CSULB Math 697

Rapid Distortion Theory and Anisotropic Flow

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1 Introduction to Fluid Dynamics

1.1 Navier-Stokes

The Navier-Stokes equations describe the motion of viscous fluid flow in which the derivations are simply a matter of applying Newton's second law to fluid conservation laws. For the simplest case, these equations can be reduced by assuming *incompressibility* of the fluid, that is, the fluid density does not change. Such fluids are called Newtonian fluids, and include water, air, and alcohol – all incompressible fluids. In this case of incompressible flow, we have as the Navier-Stokes equations:

$$\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{u_t} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = \frac{\mu}{\rho_0}\Delta \boldsymbol{u} - \frac{1}{\rho_0}\nabla p + \vec{F}_{ext} \tag{1.1}$$

where μ is the dynamic viscosity, ρ_0 is the uniform fluid density, p is the pressure, and $\vec{F}_{ext}$ represents external forces, such as gravity. The solution vector $\boldsymbol{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ is the three-dimensional flow velocity. The notation $\frac{D\vec{u}}{Dt}$ is referred to as the *substantial* or *material* derivative.

The condition for incompressible flow – also known as a *solenoidal* vector field – comes from the constant density of the fluid in the continuity equation. The continuity equation states that the rate at which mass enters a system is equal to the rate at which mass leaves the system plus the accumulation of mass within the system [5].

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) &= 0 && \text{(1.2)}\\
0 + (\nabla \cdot \rho)\,\boldsymbol{u} + \rho\,(\nabla \cdot \boldsymbol{u}) &= 0 && \text{(1.3)}\\
\rho\,(\nabla \cdot \boldsymbol{u}) &= 0 && \text{(1.4)}\\
\nabla \cdot \boldsymbol{u} &= 0 \qquad \text{the zero-divergence condition} && \text{(1.5)}
\end{aligned}$$

Thus, for incompressible flow, we have the zero-divergence condition derived above.

In the next few sections, we will explore background ideas behind turbulent flow to set up for derivation of the Rapid Distortion Theory (RDT) equations.

1.2 Turbulence

Turbulent fluid flow is characterized by chaotic or rapid changes in pressure and velocity, which is also seemingly random and unpredictable [1]. Turbulent flow is in contrast to laminar (or streamline) flow where fluid flows in parallel layers, with little to no disruptions or mixing between layers. An example of both types of flow can be seen in Figure 1.1.

Fluid turbulence can be characterized by the Reynolds number (Re), a dimensionless quantity defined by [2]:

$$Re = \frac{\rho u L}{\mu} \tag{1.6}$$

where u is the relative velocity of the fluid, L is a characteristic linear dimension, and μ is the dynamic viscosity of the fluid. The critical Reynolds number, Re_{crit}, is the threshold where laminar flow ceases to exist (the flow becomes turbulent), and is different depending on the shape and friction coefficient of the boundary layer [3].

For turbulent flow, we typically decompose the flow field into its mean and fluctuating components:

$$\boldsymbol{u} = \langle \boldsymbol{u} \rangle + \vec{f} \tag{1.7}$$

where it is assumed that the time-averaged mean of the fluctuating component $\vec{f}$ is zero. This is representation is known as the Reynolds decomposition.

1.3 Vorticity

In both laminar and turbulent flows, we have rotational velocity of the fluid – i.e. we have non-zero vorticity ($\boldsymbol{\omega}$). The vorticity can be obtained by taking the curl of the

Figure 1.1: Laminar and turbulent water flow over the hull of a submarine. As the relative velocity of the water increases turbulence occurs.

velocity

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \tag{1.8}$$

The purpose of including this section is for completeness. Vorticity is used in the derivation of the Rapid Distortion Theory equations, which we do not derive in this paper.

1.4 Strain

The deformation of a viscous fluid is called *strain* – the response of a system to an applied force (*stress*). There are generally three types of stress: tensional, compression, and shear. For the purposes of this project, we will explore rapid shear stresses.

The rate of change of this deformation is represented in the strain rate tensor (the Jacobian of the velocity field):

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \tag{1.9}$$

For simple shear stress in fluids (Figure 1.2), we have the following:

$$\frac{\partial \langle u_1 \rangle}{\partial x_1} = \frac{\partial \langle u_2 \rangle}{\partial x_2} = \frac{\partial \langle u_3 \rangle}{\partial x_3} = 0 \tag{1.10}$$

$$\frac{\partial \langle u_1 \rangle}{\partial x_2} = \mathcal{S} \tag{1.11}$$

$$\text{i.e. the shear rate tensor is:} \quad \begin{bmatrix} 0 & \mathcal{S} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.12}$$

The *mean* strain rate tensor is defined as the average of the Jacobian matrix of the velocity field and the transpose of the Jacobian.

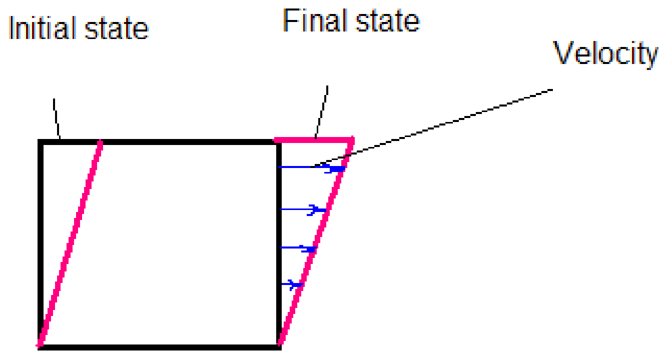

Figure 1.2: Simple shear deformation

$$S^{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right) \tag{1.13}$$

$$S = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right) \tag{1.14}$$

1.5 Pressure

The pressure rate-of-strain tensor comes from the Poisson equation for pressure. The Poisson equation for $p^{'}$ is:

$$\frac{1}{\rho}\Delta p^{'} = -2\frac{\partial\langle u_i\rangle}{\partial x_j}\frac{\partial f_j}{\partial x_i} - \frac{\partial^2}{\partial x_i \partial x_j}\left(f_i f_j - \langle f_i f_j\rangle\right) \tag{1.15}$$

$$p^{'} = p^{(r)} + p^{(s)} + p^{(h)} \tag{1.16}$$

The three terms of $p^{'}$ represent the rapid, slow, and harmonic pressure contributions, respectively.

$$\frac{1}{\rho}\Delta p^{(r)} = -2\frac{\partial\langle u_i\rangle}{\partial x_j}\frac{\partial f_j}{\partial x_i} \tag{1.17}$$

$$\frac{1}{\rho}\Delta p^{(s)} = -\frac{\partial^2}{\partial x_i \partial x_j}\left(f_i f_j - \langle f_i f_j\rangle\right) \tag{1.18}$$

$$\Delta p^{(h)} = 0 \tag{1.19}$$

1.6 Visualizing Flow Field Data

The fluid flow consists of a velocity cube for each direction (x, y, z).

$$\boldsymbol{u}(x, y, z, t = 0) = randn\sin(\pi z) + y\cos(\pi y) + \frac{1}{10}\sin(10\pi x) \tag{1.20}$$

Matlab code for plotting a flow field at a single time point (an example of which is in Figure 1.3).

```
function fh = plotField(X,Y,Z, u)

    if nargin == 1
        u = X;
        siz = size(u);
        [X,Y,Z] = meshgrid(1:siz(2), 1:siz(1), 1:siz(3));
    end

    fh = figure(gcf);

    xmin = min(X(:));
    xmax = max(X(:));
    ymax = max(Y(:));
    ymin = min(Y(:));
    zmin = min(Z(:));

    hs = slice(X,Y,Z,u,[xmin,(xmax+xmin)/2,xmax],[ymin,
        ymax],zmin);
    set(hs,'FaceColor','interp','EdgeColor','none')
    colormap jet
    xlabel x; ylabel y; zlabel z;

end
```

This code has been provided for future reference.

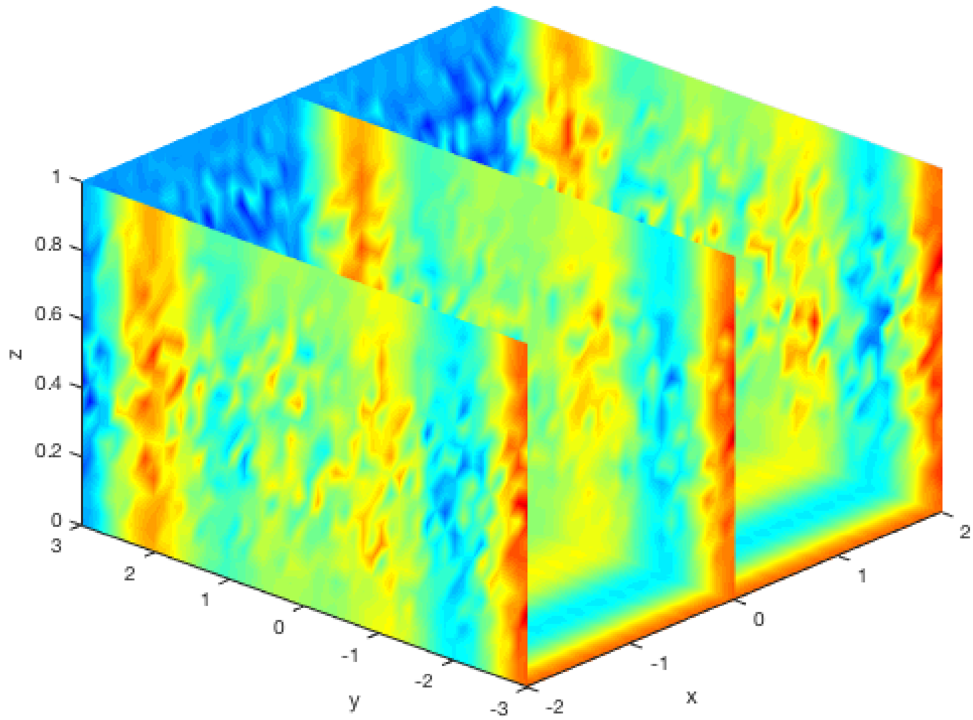

Figure 1.3: Example flow field from eqn. 1.20

2 Rapid Distortion Theory

Rapid Distortion Theory, or RDT, comes from the evolution of turbulence as the turbulence-to-mean-shear ratio becomes arbitrarily large. The derivation for the RDT equations can be found in [1].

2.1 Fourier Modes

In place of vorticity, we can express the initial flow as the sum of Fourier modes. The transformation between the flow field in Fourier space and the real field is:

$$\mathbf{u}(\mathbf{x},t) = \sum_{\boldsymbol{\kappa}^o} \hat{\mathbf{u}}(\boldsymbol{\kappa}^{\boldsymbol{o}},t)\cdot e^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}} \tag{2.1}$$

where $\boldsymbol{\kappa}(0) = \boldsymbol{\kappa}^{\boldsymbol{o}}$ is the intial condition for the wavenumber vector, and denotes one of a set of potentially random wavenumbers. Furthermore, the modes have associated conjugate pairs to ensure $\mathbf{u}(\mathbf{x},t)$ is real, e.g. for a single mode:

$$\mathbf{u}(\mathbf{x},t) = \hat{\mathbf{u}}(\boldsymbol{\kappa},t)\cdot e^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}} + \hat{\mathbf{u}}^*(\boldsymbol{\kappa},t)\cdot e^{-i\boldsymbol{\kappa}(t)\cdot\mathbf{x}} \tag{2.2}$$

where $e^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}} = e^{i\boldsymbol{\kappa}\cdot\mathbf{x}} = e^{i\kappa_1 x_1}e^{i\kappa_2 x_2}e^{i\kappa_3 x_3}$, and $\hat{\boldsymbol{u}}^*$ denotes the complex conjugate of $\hat{\boldsymbol{u}}$.

A divergence-free field in the Fourier space follows as:

$$\begin{aligned}
0 = \nabla\cdot\mathbf{u}(\mathbf{x},t) &= \nabla\cdot\left(\sum_{\boldsymbol{\kappa}^o}\hat{\mathbf{u}}(\boldsymbol{\kappa}^{\boldsymbol{o}},t)e^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}}\right) && (2.3)\\
&= \sum_{\boldsymbol{\kappa}^o}\nabla\cdot\left(\hat{\mathbf{u}}(\boldsymbol{\kappa}^{\boldsymbol{o}},t)e^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}}\right) && (2.4)\\
&= \sum_{\boldsymbol{\kappa}^o}\hat{\mathbf{u}}(\boldsymbol{\kappa}^{\boldsymbol{o}},t)\left(\frac{\partial}{\partial x_1}e^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}} + \frac{\partial}{\partial x_2}e^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}} + \frac{\partial}{\partial x_3}e^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}}\right) && (2.5)\\
&= \sum_{\boldsymbol{\kappa}^o}\hat{\mathbf{u}}(\boldsymbol{\kappa}^{\boldsymbol{o}},t)\left[\kappa_1\hat{u}_1 + \kappa_2\hat{u}_2 + \kappa_3\hat{u}_3\right]\cdot ie^{i\boldsymbol{\kappa}(t)\cdot\mathbf{x}} && (2.6)
\end{aligned}$$

For a single mode, this implies:

$$0 = \kappa_1\hat{u}_1 + \kappa_2\hat{u}_2 + \kappa_3\hat{u}_3 = \boldsymbol{\kappa}(t)\cdot\hat{\mathbf{u}}(t) \tag{2.7}$$

2.1.1 Orthonormality

Given two wavenumber vectors $\boldsymbol{\kappa}_n, \boldsymbol{\kappa}_m$, the inner product is

$$\langle e^{i\boldsymbol{\kappa}_n \cdot \boldsymbol{x}}, e^{i\boldsymbol{\kappa}_m \cdot \boldsymbol{x}} \rangle_{\mathcal{L}} = \int_{\mathcal{L}} e^{i\boldsymbol{\kappa}_n \cdot \boldsymbol{x}} e^{i\boldsymbol{\kappa}_m \cdot \boldsymbol{x}} dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases} \tag{2.8}$$

2.1.2 Evolution of the Wavenumber

$$\frac{d\kappa_l}{dt} = -\kappa_j \frac{\partial \langle U_j \rangle}{\partial x_l} = -\sum_j \kappa_j \frac{\partial \langle U_j \rangle}{\partial x_l} \tag{2.9}$$

which gives us the differential equations for the evolutions of the wavenumbers, with $\boldsymbol{\kappa}(0) = \boldsymbol{\kappa}^o = \begin{bmatrix} \kappa_1^o & \kappa_2^o & \kappa_3^o \end{bmatrix}^T$.

$$\kappa_1(t) = \kappa_1^o \tag{2.10}$$

$$\kappa_2(t) = \kappa_2^o - \mathcal{S}\kappa_1^o t \tag{2.11}$$

$$\kappa_3(t) = \kappa_3^o \tag{2.12}$$

2.2 RDT Equations

The vectors in Fourier space evolve as:

$$\frac{d\hat{u}_j}{dt} = -\,\hat{u}_k \frac{\partial \langle U_l \rangle}{\partial x_k} \left(\delta_{jl} - 2\frac{\kappa_j \kappa_l}{|\kappa|^2} \right) \qquad \text{sum over } k, l \tag{2.13}$$

With δ_{jl} being the Kronecker delta. For simple shear flow, this reduces to:

$$\frac{d\hat{u}_j}{dt} = \hat{u}_2 \frac{\partial \langle U_1 \rangle}{\partial x_2} \left(2\frac{\kappa_j \kappa_1}{|\kappa|^2} - \delta_{j,1} \right) \tag{2.14}$$

Leading to the differential equations below. Again, this is for shear flow as defined in Equation 1.12.

$$\frac{d\hat{u}_1}{dt} = \mathcal{S}\hat{u}_2\left(\frac{2\left(\kappa_1^o\right)^2}{|\boldsymbol{\kappa}|^2} - 1\right) \tag{2.15}$$

$$\frac{d\hat{u}_2}{dt} = 2\mathcal{S}\hat{u}_2\frac{\kappa_1^o\kappa_2(t)}{|\boldsymbol{\kappa}|^2} \tag{2.16}$$

$$\frac{d\hat{u}_3}{dt} = 2\mathcal{S}\hat{u}_2\frac{\kappa_1^o\kappa_3^o}{|\boldsymbol{\kappa}|^2} \tag{2.17}$$

2.2.1 Solenoidal Condition for RDT

For a sanity check, we will explore the solenoidal field condition for the RDT equations in the Fourier space.

$$0 = \nabla \cdot \boldsymbol{u} \tag{2.18}$$

$$0 = \kappa_1\hat{u}_1 + \kappa_2\hat{u}_2 + \kappa_3\hat{u}_3 \tag{2.19}$$

For a single mode, from equation 2.7. Taking the time derivative, we can explore constraints on the wavevector.

$$
\begin{aligned}
0 &= \frac{d}{dt}\left(\nabla \cdot \boldsymbol{u}\right) && (2.20)\\
0 &= \frac{d}{dt}\left(\kappa_1 \hat{u}_1 + \kappa_2 \hat{u}_2 + \kappa_3 \hat{u}_3\right) && (2.21)\\
0 &= \kappa_1^o \frac{d\hat{u}_1}{dt} + \kappa_2(t)\frac{d\hat{u}_2}{dt} + \kappa_3^o \frac{d\hat{u}_3}{dt} + \dot{\kappa_2}(t)\hat{u}_1 && (2.22)\\
0 &= \kappa_1^o \mathcal{S}\hat{u}_2 \left(\frac{2\left(\kappa_1^o\right)^2}{|\boldsymbol{\kappa}|^2} - 1\right) + \kappa_2(t) 2\mathcal{S}\hat{u}_2 \frac{\kappa_1^o \kappa_2(t)}{|\boldsymbol{\kappa}|^2} && (2.23)\\
&\quad + \kappa_3^o 2\mathcal{S}\hat{u}_2 \frac{\kappa_1^o \kappa_3^o}{|\boldsymbol{\kappa}|^2} - \mathcal{S}\kappa_1^o \hat{u}_2 && (2.24)\\
0 &= \frac{\mathcal{S}\kappa_1^o \hat{u}_2}{|\boldsymbol{\kappa}|^2}\left(2\left(\kappa_1^o\right)^2 - |\boldsymbol{\kappa}|^2 + 2\left(\kappa_2(t)\right)^2 + \left(\kappa_3^o\right)^2 - |\boldsymbol{\kappa}|^2\right) && (2.25)\\
0 &= \frac{\mathcal{S}\kappa_1^o \hat{u}_2}{|\boldsymbol{\kappa}|^2}\left(2\left(\kappa_1^o\right)^2 - 2|\boldsymbol{\kappa}|^2 + 2\left(\kappa_2(t)\right)^2 + \left(\kappa_3^o\right)^2\right) && (2.26)\\
0 &= \frac{\mathcal{S}\kappa_1^o \hat{u}_2}{|\boldsymbol{\kappa}|^2}\left(2|\boldsymbol{\kappa}|^2 - 2|\boldsymbol{\kappa}|^2 - 2\left(\kappa_3^o\right)^2 + \left(\kappa_3^o\right)^2\right) && (2.27)\\
0 &= \frac{\mathcal{S}\kappa_1^o \hat{u}_2}{|\boldsymbol{\kappa}|^2}\left(-\left(\kappa_3^o\right)^2\right) && (2.28)\\
0 &= \frac{\mathcal{S}\hat{u}_2}{|\boldsymbol{\kappa}|^2}\kappa_1^o \left(\kappa_3^o\right)^2 && (2.29)\\
\implies 0 &= \kappa_1^o \left(\kappa_3^o\right)^2 && (2.30)
\end{aligned}
$$

2.3 Simulations

In this section, we present the results of a few simulations of RDT using 4th-order Runge-Kutta as the differential equation solver. Persistent values between runs are outlined in Table 2.1 below.

RK4 step size h	0.05
time vector	0:0.01:1
grid size n (number of points in each dimension)	10
units for each dimension of cube	$\frac{2\pi}{n}[-1, 1]$
wavevector $\boldsymbol{\kappa}$	$[0.107, 0.144, 1.215]^T$
Shear force $\mathcal{S}$	100
initial condition $u(0)$	random, seeded
initial condition $v(0)$	random, seeded
initial condition $w(0)$	0

Table 2.1: Simulation Parameters

Throughout the following simulations, we vary the single-mode wavevector, $\boldsymbol{\kappa}$, to explore the evolution of the fluid velocity. The purpose of this is to reconcile the outcome from Section 2.2.1, which will be explored further beyond the timeline of this course.

The following figures show the $x-, y-$, and $z-$velocity components of the fluid flow, respectively, for the time point specified in the figure caption. The color axis is fixed for each velocity field for the duration of a simulation. Different points in time are presented to the reader to get a sense of the velocity profile, not to compare each simulation from frame-to-frame.

2.3.1 Simulation 1: $\kappa_1 = 0$

In this simulation, $\boldsymbol{\kappa} = [0, 0.144, 1.215]^T$.

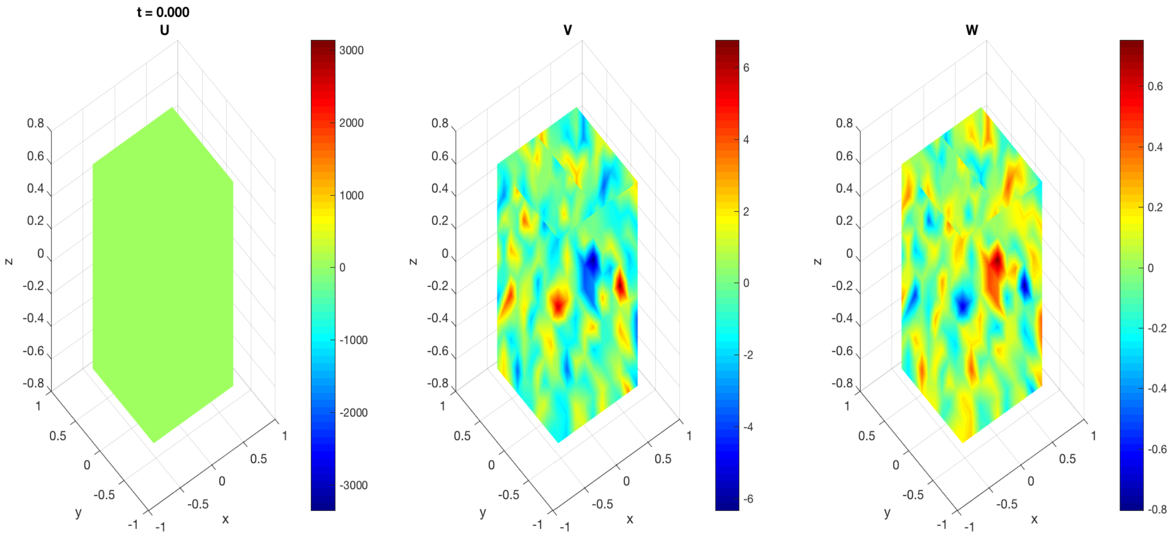

Figure 2.1: t = 0

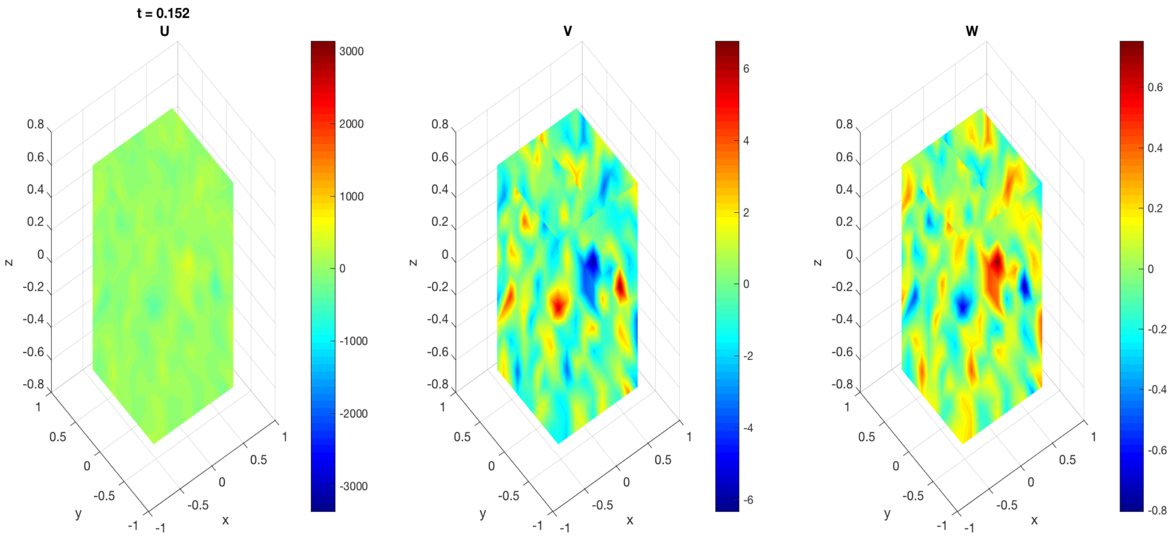

Figure 2.2: t = 0.152

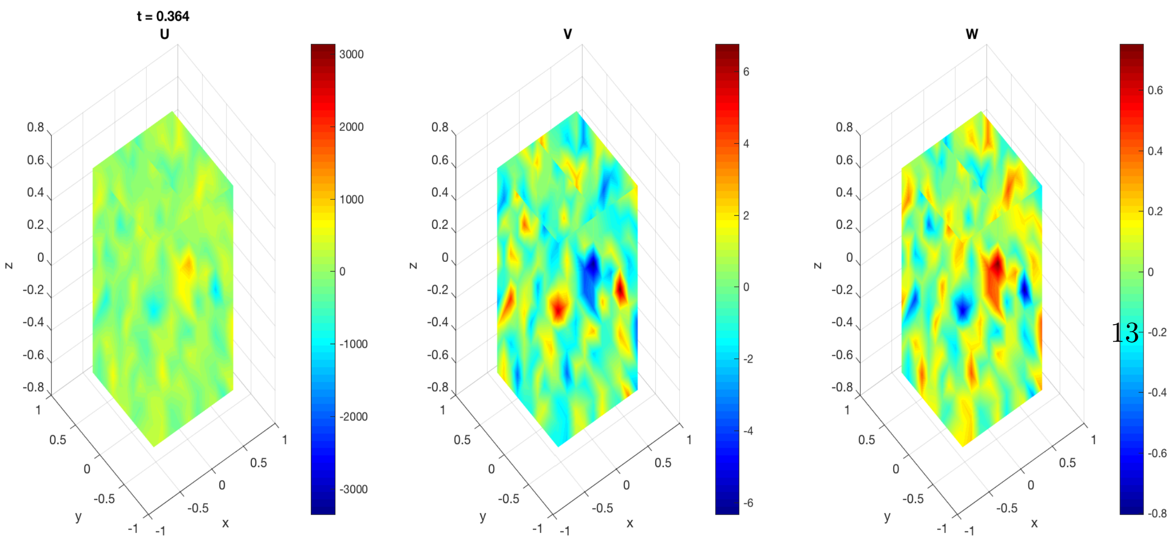

Figure 2.3: t = 0.364

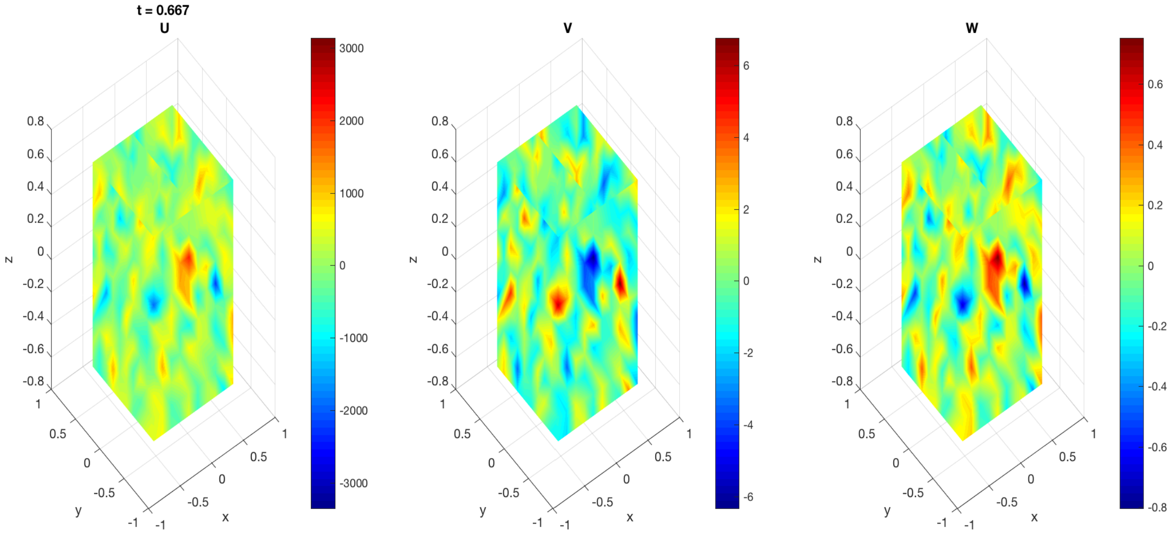

Figure 2.4: t = 0.667

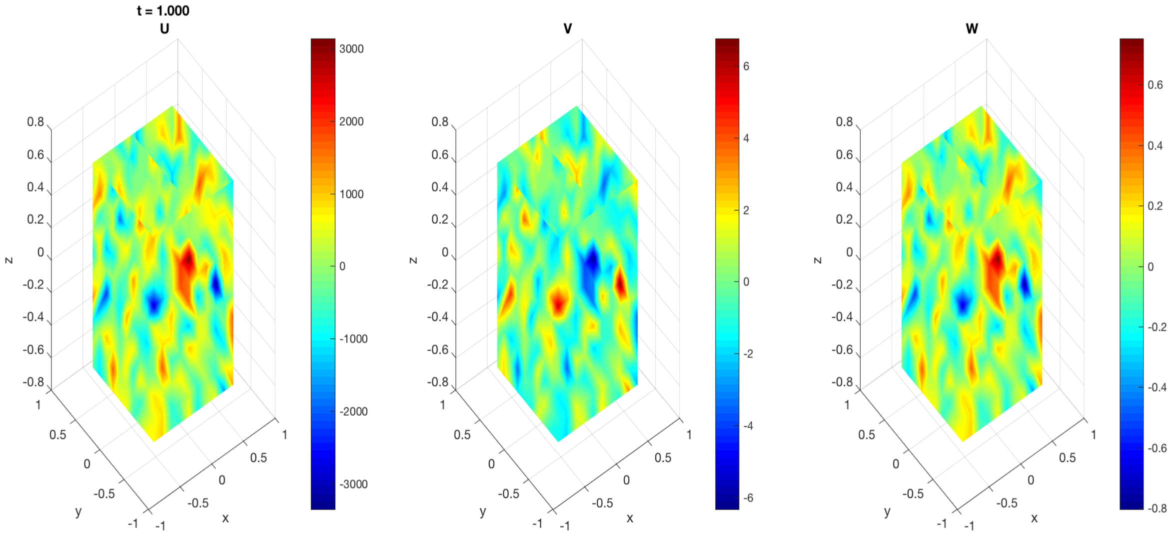

Figure 2.5: t = 1.0

2.3.2 Simulation 2: $\kappa_3 = 0$

In this simulation, $\boldsymbol{\kappa} = [0.107, 0.144, 0]^T$.

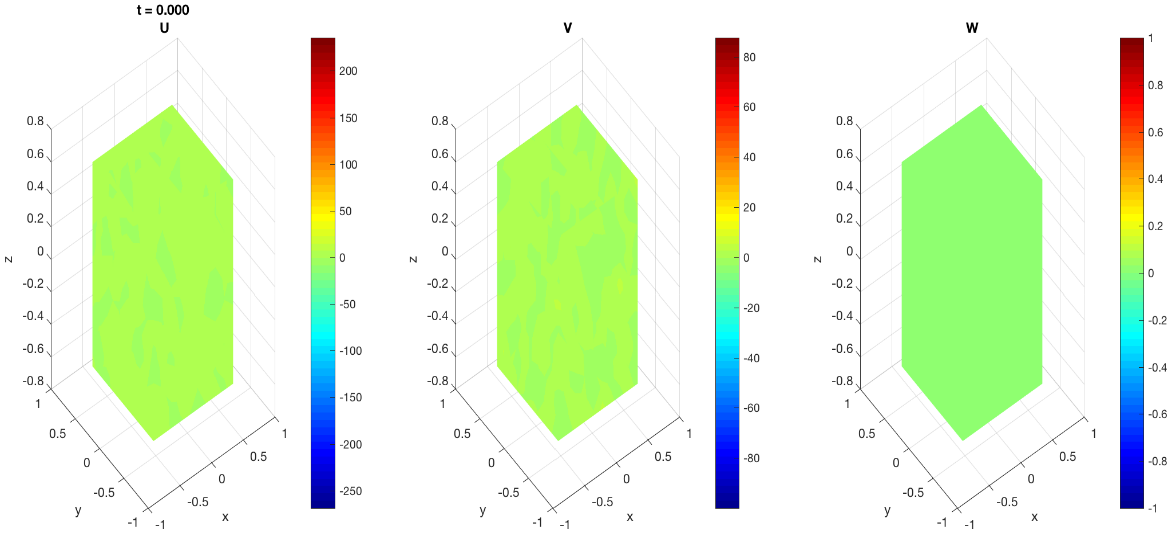

Figure 2.6: t = 0

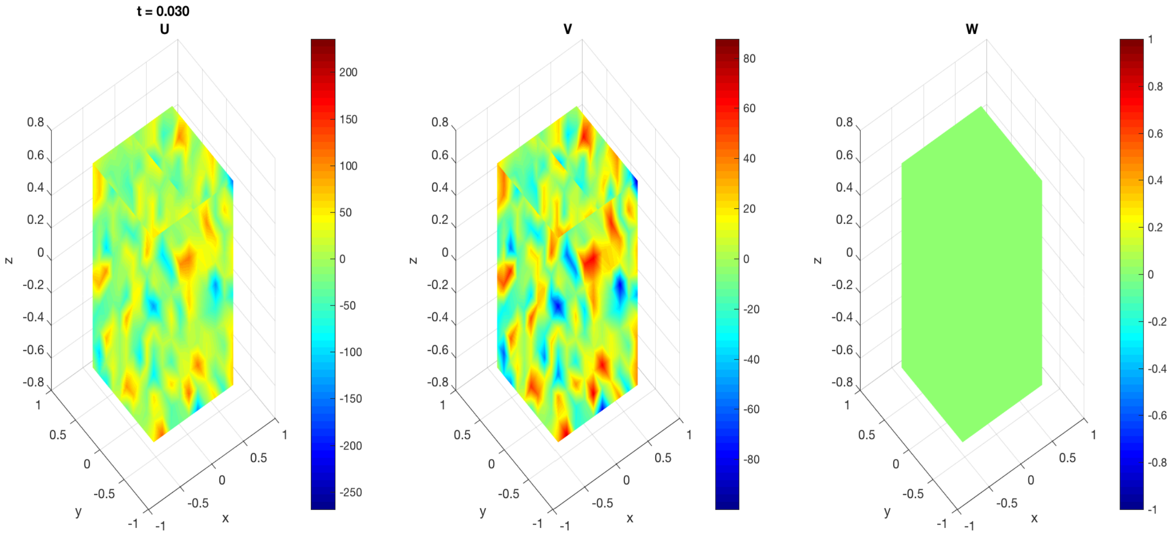

Figure 2.7: t = 0.030

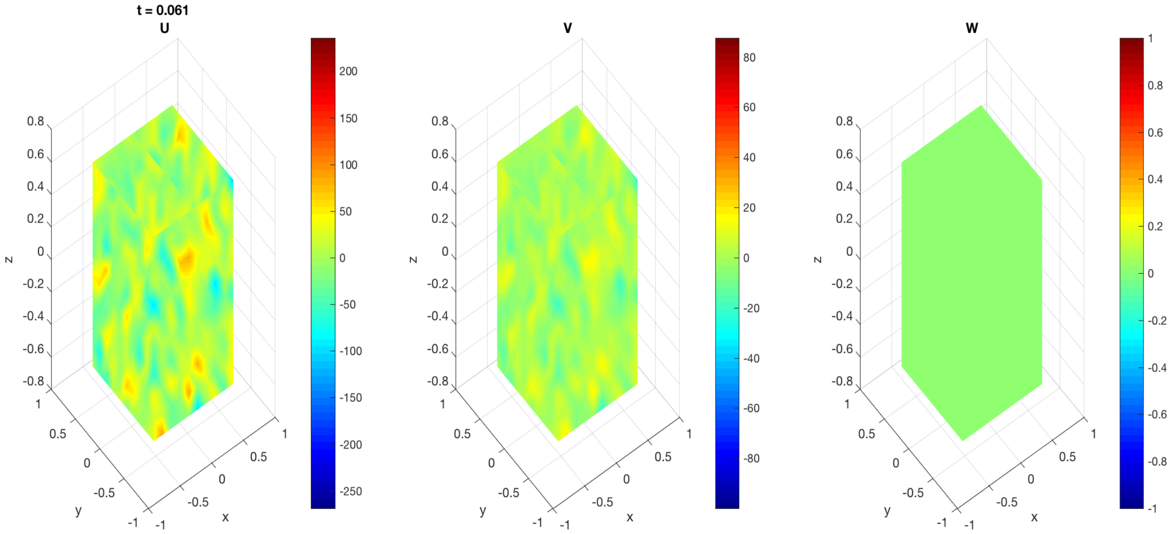

Figure 2.8: t = 0.061

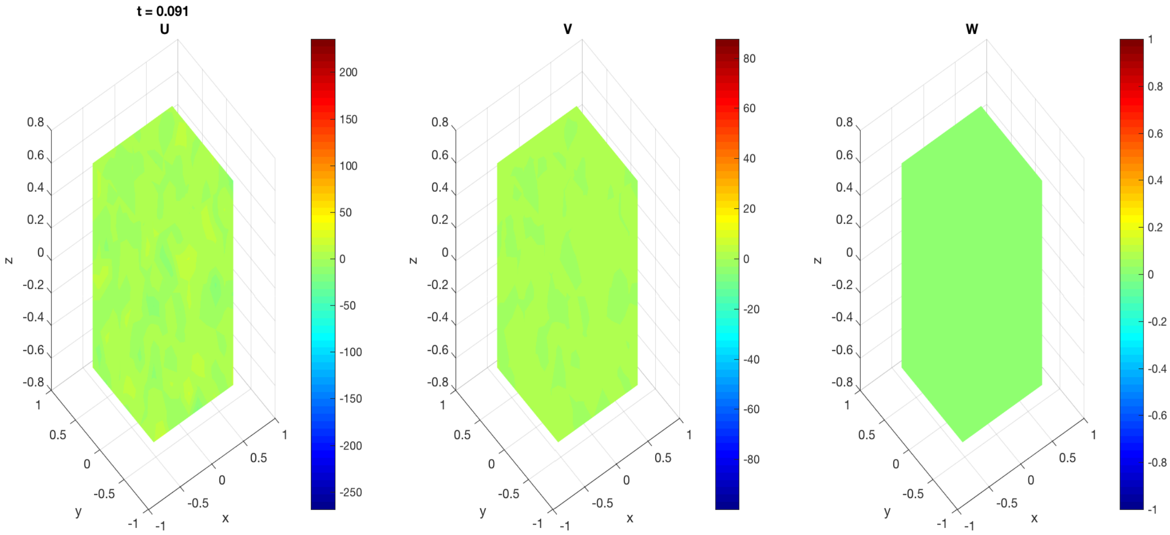

Figure 2.9: t = 0.091

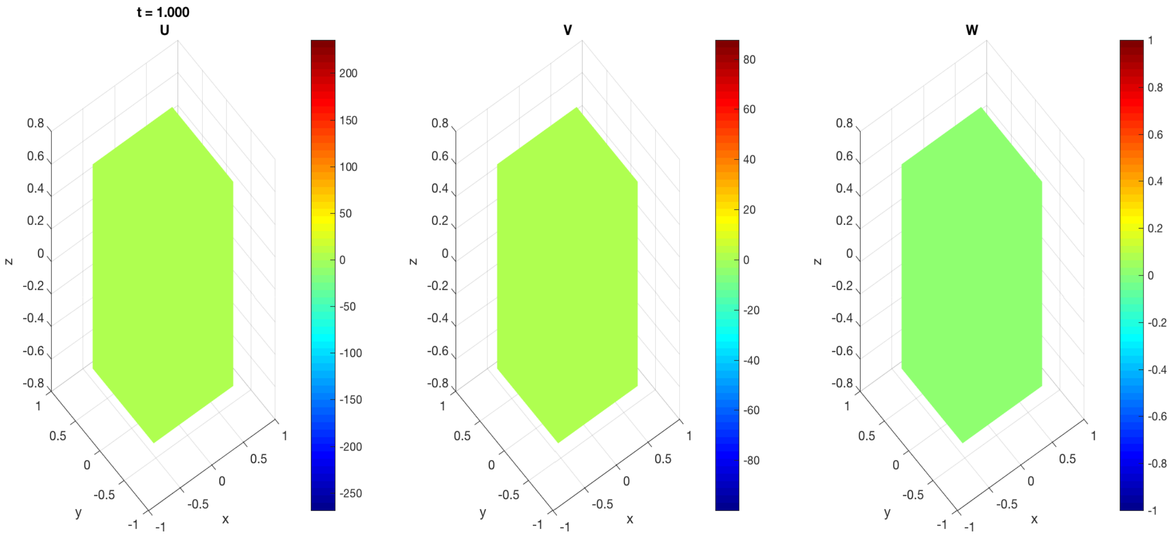

Figure 2.10: t = 1.0

2.3.3 Simulation 3: κ as-is

In this simulation, $\boldsymbol{\kappa} = [0.107, 0.144, 1.215]^T$.

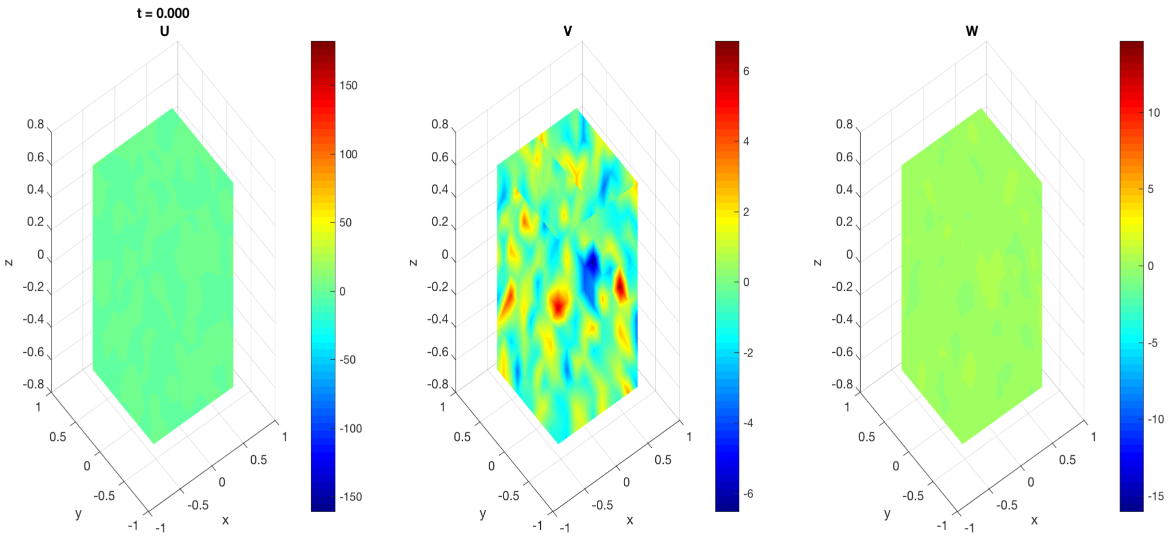

Figure 2.11: t = 0

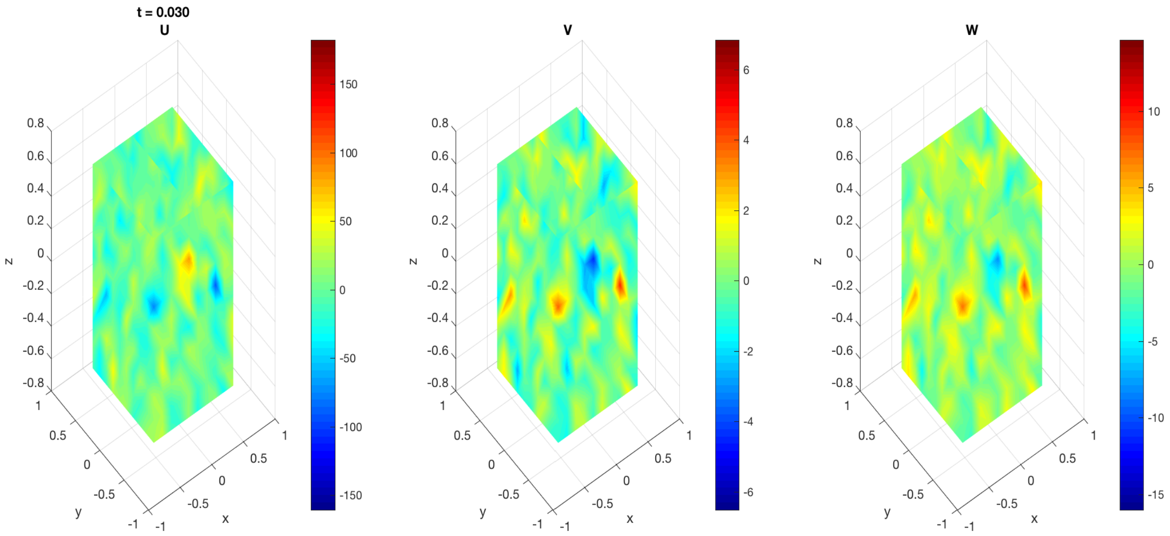

Figure 2.12: t = 0.030

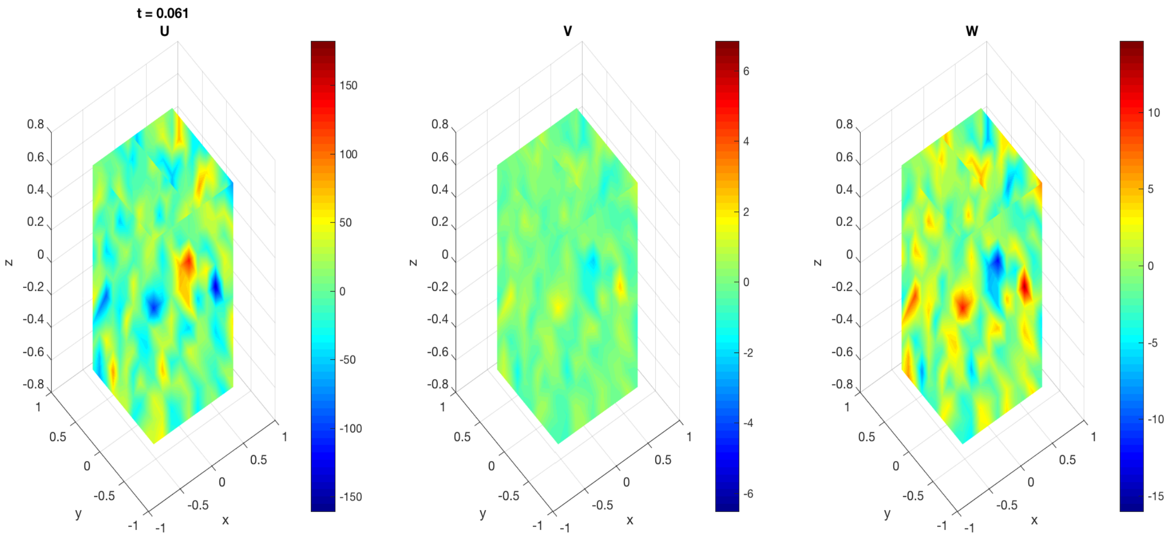

Figure 2.13: t = 0.061

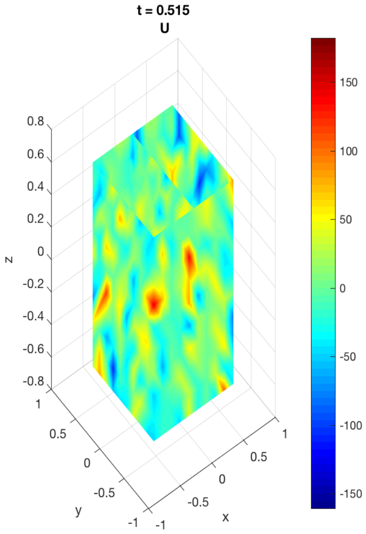

Figure 2.14: t = 0.515

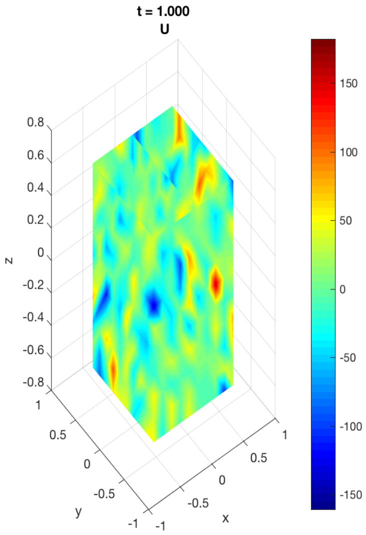

Figure 2.15: t = 1.0

3 Summary of Work

The first item action for beginning this independent study course involved studying the Navier-Stokes equations and the many definitions needed for an introduction to fluid mechanics and dynamics. This included becoming familiar with the notion of incompressibility, an intuitive sense of the difference between isotropic and anisotropic flow, and the differences between turbulent and non-turbulent flow. These basic concepts were not familiar to begin with, and it is essential to have a grasp of the underlying ideas before I can begin to apply these to numerical simulations.

After a few weeks of familiarizing myself with the background concepts – and dealing with each authors' slightly different notation – a Matlab function was written to visualize the three-dimensional fluid flow at a given time. For my style of learning, visualization is an important first step even if the initial conditions are unusable (for example, completely random instead of following the conditions for a zero-divergence flow field). The function for plotting the fluid field took about three hours total over a few days. Once this was in place, we could implement a basic numerical scheme (Fourth-Order Runge-Kutta) for solving the RDT differential equations.

The RDT differential equations are defined in the Fourier space, and already spelled out clearly in the Pope text [1]. The assumptions we made to break the problem down even further was to use shear force. Dr. Lee derived these equations previously, and after following her notes, I was able to implement the equations.

Numerically, roundoff error can cause an initially solenoidal field to develop non-zero divergence over time. In order to correct for buildup of error, we included a short "forcing function" to ensure that the field remained divergence-free for the duration of the simulation. A considerable amount of time was spent (and currently is being spent) here since zero-divergence in physical space is different than in Fourier space. Converting between the two can cause some errors that we are working on.

The most challenging part of this course was getting a sense of the conversion between physical space and Fourier space. Theoretically, I understand the difference, but to identify the different moving parts for how the difference perspectives related to Rapid Distortion Theory was and still is a challenge.

With the encouragement of Dr. Lee, we applied for a poster session at the American Physical Society Conference in March 2018. I would like to thank Dr. Lee for her responsiveness, continuous encouragement, and insight in keeping this project grounded and directed.

References

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