

STAT3811/3955 Survival Analysis

Assignment 1

Kei Ikegami

February 22, 2017

1 Q1

1.1 (a)

$$\begin{aligned} E[T|T > t] &= \int_t^\infty T f(T|T > t) dT = \int_t^\infty T \frac{f(T)}{1 - F(t)} dT \\ &= \frac{1}{1 - F(t)} \left(\int_0^\infty T f(T) dT - \int_0^t T f(T) dT \right) = \frac{1}{1 - F(t)} \left(\mu - tF(t) + \int_0^t F(T) dT \right) \end{aligned}$$

So I get the below derivative of $m(t)$.

$$m'(t) = \frac{1}{(1 - F(t))^2} \left\{ (-F(t) - t f(t) + F(t)) (1 - F(t)) + (\mu - tF(t) + \int_0^t F(T) dT) f(t) \right\} - 1$$

Then the following calculation leads to the result.

$$\frac{1 + m'(t)}{m(t)} = \frac{f(t) \left(\mu - tF(t) + \int_0^t F(T) dT - t(1 - F(t)) \right)}{(1 - F(t)) \left(\mu - tF(t) + \int_0^t F(T) dT \right) - t(1 - F(t))^2} = \frac{f(t)}{1 - F(t)} = \lambda(t)$$

1.2 (b)

Since $\int_0^t F(T) dT = \int_0^t (-S(T) + 1) dT = -\int_0^t S(T) dT + t$, then

$$\begin{aligned} m(t) &= \frac{1}{S(t)} \int_0^\infty (T - t) f(T) dT = \frac{1}{S(t)} \left(\int_0^\infty (T - t) f(T) dT - \int_0^t (T - t) f(T) dT \right) \\ &= \frac{1}{S(t)} \left(\mu - t + \int_0^t F(T) dT \right) = \frac{1}{S(t)} \left(\mu - \int_0^t S(u) du \right) \end{aligned}$$

Now, when T has an exponential distribution with $\mu = \frac{1}{\lambda}$,

$$m(t) = \exp(\lambda t) \left(\mu + \frac{1}{\lambda} \exp(-\lambda t) - \frac{1}{\lambda} \right) = \frac{1}{\lambda} = \mu$$

because $\int_0^\infty t \lambda \exp(-\lambda t) dt = \frac{1}{\lambda}$.

1.3 (c)

First I consider the mean,

$$\lim_{t \rightarrow 0} m(t) = \lim_{t \rightarrow 0} E[T|T > t] = E[T] = 1$$

Now let $\delta = \text{med}(T)$, then $F(\delta) = \frac{1}{2}$ and $\lambda(\delta) = \frac{2}{\delta+1}$ due to (a). Then by using (b) I get the below calculation.

$$\frac{2}{\delta+1} = 2 \left(1 - \int_0^\delta (1 - F(u)) du \right) \Leftrightarrow \int_0^\delta (1 - F(u)) du = \frac{\delta}{\delta+1}$$

By taking derivative of both sides about δ , I get the result as follows.

$$1 - F(\delta) = \frac{1}{(\delta+1)^2} \Leftrightarrow \frac{1}{2} = \frac{1}{(\delta+1)^2} \Leftrightarrow \delta = \sqrt{2} - 1$$

1.4 (d)

First I have the below representation of $m(t)$.

$$m(t) \frac{\mu - \int_0^t S(u) du}{S(t)} = \frac{\int_t^\infty S(u) du}{S(t)}$$

Since the limits of the both of enumerator and denominator are 0 as $t \rightarrow \infty$. By using L'Hopital's rule twice, I get the below result,

$$\lim_{t \rightarrow \infty} m(t) = \lim_{t \rightarrow \infty} \frac{-S(t)}{-f(t)} = \lim_{t \rightarrow \infty} \frac{f(t)}{-f'(t)} = \lim_{t \rightarrow \infty} \left(-\frac{d}{dt} \log f(t) \right)^{-1}$$

1.5 (e)

In this case, $f(t) = \frac{1}{\sqrt{2\pi\sigma t}} \exp(\frac{\log t - \mu}{2\sigma^2})$, I use (d) to get the result.

$$\begin{aligned} \left(-\frac{d}{dt} \log f(t) \right)^{-1} &= -\frac{f(t)}{f'(t)} = -\frac{\sigma^2 t}{\mu - \log t - \sigma^2} \\ \lim_{t \rightarrow \infty} -\frac{\sigma^2 t}{\mu - \log t - \sigma^2} &= \lim_{t \rightarrow \infty} -\frac{1}{-\frac{1}{t}} = \infty \end{aligned}$$

2 Q3

2.1 (a)

By definition, $S(t|z) = 1 - F(t|z)$. Thus I calculate $F(t|z)$ as follows.

$$F(t|z) = \Pr(Y \leq \log t | z) = \Pr(w \leq \frac{\log t - \mu - \beta z}{\sigma} | z)$$

Because $\int_{-\infty}^\omega \frac{\exp(u)}{(1+\exp(u))^2} du = \frac{\exp(\omega)}{1+\exp(\omega)}$, then by the above calculation,

$$S(t|z) = 1 - F(t|z) = \frac{1}{1 + \exp(\frac{\log t - \mu - \beta z}{\sigma})}$$

2.2 (b)

By (a),

$$\frac{S(t|z)}{1 - S(t|z)} = \frac{1}{\exp(\frac{\log t - \mu - \beta z}{\sigma})} = \exp\left(-\frac{\log t - \mu - \beta z}{\sigma}\right)$$

2.3 (c)

By (b), let Odds_i be the odds for z_i ,

$$\frac{\text{Odds}_1}{\text{Odds}_2} = \exp\left(\frac{\beta}{\sigma}\right)$$

And this odds ratio is independent of t .

3 Q5

3.1 (a)

Just calculate as follows,

$$\begin{aligned} P(T_i < C_i) &= \int_0^\infty \left(\int_0^c \lambda \exp(-\lambda t) dt \right) \theta \exp(-\theta c) dc = 1 - \int_0^\infty \theta \exp(-(\lambda + \theta)c) dc \\ &= 1 - \frac{\theta}{\theta + \lambda} = \frac{\lambda}{\theta + \lambda} \end{aligned}$$

Then the probability distribution of δ is

$$\delta = \begin{cases} 1 & \text{with probability } \frac{\lambda}{\lambda + \theta} \\ 0 & \text{with probability } \frac{\theta}{\lambda + \theta} \\ \text{otherwise} & \text{with probability } 0 \end{cases}$$

3.2 (b)

Let $F_Y(y)$, $f_Y(y)$ be the distribution function and probability distribution function of Y . Then, due to the independence of T and C ,

$$1 - F_Y(y) = 1 - \Pr(\min(T, C) < y) = \Pr(y \leq \min(T, C)) = \Pr(y \leq T) \Pr(y \leq C) = \exp(-(\lambda + \theta)y)$$

Thus I get $F_Y(y) = 1 - \exp(-(\lambda + \theta)y)$, which means Y has a exponential distribution with parameter $\lambda + \theta$.

3.3 (c)

Consider the marginal distribution of Y when $\delta = 1$ as follows,

$$f(Y, \delta = 1) = \lim_{h \rightarrow 0} \frac{\Pr(y \leq Y \leq y + h, \delta = 1)}{h}$$

Now the denominator of this can be decomposed, because of the independence of T, C ,

$$\begin{aligned} \Pr(y \leq Y \leq y + h, \delta = 1) &= \Pr(y \leq Y \leq y + h, T < C) = \Pr(y \leq T \leq y + h, y \leq C) \\ &= \Pr(y \leq T \leq y + h) \Pr(y \leq C) = \exp(-\lambda y) (1 - \exp(-\lambda h)) \exp(-\theta y) \end{aligned}$$

Then, by using L'Hopital rule, the marginal distribution is

$$\begin{aligned} f(Y, \delta = 1) &= \exp(-(\lambda + \theta)y) \lim_{h \rightarrow 0} \frac{1 - \exp(-\lambda h)}{h} \\ &= \lambda \exp(-(\lambda + \theta)y) = \left(\frac{\lambda}{\lambda + \theta} \right) (\lambda + \theta) \exp(-(\lambda + \theta)y) \end{aligned}$$

The same is true of $\delta = 0$ case, so the joint probability distribution function is expressed as the multiplication of random variable's probability distribution function. This means that the two random variables are independent from each other.

3.4 (d)

Consider the distribution function of W_2 , denote its distribution function as $F_W(w)$,

$$\begin{aligned} F_W(w) &= \Pr(T_1 + T_2 < w) = \Pr(T_1 < w - T_2) \\ &= \int_0^w \left(\int_0^{w-t_2} \lambda \exp(-\lambda t_1) dt_1 \right) \lambda \exp(-\lambda t_2) dt_2 = (1 - \exp(-\lambda w)) - \int_0^w \lambda \exp(-\lambda w) dt_2 \\ &= 1 - \exp(-\lambda w) - w \lambda \exp(-\lambda w) \end{aligned}$$

Then, by taking derivative of the above, I get the pdf $f(w) = \lambda^2 w \exp(-\lambda w)$.

3.5 (e)

3.6 (f)

3.7 (g)

3.8 (h)

3.9 (i)

3.10 (j)