Rapid Distortion Theory under Anisotropic Flow

Intro

The Navier-Stokes equations describe the motion of Newtonian fluid flow. Newtonian fluids are essentially simplifications of "real-life" fluids -- **define here**.

In this study, we eventually address the difference of isotropic vs. anisotropic flow. Anisotropic fluids are directionally dependent (different properties in different directions). In isotropic flow, the statistical characteristics of turbulence are independent of the direction of space. For incompressible flow, we have:

$$\vec{u}_t + (\vec{u} \cdot \nabla\vec{u})\vec{u} = \frac{\mu}{\rho_0}\Delta\vec{u} - \frac{\nabla p}{\rho_0} + g$$

where μ is the dynamic viscosity, p is the pressure, ρ_0 is the uniform fluid density, and g represents an external force (e.g. gravity). The solution vector $\vec{u} = [u, v, w]^T$ represents the three-dimensional flow velocity. Looking to simplify the convection terms:

$$\begin{aligned}
(\vec{u} \cdot \nabla\vec{u})\vec{u} &= \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\begin{bmatrix} u \\ v \\ w \end{bmatrix} \\
&= \begin{bmatrix} uu_x + vu_y + wu_z \\ uv_x + vv_y + wv_z \\ uw_x + vw_y + ww_z \end{bmatrix} \\
&= \begin{bmatrix} (u^2)_x - uu_x + vu_y + wu_z \\ (v^2)_x - vv_y + uv_x + wv_z \\ (w^2)_x - ww_z + uw_x + vw_y \end{bmatrix} \\
&= \begin{bmatrix} (u^2)_x - uu_x + \left((uv)_y - uv_y\right) + ((uw)_z - uw_z) \\ (v^2)_x - vv_y + ((uv)_x - vu_x) + ((vw)_z - vw_z) \\ (w^2)_x - ww_z + ((uw)_x - wu_x) + \left((vw)_y - wv_y\right) \end{bmatrix} \\
&= \begin{bmatrix} (u^2)_x + (uv)_y + (uw)_z - u\left(u_x + v_y + w_z\right) \\ (v^2)_x + (uv)_x + (vw)_z - v\left(u_x + v_y + w_z\right) \\ (w^2)_x + (uw)_x + (vw)_y - w\left(u_x + v_y + w_z\right) \end{bmatrix} \\
&= \begin{bmatrix} (u^2)_x + (uv)_y + (uw)_z - u(\nabla \cdot \overrightarrow{u}) \\ (v^2)_x + (uv)_x + (vw)_z - v(\nabla \cdot \overrightarrow{u}) \\ (w^2)_x + (uw)_x + (vw)_y - w(\nabla \cdot \overrightarrow{u}) \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} (u^2)_x + (uv)_y + (uw)_z \\ (v^2)_x + (uv)_x + (vw)_z \\ (w^2)_x + (uw)_x + (vw)_y \end{bmatrix}$$

This can make some of the computations easier (?). Onto RDT …

The mean shear rate matrix (or mean shear rate of strain?) is as follows (p.154, sec 5.4.5 of Pope):

$$S = \frac{1}{2}\begin{bmatrix} u_x + u_x & u_y + v_x & u_z + w_x \\ v_x + u_y & v_y + v_y & v_z + w_y \\ w_x + u_z & w_y + v_z & w_z + w_z \end{bmatrix}$$

$$S = \Sigma_i \Sigma_j$$

The pressure rate-of-strain tensor comes from the Poisson equation for pressure. The Poisson equation for p' is:

$$\frac{1}{\rho}\Delta p' = -2\frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial u_j}{\partial x_i} - \frac{\partial^2}{\partial x_i \partial x_j}\left(u_i u_j - \langle u_i u_j\rangle\right)$$

$$p' = p^{(r)} + p^{(s)} + p^{(h)}$$

Where the three terms represent the rapid pressure, slow pressure, and harmonic contribution, respectively:

(sum over *i, j*)

$$\frac{1}{\rho}\Delta p^{(r)} = -2\frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

$$\frac{1}{\rho}\Delta p^{(s)} = -\frac{\partial^2}{\partial x_i \partial x_j}\left(u_i u_j - \langle u_i u_j\rangle\right)$$

$$\Delta p^{(h)} = 0$$

Definitions:

Homogenous fluid – properties (e.g. viscosity) are independent of position.

Visualizing Fluid Velocity Data

For a quick example of what some data might look like …

$$u(x, y, z) = 10\ randn \sin(\pi x) + 10 \sin(10\ \pi z) + 10\ y \cos(\pi y)$$