STAT3811/3955 Survival Analysis
Assignment 1

Kei Ikegami

February 22, 2017

1 Q1

1.1 (a)

$$
\begin{aligned}
E[T|T>t] &= \int_t^\infty Tf(T|T>t)\mathrm{d}T = \int_t^\infty T\frac{f(T)}{1-F(t)}\mathrm{d}T \\
&= \frac{1}{1-F(t)}\left(\int_0^\infty Tf(T)\mathrm{d}T - \int_0^t Tf(T)\mathrm{d}T\right) = \frac{1}{1-F(t)}\left(\mu - tF(t) + \int_0^t F(T)\mathrm{d}T\right)
\end{aligned}
$$

So I get the below derivative of $m(t)$.

$$
m^{'}(t) = \frac{1}{(1-F(t))^2}\left\{\left(-F(t) - tf(t) + F(t)\right)(1-F(t)) + (\mu - tF(t) + \int_0^t F(T)\mathrm{d}T)f(t)\right\} - 1
$$

Then the following calculation leads to the result.

$$
\frac{1+m^{'}(t)}{m(t)} = \frac{f(t)\left(\mu - tF(t) + \int_0^t F(T)\mathrm{d}T - t(1-F(t))\right)}{(1-F(t))\left(\mu - tF(t) + \int_0^t F(T)\mathrm{d}T\right) - t(1-F(t))^2} = \frac{f(t)}{1-F(t)} = \lambda(t)
$$

1.2 (b)

Since $\int_0^t F(T)\mathrm{d}T = \int_0^t(-S(T)+1)\mathrm{d}T = -\int_0^t S(T)\mathrm{d}T + t$, then

$$
\begin{aligned}
m(t) &= \frac{1}{S(t)}\int_0^\infty (T-t)f(T)\mathrm{d}T = \frac{1}{S(t)}\left(\int_0^\infty (T-t)f(T)\mathrm{d}T - \int_0^t (T-t)f(T)\mathrm{d}T\right) \\
&= \frac{1}{S(t)}\left(\mu - t + \int_0^t F(T)\mathrm{d}T\right) = \frac{1}{S(t)}\left(\mu - \int_0^t S(u)\mathrm{d}u\right)
\end{aligned}
$$

Now, when T has an exponential distribution with $\mu = \frac{1}{\lambda}$,

$$
m(t) = \exp(\lambda t)\left(\mu + \frac{1}{\lambda}\exp(-\lambda t) - \frac{1}{\lambda}\right) = \frac{1}{\lambda} = \mu
$$

because $\int_0^\infty t\lambda\exp(-\lambda t)\mathrm{d}t = \frac{1}{\lambda}$.

1.3 (c)

First I consider the mean,

$$
\lim_{t\to 0} m(t) = \lim_{t\to 0} E[T|T>t] = E[T] = 1
$$

Now let $\delta = med(T)$, then $F(\delta) = \frac{1}{2}$ and $\lambda(\delta) = \frac{2}{\delta+1}$ due to (a). Then by using (b) I get the below calculation.

$$\frac{2}{\delta+1} = 2\left(1 - \int_0^\delta (1-F(u))\mathrm{d}u\right) \Leftrightarrow \int_0^\delta (1-F(u))\mathrm{d}u = \frac{\delta}{\delta+1}$$

By taking derivative of both sides about δ, I get the result as follows.

$$1 - F(\delta) = \frac{1}{(\delta+1)^2} \quad \Leftrightarrow \quad \frac{1}{2} = \frac{1}{(\delta+1)^2} \quad \Leftrightarrow \quad \delta = \sqrt{2} - 1$$

1.4 (d)

First I have the below representation of $m(t)$.

$$m(t)\frac{\mu - \int_0^t S(u)\mathrm{d}u}{S(t)} = \frac{\int_t^\infty S(u)\mathrm{d}u}{S(t)}$$

Since the limits of the both of enumerator and denominator are 0 as $t \to \infty$. By using L'Hopital's rule twice, I get the below result,

$$\lim_{t\to\infty} m(t) = \lim_{t\to\infty} \frac{-S(t)}{-f(t)} = \lim_{t\to\infty} \frac{f(t)}{-f'(t)} = \lim_{t\to\infty} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\log f(t)\right)^{-1}$$

1.5 (e)

In this case, $f(t) = \frac{1}{\sqrt{2\pi}\sigma t}\exp(\frac{\log t - \mu}{2\sigma^2})$, I use (d) to get the result.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\log f(t)\right)^{-1} = -\frac{f(t)}{f'(t)} = -\frac{\sigma^2 t}{\mu - \log t - \sigma^2}$$
$$\lim_{t\to\infty} -\frac{\sigma^2 t}{\mu - \log t - \sigma^2} = \lim_{t\to\infty} -\frac{1}{-\frac{1}{t}} = \infty$$

2 Q3

2.1 (a)

By definition, $S(t|z) = 1 - F(t|z)$. Thus I calculate $F(t|z)$ as follows.

$$F(t|z) = \Pr(Y \le \log t|z) = \Pr(w \le \frac{\log\, t - \mu - \beta z}{\sigma}|z)$$

Because $\int_{-\infty}^{\omega} \frac{\exp(u)}{(1+\exp(u))^2}\mathrm{d}u = \frac{\exp(\omega)}{1+\exp(\omega)}$, then by the above calculation,

$$S(t|z) = 1 - F(t|z) = \frac{1}{1+\exp(\frac{\log\, t - \mu - \beta z}{\sigma})}$$

2.2 (b)

By (a),

$$\frac{S(t|z)}{1 - S(t|z)} = \frac{1}{\exp(\frac{\log\, t - \mu - \beta z}{\sigma})} = \exp\left(-\frac{\log\, t - \mu - \beta z}{\sigma}\right)$$

2.3 (c)

By (b), let $Odds_i$ be the odds for z_i,

$$\frac{Odds_1}{Odds_2} = \exp\left(\frac{\beta}{\sigma}\right)$$

And this odds ratio is independent of t.

3 Q5

3.1 (a)

Just calculate as follows,

$$P(T_i < C_i) = \int_0^\infty \left(\int_0^c \lambda \exp(-\lambda t)\, \mathrm{d}t \right) \theta \exp(-\theta c)\, \mathrm{d}c = 1 - \int_0^\infty \theta \exp(-(\lambda+\theta)c)\, \mathrm{d}c$$
$$= 1 - \frac{\theta}{\theta+\lambda} = \frac{\lambda}{\theta+\lambda}$$

Then the probability distribution of δ is

$$\delta = \begin{cases} 1 & \text{with probability } \frac{\lambda}{\lambda+\theta} \\ 0 & \text{with probability } \frac{\theta}{\lambda+\theta} \\ \text{otherwise} & \text{with probability } 0 \end{cases}$$

3.2 (b)

Let $F_Y(y)$, $f_Y(y)$ be the distribution function and probability distribution function of Y. Then, due to the independence of T and C,

$$1 - F_Y(y) = 1 - \Pr(\min(T, C) < y) = \Pr(y \leq \min(T, C)) = \Pr(y \leq T)\Pr(y \leq C) = \exp(-(\lambda+\theta)y)$$

Thus I get $F_Y(y) = 1 - \exp(-(\lambda+\theta)y)$, which means Y has a exponential distribution with parameter $\lambda+\theta$.

3.3 (c)

Consider the marginal distribution of Y when $\delta = 1$ as follows,

$$f(Y, \delta = 1) = \lim_{h \to 0} \frac{\Pr(y \leq Y \leq y+h, \delta = 1)}{h}$$

Now the denominator of this can be decomposed, because of the independence of T, C,

$$\Pr(y \leq Y \leq y+h, \delta = 1) = \Pr(y \leq Y \leq y+h, T < C) = \Pr(y \leq T \leq y+h, y \leq C)$$
$$= \Pr(y \leq T \leq y+h)\Pr(y \leq C) = \exp(-\lambda y)(1 - \exp(-\lambda h))\exp(-\theta y)$$

Then, by using L'Hopital rule, the marginal distribution is

$$f(Y, \delta = 1) = \exp(-(\lambda+\theta)y) \lim_{h \to 0} \frac{1 - \exp(-\lambda h)}{h}$$
$$= \lambda \exp(-(\lambda+\theta)y) = \left(\frac{\lambda}{\lambda+\theta} \right) (\lambda+\theta)\exp(-(\lambda+\theta)y)$$

3.4 (d)

3.5 (e)

3.6 (f)

3.7 (g)

3.8 (h)

3.9 (i)

3.10 (j)