

応用関数解析特論レポート

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\mathbb{C}^2 の標準基底

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

について He_1, He_2 を計算すると

$$\begin{aligned} He_1 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ He_2 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

となる. それぞれの内積は

$$\begin{aligned} \langle He_1, He_1 | He_1, He_1 \rangle &= \frac{1}{2}(1+1) = 1 = \langle e_1, e_1 | e_1, e_1 \rangle \\ \langle He_1, He_2 | He_1, He_2 \rangle &= \frac{1}{2}(1-1) = 0 = \langle e_1, e_2 | e_1, e_2 \rangle \\ \langle He_2, He_1 | He_2, He_1 \rangle &= \frac{1}{2}(1-1) = 0 = \langle e_2, e_1 | e_2, e_1 \rangle \\ \langle He_2, He_2 | He_2, He_2 \rangle &= \frac{1}{2}(1+1) = 1 = \langle e_2, e_2 | e_2, e_2 \rangle \end{aligned}$$

となり, 標準基底について H はユニタリ作用素の条件を満たす. $\forall x, y \in \mathbb{C}^2$ は e_1, e_2 の線形結合で表せるので,

$$\langle Hx, Hy | Hx, Hy \rangle = \langle x, y | x, y \rangle$$

が得られる. よって H はユニタリ作用素である.

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\mathbb{C}^3 の正規直交基底の 1 つとして,

$$e_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, e_2 = \frac{1}{\sqrt{182}} \begin{bmatrix} 13 \\ -2 \\ -3 \end{bmatrix}, e_3 = \frac{1}{\sqrt{13}} \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

が挙げられる.

3

状態と正規直行基底の内積を計算すると

$$\begin{aligned} |\langle \xi_1, \psi | \xi_1, \psi \rangle|^2 &= \left(\frac{1}{6}(-2 + 2 + 1 + 0) \right)^2 = \left(\frac{1}{6} \right)^2 = \frac{1}{36} \\ |\langle \xi_2, \psi | \xi_2, \psi \rangle|^2 &= \left(\frac{1}{6}(2 - 2 + 1 + 0) \right)^2 = \left(\frac{1}{6} \right)^2 = \frac{1}{36} \\ |\langle \xi_3, \psi | \xi_3, \psi \rangle|^2 &= \left(\frac{1}{6}(2 + 2 - 1 + 0) \right)^2 = \left(\frac{1}{2} \right)^2 = \frac{1}{4} \\ |\langle \xi_4, \psi | \xi_4, \psi \rangle|^2 &= \left(\frac{1}{6}(2 + 2 + 1 - 0) \right)^2 = \left(\frac{5}{6} \right)^2 = \frac{25}{36} \end{aligned}$$

となる. 公理 2 より, 数値 1, 2, 3, 4 が検出される確率はそれぞれ $\frac{1}{36}, \frac{1}{36}, \frac{1}{4}, \frac{25}{36}$ である.

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$m, n \in \mathbb{N}$ とし, 集合の元どうしで内積を計算する. $x \in [0, 2\pi]$ のとき $\cos nx, \sin nx \in [0, 1]$ なので,

$$\begin{aligned} \overline{\cos nx} &= \cos nx \\ \overline{\sin nx} &= \sin nx \end{aligned}$$

が成立する. まず, 元が同じ場合の内積を計算する.

• \cos

$$\begin{aligned} \left\langle \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos nx} \frac{1}{\sqrt{\pi}} \cos nx dx \\ &= \frac{1}{\pi} \int_0^{2\pi} \cos^2 nx dx = \frac{1}{2\pi} \int_0^{2\pi} (1 + \cos 2nx) dx \\ &= \frac{1}{2\pi} \left[x + \frac{1}{2} \sin 2nx \right]_0^{2\pi} = \frac{1}{2\pi} (2\pi - 0) = 1 \end{aligned}$$

- sin

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{\pi}} \sin nx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin nx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \sin^2 nx dx = \frac{1}{2\pi} \int_0^{2\pi} (1 - \cos 2nx) dx \\
&= \frac{1}{2\pi} \left[x - \frac{1}{2} \sin 2nx \right]_0^{2\pi} = \frac{1}{2\pi} (2\pi - 0) = 1
\end{aligned}$$

続いて、元が異なる場合の内積を計算する.

- cos ($m \neq n$)

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{\pi}} \cos mx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos mx} \frac{1}{\sqrt{\pi}} \cos nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \cos mx \cos nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\cos(m+n)x + \cos(m-n)x) dx \\
&= \frac{1}{2\pi} \left[\frac{1}{m+n} \sin(m+n)x + \frac{1}{m-n} \sin(m-n)x \right]_0^{2\pi} = \frac{1}{2\pi} (0 - 0) = 0
\end{aligned}$$

- sin ($m \neq n$)

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{\pi}} \sin mx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin mx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \sin mx \sin nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\cos(m-n)x - \cos(m+n)x) dx \\
&= \frac{1}{2\pi} \left[\frac{1}{m-n} \sin(m-n)x - \frac{1}{m+n} \sin(m+n)x \right]_0^{2\pi} = \frac{1}{2\pi} (0 - 0) = 0
\end{aligned}$$

- cos と sin

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{\pi}} \cos mx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos mx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \cos mx \sin nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\sin(m+n)x + \sin(m-n)x) dx \\
&= \frac{1}{2\pi} \left[-\frac{1}{m+n} \cos(m+n)x - \frac{1}{m-n} \cos(m-n)x \right]_0^{2\pi} \\
&= \frac{1}{2\pi} \left(\left(-\frac{1}{m+n} - \frac{1}{m-n} \right) - \left(-\frac{1}{m+n} - \frac{1}{m-n} \right) \right) = 0
\end{aligned}$$

- sin と cos

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{\pi}} \sin mx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin mx} \frac{1}{\sqrt{\pi}} \cos nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \cos nx \sin mx dx = 0
\end{aligned}$$

以上より, 同じ元どうしの内積が 1 になり, 異なる元どうしの内積が 0 になるので,

$$\left\{ \frac{1}{\sqrt{\pi}} \cos nx \mid n \in \mathbb{N} \right\} \cup \left\{ \frac{1}{\sqrt{\pi}} \sin nx \mid n \in \mathbb{N} \right\}$$

は正規直交系である.

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$$\begin{aligned} \langle \xi, \zeta \rangle &= \left\langle \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes e_i, \frac{1}{n} \sum_{j=1}^n \sum_{k=1}^n e_j \otimes e_k \right\rangle \\ &= \frac{1}{n\sqrt{n}} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle \end{aligned}$$

$\{e_i\}_{i=1}^n$ は \mathbb{C}^n の正規直交基底なので, $\langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle$ は $e_i = e_j = e_k$ のときのみ 1 になり, それ以外は 0 となる. よって

$$\begin{aligned} \langle \xi, \zeta | \xi, \zeta \rangle &= \frac{1}{n\sqrt{n}} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle \\ &= \frac{1}{n\sqrt{n}} \sum_{i=1}^n 1 = \frac{1}{n\sqrt{n}} \cdot n = \frac{1}{\sqrt{n}} \end{aligned}$$

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$$\begin{aligned} Ty &= \left(\sum_{i=1}^n |x_i\rangle \langle x_i| \right) y = \sum_{i=1}^n |x_i\rangle \langle x_i| y \\ &= \sum_{i=1}^n \langle x_i, y | x_i, y \rangle x_i = \sum_{i=1}^n \left\langle x_i, \sum_{j=1}^n \alpha_j x_j \right\rangle x_i \\ &= \sum_{i=1}^n \sum_{j=1}^n \alpha_j \langle x_i, x_j \rangle x_i = \sum_{i=1}^n \alpha_i x_i = y \end{aligned}$$

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まず,

$$\begin{aligned} \frac{d}{dx}^* \cos x &= a_1 \cos x + b_1 \sin x \\ \frac{d}{dx}^* \sin x &= a_2 \cos x + b_2 \sin x \end{aligned}$$

と置く. \cos と \sin の内積を計算すると,

$$\begin{aligned}\langle \cos x, \cos x \rangle &= \int_0^{2\pi} \cos^2 x dx = \int_0^{2\pi} \frac{1 + \cos 2x}{2} dx \\ &= \frac{1}{2} \left[x + \frac{1}{2} \sin 2x \right]_0^{2\pi} = \frac{1}{2} (2\pi + 0) = \pi\end{aligned}$$

$$\begin{aligned}\langle \sin x, \sin x \rangle &= \int_0^{2\pi} \sin^2 x dx = \int_0^{2\pi} \frac{1 - \cos 2x}{2} dx \\ &= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{2\pi} = \frac{1}{2} (2\pi - 0) = \pi\end{aligned}$$

$$\begin{aligned}\langle \cos x, \sin x \rangle &= \int_0^{2\pi} \cos x \sin x dx = \int_0^{2\pi} \frac{1}{2} \sin 2x dx \\ &= \frac{1}{2} \left[-\frac{1}{2} \cos 2x \right]_0^{2\pi} = \frac{1}{2} \left(-\frac{1}{2} + \frac{1}{2} \right) = 0\end{aligned}$$

$$\langle \sin x, \cos x \rangle = \int_0^{2\pi} \sin x \cos x dx = 0$$