LEVEL-k AUCTIONS: CAN A NONEQUILIBRIUM MODEL OF STRATEGIC THINKING EXPLAIN THE WINNER'S CURSE AND OVERBIDDING IN PRIVATE-VALUE AUCTIONS?

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General Model

- N bidders bid for a single object.
- X_i is bidder i's private signal. $X = (X_1, \ldots, X_N)$.
- S_j is additional random variable which is informative about the value of the object. $S = (S_1, \ldots, S_M)$.
- $V_i = u_i(S, X)$ is bidder i's value of the object, where u_i is symmetric across i.
- $V_i - p$ is the payoff for the bidder i winning the auction by paying p.
- Y is the highest signal among bidders other than i.
- $v(x, y) = E\left[V_i | X_i = x, Y = y\right]$ is the expected value conditional on winning.
- $r(x) = E\left[V_i | X_i = x\right]$ is the unconditional expected value.

Classification of Auctions

- First price auction vs Second price auction
- Independent private value auction(i.p.v) vs Common value auction(c.v)
- In i.p.v, the signals and values are independent among bidders. And then $v(x, x) = r(x) = x$.
- In c.v, the information of i and j is not independent and learning about the other bidders' information can cause the bidder to reassess his estimate of the value of the object (e.g. Timber auction). And so $v(x, x) \neq r(x)$ normally with $r(x) > v(x, x)$.

First Price Auction

- In c.v, the optimal bidding strategy is calculated as (1).
- In i.p.v, the optimal bidding strategy is calculated as (2), because $v(x,x)=x$ and Y is independent of X.
- Using $v(x,x)$ is called "value adjustment for the information revealed by winning".
- The latter terms in (1) and (2) are due to "bidding trade-off", which means higher bidding raises both the probability of winning and the cost.

$$a_*(x) = v(x,x) - \int_{\underline{x}}^{x} \exp\left(-\int_{y}^{x} \frac{f_Y(t|t)}{F_Y(t|t)}\mathrm{d}t\right)\mathrm{d}(v(y,y)) \quad (1)$$

$$a_*(x) = x - \int_{\underline{x}}^{x} \frac{F_Y(y)}{F_Y(x)}\mathrm{d}y = E\left[Y|Y<X\right] \quad (2)$$

Second Price Auction

- In c.v, the optimal bidding strategy is calculated as (3). This is not a weakly dominant strategy.
- In i.p.v, the optimal bidding strategy is calculated as (4), because $\upsilon(x, x) = x$. This is a weakly dominant strategy.

$$b_*(x) = \upsilon(x, x) \tag{3}$$
$$b_*(x) = x \tag{4}$$

Points

- χ is the parameter denoting the probability that the bidder think the other bidders bid independently of signals.
- Cursed equilibrium for a given χ value is called χ-cursed equilibrium.
- In ER's(2002, 2006), χ-cursed equilibrium is the same as one in a hypothetical "χ-virtual game", in which players believe that, with probability χ, other's bid is independent of types.
- In i.p.v, the players bid independently of others' signals by definition, then the optimal bidding strategy in χ-cursed equilibrium is the same as one in equilibrium $(\upsilon(x, x) = r(x) = x)$.
- In c.v, since $\upsilon(x, x) \neq r(x)$, cursed equilibrium differs from equilibrium.

Common Value Auction

- In first price auction, the optimal bidding strategy is calculated as (5).
- In second price auction, the optimal bidding strategy is calculated as (6).
- These calculations are exactly the same as ones in equilibrium.

$$
\begin{aligned}
a_\chi(x) = {} & \{(1-\chi)\upsilon(x,x) + \chi r(x)\} \\
& - \int_{\underline{x}}^{x} \exp\left(-\int_{y}^{x} \frac{f_Y(t|t)}{F_Y(t|t)} \mathrm{d}t\right) \mathrm{d}\{(1-\chi)\upsilon(y,y) + \chi r(y)\} \qquad (5)
\end{aligned}
$$

$$
b_\chi(x) = (1-\chi)\upsilon(x,x) + \chi r(x) \qquad (6)
$$

Points

- Level-k model allows behavior to be heterogeneous, but assumes that each player's behavior is drawn from the common distribution over the k types.
- In this paper there are 3 types (denoted by L k) which best response to the lower type. i.e. L1 best responses to L0, and L2 best responses to L1.
- The key assumption is the behavior of L0. One is Random L0, in which the L0 bids uniformly randomly independent of its own signal. The other is Truthful L0, in which L0 bids the value suggested by its own signal.
- L1 and L2 are called Random L1 and L2 when L0 is set to Random L0.
- L1 and L2 are called Truthful L1 and L2 when L0 is set to Truthful L0.
- In the latter slides I show the optimal bidding strategies of each player in each case one by one.

Random L0

- This player bids i.i.d. uniformly over the range $[\underline{z}, \bar{z}]$, which is determined by its private signal and the value $V_i = u_i(S, X)$.

Random L1 in First Price Auction

- Random L1 plays in the belief that all other players follow Random L0.
- Let Z be the highest bid among the others, the distribution function of Z be $F_z(z)$, and the pdf be $f_z(z)$ (these two are from the ordered statistics).
- The optimal bidding strategy of Random L1 ($a_1^r(x)$) solves (7) and is characterized by (8).

$$\max_a \int_{\underline{z}}^{a} (r(x) - a) f_z(z) \mathrm{d}z \tag{7}$$

$$(r(x) - a) f_z(a) - F_z(a) = 0 \tag{8}$$

Random L1 in First Price Auction

- This optimal bidding strategy is common in i.p.v and c.v.
- There are two differences from the optimal bidding strategy in equilibrium.
- One is that $r(x)$ replaces $v(x, x)$, which reflects the fact that Random L1 think winning conveys no information about the value of the object even in c.v. (difference in value adjustment). Since in general $r(x) > v(x, x)$, Random L1 tends to overbid due to the value adjustment.
- Second is that the integral in (7) is over Z rather than Y (see (1) in paper). This is caused by L1 use nonequilibrium belief to evaluate the bidding trade-off. (difference in bidding trade-off)

Random L1 in Second Price Auction

- The optimal bidding strategy $(b_1^r(x))$ solves (9)'s maximization problem. And get (10).

$$\max_b \int_{\underline{z}}^{b} (r(x) - z) f_z(z) \mathrm{d}z \tag{9}$$

$$b_1^r(x) = r(x) \tag{10}$$

Random L1 in Second Price Auction

- One difference from equilibrium case is $r(x)$ replaces $\upsilon(x, x)$. (difference in value adjustment). And this leads to overbidding in c.v.
- Second difference from equilibrium case is that the player use nonequilibrium belief. However note that this does not result in the deviating from truthful bidding as in first price auction.
- In other words, we have no bidding trade-off term in the optimal bidding strategy just as in the equilibrium case.
- (10) coincides with (6) when $\chi = 1$, i.e. fully cursed equilibrium.
- (10) coincides with equilibrium in i.p.v, where $r(x) = x$.

Random L2 , Truthful L1 and Truthful L2

- Random L2 best responses to Random L1.
- (8) and (10) tell that Random L1's bidding strategy is an increasing function of its private signal both in first and second price auction.
- For Random L2 in this case, the highest bid among the other players conveys the information about the value of the object, then Random L2 adjusts its own estimated value according to the information. This means that it uses $\upsilon(x, y)$ rather than $r(x)$.
- This structure is common in Truthful L1 and Truthful L2. Thus I provide the general bidding strategy which can be applied to the three cases.

General Bidding Strategy in First Price Auction

- Suppose that a level-k bidder expects others to bid according to the monotonically increasing bidding strategy $a_{k-1}(x)$.
- The bidder's optimal bidding strategy with V_i and X_i solves (11), and is characterized by the first order condition (12).

$$\max_a \int_{\underline{x}}^{a_{k-1}^{-1}(a)} (\upsilon(x,y) - a) f_Y(y|x)\mathrm{y} \tag{11}$$

$$(\upsilon(x, a_{k-1}^{-1}(a)) - a) f_Y(a_{k-1}^{-1}(a)|x) \frac{\partial a_{k-1}^{-1}(a)}{\partial a} - F_Y(a_{k-1}^{-1}(a)|x) = 0 \tag{12}$$

General Bidding Strategy in Second Price Auction

- Suppose that a level-k bidder expects others to bid according to the monotonically increasing bidding strategy $b_{k-1}(x)$.
- The bidder's optimal bidding strategy with V_i and X_i solves (13), and is characterized by the first order condition (14).

$$\max_b \int_{\underline{x}}^{b_{k-1}^{-1}(b)} (v(x,y) - b_{k-1}(y)) f_Y(y|x) \mathrm{y} \tag{13}$$

$$b = v(x, b_{k-1}^{-1}(b)) \tag{14}$$

Strategic Substitutability

- "Value adjustment (using $v(x,x)$) tends to make bidders' bids strategic substitutes" means that a player underbids when the others overbid and vice versa.
- The player who thinks the others overbid regards their private signals as smaller than ones estimated when the bidding is equilibrium.
- He adjusts the value of the object following his own estimated private signals of others. Then the bidding is less than equilibrium due to the small private signals.
- Thus He underbids when he thinks that the others overbid.

Random L2 in First Price Auction

- $a_2^r(x)$ is determined by (12) with $a_1^{r^{-1}}(a)$ replacing $a_{k-1}^{-1}(a)$.
- Given the strategic substitutability of value adjustment, Random L2 underbid because Random L1 overbid relative to the equilibrium level.

$$(\upsilon(x, a_1^{r^{-1}}(a)) - a) f_Y(a_1^{r^{-1}}(a)|x) \frac{\partial a_1^{r^{-1}}(a)}{\partial a} - F_Y(a_1^{r^{-1}}(a)|x) = 0 \tag{15}$$

Random L2 in Second Price Auction

- $b_2^r(x)$ is determined by (14) with $b_1^{r^{-1}}(b)$ replacing $b_{k-1}^{-1}(b)$.
- Given the strategic substitutability of value adjustment, Random L2 underbid because Random L1 overbid relative to the equilibrium level.
- Bidding strategy is again truthful ("truthful" means that the bidder does not consider the bidding trade-off).
- Note that $b_1^r(x) = r(x)$ from (10).

$$b = \upsilon(x, b_1^{r^{-1}}(b)) = \upsilon(x, r^{-1}(b)) \tag{16}$$

Truthful L1 in First Price Auction

- Truthful L0's bidding strategy is denoted by $a_0^t(x) \equiv r(x)$, which is from the definition of Truthful L0.
- Truthful L1's bidding strategy $(a_1^t(x))$ can be characterized by the special case of (12) with $r(x)$ replacing $a_{k-1}(x)$.
- Since Truthful L0 overbids relative to the equilibrium (see (1)), the strategic substitutability tends to make Truthful L1 underbid.

$$(v(x, r^{-1}(a)) - a) f_Y(r^{-1}(a)|x) \frac{\partial r^{-1}(a)}{\partial a} - F_Y(r^{-1}(a)|x) = 0 \tag{17}$$

Truthful L1 in Second Price Auction

- The optimal bidding strategy of Truthful L1 ($b_1^t(x)$) can be characterized by (14) with $b_0^t(x) \equiv r(x)$ replacing $b_{k-1}(x)$.
- Since Truthful L0 normally overbids relative to the equilibrium due to using $r(x)$ rather than $\upsilon(x, x)$, the strategic substitutability tends to make Truthful L1 underbid.

$$b = \upsilon(x, r^{-1}(b)) \tag{18}$$

Truthful L2 in First Price Auction

- By $a_1^t(x)$ replacing $a_{k-1}^t(x)$ in (12), the optimal bidding strategy $(a_2^t(x))$ is characterized.
- The strategic substitutability tends to make Truthful L2 overbid reflecting Truthful L1's underbidding.

$$(\upsilon(x, a_1^{t^{-1}}(a)) - a) f_Y(a_1^{t^{-1}}(a)|x) \frac{\partial a_1^{t^{-1}}(a)}{\partial a} - F_Y(a_1^{t^{-1}}(a)|x) = 0 \tag{19}$$

Truthful L2 in Second Price Auction

- By $b_1^t(x)$ replacing $b_{k-1}^t(x)$ in (14), the optimal bidding strategy $(b_2^t(x))$ is characterized.
- The strategic substitutability tends to make Truthful L2 overbid reflecting Truthful L1's underbidding.

$$b = \upsilon(x, b_1^{t^{-1}}(b)) \tag{20}$$

Auction Example: KL

- KL(Kagel and Levin 1986) is an example of first price common value auction.
- $V_i = S$ is uniformly distributed on a subset of $[\underline{s}, \bar{s}]$.
- $X|S$ is conditionally uniformly i.i.d. on the interval $[s - \frac{a}{2}, s + \frac{a}{2}]$ with $a > 0$.
- Thus $f_{X|S} = \frac{1}{a}$ and $F_{X|S} = \frac{x-s}{a} + \frac{1}{2}$. And $E[X|S] = s$.
- Then the value adjustment for no information by winning is calculated as (21).

$$r(x) \equiv E[S|X = x] = x \tag{21}$$

Derivation of (21)

Let the range of S be R_S and its length be l_S.

$$E[S|X=x] = \int_{R_S} s \cdot f_{S|X=x}(s)\mathrm{d}s = \int_{R_S} S \frac{f_{X=x|S}(x) f_S(s)}{\int_{R_S} f_{X=x|S}(x) f_S(s)\mathrm{d}s}\mathrm{d}s \quad (22)$$

Now $\int_{R_S} f_{X=x|S}(x) f_S(s)\mathrm{d}s = \int_{x-\frac{a}{2}}^{x+\frac{a}{2}} \frac{f_S(s)}{a}\mathrm{d}s = \frac{1}{a \cdot l_S} a = \frac{1}{l_S}$

Then

$$(22) = \int_{R_S} s \cdot l_S \cdot f_{X=x|S}(x) f_S(s)\mathrm{d}s = \int_{R_S} s \cdot f_{X=x|S}(x)\mathrm{d}s$$

$$= \frac{1}{a} \int_{x-\frac{a}{2}}^{x+\frac{a}{2}} s \ \mathrm{d}s = \frac{1}{a} ax = x$$

Auction Example: KL

- The value adjustment for the information revealed by winning is calculated as in (23).
- The derivation of (23) is in the web appendix.
- With $N > 2$, $\upsilon(x,x) \leq r(x) = x$ holds. Thus the cursed equilibrium bidders overbid relative to the equilibrium.

$$\upsilon(x,y) = \begin{cases} x - \frac{a}{2} + \frac{a}{N} - \frac{x-y}{N} & x - a \leq y \leq x \\ y - \frac{a}{2} + \frac{a}{N} - \frac{\left(\frac{y-x}{a}\right)^{N-1}}{1-\left(\frac{y-x}{a}\right)^{N-1}} \left(\frac{N-1}{N}\right)(x + a - y) & x < y \leq x + a \end{cases} \tag{23}$$

Auction Example: AK

- AK (Avery and Kagel 1997) is an example of second price common value auction.
- $V_i = \sum_{i=1}^{N} X_i$, with X_i i.i.d. uniformly distributed on the interval $[\underline{x}, \bar{x}]$.
- Each value adjustment can be calculated as follows.
- $\upsilon(x,x) > (<) r(x) \Leftrightarrow x > (<) \frac{(N-1)\bar{x}+\underline{x}}{N}$. i.e. $\upsilon(x,x) > r(x)$ with high signals and $\upsilon(x,x) < r(x)$ with low signals.
- Then, in cursed equilibrium, high signal bidders underbid relative to equilibrium and low signal bidders overbid relative to equilibrium.

$$r(x) = x + (N-1)\frac{\bar{x}+\underline{x}}{2} \tag{24}$$

$$\upsilon(x,y) = x + y\frac{N}{2} + \frac{N-2}{2}\underline{x} \tag{25}$$

Auction Example: AK

- In AK, N is set to be 2 and $[\underline{x}, \bar{x}] = [1, 4]$.
- Thus $r(x) = x + \frac{5}{2}$ and $\upsilon(x, x) = 2x$.
- $r(x) < \upsilon(x, x)$ when $x > \frac{5}{2}$.
- $r(x) > \upsilon(x, x)$ when $x < \frac{5}{2}$.

Auction Example: GHP

- GHP (Goeree, Holt, and Palfrey 2002) is an example of independent private value first price auction.
- $N = 2$ and $V_i = X_i$.
- In a low-value treatment, bids are restricted to $\{0, 2, 4, 6, 8, 11\}$ with equal probability for each.
- In a high-value treatment, bids are restricted to $\{0, 3, 5, 7, 9, 12\}$ with equal probability for each.

Summary Table

TABLE I

TYPES' BIDDING STRATEGIES[a,b]

Auction/Type	Equilibrium	χ-Cursed Equilibrium	Random $L1$	Random $L2$	Truthful $L1$	Truthful $L2$
2nd-price i.p.v.	x	x	$b_1^r(x) = x$	$b_2^r(x) = x$	$b_1^t(x)$ from (24) with $v(x,\cdot) \equiv x$	$b_2^t(x)$ from (26) with $v(x,\cdot) \equiv x$
2nd-price c.v.	$b_*(x) = v(x, x)$	$b_\chi(x) = (1-\chi)v(x, x) + \chi r(x)$	$b_1^r(x) = r(x)$	$b_2^r(x)$ from (22): $b = v(x, b_1^{r-1}(b))$	$b_1^t(x)$ from (24): $b = v(x, r^{-1}(b))$	$b_2^t(x)$ from (26): $b = v(x, b_1^{t-1}(b))$
2nd-price c.v.: KL	$x - \frac{a}{2} + \frac{a}{N}$	$x - (1-\chi)a\frac{N-2}{2N}$	x	$x - \frac{a}{2}(\frac{N-2}{N-1})$	$x - \frac{a}{2}(\frac{N-2}{N-1})$	No closed-form solution
2nd-price c.v.: AK	$2x$	$\chi(x + \frac{5}{2}) + (1-\chi)2x$	$x + \frac{5}{2}$	3.5 if $x \leq 2.5$; 6.5 if $x > 2.5$	3.5 if $x \leq 2.5$; 6.5 if $x > 2.5$	No closed-form solution
1st-price i.p.v.	$a_*(x)$ from (4)	$a_*(x)$ from (4)	$a_1^r(x)$ from (14)	$a_2^r(x)$ from (21) with $v(x,\cdot) \equiv x$	$a_1^t(x)$ from (23) with $v(x,\cdot) \equiv x$	$a_2^t(x)$ from (25) with $v(x) = x$
1st-price c.v.	$a_*(x)$ from (3)	$a_\chi(x)$ from (10)	$a_1^r(x)$ from (14)	$a_2^r(x)$ from (21)	$a_1^t(x)$ from (23)	$a_2^t(x)$ from (25)
1st-price c.v.: KL	$x - \frac{a}{2} + \frac{a}{N+1} \times \exp(-\frac{N(x-\underline{x}-\frac{a}{2})}{a})$	$[\chi x + (1-\chi)(x - \frac{a}{2} + \frac{a}{N} - \frac{a}{N}] + \frac{a}{N+1}\exp(-\frac{N(x-\underline{x}-\frac{a}{2})}{a})$	$x - \frac{a}{N}$	$x - \frac{a}{2}$	$x - \frac{a}{2}$	$x - \frac{a}{2}$

[a]If there is no general closed-form expression, Table I refers to the equation in the text that determines the bidding strategy.
[b]The abbreviation i.p.v. denotes independent private value and c.v. denotes common value.

Comments on Summary Table

- In first price independent private value auction (GHP), only bidding trade-off matters.
- In second price common value auction (AK), only value adjustment matters.
- In first price common value auction (first price version KL), both of them matter in straightforward ways.
- The table also includes second price auction version KL example.

Level-k Improvements in First Price Auction

- In i.p.v, when the value is uniformly i.i.d., the equilibrium bidding strategy is a best response to any beliefs derived from others' bidding strategies, therefore L1 and L2 in both of types coincide the equilibrium.
- In i.p.v, when the value is distributed nonuniformly, level-k model can explain the overbidding, which is shown in GHP.
- In GHP, in high value treatment, Random L1 slightly overbids when he has the highest valuation and Random L2 underbids. Furthermore Truthful L1 overbids while Truthful L2 underbids.
- In GHP, in low value treatment, both of Truthful L1 and Truthful L2 underbid, when Random L1 and L2 coincides with equilibrium.
- Details are in web appendix.

Level-k Improvements in First Price Auction

- In c.v, the experiment is KL, both of Random L1 and Random L2 approximately coincide with equilibrium and fully cursed equilibrium biddings when $N = 2$.
- When $N > 2$, Random L1 approximately coincides with fully cursed equilibrium bidding, and overbids relative to equilibrium.
- When $N > 2$, value evaluation and bidding trade-off offset each other for both Random L2 and Truthful L1, then they coincide with equilibrium. So does Truthful L2 because it responds to Truthful L1 coinciding with equilibrium.

Level-k Improvements in Second Price Auction

- In i.p.v, all model have weakly dominant strategies and all players follow them, thus level-k model cannot explain the deviation from equilibrium.
- In c.v, both types of level-k model have the potential to explain the deviation.
- In KL (with $N > 2$), Random L1 overbids as N and a increase, and random L2 underbids as N decreases and a increases (This is clear from Table 1).

Level-k Improvements in Second Price Auction

- In AK, Random L1 with high signal underbids and one with low signal overbids, since Random L1 bids $r(x)$ in 2nd price auction when $v(x,x) > r(x)$ for bidders with high signals and $v(x,x) < r(x)$ for ones with low signals.
- In AK, Random L2 with high (low) signal bids the same amount as Random L1 with highest (lowest) possible signal does.
- This is because Random L2 with high signal thinks that other players are Random L1, and so he must win the auction by biding as Random L1 with highest possible signal does. Since $v(x,x) > r(x)$, the equilibrium expected value of the object is higher than any determined cost. And the same logic can be applied to Random L2 with low signal.

Level-k Improvements in Second Price Auction

- In KL and AK, Truthful L1 behave as Random L2, since Truthful L0's bidding strategy is identical with the one of Random L1.
- In KL, Truthful L2 overbids relative to the cursed equilibrium and increases with N.
- In AK, Truthful L2 overbids for some value and underbids for others.
- The last two follow from some computations.

Level-k Improvements Summary

- In 1st price i.p.v (GHP) with nonuniform value, level-k model can improve weakly
- In 1st price c.v (KL), Random L1 of level-k model can deviate from equilibrium when $N > 2$.
- In 2nd price i.p.v, there is no possibility.
- In 2nd price c.v (AK, KL), level-k model has the potential to improve upon cursed equilibrium in both of experiments.

Preparation for Comparing

- Because learning can lead even unsophisticated persons to equilibrium, strategic thinking appears most clearly before they see the others' responses. So in this paper they focus on the unexperienced subjects in all the three experiments.
- Due to the small sample size, the first 5 responses are regarded as intial responses for each individual.
- In first price indipendent private value auction (GHP), the cursed equilibrium coincides with equilibrium, so they use QRE (quantal response equilibrium) instead of cursed equilibrium to explain overbidding.

How to Compare

- There are three types, i.e. k indexing level-k types $(k = 1, 2, \ldots, K)$, cursed types, and QRE types. And denote each type by k.
- Denote each game by $g = 1, 2, 3, 4$.
- λ_{ik} is the logistic error term indexed by subject i and type k.
- There are three error structures, i.e. the error varying with i and k, the error specific to k, and the constant error.
- c_{it}^{g} is the observed bidding of subject i in game g at time t. And $c_{k}^{g}(x)$ is the calculated optimal bidding strategy of the player with type$k$ whose signal is x in game g.

How to Compare

- Let $S_k^g(c|x)$ be the expected payoff of type k player with signal x by bidding c in game g, which is defined in advance.
- The probability of observing bidding c for type k is computed as follows.

$$Pr(c|k, x, g, \lambda) = \frac{\exp(\lambda S_k^g(c|x))}{\int_{\underline{c}}^{\bar{c}} \exp(\lambda S_k^g(e|x))\mathrm{d}e} \qquad (26)$$

How to Compare

- Let $c_i^g = (c_{i1}^g, c_{i2}^g, c_{i3}^g, c_{i4}^g, c_{i5}^g)$.
- Let π_k denote the proportion of type k in the population.
- With independent errors, by using (26), the type specific likelihood of c_i^g for subject i of type k with signal x and error λ_{ik} in game g is (27).
- Furthermore the same likelihood unconditional on type is (28).

$$L_k(c_i^g|k,x,g,\lambda_{ik}) = \Pi_{t=1}^5 Pr(c_{it}^g|k,x,g,\lambda) \tag{27}$$

$$\sum_{k=1}^{K} \pi_k L_k(c_i^g|k,x,g,\lambda_{ik}) = \sum_{k=1}^{K} \pi_k \Pi_{t=1}^5 Pr(c_{it}^g|k,x,g,\lambda) \tag{28}$$

How to Compare

- N_g is the number of subjects in the game g.
- Let $c^g = (c_1^g, c_2^g, \dots, c_{N_g}^g)$, and the model's likelihood is (29).

$$L(\pi, \Lambda | c^g) = \Pi_{i=1}^{N_g} \sum_{k=1}^{K} \pi_k L_k(c_i^g | k, x, g, \lambda_{ik}) \qquad (29)$$

Verification

- The choice probability function (26) generates the highest value when $c_k^g(x)$ is inputed.
- Since $S_k^g(c|x)$ is quasi concave, the higher the probability is, the $c_{it}^g$is closer to the optimal bidding specific to type and game. Thus the likelihood function works.

Note

- Truthful L0 is omitted because truthful bidding is very rare in first price auction and truthful bidding can not be optimal under any belief.
- $(\sim X)$ means the bidding strategy of the type is not separated from one of type X, so the estimation of the parameters is done about the mixed type. This notation is used in the following tables.
- BIC is used because there are a lot of parameters in subject specific precisions specification.

Table3a

- KL first price example.
- Likelihood ratio test reject the constant or type specific precisions.
- Cursed equilibrium has a modest advantage over level-k plus equilibrium model in both likelihood and BIC.
- This is due to the tractability of cursed equilibrium model about the types with intermediate cursedness.

Table3c

- Ak second price example.
- The estimated frequency of Random L0 is 0%, which implies that Random L0 exists in the minds of Random L1.
- Level-k plus equilibrium model has substantial advantages in likelihood and BIC over cursed equilibrium model.

Table3d

- GHP first price example.
- Level-k plus equilibrium model has substantial advantages in likelihood and BIC over cursed equilibrium model.
- The frequencies of each type in level-k model in Table a, c, d are very stable.

Table3b

- KL second price example.
- Level-k plus equilibrium model has substantial advantages in likelihood and BIC over cursed equilibrium model.
- The estimated frequencies of each type are very different from those of the other examples. This is mainly because the frequency of Truthful L2 is estimated highly.
- This wrong estimation occurs since, through estimation procedure, the subjects bidding above equilibrium who just make a mistake about the optimal bidding strategy in second price auction are captured as Truthful L2, which thinks the other players underbid relative to the equilibrium.

Conclusion

- Level-k model has the potential to improve the explanation about the non optimal biddings in the variable types of auctions which were previously studied.
- And the estimation of the parameters coincides with the results of previous studies.