応用関数解析特論レポート

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1

$\mathbb{C}^2$ の標準基底

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

について $\mathrm{H}e_1, \mathrm{H}e_2$ を計算すると

$$\begin{aligned} \mathrm{H}e_1 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \mathrm{H}e_2 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

となる．それぞれの内積は

$$\begin{aligned} \langle \mathrm{H}e_1, \mathrm{H}e_1 \rangle &= \frac{1}{2}(1+1) = 1 = \langle e_1, e_1 \rangle \\ \langle \mathrm{H}e_1, \mathrm{H}e_2 \rangle &= \frac{1}{2}(1-1) = 0 = \langle e_1, e_2 \rangle \\ \langle \mathrm{H}e_2, \mathrm{H}e_1 \rangle &= \frac{1}{2}(1-1) = 0 = \langle e_2, e_1 \rangle \\ \langle \mathrm{H}e_2, \mathrm{H}e_2 \rangle &= \frac{1}{2}(1+1) = 1 = \langle e_2, e_2 \rangle \end{aligned}$$

となり，標準基底について H はユニタリ作用素の条件を満たす．$^\forall x, y \in \mathbb{C}^2$ は e_1, e_2 の線形結合で表せるので，

$$\langle \mathrm{H}x, \mathrm{H}y \rangle = \langle x, y \rangle$$

が得られる．よって H はユニタリ作用素である．

2

$\mathbb{C}^3$ の正規直交基底の 1 つとして,

$$e_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, e_2 = \frac{1}{\sqrt{182}} \begin{bmatrix} 13 \\ -2 \\ -3 \end{bmatrix}, e_3 = \frac{1}{\sqrt{13}} \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

が挙げられる.

3

状態と正規直行基底の内積を計算すると

$$|\langle \xi_1, \psi \rangle|^2 = \left(\frac{1}{6}(-2+2+1+0) \right)^2 = \left(\frac{1}{6} \right)^2 = \frac{1}{36}$$
$$|\langle \xi_2, \psi \rangle|^2 = \left(\frac{1}{6}(2-2+1+0) \right)^2 = \left(\frac{1}{6} \right)^2 = \frac{1}{36}$$
$$|\langle \xi_3, \psi \rangle|^2 = \left(\frac{1}{6}(2+2-1+0) \right)^2 = \left(\frac{1}{2} \right)^2 = \frac{1}{4}$$
$$|\langle \xi_4, \psi \rangle|^2 = \left(\frac{1}{6}(2+2+1-0) \right)^2 = \left(\frac{5}{6} \right)^2 = \frac{25}{36}$$

となる. 公理 2 より, 数値 1, 2, 3, 4 が検出される確率はそれぞれ $\frac{1}{36}, \frac{1}{36}, \frac{1}{4}, \frac{25}{36}$ である.

4

$m, n \in \mathbb{N}$ とし, 集合の元どうしで内積を計算する. $x \in [0, 2\pi]$ のとき $\cos nx, \sin nx \in [0, 1]$ なので,

$$\overline{\cos nx} = \cos nx$$
$$\overline{\sin nx} = \sin nx$$

が成立する. まず, 元が同じ場合の内積を計算する.

- cos

$$\left\langle \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos nx} \frac{1}{\sqrt{\pi}} \cos nx dx$$
$$= \frac{1}{\pi} \int_0^{2\pi} \cos^2 nx dx = \frac{1}{2\pi} \int_0^{2\pi} (1 + \cos 2nx) dx$$
$$= \frac{1}{2\pi} \left[x + \frac{1}{2} \sin 2nx \right]_0^{2\pi} = \frac{1}{2\pi} (2\pi - 0) = 1$$

- sin

$$
\begin{aligned}
&\left\langle \frac{1}{\sqrt{\pi}} \sin nx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin nx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \sin^2 nx dx = \frac{1}{2\pi} \int_0^{2\pi} (1 - \cos 2nx) dx \\
&= \frac{1}{2\pi} \left[x - \frac{1}{2} \sin 2nx \right]_0^{2\pi} = \frac{1}{2\pi} (2\pi - 0) = 1
\end{aligned}
$$

続いて, 元が異なる場合の内積を計算する.

- cos $(m \neq n)$

$$
\begin{aligned}
&\left\langle \frac{1}{\sqrt{\pi}} \cos mx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos mx} \frac{1}{\sqrt{\pi}} \cos nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \cos mx \cos nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\cos (m+n)x + \cos (m-n)x) dx \\
&= \frac{1}{2\pi} \left[\frac{1}{m+n} \sin (m+n)x + \frac{1}{m-n} \sin (m-n)x \right]_0^{2\pi} = \frac{1}{2\pi} (0 - 0) = 0
\end{aligned}
$$

- sin $(m \neq n)$

$$
\begin{aligned}
&\left\langle \frac{1}{\sqrt{\pi}} \sin mx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin mx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \sin mx \sin nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\cos (m-n)x - \cos (m+n)x) dx \\
&= \frac{1}{2\pi} \left[\frac{1}{m-n} \sin (m-n)x - \frac{1}{m+n} \sin (m+n)x \right]_0^{2\pi} = \frac{1}{2\pi} (0 - 0) = 0
\end{aligned}
$$

- cos と sin

$$
\begin{aligned}
&\left\langle \frac{1}{\sqrt{\pi}} \cos mx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos mx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \cos mx \sin nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\sin (m+n)x + \sin (m-n)x) dx \\
&= \frac{1}{2\pi} \left[-\frac{1}{m+n} \cos (m+n)x - \frac{1}{m-n} \cos (m-n)x \right]_0^{2\pi} \\
&= \frac{1}{2\pi} \left(\left(-\frac{1}{m+n} - \frac{1}{m-n} \right) - \left(-\frac{1}{m+n} - \frac{1}{m-n} \right) \right) = 0
\end{aligned}
$$

- sin と cos

$$
\begin{aligned}
&\left\langle \frac{1}{\sqrt{\pi}} \sin mx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin mx} \frac{1}{\sqrt{\pi}} \cos nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \cos nx \sin mx dx = 0
\end{aligned}
$$

以上より, 同じ元どうしの内積が 1 になり, 異なる元どうしの内積が 0 になるので,

$$\left\{ \frac{1}{\sqrt{\pi}} \cos nx \;\middle|\; n \in \mathbb{N} \right\} \cup \left\{ \frac{1}{\sqrt{\pi}} \sin nx \;\middle|\; n \in \mathbb{N} \right\}$$

は正規直交系である.

5

$$\begin{aligned} \langle \xi, \zeta \rangle &= \left\langle \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i \otimes e_i, \frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{n} e_j \otimes e_k \right\rangle \\ &= \frac{1}{n\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle \end{aligned}$$

$\{\, e_i \,\}_{i=1}^{n}$ は $\mathbb{C}^n$ の正規直交基底なので, $\langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle$ は $e_i = e_j = e_k$ のときのみ 1 になり, それ以外は 0 となる. よって

$$\begin{aligned} \langle \xi, \zeta \rangle &= \frac{1}{n\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle \\ &= \frac{1}{n\sqrt{n}} \sum_{i=1}^{n} 1 = \frac{1}{n\sqrt{n}} \cdot n = \frac{1}{\sqrt{n}} \end{aligned}$$

6

$$\begin{aligned} Ty &= \left(\sum_{i=1}^{n} |x_i\rangle \langle x_i| \right) y = \sum_{i=1}^{n} |x_i\rangle \langle x_i| \, y \\ &= \sum_{i=1}^{n} \langle x_i, y \rangle \, x_i = \sum_{i=1}^{n} \left\langle x_i, \sum_{j=1}^{n} \alpha_j x_j \right\rangle x_i \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_j \langle x_i, x_j \rangle \, x_i = \sum_{i=1}^{n} \alpha_i x_i = y \end{aligned}$$

7

まず,

$$\begin{aligned} \frac{d}{dx}^{*} \cos x &= a_1 \cos x + b_1 \sin x \\ \frac{d}{dx}^{*} \sin x &= a_2 \cos x + b_2 \sin x \end{aligned}$$

と置く. cos と sin の内積を計算すると, それぞれ

$$\begin{aligned}\langle \cos x, \cos x\rangle &= \int_0^{2\pi} \cos^2 x dx = \int_0^{2\pi} \frac{1+\cos 2x}{2} dx \\ &= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{2\pi} = \frac{1}{2}\left(2\pi + 0\right) = \pi\end{aligned}$$

$$\begin{aligned}\langle \sin x, \sin x\rangle &= \int_0^{2\pi} \sin^2 x dx = \int_0^{2\pi} \frac{1-\cos 2x}{2} dx \\ &= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{2\pi} = \frac{1}{2}\left(2\pi - 0\right) = \pi\end{aligned}$$

$$\begin{aligned}\langle \cos x, \sin x\rangle &= \int_0^{2\pi} \cos x \sin x dx = \int_0^{2\pi} \frac{1}{2}\sin 2x dx \\ &= \frac{1}{2}\left[-\frac{1}{2}\cos 2x\right]_0^{2\pi} = \frac{1}{2}\left(-\frac{1}{2} + \frac{1}{2}\right) = 0\end{aligned}$$

$$\langle \sin x, \cos x\rangle = \int_0^{2\pi} \sin x \cos x dx = 0$$

となる.

a_1, b_1 を求めるために $\left\langle \frac{d}{dx}\cos x, \cos x\right\rangle$, $\left\langle \cos x, \frac{d}{dx}^* \cos x\right\rangle$, $\left\langle \frac{d}{dx}\sin x, \cos x\right\rangle$, $\left\langle \sin x, \frac{d}{dx}^* \cos x\right\rangle$ を計算する.

$$\left\langle \frac{d}{dx}\cos x, \cos x\right\rangle = \langle -\sin x, \cos x\rangle = -\langle \sin x, \cos x\rangle = 0 \tag{1}$$

$$\begin{aligned}\left\langle \cos x, \frac{d}{dx}^* \cos x\right\rangle &= \langle \cos x, a_1 \cos x + b_1 \sin x\rangle \\ &= a_1 \langle \cos x, \cos x\rangle + b_1 \langle \cos x, \sin x\rangle = \pi a_1\end{aligned} \tag{2}$$

随伴作用素の定義と (1), (2) より, $a_1 = 0$ である.

$$\left\langle \frac{d}{dx}\sin x, \cos x\right\rangle = \langle \cos x, \cos x\rangle = \pi \tag{3}$$

$$\begin{aligned}\left\langle \sin x, \frac{d}{dx}^* \cos x\right\rangle &= \langle \sin x, a_1 \cos x + b_1 \sin x\rangle \\ &= a_1 \langle \sin x, \cos x\rangle + b_1 \langle \sin x, \sin x\rangle = \pi b_1\end{aligned} \tag{4}$$

随伴作用素の定義と (3), (4) より, $b_1 = 1$ である. 以上より,

$$\frac{d}{dx}^* \cos x = \sin x$$

が示された.