応用関数解析特論レポート

園田継一郎

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1

$^\forall x, y \in \mathbb{C}^2$ を $x_1, x_2, y_1, y_2 \in \mathbb{C}$ を用いて

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

とすると, 内積は

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2$$

となる. x, y に左から H をかけると

$$\begin{aligned} Hx &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} \\ Hy &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \end{bmatrix} \end{aligned}$$

となる. これらの内積を計算すると

$$\begin{aligned} \langle Hx, Hy \rangle &= \frac{1}{2} \left((x_1 + x_2)(y_1 + y_2) + (x_1 - x_2)(y_1 - y_2) \right) \\ &= \frac{1}{2} \left(x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2 + x_1 y_1 - x_1 y_2 - x_2 y_1 + x_2 y_2 \right) \\ &= \frac{1}{2} \left(2 x_1 y_1 + 2 x_2 y_2 \right) = x_1 y_1 + x_2 y_2 = \langle x, y \rangle \end{aligned}$$

となるので, H はユニタリ作用素の条件を満たす.

2

$\mathbb{C}^3$ の正規直交基底の 1 つとして,

$$e_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, e_2 = \frac{1}{\sqrt{182}} \begin{bmatrix} 13 \\ -2 \\ -3 \end{bmatrix}, e_3 = \frac{1}{\sqrt{13}} \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

が挙げられる.

3

各正規直交基底と状態の内積を計算すると

$$
\begin{aligned}
|\langle \xi_1, \psi\rangle|^2 &= \left|\frac{1}{6}(-2+2+1+0)\right|^2 = \left|\frac{1}{6}\right|^2 = \frac{1}{36} \\
|\langle \xi_2, \psi\rangle|^2 &= \left|\frac{1}{6}(2-2+1+0)\right|^2 = \left|\frac{1}{6}\right|^2 = \frac{1}{36} \\
|\langle \xi_3, \psi\rangle|^2 &= \left|\frac{1}{6}(2+2-1+0)\right|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4} \\
|\langle \xi_4, \psi\rangle|^2 &= \left|\frac{1}{6}(2+2+1-0)\right|^2 = \left|\frac{5}{6}\right|^2 = \frac{25}{36}
\end{aligned}
$$

となる．公理 2 より, 数値 1, 2, 3, 4 が検出される確率はそれぞれ $\frac{1}{36}, \frac{1}{36}, \frac{1}{4}, \frac{25}{36}$ である．

4

$m, n \in \mathbb{N}$ とし, 集合の元どうしで内積を計算する． $x \in [0, 2\pi]$ のとき $\cos nx, \sin nx \in [0, 1]$ なので,

$$
\begin{aligned}
\overline{\cos nx} &= \cos nx \\
\overline{\sin nx} &= \sin nx
\end{aligned}
$$

が成立する．まず, 元が同じ場合の内積を計算する．

- cos

$$
\begin{aligned}
&\left\langle \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\cos nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}}\cos nx}\frac{1}{\sqrt{\pi}}\cos nx dx \\
&= \frac{1}{\pi}\int_0^{2\pi}\cos^2 nx dx = \frac{1}{2\pi}\int_0^{2\pi}(1+\cos 2nx)dx \\
&= \frac{1}{2\pi}\left[x + \frac{1}{2}\sin 2nx\right]_0^{2\pi} = \frac{1}{2\pi}(2\pi - 0) = 1
\end{aligned}
$$

- sin

$$
\begin{aligned}
&\left\langle \frac{1}{\sqrt{\pi}}\sin nx, \frac{1}{\sqrt{\pi}}\sin nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}}\sin nx}\frac{1}{\sqrt{\pi}}\sin nx dx \\
&= \frac{1}{\pi}\int_0^{2\pi}\sin^2 nx dx = \frac{1}{2\pi}\int_0^{2\pi}(1-\cos 2nx)dx \\
&= \frac{1}{2\pi}\left[x - \frac{1}{2}\sin 2nx\right]_0^{2\pi} = \frac{1}{2\pi}(2\pi - 0) = 1
\end{aligned}
$$

続いて, 元が異なる場合の内積を計算する.

- cos $(m \neq n)$

$$
\begin{aligned}
&\left\langle \frac{1}{\sqrt{\pi}} \cos mx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos mx} \frac{1}{\sqrt{\pi}} \cos nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \cos mx \cos nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\cos{(m+n)x} + \cos{(m-n)x}) dx \\
&= \frac{1}{2\pi} \left[\frac{1}{m+n} \sin{(m+n)x} + \frac{1}{m-n} \sin{(m-n)x} \right]_0^{2\pi} = \frac{1}{2\pi} (0-0) = 0
\end{aligned}
$$

- sin $(m \neq n)$

$$
\begin{aligned}
&\left\langle \frac{1}{\sqrt{\pi}} \sin mx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin mx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \sin mx \sin nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\cos{(m-n)x} - \cos{(m+n)x}) dx \\
&= \frac{1}{2\pi} \left[\frac{1}{m-n} \sin{(m-n)x} - \frac{1}{m+n} \sin{(m+n)x} \right]_0^{2\pi} = \frac{1}{2\pi} (0-0) = 0
\end{aligned}
$$

- cos と sin

$$
\begin{aligned}
&\left\langle \frac{1}{\sqrt{\pi}} \cos mx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \cos mx} \frac{1}{\sqrt{\pi}} \sin nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \cos mx \sin nx dx = \frac{1}{2\pi} \int_0^{2\pi} (\sin{(m+n)x} + \sin{(m-n)x}) dx \\
&= \frac{1}{2\pi} \left[-\frac{1}{m+n} \cos{(m+n)x} - \frac{1}{m-n} \cos{(m-n)x} \right]_0^{2\pi} \\
&= \frac{1}{2\pi} \left(\left(-\frac{1}{m+n} - \frac{1}{m-n} \right) - \left(-\frac{1}{m+n} - \frac{1}{m-n} \right) \right) = 0
\end{aligned}
$$

- sin と cos

$$
\begin{aligned}
&\left\langle \frac{1}{\sqrt{\pi}} \sin mx, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle = \int_0^{2\pi} \overline{\frac{1}{\sqrt{\pi}} \sin mx} \frac{1}{\sqrt{\pi}} \cos nx dx \\
&= \frac{1}{\pi} \int_0^{2\pi} \cos nx \sin mx dx = 0
\end{aligned}
$$

以上より, 同じ元どうしの内積が 1 になり, 異なる元どうしの内積が 0 になるので,

$$
\left\{ \frac{1}{\sqrt{\pi}} \cos nx \;\middle|\; n \in \mathbb{N} \right\} \cup \left\{ \frac{1}{\sqrt{\pi}} \sin nx \;\middle|\; n \in \mathbb{N} \right\}
$$

は正規直交系である.

5

$$
\begin{aligned}
\langle \xi, \zeta \rangle &= \left\langle \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i \otimes e_i, \frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{n} e_j \otimes e_k \right\rangle \\
&= \frac{1}{n\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle
\end{aligned}
$$

$\{ e_i \}_{i=1}^{n}$ は $\mathbb{C}^n$ の正規直交基底なので, $\langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle$ は $e_i = e_j = e_k$ のときのみ 1 になり, それ以外は 0 となる. よって

$$
\begin{aligned}
\langle \xi, \zeta \rangle &= \frac{1}{n\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \langle e_i, e_j \rangle \cdot \langle e_i, e_k \rangle \\
&= \frac{1}{n\sqrt{n}} \sum_{i=1}^{n} 1 = \frac{1}{n\sqrt{n}} \cdot n = \frac{1}{\sqrt{n}}
\end{aligned}
$$

6

$$
\begin{aligned}
Ty &= \left(\sum_{i=1}^{n} |x_i\rangle \langle x_i| \right) y = \sum_{i=1}^{n} |x_i\rangle \langle x_i| \, y \\
&= \sum_{i=1}^{n} \langle x_i, y \rangle \, x_i = \sum_{i=1}^{n} \left\langle x_i, \sum_{j=1}^{n} \alpha_j x_j \right\rangle x_i \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_j \langle x_i, x_j \rangle \, x_i = \sum_{i=1}^{n} \alpha_i x_i = y
\end{aligned}
$$

7

まず,

$$
\begin{aligned}
\frac{d}{dx}^{*} \cos x &= a_1 \cos x + b_1 \sin x \\
\frac{d}{dx}^{*} \sin x &= a_2 \cos x + b_2 \sin x
\end{aligned}
$$

と置く. cos と sin の内積を計算すると, それぞれ

$$\begin{aligned}\langle \cos x, \cos x \rangle &= \int_0^{2\pi} \cos^2 x dx = \int_0^{2\pi} \frac{1+\cos 2x}{2} dx \\ &= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{2\pi} = \frac{1}{2}(2\pi + 0) = \pi\end{aligned}$$

$$\begin{aligned}\langle \sin x, \sin x \rangle &= \int_0^{2\pi} \sin^2 x dx = \int_0^{2\pi} \frac{1-\cos 2x}{2} dx \\ &= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{2\pi} = \frac{1}{2}(2\pi - 0) = \pi\end{aligned}$$

$$\begin{aligned}\langle \cos x, \sin x \rangle &= \int_0^{2\pi} \cos x \sin x dx = \int_0^{2\pi} \frac{1}{2}\sin 2x dx \\ &= \frac{1}{2}\left[-\frac{1}{2}\cos 2x\right]_0^{2\pi} = \frac{1}{2}\left(-\frac{1}{2} + \frac{1}{2}\right) = 0\end{aligned}$$

$$\langle \sin x, \cos x \rangle = \int_0^{2\pi} \sin x \cos x dx = 0$$

となる.

a_1, b_1 を求めるために $\left\langle \frac{d}{dx}\cos x, \cos x \right\rangle$, $\left\langle \cos x, \frac{d}{dx}^* \cos x \right\rangle$, $\left\langle \frac{d}{dx}\sin x, \cos x \right\rangle$, $\left\langle \sin x, \frac{d}{dx}^* \cos x \right\rangle$ を計算する.

$$\left\langle \frac{d}{dx}\cos x, \cos x \right\rangle = \langle -\sin x, \cos x \rangle = -\langle \sin x, \cos x \rangle = 0 \tag{1}$$

$$\begin{aligned}\left\langle \cos x, \frac{d}{dx}^* \cos x \right\rangle &= \langle \cos x, a_1 \cos x + b_1 \sin x \rangle \\ &= a_1 \langle \cos x, \cos x \rangle + b_1 \langle \cos x, \sin x \rangle = \pi a_1\end{aligned} \tag{2}$$

随伴作用素の定義と (1), (2) より, $a_1 = 0$ である.

$$\left\langle \frac{d}{dx}\sin x, \cos x \right\rangle = \langle \cos x, \cos x \rangle = \pi \tag{3}$$

$$\begin{aligned}\left\langle \sin x, \frac{d}{dx}^* \cos x \right\rangle &= \langle \sin x, a_1 \cos x + b_1 \sin x \rangle \\ &= a_1 \langle \sin x, \cos x \rangle + b_1 \langle \sin x, \sin x \rangle = \pi b_1\end{aligned} \tag{4}$$

随伴作用素の定義と (3), (4) より, $b_1 = 1$ である. 以上より,

$$\frac{d}{dx}^* \cos x = \sin x$$

が示された.

a_2, b_2 を求めるために $\left\langle \frac{d}{dx}\sin x, \sin x \right\rangle$, $\left\langle \sin x, \frac{d}{dx}^* \sin x \right\rangle$, $\left\langle \frac{d}{dx}\cos x, \sin x \right\rangle$, $\left\langle \cos x, \frac{d}{dx}^* \sin x \right\rangle$ を計算する.

$$\left\langle \frac{d}{dx}\sin x, \sin x \right\rangle = \langle \cos x, \sin x \rangle = 0 \tag{5}$$

$$\begin{aligned} \left\langle \sin x, \frac{d}{dx}^* \sin x \right\rangle &= \langle \sin x, a_2 \cos x + b_2 \sin x \rangle \\ &= a_2 \langle \sin x, \cos x \rangle + b_2 \langle \sin x, \sin x \rangle = \pi b_2 \end{aligned} \tag{6}$$

随伴作用素の定義と (5), (6) より, $b_2 = 0$ である.

$$\left\langle \frac{d}{dx}\cos x, \sin x \right\rangle = \langle -\sin x, \sin x \rangle = -\langle \sin x, \sin x \rangle = -\pi \tag{7}$$

$$\begin{aligned} \left\langle \cos x, \frac{d}{dx}^* \sin x \right\rangle &= \langle \cos x, a_2 \cos x + b_2 \sin x \rangle \\ &= a_2 \langle \cos x, \cos x \rangle + b_2 \langle \cos x, \sin x \rangle = \pi a_2 \end{aligned} \tag{8}$$

随伴作用素の定義と (7), (8) より, $a_2 = -1$ である. 以上より,

$$\frac{d}{dx}^* \sin x = -\cos x$$

が示された. V の基底について

$$\begin{aligned} \frac{d}{dx}^* \cos x = \sin x = -\frac{d}{dx}\cos x \\ \frac{d}{dx}^* \sin x = -\cos x = -\frac{d}{dx}\sin x \end{aligned}$$

が成立するので,

$$\frac{d}{dx}^* = -\frac{d}{dx}$$

である.