THE UNIVERSITY OF HONG KONG
DEPARTMENT OF STATISTICS AND ACTUARIAL SCIENCE

STAT6011/7611/6111/3317 COMPUTATIONAL STATISTICS
(2016 Fall)

Assignment 3, due on October 27

All numerical computation MUST be conducted in Python, and attach the Python code.

1. Assume $Y_i|\theta_i \sim N(\theta_i, 1)$ i.i.d. for $i = 1, \ldots, n$, and $\theta_i \sim N(0, \sigma^2)$.
   (a) Derive the marginal (integrated) likelihood function $p(\mathbf{y}|\sigma^2)$.
   (b) The next two parts involve the data set q1.csv with $n = 100$. Assume a prior $p(\sigma^2) \propto 1/\sigma^2$, derive the posterior distribution of σ^2 and apply the random walk Metropolis–Hasting algorithm to draw 10,000 samples (show the trace plot).
   (c) Derive the full conditional distributions for σ^2 and θ_i ($i = 1, \ldots, n$) and implement Gibbs sampling for all the parameters $(\sigma^2, \theta_1, \ldots, \theta_n)$.

2. Assume $Y_{ij}|\theta_i \sim N(\theta_i, 1)$ i.i.d. for $i = 1, \ldots, n$ and $j = 1, \ldots, J$, and $\theta_i \sim N(0, \sigma^2)$.
   (a) Derive the marginal (integrated) likelihood function $p(\mathbf{y}|\sigma^2)$.
   (b) Show that the posterior distribution of σ^2 is not proper through mathematical derivations.
   (c) The next two parts involve the data set q2.csv with $n = 100$ and $J = 5$. Assume a prior $p(\sigma^2) \propto 1/\sigma^2$, derive the posterior distribution of σ^2 and apply the random walk Metropolis–Hasting algorithm to draw 10,000 samples (show the trace plot).
   (d) Derive the full conditional distributions for σ^2 and θ_i ($i = 1, \ldots, n$) and implement Gibbs sampling for all the parameters $(\sigma^2, \theta_1, \ldots, \theta_n)$.

3. Consider linear regression: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\mathbf{y}$ is an $n \times 1$ vector of outcomes, $\mathbf{X}$ is an $n \times k$ matrix of covariates, and $\boldsymbol{\epsilon}$ is an $n \times 1$ vector of i.i.d. $N(0, \sigma^2)$ random errors. Assign a joint prior distribution of $p(\boldsymbol{\beta}, \sigma^2) \sim \sigma^{-2}$. Follow the notation of the lecture notes "Bayes Review Examples" for all derivations.
   (a) Derive the joint posterior distribution of $(\boldsymbol{\beta}, \sigma^2)$.
   (b) Derive the marginal posterior distributions of $\boldsymbol{\beta}$ and σ^2, and compute their respective posterior means.