by $d(t)$.
Can you exactly tell how?

$\langle v_c(t)\rangle_{T_s}$ . . . . $d(t)$.

. . . . ?

$v_c = \frac{V_{in}\, D}{D'}$

Buck converter



$V_o = \overset{10V}{V_{in}} \cdot D$

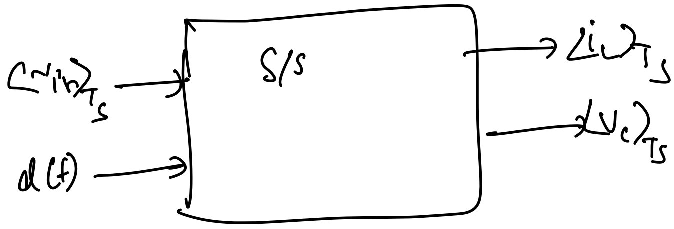
$\downarrow$

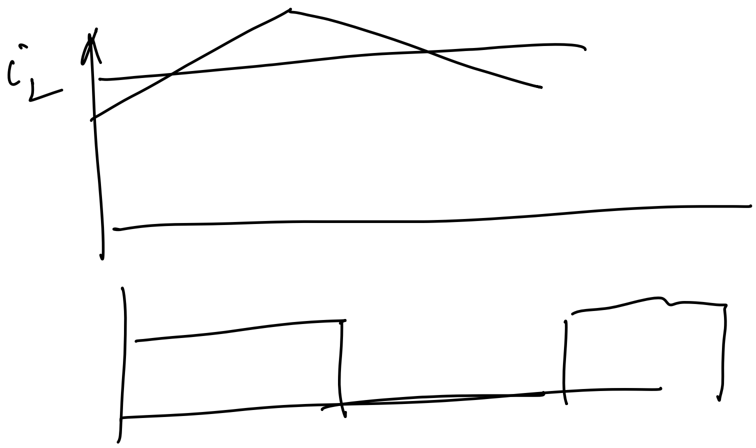
$= 7$ $\boxed{D = 0.7}$

I want a relationship between

$\langle v_c(t)\rangle_{T_s}$, $\langle i_L(t)\rangle_{T_s}$ & $d(t)$

Q $\langle v_c(t)\rangle_{T_s}$? & not $\langle d(t)\rangle_{T_s}$





$d(t)$ has no variation in 1 sw. cycle hence,

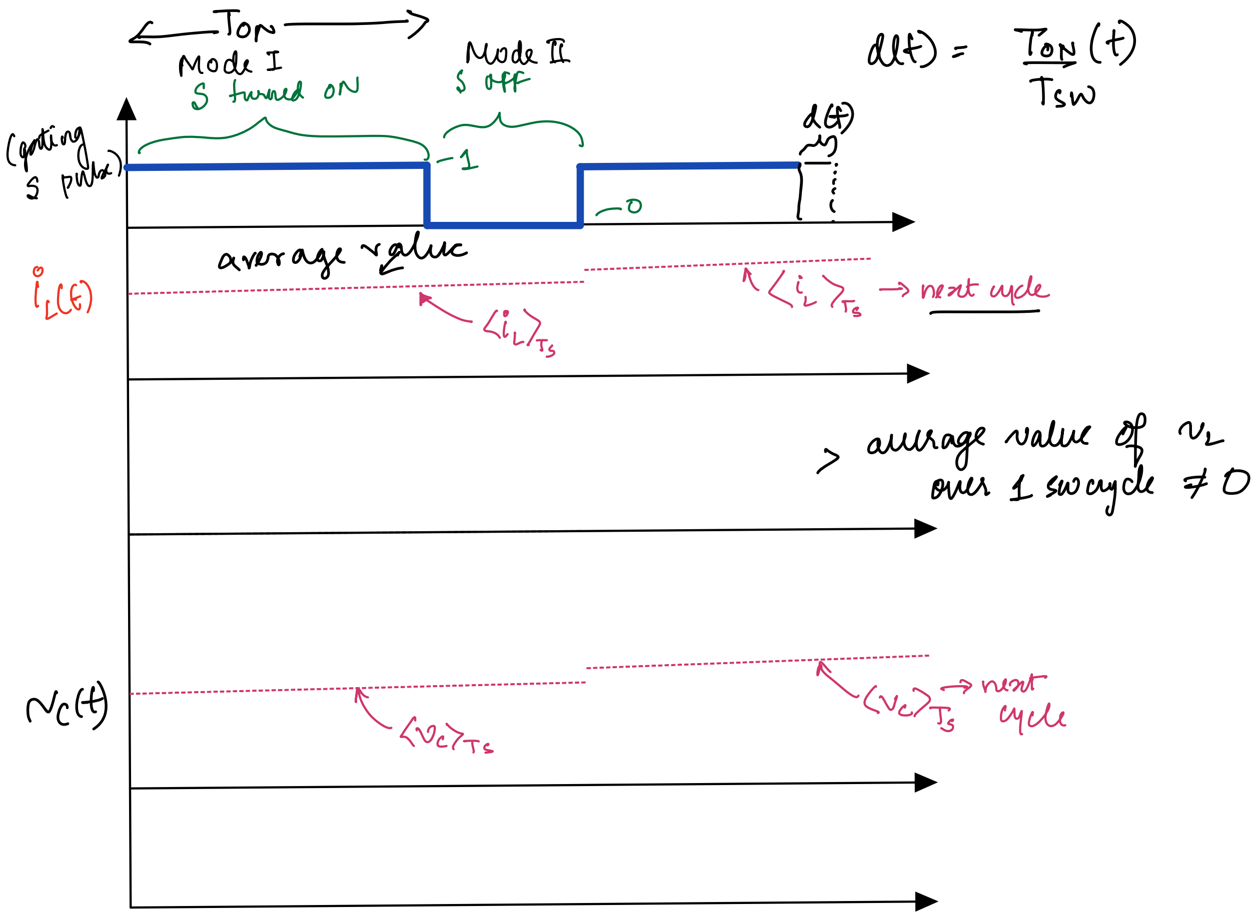
$$d(t) = \langle d(t) \rangle_{T_s}$$

D ; $d(t)$

$$\langle v_L(t)\rangle_{T_s} = L\frac{d\langle i_L(t)\rangle_{T_s}}{dt} = \langle v_{in}(t)\rangle_{T_s} d(t) - \langle v_c(t)\rangle_{T_s} d'(t)$$

$$\langle i_c(t)\rangle_{T_s} = C\frac{d\langle v_c(t)\rangle_{T_s}}{dt} =$$

$$-\frac{\langle v_c(t)\rangle_{T_s}}{R}\cdot d(t) + \left(-\frac{\langle v_c(t)\rangle_{T_s}}{R} + \langle i_L(t)\rangle_{T_s}\right) d'(t)$$



Q If I slightly change my $d(t)$, how is it going to affect my $\langle i_L(t) \rangle_{T_S}$ & $\langle v_c(t) \rangle_{T_S}$?

A To get to that,
we will consider every signal to have a steady state + perturbation.

$$x(t) = X + \tilde{x}$$

eg. Buck conv.

$$v_o = v_{in} \cdot D$$

$v_{in} = 10\,V$, $D = 0.7$, $v_o = 7V$

$D = 0.71$ $\quad v_o = 7.1\,V$

$$\underbrace{D}_{0.7} + \underbrace{\tilde{d}}_{0.01} = 0.71 \qquad v_o = \underbrace{V_o}_{7V} + \underbrace{\tilde{v}_o}_{0.1V}$$

Step I Introduce perturbation

$$\langle i_L(t) \rangle_{T_S} = I_L + \tilde{i}_L(t) \qquad \tilde{i}_L(t) \ll I_L$$

$$\langle v_c(t) \rangle_{T_S} = V_c + \tilde{v}_c(t) \qquad \tilde{v}_c(t) \ll V_c$$

$$d(t) = D + \tilde{d}(t) \qquad \tilde{d}(t) \ll D$$

$$\langle v_{in}(t) \rangle_{T_S} = \underbrace{V_{in}}_{\text{large signal (or DC)}} + \underbrace{\tilde{v}_{in}(t)}_{\text{small signal (or perturbation)}}$$

$$\langle v_L(t)\rangle_{T_s} = L\frac{d\langle i_L(t)\rangle_{T_s}}{dt} = \langle v_{in}(t)\rangle_{T_s} d(t) - \langle v_c(t)\rangle_{T_s} d'(t)$$

$$L\frac{d}{dt}\left[I_L + \tilde{i}_L(t)\right] = \left(V_{in} + \tilde{v}_{in}(t)\right)\left(D + \tilde{d}(t)\right) - \left(V_c + \tilde{v}_c(t)\right)\left(D' - \tilde{d}(t)\right)$$

$$d'(t) \Rightarrow 1 - d(t) \longrightarrow \text{By definition}$$
$$= 1 - \left(D + \tilde{d}(t)\right)$$
$$= 1 - D - \tilde{d}(t)$$
$$= D' - \tilde{d}(t)$$

from 4.52

1

$$L\frac{d}{dt}I_L + L\frac{d\tilde{i}_L(t)}{dt} = V_{in}D + V_{in}\tilde{d}(t) + D\tilde{v}_{in}(t) + \tilde{v}_{in}(t)\tilde{d}(t) - V_cD' + V_c\tilde{d}(t) - \tilde{v}_c(t)D' + \tilde{v}_c(t)\tilde{d}(t)$$

terms become very small

Collect similar terms

1. Large signal terms. / DC terms

$$L\frac{dI_L}{dt} = V_{in}D - V_cD' = 0 \Rightarrow V_{in}D = V_cD'$$

$$\boxed{\therefore V_c = \frac{V_{in}D}{D'}}$$

2. Linear small signal.

$$\boxed{L\frac{d\tilde{i}_L(t)}{dt} = V_{in}\tilde{d}(t) + D\tilde{v}_{in}(t) + V_c\tilde{d}(t) - D'\tilde{v}_c(t)}$$

2

1 — $\boxed{V_o = V_{in}\frac{D}{D'}}$

Cap. voltage

$$\langle i_c(t)\rangle_{T_s} = C\frac{d}{dt}\langle v_c(t)\rangle_{T_s} = \underbrace{-\frac{\langle v_c(t)\rangle_{T_s}}{R}\cdot d(t)} + \overbrace{\left(-\frac{\langle v_c(t)\rangle_{T_s}}{R} + \langle i_L(t)\rangle_{T_s}\right)d'(t)}$$

$$C\frac{d}{dt}\langle v_c(t)\rangle_{T_s} = -\frac{\langle v_c(t)\rangle_{T_s}}{R}\underbrace{\left(d(t)+d'(t)\right)}_{=1} + \langle i_L(t)\rangle_{T_s}\,d'(t)$$

$$= -\frac{\langle v_c(t)\rangle_{T_s}}{R} + \langle i_L(t)\rangle_{T_s}\,d'(t)$$

$$\therefore\ C\frac{d}{dt}\left(V_c+\tilde{v}_c(t)\right) = -\left(\frac{V_c+\tilde{v}_c(t)}{R}\right) + \left(I_L+\tilde{i}_L(t)\right)\left(D'-\tilde{d}(t)\right)$$

① DC / Large signal term.

$$C\frac{dV_c}{dt} = -\frac{V_c}{R} + I_L D' = 0 \Rightarrow I_L = +\frac{V_c}{RD'}$$

$$\boxed{I_L = \frac{V_o}{RD'} = \frac{I_o}{D'}} \quad -③$$

② Small signal linear term.

$$C\frac{d}{dt}\tilde{v}_c(t) = -\frac{\tilde{v}_c(t)}{R} + D'\tilde{i}_L(t) - I_L\tilde{d}(t)$$

④

$$① \rightarrow V_o = V_c = \frac{V_{in}D}{D'} \qquad ③ \Rightarrow I_L = \frac{I_o}{D'}$$

$$L\frac{d\,\tilde{i}_L(t)}{dt} = V_{in}\tilde{d}(t) + D\tilde{v}_{in}(t) + V_c\tilde{d}(t) - D'\tilde{v}_c(t)$$

$$C\frac{d}{dt}\tilde{v}_c(t) = -\frac{\tilde{v}_c(t)}{R} + D'\tilde{i}_L(t) - I_L\tilde{d}(t)$$

Q: How $\tilde{d}, \tilde{v}_{in}$ affects $\tilde{i}_L$, $\tilde{v}_c$

$\dot{x} = f(x,t) \longrightarrow$ Solve PDE/ODE $\longrightarrow x(t) = g(x,t)$

$\downarrow \mathcal{L}$

$\boxed{X(s) = F(x)} \xrightarrow{\mathcal{L}^{-1}} \boxed{x(t) = g(x,t)} \longrightarrow$ is what we want

$$\frac{\tilde{i}_L(t)}{\tilde{d}(t)} = \boxed{?}$$

$u(t) \xrightarrow{\mathcal{L}} \frac{1}{s}$

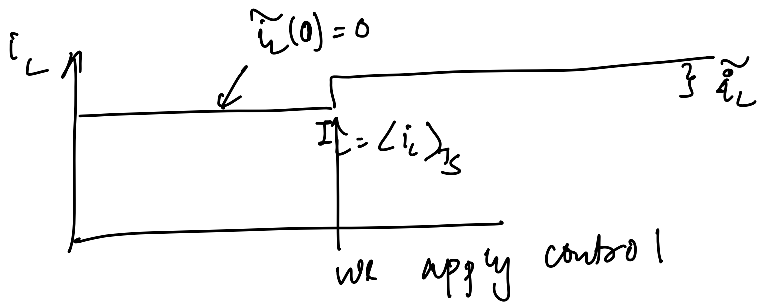
$\frac{d}{dt}(x(t)) \xrightarrow{\mathcal{L}} sX(s) - x(0)$ initial value.

$\int_0^t x(\tau)d\tau \xrightarrow{\mathcal{L}} \frac{X(s)}{s}$

$x(t) \xrightarrow{\mathcal{L}} X(s)$ cap letter

$$L\frac{d\,\tilde{i}_L(t)}{dt} = V_{in}\tilde{d}(t) + D\tilde{v}_{in}(t) + V_c\tilde{d}(t) - D'\tilde{v}_c(t)$$

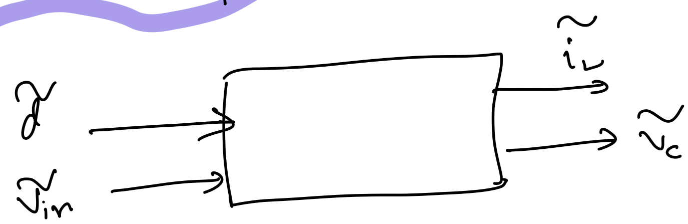
$$L\left[s\tilde{I}_L(s) - \underbrace{\tilde{i}_L(0)}_{=?}\right] = V_{in}\tilde{d}(s) + D\tilde{V}_{in}(s) + V_c\tilde{d}(s) - D'\tilde{V}_c(s)$$



$$sL\tilde{i}_L(s) = V_{in}\tilde{d}(s) + D\tilde{v}_{in}(s) + V_c\tilde{d}(s) - D'\tilde{v}_c(s) \quad (5)$$

$$C\frac{d}{dt}\tilde{v}_c(t) = -\frac{\tilde{v}_c(t)}{R} + D'\tilde{i}_L(t) - I_L\tilde{d}(t)$$

$$sC\tilde{v}_c(s) = -\frac{\tilde{v}_c(s)}{R} + D'\tilde{i}_L(s) - I_L\tilde{d}(s) \quad (6)$$



① I want to find $\frac{\tilde{i}_L}{\tilde{d}}, \frac{\tilde{v}_c}{\tilde{d}}$.

$\underline{\tilde{v}_{in}=0}$

$$sL\tilde{i}_L(s) = V_{in}\tilde{d}(s) + V_c\tilde{d}(s) - D'\tilde{v}_c(s)$$

$$\left(sC+\frac{1}{R}\right)\tilde{v}_c(s) = D'\tilde{i}_L(s) - I_L\tilde{d}(s)$$

② Let's find $\frac{\tilde{i}_L(s)}{\tilde{d}(s)}$

remove $\tilde{v}_c(s)$ $\quad \tilde{v}_c(s) = \dfrac{D'\tilde{i}_L(s) - I_L\tilde{d}(s)}{\left(sC+\frac{1}{R}\right)}$

$$sL\,\tilde{i}_L(s) = (V_{in}+V_c)\,\tilde{d}(s) - D'\cdot\frac{D'\,\tilde{i}_L(s) - I_L\,\tilde{d}(s)}{\left(sC+\frac{1}{R}\right)}$$

or, $$sL\,\tilde{i}_L(s) = (V_{in}+V_c)\,\tilde{d}(s) - \frac{D'^2\,\tilde{i}_L(s)}{\left(sC+\frac{1}{R}\right)} + \frac{D' I_L\,\tilde{d}(s)}{sC+\frac{1}{R}}$$

or, $$\overbrace{\left(\overset{\Omega}{sL} + \frac{\overset{\text{unitless}}{D'^2}}{\underset{\mho/S}{sC+\frac{1}{R}}}\right)\overset{A}{\tilde{i}_L(s)}}^{V.} = \underbrace{\left[\overset{V}{(V_{in}+V_c)} + \frac{\overset{\text{unitless}}{D'}\,\overset{A}{I_L}}{\boxed{sC+\frac{1}{R}}}\right]}_{V}\cdot\overset{\text{unitless}}{\tilde{d}(s)}$$

siemens / ℧ → V

$$\therefore\quad \frac{\tilde{i}_L(s)}{\tilde{d}(s)} = \frac{\left[(V_{in}+V_c) + \frac{D' I_L}{sC+\frac{1}{R}}\right]\cdot\left(sC+\frac{1}{R}\right)}{\left(s^2LC + \frac{sL}{R} + D'^2\right)}$$

$$\boxed{\frac{\tilde{i}_L(s)}{\tilde{d}(s)} = \frac{(V_{in}+V_c)\left(sC+\frac{1}{R}\right) + D' I_L}{s^2LC + \frac{sL}{R} + D'^2}}$$

Similarly you could find

$$\frac{\tilde{v}_c(s)}{\tilde{d}(s)} = \ldots$$

① Two other alternative methods to do the same derivation

② Why did we do this at all?

→ Basis of closed loop control, bode plot & controller design.