Math 521: Homework 2

Due on Thursday, February 15, 2018

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Theory

1. Unique Decomposition

Let W_1, W_2 be vector subspaces and $W = W_1 + W_2, W_1 \neq W_2$. Show, by giving an example, that the decomposition of a vector $\boldsymbol{x} \in W$ is not unique.

The requirements for a subspace $\hat{W}$ include :

- The zero vector is in $\hat{W}$
- If $\boldsymbol{u}, \boldsymbol{v} \in \hat{W}$, then $\boldsymbol{u} + \boldsymbol{v} \in \hat{W}$
- If $\boldsymbol{u} \in \hat{W}, c \in \mathbb{R}, c\boldsymbol{u} \in \hat{W}$

If we let $W = \{\begin{bmatrix} x & y & 0 \end{bmatrix}^T \ni x, y \in \mathbb{R}\} \subset \mathbb{R}^3$, then we can decompose W into $W_1 = \{\begin{bmatrix} x & 0 & 0 \end{bmatrix}^T \ni x, y \in \mathbb{R}\}$ and $W_2 = \{\begin{bmatrix} x & y & 0 \end{bmatrix}^T \ni x, y \in \mathbb{R}\}$. Then, we can express a vector $\boldsymbol{x} \in W$ non-uniquely. For example:

$$\boldsymbol{x} = \begin{bmatrix} 2 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

2. Bases for A

Consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix}$$

Determine bases for the column space, row space, null space, and left null space of A.

- The column space of A is the linearly independent columns in A. Since the second column is a scalar multiple of the first (by -1), the column space of A is:
  $\text{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$.

- The row space of A is the linearly independent rows of A. Notice that rows 2 and 3 are scalar multiples of the first row. Hence, the row space is:
  $\text{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$.

- The null space of A includes the vectors that solve $Ax = \mathbf{0}$. Then it is easy to see that $A\begin{bmatrix}x_1\\x_1\end{bmatrix} = \mathbf{0}$ solves this. In other words,
  null space of $A = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

- The *left* null space of A are the vectors that solve $x^T A = 0$.
  $\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}1 & -1\\2 & -2\\3 & -3\end{bmatrix} = \begin{bmatrix}x_1 + 2x_2 + 3x_3 & -x_1 - 2x_2 - 3x_3\end{bmatrix} = \mathbf{0}$
  The most direct way to solve this is to convert $[A \quad | \quad I_3]$ to reduced-row echelon form. Performing this calculation, we obtain the basis for the left null space:
  $\text{span} = \left\{\begin{bmatrix}1\\0\\-\frac{1}{3}\end{bmatrix}, \begin{bmatrix}0\\1\\-\frac{2}{3}\end{bmatrix}\right\}$

3. Projections

Let $V = \mathbb{R}^3$, let

$$u^{(1)} = \begin{bmatrix}1\\2\\0\end{bmatrix}, u^{(2)} = \begin{bmatrix}-1\\0\\1\end{bmatrix}, x = \begin{bmatrix}0\\2\\1\end{bmatrix},$$

and define $W = \text{span}(u^{(1)}, u^{(2)})$. Find the orthogonal projection of x onto W. Also find the projection matrix $\mathbb{P}$ associated with this mapping.

For orthogonal projections, the Gram-Schmidt process is used. The Gram-Schmidt process is a method for generating an orthonormal basis from a set of vectors. A key part of this method is the projection step.
First, we determine an orthonormal basis for W using the Gram-Schmidt algorithm (see the code in the Code section). We obtain the basis vectors:

$$e^{(1)} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \qquad e^{(2)} = \frac{1}{3\sqrt{5}} \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix}$$

A projection matrix $\mathbb{P}_i$ onto $e^{(i)}$, is given by $e^{(i)}e^{(i)^T}$. From this, we have the two projection matrices for the basis vectors above:

$$\mathbb{P}_1 = \frac{1}{5} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbb{P}_2 = \frac{1}{45} \begin{bmatrix} 16 & -8 & -20 \\ -8 & 4 & 10 \\ -20 & 10 & 25 \end{bmatrix}$$

$$\text{then the total projection is: } \mathbb{P} = \mathbb{P}_1 + \mathbb{P}_2 = \frac{1}{9} \begin{bmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

4. Orthonormal Basis Vectors

Reconsider Problem 3. Find vectors such that $x = UU^T x$ and $x \neq UU^T x$ where the matrix U consists of the orthonormal basis vectors of W from Problem 3.

Note that $UU^T = \mathbb{P}$.

5. SVD

Determine the SVD of the data matrix

$$A = \begin{bmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

and compute rank-one, -two, and -three approximations to A.

Computing

1. Kohonen's Novelty Filter

Consider the training set consisting of the following three patterns consisting of 5 x 4 arrays of black squares.

Proceed by assuming that the black square entries have numerical value one and the blank entries have numerical value zero. Concatenate the columns of each pattern to make vectors in $\mathbb{R}^{20}$. Does your result make sense? Why or why not?

2. SVD

Compute the SVD of the matrix A whose entries come from the pattern in ¡reference here¿ and display (in terms of an image) the reconstructions A_1, A_2, A_3, A_4. Again, treat the squares as ones and the blanks as zeros. Your reconstructions should be matrices with numerical values. Interpret your results.

Code

Gram-Schmidt

```
clear;

A = [1 -1; 2 0 ; 0 1];
x = [0; 2; 1];

%
% Perform the Gram-Schmidt algorithm
%

% take the first normalized col vector to be the first orth. basis vector
u1 = A(:,1);
u1 = u1 ./ norm(A(:,1));

B = zeros(size(A));
B(:,1) = u1;

% Define the projection function
proj = @(u,v) (u'*v) / (u'*u) * u;

for ii = 2:size(A,2)
    tmp = 0;
    % sum the projections
    for jj = 2:ii
        tmp = tmp + proj(B(:,jj-1), A(:,ii));
    end
    % tmp = (current vector) - (sum of projections)
    tmp = A(:,ii) - tmp;

    % normalize, and set as new orth. basis vector
    B(:,ii) = tmp / norm(tmp);
end
```

References

[1] T. Kohonen. Self-Organization and Associative Memory. Springer-Verlag, Berlin, 1984.