Online Appendix for "Identification of Time and Risk Preferences in Buy Price Auctions"

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This online appendix contains proofs for some results in the paper, and additional discussion of identification of utility functions from certainty equivalents.

A Appendix: Proofs of selected results from the main text

A.2 Proof of Proposition 2

Consider the term

$$e^{-\gamma}U(v-r)+\sum_{n=1}^{\infty}\frac{\gamma^n e^{-\gamma}}{n!}F_V^n(v)E_n[U(v-\max\{r,Y\})|Y\leq v]. \tag{A.1}$$

Note that

$$\begin{aligned}
&F_V^n(v)E_n[U(v-\max(r,Y))|Y\leq v]\\
&=\int_0^v U(v-\max\{r,y\})nF_V^{n-1}(y)f_V(y)dy\\
&=\int_0^r U(v-r)nF_V^{n-1}(y)f_V(y)dy+\int_r^v U(v-y)nF_V^{n-1}(y)f_V(y)dy\\
&=U(v-r)F_V^n(r)+\int_r^v U(v-y)nF_V^{n-1}(y)f_V(y)dy.
\end{aligned}$$

So we can write (A.1) as:

$$\begin{aligned}
&e^{-\gamma}U(v-r)+\sum_{n=1}^{\infty}\frac{\gamma^n e^{-\gamma}}{n!}U(v-r)F_V^n(r)+\sum_{n=1}^{\infty}\frac{\gamma^n e^{-\gamma}}{n!}\int_r^v U(v-y)nF_V^{n-1}(y)f_V(y)dy\\
&=e^{-\gamma}U(v-r)\left[1+\sum_{n=1}^{\infty}\frac{\gamma^n F_V^n(r)}{n!}\right]+\sum_{n=1}^{\infty}\frac{\gamma^n e^{-\gamma}}{n!}\int_r^v U(v-y)nF_V^{n-1}(y)f_V(y)dy\\
&=e^{-\gamma}U(v-r)\exp(\gamma F_V(r))+e^{-\gamma}\sum_{n=1}^{\infty}\frac{\gamma^n}{n!}\int_r^v U(v-y)nF_V^{n-1}(y)f_V(y)dy\\
&=U(v-r)\exp[\gamma F_V(r)-\gamma]+e^{-\gamma}\int_r^v U(v-y)f_V(y)\left[\sum_{n=1}^{\infty}\frac{\gamma^n nF_V^{n-1}(y)}{n!}\right]dy,
\end{aligned}$$

where the last equality follows from the dominated convergence theorem. We also have

$$\sum_{n=1}^{\infty}\frac{\gamma^n nF_V^{n-1}(y)}{n!}=\sum_{n=1}^{\infty}\frac{\gamma\gamma^{n-1}F_V^{n-1}(y)}{(n-1)!}=\gamma\exp(\gamma F_V(y)),$$

so

$$U^R(v,t)=\delta(T-t)\left\{U(v-r)\exp[\gamma F(r)-\gamma]+\int_r^v U(v-y)\exp(\gamma F_V(y)-\gamma)\gamma f_V(y)dy\right\}.$$

The other parts of the Proposition are straightforward to verify.
□

A.4 Proof of Proposition 4

We show properties of the inverse cutoff function defined by

$$\begin{aligned} p(c, r, \tau, t) &= c - U^{-1}\left(\delta(\tau)\left(\alpha(r, \tau, t)U(c - r) + \int_r^c U(c - y)h(y, \tau, t)dy\right)\right) \\ &= c - M(c, r, \tau, t) \end{aligned}$$

over the support $r \in [0, \infty)$, $c \in [r, \infty)$, $\tau \in (0, \infty)$, and $t \in (0, \infty)$.

We start by deriving some useful properties of $U^{-1\prime}(x)$ and $U^{-1\prime\prime}(x)$ given Assumption 1. Starting with the identity

$$z = U^{-1}\left(U(z)\right),$$

differentiate w.r.t. z to get

$$1 = U^{-1\prime}(U(z))U'(z).$$

Evaluating this expression at $z = U^{-1}(x)$ obtain

$$U^{-1\prime}(x) = \frac{1}{U'(U^{-1}(x))} = \left(U'(U^{-1}(x))\right)^{-1}.$$

Differentiating this results in

$$\begin{aligned} U^{-1\prime\prime}(x) &= -\left(U'(U^{-1}(x))\right)^{-2} U''(U^{-1}(x))U^{-1\prime}(x) \\ &= -\left(U'(U^{-1}(x))\right)^{-2} U''(U^{-1}(x))\left(U'(U^{-1}(x))\right)^{-1} \\ &= -\left(U'(U^{-1}(x))\right)^{-3} U''(U^{-1}(x)). \end{aligned}$$

Given our assumptions on $U(x)$, these results imply that:

1. $U^{-1\prime}(0) = 1$;
2. $U^{-1\prime\prime}(0) = -U''(0)$;
3. $U^{-1\prime}(\cdot) > 0$ and is bounded away from 0 and ∞;
4. $U^{-1\prime\prime}(\cdot) \geq 0$ and is bounded away from ∞.

With these results in hand, consider the statements in the Proposition one by one. First, $p_c(c, r, \tau, t) > 0$ because by Assumption 3 , the derivative of M w.r.t. its first argument is strictly less than 1.

$p_c(c,r,\tau,t) < 1$, since

$$p_c(c,r,\tau,t) = 1 - U^{-1\prime}\left(\delta(\tau)\left(\alpha(r,\tau,t)U(c-r) + \int\limits_r^c U(c-y)h(y,\tau,t)dy\right)\right)$$
$$\cdot\, \delta(\tau)\left(\alpha(r,\tau,t)U'(c-r) + \int\limits_r^c U'(c-y)h(y,\tau,t)dy\right),$$

and because under our assumptions, $U^{-1\prime}(\cdot) > 0$, $\delta(\cdot) > 0$, $U'(\cdot) > 0$, $\alpha(y,\tau,t) > 0$, and $h(y,\tau,t) > 0$ for $y > r$.

$p_r(c,r,\tau,t) > 0$, since

$$\begin{aligned}
p_r(c,r,\tau,t) &= -U^{-1\prime}\left(\delta(\tau)\left(\alpha(r,\tau,t)U(c-r) + \int\limits_r^c U(c-y)h(y,\tau,t)dy\right)\right) \\
&\quad \cdot\, \delta(\tau)\left(\frac{\partial\alpha(r,\tau,t)}{\partial r}U(c-r) - \alpha(r,\tau,t)U'(c-r) + U(c-r)h(r,\tau,t)\right) \\
&= U^{-1\prime}\left(\delta(\tau)\left(\alpha(r,\tau,t)U(c-r) + \int\limits_r^c U(c-y)h(y,\tau,t)dy\right)\right)\delta(\tau)\alpha(r,\tau,t)U'(c-r).
\end{aligned}$$

The second line follows since $\frac{\partial\alpha(r,\tau,t)}{\partial r} = h(r,\tau,t)$, and the term is strictly positive since under our assumptions, $U^{-1\prime}(\cdot) > 0$, $\delta(\cdot) > 0$, $U'(\cdot) > 0$, and $\alpha(y,\tau,t) > 0$.

$p_\tau(c,r,\tau,t) \geq 0$, since

$$p_\tau(c,r,\tau,t) = -U^{-1\prime}\left(\delta(\tau)\left(\alpha(r,\tau,t)U(c-r) + \int\limits_r^c U(c-y)h(y,\tau,t)dy\right)\right)$$
$$\cdot\left[\begin{array}{c} \delta'(\tau)\left(\alpha(r,\tau,t)U(c-r) + \int\limits_r^c U(c-y)h(y,\tau,t)dy\right) \\ +\delta(\tau)\dfrac{\partial\left(\alpha(r,\tau,t)U(c-r) + \int\limits_r^c U(c-y)h(y,\tau,t)dy\right)}{\partial\tau} \end{array}\right].$$

The first term in the square brackets is weakly negative since Assumption 1 implies $\delta'(\cdot) < 0$, $\alpha(\cdot,\cdot,\cdot) > 0$, $h(\cdot,\cdot,\cdot) > 0$, and $U(\cdot) \geq 0$. The second term in the square brackets is weakly negative since $\delta(\tau) > 0$ and the derivative of the expected utility from rejecting the BP w.r.t. τ is weakly negative (since the distribution of the highest competitor valuation is stochastically increasing in the length of the bidding phase τ (this derivative is zero when $c = r$)). Since $U^{-1\prime}(x) > 0$, this implies $p_\tau(c,r,\tau,t) \geq 0$.

$p(c,r,\tau,t)=r$ iff $c=r$, because

$$\begin{aligned}
p(c,c,\tau,t) &= c-U^{-1}\left(\delta(\tau)\left(\alpha(c,\tau,t)U(c-c)+\int_c^c U(c-y)h(y,\tau,t)dy\right)\right)\\
&= c-U^{-1}(0)=c=r.
\end{aligned}$$

The "only if" follows because $p_c(c,r,\tau,t)>0$, and because $p(c,r,\tau,t)$ is only defined for $c\geq r$.

$p(c,r,\tau,t)\geq r$ from a similar argument, since $p(c,r,\tau,t)=r$ when $c=r$ and $p_c(c,r,\tau,t)>0$.

$p(c,r,\tau,t)\leq c$, since

$$p(c,r,\tau,t)=c-U^{-1}\left(\delta(\tau)\left(\alpha(r,\tau,t)U(c-r)+\int_r^c U(c-y)h(y,\tau,t)dy\right)\right)$$

and $U^{-1}(\cdot)\geq 0$.

$p_c(z,z,\tau,t)=1-\delta(\tau)\alpha(z,\tau,t)$, since

$$\begin{aligned}
p_c(z,z,\tau,t) &= 1-U^{-1\prime}\left(\delta(\tau)\left(\alpha(z,\tau,t)U(z-z)+\int_z^z U(z-y)h(y,\tau,t)dy\right)\right)\\
&\quad\cdot\delta(\tau)\left(\alpha(z,\tau,t)U'(z-z)+\int_z^z U'(z-y)h(y,\tau,t)dy\right)\\
&= 1-U^{-1\prime}(0)\,\delta(\tau)\alpha(z,\tau,t)U'(0)\\
&= 1-\delta(\tau)\alpha(z,\tau,t).
\end{aligned}$$

$p_r(z,z,\tau,t)=\delta(\tau)\alpha(z,\tau,t)$, since

$$\begin{aligned}
p_r(z,z,\tau,t) &= -U^{-1\prime}\left(\delta(\tau)\left(\alpha(z,\tau,t)U(z-z)+\int_z^z U(z-y)h(y,\tau,t)dy\right)\right)\\
&\quad\cdot\delta(\tau)\left(\frac{\partial\alpha(z,\tau,t)}{\partial z}U(z-z)-\alpha(z,\tau,t)U'(z-z)+U(z-z)h(z,\tau,t)\right)\\
&= U^{-1\prime}\left(\delta(\tau)\left(\alpha(z,\tau,t)U(z-z)+\int_z^z U(z-y)h(y,\tau,t)dy\right)\right)\delta(\tau)\alpha(z,\tau,t)U'(z-z)\\
&= U^{-1\prime}(0)\,\delta(\tau)\alpha(z,\tau,t)U'(0)\\
&= \delta(\tau)\alpha(z,\tau,t),
\end{aligned}$$

where the last line follows because $U^{-1\prime}(0)=U'(0)=1$.

Next, we consider the second derivatives of the inverse cutoff function w.r.t. c and r, i.e. $p_{cc}(c,r)$, $p_{rr}(c,r)$, and $p_{cr}(c,r)$. We drop the τ and t arguments for compactness.

For $p_{cc}(c,r)$, we have

$$p_c(c,r) = 1 - U^{-1\prime}\left(\delta\left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy\right)\right)\delta\left[\alpha(r)U'(c-r) + \int_r^c U'(c-y)h(y)dy\right],$$

so

$$\begin{aligned}
p_{cc}(c,r) = &-U^{-1\prime\prime}\left(\delta\left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy\right)\right)\delta^2\left[\alpha(r)U'(c-r) + \int_r^c U'(c-y)h(y)dy\right]^2 \\
&- U^{-1\prime}\left(\delta\left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy\right)\right)\delta\left[\alpha(r)U''(c-r) + \int_r^c U''(c-y)h(y)dy + h(c)\right].
\end{aligned}$$

Under our assumptions, all these terms are bounded away from ∞ and $-\infty$, so $p_{cc}(c,r)$ is bounded away from ∞ and $-\infty$. Moreover, if we evaluate this expression at $c = r = z$, we get

$$\begin{aligned}
p_{cc}(z,z) = &-U^{-1\prime\prime}\left(\delta\left(\alpha(z)U(z-z) + \int_z^z U(z-y)h(y)dy\right)\right)\delta^2\left[\alpha(z)U'(z-z) + \int_z^z U'(z-y)h(y)dy\right]^2 \\
&- U^{-1\prime}\left(\delta\left(\alpha(z)U(z-z) + \int_z^z U(z-y)h(y)dy\right)\right)\delta\left[\alpha(z)U''(z-z) + \int_z^z U''(z-y)h(y)dy + h(z)\right] \\
= &-U^{-1\prime\prime}(0)\,\delta^2\alpha(z)^2 - U^{-1\prime}(0)\,\delta\left[\alpha(z)U''(0) + h(z)\right] \\
= &-U''(0)\,\delta\alpha(z)\left(1 - \delta\alpha(z)\right) - \delta h(z).
\end{aligned}$$

For $p_{rr}(c,r)$, we have

$$p_r(c,r) = U^{-1\prime}\left(\delta\left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy\right)\right)\delta\alpha(r)U'(c-r),$$

so

$$\begin{aligned}
p_{rr}(c,r) = &-U^{-1\prime\prime}\left(\delta\left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy\right)\right)\delta^2\alpha(r)^2U'(c-r)^2 \\
&+ U^{-1\prime}\left(\delta\left(\alpha(r)U(c-r) + \int_r^c U(c-y)h(y)dy\right)\right)\delta\left[\alpha'(r)U'(c-r) - \alpha(r)U''(c-r)\right].
\end{aligned}$$

Again, under our assumptions, all the terms in this expression are bounded away from ∞ and $-\infty$, so $p_{rr}(c,r)$ is bounded away from ∞ and $-\infty$. If we evaluate this expression at $c=r=z$, we get

$$\begin{aligned}
p_{rr}(z,z) &= -U^{-1\prime\prime}\left(\delta\left(\alpha(z)U(z-z)+\int_z^z U(z-y)h(y)dy\right)\right)\delta^2\alpha(z)^2U'(z-z)^2 \\
&\quad + U^{-1\prime}\left(\delta\left(\alpha(z)U(z-z)+\int_z^z U(z-y)h(y)dy\right)\right)\delta\left[\alpha'(z)U'(z-z)-\alpha(z)U''(z-z)\right] \\
&= -U^{-1\prime\prime}(0)\,\delta^2\alpha(z)^2U'(0)^2 + U^{-1\prime}(0)\,\delta\left[\alpha'(z)U'(0)-\alpha(z)U''(0)\right] \\
&= U''(0)\,\delta^2\alpha(z)^2 + \delta\left[\alpha'(z)-\alpha(z)U''(0)\right] \\
&= -U''(0)\,\delta\alpha(z)\left(1-\delta\alpha(z)\right)+\delta\alpha'(z).
\end{aligned}$$

For $p_{rc}(c,r)=p_{cr}(c,r)$, we have

$$p_r(c,r) = U^{-1\prime}\left(\delta\left(\alpha(r)U(c-r)+\int_r^c U(c-y)h(y)dy\right)\right)\delta\alpha(r)U'(c-r),$$

so

$$\begin{aligned}
p_{rc}(c,r) &= U^{-1\prime}\left(\delta\left(\alpha(r)U(c-r)+\int_r^c U(c-y)h(y)dy\right)\right)\delta\alpha(r)U''(c-r) \\
&\quad + U^{-1\prime\prime}\left(\delta\left(\alpha(r)U(c-r)+\int_r^c U(c-y)h(y)dy\right)\right)\delta\alpha(r)U'(c-r) \\
&\quad \cdot\left[\delta\left(\alpha(r)U'(c-r)+\int_r^c U'(c-y)h(y)dy\right)\right].
\end{aligned}$$

Again, all the terms are bounded away from ∞ and $-\infty$, so $p_{rc}(c,r)$ is bounded away from ∞ and

$-\infty$. Evaluated at $c = r = z$, we get

$$
\begin{aligned}
p_{rc}(z,z) = U^{-1\prime} & \left(\delta \left(\alpha(z)U(z-z) + \int\limits_z^z U(z-y)h(y)dy \right) \right) \delta\alpha(z)U''(z-z) \\
& + U^{-1\prime\prime} \left(\delta \left(\alpha(z)U(z-z) + \int\limits_z^z U(z-y)h(y)dy \right) \right) \delta\alpha(z)U'(z-z) \\
& \cdot \left[\delta \left(\alpha(z)U'(z-z) + \int\limits_z^z U'(z-y)h(y)dy \right) \right] \\
& = U^{-1\prime}(0)\, \delta\alpha(z)U''(0) + U^{-1\prime\prime}(0)\, \delta\alpha(z)U'(0)\delta\alpha(z)U'(0) \\
& = \delta\alpha(z)U''(0) - U''(0)\, \delta\alpha(z)\delta\alpha(z) \\
& = U''(0)\delta\alpha(z)\left(1 - \delta\alpha(z)\right).
\end{aligned}
$$

□

A.6 Proof of Proposition 8

Since the hazard rate of the first action (accept or reject the BP) is observed in the data and satisfies

$$\theta(t_1|p, r, \tau_0) = \lambda(t_1)(1 - F_V(r)), \tag{A.2}$$

it is clear that $\lambda(t_1)(1 - F_V(r))$ is identified on $r \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, \overline{T})$.

We next show that this implies that $\alpha(r, \tau_0, t_1)$ is identified on $r \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T - \tau_0)$. By definition

$$\alpha(r, \tau_0, t_1) = \exp(\gamma F_V(r) - \gamma)$$

where

$$\gamma = \int_t^{t+\tau_0} \lambda(s)ds.$$

Therefore

$$
\begin{aligned}
\alpha(r, \tau_0, t_1) &= \exp\left(-(1 - F_V(r)) \int_t^{t+\tau_0} \lambda(s)ds\right) \\
&= \exp\left(- \int_t^{t+\tau_0} \lambda(s)(1 - F_V(r))ds\right).
\end{aligned}
$$

Since $\lambda(t_1)(1 - F_V(r))$ is identified on $r \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T)$, this implies that $\alpha(r, \tau_0, t_1)$ is identified on $r \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T - \tau_0)$.

Next we show that $h(y, \tau_0, t_1)$ is identified on $y \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T - \tau_0)$. Again, by definition

$$\begin{aligned} h(y, \tau_0, t_1) &= \exp(\gamma F_V(y) - \gamma)\gamma f_V(y) \\ &= \alpha(r, \tau_0, t_1) \int_t^{t+\tau_0} \lambda(s) f_V(y) ds. \end{aligned}$$

Since $\lambda(t_1)(1 - F_V(y))$ is identified on $y \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T)$, its derivative $-\lambda(t_1) f_V(y)$, is also identified on $y \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T)$. This implies $\int_t^{t+\tau_0} \lambda(s) f_V(y) ds$ is identified on $y \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T - \tau_0)$. Therefore, $h(y, \tau_0, t)$ is identified on $y \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T - \tau_0)$.

Next, we consider identification of $c(p, r, \tau_0, t_1)$. From Section 3.3, we know

$$\Pr(B = 1 | p, r, \tau_0, t_1) = \frac{1 - F_V(c(p, r, \tau_0, t_1))}{1 - F_V(r)},$$

where $\Pr(B = 1 | p, r, \tau_0, t_1)$ is observed on the support $r \in [\underline{r}, \overline{r}]$, $p \in [r, \overline{p}]$ and $t_1 \in [0, T)$ (at τ_0). Therefore,

$$\begin{aligned} \Pr(B = 1 | p, r, \tau_0, t_1) &= \frac{\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))}{\lambda(t_1)(1 - F_V(r))} \\ &= \frac{\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))}{\theta(t_1 | p, r, \tau_0)}, \end{aligned}$$

and therefore $\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))$ is identified on the same support. Note that this term is the hazard rate of the BP being accepted.

Since we have already identified $\lambda(t_1)(1 - F_V(r))$ on $r \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T)$, this implies that

$$c(p, r, \tau_0, t_1) = z,$$

where z satisfies

$$\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1))) = \lambda(t_1)(1 - F_V(z)). \tag{A.3}$$

Intuitively, this says that the cutoff at (p, r, τ_0, t_1) is equal to the hypothetical reserve price that would imply that the hazard rate of first action is equal $\lambda(t_1)(1 - F_V(c(p, r, \tau_0, t_1)))$.

It remains to be verified that we can identify the z that satisfies (A.3). Note that the r.h.s. of (A.3) is strictly decreasing in z. Since $c(p, r, \tau_0, t_1) \geq \underline{r}$, the l.h.s. is $\leq$ the r.h.s. at $z = \underline{r}$. Hence, we want to increase z above $\underline{r}$ to satisfy (A.3). The problem is that we only observe the r.h.s. for $z \in [\underline{r}, \overline{r}]$. However, as long as $c(p, r, \tau_0, t_1) \leq \overline{r}$, we can find a $z \in [\underline{r}, \overline{r}]$ that satisfies (A.3). This implies that $c(p, r, \tau_0, t_1)$ is identified on the set (p, r, t_1) such that $c(p, r, \tau_0, t_1) \leq \overline{r}$.[1] This immediately implies that inverse cutoff function $p(c, r, \tau_0, t_1)$ is identified on the set $r \in [\underline{r}, \overline{r}]$, $t_1 \in [0, T - \tau_0)$,

[1] This set exists. To show this, consider a situation where $r = \underline{r}$ and $p = \underline{r} + \epsilon$ for some arbitrarily small ϵ. For small enough ϵ, $c(p, r, \tau_0, t_1)$ will be below $\overline{r}$ (since c is continuous and $c(\overline{r}, \overline{r}, \tau_0, t_1) = \overline{r}$). Obviously the size of this set will depend on the range $[\underline{r}, \overline{r}]$.

and $c \in [\underline{r}, \overline{r}]$.

Thus, we have shown that:

1. $\alpha(r, \tau_0, t_1)$ is identified on $r \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T - \tau_0)$;
2. $h(y, \tau_0, t_1)$ is identified on $y \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, T - \tau_0)$;
3. $p(c, r, \tau_0, t_1)$ is identified on the set $r \in [\underline{r}, \overline{r}]$, $t_1 \in [0, T - \tau_0)$, and $c \in [\underline{r}, \overline{r}]$.

Recall that our integral equation

$$U(c - p(c, r, \tau_0, t_1)) = \delta(\tau_0) \left(\alpha(r, \tau_0, t_1) U(c - r) + \int_r^c U(c - y) h(y, \tau_0, t_1) dy \right) \tag{A.4}$$

can be reduced to

$$U''(c - r) = \frac{\Phi_r(c, r, \tau_0, t_1) + h(r, \tau_0, t_1)}{\Phi(c, r, \tau_0, t_1)} U'(c - r), \tag{A.5}$$

where

$$\Phi(c, r, \tau_0, t_1) = \alpha(r, \tau_0, t_1) \left[\frac{(1 - p_c(c, r, \tau_0, t_1))}{p_r(c, r, \tau_0, t_1)} - 1 \right].$$

Identification of $\alpha(r, \tau_0, t_1)$, $h(y, \tau_0, t_1)$, and $p(c, r, \tau_0, t_1)$ implies that we can identify $\frac{\Phi_r(c,r,\tau_0,t_1)+h(r,\tau_0,t_1)}{\Phi(c,r,\tau_0,t_1)}$ on $r \in [\underline{r}, \overline{r}]$, $t_1 \in [0, T - \tau_0)$, and $c \in [\underline{r}, \overline{r}]$. Hence, by arguments similar to Proposition 3, Equation (A.5) identifies $U(\cdot)$ on $[0, \underline{r} - \overline{r}]$. By the same arguments as in Section 3.4, $\delta(\cdot)$ is identified at τ_0. $\square$

A.7 Proof of Proposition 9

Assumption 8 further restricts the support of p to $[p_0 - \epsilon, p_0 + \epsilon]$. We also assume that p_0 is such that there exists a $r^* \in (\underline{r}, \overline{r})$ and a t_1^* such that $c(p_0, r^*, \tau_0, t_1^*) \in (\underline{r}, \overline{r})$. By the same arguments as in the proof of Proposition 8, we know:

1. $\alpha(r, \tau, t_1)$ is identified on $r \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, \overline{T} - \tau_0)$;
2. $h(y, \tau_0, t_1)$ is identified on $y \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, \overline{T} - \tau_0)$.

By the same arguments as above (and the condition that $c \in (\underline{r}, \overline{r})$), one can see that $c(p, r, \tau_0, t_1)$ will be identified for $p \in (p_0 - \epsilon, p_0 + \epsilon)$, $r \in (r - \eta, r + \eta)$, $t_1 = t_1^*$, and $\tau = \tau_0$, for η sufficiently small. Therefore, the inverse cutoff function $p(c, r, \tau_0, t_1)$ will be identified at $t_1 = t_1^*$, and $\tau = \tau_0$ in a ball centered at $(c(p_0, r^*, \tau_0, t_1^*), r^*)$. This implies that $p_r(c, r, \tau_0, t_1)$ and $p_c(c, r, \tau_0, t_1)$ are identified over that same region, as are $\Phi(c, r, \tau_0, t_1)$ and $\Phi_r(c, r, \tau_0, t_1)$. We have

$$\frac{U''(c - r)}{U'(c - r)} = \frac{\Phi_r(c, r, \tau_0, t_1) + h(r, \tau_0, t_1)}{\Phi(c, r, \tau_0, t_1)} \tag{A.6}$$

Hence, the Arrow-Pratt measure of risk aversion $\frac{U''}{U'}$ is identified at the point $c(p_0, r^*, \tau_0, t_1^*) - r^*$. Again, by the same arguments as Section 3.4, $\delta(\cdot)$ is identified at τ_0.

$\square$

B Appendix: Proof that $U''' \leq 0$ is a sufficient condition for Assumption 3

We have

$$M(v,r,\tau,t) = U^{-1}\left(\delta(\tau)\left(\alpha(r,\tau,t)U(v-r) + \int\limits_r^v U(v-y)h(y,\tau,t)dy\right)\right)$$

so

$$\begin{aligned}
M_v(v,r,\tau,t) &= U^{-1\prime}\left(\delta(\tau)\left(\alpha(r,\tau,t)U(v-r) + \int\limits_r^v U(v-y)h(y,\tau,t)dy\right)\right) \\
&\quad \cdot \delta(\tau)\left(\alpha(r,\tau,t)U'(v-r) + \int\limits_r^v U'(v-y)h(y,\tau,t)dy\right) \\
&= \frac{\delta(\tau)\left(\alpha(r,\tau,t)U'(v-r) + \int\limits_r^v U'(v-y)h(y,\tau,t)dy\right)}{U'\left(U^{-1}\left(\delta(\tau)\left(\alpha(r,\tau,t)U(v-r) + \int\limits_r^v U(v-y)h(y,\tau,t)dy\right)\right)\right)} \\
&< \frac{\delta(\tau)\left(\alpha(\cdot)U'(v-r) + \int\limits_r^v U'(v-y)h(\cdot)dy + \left(1-\alpha(\cdot) - \int\limits_r^v h(\cdot)dy\right)U'(0)\right)}{U'\left(U^{-1}\left(\delta(\tau)\left(\alpha(\cdot)U(v-r) + \int\limits_r^v U(v-y)h(\cdot)dy + \left(1-\alpha(\cdot) - \int\limits_r^v h(\cdot)dy\right)U(0)\right)\right)\right)}.
\end{aligned}$$

The strict inequality holds because of our normalizations that $U(0) = 0$ and $U'(0) = 1$, and because $1 - \alpha(r,\tau,t) - \int\limits_r^v h(y,\tau,t)dy > 0$ for any finite v.

Therefore, we have

$$M_v(v,r,\tau,t) < \frac{\delta(\tau)EU'(x)}{U'\left(U^{-1}(\delta(\tau)EU(x))\right)},$$

where the random variable x has a mixed-continuous distribution, taking the value 0 with probability $1 - \alpha(r,\tau,t) - \int_r^v h(y,\tau,t)dy$, the value $v - r$ with probability $\alpha(r,\tau,t)$, and having density $h(y,\tau,t)$ over the interval $(0, v-r)$. Because $U'' \leq 0$ and $\delta(\tau) < 1$, Jensen's Inequality implies that

$\delta(\tau) EU(x) < U(Ex)$. Therefore

$$\begin{aligned} M_v(v, r, \tau, t) &< \frac{\delta(\tau) EU'(x)}{U'\left(U^{-1}(U(Ex))\right)} \\ &= \frac{\delta(\tau) EU'(x)}{U'(Ex)}. \end{aligned}$$

Since $U''' \leq 0$, Jensen's inequality implies $EU'(x) \leq U'(Ex)$. Hence,

$$M_v(v, r, \tau, t) < \delta(\tau) < 1.$$

□

C Appendix: Identification of Utility Functions from Certainty Equivalents

Suppose that U is a utility function defined on $\mathcal{X} \subset \mathbb{R}$, and $\mathcal{F}$ is a collection of distributions with supports contained in $\mathcal{X}$. This generates a certainty equivalent functional (also called a quasilinear mean)

$$m(F) = U^{-1}\left(\int U(x) dF(x)\right), \quad F \in \mathcal{F}.$$

Now suppose that we are given a collection of lotteries $\mathcal{F}$ and a quaslinear mean functional m. If $\mathcal{F}$ is sufficiently rich, it is plausible that the utility function U is uniquely determined (up to affine transformations) by m. We show that this is true even for a well-chosen one-dimensional family of lotteries.

Our example is adapted from the proof of Theorem 83 in Hardy, Littlewood, and Polya (1952). Let $\mathcal{X} = [a, b]$ and consider the collection of lotteries $\mathcal{F} = \{F_t(x), t \in [0, 1]\}$, where the F_t are mixtures of point masses at the endpoints a and b:

$$F_t(x) = (1 - t)\delta_a(x) + t\delta_b(x).$$

Note that

$$\begin{aligned} m(F_0) &= m(\delta_a) = a \\ m(F_1) &= m(\delta_b) = b \end{aligned}$$

and since m is continuous and strictly increasing, $m(F_t)$ takes every value in $[a, b]$.

Suppose that there is another function V satisfying

$$m(F) = V^{-1}\left(\int V(x)dF(x)\right), \quad F \in \mathcal{F}.$$

Let

$$\tilde{x}(t) = m(F_t) = U^{-1}\Big[(1-t)U(a) + tU(b)\Big] = V^{-1}\Big[(1-t)V(a) + tV(b)\Big].$$

We have

$$U(\tilde{x}(t)) = (1-t)U(a) + tU(b),$$

and we can solve for t and $(1-t)$:

$$t = \frac{U(\tilde{x}(t)) - U(a)}{U(b) - U(a)}, \quad (1-t) = \frac{U(b) - U(\tilde{x}(t))}{U(b) - U(a)}$$

Now

$$\begin{aligned} V(\tilde{x}(t)) &= (1-t)V(a) + tV(b) \\ &= \frac{U(b) - U(\tilde{x}(t))}{U(b) - U(a)} \cdot V(a) + \frac{U(\tilde{x}(t)) - U(a)}{U(b) - U(a)} \cdot V(b). \end{aligned}$$

This is a linear (in fact, affine) function of $U(\tilde{x}(t))$, so we can write

$$V(\tilde{x}(t)) = \alpha + \beta U(\tilde{x}(t)),$$

where α and β do not depend on t and $\beta > 0$. Since this holds for all $t \in [0, 1]$, we have

$$V(x) = \alpha + \beta U(x), \quad \forall x \in [a, b].$$

Thus V must be an affine transformation of U. $\square$