Affine Distortions

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1 introduction

As mentioned in [2],

> If a diagrammer were to draw a model in a mathematically ideal way, it would convey very little information: all edges would line up, all creases would run all the way to corners; multiple layers would perfectly overlap. In practice, then, the diagrammer must introduce small distortions: gaps between edges, layers that do not line up. Such distortions convey far more information and should be included wherever appropriate (of course, they are also harder to draw)[5]

a conventional diagram is not drawn in a "mathematically-ideal" way. For more illustration of the problem, see Fig.1.

The study in [3] offers a way to distort a diagram with a set of few parameters. The scheme proposed here is based on the "layer structures" of

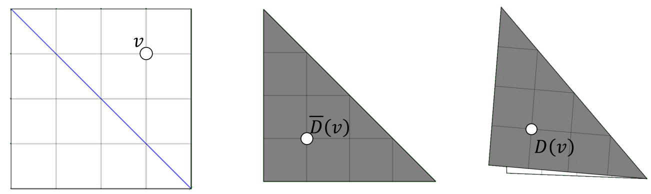

Figure 1: An example of distortion

a diagram. Hence, we need to calculate the collision constraints [1] to get a diagram. As we see later in this paper, the scheme shown in this paper does not assume the layer orders of faces in the diagram.

In this paper, we construct mathematically general framework to obtain distorted diagrams based on affine transforms of the "crease patterns".

2 Notations, Definitions and Lemmas

In this paper, we denote the set of real numbers as $\mathbb{R}$. Any vertex v on the 2-d plane is an element of $\mathbb{R} \times \mathbb{R} =: \mathbb{R}^2$.

$$v := (v_x, v_y)^T \in \mathbb{R}^2, \tag{2.1}$$

where T denotes the transpose operation.

For any pair of $v, w \in \mathbb{R}^2$,

$$v^T w := v_x w_x + v_y w_y \in \mathbb{R}, \tag{2.2}$$

is the inner product, and

$$vw^T := \begin{pmatrix} v_x w_x & v_y w_x \\ v_x w_y & v_y w_y \end{pmatrix}, \tag{2.3}$$

is a 2×2 matrix.

In this paper, we denote vertices in $\mathbb{R}^2$ with lowercase of alphabets e.g. v, w, p, q. Operators such as matrices are denoted by uppercase of alphabets e.g. R, M, D. Subsets of $\mathbb{R}^2$ is denoted as calligraphic alphabets e.g. $\mathcal{F}, \mathcal{E}$. For families of those subsets, we use Fraktur alphabets e.g. $\mathfrak{C}$ and they are labeled with Greek characters e.g. λ, α, β.

2.1 Linear Algebra

For any 2×2 real matrix (operator) Y:

$$Y := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.4}$$

the determinant and the trace are defined as

$$\det(Y) := ad - bc \tag{2.5}$$
$$\mathrm{Tr}(Y) := a + d. \tag{2.6}$$

The group of those matrices with $\det(Y) \neq 0$:

$$\mathrm{GL}_2(\mathbb{R}) := \left\{ Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det(Y) \neq 0, \quad a, b, c, d \in \mathbb{R} \right\} \tag{2.7}$$

is called general linear group $\mathrm{GL}_2(\mathbb{R})$.

We also introduce notations for typical $\mathrm{GL}_2(\mathbb{R})$ elements:

$$\mathbb{I} \quad := \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.8}$$

is the identity operator.

$$M \quad := \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.9}$$

is the mirror operator, and

$$R_\theta \quad := \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.10}$$

is the rotational operator. Especially,

$$R := R_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.11}$$

is the 90-degree rotation.

Lemma 1.

$$R_\theta = \cos\theta\mathbb{I} + \sin\theta R. \tag{2.12}$$

Lemma 2. We can show identities below $\forall\theta, \theta_1, \theta_2 \in \mathbb{R}$:

$$(1) \qquad M^2 = \mathbb{I} \tag{2.13}$$

$$(2) \qquad R_{\theta_1} R_{\theta_2} = R_{\theta_2} R_{\theta_1} = R_{\theta_1+\theta_2} \tag{2.14}$$

$$(3) \qquad R_\theta^{-1} = R_{-\theta} = R_\theta^T \tag{2.15}$$

$$(4) \qquad MR_\theta = R_{-\theta}M \tag{2.16}$$

$$(5) \qquad \det(\mathbb{I}) = \det(R_\theta) = 1 \tag{2.17}$$

$$(6) \qquad \det(M) = -1 \tag{2.18}$$

Therefore, $\langle \mathbb{I}, M, R_\theta, MR_\theta \rangle$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

2.2 Geometry

Definition 3 (Face)**.** A face is a subset of $\mathbb{R}^2$

$$\mathcal{F} \subset \mathbb{R}^2. \tag{2.19}$$

Definition 4 (Edges)**.** For a family of faces,

$$\mathfrak{C} := \{\mathcal{F}_\lambda\}_{\lambda\in\Lambda} \tag{2.20}$$

we define the edges:

$$\mathfrak{S}(\mathfrak{C}) := \left\{\mathcal{E} = \mathcal{F}_\alpha \cap \mathcal{F}_\beta \mid \exists p \neq q \in \mathbb{R}^2 \quad \text{s.t.} \quad \mathcal{E} = \mathrm{Conv}(p, q)\right\}. \tag{2.21}$$

Definition 5. A pair of faces $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ is adjacent

$$\mathcal{F}_\alpha \mid \mathcal{F}_\beta \tag{2.22}$$

if and only if

$$\mathcal{F}_\alpha \cap \mathcal{F}_\beta \in \mathfrak{S}(\mathfrak{C}). \tag{2.23}$$

2.3 Crease Patterns

Definition 6 (Crease Pattern)**.** A family of faces

$$\mathfrak{C} := \{\mathcal{F}_\lambda\}_{\lambda\in\Lambda} \tag{2.24}$$

is a crease pattern if and only if:

$$\begin{aligned} &\forall\alpha, \forall\beta \in \Lambda, \quad \exists\mathcal{F}_{\lambda_1}, \dots, \mathcal{F}_{\lambda_n} \in \mathfrak{C} \\ &\text{s.t.} \quad \forall 0 \le i \le n, \quad \mathcal{F}_{\lambda_i} \mid \mathcal{F}_{\lambda_{i+1}}, \end{aligned} \tag{2.25}$$

where $\lambda_0 := \alpha$ and $\lambda_{n+1} := \beta$.

Definition 7 (The Crease Projections)**.** For a edge of a crease pattern $\forall\mathcal{E} \in \mathfrak{S}(\mathfrak{C})$, we define the crease projections as

$$P_\mathcal{E} := \frac{dd^T}{|d|^2}, \quad P_\mathcal{E}^\perp := R\frac{dd^T}{|d|^2}R^T, \quad d := p - q, \tag{2.26}$$

where $p \neq q \in \mathcal{E}$ are arbitrary vertices on the edge.

Note that the definition is independent of the choices of $p, q \in \mathcal{E}$ since $\forall p_1, p_2, q_1, q_2 \in \mathcal{E}$,

$$p_1 - q_1 \propto p_2 - q_2. \tag{2.27}$$

Lemma 8. For any $\mathcal{E} \in \mathfrak{S}(\mathfrak{C})$,

$$P_\mathcal{E} + P_\mathcal{E}^\perp = \mathbb{I} \tag{2.28}$$

is always consistent.

Lemma 9. For a pair of $\forall v, w \in \mathcal{E}$,

$$P_\mathcal{E}R(v - w) = \vec{0} = P_\mathcal{E}^\perp(v - w). \tag{2.29}$$

is always consistent.

Theorem 10. For an arbitrary edge $\mathcal{E} \in \mathfrak{S}(\mathfrak{C})$, there exists $\exists\theta \in [0, 2\pi)$ such that

$$P_\mathcal{E}^\perp = R_{2\theta}M. \tag{2.30}$$

Proof. We can suppose that $\exists\theta \in [0, 2\pi)$ for $\forall p \neq q \in \mathcal{E} \in \mathfrak{S}(\mathfrak{C})$

$$d = (p - q)/|p - q| =: (\cos\theta, \sin\theta)^T \tag{2.31}$$

without loss of generality. Therefore,

$$\begin{aligned} \mathbb{I} - 2P_e^\perp &= R\left(\mathbb{I} - 2\frac{(p - q)(p - q)^T}{|p - q|^2}\right)R^T &(2.32)\\ &= R\left(\mathbb{I} - 2(\cos\theta, \sin\theta)(\cos\theta, \sin\theta)^T\right)R^T &(2.33)\\ &= -RR_{2\theta}MR^T &(2.34)\\ &= -RR_{2\theta}RM = R_{2\theta}M &(2.35) \end{aligned}$$

■

2.4 Flatness

Definition 11 (Loop on a Crease Pattern)**.** The loops on a crease pattern is defined as

$$\mathcal{L}(\mathfrak{C}) := \left\{\{\lambda_i\}_{i=1}^{n_L} \subset \Lambda \mid \mathcal{F}_{\lambda_i} \mid \mathcal{F}_{\lambda_{i+1}}, \quad \lambda_{n_L+1} = \lambda_1, \quad 1 \le i \le n_L < \infty\right\} \tag{2.36}$$

By definition 6, any faces of $\mathfrak{C}$ are included in a loop $L \in \mathcal{L}(\mathfrak{C})$.

Definition 12 (Pi Operators of a Loop)**.** For a loop $L = \{\lambda_i\}_{i=1}^{n_L} \in \mathcal{L}(\mathfrak{C})$, we define the Pi operators:

$$\Pi_i(L) := \left(\mathbb{I} - 2P_{\mathcal{E}_i}^{\perp}\right)\left(\mathbb{I} - 2P_{\mathcal{E}_{i-1}}^{\perp}\right)\cdots\left(\mathbb{I} - 2P_{\mathcal{E}_1}^{\perp}\right) \quad (1 \le i \le n_L), \tag{2.37}$$

where $\mathcal{E}_i := \mathcal{F}_{\lambda_i} \cap \mathcal{F}_{\lambda_{i+1}}$.

Definition 13 (Flat Crease Patterns)**.** A crease pattern $\mathfrak{C}$ is flat if and only if

$$\forall L = \{\lambda_i\}_{i=1}^{n_L} \in \mathcal{L}(\mathfrak{C}), \qquad \Pi_{n_L}(L) = \mathbb{I}, \tag{2.38}$$

$$\exists v_i \in \mathcal{E}_i \qquad \text{s.t.} \quad \sum_{i=1}^{n_L} \Pi_i(L)\, P_{\mathcal{E}_i}^{\perp} v_i = \vec{0}, \tag{2.39}$$

where $\mathcal{E}_i := \mathcal{F}_{\lambda_i} \cap \mathcal{F}_{\lambda_{i+1}}$.

Lemma 14. If a crease pattern $\mathfrak{C}$ is flat, $\forall L \in \mathcal{L}(\mathfrak{C})$, $n_L < \infty$ is an even number.

Proof. Through 10, $\forall e_i$, $\exists \theta_i \in \mathfrak{S}(\mathfrak{C})$ such that

$$\begin{aligned}&\left(\mathbb{I} - 2P_{e_{n_L}}^{\perp}\right)\left(\mathbb{I} - 2P_{e_{n_L-1}}^{\perp}\right)\cdots\left(\mathbb{I} - 2P_{e_1}^{\perp}\right)\\ &= \left(R_{2\theta_{n_L}} M\right)\left(R_{2\theta_{n_L-1}} M\right)\cdots\left(R_{2\theta_1} M\right).\end{aligned} \tag{2.40}$$

By taking the determinant,

$$\begin{aligned}1 = \det(\mathbb{I}) &= \det\left(\left(\mathbb{I} - 2P_{e_{n_L}}^{\perp}\right)\left(\mathbb{I} - 2P_{e_{n_L-1}}^{\perp}\right)\cdots\left(\mathbb{I} - 2P_{e_1}^{\perp}\right)\right) && (2.41)\\ &= \det\left(\left(R_{2\theta_{n_L}} M\right)\left(R_{2\theta_{n_L-1}} M\right)\cdots\left(R_{2\theta_1} M\right)\right) && (2.42)\\ &= \det\left(R_{2\theta_{n_L}} M\right)\det\left(R_{2\theta_{n_L-1}} M\right)\cdots\det\left(R_{2\theta_1} M\right) && (2.43)\\ &= (1)^{n_L}(-1)^{n_L}. && (2.44)\end{aligned}$$

Therefore, the length of the loop n_L is an even number. ∎

2.5 Classes of Diagrams

Definition 15 (Diagrams of a Crease Pattern)**.** A diagram of a crease pattern $\mathfrak{C} = \{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ is defined as a set of maps:

$$D(\mathfrak{C}) := \left\{D_\lambda : \mathcal{F}_\lambda \to \mathbb{R}^2\right\}_{\lambda\in\Lambda}, \tag{2.45}$$

Definition 16 (Continuous Diagrams)**.** A diagram of a crease pattern

$\mathfrak{C} = \{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ is continuous if and only if $\forall\alpha,\beta \in \Lambda$ such that $\mathcal{F}_\alpha \mid \mathcal{F}_\beta$,

$$\forall v \in \mathcal{F}_\alpha \cap \mathcal{F}_\beta, \quad D_\alpha(v) = D_\beta(v). \tag{2.46}$$

Definition 17 (An Affine Diagram)**.** A diagram $D(\mathfrak{C})$ is affine if and only if $\forall\lambda \in \Lambda$ and $\forall v \in \mathcal{F}_\lambda$,

$$\begin{aligned} &\exists A_\lambda \in \mathrm{GL}_2(\mathbb{R}), \quad \exists b_\lambda \in \mathbb{R}^2 \\ &\text{s.t.} \quad D_\lambda(v) = A_\lambda v - b_\lambda. \end{aligned} \tag{2.47}$$

Definition 18 (A Regular Diagram)**.** An affine diagram $D(\mathfrak{C})$ is regular if and only if $\forall\alpha,\beta \in \Lambda$ such that $\mathcal{F}_\alpha \mid \mathcal{F}_\beta$,

$$\det(A_\alpha)\det(A_\beta) < 0 \tag{2.48}$$

Definition 19 (The Flat Diagram)**.** An affine diagram $D(\mathfrak{C})$ is flat if and only if $\forall\alpha,\beta \in \Lambda$ such that $\mathcal{F}_\alpha \mid \mathcal{F}_\beta$,

$$A_\beta = A_\alpha\left(\mathbb{I} - 2P_{\mathcal{E}}^\perp\right), \tag{2.49}$$

$$b_\beta = b_\alpha + (A_\beta - A_\alpha)\, v, \tag{2.50}$$

where $\mathcal{E} =: \mathcal{F}_\alpha \cap \mathcal{F}_\beta$, $\forall v \in \mathcal{E}$.

Theorem 20. The flat diagram $D(\mathfrak{C})$ of a crease pattern $\mathfrak{C} = \{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ exists if and only if the crease pattern $\mathfrak{C}$ is flat.

Proof. For a loop $\forall L \in \mathcal{L}(\mathfrak{C})$,

$$A_{\lambda_i} = A_{\lambda_{i+1}}\left(\mathbb{I} - 2P_{\mathcal{E}_i}^\perp\right) \tag{2.51}$$

for $\forall\lambda_i \in L$. By accumulating the expression,

$$\begin{aligned}
A_{\lambda_1} &= A_{\lambda_2}\left(\mathbb{I} - 2P_{e_1}^\perp\right) && (2.52)\\
&= A_{\lambda_3}\left(\mathbb{I} - 2P_{\mathcal{E}_2}^\perp\right)\left(\mathbb{I} - 2P_{\mathcal{E}_1}^\perp\right) && (2.53)\\
&\cdots \\
&= A_{\lambda_{n_L+1}}\left(\mathbb{I} - 2P_{\mathcal{E}_{n_L}}^\perp\right)\cdots\left(\mathbb{I} - 2P_{\mathcal{E}_1}^\perp\right) && (2.54)\\
&= A_{\lambda_1}\left(\mathbb{I} - 2P_{\mathcal{E}_{n_L}}^\perp\right)\cdots\left(\mathbb{I} - 2P_{\mathcal{E}_1}^\perp\right) && (2.55)\\
\Leftrightarrow \mathbb{I} &= \Pi_{n_L}(L). && (2.56)
\end{aligned}$$

since $A_{\lambda_1} = A_{n_L+1} \in \mathrm{GL}_2(\mathbb{R})$. Similarly, for any $v_i \in \mathcal{E}_i$,

$$\begin{aligned}
b_{\lambda_1} &= b_{\lambda_2} - 2A_{\lambda_1}\Pi_1(L)P_{\mathcal{E}_1}^\perp v_1 && (2.57)\\
&= b_{\lambda_3} - 2A_{\lambda_1}\Pi_2(L)P_{\mathcal{E}_2}^\perp v_2 - 2A_{\lambda_1}\Pi_1(L)P_{\mathcal{E}_1}^\perp v_1 && (2.58)\\
&\cdots \\
&= b_{\lambda_{n_L+1}} - 2A_{\lambda_1}\sum_{i=1}^{n_L}\Pi_i(L)P_{\mathcal{E}_i}^\perp v_i && (2.59)\\
\Leftrightarrow \vec{0} &= \sum_{i=1}^{n_L}\Pi_i(L)P_{\mathcal{E}_i}^\perp v_i, && (2.60)
\end{aligned}$$

since $b_{\lambda_1} = b_{n_L+1}$ and since $A_{\lambda_1} \in \mathrm{GL}_2(\mathbb{R})$. ■

Theorem 21. The flat diagram $D(\mathfrak{C})$ of a crease pattern $\mathfrak{C} = \{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ is a regular diagram if it exists.

Proof. Through 10,

$$\begin{aligned}
\forall \alpha, \beta \in \Lambda, \quad \mathcal{F}_\alpha \mid \mathcal{F}_\beta \Rightarrow A_\beta &= A_\alpha (\mathbb{I} - 2P_{\mathcal{E}}^{\perp}) && (2.61)\\
\Rightarrow A_\alpha A_\beta &= A_\alpha A_\alpha R_{2\theta} M && (2.62)\\
\Rightarrow \det(A_\alpha A_\beta) &= \det(A_\alpha A_\alpha R_{2\theta} M) && (2.63)\\
\Leftrightarrow \det(A_\alpha)\det(A_\beta) &= \det(A_\alpha)^2 \det(R_{2\theta})\det(M) && (2.64)\\
&= -\det(A_\alpha)^2 < 0, && (2.65)
\end{aligned}$$

since $A_\alpha \in \mathrm{GL}_2(\mathbb{R})$ and therefore, $\det(A_\alpha) \neq 0$. ∎

Theorem 22. The flat diagram $D(\mathfrak{C})$ of a crease pattern $\mathfrak{C} = \{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ is continuous if it exists.

Proof. For $\forall v, w \in \mathcal{E} := \mathcal{F}_\alpha \cap \mathcal{F}_\beta$, the expression (2.49) shows

$$\begin{aligned}
A_\alpha - A_\beta &= 2A_\alpha P_{\mathcal{E}}^{\perp} && (2.66)\\
\Rightarrow (A_\alpha - A_\beta)(v - w) &= 2A_\alpha P_{\mathcal{E}}^{\perp}(v - w) = \vec{0} && (2.67)\\
\Rightarrow (A_\alpha - A_\beta)v &= (A_\alpha - A_\beta)w && (2.68)
\end{aligned}$$

is always consistent through 9. Therefore, the expression (2.50) directly shows that $D(\mathfrak{C})$ is continuous. ∎

Definition 23 (Equivalent Diagrams)**.** A pair of Diagrams $D(\mathfrak{C})$ and $D'(\mathfrak{C})$ is equivalent

$$D(\mathfrak{C}) \sim D'(\mathfrak{C}) \qquad (2.69)$$

if and only if $\exists X \in \mathrm{GL}_2(\mathbb{R})$ and $\exists \xi \in \mathbb{R}^2$ such that

$$\forall \lambda \in \Lambda, \quad \forall v \in \mathcal{F}_\lambda, \quad D'_\lambda(v) = X(D_\lambda(v)) + \xi. \qquad (2.70)$$

Theorem 24. A pair of flat diagrams of a flat crease pattern is equivalent.

Proof. Suppose that two different flat diagrams D and D' are given for a crease pattern $\mathfrak{C} = \{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$. We denote those affine diagrams as

$$\begin{aligned}
D_\lambda(v) &:= A_\lambda v + b_\lambda && (2.71)\\
D'_\lambda(v) &:= A'_\lambda v + b'_\lambda. && (2.72)
\end{aligned}$$

Since they are flat diagrams, $\forall \alpha, \beta \in \Lambda$ such that $\mathcal{F}_\alpha \mid \mathcal{F}_\beta$,

$$\begin{aligned}
A_\beta &= A_\alpha (\mathbb{I} - 2P_{\mathcal{E}}^{\perp}) && (2.73)\\
A'_\beta &= A'_\alpha (\mathbb{I} - 2P_{\mathcal{E}}^{\perp}) && (2.74)\\
\Rightarrow A'_\beta A_\beta^{-1} &= A'_\alpha A_\alpha^{-1} =: X, && (2.75)
\end{aligned}$$

where $e := \mathcal{F}_\alpha \cap \mathcal{F}_\beta$. Here, A_α^{-1} and A_β^{-1} always exist since $A_\alpha, A_\beta \in \mathrm{GL}_2(\mathbb{R})$. Through

$$\det(X) = \det(A'_\alpha A_\alpha^{-1}) = \det(A'_\alpha)\det(A_\alpha^{-1}) \neq 0, \qquad (2.76)$$

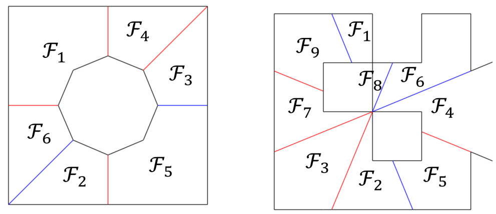

Figure 2: Examples of "odd" but flat crease patterns. The faces do not need to be closed nor compact.

we conclude $X \in \mathrm{GL}_2(\mathbb{R})$. By definition 6, $\forall \lambda \in \Lambda$,

$$\exists \{\mathcal{F}_{\lambda_i}\}_{i=1}^{n}, \quad \mathcal{F}_{\lambda_0} = \mathcal{F}_{\alpha}, \quad \mathcal{F}_{\lambda_{n+1}} = \mathcal{F}_{\lambda} \quad \text{s.t.} \quad \mathcal{F}_{\lambda_i} \mid \mathcal{F}_{\lambda_{i+1}}. \tag{2.77}$$

Therefore, $A'_{\lambda_i} A^{-1}_{\lambda_i} = A'_{\lambda_{i+1}} A^{-1}_{\lambda_{i+1}} = X$ for $i = 1, \dots n$. Hence, for all $\lambda \in \Lambda$,

$$A'_{\lambda} A^{-1}_{\lambda} = X\colon \text{const.} \tag{2.78}$$

Moreover, by definition 19, $\forall v_i \in \mathcal{F}_{\lambda_i} \cap \mathcal{F}_{\lambda_{i+1}}$,

$$\begin{aligned}
b_{\lambda_{i+1}} - b_{\lambda_i} &= (A_{\lambda_{i+1}} - A_{\lambda_i}) v_i && (2.79)\\
b'_{\lambda_{i+1}} - b'_{\lambda_i} &= (A'_{\lambda_{i+1}} - A'_{\lambda_i}) v_i = X (A_{\lambda_{i+1}} - A_{\lambda_i}) v_i && (2.80)\\
&= X (b_{\lambda_{i+1}} - b_{\lambda_i}) && (2.81)\\
\Leftrightarrow b'_{\lambda_{i+1}} - X b_{\lambda_{i+1}} &= b'_{\lambda_i} - X b_{\lambda_i}. && (2.82)
\end{aligned}$$

Therefore, for all $\lambda \in \Lambda$,

$$b'_{\lambda} - X b_{\lambda} =\colon \xi\colon \text{const.} \tag{2.83}$$

Hence,

$$\begin{aligned}
D'_{\lambda}(v) = A'_{\lambda} v + b'_{\lambda} &= X A_{\lambda} v + \xi + X b_{\lambda} && (2.84)\\
&= X (A_{\lambda} v + b_{\lambda}) + \xi && (2.85)\\
&= X (D_{\lambda}(v)) + \xi. && (2.86)
\end{aligned}$$

■

3 Implications of the Definitions

Since crease patterns are defined simply as a connected family of faces 6, it allows "odd" crease patterns as you can see in Fig. 2. Note also that the MV assignments (red/blue) of the edges are not included in this paper. A diagram is defined as a transform of faces in the crease pattern through the definition 15. Note that the diagrams in this paper do not have the layer orders of faces but just overlapping faces. Still, we can apply the collision constraints such as "taco-taco constraints" in [1] to estimate the layer orders for any diagrams in this paper.

3.1 The Crease Projections

The flatness defined in the definition 13 is an abstraction and generalization of Kawasaki–Justin theorem [4]. Kawasaki-Justin theorem states the local flat-foldable conditions of a vertex in the interior of the crease pattern. However, it cannot be applied to the crease pattern with holes such as Fig.3. It is the reason it requires (2.39) in the latter half of the definition.

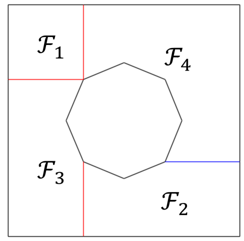

Figure 3: An example of non-flat crease pattern that satisfies Kawasaki-Justin theorem.

We have installed the definition of "mathematically ideal"[5] diagrams as the flat diagram 19. The definition refers the well studied estimation algorithm through the mirror transforms shown in [6] since

$$\mathbb{I} - 2P_{\mathcal{E}}^{\perp} = R_{2\theta}M \tag{3.1}$$

in (10) is the mirror transform along the edge of the crease pattern with angle θ.

Generally speaking, we have a difficulty that the angle of a segment depends on which end point is set to the origin to measure the angle, which leads to ambiguity between θ and $\theta + \pi$. However, we can also see that the definition in 7 avoids such issues since the difference of π is automatically immune:

$$R_{2(\theta+\pi)}M = R_{2\theta+2\pi}M = R_{2\theta}M. \tag{3.2}$$

This property is the reason why the definitions of flatness is also based on those projection operators $P_{\mathcal{E}}^{\perp}$.

3.2 The Flat Diagram

Followings in this paper, we suppose that $\mathfrak{C}$ is always flat so that the flat diagram is always given.

Since the flat diagram is unique in terms of 24, We denote the flat diagram as

$$\overline{D}(\mathfrak{C}) \coloneqq \left\{\overline{D}_\lambda : v \mapsto \overline{A}_\lambda v + \overline{b}_\lambda\right\}_{\lambda\in\Lambda}, \tag{3.3}$$

$$\exists\lambda_0 \in \Lambda \quad \text{s.t.} \quad \overline{A}_{\lambda_0} = \mathbb{I}, \quad \overline{b}_{\lambda_0} = \vec{0}. \tag{3.4}$$

$\mathcal{F}_{\lambda_0}$ is the base face.

The affine property 17 of the flat diagram ensures:

- any 3 aligned vertices in the crease pattern are also aligned in the diagram.
- any vertices that divide a segment with $1 : t$ in the crease pattern, also divide the segment after the transformation with the same proportion $1 : t$ in the diagram.

Those properties refer the practical demands on origami diagrams: the crease lines on a face must be also linear, but must not bend in the middle of it (see also Fig.4).

The most evident, continuous and affine diagram is

$$D_\lambda(v) = \mathbb{I}v + \vec{0}, \tag{3.5}$$

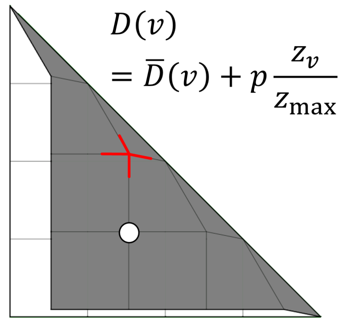

Figure 4: an example for non-affine diagram proposed by Hugo et.al.[3]. Discover that those red crease lines on the face are not linear in the diagram.

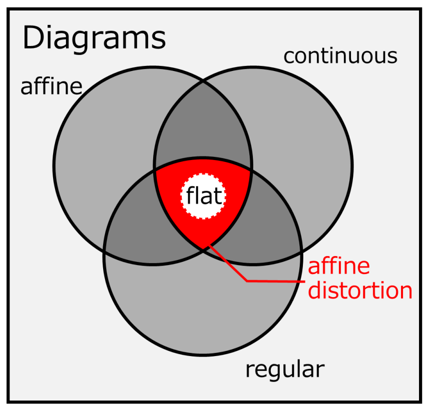

Figure 5: The classes of diagrams

where the diagram is the crease pattern itself. Hence, the classes of diagrams between affine and flat have a lot of gaps. The regularity 18 of the flat diagram lies in the middle of those classes (see also Fig. 5). It reflects the properties that any pairs of adjacent faces are flipped each other since any matrices with negative determinant

$$\det(A_\lambda) < 0 \tag{3.6}$$

map any clockwise relationships in the crease pattern into anti-clockwise relationships in the diagram, and vise-versa.

Note that maps with $\det(A_\lambda) = 0$ are apparently excluded since $A_\lambda \in \mathrm{GL}_2(\mathbb{R})$. It means that any of faces in the crease pattern are not squashed onto a segment in the diagram. In other words, we do not see the faces from their sides in the diagram.

4 Regular Distortions

Now we study the class of affine diagrams with regularity for a given crease pattern $\mathfrak{C} = \{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$. Since for an arbitrary $\alpha, \beta \in \Lambda$ that satisfies $\mathcal{F}_\alpha \mid \mathcal{F}_\beta$,

$$\overline{A}_\beta = \overline{A}_\alpha R_{2\theta} M. \tag{4.1}$$

By definition 6, we can form a series of labels $\{\lambda_i\}_{i=1}^n$ such that

$$\mathcal{F}_{\lambda_i} \mid \mathcal{F}_{\lambda_{i+1}} \quad (0 \le i \le n), \tag{4.2}$$

where λ_0 is the base face in (3.4) and $\lambda_{n+1} := \alpha$. Therefore,

$$\begin{aligned}\overline{A}_\alpha &= \overline{A}_{\lambda_0}\left(R_{2\theta_1}M\right)\cdots\left(R_{2\theta_n}M\right) && (4.3)\\ &= \mathbb{I}\left(R_{2\theta_1}M\right)\left(MR_{-2\theta_2}\right)\cdots\left(R_{2\theta_n}M\right) && (4.4)\\ &= \mathbb{I}\left(R_{2(\theta_1-\theta_2)}\right)\cdots\left(R_{2\theta_n}M\right). && (4.5)\end{aligned}$$

Thus there exists $\theta_\alpha \in [0, 2\pi)$ and $s_\alpha \in \{0, 1\}$ such that

$$\overline{A}_\alpha = R_{\theta_\alpha}M^{s_\alpha}, \quad \overline{A}_\beta = R_{\theta_\alpha+(-1)^{s_\alpha}2\theta}M^{s_\alpha+1}. \tag{4.6}$$

where $R_{2\theta_i}M = P^{\perp}_{\mathcal{E}_i}$ and $\mathcal{E}_i := \mathcal{F}_{\lambda_i} \cap \mathcal{F}_{\lambda_{i+1}}$.

4.1 Distortion

We denote an arbitrary diagram as

$$\forall \lambda \in \Lambda, \forall v \in \mathcal{F}_\lambda, \quad A_\lambda = \overline{A}_\lambda + H_\lambda \tag{4.7}$$

and the case where $H_\lambda = 0$ is the flat diagram. Hence, $H_\lambda \neq 0$ gives a "distorted" diagram which is slightly different from "mathematically predicted" diagram.

Definition 25. An affine diagram

$$D(\mathfrak{C}) = \left\{D_\lambda : v \mapsto \left(\overline{A}_\lambda + H_\lambda\right)v + b_\lambda\right\}_{\lambda\in\Lambda} \tag{4.8}$$

is distorted if and only if

$$\exists \lambda \in \Lambda \quad \text{s.t.} \quad H_\lambda \neq 0. \tag{4.9}$$

We call $\{H_\lambda\}_{\lambda\in\Lambda}$ distortion.

4.2 Regularity for the Faces Up

The regularity of a diagram in definition 18 can be examined directly. Suppose that $\overline{A}_\alpha = R_\theta$. It means $\mathcal{F}_\alpha$ is "face up" in the diagram since $\det\left(\overline{A}_\alpha\right) > 0$. And the distortion

$$H_\alpha := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.10}$$

does not flip the sign of the determinant of the face $\mathcal{F}_\alpha$:

$$0 < \det\left(R_\theta + H_\alpha\right). \tag{4.11}$$

The inequality leads

$$\begin{aligned} 0 &< \det\begin{pmatrix} \cos\theta + a & -\sin\theta + b \\ \sin\theta + c & \cos\theta + d \end{pmatrix} && (4.12)\\ &= \cos^2\theta + ad + (a+d)\cos\theta \\ &\quad + \sin^2\theta - bc + (c-b)\sin\theta && (4.13)\\ &= 1 + \det(H) + \sqrt{(a+d)^2 + (c-b)^2}\cos(\theta - \phi), && (4.14)\end{aligned}$$

where $\tan\phi := (c-b)/(a+d)$. We examine the formula for $a, b, c, d \in \mathbb{R}$ so that the inequality holds for any θ:

$$1 + \det(H) > \sqrt{(a+d)^2 + (c-b)^2}. \tag{4.15}$$

Now we apply variable transforms:

$$\begin{aligned} a+d &=: x_+, \quad d-a =: x_- \\ b+c &=: y_+, \quad b-c =: y_-, \\ x_+^2 + y_-^2 &=: r_+^2, \quad x_-^2 + y_+^2 =: r_-^2. \end{aligned} \tag{4.16}$$

By introducing $\phi_\pm \in [0, 2\pi)$,

$$x_\pm =: r_\pm \cos\phi_\pm, \quad y_\pm =: r_\pm \sin\phi_\pm. \tag{4.17}$$

Therefore, H is expressed as

$$H = \frac{r_+}{2} R_{\phi_+} + \frac{r_-}{2} R_{\phi_-} M. \tag{4.18}$$

The inequality (4.15) leads

$$\begin{aligned} 1 + \frac{x_+^2 - x_-^2 - y_+^2 + y_-^2}{4} &> r_+ && (4.19) \\ \Leftrightarrow 4 + r_+^2 - r_-^2 &> 4r_+ && (4.20) \\ \Leftrightarrow (r_+ - 2)^2 &> r_-^2 \geq 0 && (4.21) \\ \Leftrightarrow \qquad & \frac{r_-}{2} < \left| \frac{r_+}{2} - 1 \right|. && (4.22) \end{aligned}$$

4.3 Regularity for the Faces Down

We apply similar calculations for the case with $\overline{A}_\alpha = R_\theta M$:

$$\begin{aligned} 0 &> \det\left(R_\theta M + H_\alpha\right) = \det(M)\det\left(R_\theta + H_\alpha M\right) && (4.23) \\ \Leftrightarrow 0 &< \det\begin{pmatrix} \cos\theta - a & -\sin\theta + b \\ \sin\theta - c & \cos\theta + d \end{pmatrix}. && (4.24) \end{aligned}$$

Since the case is now where $a \to -a$ and $c \to -c$ in 4.2, we simply infer the conclusion in 4.22 with $+ \leftrightarrow -$:

$$\frac{r_+}{2} < \left| \frac{r_-}{2} - 1 \right|. \tag{4.25}$$

5 The Global Affine Distortion

Let us suppose the case where H_λ in (4.7) is λ-independent:

$$H_\lambda = H := r^+ R_{\phi^+} + r^- R_{\phi^-} M, \tag{5.1}$$

where it has the constraints (4.22) and (4.25):

$$r^- < |r^+ - 1|, \quad r^+ < |r^- - 1|,$$

since H belongs to both sides of any adjacent pair of faces. Those conditions are equivalent to

$$0 \leq r^+ + r^- < 1. \tag{5.2}$$

5.1 Continuity

We can show that the regular affine diagram

$$D(\mathfrak{C}) := \left\{ D_\lambda : v \mapsto \left(\overline{A}_\lambda + r^+ R_{\phi^+} + r^- R_{\phi^-} M\right) v + \overline{b}_\lambda \right\}_{\lambda \in \Lambda} \tag{5.3}$$

is continuous in terms of definition 16 since $\forall \alpha, \beta \in \Lambda$ such that $\mathcal{F}_\alpha \mid \mathcal{F}_\beta$,

$$\begin{aligned} \forall v \in \mathcal{F}_\alpha \mid \mathcal{F}_\beta, \quad D_\alpha(v) &= \left(\overline{A}_\alpha + r^+ R_{\phi^+} + r^- R_{\phi^-} M\right) v + \overline{b}_\alpha && (5.4) \\ &= \overline{D}_\alpha(v) + \left(r^+ \mathbb{I} R_{\phi^+} + r^- R_{\phi^-} M\right) v && (5.5) \\ &= \overline{D}_\beta(v) + \left(r^+ \mathbb{I} R_{\phi^+} + r^- R_{\phi^-} M\right) v && (5.6) \\ &= D_\beta(v). && (5.7) \end{aligned}$$

5.2 Equivalent Distortions

Th expression (5.3) gives

$$\begin{aligned} D_\lambda(v) &= \overline{A}_\lambda + r^+ R_{\phi^+} + r^- R_{\phi^-} M + \bar{b}_\lambda & (5.8)\\ &= R_{\phi^+}\left(R_{\phi^+}^{-1}\left(\overline{A}_\lambda(v) + \bar{b}_\lambda\right) + \left(r^+ + r^- R_{\phi^- - \phi^+} M\right) v\right) & (5.9)\\ &= R_{\phi^+}\left(R_{\phi^+}^{-1}\overline{D}_\lambda(v) + \left(r^+ + r^- R_{\phi^- - \phi^+} M\right) v\right). & (5.10)\end{aligned}$$

By definition 23,

$$\begin{aligned} D(\mathfrak{C}) &\sim \left\{R_{\phi^+}^{-1}\overline{D}_\lambda(v) + \left(r^+ + r^- R_{\phi^- - \phi^+} M\right) v\right\}_{\lambda\in\Lambda} & (5.11)\\ &=: \left\{\overline{D}'_\lambda(v) + \left(r^+ + r^- R_{\phi^- - \phi^+} M\right) v\right\}_{\lambda\in\Lambda}, & (5.12)\end{aligned}$$

since $\{R_{\phi^+}^{-1}\overline{D}_\lambda\}_{\lambda\in\Lambda}$ is also a flat diagram $\{\overline{D}'_\lambda\}_{\lambda\in\Lambda}$.

5.3 The Formalism

Through the arguments so far, we have a theorem below.

Theorem 26 (The Global Affine Distortion)**.** A diagram of a flat crease pattern $\mathfrak{C} = \{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$:

$$\begin{aligned} D(\mathfrak{C}) &=: \left\{D_\lambda : v \mapsto \overline{D}(v) + Hv\right\}_{\lambda\in\Lambda} & (5.13)\\ H &:= r^+\mathbb{I} + r^- R_\phi M, & (5.14)\end{aligned}$$

where $0 \le r^+ + r^- < 1$, $0 \le r^\pm < 1$ and $0 \le \phi < 2\pi$. is regular, continuous and affine.

Here, we can see that H is symmetric:

$$\begin{aligned} H^T &= r^+\mathbb{I} + r^- M^T R_\phi^T & (5.15)\\ &= r^+\mathbb{I} + r^- R_\phi M = H. & (5.16)\end{aligned}$$

Hence, H is a Hermitian matrix. Since the trace and the determinant of H are

$$\begin{aligned} \mathrm{Tr}(H) &= r^+\mathrm{Tr}(\mathbb{I}) + r^-\mathrm{Tr}(R_\phi M) & (5.17)\\ &= 2r^+, & (5.18)\\ \det(H) &= (r^+ + r^-)(r^+ - r^-), & (5.19)\end{aligned}$$

the two eigen values of H are

$$r^+ \pm r^-. \tag{5.20}$$

Therefore, H has the unique "spectrum decomposition":

$$H = (r^+ + r^-) P_+ + (r^+ - r^-) P_-, \tag{5.21}$$

where $P_\pm$ are the projections onto their eigen spaces:

$$P_\pm := \frac{1}{2}\begin{pmatrix} 1 \pm \cos\phi & \pm\sin\phi \\ \pm\sin\phi & 1 \mp \cos\phi \end{pmatrix}, \tag{5.22}$$

that satisfy the orthogonal relations:

$$P_\pm P_\pm = P_\pm, \quad P_\pm P_\mp = 0, \quad P_+ + P_- = \mathbb{I}. \tag{5.23}$$

Therefore, an implication for the parameters in (5.14) is:

- r^+ is the mean eigen-value of the distortion.
- r^- is its variance of the distortion,
- ϕ is the angle of eigen-spaces of the distortion.

6 The Local Affine Distortion

6.1 Binary Affine Distortion

Consider the case where there are two types of distortion:

$$H_\lambda = \begin{cases} H_+ := r_+^+ R_{\phi_+^+} + r_+^- R_{\phi_+^-} M & \left(\det\left(\overline{A}_\lambda\right) > 0\right) \\ H_- := r_-^+ R_{\phi_-^+} + r_-^- R_{\phi_-^-} M & \left(\det\left(\overline{A}_\lambda\right) < 0\right), \end{cases} \tag{6.1}$$

where $0 \le \phi_+^\pm, \phi_-^\pm < 2\pi$ and

$$r_+^- < |r^+ - 1|, \quad r_-^+ < |r_-^- - 1|. \tag{6.2}$$

Now we consider $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ such that $\mathcal{F}_{\lambda_1} \mid \mathcal{F}_{\lambda_2}$ and $\mathcal{F}_{\lambda_2} \mid \mathcal{F}_{\lambda_3}$. The continuity in definition 16 requires

$$\begin{aligned} \left(H_{\lambda_1} - H_{\lambda_2}\right) d_1 &= 0 \Rightarrow \left(H_+ - H_-\right) d_1 = 0, \\ \left(H_{\lambda_2} - H_{\lambda_3}\right) d_2 &= 0 \Rightarrow \left(H_- - H_+\right) d_2 = 0, \end{aligned} \tag{6.3}$$

where $d_i = p_i - q_i$ and $p_i, q_i \in \mathcal{E}_i := \mathcal{F}_{\lambda_i} \cap \mathcal{F}_{\lambda_{i+1}}$. The expression (6.3) shows that $H_+ - H_-$ has 0 eigen-value with two (generally) different eigen-vectors d_1 and d_2. This property directly leads

$$H_+ - H_- = 0. \tag{6.4}$$

Eventually, the case considered here coincides with the global case 5.1 if the number of faces $|\Lambda|$ is greater than 2.

6.2 General Continuous Regular Affine Distortion

It is not proven but we have a conjecture:

Conjecture 27. Any regular, affine and continuous diagram are equivalent to a global affine distortion: (5.14).

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