Bosonic Qiskit for Error Correction

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Abstract

Quantum noise is a critical issue in quantum computing, compromising the accuracy and fidelity of quantum operations. This paper presents a comprehensive study on leveraging Bosonic Qiskit, cat states, Gottesman-Kitaev-Preskill (GKP) encoding, and Wigner functions that utilize phase space analysis techniques to combat quantum noise errors. Bosonic Qiskit, an extension of the Qiskit framework, facilitates the simulation and manipulation of continuous-variable quantum systems based on bosonic modes. We explore the generation and control of cat states, which are coherent superpositions with noise-resilient characteristics. By utilizing GKP encoding, which exploits the infinite-dimensional Hilbert space of qumodes, we enhance the fault-tolerance of quantum information against noise. Furthermore, we employ phase space analysis techniques to characterize and understand the impact of quantum noise on encoded states. Through comprehensive simulations and evaluations, we demonstrate the efficacy of this integrated approach in mitigating quantum noise errors. The findings from this study provide valuable insights for developing robust and error-tolerant quantum computing architectures, enabling advancements in the field of continuous-variable quantum error correction and noise mitigation.

INTRODUCTION TO BOSONIC QISKIT

Qiskit is a freely available toolkit that allows users to create and manipulate quantum circuits using qubits, using the Python programming language. It provides the ability to design circuits, optimize them for specific hardware, and then either simulate their behavior or run them on actual quantum devices. Bosonic Qiskit is an extension of the Qiskit framework tailored for simulating and manipulating bosonic quantum systems. Bosonic Qiskit provides researchers and developers with a comprehensive set of tools to explore the unique properties of bosonic particles, such as photons and certain atoms, enabling investigations in quantum optics, quantum simulations, and quantum communication.

The use of bosons, rather than qubits, as building blocks for quantum operations can be very useful. Bosonic modes, also known as qumodes, contain multiple levels. This differs from qubits that are limited to two states, $|0>$ and $|1>$. A qumode has an infinite number of basis states. Bosonic Qiskit allows users to simulate quantum circuits with both qubits and qumodes.

Qumodes play a crucial role in continuous-variable (CV) quantum error correction. Qumodes, also known as quantum modes or bosonic modes, play a crucial role in certain types of quantum error correction schemes, particularly in continuous-variable (CV) quantum error correction. Qumodes refer to the infinite-dimensional Hilbert space associated with continuous-variable quantum systems, such as bosonic modes of light or mechanical oscillators. In CV quantum error correction, qumodes are used to encode and manipulate quantum information in a continuous-variable manner. Qumodes offer an advantage over discrete-variable quantum systems, such as qubits, by providing an infinite-dimensional state space. This larger state space allows for more degrees of freedom to encode and manipulate quantum information, potentially leading to more robust error correction codes. Continuous-variable quantum error correction schemes, such as the Gottesman-Kitaev-Preskill (GKP) code, utilize qumodes to spread quantum information over a larger state space. This spreading of information makes the code more resilient to certain types of errors, such as small displacements or fluctuations in the quadrature amplitudes. By encoding quantum information in qumodes, these errors can be detected and corrected more effectively. Qumodes are amenable to precise measurements of continuous variables, such as quadrature amplitudes. This is crucial for error detection in CV quantum error correction schemes. By performing measurements on qumodes, it is possible to extract information about the encoded quantum state and identify errors that have occurred during the computation. Qumodes allow for the implementation of fault-tolerant operations in CV quantum error correction. Error correction protocols can be designed to manipulate and correct errors while preserving the continuous-variable nature of the encoded quantum information. This is important for maintaining the integrity of the encoded state during the error correction process. Overall, qumodes provide a framework for encoding and manipulating continuous-variable quantum information, making them useful for quantum error correction schemes tailored to continuous-variable quantum systems.

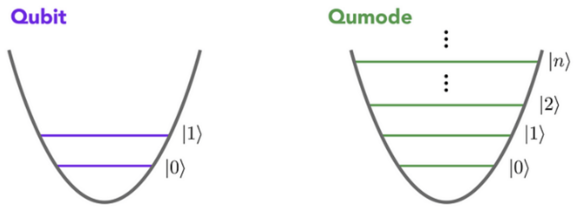

FIG. 1. Energy level diagram of a Qubit and a Qumode.

The Bosonic Qiskit software allows the user to incorporate qumodes, qubits, and classical bits in their Qiskit circuit. The qumodes are represented by the QumodeRegister class. The CVCircuit class is represented by using a combination of the QumodeRegister, QuantumRegister, and ClassicalRegister classes. The CVCircuit class is where bosonic gates can be implemented.

Bosonic Qiskit allows for the implementation of single qumode gates (displacement, squeezing), multi-qumode gates (beamsplitter, two mode squeezing), and hybrid qumode gates (controlled displacement, SNAP). Users can also define their own qubit-qumode gates. The package also provides visualization tools for Fock basis measurement, Wigner functions, and more phase space visualization.

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| Operator Name          | Symbol    | Mathematical Expression                 |
|------------------------|-----------|-----------------------------------------|
| Displacement Gate      | D(α)      | D(α) = exp(α * a† - α^* * a)          |
| Squeezing Gate         | S(r, ϕ)   | S(r, ϕ) = exp(0.5 * (r * (a†)^2 - r^* * a^2) * exp(i * ϕ)) |
| Beamsplitter Gate      | BS(θ)     | BS(θ) = |cos(θ)    -sin(θ)|                |
|                        |           |         |sin(θ)     cos(θ)|                |
| Two-Mode Squeezing     | SMS(r)    | SMS(r) = exp(r * (a†⊗a† - a⊗a))         |
| Controlled Displacement| -          | Controlled-D(α, c) = |I    0|              |
| Gate                   |           |                      |0   D(α)|           |
| SNAP Gate              | SNAP(κ)   | SNAP(κ) = exp(i * κ * (a†⊗a + a⊗a†))    |
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FIG. 2. Bosonic Qiskit Operation Examples

Optimal control techniques play a crucial role in quantum error correction by enabling the design and implementation of high-fidelity error correction protocols. However, it is essential to clarify that when we refer to 'error correction' in this context, we are not discussing quantum error correcting codes. Instead, we are focused on the use of optimal control for the implementation of dynamically corrected gates. Quantum error correcting codes deal with the protection of quantum information against errors during quantum computations. On the other hand, in our work, we emphasize the utilization of optimal control techniques to achieve dynamically corrected gates, which improve the accuracy and reliability of quantum operations. By applying optimal control methods, it becomes possible to optimize the control pulses and sequences used in error correction protocols to mitigate the impact of errors.

Bosonic Qiskit offers a valuable toolset for optimal control in bosonic quantum systems. Optimal control techniques aim to find control parameters that enable the precise manipulation and evolution of quantum systems to achieve desired objectives. In the context of bosonic systems, such as quantum optics and quantum simulations, the ability to efficiently control and shape the behavior of bosonic particles, such as photons, is of utmost importance. Bosonic Qiskit provides a framework that combines the principles of optimal control with the power of quantum computing. By leveraging the capabilities of Qiskit and its associated tools, researchers and developers can utilize optimization algorithms and quantum circuits to explore and optimize control strategies for bosonic systems. This enables the design of tailored control pulses, optimal gate sequences, and other control strategies to achieve specific goals, such as high-fidelity operations, enhanced quantum state preparation, and improved entanglement generation. The integration of optimal control techniques into Bosonic Qiskit empowers the development of advanced protocols and algorithms, facilitating the efficient and precise control of bosonic quantum systems for a wide range of applications.[1] [2]

PHASE SPACE

The phase-space is a way of representing quantum states that allows for the ability to create an analogue between classical and quantum dynamics. Utilizing phase-space representation for quantum mechanics was first proposed by Wigner; Wigner created the Wigner function that takes a quantum wave function and represents it in phase-space. Phase-space is a space where all possible states of a system are represented where a coordinate corresponds to variables that fully characterize the system's state. In classical mechanics the axis of the phase-space corresponds to its position coordinates and momentum coordinates. Phase-space allows for the trajectory of a system's evolution to be visualized. Therefore, phase-space is useful in quantum mechanics and may have additional dimensions corresponding to quantum mechanical observables. The phase-space provides a framework to understand the dynamics of quantum states and/or systems, while quantum optimal control makes use of this framework to develop controls that achieve desired results and improve performance.

By taking the expectation value of any arbitrary (quantum mechanical) operator, phase-space representation creates a transformation from the operator to the phase space. Direct measurement of the phase-space has become more possible and useful in places such as the displace parity formulation used in the measurement of optical states in QED (Quantum electrodynamic) experiments. Many metrics that exist in quantum mechanics, such as the purity of a state and angular momentum operators, can be mapped to and calculated in phase space. Phase space methods have also been useful in characterizing a quantum state in ways like measuring coherence, phase space inequalities, and the negativity in the Wigner function. [3]

In the context of quantum optimal control, the phase space representation has been useful in creating control strategies. Phase-space variables, such as position and momentum, have facilitated the use of optimization techniques in quantum optimal control problems. Optimal control in quantum mechanics focuses on determining optimal controls in order to push the system towards a desired quantum state or perform certain quantum operations. Things such as molecular rotation and vibrational motion can be described classically much like how it is done when considering large polyatomic molecules. In Joe-Wong et al. an analogy between a classical system of "n" particles and a quantum of N states shows they share similar landscape topologies, "the classical mechanical phase space state-to-state control results found here are clear analogues with the goal of maximizing quantum mechanical pure state transition probabilities." [4]

To improve the traditional gate-based variational methods by finding optimal pulses, it was found that the laser pulses "provide more flexibility in the evolution of the initial state, resulting in a broader exploration of the state space and allowing for the total evolution time T to be minimized."[5] The laser pulse allows for the implementation of gates via a laser pulse. These advantages allow us to control unwanted effects on qubit decoherence and lifetimes, becoming important for the NISQ error of QEC (Quantum Error Correction). With the freedom of being able to move through the phase-space, the phase-space offers simpler implementation of quantum optimal control.

WIGNER FUNCTION

The Wigner function serves as a connection between quantum optimal control and the phase space; the function brings phase space representation into quantum mechanics. The function offers a way to transform the wave function in order to introduce momentum variables and to represent probability in the phase space: . The transformation makes it possible to treat classical mechanics and quantum mechanics on the same level, thus making it easier to discover the behavior in quantum mechanics. The function is distinctly different from the Schrödinger equation as it contains both real space and momentum variables. It is interesting to be used for classical approaches and for identifying the deviations from classical behavior, and the entanglement that can occur in quantum systems, as seen in Fig.1. [6]

The function's representation of quantum mechanics in phase space causes some properties from a probability distribution must be sacrificed, making the Wigner function a quasi-probability distribution. However, it can't be interpreted as a joint distribution of particle position and momentum. [7] The function must sacrifice positive-definiteness and allows for negative values, therefore making it not a probability function. The function allows for a comprehensive description of the system's quantum properties such as coherence, superposition, and entanglement. The Wigner function is a phase space counterpart of the density matrix and a quantum counterpart of the classical distribution function; through maintaining classical statistical mechanics, mean values and probabilities are evaluated in the phase space.

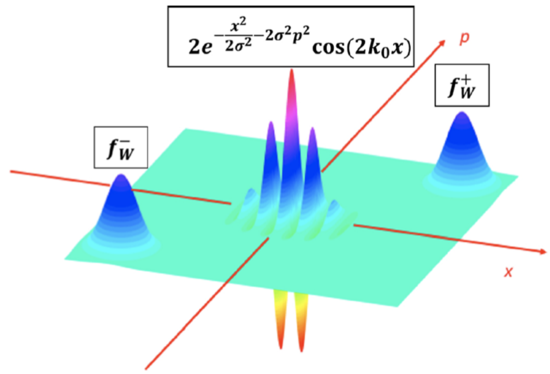

FIG. 3. Depicted are two Gaussian wave packets, using the assumption that they form a single wave function. Shown is the resulting Wigner function with the entanglement occurring at x=p=0. Figure taken from [6]

As a phase space representation of quantum states, the Wigner function plays a role in optimal control by enabling the characterization of quantum states and operations. It facilitates the formulation and solving of optimal problems with a classical representation. As mentioned with phase space, the use of the phase space in terms of the Wigner function is useful to create optimal control strategies. Furthermore, to experimentally reconstruct the Wigner function provides more information on the system that any other quantum approach can obtain. The distribution's phase space nature allows to identify where quantum corrections enter a problem by highlighting the difference from the classical version.

The Wigner function becomes related to a quantum wave function through the Wigner-Weyl transform. This transform maps the quantum density operator (or the quantum state represented by the wave function) to the Wigner function. The Weyl transform $\widetilde{A}$ of an operator $\hat{A}$ is

$$\widetilde{A}(x,p) = \int e^{\frac{-ipx}{\hbar}} < x + \frac{y}{2}|\hat{A}|x - \frac{y}{2} > dy. \quad (1)$$

Or in terms of matrix elements of the operator in the momentum basis

$$\widetilde{A}(x,p) = \int e^{\frac{ixu}{\hbar}} < p + \frac{u}{2}|\hat{A}|p - \frac{u}{2} > du. \quad (2)$$

The Wigner-Weyl transform converts the operator into a function of x (position) and p (momentum). Therefore, one could go from a wave function, that describes a quantum system in terms of position and momentum, to a Wigner function, that represents a quantum state in phase space where the phases space consists of both position and momentum variables, through the use of the Wigner-Weyl transform.

To recover the original wave function $\psi(x)$from the Wigner function requires integrating over the momentum degrees of freedom. Further, its Fourier Transform can be obtained by integrating out space degrees of freedom. By multiplying the Wigner function by $e^{\frac{-ipx}{\hbar}}$ and integrating over momentum, one can obtain the original wave function. Using the example Wigner function

$$W(x,p) = \frac{\widetilde{p}}{h} = \frac{1}{h}\int e^{\frac{-ipy}{\hbar}}\psi(x+\frac{y}{2}\psi^*(x-\frac{y}{2})dy \qquad (3)$$

, following these steps, and then setting x=x/2 and x'=x/2, the original wave function

$$\psi(x) = \frac{1}{\psi^*(0)}\int W(\frac{x}{2},p)e^{\frac{ipx}{\hbar}}dp \qquad (4)$$

can be obtained, up to an overall constant. By normalizing $\psi(x)$, the constant $\psi^*(0)$ can be determined. [8]

Wigner functions have a same property as classical probability density: the integral along any phase-space strip (the region between any two parallel lines) is a positive number bounded between zero and one. Defining the Wigner function for discrete system has difficulties unlike for continuous systems. Wooter defined the discrete version of the Wigner function for N-dimensional quantum systems when N is a prime number; the phase space is an N x N grid. Wooter's method can be applied when N is a composite number, but the Wigner function must be defined in a phase-space grid which is the Cartesian product of the phase spaces corresponding to the prime factors of N. Furthermore, for the case where we encode one qubit of quantum information using n physical qubits, the stabilizers are chosen so that the code corrects a set of errors E_i which are also translation operators; encoded states are eigenstates of translation operators, therefore their corresponding Wigner functions have to be invariant under the same translations. When translating the Wigner function of an encoded stated by a correctable error, you should obtain a Wigner function that is orthogonal to the original one. This points to the phase-space representation and the Wigner function providing information on the encoded states. However, there is more work to be done. Some quantum information problems, such as the mean kind problem, show promise for the phase-space approach. [9]

In quantum computing, Wigner functions have been utilized in various way. Two-mode qubit-like entangles squeezed states were compared to entangled coherent state with the use of Wigner functions for the two states. The negativity of the Wigner function was used to convey the entanglement of upwards of 3000 atoms by a single photon. The creation of the negative parts of the Wigner function was also studied through entanglement for multi-qubit squeezed states. Another use was the representation of optical qubits in a qubit-oscillator system. The two-state atom was replaced by the qubit and the Wigner distribution's negativity was used for pointing towards the non-classicality of the developing states.[6]

CAT STATES

While a discrete-variable system and a continuous-variable quantum computer possess the same theoretical computational power, there are scenarios where the latter may exhibit greater efficiency. For instance, a single oscillator has the potential to hold an unlimited amount of information due to its infinitely large Hilbert space. Regarding hardware requirements, using linear elements and photon detectors in optical platforms and simple microwave resonators with long coherence times in superconducting QED systems are much more favorable and easier to work with. Moreover, the inherent relationship between continuous variables and communication enables simplified transmission of quantum information, benefiting applications such as teleportation, cryptography, and dense coding. However, there are trade-offs to consider. Experimental realizations may encounter challenges due to non-orthogonal basis states, potential excursions from a logical sub-space, and the need to manipulate encoded states with high fidelity. Nevertheless, there are several promising continuous variable quantum error correction (QEC) protocols available.

The development of the cat state comes from the famous Schrodinger's Cat thought experiment: the experiment contains a single radioactive particle and a cat along with a mechanism such that if the particle decays, poison is released, killing the cat; if not, the cat lives. We can represent this experiment quantum mechanically utilizing states such as undecayed particle = $|\uparrow>$, decayed particle = $|\downarrow>$, alive cat = $|A>$ and dead cat = $|D>$. When the particle reaches its half-life, the system evolves into a superposition state that can be represented as

$$\Psi = \frac{1}{\sqrt{2}}(|\uparrow>|A>+|\downarrow>|D>). \qquad (5)$$

Thus, to create a cat state is to generate a quantum superposition of classically distinct states. Physical cat states have been achieved in microwave fields, oscillation fields of trapped ions, and, most recently, optical pulses.

For microwave fields, Nobel Prize winner Serge Haroche made physical cat states, "with the preparation of a coherent field in the cavity, whose Wigner function is a Gaussian. A single nonresonant atom is then prepared in a coherent superposition of two states... This atom crosses the cavity, and each of its two components shifts the phase of the field in opposite directions by a simple dispersive index effect. Here again, we take advantage of the huge coupling of Rydberg atoms to microwaves which makes a single atom index large enough to have a macroscopic effect on the field phase. At the cavity exit, the atom and the field are entangled, each atomic state being correlated to a field state with a different phase ... After the atom has been exposed to the second Ramsey pulse and detected, there is no way to know in which state the atom crossed the cavity and the field collapses into a Schrödinger cat superposition. In other words, the atomic Schrödinger kitten has produced a photonic Schrödinger cat which contains several photons on average." For oscillation fields of trapped ions,

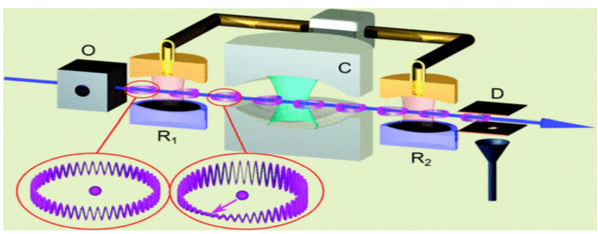

FIG. 4. Experimental set up to create a cat state in a microwave system [10]

Nobel Prize winner David Wineland made physical cat states using,

"optical-dipole force ... because the strength of the force can depend on the ion's internal state ... using state-dependent optical-dipole forces, we were able to produce an analog to the Schrödinger's cat state... the spin states of the ion are like the states of the single radioactive particle and the coherent states of the ion, which follow more macroscopic classical trajectories, are like the state of the cat."

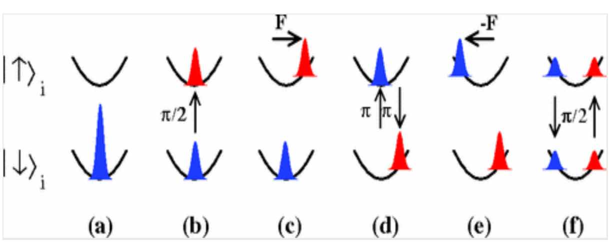

FIG. 5. Experimental set up to create a cat state using trapped ions [11]

Lastly, Rempe's group achieved cat states through optical pulses by,

"put[ting] the atom in the cavity in an equal superposition of its on and off states, the reflection of the laser pulse from the cavity then entangles this state with different phase shifts of the reflected optical field ... If one detects the atom in an equal superposition of its on and off states, the reflected optical field is correspondingly in an equal superposition of classically distinct states of 0 or π radians of phase shifts. Such states are called the even or odd Schrödinger-cat states, depending on whether there is a positive or negative sign between the two superposition components."

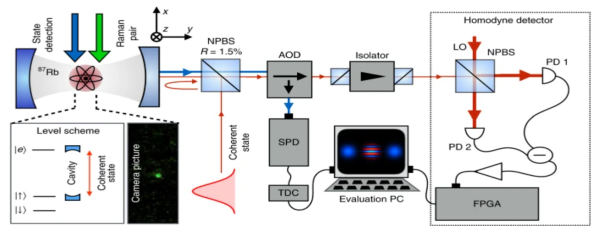

FIG. 6. Experimental set up to create a cat state using optical pulses [12]

These physical cat states are then measured using the Wigner function in order to characterize the quantum states created. As an example, we will focus on how Rempe's group calculated and utilized the Wigner function. To construct the function from the physical cat states created, the group used a homodyne detector with two photodiodes placed perpendicularly to each other. From there, a field-programmable gate array picks up the signal to be read out by a PC. The signals contain information on the amplitude of the optical field for a given phase value. With the compilation of these measurements, the full optical state can then be defined by its distribution in phase space, also known as the Wigner function (W (q, p): q and p are the field quadratures spanning phase space). The even, odd, and entangled states can then be graphically represented through Gaussian distributions and interference fringes. These graphs are used to calculate loss because loss reduces the visibility of the non-classical coherences, which appears as fringes in the Wigner function. For our specific area of interest, the experiment has a propagation and detection loss of 25 percent which can be mitigated through loss-correction codes.

The author recommends loss-correction codes employed by Shoelkopf's group, which tested various QEC protocols on cat states. Their protocol was able to maintain a corrected qubit lifetime of 320 microseconds, which was much longer than the lifetime of the other parts in their system. To do so, they tested a 3D circuit (QED architecture) consisting of a single transmon qubit coupled to two waveguide resonators. One resonator stores the logical states, while the other is used for ancilla readout and control. The transmon was used as an ancilla which provided the error syndrome and encoded/decoded the logical states. The error syndrome is measured using a

Ramsey-style pulse sequence. The process begins with the ancilla in the ground state while the resonator is in the vacuum state. The qubit is then initiated by applying a single-qubit gate to the ancilla. Then a controller encodes one of the six cardinal points on a Bloch sphere in the even logical basis states by transferring the qubit from the ancilla into a superposition of cat states in the resonator. This step results in the ancilla returning to its ground state, causing complete disentanglement from the resonator state. Meanwhile, they are detecting photon jumps by monitoring the parity so that they may decode and correct any errors throughout this process.

GKP STATES

GKP states were developed by Gottesman, Kitaev, and Preskill (GKP)[13] in pursuit of creating error correction codes that exploit the noncommutative geometry of phase space to protect against errors that shift the values of position and momentum (the conjugate variables q and p); or for quantum optical systems, these codes protect against losses that cause the amplitude of an oscillator to decay. That's not all; GKP error correction codes can be used in long-distance quantum communication schemes implementing quantum repeaters using only beamsplitters, homodyne detectors, and GKP ancilla resource states. These motivating factors have encouraged numerous scientists to experimentally develop GKP states in order to implement these error correction protocols; however, it's proven to be extremely difficult, but not impossible!

How is it done? GKP codes encode a d-dimensional logical subspace in n bosonic modes.[14]. Following the same principles found in a harmonic oscillator, we can define a GKP code using Pauli Operator. Using the example presented by Shurti's group let's define our GKP code with the operators $\bar{X} = \hat{D}(\alpha)$ and $\bar{Z} = \hat{D}(\beta)$ where α and β are two complex numbers. $\hat{D}$ is the displacement operator which allows us to shift our state in the logical subspace without changing the order of moments.

$$\hat{D} = e^{\alpha^{\dagger}\hat{a} - \alpha^{*}\hat{a}} \tag{6}$$

Displacements also commute/anti-commute up to a phase, signifying

$$\alpha\beta^{*} - \alpha\beta^{*} = 2i\pi \tag{7}$$

$$\beta\alpha^{*} - \beta^{*}\alpha = i\pi \tag{8}$$

Thus, $\bar{X}\bar{Z} = -\bar{Z}\bar{X}$.

However, given that we are working with pauli matrices in order to ensure that the matrices operate as expected, we must define the GKP code space to be the simultaneous +1 eigenspace of $\bar{X}, \bar{Z}$, where

$$\hat{S}_x = \bar{X}^2 = \hat{D}(2\alpha), \hat{S}_z = \bar{Z}^2 = \hat{D}(2\beta) \tag{9}$$

Now that we have defined our GKP code space, we can now define GKP states.

GKP states, also known as grid states, are non-gaussian and described as single bosonic modes. These modes have annihilation and creation operators with corresponding quadrature operators

$$\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^{\dagger}), \hat{x} = \frac{1}{\sqrt{2}i}(\hat{a} + \hat{a}^{\dagger}) \tag{10}$$

that satisfy $[\hat{x}, \hat{p}] = i$.

Since GKP states are non-Gaussian it is required that non - Gaussian elements are used to generate them.

Anderson's group achieved this through "us[ing] a cavity QED system as the central and only non-Gaussian element. In particular, we consider the reflection of an incoming optical field onto a single-mode cavity containing a 3-level system.The 3-level system consists of two low energy states, $|0>$ and $|1>$, and one high energy state, $|e>$, which can be optically excited from the state $|1>$ through a Jaynes Cumming Hamiltonian with coupling strength g."

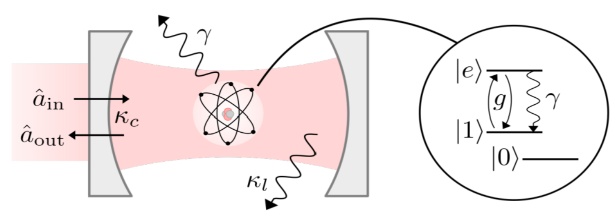

FIG. 7. Experimental set up to create a GKP state within an optical system [15]

Glancy's group developed GKP states through "using superpositions of optical coherent states (sometimes called "Schrödinger cat states"), linear optical devices, squeezing, and homodyne detection. We initially considered two optical modes containing Schrödinger cat states. A displacement followed by a squeezing is applied to both modes and then the two modes are sent into a beam splitter. The action of the beam splitter entangles the two modes. An approximate GKP state is obtained when we perform a measurement of the p-quadrature in one of the beam splitter output modes"[16]

These physical GKP states are difficult to achieve, however we can simulate them using the Bosonic Qiskit package. If we follow the steps within the Bosonic Qiskit tutorial as well as recreate the circuits developed by Sabathy's group [17]. We can gain a deep understanding of GKP states and its ability to reduce error within a quantum computer.

CONCLUSIONS

Bosonic Qiskit is a new Qiskit framework for quantum computing that incorporates cat and GKP states, representing a significant advancement in the field of error correction. Both cat and GKP states have demonstrated their potential in reducing noise and error quantum computing. They show promise in the advancement of increasing reliability and efficiency of quantum computations once cat and GKP states can be experimentally created. It has been discovered that GKP states are more challenging to create than the cat phase. Nonetheless, a thorough understanding of these states is beneficial for the progress of error reduction. Furthermore, phase-space representation through the use of Wigner functions offers a simpler approach to studying quantum systems and states, such as cat and GKP states. Representing quantum states with classical variables, such as momentum and position, in the phase-space provides insights into their behavior.

For Error Corp., further delving into the study and implementation of cat and GKP states with phase-space representation shows promise. In the interest of the company, it is important to understand cat and GKP states to have the ability to implement algorithms and programs to reduce noise and error.

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