

応用関数解析特論レポート

園田継一郎

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1

\mathbb{C}^2 の標準基底

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

について He_1, He_2 を計算すると

$$\begin{aligned} He_1 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ He_2 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

となる. それぞれの内積は

$$\begin{aligned} \langle He_1, He_1 \rangle &= \frac{1}{2}(1+1) = 1 = \langle e_1, e_1 \rangle \\ \langle He_1, He_2 \rangle &= \frac{1}{2}(1-1) = 0 = \langle e_1, e_2 \rangle \\ \langle He_2, He_1 \rangle &= \frac{1}{2}(1-1) = 0 = \langle e_2, e_1 \rangle \\ \langle He_2, He_2 \rangle &= \frac{1}{2}(1+1) = 1 = \langle e_2, e_2 \rangle \end{aligned}$$

となり, 標準基底について H はユニタリ作用素の条件を満たす. $\forall x, y \in \mathbb{C}^2$ は e_1, e_2 の線形結合で表せるので,

$$\langle Hx, Hy \rangle = \langle x, y \rangle$$

が得られる. よって H はユニタリ作用素である.

2

\mathbb{C}^3 の正規直交基底の 1 つとして,

$$e_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, e_2 = \frac{1}{\sqrt{182}} \begin{bmatrix} 13 \\ -2 \\ -3 \end{bmatrix}, e_3 = \frac{1}{\sqrt{13}} \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

が挙げられる.

3

状態と正規直行基底の内積を計算すると

$$|\langle \xi_1, \psi \rangle|^2 = \left(\frac{1}{6}(-2 + 2 + 1 + 0) \right)^2 = \left(\frac{1}{6} \right)^2 = \frac{1}{36}$$

$$|\langle \xi_2, \psi \rangle|^2 = \left(\frac{1}{6}(2 - 2 + 1 + 0) \right)^2 = \left(\frac{1}{6} \right)^2 = \frac{1}{36}$$

$$|\langle \xi_3, \psi \rangle|^2 = \left(\frac{1}{6}(2 + 2 - 1 + 0) \right)^2 = \left(\frac{1}{2} \right)^2 = \frac{1}{4}$$

$$|\langle \xi_4, \psi \rangle|^2 = \left(\frac{1}{6}(2 + 2 + 1 - 0) \right)^2 = \left(\frac{5}{6} \right)^2 = \frac{25}{36}$$

となる. 公理 2 より, 数値 1, 2, 3, 4 が検出される確率はそれぞれ $\frac{1}{36}, \frac{1}{36}, \frac{1}{4}, \frac{25}{36}$ である.

4

$m, n \in \mathbb{N}$ とし, 集合の元どうしで内積を計算する. $x \in [0, 2\pi]$ のとき $\cos nx, \sin nx \in [0, 1]$ なので,

$$\overline{\cos nx} = \cos nx$$

$$\overline{\sin nx} = \sin nx$$

が成立する. まず, 元が同じ場合の内積を計算する.

• \cos

$$\begin{aligned} & \left\langle \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos nx} \frac{1}{\sqrt{\pi}} \cos nx dx \\ &= \frac{1}{\pi} \int_0^{2\pi} \cos^2 nx dx = \frac{1}{2\pi} \int_0^{2\pi} (1 + \cos 2nx) dx \\ &= \frac{1}{2\pi} \left[x + \frac{1}{2} \sin 2nx \right]_0^{2\pi} = \frac{1}{2\pi} (2\pi - 0) = 1 \end{aligned}$$

- \sin

$$\begin{aligned}
 \left\langle \frac{1}{\sqrt{\pi}} \sin nx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin nx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
 &= \frac{1}{\pi} \int_0^{2\pi} \sin^2 nx dx = \frac{1}{2\pi} \int_0^{2\pi} (1 - \cos 2nx) dx \\
 &= \frac{1}{2\pi} \left[x - \frac{1}{2} \sin 2nx \right]_0^{2\pi} = \frac{1}{2\pi} (2\pi - 0) = 1
 \end{aligned}$$

続いて, 元が異なる場合の内積を計算する.

- $\cos (m \neq n)$

$$\begin{aligned}
 \left\langle \frac{1}{\sqrt{\pi}} \cos mx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos mx} \frac{1}{\sqrt{\pi}} \cos nx dx \\
 &= \frac{1}{\pi} \int_0^{2\pi} \cos mx \cos nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\cos (m+n)x + \cos (m-n)x) dx \\
 &= \frac{1}{2\pi} \left[\frac{1}{m+n} \sin (m+n)x + \frac{1}{m-n} \sin (m-n)x \right]_0^{2\pi} = \frac{1}{2\pi} (0 - 0) = 0
 \end{aligned}$$

- $\sin (m \neq n)$

$$\begin{aligned}
 \left\langle \frac{1}{\sqrt{\pi}} \sin mx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin mx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
 &= \frac{1}{\pi} \int_0^{2\pi} \sin mx \sin nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\cos (m-n)x - \cos (m+n)x) dx \\
 &= \frac{1}{2\pi} \left[\frac{1}{m-n} \sin (m-n)x - \frac{1}{m+n} \sin (m+n)x \right]_0^{2\pi} = \frac{1}{2\pi} (0 - 0) = 0
 \end{aligned}$$

- \cos と \sin

$$\begin{aligned}
 \left\langle \frac{1}{\sqrt{\pi}} \cos mx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos mx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
 &= \frac{1}{\pi} \int_0^{2\pi} \cos mx \sin nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\sin (m+n)x + \sin (m-n)x) dx \\
 &= \frac{1}{2\pi} \left[-\frac{1}{m+n} \cos (m+n)x - \frac{1}{m-n} \cos (m-n)x \right]_0^{2\pi} \\
 &= \frac{1}{2\pi} \left(\left(-\frac{1}{m+n} - \frac{1}{m-n} \right) - \left(-\frac{1}{m+n} - \frac{1}{m-n} \right) \right) = 0
 \end{aligned}$$

- \sin と \cos

$$\begin{aligned}
 \left\langle \frac{1}{\sqrt{\pi}} \sin mx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin mx} \frac{1}{\sqrt{\pi}} \cos nx dx \\
 &= \frac{1}{\pi} \int_0^{2\pi} \cos nx \sin mx dx = 0
 \end{aligned}$$