Problem Set #4

MACS 40200, Dr. Evans
Due Monday, Feb. 18 at 1:30pm

1. **Estimating the Brock and Mirman (1972) model by SMM (10 points).** You can observe time series data in an economy for the following variables: $(c_t, k_t, w_t, r_t, y_t)$. The Data can be loaded from the file `NewMacroSeries.txt` in the `PS4/data` folder. This file is a comma separated text file with no labels. The variables are ordered as $(c_t, k_t, w_t, r_t, y_t)$. These data have 100 periods, which are quarterly (25 years). Suppose you think that the data are generated by a process similar to the Brock and Mirman (1972). A simplified set of characterizing equations of the Brock and Mirman model are the following.

$$(c_t)^{-1} - \beta E\left[r_{t+1}(c_{t+1})^{-1}\right] = 0 \tag{1}$$

$$c_t + k_{t+1} - w_t - r_t k_t = 0 \tag{2}$$

$$w_t - (1-\alpha)e^{z_t}\left(k_t\right)^{\alpha} = 0 \tag{3}$$

$$r_t - \alpha e^{z_t}\left(k_t\right)^{\alpha-1} = 0 \tag{4}$$

$$z_t = \rho z_{t-1} + (1-\rho)\mu + \varepsilon_t \quad \text{where} \quad E\left[\varepsilon_t\right] = 0 \tag{5}$$

$$y_t = e^{z_t}(k_t)^{\alpha} \tag{6}$$

The variable c_t is aggregate consumption in period t, k_{t+1} is total household savings and investment in period t for which they receive a return in the next period (this model assumes full depreciation of capital). The wage per unit of labor in period t is w_t and the interest rate or rate of return on investment is r_t. Total factor productivity is z_t, which follows an AR(1) process given in (5). y_t is GDP. The rest of the symbols in the equations are parameters that must be estimated $(\alpha, \beta, \rho, \mu, \sigma)$. The constraints on these parameters are the following.

$$\alpha, \beta \in (0,1), \quad \mu, \sigma > 0, \quad \rho \in (-1,1)$$

Assume that the first observation in the data file variables is $t = 1$. Let k_1 be the first observation in the data file for the variable k_t. One nice property of the Brock and Mirman (1972) model is that the household decision has a known analytical solution in which the optimal savings decision k_{t+1} is a function of the productivity shock today z_t and the amount of capital today k_t.

$$k_{t+1} = \alpha\beta e^{z_t} k_t^{\alpha} \tag{7}$$

With this solution and equations (2) through (5), it is straightforward to simulate the data of the Brock and Mirman (1972) model given parameters $(\alpha, \beta, \rho, \mu, \sigma)$. First, assume that $z_0 = \mu$ and that $k_1 = \text{mean}(k_t)$ from the data. These are initial values that will not change across simulations. Also assume that $\beta = 0.99$. Next, draw a matrix of $S = 1{,}000$ simulations (columns) of $T = 100$ (rows) from a uniform $u_{s,t} \sim U(0,1)$ distribution. These draws will not change across this SMM estimation procedure.

- For each guess of the parameter vector $(\alpha, \beta = 0.99, \rho, \mu, \sigma)$, you can use $u_{s,t}$ to generate normally distributed errors $\varepsilon_{s,t} \sim N(0, \sigma)$ using the inverse cdf of the normal distribution, where s is the index of the simulation number (columns).
- With $\varepsilon_{s,t}$, ρ, μ, and $z_0 = \mu$ you can use (5) to generate the simulations for $z_{s,t}$.
- With α, β, $z_{s,t}$, and k_1, you can use (7) to generate simulated values for $k_{s,t+1}$.
- With α, $z_{s,t}$ and $k_{s,t}$, you can use (3) and (4) to generate simulated values for $w_{s,t}$ and $r_{s,t}$, respectively.
- With $w_{s,t}$, $r_{s,t}$, and $k_{s,t}$, you can use (2) to generate simulated values for $c_{s,t}$.
- With α, $z_{s,t}$, and $k_{s,t}$, you can use (6) to generate simulated values for $y_{s,t}$.

(a) Estimate four parameters $(\alpha, \rho, \mu, \sigma)$ given $\beta = 0.99$ of the Brock and Mirman (1972) model described by equations (1) through (6) by SMM. Choose the four parameters to match the following six moments from the 100 periods of empirical data $\{c_t, k_t, w_t, r_t, y_t\}_{t=1}^{100}$ in `NewMacroSeries.txt`: mean(c_t), mean(k_t), mean(c_t/y_t), var(y_t), corr(c_t, c_{t-1}), corr(c_t, k_t). In your simulations of the model, set $T = 100$ and $S = 1,000$. Input the bounds to be $\alpha \in [0.01, 0.99]$, $\rho \in [-0.99, 0.99]$, $\mu \in [5, 14]$, and $\sigma \in [0.01, 1.1]$. Also, use the identity matrix as your weighting matrix $\boldsymbol{W}$. Report your solution $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\rho}, \hat{\mu}, \hat{\sigma})$, the vector of moment differences at the optimum, and the criterion function value. Also report your standard errors for the estimated parameter vector $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\rho}, \hat{\mu}, \hat{\sigma})$ based on the identity matrix for the optimal weighting matrix.

(b) Perform the estimation using the two-step estimator for the optimal weighting matrix $\boldsymbol{W}_{2step}$. Report your solution $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\rho}, \hat{\mu}, \hat{\sigma})$, the vector of moment differences at the optimum, and the criterion function value. Also report your standard errors for the estimated parameter vector $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\rho}, \hat{\mu}, \hat{\sigma})$ based on the two-step optimal weighting matrix $\boldsymbol{W}_{2step}$.

References

Brock, William A. and Leonard J. Mirman, "Optimal economic growth and uncertainty: The discounted case," *Journal of Economic Theory*, June 1972, *4* (3), 479–513.