Chapter 2

Determinants

2.1 The Determinant of a Matrix

- The determinant of an $n \times n$ matrix A, $\det(A)$, will tell us whether the matrix is nonsingular (its multiplicative inverse exists or not).

Case 1. 1×1 matrix

- If $A = [a]$ is a 1×1 matrix then A will have a multiplicative inverse iff $a \neq 0$ (i.e., $\det(A) \neq 0$)
- Define $\det(A) = a$
- A is nonsingular iff $\det(A) \neq 0$

Case 2. 2×2 matrix

- Let $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

  By Theorem 1.5.2, A will be nonsingular iff it is row equivalent to I

  (1) If $a_{11} \neq 0$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{11}a_{21} & a_{11}a_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{11}a_{22} - a_{21}a_{12} \end{bmatrix}$$

A is row equivalent to I iff $a_{11}a_{22} - a_{21}a_{12} \neq 0$

(2) If $a_{11} = 0$

$$A' = \begin{bmatrix} a_{21} & a_{22} \\ 0 & a_{12} \end{bmatrix}$$

Switching the two rows of A, we get A' is row equivalent to I iff $a_{21}a_{12} \neq 0$ ($a_{21} \neq 0$ and $a_{12} \neq 0$)

- Define $\det(A) = a_{11}a_{22} - a_{21}a_{12}$

Notation

- We will refer to the determinant of a specific matrix by enclosing the array between vertical lines.

- For example, if $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$, then $\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix}$ represents the determinant of A.

Case 3. 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- If $a_{11} \neq 0$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \dfrac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \dfrac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} \\ 0 & \dfrac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}} & \dfrac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} \end{bmatrix}$$

- The matrix will be row equivalent to I if and only if

$$a_{11}\begin{vmatrix} \dfrac{a_{11}a_{22}-a_{21}a_{12}}{a_{11}} & \dfrac{a_{11}a_{22}-a_{21}a_{12}}{a_{11}} \\ \dfrac{a_{11}a_{32}-a_{31}a_{12}}{a_{11}} & \dfrac{a_{11}a_{33}-a_{31}a_{13}}{a_{11}} \end{vmatrix} \neq 0$$

- $a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \neq 0$

If a_{11}=0

- Three cases should be considered:
  - (1) $a_{11}=0$, $a_{21}\neq 0$
  - (2)$a_{11}=a_{21}=0$, $a_{31}\neq 0$
  - (3) $a_{11}=a_{21}=a_{31}=0$,
- *In case (1):* $a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}\neq 0$
- $\Rightarrow -a_{12}a_{21}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}\neq 0$

In case (2)

(2) $\mathbf{a_{11}=a_{21}=0,\ a_{31}\neq 0}$

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Is row equivalent to *I* if and only if

$$\Rightarrow -a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \neq 0$$

$$\Rightarrow a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} \neq 0$$

$$\Rightarrow a_{31}(a_{12}a_{23} - a_{13}a_{22}) \neq 0$$

In case (3)

- Clearly, in case (iii) the matrix A cannot be row equivalent to $\boldsymbol{I}$ and hence must be singular.
- In this case, if we set a_{11}, a_{21}, and a_{31} equal to 0 in formula (3), the result will be $\det(A) = 0$.
- In general, then, formula (2) gives a necessary and sufficient condition for a 3×3 matrix A to be nonsingular (regardless of the value of a_{11}).

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\det(A)$

$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$

$= a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + a_{13}\det(M_{13})$

where $M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$, $M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$, $M_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

Definition

Let $A = (a_{ij})$ be an $n \times n$ matrix. Let M_{ij} be the $(n\text{-}1)\times(n\text{-}1)$ matrix obtained from A by deleting the row and column containing a_{ij}.

The determinant of M_{ij} is called the **minor** of a_{ij}.

We define the **cofactor** A_{ij} of a_{ij} by

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

$$\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + a_{13}\det(M_{13})$$
$$= -a_{21}\det(M_{21}) + a_{22}\det(M_{22}) - a_{23}\det(M_{23})$$

Example

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- $\det(A) = a_{11}(a_{22}) - a_{12}(a_{21}) = a_{11}A_{11} + a_{12}A_{12} \quad (n = 2)$
- The equation is called the ***cofactor expansion*** of $\det(A)$ along the first row of A.

Note

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$\det(A)$

$= a_{11}a_{22} - a_{21}a_{12} = a_{21}(-a_{12}) + a_{22}(a_{11})$

$= a_{21}A_{21} + a_{22}A_{22}$ (2nd row)

$= a_{11}(a_{22}) + a_{21}(-a_{12}) = a_{11}A_{11} + a_{21}A_{21}$ (1st column)

$= a_{12}(-a_{21}) + a_{22}(a_{11}) = a_{12}A_{12} + a_{22}A_{22}$ (2nd column)

Example 1

- If $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$

det(A) $= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$$= 2\times\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} - 5\times\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 4\times\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} = (-4) - 5(8) + 4(7) = -16$$

$$= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$$

$$= 2\times\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} - 3\times\begin{vmatrix} 5 & 4 \\ 4 & 6 \end{vmatrix} + 5\times\begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = (-4) - 3(14) + 5(6) = -16$$

$= \dots$

Example 2

- The cofactor expansion of det(A)along the second column is given by

$$A=\begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$$

$$\det(A) = -5\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$$

$$= -5(18-10) + 1(12-20) - 4(4-12) = -16$$

Definition

The determinant of an $n \times n$ matrix A, denoted $\det(A)$, is a scalar associated with the matrix A that is defined inductively as follows:

$$\det(A) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \text{if } n > 1 \end{cases}$$

where

$$A_{1j} = (-1)^{1+j} \det(M_{1j}), \qquad j = 1, 2, \ldots, n$$

are the cofactors associated with the entries in the first row of A.

Theorem 2.1.1

If A is an $n \times n$ matrix with $n \geq 2$, then $\det(A)$ can be expressed as a cofactor expansion using <u>any row or column</u> of A

$$\begin{aligned}\det(A) &= a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in} \\ &= a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj}\end{aligned}$$

for $i, j = 1, 2, \ldots, n$

- The cofactor expansion can be performed along the row or column that <u>contains the most zeros</u> to save work. For example,

$$A = \begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{vmatrix} = (-2)\begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix} = (-2)\cdot 3 \cdot \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$$

$$= (-2)\cdot 3 \cdot (-2) = 12$$

Theorem 2.1.2

If A is an $n{\times}n$ matrix, then $\det(A^T) = \det(A)$

Proof

Think:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

(Hint: prove by induction!)

Proof by Induction

The proof is by induction on n. Clearly, the result holds if $n = 1$, since a 1×1 matrix is necessarily symmetric. Assume that the result holds for all $k \times k$ matrices and that A is a $(k+1) \times (k+1)$ matrix. Expanding $\det(A)$ along the first row of A, we get

$$\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + - \cdots \pm a_{1,k+1}\det(M_{1,k+1})$$

Since the M_{ij}'s are all $k \times k$ matrices, it follows from the induction hypothesis that

$$\det(A) = a_{11}\det(M_{11}^T) - a_{12}\det(M_{12}^T) + - \cdots \pm a_{1,k+1}\det(M_{1,k+1}^T) \tag{9}$$

The right-hand side of (9) is just the expansion by minors of $\det(A^T)$ using the first column of A^T. Therefore,

$$\det(A^T) = \det(A) \qquad \blacksquare$$

Example det(A)=det(A^T)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \Rightarrow A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Theorem 2.1.3

If A is an $n \times n$ <u>triangular matrix</u>, then $\det(A)$ = the product of the diagonal elements of A.

proof

Think:

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 5 & 5 & 5 \\ 0 & 2 & 5 & 5 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Theorem 2.1.4

Let A be an $n \times n$ matrix.

(i) If A has a row or column consisting entirely of zeros, then $\det(A) = 0$

(ii) If A has two identical rows or two identical columns, then $\det(A) = 0$

Theorem 2.1.4 *proof*

- Think:

$$A = \begin{bmatrix} a_{11} & \cdots & 0 & \cdots & a_{1n} \\ a_{21} & & 0 & & a_{2n} \\ a_{31} & & 0 & & a_{3n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & 0 & \cdots & a_{nn} \end{bmatrix}$$

and

$$A' = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & \vdots & & & \vdots \\ a_{11} & a_{12} & a_{13} & & a_{1n} \\ & \vdots & & & \vdots \\ a_{n1} & \cdots & & \cdots & a_{nn} \end{bmatrix}$$

(Hint: prove by induction)

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\det(A) = 4 \times \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} + 5 \times \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} + 6 \times \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

2.2 Properties of Determinants

Lemma 2.2.1

Let A be an $n\times n$ matrix. If A_{jk} denotes the cofactor of a_{jk} for $k = 1, 2, \ldots, n$, then

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

proof

(i) If $i = j$, it is just a cofactor expansion along the i-th row of A.

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = 0 \qquad \text{if } i \neq j$$

(i) If $i \neq j$, Let A^* be the matrix obtained by replacing the j-th row of A by the i-th row of A:

$$\boldsymbol{A}^* = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \end{array} \begin{array}{l} \\ \\ j-\text{th row : replaced by } i\text{ - th row} \\ \\ \\ \\ \\ \end{array}$$

- Since A^* has two rows are of the same, $\det(A^*) = 0$. If we expand A^* along the j-th row:

$$0 = \det(A^*) = a_{i1}A^*_{j1} + a_{i2}A^*_{j2} + \ldots + a_{in}A^*_{jn}$$
$$= a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn}$$

$$A^* = \begin{array}{cl} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} & \begin{array}{l} \\ \\ j^{th}\text{ row} \\ \\ \\ \\ \\ \end{array} \end{array} \quad A^*_{j1} = \begin{array}{cl} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} & \begin{array}{l} \\ \\ j^{th}\text{ row} \\ \\ \\ \\ \\ \end{array} \end{array} \quad A_{j1} = \begin{array}{cl} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} & \begin{array}{l} \\ \\ j^{th}\text{ r} \\ \\ \\ \\ \\ \end{array} \end{array}$$

Row Operation I

- *Two rows of A are interchanged.*
- *If A is 2×2 matrix and*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathrm{EA} = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} \qquad A\mathrm{E} = \begin{bmatrix} a_{21} & a_{11} \\ a_{22} & a_{21} \end{bmatrix}$$

$$\det(EA) = -\det(A) = \det(E)\det(A)$$

- Let E_{ij} is formed from I by interchanging the i-th row and j-th row of I, then

  $\det(E_{ij}) = \det(E_{ij}I) = -\det(I) = -1$

  $\det(E_{ij}A) = -\det(A) = \det(E)\det(A)$

3×3 Example

$$\det(E_{13}A) = \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}$$

$$= -a_{21}\begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix} + a_{22}\begin{vmatrix} a_{31} & a_{33} \\ a_{11} & a_{13} \end{vmatrix} - a_{23}\begin{vmatrix} a_{31} & a_{31} \\ a_{11} & a_{12} \end{vmatrix}$$

$$= a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -\det(A)$$

Row Operation II

- *A row of A_2 is multiplied by a nonzero constant.*

$$A_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & \vdots & & & \vdots \\ \alpha a_{i1} & \alpha a_{i2} & \alpha a_{i3} & & \alpha a_{in} \\ & \vdots & & & \vdots \\ a_{n1} & \cdots & & \cdots & a_{nn} \end{bmatrix}$$

- If $\det(EA)$ is expanded by cofactors along the i-th row:
- $\det(A_2)$

$= \det(EA) = \alpha\, a_{i1}A_{i1} + \alpha\, a_{i2}A_{i2} + \ldots + \alpha\, a_{in}A_{in}$

$= \alpha\,(a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in})$

$= \alpha \det(A)$

- $\det(EA) = \alpha \det(A) = \det(E)\det(A)$

Row Operation II

- Let E denote the elementary matrix formed from I by multiplying the i-th row by the nonzero constant α
- $\det(E) = \det(EI) = \alpha \det(I) = \alpha$
- $det(EA) = \alpha\, det(A) = det(E)det(A)$

Row Operation III

- *A multiple of one row is added to another row.*
- Let E is formed from I by adding c times the i-th row to the j-th row, $\det(E) = 1$.
- If $\det(EA)$ is expanded by cofactors along the j-th row:

$$\det(EA)$$
$$= (a_{j1} + c\, a_{i1})A_{j1} + (a_{j2} + c\, a_{i2})A_{j2} + \ldots + (a_{jn} + c\, a_{in})A_{jn}$$
$$= (a_{j1}A_{j1} + a_{j2}A_{j2} + \ldots + a_{jn}A_{jn}) + c\,(a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn})$$
$$= \det(A) + c \times 0$$
$$= \det(A)$$

- $\det(EA) = \det(A) = \det(E)\det(A)$

Summary

Let E denote the elementary matrix of a row operation, then

$$\det(EA) = \det(E)\det(A)$$

where

$$\det(E) = \begin{cases} -1 & \text{if } E \text{ is of type I} \\ \alpha \neq 0 & \text{if } E \text{ is of type II} \\ 1 & \text{if } E \text{ is of type III} \end{cases}$$

Similar results hold for column operations. Indeed, if E is an elementary matrix, then E^T is also an elementary matrix (see Exercise 8 at the end of the section) and

$$\begin{aligned} \det(AE) &= \det((AE)^T) = \det(E^T A^T) \\ &= \det(E^T)\det(A^T) = \det(E)\det(A) \end{aligned}$$

Summary

I. Interchanging two rows of a matrix changes the sign of the determinant.

II. Multiplying a single row of a matrix by a scalar has the effect of multiplying the value of the determinant by that scalar.

III. Adding a multiple of one row to another does not change the value of the determinant.

Row Operation I implemented by Row operations III and II

- *Two rows of A are interchanged can be proved by using row operations III and II.*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{2nd row is subtracted from 3rd row}}$$

$$A^{(1)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - a_{31} & a_{22} - a_{32} & a_{23} - a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\xrightarrow{\text{add 2nd row to 3rd row}} A^{(2)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - a_{31} & a_{22} - a_{32} & a_{23} - a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$\xrightarrow{\text{2nd row is subtracted from 3rd row}} A^{(3)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ -a_{31} & -a_{32} & -a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$\xrightarrow{\text{2nd row is multiplied by (-1)}} A^{(4)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

- $\det(A)=\det(A^{(1)})=\det(A^{(2)})=\det(A^{(3)})$ row operation III
  $\det(A^{(4)}) = (-1) \det(A^{(3)}) = - \det(A)$ row operation II
- Let E_{ij} is formed from I by interchanging the i-th row and j-th row of I, then

$$\det(E_{ij}) = \det(E_{ij}I) = - \det(I) = -1$$

$$\det(E_{ij}A) = - \det(A) = \det(E) \det(A)$$

Theorem 2.2.2

An $n \times n$ matrix A is singular iff $\det(A) = 0$

proof

- Let $U = E_k E_{k\text{-}1} \ldots E_1 A$ is in reduced echelon form, then
  $\det(U) = \det(E_k) \det(E_{k\text{-}1}) \ldots \det(E_1) \det(A)$
- Since $\det(E_i) \neq 0$, then $\det(U) = 0$ iff $\det(A) = 0$

Theorem 2.2.2 *proof*

- If A is singular, then U has at least one row consisting entirely of zeros and hence $\det(U) = 0$
- If A is nonsingular, then U is triangular with 1's along the diagonal and hence $\det(U) = 1$

- Another way to compute det(A)
  - reduce A to row echelon form:

$$U = E_k E_{k-1} \dots E_1 A$$

  - If the last row of U consists entirely of zeros, A is singular and det(A) = 0
  - Otherwise, A is nonsingular and

$$\det(A) = [\det(E_k)\det(E_{k-1}) \dots \det(E_1)]^{-1}$$

– If A is nonsingular, it is simpler to reduce A to triangular form using only row operation I and III:

$$\boldsymbol{T = E_m E_{m\text{-}1} \ldots E_1 A}$$

and hence

$$\det(A) = \pm \det(T) = \pm\, t_{11}\, t_{22} \ldots t_{nn}$$

where t_{ii}'s are the diagonal entries of T.

Example 1

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} \underline{\underline{\begin{array}{c}\text{(-2)}\times\text{1st row and then added to 2nd row} \\ \text{(-3)}\times\text{1st row and then added to 3rd row}\end{array}}} \begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{vmatrix}$$

$$\underline{\underline{\text{interchang e 2nd and 3rd row}}} \quad (-1)\begin{vmatrix} 2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5 \end{vmatrix}$$

$$= (-1)(2)(-6)(-5) = -60$$

Time complexity for computing |A|

Table I Operation Counts

	Cofactors		Elimination	
n	Additions	Multiplications	Additions	Multiplications and Divisions
2	1	2	1	3
3	5	9	5	10
4	23	40	14	23
5	119	205	30	44
10	3,628,799	6,235,300	285	339

$$\det(AE)=\det(A)\det(E),$$

for type I, II, III column operations

$$\det(A^T)=\det(A)$$

Theorem 2.2.3

If A and B are $n\times n$ matrices, then det(AB) = det(A) det(B)

- Case 1: If B is singular det(B)=0 =>AB is singular,
  - 0=det(AB) = det(A) det(B)=det(A)*0=0
- Case 2: If B is nonsingular, B can be written as a product of elementary matrices.
- Note det(AE)=det(A)det(E)

$$
\begin{aligned}
\det(AB) &= \det(AE_k E_{k-1} \cdots E_1) \\
&= \det(A)\det(E_k)\det(E_{k-1})\cdots\det(E_1) \\
&= \det(A)\det(E_k E_{k-1} \cdots E_1) \\
&= \det(A)\det(B)
\end{aligned}
$$

2.3 Additional Topics and Applications

- Cramer’s rule:
- Computing the <u>inverse</u> of a nonsingular matrix using determinants.
- Solving $A\mathbf{x} = \mathbf{b}$ using determinants

The Adjoint of a Matrix

- Let A be an $n \times n$ matrix, define the *adjoint* of A by

$$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

- By Lemma 2.2.1

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$A(\mathrm{adj}\ A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}A_{11} + \cdots + a_{1n}A_{1n} & 0 & \cdots & 0 \\ 0 & a_{21}A_{21} + \cdots + a_{2n}A_{2n} & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & a_{n1}A_{n1} + \cdots + a_{nn}A_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \det(A) \end{bmatrix} = \det(A) \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \det(A)\, I$$

- From the above derivation, we get

$$A\,(\operatorname{adj} A) = \det(A)\, I$$

$$A\left(\frac{1}{\det(A)}\operatorname{adj} A\right) = I$$

$$\boxed{A^{-1} = \frac{1}{\det(A)}\operatorname{adj} A \qquad \text{when } \det(A) \neq 0}$$

Example 1

- For a 2×2 matrix

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example 2

- Compute adj A and A^{-1}, $A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & -7 & 4 \\ 1 & 4 & -3 \\ -2 & 2 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A = \frac{1}{5} \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

Cramer's Rule Example 3

- Use Cramer's rule to solve

$$\begin{aligned} x_1 &+ 2x_2 + x_3 = 5 \\ 2x_1 &+ 2x_2 + x_3 = 6 \\ x_1 &+ 2x_2 + 3x_3 = 9 \end{aligned}$$

- *Sol*

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad \det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -4$$

$$\det(A_1) = \begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix} = -4, \quad \det(A_2) = \begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix} = -4,$$

$$\det(A_3) = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix} = -8$$

$$\therefore x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-4}{-4} = 1, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-4}{-4} = 1,$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-8}{-4} = 2$$

Theorem 2.3.1 (Cramer's Rule)

Let A be an $n \times n$ nonsingular matrix and let $\mathbf{b} \in R^n$, and let A_i be the matrix obtained by replacing the i-th column of A by $\mathbf{b}$. If $\mathbf{x}$ is the unique solution to $A\mathbf{x} = \mathbf{b}$, then

$$x_i = \frac{\det(A_i)}{\det(A)} \qquad \text{for } i = 1, 2, \cdots, n$$

Theorem 2.3.1 *proof*

- Since $\mathbf{x} = A^{-1}\mathbf{b} = \dfrac{1}{\det(A)}(\operatorname{adj} A)\,\mathbf{b}$

$$= \frac{1}{\det(A)}\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

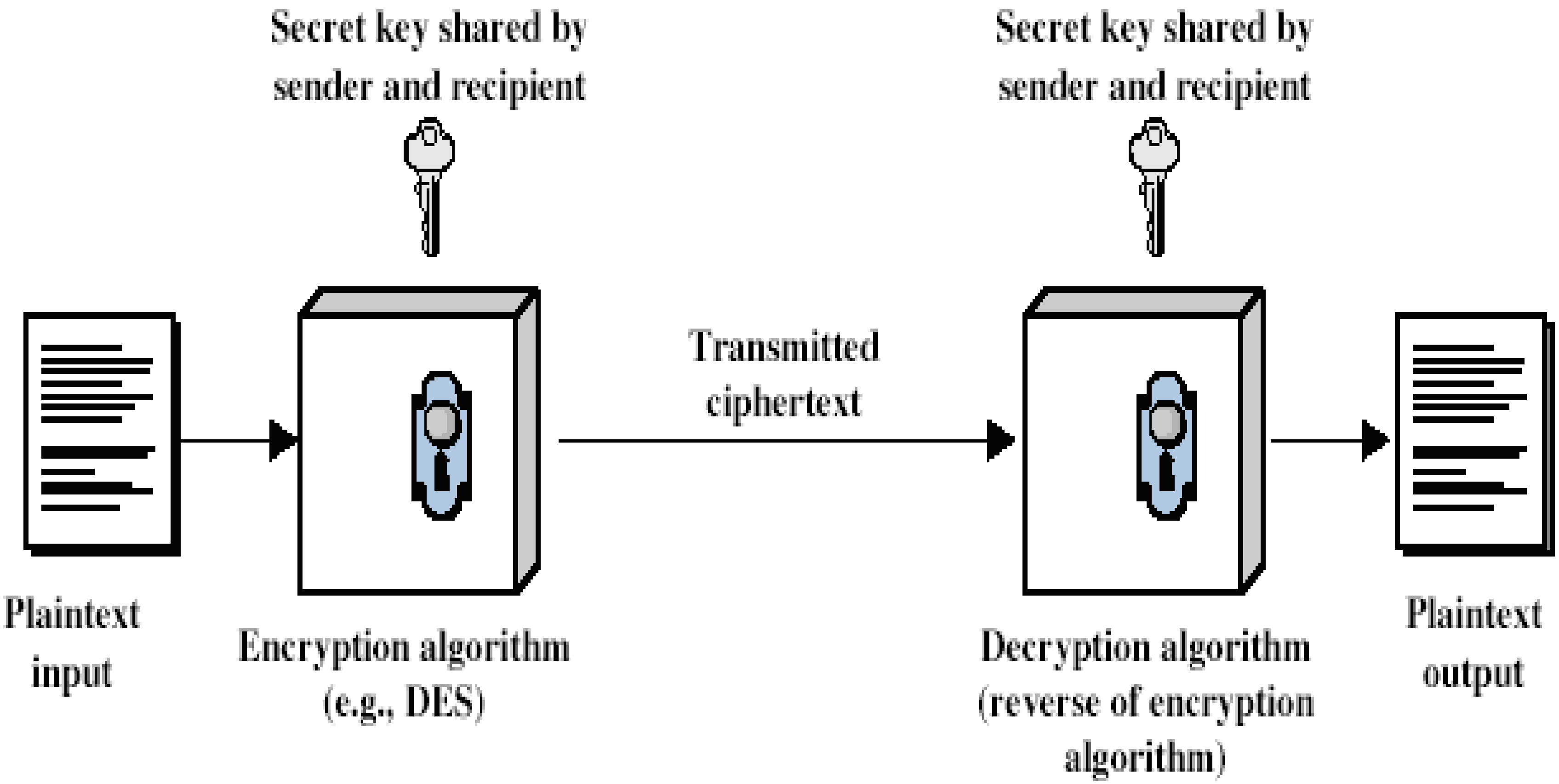
$$x_i = \frac{1}{\det(A)}\left(b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni}\right) = \frac{\det(A_i)}{\det(A)}$$

Note

$$A_i = \begin{bmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & & b_2 & & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix},$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & & a_{2i} & & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix}$$

Application: Symmetric Cipher Model



Basic Terminology

- **Plaintext**
  - **The original message**
- **Ciphertext**
  - **The coded message**
- **Enciphering or encryption**
  - **Process of converting from plaintext to ciphertext**
- **Deciphering or decryption**
  - **Restoring the plaintext from the ciphertext**
- **Cryptography**
  - **Study of encryption**
- **Cryptographic system or cipher**
  - **Schemes used for encryption**
- **Cryptanalysis**
  - **Techniques used for deciphering a message without any knowledge of the enciphering details**
- **Cryptology**
  - **Areas of cryptography and cryptanalysis together**

Caesar Cipher Algorithm

- Can define transformation as:

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C

- Mathematically give each letter a number

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

- Algorithm can be expressed as:

$$c = \mathrm{E}(3, p) = (p + 3) \bmod 26$$

  – A shift may be of any amount, so that the general Caesar algorithm is:

$$C = E(k, p) = (p + k) \mathit{mod}\ 26$$

- Where k takes on a value in the range 1 to 25; the decryption algorithm is simply:

$$p = D(k, C) = (C - k) \mathit{mod}\ 26$$

Application: Coded Messages

- Assign an integer value to each letter of the alphabet and to send the message as a string of integers, for example

  SEND MONEY $\Rightarrow$ <u>5, 8, 10, 21, 7, 2, 10, 8, 3</u>

  such message is easy to break.

  How to disguise the message?

Hill Cipher

- K=$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$ P=$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$

Encryption

C=KP mod 26

$$=\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$=\begin{bmatrix} (k_{11} \times p_1 + k_{12} \times p_2 + k_{13} \times p_3) mod\ 26 \\ (k_{21} \times p_1 + k_{22} \times p_2 + k_{23} \times p_3) mod\ 26 \\ (k_{31} \times p_1 + k_{32} \times p_2 + k_{33} \times p_3) mod\ 26 \end{bmatrix}$$

Decryption

$$\boldsymbol{P} = \boldsymbol{K}^{-1}\boldsymbol{C} \text{ mod } 26$$
$$= \boldsymbol{K}^{-1}\boldsymbol{KP} \text{ mod } 26$$

How to find invertible matrix

- Let A be a matrix whose entries are **all integers** and **$\det(A) = \pm 1$**, since

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A = \pm \operatorname{adj} A$$

- then the entries of A^{-1} will be integers.

- Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 2 & 3 & 2 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ -4 & 1 & 1 \end{bmatrix}$
- The coded message is put into the matrix B:
- SEND MONEY $\Rightarrow$ <u>5, 8, 10, 21, 7, 2, 10, 8, 3</u>

$$B = \begin{bmatrix} 5 & 21 & 10 \\ 8 & 7 & 8 \\ 10 & 2 & 3 \end{bmatrix}$$

- The product of AB gives the sent message:

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 5 & 21 & 10 \\ 8 & 7 & 8 \\ 10 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 31 & 37 & 29 \\ 80 & 83 & 69 \\ 54 & 67 & 50 \end{bmatrix}$$

$\Rightarrow$ The sent message is <u>31, 80, 54, 37, 83, 67, 29, 69, 50</u>

- The receiver can decode the message by multiplying by A^{-1}

$$B = A^{-1}(AB) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 31 & 37 & 29 \\ 80 & 83 & 69 \\ 54 & 67 & 50 \end{bmatrix} = \begin{bmatrix} 5 & 21 & 10 \\ 8 & 7 & 8 \\ 10 & 2 & 3 \end{bmatrix}$$

- To construct a coding matrix A, we can begin with the identity I and successively apply **row operation III**, being careful to add integer multiples of one row to another.
- Row operation I can also be used. The resulting matrix A will have integer entries, and since $\det(A) = \pm\det(I) = \pm 1$, A^{-1} **will also have integer entries**.

The Cross Product

Given two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$, one can define a third vector, the *cross product*, denoted $\mathbf{x} \times \mathbf{y}$, by

$$\mathbf{x} \times \mathbf{y} = \begin{pmatrix} x_2y_3 - y_2x_3 \\ y_1x_3 - x_1y_3 \\ x_1y_2 - y_1x_2 \end{pmatrix} \tag{1}$$

If C is any matrix of the form

$$C = \begin{pmatrix} w_1 & w_2 & w_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

then

$$\mathbf{x} \times \mathbf{y} = C_{11}\mathbf{e}_1 + C_{12}\mathbf{e}_2 + C_{13}\mathbf{e}_3 = \begin{pmatrix} C_{11} \\ C_{12} \\ C_{13} \end{pmatrix}$$

Expanding $\det(C)$ by cofactors along the first row, we see that

$$\det(C) = w_1C_{11} + w_2C_{12} + w_3C_{13} = \mathbf{w}^T(\mathbf{x} \times \mathbf{y})$$

In particular, if we choose $\mathbf{w} = \mathbf{x}$ or $\mathbf{w} = \mathbf{y}$, then the matrix C will have two identical rows, and hence its determinant will be 0. We then have

$$\mathbf{x}^T(\mathbf{x} \times \mathbf{y}) = \mathbf{y}^T(\mathbf{x} \times \mathbf{y}) = 0 \tag{2}$$

In calculus books, it is standard to use row vectors

$$\mathbf{x} = (x_1, x_2, x_3) \quad \text{and} \quad \mathbf{y} = (y_1, y_2, y_3)$$

and to define the cross product to be the row vector

$$\mathbf{x} \times \mathbf{y} = (x_2y_3 - y_2x_3)\mathbf{i} - (x_1y_3 - y_1x_3)\mathbf{j} + (x_1y_2 - y_1x_2)\mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the row vectors of the 3×3 identity matrix. If one uses $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in place of w_1, w_2, and w_3, respectively, in the first row of the matrix M, then the cross product can be written as a determinant:

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

where **i**, **j**, and **k** are the row vectors of the 3×3 identity matrix. If one uses **i**, **j**, and **k** in place of w_1, w_2, and w_3, respectively, in the first row of the matrix M, then the cross product can be written as a determinant:

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

In linear algebra courses, it is generally more standard to view $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x} \times \mathbf{y}$ as column vectors. In this case, we can represent the cross product in terms of the determinant of a matrix whose entries in the first row are $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, the column vectors of the 3×3 identity matrix:

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

The relation given in equation (2) has applications in Newtonian mechanics. In particular, the cross product can be used to define a *binormal* direction, which Newton used to derive the laws of motion for a particle in 3-space.