Linear Algebra

線性代數

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Course Information

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- 助教：研一

Office Hours

資訊工程學系　專任教師　丁德榮　　　　　　　　　　　　　　　　印表日期

	星期一	星期二	星期三	星期四	星期五	星期六
第1節 08：10-09：00			遠距教學			54023 資工三 網路通訊專題(二) 資工研討室(二)
第2節 09：05-09：55		54003 資工一 線性代數 T104	54020 資工三 網路安全 34303	54030 資工碩一 智慧物聯網實務 資工電腦教室(一)		54023 資工三 網路通訊專題(二) 資工研討室(二)
第3節 10：15-11：05		54003 資工一 線性代數 T104	54020 資工三 網路安全 34303	54030 資工碩一 智慧物聯網實務 資工電腦教室(一)		54023 資工三 網路通訊專題(二) 資工研討室(二)
第4節 11：10-12：00		54003 資工一 線性代數 T104	54020 資工三 網路安全 34303	54030 資工碩一 智慧物聯網實務 資工電腦教室(一)		54023 資工三 網路通訊專題(二) 資工研討室(二)
第14節 12：05-12：55			研究生 開會			
第5節 13：10-14：00			研究生 開會			
第6節 14：05-14：55		Office Hour 資工系二樓 分機:8445		Office Hour 資工系二樓 分機:8445		
第7節 15：15-16：05		Office Hour 資工系二樓 分機:8445		Office Hour 資工系二樓 分機:8445		

課程資訊

- Textbook:
  - Linear Algebra with Applications 10/e , Steven J. Leon, Lisette de Pillis, 2021, 滄海書局, 04-27088787
- Reference Books:
  - Elementary Linear Algebra, Larson, Edwards and Falvo, 8e, 2017, 高立
  - Elementary Linear Algebra-Application, Anton and Rorres, 10e, 2017, 新月
  - Introduction to Linear Algebra, Fifth Edition (2016) Gilbert Strang

課程評分

- 小考 30%

  (至少5次，每次五題，30分鐘，最後一節考)
- 期中考 35%
- 期末考 35%

如何學好線性代數

- 看懂定義
- 記好性質
- 證明技巧訓練
- 多做習題
- 再多看幾本書
- 課程錄影：dlearn.ncue.edu.tw

課程大綱

- Chapter 1. MATRICES AND SYSTEMS OF EQUATIONS
  - Systems of linear equations
  - Row echelon form
  - Matrix Arithmetic
  - Matrix algebra
  - Elementary matrices
  - Partitioned Matrices
- Chapter 2. DETERMINANTS
  - The determinant of a matrix
  - Properties of determinants
  - Additional Topics and Applications
- Chapter 3. VECTOR SPACES
  - Definition and examples
  - Subspaces
  - Linear independence
  - Basis and dimension
  - Change of basis
  - Row space and column space

課程大綱

- Chapter 4. LINEAR TRANSFORMATIONS
  - Definition and examples
  - Matrix representation
  - Similarity
- Chapter 5. ORTHOGONALITY
  - The scalar product in Rn
  - Orthogonal subspaces
  - Least Squares Problem
  - Inner product spaces
  - Least squares problem
  - Orthonormal sets
  - Gram-Schmidt Orthogonalization
- Chapter 6. EIGENVALUES
  - Eigenvalues and eigenvectors
  - Diagonalization/Hermitian Matrices/SVD/Quardratic Forms

教學進度

教學內容與進度：

週次	上課日期	教學單元與進度	學生應預習之章節	作業評量與檢討
1	02/23	1.MATRICES AND SYSTEMS OF EQUATIONS		
2	03/02	1.MATRICES AND SYSTEMS OF EQUATIONS		
3	03/09	1.MATRICES AND SYSTEMS OF EQUATIONS		
4	03/16	2.DETERMINANTS		
5	03/23	2.DETERMINANTS		
6	03/30	3. VECTOR SPACES		
7	04/06	校際交流日		校際交流日
8	04/13	3. VECTOR SPACES		
9	04/20	期中考 **期中考**		期中考（Mid－term）
10	04/27	3. VECTOR SPACES		
11	05/04	3. VECTOR SPACES		
12	05/11	4. LINEAR TRANSFORMATIONS		
13	05/18	4. LINEAR TRANSFORMATIONS		
14	05/25	5. ORTHOGONALITY		
15	06/01	5. ORTHOGONALITY		
16	06/08	6. EIGENVALUES		
17	06/15	6. EIGENVALUES		
18	06/22	期末考 **期末考**		期末考（Final exam）

Chapter 1

Matrixes and Systems of Equations

1-1 Linear Systems

- Learning Goals
  - **Linear Systems**
  - **Elementary Row Operations**
  - **Strictly triangular system**
  - **Back substitution (倒退代入法)**

1.1 Systems of Linear Equations

- A linear equation in n unknowns is of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers and x_1, $x_2, \ldots, x_n$ are variables.

- A linear system of m equations in n unknowns is of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&. \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}$$

where the a_{ij}'s and the b_i's are all real numbers.

- *Refer as $m \times n$ linear system*

Linear system

- This form is referred to as an ***m*×*n* linear system**.
- **Ex**

(a) $x_1 + 2x_2 = 5$ $2x_1 + 3x_2 = 8$ is a 2×2 system	(b) $x_1 - x_2 + x_3 = 2$ $2x_1 + x_2 - x_3 = 4$ is a 2×3 system	(c) $x_1 + x_2 = 2$ $x_1 - x_2 = 1$ $x_1 \quad = 4$ is a 3×2 system
Sol. (1, 2)	Sol. (2, 0, 0)	no solution

  – Try to find the solutions of (a) (b) (c).

  – A solution to an $m \times n$ linear system is an **ordered *n*-tuple of numbers** $(x_1, x_2, \ldots, x_n)$ that satisfies all the equations of the system.

Note

- If a linear system has no solution, the system is **inconsistent**.
- If the system has at least one solution, it is **consistent.**
- The set of all solutions to a linear system is called the **solution set** of the system.
  - The system (c) is inconsistent
  - The systems (a) and (b) are consistent with solution sets being (1, 2) and (2, α, α).

2×2 Systems

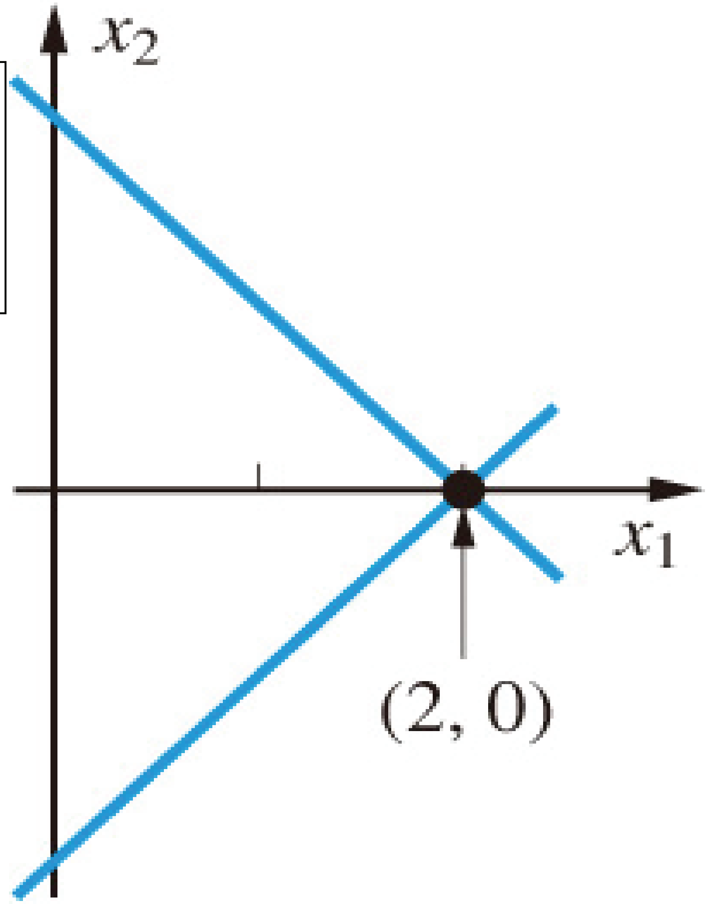
- A 2×2 linear system is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

- Each equation can be represented graphically as a line in the plane.
- An ordered pair (x_1, x_2) will be a solution ***iff*** it lies on both lines.

$$x_1 + x_2 = 2$$
$$x_1 - x_2 = 2$$

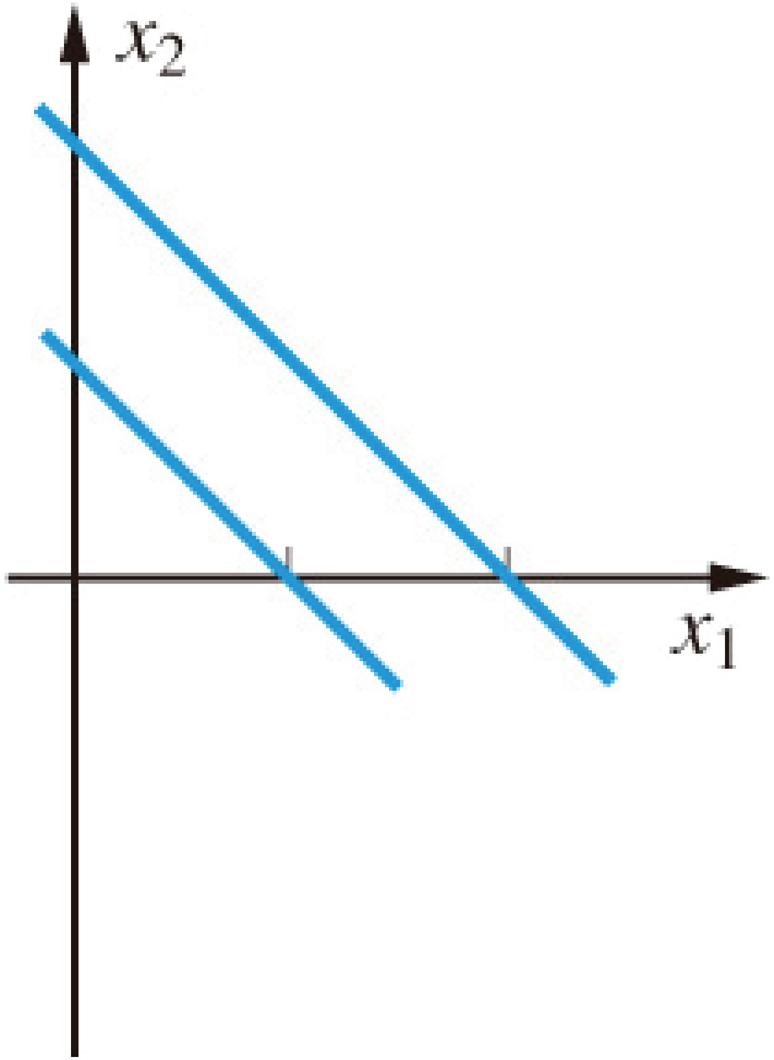
consistent (唯一解)



Intersect at a point

⇨ Exactly one solution

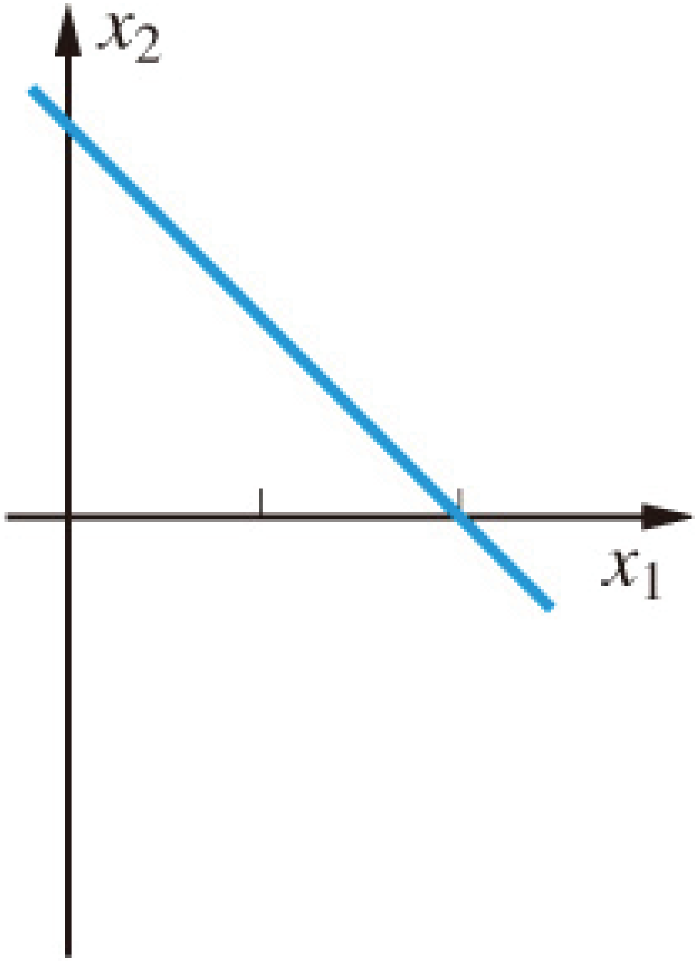
$$x_1 + x_2 = 2$$
$$x_1 + x_2 = 1$$
inconsistent (無解)



Parallel lines
⇨ No solution

$$x_1 + x_2 = 2$$
$$-x_1 - x_2 = -2$$

consistent (無限多解)



Same line

⇨ Infinitely many solutions

Equivalent Systems (等價系統)

- Consider the two systems:

(a)
$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ x_2 \qquad &= 3 \\ 2x_3 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ -3x_1 - x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 + x_3 &= 2 \end{aligned}$$

System (b) seems to be more difficult to solve.

⇨Add the first two eqs. of system (b): Thus, the solution of the system is (−2, 3, 2).

$$\begin{array}{rrrrl} & 3x_1 & + 2x_2 & - x_3 & = -2 \\ + & -3x_1 & - x_2 & + x_3 & = 5 \\ \hline & & x_2 & & = 3 \end{array}$$

- ⇨ Subtracting the first eq. from the third:

$$
\begin{array}{rl}
 & 3x_1 + 2x_2 + x_3 = 2 \\
\text{-)} & 3x_1 + 2x_2 - x_3 = -2 \\
\hline
 & 2x_3 = 4
\end{array}
$$

- So, you will find that the two systems are the same.

Definition

Two systems of equations involving the same variables are said to be **equivalent** if they have the same solution set.

Exchange two rows

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 - x_2 &= 2 \\ 4x_1 + x_2 &= 6 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 4x_1 + x_2 &= 6 \\ 3x_1 - x_2 &= 2 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

An equation is multiplied through by a nozero real number

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ -2x_1 - x_2 + 4x_3 &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 6 \\ -2x_1 - x_2 + 4x_3 &= 1 \end{aligned}$$

- **If multiple of one equation is added to another equation,** the new system will be equivalent to the original system.

$$
\begin{aligned}
a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\
a_{j1}x_1 + \cdots + a_{jn}x_n &= b_j
\end{aligned}
$$

If and only if it satisfies the equations

$$
\begin{aligned}
a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\
(a_{j1} + \alpha a_{i1})x_1 + \cdots + (a_{jn} + \alpha a_{in})x_n &= b_j + \alpha b_i
\end{aligned}
$$

Note

- The following three operations can be used on a system to get an equivalent system:

  I. The order in which any two equations are written may be interchanged.

  II. Both sides of an equation may be multiplied by the same **nonzero** real numbers.

  III. A multiple of one equation may be added to (or subtracted from) another.

$n \times n$ Systems Definition

A system is said to be in **strict triangular form** if in the kth equation the coefficients of the first k-1 variables are all zero and the coefficient of x_k is nonzero ($k = 1, 2, \ldots, n$)

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ 2x_3 &= 4 \end{aligned}$$

Example 1

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \qquad \text{(a)} \\ x_2 - x_3 &= 2 \qquad \text{(b)} \\ 2x_3 &= 4 \qquad \text{(c)} \end{aligned}$$

- ***Sol***: (Using this strictly triangular form, you can find the result fast.)

  (c) $2x_3 = 4$, $x_3 = 2$

  代入 (b) x_2 -2 = 2, $x_2 = 4$

  代入 (a) $3x_1 + 2\times4 + 2 = 1$

  $3x_1 = -9 \Rightarrow x_1 = -3$

  Thus, the solution is (-3, 4, 2)

Note

- The method of solving a **strictly triangular system** is referred to as **back substitution**.
- Any $n \times n$ strictly triangular system can be solved in the same manner.

Example 2

- Solve the system

$$\boxed{\begin{array}{rrrrr} 2x_1 & -x_2 & +3x_3 & -2x_4 & =1 \\ & x_2 & -2x_3 & +3x_4 & =2 \\ & & 4x_3 & +3x_4 & =3 \\ & & & 4x_4 & =4 \end{array}}$$

- ***Sol***:

$$\begin{array}{rr} 4x_4 = 4 & x_4 = 1 \\ 4x_3 + 3\cdot 1 = 3 & x_3 = 0 \\ x_2 - 2\cdot 0 + 3\cdot 1 = 2 & x_2 = -1 \\ 2x_1 - (-1) + 3\cdot 0 - 2\cdot 1 = 0 & x_1 = 1 \end{array}$$

Thus, the solution is (1,−1, 0, 1).

Example 3

- Solve the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

- How to get the strict triangular form?

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ -7x_2 - 6x_3 &= -10 \\ -x_2 - x_3 &= -2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ -7x_2 - 6x_3 &= -10 \\ -1/7\, x_3 &= -4/7 \end{aligned}$$

Using back substitution, we get

$$x_3 = 4, \qquad x_2 = -2, \qquad x_1 = 3$$

Example 3 (conti.)

- The **coefficient matrix** of the system:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

- if m=n square matrix $\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}$

- The **augmented matrix** of the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right]$$

Elementary Row Operations

I. Interchange two rows.

II. Multiply a row by a nonzero real number.

III. Replace a row by it’s sum with a multiple of another row.

Pivot

$a_{11} = 1$

The first nonzero entry in the pivot row.

$$\left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{3} \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right] \begin{matrix} \text{Pivotal row} \\ \times\,-3 \\ \times\,-2 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ \mathbf{0} & \mathbf{-7} & \mathbf{-6} & \mathbf{-10} \\ 0 & -1 & -1 & -2 \end{array}\right] \begin{matrix} \\ \text{Pivotal row} \\ \times\,-1/7 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} \end{array}\right]$$

This is an augmented matrix for the strictly triangular system.

Matrix representation (A|B) Augment Matrix (擴充矩陣)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & & & \\ b_{m1} & b_{m2} & \cdots & b_{mr} \end{pmatrix}$$

then

$$(A|B) = \left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & b_{11} & \cdots & b_{1r} \\ \vdots & & & \vdots & & \\ a_{m1} & \cdots & a_{mn} & b_{m1} & \cdots & b_{mr} \end{array}\right)$$

With each system of equations we may associate an augmented matrix of the form

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

Example 4

- Solve the system:

$$\begin{aligned} & - x_2 - x_3 + x_4 = 0 \\ x_1 &+ x_2 + x_3 + x_4 = 6 \\ 2x_1 &+ 4x_2 + x_3 - 2x_4 = -1 \\ 3x_1 &+ x_2 - 2x_3 + 2x_4 = 3 \end{aligned}$$

The **augment matrix** (**A|b**) is

$$\left(\begin{array}{cccc|c} 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right)$$

Example 4 (con.)

Pivot $a_{11} = 1$ $\longrightarrow$

$$\left[\begin{array}{cccc|c} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{6} \\ 0 & -1 & -1 & 1 & 0 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right]$$

$\longleftarrow$ Pivotal row

$\times -2$

$\times -3$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ \mathbf{0} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} & \mathbf{0} \\ 0 & 2 & -1 & -4 & -13 \\ 0 & -2 & -5 & -1 & -15 \end{array}\right]$$

$\longleftarrow$ Pivotal row

$\times 2$

$\times -2$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{-3} & \mathbf{-2} & \mathbf{-13} \\ 0 & 0 & -3 & -3 & -15 \end{array}\right]$$

$\longleftarrow$ Pivotal row

$\times -2$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right]$$

solution $(2, -1, 3, 2)$

General Steps

- General steps for solving $n \times n$ linear systems:
  - Step 1

$$\left[\begin{array}{cccc|c} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \end{array}\right]$$

  - Step 2

$$\left[\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & x & x & x \end{array}\right]$$

  - Step 3

$$\left[\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & x & x & x \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \end{array}\right]$$

1-2 Row Echelon Form

- **Learning Goals**
  - Row Echelon Form
  - Gaussian elimination
  - Overdetermined Systems, underdetermined systems
  - Reduced Row Echelon Form
  - Gauss-Jordan reduction

Example 1

- Consider the system represented by the following augmented matrix:

$$\left[\begin{array}{ccccc|c} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right] \leftarrow \textbf{Pivotal row} \Rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{3} \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{0} \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right] \leftarrow \textbf{Pivotal row} \Rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right]$$

Row Echelon Form

Example 1 (con.)

- Thus equations represented by the last two rows are:

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = -4$$
$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = -3$$

- The system is **inconsistent!**

- Consider the system represented by the following augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{array}\right]$$

- Using the "elementary row operations", we get

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

- Thus the solution set will be the set of all 5-tuples satisfying the first three equations:

$$\begin{array}{ccccccccccc} x_1 & + & \boxed{x_2} & + & x_3 & + & x_4 & + & x_5 & = & 1 \\ & & & & x_3 & + & \boxed{x_4} & + & 2x_5 & = & 0 \\ & & & & & & & & x_5 & = & 3 \end{array}$$

  - x_1, x_3, x_5 are the **lead variable** (the variables corresponding to the first nonzero elements in each row of the augmented matrix)
  - x_2, x_4 are the **free variables** (the remaining variables corresponding to the columns skipped in the reduction process)

- If we transfer the free variables over to the right-hand side, we get

$$\begin{aligned} x_1 + x_3 + x_5 &= 1 - x_2 - x_4 \\ x_3 + 2x_5 &= -x_4 \\ x_5 &= 3 \end{aligned}$$

  - Set free variables $x_2 = \alpha$, $x_4 = \beta$, the solution set is

$$(-\alpha+4, \alpha, -\beta-6, \beta, 3)$$

- We find that if all the coefficients of leading variables are one, we can get the solution faster.

Definition

A matrix is said to be in **row echelon form** if

(i) The first nonzero entry in each row is **1**

(ii) If row k does not consist entirely of zeros, the number of **leading zero** entries in row $k+1$ is greater than the number of leading zero entries in row k.

(iii) If there are rows whose entries are all zero, they are below the rows having nonzero entries.

Example 2

- The following matrices are in row echelon form:

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 3

- The following matrices are not in row echelon form:

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- **Why**? Is which in strict triangular form?

Definition

- The process of using elementary row operations **I**, **II**, and **III** to transform a linear system into one whose augmented matrix is in row echelon form is called **Gaussian elimination**.

Note

- If the row echelon form of the augmented matrix contains a row of the form

$$[\, 0 \;\; 0 \;\; \ldots \;\; 0 \,|\, 1 \,]$$

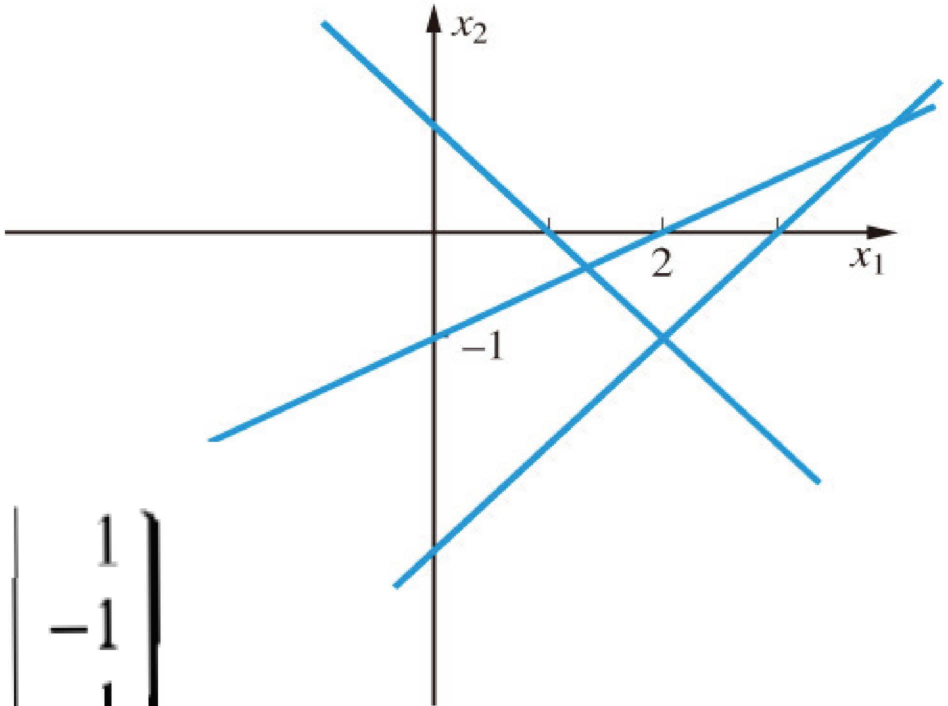
the system is **inconsistent**. Otherwise the system will be consistent. If the system is consistent and the nonzero rows of the row echelon form of the matrix form a **strictly triangular system**, the system will have a **unique solution**.

Overdetermined Systems

- A linear system of m equations and n unknowns is said to be **overdetermined** if there are more equations than unknowns (***m > n***).

$$\textbf{(a)}\quad \begin{array}{rcrcr} x_1 & + & x_2 & = & 1 \\ x_1 & - & x_2 & = & 3 \\ -x_1 & + & 2x_2 & = & 2 \end{array}$$

$$m = 3, \quad n = 2$$



$$\left[\begin{array}{rr|r} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right]$$

Example 4

- Solve each of the following overdetermined systems:

(b)

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ 2x_1 & - & x_2 & + & x_3 & = & 2 \\ 4x_1 & + & 3x_2 & + & 3x_3 & = & 4 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 5 \end{array}$$

$m = 4, \quad n = 3$

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(c)
$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ 2x_1 & - & x_2 & + & x_3 & = & 2 \\ 4x_1 & + & 3x_2 & + & 3x_3 & = & 4 \\ 3x_1 & + & x_2 & + & 2x_3 & = & 3 \end{array}$$

$m = 4, \quad n = 3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = -0.2x_3$

$x_1 = 1 - 2x_2 - x_3 = 1 - 0.6x_3$

the solution set is the set of all ordered triples of the form $(1 - 0.6\alpha, -0.2\alpha, \alpha)$, where α is a real number.

- **An *overdetermined* system is usually (but not always) inconsistent.**

Underdetermined Systems

- A linear system is said to be **underdetermined** if there are fewer equations than unknowns (***m*** < ***n***).
- **Example 5**. Solve the following underdetermined systems:

(a)
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 4x_2 & + & 2x_3 & = & 3 \end{array}$$

(b)
$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 2 \\ x_1 & + & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 3 \\ x_1 & + & x_2 & + & x_3 & + & 2x_4 & + & 3x_5 & = & 2 \end{array}$$

Note

- **A <u>consistent</u> *underdetermined* system will have <u>infinitely many solutions.</u>**

(a) $$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 4x_2 & + & 2x_3 & = & 3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$
\textbf{(b)}\quad
\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 &= 2\\
x_1 + x_2 + x_3 + 2x_4 + 2x_5 &= 3\\
x_1 + x_2 + x_3 + 2x_4 + 3x_5 &= 2
\end{aligned}
$$

$$
\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2\\ 1 & 1 & 1 & 2 & 2 & 3\\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2\\ 0 & 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]
$$

$$
\to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 0 & 3\\ 0 & 0 & 0 & 1 & 0 & 2\\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 & 2\\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \quad \begin{aligned} x_1 &= 1 - x_2 - x_3\\ x_4 &= 2\\ x_5 &= -1 \end{aligned}
$$

for any real numbers α and β, the 5-tuple $(1 - \alpha - \beta, \alpha, \beta, 2, -1)$ is a solution of the system.

Reduced Row Echelon Form

Definition

A matrix is said to be in **reduced row echelon form** if:

(1) the matrix is in row echelon form

(2) the first nonzero entry is the only nonzero entry in its column

- The following matrices are reduced row echelon form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note

- The process of using elementary row operations to transform a matrix into reduced row echelon form is called **Gauss-Jordan reduction**.

Example 6

- Using Gauss-Jordan reduction to solve the system

$$\begin{array}{rcrcrcrcl} -x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & 0 \\ 3x_1 & + & x_2 & - & x_3 & - & x_4 & = & 0 \\ 2x_1 & - & x_2 & - & 2x_3 & - & x_4 & = & 0 \end{array}$$

- ***Sol***:

$$\left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$$

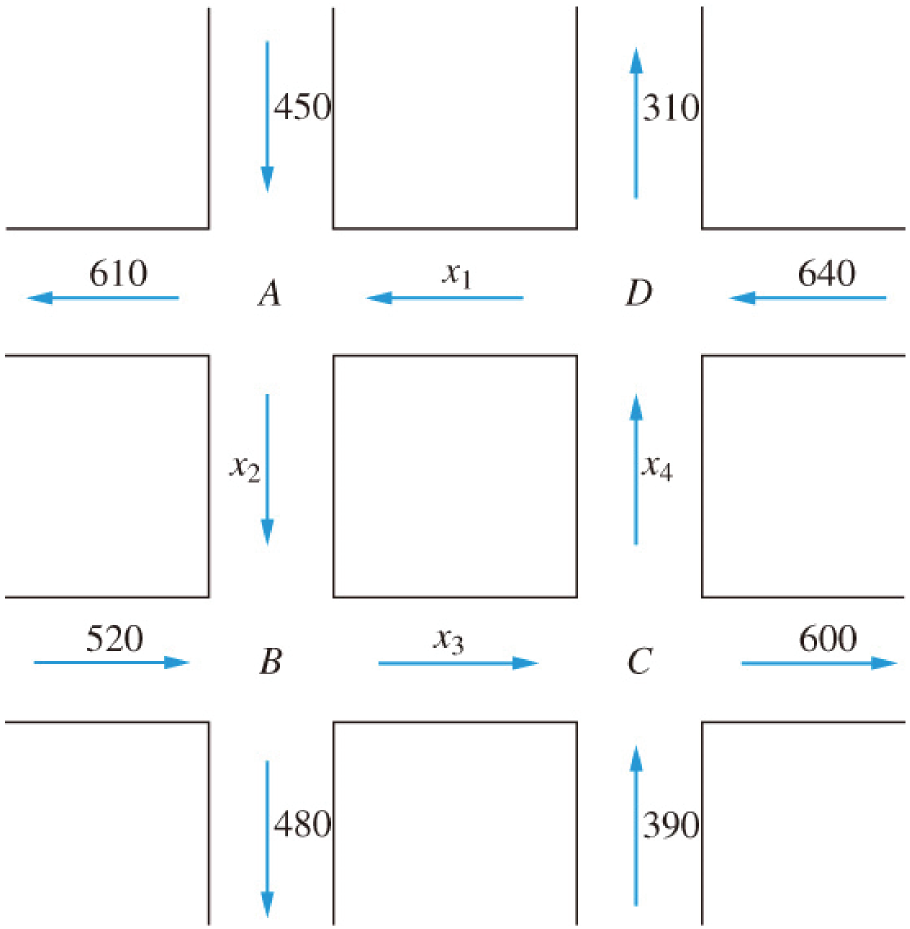
$$\rightarrow \left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right] \rightarrow \begin{array}{l} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \end{array} \begin{array}{l} \text{row} \\ \text{echelon} \\ \text{form} \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} \text{reduced} \\ \text{row echelon} \\ \text{form} \end{array}$$

$$\Rightarrow \begin{array}{ccccc} x_1 & - & x_4 & = & 0 \\ x_2 & + & x_4 & = & 0 \\ x_3 & - & x_4 & = & 0 \end{array}$$

- Is this an overdetermined system or underdetermined system?
- Which are the lead variables? free variables?
- The solution set is (α,- α, α, α)

Application 1 Traffic Flow



$$x_1 + 450 = x_2 + 610 \quad \text{(intersection } A)$$
$$x_2 + 520 = x_3 + 480 \quad \text{(intersection } B)$$
$$x_3 + 390 = x_4 + 600 \quad \text{(intersection } C)$$
$$x_4 + 640 = x_1 + 310 \quad \text{(intersection } D)$$

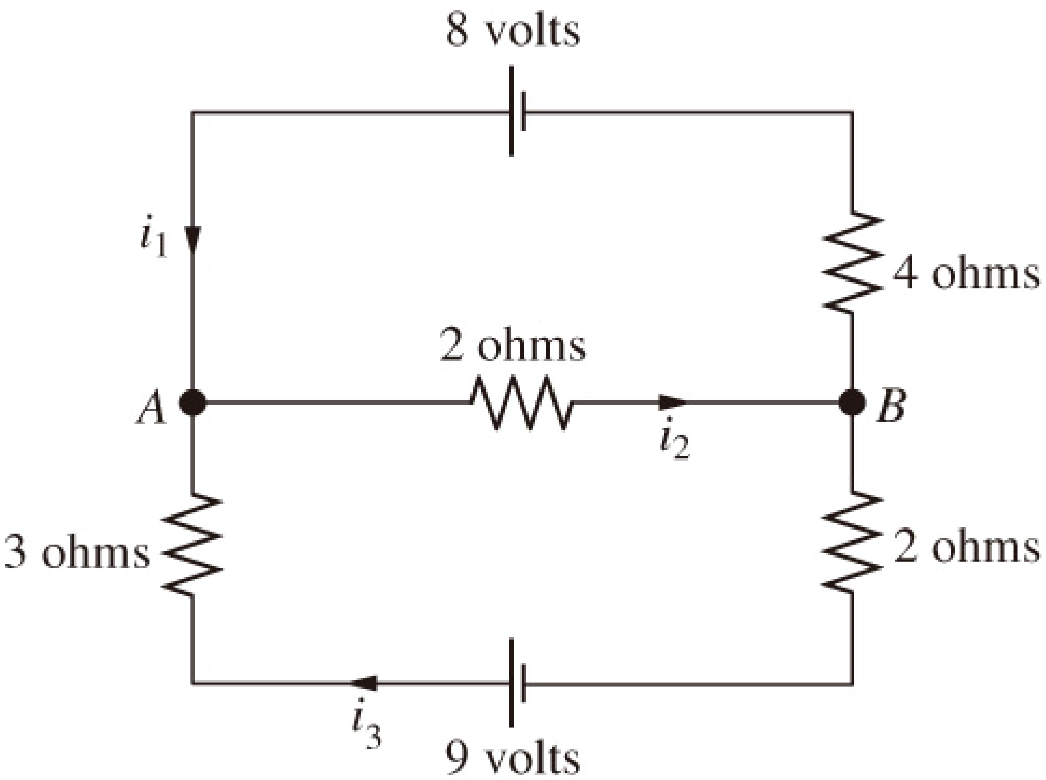
$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ -1 & 0 & 0 & 1 & -330 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 330 \\ 0 & 1 & 0 & -1 & 170 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Application 2 Electrical Networks



$$\begin{aligned} i_1 - i_2 + i_3 &= 0 && \text{(node } A\text{)} \\ -i_1 + i_2 - i_3 &= 0 && \text{(node } B\text{)} \\ 4i_1 + 2i_2 &= 8 && \text{(top loop)} \\ 2i_2 + 5i_3 &= 9 && \text{(bottom loop)} \end{aligned}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 4 & 2 & 0 & 8 \\ 0 & 2 & 5 & 9 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{4}{3} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Kirchhoff's Laws

- At every node, the sum of the incoming currents equals the sum of the outgoing currents.
- Around every closed loop, the algebraic sum of the voltage gains must equal the algebraic sum of the voltage drops.
- The voltage drops E for each resistor are given by ***Ohm's law***: ***$E = iR$***, where i represents the current in amperes and R the resistance in ohms.

Homogeneous Systems Definition

A system of linear equations is said to be **homogeneous** if the constants on the right–hand side are all zero.

Homogeneous systems are always consistent since it must have the trivial solution $(0, 0, \ldots, 0)$.

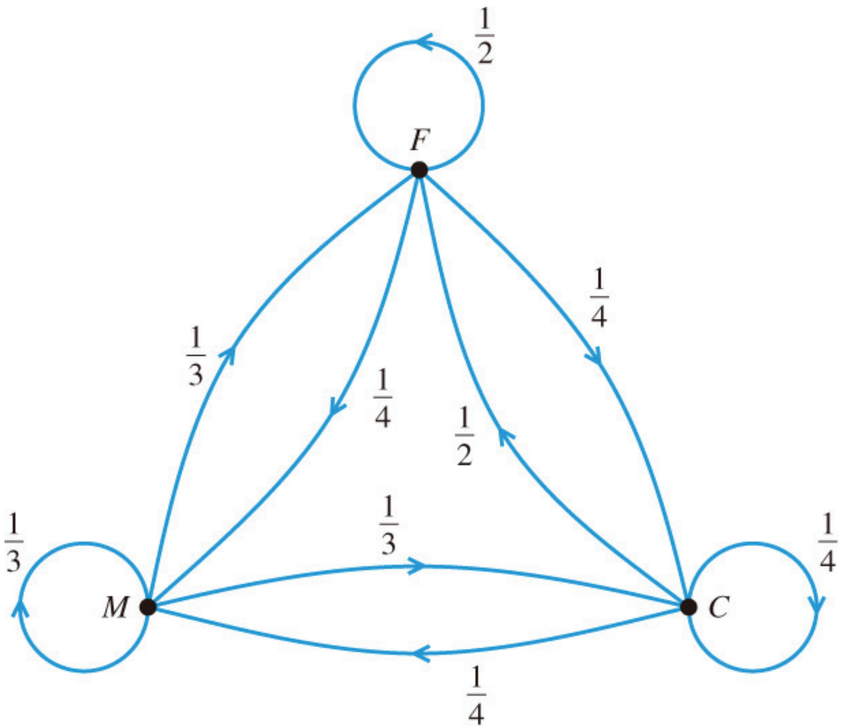
Theorem 1.2.1

- An $m \times n$ homogeneous system of linear equations has a nontrivial solution if $\boldsymbol{n > m}$ (underdetermined).
- ***Pf***:
  - A homogeneous system is always consistent. There are at most m lead variables. Since there are n variables altogether and $n>m$, there must be some free variables and they can be assigned arbitrary values. So, we can get a nontrivial solution if $n>m$.

Example $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$

Application 4 Economic Model

	F	M	C
F	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
M	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{4}$
C	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{4}$



$$\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 = x_1$$

$$\frac{1}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = x_2$$

$$\frac{1}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = x_3$$

$$\begin{aligned} -\tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{2}x_3 &= 0 \\ \tfrac{1}{4}x_1 - \tfrac{2}{3}x_2 + \tfrac{1}{4}x_3 &= 0 \\ \tfrac{1}{4}x_1 + \tfrac{1}{3}x_2 - \tfrac{3}{4}x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 : x_2 : x_3 = 5 : 3 : 3$$

1-3 Matrix Algebra

- Learning Goals
  - Matrix operation
  - Matrix Algebra
  - Consistency Theorem for Linear Systems

1.3 Matrix Notation

- An $m \times n$ matrix $\boldsymbol{A}$ can be represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- The entries of a matrix are called ***scalars*** (real or complex number). In general, a_{ij} will denote the entry of the matrix $\boldsymbol{A}$ that is in the ith row and the jth column.
- We will sometimes shorten this matrix to $\boldsymbol{A}=(a_{ij})$.

Vectors

- An n-tuple of real number is referred to as a **vector.**
  - **row vector:** a $1 \times n$ matrix, e.g., $[a_1, a_2, \ldots, a_n]$
  - **column vector:** an $n \times 1$ matrix , e.g., $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$
- The set of all $n \times 1$ matrices of real numbers is called **Euclidean *n*-space** and is denoted by $\boldsymbol{R}^n$.

- The ith row vector of a $m\times n$ matrix A is denoted by $\vec{\mathbf{a}}_i$ and the jth column vector is denoted by $\vec{\mathbf{a}}_j$, where
  - $\vec{\mathbf{a}}_i = (a_{i1}, a_{i2}, \cdots, a_{in})$ $i = 1, 2, \ldots, m$, and
  - $\vec{\mathbf{a}}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ $, j = 1, 2, \ldots, n$

Example 1

$$A = \begin{bmatrix} 3 & 2 & 5 \\ -1 & 8 & 4 \end{bmatrix}$$

then $\mathbf{a}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

and $\vec{\mathbf{a}}_1 = (3, \quad 2, \quad 5), \quad \vec{\mathbf{a}}_2 = (-1, \quad 8, \quad 4)$

- The matrix $\boldsymbol{A}$ can be represented in terms of its column vectors or row vectors

  $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ or

$$A = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}$$

Equality

Two $m \times n$ matrices A and B are said to be **equal** if $a_{ij} = b_{ij}$ for each i and j.

Scalar Multiplication Definition

If A is an $m \times n$ matrix and α is a scalar, then αA is the $m \times n$ matrix whose (i, j) entry is αa_{ij}.

- Example

$$A = \begin{bmatrix} 4 & 8 & 2 \\ 6 & 8 & 10 \end{bmatrix}$$

then $$\frac{1}{2} A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 4 & 5 \end{bmatrix}$$

and $$3A = \begin{bmatrix} 12 & 24 & 6 \\ 18 & 24 & 30 \end{bmatrix}$$

Matrix Addition Definition

If $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ are both $m \times n$ matrices, then the sum $\boldsymbol{A}+\boldsymbol{B}$ is the $m \times n$ matrix whose (i, j) entry is $a_{ij}+b_{ij}$ for each ordered pair (i, j).

- Example

$$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 3 \\ 5 & 7 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} + \begin{bmatrix} -8 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \\ 10 \end{bmatrix}$$

- We can then define $\boldsymbol{A} - \boldsymbol{B}$ to be $\boldsymbol{A} + (-1)\boldsymbol{B}$
- If $\boldsymbol{O}$ represent a matrix, with the same dimension as $\boldsymbol{A}$, whose entries are all 0, then the following properties must hold

  (1) $\boldsymbol{O}$ acts as the ***additive identity***, i.e., $\boldsymbol{A} + \boldsymbol{O} = \boldsymbol{O} + \boldsymbol{A} = \boldsymbol{A}$

  (2) each matrix $\boldsymbol{A}$ has an ***additive inverse***, $\boldsymbol{A} + (-1)\boldsymbol{A} = \boldsymbol{O} = (-1)\boldsymbol{A} + \boldsymbol{A}$
- It is commonly to denote the additive inverse by $-\boldsymbol{A}$, thus $-\boldsymbol{A} = (-1)\boldsymbol{A}$

Example for A-B

$$\begin{aligned}\begin{bmatrix}2 & 4\\3 & 1\end{bmatrix}-\begin{bmatrix}4 & 5\\2 & 3\end{bmatrix} &= \begin{bmatrix}2 & 4\\3 & 1\end{bmatrix}+(-1)\begin{bmatrix}4 & 5\\2 & 3\end{bmatrix}\\ &= \begin{bmatrix}2 & 4\\3 & 1\end{bmatrix}+\begin{bmatrix}-4 & -5\\-2 & -3\end{bmatrix}\\ &= \begin{bmatrix}2-4 & 4-5\\3-2 & 1-3\end{bmatrix}\\ &= \begin{bmatrix}-2 & -1\\1 & -2\end{bmatrix}\end{aligned}$$

National Rules

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$A + BC = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 7 & 7 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 11 \\ 0 & 6 \end{bmatrix}$$

$$3A + B = \begin{bmatrix} 9 & 12 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 5 & 7 \end{bmatrix}$$

Matrix Multiplication and Linear Systems

- **CASE 1 One Equation in Several Unknowns**
- Consider the equation: $3x_1 + 2x_2 + 5x_3 = 4$

  – If we set $\boldsymbol{A} = [3 \ \ 2 \ \ 5]$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and define the product $\boldsymbol{A}\mathbf{x}$ by

$$\boldsymbol{A}\mathbf{x} = [3 \quad 2 \quad 5]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3x_1 + 2x_2 + 5x_3$$

then the equation can be written as the matrix equation: $\boldsymbol{A}\mathbf{x} = \mathbf{4}$.

- For a linear equation of n unknowns of the form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

  – If we let $\boldsymbol{A} = [a_1, a_2, \ldots, a_n]$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, and $\boldsymbol{A}\mathbf{x}$ is defined by

$$\boldsymbol{A}\mathbf{x} = a_1x_1 + a_2x_2 + \ldots + a_nx_n$$

  then the system can be written in the form $\boldsymbol{A}\mathbf{x} = \mathbf{b}$.

Note

- The result of multiplying a <u>row vector on the left</u> times a <u>column vector on the right</u> is a <u>scalar</u>. This type of multiplication is often referred to as a **scalar product**.

<u>CASE 2</u> *M* Equations in *N* Unknowns

- Consider an $m \times n$ linear system

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

It can be represented as a matrix equation: $\boldsymbol{A}\mathbf{x} = \mathbf{b}$

- It can be represented as a matrix equation: $\boldsymbol{A}\mathbf{x} = \mathbf{b}$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

- and define the product $\boldsymbol{A}\mathbf{x}$ by

$$\boldsymbol{A}\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}$$

- The ith entry of $A\mathbf{x}$ is

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = \vec{\mathbf{a}}_i\mathbf{x}$$

which is the scalar product of the ith row vector of A times the column vector $\mathbf{x}$,

$$A\mathbf{x} = \begin{bmatrix} \vec{\mathbf{a}}_1\mathbf{x} \\ \vec{\mathbf{a}}_2\mathbf{x} \\ \vdots \\ \vec{\mathbf{a}}_m\mathbf{x} \end{bmatrix}$$

Example 2

$$\boldsymbol{A} = \begin{bmatrix} 4 & 2 & 1 \\ 5 & 3 & 7 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\rightarrow \quad \boldsymbol{A}\mathbf{x} = \begin{bmatrix} 4x_1 + 2x_2 + 1x_3 \\ 5x_1 + 3x_2 + 7x_3 \end{bmatrix}$$

Example 3

$$A = \begin{bmatrix} -3 & 1 \\ 2 & 5 \\ 4 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\rightarrow \qquad A\mathbf{x} = \begin{bmatrix} -3\cdot 2 + 1\cdot 4 \\ 2\cdot 2 + 5\cdot 4 \\ 4\cdot 2 + 2\cdot 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 24 \\ 16 \end{bmatrix}$$

Example 4

- The system of linear equation

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & + & x_3 & = & 5 \\ x_1 & - & 2x_2 & + & 5x_3 & = & -2 \\ 2x_1 & + & x_2 & - & 3x_3 & = & 1 \end{array}$$

- can be written as a matrix equation

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -2 & 5 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

Alternative representation

- The product $\boldsymbol{A}\mathbf{x}$ can also be expressed as ***a sum of column vectors***:

$$\boldsymbol{A}\mathbf{x} = \begin{bmatrix} \boxed{a_{11}x_1} + \boxed{a_{12}x_2} + \cdots + \boxed{a_{1n}x_n} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

So, the system of equation $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ can be expressed as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

Example 5

- The linear system

$$\begin{aligned} 2x_1 \quad &+ 3x_2 \quad - 2x_3 \quad = 5 \\ 5x_1 \quad &- 4x_2 \quad + 2x_3 \quad = 6 \end{aligned}$$

- can be written as s matrix equation

$$x_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Definition

If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are vectors in $\boldsymbol{R}^m$ and $c_1, c_2, \ldots, c_n$ are scalars, then a sum of the form

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots + c_n\mathbf{a}_n$$

is said to be a **linear combination** of the vectors of $\mathbf{a}_1$, $\mathbf{a}_2, \ldots, \mathbf{a}_n$.

- The product $A\mathbf{x}$ is a linear combination of the column vectors of A:

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

Example 6

$$x_1\begin{bmatrix}2\\5\end{bmatrix}+x_2\begin{bmatrix}3\\-4\end{bmatrix}+x_3\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}5\\6\end{bmatrix}$$

- We choose $x_1 = 2$, $x_2 = 3$, and $x_3 = 4$ in Example 5, then

$$\begin{bmatrix}5\\6\end{bmatrix}=2\begin{bmatrix}2\\5\end{bmatrix}+3\begin{bmatrix}3\\-4\end{bmatrix}+4\begin{bmatrix}-2\\2\end{bmatrix}$$

$$\rightarrow \qquad \mathbf{x} = \begin{bmatrix}2\\3\\4\end{bmatrix}$$

Theorem 1.3.1
Consistency Theorem for Linear Systems

A linear system $A\mathbf{x} = \mathbf{b}$ is consistent **if and only if** $\mathbf{b}$ can be written as a linear combination of the column vectors of A.

Example 7

$$x_1 + 2x_2 = 1$$
$$2x_1 + 4x_2 = 1$$

$$\rightarrow \qquad x_1\begin{bmatrix}1\\2\end{bmatrix} + x_2\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}x_1 + 2x_2\\2x_1 + 4x_2\end{bmatrix}$$

Matrix Multiplication

- It is possible to multiply a matrix $\boldsymbol{A}$ times a matrix $\boldsymbol{B}$ if the number of columns of $\boldsymbol{A}$ equals to the number of rows of $\boldsymbol{B}$ and the product $\boldsymbol{AB}$ is the matrix whose columns are $\boldsymbol{A}\mathbf{b}_1, \boldsymbol{A}\mathbf{b}_2, \ldots, \boldsymbol{A}\mathbf{b}_n$

$$\boldsymbol{AB} = \boldsymbol{A}(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n) = (\boldsymbol{A}\mathbf{b}_1, \boldsymbol{A}\mathbf{b}_2, \ldots, \boldsymbol{A}\mathbf{b}_n)$$

The (i, j) entry of $\boldsymbol{AB}$ is the ith entry of the column vector $\boldsymbol{A}\mathbf{b}j$. It is determined by multiplying the ith row vector of $\boldsymbol{A}$ times the jth column vector of $\boldsymbol{B}$.

Definition

If $\boldsymbol{A} = (a_{ij})$ is a $m\times n$ matrix and $\boldsymbol{B} = (b_{ij})$ is a $n\times r$ matrix, then the product $\boldsymbol{AB} = \boldsymbol{C} = (c_{ij})$ is the $m\times r$ matrix whose entries are defined by

$$c_{ij} = \vec{\mathbf{a}}_i \mathbf{b}_j = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Example 8

If $A = \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix}$

$$AB = \begin{bmatrix} 3\cdot(-2)+(-2)\cdot 4 & 3\cdot 1+(-2)\cdot 1 & 3\cdot 3+(-2)\cdot 6 \\ 2\cdot(-2)+4\cdot 4 & 2\cdot 1+4\cdot 1 & 2\cdot 3+4\cdot 6 \\ 1\cdot(-2)+(-3)\cdot 4 & 1\cdot 1+(-3)\cdot 1 & 1\cdot 3+(-3)\cdot 6 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 1 & -3 \\ 12 & 6 & 30 \\ -14 & -2 & -15 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2\cdot 3+1\cdot 2+3\cdot 1 & -2\cdot(-2)+1\cdot 4+3\cdot(-3) \\ 4\cdot 3+1\cdot 2+6\cdot 1 & 4\cdot(-2)+1\cdot 4+6\cdot(-3) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 20 & -22 \end{bmatrix}$$

Note

- Multiplication of matrices **is not** commutative (i.e., $AB \neq BA$).

Example 9

AB *not exist*

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 6 \end{bmatrix}$$

then $$BA = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 17 & 26 \\ 15 & 24 \end{bmatrix}$$

Example 10

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

then $\quad AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$

and $\quad BA = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Hence, $\quad AB \neq BA$.

If A and B are both $n \times n$ matrices, then AB and BA will also be $n \times n$ matrices, but, in general, they will not be equal. *Multiplication of matrices is not commutative*.

Transpose

- The **transpose** of a m $\times$ n matrix $\boldsymbol{A}$ is the n $\times$ m matrix $\boldsymbol{B}$ defined by

  $b_{ji}=a_{ij}$, for i =1, 2, …, m and j =1, 2, …, n.

- The transport of $\boldsymbol{A}$ is denoted by $\boldsymbol{A^T}$.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$$

Example 11

(a) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

(b) If $B = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix}$, then $B^T = \begin{bmatrix} -3 & 4 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix}$.

(c) If $C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, then $C^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

Symmetric

- An n×n matrix $\boldsymbol{A}$ is said to be symmetric if by $\boldsymbol{A^T=A}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & -2 \\ 2 & -2 & -3 \end{pmatrix}$$

1-4 Matrix Algebra

- Learning Goals
  - Matrix Algebra Rules
  - Inverse of matrix
  - Application

1.4 Matrix Algebra Algebraic Rules
Theorem 1.4.1

Each of the following statements is valid for any scalars α and β and for any matrices A, B, and C for which the indicated operations are defined.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $(AB)C = A(BC)$
4. $A(B + C) = AB + AC$
5. $(A + B)C = AC + BC$
6. $(\alpha\beta)A = \alpha(\beta A)$
7. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
8. $(\alpha + \beta)A = \alpha A + \beta A$
9. $\alpha(A+B) = \alpha A + \alpha B$

Proof of Rule 4

- $A=(a_{ij})$ is a $m \times n$ matrix, and $B=(b_{ij})$ and $C=(c_{ij})$ are both $n \times r$ matrices. Let $D=A(B+C)$ and $E=AB+AC$

$$d_{ij}=\sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj})$$

and

$$e_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}+\sum_{k=1}^{n} a_{ik}c_{kj}$$

- But

$$\sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj})=\sum_{k=1}^{n} a_{ik}b_{kj}+\sum_{k=1}^{n} a_{ik}c_{kj}$$

so that $d_{ij}=e_{ij}$ and hence $A(B + C) = AB + AC$.

Proof of Rule 3

- A be a $m \times n$ matrix, and B an $n \times r$ matrix and C an $r \times s$ matrix. Let $D = AB$ and $E = BC$

$$d_{il} = \sum_{k=1}^{n} a_{ik} b_{kl} \quad \text{and} \quad e_{kj} = \sum_{l=1}^{r} b_{kl} c_{lj}$$

- The (i, j) entry of DC is

$$\sum_{l=1}^{r} d_{il} c_{lj} = \sum_{l=1}^{r} \left(\sum_{k=1}^{n} a_{ik} b_{kl} \right) c_{lj}$$

and the (i, j) entry of AE is

$$\sum_{k=1}^{n} a_{ik} e_{kj} = \sum_{k=1}^{n} a_{ik} \left(\sum_{l=1}^{r} b_{kl} c_{lj} \right)$$

- Since

$$\sum_{l=1}^{r}\left(\sum_{k=1}^{n} a_{ik} b_{kl}\right) c_{lj} = \sum_{l=1}^{r}\left(\sum_{k=1}^{n} a_{ik} b_{kl} c_{lj}\right) = \sum_{k=1}^{n} a_{ik}\left(\sum_{l=1}^{r} b_{kl} c_{lj}\right)$$

it follows that

$$(AB)C = DC = AE = A(BC)$$

Example 1

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix},\ B=\begin{bmatrix}2 & 1\\ -3 & 2\end{bmatrix},\ \text{and}\ C=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$$

$$\rightarrow \qquad A(BC)=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}=\begin{bmatrix}6 & 5\\ 16 & 11\end{bmatrix}$$

$$\rightarrow \qquad (AB)C=\begin{bmatrix}-4 & 5\\ -6 & 11\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}=\begin{bmatrix}6 & 5\\ 16 & 11\end{bmatrix}$$

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix},\ B=\begin{bmatrix}2 & 1\\ -3 & 2\end{bmatrix},\ \text{and}\ C=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$$

- Thus

$$A(BC)=\begin{bmatrix}6 & 5\\ 16 & 11\end{bmatrix}=(AB)C$$

$$A(B+C)=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}3 & 1\\ -1 & 3\end{bmatrix}=\begin{bmatrix}1 & 7\\ 5 & 15\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}-4 & 5\\ -6 & 11\end{bmatrix}+\begin{bmatrix}5 & 2\\ 11 & 4\end{bmatrix}=\begin{bmatrix}1 & 7\\ 5 & 15\end{bmatrix}$$

- Therefore

$$A(B+C)=AB+AC$$

Notation

- If A is an $n\times n$ matrix and k is a positive integer, then

$$A^k = \underbrace{AA\cdots A}_{k \text{ times}}$$

Example 2

- If $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$, then $A^2=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}$
- In general

$$A^3 = AAA = AA^2 = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}=\begin{bmatrix}4 & 4\\ 4 & 4\end{bmatrix}$$

$$A^n = \begin{bmatrix}2^{n-1} & 2^{n-1}\\ 2^{n-1} & 2^{n-1}\end{bmatrix}$$ Proof by induction

The Identity Matrix

- The identity matrix I for matrix multiplication will serve as

$$IA = AI = A$$

The **identity matrix** is the $n \times n$ matrix $I = (\delta_{ij})$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Example

- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3×3 identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{bmatrix} \quad (IA = A)$$

$$\begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{bmatrix} \quad (AI = A)$$

- In general, if B is any $m \times n$ matrix and C is any $n \times m$, then

$$BI = B \quad \text{and} \quad IC = C$$

- The column vectors of the $n \times n$ identity matrix I are the standard vectors used to define a coordinate system in Euclidean n-space and its standard notation for the jth column vector of I is $\mathbf{e}_j$, that is

$$I = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$$

Matrix Inversion

- A real number a is said to have a <u>multiplicative inverse</u> if there exists a number b such that $ab = 1$.

- Any nonzero number a has a multiplicative inverse $b=1/a$.

Definition

An $n \times n$ matrix A is said to be **nonsingular** or **invertible** if there exists a matrix B such that $AB = BA = I$. The matrix B is said to be a **multiplicative inverse** of A.

- If B and C are both multiplicative inverse of A (i.e., $BA = AB = I$ and $CA = AC = I$),

  then $B = BI = B(AC) = (BA)C = IC = C$

- A matrix can have at most one multiplicative inverse.
- **Notation:** The inverse of A is denoted by A^{-1}.

Example 3

- The matrices $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}$ are inverse of each other, since

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 4

- The 3×3 matrices $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$

  are inverse, since

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 5

- The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has no inverse.

- ***Sol***: Let

$$A^{-1} = B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

then

$$BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} b_{11} & 0 \\ b_{21} & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition

An $n \times n$ matrix is said to be **singular** if it does not have a multiplicative inverse.

Theorem 1.4.2

If A and B are nonsingular n×n matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$

- Proof

$$(B^{-1}A^{-1})\,AB = B^{-1}(A^{-1}A)B = B^{-1}\,I\,B = B^{-1}B = I$$

-

$$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A\,I\,A^{-1} = AA^{-1} = I$$

Note

- If A_1, A_2, ..., A_k are all nonsingular $n \times n$ matrices, then the product $A_1 A_2 \ldots A_k$ is nonsingular and

$$(A_1 A_2 \ldots A_k)^{-1} = A_k^{-1} \ldots A_2^{-1} A_1^{-1}$$

Algebraic Rules for Transposes

1. $(A^T)^T = A$

2. $(\alpha A)^T = \alpha A^T$

3. $(A + B)^T = A^T + B^T$

4. $(AB)^T = B^T A^T$

Example 6

- Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 5 \\ 34 & 23 & 14 \\ 15 & 8 & 9 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 4 \\ 1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 34 & 15 \\ 6 & 23 & 8 \\ 5 & 14 & 9 \end{bmatrix}$$

The first three rules are straightforward. We leave it to the reader to verify that they are valid. To prove the fourth rule, we need only show that the (i, j) entries of $(AB)^T$ and B^TA^T are equal. If A is an $m \times n$ matrix, then, for the multiplications to be possible, B must have n rows. The (i, j) entry of $(AB)^T$ is the (j, i) entry of AB. It is computed by multiplying the jth row vector of A times the ith column vector of B:

$$\vec{\mathbf{a}}_j \mathbf{b}_i = (a_{j1}, a_{j2}, \ldots, a_{jn}) \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jn}b_{ni} \tag{5}$$

The (i, j) entry of B^TA^T is computed by multiplying the ith row of B^T times the jth column of A^T. Since the ith row of B^T is the transpose of the ith column of B and the jth column of A^T is the transpose of the jth row of A, it follows that the (i, j) entry of B^TA^T is given by

$$\mathbf{b}_i^T \vec{\mathbf{a}}_j^T = (b_{1i}, b_{2i}, \ldots, b_{ni}) \begin{bmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jn} \end{bmatrix} = b_{1i}a_{j1} + b_{2i}a_{j2} + \cdots + b_{ni}a_{jn} \tag{6}$$

It follows from (5) and (6) that the (i, j) entries of $(AB)^T$ and B^TA^T are equal.

The next example illustrates the idea behind the last proof.

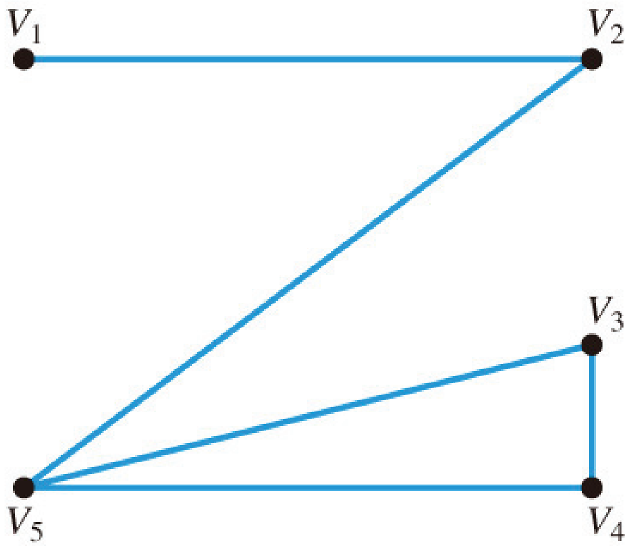
Definition

An $n \times n$ matrix A is said to be **symmetric** if $A^T = A$ (i.e., $a_{ij} = a_{ji}$)

- Example

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix}$$

Figure 1.4.2



Graph G(V, E)

V={V_1, V_2, V_3, V_4, V_5}

adjacency matrix 相鄰矩陣

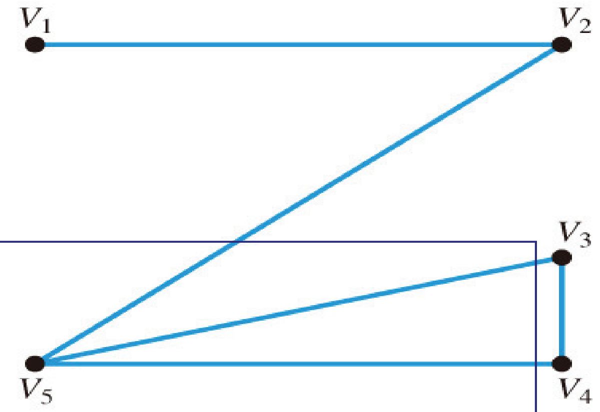
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The line segments joining the vertices correspond to the edges:

E={$\{V_1, V_2\}$, $\{V_2, V_5\}$, $\{V_3, V_4\}$, $\{V_3, V_5\}$, $\{V_4, V_5\}$ }

$$a_{ij} = \begin{cases} 1 & \text{if } \{V_i, V_j\} \text{ is an edge of the graph} \\ 0 & \text{if there is no edge joining } V_i \text{ and } V_j \end{cases}$$

Adjacency Matrix

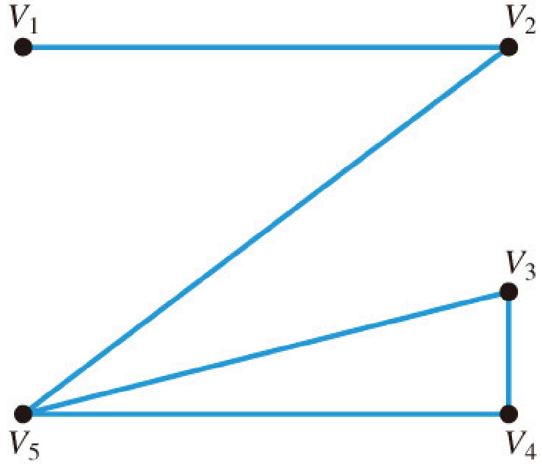


- **Adjacency Matrix is symmetric**.
- ***Walk*** is a sequence of edges linking one vertex to another.
  - $\{V_1, V_2\}, \{V_2, V_5\}$ represents a **walk** from V_1 to V_5.
  - $V_1 \rightarrow V_2 \rightarrow V_5$
  - ***Length of walk*** is the number of edges
  - eg. $V_1 \rightarrow V_2 \rightarrow V_5$ length=2
- It is possible to traverse the same edges **more than once** in a walk.
  - $V_5 \rightarrow V_3 \rightarrow V_5 \rightarrow V_3$ length =3

Theorem 1.4.3

- **If A is an $n \times n$ adjacency matrix of a graph and $a_{ij}^{(k)}$ represents the (i, j) entry of A^k, then $a_{ij}^{(k)}$ is equal to the number of walks of length k from V_i to V_j.**
- **Proof by mathematical induction.**
  - **Basic step. Prove that k=1 is true.**
  - **Inductive step. Assume k=m is true $\Rightarrow$ prove that k=m+1 is true.**

Example 7

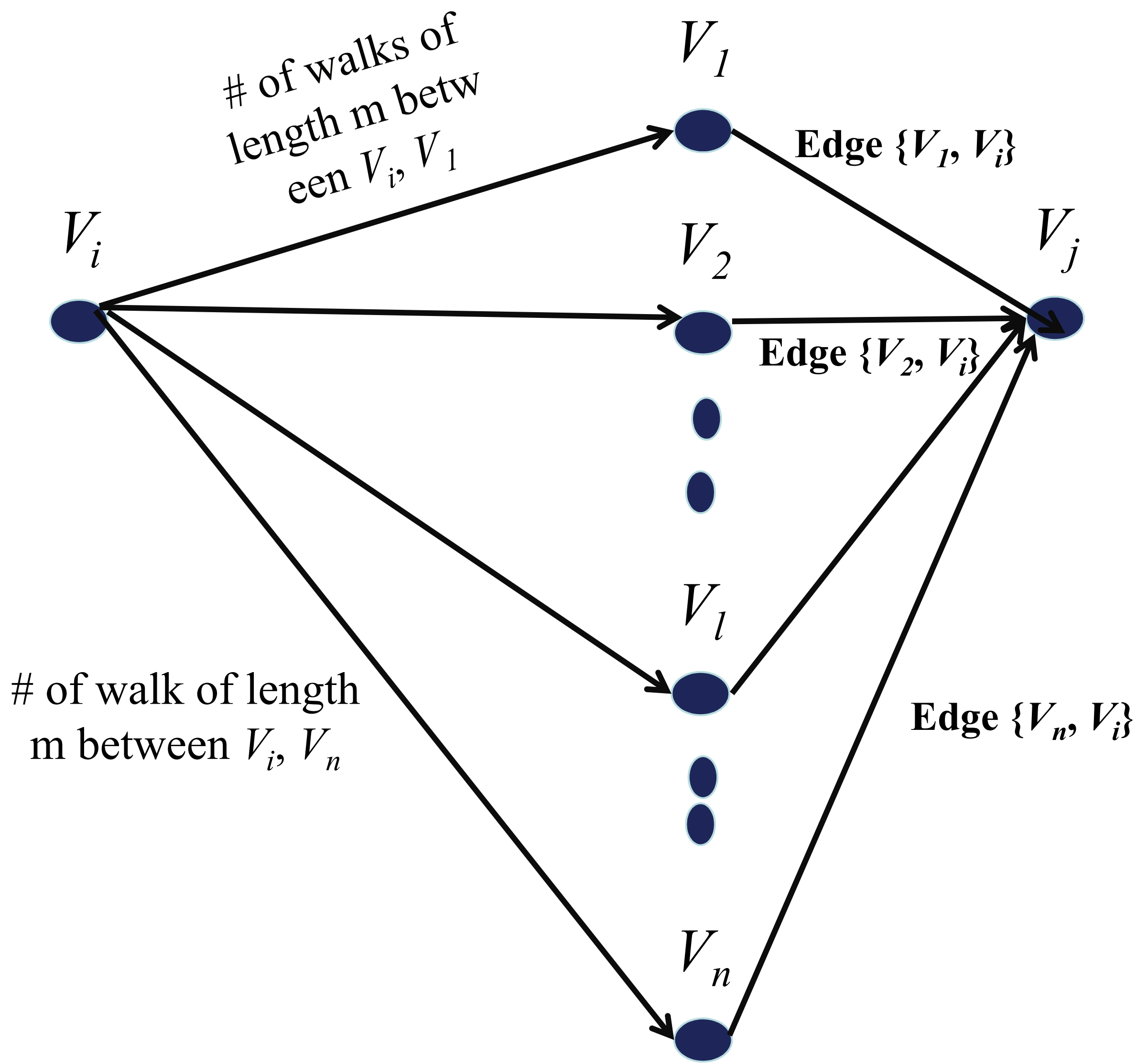


$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

To determine the number of walks of length 3 between any two vertices of the graph in Figure 1.4.2, we need only compute

$$A^2 = \qquad\qquad A^3 = \begin{bmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 & 4 \\ 1 & 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 2 & 4 \\ 0 & 4 & 4 & 4 & 2 \end{bmatrix}$$

Thus, the number of walks of length 3 from V_3 to V_5 is $a_{35}^{(3)}$. Note that the matrix A_3 is symmetric. This reflects the fact that there are the same number of walks of length 3 from V_i to V_j as there are from V_j to V_i.



# of walks of length m betw een V_i, V_1
V_1
V_2
V_i
V_j
V_l
V_n
Edge $\{V_1, V_i\}$
Edge $\{V_2, V_i\}$
Edge $\{V_n, V_i\}$
# of walk of length m between V_i, V_n

Theorem 1.4.3 *proof*

Proof.

- In the case $k = 1$, it follows from the definition of the adjacency matrix that a_{ij} represents the number of walks of length 1 from V_i to V_j.
- Assume for some m that each entry of A^m is equal to the number of walks of length m between the corresponding vertices. Thus, $a_{il}^{(m)}$ is the number of walks of length m from V_i to V_l.
- Now on the one hand, if there is an edge $\{V_l, V_j\}$, then $a_{il}^{(m)} a_{lj} = a_{il}^{(m)}$ is the number of walks of length $m+1$ from V_i to V_j of the form $V_i \rightarrow \cdots \rightarrow V_l \rightarrow V_j$

Theorem 1.4.3 *proof.*

- On the other hand, if $\{V_l, V_j\}$ is not an edge, then there are no walks of length m+1 of this form from V_i to V_j and

$$a_{il}^{(m)} a_{lj} = a_{il}^{(m)} \cdot 0 = 0$$

- It follows that the total number of walks of length m+1 from V_i to V_j is given by

$$a_{i1}^{(m)} a_{1j} + a_{i2}^{(m)} a_{2j} + \cdots + a_{in}^{(m)} a_{nj}$$

- But this is just the (i, j) entry of A^{m+1}.

1-5 Elementary Matrices

- Learning Goals
  - Elementary Matrices
  - Row equivalent relation
  - Equivalent conditions for Nonsingularity
  - Find Inverse
  - Triangular Factorization

1.5 Equivalent Systems

- Given an $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$, if we multiply both sides of the equation by a nonsingular $m \times m$ matrix M:

$$A\mathbf{x} = \mathbf{b}$$

$$MA\mathbf{x} = M\mathbf{b}$$

- If $\hat{\mathbf{x}}$ is a solution to (2), then

$$MA\hat{\mathbf{x}} = M\mathbf{b}$$

$$M^{-1}(MA\hat{\mathbf{x}}) = M^{-1}(M\mathbf{b})$$

$$A\hat{\mathbf{x}} = \mathbf{b}$$

⇨ $\hat{\mathbf{x}}$ is also a solution to (1)

∴ (1) and (2) are equivalent.

- If we apply a sequence of **nonsingular** matrices, E_1, E_2, …, E_k to both sides of the equation $A\mathbf{x} = \mathbf{b}$:

$$A\mathbf{x} = \mathbf{b}$$
$$E_1 A\mathbf{x} = E_1\mathbf{b}$$
$$E_2 E_1 A\mathbf{x} = E_2 E_1\mathbf{b}$$
$$\vdots$$
$$E_k \ldots E_2 E_1 A\mathbf{x} = E_k \ldots E_2 E_1\mathbf{b}$$
$$MA\mathbf{x} = M\mathbf{b}$$
$$U\mathbf{x} = \mathbf{c}$$

- All of these equations are equivalent!
- $\therefore$ All of these equations has the same solution set.

Elementary Matrices

- A matrix obtained by performing exactly one elementary row operation on the identity matrix I is called an **elementary matrix**.

- **Type I**: Interchanging two rows of I
- **Type II**: Multiplying a row of I by a nonzero constant
- **Type III**: Adding a multiple of one row of I to another row

Example 1

<u>Type I</u>: Interchanging two rows of *I*

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \; E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$AE_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}$$

Example 2

<u>Type II</u>: Multiplying a row of I by a nonzero constant

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\rightarrow E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$$

$$AE_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 3a_{13} \\ a_{21} & a_{22} & 3a_{23} \\ a_{31} & a_{32} & 3a_{33} \end{bmatrix}$$

Example 3

<u>Type III</u>: Adding a multiple of one row of I to another row

$$E_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row operation:

$$E_3 A = \begin{bmatrix} a_{11} + 3a_{31} & a_{12} + 3a_{32} & a_{13} + 3a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Column operation:

$$AE_3 = \begin{bmatrix} a_{11} & a_{12} & 3a_{11} + a_{13} \\ a_{21} & a_{22} & 3a_{21} + a_{23} \\ a_{31} & a_{32} & 3a_{31} + a_{33} \end{bmatrix}$$

Note

- Suppose that E is an $n \times n$ elementary matrix. If A is an $n \times r$ matrix, premultiplying A by E has the effect of performing that same row operation on A. If B is an $m \times n$ matrix, postmultiplying B by E is equivalent to performing that same column operation on B.

Theorem 1.5.1

If E is an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type.

- Proof
  - **Type I**: interchange of two rows

$$E_1 E_1 = I \Rightarrow E_1^{-1} = E_1$$

– **<u>Type II</u>**: multiplying the ith row of $\boldsymbol{I}$ by a nonzero scalar α

$$\boldsymbol{E}_2 = \begin{array}{cc} \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & \mathbf{0} & \\ & & 1 & & & & \\ & & & \alpha & & & \\ & & & & 1 & & \\ & \mathbf{0} & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ i\text{th row} \\ \\ \\ \\ \end{array} \end{array} \quad \boldsymbol{E}_2^{-1} = \begin{array}{cc} \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & \mathbf{0} & \\ & & 1 & & & & \\ & & & \dfrac{1}{\alpha} & & & \\ & & & & 1 & & \\ & \mathbf{0} & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ i\text{th row} \\ \\ \\ \\ \end{array} \end{array}$$

– **<u>Type III</u>**: adding m times the ith row to the jth row

$$\boldsymbol{E}_3 = \begin{bmatrix} 1 & & & & & & \\ \vdots & \ddots & & & & \mathbf{0} & \\ 0 & \cdots & 1 & & & & \\ \vdots & & & \ddots & & & \\ 0 & \cdots & m & \cdots & 1 & & \\ \vdots & & & & & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{matrix} \\ \\ i\text{th row} \\ \\ j\text{th row} \\ \\ \\ \end{matrix}$$

$$\boldsymbol{E}_3^{-1} = \begin{bmatrix} 1 & & & & & & \\ \vdots & \ddots & & & & \mathbf{0} & \\ 0 & \cdots & 1 & & & & \\ \vdots & & & \ddots & & & \\ 0 & \cdots & -m & \cdots & 1 & & \\ \vdots & & & & & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{matrix} \\ \\ i\text{th row} \\ \\ j\text{th row} \\ \\ \\ \end{matrix}$$

Definition

A matrix B is row equivalent to A if there exists a finite sequence $E_1, E_2, \ldots, E_k$ of elementary matrices such that

$$B = E_k E_{k-1} \ldots E_1 A$$

Note

- B is row equivalent to A if B can be obtained from A by a finite number of row operations.
- Two augmented matrices $(A \mid \mathbf{b})$ and $(B \mid \mathbf{c})$ are row equivalent ***iff*** $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{c}$ are equivalent systems.
  - A is row equivalent to A *(reflesive)反身性*
  - If A is row equivalent to B, B is row equivalent to A. *(symmetric)對稱性*
  - If A is row equivalent to B, and B is row equivalent to C, then A is row equivalent to C. *(transitive) 遞移性*

Theorem 1.5.2
Equivalent conditions for Nonsingularity

Let A be an $n\times n$ matrix. The following are equivalent:

(a) A is nonsingular.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$.

(c) A is row equivalent to I.

- ***Pf***: (a) $\Rightarrow$ (b)

If A is nonsingular (i.e., A^{-1} exists), then for $A\mathbf{x} = \mathbf{0}$

$$\hat{\mathbf{x}} = I\hat{\mathbf{x}} = (A^{-1}A)\hat{\mathbf{x}} = A^{-1}(A\hat{\mathbf{x}}) = A^{-1}\mathbf{0} = \mathbf{0}$$

(b) $\Rightarrow$ (c)

If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$

$A\mathbf{x} = \mathbf{0} \Rightarrow U\mathbf{x} = \mathbf{0}$ (using elementary row operation)

where U is in row echelon form and $U = E_k \ldots E_1 \boldsymbol{A}$

From Theorem 1.2.1.

In U, the number of nonzero rows must be the same as unknowns ($m = n$)

Thus,

U must be a strictly triangular matrix with diagonal elements all equal to 1.

$\therefore$ I will be the reduced row echelon form of A

$\therefore$ A is row equivalent to I

(c) $\Rightarrow$ (a)

If A is row equivalent to I, there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$A = E_k E_{k-1} \ldots E_1 I = E_k E_{k-1} \ldots E_1$$

$\because$ $E_1, E_2, \ldots, E_k$ are all invertible

$\therefore$ $A^{-1} = (E_k E_{k-1} \ldots E_1)^{-1} = E_1^{-1} E_2^{-1} \ldots E_{k-1}^{-1} E_k^{-1}$

$\therefore$ A is nonsingular (invertible)

Corollary 1.5.3

The system of n linear equations in n unknown $A\mathbf{x} = \mathbf{b}$ has a unique solution ***iff*** A is nonsingular.

- ***Pf***: ($\Leftarrow$) If A is nonsingular, and $\hat{\mathbf{x}}$ is any solution of $A\mathbf{x} = \mathbf{b}$, then

$$A\hat{\mathbf{x}} = \mathbf{b}$$

$$A^{-1}(A\hat{\mathbf{x}}) = A^{-1}\mathbf{b}$$

$\therefore \quad \hat{\mathbf{x}} = A^{-1}\mathbf{b}$ is the unique solution.

$(\Rightarrow)$ Suppose that $A\mathbf{x} = \mathbf{b}$ has a unique solution $\hat{\mathbf{x}}$ $(A\hat{\mathbf{x}} = \mathbf{b})$, and ***A* is singular**

$\therefore$ $A\mathbf{x} = \mathbf{0}$ has a solution $\mathbf{z} \neq 0$ (i.e., $A\mathbf{z} = \mathbf{0}$) **why?**

$\therefore$ Let $\mathbf{y} = \hat{\mathbf{x}} + \mathbf{z}$, clear $\mathbf{y} \neq \hat{\mathbf{x}}$

$$A\mathbf{y} = A(\hat{\mathbf{x}} + \mathbf{z}) = A\hat{\mathbf{x}} + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

$\therefore$ $\mathbf{y}$ is also the solution to $A\mathbf{x} = \mathbf{b}$

$\therefore$ contradiction

$\therefore$ A must be nonsingular

- If A is nonsingular, A is row equivalent to I, so there exist elementary matrices E_1, E_2, …, E_k such that

$$E_k\, E_{k\text{-}1} \ldots E_1\, A = I$$

$$(E_k\, E_{k\text{-}1} \ldots E_1\, A)A^{-1} = (I)A^{-1}$$

$$E_k\, E_{k\text{-}1} \ldots E_1\, I = A^{-1}$$

Finding Inverse of *A*

- The same series of elementary row operations that **transform a nonsingular matrix** A **into** I **will transform** I **into** A^{-1}.
- That is, the reduced row echelon form of the augmented matrix $(A \mid I)$ will be $(I \mid A^{-1})$.

Example 4

- Compute A^{-1} if

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$

- ***Sol***:

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right]$$

Example 5

- Solve the system:

$$\begin{aligned} x_1 &+ 4x_2 + 3x_3 = 12 \\ -x_1 &- 2x_2 \qquad\quad = -12 \\ 2x_1 &+ 2x_2 + 3x_3 = 8 \end{aligned}$$

- ***Sol***: $A\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b}$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 12 \\ -12 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -\frac{8}{3} \end{bmatrix}$$

Diagonal and Triangular Matrices

- An $n \times n$ matrix A is said to be **upper triangular** if $a_{ij} = 0$ for $i > j$, eg.,

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

- An $n \times n$ matrix A is said to be **lower triangular** if $a_{ij} = 0$ for $i < j$. e.g.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 6 & 0 & 0 \\ 1 & 4 & 3 \end{bmatrix}$$

- A matrix is said to be **triangular** if it is either upper triangular or lower triangular
- An $n \times n$ matrix $\boldsymbol{A}$ is said to be **diagonal** if $a_{ij} = 0$ whenever $i \neq j$. *e.g.*,

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- **Note**: A diagonal matrix is both upper triangular and lower triangular.

Triangular Factorization

- If an $n \times n$ matrix A can be reduced to strict upper triangular form using **only row operation III**, then it is possible to represent the reduction process in terms of a **matrix factorization**.
- ***LU factorization***.

Example 6

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow[l_{31}=2]{l_{21}=1/2} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} \xrightarrow{l_{32}=-3} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} = A$$

Note

- L is unit lower triangular.

- The factorization of the matrix A into a product of a unit lower triangular matrix L times a strictly upper triangular matrix U is often referred to as an *LU factorization.*

- Why LU factorization works?
- Since $E_3 E_2 E_1 A = U$, where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \boldsymbol{E}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

$$E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = L$$

1.6 Partitioned Matrices

- A matrix can be partitioned into small submatrices (called **blocks**), e.g.,

$$C=\begin{bmatrix}1 & -2 & 4 & 1 & 3\\ 2 & 1 & 1 & 1 & 1\\ 3 & 3 & 2 & -1 & 2\\ 4 & 6 & 2 & 2 & 4\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}=\left[\begin{array}{ccc|cc}1 & -2 & 4 & 1 & 3\\ 2 & 1 & 1 & 1 & 1\\ \hline 3 & 3 & 2 & -1 & 2\\ 4 & 6 & 2 & 2 & 4\end{array}\right]$$

$$B=\begin{bmatrix}-1 & 2 & 1\\ 2 & 3 & 1\\ 1 & 4 & 1\end{bmatrix}=[\mathbf{b}_1, \quad \mathbf{b}_2, \quad \mathbf{b}_3]=\left[\begin{array}{c|c|c}-1 & 2 & 1\\ 2 & 3 & 1\\ 1 & 4 & 1\end{array}\right]$$

- If A is an $m \times n$ matrix and B is $n \times r$ which has been partitioned into columns $[\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r]$, then the block multiplication of A times B is given by

$$AB = A\,[\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r] = [A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_r]$$

- For example

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$\Rightarrow A\mathbf{b}_1 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, A\mathbf{b}_2 = \begin{bmatrix} 15 \\ -1 \end{bmatrix}, A\mathbf{b}_3 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\text{hence } A(b_1, b_2, b_3) = \left[\begin{array}{c|c|c} 6 & 15 & 5 \\ -2 & -1 & 1 \end{array}\right]$$

- In particular,

$$[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] = \boldsymbol{A} = \boldsymbol{AI} = [\boldsymbol{A}\mathbf{e}_1, \boldsymbol{A}\mathbf{e}_2, \ldots, \boldsymbol{A}\mathbf{e}_n]$$

- For example,

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 4 \\ 1 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & -3 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} & \vec{\mathbf{a}}_1 B = [1 \quad 9 \quad -1] \\ \Rightarrow\ & \vec{\mathbf{a}}_2 B = [5 \quad 10 \quad -5] \\ & \vec{\mathbf{a}}_3 B = [-4 \quad 9 \quad 4] \end{aligned}$$

$$\Rightarrow AB = \begin{bmatrix} \hat{\mathbf{a}}_1 B \\ \hat{\mathbf{a}}_2 B \\ \hat{\mathbf{a}}_3 B \end{bmatrix} = \left[\begin{array}{ccc} 1 & 9 & -1 \\ \hline 5 & 10 & -5 \\ \hline -4 & 9 & 4 \end{array}\right]$$

Block Multiplication CASE 1

- Let A be an $m\times n$ matrix and B an $n\times r$ matrix, consider the following 4 cases
- **CASE 1**
- $B = [B_1, B_2]$, where B_1 is an $n\times t$ matrix and B_2 is an $n\times(r\text{-}t)$ matrix, then

$$\begin{aligned} AB &= A[\mathbf{b}_1, \ldots, \mathbf{b}_t, \mathbf{b}_{t+1}, \ldots, \mathbf{b}_r] \\ &= [A\mathbf{b}_1, \ldots, A\mathbf{b}_t, A\mathbf{b}_{t+1}, \ldots, A\mathbf{b}_r] \\ &= [A[\mathbf{b}_1, \ldots, \mathbf{b}_t], A[\mathbf{b}_{t+1}, \ldots, \mathbf{b}_r]] = [AB_1, AB_2] \end{aligned}$$

Thus, $A[B_1, B_2] = [AB_1, AB_2]$

CASE 2

- $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \end{bmatrix}$, where A_1 is a $k \times n$ matrix and A_2 is a $(m\text{-}k) \times n$ matrix,
- Thus

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} B = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vdots \\ \vec{\mathbf{a}}_k \\ \hline \vec{\mathbf{a}}_{k+1} \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix} B = \begin{bmatrix} \vec{\mathbf{a}}_1 B \\ \vdots \\ \vec{\mathbf{a}}_k B \\ \hline \vec{\mathbf{a}}_{k+1} B \\ \vdots \\ \vec{\mathbf{a}}_m B \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vdots \\ \vec{\mathbf{a}}_k \end{bmatrix} B \\ \begin{bmatrix} \vec{\mathbf{a}}_{k+1} \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix} B \end{bmatrix} = \begin{bmatrix} A_1 B \\ A_2 B \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} B = \begin{bmatrix} A_1 B \\ A_2 B \end{bmatrix}$$

CASE 3

- $A = [A_1, A_2]$ and $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, where A_1 is an $m \times s$ matrix, A_2 is an $m \times (n\text{-}s)$ matrix, B_1 is an $s \times r$ matrix, and B_2 is an $(n\text{-}s) \times r$ matrix, if $C = AB$, then

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj} = \sum_{l=1}^{s} a_{il} b_{lj} + \sum_{l=s+1}^{n} a_{il} b_{lj}$$

- Thus c_{ij} is the sum of the (i, j) entry of A_1B_1 and the (i, j) entry of A_2B_2.
- Therefore

$$C = AB = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = A_1B_1 + A_2B_2$$

CASE 4

- Let A and B be partitioned as follows:

$$A = \begin{array}{cc} \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] & \begin{array}{c} k \\ m-k \end{array} \\ \begin{array}{cc} s & n-s \end{array} & \end{array}, \quad B = \begin{array}{cc} \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] & \begin{array}{c} s \\ n-s \end{array} \\ \begin{array}{cc} t & r-t \end{array} & \end{array}$$

- Let

$$A_1 = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}, \quad A_2 = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}, B_1 = \begin{bmatrix} B_{11} & B_{12} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{21} & B_{22} \end{bmatrix}$$

then

$$AB = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = A_1B_1 + A_2B_2$$

- And

$$A_1B_1 = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} B_1 = \begin{bmatrix} A_{11}B_1 \\ A_{21}B_1 \end{bmatrix} = \begin{bmatrix} A_{11}\begin{bmatrix} B_{11} & B_{12} \end{bmatrix} \\ A_{21}\begin{bmatrix} B_{11} & B_{12} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} \end{bmatrix}$$

$$A_2B_2 = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} B_2 = \begin{bmatrix} A_{12}B_2 \\ A_{22}B_2 \end{bmatrix} = \begin{bmatrix} A_{12}\begin{bmatrix} B_{21} & B_{22} \end{bmatrix} \\ A_{22}\begin{bmatrix} B_{21} & B_{22} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A_{12}B_{21} & A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix}$$

- Therefore

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

- In summary, if the blocks have the proper dimensions, the block multiplication can be carried out in the same manner as ordinary matrix multiplication

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1t} \\ \vdots & & \\ A_{s1} & \cdots & A_{st} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{11} & \cdots & B_{1r} \\ \vdots & & \\ B_{t1} & \cdots & B_{tr} \end{bmatrix},$$

$$\text{then } C = AB = \begin{bmatrix} C_{11} & \cdots & C_{1r} \\ \vdots & & \\ C_{s1} & \cdots & C_{sr} \end{bmatrix}, \text{ where } C_{ij} = \sum_{k=1}^{t} A_{ik} B_{kj}$$

Example 1

- Let

$$A=\begin{bmatrix}1&1&1&1\\2&2&1&1\\3&3&2&2\end{bmatrix}\text{ and } B=\begin{bmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\end{bmatrix}=\left[\begin{array}{cc|cc}1&1&1&1\\1&2&1&1\\\hline 3&1&1&1\\3&2&1&2\end{array}\right]$$

- ***Sol***: (1) If

$$A=\begin{bmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{bmatrix}=\left[\begin{array}{cc|cc}1&1&1&1\\\hline 2&2&1&1\\3&3&2&2\end{array}\right]$$

then

$$AB=\left[\begin{array}{cc|cc}1&1&1&1\\\hline 2&2&1&1\\3&3&2&2\end{array}\right]\left[\begin{array}{cc|cc}1&1&1&1\\1&2&1&1\\\hline 3&1&1&1\\3&2&1&2\end{array}\right]=\left[\begin{array}{cc|cc}8&6&4&5\\\hline 10&9&6&7\\18&15&10&12\end{array}\right]$$

(2) If

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 3 & 2 & 2 \end{array}\right]$$

then

$$AB = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ \hline 3 & 3 & 2 & 2 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right] = \left[\begin{array}{rr|rr} 8 & 6 & 4 & 5 \\ 10 & 9 & 6 & 7 \\ \hline 18 & 15 & 10 & 12 \end{array}\right]$$

Example 2

- Let A be an $n \times n$ matrix of the form $A = \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}$ where A_{11} is a $k \times k$ matrix ($k < n$). Then A is nonsingular iff A_{11} and A_{22} are nonsingular.
- ***Sol***: ($\Leftarrow$) If A_{11} and A_{22} are nonsingular, then

$$\begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix} = \begin{bmatrix} I_k & O \\ O & I_{n-k} \end{bmatrix} = I \text{ and}$$

$$\begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} I_k & O \\ O & I_{n-k} \end{bmatrix} = I$$

Thus A is nonsingular.

- ($\Rightarrow$) If A is nonsingular, let $B = A^{-1}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$
  Since $BA = I = AB$, i.e.,

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix} = \begin{bmatrix} I_k & O \\ O & I_{n\text{-}k} \end{bmatrix} = \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{bmatrix} B_{11}A_{11} & B_{12}A_{22} \\ B_{21}A_{11} & B_{22}A_{22} \end{bmatrix} = \begin{bmatrix} I_k & O \\ O & I_{n\text{-}k} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix}$$

- Thus, $$B_{11}A_{11} = I_k = A_{11}B_{11}$$ $$B_{22}A_{22} = I_{n\text{-}k} = A_{22}B_{22}$$
  And hence A_{11} and A_{22} are nonsingular, with inverse B_{11} and B_{22}.

Outer Product Expansions

- Given two vectors **x** and **y** in R^n, the scalar product or the inner product is defined as the matrix product $\mathbf{x}^T\mathbf{y}$, which is the product of a row vector (a $1\times n$ matrix) times a column vector (an $n\times 1$ matrix) and results in a 1×1 matrix or simply a scalar

$$\mathbf{x}^T\mathbf{y} = [x_1, x_2, \cdots, x_n]\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

- The **outer product** is defined as the matrix product $\mathbf{xy}^T$, which is the product of an $n\times1$ matrix times an $1\times n$ matrix and results in an $n\times n$ matrix:

$$\mathbf{xy}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} [y_1, y_2, \cdots, y_n] = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & & & \\ x_ny_1 & x_ny_2 & \cdots & x_ny_n \end{bmatrix}$$

  – Each of the rows is a multiple of $\mathbf{y}^T$
  – Each of the column vectors is a multiple of $\mathbf{x}$

- For example

$$\mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

$$\Rightarrow \mathbf{x}\mathbf{y}^T = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 20 & 8 \\ 3 & 5 & 2 \\ 9 & 15 & 6 \end{bmatrix}$$

- Let X be an $m \times n$ matrix and Y a $k \times n$ matrix, the outer product expansion of X and Y is a matrix product XY^T. XY^T can be calculated by partitioning X into columns and Y^T into rows and then perform the block multiplication:

$$XY^T = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n] \begin{bmatrix} \mathbf{y}_1^T \\ \mathbf{y}_2^T \\ \vdots \\ \mathbf{y}_n^T \end{bmatrix} = \mathbf{x}_1 \mathbf{y}_1^T + \mathbf{x}_2 \mathbf{y}_2^T + \cdots + \mathbf{x}_n \mathbf{y}_n^T$$

Example 3

- Given

$$\boldsymbol{X} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \text{ and } \boldsymbol{Y} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{bmatrix}$$

$$\Rightarrow XY^T = \left[\begin{array}{c:c} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{array}\right] \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hdashline 2 & 4 & 1 \end{array}\right] = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 1 \\ 8 & 16 & 4 \\ 4 & 8 & 2 \end{bmatrix}$$