

STAT3811/3955 Survival Analysis

Assignment 1

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February 22, 2017

1 Q1

1.1 (a)

$$\begin{aligned} E[T|T > t] &= \int_t^\infty T f(T|T > t) dT = \int_t^\infty T \frac{f(T)}{1 - F(t)} dT \\ &= \frac{1}{1 - F(t)} \left(\int_0^\infty T f(T) dT - \int_0^t T f(T) dT \right) = \frac{1}{1 - F(t)} \left(\mu - tF(t) + \int_0^t F(T) dT \right) \end{aligned}$$

So I get the below derivative of $m(t)$.

$$m'(t) = \frac{1}{(1 - F(t))^2} \left\{ (-F(t) - t f(t) + F(t)) (1 - F(t)) + (\mu - tF(t) + \int_0^t F(T) dT) f(t) \right\} - 1$$

Then the following calculation leads to the result.

$$\frac{1 + m'(t)}{m(t)} = \frac{f(t) \left(\mu - tF(t) + \int_0^t F(T) dT - t(1 - F(t)) \right)}{(1 - F(t)) \left(\mu - tF(t) + \int_0^t F(T) dT \right) - t(1 - F(t))^2} = \frac{f(t)}{1 - F(t)} = \lambda(t)$$

1.2 (b)

Since $\int_0^t F(T) dT = \int_0^t (-S(T) + 1) dT = -\int_0^t S(T) dT + t$, then

$$\begin{aligned} m(t) &= \frac{1}{S(t)} \int_0^\infty (T - t) f(T) dT = \frac{1}{S(t)} \left(\int_0^\infty (T - t) f(T) dT - \int_0^t (T - t) f(T) dT \right) \\ &= \frac{1}{S(t)} \left(\mu - t + \int_0^t F(T) dT \right) = \frac{1}{S(t)} \left(\mu - \int_0^t S(u) du \right) \end{aligned}$$

Now, when T has an exponential distribution with $\mu = \frac{1}{\lambda}$,

$$m(t) = \exp(\lambda t) \left(\mu + \frac{1}{\lambda} \exp(-\lambda t) - \frac{1}{\lambda} \right) = \frac{1}{\lambda} = \mu$$

because $\int_0^\infty t \lambda \exp(-\lambda t) dt = \frac{1}{\lambda}$.

1.3 (c)

First I consider the mean,

$$\lim_{t \rightarrow 0} m(t) = \lim_{t \rightarrow 0} E[T|T > t] = E[T] = 1$$

Now let $\delta = \text{med}(T)$, then $F(\delta) = \frac{1}{2}$ and $\lambda(\delta) = \frac{2}{\delta+1}$ due to (a). Then by using (b) I get the below calculation.

$$\frac{2}{\delta+1} = 2 \left(1 - \int_0^\delta (1 - F(u)) du \right) \Leftrightarrow \int_0^\delta (1 - F(u)) du = \frac{\delta}{\delta+1}$$

By taking derivative of both sides about δ , I get the result as follows.

$$1 - F(\delta) = \frac{1}{(\delta+1)^2} \Leftrightarrow \frac{1}{2} = \frac{1}{(\delta+1)^2} \Leftrightarrow \delta = \sqrt{2} - 1$$

1.4 (d)

First I have the below representation of $m(t)$.

$$m(t) \frac{\mu - \int_0^t S(u) du}{S(t)} = \frac{\int_t^\infty S(u) du}{S(t)}$$

Since the limits of the both of enumerator and denominator are 0 as $t \rightarrow \infty$. By using L'Hopital's rule twice, I get the below result,

$$\lim_{t \rightarrow \infty} m(t) = \lim_{t \rightarrow \infty} \frac{-S(t)}{-f(t)} = \lim_{t \rightarrow \infty} \frac{f(t)}{-f'(t)} = \lim_{t \rightarrow \infty} \left(-\frac{d}{dt} \log f(t) \right)^{-1}$$

1.5 (e)

In this case, $f(t) = \frac{1}{\sqrt{2\pi\sigma t}} \exp(\frac{\log t - \mu}{2\sigma^2})$, I use (d) to get the result.

$$\begin{aligned} \left(-\frac{d}{dt} \log f(t) \right)^{-1} &= -\frac{f(t)}{f'(t)} = -\frac{\sigma^2 t}{\mu - \log t - \sigma^2} \\ \lim_{t \rightarrow \infty} -\frac{\sigma^2 t}{\mu - \log t - \sigma^2} &= \lim_{t \rightarrow \infty} -\frac{1}{-\frac{1}{t}} = \infty \end{aligned}$$

2 Q3

2.1 (a)

By definition, $S(t|z) = 1 - F(t|z)$. Thus I calculate $F(t|z)$ as follows.

$$F(t|z) = \Pr(Y \leq \log t | z) = \Pr(w \leq \frac{\log t - \mu - \beta z}{\sigma} | z)$$

Because $\int_{-\infty}^\omega \frac{\exp(u)}{(1+\exp(u))^2} du = \frac{\exp(\omega)}{1+\exp(\omega)}$, then by the above calculation,

$$S(t|z) = 1 - F(t|z) = \frac{1}{1 + \exp(\frac{\log t - \mu - \beta z}{\sigma})}$$

2.2 (b)

By (a),

$$\frac{S(t|z)}{1 - S(t|z)} = \frac{1}{\exp(\frac{\log t - \mu - \beta z}{\sigma})} = \exp\left(-\frac{\log t - \mu - \beta z}{\sigma}\right)$$

2.3 (c)

By (b), let $Odds_i$ be the odds for z_i ,

$$\frac{Odds_1}{Odds_2} = \exp\left(\frac{\beta}{\sigma}\right)$$

And this odds ratio is independent of t .

3 Q5

3.1 (a)

Just calculate as follows,

$$\begin{aligned} P(T_i < C_i) &= \int_0^\infty \left(\int_0^c \lambda \exp(-\lambda t) dt \right) \theta \exp(-\theta c) dc = 1 - \int_0^\infty \theta \exp(-(\lambda + \theta)c) dc \\ &= 1 - \frac{\theta}{\theta + \lambda} = \frac{\lambda}{\theta + \lambda} \end{aligned}$$

Then the probability distribution of δ is

$$\delta = \begin{cases} 1 & \text{with probability } \frac{\lambda}{\lambda + \theta} \\ 0 & \text{with probability } \frac{\theta}{\lambda + \theta} \\ \text{otherwise} & \text{with probability } 0 \end{cases}$$

3.2 (b)

Let $F_Y(y)$, $f_Y(y)$ be the distribution function and probability distribution function of Y . Then, due to the independence of T and C ,

$$1 - F_Y(y) = 1 - \Pr(\min(T, C) < y) = \Pr(y \leq \min(T, C)) = \Pr(y \leq T) \Pr(y \leq C) = \exp(-(\lambda + \theta)y)$$

Thus I get $F_Y(y) = 1 - \exp(-(\lambda + \theta)y)$, which means Y has a exponential distribution with parameter $\lambda + \theta$.

3.3 (c)

Consider the marginal distribution of Y when $\delta = 1$ as follows,

$$f(Y, \delta) = \lim_{h \rightarrow 0} \frac{\Pr(y \leq Y \leq y + h, \delta = 1)}{h}$$

Now the denominator of this can be decomposed, because of the independence of T, C ,

$$\begin{aligned} \Pr(y \leq Y \leq y + h, \delta = 1) &= \Pr(y \leq Y \leq y + h, T < C) = \Pr(y \leq T \leq y + h, y \leq C) \\ &= \Pr(y \leq T \leq y + h) \Pr(y \leq C) = \exp(-\lambda y) (1 - \exp(-\lambda h)) \exp(-\theta y) \end{aligned}$$

3.4 (d)

3.5 (e)

3.6 (f)

3.7 (g)

3.8 (h)

3.9 (i)

3.10 (j)