Problem Set #1
OSE Math: Introduction to Measure Theory
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S1 Measure Spaces

Exercise 1.3

1. Consider A $\in \mathcal{G}_1$. Then A is open and $\in \mathcal{G}_1$. However, A^c is closed. Therefore, A^c $\notin \mathcal{G}_1$. Therefore, A is not closed under complements. Therefore, $\mathcal{G}_1$ is not an algebra (and hence not a $\sigma-$algebra as well).

2. $\mathcal{G}_2$ is an algebra because it satisfies (a) - (c):

   (a) When $a = b \in \mathbb{R}$, $(a, b] = \emptyset \in \mathcal{G}_2$.

   (b) $\mathcal{G}_2$ is closed under complements. Consider $a, b \in \mathbb{R}$. Then, $(a, b]^c = (-\infty, a] \cup (b, \infty) \in \mathcal{G}_2$, $(-\infty, b]^c = (b, \infty) \in \mathcal{G}_2$, $(a, \infty)^c = (-\infty, a] \in \mathcal{G}_2$.

   (c) $\mathcal{G}_2$ is closed under finite unions because finite union of the form $(a, b], (-\infty, b]$, and (a, ∞) results in a finite union of the same form.

   However, $\mathcal{G}_2$ is not a $\sigma-$algebra because it does not contain a infinite countable union of the forms $(a, b], (-\infty, b], and(a, \infty)$.

3. We know that $\mathcal{G}_3$ is an algebra for the same proof as (ii). However, $\mathcal{G}_3$ contains countable unions of of the forms $(a, b], (-\infty, b], and(a, \infty)$ as well. Therefore, $\mathcal{G}_3$ is also a $\sigma-$algebra.

Exercise 1.7

If $\mathcal{A}$ is any σ-algebra, it contains empty set, ie $\emptyset \in \mathcal{A}$. Moreover, $\emptyset^c = X \in \mathcal{A}$. Therefore, $\{\emptyset, X\} \in \mathcal{A}$. Moreover, since $\mathcal{P}(X)$ contains every subset of X and A contains subsets of X, $\mathcal{A} \in \mathcal{P}(X)$. So, $\{\emptyset, X\} \subset \mathcal{A} \subset \mathcal{P}(X)$. Given this set structure, $\mathcal{P}(X)$ is the largest possible σ-algebra because it contains every possible combinations of the subsets of X. Moreover, $\mathcal{P}(X)$ is the smallest possible σ-algebra because it contains minimal subsets of X that ensures the definition of σ-algebra. $\square$

Exercise 1.10

1. Since $\emptyset \in S_\alpha$ for all α. Therefore, by set theory we know that $\emptyset \in \cap_\alpha S_\alpha$.

2. Consider $A \in \cap_\alpha S_\alpha$. From set theory we know that $A \in S_\alpha$ for all α. Since S_α is a $\sigma-$algebra, it is closed under complements. Therefore, $A^c \in S_\alpha$ for all α. From this we know that $A^c \in \cap_\alpha S_\alpha$. Therefore, $\cap_\alpha S_\alpha$ is closed under complements.

3. Consider $A_1, A_2, A_3 \ldots \in \cap_\alpha S_\alpha$. From set theory we know that $A_1, A_2, A_3 \ldots \in S_\alpha$ for all α. Therefore, $\cup_{n=1}^\infty A_n \in S_\alpha$ for all α. Thus, from set theory, we know that $\cup_{n=1}^\infty A_n \in \cap_\alpha S_\alpha$. Therefore, $\cap_\alpha S_\alpha$ closed under countable unions.

(1), (2), (3) proves the proposition. $\square$

Exercise 1.22

1. Will show that μ is monotone. Since $A \subset B$, we can represent $B = A \cup (A^c \cap B)$. Here, $A \cap (A^c \cap B) = \emptyset$. Therefore, $\mu(B) = \mu(A) + \mu(A^c \cap B)$. Since by Remark 1.14, we are restricted to non-negative measures, ie $\mu(A^c \cap B) \geq 0$. So we have $\mu(A) \leq \mu(B)$. $\square$.

2. Will show that μ is countably subadditive. Let, $B_1 = A_1, B_2 = A_2 \cap A_1^c, B_3 = A_3 \cap (A_1 \cup A_2)^c, \ldots, B_n = A_n \cap (\cup_{n=1}^{i-1} A_i)$. Here, $\cup_{n=1}^{\infty} A_i = \cup_{n=1}^{\infty} B_i$, and thus $\mu(\cup_{i=1}^{\infty} A_i) = \mu(\cup_{i=1}^{\infty} B_i)$. Moreover, $B_i \neq B_j$ for all $i \neq j$, and thus $\mu(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i)$. Finally, since $B_i \subset A_i$, $\cup_{n=1}^{\infty} B_i \subset \cup_{n=1}^{\infty} A_i$. Therefore, by monotonicity of μ proved in (i), we have $\sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$. $\square$

Exercise 1.23

1. $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$ because μ is a measure. Therefore, $\lambda(\emptyset) = 0$

2. Consider $\{A_i\}_{i=1}^{\infty} \subset S$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$. Then,

$$\begin{aligned}
\lambda(\cup_{i=1}^{\infty} A_i) &= \mu((\cup_{i=1}^{\infty} A_i) \cap B) \\
&= \mu(\cup_{i=1}^{\infty} (A_i \cap B)) \\
&= \sum_{i=1}^{\infty} \mu(A_i \cap B) \\
&= \sum_{i=1}^{\infty} \lambda(A_i \cap B)
\end{aligned}$$

This second to last inequality holds because $(A_i \cap B) \cap (A_i \cap B) = \emptyset$ as $A_i \cap A_j = \emptyset$ for all $i \neq j$. Moreover, the last equality holds because $(A_i \cap B) \cap B = (A_i \cap B)$.

(i) and (ii) proves that λ is also a measure. $\square$

Exercise 1.26

Let $B_1 = \emptyset, B_2 = A_1 \cap A_2^c, , B_i = A_i^c \cap A_{i-1}$. Then note that,

$$\begin{cases}
\mathrm{A}_N = A_1 - \cup_{n=1}^{N} B_n, & \forall n \geq 2 \\
\mathrm{A}_n \cap B_n = \emptyset, B_m \cap B_n = \emptyset, & \mathrm{m} \neq n \\
\cap_{n=1}^{\infty} A_n = A_1 - \cup_{n=1}^{\infty} B_n
\end{cases}$$

This means that $\cup_{n=1}^{\infty} B_n$ is a countable union of pairwise disjoint sets in S. Hence,

$$\begin{aligned}
\mu(\cap_{n=1}^{\infty} A_n) &= \mu(A_i) - \mu(\cup_{n=1}^{\infty} B_n) \\
&= \mu(A_1) - \sum_{n=1}^{\infty} \mu(B_n) \\
&= \mu(A_1) - \lim_{N \to \infty} \sum_{n=1}^{N} \mu(B_n) \\
&= \mu(A_1) - \lim_{N \to \infty} \mu(A_1 - A_n) \\
&= \mu(A_1) - \mu(A_1) + \lim_{N \to \infty} \mu(A_n) = \lim_{N \to \infty} \mu(A_n) \quad \square
\end{aligned}$$

S2 Construction of Lebesgue Measure

Exercise 2.10

From the definition of outer measure, we know that it is countably subadditive, ie $\mu^*(B) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c)$. We also are given that $\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^c)$. Therefore, the preceding theorem can be replaced by $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$. $\square$

Exercise 2.14

From (Definition 1.11), let O denote the collection of open sets of R. Then Borel σ-algebra $\sigma(O)$ is the smallest σ-algebra containing all open sets of R. From Caratheodory Extension Theorem (Theorem 2.12), we know that $\sigma(A) \subset M$. Therefore, to prove that $B(R) \subset M$, it suffices to show that $\sigma(A) = \sigma(O)$.

Consider $\mathcal{G} = \{A : A \text{ is a countable union of } (a, b], (-\infty, b], \text{ and } (a, \infty)\}$. We showed in (Exercise 1.3) that $\mathcal{G}$ is a σ-algebra, $\sigma(A)$. Therefore, we will use this family of sets $\mathcal{G}$.

We will first show that $\sigma(A) \subset \sigma(O)$. Notice the following:

$$(a, b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n})$$

$$(-\infty, b] = (\bigcup_{n=1}^{\infty} (-n, a + \frac{1}{n})) \bigcup (\bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n}))$$

$$(a, \infty) = \bigcup_{n=1}^{\infty} (a, n)$$

Therefore, every set in A can be represented as a countable union or intersection of open sets. Therefore, $A \subset \sigma(O)$. It follows that $\sigma(A) \subset \sigma(\sigma(O)) = \sigma(O)$.

Next, we will show that $\sigma(O) \subset \sigma(A)$. Notice the following:

$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}]$$

Therefore, every open set in O can be represented as a countable union of $(a, b]$. Therefore, $O \subset \sigma(A)$. It follows that $\sigma(O) \subset \sigma(\sigma(A)) = \sigma(A)$.

In conclusion, $\sigma(A) = \sigma(O)$ $\square$

S3 Measurable Functions

Exercise 3.1

Consider a countable subset of the real line, $X = \{x_1, x_2, \ldots\}$. For each subset x_i, fix $\epsilon > 0$ and consider a cover $I_i = \{x_i - \frac{\epsilon}{2^i}, x_i + \frac{\epsilon}{2^i}\}$, which has length $\frac{\epsilon}{2^{i-1}}$. Here we know that $X \subseteq \bigcup_{x_i \in X} I_i$ and thus covers the entire countable subset. Here The total length of this cover, $\bigcup_{x_i \in X} I_i$, has length:

$$\sum_{n=1}^{\infty} \frac{\epsilon}{2^{n-1}} = 2\epsilon$$

From the definition, Lebesgue (outer) measure of X is:

$$\mu^*(X) = \inf \left\{ \sum_{n=1}^{\infty} I_i : A \subset \bigcup_{x_i \in X} I_i \right\}$$

Here, cover, $\bigcup_{x_i \in X} I_i$ can be made arbitrarily small. Therefore, $\mu^*(X) = 0 \quad \square$

Exercise 3.7

We want to show equivalence between:

1. $\{x \in X : f(x) < a\} \in \mathcal{M}$
2. $\{x \in X : f(x) \leq a\} \in \mathcal{M}$
3. $\{x \in X : f(x) > a\} \in \mathcal{M}$
4. $\{x \in X : f(x) \geq a\} \in \mathcal{M}$

We know that M is a Borel σ-algebra generated by X, ie $B(X)$. We will use this property to show equivalence.

(Show 1 $\Leftrightarrow$ 4): We know that σ-algebra is closed under complements. Therefore, $\{x \in X : f(x) < a\} \in \mathcal{M}$ implies $\{x \in X : f(x) < a\}^c = \{x \in X : f(x) \geq a\} \in \mathcal{M}$. The opposite direction can be proved in a similar manner.

(Show 1 $\Leftrightarrow$ 2): We will use a limit argument. Consider sequence $a_n = a - \frac{1}{n}$ with a limit point a. Consider $A_n = \{x \in X : f(x) < a - \frac{1}{n}\}$. Since for all a $\{x \in X : f(x) < a\} \in \mathcal{M}$, for all $n \in \mathcal{N}$ $A_n \in \mathcal{M}$. Since σ-algebra is closed under countable unions, $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} \{x \in X : f(x) < a - \frac{1}{n}\} = \{x \in X : f(x) \leq a\} \in \mathcal{M}$. The opposite direction is trivial.

(Show 2 $\Leftrightarrow$ 3): We can use the same proof used to show 1 $\Leftrightarrow$ 4. $\square$

Exercise 3.10

Since f and g are continuous functions, f + g and f - g are continuous functions as well. Thus, we can consider $F(f(x), g(x)) = f(x) + g(X)$ and $F(f(x), g(x)) = f(x) * g(X)$. Thus, from property (4), f + g and f - g are measurable.

Since f and g are measurable functions, we know that for all $a \in \mathbb{R}, \{x \in X : f(x) < a\} \in \mathcal{M}$ and $\{x \in X : g(x) < a\} \in \mathcal{M}$. Here, $\{x \in X : max(f(x), g(x)) < a\} = \{x \in X : f(x) < a\} \cap \{x \in X : g(x) < a\}$. We know that intersection of measurable sets is measurable. Thus, $\{x \in X : max(f(x), g(x)) < a\} \in M$. So $max(f, g)$ is measurable.

To show that min(f, g) is measurable we can use similar argument as showing max(f,g) is measurable. Since f and g are measurable functions and from (exercise 3.7), for all $a \in \mathbb{R}, \{x \in \mathbb{X} : f(x) > a\} \in \mathcal{M}$ and $\{x \in \mathbb{X} : g(x) > a\} \in \mathcal{M}$. Here, $\{x \in \mathbb{X} : min(f(x), g(x)) > a\} = \{x \in \mathbb{X} : f(x) > a\} \cap \{x \in X : g(x) > a\}$. We know that intersection of measurable sets is measurable. Thus, $\{x \in \mathbb{X} : min(f(x), g(x)) > a\} \in M$. So $min(f, g)$ is measurable.

We know that $\{x \in \mathbb{X} : |f(x)| > a\} = \{x \in X : f(x) < -a\} \cup \{x \in \mathbb{X} : f(x) > a\}$. Here, $\{x \in \mathbb{X} : f(x) < -a\} \in M$ and $\{x \in \mathbb{X} : f(x) > a\} \in M$. Since M is a Borel σ-algebra in X, it is closed under countable unions. Therefore, $\{x \in \mathbb{X} : |f(x)| > a\} \in M$. Thus, $|f(x)|$ is measurable. □

Exercise 3.17

Fix $\epsilon > 0$. If f is bounded, there exists an M such that $|f(x)| \leq M$ for all $x \in \mathbb{X}$. Therefore, $x \in E_i^M$ for some i and all $x \in \mathbb{X}$. Note that there is an $N \geq M$ such that $\frac{1}{2^N} < \epsilon$. Then for any $n \geq N$ and all $x \in \mathbb{X}$, $|f(x) - s_n(x)| < \epsilon$. Therefore, the convergence in (1) is uniform. □

S4 Lebesgue Integration

Exercise 4.13

To prove that $f \in \mathcal{L}^1(\mu, E)$, we have to show that $\int_E f^+ d\mu < \infty$ and $\int_E f^- d\mu < \infty$. Here from defenition, $|f| = f^+ + f^-$, $0 \leq f^+$ and $0 \leq f^-$. Moreover, since $|f| < M$ on $E \in M$, we have that $0 \leq f^+ < M$ and $0 \leq f^- < M$ on $E \in M$. Thus, by proposition 4.5 and 4.6,

$$0 \leq \int_E f^+ d\mu < M\mu(E) < \infty$$

$$0 \leq \int_E f^+ d\mu < M\mu(E) < \infty$$

So from definition 4.9, $f \in \mathcal{L}^1(\mu, E) \quad \square$

Exercise 4.14

Will prove by contradiction. Assume that f is not finite almost everywhere on E. This means that there is a set in E that makes f inifniite. Therefore, it follows that,

$$\infty = \int_E f d\mu \leq \int_E |f| d\mu$$

This implies that $f \notin L^1(\mu, E)$. Therefore, it must be that f is finite almost everywhere on E $\quad \square$

Exercise 4.15

If $f \leq g$ on E, we know that $f^+ \leq g^+$ and $f^- \geq g^-$. Here,

$$\{s : 0 \leq s \leq f^+, s \text{ simple, measurable}\} \subset \{s : 0 \leq s \leq g^+, s \text{ simple, measurable}\}$$

.

$$\{s : 0 \leq s \leq g^-, s \text{ simple, measurable}\} \subset \{s : 0 \leq s \leq f^-, s \text{ simple, measurable}\}$$

Therefore, by definition of Lebesgue Integral (Definition 4.3),

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu \leq \int_E g^+ d\mu - \int_E g^- d\mu = \int_E g d\mu$$

Thus, $\int_E f d\mu \leq \int_E g d\mu \quad \square$

Exercise 4.16

From definition 4.1, $s(x) = \sum_{i=1}^N c_i \chi_{E_i}$, E_i measurable, then for any $E \in M$ we define $\int_E s d\mu = \sum_{i=1}^N c_i \mu(E \cap E_i)$. Here, $A \subset E$, so

$$\int_A s d\mu = \sum_{i=1}^N c_i \mu(A \cap E_i) \leq \sum_{i=1}^N c_i \mu(E \cap E_i) = \int_E s d\mu$$

.
Therefore,

$$\begin{aligned}\int_A f^+ d\mu &= \sup\left\{\int_A s d\mu : 0 \le s \le f^+, s \text{ simple, measurable}\right\} \\ &\le \sup\left\{\int_E s d\mu : 0 \le s \le f^+, s \text{ simple, measurable}\right\} \\ &= \int_E f^+ d\mu\end{aligned}$$

$$\begin{aligned}\int_A f^- d\mu &= \sup\left\{\int_A s d\mu : 0 \le s \le f^-, s \text{ simple, measurable}\right\} \\ &\le \sup\left\{\int_E s d\mu : 0 \le s \le f^-, s \text{ simple, measurable}\right\} \\ &= \int_E f^- d\mu\end{aligned}$$

Here, $f \in \mathcal{L}^1(\mu, E)$, we have $\int_E f^- d\mu < \infty$ and $\int_E f^+ d\mu < \infty$. Thus, it follows that $\int_A f^- d\mu < \infty$ and $\int_A f^+ d\mu < \infty$. So, $f \in \mathcal{L}^1(\mu, A)$ $\square$

Exercise 4.21

If $A, B \in \mathcal{M}, B \subset A, \mu(A - B) = 0,$ and $f \in L^1$, from (Proposition 4.6) we know that:

$$\int_{A-B} f d\mu = 0$$

For general f, observe that $f = f^+ - f^-$. Here, f^+ and f^- are non-negative. So we can apply (Theorem 4.19) so that $\mu_+(A) = \int_A f^+ d\mu$ and $\mu_-(A) = \int_A f^- d\mu$ are measure on M. Similarly, $\mu_+(B) = \int_A f^+ d\mu$ and $\mu_-(B) = \int_A f^- d\mu$ are measure on M.

We also know from (Definition 4.3),

$$\int_A f d\mu = \int_A f^+ d\mu - \int_A f^- d\mu = \mu_+(A) - \mu_-(A)$$

$$\int_B f d\mu = \int_B f^+ d\mu - \int_B f^- d\mu = \mu_+(B) - \mu_-(B)$$

Here, we will use the property of measure (Definition 1.13). Consider $A = (A - B) \cup B$, where $(A - B) \cap B = \emptyset$. Therefore, $\mu(A) = \mu(A - B) + \mu(B)$. Here, $\mu(A - B) = 0$, so $\mu(A) = \mu(B)$. In other words, $\mu_+(A) = \mu_+(B)$ and $\mu_-(A) = \mu_-(B)$. Therefore,

$$\int_A f d\mu = \mu_+(A) - \mu_-(A) = \mu_+(B) - \mu_-(B) = \int_B f d\mu \quad \square$$

Exercise 4.28

From (Theorem 4.27), if $f : E \subset \mathbb{R}$ is Riemann-integrable, then $f \in L(E, \mu)$, and $\int_E f d\mu = \int_E f dx$. Therefore, we can use the same proof as Exercise 4.14 to prove this statement. $\square$