Problem Set #2
OSE Econ
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DSGE Model

Exercise 1

The Euler equation of the Brock and Mirman model is:

$$\frac{1}{e^{z_t}K_t^\alpha - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^\alpha - K_{t+2}} \right\}$$

We guess the policy function to take the form of: $K_{t+1} = Ae^{z_t}K_t^\alpha$. We replace K_{t+2} in the Euler equation with this functional form.

$$\frac{1}{\frac{1}{A}K_{t+1} - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^\alpha - Ae^{z_{t+1}}K_{t+1}^\alpha} \right\}$$
$$\Leftrightarrow A^2 - (1 + \alpha\beta)A + \alpha\beta = 0$$
$$\Leftrightarrow A = 1, \alpha\beta$$

Therefore, $\boxed{A = \alpha\beta}$.

Exercise 2

Consider the following functional forms:

$$u(c_t, l_t) = \ln c_t + a \ln(1 - l_t)$$
$$F(K_t, L_t, z_t) = e^{z_t}K_t^\alpha L_t^{1-\alpha}$$

Then, the seven characterizing equations in seven unknowns, $\{c_t, k_t, l_t, w_t, r_t, T_t, z_t\}$ for the model are the following:

$$c_t = (1 - \tau)[w_t l_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \tag{1}$$

$$\frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}}[(r_{t+1} - \delta)(1 - \tau) + 1] \right] \tag{2}$$

$$\frac{a}{1 - l_t} = \frac{1}{c_t}w_t(1 - \tau) \tag{3}$$

$$r_t = \alpha e^{z_t}k_t^{\alpha-1}l_t^{1-\alpha} \tag{4}$$

$$w_t = (1 - \alpha)e^{z_t}k_t^\alpha l_t^{-\alpha} \tag{5}$$

$$T_t = \tau[w_t l_t + (r_t - \delta)k_t] \tag{6}$$

$$z_t = (1 - \rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2) \tag{7}$$

We cannot use the same tricks as in Exercise 1 to solve for the policy function because we realize that equations don't cancel out nicely.

Exercise 3

Consider the following functional forms:

$$u(c_t, l_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + a \ln(1 - l_t)$$

$$F(K_t, L_t, z_t) = e^{z_t} K_t^\alpha L_t^{1-\alpha}$$

Then, the seven characterizing equations in seven unknowns, $\{c_t, k_t, l_t, w_t, r_t, T_t, z_t\}$ for the model are the following:

$$c_t = (1 - \tau)[w_t l_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \tag{1}$$

$$c_t^{-\gamma} = \beta E_t \left[c_{t+1}^{-\gamma}[(r_{t+1} - \delta)(1 - \tau) + 1] \right] \tag{2}$$

$$\frac{a}{1 - l_t} = c_t^{-\gamma} w_t (1 - \tau) \tag{3}$$

$$r_t = \alpha e^{z_t} k_t^{\alpha-1} l_t^{1-\alpha} \tag{4}$$

$$w_t = (1 - \alpha) e^{z_t} k_t^\alpha l_t^{-\alpha} \tag{5}$$

$$T_t = \tau[w_t l_t + (r_t - \delta)k_t] \tag{6}$$

$$z_t = (1 - \rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2) \tag{7}$$

Exercise 4

Consider the following functional forms:

$$u(c_t, l_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + a \frac{(1 - l_t)^{1-\xi} - 1}{1 - \xi}$$

$$F(K_t, L_t, z_t) = e^{z_t}[\alpha K_t^\eta + (1 - \alpha)L_t^\eta]^{\frac{1}{\eta}}$$

Then, the seven characterizing equations in seven unknowns, $\{c_t, k_t, l_t, w_t, r_t, T_t, z_t\}$ for the model are the following:

$$c_t = (1 - \tau)[w_t l_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \tag{1}$$

$$c_t^{-\gamma} = \beta E_t \left[c_{t+1}^{-\gamma}[(r_{t+1} - \delta)(1 - \tau) + 1] \right] \tag{2}$$

$$\frac{a}{(1 - l_t)^\xi} = c_t^{-\gamma} w_t (1 - \tau) \tag{3}$$

$$r_t = \alpha e^{z_t} k_t^{\eta-1}[\alpha k_t^\eta + (1 - \alpha)l_t^\eta]^{\frac{1-\eta}{\eta}} \tag{4}$$

$$w_t = (1 - \alpha) e^{z_t} l_t^{\eta-1}[\alpha k_t^\eta + (1 - \alpha)l_t^\eta]^{\frac{1-\eta}{\eta}} \tag{5}$$

$$T_t = \tau[w_t l_t + (r_t - \delta)k_t] \tag{6}$$

$$z_t = (1 - \rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2) \tag{7}$$

Exercise 5

Consider the following functional forms:

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma}$$

$$F(K_t, L_t, z_t) = K_t^\alpha (L_t e^{z_t})^{1-\alpha}$$

We assume $l_t = 1$. Then, by the labor market clearance condition, $L_t = l_t = 1$. The six characterizing equations in six unknowns, $\{c_t, k_t, l_t, w_t, r_t, T_t, z_t\}$ for the model are the following:

$$c_t = (1 - \tau)[w_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \tag{1}$$

$$c_t^{-\gamma} = \beta E_t \left[c_{t+1}^{-\gamma}[(r_{t+1} - \delta)(1 - \tau) + 1] \right] \tag{2}$$

$$r_t = \alpha k_t^{\alpha-1}(e_t^{z_t})^{1-\alpha} \tag{3}$$

$$w_t = (1 - \alpha)k_t^\alpha (e_t^{z_t})^{1-\alpha} \tag{4}$$

$$T_t = \tau[w_t + (r_t - \delta)k_t] \tag{5}$$

$$z_t = (1 - \rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d(0, \sigma_z^2) \tag{6}$$

The steady state version of these equations are the following:

$$\bar{c} = (1 - \tau)[\bar{w} + (\bar{r} - \delta)\bar{k}] + \bar{k} + \bar{T} - \bar{k} \tag{1}$$

$$\bar{T} = \tau[\bar{w} + (\bar{r} - \delta)\bar{k}] \tag{2}$$

$$\bar{c}^{-\gamma} = \beta E_t \left[\bar{c}^{-\gamma}[(\bar{r} - \delta)(1 - \tau) + 1] \right] \tag{3}$$

$$\bar{r} = \alpha \bar{k}^{\alpha-1}(e^{\bar{z}})^{1-\alpha} \tag{4}$$

$$\bar{w} = (1 - \alpha)\bar{k}^\alpha (e^{\bar{z}})^{1-\alpha} \tag{5}$$

The analytic solution to these steady state equations are the following (we know that $\bar{z} = 0$):

$$\bar{r} = \frac{1 - \beta}{\beta(1 - \tau)} + \delta$$

$$\bar{k} = \left(\frac{\bar{r}}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

$$\bar{w} = (1 - \alpha)\bar{k}^\alpha$$

$$\bar{c} = (1 - \tau)[\bar{w} + (\bar{r} - \delta)\bar{k}]\bar{T}$$

$$\bar{T} = \tau[\bar{w} + (\bar{r} - \delta)\bar{k}]$$

The comparison of the steady state values of all variables are done in the jupyter notebook in the same directory.

Exercise 6

Consider the following functional forms:

$$u(c_t, l_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + a\frac{(1 - l_t)^{1-\xi} - 1}{1 - \xi}$$

$$F(K_t, L_t, z_t) = K_t^{\alpha}(L_t e^{z_t})^{1-\alpha}$$

The seven characterizing equations in seven unknowns, $\{c_t, k_t, l_t, w_t, r_t, T_t, z_t\}$ for the model are the following:

$$c_t = (1 - \tau)[w_t l_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \tag{1}$$

$$c_t^{-\gamma} = \beta E_t\left[c_{t+1}^{-\gamma}[(r_{t+1} - \delta)(1 - \tau) + 1]\right] \tag{2}$$

$$\frac{a}{(1 - l_t)^{\xi}} = c_t^{-\gamma}w_t(1 - \tau) \tag{3}$$

$$r_t = \alpha\left(\frac{l_t e^{z_t}}{k_t}\right)^{1-\alpha} \tag{4}$$

$$w_t = (1 - \alpha)e^{z_t}\left(\frac{k_t}{l_t e^{z_t}}\right)^{\alpha} \tag{5}$$

$$T_t = \tau[w_t l_t + (r_t - \delta)k_t] \tag{6}$$

$$z_t = (1 - \rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2) \tag{7}$$

The analytic solution to these stead state equations are the following (we know that $\bar{z} = 0$):

$$\bar{c} = (1 - \tau)[\bar{w}\bar{l} + (\bar{r} - \delta)\bar{k}] + \bar{k} + \bar{T} - \bar{k} \tag{1}$$

$$\bar{c}^{-\gamma} = \beta E_t\left[\bar{c}^{-\gamma}[(\bar{r} - \delta)(1 - \tau) + 1]\right] \tag{2}$$

$$\frac{a}{(1 - \bar{l})^{\xi}} = \bar{c}^{-\gamma}\bar{w}(1 - \tau) \tag{3}$$

$$\bar{r} = \alpha\left(\frac{\bar{l}e^{\bar{z}}}{\bar{k}}\right)^{1-\alpha} \tag{4}$$

$$\bar{w} = (1 - \alpha)e^{\bar{z}}\left(\frac{\bar{k}}{\bar{l}e^{\bar{z}}}\right)^{\alpha} \tag{5}$$

$$\bar{T} = \tau[\bar{w}\bar{l} + (\bar{r} - \delta)\bar{k}] \tag{6}$$

Numerical steady state values are computed in the jupyter notebook in the same directory.

Exercise 7

Please refer to jupyter notebook in the same directory.

Linearization Methods

Exercise 3

We can represent the characterizing equations for the model as a vector of functions in the following form:

$$\Gamma = E_t \left[F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t \right] = 0$$

With this notation, we can write, $\tilde{Z}_t = N\tilde{Z}_{t-1} + \epsilon_t$. Moreover, we hypothesize the transition function $X_t = f(X_{t-1}, Z_t)$ can also be linearizerly approximated as:

$$\tilde{X}_t = P\tilde{X}_{t-1} + Q\tilde{Z}_t$$

We substitute these to the Γ equation and using the property that $E_t\left[\epsilon_t\right] = 0$:

$$
\begin{aligned}
& E_t \left[F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t \right] \\
&= E_t \left[F(P\tilde{X}_t + Q\tilde{Z}_{t+1}) + G(P\tilde{X}_{t-1} + Q\tilde{Z}_t) + H\tilde{X}_{t-1} + L(N\tilde{Z}_t + \epsilon_{t+1}) + M\tilde{Z}_t \right] \\
&= E_t \left[F(P(P\tilde{X}_{t-1} + Q\tilde{Z}_t) + Q(N\tilde{Z}_t + \epsilon_{t+1})) + G(P\tilde{X}_{t-1} + Q\tilde{Z}_t) \right. \\
&\qquad \left. + H\tilde{X}_{t-1} + L(N\tilde{Z}_t + \epsilon_{t+1}) + M\tilde{Z}_t \right] \\
&= [(FP + G)P + H]\overline{X}_{t-1} + [(FQ + L)N + (FP + G)Q + M]\overline{Z}_t \quad \square
\end{aligned}
$$

Perturbation Methods

Exercise 1

The solution of the cubic term $x_{uuu}(u_0)$ is the following:

$$x_{uuu} = -\frac{F_{xxx}x_u^3 + 3\left(F_{xxu}x_u^2 + F_{uux}x_u + F_{xx}x_{uu} + F_{xx}x_u x_{uu}\right) + F_{uau}}{F_x}$$