Problem Set #2
OSE Econ
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DSGE Model

Exercise 1

The Euler equation of the Brock and Mirman model is:

$$\frac{1}{e^{z_t}K_t^{\alpha} - K_{t+1}} = \beta E_t\left\{\frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^{\alpha} - K_{t+2}}\right\}$$

We guess the policy function to take the form of: $K_{t+1} = Ae^{z_t}K_t^{\alpha}$. We replace K_{t+2} in the Euler equation with this functional form.

$$\begin{aligned}\frac{1}{\frac{1}{A}K_{t+1} - K_{t+1}} &= \beta E_t\left\{\frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^{\alpha} - Ae^{z_{t+1}}K_{t+1}^{\alpha}}\right\} \\ &\Leftrightarrow A^2 - (1+\alpha\beta)A + \alpha\beta = 0 \\ &\Leftrightarrow A = 1, \alpha\beta\end{aligned}$$

Therefore, $\boxed{\text{A} = \alpha\beta}$.

Exercise 2

Consider the following functional forms:

$$\begin{aligned}u(c_t, l_t) &= \ln c_t + a\ln(1-l_t) \\ F(K_t, L_t, z_t) &= e^{z_t}K_t^{\alpha}L_t^{1-\alpha}\end{aligned}$$

Then, the seven characterizing equations in seven unknowns, $\{c_t, k_t, l_t, w_t, r_t, T_t, z_t\}$ for the model are the following:

$$c_t = (1-\tau)[w_t l_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \tag{1}$$

$$\frac{1}{c_t} = \beta E_t\left[\frac{1}{c_{t+1}}[(r_{t+1}-\delta)(1-\tau)+1]\right] \tag{2}$$

$$\frac{a}{1-l_t} = \frac{1}{c_t}w_t(1-\tau) \tag{3}$$

$$r_t = \alpha e^{z_t}k_t^{\alpha-1}l_t^{1-\alpha} \tag{4}$$

$$w_t = (1-\alpha)e^{z_t}k_t^{\alpha}l_t^{-\alpha} \tag{5}$$

$$T_t = \tau[w_t l_t + (r_t-\delta)k_t] \tag{6}$$

$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2) \tag{7}$$

We cannot use the same tricks as in Exercise 1 to solve for the policy function because we realize that equations don't cancel out nicely.

Exercise 3

Consider the following functional forms:

$$
\begin{aligned}
u(c_t, l_t) &= \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\ln(1 - l_t) \\
F_(K_t, L_t, z_t) &= e^{z_t} K_t^{\alpha} L_t^{1-\alpha}
\end{aligned}
$$

Then, the seven characterizing equations in seven unknowns, $\{c_t, k_t, l_t, w_t, r_t, T_t, z_t\}$ for the model are the following:

$$
\begin{aligned}
c_t &= (1-\tau)[w_t l_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} && (1) \\
c_t^{-\gamma} &= \beta E_t \left[c_{t+1}^{-\gamma}[(r_{t+1} - \delta)(1-\tau) + 1]\right] && (2) \\
\frac{a}{1 - l_t} &= c_t^{-\gamma} w_t (1-\tau) && (3) \\
r_t &= \alpha e^{z_t} k_t^{\alpha - 1} l_t^{1-\alpha} && (4) \\
w_t &= (1-\alpha) e^{z_t} k_t^{\alpha} l_t^{-\alpha} && (5) \\
T_t &= \tau[w_t l_t + (r_t - \delta)k_t] && (6) \\
z_t &= (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2) && (7)
\end{aligned}
$$

Exercise 4

Consider the following functional forms:

$$
\begin{aligned}
u(c_t, l_t) &= \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\frac{(1 - l_t)^{1-\xi} - 1}{1-\xi} \\
F(K_t, L_t, z_t) &= e^{z_t} [\alpha K_t^{\eta} + (1-\alpha) L_t^{\eta}]^{\frac{1}{\eta}}
\end{aligned}
$$

Then, the seven characterizing equations in seven unknowns, $\{c_t, k_t, l_t, w_t, r_t, T_t, z_t\}$ for the model are the following:

$$
\begin{aligned}
c_t &= (1-\tau)[w_t l_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} && (1) \\
c_t^{-\gamma} &= \beta E_t \left[c_{t+1}^{-\gamma}[(r_{t+1} - \delta)(1-\tau) + 1]\right] && (2) \\
\frac{a}{(1 - l_t)^{\xi}} &= c_t^{-\gamma} w_t (1-\tau) && (3) \\
r_t &= \alpha e^{z_t} k_t^{\eta - 1} [\alpha k_t^{\eta} + (1-\alpha) l_t^{\eta}]^{\frac{1-\eta}{\eta}} && (4) \\
w_t &= (1-\alpha) e^{z_t} l_t^{\eta - 1} [\alpha k_t^{\eta} + (1-\alpha) l_t^{\eta}]^{\frac{1-\eta}{\eta}} && (5) \\
T_t &= \tau[w_t l_t + (r_t - \delta)k_t] && (6) \\
z_t &= (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2) && (7)
\end{aligned}
$$

Exercise 5

Consider the following functional forms:

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$
$$F(K_t, L_t, z_t) = K_t^{\alpha}(L_t e_t^{z_t})^{1-\alpha}$$

We assume $l_t = 1$. Then, by the labor market clearance condition, $L_t = l_t = 1$. The six characterizing equations in six unknowns, $\{c_t, k_t, l_t, w_t, r_t, T_t, z_t\}$ for the model are the following:

$$c_t = (1-\tau)[w_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \tag{1}$$
$$c_t^{-\gamma} = \beta E_t\left[c_{t+1}^{-\gamma}[(r_{t+1} - \delta)(1-\tau) + 1]\right] \tag{2}$$
$$r_t = \alpha k_t^{\alpha-1}(e_t^{z_t})^{1-\alpha} \tag{3}$$
$$w_t = (1-\alpha)k_t^{\alpha}(e_t^{z_t})^{1-\alpha} \tag{4}$$
$$T_t = \tau[w_t + (r_t - \delta)k_t] \tag{5}$$
$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d(0, \sigma_z^2) \tag{6}$$

The steady state version of these equations are the following:

$$\bar{c} = (1-\tau)[\bar{w} + (\bar{r} - \delta)\bar{k}] + \bar{k} + \bar{T} - \bar{k} \tag{1}$$
$$\bar{T} = \tau[\bar{w} + (\bar{r} - \delta)\bar{k}] \tag{2}$$
$$\bar{c}^{-\gamma} = \beta E_t\left[\bar{c}^{-\gamma}[(\bar{r} - \delta)(1-\tau) + 1]\right] \tag{3}$$
$$\bar{r} = \alpha\bar{k}^{\alpha-1}(e^{\bar{z}})^{1-\alpha} \tag{4}$$
$$\bar{w} = (1-\alpha)\bar{k}^{\alpha}(e^{\bar{z}})^{1-\alpha} \tag{5}$$

The analytic solution to these steady state equations are the following (we know that $\bar{z} = 0$):

$$\bar{r} = \frac{1-\beta}{\beta(1-\tau)} + \delta$$
$$\bar{k} = \left(\frac{\bar{r}}{\alpha}\right)^{\frac{1}{\alpha-1}}$$
$$\bar{w} = (1-\alpha)\bar{k}^{\alpha}$$
$$\bar{c} = (1-\tau)[\bar{w} + (\bar{r} - \delta)\bar{k}]\bar{T}$$
$$\bar{T} = \tau[\bar{w} + (\bar{r} - \delta)\bar{k}]$$

The comparison of the steady state values of all variables are done in the jupyter notebook in the same directory.

Exercise 6

Consider the following functional forms:

$$u(c_t, l_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\frac{(1-l_t)^{1-\xi} - 1}{1-\xi}$$
$$F(K_t, L_t, z_t) = K_t^{\alpha}(L_t e^{z_t})^{1-\alpha}$$

The seven characterizing equations in seven unknowns, $\{c_t, k_t, l_t, w_t, r_t, T_t, z_t\}$ for the model are the following:

$$c_t = (1-\tau)[w_t l_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \tag{1}$$
$$c_t^{-\gamma} = \beta E_t\left[c_{t+1}^{-\gamma}[(r_{t+1} - \delta)(1-\tau) + 1]\right] \tag{2}$$
$$\frac{a}{(1-l_t)^{\xi}} = c_t^{-\gamma} w_t (1-\tau) \tag{3}$$
$$r_t = \alpha\left(\frac{l_t e^{z_t}}{k_t}\right)^{1-\alpha} \tag{4}$$
$$w_t = (1-\alpha)e^{z_t}\left(\frac{k_t}{l_t e^{z_t}}\right)^{\alpha} \tag{5}$$
$$T_t = \tau[w_t l_t + (r_t - \delta)k_t] \tag{6}$$
$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2) \tag{7}$$

The analytic solution to these stead state equations are the following (we know that $\bar{z} = 0$):

$$\bar{c} = (1-\tau)[\bar{w}\bar{l} + (\bar{r} - \delta)\bar{k}] + \bar{k} + \bar{T} - \bar{k} \tag{1}$$
$$\bar{c}^{-\gamma} = \beta E_t\left[\bar{c}^{-\gamma}[(\bar{r} - \delta)(1-\tau) + 1]\right] \tag{2}$$
$$\frac{a}{(1-\bar{l})^{\xi}} = \bar{c}^{-\gamma}\bar{w}(1-\tau) \tag{3}$$
$$\bar{r} = \alpha\left(\frac{\bar{l}e^{\bar{z}}}{\bar{k}}\right)^{1-\alpha} \tag{4}$$
$$\bar{w} = (1-\alpha)e^{\bar{z}}\left(\frac{\bar{k}}{\bar{l}e^{\bar{z}}}\right)^{\alpha} \tag{5}$$
$$\bar{T} = \tau[\bar{w}\bar{l} + (\bar{r} - \delta)\bar{k}] \tag{6}$$

Numerical steady state values are computed in the jupyter notebook in the same directory.

Exercise 7

Please refer to jupyter notebook in the same directory.

Linearization Methods

Exercise 3

We can represent the characterizing equations for the model as a vector of functions in the following form:

$$\Gamma = E_t\left[F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t\right] = 0$$

With this notation, we can write, $\tilde{Z}_t = N\tilde{Z}_{t-1} + \epsilon_t$. Moreover, we hypothesize the transition function $X_t = f(X_{t-1}, Z_t)$ can also be linearizerly approximated as:

$$\tilde{X}_t = P\tilde{X}_{t-1} + Q\tilde{Z}_t$$

We substitute these to the Γ equation and using the property that $E_t\left[\epsilon_t\right] = 0$:

$$\begin{aligned}
&E_t\left[F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t\right] \\
&= E_t\left[F(P\tilde{X}_t + Q\tilde{Z}_{t+1}) + G(P\tilde{X}_{t-1} + Q\tilde{Z}_t) + H\tilde{X}_{t-1} + L(N\tilde{Z}_t + \epsilon_{t+1}) + M\tilde{Z}_t\right] \\
&= E_t\Big[F(P(P\tilde{X}_{t-1} + Q\tilde{Z}_t) + Q(N\tilde{Z}_t + \epsilon_{t+1})) + G(P\tilde{X}_{t-1} + Q\tilde{Z}_t) \\
&\qquad + H\tilde{X}_{t-1} + L(N\tilde{Z}_t + \epsilon_{t+1}) + M\tilde{Z}_t\Big] \\
&= [(FP+G)P + H]\overline{X}_{t-1} + [(FQ+L)N + (FP+G)Q + M]\overline{Z}_t \quad \square
\end{aligned}$$

Perturbation Methods

Exercise 1

The solution of the cubic term $x_{uuu}(u_0)$ is the following:

$$x_{uuu} = -\frac{F_{xxx}x_u^3 + 3\left(F_{xxu}x_u^2 + F_{uux}x_u + F_{xu}x_{uu} + F_{xx}x_u x_{uu}\right) + F_{uau}}{F_x}$$