STAT3811/3955 Survival Analysis
Assignment 1

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1 Q1

1.1 (a)

$$
\begin{aligned}
E[T|T>t] &= \int_t^{\infty} Tf(T|T>t)\mathrm{d}T = \int_t^{\infty} T\frac{f(T)}{1-F(t)}\mathrm{d}T \\
&= \frac{1}{1-F(t)}\left(\int_0^{\infty} Tf(T)\mathrm{d}T - \int_0^{t} Tf(T)\mathrm{d}T\right) = \frac{1}{1-F(t)}\left(\mu - tF(t) + \int_0^{t} F(T)\mathrm{d}T\right)
\end{aligned}
$$

So I get the below derivative of $m(t)$.

$$
m^{'}(t) = \frac{1}{(1-F(t))^2}\left\{\left(-F(t) - tf(t) + F(t)\right)\left(1-F(t)\right) + \left(\mu - tF(t) + \int_0^t F(T)\mathrm{d}T\right)f(t)\right\} - 1
$$

Then the following calculation leads to the result.

$$
\frac{1+m^{'}(t)}{m(t)} = \frac{f(t)\left(\mu - tF(t) + \int_0^t F(T)\mathrm{d}T - t(1-F(t))\right)}{(1-F(t))\left(\mu - tF(t) + \int_0^t F(T)\mathrm{d}T\right) - t(1-F(t))^2} = \frac{f(t)}{1-F(t)} = \lambda(t)
$$

1.2 (b)

Since $\int_0^t F(T)\mathrm{d}T = \int_0^t(-S(T)+1)\mathrm{d}T = -\int_0^t S(T)\mathrm{d}T + t$, then

$$
\begin{aligned}
m(t) &= \frac{1}{S(t)}\int_0^{\infty}(T-t)f(T)\mathrm{d}T = \frac{1}{S(t)}\left(\int_0^{\infty}(T-t)f(T)\mathrm{d}T - \int_0^{t}(T-t)f(T)\mathrm{d}T\right) \\
&= \frac{1}{S(t)}\left(\mu - t + \int_0^t F(T)\mathrm{d}T\right) = \frac{1}{S(t)}\left(\mu - \int_0^t S(u)\mathrm{d}u\right)
\end{aligned}
$$

Now, when T has an exponential distribution with $\mu = \frac{1}{\lambda}$,

$$
m(t) = \exp(\lambda t)\left(\mu + \frac{1}{\lambda}\exp(-\lambda t) - \frac{1}{\lambda}\right) = \frac{1}{\lambda} = \mu
$$

because $\int_0^{\infty} t\lambda\exp(-\lambda t)\mathrm{d}t = \frac{1}{\lambda}$.

1.3 (c)

First I consider the mean,

$$
\lim_{t\to 0} m(t) = \lim_{t\to 0} E[T|T>t] = E[T] = 1
$$

Now let $\delta = med(T)$, then $F(\delta) = \frac{1}{2}$ and $\lambda(\delta) = \frac{2}{\delta+1}$ due to (a). Then by using (b) I get the below calculation.

$$\frac{2}{\delta+1} = 2\left(1 - \int_0^\delta (1-F(u))\mathrm{d}u\right) \Leftrightarrow \int_0^\delta (1-F(u))\mathrm{d}u = \frac{\delta}{\delta+1}$$

By taking derivative of both sides about δ, I get the result as follows.

$$1 - F(\delta) = \frac{1}{(\delta+1)^2} \quad \Leftrightarrow \quad \frac{1}{2} = \frac{1}{(\delta+1)^2} \quad \Leftrightarrow \quad \delta = \sqrt{2} - 1$$

1.4 (d)

First I have the below representation of $m(t)$.

$$m(t)\frac{\mu - \int_0^t S(u)\mathrm{d}u}{S(t)} = \frac{\int_t^\infty S(u)\mathrm{d}u}{S(t)}$$

Since the limits of the both of enumerator and denominator are 0 as $t \to \infty$. By using L'Hopital's rule twice, I get the below result,

$$\lim_{t\to\infty} m(t) = \lim_{t\to\infty} \frac{-S(t)}{-f(t)} = \lim_{t\to\infty} \frac{f(t)}{-f'(t)} = \lim_{t\to\infty} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\log f(t)\right)^{-1}$$

1.5 (e)

In this case, $f(t) = \frac{1}{\sqrt{2\pi}\sigma t}\exp(\frac{\log t - \mu}{2\sigma^2})$, I use (d) to get the result.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\log f(t)\right)^{-1} = -\frac{f(t)}{f'(t)} = -\frac{\sigma^2 t}{\mu - \log t - \sigma^2}$$
$$\lim_{t\to\infty} -\frac{\sigma^2 t}{\mu - \log t - \sigma^2} = \lim_{t\to\infty} -\frac{1}{-\frac{1}{t}} = \infty$$

2 Q3

2.1 (a)

By definition, $S(t|z) = 1 - F(t|z)$. Thus I calculate $F(t|z)$ as follows.

$$F(t|z) = \Pr(Y \leq \log t|z) = \Pr(w \leq \frac{\log\, t - \mu - \beta z}{\sigma}|z)$$

Because $\int_{-\infty}^{\omega} \frac{\exp(u)}{(1+\exp(u))^2}\mathrm{d}u = \frac{\exp(\omega)}{1+\exp(\omega)}$, then by the above calculation,

$$S(t|z) = 1 - F(t|z) = \frac{1}{1+\exp(\frac{\log\, t-\mu-\beta z}{\sigma})}$$

2.2 (b)

By (a),

$$\frac{S(t|z)}{1-S(t|z)} = \frac{1}{\exp(\frac{\log\, t-\mu-\beta z}{\sigma})} = \exp\left(-\frac{\log\, t - \mu - \beta z}{\sigma}\right)$$

2.3 (c)

By (b), let $Odds_i$ be the odds for z_i,

$$\frac{Odds_1}{Odds_2} = \exp\left(\frac{\beta}{\sigma}\right)$$

And this odds ratio is independent of t.

3 Q5

3.1 (a)

Just calculate as follows,

$$\begin{aligned}
P(T_i < C_i) &= \int_0^\infty \left(\int_0^c \lambda \exp(-\lambda t)\,\mathrm{d}t \right) \theta \exp(-\theta c)\,\mathrm{d}c = 1 - \int_0^\infty \theta \exp(-(\lambda+\theta)c)\,\mathrm{d}c \\
&= 1 - \frac{\theta}{\theta+\lambda} = \frac{\lambda}{\theta+\lambda}
\end{aligned}$$

Then the probability distribution of δ is

$$\delta = \begin{cases} 1 & \text{with probability } \frac{\lambda}{\lambda+\theta} \\ 0 & \text{with probability } \frac{\theta}{\lambda+\theta} \\ \text{otherwise} & \text{with probability } 0 \end{cases}$$

3.2 (b)

Let $F_Y(y)$, $f_Y(y)$ be the distribution function and probability distribution function of Y. Then, due to the independence of T and C,

$$1 - F_Y(y) = 1 - \Pr(\min(T, C) < y) = \Pr(y \le \min(T, C)) = \Pr(y \le T)\Pr(y \le C) = \exp(-(\lambda+\theta)y)$$

Thus I get $F_Y(y) = 1 - \exp(-(\lambda+\theta)y)$, which means Y has a exponential distribution with parameter $\lambda + \theta$.

3.3 (c)

Consider the marginal distribution of Y when $\delta = 1$ as follows,

$$f(Y, \delta = 1) = \lim_{h\to 0} \frac{\Pr(y \le Y \le y+h, \delta = 1)}{h}$$

Now the denominator of this can be decomposed, because of the independence of T, C,

$$\begin{aligned}
\Pr(y \le Y \le y+h, \delta = 1) &= \Pr(y \le Y \le y+h, T < C) = \Pr(y \le T \le y+h, y \le C) \\
&= \Pr(y \le T \le y+h)\Pr(y \le C) = \exp(-\lambda y)(1 - \exp(-\lambda h))\exp(-\theta y)
\end{aligned}$$

Then, by using L'Hopital rule, the marginal distribution is

$$\begin{aligned}
f(Y, \delta = 1) &= \exp(-(\lambda+\theta)y) \lim_{h\to 0} \frac{1 - \exp(-\lambda h)}{h} \\
&= \lambda \exp(-(\lambda+\theta)y) = \left(\frac{\lambda}{\lambda+\theta} \right) (\lambda+\theta)\exp(-(\lambda+\theta)y)
\end{aligned}$$

The same is true of $\delta = 0$ case, so the joint probability distribution function is expressed as the multiplication of random variable's probability distribution function. This means that the two random variables are independent from each pther.

3.4 (d)

Consider the distribution function of W_2, denote its distribution function as $F_W(w)$,

$$\begin{aligned}
F_W(w) &= \Pr(T_1 + T_2 < w) = \Pr(T_1 < w - T_2) \\
&= \int_0^w \left(\int_0^{w-t_2} \lambda \exp(-\lambda t_1)\mathrm{d}t_1 \right) \lambda \exp(-\lambda t_2)\mathrm{d}t_2 = (1 - \exp(-\lambda w)) - \int_0^w \lambda \exp(-\lambda w)\mathrm{d}t_2 \\
&= 1 - \exp(-\lambda w) - w\lambda \exp(-\lambda w)
\end{aligned}$$

Then, by taking derivative of the above, I get the pdf $f(w) = \lambda^2 w \exp(-\lambda w)$. And the same way shows that W_m has an gamma distribution with the parameter m, λ. i.e.

$$f(w_m) = \frac{\lambda^m (w_m)^{m-1} \exp(-\lambda w_m)}{\Gamma(m)}$$

3.5 (e)

Let L be the likelihood, then

$$L = \Pi_{i=1}^n \left[(\lambda \exp(-\lambda y_i))^{\delta_i}(\exp(-\lambda y_i))^{1-\delta_i}\right]\left[(\theta \exp(-\theta y_i))^{\delta_i}(\exp(-\theta y_i))^{1-\delta_i}\right]$$

Let l be the loglikelihood, then

$$\begin{aligned} l &= \sum_{i=1}^n \delta_i(\log\,\lambda - \lambda y_i - \theta y_i) + (1-\delta_i)(\log\,\theta - \lambda y_i - \theta y_i) \\ &= n\log\,\theta - (\lambda+\theta)\sum_{i=1}^n y_i + \log\frac{\lambda}{\theta}\sum_{i=1}^n \delta_i \end{aligned}$$

Thus the MLE is obtained as follows.

$$\begin{aligned} \frac{\partial l}{\partial \lambda} = 0 \quad &\Leftrightarrow \quad -\sum_{i=1}^n y_i + \frac{\theta}{\lambda}\frac{1}{\theta}\sum_{i=1}^n \delta_i = 0 \\ &\Leftrightarrow \quad \lambda = \frac{\sum_{i=1}^n \delta_i}{\sum_{i=1}^n y_i} \end{aligned}$$

3.6 (f)

$$E[\hat{\lambda}] = E\left[\frac{\sum_{i=1}^n \delta_i}{\sum_{i=1}^n y_i}\right] = E\left[\frac{1}{\sum_{i=1}^n y_i}\right]\,(E[\delta_1]+E[\delta_2]+\cdots+E[\delta_n]) = E\left[\frac{1}{\sum_{i=1}^n y_i}\right]\frac{n\lambda}{\lambda+\theta}$$

The last equality is from that $E[\delta_i] = \frac{\lambda}{\lambda+\theta}$ for all i. Now, from (d), the last expectation can be calculated as follows. (let z represent the summation of y)

$$\begin{aligned} E\left[\frac{1}{\sum_{i=1}^n y_i}\right] &= \int_0^\infty \frac{1}{z}\frac{(\lambda+\theta)^n z^{n-1}\exp(-(\lambda+\theta)z)}{\Gamma(n)}\mathrm{d}z \\ &= \frac{\Gamma(n-1)}{\Gamma(n)}(\lambda+\theta)\int_0^\infty \frac{(\lambda+\theta)^{n-1}z^{(n-1)-1}\exp(-(\lambda+\theta)z)}{\Gamma(n-1)}\mathrm{d}z \\ &= \frac{\lambda+\theta}{n-1} \end{aligned}$$

By the above computation, we get $E[\hat{\lambda}] = \frac{\lambda+\theta}{n-1}\frac{n\lambda}{\lambda+\theta} = \frac{n\lambda}{n-1}$. And the limit is clearly λ.

3.7 (g)

By using (f)'s notation, and from the previous results, I get

$$Var(\delta_1) = \frac{\lambda\theta}{(\lambda+\theta)^2}$$

$$E[(\sum_{i=1}^n \delta_i)^2] = Var(\sum_{i=1}^n \delta_i) + (E[\sum_{i=1}^n \delta_i])^2 = n\;Var(\delta_1) + (\frac{n\lambda}{\lambda+\theta})^2 = \frac{n\lambda\theta + n^2\lambda^2}{(\lambda+\theta)^2}$$

$$E\left[\frac{1}{z^2}\right] = \int_0^\infty \frac{1}{z^2}\frac{(\lambda+\theta)^n z^{n-1}\exp(-(\lambda+\theta)z)}{\Gamma(n)}\mathrm{d}z = \frac{(\lambda+\theta)^2}{(n-1)(n-2)}$$

Then the result is as follows.

$$\begin{aligned} Var(\hat{\lambda}) = E[\hat{\lambda}^2] - (E[\hat{\lambda}])^2 &= E\left[\frac{1}{(\sum_{i=1}^n y_i)^2}\right]E[(\sum_{i=1}^n \delta_i)^2] - (E[\hat{\lambda}])^2 \\ &= \frac{n\lambda\theta + n^2\lambda^2}{(n-1)(n-2)} - \frac{n^2\lambda^2}{(n-1)^2} \end{aligned}$$

3.8 (h)

First I have to calculate the asymptotic variance of this estimator, which is the inverse of the expected Fisher information from the available data. By using the loglikelihood, the Fisher information of a data point about λ is as follows.

$$\begin{aligned} I(\lambda) &= E\left[-\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}(\log\theta - \lambda y_i - \theta y_i + \delta_i \log\frac{\lambda}{\theta})\right] \\ &= E\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}(-y_i + \frac{\delta_i}{\lambda})\right] \\ &= E[\delta_i \lambda^{-2}] = \frac{1}{\lambda(\lambda+\theta)} \end{aligned}$$

Then the Fisher information of full data is the summation of a data point ones, and from consistency and asymptotic normality of MLE, the asymptotic distribution of $\hat{\lambda}$ is

$$\hat{\lambda} \xrightarrow{d} N\left(\lambda, \frac{\lambda(\lambda+\theta)}{n}\right)$$

Then by standardization,

$$\frac{\hat{\lambda} - \lambda}{\sqrt{\frac{\lambda(\lambda+\theta)}{n}}} \xrightarrow{d} N(0,1)$$

Then, let z_α be the 97.5% quantile, since normal distribution is symmetric, in large sample,

$$\Pr\left(-z_\alpha \leq \frac{\sqrt{n}(\hat{\lambda}-\lambda)}{\sqrt{\lambda(\lambda+\theta)}} \leq z_\alpha\right) = 0.95$$

Thus the 95% confidence interval for λ is $\left[\hat{\lambda} - z_\alpha\sqrt{\frac{\hat{\lambda}(\hat{\lambda}+\theta)}{n}}, \hat{\lambda} + z_\alpha\sqrt{\frac{\hat{\lambda}(\hat{\lambda}+\theta)}{n}}\right]$, because of continuous mapping theorem.

3.9 (i)

3.10 (j)