Macroeconomics 1 2018 S1S2
Homework 1

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1 Problem1

We need an additional assumption for stationarity.

$$\begin{aligned} & y_t = c + \phi y_{t-1} + \epsilon_t \\ \Leftrightarrow\ & (1-\phi L) y_t = c + \epsilon_t \\ \Leftrightarrow\ & y_t = (1-\phi L)^{-1}(c+\epsilon_t) = c + \sum_{u=0}^{\infty} \phi^u L^u \epsilon_t = c + \sum_{u=0}^{\infty} \phi^u \epsilon_{t-u} \end{aligned} \tag{1}$$

For stationarity, it is necessary the above limit exists. By Chaucy's convergence judgement, we know that it is necessary and sufficient for the convergence of series that the residual, i.e. $\sum_{u=n+1}^{\infty} \phi^u \epsilon_{t-u}$, converges to 0. Usually, in time series analysis, the Hilbert space, whose product is defined by covariance, is used when the convergence is discussed. Thus if we want to know whether $\sum_{u=n+1}^{\infty} \phi^u \epsilon_{t-u} \to 0$ or not, we must check the convergence w.r.t the norm induced by the covariance product. And for the below calculation we need as assumption that states $Var(y_t) < \infty$.

$$\| \sum_{u=n+1}^{\infty} \phi^u \epsilon_{t-u} \| = Var(\sum_{u=n+1}^{\infty} \phi^u \epsilon_{t-u}) = \sum_{n+1}^{\infty} |\phi|^{2u} \sigma^2 = 0$$

The second equality is followed by the additional assumption. The third equality is by the assumption $|\phi| < 1$. Now we have the result that $\sum_{u=0}^{\infty} \phi^u \epsilon_{t-u}$ exists, in other words, the time series is stationary.

Next, I calculate the mean, variance, j-th autocovariance.

Mean By (1), $E[y_t] = E\left[c + \sum_{u=0}^{\infty} \phi^u \epsilon_{t-u}\right] = c + 0 = c$

Variance By (1), $Var(y_t) = Var\left(\sum_{u=0}^{\infty} \phi^u \epsilon_{t-u}\right) = \sum_{u=0}^{\infty} |\phi|^{2u} \sigma^2 = \frac{\sigma^2}{1-\phi^2}$

Autocovariance $\gamma(t, t-j) = \phi Cov(Y_{t-1}, Y_{t-j}) = \cdots = \phi^j Var(Y_{t-j}) = \frac{\phi^u \sigma^2}{1-\phi^2}$

2 Problem2

2.1 (1)

The Python code is as follows.

```
# import relevant package
import pandas as pd
import matplotlib.pyplot as plt
import numpy as np
from math import pow
% matplotlib inline

# make the three time series
# original series
original = [100*(CPI[i] - CPI[i-1])/CPI[i-1] for i in range(1, len(CPI))]
original_series = original[12:]
#ma
ma_series = [sum([original[i-j] for j in range(12)])/12 for i in range(12, len(original))]
#log
log_series = [100*(pow(1+(np.log(CPI[i]) - np.log(CPI[i-12])), 1/12)-1) for i in range(13, len(CPI))]

# plot the data
date = data[["year"]].values[:, 0][13:]
plt.figure(figsize=(20,10))
plt.plot(date, original_series, label = 'original')
plt.plot(date, ma_series, label = 'MA')
plt.plot(date, log_series, label = 'Log')
plt.legend()
plt.xlabel("date")
plt.ylabel("inflation_rate")
plt.savefig('inflation.png')
plt.show()

# csv output
df = pd.DataFrame({"date": date, "original":original_series, "ma":ma_series, "log":log_series})
df.to_csv("corrected_inflation.csv")
```

2.2 (3)

The R code is as follows.

```
# read data
corrected_inflation <- read_csv("~/Desktop/2018summer/macro/hw1/empirics/corrected_inflation.csv")

# time series format
ORI <- ts(corrected_inflation[["original"]])
LOG <- ts(corrected_inflation[["log"]])
MA <- ts(corrected_inflation[["ma"]])

```

```
# ma
AIC_ma <- 1:24
for (i in 1:24){
  result <- arima(MA, order = c(i,0,0))
  AIC_ma[i] <- result$aic
}

ma_optimal_p <- which.min(AIC_ma)

# log
AIC_log <- 1:24
for (i in 1:24){
  result <- arima(LOG, order = c(i,0,0))
  AIC_log[i] <- result$aic
}

log_optimal_p <- which.min(AIC_log)
```

and $p = 13$ is chosen in both of seasonal adjustments.

3 Problem3

I use state-space model for solving this problem. When the first difference series follow AR(2) model, i.e. $\Delta Y_t = \beta_1 \Delta Y_{t-1} + \beta_2 \Delta Y_{t-2}$, let the state vector and the disturbance vector be

$$X_t = \begin{pmatrix} \Delta Y_t \\ \Delta Y_{t-1} \end{pmatrix},\ e_t = \begin{pmatrix} \epsilon_t \\ 0 \end{pmatrix}$$

and let the update matrix be

$$F = \begin{pmatrix} \beta_1 & \beta_2 \\ 1 & 0 \end{pmatrix}$$

then I have the below description of this model.

$$X_{t+1} = FX_t + e_{t+1},\ \Delta Y_{t+1} = [1\ 0]X_{t+1}$$

Thus

$$\begin{aligned} E_t\left[\Delta Y_{t+j}\right] &= [1\ 0]E_t\left[X_{t+j}\right] \\ &= [1\ 0]E_t\left[FX_{t+j-1} + e_{t+j}\right] \\ &= [1\ 0]E_t\left[F^2 X_{t+j-2} + Fe_{t+j-1} + e_{t+j}\right] \\ &= \cdots \\ &= [1\ 0]F^j E_t\left[X_t\right] \\ &= [1\ 0]F^j \begin{pmatrix} \Delta Y_t \\ \Delta Y_{t-1} \end{pmatrix} \end{aligned}$$

and then BN trend at t, denoted as τ_t, is derived as follows,

$$\tau_t = y_t + \lim_{J\to\infty} \sum_{j=i}^{J} E_t\left[\Delta Y_{t+j}\right] = y_t + \lim_{J\to\infty} \sum_{j=i}^{J} [1\ 0]F^j \begin{pmatrix} \Delta Y_t \\ \Delta Y_{t-1} \end{pmatrix} = y_t + [1\ 0]\left(\sum_{j=1}^{\infty} F^j\right)\begin{pmatrix} \Delta Y_t \\ \Delta Y_{t-1} \end{pmatrix}$$

I can calculate the limit of series of powered matrices as follows, note that I is the two-dimensional identity matrix,

$$\left(\sum_{j=1}^{\infty} F^j\right) = F\left(I - F\right)^{-1} = \frac{1}{1-\beta_1-\beta_2}\begin{pmatrix} \beta_1+\beta_2 & \beta_2 \\ 1 & \beta_2 \end{pmatrix}$$

Then, denoting BN cycle term as c_t,

$$\begin{aligned} \tau_t &= y_t + \frac{1}{1-\beta_1-\beta_2}\left\{(\beta_1+\beta_2)\Delta Y_t + \beta_2 \Delta Y_{t-1}\right\} \\ c_t &= -\frac{1}{1-\beta_1-\beta_2}\left\{(\beta_1+\beta_2)\Delta Y_t + \beta_2 \Delta Y_{t-1}\right\} \end{aligned}$$

4 Problem4

5 Problem5

5.1 (1)

$$\begin{cases} y_t = a_{10} + a_{11}y_{t-1} + a_{12}z_{t-1} + e_{1t} \\ z_t = a_{20} + a_{21}y_{t-1} + a_{22}z_{t-1} + e_{2t} \end{cases} \tag{2}$$

By (2), I get the below,

$$\begin{cases} (1 - a_{11}L)y_t = a_{10} + a_{12}z_{t-1} + e_{1t} \\ (1 - a_{22}L)z_t = a_{20} + a_{21}y_{t-1} + e_{2t} \end{cases} \Rightarrow \begin{cases} y_t = (1 - a_{11}L)^{-1}\left(a_{10} + a_{12}z_{t-1} + e_{1t}\right) \\ z_t = (1 - a_{22}L)^{-1}\left(a_{20} + a_{21}y_{t-1} + e_{2t}\right) \end{cases} \tag{3}$$

Thus I have the following AR(2) form for each time series,

$$\begin{cases} y_t = a_{10} + a_{11}y_{t-1} + a_{12}(1 - a_{22}L)^{-1}\left(a_{20} + a_{21}y_{t-2} + e_{2t}\right) + e_{1t} \\ z_t = a_{20} + a_{21}(1 - a_{11}L)^{-1}\left(a_{10} + a_{12}z_{t-2} + e_{1t}\right) + a_{22}z_{t-1} + e_{2t} \end{cases} \tag{4}$$

5.2 (2)

Because of symmetry, I show the procedure only for y_t and just show the result w.r.t z_t. By $(2), (3)$, I have the following representation.

$$\begin{aligned} y_t =& (1 - a_{11}L)^{-1}\left\{a_{10} + (1 - a_{22}L)^{-1}a_{12}a_{20}\right\} \\ &+ a_{12}a_{21}(1 - a_{11}L)^{-1}(1 - a_{22}L)^{-1}y_{t-2} \\ &+ a_{12}(1 - a_{11}L)^{-1}(1 - a_{22}L)^{-1}e_{2.t-1} \\ &+ (1 - a_{11}L)^{-1}e_{1,t} \end{aligned}$$

Then I have the following explicit form. Note that λ_1, λ_2 are the inverse of the solutions of $1 - (a_{11} + a_{22})x + (a_{11}a_{22} - a_{12}a_{21})x^2$, and I assume that this equation has the real number solution, in other words $(a_{11} + a_{22})^2 - 4(a_{11}a_{22} -$

$a_{12}a_{21}) = (a_{11} - aa_{22})^2 + 4a_{12}a_{21} \geq 0$ and $\lambda_1, \lambda_2 < 1$.

$$
\begin{aligned}
y_t =& \left(1 - a_{12}a_{21}(1-a_{11}L)^{-1}(1-a_{22}L)^{-1}L^2\right)^{-1} \\
& \left[(1-a_{11}L)^{-1}\left\{a_{10} + (1-a_{22}L)^{-1}a_{12}a_{20}\right\} + a_{12}(1-a_{11}L)^{-1}(1-a_{22}L)^{-1}e_{2.t-1} + (1-a_{11}L)^{-1}e_{1,t}\right] \\
=& \left(1 - \frac{a_{12}a_{21}L^2}{(1-a_{11}L)(1-a_{22}L)}\right)^{-1} \\
& \left[(1-a_{11}L)^{-1}\left\{a_{10} + (1-a_{22}L)^{-1}a_{12}a_{20}\right\} + a_{12}(1-a_{11}L)^{-1}(1-a_{22}L)^{-1}e_{2.t-1} + (1-a_{11}L)^{-1}e_{1,t}\right] \\
=& \left(1 - (a_{11}+a_{22})L + (a_{11}a_{22} - a_{12}a_{21})L^2\right)^{-1}(1-a_{11}L)(1-a_{22}L) \\
& \left[(1-a_{11}L)^{-1}\left\{a_{10} + (1-a_{22}L)^{-1}a_{12}a_{20}\right\} + a_{12}(1-a_{11}L)^{-1}(1-a_{22}L)^{-1}e_{2.t-1} + (1-a_{11}L)^{-1}e_{1,t}\right] \\
=& (1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}(1-a_{11}L)(1-a_{22}L) \\
& \left[(1-a_{11}L)^{-1}\left\{a_{10} + (1-a_{22}L)^{-1}a_{12}a_{20}\right\} + a_{12}(1-a_{11}L)^{-1}(1-a_{22}L)^{-1}e_{2.t-1} + (1-a_{11}L)^{-1}e_{1,t}\right] \\
=& (1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}\left\{a_{10} - a_{22}a_{10} + a_{12}a_{20}\right\} \\
& + a_{12}(1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}e_{2,t-1} \\
& + (1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}(e_{1,t} - a_{22}e_{1,t-1}) \\
=& \left\{a_{10} - a_{22}a_{10} + a_{12}a_{20}\right\}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{j}\lambda_1^k\lambda_2^{j-k}\right) \\
& + a_{12}\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\lambda_1^k\lambda_2^{j-k}\right)e_{2,t-1-j} \\
& + \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\lambda_1^k\lambda_2^{j-k}\right)e_{1,t-j} - a_{22}\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\lambda_1^k\lambda_2^{j-k}\right)e_{1,t-1-j}
\end{aligned}
$$

5.3 (3)

Now I have the explicit representation as follows, since in this case $\lambda_1, \lambda_2 = \frac{3}{5}, 1$.

$$
\begin{aligned}
y_t &= 0.2\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\frac{3}{5}^k\right)e_{2,t-1-j} + \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\frac{3}{5}^k\right)e_{1,t-j} + 0.8\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\frac{3}{5}^k\right)e_{1,t-1-j} \\
&= 0.5\sum_{j=0}^{\infty}\left(1 - \frac{3}{5}^j\right)e_{2,t-1-j} + 2.5\sum_{j=0}^{\infty}\left(1 - \frac{3}{5}^j\right)e_{1,t-j} + 2\sum_{j=0}^{\infty}\left(1 - \frac{3}{5}^j\right)e_{1,t-1-j}
\end{aligned}
$$

Now each term in the above does not exist, so this is not a stationary process.

5.4 (4)

$$
\frac{\partial y_{t+s}}{\partial e_{1,t}} = \sum_{k=0}^{s}\frac{3}{5}^k - 0.8\sum_{k=0}^{s-1}\frac{3}{5}^k = \frac{1}{2} + \frac{1}{2}\left(\frac{3}{5}\right)^{s-1}
$$

$$
\frac{\partial y_{t+s}}{\partial e_{2,t}} = 0.2\sum_{k=0}^{s-1}\frac{3}{5}^k = \frac{1}{2} - \frac{1}{2}\left(\frac{3}{5}\right)^{s-1}
$$

5.5 (5)

$$\frac{\partial y_{t+s}}{\partial \epsilon_{y,t}} = \frac{\partial e_{1,t}}{\partial \epsilon_{y,t}}\frac{\partial y_{t+s}}{\partial e_{1,t}} + \frac{\partial e_{2,t}}{\partial \epsilon_{y,t}}\frac{\partial y_{t+s}}{\partial e_{2,t}} = \frac{\partial y_{t+s}}{\partial e_{1,t}} = \frac{1}{2} + \frac{1}{2}\left(\frac{3}{5}\right)^{s-1}$$

$$\frac{\partial y_{t+s}}{\partial \epsilon_{z,t}} = \frac{\partial e_{1,t}}{\partial \epsilon_{z,t}}\frac{\partial y_{t+s}}{\partial e_{1,t}} + \frac{\partial e_{2,t}}{\partial \epsilon_{z,t}}\frac{\partial y_{t+s}}{\partial e_{2,t}} = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\left(\frac{3}{5}\right)^{s-1}\right) + \frac{1}{2} - \frac{1}{2}\left(\frac{3}{5}\right)^{s-1} = \frac{3}{4} - \frac{1}{4}\left(\frac{3}{5}\right)^{s-1}$$

5.6 (6)

$$\frac{\partial y_{t+s}}{\partial \epsilon_{y,t}} = \frac{\partial e_{1,t}}{\partial \epsilon_{y,t}}\frac{\partial y_{t+s}}{\partial e_{1,t}} + \frac{\partial e_{2,t}}{\partial \epsilon_{y,t}}\frac{\partial y_{t+s}}{\partial e_{2,t}} = \frac{\partial y_{t+s}}{\partial e_{1,t}} + \frac{1}{2}\frac{\partial y_{t+s}}{\partial e_{2,t}} = \frac{1}{2} + \frac{1}{2}\left(\frac{3}{5}\right)^{s-1} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\left(\frac{3}{5}\right)^{s-1}\right) = \frac{3}{4} + \frac{1}{4}\left(\frac{3}{5}\right)^{s-1}$$

$$\frac{\partial y_{t+s}}{\partial \epsilon_{z,t}} = \frac{\partial e_{1,t}}{\partial \epsilon_{z,t}}\frac{\partial y_{t+s}}{\partial e_{1,t}} + \frac{\partial e_{2,t}}{\partial \epsilon_{z,t}}\frac{\partial y_{t+s}}{\partial e_{2,t}} = \frac{1}{2} - \frac{1}{2}\left(\frac{3}{5}\right)^{s-1}$$

5.7 (7)