

# 応用関数解析特論レポート

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## 1

$\forall x, y \in \mathbb{C}^2$  を  $x_1, x_2, y_1, y_2 \in \mathbb{C}$  を用いて

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

とすると, 内積は

$$\langle x, y \rangle = \overline{x_1}y_1 + \overline{x_2}y_2$$

となる.  $x, y$  に左から  $H$  をかけると

$$\begin{aligned} Hx &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} \\ Hy &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \end{bmatrix} \end{aligned}$$

となる. これらの内積を計算すると

$$\begin{aligned} \langle Hx, Hy \rangle &= \frac{1}{2} \left( \overline{(x_1 + x_2)}(y_1 + y_2) + \overline{(x_1 - x_2)}(y_1 - y_2) \right) \\ &= \frac{1}{2} (\overline{x_1}y_1 + \overline{x_1}y_2 + \overline{x_2}y_1 + \overline{x_2}y_2 + \overline{x_1}y_1 - \overline{x_1}y_2 - \overline{x_2}y_1 + \overline{x_2}y_2) \\ &= \frac{1}{2} (2\overline{x_1}y_1 + 2\overline{x_2}y_2) = \overline{x_1}y_1 + \overline{x_2}y_2 = \langle x, y \rangle \end{aligned}$$

となるので,  $H$  はユニタリ作用素の条件を満たす.

## 2

$\mathbb{C}^3$  の正規直交基底の 1 つとして,

$$e_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, e_2 = \frac{1}{\sqrt{182}} \begin{bmatrix} 13 \\ -2 \\ -3 \end{bmatrix}, e_3 = \frac{1}{\sqrt{13}} \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

が挙げられる.

### 3

正規直交基底と状態の内積を計算すると

$$\begin{aligned} |\langle \xi_1, \psi \rangle|^2 &= \left| \frac{1}{6}(-2 + 2 + 1 + 0) \right|^2 = \left| \frac{1}{6} \right|^2 = \frac{1}{36} \\ |\langle \xi_2, \psi \rangle|^2 &= \left| \frac{1}{6}(2 - 2 + 1 + 0) \right|^2 = \left| \frac{1}{6} \right|^2 = \frac{1}{36} \\ |\langle \xi_3, \psi \rangle|^2 &= \left| \frac{1}{6}(2 + 2 - 1 + 0) \right|^2 = \left| \frac{1}{2} \right|^2 = \frac{1}{4} \\ |\langle \xi_4, \psi \rangle|^2 &= \left| \frac{1}{6}(2 + 2 + 1 - 0) \right|^2 = \left| \frac{5}{6} \right|^2 = \frac{25}{36} \end{aligned}$$

となる. 公理 2 より, 各数値が観測される確率は表 1 のようになる.

表 1 各数値が観測される確率

数値	確率
1	$\frac{1}{36}$
2	$\frac{1}{36}$
3	$\frac{1}{4}$
4	$\frac{25}{36}$

### 4

$m, n \in \mathbb{N}$  とし, 集合の元どうしで内積を計算する.  $x \in [0, 2\pi]$  のとき  $\cos nx, \sin nx \in [0, 1]$  なので,

$$\begin{aligned} \overline{\cos nx} &= \cos nx \\ \overline{\sin nx} &= \sin nx \end{aligned}$$

が成立する. まず, 元が同じ場合の内積を計算する.

●  $\cos$

$$\begin{aligned} \left\langle \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos nx} \frac{1}{\sqrt{\pi}} \cos nx dx \\ &= \frac{1}{\pi} \int_0^{2\pi} \cos^2 nx dx = \frac{1}{2\pi} \int_0^{2\pi} (1 + \cos 2nx) dx \\ &= \frac{1}{2\pi} \left[ x + \frac{1}{2} \sin 2nx \right]_0^{2\pi} = \frac{1}{2\pi} (2\pi - 0) = 1 \end{aligned}$$

- sin

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{\pi}} \sin nx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin nx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \sin^2 nx dx = \frac{1}{2\pi} \int_0^{2\pi} (1 - \cos 2nx) dx \\
&= \frac{1}{2\pi} \left[ x - \frac{1}{2} \sin 2nx \right]_0^{2\pi} = \frac{1}{2\pi} (2\pi - 0) = 1
\end{aligned}$$

続いて, 元が異なる場合の内積を計算する.

- cos ( $m \neq n$ )

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{\pi}} \cos mx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos mx} \frac{1}{\sqrt{\pi}} \cos nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \cos mx \cos nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\cos(m+n)x + \cos(m-n)x) dx \\
&= \frac{1}{2\pi} \left[ \frac{1}{m+n} \sin(m+n)x + \frac{1}{m-n} \sin(m-n)x \right]_0^{2\pi} = \frac{1}{2\pi} (0 - 0) = 0
\end{aligned}$$

- sin ( $m \neq n$ )

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{\pi}} \sin mx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin mx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \sin mx \sin nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\cos(m-n)x - \cos(m+n)x) dx \\
&= \frac{1}{2\pi} \left[ \frac{1}{m-n} \sin(m-n)x - \frac{1}{m+n} \sin(m+n)x \right]_0^{2\pi} = \frac{1}{2\pi} (0 - 0) = 0
\end{aligned}$$

- cos と sin ( $m = n$ )

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos nx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \cos nx \sin nx dx = \frac{1}{2\pi} \int_0^{2\pi} \sin 2nx dx \\
&= \frac{1}{2\pi} \left[ -\frac{1}{2n} \cos 2nx \right]_0^{2\pi} = -\frac{1}{4n\pi} (1 - 1) = 0
\end{aligned}$$

- sin と cos ( $m = n$ )

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{\pi}} \sin nx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle &= \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin nx} \frac{1}{\sqrt{\pi}} \cos nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \sin nx \cos nx dx = 0
\end{aligned}$$

- $\cos$  と  $\sin$  ( $m \neq n$ )

$$\begin{aligned}
& \left\langle \frac{1}{\sqrt{\pi}} \cos mx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos mx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \cos mx \sin nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\sin(m+n)x + \sin(m-n)x) dx \\
&= \frac{1}{2\pi} \left[ -\frac{1}{m+n} \cos(m+n)x - \frac{1}{m-n} \cos(m-n)x \right]_0^{2\pi} \\
&= \frac{1}{2\pi} \left( \left( -\frac{1}{m+n} - \frac{1}{m-n} \right) - \left( -\frac{1}{m+n} - \frac{1}{m-n} \right) \right) = 0
\end{aligned}$$

- $\sin$  と  $\cos$  ( $m \neq n$ )

$$\begin{aligned}
& \left\langle \frac{1}{\sqrt{\pi}} \sin mx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin mx} \frac{1}{\sqrt{\pi}} \cos nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \cos nx \sin mx dx = 0
\end{aligned}$$

以上より, 同じ元どうしの内積が 1 になり, 異なる元どうしの内積が 0 になるので,

$$\left\{ \frac{1}{\sqrt{\pi}} \cos nx \mid n \in \mathbb{N} \right\} \cup \left\{ \frac{1}{\sqrt{\pi}} \sin nx \mid n \in \mathbb{N} \right\}$$

は正規直交系である.

## 5

$$\begin{aligned}
\langle \xi, \zeta \rangle &= \left\langle \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes e_i, \frac{1}{n} \sum_{j=1}^n \sum_{k=1}^n e_j \otimes e_k \right\rangle \\
&= \frac{1}{n\sqrt{n}} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle
\end{aligned}$$

$\{e_i\}_{i=1}^n$  は  $\mathbb{C}^n$  の正規直交基底なので,  $\langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle$  は  $e_i = e_j = e_k$  のときのみ 1 になり, それ以外は 0 となる. よって

$$\begin{aligned}
\langle \xi, \zeta \rangle &= \frac{1}{n\sqrt{n}} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle \\
&= \frac{1}{n\sqrt{n}} \sum_{i=1}^n 1 = \frac{1}{n\sqrt{n}} \cdot n = \frac{1}{\sqrt{n}}
\end{aligned}$$

6

$$\begin{aligned}
 Ty &= \left( \sum_{i=1}^n |x_i\rangle \langle x_i| \right) y = \sum_{i=1}^n |x_i\rangle \langle x_i| y \\
 &= \sum_{i=1}^n \langle x_i, y \rangle x_i = \sum_{i=1}^n \left\langle x_i, \sum_{j=1}^n \alpha_j x_j \right\rangle x_i \\
 &= \sum_{i=1}^n \sum_{j=1}^n \alpha_j \langle x_i, x_j \rangle x_i = \sum_{i=1}^n \alpha_i x_i = y
 \end{aligned}$$

7

まず,

$$\begin{aligned}
 \frac{d}{dx} \cos x &= -\sin x \\
 \frac{d}{dx} \sin x &= \cos x
 \end{aligned}$$

と置く.  $\cos$  と  $\sin$  の内積を計算すると, それぞれ

$$\begin{aligned}
 \langle \cos x, \cos x \rangle &= \int_0^{2\pi} \cos^2 x dx = \int_0^{2\pi} \frac{1 + \cos 2x}{2} dx \\
 &= \frac{1}{2} \left[ x + \frac{1}{2} \sin 2x \right]_0^{2\pi} = \frac{1}{2} (2\pi + 0) = \pi
 \end{aligned}$$

$$\begin{aligned}
 \langle \sin x, \sin x \rangle &= \int_0^{2\pi} \sin^2 x dx = \int_0^{2\pi} \frac{1 - \cos 2x}{2} dx \\
 &= \frac{1}{2} \left[ x - \frac{1}{2} \sin 2x \right]_0^{2\pi} = \frac{1}{2} (2\pi - 0) = \pi
 \end{aligned}$$

$$\begin{aligned}
 \langle \cos x, \sin x \rangle &= \int_0^{2\pi} \cos x \sin x dx = \int_0^{2\pi} \frac{1}{2} \sin 2x dx \\
 &= \frac{1}{2} \left[ -\frac{1}{2} \cos 2x \right]_0^{2\pi} = \frac{1}{2} \left( -\frac{1}{2} + \frac{1}{2} \right) = 0
 \end{aligned}$$

$$\langle \sin x, \cos x \rangle = \int_0^{2\pi} \sin x \cos x dx = 0$$

となる.

- (i)  $a_1, b_1$  を求めるために  $\left\langle \frac{d}{dx} \cos x, \cos x \right\rangle, \left\langle \cos x, \frac{d}{dx}^* \cos x \right\rangle, \left\langle \frac{d}{dx} \sin x, \cos x \right\rangle, \left\langle \sin x, \frac{d}{dx}^* \cos x \right\rangle$  を計算する.

$$\left\langle \frac{d}{dx} \cos x, \cos x \right\rangle = \langle -\sin x, \cos x \rangle = -\langle \sin x, \cos x \rangle = 0 \quad (1)$$

$$\begin{aligned} \left\langle \cos x, \frac{d}{dx}^* \cos x \right\rangle &= \langle \cos x, a_1 \cos x + b_1 \sin x \rangle \\ &= a_1 \langle \cos x, \cos x \rangle + b_1 \langle \cos x, \sin x \rangle = \pi a_1 \end{aligned} \quad (2)$$

随伴作用素の定義と (1), (2) より,  $a_1 = 0$  である.

$$\left\langle \frac{d}{dx} \sin x, \cos x \right\rangle = \langle \cos x, \cos x \rangle = \pi \quad (3)$$

$$\begin{aligned} \left\langle \sin x, \frac{d}{dx}^* \cos x \right\rangle &= \langle \sin x, a_1 \cos x + b_1 \sin x \rangle \\ &= a_1 \langle \sin x, \cos x \rangle + b_1 \langle \sin x, \sin x \rangle = \pi b_1 \end{aligned} \quad (4)$$

随伴作用素の定義と (3), (4) より,  $b_1 = 1$  である. 以上より,

$$\frac{d}{dx}^* \cos x = \sin x$$

- (ii)  $a_2, b_2$  を求めるために  $\left\langle \frac{d}{dx} \sin x, \sin x \right\rangle, \left\langle \sin x, \frac{d}{dx}^* \sin x \right\rangle, \left\langle \frac{d}{dx} \cos x, \sin x \right\rangle, \left\langle \cos x, \frac{d}{dx}^* \sin x \right\rangle$  を計算する.

$$\left\langle \frac{d}{dx} \sin x, \sin x \right\rangle = \langle \cos x, \sin x \rangle = 0 \quad (5)$$

$$\begin{aligned} \left\langle \sin x, \frac{d}{dx}^* \sin x \right\rangle &= \langle \sin x, a_2 \cos x + b_2 \sin x \rangle \\ &= a_2 \langle \sin x, \cos x \rangle + b_2 \langle \sin x, \sin x \rangle = \pi b_2 \end{aligned} \quad (6)$$

随伴作用素の定義と (5), (6) より,  $b_2 = 0$  である.

$$\left\langle \frac{d}{dx} \cos x, \sin x \right\rangle = \langle -\sin x, \sin x \rangle = -\langle \sin x, \sin x \rangle = -\pi \quad (7)$$

$$\begin{aligned} \left\langle \cos x, \frac{d}{dx}^* \sin x \right\rangle &= \langle \cos x, a_2 \cos x + b_2 \sin x \rangle \\ &= a_2 \langle \cos x, \cos x \rangle + b_2 \langle \cos x, \sin x \rangle = \pi a_2 \end{aligned} \quad (8)$$

随伴作用素の定義と (7), (8) より,  $a_2 = -1$  である. 以上より,

$$\frac{d}{dx}^* \sin x = -\cos x$$

$V$  の基底について

$$\begin{aligned}\frac{d^*}{dx} \cos x &= \sin x = -\frac{d}{dx} \cos x \\ \frac{d^*}{dx} \sin x &= -\cos x = -\frac{d}{dx} \sin x\end{aligned}$$

が成立するので,

$$\frac{d^*}{dx} = -\frac{d}{dx}$$

である.