General Distortion

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1 introduction

In this paper, we construct the general framework to "distort" the faces of a flat-foldable model (distortion problem). Formally, it is described as a function that transforms the vertices of the CP $v \in \mathbb{R}^2$, to $v_d \in \mathbb{R}^2$:

$$T : v \mapsto v_d. \tag{1.1}$$

Especially, we denote usual transform (i.e. not distorted) as over-lined symbols:

$$\overline{T} : v \mapsto v_f. \tag{1.2}$$

For more illustration of the problem, see 1.

1.1 the form of the transforms

Typically, $\overline{T}$ is given as an Affine transform. And it is uniform ($\overline{T}(v_1) = \overline{T}(v_2)$) if $v_1, v_2 \in \mathbb{R}^2$ are on the same face F_n of the CP. Namely, $\overline{T}$ is characterized by a countable set of Affine transforms:

$$\overline{T} := \left\{\overline{T}_n\right\}, \tag{1.3}$$

where $n \in \mathbb{N}$ runs all over suffix of the faces. The action of an Affine transform $\overline{T}_n$ is given as

$$\overline{T}_n(v) := \overline{A}_n |v\rangle + \bar{b}_n \tag{1.4}$$

, where $\overline{A}_n$ is a 2×2 matrix, $\bar{b}_n$ and $|v\rangle$ is a 2 dimensional vertical vector in $\mathbb{R}^2$. The explicit expressions for them are given through Prim's method[**?**] by giving the initial transform for arbitrary chosen "base face F_0":

$$\overline{A}_0 \quad := \mathbb{I},\ \bar{b}_0 := \vec{0}. \tag{1.5}$$

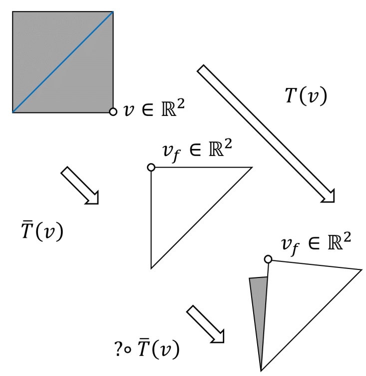

Figure 1: an illustration of Distortion Problem

1.1.1 Uniqueness

Although the selection of the base face is arbitrary, it can be proven that Prim's algorithm defines unique solutions for $\overline{T}$ in terms that $\overline{T}, \overline{S}$ from different base faces (or different spanning trees) can be inferred by simple mirroring and rotation:

$$\begin{aligned} & \exists k = \pm 1,\ \exists \theta \in [0, 2\pi] \text{ s.t. } \forall n \\ \overline{T}_n \quad & = M_k R_\theta \circ \overline{S}_n && (1.6) \\ M_k \quad & := \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix},\ R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. && (1.7) \end{aligned}$$

$\because$ TBD.

1.1.2 distorted transforms

Similarly, distorted transforms are given also as face-by-face Affine transforms $T = \{T_n\}$:

$$T_n : v \mapsto v_d = A_n |v\rangle + b_n \tag{1.8}$$

if we impose T_n to map any segments on the face F_n to segments as well (see 1.9). This requirement is critical when we consider diagramming, where crease lines can be put anywhere on the face.

For this reason, we can treat non-distorted transformation $\overline{T}$ as a special case of distorted transformation T.

1.2 constraints for the transforms

Here we suppose F_n and F_{n-1} are adjacent and share a crease line defined by its extreme points $v_n, w_n \in \mathbb{R}^2$. Suppose also that F_n places on the right hand side of the segment in the direction of $v_n \to w_n$ and F_{n-1} does vise versa.

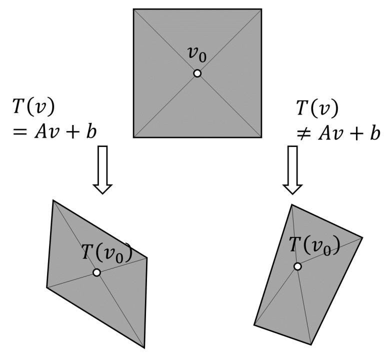

Figure 2: segments inside the face under Affine transform and other transform

An important requirement for the general distortion is

$$v_n^f := T_n(v_n) = T_{n-1}(v_n)\,,\ w_n^f := T_n(w_n) = T_{n-1}(w_n) \tag{1.9}$$

since the segment transforms to the same segment from both view of F_n, F_{n-1}. Here (1.9) implies

$$A_{n-1}|v_n\rangle = A_n|v_n\rangle\,,\ A_{n-1}|w_n\rangle = A_n|w_n\rangle\,, \tag{1.10}$$

which leads

$$(A_n - A_{n-1})|x_n\rangle = \vec{0}, \tag{1.11}$$

where

$$x_n := w_n - v_n. \tag{1.12}$$

By introducing

$$|y_n\rangle \quad := R|x_n\rangle \tag{1.13}$$

$$:= R_{-\pi/2}|x_n\rangle\,, \tag{1.14}$$

(1.11) is equivalent to

$$\exists \alpha_n \in \mathbb{R}^2 \text{ s.t. } A_n - A_{n-1} = |\alpha_n\rangle\langle y_n|\,. \tag{1.15}$$

1.2.1 applying Prim's algorithm

From (1.15), if we again apply Prim's algorithm to determine T_n for given T_{n-1}, the first equation of (1.9) imposes

$$A_n \quad = A_{n-1} + |\alpha_n\rangle\langle y_n| \tag{1.16}$$

$$b_n \quad = b_{n-1} - (A_n - A_{n-1})|v_n\rangle \tag{1.17}$$

$$= v_n^f - A_n|v_n\rangle \tag{1.18}$$

$$= b_0 - \sum_{i=1}^{n} \langle y_i|v_i\rangle|\alpha_i\rangle\,, \tag{1.19}$$

where $\langle a|b\rangle$ denotes the dot product $a\cdot b$ in $\mathbb{R}^2$ space. The last equation shows that for given parameter $\alpha_n \in \mathbb{R}^2$, b_n is completely fixed by A_n, which is also determined by α_n. Alternatively, it can be expressed as

$$T_n(v) = v_n^f + \left(A_{n-1} + |\alpha_n\rangle\langle y_n|\right)|v - v_n\rangle\,. \tag{1.20}$$

Here the property $\langle y_n|x_n\rangle = 0$ allows us to expand arbitrary $v \in \mathbb{R}^2$ with orthogonal basis $\{|x_n\rangle, |y_n\rangle\}$:

$$|v\rangle = \frac{\langle x_n|v\rangle}{\langle x_n|x_n\rangle}|x_n\rangle + \frac{\langle y_n|v\rangle}{\langle y_n|y_n\rangle}|y_n\rangle\,, \tag{1.21}$$

This leads

$$\begin{aligned} A_{n-1}|v - v_n\rangle &= \frac{\langle x_n|v - v_n\rangle}{\langle x_n|x_n\rangle}\left|x_n^f\right\rangle + \frac{\langle y_n|v - v_n\rangle}{\langle y_n|y_n\rangle}\left|y_n^f\right\rangle && (1.22) \\ & && (1.23) \end{aligned}$$

where

$$\begin{aligned} \left|x_n^f\right\rangle &:= A_{n-1}|x_n\rangle\,, && (1.24) \\ &= w_n^f - v_n^f && (1.25) \\ \left|y_n^f\right\rangle &:= A_{n-1}|y_n\rangle && (1.26) \end{aligned}$$

and thus

$$\begin{aligned} T_n(v) = \quad & v_n^f + \frac{\langle x_n|v - v_n\rangle}{\langle x_n|x_n\rangle}\left|x_n^f\right\rangle \\ & + \frac{\langle y_n|v - v_n\rangle}{\langle y_n|y_n\rangle}\left|y_n^f\right\rangle + \langle y_n|v - v_n\rangle|\alpha_n\rangle\,. \end{aligned} \tag{1.27}$$

1.2.2 constraints for the parameters

Thus distortion problem is now equivalent to put 2-real parameter for each crease lines $\{\alpha_i\}$ so that k alphas for face F_n (k-polygon) are all consistent to define the same T_n. This constraint is equivalent to considering all possible sets of faces $\{F_i\}_{n=1}^{N}$ such that F_i and F_{i+1} are adjacent by sharing the crease line with $v, v_{i+1} \in \mathbb{R}^2$ (i.e. v is surrounded by the faces clockwise) and they give $T_{N+1} = T_1$. Through (1.15), it is described as

$$\begin{aligned} 0 = A_{N+1} - A_1 &= \sum_{i=1}^{N} A_{i+1} - A_i && (1.28) \\ \Leftrightarrow 0 &= \sum_{i=1}^{N} |\alpha_i\rangle\langle v_i - v|\,R^T, && (1.29) \end{aligned}$$

where $\langle a|$ denotes the horizontal vector of a and thus $|b\rangle\langle a|$ is a matrix. And X^T denotes for the transpose of the matrix X.

2 typical cases of distortion

In general, (1.29) is hard to be satisfied not only because of its form but also because the equation for each center vertex v is not completely independent each other. Due to this issue, we first examine non-distorted transform $\overline{T_n}$ in our framework. And then we move to some nontrivial cases of T_n, which is given by a sort of the "difference" from $\overline{T_n}$.

2.1 non distorted case

Through (1.29), the non-distorted case $\overline{T_n}$ is defined as

$$|\alpha_n\rangle := -\frac{2}{\langle y_n|y_n\rangle}\overline{A_{n-1}}\,|y_n\rangle\,. \tag{2.1}$$

In fact,

$$\overline{A_n} = \overline{A_{n-1}}\left(\mathbb{I} - 2\frac{|y_n\rangle\langle y_n|}{\langle y_n|y_n\rangle}\right) \tag{2.2}$$

$$= \overline{A_{n-1}}M_{-1}R_{-2\theta_n}\,, \tag{2.3}$$

where $\theta_n := \arg y_n$. This means the transforms are given by recursively applied "flips along the crease line". Also from (1.19),

$$\overline{b_n} = \overline{b_0} - \left(\overline{A_n} - \overline{A_{n-1}}\right)|v_n\rangle \tag{2.4}$$

$$= \vec{0} + 2\sum_{i=1}^{n}\overline{A_{i-1}}\frac{|y_n\rangle\langle y_n|}{\langle y_n|y_n\rangle} \tag{2.5}$$

$$= 2\sum_{i=1}^{n}\overline{A_{i-1}}\frac{|y_n\rangle\langle y_n|}{\langle y_n|y_n\rangle}. \tag{2.6}$$

Although we can derive the expressions above from the definition of "flip", we just try to justify them by examining several properties of $\overline{A_n}$. One can check that

$$\det\overline{A_n} = \det\overline{A_{n-1}}\det M_{-1}\det R_{\theta_n} \tag{2.7}$$

$$= -\det\overline{A_{n-1}} \tag{2.8}$$

$$= (-1)^n\det\overline{A_0} \tag{2.9}$$

$$= (-1)^n = \pm 1, \tag{2.10}$$

which means the transformation for each face preserves the area of it. Moreover, it can be proven that $\overline{A_n}$ is angle preserving.

$\because$ TBD. Those suffices to justify that $\overline{T_n}$ is constructed from simple flips, which is the very non-distorted transformation.

2.1.1 Kawasaki property assures the constraint

We have to investigate (1.29) is truly saturated by (2.1). Fortunately, if all the vertices on the CP are locally flat foldable, especially satisfy Kawasaki's theorem:

$$\exists M \in \mathbb{N} \text{ s.t. } N = 2M, \ \sum_{i=1}^{M} \theta_{2i} - \theta_{2i-1} = \pi, \tag{2.11}$$

we have

$$\begin{aligned}
\overline{A_n} &= \overline{A_{n-1}} M_{-1} R_{\theta_n} &(2.12)\\
&= \overline{A_0} M_{-1} R_{-2\theta_1} M_{-1} R_{-2\theta_2} \cdots M_{-1} R_{-2\theta_{n-1}} M_{-1} R_{-2\theta_n} &(2.13)\\
&= \overline{A_0} \left(M_{-1}\right)^2 R_{2(\theta_1-\theta_2)} \cdots \left(M_{-1}\right)^2 R_{2(\theta_{n-1}-\theta_n)} &(2.14)\\
&= \overline{A_0} R_{-2\sum_{i=1}^{M} \theta_{2i}-\theta_{2i-1}} &(2.15)\\
&= \overline{A_0} R_{-2\pi} = \overline{A_0}. &(2.16)
\end{aligned}$$

Here we have used properties like

$$\begin{aligned}
M_{-1} R_{\theta} &= R_{-\theta} M_{-1} &(2.17)\\
M_{-1} M_{-1} &= \mathbb{I} &(2.18)
\end{aligned}$$

This concludes (1.29) is always satisfied for any vertices if they are locally flat foldable.

2.1.2 Practical form of the transforms

Through (1.27), the transform is practically given as

$$\overline{T_n}(v) = v_n^f + \frac{\langle x_n | v - v_n \rangle}{\langle x_n | x_n \rangle} \left| x_n^f \right\rangle - \frac{\langle y_n | v - v_n \rangle}{\langle y_n | y_n \rangle} \left| y_n^f \right\rangle. \tag{2.19}$$

Moreover, for y_n^f, the expression s given more simply:

$$\begin{aligned}
\left| y_n^f \right\rangle &= \overline{A_{n-1}} \left| y_n \right\rangle &(2.20)\\
&= \overline{A_{n-1}} R \left| x_n \right\rangle &(2.21)\\
&= (-1)^{n-1} R \overline{A_{n-1}} \left| x_n \right\rangle &(2.22)\\
&= (-1)^{n-1} R \left| x_n^f \right\rangle. &(2.23)
\end{aligned}$$

Note that now A_{n-1} in (1.26) is not needed to get y_n^f. Thus (2.19) is now

$$\overline{T_n}(v) = v_n^f + \frac{\langle v - v_n | x_n \rangle}{\langle x_n | x_n \rangle} \left| x_n^f \right\rangle + (-1)^n \frac{\langle v - v_n | R | x_n \rangle}{\langle x_n | x_n \rangle} R \left| x_n^f \right\rangle. \tag{2.24}$$

2.2 globally Perturbed case

Suppose the case where

$$\begin{aligned}
\forall n\ A_n &= \overline{A_n} + \mathcal{E}, && (2.25)\\
b_0 &\neq \overline{b_0} = \vec{0} && (2.26)
\end{aligned}$$

this requires

$$\begin{aligned}
\mathcal{E} &= A_0 - \mathbb{I}, && (2.27)\\
b_n &= b_0 - \left(\overline{A_n} + \mathcal{E} - \overline{A_{n-1}} - \mathcal{E}\right)|v_n\rangle && (2.28)\\
&= b_0 + \left(\overline{A_n} - \overline{A_{n-1}}\right)|v_n\rangle && (2.29)\\
&= b_0 + \overline{b_n}, && (2.30)\\
A_n - A_{n-1} &= \overline{A_n} - \overline{A_{n-1}}. && (2.31)
\end{aligned}$$

The last equation saturates the requirement (1.29) through 2.1.1. Hence,

$$\begin{aligned}
T_n(v) &= \overline{A_n}|v\rangle + \mathcal{E}|v\rangle + b_0 + \overline{b_n} && (2.32)\\
&= \overline{T_n}(v) + \mathcal{E}|v\rangle + b_0 && (2.33)\\
&= \overline{T_n}(v) + (A_0 - \mathbb{I})|v\rangle + b_0. && (2.34)
\end{aligned}$$

2.3 globally transformed case

Suppose the case

$$\begin{aligned}
\forall n\ A_n &= X\overline{A_n} + \mathcal{E}, && (2.35)\\
b_0 &\neq \overline{b_0} = \vec{0}. && (2.36)
\end{aligned}$$

They require

$$\begin{aligned}
\mathcal{E} &= A_0 - X, && (2.37)\\
b_n &= b_0 - \left(X\overline{A_n} + \mathcal{E} - X\overline{A_{n-1}} - \mathcal{E}\right)|v_n\rangle && (2.38)\\
&= b_0 + X\left(\overline{A_n} - \overline{A_{n-1}}\right)|v_n\rangle && (2.39)\\
&= b_0 + X\overline{b_n}, && (2.40)\\
A_n - A_{n-1} &= X\left(\overline{A_n} - \overline{A_{n-1}}\right). && (2.41)
\end{aligned}$$

The last equation again assures the requirement (1.29) again through 2.1.1. Hence,

$$\begin{aligned}
T_n(v) &= X\overline{A_n}|v\rangle + \mathcal{E}|v\rangle + b_0 + X\overline{b_n} && (2.42)\\
&= X\overline{T_n}(v) + \mathcal{E}|v\rangle + b_0 && (2.43)\\
&= X\overline{T_n}(v) + (A_0 - X)|v\rangle + b_0. && (2.44)
\end{aligned}$$

If $\det X \neq 0$, the last expression can be reinterpreted as the case where (2.34) with $b_0 = \vec{0}$ is globally affected by X:

$$\begin{aligned}
T_n(v) &= X\left(\overline{T_n}(v) + (A'_0 - \mathbb{I})|v\rangle\right) + b_0 && (2.45)\\
&=: T_X\left(\overline{T_n}(v) + (A'_0 - \mathbb{I})|v\rangle\right) && (2.46)
\end{aligned}$$

where $A'_0 := X^{-1}A_0,\ T_X(v) := X|v\rangle + b_0$.

Note also that those distortions given as

$$T_n(v) = \overline{T_n}(v) + (A_0 - \mathbb{I})|v\rangle \tag{2.47}$$

is generated from non-distorted transformation, and hence it is again unique even that choosing the base face and spanning tree is arbitrary.

3 CP distortion

Now distortion (2.46) is rewritten as

$$\begin{aligned} v_d &= T_X(v_f + dv) && (3.1)\\ dv &:= v' - v && (3.2)\\ v' &:= A_0(v), && (3.3) \end{aligned}$$

where T_X, A_0 can be any Affine transform or linear operator respectively. Since T_X is a global transformation (i.e. rotate or shearing all the vertices at once), non-trivial part is dv and it is given by difference between original CP position v and "globally vended CP" position v'. Thus These arbitrary actions are better to be considered as geometrical parameters like "rotation" $\theta \in [0, 2\pi]$, "shears" $s, t \neq 0$, and its "direction" $\phi \in [0, \pi]$:

$$A_0 := R_{\theta+\phi}\begin{pmatrix} s & \\ & t \end{pmatrix} R_{-\phi}. \tag{3.4}$$

for instance, assuming T_X is identity and

$$\begin{aligned} s = t &=: (1-k) && (3.5)\\ \theta = \phi &= 0, && (3.6) \end{aligned}$$

gives an one parameter distortion where the CP is scaled with $k \in \mathbb{R}$.

$$v_d = v_f - kv \tag{3.7}$$

3.1 non global CP distortion

From this point, it is worth to study how to vend CP non globally but locally Affine because it directly implies distortion of the folded model.