Econometrics 2 2017
Problem set 1

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1 Problem 1

1.1 (a)

Let $\epsilon = y - \alpha - x^{'}\beta$, where $\alpha = E[y] - E[x^{'}]\beta$ and $\beta = \Sigma^{-1}\delta$, then I show $E[\epsilon] = 0$ and $E[\epsilon x] = 0$.

$$E[\epsilon] = E[y - E[y] + E[x^{'}]\beta - x^{'}\beta] = E[(y - E[y]) - (x^{'} - E[x^{'}])\beta] = (E[y] - E[y]) - (E[x^{'}] - E[x^{'}])\beta = 0$$

$$\begin{aligned} E[\epsilon x] = E[x\epsilon] &= E[x(y - E[y] + E[x^{'}]\beta - x^{'}\beta)] = E[x(y - E[y]) - x(x^{'} - E[x^{'}])\beta] \\ &= E[(x - E[x])(y - E[y]) + E[x](y - E[y]) - (x - E[x])(x^{'} - E[x^{'}])\beta - E[x](x^{'} - E[x^{'}])\beta] \\ &= \delta - E[(x - E[x])(x^{'} - E[x^{'}])]\Sigma^{-1}\delta = \delta - \delta = 0 \end{aligned}$$

So now I get the result.

1.2 (b)

This transformation is useful.

$$E[(y - a - x^{'}b)^2] = E[(y - E[y])^2] + 2E[(y - E[y])(E[y] - (a + x^{'}b))] + E[(E[y] - (a + x^{'}b))^2]$$

Then I get the FOC by differentiating by a as follows.

$$E[-2(y - E[y])] + E[-2(E[y] - (a + x^{'}b))] = 0 \;\Rightarrow\; a = E[y] - E[x^{'}]b$$

Next, after inserting the above relationship to the original, I get the FOC by differentiating by b as follows.

$$-2E[(y - E[y])(x^{'} - E[x^{'}])]^{'} + 2E[(x^{'} - E[x^{'}])^{'}(x^{'} - E[x^{'}])b] = 0 \;\Leftrightarrow\; b = \Sigma^{-1}\delta$$

And the second derivative by b is 2Σ, which is positive semi definite, then the second order condition for minimization is fulfilled. Thus I show α, β in (a) solves this minimization problem.

Then I show the second part. First I show the important property of the conditional expectation. If $y = E[y|x] + \epsilon$, then $E[\epsilon|x] = E[y - E[y|x]|x] = 0$ and for any function $h(x)$, $E[h(x)\epsilon] = E[E[h(x)\epsilon|x]] = E[h(x)E[\epsilon|x]] = 0$. Using this second property can easily prove the argument.

$$\begin{aligned} E[(y - a - x^{'}b)^2] &= E[(y - E[y|x] + E[y|x] - a - x^{'}b)^2] \\ &= E[(y - E[y|x])^2] + 2E[(y - E[y])(E[y|x] - a - x^{'}b)] + E[(E[y|x] - a - x^{'}b)^2] \\ &= E[(y - E[y|x])^2] + E[(E[y|x] - a - x^{'}b)^2] \end{aligned}$$

The second term in the first line vanishes since $E[(y - E[y])(E[y|x] - a - x^{'}b)] = E[\epsilon(E[y|x] - a - x^{'}b)] = 0$. This is because $E[y|x] - a - x^{'}b$ is just a function of x.

1.3 (c)

In the real econometric analysis, the relationship $E[\epsilon x] = 0$ is just an assumption. And that the correlation between the regressor and the error is zero needs the situation where all relevant variables are in the regression model, but this is unrealistic because certainly many unobservable variables exist. We can use IV for solving such a situation and get the consistent estimator of the coefficient of interesting variables.

1.4 (d)

(b) says that the population regression function is the best linear approximate of the conditional mean of the dependent variable even if the form is not linear among the support, when we use squared loss function. So we have to show the OLS estimator consistently estimates the coefficients. Let β_{OLS} be the OLS estimator of the constant and coefficients.

$$\beta_{OLS} = \left(\mathbf{X}'\mathbf{X}\right)^{-1} \mathrm{X}'\mathrm{y}$$

where $\mathbf{X} = (x_1, \cdots, x_n)'$, $\mathrm{y} = (y_1, \cdots, y_n)'$. So this is the sample analogue of $E\left[X_i X_i'\right]^{-1} E\left[X_i y_i\right]$. By LLN, we can estimate consistently this coefficients. So the remain is to prove the equality between $E\left[X_i X_i'\right]^{-1} E\left[X_i y_i\right]$ and $\begin{pmatrix} E[y] - E[x']\beta \\ \Sigma^{-1}\delta \end{pmatrix}$. Note that x_i contains 1 whose coefficient is the constant term, while x contains only variables.

We show $E\left[X_i y_i\right] = E\left[X_i X_i'\right] \begin{pmatrix} E[y] - E[x']\beta \\ \Sigma^{-1}\delta \end{pmatrix}$.

$$\begin{aligned}
E\left[X_i X_i'\right] \begin{pmatrix} E[y] - E[x']\beta \\ \Sigma^{-1}\delta \end{pmatrix} &= \begin{pmatrix} 1 & E[x'] \\ E[x] & \Sigma + E[x]E[x'] \end{pmatrix} \begin{pmatrix} E[y] - E[x']\beta \\ \Sigma^{-1}\delta \end{pmatrix} \\
&= \begin{pmatrix} E[y - E[x']]\beta + E[x']\beta \\ E[x](E[y] - E[x']\beta) + (\Sigma + E[x]E[x'])\beta \end{pmatrix} \\
&= \begin{pmatrix} E[y] \\ E[y]E[x] + \delta \end{pmatrix} \\
&= \begin{pmatrix} E[y_i] \\ E[x_{i1}, y_i] \\ \vdots \\ E[x_{ik} y_i] \end{pmatrix}
\end{aligned}$$

Now we have the ideal result.

1.5 (e)

(1) We have no evidence for the accuracy of approximation outside the support.

(2)

2 Problem 2

2.1 (a)

$$Cov(Az_i, \epsilon_i) = E[(Az_i - E[Az_i])(\epsilon_i - E[\epsilon_i])] = AE[(z_i - E[z_i])(\epsilon_i - E[\epsilon_i])] = 0$$

Therefore Az_i does not correlate with ϵ_i. And clearly the correlation exist between Az_i and x_i. Thus Az_i is a valid IV for x_i. And $rank(E[Az_i x_i']) = K$ allows it to have inverse matrix, so IV estimator can be constructed.

2.2 (b)

Multiplying Az_i to the first model by left.

$$Az_iy_i = Az_ix_i^{'}\beta + Az_i\epsilon_i$$

Sum up the model by individuals. And divide by n.

$$\sum_{i=1}^{n} Az_iy_i = (\sum_{i=1}^{n} Az_ix_i^{'})\beta + \sum_{i=1}^{n} Az_i\epsilon_i \;\Leftrightarrow\; \left(\frac{1}{n}\sum_{i=1}^{n} Az_iy_i\right) = \left(\frac{1}{n}\sum_{i=1}^{n} Az_ix_i^{'}\right)\beta + \left(\frac{1}{n}\sum_{i=1}^{n} Az_i\epsilon_i\right)$$

Consider the moment condition $E[Az_i\epsilon_i] = 0$. And we have the assumption that $rank(E[Az_ix_i^{'}]) = K$, so the sample analogue of this moment can be inversed. Then I have the IV estimator as the method of moment estimator as follows.

$$\hat{\beta_A} = \left(\frac{1}{n}\sum_{i=1}^{n} Az_ix_i^{'}\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} Az_iy_i\right)$$

2.3 (c)

Using the matrix A, I can write the sample moment condition as follows.

$$\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{i=1}^{n} x_iz_i^{'}\right)\left(\sum_{i=1}^{n} z_iz_i^{'}\right)^{-1} z_i(y_i - x_i^{'}b) = 0$$

And I use the following matrix notation to make it easy to see, $\sum_{i=1}^{n} x_iz_i^{'} = X^{'}Z$ and $\sum_{i=1}^{n} z_iz_i^{'} = Z^{'}Z$. Then the condition leads to the IV estimator.

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left(X^{'}Z\right)\left(Z^{'}Z\right)^{-1} z_i(y_i - x_i^{'}b) = 0\\
\Leftrightarrow\quad &\sum_{i=1}^{n}\left(X^{'}Z\right)\left(Z^{'}Z\right)^{-1} z_iy_i = \sum_{i=1}^{n}\left(X^{'}Z\right)\left(Z^{'}Z\right)^{-1} z_ix_i^{'}b\\
\Leftrightarrow\quad &\left(X^{'}Z\right)\left(Z^{'}Z\right)^{-1}\left(Z^{'}y\right) = \left(X^{'}Z\right)\left(Z^{'}Z\right)^{-1}\left(Z^{'}X\right)b\\
\Leftrightarrow\quad &b = (X^{'}P_ZX)^{-1}X^{'}P_Zy
\end{aligned}$$

2.4 (d)

Using matrix notations lead to the result. Note that $Z^{'}X$ has its inverse matrix because $L = K$.

$$\hat{\beta_A} = (AZ^{'}X)^{-1}AZ^{'}y = (Z^{'}X)^{-1}A^{-1}AZ^{'}y = (Z^{'}X)^{-1}Z^{'}y$$

The last term shows it does not depend on A.

3 Problem 3

I show the result of the regression. Column 1 is the result of the first type of regression, which I use just a one year data and OLS. Column 2 is the second one, which I use all the data and fixed effect estimator. Column 3 is the third one, which I use different data in that $x_{2it} = \alpha_i + 0.1u_{2it}$.

3.1 (a)

I use Python in this problem for convenience.

	(1)	(2)	(3)
VARIABLES	y	y	y
x_1	1.341***	1.018***	1.026***
	(0.0276)	(0.0316)	(0.0328)
x_2	1.348***	1.046***	1.418***
	(0.0275)	(0.0320)	(0.319)
z	0.940***		
	(0.0348)		
o.z		-	-
Constant	1.046***	1.019***	0.933***
	(0.0347)	(0.0187)	(0.0205)
Observations	1,000	3,000	3,000
R-squared	0.907	0.678	0.500
Number of v1		2,000	2,000
Standard errors in parentheses			
*** p<0.01, ** p<0.05, * p<0.1			

Figure 1: Results of the three regressions

3.2 (b)

Constant term and the coefficient of z_i is consistently estimated in this case and the coefficients of x_{1it}, x_{2it} is biased. This is because the first two components do not correlate with the error term while the last two correlates through α_i. I see the coefficients of the last two is positively biased, in other words they are bigger than true value 1. And this is caused by the positive correlations with error term.

3.3 (c)

This is done in the first table.

3.4 (d)

In this regression, while the estimate of the coefficient of x_{1it} seems to be correct, the coefficient of x_{2it} is positively biased and furthermore has a large standard error as in the first table. I explain this.

In the original data, $y_{it} = 1 + x_{1it} + x_{2it} + z_i + \alpha_i + \nu_{it} = y_{it} = 1 + (\alpha_i + u_{1it}) + (\alpha_i + u_{2it}) + z_i + \alpha_i + \nu_{it}$. And in this problem, $y_{it} = 1 + (\alpha_i + u_{1it}) + (\alpha_i + 0.1u_{2it}) + z_i + \alpha_i + \nu_{it}$. Remember the fixed effect estimator is the same as the within estimator, which is given by the regression using each observation's gap from each individual mean . Then I have the variance of the estimator as in the below form, when β_k^l denotes the estimated coefficient of x_{kit} in data $l \ \in \{c, d\}$.

$$Var\left(\beta_2^c\right) = \frac{\sigma^2}{\sum_i (u_{2it} - \bar{u_{2i}})^2}$$
$$Var\left(\beta_2^d\right) = \frac{100\sigma^2}{\sum_i (u_{2it} - \bar{u_{2i}})^2}$$

where $\sigma^2 = Var(\nu_{it} - \bar{\nu_i})$. 100 in the second one is from $0.1u_{2it}$. This reveals the standard deviation in the data d is ten times larger than one in the data c. And this is true as you see in the first table.

3.5 (e)

There is an estimated constant in STATA result as in the first table, but this is just for convenience and calculated as the mean of individual effect. Constant term and the coefficients of variables which take the same value for each individual cannot be estimated by FE estimator, because the effect of such variables to the dependent variable cannot be separated by the individual effects. This is clear as you see in the below within regression, which produces the same result of FE estimator.

$$(y_{it} - \bar{y_i}) = \beta_1(u_{1it} - \bar{u_{1i}}) + \beta_2(u_{2it} - \bar{u_2 i}) + (\nu_{it} - \bar{\nu_i})$$

Constant term and z_i is not in the above regression, so it is natural that you cannot obtain the estimated coefficients of them.

4 Problem 4

4.1 (a)

The model tells us that

$$Y_1 = X\beta_1 + E_1$$
$$Y_2 = X\beta_2 + E_2$$

Therefore it is clear that all in one matrix formulation is the mentioned form.

4.2 (b)

σ_{ab} denotes the covariance between $\epsilon_{i,a}$ and $\epsilon_{i,b}$. Note that the errors of different individuals do not correlate. The matrix Σ is

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

Let E be the joint errors. Therefore the variance covariance matrix in this model is as follows.

$$Var\,(E) = \begin{pmatrix} \sigma_{11} & 0 & \cdots & 0 & \sigma_{12} & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \sigma_{11} & 0 & 0 & \cdots & \sigma_{12} \\ \sigma_{21} & 0 & \cdots & 0 & \sigma_{22} & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \sigma_{21} & 0 & 0 & \cdots & \sigma_{22} \end{pmatrix} = \Sigma \otimes I_N$$

4.3 (c)

By the definition of GLS estimator, I get in this case as follows.

$$\beta_{GLS} = \left(\left(\begin{array}{cc} X & 0 \\ 0 & X \end{array} \right)' (\Sigma \otimes I_N)^{-1} \left(\begin{array}{cc} X & 0 \\ 0 & X \end{array} \right) \right)^{-1} \left(\begin{array}{cc} X & 0 \\ 0 & X \end{array} \right)' (\Sigma \otimes I_N)^{-1} Y$$

when $Y = \left(\begin{array}{c} Y_1 \\ Y_2 \end{array} \right)$

4.4 (d)

By (c)

$$\begin{aligned}
\beta_{GLS} &= \left(\begin{pmatrix} X' & 0 \\ 0 & X' \end{pmatrix} (\Sigma^{-1} \otimes I_N) \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \right)^{-1} \begin{pmatrix} X' & 0 \\ 0 & X' \end{pmatrix} (\Sigma^{-1} \otimes I_N) Y \\
&= \left(\begin{pmatrix} X' & 0 \\ 0 & X' \end{pmatrix} \begin{pmatrix} \sigma_{22} I_N & -\sigma_{12} I_N \\ -\sigma_{21} I_N & \sigma_{11} I_N \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \right)^{-1} \begin{pmatrix} X' & 0 \\ 0 & X' \end{pmatrix} \begin{pmatrix} \sigma_{22} I_N & -\sigma_{12} I_N \\ -\sigma_{21} I_N & \sigma_{11} I_N \end{pmatrix} Y \\
&= \begin{pmatrix} \sigma_{22} X'X & -\sigma_{12} X'X \\ -\sigma_{21} X'X & \sigma_{11} X'X \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{22} X' & -\sigma_{12} X' \\ -\sigma_{21} X' & \sigma_{11} X' \end{pmatrix} y \\
&= \left(\begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{pmatrix} \otimes X'X \right)^{-1} \left(\begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{pmatrix} \otimes X' \right) y \\
&= \left(I_2 \otimes (X'X)^{-1} X' \right) y \\
&= \begin{pmatrix} (X'X)^{-1} X' & 0 \\ 0 & (X'X)^{-1} X' \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\
&= \begin{pmatrix} (X'X)^{-1} X' Y_1 \\ (X'X)^{-1} X' Y_2 \end{pmatrix}
\end{aligned}$$

The second equality is from $\Sigma^{-1} = \frac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{pmatrix}$. By the above I get the result.

5 Problem 5

5.1 (a)

In this case, $\Sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1T} \\ \vdots & & \vdots \\ \sigma_{1T} & \cdots & \sigma_{TT} \end{pmatrix}$, where $\sigma_{tt'} = E[\epsilon_{it}\epsilon_{it'} | x_{it}, z_{it'}]$. Using this notations,

$$Var(E|X) = I_N \otimes \Sigma$$

5.2 (b)

$$\begin{aligned}
M &= I_{NT} - (I_N \otimes \iota_T) \left((I_N \otimes \iota_T)' (I_N \otimes \iota_T) \right)^{-1} (I_N \otimes \iota_T)' \\
&= I_{NT} - (I_N \otimes \iota_T) (T \cdot I_N)^{-1} (I_N \otimes \iota_T)' \\
&= I_{NY} - \frac{1}{T} \left(I_N \otimes \iota_T \iota_T' \right)
\end{aligned}$$

Then,

$$My = y - \frac{1}{T} \left(I_N \otimes \iota_T \iota_T' \right) y$$

Now I focus on the first $T \times T$ elements in $I_N \otimes \iota_T \iota_T'$, which is $\iota_T \iota_T'$. When the first T elements in y is denoted as y_1, the below equation holds.

$$y_1 - \frac{1}{T} \left(\iota_T \iota_T' \right) y_1 = y_1 - \frac{1}{T} \iota_T \left(\iota_T' y_1 \right) = y_1 - \bar{y_1} \iota_T$$

This is true for all individuals, i.e. $\forall i \in \{1, 2, \cdots, N\}$. So we are done.