FRACTIONAL CHROMATIC POLYNOMIALS

Paul Cusson, Ke Han Xiao and Cunyan Zhao, under the supervision of Prof. D. Jakobson.

Presented by Ke Han Xiao

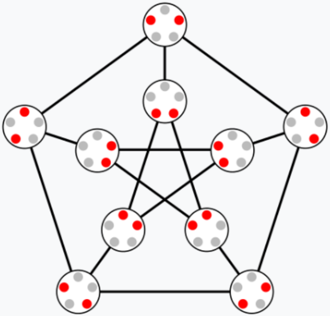
- **Introduction** We study fractional colourings of finite graphs, and discuss their asymptotic behaviour.
  Let $a \geq b > 0$ be positive integers, and let G be a finite graph.
- **Fractional coloring** A (fractional) $(a : b)$ vertex colouring of G is an assignment to each vertex of a set of b colours, out of available set of a colours, so that adjacent vertices are assigned disjoint sets of colours (they have no colours in common).
- An $(a : b)$ colouring of G is a homomorphism from G to the **Kneser graph** $KG_{a,b}$: its vertices are b-element subsets of a set of a elements, and the two vertices are adjacent **iff** the corresponding sets are disjoint.
  The usual colouring of G corresponds to taking $b = 1$ (every vertex is assigned a single colour).

- **Introduction** We study fractional colourings of finite graphs, and discuss their asymptotic behaviour.
  Let $a \geq b > 0$ be positive integers, and let G be a finite graph.
- **Fractional coloring** A (fractional) $(a : b)$ vertex colouring of G is an assignment to each vertex of a set of b colours, out of available set of a colours, so that adjacent vertices are assigned disjoint sets of colours (they have no colours in common).
- An $(a : b)$ colouring of G is a homomorphism from G to the **Kneser graph** $KG_{a,b}$: its vertices are b-element subsets of a set of a elements, and the two vertices are adjacent **iff** the corresponding sets are disjoint.
  The usual colouring of G corresponds to taking $b = 1$ (every vertex is assigned a single colour).

- **Introduction** We study fractional colourings of finite graphs, and discuss their asymptotic behaviour.
  Let $a \geq b > 0$ be positive integers, and let G be a finite graph.
- **Fractional coloring** A (fractional) $(a : b)$ vertex colouring of G is an assignment to each vertex of a set of b colours, out of available set of a colours, so that adjacent vertices are assigned disjoint sets of colours (they have no colours in common).
- An $(a : b)$ colouring of G is a homomorphism from G to the **Kneser graph** $KG_{a,b}$: its vertices are b-element subsets of a set of a elements, and the two vertices are adjacent **iff** the corresponding sets are disjoint.
  The usual colouring of G corresponds to taking $b = 1$ (every vertex is assigned a single colour).

- **Kneser graph**: Kneser graph $KG_{a,b}$ is the graph whose vertices correspond to the b-element subsets of a set of a elements, and where two vertices are adjacent **if and only if** the two corresponding sets are disjoint.
- **Graph homomorphism**: a graph homomorphism is a mapping between two graphs that respects their structure. More concretely, it is a function between the vertex sets of two graphs that maps adjacent vertices to adjacent vertices.

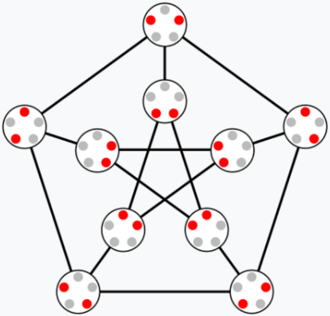
Kneser graph



The Kneser graph $K(5, 2)$,

- **Kneser graph**: Kneser graph $KG_{a,b}$ is the graph whose vertices correspond to the b-element subsets of a set of a elements, and where two vertices are adjacent **if and only if** the two corresponding sets are disjoint.
- **Graph homomorphism**: a graph homomorphism is a mapping between two graphs that respects their structure. More concretely, it is a function between the vertex sets of two graphs that maps adjacent vertices to adjacent vertices.

Kneser graph



The Kneser graph $K(5, 2)$,

- We denote by $P_G(a, b)$ the number of $(a : b)$ vertex colourings of G.
- **Proposition:** Let b be fixed, then $P_G(a, b)$ is a polynomial in a.
- **Proof:** Let each vertex u of G is replaced by a complete graph $K_b(u)$, for a fixed b we have

$$P_G(a, b) = P_{G(K_b)}(a),$$

where $P_H(a)$ denotes the chromatic polynomial for the graph H, and the result follows.

- We denote by $P_G(a, b)$ the number of $(a : b)$ vertex colourings of G.
- **Proposition:** Let b be fixed, then $P_G(a, b)$ is a polynomial in a.
- **Proof:** Let each vertex u of G is replaced by a complete graph $K_b(u)$, for a fixed b we have

$$P_G(a, b) = P_{G(K_b)}(a),$$

where $P_H(a)$ denotes the chromatic polynomial for the graph H, and the result follows.

- We denote by $P_G(a, b)$ the number of $(a : b)$ vertex colourings of G.
- **Proposition:** Let b be fixed, then $P_G(a, b)$ is a polynomial in a.
- **Proof:** Let each vertex u of G is replaced by a complete graph $K_b(u)$, for a fixed b we have

  $$P_G(a, b) = P_{G(K_b)}(a),$$

  where $P_H(a)$ denotes the chromatic polynomial for the graph H, and the result follows.

- We next consider the rate of growth of $P_G(a, b)$. It is easy to show that

$$P_G(a + c, b + d) \geq P_G(a, b)P_G(c, d).$$

- It follows that :

$$\ln P_G((k + l)a, (k + l)b) \geq \ln P_G(ka, kb) + \ln P_G(la, lb)$$

Thus,the function $\ln P_G(ka, kb)$ is a superadditive function of k.

- We next consider the rate of growth of $P_G(a, b)$. It is easy to show that

$$P_G(a + c, b + d) \geq P_G(a, b) P_G(c, d).$$

- It follows that :

$$\ln P_G((k + l)a, (k + l)b) \geq \ln P_G(ka, kb) + \ln P_G(la, lb)$$

Thus,the function $\ln P_G(ka, kb)$ is a superadditive function of k.

- By Fekete Lemma, there exists the following limit, which we call ***fractional colouring growth rate* function** or simply ***colouring rate* function**, and which we denote by $\mathcal{CR}_G(x)$, where $x = a/b$:

$$\mathcal{CR}_G(x) := \lim_{k\to\infty} \frac{\ln P_G(ka, kb)}{kb|G|} = \sup_k \frac{\ln P_G(ka, kb)}{kb|G|}.$$

  That function is well-defined, since the limit only depends on the ratio a/b. Until now, $\mathcal{CR}_G(x)$ is only defined for a dense set of $x \in \mathbb{Q}$, and for $x \geq \chi_f(G)$.

- **Remark1** Let $G = G_1 + G_2$ be a disjoint union of two graphs, and let $a/b \geq \chi_f(G_i), i = 1, 2$. Then clearly $P_G(a, b) = P_{G_1}(a, b) \cdot P_{G_2}(a, b)$, so

$$\mathcal{CR}_{G_1+G_2}(x) = \mathcal{CR}_{G_1}(x) + \mathcal{CR}_{G_2}(x).$$

  on the common interval of definition.
- **Remark2** Let G be a subgraph of H. Then $\chi_f(G) \leq \chi_f(H)$, and

$$\mathcal{CR}_G(x) \geq \mathcal{CR}_H(x)$$

  on the common interval of definition.

- **Remark1** Let $G = G_1 + G_2$ be a disjoint union of two graphs, and let $a/b \geq \chi_f(G_i), i = 1, 2$. Then clearly $P_G(a, b) = P_{G_1}(a, b) \cdot P_{G_2}(a, b)$, so

$$\mathcal{CR}_{G_1+G_2}(x) = \mathcal{CR}_{G_1}(x) + \mathcal{CR}_{G_2}(x).$$

  on the common interval of definition.
- **Remark2** Let G be a subgraph of H. Then $\chi_f(G) \leq \chi_f(H)$, and

$$\mathcal{CR}_G(x) \geq \mathcal{CR}_H(x)$$

  on the common interval of definition.

- **Proposition:** The function $\mathcal{CR}_G(x)$ is non-decreasing in x.
- **Proof:** Let $\chi_f < x < y < \infty$ be such that $x = \frac{a_1}{b_1} < \frac{a_2}{b_2} = y$. As $kb_1 a_2 > ka_1 b_2$, it is clear by definition of $P(a, b)$ that $P(kb_1 a_2, kb_1 b_2) \geq P(ka_1 b_2, kb_1 b_2)$
- Thus it follows

$$\frac{\ln P(kb_1 a_2, kb_1 b_2)}{kb_1 b_2 |G|} \geq \frac{\ln P(ka_1 b_2, kb_1 b_2)}{kb_1 b_2 |G|}.$$

- **Proposition:** The function $\mathcal{CR}_G(x)$ is non-decreasing in x.
- **Proof:** Let $\chi_f < x < y < \infty$ be such that $x = \frac{a_1}{b_1} < \frac{a_2}{b_2} = y$. As $kb_1 a_2 > ka_1 b_2$, it is clear by definition of $P(a, b)$ that $P(kb_1 a_2, kb_1 b_2) \geq P(ka_1 b_2, kb_1 b_2)$
- Thus it follows

$$\frac{\ln P(kb_1 a_2, kb_1 b_2)}{kb_1 b_2 |G|} \geq \frac{\ln P(ka_1 b_2, kb_1 b_2)}{kb_1 b_2 |G|}.$$

- **Proposition:** The function $\mathcal{CR}_G(x)$ is non-decreasing in x.
- **Proof:** Let $\chi_f < x < y < \infty$ be such that $x = \frac{a_1}{b_1} < \frac{a_2}{b_2} = y$. As $kb_1 a_2 > ka_1 b_2$, it is clear by definition of $P(a, b)$ that $P(kb_1 a_2, kb_1 b_2) \geq P(ka_1 b_2, kb_1 b_2)$
- Thus it follows

$$\frac{\ln P(kb_1 a_2, kb_1 b_2)}{kb_1 b_2 |G|} \geq \frac{\ln P(ka_1 b_2, kb_1 b_2)}{kb_1 b_2 |G|}.$$

- We next let $k \to \infty$. By definition of the function g, we conclude that
$$\mathcal{CR}_G(y) \geq \mathcal{CR}_G(x).$$
- Finally, We extend $\mathcal{CR}_G(x)$ to a function on the interval $I_G = (\chi_f(G), +\infty)$ as follows:
$$\mathcal{CR}_G(x) := \sup_{y \in I_G : y \in \mathbb{Q}, y \leq x} \mathcal{CR}_G(y).$$

- We next let $k \to \infty$. By definition of the function g, we conclude that
$$\mathcal{CR}_G(y) \geq \mathcal{CR}_G(x).$$
- Finally, We extend $\mathcal{CR}_G(x)$ to a function on the interval $I_G = (\chi_f(G), +\infty)$ as follows:
$$\mathcal{CR}_G(x) := \sup_{y \in I_G : y \in \mathbb{Q}, y \leq x} \mathcal{CR}_G(y).$$

- **Tree:**

$$\mathcal{CR}_T(x) = \frac{x}{n} \cdot \ln x + \frac{(n-2)(x-1)}{n} \cdot \ln(x-1) - \frac{(n-1)(x-2)}{n} \cdot \ln(x-2).$$

- **Complement of a complete graph:**

$$\mathcal{CR}_{\overline{\mathrm{K_n}}}(x) = x \ln x + (1-x) \ln(x-1).$$

- **Complete graph:**

$$\mathcal{CR}_{K_n}(x) = \frac{x}{n} \cdot \ln x - \frac{x-n}{n} \cdot \ln(x-n).$$

- **Tree:**

$$\mathcal{CR}_T(x) = \frac{x}{n} \cdot \ln x + \frac{(n-2)(x-1)}{n} \cdot \ln(x-1) - \frac{(n-1)(x-2)}{n} \cdot \ln(x-2).$$

- **Complement of a complete graph:**

$$\mathcal{CR}_{\overline{\mathrm{K_n}}}(x) = x \ln x + (1-x)\ln(x-1).$$

- **Complete graph:**

$$\mathcal{CR}_{K_n}(x) = \frac{x}{n} \cdot \ln x - \frac{x-n}{n} \cdot \ln(x-n).$$

- **Tree:**

$$\mathcal{CR}_T(x) = \frac{x}{n} \cdot \ln x + \frac{(n-2)(x-1)}{n} \cdot \ln(x-1) - \frac{(n-1)(x-2)}{n} \cdot \ln(x-2).$$

- **Complement of a complete graph:**

$$\mathcal{CR}_{\overline{\mathrm{K_n}}}(x) = x \ln x + (1-x) \ln(x-1).$$

- **Complete graph:**

$$\mathcal{CR}_{K_n}(x) = \frac{x}{n} \cdot \ln x - \frac{x-n}{n} \cdot \ln(x-n).$$

- **Properties of colouring rate function**
  By our previous definition, it suffices to consider $\mathcal{CR}_G$ restricted to $\mathbb{Q}$. We know that $P_{G(K_{kb})}$ is a polynomial of degree nkb, where n is the number of vertices of G, with the highest coefficient equal to 1. To study the asymptotics of $\mathcal{CR}_G(x)$, we shall use the following result of Sokal,
- **proposition**
  Let G be a graph of maximal degree D. Then all roots ρ of the chromatic polynomial $P_G(x)$ satisfy the inequality

$$|\rho| \leq KD,$$

  where K is a constant strictly smaller than 8.

- **Properties of colouring rate function**
  By our previous definition, it suffices to consider $\mathcal{CR}_G$ restricted to $\mathbb{Q}$. We know that $P_{G(K_{kb})}$ is a polynomial of degree nkb, where n is the number of vertices of G, with the highest coefficient equal to 1. To study the asymptotics of $\mathcal{CR}_G(x)$, we shall use the following result of Sokal,
- **proposition**
  Let G be a graph of maximal degree D. Then all roots ρ of the chromatic polynomial $P_G(x)$ satisfy the inequality

$$|\rho| \leq KD,$$

where K is a constant strictly smaller than 8.

- **Properties of colouring rate function(2)**
  Let d be the maximal degree of G. Consider two rational numbers $a/b < c/b$, where

$$a/b > M \cdot d,$$

  where M is a large constant to be chosen later on.
- By definition of $\mathcal{CR}_G$, it suffices to estimate the ratio

$$\frac{\ln P_G(kc, kb) - \ln P_G(ka, kb)}{kbn} = \frac{\ln\left(\frac{P_G(kc,kb)}{P_G(ka,kb)}\right)}{kbn} = \frac{\ln\left(\frac{P_{G(K_{kb})}(kc)}{P_{G(K_{kb})}(ka)}\right)}{kbn}.$$

- Let $\rho_1, \ldots, \rho_{kbn}$ be the roots of $P_{G(K_{kb})}(x)$. Now,

$$\frac{P_{G(K_{kb})}(kc)}{P_{G(K_{kb})}(ka)} = \frac{\prod_{j=1}^{kbn}(kc - \rho_j)}{\prod_{j=1}^{kbn}(ka - \rho_j)} = \left(\frac{c}{a}\right)^{kbn} \frac{\prod_{j=1}^{kbn}(1 - \frac{\rho_j}{kc})}{\prod_{j=1}^{kbn}(1 - \frac{\rho_j}{ka})}$$

- **Properties of colouring rate function(2)**
  Let d be the maximal degree of G. Consider two rational numbers $a/b < c/b$, where

$$a/b > M \cdot d,$$

  where M is a large constant to be chosen later on.
- By definition of $\mathcal{CR}_G$, it suffices to estimate the ratio

$$\frac{\ln P_G(kc, kb) - \ln P_G(ka, kb)}{kbn} = \frac{\ln\left(\frac{P_G(kc,kb)}{P_G(ka,kb)}\right)}{kbn} = \frac{\ln\left(\frac{P_{G(K_{kb})}(kc)}{P_{G(K_{kb})}(ka)}\right)}{kbn}.$$

- Let $\rho_1, \ldots, \rho_{kbn}$ be the roots of $P_{G(K_{kb})}(x)$. Now,

$$\frac{P_{G(K_{kb})}(kc)}{P_{G(K_{kb})}(ka)} = \frac{\prod_{j=1}^{kbn}(kc - \rho_j)}{\prod_{j=1}^{kbn}(ka - \rho_j)} = \left(\frac{c}{a}\right)^{kbn} \frac{\prod_{j=1}^{kbn}(1 - \frac{\rho_j}{kc})}{\prod_{j=1}^{kbn}(1 - \frac{\rho_j}{ka})}$$

- **Properties of colouring rate function(2)**
  Let d be the maximal degree of G. Consider two rational numbers $a/b < c/b$, where

$$a/b > M \cdot d,$$

  where M is a large constant to be chosen later on.
- By definition of $\mathcal{CR}_G$, it suffices to estimate the ratio

$$\frac{\ln P_G(kc, kb) - \ln P_G(ka, kb)}{kbn} = \frac{\ln\left(\frac{P_G(kc,kb)}{P_G(ka,kb)}\right)}{kbn} = \frac{\ln\left(\frac{P_{G(K_{kb})}(kc)}{P_{G(K_{kb})}(ka)}\right)}{kbn}.$$

- Let $\rho_1, \ldots, \rho_{kbn}$ be the roots of $P_{G(K_{kb})}(x)$. Now,

$$\frac{P_{G(K_{kb})}(kc)}{P_{G(K_{kb})}(ka)} = \frac{\prod_{j=1}^{kbn}(kc - \rho_j)}{\prod_{j=1}^{kbn}(ka - \rho_j)} = \left(\frac{c}{a}\right)^{kbn} \frac{\prod_{j=1}^{kbn}(1 - \frac{\rho_j}{kc})}{\prod_{j=1}^{kbn}(1 - \frac{\rho_j}{ka})}$$

- By previous Proposition, we have $|\rho_j| < 8kbd$ for all j, since the maximal degree of $G(K_{kb})$ is equal to kbd.

$$|\rho_j/kc| < |\rho_j/ka| < \frac{8kbd}{Mkbd} = \frac{8}{M}.$$

Let $\epsilon > 0$; we shall choose M large enough so that $8/M < \epsilon$. It follows that

$$\ln\left(\frac{1-\epsilon}{1+\epsilon}\right) \le \left(\frac{\ln\left(\frac{P_G(kc,kb)}{P_G(ka,kb)}\right)}{kbn} - \ln(c/a)\right) \le \ln\left(\frac{1+\epsilon}{1-\epsilon}\right)$$

- **Proposition** We pass to the limit $k \to \infty$, since ϵ was arbitrary, the previous estimate implies:
As $x, y \to \infty$, we have

$$\mathcal{CR}_G(y) - \mathcal{CR}_G(x) \asymp \ln y - \ln x,$$

or, equivalently, $\mathcal{CR}_G(x)$ grows at the rate of $\ln x$.

- By previous Proposition, we have $|\rho_j| < 8kbd$ for all j, since the maximal degree of $G(K_{kb})$ is equal to kbd.

$$|\rho_j/kc| < |\rho_j/ka| < \frac{8kbd}{Mkbd} = \frac{8}{M}.$$

Let $\epsilon > 0$; we shall choose M large enough so that $8/M < \epsilon$. It follows that

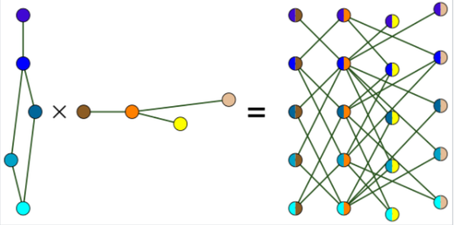
$$\ln\left(\frac{1-\epsilon}{1+\epsilon}\right) \leq \left(\frac{\ln\left(\frac{P_G(kc,kb)}{P_G(ka,kb)}\right)}{kbn} - \ln(c/a)\right) \leq \ln\left(\frac{1+\epsilon}{1-\epsilon}\right)$$

- **Proposition** We pass to the limit $k \to \infty$, since ϵ was arbitrary, the previous estimate implies:
As $x, y \to \infty$, we have

$$\mathcal{CR}_G(y) - \mathcal{CR}_G(x) \asymp \ln y - \ln x,$$

or, equivalently, $\mathcal{CR}_G(x)$ grows at the rate of $\ln x$.

- **Tensor products of graphs**
  Let G_1, G_2 be two simple graphs. The *tensor product* $G_1 \times G_2$ is the graph whose vertex set is the Cartesian product $V(G_1) \times V(G_2)$, and whose edge set is defined as follows: (v_1, w_1) is adjacent to (v_2, w_2) iff v_1 is adjacent to v_2 in G_1, *and* w_1 is adjacent to w_2 in G_2. Here v_1, v_2 are vertices or G_1, and w_1, w_2 are vertices of G_2.



The tensor product of graphs.

- **Proposition:** Let $a, b, c, d \in \mathbb{N}$. Then

$$P_{G_1 \times G_2}(a + c, b + d) \geq P_{G_1}(a, b) P_{G_2}(c, d).$$

- **Proof:** Let $\mathcal{C}$ be an $(a : b)$ colouring of G_1, and let $\mathcal{D}$ be an $(c : d)$ colouring of G_2.
  For $v \in V(G_1)$, denote by $A(v)$ the set of b colours assigned to v in $\mathcal{C}$. For $w \in V(G_2)$, denote by $B(w)$ the set of d colours assigned to w in $\mathcal{D}$.
  Denote by $\mathcal{C} \times \mathcal{D}$ the colouring that assigns the set $A(v) \cup B(w)$ to the vertex (v, w) of $G_1 \times G_2$.
  This defines a proper $(a + c : b + d)$ colouring of $G_1 \times G_2$.

- **Proposition:** Let $a, b, c, d \in \mathbb{N}$. Then

$$P_{G_1 \times G_2}(a+c, b+d) \geq P_{G_1}(a,b) P_{G_2}(c,d).$$

- **Proof:** Let $\mathcal{C}$ be an $(a:b)$ colouring of G_1, and let $\mathcal{D}$ be an $(c:d)$ colouring of G_2.
For $v \in V(G_1)$, denote by $A(v)$ the set of b colours assigned to v in $\mathcal{C}$. For $w \in V(G_2)$, denote by $B(w)$ the set of d colours assigned to w in $\mathcal{D}$.
Denote by $\mathcal{C} \times \mathcal{D}$ the colouring that assigns the set $A(v) \cup B(w)$ to the vertex (v, w) of $G_1 \times G_2$.
This defines a proper $(a+c : b+d)$ colouring of $G_1 \times G_2$.

- **Convergence of fractional chromatic polynomials**
  Let $\hom(G_n, H)$ denote the number of homomorphisms (adjacency preserving maps) from G_n to H. A sequence $\{G_n\}$ of graphs is called *right convergent* if $\ln \hom(G_n, H)/|G_n|$ converges as $n \to \infty$ for any graph H in a "reasonable class" of graphs.
- Let $\{G_n\}$ be a right convergent sequence of graphs. Fix two natural numbers a, b so that $a \geq 2b + 1 > 0$.
  It follows that $\ln P_{G_n}(a, b)/|G_n|$ converges as $n \to \infty$.
  It follows that $\ln P_{G_n}(a, b)/(b|G_n|)$ converges as well for fixed a, b.
- Question: Can we interchange the limits $k \to \infty$ and $n \to \infty$, and conclude that $\lim_{n\to\infty} \mathcal{CR}_{G_n}(x)$ exists (in some suitable sense)?

- **Convergence of fractional chromatic polynomials**
  Let $\hom(G_n, H)$ denote the number of homomorphisms (adjacency preserving maps) from G_n to H. A sequence $\{G_n\}$ of graphs is called *right convergent* if $\ln \hom(G_n, H)/|G_n|$ converges as $n \to \infty$ for any graph H in a "reasonable class" of graphs.
- Let $\{G_n\}$ be a right convergent sequence of graphs. Fix two natural numbers a, b so that $a \geq 2b + 1 > 0$.
  It follows that $\ln P_{G_n}(a, b)/|G_n|$ converges as $n \to \infty$.
  It follows that $\ln P_{G_n}(a, b)/(b|G_n|)$ converges as well for fixed a, b.
- Question: Can we interchange the limits $k \to \infty$ and $n \to \infty$, and conclude that $\lim_{n\to\infty} \mathcal{CR}_{G_n}(x)$ exists (in some suitable sense)?

- **Convergence of fractional chromatic polynomials**
  Let $\hom(G_n, H)$ denote the number of homomorphisms (adjacency preserving maps) from G_n to H. A sequence $\{G_n\}$ of graphs is called *right convergent* if $\ln \hom(G_n, H)/|G_n|$ converges as $n \to \infty$ for any graph H in a “reasonable class” of graphs.
- Let $\{G_n\}$ be a right convergent sequence of graphs. Fix two natural numbers a, b so that $a \geq 2b + 1 > 0$.
  It follows that $\ln P_{G_n}(a, b)/|G_n|$ converges as $n \to \infty$.
  It follows that $\ln P_{G_n}(a, b)/(b|G_n|)$ converges as well for fixed a, b.
- Question: Can we interchange the limits $k \to \infty$ and $n \to \infty$, and conclude that $\lim_{n\to\infty} \mathcal{CR}_{G_n}(x)$ exists (in some suitable sense)?

- **References:**

[Bol] B. Bollobás. *Random Graphs* Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge, Second Edition, 2001.

[BCKL] C. Borgs, J. Chayes, J. Kahn, and L. Lovász. *Left and right convergence of graphs with bounded degree.* Random Structures & Algorithms 42 (2013), 1–28.

[BCLSV] C. Borgs, J. Chayes, L. Lovász, V. Sós and K. Vesztergombi. *Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing.* Advances in Mathematics 219 (2008), 1801–1851.

[HR] J. Hladký and I. Rocha. *Independent sets, cliques and colorings in graphons.* European Journal of Combinatorics, Volume 88, August 2020, 103108.

[JLRT] D. Jakobson, T. Langsetmo, I. Rivin and L. Turner. *Rank and Bollobás-Riordan Polynomials: Coefficient measures and zeros.* To appear in AMS Contemporary Mathematics series.

[JMNT] D. Jakobson, C. MacRury, S. Norin and L. Turner. *Limiting distribution of coefficients of rank polynomials for random sparse graphs.* Electronic Journal of Combinatorics vol. 25, issue 4 (2018), P 4.50.

[Lea] I. Leader. *The fractional chromatic number of infinite graphs.* Journal of Graph Theory, Vol. 20, No. 4 (1995), 411–417.

[Rec] P. M. Recuero. *Closed formulas for fractional chromatic polynomials of some common classes of graphs.* arXiv:1610.05793

[S-U] E.R. Scheinerman and D.H. Ullman. *Fractional Graph Theory: A Rational Approach to the Theory of Graphs.* Courier Corporation, 2011.

[Sok] A. Sokal. *Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions.* Combin. Probab. Comput., 10 (1) (2001), pp. 41–77.

[St] S. Stahl. *n-tuple colourings and associated graphs.* Jour. of Comb. Theory B 20 (1976), 185–203.

[Tut] W. T. Tutte. *On dichromatic polynomials.* Jour. of Comb. Theory 2 (1967), 301–320.

[VV] M. Valencio-Pabon and J.-C. Vera. *On the diameter of Kneser graphs.*. Discrete Mathematics 305 (2005) 383–385.

[Wel] D. Welsh. *The Tutte Polynomial.* Random Structures and Algorithms, (1999), 210–228.

[ZHLS] L. Zhang, K. Hennayake, H.-J. Lai and Y. Shao. *A lower bound on the l-edge connectivity and optimal graphs.* Jour. of Combinatorial Mathematics and Combinatorial Computing 66 (2008), 79–95.