Chapter 3

Vector Spaces

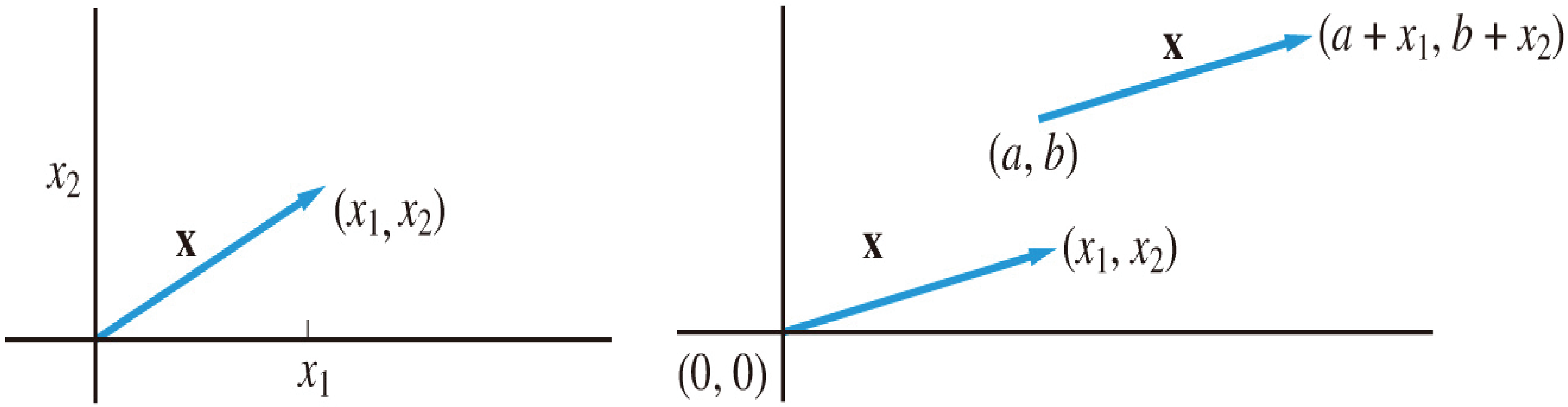
3.1 Definition and Examples Euclidean Vector Spaces

- The most elementary vector spaces are the Euclidean vector spaces R^n, $n = 1, 2, \ldots$

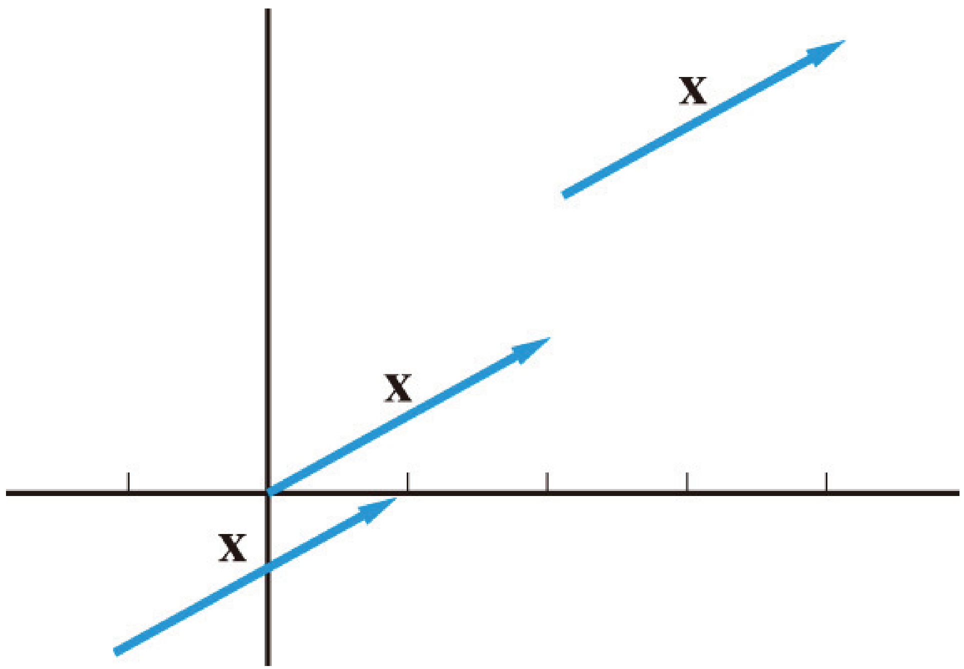
- For example, R^2, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ denote the directed line segment from (0, 0) to (x_1, x_2)

- Two equal line segments will have the same <u>length</u> and <u>direction</u>

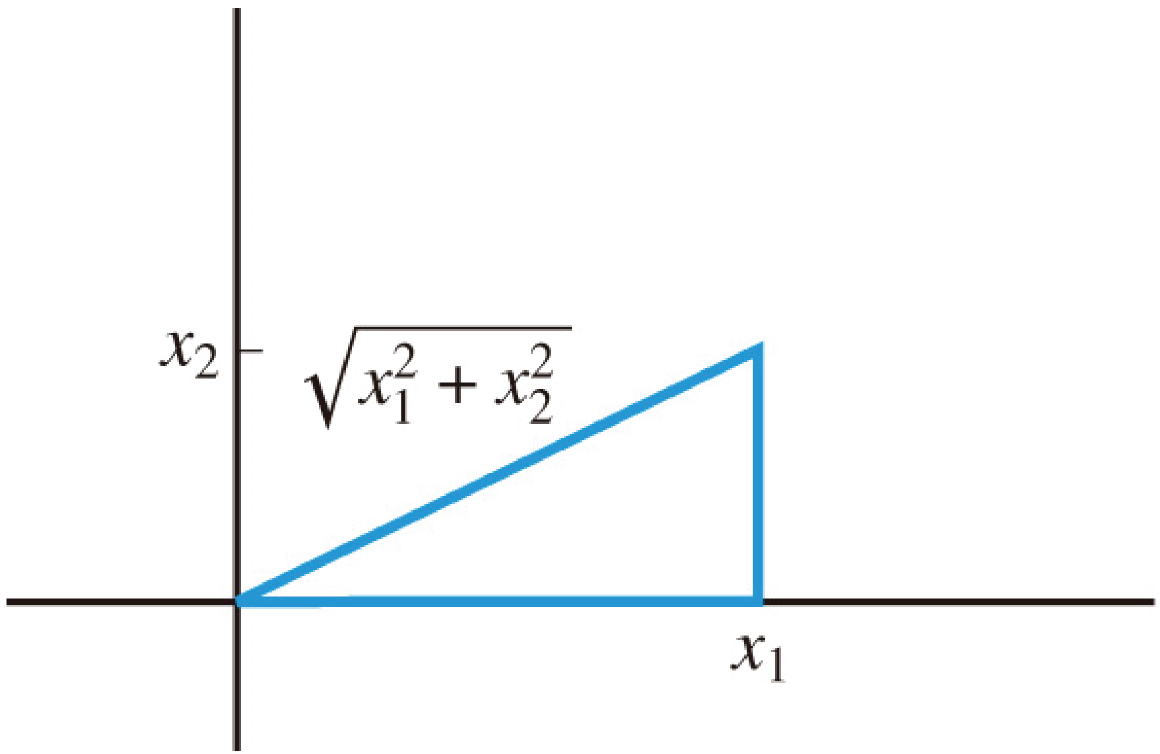
- $\mathbf{x}$ can be represented by any line segment from (a, b) to $(a + x_1, b + x_2)$



- For example, the vector $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in R^2 could just as well be represented by the directed line segment from (2, 2,) to (4, 3) or from (-1, -1) to (1, 0)



- The Euclidean length of the line segment from (0, 0) to (x_1, x_2) is $\sqrt{x_1^2 + x_2^2}$

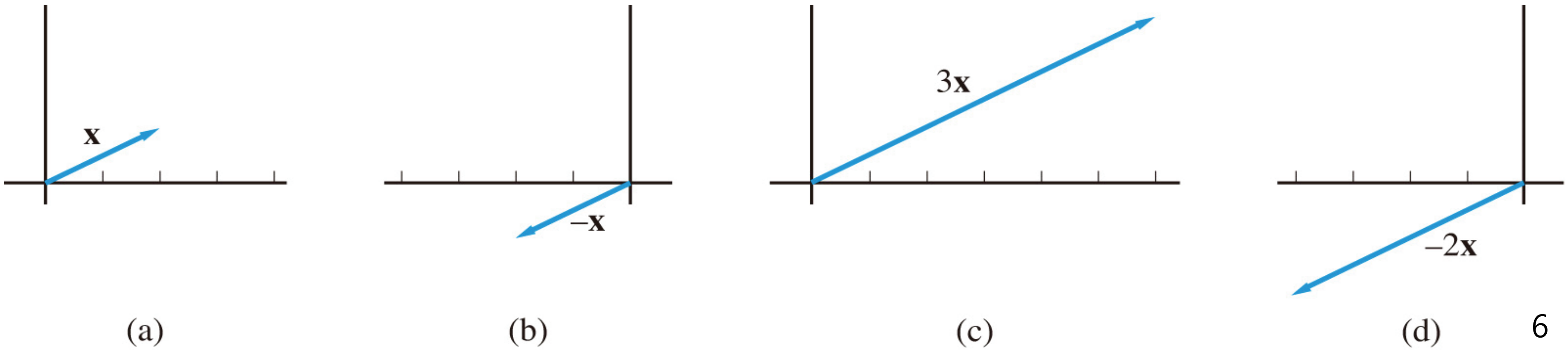


- Given a scalar α, the product $\alpha\mathbf{x}$ is given by

$$\alpha\,\mathbf{x} = \alpha\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha\, x_1 \\ \alpha\, x_2 \end{bmatrix}$$

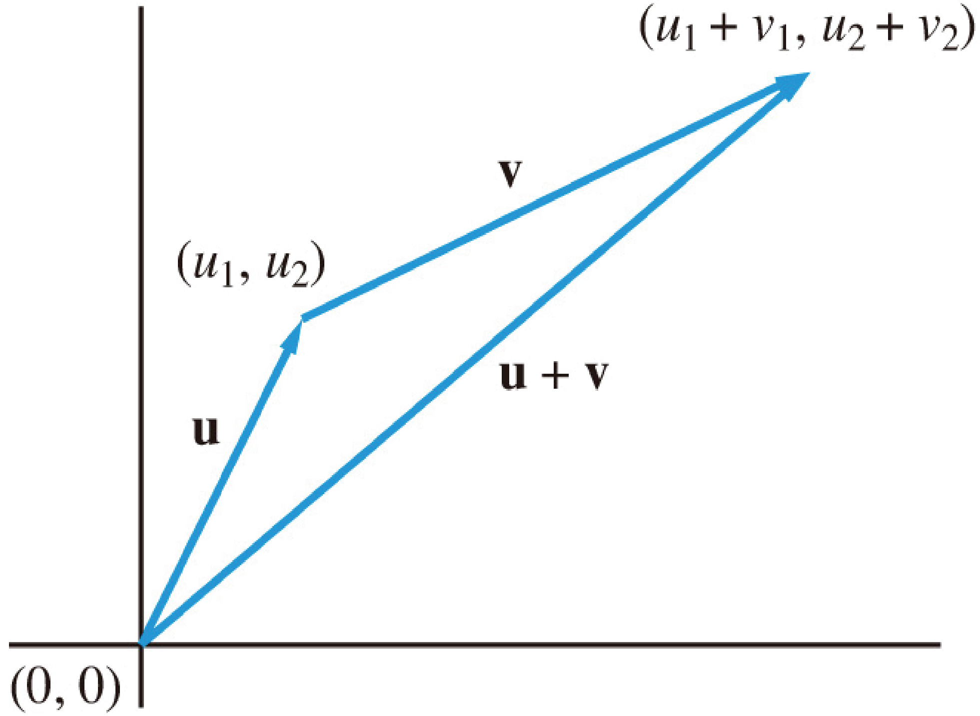
- For example, if $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then

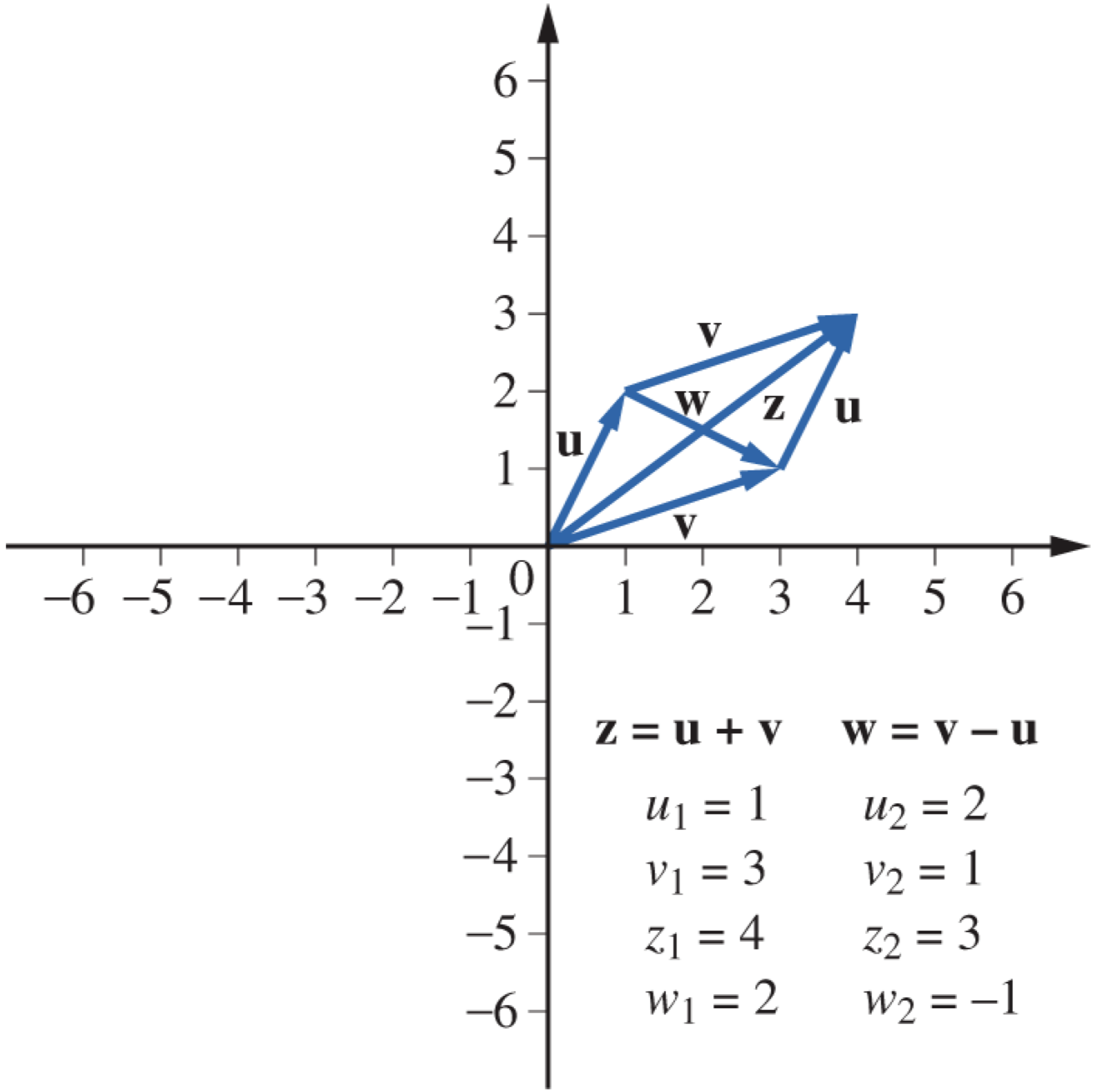
$$-\mathbf{x} = \begin{bmatrix} -2 \\ -1 \end{bmatrix},\quad 3\mathbf{x} = \begin{bmatrix} 6 \\ 3 \end{bmatrix},\quad -2\mathbf{x} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$



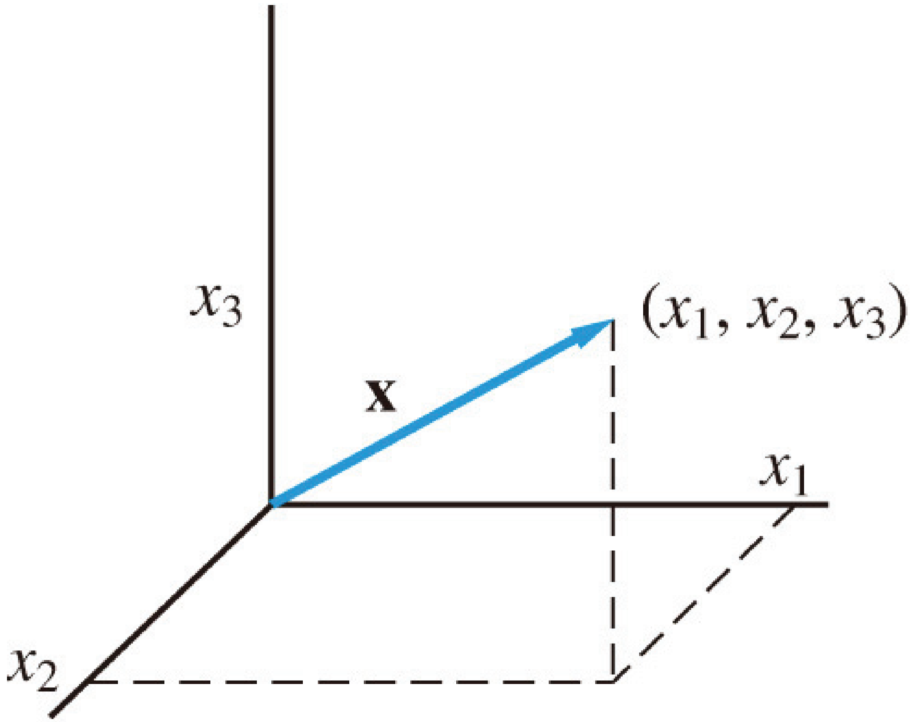
- The sum of two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is defined by

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

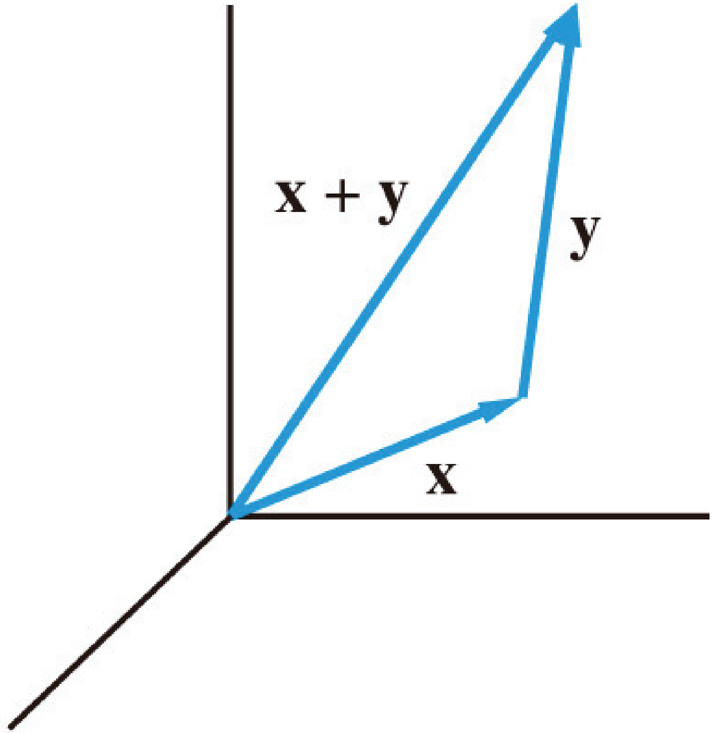




u
v
w
z
0
z = u + v
w = v − u
u1 = 1
u2 = 2
v1 = 3
v2 = 1
z1 = 4
z2 = 3
w1 = 2
w2 = −1



(a)



(b)

- In general, scalar multiplication and addition in R^n are defined by

$$\alpha\,\mathbf{x} = \begin{bmatrix} \alpha\, x_1 \\ \alpha\, x_2 \\ \vdots \\ \alpha\, x_n \end{bmatrix} \quad \text{and} \quad \mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

for $\mathbf{u}$, $\mathbf{v} \in R^n$ and any scalar α.

Vector Spaces Axioms
Definition

Let V be a set on which the operations of addition and scalar multiplication are defined. By this we mean:

(1) for each pair of elements $\mathbf{x}$ and $\mathbf{y}$ in V, one can associate a unique element $\mathbf{x}+\mathbf{y}$ that is also in V,

(2) for each vector $\mathbf{x}$ in V and a scalar α, one can associate a unique element $\alpha\mathbf{x}$ in V.

The set V together with the operations of addition and scalar multiplication is said to form a **vector space** if the following axioms are satisfied. **(V, +, ×)**

公設

A1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for any $\mathbf{x}$ and $\mathbf{y}$ in V

A2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for any $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in V

A3. There exist an element $\mathbf{0}$ in V such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for each $\mathbf{x} \in V$

A4. For each $\mathbf{x} \in V$, there exist an element $-\mathbf{x}$ in V such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

A5. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ for each scalar α and any $\mathbf{x}$ and $\mathbf{y}$ in V

A6. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ for any scalars α and β and any $\mathbf{x} \in V$

A7. $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for any scalars α and β and any $\mathbf{x} \in V$

A8. $1 \cdot \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$

- The **closure properties** of addition and scalar multiplication operations can be summarized as follows:

C1. If $\mathbf{x} \in V$ and α is a scalar, then $\alpha\mathbf{x} \in V$

C2. If $\mathbf{x}, \mathbf{y} \in V$, then $\mathbf{x}+\mathbf{y} \in V$

The Vector Spaces $R^{m\times n}$

- The vector spaces R^n can be viewed as the set of all $n\times 1$ matrices with real entries.
- $R^{m\times n}$ can denote the set of all $m\times n$ matrices with real entries.
- If $A = (a_{ij})$ and $B = (b_{ij})$, the sum $A+B$ is the $m\times n$ matrix $C = (c_{ij})$, where $c_{ij} = a_{ij} + b_{ij}$
- αA is the $m\times n$ matrix whose ij-th entry is αa_{ij}

Example

- Let $W=\{(a, 1) \mid a \text{ real}\}$
- By C1. $(a, 1) \in W$

$$\alpha(a, 1) = (\alpha a, \alpha) \notin W$$

- By C2. $(a, 1) \in W$ and $(b, 1) \in W$

$$(a, 1) + (b, 1) = (a+b, 2) \notin W$$

$\therefore$ W together with the operations of addition and scalar multiplication is ***not*** a vector space

Given a set V on which the operations of addition and scalar multiplication have been defined and satisfy properties C1 and C2, we can check the eight axioms are valid.

The Vector Space *C*[*a*, *b*]

- Let $C[a, b]$ denote the set of all real-valued functions that are defined and continuous on the closed intervals $[a, b]$. In this case, our universal set is a set of functions. Thus, our vectors are the functions in $C[a, b]$.
- If f and g are functions in $C[a, b]$ and α is a real number:

$$(f+g)(x) = f(x) + g(x)$$
$$(\alpha f)(x) = \alpha f(x)$$

for all x in $[a, b]$.

- Clearly, αf is in $C[a, b]$, since a constant times a continuous function is always continuous.

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$$

$$\text{for every } x \; in \, [a, b]$$

- Since the function

$$z(x) = 0 \qquad \text{for all } x \text{ in } [a, \; b]$$

acts as the zero vector; that is,

$$f + z = f \qquad \text{for all } f \text{ in } C[a, b]$$

The Vector Spaces P_n

- Let P_n denote the set of all polynomials of degree less than n. Define $p+q$ and αp by

  $$(p+q)(x) = p(x) + q(x)$$

  and

  $$(\alpha p)(x) = \alpha p(x)$$

  for all real numbers x.
- P_n is a vector space

Additional Properties of Vector Spaces

Theorem 3.1.1

If V is a vector space and $\mathbf{x}$ is any element of V, then

(i) $0\mathbf{x} = \mathbf{0}$

(ii) $\mathbf{x}+\mathbf{y} = \mathbf{0}$ implies that $\mathbf{y} = -\mathbf{x}$ (i.e., the additive inverse of $\mathbf{x}$ is unique)

(iii) $(-1)\mathbf{x} = -\mathbf{x}$

Theorem 3.1.1 *proof*

(i)

$$\mathbf{x} = 1\mathbf{x} = (1+0)\mathbf{x} = 1\mathbf{x} + 0\mathbf{x} = \mathbf{x} + 0\mathbf{x}$$

$$-\mathbf{x} + \mathbf{x} = -\mathbf{x} + (\mathbf{x} + 0\mathbf{x}) = (-\mathbf{x} + \mathbf{x}) + 0\mathbf{x}$$ **(A2)**

$$\mathbf{0} = \mathbf{0} + 0\mathbf{x} = 0\mathbf{x}$$ **(A1, A3, and A4)**

(ii) Suppose $\mathbf{x} + \mathbf{y} = \mathbf{0}$, then

$$-\mathbf{x} = -\mathbf{x} + \mathbf{0} = -\mathbf{x} + (\mathbf{x} + \mathbf{y})$$ **(A1, A3, and A4)**

$$-\mathbf{x} = (-\mathbf{x} + \mathbf{x}) + \mathbf{y} = \mathbf{0} + \mathbf{y} = \mathbf{y}$$

(iii)

(i and A6)

$$0 = 0\mathbf{x} = (1 + (-1))\mathbf{x} = 1\mathbf{x} + (-1)\mathbf{x}$$

(A8)

$$\mathbf{x} + (-1)\mathbf{x} = 0$$

$$(-1)\mathbf{x} = -\mathbf{x}$$

3.2 Subspaces Example 1

- Let $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, x_2 = 2x_1 \right\}$, then S is a subset of R^2
- If $\mathbf{x} = \begin{bmatrix} c \\ 2c \end{bmatrix}$ is any element of S and α is any scalar, then
- By C1:

$$\alpha \mathbf{x} = \alpha \begin{bmatrix} c \\ 2c \end{bmatrix} = \begin{bmatrix} \alpha c \\ 2\alpha c \end{bmatrix} \in S$$

- By C2: if $\begin{bmatrix} a \\ 2a \end{bmatrix} \in S$ and $\begin{bmatrix} b \\ 2b \end{bmatrix} \in S$, then

$$\begin{bmatrix} a \\ 2a \end{bmatrix} + \begin{bmatrix} b \\ 2b \end{bmatrix} = \begin{bmatrix} a+b \\ 2(a+b) \end{bmatrix} \in S$$

- $\therefore$ S is a <u>vector space</u>

Definition

If S is a nonempty subset of a vector space V, and S satisfies the following conditions:

(i) $\alpha\mathbf{x} \in S$ whenever $\mathbf{x} \in S$ for any scalar α (closed under scalar multiplication)

(ii) $\mathbf{x} + \mathbf{y} \in S$ whenever $\mathbf{x} \in S$ and $\mathbf{y} \in S$ (closed under addition)

then S is said to be a **subspace** of V

- A subapace of V is a subset S that is closed under the operations of V.
- Every subspace of a vector space is a vector space in its own right.

Remark

- If V is a vector space, then $\{\mathbf{0}\}$ and V are subspaces of V. All other subspaces are referred to as *proper subspaces*, $\{\mathbf{0}\}$ is referred to as the *zero subspace*.
- To show that a subset S of a vector space forms a subspace, we must show that S is nonempty and that the closure properties (i) and (ii) in the definition are satisfied. Since every subspace must contain the zero vector, we can verify that S is nonempty by showing that $\mathbf{0} \in S$.
-

Example 2

- Let $S = \{(x_1, x_2, x_3)^T \mid x_1 = x_2\}$, is S a subspace of R^3?
- ***Sol:***

(i) If $\mathbf{x} = (a, a, b)^T \in S$ *then*

$$\alpha\mathbf{x} = (\alpha a, \alpha a, \alpha b)^{\mathrm{T}} \in S$$

(ii) If $\mathbf{x} = (a, a, b)^T \in S$ and $\mathbf{y} = (c, c, d)^T \in S$ ***then***

$$\mathbf{x} + \mathbf{y} = (a+c, a+c, b+d)^T \in S$$

Therefore, S is a subspace of R^3

Example 3

- Let $S = \{\begin{bmatrix} x \\ 1 \end{bmatrix} \mid x \text{ is a real number}\}$, is S a subspace of R^2?
- ***Sol:***

By C1: $\alpha \begin{bmatrix} a \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha \end{bmatrix}$, if $\alpha \neq 1$, $\begin{bmatrix} \alpha a \\ \alpha \end{bmatrix} \notin S$

By C2: $\begin{bmatrix} a \\ 1 \end{bmatrix} + \begin{bmatrix} b \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ 2 \end{bmatrix} \notin S$

So, S is not a subspace of R^2.

Example 4

- Let $S = \{A \in R^{2\times2} \mid a_{12} = -a_{21}\}$, is S a subspace of $R^{2\times2}$?
- ***Sol:***

  (i) By C1:

  If $A \in S$, $A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$

  $\therefore\ \alpha A = \begin{bmatrix} \alpha a & \alpha b \\ -\alpha b & \alpha c \end{bmatrix} \in S$

(ii) By C2:

If A, $B \in S$, that is

$$A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}, B = \begin{bmatrix} d & e \\ -e & f \end{bmatrix}$$

$$\therefore A + B = \begin{bmatrix} a+d & b+e \\ -(b+e) & c+f \end{bmatrix} \in S$$

So, S is a subspace of $R^{2\times 2}$

Example 5

- Let S be the set of all polynomials of degree less than n with the property that $p(0) = 0$. The set S is nonempty since it contains the zero polynomial. We claim that S is a subspace of P_n. This follows because

(i) If $p(x) \in S$ and α is a scalar, then

$$\alpha p(0) = \alpha \cdot 0 = 0$$

and hence $\alpha p \in S$.

(ii) If $p(x)$ and $q(x)$ are elements of S, then

$$(p + q)(0) = p(0) + q(0) = 0 + 0 = 0$$

and hence $p + q \in S$.

Example 6

- Let $C^n[a, b]$ be the set of all functions f that have a continuous nth derivative on $[a, b]$, then $C^n[a, b]$ is a subspace of $C^n[a, b]$.

Example 7

- The function $f(x) = |x|$ is in $C[-1, 1]$, but it is not differentiable at $x = 0$ and hence it is not in $C^1[-1, 1]$.
- The function $g(x) = x|x|$ is in $C^1[-1, 1]$, since it is differentiable at every point in $[-1, 1]$ and $g'(x) = 2|x|$ is continuous on $[-1, 1]$.
- However, $g \notin C^2[-1,1]$, since $g''(x)$ is not defined when $x = 0$. Thus, the vector space $C^2[-1, 1]$ is a proper subspace of both $C[-1, 1]$ and $C^1[-1, 1]$.

Example 8

- Let S be the set of all f in $C^2[a, b]$ such that

$$f''(x) + f(x) = 0$$

for all x in $[a, b]$. The set S is nonempty, since the zero function is in S. If $f \in S$ and a is any scalar, then, for any x in $[a, b]$,

$$\begin{aligned}(\alpha f)''(x) + (\alpha f)(x) &= \alpha f''(x) + \alpha f(x) \\ &= \alpha(f''(x) + f(x)) = \alpha \cdot 0 = 0\end{aligned}$$

Thus, $\alpha f \in S$.

Example 8 (con.)

- If f and g are both in S, then

$$
\begin{aligned}
(f+g)''(x)+(f+g)(x) &= f''(x)+g''(x)+f(x)+g(x) \\
&= [f''(x)+f(x)]+[g''(x)+g(x)] \\
&= 0+0=0
\end{aligned}
$$

- Hence, the set of all solutions on $[a, b]$ of the differential equation $y'' + y = 0$ forms a subspaces of $C^2[a, b]$. Note that $f(x) = \sin x$ and $g(x) = \cos x$ are both in S. Since S is a subspace, it follows that any function of the form $c_1 \sin x + c_2 \cos x$ must also be in S.

The Null Space of a Matrix

- Let A be an $m \times n$ matrix. Let $N(A)$ denote the set of all solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Thus

$$N(A) = \{\mathbf{x} \in R^n \mid A\mathbf{x} = 0\}$$

- $N(A)$ is called the **nullspace** of A

- **Is N(A) a subspace of R^n?**
  - If $\mathbf{x} \in N(A)$ and α is a scalar, then

$$A(\alpha\mathbf{x}) = \alpha(A\mathbf{x}) = \alpha\,\mathbf{0} = \mathbf{0}$$

$$\therefore \alpha\mathbf{x} \in \boldsymbol{N}(A)$$

  - If $\mathbf{x} \in \boldsymbol{N}(A)$ and $\mathbf{y} \in \boldsymbol{N}(A)$, then

$$A\,(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

$$\therefore\ \mathbf{x} + \mathbf{y} \in \boldsymbol{N}(A)$$

$\therefore\ N(A)$ is a subspace of R^n

Example 9

- Determine $N(A)$ if $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$
- *Sol:*

  Using the Gauss-Jordan reduction to solve $A\mathbf{x} = \mathbf{0}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

Example 9 (con.)

lead variable: x_1, x_2, and free variable: x_3, x_4

x_1 - x_3+ $x_4 = 0 \Rightarrow x_1 = x_3 - x_4$

$x_2 + 2x_3$ - $x_4 = 0 \Rightarrow x_2 =$ -$2x_3 + x_4$

set $x_3 = \alpha$, $x_4 = \beta$, then

$$\begin{bmatrix} \alpha-\beta \\ -2\alpha+\beta \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -2\alpha \\ \alpha \\ 0 \end{bmatrix} + \begin{bmatrix} -\beta \\ \beta \\ 0 \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

is a solution to $A\mathbf{x} = \mathbf{0}$

The vector space *N*(*A*) consists of all vectors

The Span of a Set of Vectors
Definition

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in a vector space V.
A sum of the form $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars, is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is called the **span** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, denoted by $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$.

- In example 9, the null space of A, $N(A)$, was the span of the vectors

  $(1, -2, 1, 0)^T$ and $(-1, 1, 0, 1)^T$

Example 10

- In R^3, the span of $\mathbf{e}_1$ and $\mathbf{e}_2$ is the set of all vectors of the form

$$\alpha\mathbf{e}_1 + \beta\mathbf{e}_2 = \alpha\begin{bmatrix}1\\0\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}\alpha\\\beta\\0\end{bmatrix}$$

Span($\mathbf{e}_1$, $\mathbf{e}_2$) is the set of all vectors in $\boldsymbol{R}^3$ that lie in the x_1x_2-plane.

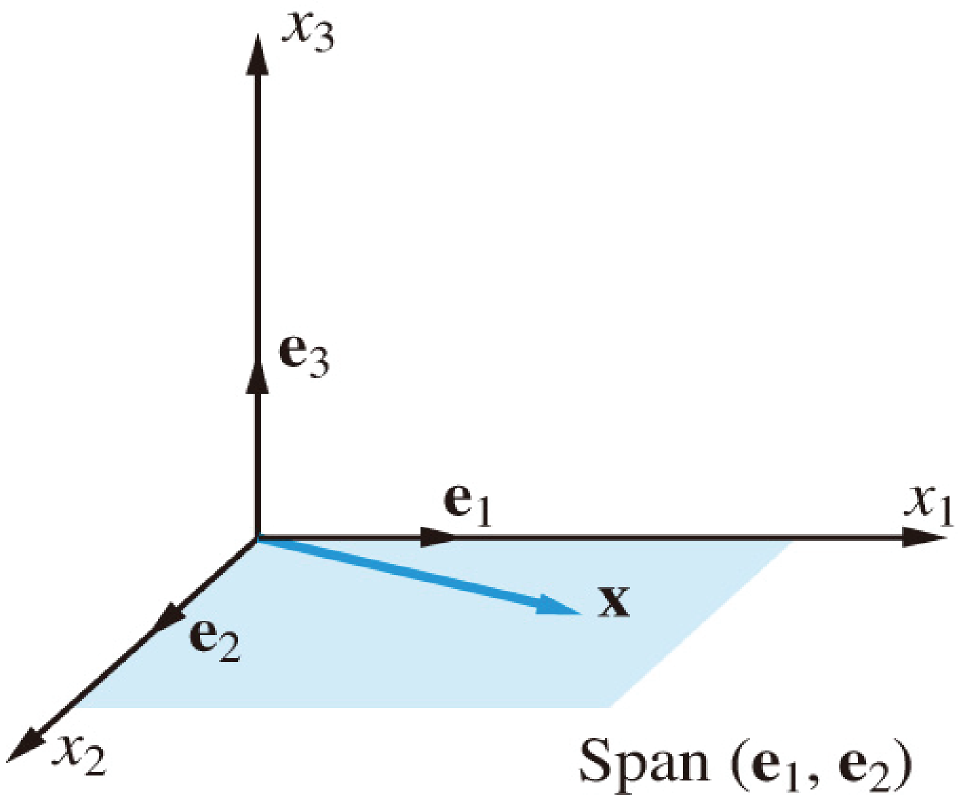
Example 10 (con.)

- Verify that Span($\mathbf{e}_1$, $\mathbf{e}_2$) is a subspace of R^3.
- The span of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is the set of all vectors of the form

$$\alpha\mathbf{e}_1 + \beta\mathbf{e}_2 + \gamma\mathbf{e}_3 = \alpha\begin{bmatrix}1\\0\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\\0\end{bmatrix} + \gamma\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}$$

- Thus, Span($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) = R^3
- A vector in R^3 is in **Span($\mathbf{e}_1$, $\mathbf{e}_2$)** iff it lies in the x_1x_2-plane in 3-space.

- We can think of the x_1x_2-plane as the geometrical representation of the subspace Span($\mathbf{e}_1$, $\mathbf{e}_2$).



Theorem 3.2.1

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are elements of a vector space V, then Span($\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$) is a <u>subspace</u> of V.

- *proof*
- Let $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$ be an arbitrary element of Span($\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$)
- By C1: $\beta\mathbf{v} = (\beta\alpha_1)\,\mathbf{v}_1 + (\beta\alpha_2)\,\mathbf{v}_2 + \ldots + (\beta\alpha_n)\,\mathbf{v}_n$

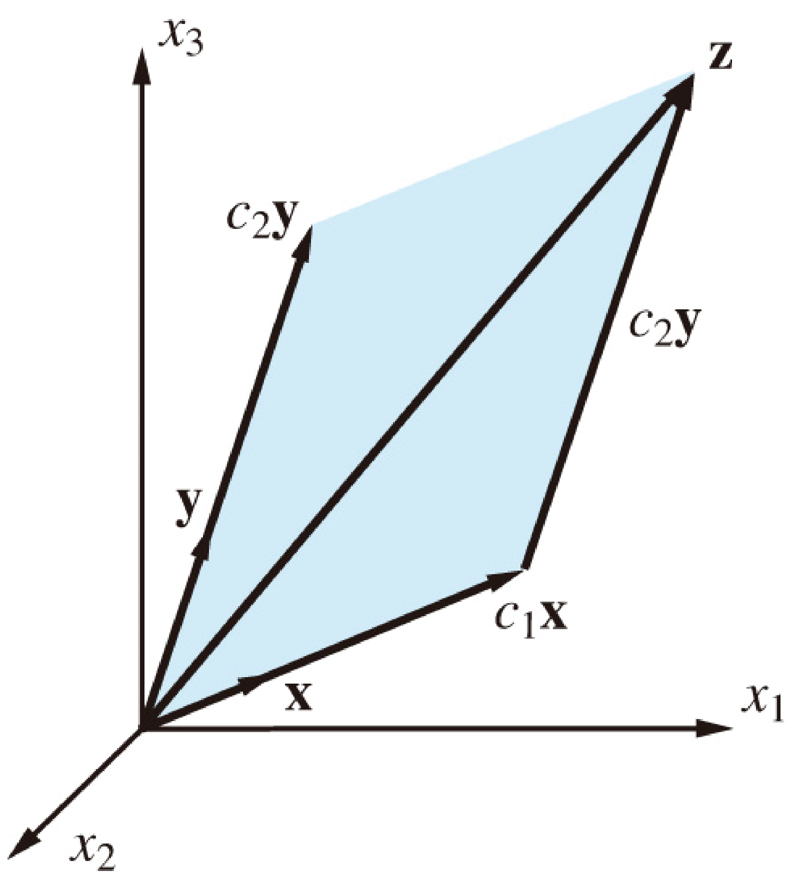
  $= \gamma_1\mathbf{v}_1 + \gamma_2\mathbf{v}_2 + \ldots + \gamma_n\mathbf{v}_n$

  Therefore $\beta\mathbf{v} \in$ Span($\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$)

- Let $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n \in$ Span($\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$) and
  $\mathbf{w} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \ldots + \beta_n\mathbf{v}_n \in$ Span($\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$)
  $\mathbf{v} + \mathbf{w} = (\alpha_1+\beta_1)\,\mathbf{v}_1 + (\alpha_2+\beta_2)\,\mathbf{v}_2 + \ldots + (\alpha_n+\beta_n)\,\mathbf{v}_n \in$ Span($\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$)
- Therefore, Span($\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$) is a subspace of V.

- Two vectors **x** and **y** in R^3, if (0, 0, 0), (x_1, x_2, x_3) and (y_1, y_2, y_3) are not collinear, these points determines a plane.
- If $\mathbf{z} = c_1\mathbf{x} + c_2\mathbf{y}$, then **z** is a sum of vectors parallel to **x** and **y** and hence must lie on the plane determined by the two vectors.

- In general, if two vectors **x** and **y** can be used to determine a plane in 3-space R^3, that plane is the geometrical representation of Span(**x**, **y**).

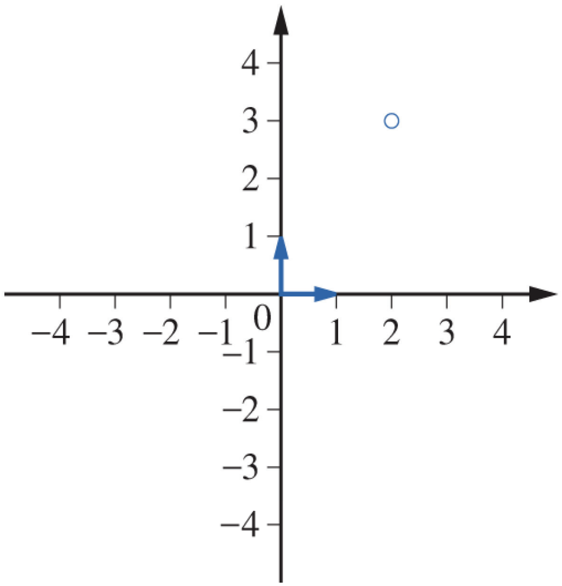


Spanning Set for a Vector Space

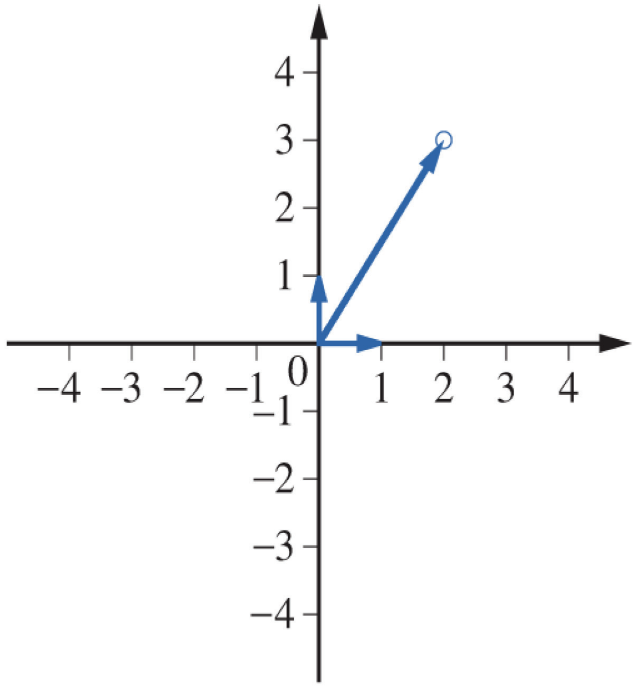
- Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in a vector space V. Span($\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$) is referred to as the subspace *spanned* by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
- If Span($\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$) = V, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is said to *span* V or that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a *spanning set* for V.

Definition

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a **spanning set** for V iff every vector in V can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.



Terminal point of first vector (1, 0)
Terminal point of second vector (0, 1)
Target point (2, 3)



$c_1 = 2 \qquad c_2 = 3$

Example 11

- Which of the following are spanning sets for R^3?

(a) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1, 2, 3)^T\}$

(b) $\{(1, 1, 1)^T, (1, 1, 0)^T, (1, 0, 0)^T\}$

(c) $\{(1, 0, 1)^T, (0, 1, 0)^T\}$

(d) $\{(1, 2, 4)^T, (2, 1, 3)^T, (4, -1, 1)^T\}$

Example 11(a) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1, 2, 3)^T\}$

- let $(a, b, c)^T \in R^3$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + c\,\mathbf{e}_3 + 0\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ (a) is a spanning set for R^3.

Note

- The **"standard" spanning set** for R^3:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Example 11(b)
$\{(1, 1, 1)^T, (1, 1, 0)^T, (1, 0, 0)^T\}$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 \\ \alpha_1 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= a \\ \alpha_1 + \alpha_2 \qquad &= b \\ \alpha_1 \qquad\qquad &= c \end{aligned}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} c \\ b - c \\ a - b \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (b-c)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (a-b)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c \\ c \\ c \end{bmatrix} + \begin{bmatrix} b-c \\ b-c \\ 0 \end{bmatrix} + \begin{bmatrix} a-b \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore$ (b) is a spanning set for R^3.

Example 11(c)
$\{(1, 0, 1)^T, (0, 1, 0)^T\}$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_1 \end{bmatrix}$$

if $a \neq c$, then $\notin$ $\text{Span}\{(1, 0, 1)^T, (0, 1, 0)^T\}$

$\therefore$ (c) is **not** a spanning set for R^3

Example 11(d)
$\{(1, 2, 4)^T, (2, 1, 3)^T, (4, -1, 1)^T\}$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 + 4\alpha_3 \\ 2\alpha_1 + \alpha_2 - \alpha_3 \\ 4\alpha_1 + 3\alpha_2 + \alpha_3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} \alpha_1 + 2\alpha_2 + 4\alpha_3 &= a \\ 2\alpha_1 + \alpha_2 - \alpha_3 &= b \\ 4\alpha_1 + 3\alpha_2 + \alpha_3 &= c \end{aligned}$$

$$\left[\begin{array}{rrr|c} 1 & 2 & 4 & a \\ 2 & 1 & -1 & b \\ 4 & 3 & 1 & c \end{array}\right] \Rightarrow \left[\begin{array}{rrr|c} 1 & 2 & 4 & a \\ 0 & -3 & -9 & -2a+b \\ 0 & -5 & -15 & -4a+c \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{rrr|c} 1 & 2 & 4 & a \\ 0 & 1 & 3 & (2a-b)/3 \\ 0 & 0 & 0 & -(2a-5b+3c)/3 \end{array}\right]$$

if $2a - 3c + 5b \neq 0$, then the system is <u>inconsistent</u>

$\therefore$ (d) is **not** a spanning set for R^3

Example 12

- The vectors $1 - x^2$, $x + 2$, and x^2 span P_3?
- *Sol:*

If $ax^2 + bx + c$ is any polynomial in P_3.

$$ax^2 + bx + c = \alpha_1(1 - x^2) + \alpha_2(x + 2) + \alpha_3 x^2$$

$$= (\alpha_3 - \alpha_1) x^2 + (\alpha_2) x + (\alpha_1 + 2\alpha_2)$$

$$\therefore \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \alpha_3 - \alpha_1 \\ \alpha_2 \\ \alpha_1 + 2\alpha_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} c - 2b \\ b \\ a + c - 2b \end{bmatrix}$$

$\therefore 1 - x^2$, $x + 2$, and x^2 span P_3

- **Theorem 3.2.2** *If the linear system* $A\mathbf{x} = \mathbf{b}$ *is consistent and* $\mathbf{x}_0$ *is a particular solution, then a vector* $\mathbf{y}$ *will also be a solution if and only if* $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$ *where* $\mathbf{z} \in N(A)$.

Let S be the solution set to a consistent $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$. In the case that $\mathbf{b} = \mathbf{0}$ we have that $S = N(A)$ and consequently the solution set forms a subspace of $\mathbb{R}^n$. If $\mathbf{b} \neq \mathbf{0}$, then S does not form a subspace of $\mathbb{R}^n$; however, if one can find a particular solution $\mathbf{x}_0$, then it is possible to represent any solution vector in terms of $\mathbf{x}_0$ and a vector $\mathbf{z}$ from the null space of A.

Let $A\mathbf{x} = \mathbf{b}$ be a consistent linear system and let $\mathbf{x}_0$ be a particular solution to the system. If there is another solution $\mathbf{x}_1$ to the system, then the difference vector $\mathbf{z} = \mathbf{x}_1 - \mathbf{x}_0$ must be in $N(A)$ since

$$A\mathbf{z} = A\mathbf{x}_1 - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

Thus if there is a second solution, it must be of the form $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{z}$ where $\mathbf{z} \in N(A)$.

In general, if $\mathbf{x}_0$ is a particular solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{z}$ is any vector in $N(A)$, then setting $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$, we have

$$A\mathbf{y} = A\mathbf{x}_0 + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

So $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$ must also be a solution to the system $A\mathbf{x} = \mathbf{b}$.

3.3 Linear Independence

- Each vector in the vector space can be built up from the elements in a generating set (spanning set) using only the operations of addition and scalar multiplication.
- In general, it is desirable to find a *minimum spanning set*.
- How the vectors in the generating set *depend* on each other.

Linear dependence/independence

- Consider the following vectors in R^3:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix}$$

- Let S be the subspace of R^3 spanned by $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$.

$$\Rightarrow 3\mathbf{x}_1 + 2\mathbf{x}_2 = \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix} + \begin{bmatrix} -4 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix} = \mathbf{x}_3$$

$\Rightarrow$Any linear Combination of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ can be reduced to a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3$$

$$= \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3(3\mathbf{x}_1 + 2\mathbf{x}_2)$$

$$= (\alpha_1 + 3\alpha_3)\,\mathbf{x}_1 + (\alpha_2 + 2\alpha_3)\,\mathbf{x}_2$$

Thus, $\mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2)$

- Eq. (1) can be rewritten as

$$3\mathbf{x}_1 + 2\mathbf{x}_2 - 1\mathbf{x}_3 = \mathbf{0}$$

$$\mathbf{x}_1 = -\frac{2}{3}\mathbf{x}_2 + \frac{1}{3}\mathbf{x}_3, \quad \mathbf{x}_2 = -\frac{3}{2}\mathbf{x}_1 + \frac{1}{2}\mathbf{x}_3, \quad \mathbf{x}_3 = 3\mathbf{x}_1 + 2\mathbf{x}_2$$

- Span($\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$)
  = Span($\mathbf{x}_1$, $\mathbf{x}_2$) = Span($\mathbf{x}_2$, $\mathbf{x}_3$) = Span($\mathbf{x}_1$, $\mathbf{x}_3$)

- Because of the **dependency relation**:

$$3\mathbf{x}_1 + 2\mathbf{x}_2 - \mathbf{x}_3 = \mathbf{0}$$

  the subspace *S* can be represented as the span of *any two* of the given vectors.

- Suppose there are scalars c_1 and c_2, not both 0, such that:

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = 0$$

then we could solve for one of the vectors in terms of the other:

$$\mathbf{x}_1 = -\frac{c_2}{c_1}\mathbf{x}_2 \ \ (c_1 \neq 0), \quad \mathbf{x}_2 = -\frac{c_1}{c_2}\mathbf{x}_1 \ \ (c_2 \neq 0)$$

- However, neither of the two vectors in question is a multiple of the others
- The only way that (3) can hold is if $c_1 = c_2 = 0$.

Summarization

(I) If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span a vector space V and one of these vectors can be written as a linear combination of the other $n - 1$ vectors, then these $n - 1$ vectors span V.

(II) Given n vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, it is possible to write one of the vectors as a linear combination of the other $n - 1$ vectors iff there exist scalars $c_1, c_2, \ldots, c_n$ <u>not all zero</u> such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

Proof (I)

- Suppose $\mathbf{v}_n$ can be written as a linear combination of the other $n - 1$ vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$:

$$\mathbf{v}_n = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \ldots + \beta_{n-1}\mathbf{v}_{n-1}$$

- Let $\mathbf{v}$ be any vectors in V, since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span V, we can write

$$\begin{aligned}\mathbf{v} &= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_{n-1}\mathbf{v}_{n-1} + \alpha_n\mathbf{v}_n \\ &= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_{n-1}\mathbf{v}_{n-1} + \alpha_n(\beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \ldots + \beta_{n-1}\mathbf{v}_{n-1}) \\ &= (\alpha_1+\alpha_n\beta_1)\mathbf{v}_1 + (\alpha_2+\alpha_n\beta_2)\mathbf{v}_2 + \ldots + (\alpha_{n-1}+\alpha_n\beta_{n-1})\mathbf{v}_{n-1}\end{aligned}$$

- Thus, any vectors $\mathbf{v}$ in V can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$

Proof (II)

($\Rightarrow$)

Suppose $\mathbf{v}_n$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$:

$$\mathbf{v}_n = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_{n-1}\mathbf{v}_{n-1}$$

Subtracting $\mathbf{v}_n$ from both sides of the equation, we get

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_{n-1}\mathbf{v}_{n-1} - \mathbf{v}_n = \mathbf{0}$$

If we set $c_i = \alpha_i$ for $i = 1, 2, \ldots, n\text{-}1$ and $c_n = -1$, then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

(⇦) Conversely, if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

and at least one of the c_i's, say c_n, is nonzero, then

$$\mathbf{v}_n = \frac{-c_1}{c_n}\mathbf{v}_1 + \frac{-c_2}{c_n}\mathbf{v}_2 + \cdots + \frac{-c_{n-1}}{c_n}\mathbf{v}_{n-1}$$

Definition

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in a vector space V are said to be **linearly independent** if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

implies that $c_1 = c_2 = \ldots = c_n = 0$

- If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a *minimum spanning set*, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent
- Conversely, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent and span V, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a *minimum spanning set* for V.
- A *minimum spanning set* is called a **basis**.

Example 1

- Are the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ linearly independent or linearly dependent?
- *Sol:*

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow c_1 + c_2 = 0$$

$$c_1 + 2c_2 = 0$$

$$\Rightarrow c_1 = c_2 = 0$$

$\therefore$ linearly independent

Definition

The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$ in a vector space V are said to be **linearly dependent** if there exist scalars c_1, c_2,…, c_n not all zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

Example 2

- Are the vectors $\mathbf{x} = (1, 2, 3)^T$, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ linearly independent or linearly dependent?
- *Sol:*

$$c_1\begin{bmatrix}1\\2\\3\end{bmatrix}+c_2\begin{bmatrix}1\\0\\0\end{bmatrix}+c_3\begin{bmatrix}0\\1\\0\end{bmatrix}+c_4\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow\begin{bmatrix}c_1+c_2\\2c_1+c_3\\3c_1+c_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow c_1 = 1,\ c_2 = -1,\ c_3 = -2,\ c_4 = -3$

$\therefore$ linearly dependent

Note

- Give a set of vectors $\{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n\}$ in a vector space V, it is trivial to find scalars $c_1, c_2, \ldots, c_n$ such that

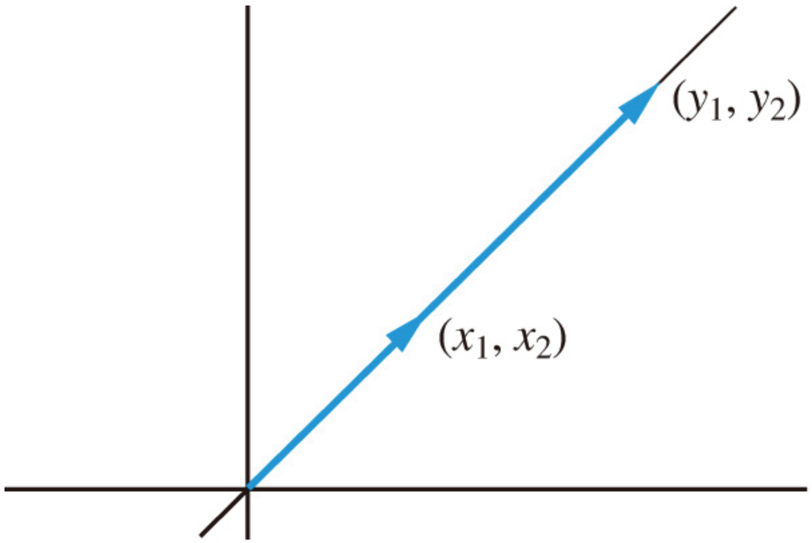
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

Just take

$$c_1 = c_2 = \ldots = c_n = 0$$

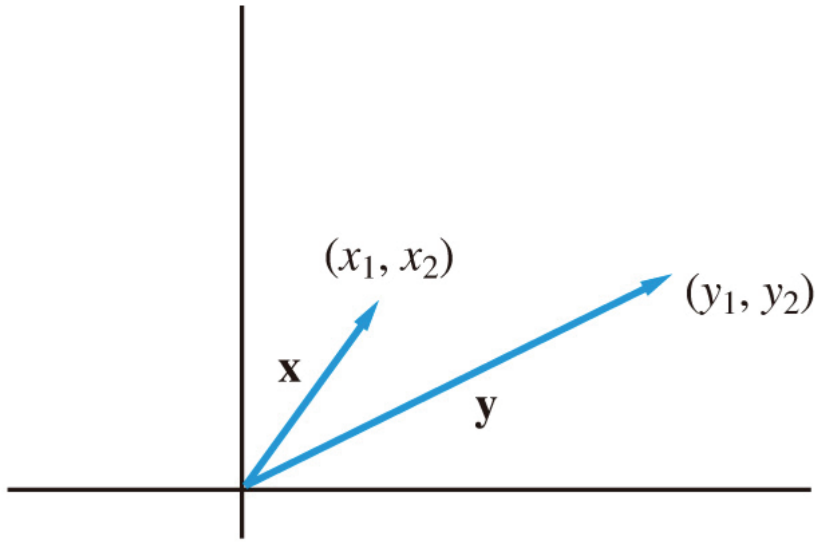
If there are no trivial choices of scalars for which the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ equals the zero vector, then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ are linearly dependent. If the only way the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ can equal the zero vector is for all the scalars $c_1, c_2, \ldots, c_n$ to be 0, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.

Geometric Interpretation



(a) **x** and **y** linearly dependent

$c_1\mathbf{x} + c_2\mathbf{y} = \mathbf{0}$
c_1 and c_2 are not both 0,
say $c_1 \neq 0$, then
$\mathbf{x} = (-c_2/c_1)\mathbf{y} = k\,\mathbf{y}$



(b) **x** and **y** linearly independent

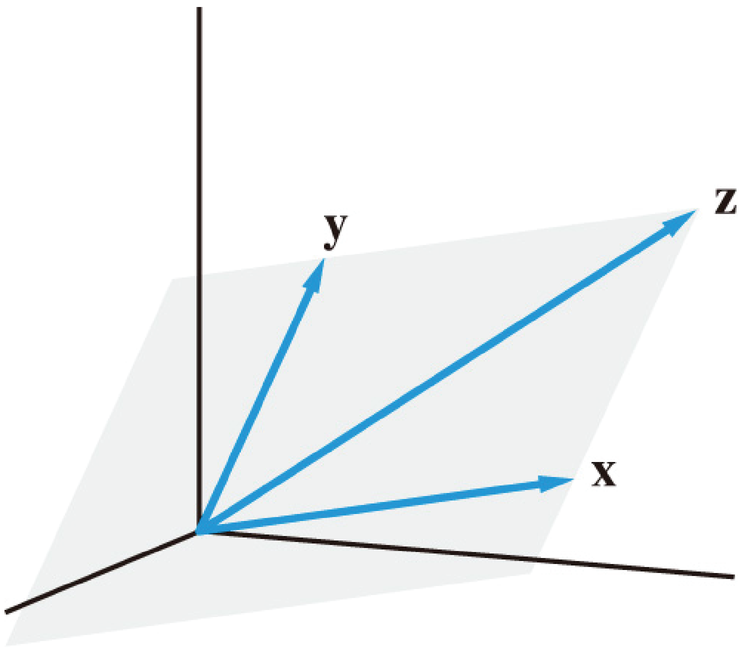
$\mathbf{x} \neq k\,\mathbf{y}$

- If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_2 \end{bmatrix}$

  are linearly independent in R^3, then the two points (x_1, x_2, x_3) and (y_1, y_2, y_3) will not lie on the same line through the origin in 3-space.

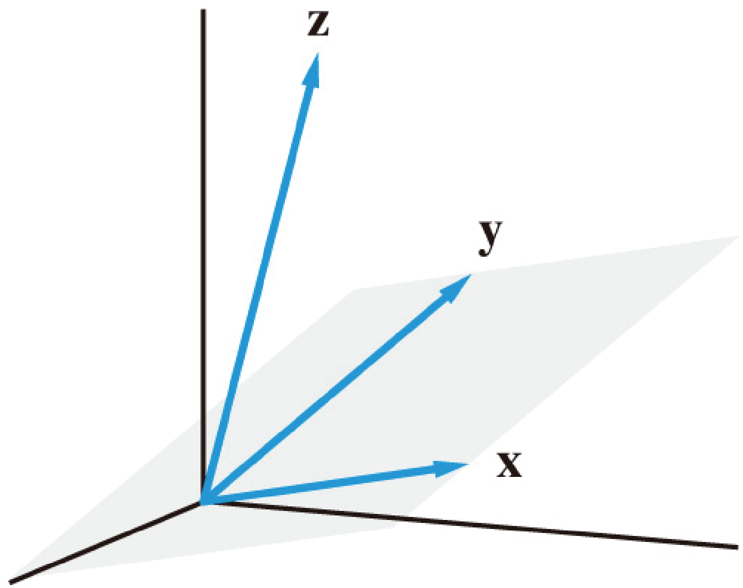
- Since (0, 0, 0), (x_1, x_2, x_3), and (y_1, y_2, y_3) are not collinear, they determine a plane.

- If (z_1, z_2, z_3) lies on this plane, the vector $\mathbf{z} = (z_1, z_2, z_3)^T$ can be written as a linear combination of $\mathbf{x}$ and $\mathbf{y}$, and hence $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are linearly dependent.



(a)

- If (z_1, z_2, z_3) does not lie on the plane, the three vectors will be linearly independent.



(b)

Theorems and Examples
Example 3

- Which of the following collections of vectors are linearly independent in R^3?

(a) $(1, 1, 1)^T$, $(1, 1, 0)^T$, $(1, 0, 0)^T$

(b) $(1, 0, 1)^T$, $(0, 1, 0)^T$

(c) $(1, 2, 4)^T$, $(2, 1, 3)^T$, $(4, -1, 1)^T$

Example 3 (a)

- We must show tjat the only way for

  $$c_1(1, 1, 1)^T + c_2(1, 1, 0)^T + c_3(1, 0, 0)^T = (0, 0, 0)^T$$

  is if the scalars c_1, c_2, c_3 are all zero.

  $$\Rightarrow \begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 + c_2 \quad\; &= 0 \\ c_1 \qquad\quad\; &= 0 \end{aligned}$$

  $\Rightarrow c_1 = 0,\ c_2 = 0,\ c_3 = 0.$

Example 3 (b)

- If

$$c_1\,(1, 0, 1)^T + c_2\,(0, 1, 0)^T = (0, 0, 0)^T$$

  then

$$(c_1, c_2, c_1)^T = (0, 0, 0)^T$$

  so $c_1 = c_2 = 0$.
- Therefore, the two vectors are linearly independent.

Example 3 (c)

- If

  $c_1(1, 2, 4)^T + c_2(2, 1, 3)^T + c_3(4, -1, 1)^T = (0, 0, 0)^T$

  then

$$\begin{array}{rcrcrcl} c_1 & + & 2c_2 & + & 4c_3 & = & 0 \\ 2c_1 & + & c_2 & - & c_3 & = & 0 \\ 4c_1 & + & 3c_2 & + & c_3 & = & 0 \end{array}$$

- The coefficient matrix of the system is singular and hence the system has nontrivial solution. Therefore, the vectors are linearly dependent.

Theorem 3.3.1

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n vectors in R^n and let $X = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ then the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ will be *linearly dependent* iff *X is singular* (i.e., $\det(X) = 0$)

proof: $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n = \mathbf{0}$

$$c_1\begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix} + c_2\begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix} + \cdots + c_n\begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow c_1 x_{11} + c_2 x_{12} + \ldots + c_n x_{1n} = 0$$

$$c_1 x_{21} + c_2 x_{22} + \ldots + c_n x_{2n} = 0$$

$$\vdots$$

$$c_1 x_{n1} + c_2 x_{n2} + \ldots + c_n x_{nn} = 0$$

Let $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T$, then the system can be written as

$$X\mathbf{c} = \mathbf{0}$$

c has a nontrivial solution iff ***X*** is singular

$\Rightarrow \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ will be linearly dependent iff X is singular.

To test whether n vectors are linearly independent in R^n:

(1) Form a matrix X whose columns are the vectors being tested.

(2) To determine whether X is singular, calculate the value of $\det(X)$.

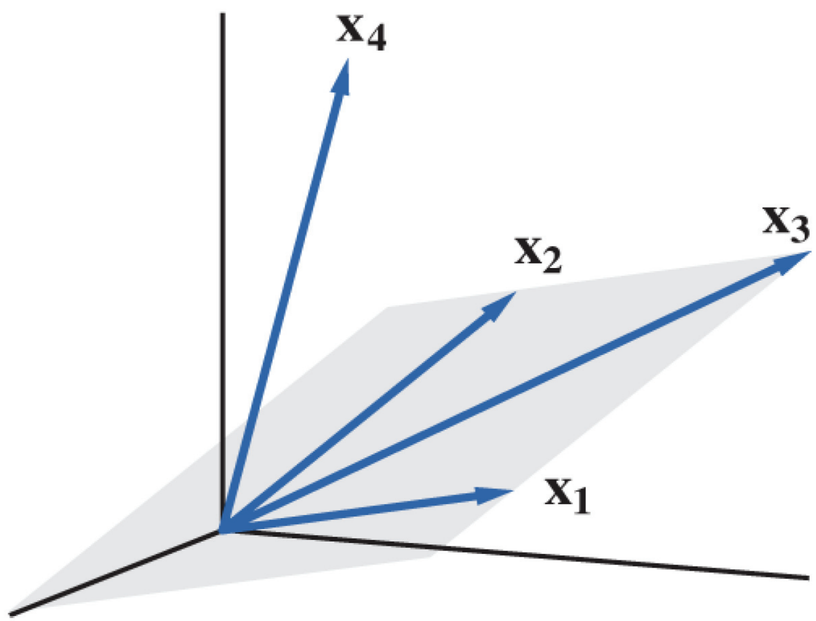
(a) If $\det(X) = 0$ (X is singular) ⇨ the vectors are linearly dependent

(b) If $\det(X) \neq 0$ (X is nonsingular) ⇨ the vectors are linearly independent

Example 4

- The following vectors are pictured in Figure 3.3.3.

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$



Example 4 (con.)

- We can see a dependency relation among the first three of the vectors since

$$\mathbf{x}_3 = \mathbf{x}_1 + \mathbf{x}_2$$

- In this case, the vector $\mathbf{x}_3$ lies in the plane spanned by $\mathbf{x}_1$ and $\mathbf{x}_2$. It follows then that

$$\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3 + 0\mathbf{x}_4 = 0$$

- The collection of four vectors must be linearly dependent since the scalars $c_1 = 1$, $c_2 = 1$, $c_3 = -1$, $c_4 = 0$ are not all 0.

Example 5

- Are $(4, 2, 3)^T$, $(2, 3, 1)^T$, $(2, -5, 3)^T$ linearly independent or dependent?
- *Sol:*

$$\begin{vmatrix} 4 & 2 & 2 \\ 2 & 3 & -5 \\ 3 & 1 & 3 \end{vmatrix} = 0 \Rightarrow \text{linearly dependent!}$$

Note

- To test whether k vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly independent in R^n, we can rewrite the equation

  $$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_k\mathbf{x}_k = \mathbf{0}$$

  as a linear system $X\mathbf{c} = \mathbf{0}$, where $X = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)$.
- If $k \neq n$, the matrix X is not square, $\det(X) = ?$
- The system is homonegeous, so it has the trivial solution $\mathbf{c} = \mathbf{0}$.

- It will have nontrivial solutions iff *the row echelon form of X involve free variables*.
- If there are nontrivial solutions, then the vectors are linearly dependent.
- If there are no free variables, then $\mathbf{c} = \mathbf{0}$ is the only solution ⇨ linearly independent

Example 6

- Given

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 7 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ -1 & 3 & 0 & 0 \\ 2 & 1 & 7 & 0 \\ 3 & -2 & 7 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There exist a free variable c_3 ⇨ there are nontrivial solutions ⇨ linearly dependent.

Theorem 3.3.2

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in a vector space V. A vector $\mathbf{v} \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ can be written <u>uniquely</u> as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ iff $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are <u>linearly independent</u>.

- *Linear combinations of linearly independent vectors are unique.*

Theorem 3.3.2 *proof*

- If $\mathbf{v} \in$ Span($\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$), then v can be written as a linear combination

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$$

- Suppose that $\mathbf{v}$ can also be expressed as a linear combination

$$\mathbf{v} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \ldots + \beta_n\mathbf{v}_n$$

- We will show that , if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, that $\beta_i = \alpha_i$, $i = 1, 2, \ldots, n$ and if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, then it is possible to choose the β_i's different from the α_i's.

Theorem 3.3.2 *proof*

- If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, then subtracting (6) from (5) yield
  $0 = (\alpha_1 - \beta_1)\,\mathbf{v}_1 + (\alpha_2 - \beta_2)\,\mathbf{v}_2 + \ldots + (\alpha_\mathrm{n} - \beta_\mathrm{n})\,\mathbf{v}_n$
  By the linearly independent of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, the coefficients of (7) must will be all 0. Hence,
  $$\alpha_1 = \beta_1,\ \alpha_2 = \beta_2,\ \ldots,\ \alpha_n = \beta_n$$
  Thus, the representation (5) is unique then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.

Theorem 3.3.2 *proof*

- On the other hand, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, then there exist $c_1, c_2, \ldots, c_n$, not all 0, such that

$$\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

- Now if we set

$$\beta_1 = \alpha_1 + c_1,\ \beta_2 = \alpha_2 + c_2,\ \ldots,\ \beta_n = \alpha_n + c_n$$

then, adding (5) and (8), we get

$$\begin{aligned}\mathbf{v} &= (\alpha_1 + c_1)\mathbf{v}_1 + \cdots + (\alpha_n + c_n)\mathbf{v}_n \\ &= \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots \beta_n\mathbf{v}_n\end{aligned}$$

Theorem 3.3.2 *proof*

- Since the c_i's are not all 0, $\beta_i \neq \alpha_i$ for at least one value of i. Thus, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, the representation of a vector as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is not unique.

Vector Spaces of Functions – the Vector Space P_n

- To test whether the following polynomials $p_1, p_2, \ldots,$ p_k are linearly independent in P_n, we set

  $$c_1 p_1 + c_2 p_2 + \ldots + c_k p_k = z$$

  where **z** represent the zero polynomial:

  $$z(x) = 0x^{n-1} + 0x^{n-2} + \ldots + 0x + 0$$

- Let $c_1 p_1 + c_2 p_2 + \ldots + c_k p_k = a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$, then

  $$a_i = 0, \;\; 1 \le i \le n$$

- Each of the a_i's is a linear combination of the c_j's
- A homogeneous linear system with unknowns c_1, $c_2, \ldots, c_k$
- If the system has only the trivial solution, the polynomials are linearly independent; otherwise, they are linearly dependent.

Example 7

- Are the following polynomials linearly independent?

  $p_1(x) = x^2 - 2x + 3$, $p_2(x) = 2x^2 + x + 8$,

  $p_3(x) = x^2 + 8x + 7$

- *Sol:*

  $c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) = 0x^2 + 0x + 0$

  $c_1\ (x^2 - 2x + 3) + c_2(2x^2 + x + 8) + c_3(x^2 + 8x + 7)$

  $= (c_1 + 2c_2 + c_3)\ x^2 + (-2c_1 + c_2 + 8c_3)\ x + (3c_1 + 8c_2 +$
  $7c_3)$

- $= 0x^2 + 0x + 0$

Example 7 (con.)

$$\Rightarrow \quad \begin{aligned} c_1 + 2c_2 + c_3 &= 0 \\ -2c_1 + c_2 + 8c_3 &= 0 \\ 3c_1 + 8c_2 + 7c_3 &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $\begin{vmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{vmatrix} = 0 \Rightarrow$ linearly dependent.

The Vector Space $C^{(n-1)}[a, b]$

- Let $f_1, f_2, \ldots, f_n$ be elements of $C^{(n-1)}[a, b]$, if these vectors are linearly dependent, then exist scalars c_1, $c_2, \ldots, c_n$, not all zero, such that

  $$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

  for each x in $[a, b]$. Taking the derivatives with respect to x of both sides of (10) yields

  $$c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_n f_n'(x) = 0$$

- For each continue taking of both sides, we end up with the system

$$
\begin{aligned}
&c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \\
&c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_n f_n'(x) = 0 \\
&\vdots \\
&c_1 f_1^{(n-1)}(x) + c_2 f_2^{(n-1)}(x) + \cdots + c_n f_n^{(n-1)}(x) = 0
\end{aligned}
$$

- For each fixed x in $[a, b]$, the matrix equation

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

will have the same nontrivial solution $(c_1, c_2, \ldots, c_n)^T$. Thus, if $f_1, f_2, \ldots, f_n$ are linearly dependent in $C^{(n-1)}[a, b]$, then, for each fixed x in $[a, b]$, the coefficient matrix of system (11) is singular. It the matrix is singular, its determinant is zero.

Definition

Let $f_1, f_2, \ldots, f_n$ be functions in $C^{(n\text{-}1)}[a, b]$, and define the function $W[f_1, f_2, \ldots, f_n](x)$ in $[a, b]$ by

$$W[f_1, f_2, \cdots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

The function $W[f_1, f_2, \ldots, f_n](x)$ is called the **Wronskian** of $f_1, f_2, \ldots, f_n$.

Theorem 3.3.3

Let $f_1, f_2, \ldots, f_n$ be elements of $C^{(n\text{-}1)}[a, b]$. If there exists a points x_0 in [a, b] such that $W[f_1, f_2, \ldots, f_n](x_0) \neq 0$, then $f_1, f_2, \ldots, f_n$ are linearly independent.

- *Proof*

  If $f_1, f_2, \ldots, f_n$ were linearly dependent, then by the preceding discussion, the coefficient matrix in (11) would be singular for each x in $[a, b]$ and hence $W[f_1, f_2, \ldots, f_n](x)$ would be identically zero on $[a, b]$.

Example 8

- Show that e^x and e^{-x} are linearly independent in $C(-\infty, \infty$.
- *Sol:*

$$W\left[e^x, e^{-x}\right] = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^x \end{vmatrix} = -2$$

Since W[e^x, e^{-x}] is not identically zero, e^x and e^{-x} are linearly independent.

Example 9

- Consider the function x^2 and $x|x|$ in $C[-1, 1]$. Both functions are in the subspace $C^1[-1, 1]$, so we can compute the Wronskian

$$W[x^2, x|x|] = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} \equiv 0$$

- Since the Wronskian is identically zero, it gives no information as to whether the functions are linearly independent.

Example 9 (con.)

- To answer the question, suppose that

$$c_1 x^2 + c_2 x|x| = 0$$

for all x in [-1, 1]. Then, in particular for $x = 1$ and $x = -1$, we have

$$c_1 + c_2 = 0$$
$$c_1 - c_2 = 0$$

and the only solution of this system is $c_1 = c_2 = 0$. Thus, the functions x^2 and $x|x|$ are linearly independent in $C[-1, 1]$ even through $W[x^2, x|x|] \equiv 0$.

Example 10

- Show that the vectors 1, x, x^2, and x^3 are linearly independent in $C((-\infty, \infty))$.
- *Sol:*

$$W[1, x, x^2, x^3] = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12$$

Since $W[1, x, x^2, x^3] \neq 0$, the vectors are linear independent.

3.4 Basis and Dimension

Definition

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a **basis** for a vector space V if and only if

(1) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent (minimal spanning set)

(2) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span V (spanning set)

Example 1

- The *standard basis* for R^3 is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$; however, there are many bases that we could choose for R^3. For example,

$$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}2\\0\\1\end{bmatrix}\right\} \text{ and } \left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$$

are both bases for R^3. We will see shortly that any basis for R^3 must have exactly three elements.

Example 2

- In $R^{2\times 2}$, consider the set $\{E_{11}, E_{12}, E_{21}, E_{22}\}$, where

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

If $c_1E_{11} + c_2E_{12} + c_3E_{21} + c_4E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

then $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So, E_{11}, E_{12}, E_{21}, E_{22} are linearly independent.

- If A $=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in $R^{2\times 2}$, then

A= $aE_{11} + bE_{12} + cE_{21} + dE_{22}$

- Thus, E_{11}, E_{12}, E_{21}, E_{22} span $R^{2\times 2}$, and hence form a basis for $R^{2\times 2}$.

Theorem 3.4.1

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for a vector space V, then any collection of m vectors in V, where $m > n$, is linearly dependent.

- ***Proof***
- **Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be m vectors in V, when $m > n$. Then, since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span V, we have**

$$\mathbf{u}_i = a_{i1}\mathbf{v}_1 + a_{i2}\mathbf{v}_2 + \cdots a_{in}\mathbf{v}_n \quad \text{for} \quad i = 1, 2, \ldots, m$$

Theorem 3.4.1 *proof* (con.)

- A linearly combination $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_m\mathbf{u}_m$ can be written in the form

$$c_1\sum_{j=1}^{n} a_{1j}\mathbf{v}_j + c_2\sum_{j=1}^{n} a_{2j}\mathbf{v}_j + \cdots + c_m\sum_{j=1}^{n} a_{mj}\mathbf{v}_j$$

Rearranging the terms, we see that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_m\mathbf{u}_m = \sum_{i=1}^{m}\left[c_i\left(\sum_{j=1}^{n} a_{ij}\mathbf{v}_j\right)\right] = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}c_i\right)\mathbf{v}_j$$

Theorem 3.4.1 *proof* (con.)

- Now consider the system of equation

$$\sum_{i=1}^{m} a_{ij} c_i = 0 \qquad j = 1, 2, \ldots, n$$

- This is a homogeneous system with more unknowns than equations. Therefore, by Theorem 1.2.1, the system must have a nontrivial solution $(\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_m)^T$
- But then

$$\hat{c}_1 \mathbf{u}_1 + \hat{c}_2 \mathbf{u}_2 + \ldots + \hat{c}_m \mathbf{u}_m = \sum_{j=1}^{n} 0 \mathbf{v}_j = \mathbf{0}$$

Hence, $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_m$ are linearly deoendent.

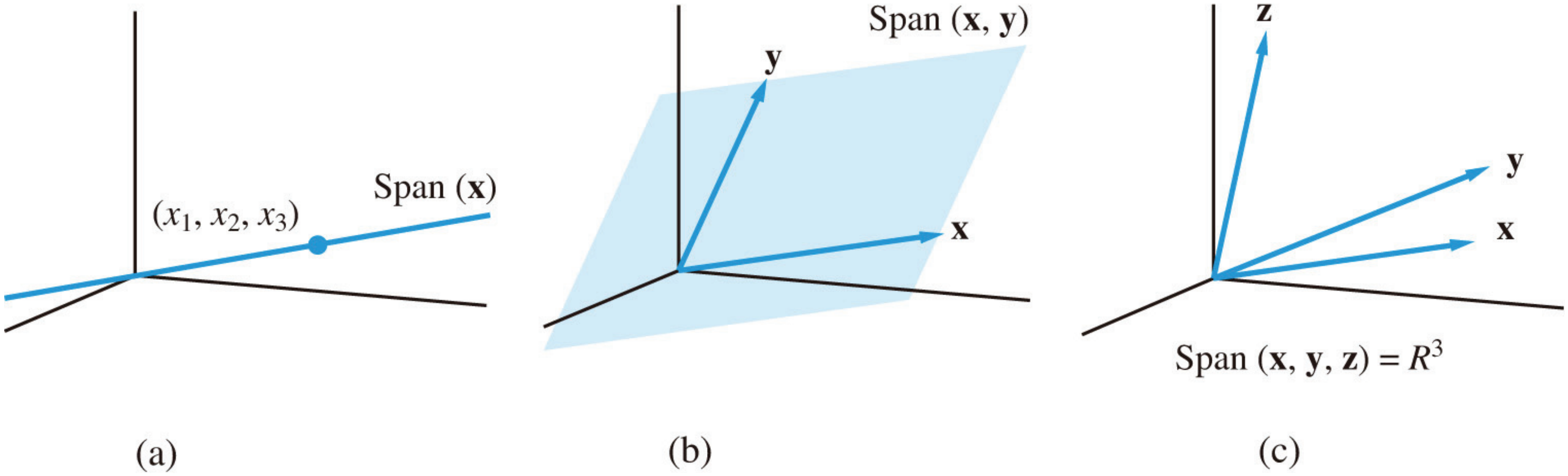
Corollary 3.4.2

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ are both bases for a vector space V, then $n = m$.

- *Proof*
- Let both $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be abses for V. Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span V and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are linearly independent, it follows from Theorem 3.4.1 that $m \le n$. By same reasoning, $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ span V and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, so $n \le m$.

Definition

Let V be vector space. If V has a basis consisting of n vectors, we say that V has **dimension** n. The subspace $\{\mathbf{0}\}$ of V is said to have dimension 0. V is said to be **finite-dimensional** if there is a finite set of vectors that spans V; otherwise we say that V is **infinite-dimensional**.



(a) Span($\mathbf{x}$) = $\{\alpha\mathbf{x} \mid \alpha$ is a scalar$\}$: line

(b) Span($\mathbf{x}, \mathbf{y}$) = $\{\alpha\mathbf{x} + \beta\mathbf{y} \mid \alpha, \beta$ are scalars$\}$: plane

(c) Span($\mathbf{x}, \mathbf{y}, \mathbf{z}$) = R^3

Example 3

- Let P be the vector space of all polynomial. We claim that P is infinite dimensional. If P were finite dimensional, say, if dimension n, any set of n+1 vectors would be linearly dependent.
- However, 1, x, x^2, …, x^n are linearly independent, since $W[1, x, x^2, \ldots, x^n] > 0$. Therefore, P cannot be of dimension n. Since m was arbitrary, P must be infinite dimension. The same argument shows that $C[a, b]$ is infinite dimensional.

Theorem 3.4.3

If V is a vector space of dimension $n > 0$

(I) Any set of n linearly independent vectors spans V;

(II) Any n vectors that span V are linearly independent.

Example 4

- Show that $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for R^3

- *Sol:*

- Since dim $R^3 = 3$, we need only show that these vectors are linearly independent.

$$\begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix} = 2 \neq 0 \Rightarrow \text{linearly independent}$$

Theorem 3.4.4

If V is a vector space of dimension $n > 0$, then

(I) No set of less then n vectors can span V

(II) Any subset of less then n linearly independent vectors can be extended to form a basis for V

(III) Any spanning set containing more than n vectors can be pared down to form a basis for V.

Standard Bases

- The standard basis for R^n is $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$
- The standard basis for $R^{2\times 2}$ is $\{E_{11}, E_{12}, E_{21}, E_{22}\}$
- The standard basis for P_n is $\{1, x, x^2, x^3, \ldots, x^{n-1}\}$

3.5 Change of Basis
Changing Coordinates in R^2

- The standard basis for R^2 is $\{\mathbf{e}_1, \mathbf{e}_2\}$, any vector $\mathbf{x} \in R^2$ can be expressed as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

$\Rightarrow$ The scalars x_1, x_2 are **coordinates** of $\mathbf{x}$ with respect to the standard basis.

- For another basis $\{\mathbf{y}, \mathbf{z}\}$ for R^2

$$\mathbf{x} = \alpha\mathbf{y} + \beta\mathbf{z}$$

$\Rightarrow$ The scalars α, β are the coordinates of $\mathbf{x}$ with respect to the basis $\{\mathbf{y}, \mathbf{z}\}$

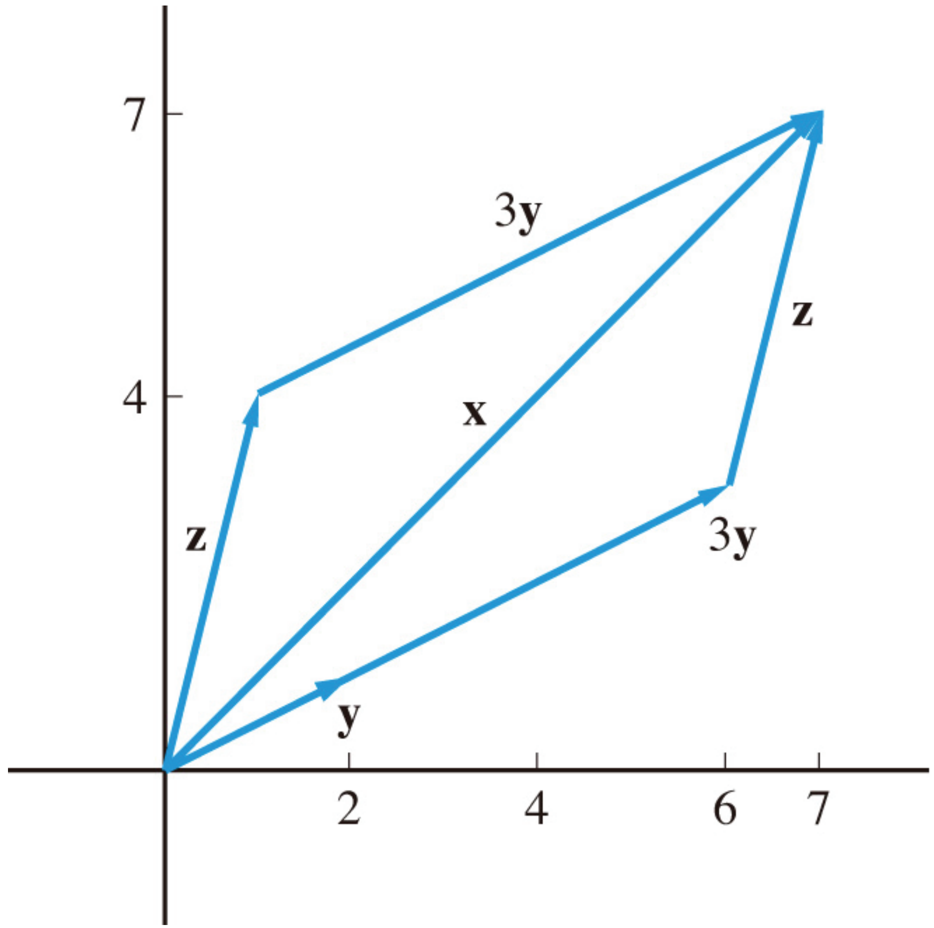
- The vector $(\alpha, \beta)^T$ is referred to as the **coordinate vector** of $\mathbf{x}$ with respect to the **ordered basis** $[\mathbf{y}, \mathbf{z}]$

Example 1

- Let $\mathbf{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$
- Since $\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7 \neq 0$, the vector **y** and **z** are <u>linearly independent</u> and hence form a basis for R^2
- For example, $$\mathbf{x} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 4 \end{bmatrix} = 3\mathbf{y} + \mathbf{z}$$

  $\Rightarrow$ The coordinate vector of **x** with respect to [**y**, **z**] is $(3, 1)^T$

Figure 3.5.1



Changing Coordinates Example

- Let $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be the new basis
- Let us consider two problems:

I. $[\mathbf{e}_1, \mathbf{e}_2] \Rightarrow [\mathbf{u}_1, \mathbf{u}_2]$

Given a vector $\mathbf{x} = (x_1, x_2)^T$, find its coordinates with respect to $\mathbf{u}_1$ and $\mathbf{u}_2$

II. $[\mathbf{u}_1, \mathbf{u}_2] \Rightarrow [\mathbf{e}_1, \mathbf{e}_2]$

Given a vector $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$, find its coordinates with respect to $\mathbf{e}_1$ and $\mathbf{e}_2$

- Since $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3\mathbf{e}_1 + 2\mathbf{e}_2$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{e}_1 + \mathbf{e}_2$

$$\begin{aligned} \Rightarrow \mathbf{x} &= c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = c_1(3\mathbf{e}_1 + 2\mathbf{e}_2) + c_2(\mathbf{e}_1 + \mathbf{e}_2) \\ &= (3c_1 + c_2)\mathbf{e}_1 + (2c_1 + c_2)\mathbf{e}_2 \end{aligned}$$

$$\Rightarrow x_1 = 3c_1 + c_2, x_2 = 2c_1 + c_2$$

Example (con.)

- Thus the coordinate vector of $c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ with respect to $[\mathbf{e}_1, \mathbf{e}_2]$ is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3c_1 + c_2 \\ 2c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$U = (\mathbf{u}_1, \mathbf{u}_2) = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$\Rightarrow$ $\mathbf{x} = U\,\mathbf{c}$

- U is called the **transition matrix** from the ordered basis $[\mathbf{u}_1, \mathbf{u}_2]$ to the standard basis $[\mathbf{e}_1, \mathbf{e}_2]$
- Since U is nonsingular (why?)

  $\Rightarrow$ $\mathbf{c} = U^{-1}\,\mathbf{x}$
- U^{-1} is the **transition matrix** from $[\mathbf{e}_1, \mathbf{e}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$

$$[\mathbf{u}_1, \mathbf{u}_2] \xrightarrow{\quad\quad\quad U \quad\quad\quad} [\mathbf{e}_1, \mathbf{e}_2]$$

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \begin{gathered} \mathbf{x} = U\mathbf{c}\ (\mathbf{c} = U^{-1}\mathbf{x}) \\ c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \end{gathered} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 2

- Let $\mathbf{u}_1 = (3, 2)^T{}_, \mathbf{u}_2 = (1, 1)^T$, $\mathbf{x} = (7, 4)^T$, find the coordinates of $\mathbf{x}$ with respect to $\mathbf{u}_1$ and $\mathbf{u}_2$
- *Sol:*
- The transition matrix from $[\mathbf{u}_1, \mathbf{u}_2]$ to $[\mathbf{e}_1, \mathbf{e}_2]$ is

$$U = (\mathbf{u}_1, \mathbf{u}_2) = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

- Thus, the transition matrix from $[\mathbf{e}_1, \mathbf{e}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$ is

$$U^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Example 2 (con.)

$$\Rightarrow \mathbf{c} = U^{-1}\mathbf{x} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = 3\mathbf{u}_1 - 2\mathbf{u}_2$$

Verification: $3\mathbf{u}_1 - 2\mathbf{u}_2 = 3\begin{bmatrix} 3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \mathbf{x}$

Example 3

- Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, find the transition matrix from $[\mathbf{e}_1, \mathbf{e}_2]$ to $[\mathbf{b}_1, \mathbf{b}_2]$ and determine the coordinates of $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with respect to $[\mathbf{b}_1, \mathbf{b}_2]$
- *Sol:*
- The transition matrix from $[\mathbf{b}_1, \mathbf{b}_2]$ to $[\mathbf{e}_1, \mathbf{e}_2]$ is

$$\mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_2] = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

- Thus, the transition matrix from $[\mathbf{e}_1, \mathbf{e}_2]$ to $[\mathbf{b}_1, \mathbf{b}_2]$ is

$$\mathbf{B}^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Example 3 (con.)

$$\Rightarrow \mathbf{c} = \mathbf{B}^{-1}\mathbf{x} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \mathbf{x} = 7\mathbf{b}_1 + 3\mathbf{b}_2$$

$$[\mathbf{v}_1, \mathbf{v}_2] \xrightarrow{\quad S \quad} [\mathbf{u}_1, \mathbf{u}_2]$$

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \qquad \mathbf{c} = S\mathbf{d} \ (\mathbf{d} = S^{-1}\mathbf{c}) \qquad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$$

- Let $S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$ be the transition matrix from an ordered basis $[\mathbf{v}_1, \mathbf{v}_2]$ of R^2 to another ordered basis $[\mathbf{u}_1, \mathbf{u}_2]$, then since $\mathbf{v}_1 = 1\ \mathbf{v}_1 + 0\ \mathbf{v}_2$

$\Rightarrow$ The coordinate vector of $\mathbf{v}_1$ with respect to $[\mathbf{u}_1, \mathbf{u}_2]$ is

$$s_1 = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} s_{11} \\ s_{21} \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{v}_1 = s_{11}\ \mathbf{u}_1 + s_{21}\ \mathbf{u}_2$$

Similarly, $\mathbf{v}_2 = 0\ \mathbf{v}_1 + 1\ \mathbf{v}_2$

$\Rightarrow$ The coordinate vector of $\mathbf{v}_2$ with respect to $[\mathbf{u}_1, \mathbf{u}_2]$ is

$$s_2 = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} s_{12} \\ s_{22} \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{v}_2 = s_{12}\,\mathbf{u}_1 + s_{22}\,\mathbf{u}_2$$

Thus, $\mathbf{v}_1 = s_{11}\,\mathbf{u}_1 + s_{21}\,\mathbf{u}_2$ and $\mathbf{v}_2 = s_{12}\,\mathbf{u}_1 + s_{22}\,\mathbf{u}_2$

Represent v_1 and v_2 as the linear combination of u_1 and u_2 will get the transition matrix S from $[v_1, v_2]$ to $[u_1, u_2]$

- Assume a given vector $\mathbf{x}$, its coordinates with respect to $\{\mathbf{v}_1, \mathbf{v}_2\}$ are known:

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

we must find scalars d_1 and d_2 so that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 \qquad \textbf{(3)}$$

- If we set $V = (\mathbf{v}_1, \mathbf{v}_2)$ and $U = (\mathbf{u}_1, \mathbf{u}_2)$, then equation (3) can be written in matrix form

$$V\mathbf{c} = U\mathbf{d}$$

It follows that

$$\mathbf{d} = U^{-1}V\mathbf{c}$$

$\Rightarrow$Thus, given a vector $\mathbf{x}$ in R^2 and its coordinate vector $\mathbf{c}$ with respect to the ordered basis $\{\mathbf{v}_1, \mathbf{v}_2\}$, to find the coordinate vector of x with respect to the new basis $\{\mathbf{u}_1, \mathbf{u}_2\}$, we simply multiply c by the transition matrix $S = U^{-1}V$.

Example 4

- Represent $\mathbf{v}_1$ and $\mathbf{v}_2$ as the linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ will get the transition matrix S from $[\mathbf{v}_1, \mathbf{v}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$, where $\mathbf{v}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- *Sol:*

$$\mathbf{v}_1 = s_{11}\,\mathbf{u}_1 + s_{21}\,\mathbf{u}_2 = s_{11}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + s_{21}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3s_{11} + s_{21} \\ 2s_{11} + s_{21} \end{bmatrix}$$

$$\mathbf{v}_2 = s_{12}\,\mathbf{u}_1 + s_{22}\,\mathbf{u}_2 = s_{12}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + s_{22}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3s_{12} + s_{22} \\ 2s_{12} + s_{22} \end{bmatrix}$$

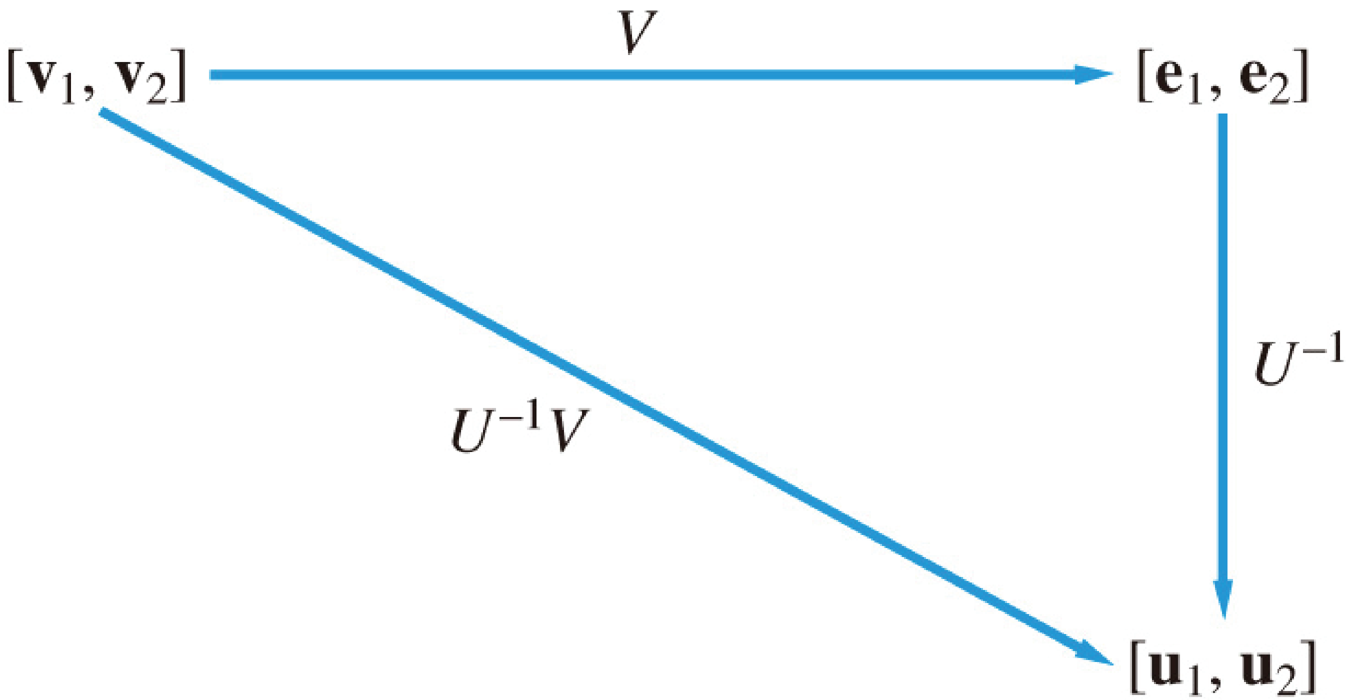
$$\Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 3s_{11} + s_{21} \\ 2s_{11} + s_{21} \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} s_{11} \\ s_{21} \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 3s_{12} + s_{22} \\ 2s_{12} + s_{22} \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} s_{12} \\ s_{22} \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

$$\Rightarrow \quad S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix}$$

is the transition matrix from $[\mathbf{v}_1, \mathbf{v}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$

Figure 3.5.2



- $V\mathbf{d} = \mathbf{x}$ and $U^{-1}\mathbf{x} = \mathbf{c} \Rightarrow U^{-1}V\mathbf{d} = \mathbf{c}$
- $U^{-1}V$ is the transition matrix from $[\mathbf{v}_1, \mathbf{v}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$

Example

- The transition matrix from $[\mathbf{v}_1, \mathbf{v}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$ is given by

$$U^{-1}V = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix} = S$$

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \text{ and } \mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Change of Basis for a General Vector Space Definition

Let V be a vector space and let $E = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ be an ordered basis for V. If $\mathbf{v}$ is any element of V, then $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

where $c_1, c_2, \ldots, c_n$ are scalars. Thus we can associate with each vector $\mathbf{v}$ a unique vector $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T$ in $\boldsymbol{R}^n$. The vector $\mathbf{c}$ defined in this way is called the ***coordinate vector*** of $\mathbf{v}$ with respect to the ordered basis E and is denoted $[\mathbf{v}]_E$. The c_i's are called the ***coordinates*** of $\mathbf{v}$ relative to E.

Example 5

Let $E = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = [(1, 1, 1)^T, (2, 3, 2)^T, (1, 5, 4)^T]$,

$F = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = [(1, 1, 0)^T, (1, 2, 0)^T, (1, 2, 1)^T]$,

(1) Find the transition matrix from E to F.

(2) If $\mathbf{x} = 3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$ and $\mathbf{y} = \mathbf{v}_1 - 3\mathbf{v}_2 + 2\mathbf{v}_3$

Find the coordinates of $\mathbf{x}$ and $\mathbf{y}$ with respect to the ordered basis F.

Example 5

- *Sol:*

$$\begin{tikzcd}
\boldsymbol{E} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \arrow[r, "V"] \arrow[dr, "U^{-1}V"'] & [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \arrow[d, "U^{-1}"] \\
& \boldsymbol{F} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]
\end{tikzcd}$$

(1)

$$U^{-1}V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -3 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2)

$$[\mathbf{x}]_F = \begin{bmatrix} 1 & 1 & -3 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 3 \end{bmatrix}$$

$$[\mathbf{y}]_F = \begin{bmatrix} 1 & 1 & -3 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$$

Verify:

$$8\mathbf{u}_1 - 5\mathbf{u}_2 + 3\mathbf{u}_3 = 3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$$

$$-8\mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_3 = \mathbf{v}_1 - 3\mathbf{v}_2 + 2\mathbf{v}_3$$

Example 6

- Find the transition matrix from $[1, 2x, 4x^2 - 2]$ to $[1, x, x^2]$ and the coordinates of $P(x) = a + bx + cx^2$ with respect to $[1, 2x, 4x^2-2]$.
- *Sol:*
- Since

$$\begin{array}{rcrcrcrcr} & & 1 & = & 1\cdot 1 & + & 0\cdot x & + & 0\cdot x^2 \\ & & 2x & = & 0\cdot 1 & + & 2\cdot x & + & 0\cdot x^2 \\ 4x^2 & - & 2 & = & -2\cdot 1 & + & 0\cdot x & + & 4\cdot x^2 \end{array}$$

$\Rightarrow$The transition matrix from $[1, 2x, 4x^2\text{-}2]$ to $[1, x, x^2]$ is

$$S = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\Rightarrow$The transition matrix from $[1, x, x^2]$ to $[1, 2x, 4x^2\text{-}2]$ is

$$S^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$\Rightarrow$ The coordinates of $P(x) = a + bx + cx^2$ with respect to $[1, 2x, 4x^2-2]$ is

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + \frac{1}{2}c \\ \frac{1}{2}b \\ \frac{1}{4}c \end{bmatrix}$$

Verification:

$$(a + \tfrac{1}{2}c) \times 1 + (\tfrac{1}{2}b) \times (2x) + (\tfrac{1}{4}c) \times (4x^2 - 2)$$

$$= a + bx + cx^2$$

3.6 Row Space and Column Space

- If A is an $m\times n$ matrix, the m vectors in $R^{1\times n}$ corresponding to the rows of A is referred to as the **row vectors of A** and the n vectors in R^m corresponding to the columns of A is referred to as the **column vectors of A**.

Definition

If A is an $m \times n$ matrix, the subspace of $R^{1 \times n}$ spanned by the row vectors of A is called the ***row space of A***. The subspace of R^m spanned by the column vectors of A is called the ***column space of A***.

Example 1

- Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
- The row space of A:

$$\alpha[1, 0, 0] + \beta[0, 1, 0] = [\alpha, \beta, 0]$$

$$\Leftarrow \text{2-d subspace of } R^{1\times 3}$$

The column space of A:

$$\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Leftarrow \boldsymbol{R}^2$$

Theorem 3.6.1

Two row equivalent matrices have the same row space.

Proof.

If B is row equivalent to A, then B can be formed from A by a finite sequence of row operations. Thus, the row vectors of B must be linear combinations of the row vectors of A. Consequently, the row space of B must be a subspace of the row space of A. Since A is row equivalent to B, by the same reasoning, the row space of A is a subspace of the row space of B.

Definition

The **rank** of a matrix A is the dimension of the row space of A.

- **In Example 1, rank(A) = 2**
- **To determine the rank of a matrix, we can reduce the matrix to row echelon form. The nonzero rows of the row echelon matrix will form a basis for the row space.**

Example 2

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$\Rightarrow$ (1, -2, 3) and (0, 1, 5) will form a basis for the row space of U

$\Rightarrow$ rank(A) = rank(U) = 2

Linear Systems

- Consider the system $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\Rightarrow x_1 \underbrace{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}}_{\mathbf{a}_1} + x_2 \underbrace{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}}_{\mathbf{a}_2} + \cdots + x_n \underbrace{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{\mathbf{a}_n} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\Rightarrow A\mathbf{x} = \mathbf{b}$ is consistent if and only if <u>$\mathbf{b}$ is in the column space of A</u>

Theorem 3.6.2

(Consistency Theorem for Linear Systems)

A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Note

- If $\mathbf{b} = \mathbf{0}$, the system $A\mathbf{x} = \mathbf{b}$ becomes

  $$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$$

  $A\mathbf{x} = \mathbf{0}$ will have only the trivial solution $\mathbf{x} = \mathbf{0}$ iff the column vectors of A are linearly independent.

Theorem 3.6.3

Let A be an $m \times n$ matrix. The linear system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in R^m$ iff the column vectors of A span R^m. The system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b} \in R^m$ iff the column vector of A are linearly independent.

Note

- Let A be an $m \times n$ matrix. If the n column vector of A span R^m, then $n \geq m$. If the n columns of A are linearly independent, then $n \leq m$.
  $\Rightarrow$ If the column vectors of A form a basis for R^m, then $n = m$.

Corollary 3.6.4

An $n \times n$ matrix A is nonsingular if and only if the column vectors of A form a basis for R^n.

- ***Proof***
- **Since A is nonsingular (A is invertible)**
  Thus, all column of A are linearly independent
  These n column vectors form a basis for R^n.

Definition

The dimension of the nullspace of a matrix is called the **nullity** of the matrix (dim $N(A)$)

Theorem 3.6.5

(The Rank-Nullity Theorem)

If A is an $m \times n$ matrix, then the rank of A plus the nullity of A equals n.

- ***Proof***
- **Let U be the row echelon form of A**
  Rank(A) = r = the number of nonzero rows in U (r lead variables)
  Nullity of A = the number of free variables = n - r

Example 3

- Let $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$,

  find a basis for the row space of A and a basis for $N(A)$. Verify that dim $N(A) = n - r$

- ***Sol:***
- The reduced row echelon form of A is $U = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- Thus, $\{[1, 2, 0, 3], [0, 0, 1, 2]\}$ is a basis for the row space of A and rank$(A) = 2$

- $$\begin{aligned} x_1 + 2x_2 + \quad 3x_4 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned}$$

  lead variable: $x_1, x_3 \Rightarrow$ rank = 2

  free variable: $x_2, x_4 \Rightarrow \dim N(A) = 2$

- Let $x_2 = \alpha$, $x_4 = \beta$, then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2\alpha - 3\beta \\ \alpha \\ -2\beta \\ \beta \end{bmatrix} = \begin{bmatrix} -2\alpha \\ \alpha \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3\beta \\ 0 \\ -2\beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$\Rightarrow$ $(-2, 1, 0, 0)^T$ and $(-3, 0, -2, 1)^T$ form a basis for $N(A)$

$\Rightarrow \dim N(A) = 2 = n - r = 4 - 2$

The Column Space

- If U is the row echelon form of A, then A and U have the same row space (**Theorem 3.6.1**); But A and U have the different column space, since $A\mathbf{x} = \mathbf{0}$ if and only if $U\mathbf{x} = \mathbf{0}$, their column vectors satisfy the same dependency relations

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 3.6.6

If A is an $m \times n$ matrix, the dimension of the row space of A equals the dimension of the column space of A.

- To find the column space of A, we can use the row echelon form U of A by determining the columns of U that corresponds to the lead 1's. These same columns of A will be linearly independent and form a basis for the column space of A.

Example 4

- Let

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5)$$

- The row echelon form of A is

$$U = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The lead variables are x_1, x_2, x_5

$$\Rightarrow \qquad \mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \mathbf{a}_5 = \begin{bmatrix} 2 \\ -2 \\ 4 \\ 5 \end{bmatrix}$$

form a basis for the column space of A.

$\Rightarrow$ rank(A) = 3

Example 5

- The subspace Span($\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$) is the same as the column space of the matrix:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 2 \\ 5 \\ -3 \\ 2 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ 0 \end{bmatrix}, \ \mathbf{x}_4 = \begin{bmatrix} 3 \\ 8 \\ -5 \\ 4 \end{bmatrix}$$

- *Sol:*
- The subspace Span($\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$) is the same as the column space of the matrix:

$$X = \begin{array}{c} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 5 & 4 & 8 \\ -1 & -3 & -2 & -5 \\ 0 & 2 & 0 & 4 \end{bmatrix} \\ \begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \end{array} \end{array}$$

- The row echelon form of X is

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The lead variables are $x_1, x_2 \Rightarrow \text{rank}(X) = 2$

$\Rightarrow$ $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis of the column space of X.

$\Rightarrow \dim \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = 2$