Homomorphisms and the Fractional Edge Coloring

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1 Homomorphism for fractional coloring

Proposition 1.0.1. *Since $X_f(G)$ is the infimum of the fractions n/k such that $G \to K(n,k)$, there is a monotonicity of the fractional chromatic number in the homomorphism order. If $G \to H$ has a homomorphism, then $X_f(G) \leq X_f(H)$.*

The fractional chromatic number is the solution of a natural linear program.
We begin by formulating the ordinary chromatic number as a {0,1} linear program. Let G be a graph, and for each independent set $I \in V(G)$. Let X_I be a {0, 1} variable. An n-colouring of G is a set of n independent sets I, which partition V (G).
If we choose values $x_I = 1$ for the chosen sets I and values $x_I = 0$ for the other sets I, we have a solution to the system:

$$\sum_I x_I = n$$

$$\sum_{I \ni x} x_I = 1, \ for \ all \ v \in V(G)$$

The first equation indicates that exactly n independent sets has been chosen.
The second set of equations indicates that each vertex v of G is in exactly one chosen independent set.
Conversely, any {0, 1} solution to the above equations corresponds to an n-colouring of G.

Theorem 1.0.2. *The chromatic number $X(G)$ of a graph G is equal to the optimum value of the integer linear program.*

Now, consider the continuous relaxation of this integer linear program, by relaxing the constraint $x_I = \{0,1\}$ to $0 \leq x_I \leq 1$. We can take the value x_I to mean the 'degree' to which the independent set I is to be taken. This view is aided by the fact that a linear program with integer coefficients is known to have an optimum with rational values of the variables.

Consider then a rational optimum x_I, I independent, of the previous linear program:

$x_I \geq 0$ independent set I.

Assume that all fractions x_I have the same denominator k, and by multiplying through by k we make sure that all values of x_I are integers. If:

$$\sum_I x_I = n$$

then we have chosen n independent sets I, with repetition allowed (so at-most n non-repeated I), which cover each vertex of G exactly k times, i.e., we have a k-tuple n-colouring of G. Since we multiplied by k, the original fractional solution has value n/k.

Clearly, any k-tuple n-colouring of G gives rise to a feasible solution of this linear program: it corresponds to n independent sets I covering each vertex k times, so if each variable x_I is set to $1/k$, then we obtain a feasible solution of value n/k. Therefore:

Theorem 1.0.3. *The fractional chromatic number of a graph G is the minimum fraction n/k such that G admits a k-tuple n-colouring. In particular, $X_f(G)$ is always a rational number.*

Now consider a linear program:

$$min \sum_I x_I$$

$$\sum_{I \ni x} x_I \geq 1, \ for \ all \ v \in V(G)$$

and its dual

$$max \sum_v y_v$$

$$\sum_{v \in I} y_v \leq 1, \ for \ all \ I \ independent$$

$$y_v \geq 0, \ for \ all \ v \in V(G)$$

This linear program assigns non-negative values y_v to vertices v of G, so that the sum of the values over any independent set is at most 1, and tries to maximize the total sum of values. Clearly, if all values y_v are integers, i.e., 0 or 1 (given the constraints), the set of vertices v with $y_v = 1$ will define a maximum clique in G. Thus the solution to this linear program is called the fractional clique number of G.

Note that, a clique is a subset of vertices of an undirected graph such that every two distinct vertices in the clique are adjacent (not in the same independent set).
The importance of the fractional clique number is that it allows us to obtain lower bounds on $X_f(G)$. Indeed, any solution y_v, $v \in V(G)$, of the dual linear program yields the lower bound $\sum_v y_v \leq X_f(G)$.

Corollary 1.0.4. *Suppose G is a graph with n vertices and independence number α. Then*

$$X_f(G) \geq \frac{n}{\alpha}$$

2 Homomorphisms among the Kneser graphs

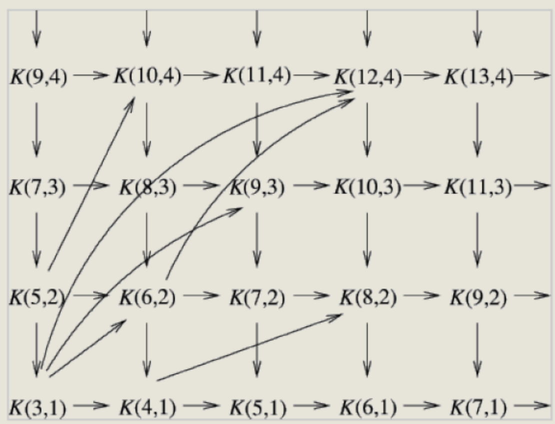

Fig. 6.5. Homomorphisms among the Kneser graphs.

Proposition 2.0.1. *For all integers n, k, with $n \leq 2k$ and $k \leq 1$,*

1. $K(n,k) \leftarrow K(n+1,k)$

2. $K(n,k) \leftarrow K(tn,tk)$, for every positive integer t

3. $K(n,k) \leftarrow K(n-2,k-1)$, for $k > 1$.

(Note that, as we suppose $n \leq 2k$, we actually have $\frac{n}{k} \geq \frac{n-2}{k-1}$, by $n(k-1) \geq (n-2)k$.)

Proof. Statement 1 is obvious from the definition, as $K(n,k)$ is actually a sub-graph of $K(n+1,k)$, if the ground sets are taken to be $\{1,2,...,n\}$ and $\{1,2,...,n,n+1\}$.
Statement 2 is best seen if the ground set of $K(n,k)$ is as above, but the ground set of $K(tn,tk)$ is taken to be $\{1,2,...,t\} \times \{1,2,...,n\}$. This allows us to map each vertex $\{x_1,x_2,...,x_k\}$ of $K(n,k)$ to the vertex $\{1,2,...,t\} \times \{x_1,x_2,...,x_k\}$ of $K(tn,tk)$. This is easily seen to be a homomorphism.

To prove 3, we shall transform each k-element subset A of $\{1, 2, ..., n\}$, into a $(k-1)$-element subset A' of $\{1, 2, ..., n-2\}$, in such a way that disjoint sets remain disjoint after the transformation.

The first idea for such a transformation might be to simply eliminate the largest element of A. Unfortunately, this will not eliminate both n and $n-1$ from those sets A that contain both of them (but will work fine for all other sets A). Hence we define A' as follows:

If A does not contain both n and $n-1$, then $A' = A - maxA$. Otherwise $A' = A - \{n, n-1\} \cup \{x\}$, where x is the maximum element absent from A.

It only remains to prove that $A_1 \cap A_2 = \emptyset$ implies $A'_1 \cap A'_2 = \emptyset$. If not, it could only have been the process of adding x that created an intersection. Obviously both sets A_1, A_2 could not have been added to (only one can contain n, $n-1$), so suppose x was added to A_1, but already lies in A_2. Then it is easy to see that x must have been the largest element of A_2 and hence does not belong to A'_2. □

It seems possible that no other homomorphisms amongst the Kneser graphs exist, apart from the above three kinds and their composition.

Theorem 2.0.2 (Erdős-Ko-Rado theorem)**.** *Let n and k be positive integers, with $n \geq 2k$. In a set of cardinality n, a family of distinct subsets of cardinality k, no two of which are disjoint, can have at most $\binom{n-1}{k-1}$ members.*

On the other hand, the set S of all k-tuples of $\{1, 2, ..., n\}$, which contain the element 1 forms an independent set in $K(n, k)$, with vertices $\binom{n-1}{k-1}$, whence the independence ratio is $i(K(n, k)) = k/n$. This leads to:

Proposition 2.0.3. *The independence number of the Kneser graph $K(n, k)$ (with $n \geq 2k$) is $\binom{n-1}{k-1}$.*

Remark1: The ratio of the independence number of a graph G to its vertex count is known as the independence ratio of G.(Bollobás 1981).

Remark2: The kenser graph $K(n, k)$ has $\binom{n}{k}$ vertices.

Lemma 2.0.4 (No-Homomorphism Lemma, Albertson and Collins)**.** *Let G, H be graphs such that H is vertex-transitive and $G \to H$. Then:*

$$i(G) \geq i(H).$$

Proof. Let $S(H)$ denote the family of independent sets of size $\alpha(H)$(independence number) in H. By symmetry,every vertex of H is in the same number, say m, of members of $S(H)$. Thus,

$$\alpha(H) \times |S(H)| = m \times |H|$$

since each expression counts the number of inclusions $u \in I$, with $u \in V(H)$ and $I \in S(H)$.
Let $\phi : G \to H$ be a homomorphism. Then, for each $I \in S(H)$, we have $|\phi^{-1}(I)| \leq \alpha(G)$. Summing this inequality for all members of $S(H)$, we get:

$$\sum_{I \in S(H)} |\phi^{-1}(I)| \leq \alpha(G) \times |S(H)|$$

However, each $u \in V(G)$ contributes exactly m to the $\sum_{I \in S(H)} |\phi^{-1}(I)|$, since $\phi(u)$ belongs to exactly m members of $S(H)$. Thus,

$$\sum_{I \in S(H)} |\phi^{-1}(I)| = m \times |G|$$

Combining above equations, we get:

$$i(G) = \alpha(G)/|G| \geq m/|S(H)| = \alpha(H)/|H| = i(H).$$

□

Proposition 2.0.5. *If $2 \geq n'/k' < n/k$, then*

$$k(n,k) \not\to k(n',k')$$

Proof. As the independence ratio is i(K(n, k)) = k/n. Since each i(K(n, k)) = k/n, the result follows by the application of the No-Homomorphism Lemma. □

Finally, by some results towards proving that there are no homomorphisms amongst the Kneser graphs other than those illustrated in the figure above. We have several conclusions:

Corollary 2.0.6. *If $n' - 2k' < n - 2k$ then $K(n,k) \not\to K(n',k')$.*

and a **Conjecture**:
$K(n,k) \not\to K(tn - 2k + 1, tk - k + 1)$

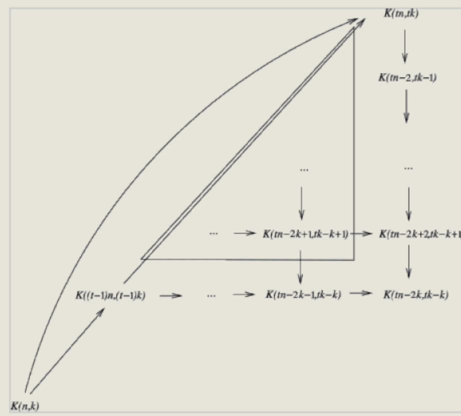

Fig. 6.6. Homomorphisms from the Kneser graph $K(n, k)$.

3 Fractional edge coloring

Definition 3.0.1. *A matching of a graph G is a subset M of $E(G)$ such that no two edges in M share an endpoint.*

In any edge colouring, a set of edges receiving the same colour is a matching. So, an edge k-colouring of a graph $G = (V; E)$ is a covering of the edge set E of G with k matchings of G. The idea is similar to the feactional coloring we discussed above, and hence, the chromatic index is the optimal value to the following Integral Program:

$$min \sum y$$

$$s.t \sum_{e \in M} y_M \geq 1, \;\; for \; all \; e \in E(G)$$

$$y \geq 0, \;\; y \in \mathbb{Z}^{M(G)}$$

where M(G) is the set of all matchings of G.

Moreover, the fractional chromatic index of a graph G, denoted $X'_f(G)$, is the optimal value to the following LP:

$$min \sum y$$

$$s.t \sum_{e \in M} y_M \geq 1, \;\; for \; all \; e \in E(G)$$

$$y \geq 0, \;\; y \in \mathbb{R}^{M(G)}$$

And its dual is:

$$max \sum z$$
$$\sum_{e \in M} z_e \leq 1, \ for \ all \ Matching \ M$$
$$z \geq 0, \ for \ all \ Z \in \mathbb{R}^E$$

Weak duality immediately yields the following lower bounds: For any graph G of maximum degree Δ, $X'_f(G) \geq \Delta$.

Proof. For any vertex $v \in V$, let $z_v^e = 1$ if $e \in \delta(v)$ and $z_v^e = 0$ otherwise. Since, z_v is dual feasible (fullfill the above requirement), $X'_f(G) \geq max \sum z_v = max_{v \in G} |\delta(v)| = \Delta$. □

3.1 Edmonds' polynomial time algorithm to solve the maximum weight matching problem

Definition 3.1.1. *For graph G with matching M, an* ***exposed vertex*** *v is a vertex that does not belong to M, but is in the graph G. That is $V(G - M)$ are exposed vertices.*

Definition 3.1.2. *Given a matching M, an* ***augmenting path*** *is an odd length path whose end-points are distinct exposed vertices, and whose edges $e_1, ..., e_{2k+1}$ alternate such that $e_{2i+1} \notin M$ for all $0 \leq i \leq k$ and $e_{2j} \in M$ for all $1 \leq j \leq k$.*

Now, by the definition of the augmenting path, we trivially can see that:

Theorem 3.1.3. *Given an augmenting path and a matching M, by removing the matched edges of the path from M, and by adding the unmatched edges of path to M, we increase the size of M by one.*

Theorem 3.1.4 (Theorem of Correctness)**.** *Given a graph G and a matching M, M is a maximum matching if and only if there is no augmenting path in G.*

Proof. $\Rightarrow$Assume towards a contradiction that G contains an M augmenting path P. Then, clearly M is not maximal. $\Leftarrow$ Suppose not, We know that there exists a larger matching M_0, so we define $H = (V, M \cup M0)$. We know that for all $v \in V(H)$, $deg_H(v) \leq 2$, so H is the union of cycles and paths.

We have two claims that imply that there exists an M-augmenting. The first one is that all cycles in H have even length. Indeed, for each cycle C in H, each vertex must be connected to one edge in $M - M_0$ and one edge in $M_0 - M$ and this cannot happen if there is an odd amount of edges.(Since they are 2 matchings)

The second claim that there is a connected component with more edges in M_0 than M follows from the fact that $|M0| > |M|$. This component must be a path because of our first claim and this implies that this path is M-augmenting. $\square$

The sequential Edmonds' Blossom Algorithm

Algorithm 1 Blossom Algorithm: Find Maximum Matching

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1: procedure SEQ_FIND_MAXIMUM_MATCHING(G, M)
2:     P = SEQ_FIND_AUG_PATH(G, M)
3:     if P == [ ] then
4:         return M
5:     else
6:         Add alternating edges of P to M
7:         return SEQ_FIND_MAXIMUM_MATCHING(G, M)
```

We consider the exposed vertices $v \in G - M$. These vertices become roots of trees within a forest. The algorithm loops through those vertices, adding pairs of unmatched-matched edges to the corresponding tree in order to build alternating paths. We loop through all vertices that are of even distance from the root, so as we add pairs of unmatched-matched edges, we always have alternating paths in the tree.

When two of these alternating paths are connected by an unmatched edge, the algorithm has found an augmenting path. That is, when the algorithm come across an edge (v, w) that connects two trees, it takes the alternating path from the the root of v to v, then adds the new edge from (v, w), then adds the alternating path from w to the root of w.The algorithm then returns this augmenting path.

Complexity Analysis:

as $|V(G)| = n$, $|V(G)| + |E(G)| = m$,

$$Total\ cost = O(n^2 m)$$

Definition 3.1.5. *The matching polytope of a graph G is the convex hull of the incidence vectors of matchings of G:*

$$MP(G) = conv(x^M : M \in M(G));$$

where $M(G)$ is the matching polytope set of all matchings of G.

Edmonds describes a equivalence formula of matching polytope of a graph G by using his algorithm.

$$\left\{ x \in \mathbb{R}^{E(G)} \quad s.t. \quad \begin{array}{cc} \sum_{e\in\delta(v)} x_e = 1 & \forall v \in V \\ \sum_{e\in\delta(H)} x_e \geq 1 & \forall H \subseteq G, |H| \ odd, |H| > 1 \\ x_e \geq 0 & \forall e \in E(G). \end{array} \right\}.$$

Theorem 3.1.6. *There exists an algorithm such that given any graph G determines the fractional chromatic index and an optimal fractional edge colouring in polynomial time.*

The theorem relies on the following consequence of the **Ellipsoid method**:

Theorem 3.1.7. *There exists an algorithm that, for any $c \in \mathbb{Z}^n$ and well-described polyhedron $(P; n, \phi)$ for which we have a polynomial time separation oracle, either:*
1) Finds an optimum dual solution which is an extreme point of the dual polytope to P,
or
2) Asserts that the dual problem is unbounded or has no solution, in a polynomial time.

Proof. Consider the criteria 2), let z be a vector in $\mathbb{R}^{E(G)}$. We can clearly check if $z \geq 0$ in polynomial time. Hence, we need only check if for each matching $M \in M(G)$, we have $\sum_{e\in M} z_e \leq 1$. By Edmonds' blossom algorithm, we can find a maximum weight matching M_0 with edge weights z in polynomial time. Now, if $\sum_{e\in M_0} z_e \leq 1$, then z is in the polytope that Edmond describes, thus such a algorithm exists. Since all procedures require a polynomial time of iterations, combine them all, we shall have a polynomial time solution for determines the fractional chromatic index and an optimal fractional edge colouring of G. $\square$

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