Macroeconomics 1 2018 S1S2
Homework 1

Kei Ikegami (29186009)

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1 Problem1

We need an additional assumption for stationarity.

$$y_t = c + \phi y_{t-1} + \epsilon_t$$
$$\Leftrightarrow (1 - \phi L)y_t = c + \epsilon_t$$
$$\Leftrightarrow y_t = (1 - \phi L)^{-1}(c + \epsilon_t) = c + \sum_{u=0}^{\infty} \phi^u L^u \epsilon_t = c + \sum_{u=0}^{\infty} \phi^u \epsilon_{t-u} \tag{1}$$

For stationarity, it is necessary the above limit exists. By Chaucy's convergence judgement, we know that it is necessary and sufficient for the convergence of series that the residual, i.e. $\sum_{u=n+1}^{\infty} \phi^u \epsilon_{t-u}$, converges to 0. Usually, in time series analysis, the Hilbert space, whose product is defined by covariance, is used when the convergence is discussed. Thus if we want to know whether $\sum_{u=n+1}^{\infty} \phi^u \epsilon_{t-u} \to 0$ or not, we must check the convergence w.r.t the norm induced by the covariance product. And for the below calculation we need as assumption that states $Var(y_t) < \infty$.

$$\| \sum_{u=n+1}^{\infty} \phi^u \epsilon_{t-u} \| = Var(\sum_{u=n+1}^{\infty} \phi^u \epsilon_{t-u}) = \sum_{n+1}^{\infty} |\phi|^{2u} \sigma^2 = 0$$

The second equality is followed by the additional assumption. The third equality is by the assumption $|\phi| < 1$. Now we have the result that $\sum_{u=0}^{\infty} \phi^u \epsilon_{t-u}$ exists, in other words, the time series is stationary.

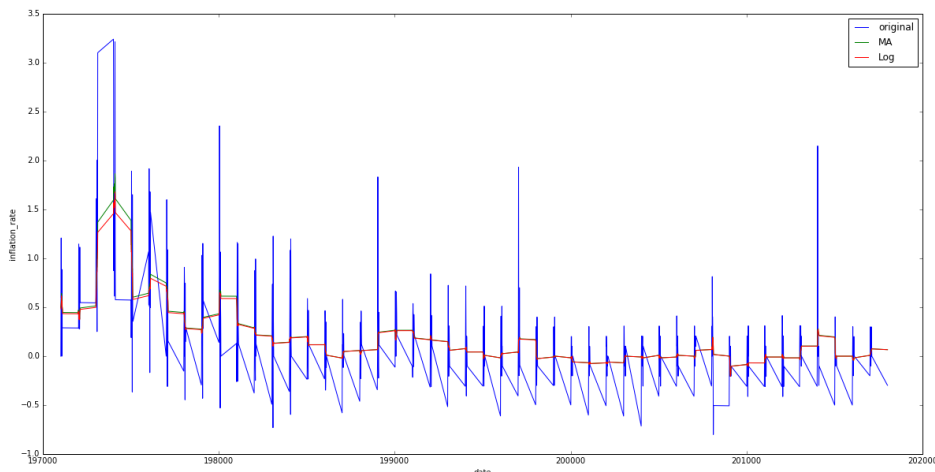Next, I calculate the mean, variance, j-th autocovariance.

Mean By (1), $E[y_t] = E\left[c + \sum_{u=0}^{\infty} \phi^u \epsilon_{t-u}\right] = c + 0 = c$

Variance By (1), $Var(y_t) = Var\left(\sum_{u=0}^{\infty} \phi^u \epsilon_{t-u}\right) = \sum_{u=0}^{\infty} |\phi|^{2u} \sigma^2 = \frac{\sigma^2}{1-\phi^2}$

Autocovariance $\gamma(t, t-j) = \phi Cov(Y_{t-1}, Y_{t-j}) = \cdots = \phi^j Var(Y_{t-j}) = \frac{\phi^u \sigma^2}{1-\phi^2}$

2 Problem2

2.1 (1)

The Python code is as follows.

```
1   # import relevant package
2   import pandas as pd
3   import matplotlib.pyplot as plt
4   import numpy as np
5   from math import pow
6   % matplotlib inline
7
8   # make the three time series
9   # original series
10  original = [100*(CPI[i] − CPI[i−1])/CPI[i−1] for i in range(1, len(CPI))]
11  original_series = original[12:]
12  #ma
13  ma_series = [sum([original[i−j] for j in range(12)])/12 for i in range(12, len(original))]
14  #log
15  log_series = [100*(pow(1+(np.log(CPI[i]) − np.log(CPI[i−12])), 1/12)−1) for i in range(13, len(CPI))]
16
17  # plot the data
18  date = data[["year"]].values[:, 0][13:]
19  plt.figure(figsize=(20,10))
20  plt.plot(date, original_series, label = 'original')
21  plt.plot(date, ma_series, label = 'MA')
22  plt.plot(date, log_series, label = 'Log')
23  plt.legend()
24  plt.xlabel("date")
25  plt.ylabel("inflation_rate")
26  plt.savefig('inflation.png')
27  plt.show()
28
29  # csv output
30  df = pd.DataFrame({"date": date, "original":original_series, "ma":ma_series, "log":log_series})
31  df.to_csv("corrected_inflation.csv")
```

2.2 (3)

The R code is as follows.

```
1   # read data
2   corrected_inflation <− read_csv("~/Desktop/2018summer/macro/hw1/empirics/corrected_inflation.csv")
3
4   # time series format
5   ORI <− ts(corrected_inflation[["original"]])
6   LOG <− ts(corrected_inflation[["log"]])
7   MA <− ts(corrected_inflation[["ma"]])
```

```
8
9    # ma
10   AIC_ma <- 1:24
11   for (i in 1:24){
12      result <- arima(MA, order = c(i,0,0))
13      AIC_ma[i] <- result$aic
14   }
15
16   ma_optimal_p <- which.min(AIC_ma)
17
18   # log
19   AIC_log <- 1:24
20   for (i in 1:24){
21      result <- arima(LOG, order = c(i,0,0))
22      AIC_log[i] <- result$aic
23   }
24
25   log_optimal_p <- which.min(AIC_log)
```

and $p = 13$ is chosen in both of seasonal adjustments.

2.3   (3)

The estimated coefficients are in the next page. By the table, I see the stochastic significance at $ar1, ar8, ar12$ and $ar13$ in both types of seasonal adjustments. The sign is minus only for $ar12$, and the values are almost the same in the two cases. This means that the inflation rate of the same month in the previous year negatively correlates with one of this year. Then I can say the monthly inflation rate has stationary property in this data.

3   Problem3

I use state-space model for solving this problem. When the first difference series follow AR(2) model, i.e. $\Delta Y_t = \beta_1 \Delta Y_{t-1} + \beta_2 \Delta Y_{t-2}$, let the state vector and the disturbance vector be

$$X_t = \begin{pmatrix} \Delta Y_t \\ \Delta Y_{t-1} \end{pmatrix}, \ e_t = \begin{pmatrix} \epsilon_t \\ 0 \end{pmatrix}$$

and let the update matrix be

$$F = \begin{pmatrix} \beta_1 & \beta_2 \\ 1 & 0 \end{pmatrix}$$

then I have the below description of this model.

$$X_{t+1} = FX_t + e_{t+1}, \ \Delta Y_{t+1} = [1 \ 0]X_{t+1}$$

Thus

$$
\begin{aligned}
E_t \left[\Delta Y_{t+j} \right] &= [1 \ 0]E_t \left[X_{t+j} \right] \\
&= [1 \ 0]E_t \left[FX_{t+j-1} + e_{t+j} \right] \\
&= [1 \ 0]E_t \left[F^2 X_{t+j-2} + Fe_{t+j-1} + e_{t+j} \right] \\
&= \cdots \\
&= [1 \ 0]F^j E_t \left[X_t \right] \\
&= [1 \ 0]F^j \begin{pmatrix} \Delta Y_t \\ \Delta Y_{t-1} \end{pmatrix}
\end{aligned}
$$

and then BN trend at t, denoted as τ_t, is derived as follows,

$$\tau_t = y_t + \lim_{J \to \infty} \sum_{j=i}^{J} E_t \left[\Delta Y_{t+j} \right] = y_t + \lim_{J \to \infty} \sum_{j=i}^{J} [1 \ 0]F^j \begin{pmatrix} \Delta Y_t \\ \Delta Y_{t-1} \end{pmatrix} = y_t + [1 \ 0] \left(\sum_{j=1}^{\infty} F^j \right) \begin{pmatrix} \Delta Y_t \\ \Delta Y_{t-1} \end{pmatrix}$$

3

Table 1: estimating AR(13)

	Dependent variable:	
	MA	LOG
	(1)	(2)
ar1	1.220***	1.209***
	(0.035)	(0.035)
ar2	−0.093	−0.078
	(0.057)	(0.056)
ar3	−0.040	−0.043
	(0.057)	(0.056)
ar4	−0.079	−0.079
	(0.057)	(0.056)
ar5	0.068	0.063
	(0.057)	(0.057)
ar6	−0.022	−0.021
	(0.057)	(0.057)
ar7	−0.078	−0.069
	(0.057)	(0.057)
ar8	0.118**	0.105*
	(0.057)	(0.057)
ar9	0.006	0.007
	(0.058)	(0.057)
ar10	−0.011	−0.006
	(0.058)	(0.057)
ar11	−0.033	−0.028
	(0.058)	(0.057)
ar12	−0.602***	−0.603***
	(0.057)	(0.057)
ar13	0.539***	0.536***
	(0.035)	(0.035)
intercept	0.211	0.203
	(0.150)	(0.143)
Observations	564	564
Log Likelihood	1,214.527	1,259.025
σ^2	0.001	0.001
Akaike Inf. Crit.	−2,399.054	−2,488.050
Note:	*p<0.1; **p<0.05; ***p<0.01	

I can calculate the limit of series of powered matrices as follows, note that I is the two-dimensional identity matrix,

$$\left(\sum_{j=1}^{\infty} F^j\right) = F\left(I - F\right)^{-1} = \frac{1}{1 - \beta_1 - \beta_2}\begin{pmatrix} \beta_1 + \beta_2 & \beta_2 \\ 1 & \beta_2 \end{pmatrix}$$

Then, denoting BN cycle term as c_t,

$$\tau_t = y_t + \frac{1}{1 - \beta_1 - \beta_2}\left\{(\beta_1 + \beta_2)\Delta Y_t + \beta_2 \Delta Y_{t-1}\right\}$$
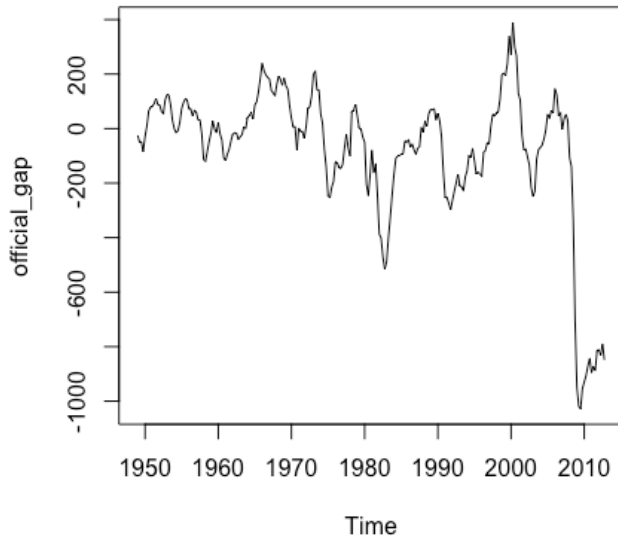
$$c_t = -\frac{1}{1 - \beta_1 - \beta_2}\left\{(\beta_1 + \beta_2)\Delta Y_t + \beta_2 \Delta Y_{t-1}\right\}$$

4   Problem4

All the code used in this problem is in the last of this section.
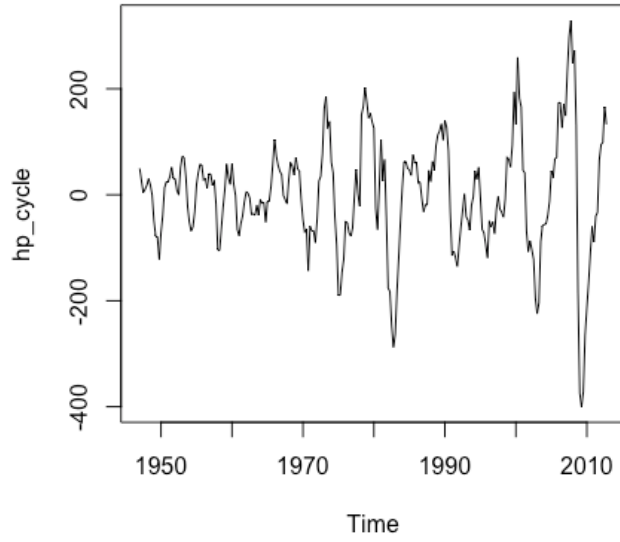
4.1   (1)

I plot the official GDP gap calculated by the data below.


4.2   (2)

I plot the HP cycle components.

5

The range of the gap shrinks by using HP filter, and the cycle is made easy to detect.

4.3 (3)

I carry out the Augmented Dickey-Fuller test for the time series. And the p-value is 0.0631, then I reject the null hypothesis. Next I make an estimation of the model for the first difference series in order to do BN-decomposition. The result says that $AR(2)$ is the best fitted model, then I decompose the time series by using the result of the problem (3). And the result cycle components is plotted below.
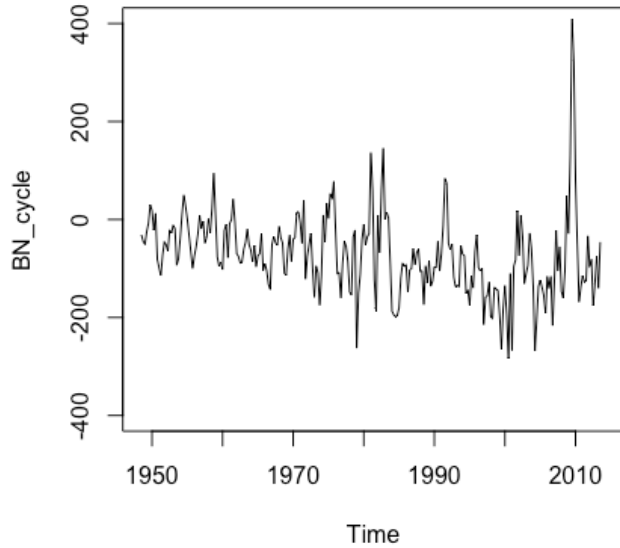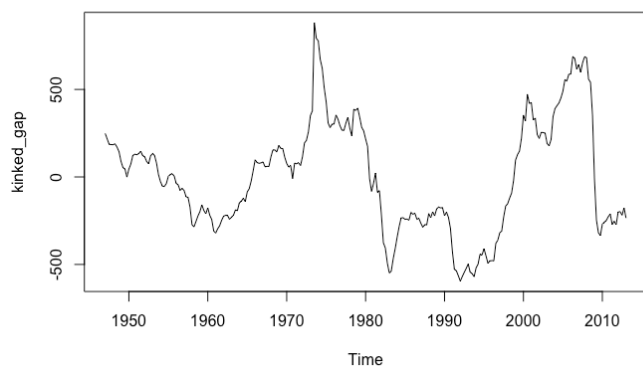

BN-cycle show the more fluctuating property than HP-filter.

4.4 (4)

First I plot the transitory component calculated by the linear regression on the trend term .
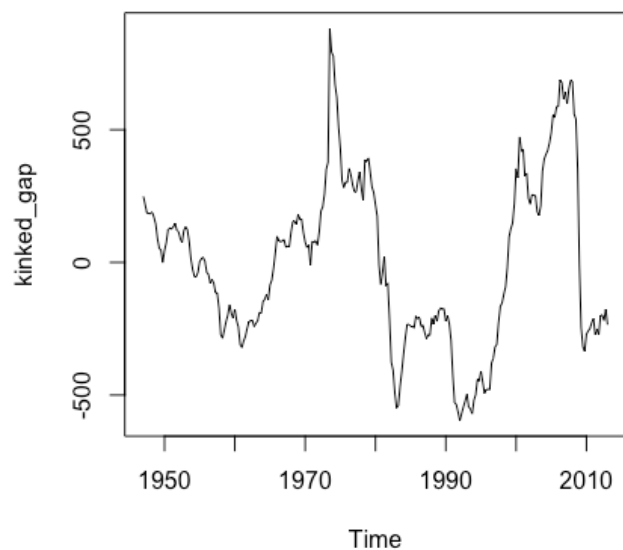
This seems meaningless, so I choose the best trend term form according to AIC.

This looks same as the previous one.

5 Problem5

5.1 (1)

$$\begin{cases} y_t = a_{10} + a_{11}y_{t-1} + a_{12}z_{t-1} + e_{1t} \\ z_t = a_{20} + a_{21}y_{t-1} + a_{22}z_{t-1} + e_{2t} \end{cases} \tag{2}$$

By (2), I get the below,

$$\begin{cases} (1 - a_{11}L)y_t = a_{10} + a_{12}z_{t-1} + e_{1t} \\ (1 - a_{22}L)z_t = a_{20} + a_{21}y_{t-1} + e_{2t} \end{cases} \Rightarrow \begin{cases} y_t = (1 - a_{11}L)^{-1}(a_{10} + a_{12}z_{t-1} + e_{1t}) \\ z_t = (1 - a_{22}L)^{-1}(a_{20} + a_{21}y_{t-1} + e_{2t}) \end{cases} \tag{3}$$

Thus I have the following AR(2) form for each time series,

$$\begin{cases} y_t = a_{10} + a_{11}y_{t-1} + a_{12}(1 - a_{22}L)^{-1}(a_{20} + a_{21}y_{t-2} + e_{2t}) + e_{1t} \\ z_t = a_{20} + a_{21}(1 - a_{11}L)^{-1}(a_{10} + a_{12}z_{t-2} + e_{1t}) + a_{22}z_{t-1} + e_{2t} \end{cases} \tag{4}$$

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5.2 (2)

Because of symmetry, I show the procedure only for y_t and just show the result w.r.t z_t. By (2), (3), I have the following representation.

$$
\begin{aligned}
y_t =&(1 - a_{11}L)^{-1}\left\{a_{10} + (1 - a_{22}L)^{-1}a_{12}a_{20}\right\} \\
&+a_{12}a_{21}(1 - a_{11}L)^{-1}(1 - a_{22}L)^{-1}y_{t-2} \\
&+a_{12}(1 - a_{11}L)^{-1}(1 - a_{22}L)^{-1}e_{2.t-1} \\
&+(1 - a_{11}L)^{-1}e_{1,t}
\end{aligned}
$$

Then I have the following explicit form. Note that λ_1, λ_2 are the inverse of the solutions of $1 - (a_{11} + a_{22})x + (a_{11}a_{22} - a_{12}a_{21})x^2$, and I assume that this equation has the real number solution, in other words $(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) = (a_{11} - aa_{22})^2 + 4a_{12}a_{21} \geq 0$.

$$
\begin{aligned}
y_t =& \left(1 - a_{12}a_{21}(1 - a_{11}L)^{-1}(1 - a_{22}L)^{-1}L^2\right)^{-1} \\
&\left[(1 - a_{11}L)^{-1}\left\{a_{10} + (1 - a_{22}L)^{-1}a_{12}a_{20}\right\} + a_{12}(1 - a_{11}L)^{-1}(1 - a_{22}L)^{-1}e_{2.t-1} + (1 - a_{11}L)^{-1}e_{1,t}\right] \\[2mm]
=& \left(1 - \frac{a_{12}a_{21}L^2}{(1 - a_{11}L)(1 - a_{22}L)}\right)^{-1} \\
&\left[(1 - a_{11}L)^{-1}\left\{a_{10} + (1 - a_{22}L)^{-1}a_{12}a_{20}\right\} + a_{12}(1 - a_{11}L)^{-1}(1 - a_{22}L)^{-1}e_{2.t-1} + (1 - a_{11}L)^{-1}e_{1,t}\right] \\[2mm]
=& \left(1 - (a_{11} + a_{22})L + (a_{11}a_{22} - a_{12}a_{21})L^2\right)^{-1}(1 - a_{11}L)(1 - a_{22}L) \\
&\left[(1 - a_{11}L)^{-1}\left\{a_{10} + (1 - a_{22}L)^{-1}a_{12}a_{20}\right\} + a_{12}(1 - a_{11}L)^{-1}(1 - a_{22}L)^{-1}e_{2.t-1} + (1 - a_{11}L)^{-1}e_{1,t}\right] \\[2mm]
=&(1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1}(1 - a_{11}L)(1 - a_{22}L) \\
&\left[(1 - a_{11}L)^{-1}\left\{a_{10} + (1 - a_{22}L)^{-1}a_{12}a_{20}\right\} + a_{12}(1 - a_{11}L)^{-1}(1 - a_{22}L)^{-1}e_{2.t-1} + (1 - a_{11}L)^{-1}e_{1,t}\right] \\[2mm]
=&(1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1}\left\{a_{10} - a_{22}a_{10} + a_{12}a_{20}\right\} \\
&+ a_{12}(1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1}e_{2,t-1} \\
&+ (1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1}(e_{1,t} - a_{22}e_{1,t-1})
\end{aligned}
$$

$$
\begin{aligned}
=& \left\{a_{10} - a_{22}a_{10} + a_{12}a_{20}\right\}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{j}\lambda_1^k\lambda_2^{j-k}\right) \\
&+ a_{12}\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\lambda_1^k\lambda_2^{j-k}\right)e_{2,t-1-j} \\
&+ \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\lambda_1^k\lambda_2^{j-k}\right)e_{1,t-j} - a_{22}\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\lambda_1^k\lambda_2^{j-k}\right)e_{1,t-1-j}
\end{aligned}
$$

The same calculation can be applied to $\{z_t\}$.

$$
\begin{aligned}
z_t =& \left\{a_{20} - a_{11}a_{20} + a_{21}a_{10}\right\}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{j}\lambda_1^k\lambda_2^{j-k}\right) \\
&+ a_{21}\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\lambda_1^k\lambda_2^{j-k}\right)e_{1,t-1-j} \\
&+ \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\lambda_1^k\lambda_2^{j-k}\right)e_{2,t-j} - a_{11}\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\lambda_1^k\lambda_2^{j-k}\right)e_{2,t-1-j}
\end{aligned}
$$

5.3   (3)

Now I have the explicit representation as follows, since in this case $\lambda_1, \lambda_2 = \frac{3}{5}, 1$.

$$y_t = 0.2 \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \frac{3}{5}^k \right) e_{2,t-1-j} + \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \frac{3}{5}^k \right) e_{1,t-j} + 0.8 \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \frac{3}{5}^k \right) e_{1,t-1-j}$$

$$= 0.5 \sum_{j=0}^{\infty} \left(1 - \frac{3}{5}^j \right) e_{2,t-1-j} + 2.5 \sum_{j=0}^{\infty} \left(1 - \frac{3}{5}^j \right) e_{1,t-j} + 2 \sum_{j=0}^{\infty} \left(1 - \frac{3}{5}^j \right) e_{1,t-1-j}$$

Now each term in the above does not exist, which means all the term diverge, so this is not a stationary process. And the standard stationarity judgement for VAR model is that all the eigenvalue of $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ are less than 1. The current parameter values give,

$$\left| \begin{pmatrix} 0.8 - \lambda & 0.2 \\ 0.2 & 0.8 - \lambda \end{pmatrix} \right| = 0 \iff \lambda = 1, 0.6$$

Then again I have non-stationarity.

5.4   (4)

$$\frac{\partial y_{t+s}}{\partial e_{1,t}} = \sum_{k=0}^{s} \frac{3}{5}^k - 0.8 \sum_{k=0}^{s-1} \frac{3}{5}^k = \frac{1}{2} + \frac{1}{2} \left(\frac{3}{5} \right)^{s-1}$$

$$\frac{\partial y_{t+s}}{\partial e_{2,t}} = 0.2 \sum_{k=0}^{s-1} \frac{3}{5}^k = \frac{1}{2} - \frac{1}{2} \left(\frac{3}{5} \right)^{s-1}$$

5.5   (5)

$$\frac{\partial y_{t+s}}{\partial \epsilon_{y,t}} = \frac{\partial e_{1,t}}{\partial \epsilon_{y,t}} \frac{\partial y_{t+s}}{\partial e_{1,t}} + \frac{\partial e_{2,t}}{\partial \epsilon_{y,t}} \frac{\partial y_{t+s}}{\partial e_{2,t}} = \frac{\partial y_{t+s}}{\partial e_{1,t}} = \frac{1}{2} + \frac{1}{2} \left(\frac{3}{5} \right)^{s-1}$$

$$\frac{\partial y_{t+s}}{\partial \epsilon_{z,t}} = \frac{\partial e_{1,t}}{\partial \epsilon_{z,t}} \frac{\partial y_{t+s}}{\partial e_{1,t}} + \frac{\partial e_{2,t}}{\partial \epsilon_{z,t}} \frac{\partial y_{t+s}}{\partial e_{2,t}} = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \left(\frac{3}{5} \right)^{s-1} \right) + \frac{1}{2} - \frac{1}{2} \left(\frac{3}{5} \right)^{s-1} = \frac{3}{4} - \frac{1}{4} \left(\frac{3}{5} \right)^{s-1}$$

5.6   (6)

$$\frac{\partial y_{t+s}}{\partial \epsilon_{y,t}} = \frac{\partial e_{1,t}}{\partial \epsilon_{y,t}} \frac{\partial y_{t+s}}{\partial e_{1,t}} + \frac{\partial e_{2,t}}{\partial \epsilon_{y,t}} \frac{\partial y_{t+s}}{\partial e_{2,t}} = \frac{\partial y_{t+s}}{\partial e_{1,t}} + \frac{1}{2} \frac{\partial y_{t+s}}{\partial e_{2,t}} = \frac{1}{2} + \frac{1}{2} \left(\frac{3}{5} \right)^{s-1} + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2} \left(\frac{3}{5} \right)^{s-1} \right) = \frac{3}{4} + \frac{1}{4} \left(\frac{3}{5} \right)^{s-1}$$

$$\frac{\partial y_{t+s}}{\partial \epsilon_{z,t}} = \frac{\partial e_{1,t}}{\partial \epsilon_{z,t}} \frac{\partial y_{t+s}}{\partial e_{1,t}} + \frac{\partial e_{2,t}}{\partial \epsilon_{z,t}} \frac{\partial y_{t+s}}{\partial e_{2,t}} = \frac{1}{2} - \frac{1}{2} \left(\frac{3}{5} \right)^{s-1}$$

5.7   (7)

By problem (5),(6), I get to know the error term composition has an impact on IRF. In problem(5), $\epsilon_{y,t}$ has no influence on $e_{2,t}$, which means that the shock to equation (1) is external to equation (6), in other words the unique shock to y_t is assumed to be exogenous to z_t. In problem (6), the reverse condition is assumed. Thus, when we analyze real data set by using VAR, we should be careful about the stacking order of all the variables so that the former variable is exogenous to all the later ones.