LEVEL-k AUCTIONS: CAN A NONEQUILIBRIUM MODEL OF STRATEGIC THINKING EXPLAIN THE WINNER'S CURSE AND OVERBIDDING IN PRIVATE-VALUE AUCTIONS?

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January 18, 2017

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Purpose

Precedents

What's new

Result

General Model

- N bidders bid for a single object.
- X_i is bidder i's private signal. $X = (X_1, \ldots, X_N)$.
- S_j is additional random variable which is informative about the value of the object. $S = (S_1, \ldots, S_M)$.
- $V_i = u_i(S, X)$ is bidder i's value of the object, where u_i is symmetric across i.
- $V_i - p$ is the payoff for the bidder i winning the auction by paying p.
- Y is the highest signal among bidders other than i.
- $\upsilon(x, y) = E\left[V_i | X_i = x, Y = y\right]$ is the expected value conditional on winning.
- $r(x) = E\left[V_i | X_i = x\right]$ is the unconditional expected value.

Classification of Auctions

- First price auction vs Second price auction
- Independent private value auction(i.p.v) vs Common value auction(c.v)
- In i.p.v, the signals and values are independent among bidders. And then $v(x,x) = r(x) = x$.
- In c.v, the information of i and j is not independent and learning about the other bidders' information can cause the bidder to reassess his estimate of the value of the object (e.g. Timber auction). And so $v(x,x) \neq r(x)$ normally with $r(x) > v(x,x)$.

First Price Auction

- In c.v, the optimal bidding strategy is calculated as (1).
- In i.p.v, the optimal bidding strategy is calculated as (2), because $v(x,x) = x$ and Y is independent of X.
- Using $v(x,x)$ is called "value adjustment for the information revealed by winning".
- The latter terms in (1) and (2) are due to "bidding trade-off", which means higher bidding raises both the probability of winning and the cost.

$$a_*(x) = v(x,x) - \int_{\underline{x}}^{x} \exp\left(-\int_{y}^{x} \frac{f_Y(t|t)}{F_Y(t|t)}\mathrm{d}t\right) \mathrm{d}(v(y,y)) \quad (1)$$

$$a_*(x) = x - \int_{\underline{x}}^{x} \frac{F_Y(y)}{F_Y(x)}\mathrm{d}y = E\left[Y|Y < X\right] \quad (2)$$

Second Price Auction

- In c.v, the optimal bidding strategy is calculated as (3). This is not a weakly dominant strategy.
- In i.p.v, the optimal bidding strategy is calculated as (4), because $v(x, x) = x$. This is a weakly dominant strategy.

$$b_*(x) = v(x, x) \tag{3}$$

$$b_*(x) = x \tag{4}$$

Points

- χ is the parameter denoting the probability that the bidder think the other bidders bid independently of signals.
- Cursed equilibrium for a given χ value is called χ-cursed equilibrium.
- In ER's(2002, 2006), χ-cursed equilibrium is the same as one in a hypothetical "χ-virtual game", in which players believe that, with probability χ, other's bid is independent of types.
- In i.p.v, the players bid independently of others' signals by definition, then the optimal bidding strategy in χ-cursed equilibrium is the same as one in equilibrium $(v(x,x) = r(x) = x)$.
- In c.v, since $v(x,x) \neq r(x)$, cursed equilibrium differs from equilibrium.

Common Value Auction

- In first price auction, the optimal bidding strategy is calculated as (5).
- In second price auction, the optimal bidding strategy is calculated as (6).
- These calculations are exactly the same as ones in equilibrium.

$$a_\chi(x) = \{(1-\chi)v(x,x) + \chi r(x)\} - \int_{\underline{x}}^{x} \exp\left(-\int_y^x \frac{f_Y(t|t)}{F_Y(t|t)}\mathrm{d}t\right) \mathrm{d}\{(1-\chi)v(y,y) + \chi r(y)\} \quad (5)$$

$$b_\chi(x) = (1-\chi)v(x,x) + \chi r(x) \quad (6)$$

Points

- Level-k model allows behavior to be heterogeneous, but assumes that each player's behavior is drawn from the common distribution over the k types.
- In this paper there are 3 types (denoted by L k) which best response to the lower type. i.e. L1 best responses to L0, and L2 best responses to L1.
- The key assumption is the behavior of L0. One is Random L0, in which the L0 bids uniformly randomly independent of its own signal. The other is Truthful L0, in which L0 bids the value suggested by its own signal.
- L1 and L2 are called Random L1 and L2 when L0 is set to Random L0.
- L1 and L2 are called Truthful L1 and L2 when L0 is set to Truthful L0.
- In the latter slides I show the optimal bidding strategies of each player in each case one by one.

Random L0

- This player bids i.i.d. uniformly over the range $[\underline{z}, \bar{z}]$, which is determined by its private signal and the value $V_i = u_i(S, X)$.

Random L1 in First Price Auction

- Random L1 plays in the belief that all other players follow Random L0.
- Let Z be the highest bid among the others, the distribution function of Z be $F_z(z)$, and the pdf be $f_z(z)$ (these two are from the ordered statistics).
- The optimal bidding strategy of Random L1 ($a_1^r(x)$) solves (7) and is characterized by (8).

$$\max_a \int_{\underline{z}}^{a} (r(x) - a) f_z(z) \mathrm{d}z \tag{7}$$

$$(r(x) - a) f_z(a) - F_z(a) = 0 \tag{8}$$

Random L1 in First Price Auction

- This optimal bidding strategy is common in i.p.v and c.v.
- There are two differences from the optimal bidding strategy in equilibrium.
- One is that $r(x)$ replaces $v(x, x)$, which reflects the fact that Random L1 think winning conveys no information about the value of the object even in c.v. (difference in value adjustment). Since in general $r(x) > v(x, x)$, Random L1 tends to overbid due to the value adjustment.
- Second is that the integral in (7) is over Z rather than Y (see (1) in paper). This is caused by L1 use nonequilibrium belief to evaluate the bidding trade-off. (difference in bidding trade-off)

Random L1 in Second Price Auction

- The optimal bidding strategy ($b_1^r(x)$) solves (9)'s maximization problem. And get (10).

$$\max_b \int_{\underline{z}}^{b} (r(x) - z) f_z(z) \mathrm{d}z \tag{9}$$

$$b_1^r(x) = r(x) \tag{10}$$

Random L1 in Second Price Auction

- One difference from equilibrium case is $r(x)$ replaces $v(x, x)$. (difference in value adjustment). And this leads to overbidding in c.v.
- Second difference from equilibrium case is that the player use nonequilibrium belief. However note that this does not result in the deviating from truthful bidding as in first price auction.
- In other words, we have no bidding trade-off term in the optimal bidding strategy just as in the equilibrium case.
- (10) coincides with (6) when $\chi = 1$, i.e. fully cursed equilibrium.
- (10) coincides with equilibrium in i.p.v, where $r(x) = x$.

Random L2 , Truthful L1 and Truthful L2

- Random L2 best responses to Random L1.
- (8) and (10) tell that Random L1's bidding strategy is an increasing function of its private signal both in first and second price auction.
- For Random L2 in this case, the high bid among the other players conveys the information about the value of the object, then Random L2 adjust its own estimated value according to the information. This means that we use $\upsilon(x, y)$ rather than $r(x)$.
- This structure is common in Truthful L1 and Truthful L2. Thus I provide the general bidding strategy which can be applied to the three cases.

General Bidding Strategy in First Price Auction

- Suppose that a level-k bidder expects others to bid according to the monotonically increasing bidding strategy $a_{k-1}(x)$.
- The bidder's optimal bidding strategy with V_i and X_i solves (11), and is characterized by the first order condition (12).

$$\max_{a} \int_{\underline{x}}^{a_{k-1}^{-1}(a)} (\upsilon(x,y) - a) f_Y(y|x) \mathrm{y} \tag{11}$$

$$(\upsilon(x, a_{k-1}^{-1}(a)) - a) f_Y(a_{k-1}^{-1}(a)|x) \frac{\partial a_{k-1}^{-1}(a)}{\partial a} - F_Y(a_{k-1}^{-1}(a)|x) = 0 \tag{12}$$

General Bidding Strategy in Second Price Auction

- Suppose that a level-k bidder expects others to bid according to the monotonically increasing bidding strategy $b_{k-1}(x)$.
- The bidder's optimal bidding strategy with V_i and X_i solves (13), and is characterized by the first order condition (14).

$$\max_{b} \int_{\underline{x}}^{b_{k-1}^{-1}(b)} (\upsilon(x, y) - b_{k-1}(y)) f_Y(y|x)\mathrm{y} \tag{13}$$

$$b = \upsilon(x, b_{k-1}^{-1}(b)) \tag{14}$$

Strategic Substitutability

- What does mean "Value adjustment (using $v(x, x)$) tends to make bidders' bids strategic substitutes" ?
- level-k が、周囲の奴らが均衡よりも overbid していると考えているとする。
- この時、もしその状態（overbid の状態）が均衡だった時に想定される彼らの private signal よりも、実際に彼らが受け取っている private signal は小さい（弱い）と考えられる。
- その分の value adjustment を level-k bidder が行う。この際の修正は、均衡 bid よりも少し低い bid となる。なぜなら質が周囲の bid 状況から想定されるほど良くないぞという情報を受け取っているから。
- こうして、周囲が overbid なら自分は underbid、という戦略的代替が成り立つことがわかった。

Random L2 in First Price Auction

- $a_2^r(x)$ is determined by (12) with $a_1^{r^{-1}}(a)$ replacing $a_{k-1}^{-1}(a)$.
- Given the strategic substitutability of value adjustment, Random L2 underbid because Random L1 overbid relative to the equilibrium level.

$$(\upsilon(x, a_1^{r^{-1}}(a)) - a) f_Y(a_1^{r^{-1}}(a)|x) \frac{\partial a_1^{r^{-1}}(a)}{\partial a} - F_Y(a_1^{r^{-1}}(a)|x) = 0 \tag{15}$$

Random L2 in Second Price Auction

- $b_2^r(x)$ is determined by (14) with $b_1^{r^{-1}}(b)$ replacing $b_{k-1}^{-1}(b)$.
- Given the strategic substitutability of value adjustment, Random L2 underbid because Random L1 overbid relative to the equilibrium level.
- Bidding strategy is again truthful ("truthful" means that the bidder does not consider the bidding trade-off).
- Note that $b_1^r(x) = r(x)$ from (10).

$$b = v(x, b_1^{r^{-1}}(b)) = v(x, r^{-1}(b)) \tag{16}$$

Truthful L1 in First Price Auction

- Truthful L0's bidding strategy is denoted by $a_0^t(x) \equiv r(x)$, which is from the definition of Truthful L0.
- Truthful L1's bidding strategy $(a_1^t(x))$ can be characterized by the special case of (12) with $r(x)$ replacing $a_{k-1}(x)$.
- Since Truthful L0 overbids relative to the equilibrium (see (1)), the strategic substitutability tends to make Truthful L1 underbid.

$$(\upsilon(x, r^{-1}(a)) - a) f_Y(r^{-1}(a)|x) \frac{\partial r^{-1}(a)}{\partial a} - F_Y(r^{-1}(a)|x) = 0 \tag{17}$$

Truthful L1 in Second Price Auction

- The optimal bidding strategy of Truthful L1 ($b_1^t(x)$) can be characterized by (14) with $b_0^t(x) \equiv r(x)$ replacing $b_{k-1}(x)$.
- Since Truthful L0 normally overbids relative to the equilibrium due to using $r(x)$ rather than $\upsilon(x, x)$, the strategic substitutability tends to make Truthful L1 underbid.

$$b = \upsilon(x, r^{-1}(b)) \tag{18}$$

Truthful L2 in First Price Auction

- By $a_1^t(x)$ replacing $a_{k-1}^t(x)$ in (12), the optimal bidding strategy $(a_2^t(x))$ is characterized.
- The strategic substitutability tends to make Truthful L2 overbid reflecting Truthful L1's underbidding.

$$(\upsilon(x, a_1^{t^{-1}}(a)) - a) f_Y(a_1^{t^{-1}}(a)|x) \frac{\partial a_1^{t^{-1}}(a)}{\partial a} - F_Y(a_1^{t^{-1}}(a)|x) = 0 \tag{19}$$

Truthful L2 in Second Price Auction

- By $b_1^t(x)$ replacing $b_{k-1}^t(x)$ in (14), the optimal bidding strategy $(b_2^t(x))$ is characterized.
- The strategic substitutability tends to make Truthful L2 overbid reflecting Truthful L1's underbidding.

$$b = \upsilon(x, b_1^{t^{-1}}(b)) \tag{20}$$

Auction Examples: KL

- KL(Kagel and Levin 1986) is an example of common value auction.
- S is uniformly distributed on a subset of $[\underline{s}, \bar{s}]$.
- $X|S$ is conditionally uniformly i.i.d. on the interval $[s - \frac{a}{2}, s + \frac{a}{2}]$ with $a > 0$.
- Thus $f_{X|S} = \frac{1}{a}$ and $F_{X|S} = \frac{x-s}{a} + \frac{1}{2}$. And $E[X|S] = s$.
- Then the value adjustment for no information by winning is calculated as (21).

$$r(x) \equiv E[S|X = x] = x \tag{21}$$

Derivation of (21)

Let the range of S be R_S and its length be l_S.

$$E[S|X=x] = \int_{R_S} s \cdot f_{S|X=x}(s)\mathrm{d}s = \int_{R_S} S \frac{f_{X=x|S}(x)f_S(s)}{\int_{R_S} f_{X=x|S}(x)f_S(s)\mathrm{d}s}\mathrm{d}s \quad (22)$$

Now $\int_{R_S} f_{X=x|S}(x)f_S(s)\mathrm{d}s = \int_{x-\frac{a}{2}}^{x+\frac{a}{2}} \frac{f_S(s)}{a}\mathrm{d}s = \frac{1}{a \cdot l_S}a = \frac{1}{l_S}$

Then

$$(22) = \int_{R_S} s \cdot l_S \cdot f_{X=x|S}(x)f_S(s)\mathrm{d}s = \int_{R_S} s \cdot f_{X=x|S}(x)\mathrm{d}s$$

$$= \frac{1}{a}\int_{x-\frac{a}{2}}^{x+\frac{a}{2}} s\ \mathrm{d}s = \frac{1}{a}ax = x$$

Auction Examples: KL

- The value adjustment for the information revealed by winning is calculated as in (23).
- The derivation of (23) is in the web appendix.
- With $N > 2$, $\upsilon(x,x) \leq r(x) = x$holds. Thus the cursed equilibrium bidders overbid relative to the equilibrium.

$$\upsilon(x,y) = \begin{cases} x - \frac{a}{2} + \frac{a}{N} - \frac{x-y}{N} & x - a \leq y \leq x \\ y - \frac{a}{2} + \frac{a}{N} - \frac{\left(\frac{y-x}{a}\right)^{N-1}}{1-\left(\frac{y-x}{a}\right)^{N-1}} \left(\frac{N-1}{N}\right)(x+a-y) & x < y \leq x + a \end{cases} \tag{23}$$

Auction Examples: AK

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Auction Examples: GHP

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Summary Table

- Table 1 を挿入

Level-k Improvements in First Price Auction

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Level-k Improvements in Second Price Auction

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Preparation for Comparing

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How to Compare

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Table3a

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Table3c

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Table3d

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Table3b

- 他と比率が違う理由もかく

Summary: Could Level-k Model really Improve?

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Summary

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Implication

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