# STAT 521: Assignment 7

**Make sure to show your computation and/or attach appropriate output.**

**Problem 1**

A researcher wishes to compare birth weights of infants among four groups of their mothers' smoking status: non-smokers, ex-smokers, $< 1/2$ pack/day, $\geq 1/2$ pack/day. Data is given as follows (See Assignment7.sas/R/py for computer analysis):

| | Non-smoker | Ex-smoker | $< \frac{1}{2}$ pack/d | $\geq \frac{1}{2}$ pack/d | |
|---|---|---|---|---|---|
| | 8.56 | 7.39 | 5.97 | 7.03 | |
| | 8.47 | 8.64 | 6.77 | 5.24 | |
| | 6.39 | 8.54 | 7.26 | 6.14 | |
| | 9.26 | 5.37 | 5.74 | 6.74 | |
| | 7.98 | 9.21 | 8.74 | 6.62 | |
| | 6.84 | 6.63 | 6.30 | 7.37 | |
| | 7.87 | | 5.52 | 4.94 | |
| | | | | 6.34 | |
| $\sum y$ | 55.37 | 45.78 | 46.30 | 50.42 | $\sum\sum y = 197.87$ |
| $\sum y^2$ | 444.00 | 359.81 | 313.68 | 322.75 | $\sum\sum y^2 = 1440.23$ |
| $\bar{y}$ | 7.91 | 7.63 | 6.61 | 6.30 | $\bar{y}_{..} = 7.07$ |
| $n$ | 7 | 6 | 7 | 8 | 28 |

a) State the null and alternative hypotheses.

b) Manually calculate sum of squares and complete the ANOVA table below.

| ***Source*** | ***df*** | ***SS*** | ***MS*** | ***F*** |
|---|---|---|---|---|
| **Treatment** | | | | |
| **Error** | | | | |
| **Total** | | | | |

c) Test the hypothesis. Use $\alpha = 0.05$. **What is your conclusion?**

d) What are the assumptions that you are making to conduct the above test?

e) Use SAS, R or Python to verify your results in part (b) and (c). Run Tukey's HSD test for multiple comparisons. Did you find any groups that were significantly different at $\alpha = 0.05$? For each of significant pairs (if any), report a 95% confidence interval of the mean difference. **Attach relevant output** and **report your findings**.

**Problem 2**

A researcher conducted a one-way ANOVA to test the null hypothesis $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$. The researcher had a balanced design with total number of subjects $N = 20$. Complete the ANOVA table below.

| ***Source*** | ***df*** | ***SS*** | ***MS*** | ***F*** |
|---|---|---|---|---|
| **Treatment** | | 6750 | | |
| **Error** | | 8000 | | |
| **Total** | | | | |

**Problem 3 (Biostats students only)**

*Uniform distribution*:

One of the simplest types of continuous distributions is the *uniform distribution*. The probability density function (PDF) of a uniform random variable $X$ is defined as:

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{elsewhere} \end{cases}$$

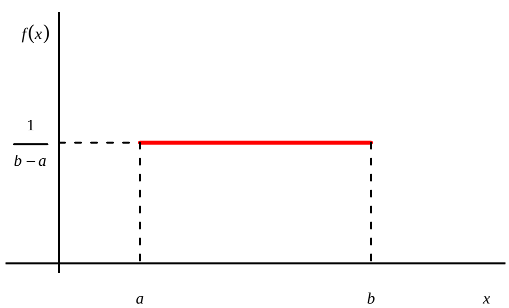


Figure 1: The probability density fucntion of $Uniform(a, b)$

A random variable $X$ can take any value between $a$ and $b$, and the density is constant over this range. We write this as:

$$X \sim Uniform(a, b)$$

A special case of the uniform distribution is when $a = 0$ and $b = 1$: $X \sim Uniform(0, 1)$. This is called the *standard uniform distribution*. The standard uniform distribution has a PDF of $f(x) = 1$ for $0 < x < 1$ and $f(x) = 0$ otherwise.

**Question**: Use the definition of the expected value $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$ (see Assignment #3) and show that $E(X) = \frac{a+b}{2}$ and $var(X) = \frac{(b-a)^2}{12}$ if $X \sim Uniform(a, b)$.

**Problem 4 (Biostats students only)**

*Monte Carlo integration using R*

Suppose we wish to solve a definite integral $I = \int_a^b g(x)\,dx$, $a < b$. Note that $g(x)$ is any one-dimensional function, not necessarily a PDF. Sometimes $g(x)$ is very complex and there may not be a closed-form solution. We can rewrite the integral as:

$$I = \int_a^b g(x)\,dx = I = \int_a^b \frac{g(x)}{f(x)} f(x)\,dx$$

If we can find a random variable that has a well-known PDF of $f(x)$ whose support is $a < x < b$, then by the definition of the expected value, the integral is equivalent to:

$$I = \int_a^b \frac{g(x)}{f(x)} f(x)\,dx = E\left[\frac{g(x)}{f(x)}\right]$$

in which a random variable $X$ has a PDF $f(x) \geq 0$ for $a < x < b$ and $f(x) = 0$ otherwise. An obvious choice of $X$ is a uniform random variable $X \sim Uniform(a, b)$ with $f(x) = \frac{1}{b-a}$ for $a < x < b$. Then it follows:

$$I = E\left[\frac{g(x)}{f(x)}\right] = E\left[\frac{g(x)}{1/(b-a)}\right] = (b-a)E\left[g(x)\right]$$

What does this imply? Remember, from Assignment #6, you can approximate $E\left[g(x)\right]$ as long as you can generate a large number of random values from $X \sim Uniform(a, b)$. If we take the average of all the random values of $g(x)$ and multiply by $(b-a)$, we can approximate the definite integral $I = \int_a^b g(x)\,dx$.

**Example**: Suppose we want to solve:

$$I = \int_0^2 \frac{2e^{-2x}}{(1+e^{-2x})^2}\,dx$$

Let $g(x) = \frac{2e^{-2x}}{(1+e^{-2x})^2}$ and $X$ be a uniform random variable $X \sim Uniform(0, 2)$ whose PDF is $f(x) = \frac{1}{2}$ for $0 < x < 2$. Then we can approximate the integral $I$ as $2E\left[g(X)\right]$.

Now let's do this with R. First I'm going to produce one million random numbers from $X \sim Uniform(0, 2)$. This can be done using `runif()` function. Its syntax is `runif(n, a, b)` where `n` is the number of random values you want to generate, and `a` and `b` are the range of $x$. In the code below, I assign values $a = 0$ and $b = 2$.

```
# Generate 1 million random number from X ~ Unif(0, 2)
a <- 0
b <- 2
x <- runif(10 ^ 6, a, b)
head(x)
```

```
## [1] 0.5751550 1.5766103 0.8179538 1.7660348 1.8809346 0.0911130
```

Then for each value of $x$, I calculate $g(x)$. For example, for the first uniform random value of 0.575155, we have $g(0.575155) = \frac{2e^{-2(0.575155)}}{\left(1+e^{-2(0.575155)}\right)^2} \approx 0.365$. For $e^x$, use `exp()` function.

```
# Calculate g(x)
g <- 2 * exp(-2 * x) / (1 + exp(-2 * x)) ^ 2
head(g)
```

```
## [1] 0.36524937 0.07857294 0.27289256 0.05521215 0.04439313 0.49587208
```

Now, to approximate $I = (b-a)E\left[g(x)\right]$, we calculate the mean of `g` and then multiply by $b - a = 2$.

```
# Take the average and multiply by (b - a)
(b - a) * mean(g)
```

```
## [1] 0.4825734
```

In this case, I got $I \approx 0.4825734$. This is close to the correct answer of $I = \int_0^2 \frac{2e^{-2x}}{(1+e^{-2x})^2}\, dx = 0.4820138$.

**Here's an exercise for you**:

Approximate:

$$I = \int_{0.2}^{0.3} 15(3x)^4 \, e^{-(3x^5)} \, dx$$

Please submit your R code as well.