

Macroeconomics 1 2018 S1S2

Homework 1

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1 Problem1

We need an additional assumption for stationarity.

$$\begin{aligned} y_t &= c + \phi y_{t-1} + \epsilon_t \\ \Leftrightarrow (1 - \phi L)y_t &= c + \epsilon_t \\ \Leftrightarrow y_t &= (1 - \phi L)^{-1}(c + \epsilon_t) = c + \sum_{u=0}^{\infty} \phi^u L^u \epsilon_t = c + \sum_{u=0}^{\infty} \phi^u \epsilon_{t-u} \end{aligned} \quad (1)$$

For stationarity, it is necessary the above limit exists. By Chauchy's convergence judgement, we know that it is necessary and sufficient for the convergence of series that the residual, i.e. $\sum_{u=n+1}^{\infty} \phi^u \epsilon_{t-u}$, converges to 0. Usually, in time series analysis, the Hilbert space, whose product is defined by covariance, is used when the convergence is discussed. Thus if we want to know whether $\sum_{u=n+1}^{\infty} \phi^u \epsilon_{t-u} \rightarrow 0$ or not, we must check the convergence w.r.t the norm induced by the covariance product. And for the below calculation we need as assumption that states $Var(y_t) < \infty$.

$$\left\| \sum_{u=n+1}^{\infty} \phi^u \epsilon_{t-u} \right\| = Var\left(\sum_{u=n+1}^{\infty} \phi^u \epsilon_{t-u} \right) = \sum_{n+1}^{\infty} |\phi|^{2u} \sigma^2 = 0$$

The second equality is followed by the additional assumption. The third equality is by the assumption $|\phi| < 1$. Now we have the result that $\sum_{u=0}^{\infty} \phi^u \epsilon_{t-u}$ exists, in other words, the time series is stationary.

Next, I calculate the mean, variance, j-th autocovariance.

Mean By (1), $E[y_t] = E[c + \sum_{u=0}^{\infty} \phi^u \epsilon_{t-u}] = c + 0 = c$

Variance By (1), $Var(y_t) = Var(\sum_{u=0}^{\infty} \phi^u \epsilon_{t-u}) = \sum_{u=0}^{\infty} |\phi|^{2u} \sigma^2 = \frac{\sigma^2}{1-\phi^2}$

Autocovariance $\gamma(t, t-j) = \phi Cov(Y_{t-1}, Y_{t-j}) = \dots = \phi^j Var(Y_{t-j}) = \frac{\phi^j \sigma^2}{1-\phi^2}$

2 Problem2

3 Problem3

I use state-space model for solving this problem. When the first difference series follow AR(2) model, i.e. $\Delta Y_t = \beta_1 \Delta Y_{t-1} + \beta_2 \Delta Y_{t-2}$, let the state vector and the disturbance vector be

$$X_t = \begin{pmatrix} \Delta Y_t \\ \Delta Y_{t-1} \end{pmatrix}, \quad e_t = \begin{pmatrix} \epsilon_t \\ 0 \end{pmatrix}$$

and let the update matrix be

$$F = \begin{pmatrix} \beta_1 & \beta_2 \\ 1 & 0 \end{pmatrix}$$

then I have the below description of this model.

$$X_{t+1} = F X_t + e_{t+1}, \quad \Delta Y_{t+1} = [1 \ 0] X_{t+1}$$

Thus

$$\begin{aligned}
E_t [\Delta Y_{t+j}] &= [1 \ 0] E_t [X_{t+j}] \\
&= [1 \ 0] E_t [F X_{t+j-1} + e_{t+j}] \\
&= [1 \ 0] E_t [F^2 X_{t+j-2} + F e_{t+j-1} + e_{t+j}] \\
&= \dots \\
&= [1 \ 0] F^j E_t [X_t] \\
&= [1 \ 0] F^j \begin{pmatrix} \Delta Y_t \\ \Delta Y_{t-1} \end{pmatrix}
\end{aligned}$$

and then BN trend at t , denoted as τ_t , is derived as follows,

$$\tau_t = y_t + \lim_{J \rightarrow \infty} \sum_{j=i}^J E_t [\Delta Y_{t+j}] = y_t + \lim_{J \rightarrow \infty} \sum_{j=i}^J [1 \ 0] F^j \begin{pmatrix} \Delta Y_t \\ \Delta Y_{t-1} \end{pmatrix} = y_t + [1 \ 0] \left(\sum_{j=1}^{\infty} F^j \right) \begin{pmatrix} \Delta Y_t \\ \Delta Y_{t-1} \end{pmatrix}$$

I can calculate the limit of series of powered matrices as follows, note that I is the two-dimensional identity matrix,

$$\left(\sum_{j=1}^{\infty} F^j \right) = F (I - F)^{-1} = \frac{1}{1 - \beta_1 - \beta_2} \begin{pmatrix} \beta_1 + \beta_2 & \beta_2 \\ 1 & \beta_2 \end{pmatrix}$$

Then, denoting BN cycle term as c_t ,

$$\begin{aligned}
\tau_t &= y_t + \frac{1}{1 - \beta_1 - \beta_2} \{(\beta_1 + \beta_2) \Delta Y_t + \beta_2 \Delta Y_{t-1}\} \\
c_t &= -\frac{1}{1 - \beta_1 - \beta_2} \{(\beta_1 + \beta_2) \Delta Y_t + \beta_2 \Delta Y_{t-1}\}
\end{aligned}$$

4 Problem4

5 Problem5

5.1 (1)

$$\begin{cases} y_t = a_{10} + a_{11}y_{t-1} + a_{12}z_{t-1} + e_{1t} \\ z_t = a_{20} + a_{21}y_{t-1} + a_{22}z_{t-1} + e_{2t} \end{cases} \quad (2)$$

By (2), I get the below,

$$\begin{cases} (1 - a_{11}L)y_t = a_{10} + a_{12}z_{t-1} + e_{1t} \\ (1 - a_{22}L)z_t = a_{20} + a_{21}y_{t-1} + e_{2t} \end{cases} \Rightarrow \begin{cases} y_t = (1 - a_{11}L)^{-1} (a_{10} + a_{12}z_{t-1} + e_{1t}) \\ z_t = (1 - a_{22}L)^{-1} (a_{20} + a_{21}y_{t-1} + e_{2t}) \end{cases} \quad (3)$$

Thus I have the following AR(2) form for each time series,

$$\begin{cases} y_t = a_{10} + a_{11}y_{t-1} + a_{12}(1 - a_{22}L)^{-1} (a_{20} + a_{21}y_{t-2} + e_{2t}) + e_{1t} \\ z_t = a_{20} + a_{21}(1 - a_{11}L)^{-1} (a_{10} + a_{12}z_{t-2} + e_{1t}) + a_{22}z_{t-1} + e_{2t} \end{cases} \quad (4)$$

5.2 (2)

Because of symmetry, I show the procedure only for y_t and just show the result w.r.t z_t . By (2),(3), I have the following representation.

$$\begin{aligned}
y_t &= (1 - a_{11}L)^{-1} \{a_{10} + (1 - a_{22}L)^{-1} a_{12} a_{20}\} \\
&\quad + a_{12} a_{21} (1 - a_{11}L)^{-1} (1 - a_{22}L)^{-1} y_{t-2} \\
&\quad + a_{12} (1 - a_{11}L)^{-1} (1 - a_{22}L)^{-1} e_{2,t-1} \\
&\quad + (1 - a_{11}L)^{-1} e_{1,t}
\end{aligned}$$

Then I have the following explicit form. Note that λ_1, λ_2 are the solutions of $1 - (a_{11} + a_{22})x + (a_{11}a_{22} - a_{12}a_{21}x^2)$, and I assume that this equation has the real number solution, in other words $(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) = (a_{11} - a_{22})^2 + 4a_{12}a_{21} \geq 0$.

$$\begin{aligned}
y_t &= (1 - a_{12}a_{21}(1 - a_{11}L)^{-1}(1 - a_{22}L)^{-1}L^2)^{-1} \\
&\quad [(1 - a_{11}L)^{-1}\{a_{10} + (1 - a_{22}L)^{-1}a_{12}a_{20}\} + a_{12}(1 - a_{11}L)^{-1}(1 - a_{22}L)^{-1}e_{2,t-1} + (1 - a_{11}L)^{-1}e_{1,t}] \\
&= \left(1 - \frac{a_{12}a_{21}L^2}{(1 - a_{11}L)(1 - a_{22}L)}\right)^{-1} \\
&\quad [(1 - a_{11}L)^{-1}\{a_{10} + (1 - a_{22}L)^{-1}a_{12}a_{20}\} + a_{12}(1 - a_{11}L)^{-1}(1 - a_{22}L)^{-1}e_{2,t-1} + (1 - a_{11}L)^{-1}e_{1,t}] \\
&= (1 - (a_{11} + a_{22})L + (a_{11}a_{22} - a_{12}a_{21})L^2)^{-1} (1 - a_{11}L)(1 - a_{22}L) \\
&\quad [(1 - a_{11}L)^{-1}\{a_{10} + (1 - a_{22}L)^{-1}a_{12}a_{20}\} + a_{12}(1 - a_{11}L)^{-1}(1 - a_{22}L)^{-1}e_{2,t-1} + (1 - a_{11}L)^{-1}e_{1,t}] \\
&= (1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1} (1 - a_{11}L)(1 - a_{22}L) \\
&\quad [(1 - a_{11}L)^{-1}\{a_{10} + (1 - a_{22}L)^{-1}a_{12}a_{20}\} + a_{12}(1 - a_{11}L)^{-1}(1 - a_{22}L)^{-1}e_{2,t-1} + (1 - a_{11}L)^{-1}e_{1,t}] \\
&= (1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1} \{a_{10} - a_{22}a_{10} + a_{12}a_{20}\} \\
&\quad + a_{12}(1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1}e_{2,t-1} \\
&\quad + (1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1}(e_{1,t} - a_{22}e_{1,t-1}) \\
&= \{a_{10} - a_{22}a_{10} + a_{12}a_{20}\} \left(\sum_{j=0}^{\infty} \sum_{k=0}^j \lambda_1^k \lambda_2^{j-k} \right) \\
&\quad + a_{12} \sum_{j=0}^{\infty} \left(\sum_{k=0}^j \lambda_1^k \lambda_2^{j-k} \right) e_{2,t-1-j} \\
&\quad + \sum_{j=0}^{\infty} \left(\sum_{k=0}^j \lambda_1^k \lambda_2^{j-k} \right) e_{1,t-j} - a_{22} \sum_{j=0}^{\infty} \left(\sum_{k=0}^j \lambda_1^k \lambda_2^{j-k} \right) e_{1,t-1-j}
\end{aligned}$$