FRACTIONAL CHROMATIC POLYNOMIALS. VERSION 21

P. CUSSON, DMITRY JAKOBSON, KE HAN XIAO, AND CUNYUAN ZHAO

ABSTRACT. We discuss the fractional colourings of graphs and fractional chromatic polynomials.

1. Introduction

We study fractional colourings of finite graphs, and discuss their asymptotic behaviour.

2. Fractional chromatic polynomials

Let $a \geq b > 0$ be positive integers, and let G be a finite graph. A (fractional) $(a : b)$ vertex colouring of G is an assignment to each vertex of a set of b colours, out of available set of a colours, so that adjacent vertices are assigned disjoint sets of colours (they have no colours in common). Equivalently, an $(a : b)$ colouring of G is a homomorphism from G to the Kneser graph $KG_{a,b}$: its vertices are b-element subsets of a set of a elements, and the two vertices are adjacent iff the corresponding sets are disjoint. The usual colouring of G corresponds to taking $b = 1$ (every vertex is assigned a single colour).

We denote by $P_G(a, b)$ the number of $(a : b)$ vertex colourings of G. The usual chromatic polynomial counts the number of ways of colouring of G in a colours, and so is equal to $P_G(a, 1)$. The fractional chromatic polynomial of G counts fractional colourings of G.

Proposition 2.1. *If b is fixed, then $P_G(a, b)$ is a polynomial in a.*

Proof: Proposition follows from the following fact: there is a bijection between the $(a : b)$ colourings of G; and the colourings by a colours of the graph $G(K_b)$, constructed as follows: each vertex u of G is replaced by a complete graph $K_b(u)$, and whenever u and v are adjacent, every vertex of $K_b(u)$ is connected to every vertex of $K_b(v)$, and vice versa. Indeed, you need exactly b colours to colour the vertices of $K_b(u)$; and the colour sets corresponding to $K_b(u)$ and $K_b(v)$ in the graph $G(K_b)$ have to be disjoint iff u and v are adjacent in G, by construction.

Accordingly, for a fixed b we have

$$P_G(a, b) = P_{G(K_b)}(a),$$

where $P_H(a)$ denotes the chromatic polynomial for the graph H, and the result follows.

□

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3. Subadditive and superadditive functions

In this section we discuss some subadditive and superadditive functions related to fractional colouring of graphs.

The b-chromatic number $\chi_b(G)$ is the smallest number a such that an $(a:b)$-colouring of G exists. It is easy to show that $\chi_{a+b}(G) \le \chi_a(G) + \chi_b(G)$. Indeed, let $x = \chi_a(G), y = \chi_b(G)$. Then every vertex of G can be properly coloured by a set of a colours chosen from the set of x colours; and by a set of b colours, chosen from the set of y colours. Take a disjoint union of the set of x colours and y colours (for a total of $x+y$ colours); for a vertex v of G, denote by $A(v)$ the a-colouring of v, and by $B(v)$ the b-colouring of v. Then the sets $\{A(v) \cup B(v)\}$ form a proper $a+b$-colouring of G, QED. By definition, this means that the b-chromatic number is a subadditive function. By Fekete Lemma (see e.g. [S-U, Lemma A.4.1] the proof for subadditive functions, and ibid. Exercise A.5.1 for super-additive functions), there exists

$$\lim_{b\to\infty} \frac{\chi_b(G)}{b} = \inf_b \frac{\chi_b(G)}{b} := \chi_f(G), \tag{3.1}$$

the *fractional chromatic number of* G. It satisfies the following inequalities:

$$\frac{\chi(G)}{1+\ln\alpha(G)} \le \chi_f(G) \le \frac{\chi_b(G)}{b} \le \chi(G). \tag{3.2}$$

Here $\alpha(G)$ is the independence number of G (the size of the largest independent set A in G, i.e. of the set $A \subset V(G)$ s.t. any two vertices in A are not adjacent in G). Kneser graphs provide examples where $\chi(G)/\chi_f(G)$ can be arbitrarily large.

We next consider the rate of growth of $P_G(a,b)$. It is easy to show that

$$P_G(a+c, b+d) \ge P_G(a,b)P_G(c,d).$$

Indeed, consider any colouring of vertices of G with b colours, chosen from the set of a colours; let $A(v)$ be the colouring of a vertex v. Similarly, consider a colouring of vertices of G with d colours, chosen from the set of c colours; let $B(v)$ be the colouring of a vertex v. Then it's clear that the sets $\{A(v) \cup B(v) : v \in V(G)\}$ provide a proper $(a+c : b+d)$-colouring of G.

It follows that

$$\ln P_G(a+c, b+d) \ge \ln P_G(a,b) + \ln P_G(c,d),$$

provided $P_G(a,b)$ and $P_G(c,d)$ are positive. Fix $a,b>0$, and assume that $a \ge \chi_b(G)$. We assume that a and b are relatively prime. Consider the line $\{(kb,ka) : k \in \mathbb{N}\}$ in the (b,a)-coordinate plane. It follows that

$$\ln P_G((k+l)a, (k+l)b) \ge \ln P_G(ka,kb) + \ln P_G(la,lb),$$

i.e. the function $\ln P_G(ka,kb)$ is a superadditive function of k.

We denote by $|G| = |V(G)|$ the number of vertices of G. By Fekete Lemma, there exists the following limit, which we call *fractional colouring growth rate* function or simply *colouring rate* function, and which we denote by $\mathcal{CR}_G(x)$, where $x = a/b$:

$$\mathcal{CR}_G(x) := \lim_{k\to\infty} \frac{\ln P_G(ka,kb)}{kb|G|} = \sup_k \frac{\ln P_G(ka,kb)}{kb|G|}. \tag{3.3}$$

That function is well-defined, since the limit only depends on the ratio a/b. The formula function $\mathcal{CR}_G(x)$ is only defined by the formula (3.3) for a dense set of

$x \in \mathbb{Q}$, and for $x \geq \chi_f(G)$. However, the following result will allow to extend $\mathcal{CR}_G$ to the whole interval $(\chi_f(G), +\infty)$.

Proposition 3.1. *The function $\mathcal{CR}_G(x)$ is non-decreasing in x.*

Proof of Proposition 3.1.

Since the graph G is fixed, we will suppress the dependence of all the quantities on G. Let $\chi_f < x < y < \infty$ be such that $x = \frac{a_1}{b_1} < \frac{a_2}{b_2} = y$. The idea is to rescale the points $p = (b_1, a_1)$ and $q = (b_2, a_2)$ so that the rescaled points lie on the same vertical line, then compare the number of the corresponding fractional colourings, and pass to the limit.

Accordingly, given $k \in \mathbb{N}$, rescale the point p by a factor of kb_2, and the point q by a factor of kb_1. We get the points $p_k = (kb_1b_2, kb_2a_1)$, and the point $q_k = (kb_1b_2, kb_1a_2)$; the point q_k lies *above* the point p_k on the vertical line $b = kb_1b_2$, i.e. $kb_1a_2 > ka_1b_2$). It is clear by definition of $P(a,b)$ that $P(kb_1a_2, kb_1b_2) \geq P(ka_1b_2, kb_1b_2)$ (since we have increased the number of available colours from ka_1b_2 to kb_1a_2). It follows after taking logarithms that

$$\frac{\ln P(kb_1a_2, kb_1b_2)}{kb_1b_2|G|} \geq \frac{\ln P(ka_1b_2, kb_1b_2)}{kb_1b_2|G|}.$$

We next let $k \to \infty$. By definition of the function g, we conclude that

$$\mathcal{CR}_G(y) \geq \mathcal{CR}_G(x).$$

□

We next extend $\mathcal{CR}_G(x)$ to a function on the interval $I_G = (\chi_f(G), +\infty)$ as follows:

$$\mathcal{CR}_G(x) := \sup_{y \in I_G : y \in \mathbb{Q}, y \leq x} \mathcal{CR}_G(y). \tag{3.4}$$

We make a few simple remarks about the colouring rate function:

Remark 3.2. Let $G = G_1 + G_2$ be a disjoint union of two graphs, and let $a/b \geq \chi_f(G_i), i = 1, 2$. Then clearly $P_G(a,b) = P_{G_1}(a,b) \cdot P_{G_2}(a,b)$, so by definition of $\mathcal{CR}$, we have

$$\mathcal{CR}_{G_1+G_2}(x) = \mathcal{CR}_{G_1}(x) + \mathcal{CR}_{G_2}(x). \tag{3.5}$$

on the common interval of definition.

Remark 3.3. Let G be a subgraph of H. Then $\chi_f(G) \leq \chi_f(H)$, and

$$\mathcal{CR}_G(x) \geq \mathcal{CR}_H(x)$$

on the common interval of definition.

4. Examples

In this section we compute $\mathcal{CR}_G(x)$ for several specific families of graphs.

4.1. **Complement of a complete graph.** Consider a graph $\overline{\mathrm{K_n}}$ with n vertices and no edges. Since each vertices in $\overline{\mathrm{K_n}}$ are independent with no other vertices connected with, we can colour each of them independently. Thus, for an $(a:b)$ colouring of $\overline{\mathrm{K_n}}$, we can colour each vertex of it in $\binom{a}{b}$ ways; and since $\overline{\mathrm{K_n}}$ has n vertices, we have:

$$P_{\overline{\mathrm{K_n}}}(a,b) = \binom{a}{b}^n.$$

We next compute the function $\mathcal{CR}_{\overline{\mathrm{K_n}}}(x)$. We use Stirling's formula

$$m! \asymp \sqrt{2\pi m}(m/e)^m$$

to compute

$$\lim_{k\to\infty} \frac{n\ln\binom{ka}{kb}}{kbn} = \lim_{k\to\infty} \frac{1}{kb} \ln\left(\frac{(ka)!}{(kb)!(ka-kb)!}\right).$$

After elementary calculations, we find that the limit is equal to

$$-(a/b)\ln((a-b)/a)+\ln((a-b/b) = -x\ln(1-1/x)+\ln(x-1) = x\ln x+(1-x)\ln(x-1),$$

where $x = a/b$. It is clear that we can extend that function by continuity from $\mathbb{Q}$ to $\mathbb{R}$ by the same formula. Accordingly,

$$\mathcal{CR}_{\overline{\mathrm{K_n}}}(x) = x\ln x + (1-x)\ln(x-1).$$

4.2. **Trees.** Consider a tree T_n with n vertices. Choose on of its leaves (terminal vertices) as a root of T_n. The root can be coloured in $\binom{a}{b}$ ways. Next, we note that in a tree, each vertex is connected to at most 1 parent, and to its child. Thus, the only restriction to covering a child comes from the colouring of its parent, giving $\binom{a-b}{b}$ ways to colour this child. We can colour the tree in the *breadth* order: we always colour the parent prior to its child. Accordingly, each of $(n-1)$ vertices besides the root can be covered in $\binom{a-b}{b}$ ways. It follows that

$$P_{T_n}(a,b) = \binom{a}{b}\binom{a-b}{b}^{n-1}$$

We next compute the function $\mathcal{CR}_{T_n}(x)$. Again by using Stirling's formula, one can compute that

$$\begin{aligned}
\lim_{k\to\infty} \frac{\ln P_{T_n}(ka,kb)}{kbn} &= \lim_{k\to\infty} \frac{\ln\binom{ka}{kb}\cdot\binom{k(a-b)}{kb}^{n-1}}{kbn} \\
&= \frac{1}{nb}\cdot(a\ln a + (n-2)(a-b)\ln(a-b) - (n-1)(a-2b)\ln(a-2b)) - \ln b \\
&= \frac{x}{n}\cdot\ln x + \frac{(n-2)(x-1)}{n}\cdot\ln(x-1) - \frac{(n-1)(x-2)}{n}\cdot\ln(x-2)
\end{aligned}$$

where $x = a/b$. Extending this function from $\mathbb{Q}$ to $\mathbb{R}$, we get

$$\mathcal{CR}_{T_n}(x) = \frac{x}{n}\cdot\ln x + \frac{(n-2)(x-1)}{n}\cdot\ln(x-1) - \frac{(n-1)(x-2)}{n}\cdot\ln(x-2). \tag{4.1}$$

Remark 4.1. If we (formally) let $n\to\infty$ in (4.1), we find that as $n\to\infty$,

$$\mathcal{CR}_{T_n}(x) \to (x-1)\cdot\ln(x-1) - (x-2)\cdot\ln(x-2).$$

4.3. **Complete graphs.** Let $G = K_n$, the complete graph. Since all the vertices are connected to each other, the first vertex can be coloured in $\binom{a}{b}$ ways, the second vertex in $\binom{a-b}{b}$ ways, ..., the $(\alpha+1)$-st vertex in $\binom{a-\alpha b}{b}$ ways, and the n-th vertex in $\binom{a-(n-1)b}{b}$ ways; here we have assumed that $a > nb$. It follows that

$$P_{K_n}(a,b) = \prod_{\alpha=0}^{n-1} \binom{a-\alpha b}{b}.$$

We next compute the function $\mathcal{CR}_{K_n}(x)$. We compute that

$$\begin{aligned}
\lim_{k\to\infty} \frac{\ln P_{K_n}(ka,kb)}{kbn} &= \lim_{k\to\infty} \frac{1}{kbn} \cdot \ln \prod_{\alpha=0}^{n-1} \binom{k(a-\alpha b)}{kb} \\
&= \frac{1}{bn} \cdot \sum_{\alpha=0}^{n-1} \Big((a-\alpha b)\ln(a-\alpha b) - (a-\alpha b - b)\ln(a-\alpha b - b) - b\ln b \Big) \\
&= \frac{1}{bn} \cdot (a \ln a - (a-nb)\ln(a-nb)) - \ln b.
\end{aligned}$$

By letting $x = a/b$, we find that the above is equal to

$$\frac{x}{n} \cdot \ln x - \frac{x-n}{n} \cdot \ln(x-n).$$

We can again extend this function from $\mathbb{Q}$ to $\mathbb{R}$ to get

$$\mathcal{CR}_{K_n}(x) = \frac{x}{n} \cdot \ln x - \frac{x-n}{n} \cdot \ln(x-n).$$

5. Properties of colouring rate function.

Recall that the colouring rate function $\mathcal{CR}_G(x)$ is defined on the interval $(\chi_f(G), +\infty)$. First, we discuss the asymptotics of $\mathcal{CR}(x)$ as $x \to \infty$.

By (3.4), it suffices to consider $\mathcal{CR}_G$ restricted to $\mathbb{Q}$. Let $|G| = |V(G)| = n$. We remark that $P_G(a,b) = P_{G(K_b)}(a)$, where the graph $G(K_b)$ has been defined in the proof of Proposition 2.1, and where $PH(a)$ denotes the chromatic polynomial of the graph H. We know that $P_{G(K_{kb})}$ is a polynomial of degree nkb, where n is the number of vertices of G, with the highest coefficient equal to 1.

To study the asymptotics of $\mathcal{CR}_G(x)$, we shall use the following result of Sokal ([Sok], see also [Bor]).

Proposition 5.1. *Let G be a graph of maximal degree D. Then all roots ρ of the chromatic polynomial $P_G(x)$ satisfy the inequality*

$$|\rho| \leq KD,$$

where K is a constant strictly smaller than 8.

Let d be the maximal degree of G. Consider two rational numbers $a/b < c/b$, where

$$a/b > M \cdot d, \tag{5.1}$$

where M is a large constant to be chosen later on. We would like to compare the values $\mathcal{CR}_G(a/b)$ and $\mathcal{CR}_G(c/b)$. By monotonicity, we know that $\mathcal{CR}_G(a/b) < \mathcal{CR}_G(c/b)$.

By definition of $\mathcal{CR}_G$, it suffices to estimate the ratio

$$\frac{\ln P_G(kc,kb)-\ln P_G(ka,kb)}{kbn}=\frac{\ln\left(\frac{P_G(kc,kb)}{P_G(ka,kb)}\right)}{kbn}=\frac{\ln\left(\frac{P_{G(K_{kb})}(kc)}{P_{G(K_{kb})}(ka)}\right)}{kbn}.$$

Let $\rho_1,\dots,\rho_{kbn}$ be the roots of $P_{G(K_{kb})}(x)$. Now,

$$\frac{P_{G(K_{kb})}(kc)}{P_{G(K_{kb})}(ka)}=\frac{\prod_{j=1}^{kbn}(kc-\rho_j)}{\prod_{j=1}^{kbn}(ka-\rho_j)}=\left(\frac{c}{a}\right)^{kbn}\frac{\prod_{j=1}^{kbn}(1-\frac{\rho_j}{kc})}{\prod_{j=1}^{kbn}(1-\frac{\rho_j}{ka})}$$

By Proposition 5.1, we have $|\rho_j|<8kbd$ for all j, since the maximal degree of $G(K_{kb})$ is equal to kbd. By (5.1) we have $a>bMd$; therefore

$$|\rho_j/kc|<|\rho_j/ka|<\frac{8kbd}{Mkbd}=\frac{8}{M}.$$

Let $\epsilon>0$; we shall choose M large enough so that $8/M<\epsilon$. It follows that

$$\left(\frac{c}{a}\right)^{kbn}\left(\frac{1-\epsilon}{1+\epsilon}\right)^{kbn}\le\frac{P_{G(K_{kb})}(kc)}{P_{G(K_{kb})}(ka)}\le\left(\frac{c}{a}\right)^{kbn}\left(\frac{1+\epsilon}{1-\epsilon}\right)^{kbn}\tag{5.2}$$

It follows from (5.2) that

$$\ln\left(\frac{1-\epsilon}{1+\epsilon}\right)\le\left(\frac{\ln\left(\frac{P_G(kc,kb)}{P_G(ka,kb)}\right)}{kbn}-\ln(c/a)\right)\le\ln\left(\frac{1+\epsilon}{1-\epsilon}\right)$$

We pass to the limit $k\to\infty$, and remark that $\ln\frac{c}{a}=\ln\frac{c/b}{a/b}$. We conclude that

$$\left|\mathcal{CR}_G(c/b)-\mathcal{CR}_G(a/b)-\ln\left(\frac{c/b}{a/b}\right)\right|\le\ln\left(\frac{1+\epsilon}{1-\epsilon}\right)$$

Since ϵ was arbitrary, the previous estimate implies the following

Proposition 5.2. *As $x,y\to\infty$, we have*

$$\mathcal{CR}_G(y)-\mathcal{CR}_G(x)\asymp\ln y-\ln x,$$

or, equivalently, $\mathcal{CR}_G(x)$ grows at the rate of $\ln x$.

6. Tensor products of graphs

Let G_1,G_2 be two simple graphs. The *tensor product* $G_1\times G_2$ is the graph whose vertex set is the Cartesian product $V(G_1)\times V(G_2)$, and whose edge set is defined as follows: (v_1,w_1) is adjacent to (v_2,w_2) iff v_1 is adjacent to v_2 in G_1, *and* w_1 is adjacent to w_2 in G_2. Here v_1,v_2 are vertices or G_1, and w_1,w_2 are vertices of G_2.

We shall prove the following result.

Proposition 6.1. *Let $a,b,c,d\in\mathbb{N}$. Then*

$$P_{G_1\times G_2}(a+c,b+d)\ge P_{G_1}(a,b)P_{G_2}(c,d).\tag{6.1}$$

Proof of Proposition 6.1. Let $\mathcal{C}$ be an $(a:b)$ colouring of G_1, and let $\mathcal{D}$ be an $(c:d)$ colouring of G_2. For $v\in V(G_1)$, denote by $A(v)$ the set of b colours assigned to v in $\mathcal{C}$. For $w\in V(G_2)$, denote by $B(w)$ the set of d colours assigned to w in $\mathcal{D}$. Denote by $\mathcal{C}\times\mathcal{D}$ the colouring that assigns the set $A(v)\cup B(w)$ to the vertex (v,w) of $G_1\times G_2$.

This defines a proper $(a+c : b+d)$ colouring of $G_1 \times G_2$. Indeed, if (v_1, w_1) is adjacent to (v_2, w_2) then $(v_1, v_2) \in E(G_1)$ and so $A(v_1) \cap A(v_2) = \emptyset$; and $(w_1, w_2) \in E(G_1)$ and so $B(w_1) \cap B(w_2) = \emptyset$.

It is clear that $\mathcal{C}_1 \times \mathcal{D}_1 = \mathcal{C}_2 \times \mathcal{D}_2$, then $\mathcal{C}_1 = \mathcal{C}_2$ and $\mathcal{D}_1 = \mathcal{D}_2$. This finishes the proof of the Proposition.

□

Denote by $G^{\times k}$ the k-fold tensor product $G \times \ldots \times G$. Note that Proposition 6.1 implies that the sequence $\ln P_{G^{\times n}}(na, nb)$ is superadditive:

$$\ln P_{(G^{\times n}) \times (G^{\times m})}((m+n)a, (m+n)b) \geq \ln P_{G^{\times m}}(ma, mb) + \ln P_{G^{\times n}}(na, nb).$$

Accordingly, using Fekete Lemma, we can naively define

$$h(a/b) := \lim_{k \to \infty} \frac{\ln P_{G^{\times k}}(ka, kb)}{bk} = \sup_{k \to \infty} \frac{\ln P_{G^{\times k}}(ka, kb)}{bk}.$$

However, the normalization seems to be incorrect in the definition above. Indeed, consider the simplest nontrivial path graph P_2. It is easy to show that $P_2^{\times 2} = P_2 + P_2 = 2P_2$ (disjoint union). Accordingly, by induction that $P_2^{\times k} = 2^k P_2$. Next, since P_2 is a tree, we find (using the calculations in section 4.2 and the calculations leading to the proof of (3.5)) that

$$\ln P_{P_2^{\times k}}(ka, kb) = 2^k \left(\ln \binom{ka}{kb} + \ln \binom{ka - kb}{kb} \right),$$

and it follows from the calculations in section 4.2 that the correct normalization that gives a finite nonzero limit is given by the following formula (where we call the limit function *tensor rate* and denote it $\mathcal{TR}_G(x), x = a/b$):

$$\mathcal{TR}(a/b) := \lim_{k \to \infty} \frac{\ln P_{G^{\times k}}(ka, kb)}{|G^{\times k}| bk}.$$

For $G = P_2$, it is easy to see that

$$\mathcal{TR}_{P_2}(x) = \mathcal{CR}_{P_2}(x).$$

7. Convergence of fractional chromatic polynomials

We shall use the terminology in [BCKL]. Let $\hom(G_n, H)$ denote the number of homomorphisms (adjacency preserving maps) from G_n to H. A sequence $\{G_n\}$ of graphs is called *right convergent* if $\ln \hom(G_n, H)/|G_n|$ converges as $n \to \infty$ for any graph H in a "reasonable class" of graphs.

Let $\{G_n\}$ be a right convergent sequence of graphs. Fix two natural numbers a, b so that $a \geq 2b + 1 > 0$. We know that $P_G(a, b)$ is equal to the number of homomorphisms of G into the Kneser graph $KG_{a,b}$: its vertices are the b-element subsets of the set $\{1, \ldots, a\}$, and two vertices are connected iff the corresponding subsets are disjoint.

It follows from the definition of right convergence and the homomorphism characterization of fractional $(a : b)$ colourings that $\ln P_{G_n}(a, b)/|G_n|$ converges as $n \to \infty$. It follows that $\ln P_{G_n}(a, b)/(b|G_n|)$ converges as well for fixed a, b.

Now, for $x = a/b$, we have

$$\mathcal{CR}_{G_n}(x) = \lim_{k \to \infty} \frac{\ln P_{G_n}(ka, kb)}{kb|G_n|}.$$

It seems interesting to understand under what additional assumptions on the graph sequence G_n we can interchange the limits $k \to \infty$ and $n \to \infty$, and conclude that $\lim_{n\to\infty} \mathcal{CR}_{G_n}(x)$ exists (in some suitable sense).

It seems clear that some additional assumptions are required. In particular, $\mathcal{CR}_G(x)$ is defined on the interval $(\chi_f(G), +\infty)$, therefore if $\mathcal{CR}_{G_n}(x)$ converges, then their intervals of definition should also converge, and hence the sequence of fractional chromatic numbers $\chi_f(G_n)$ should converge as well. However, if we make no additional assumptions about the sequence G_n, the fractional chromatic numbers may not converge, see e.g. [Lea], [HR]. Accordingly, it seems interesting to understand what additional assumptions would guarantee the convergence of fractional chromatic number.

Nevertheless, if $\{G_n = (V_n, E_n)\}$ is a sequence of connected graph with uniformly bounded degrees (say $\leq d_m ax$) such that $|G_n| \to \infty$ with relatively few number of vertices, we always have that $\mathcal{CR}_{G_n}$ converges pointwise to the chromatic rate function for trees on rational points. More precisely:

Proposition 7.1. *Let $x = a/b \in \cap_n I_{G_n}$ with $x > d_{\max}$. For each $n \in \mathbb{N}$, let $r_n = |E_n| - |V_n| + 1$ be the cycle rank of G_n. If*

$$\lim_{n\to\infty} \frac{r_n}{n} = 0,$$

then $\mathcal{CR}_{G_n}(x)$ converges and

$$\lim_{n\to\infty} \mathcal{CR}_{G_n}(x) = (x-1)\ln(x-1) - (x-2)\ln(x-2).$$

Proof. Let ST_n be some spanning tree of G_n, then by example 4.2, we clearly have

$$P_{G_n}(a,b) \leq P_{ST_n}(a,b). \tag{7.1}$$

Recall that r_n is the smallest number of edges to be deleted from G_n such that the resulting graph contains no cycle. Thus, we can choose a set $R_n \subset V_n$ of at most r_n vertices such that the subgraph induced by $V_n \setminus R_n$ is a tree or a forest. Moreover, each element of R_n has at least $\binom{a-d_{\max}\cdot b}{b}$ choices of colouring. Thus, we also have

$$P_{G_n}(a,b) \geq \binom{a}{b}\binom{a-b}{b}^{|G_n|-r_n-1}\binom{a-d_{\max}\cdot b}{b}^{r_n} =: P^-_{G_n}(a,b). \tag{7.2}$$

By following a similar process as in example 4.2, one can compute that

$$\begin{aligned}\lim_{k\to\infty} \frac{\ln P^-_{G_n}(ka,kb)}{kb|G_n|} = \frac{x\ln x}{|G_n|} &+ \frac{|G_n|-2-r_n}{|G_n|}\cdot(x-1)\ln(x-1)\\ &- \frac{|G_n|-1-r_n}{|G_n|}\cdot(x-2)\ln(x-2)\\ &+ \frac{r_n}{|G_n|}\cdot(x-d_{\max})\ln(x-d_{\max})\\ &- \frac{r_n}{|G_n|}\cdot(x-d_{\max}-1)\ln(x-d_{\max}-1).\end{aligned}$$

We have already computed in example 4.2 that

$$\begin{aligned}\lim_{k\to\infty} \frac{\ln P_{ST_n}(ka,kb)}{kb|ST_n|} = \frac{x}{|G_n|}\cdot\ln x &+ \frac{(|G_n|-2)(x-1)}{|G_n|}\cdot\ln(x-1)\\ &- \frac{(|G_n|-1)(x-2)}{|G_n|}\cdot\ln(x-2).\end{aligned}$$

Since $|G_n| \to \infty$ and $r_n/n \to 0$ as $n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty}\lim_{k\to\infty}\frac{\ln P_{ST_n}(ka,kb)}{kb|G_n|} &= \lim_{n\to\infty}\lim_{k\to\infty}\frac{\ln P^-_{G_n}(ka,kb)}{kb|G_n|}\\ &= (x-1)\ln(x-1)-(x-2)\ln(x-2)\end{aligned}$$

and the result follows from (7.1) and (7.2). □

8. Complexes of Homomorphisms

It is well-known that there exist homomorphisms from $KG_{a,b}$ to $KG_{ka,kb}$ for any $k > 0$. We are counting those homomorphisms, and the chromatic rate function gives the exponent for the exponential growth of their number.

Question: does a sequence of Kneser graphs $KG_{ka,kb}$ converge as $k \to \infty$? Maybe there is some "graph at infinity" $KG_{\infty,a/b}$, and we are counting homomorphisms into that graph?

Complexes of homomorphisms first appeared in 1978 Lovász's solution of Kneser's conjecture (where he determined the chromatic number of Kneser graphs). Babson and Kozlov further studied such complexes, proving a conjecture of Lovász in 2007.

We remark that $P_G(a,b)$ is the number of vertices in the homomorphism complex $\mathrm{Hom}(G, KG_{a,b})$ from G into the Kneser graph $KG_{a,b}$.

DIRECT LIMITS?

9. Fractional colourings and covering graphs

Let G_1 be a degree k (covering/lift ?) graph of G: there is a projection $\pi : G_1 \to G$ that is onto; $|\pi^{-1}(v)| = k$ for every vertex v of G; and whenever u is adjacent to v in G, the edges of G_1 connecting vertices in $\pi^{-1}(u)$ and $\pi^{-1}(v)$ form a perfect matching.

We remark that $(ka : kb)$ colouring of G gives rise to an $(a : b)$ colouring of *any* graph G_1 is above.

TO BE CONTINUED

10. Edge colouring

Fractional edge colouring of a graph is equivalent to fractional vertex colouring of its line graph. So, many of the results generalize.

Fractional total colouring (vertices and edges) of a graph is equivalent to a fractional vertex colouring of its total graph.

11. Symmetric graphs

What happens for symmetric (e.g. vertex transitive) graphs?

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Email address: `paulcussonchanho@gmail.com`

Email address: `dmitry.jakobson@mcgill.ca`

Email address: `ke.xiao@mail.mcgill.ca`

Email address: `cunyuan.zhao@mail.mcgill.ca`

Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Str. West, Montréal QC H3A 0B9, Canada