# 応用関数解析特論レポート

園田継一郎

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## 1

$^\forall x, y \in \mathbb{C}^2$ を $x_1, x_2, y_1, y_2 \in \mathbb{C}$ を用いて

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

とすると, 内積は

$$\langle x, y \rangle = \overline{x_1} y_1 + \overline{x_2} y_2$$

となる．$x, y$ に左から $H$ をかけると

$$\begin{aligned} Hx &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} \\ Hy &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \end{bmatrix} \end{aligned}$$

となる．これらの内積を計算すると

$$\begin{aligned} \langle Hx, Hy \rangle &= \frac{1}{2} \left( \overline{(x_1 + x_2)} (y_1 + y_2) + \overline{(x_1 - x_2)} (y_1 - y_2) \right) \\ &= \frac{1}{2} \left( \overline{x_1} y_1 + \overline{x_1} y_2 + \overline{x_2} y_1 + \overline{x_2} y_2 + \overline{x_1} y_1 - \overline{x_1} y_2 - \overline{x_2} y_1 + \overline{x_2} y_2 \right) \\ &= \frac{1}{2} \left( 2\overline{x_1} y_1 + 2\overline{x_2} y_2 \right) = \overline{x_1} y_1 + \overline{x_2} y_2 = \langle x, y \rangle \end{aligned}$$

となるので, $H$ はユニタリ作用素の条件を満たす.

## 2

$\mathbb{C}^3$ の正規直交基底の 1 つとして,

$$e_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, e_2 = \frac{1}{\sqrt{182}} \begin{bmatrix} 13 \\ -2 \\ -3 \end{bmatrix}, e_3 = \frac{1}{\sqrt{13}} \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

が挙げられる.

## 3

正規直交基底と状態の内積を計算すると

$$
\begin{aligned}
|\langle \xi_1, \psi \rangle|^2 &= \left| \frac{1}{6}(-2+2+1+0) \right|^2 = \left| \frac{1}{6} \right|^2 = \frac{1}{36} \\
|\langle \xi_2, \psi \rangle|^2 &= \left| \frac{1}{6}(2-2+1+0) \right|^2 = \left| \frac{1}{6} \right|^2 = \frac{1}{36} \\
|\langle \xi_3, \psi \rangle|^2 &= \left| \frac{1}{6}(2+2-1+0) \right|^2 = \left| \frac{1}{2} \right|^2 = \frac{1}{4} \\
|\langle \xi_4, \psi \rangle|^2 &= \left| \frac{1}{6}(2+2+1-0) \right|^2 = \left| \frac{5}{6} \right|^2 = \frac{25}{36}
\end{aligned}
$$

となる．公理 2 より, 各数値が検出される確率は表 1 のようになる．

表 1　各数値が検出される確率

| 数値 | 確率 |
|---|---|
| 1 | $\frac{1}{36}$ |
| 2 | $\frac{1}{36}$ |
| 3 | $\frac{1}{4}$ |
| 4 | $\frac{25}{36}$ |

## 4

$m, n \in \mathbb{N}$ とし, 集合の元どうしで内積を計算する．$x \in [0, 2\pi]$ のとき $\cos nx, \sin nx \in [0, 1]$ なので,

$$
\begin{aligned}
\overline{\cos nx} &= \cos nx \\
\overline{\sin nx} &= \sin nx
\end{aligned}
$$

が成立する．まず, 元が同じ場合の内積を計算する．

- cos

$$
\begin{aligned}
&\left\langle \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos nx} \frac{1}{\sqrt{\pi}} \cos nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \cos^2 nx dx = \frac{1}{2\pi} \int_0^{2\pi} (1 + \cos 2nx) dx \\
&= \frac{1}{2\pi} \left[ x + \frac{1}{2} \sin 2nx \right]_0^{2\pi} = \frac{1}{2\pi} (2\pi - 0) = 1
\end{aligned}
$$

- sin

$$\left\langle \frac{1}{\sqrt{\pi}} \sin nx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin nx} \frac{1}{\sqrt{\pi}} \sin nx dx$$
$$= \frac{1}{\pi} \int_0^{2\pi} \sin^2 nx dx = \frac{1}{2\pi} \int_0^{2\pi} (1 - \cos 2nx) dx$$
$$= \frac{1}{2\pi} \left[ x - \frac{1}{2} \sin 2nx \right]_0^{2\pi} = \frac{1}{2\pi} (2\pi - 0) = 1$$

続いて, 元が異なる場合の内積を計算する.

- cos $(m \neq n)$

$$\left\langle \frac{1}{\sqrt{\pi}} \cos mx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos mx} \frac{1}{\sqrt{\pi}} \cos nx dx$$
$$= \frac{1}{\pi} \int_0^{2\pi} \cos mx \cos nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\cos (m+n)x + \cos (m-n)x) dx$$
$$= \frac{1}{2\pi} \left[ \frac{1}{m+n} \sin (m+n)x + \frac{1}{m-n} \sin (m-n)x \right]_0^{2\pi} = \frac{1}{2\pi} (0 - 0) = 0$$

- sin $(m \neq n)$

$$\left\langle \frac{1}{\sqrt{\pi}} \sin mx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin mx} \frac{1}{\sqrt{\pi}} \sin nx dx$$
$$= \frac{1}{\pi} \int_0^{2\pi} \sin mx \sin nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\cos (m-n)x - \cos (m+n)x) dx$$
$$= \frac{1}{2\pi} \left[ \frac{1}{m-n} \sin (m-n)x - \frac{1}{m+n} \sin (m+n)x \right]_0^{2\pi} = \frac{1}{2\pi} (0 - 0) = 0$$

- cos と sin $(m = n)$

$$\left\langle \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos nx} \frac{1}{\sqrt{\pi}} \sin nx dx$$
$$= \frac{1}{\pi} \int_0^{2\pi} \cos nx \sin nx dx = \frac{1}{2\pi} \int_0^{2\pi} \sin 2nx dx$$
$$= \frac{1}{2\pi} \left[ -\frac{1}{2n} \cos 2nx \right]_0^{2\pi} = -\frac{1}{4n\pi} (1 - 1) = 0$$

- sin と cos $(m = n)$

$$\left\langle \frac{1}{\sqrt{\pi}} \sin nx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin nx} \frac{1}{\sqrt{\pi}} \cos nx dx$$
$$= \frac{1}{\pi} \int_0^{2\pi} \sin nx \cos nx dx = 0$$

- cos と sin $(m \neq n)$

$$\left\langle \frac{1}{\sqrt{\pi}}\cos mx, \frac{1}{\sqrt{\pi}}\sin nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}}\cos mx} \frac{1}{\sqrt{\pi}}\sin nx dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \cos mx \sin nx dx = \frac{1}{2\pi}\int_0^{2\pi} (\sin{(m+n)x} + \sin{(m-n)x})dx$$

$$= \frac{1}{2\pi}\left[ -\frac{1}{m+n}\cos{(m+n)x} - \frac{1}{m-n}\cos{(m-n)x} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left( \left( -\frac{1}{m+n} - \frac{1}{m-n} \right) - \left( -\frac{1}{m+n} - \frac{1}{m-n} \right) \right) = 0$$

- sin と cos $(m \neq n)$

$$\left\langle \frac{1}{\sqrt{\pi}}\sin mx, \frac{1}{\sqrt{\pi}}\cos nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}}\sin mx} \frac{1}{\sqrt{\pi}}\cos nx dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \cos nx \sin mx dx = 0$$

以上より, 同じ元どうしの内積が 1 になり, 異なる元どうしの内積が 0 になるので,

$$\left\{ \frac{1}{\sqrt{\pi}}\cos nx \;\middle|\; n \in \mathbb{N} \right\} \cup \left\{ \frac{1}{\sqrt{\pi}}\sin nx \;\middle|\; n \in \mathbb{N} \right\}$$

は正規直交系である.

# 5

$$\langle \xi, \zeta \rangle = \left\langle \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i \otimes e_i, \frac{1}{n} \sum_{j=1}^{n}\sum_{k=1}^{n} e_j \otimes e_k \right\rangle$$

$$= \frac{1}{n\sqrt{n}} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle$$

$\{\, e_i \,\}_{i=1}^{n}$ は $\mathbb{C}^n$ の正規直交基底なので, $\langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle$ は $e_i = e_j = e_k$ のときのみ 1 になり, それ以外は 0 となる. よって

$$\langle \xi, \zeta \rangle = \frac{1}{n\sqrt{n}} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle$$

$$= \frac{1}{n\sqrt{n}} \sum_{i=1}^{n} 1 = \frac{1}{n\sqrt{n}} \cdot n = \frac{1}{\sqrt{n}}$$

## 6

$$
\begin{aligned}
Ty &= \left( \sum_{i=1}^{n} |x_i\rangle \langle x_i| \right) y = \sum_{i=1}^{n} |x_i\rangle \langle x_i| \, y \\
&= \sum_{i=1}^{n} \langle x_i, y \rangle \, x_i = \sum_{i=1}^{n} \left\langle x_i, \, \sum_{j=1}^{n} \alpha_j x_j \right\rangle x_i \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_j \langle x_i, x_j \rangle \, x_i = \sum_{i=1}^{n} \alpha_i x_i = y
\end{aligned}
$$

## 7

まず,

$$
\begin{aligned}
\frac{d}{dx}^* \cos x &= a_1 \cos x + b_1 \sin x \\
\frac{d}{dx}^* \sin x &= a_2 \cos x + b_2 \sin x
\end{aligned}
$$

と置く. cos と sin の内積を計算すると, それぞれ

$$
\begin{aligned}
\langle \cos x, \cos x \rangle &= \int_0^{2\pi} \cos^2 x dx = \int_0^{2\pi} \frac{1+\cos 2x}{2} dx \\
&= \frac{1}{2} \left[ x + \frac{1}{2} \sin 2x \right]_0^{2\pi} = \frac{1}{2} (2\pi + 0) = \pi
\end{aligned}
$$

$$
\begin{aligned}
\langle \sin x, \sin x \rangle &= \int_0^{2\pi} \sin^2 x dx = \int_0^{2\pi} \frac{1-\cos 2x}{2} dx \\
&= \frac{1}{2} \left[ x - \frac{1}{2} \sin 2x \right]_0^{2\pi} = \frac{1}{2} (2\pi - 0) = \pi
\end{aligned}
$$

$$
\begin{aligned}
\langle \cos x, \sin x \rangle &= \int_0^{2\pi} \cos x \sin x dx = \int_0^{2\pi} \frac{1}{2} \sin 2x dx \\
&= \frac{1}{2} \left[ -\frac{1}{2} \cos 2x \right]_0^{2\pi} = \frac{1}{2} \left( -\frac{1}{2} + \frac{1}{2} \right) = 0
\end{aligned}
$$

$$
\langle \sin x, \cos x \rangle = \int_0^{2\pi} \sin x \cos x dx = 0
$$

となる.

$a_1, b_1$ を求めるために $\left\langle \frac{d}{dx}\cos x, \cos x\right\rangle$, $\left\langle \cos x, \frac{d}{dx}^* \cos x\right\rangle$, $\left\langle \frac{d}{dx}\sin x, \cos x\right\rangle$, $\left\langle \sin x, \frac{d}{dx}^* \cos x\right\rangle$ を計算する.

$$\left\langle \frac{d}{dx}\cos x, \cos x\right\rangle = \langle -\sin x, \cos x\rangle = -\langle \sin x, \cos x\rangle = 0 \tag{1}$$

$$\begin{aligned}\left\langle \cos x, \frac{d}{dx}^* \cos x\right\rangle &= \langle \cos x, a_1 \cos x + b_1 \sin x\rangle \\ &= a_1 \langle \cos x, \cos x\rangle + b_1 \langle \cos x, \sin x\rangle = \pi a_1\end{aligned} \tag{2}$$

随伴作用素の定義と (1), (2) より, $a_1 = 0$ である.

$$\left\langle \frac{d}{dx}\sin x, \cos x\right\rangle = \langle \cos x, \cos x\rangle = \pi \tag{3}$$

$$\begin{aligned}\left\langle \sin x, \frac{d}{dx}^* \cos x\right\rangle &= \langle \sin x, a_1 \cos x + b_1 \sin x\rangle \\ &= a_1 \langle \sin x, \cos x\rangle + b_1 \langle \sin x, \sin x\rangle = \pi b_1\end{aligned} \tag{4}$$

随伴作用素の定義と (3), (4) より, $b_1 = 1$ である. $a_1 = 0, b_1 = 1$ より

$$\frac{d}{dx}^* \cos x = \sin x$$

$a_2, b_2$ を求めるために $\left\langle \frac{d}{dx}\sin x, \sin x\right\rangle$, $\left\langle \sin x, \frac{d}{dx}^* \sin x\right\rangle$, $\left\langle \frac{d}{dx}\cos x, \sin x\right\rangle$, $\left\langle \cos x, \frac{d}{dx}^* \sin x\right\rangle$ を計算する.

$$\left\langle \frac{d}{dx}\sin x, \sin x\right\rangle = \langle \cos x, \sin x\rangle = 0 \tag{5}$$

$$\begin{aligned}\left\langle \sin x, \frac{d}{dx}^* \sin x\right\rangle &= \langle \sin x, a_2 \cos x + b_2 \sin x\rangle \\ &= a_2 \langle \sin x, \cos x\rangle + b_2 \langle \sin x, \sin x\rangle = \pi b_2\end{aligned} \tag{6}$$

随伴作用素の定義と (5), (6) より, $b_2 = 0$ である.

$$\left\langle \frac{d}{dx}\cos x, \sin x\right\rangle = \langle -\sin x, \sin x\rangle = -\langle \sin x, \sin x\rangle = -\pi \tag{7}$$

$$\begin{aligned}\left\langle \cos x, \frac{d}{dx}^* \sin x\right\rangle &= \langle \cos x, a_2 \cos x + b_2 \sin x\rangle \\ &= a_2 \langle \cos x, \cos x\rangle + b_2 \langle \cos x, \sin x\rangle = \pi a_2\end{aligned} \tag{8}$$

随伴作用素の定義と (7), (8) より, $a_2 = -1$ である. $a_2 = -1, b_2 = 0$ より

$$\frac{d}{dx}^* \sin x = -\cos x$$

$V$ の基底について

$$\frac{d}{dx}^* \cos x = \sin x = -\frac{d}{dx} \cos x$$
$$\frac{d}{dx}^* \sin x = -\cos x = -\frac{d}{dx} \sin x$$

が成立するので,

$$\frac{d}{dx}^* = -\frac{d}{dx}$$

である.