Dynamic Programming

Outline

- Introduction
- The resource allocation problem
- The traveling salesperson (TSP) problem
- Longest common subsequence problem
- 0/1 knapsack problem
- The optimal binary tree problem
- Matrix Chain-Products

學習目標

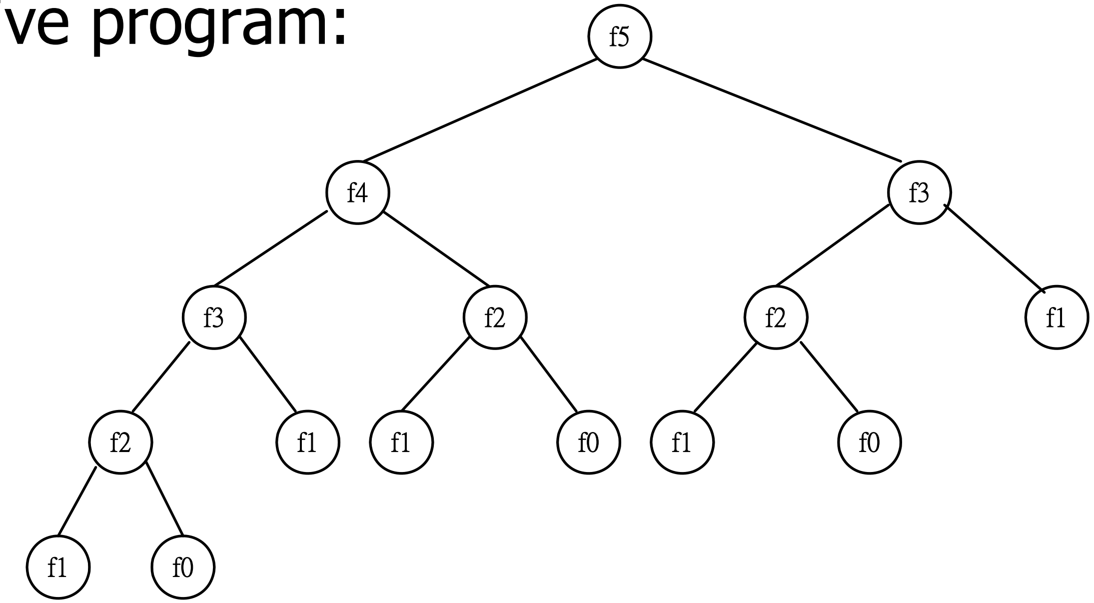
- Dynamic Programming策略設計的概念
- Dynamic Programming策略設計限制

Fibonacci sequence

- **Fibonacci sequence**: 0 , 1 , 1 , 2 , 3 , 5 , 8 , 13 , 21 , …

  $F_i = i$ if $i \leq 1$

  $F_i = F_{i-1} + F_{i-2}$ if $i \geq 2$
- Solved by a recursive program:



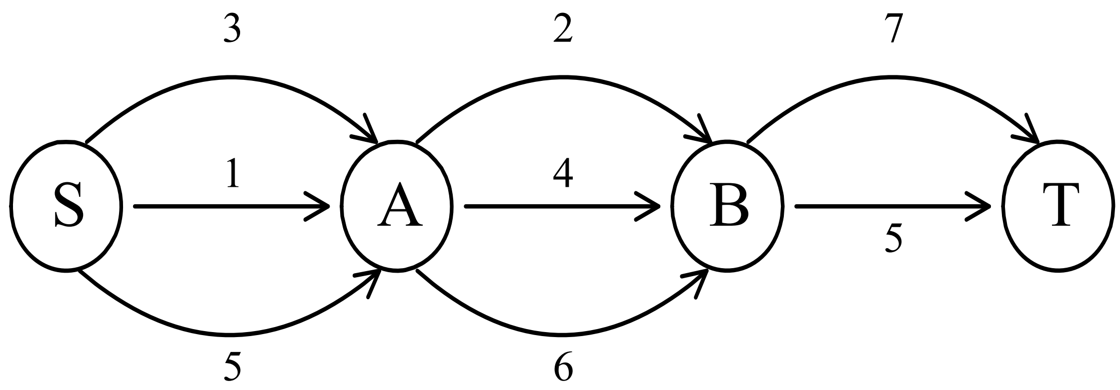
- Much replicated computation is done.
- It should be solved by a simple loop.

Dynamic Programming

- Dynamic Programming is an algorithm design method that can be used when the solution to a problem may be viewed as the result of a sequence of decisions

The shortest path

- To find a shortest path in a multi-stage graph

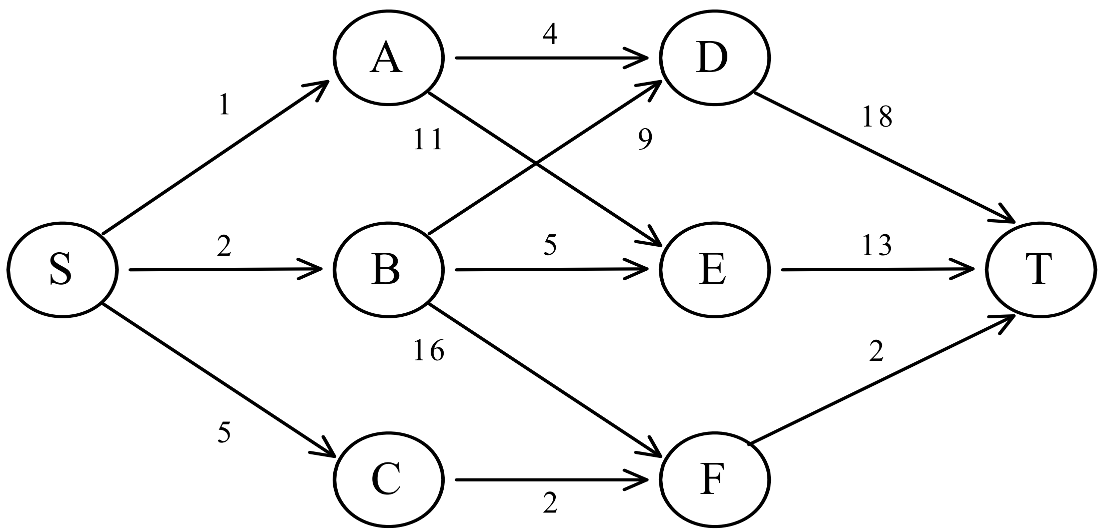


- Apply the greedy method :
  the shortest path from S to T :

  $1 + 2 + 5 = 8$

The shortest path in multistage graphs

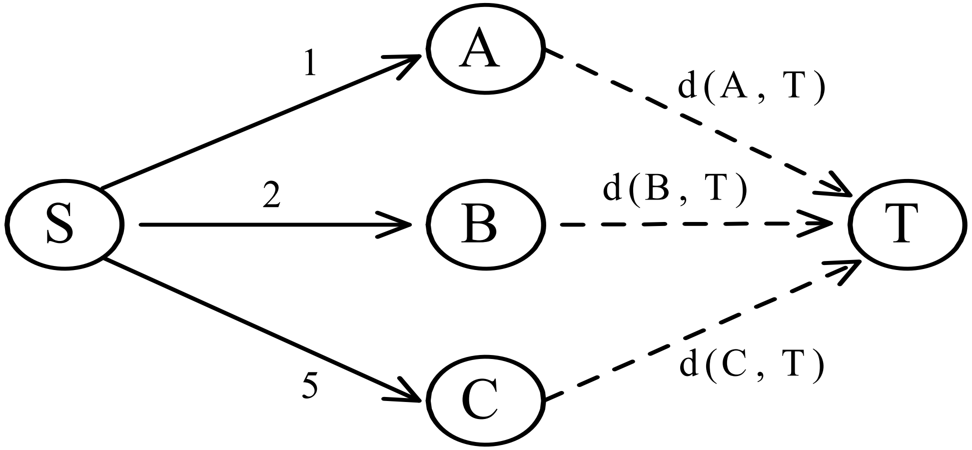
- e.g.

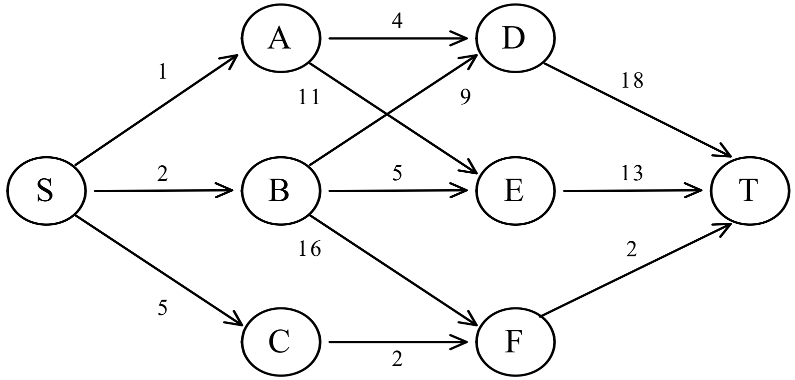


- The greedy method can not be applied to this case: (S, A, D, T) 1+4+18 = 23.
- The real shortest path is:

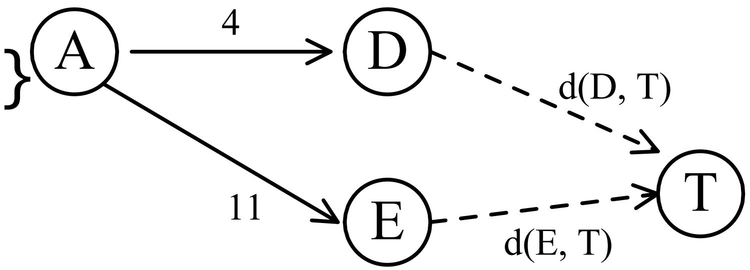
  (S, C, F, T) 5+2+2 = 9.

Dynamic programming approach



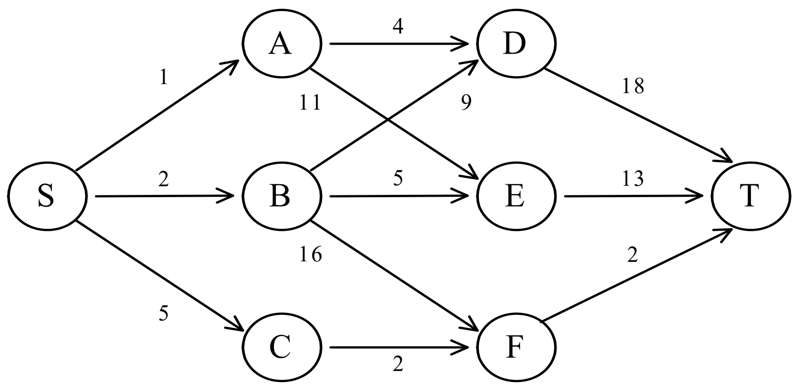


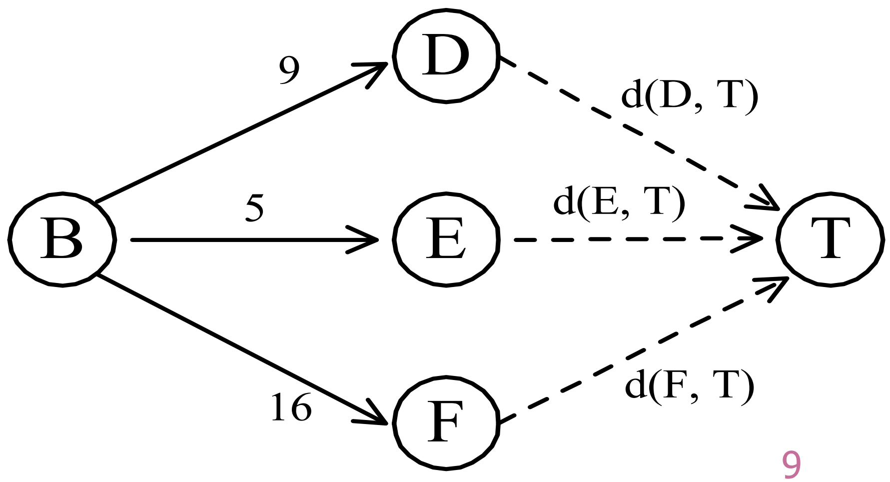
- d(S, T) = min{1+d(A, T), 2+d(B, T), 5+d(C, T)}
- d(A,T) = min{4+d(D,T), 11+d(E,T)}
  = min{4+18, 11+13} = 22.



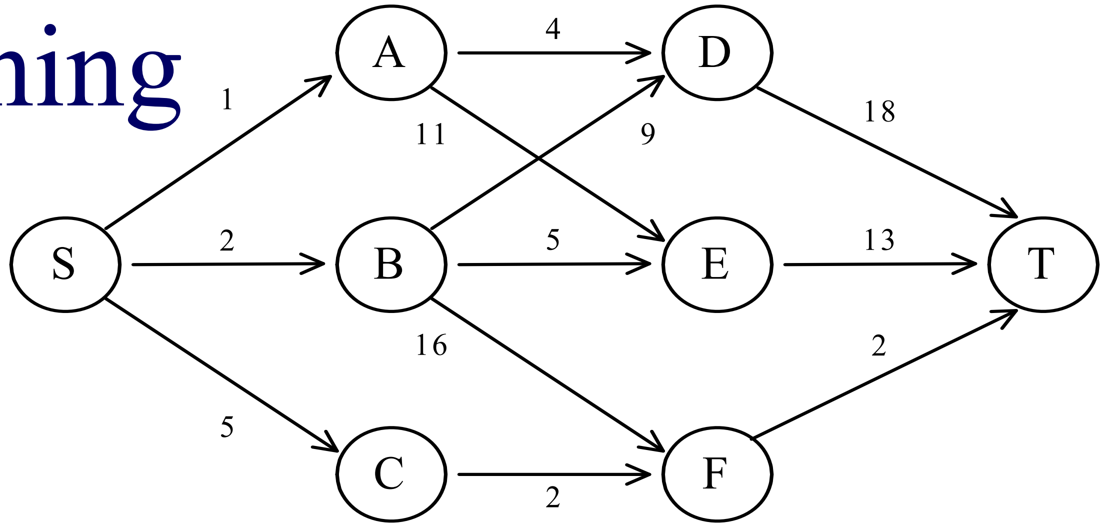
Dynamic programming

- d(B, T) = min{9+d(D, T), 5+d(E, T), 16+d(F, T)}
  = min{9+18, 5+13, 16+2} = 18.
- d(C, T) = min{ 2+d(F, T) } = 2+2 = 4
- d(S, T) = min{1+d(A, T), 2+d(B, T), 5+d(C, T)}
  = min{1+22, 2+18, 5+4} = 9.
- The above way of reasoning is called backward reasoning.





Forward reasoning



- d(S, A) = 1
  d(S, B) = 2
  d(S, C) = 5
- d(S,D)=min{d(S, A)+d(A, D),d(S, B)+d(B, D)}
  = min{ 1+4, 2+9 } = 5
  d(S,E)=min{d(S, A)+d(A, E),d(S, B)+d(B, E)}
  = min{ 1+11, 2+5 } = 7
  d(S,F)=min{d(S, A)+d(A, F),d(S, B)+d(B, F)}
  = min{ 2+16, 5+2 } = 7

- $d(S,T) = \min\{d(S, D)+d(D, T), d(S,E)+ d(E,T), d(S, F)+d(F, T)\}$
  $= \min\{ 5+18, 7+13, 7+2 \}$
  $= 9$

Principle of optimality

- Principle of optimality: Suppose that in solving a problem, we have to make a sequence of decisions D_1, D_2, …, D_n. **If this sequence is optimal, then the last k decisions, $1 < k < n$ must be optimal.**
- e.g. the shortest path problem

  If i, i_1, i_2, …, j is a shortest path from i to j, then i_1, i_2, …, j must be a shortest path from i_1 to j
- In summary, if a problem can be described by a **multistage graph**, then it can be solved by dynamic programming.

Dynamic programming

- Forward approach and backward approach:
  - Note that if the recurrence relations are formulated using the forward approach then the relations are solved backwards . i.e., beginning with the last decision
  - On the other hand if the relations are formulated using the backward approach, they are solved forwards.
- To solve a problem by using dynamic programming:
  - Prove the optimality
  - Find out the recurrence relations.
  - Represent the problem by a multistage graph.

The resource allocation problem

學習目標

- Resource allocation problem 問題定義
- 設計Dynamic Programming演算法解決 Resource allocation problem
- 演算法時間複雜度分析

The resource allocation problem

- m resources, n projects

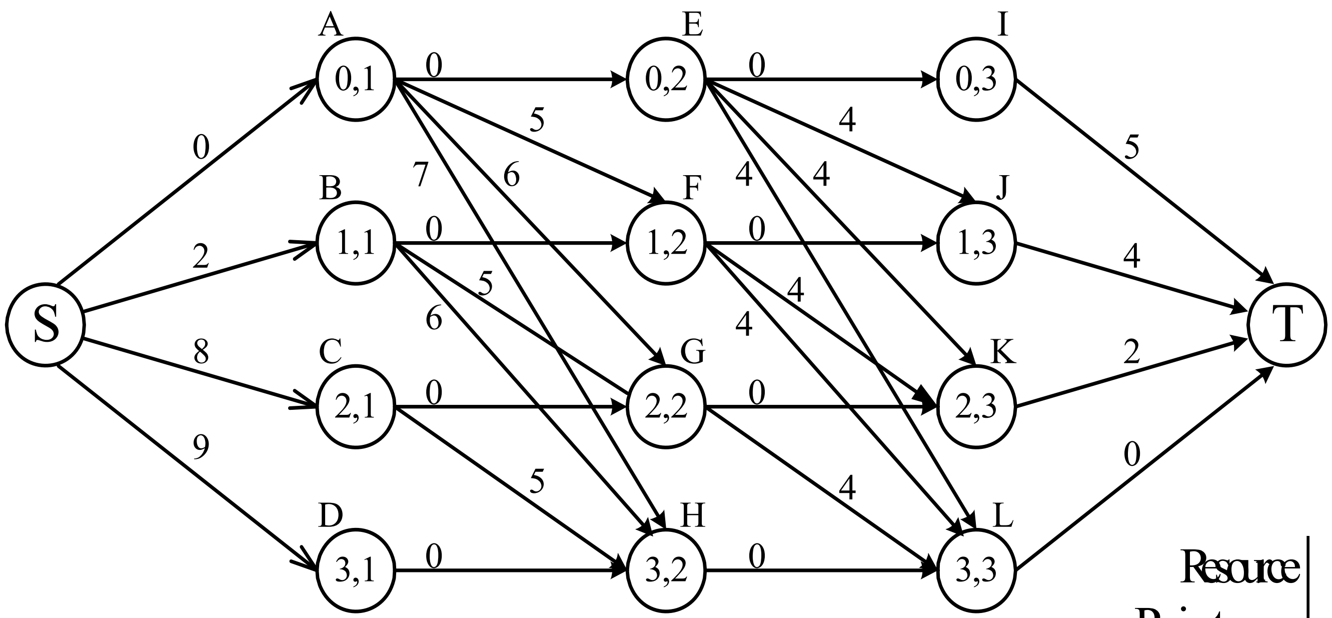
profit p(i, j) : j resources are allocated to project i. P(*i*, 0)=0 for each *i*

maximize the total profit.

Project \ Resource	1	2	3
1	2	8	9
2	5	6	7
3	4	4	4
4	2	4	5

To make a sequence of decision to determine the number Resources to be allocated to project *i*.

The multistage graph solution

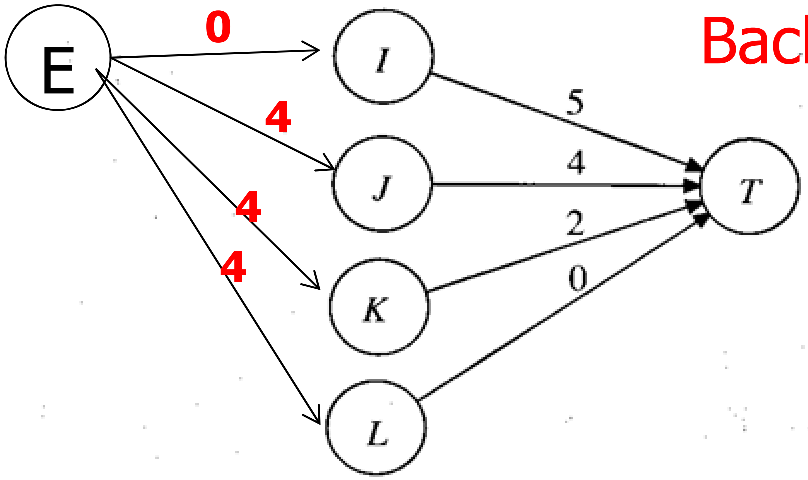


Project \ Resource	1	2	3
1	2	8	9
2	5	6	7
3	4	4	4
4	2	4	5

- The resource allocation problem can be described as a multistage graph.
- **(i, j) : i resources allocated to projects 1, 2, …, j**

e.g. node H=(3, 2) : 3 resources allocated to projects 1, 2.

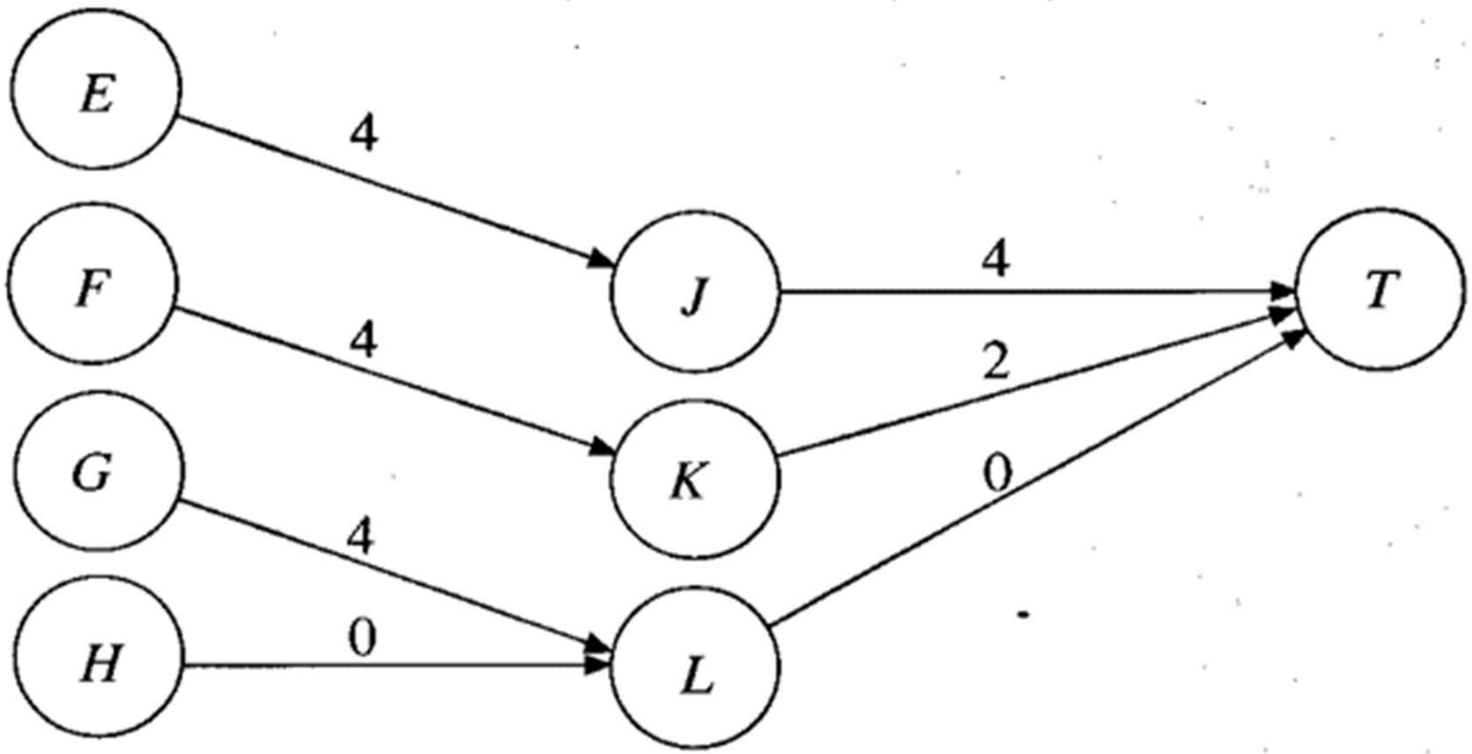
- To get the maximum profit = find the longest path from S to T .



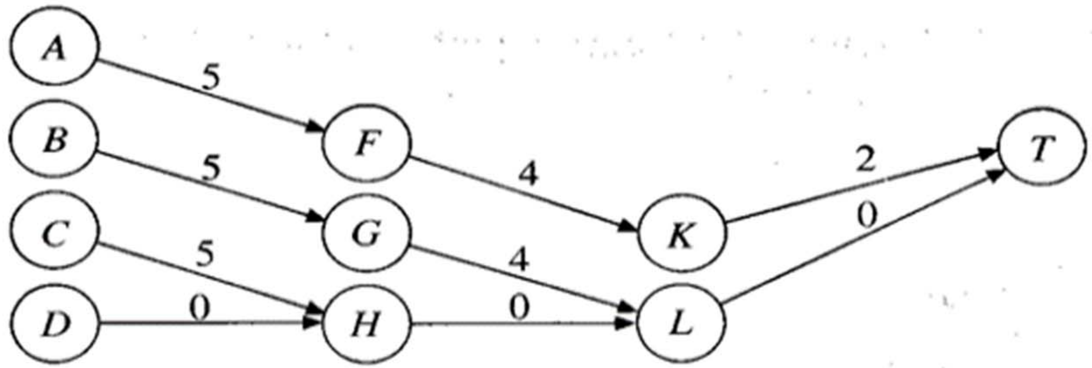
Backward reasoning approach

2) Having obtained the longest paths from I, J, K and L to T, we can obtain the longest paths from E, F, G and H to T easily. For instance, the longest path from E to T is determined as follows:

$$\begin{aligned} d(E, T) &= \max\{d(E, I) + d(I, T), d(E, J) + d(J, T), \\ &\qquad d(E, K) + d(K, T), d(E, L) + d(L, T)\} \\ &= \max\{0 + 5, 4 + 4, 4 + 2, 4 + 0\} \\ &= \max\{5, 8, 6, 4\} \\ &= 8. \end{aligned}$$


E
F
G
H
J
K
L
T
4
4
4
0
4
2
0

(3) The longest paths from A, B, C and D to T respectively are found by the same method and shown in Figure :.



(4) Finally, the longest path from S to T is obtained as follows:

$$\begin{aligned} d(S, T) &= \max\{d(S, A) + d(A, T), d(S, B) + d(B, T), \\ &\qquad\quad d(S, C) + d(C, T), d(S, D) + d(D, T)\} \\ &= \max\{0 + 11, 2 + 9, 8 + 5, 9 + 0\} \\ &= \max\{11, 11, 13, 9\} \\ &= 13. \end{aligned}$$

The longest path is

$$S \to C \to H \to L \to T.$$

- Find the longest path from S to T :

  (S, C, H, L, T), 8+5+0+0=13

  2 resources allocated to project 1.

  1 resource allocated to project 2.

  0 resource allocated to projects 3, 4.

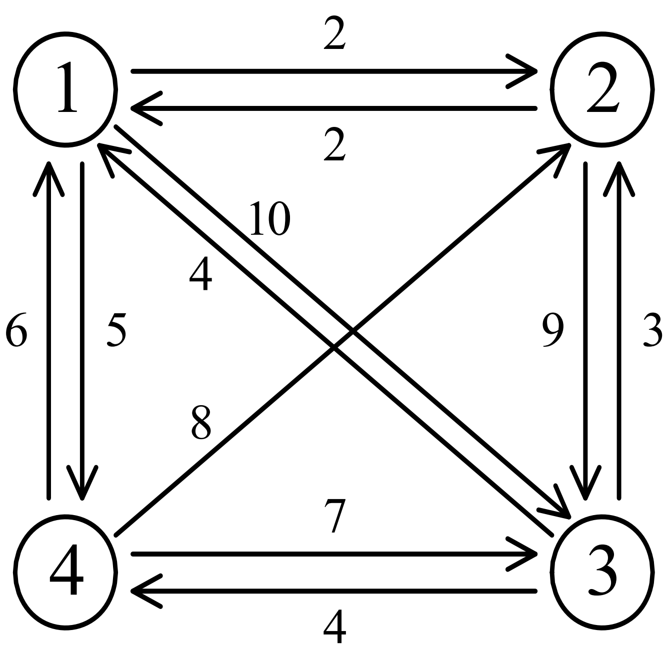
The traveling salesperson problem (TSP)

學習目標

- TSP 問題定義
- 設計Dynamic Programming演算法解決TSP
- 演算法時間複雜度分析

The traveling salesperson problem (TSP)

- e.g. a directed graph :

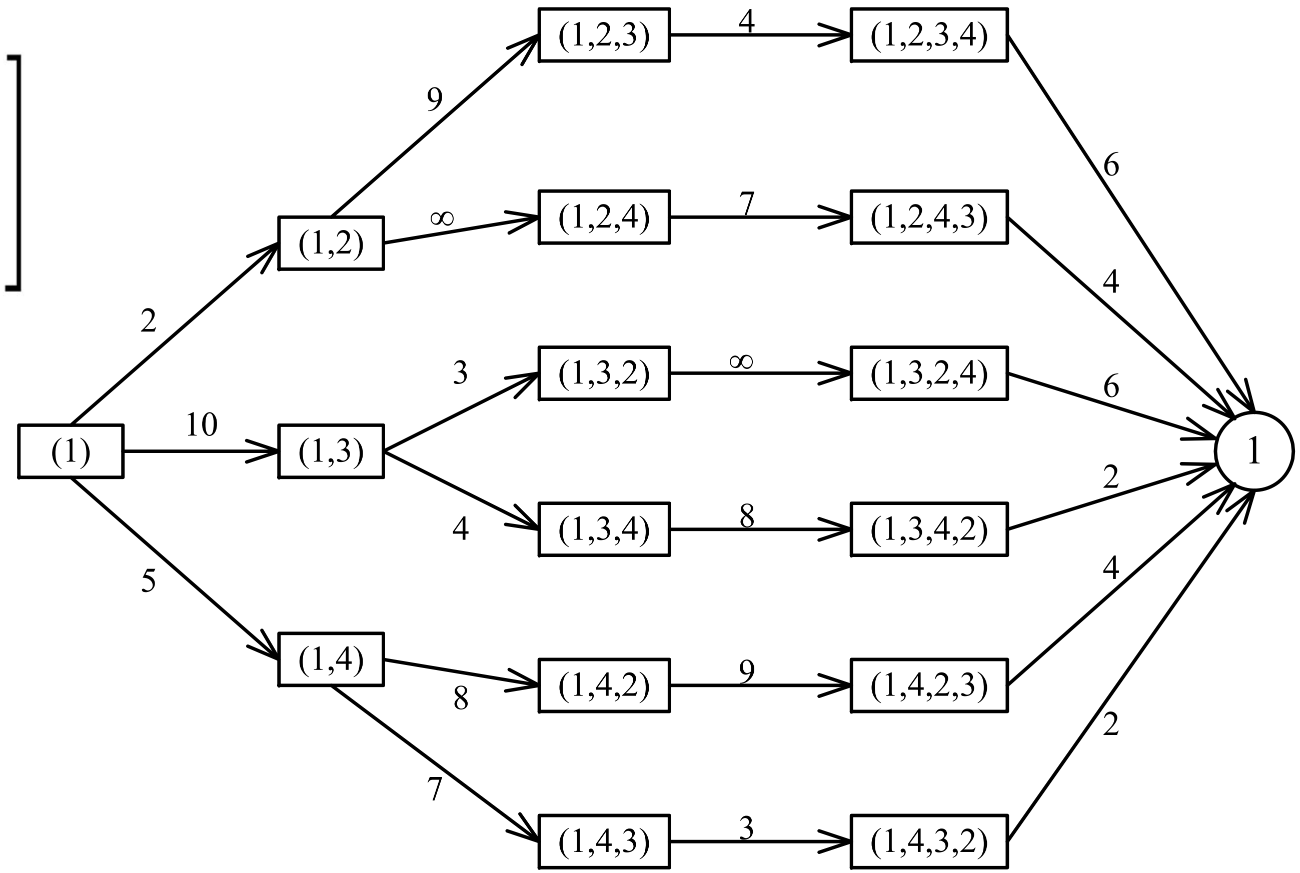


Cost matrix:

$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} &
\begin{bmatrix}
\infty & 2 & 10 & 5 \\
2 & \infty & 9 & \infty \\
4 & 3 & \infty & 4 \\
6 & 8 & 7 & \infty
\end{bmatrix}
\end{array}
$$

The multistage graph solution

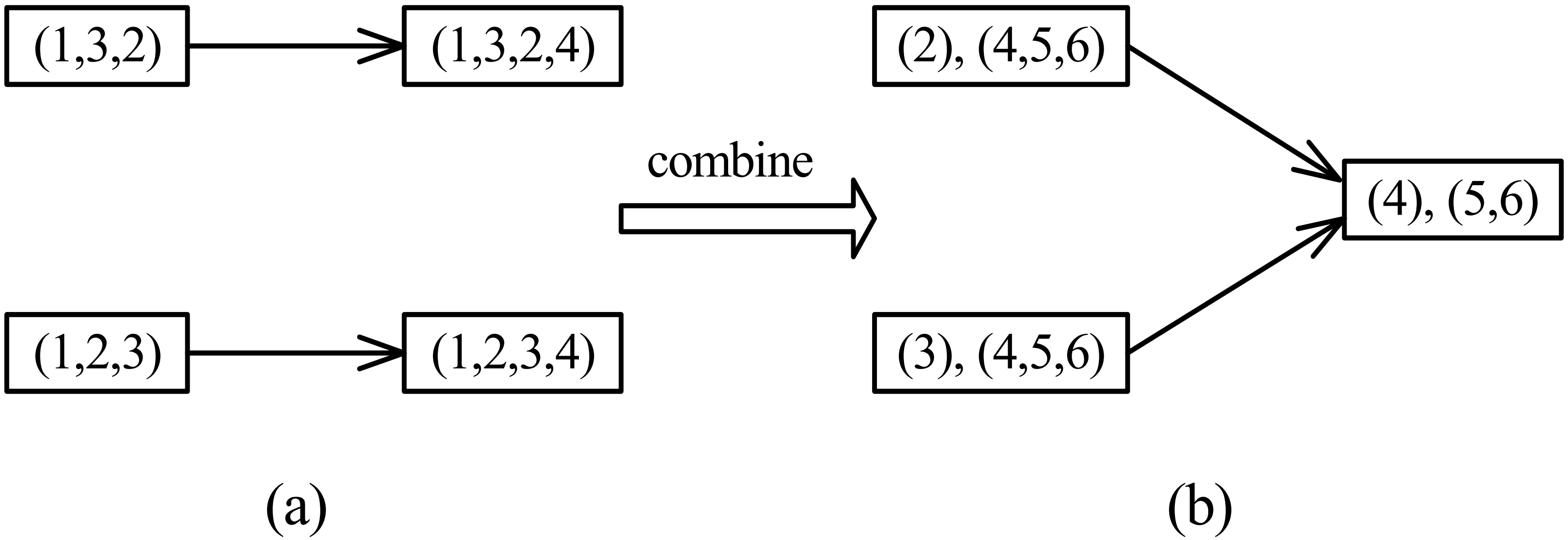
$$
\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1 & \infty & 2 & 10 & 5 \\
2 & 2 & \infty & 9 & \infty \\
3 & 4 & 3 & \infty & 4 \\
4 & 6 & 8 & 7 & \infty
\end{array}
$$



- A multistage graph can describe all possible tours of a directed graph.
- Find the shortest path:
  (1, 4, 3, 2, 1)   5+7+3+2=17

The representation of a node

- Suppose that we have 6 vertices in the graph.
- We can combine {1, 2, 3, 4} and {1, 3, 2, 4} into one node.

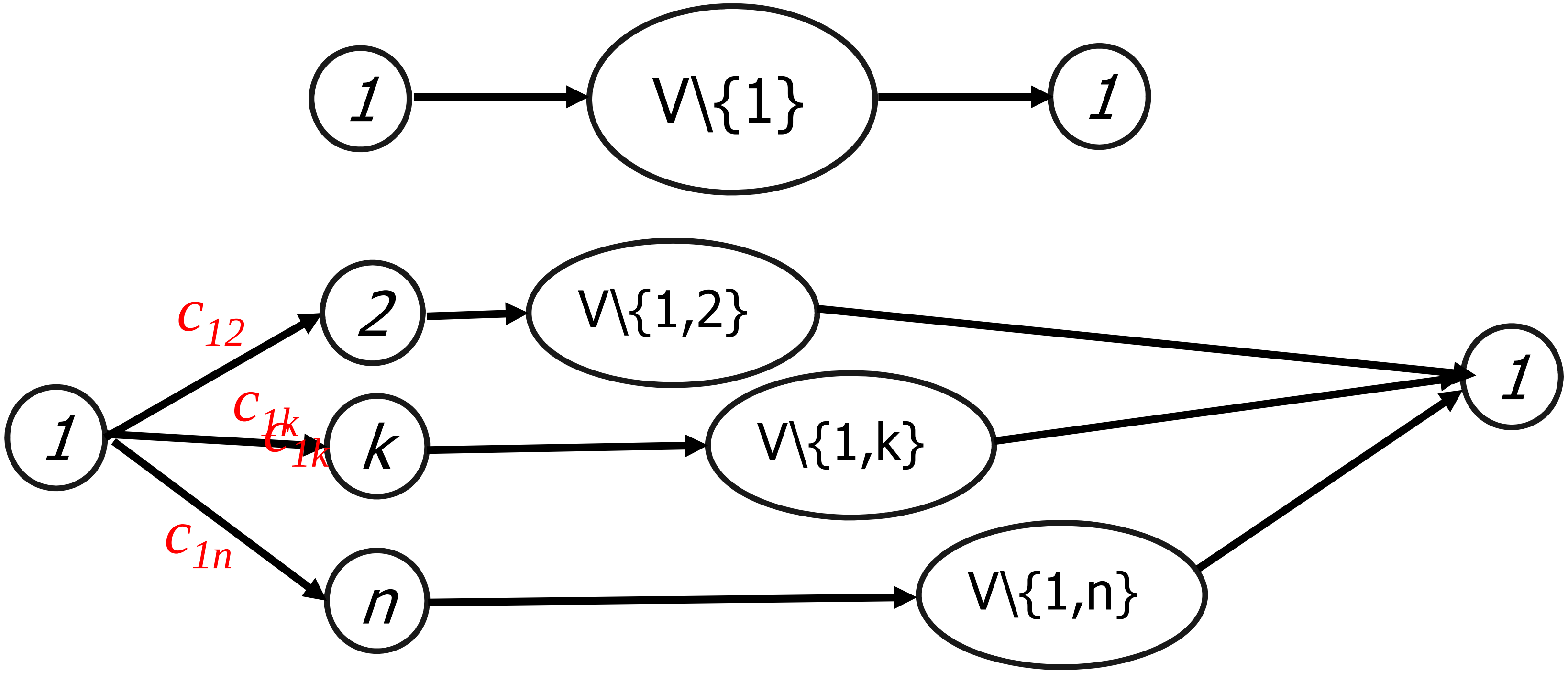


- (3),(4,5,6) means that the last vertex visited is 3 and the remaining vertices **to be visited are (4, 5, 6).**

The dynamic programming approach

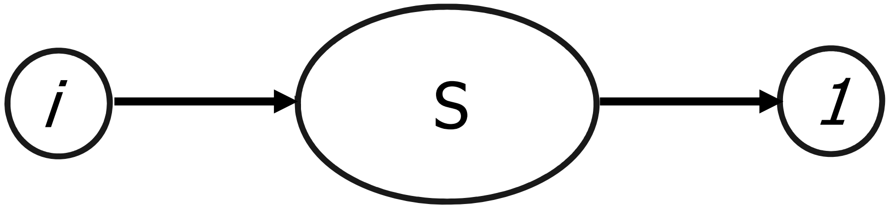
- Let **g(i, S)** be the length of a shortest path starting at vertex i, going through all vertices in S and terminating at vertex 1.
- The length of an optimal tour :

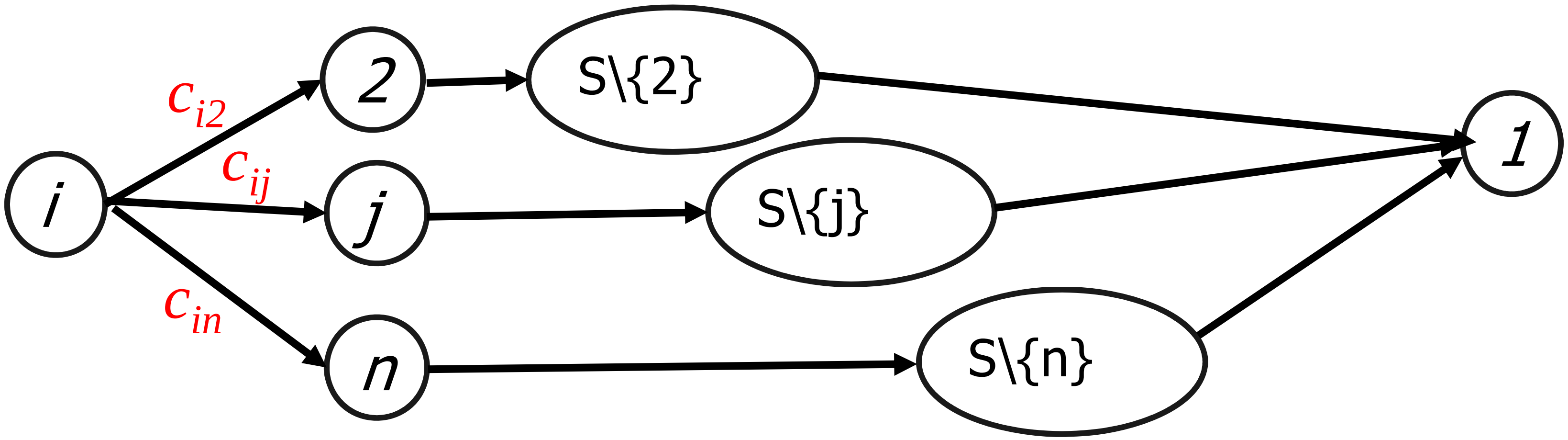
$$g(1, V\backslash\{1\}) = \min_{2 \le k \le n} \{ c_{1k} + g(k, V\backslash\{1, k\}) \}$$



- The general form:

$$g(i,S) = \min_{j\in S}\{c_{ij} + g(j, S\backslash\{j\})\}$$





- Time complexity:

$$n + \sum_{k=2}^{n} (n-1)\binom{n-2}{n-k}(n-k)$$

$$= O(n^2 2^n)$$

( ), ( )
↑ ↑
(n-1) (n-k)

The longest common subsequence (LCS) problem

學習目標

- Longest common subsequence (LCS)問題定義
- 設計Dynamic Programming演算法解決LCS問題
- 演算法時間複雜度分析

The longest common subsequence (LCS) problem

- A string : A = b a c a d
- A subsequence of A: deleting 0 or more symbols from A (not necessarily consecutive).

  e.g. ad, ac, bac, acad, bacad, bcd.
- Common subsequences of A = b a c a d and

  B = a c c b a d c b : ad, ac, bac, acad.
- The longest common subsequence (LCS) of A and B: a c a d.

Determine the length of the LCS

- Instead of finding the longest common subsequence, let us try to determine the length of the LCS.
- Then tracking back to find the LCS.
- Consider $a_1a_2\ldots a_m$ and $b_1b_2\ldots b_n$.
- **Case 1: $a_m = b_n$.** The LCS must contain a_m, we have to find the LCS of $a_1a_2\ldots a_{m-1}$ and $b_1b_2\ldots b_{n-1}$.
- **Case 2: $a_m \neq b_n$.** We have to find the LCS of $a_1a_2\ldots a_{m-1}$ and $b_1b_2\ldots b_n$, and $a_1a_2\ldots a_m$ and $b_1b_2\ldots b_{n-1}$

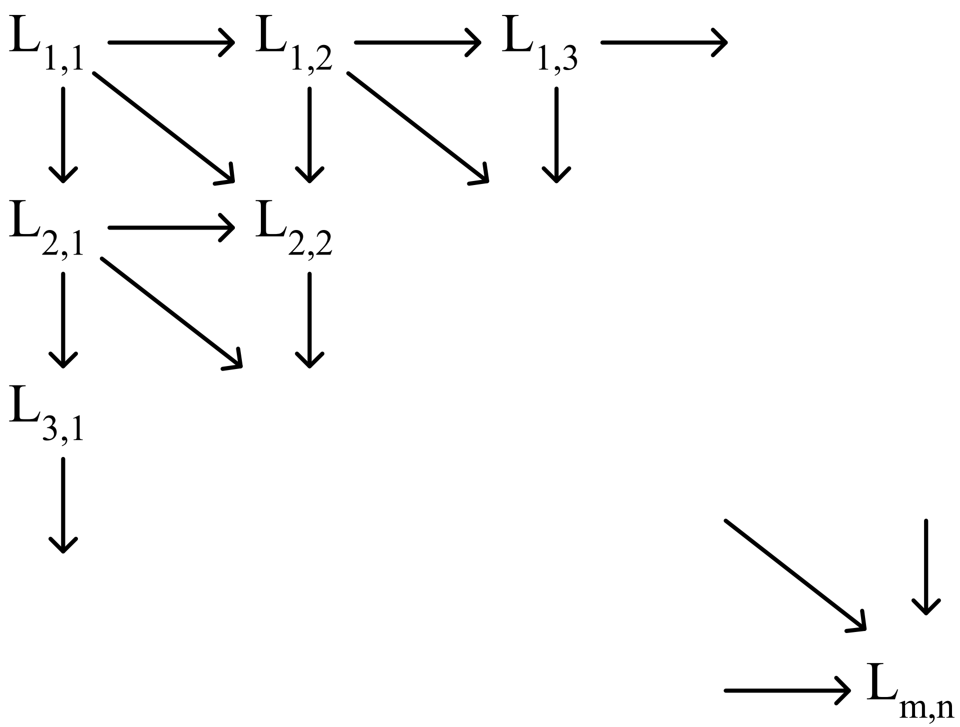
The LCS algorithm

- Let $A = a_1\ a_2 \ldots a_m$ and $B = b_1\ b_2 \ldots b_n$
- Let $L_{i,j}$ denote the length of the longest common subsequence of $a_1\ a_2 \ldots a_i$ and $b_1\ b_2 \ldots b_j$.
- $$L_{i,j} = \begin{cases} L_{i-1,j-1} + 1 & \text{if } a_i = b_j \\ \max\{ L_{i-1,j}, L_{i,j-1} \} & \text{if } a_i \neq b_j \end{cases}$$

  $L_{0,0} = L_{0,j} = L_{i,0} = 0$ for $1 \leq i \leq m$, $1 \leq j \leq n$.

Solving approach: Find $L_{1,1}$

- The dynamic programming approach for solving the LCS problem:



- Time complexity: O(mn)

Tracing back in the LCS algorithm

- e.g. A = b a c a d, B = a c c b a d c b

			B							
			a	c	c	b	a	d	c	b
		0	0	0	0	0	0	0	0	0
	b	0	0	0	0	1	1	1	1	1
	a	0	1	1	1	1	2	2	2	2
A	c	0	1	2	2	2	2	2	3	3
	a	0	1	2	2	2	3	3	3	3
	d	0	1	2	2	2	3	4	4	4

- After all $L_{i,j}$'s have been found, we can trace back to find the longest common subsequence of A and B.

0/1 knapsack problem

學習目標

- 0/1 knapsack problem 問題定義
- 設計Dynamic Programming演算法解決0/1 knapsack problem
- 演算法時間複雜度分析

0/1 knapsack problem

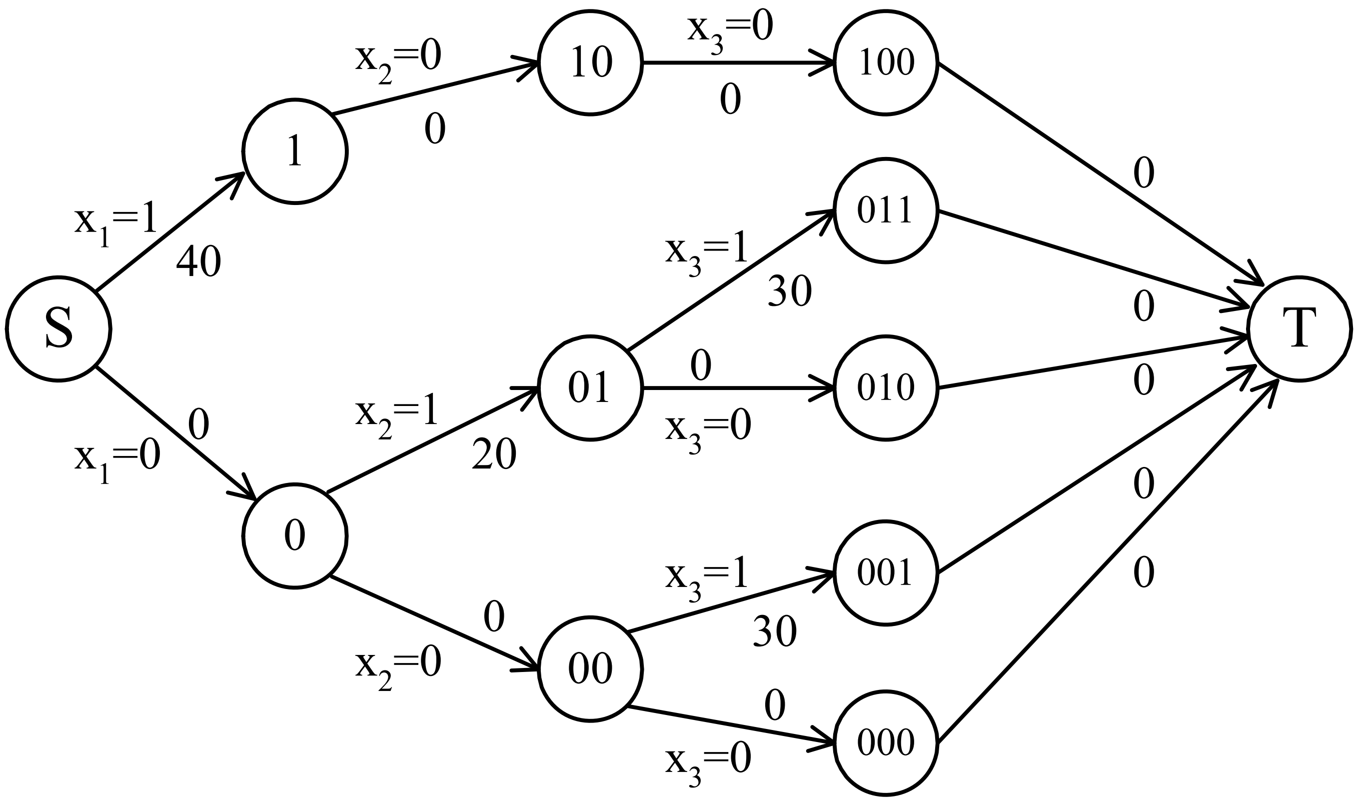
- n objects, weight $W_1, W_2, \ldots, W_n$
  profit $P_1, P_2, \ldots, P_n$
  capacity M
  maximize $\sum_{1 \le i \le n} P_i x_i$
  subject to $\sum_{1 \le i \le n} W_i x_i \le M$
  $x_i = 0$ or 1, $1 \le i \le n$
- e. g.

i	W_i	P_i
1	10	40
2	3	20
3	5	30

M=10

The multistage graph solution

- The 0/1 knapsack problem can be described by a multistage graph.



The dynamic programming approach

- The longest path represents the optimal solution:

  $x_1=0,\ x_2=1,\ x_3=1$

$\sum P_i x_i = 20+30 = 50$

- Let **$f_i(Q)$** be the value of an optimal solution to objects 1, 2, 3,…, i with capacity Q.
- $f_i(Q) = \max\{ f_{i-1}(Q), f_{i-1}(Q-W_i)+P_i \}$

  $= \max\{$ 第i個不選獲利，第i個必選獲利$\}$
- The optimal solution is $f_n(M)$.

  $f_n(M) = \max\{f_{i-1}(M), f_{i-1}(M-W_i) + P_i \}$

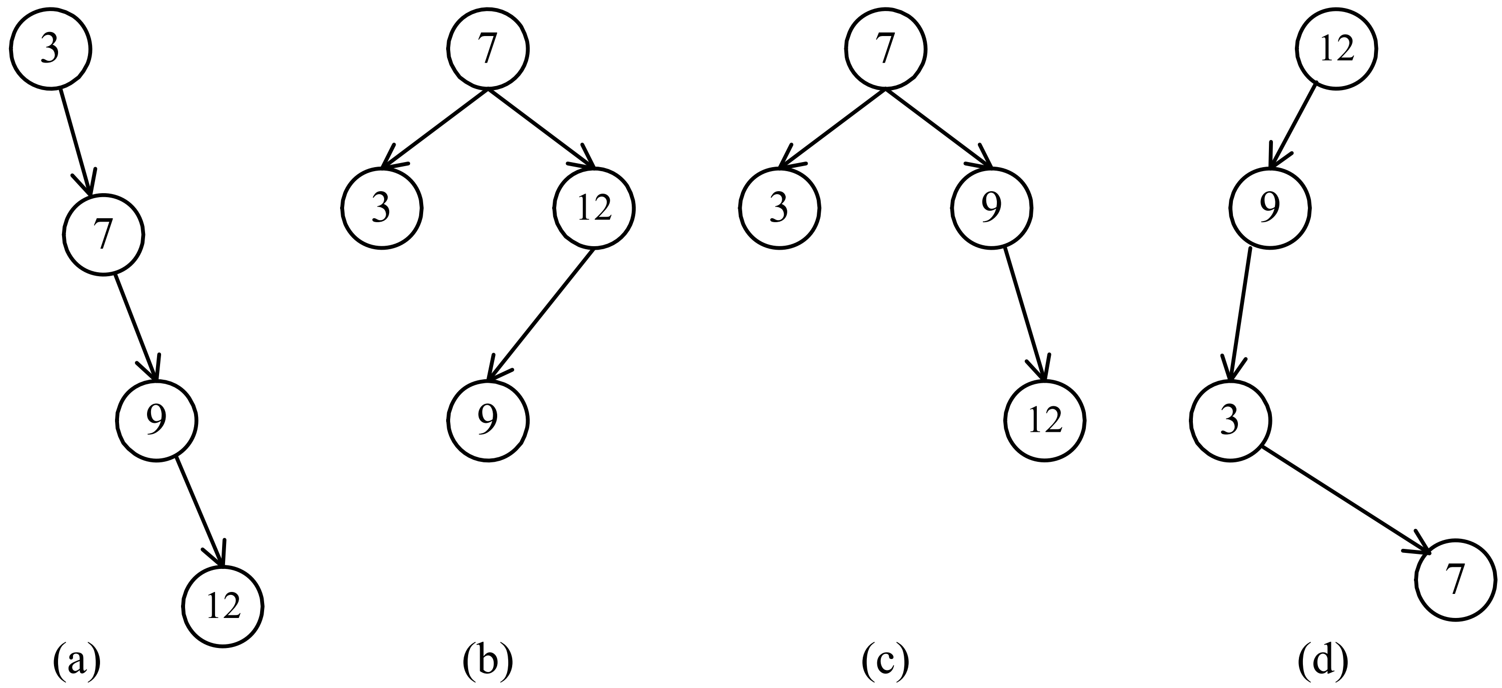
Optimal binary search trees

學習目標

- Optimal binary search Trees 問題定義
- 設計Dynamic Programming演算法解決 Optimal binary search Trees
- 演算法時間複雜度分析

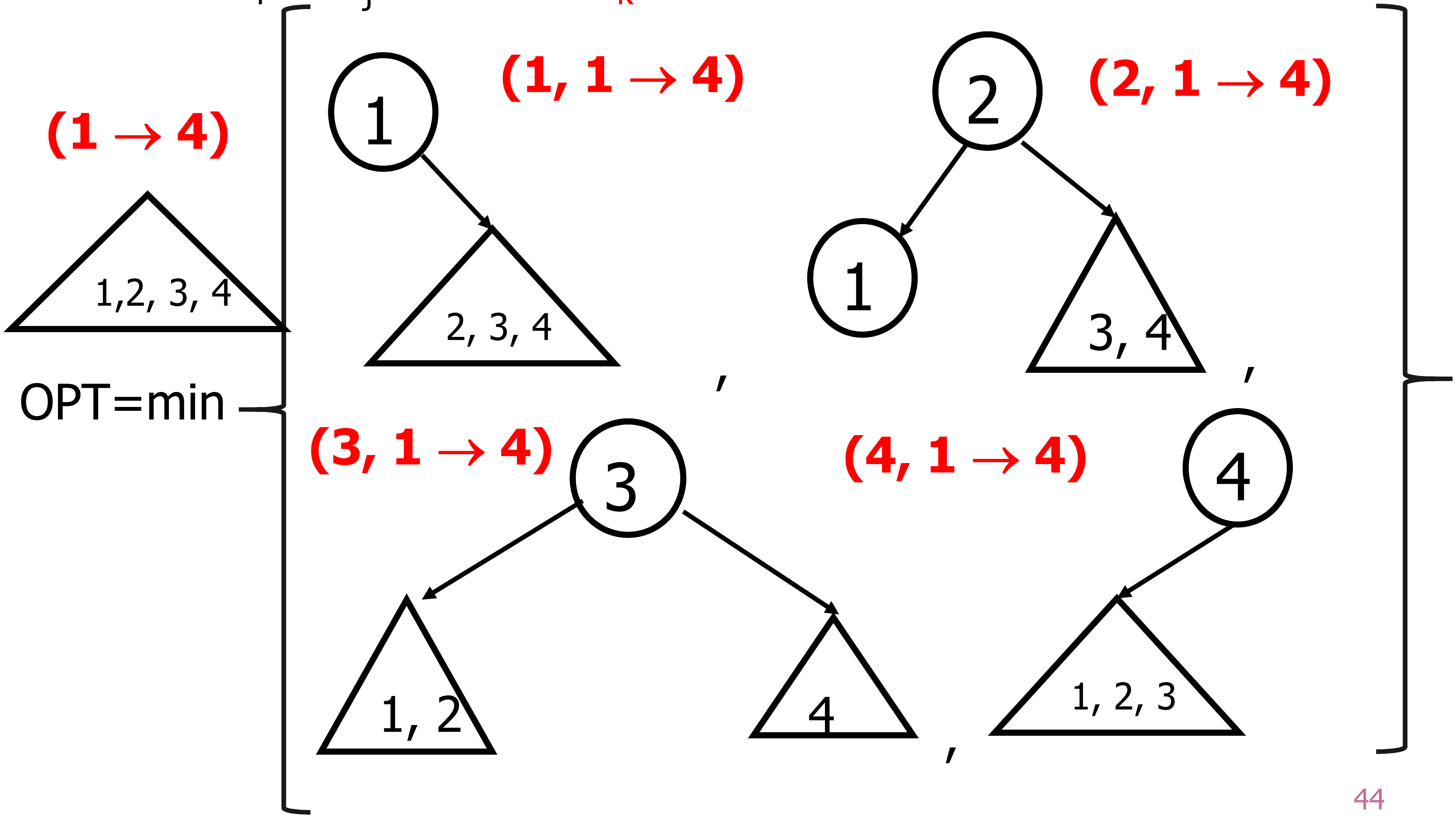
Optimal binary search trees

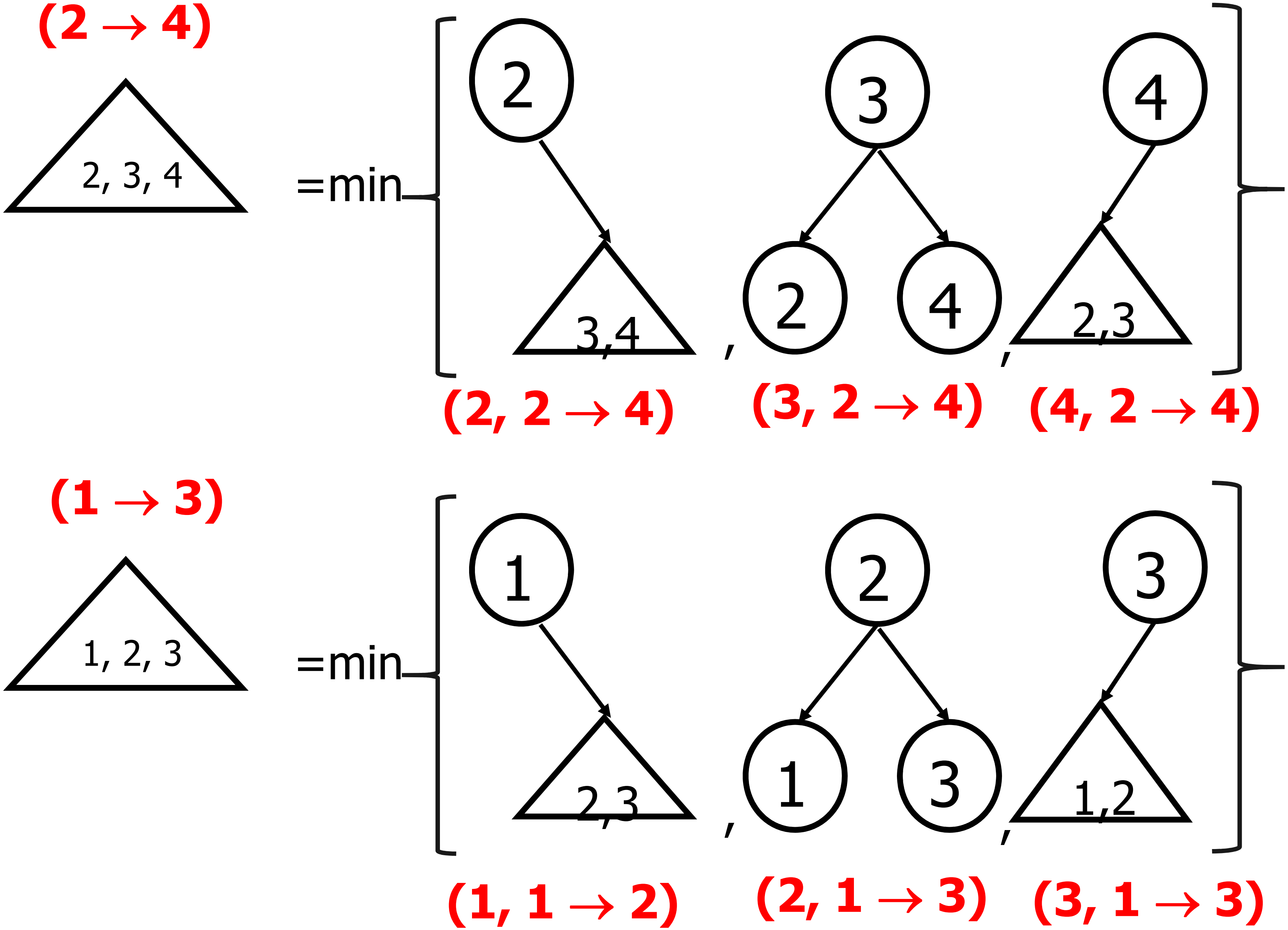
- e.g. binary search trees for 3, 7, 9, 12;

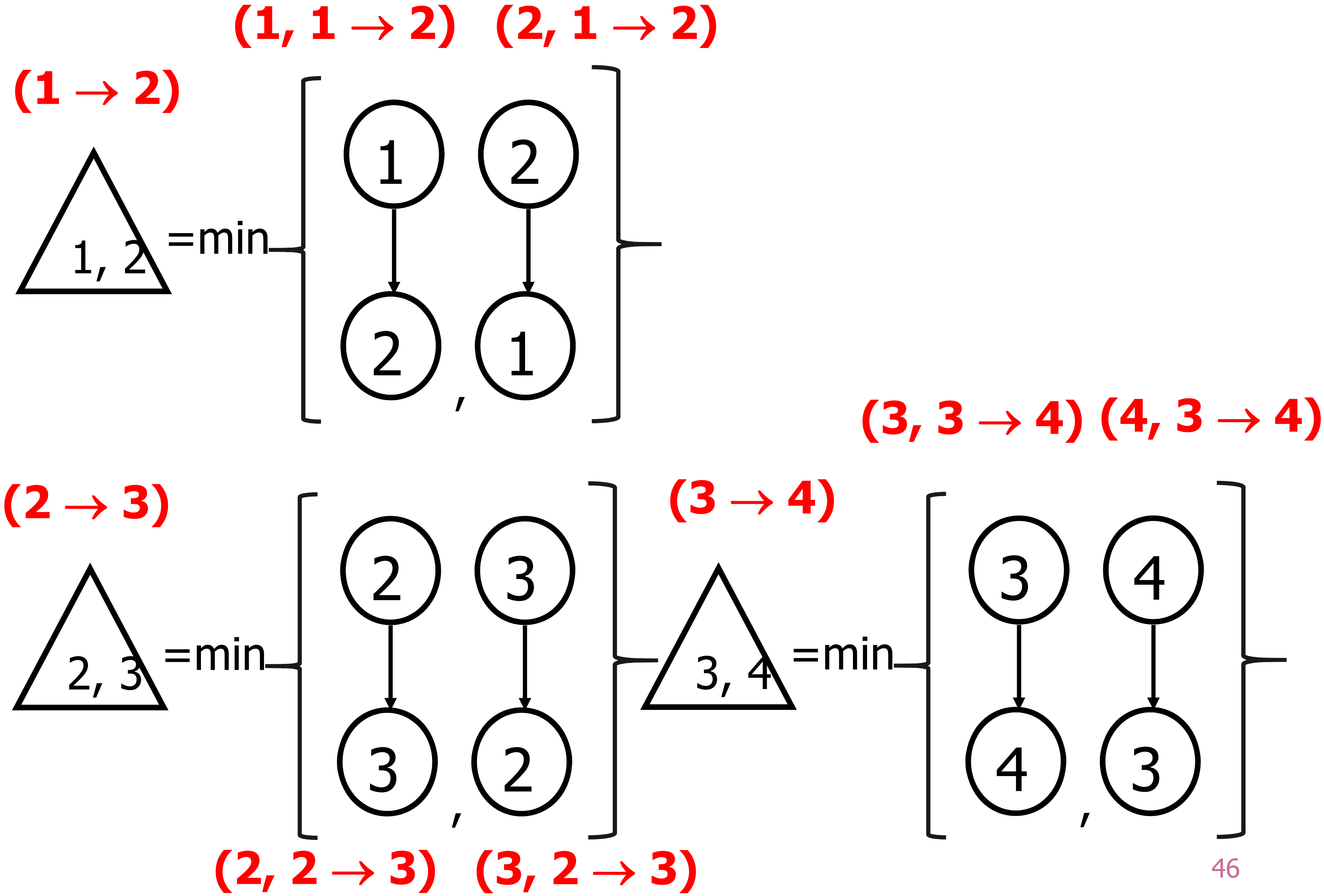


$(a_i \rightarrow a_j)$ denote the optimal binary tree containing identifiers a_i to a_j.

$(a_k, a_i \rightarrow a_j)$ denote an optimal binary tree containing identifier a_i to a_j and with a_k as its root.




(2 → 4)
2, 3, 4
=min
2
3,4
3
2
4
4
2,3
(2, 2 → 4)
(3, 2 → 4)
(4, 2 → 4)
(1 → 3)
1, 2, 3
=min
1
2,3
2
1
3
3
1,2
(1, 1 → 2)
(2, 1 → 3)
(3, 1 → 3)


(1, 1 → 2)
(2, 1 → 2)
(1 → 2)
1, 2
=min
1
2
2
1
(3, 3 → 4)
(4, 3 → 4)
(2 → 3)
(3 → 4)
2, 3
=min
2
3
3
2
3, 4
=min
3
4
4
3
(2, 2 → 3)
(3, 2 → 3)

Optimal binary tree

- Identifiers stored close to the root of the tree can be searched rather quickly.
- For each identifier a_i, associated with probability P_i.
- For each identifier not stored in tree also given probability Q_i.

Optimal binary search trees

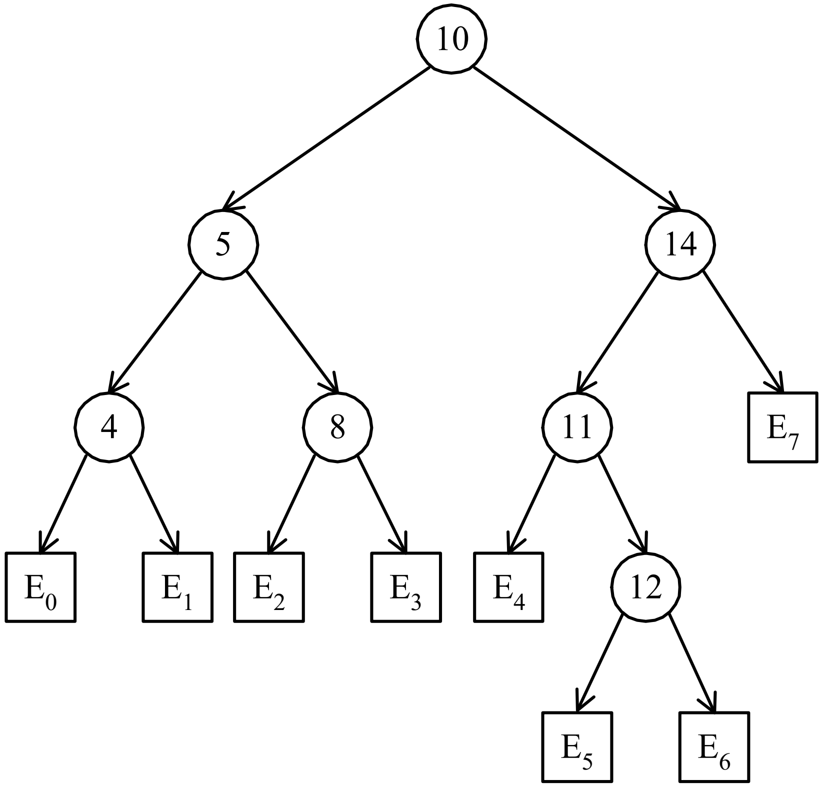
- n identifiers : $a_1 < a_2 < a_3 < \ldots < a_n$

P_i, $1 \le i \le n$: the probability that a_i is searched.

Q_i, $0 \le i \le n$: the probability that x is searched

where $a_i < x < a_{i+1}$ $(a_0 = -\infty,\ a_{n+1} = \infty)$.

$$\sum_{i=1}^{n} P_i + \sum_{i=1}^{n} Q_i = 1$$

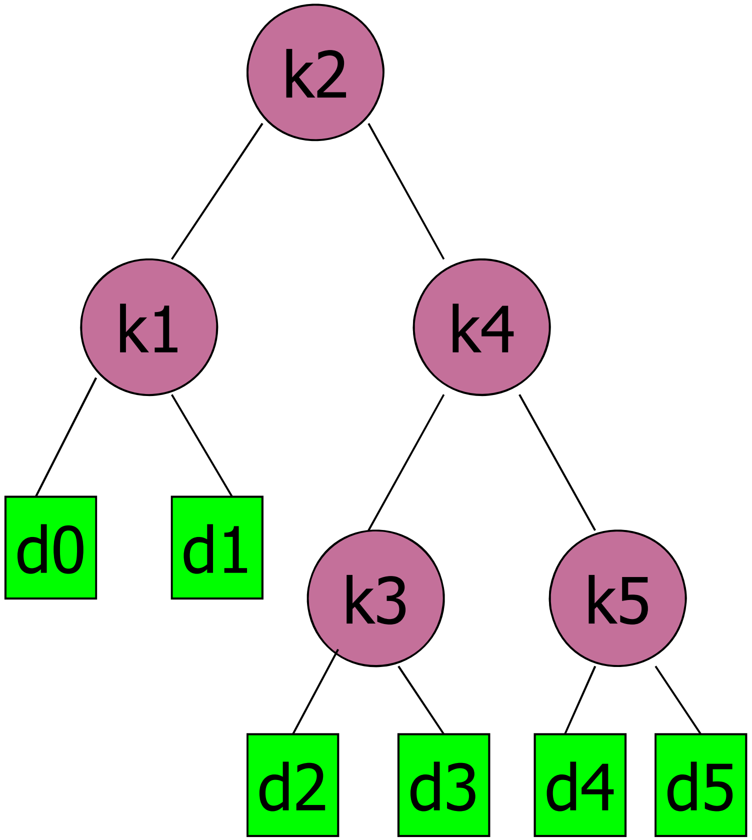


- Identifiers : 4, 5, 8, 10, 11, 12, 14
- Internal node : successful search, P_i
- External node : unsuccessful search, Q_i

- The expected cost of a binary tree:

$$\sum_{n=1}^{n} P_i * \text{level}(a_i) + \sum_{n=0}^{n} Q_i * (\text{level}(E_i) - 1)$$

- The level of the root : 1
- The optimal binary tree is a tree with minimal cost.

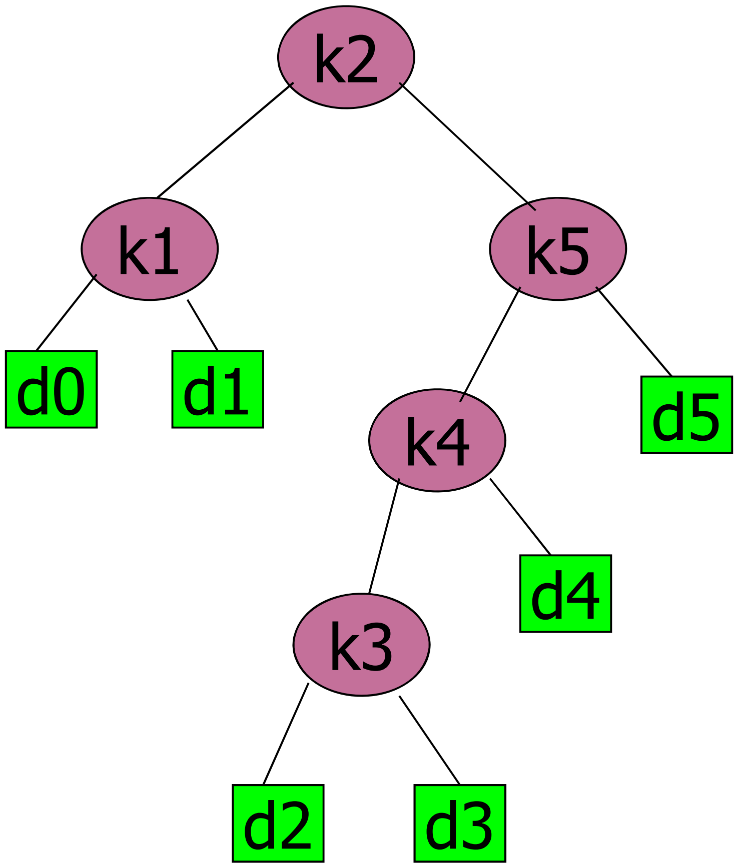


Node	level	probability	cost
k1	2	0.15	0.30
k2	1	0.10	0.10
k3	3	0.05	0.15
k4	2	0.10	0.20
K5	3	0.20	0.60
d0	3	0.05	0.10
d1	3	0.10	0.20
d2	4	0.05	0.15
d3	4	0.05	0.15
d4	4	0.05	0.15
d5	4	0.10	0.30

$$\sum_{n=1}^{n} P_i * \text{level}(a_i) + \sum_{n=0}^{n} Q_i * (\text{level}(E_i) - 1)$$

i	0	1	2	3	4	5
P_i		0.15	0.10	0.05	0.10	0.20
Q_i	0.05	0.10	0.05	0.05	0.05	0.10

Total cost=2.7



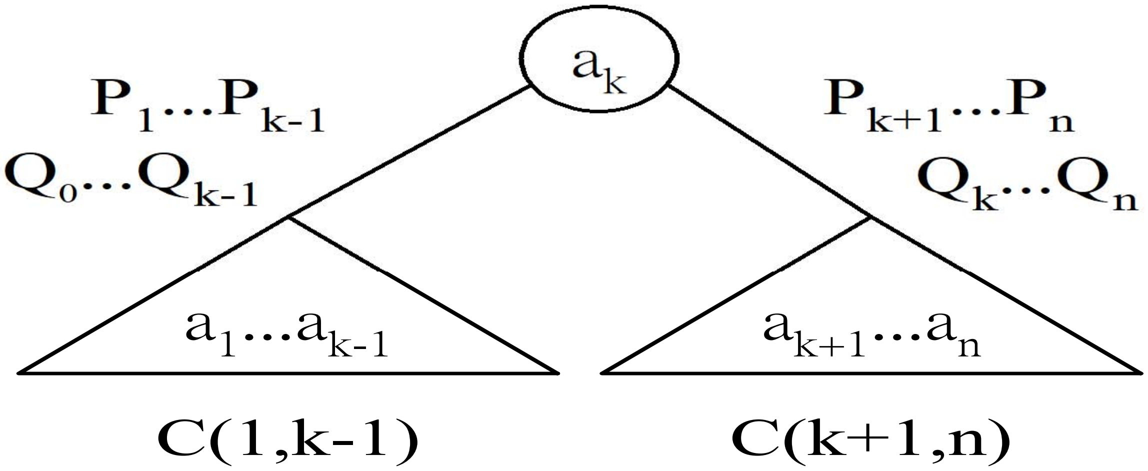
Node	level	probability	cost
k1	2	0.15	0.30
k2	1	0.10	0.10
k3	4	0.05	0.20
k4	3	0.10	0.30
K5	2	0.20	0.40
d0	3	0.05	0.10
d1	3	0.10	0.20
d2	5	0.05	0.20
d3	5	0.05	0.20
d4	4	0.05	0.15
d5	3	0.10	0.20

i	0	1	2	3	4	5
P_i		0.15	0.10	0.05	0.10	0.20
Q_i	0.05	0.10	0.05	0.05	0.05	0.10

Total cost=2.35

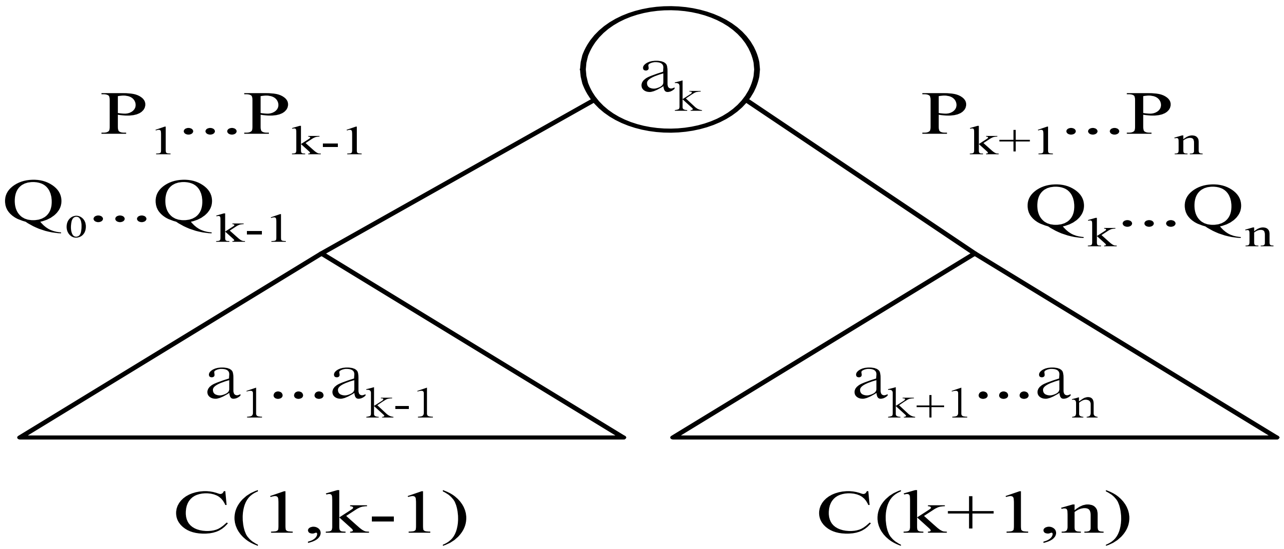
The dynamic programming approach

- Select an identifier, a_k, to be the root of the tree, all identifier < a_k (>a_k) will constitute the left (right) descendant.
- Let C(i, j) denote the cost of an optimal binary search tree containing $a_i, \ldots, a_j$.
- The cost of the optimal binary search tree with a_k as its root :
- $C(1,n) = \min\limits_{1 \le k \le n} \left\{ \begin{array}{l} P_k + \left[Q_0 + \sum_{i=1}^{k-1} (P_i + Q_i) + C(1, k-1) \right] \\ \quad + \left[Q_k + \sum_{i=k+1}^{n} (P_i + Q_i) + C(k+1, n) \right] \end{array} \right\}$



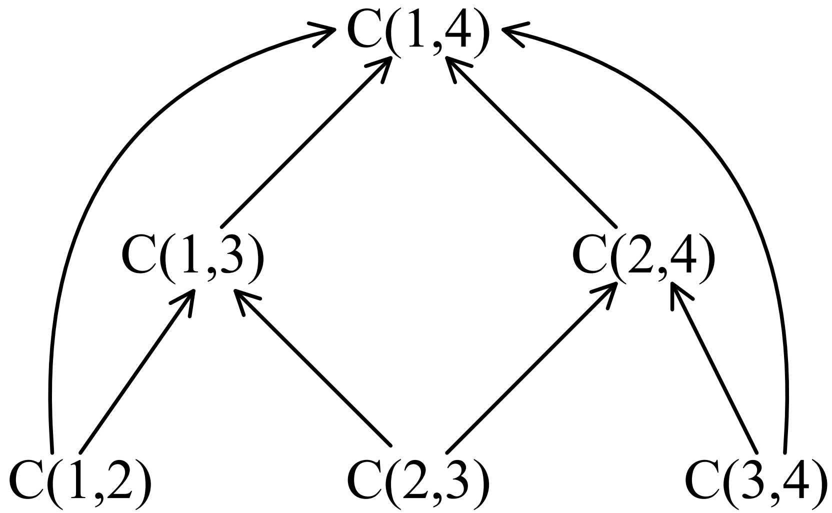
General formula

$$
\begin{aligned}
C(i, j) &= \min_{i \le k \le j} \left\{ P_k + \left[Q_{i-1} + \sum_{m=i}^{k-1} \left(P_m + Q_m\right) + C(i, k-1) \right] \right. \\
&\qquad \left. + \left[Q_k + \sum_{m=k+1}^{j} \left(P_m + Q_m\right) + C(k+1, j) \right] \right\} \\
&= \min_{i \le k \le j} \left\{ C(i, k-1) + C(k+1, j) + Q_{i-1} + \sum_{m=i}^{j} \left(P_m + Q_m\right) \right\}
\end{aligned}
$$



Computation relationships of subtrees

- e.g. n=4



- Time complexity : $O(n^3)$
  when j-i=m, there are (n-m) C(i, j)'s to compute.
  Each C(i, j) with j-i=m can be computed in O(m) time.

$$O\left(\sum_{1 \le m \le n} m(n-m)\right) = O(n^3)$$

Find the optimal binary search tree for N = 6, having keys $k_1 \dots k_6$ and weights $p_1 = 10$, $p_2 = 3$, $p_3 = 9$, $p_4 = 2$, $p_5 = 0$, $p_6 = 10$; $q_0 = 5$, $q_1 = 6$, $q_2 = 4$, $q_3 = 4$, $q_4 = 3$, $q_5 = 8$, $q_6 = 0$. The following figure shows the arrays as they would appear after the initialization and their final disposition.

Index	**0**	**1**	**2**	**3**	**4**	**5**	**6**
k		3	7	10	15	20	25
p	-	10	3	9	2	0	10
q	5	6	4	4	3	8	0

R	0	1	2	3	4	5	6
0		1					
1			2				
2				3			
3					4		
4						5	
5							6
6							

W	0	1	2	3	4	5	6
0	5	21	28	41	46	54	64
1		6	13	26	31	39	49
2			4	17	22	30	40
3				4	9	17	27
4					3	11	21
5						8	18
6							0

C	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

- To help our discussion, we define
- C_{ij} =expected time searching keys in (k_i, k_{i+1}, …, k_j; d_{i-1}, d_i, …, d_j)
- w_{ij}= sum of the probabilities of keys in in (k_i, k_{i+1}, …, k_j; d_{i-1}, d_i, …, d_j)

$$w_{i,j} = \sum_{s=i+1}^{j} P_s + \sum_{t=i}^{j} Q_t$$

$$C_{i,j} = \min_{i<k\le j}\{C_{1,k-1} + C_{k,j}\} + w_{i,j}$$

$$C_{i,i} = w_{i,i}$$

Notations for example

- OBST(i, j) denotes the optimal binary search tree containing the keys ki, ki+1, …, kj;
- $W_{i, j}$ denotes the weight matrix for OBST(i, j)
- $W_{i, j}$ can be defined using the following formula:

$$W_{i,j} = \sum_{k=i+1}^{j} p_k + \sum_{k=i}^{j} q_k$$

- $C_{i, j}$, $0 \le i \le j \le n$ denotes the cost matrix for OBST(i, j)
- $C_{i, j}$ can be defined recursively, in the following manner:

$$C_{i, i} = W_{i, j}$$

$$C_{i, j} = W_{i, j} + \min_{i<k\le j}(C_{i, k-1} + C_{k, j})$$

- $R_{i, j}$, $0 \le i \le j \le n$ denotes the root matrix for OBST(i, j)

The values of the weight matrix have been computed according to the formulas previously stated, as follows:

W (0, 0) = q0 = 5
W (1, 1) = q1 = 6
W (2, 2) = q2 = 4
W (3, 3) = q3 = 4
W (4, 4) = q4 = 3
W (5, 5) = q5 = 8
W (6, 6) = q6 = 0

W (0, 1) = q0 + q1 + p1 = 5 + 6 + 10 = 21
W (0, 2) = W (0, 1) + q2 + p2 = 21 + 4 + 3 = 28
W (0, 3) = W (0, 2) + q3 + p3 = 28 + 4 + 9 = 41
W (0, 4) = W (0, 3) + q4 + p4 = 41 + 3 + 2 = 46
W (0, 5) = W (0, 4) + q5 + p5 = 46 + 8 + 0 = 54
W (0, 6) = W (0, 5) + q6 + p6 = 54 + 0 + 10 = 64
W (1, 2) = W (1, 1) + q2 + p2 = 6 + 4 + 3 = 13

--- and so on ---
until we reach:

W (5, 6) = q5 + q6 + p6 = 18

C (0, 0) = W (0, 0) = 5
C (1, 1) = W (1, 1) = 6
C (2, 2) = W (2, 2) = 4
C (3, 3) = W (3, 3) = 4
C (4, 4) = W (4, 4) = 3
C (5, 5) = W (5, 5) = 8
C (6, 6) = W (6, 6) = 0

C	0	1	2	3	4	5	6
0	5						
1		6					
2			4				
3				4			
4					3		
5						8	
6							0

Figure 5: Cost Matrix after first step

$C(0, 1) = W(0, 1) + (C(0, 0) + C(\mathbf{1}, 1)) = 21 + 5 + 6 = 32$

$C(1, 2) = W(0, 1) + (C(1, 1) + C(\mathbf{2}, 2)) = 13 + 6 + 4 = 23$

$C(2, 3) = W(0, 1) + (C(2, 2) + C(\mathbf{3}, 3)) = 17 + 4 + 4 = 25$

$C(3, 4) = W(0, 1) + (C(3, 3) + C(\mathbf{4}, 4)) = 9 + 4 + 3 = 16$

$C(4, 5) = W(0, 1) + (C(4, 4) + C(\mathbf{5}, 5)) = 11 + 3 + 8 = 22$

$C(5, 6) = W(0, 1) + (C(5, 5) + C(\mathbf{6}, 6)) = 18 + 8 + 0 = 26$

**The bolded numbers represent the elements added in the root matrix.*

C	0	1	2	3	4	5	6
0	5	32					
1		6	23				
2			4	25			
3				4	16		
4					3	22	
5						8	26
6							0

R	0	1	2	3	4	5	6
0		1					
1			2				
2				3			
3					4		
4						5	
5							6
6							

C (0, 2) = W (0, 2) + min (C (0, 0) + C (**1**, 2), C (0, 1) + C (2, 2)) = 28 + min (**28**, 36) = 56

C (1, 3) = W (1, 3) + min (C (1, 1) + C (2, 3), C (1, 2) + C (**3**, 3)) = 26 + min (31, **27**) = 53

C (2, 4) = W (2, 4) + min (C (2, 2) + C (**3**, 4), C (2, 3) + C (4, 4)) = 22 + min (**20**, 28) = 42

C (3, 5) = W (3, 5) + min (C (3, 3) + C (4, 5), C (3, 4) + C (**5**, 5)) = 17 + min (26, **24**) = 41

C (4, 6) = W (4, 6) + min (C (4, 4) + C (5, 6), C (4, 5) + C (**6**, 6)) = 21 + min (29, **22**) = 43

C	0	1	2	3	4	5	6
0	5	32	56				
1		6	23	53			
2			4	25	42		
3				4	16	41	
4					3	22	43
5						8	26
6							0

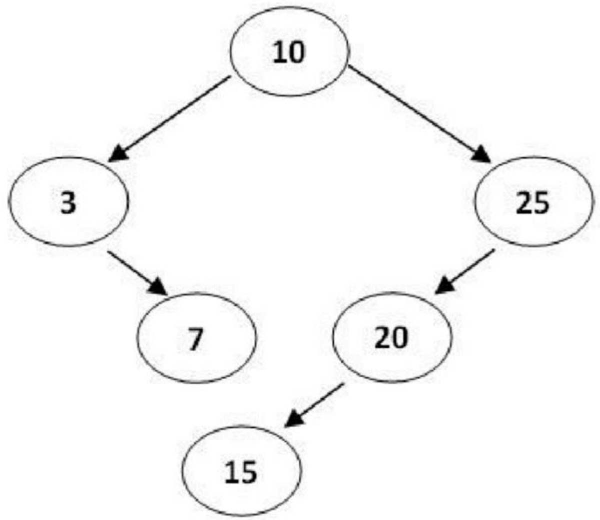
R	0	1	2	3	4	5	6
0		1	1				
1			2	3			
2				3	3		
3					4	5	
4						5	6
5							6
6							

Final array values:

C	0	1	2	3	4	5	6
0	5	32	56	98	118	151	188
1		6	23	53	70	103	140
2			4	25	42	75	108
3				4	16	41	68
4					3	22	43
5						8	26
6							0

R	0	1	2	3	4	5	6
0	0	1	1	2	3	3	3
1		0	2	3	3	3	3
2			0	3	3	3	4
3				0	4	5	6
4					0	5	6
5						0	6
6							0

The resulting optimal tree is shown in the bellow figure and has a weighted path length of 188.



Computing the node positions in the tree:

- The root of the optimal tree is R(0, 6) = k3;
- The root of the left subtree is R(0, 2) = k1;
- The root of the right subtree is R(3, 6) = k6;
- The root of the right subtree of k1 is R(1, 2) = k2
- The root of the left subtree of k6 is R(3, 5) = k5
- The root of the left subtree of k5 is R(3, 4) = k4

http://software.ucv.ro/~cmihaescu/ro/laboratoare/SDA/docs/arboriOptimali_en.pdf

Matrix Chain-Products

學習目標

- Matrix Chain-Products問題定義
- 設計Dynamic Programming演算法解決 Matrix Chain-Products
- 演算法時間複雜度分析

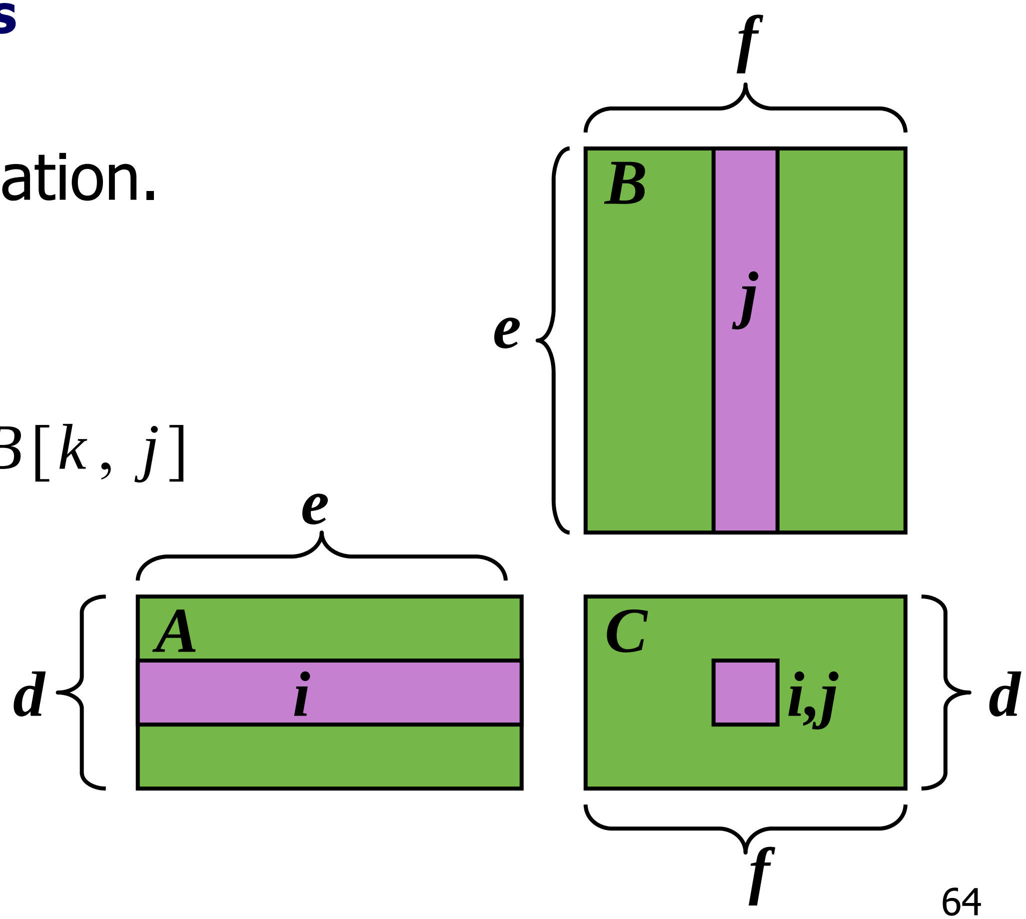
Matrix Chain-Products

- Dynamic Programming is a general algorithm design paradigm.
  - Rather than give the general structure, let us first give a motivating example:
  - **Matrix Chain-Products**

- Review: Matrix Multiplication.
  - $C = A*B$
  - A is $d \times e$ and B is $e \times f$

$$C[i,j] = \sum_{k=0}^{e-1} A[i,k] * B[k,j]$$

  - $O(def)$ time



Matrix-chain multiplication

- n matrices $A_1, A_2, \ldots, A_n$ with size
  $p_0 \times p_1,\ p_1 \times p_2,\ p_2 \times p_3,\ \ldots,\ p_{n-1} \times p_n$
  To determine the multiplication order such that # of scalar multiplications is minimized.
- To compute $A_i \times A_{i+1}$, we need $p_{i-1}p_ip_{i+1}$ scalar multiplications.

e.g. n=4, A_1: 3×5, A_2: 5×4, A_3: 4×2, A_4: 2×5

$((A_1 \times A_2) \times A_3) \times A_4$, # of scalar multiplications:

$3 * 5 * 4 + 3 * 4 * 2 + 3 * 2 * 5 = 114$

$(A_1 \times (A_2 \times A_3)) \times A_4$, # of scalar multiplications:

$3 * 5 * 2 + 5 * 4 * 2 + 3 * 2 * 5 = 100$ *

$(A_1 \times A_2) \times (A_3 \times A_4)$, # of scalar multiplications:

$3 * 5 * 4 + 3 * 4 * 5 + 4 * 2 * 5 = 160$

- **Note: n個matrix相乘有 $C_{n-1}=\binom{2(n-1)}{n-1}\Big/ n$ 種可能的配對組合(括號方式)**
  - **Ex:** 以下有四個矩陣相乘:

$$\begin{array}{cccc} A \times & B \times & C \times & D \\ 20\times 2 & 2\times 30 & 30\times 12 & 12\times 8 \end{array}$$

由**Note**得知共有五種不同的相乘順序，不同的順序需要不同的乘法次數：

$$\begin{array}{ll} A(B(CD)) & 30\times 12\times 8+\ 2\times 30\times\ 8+20\times\ 2\times 8=\ 3,680 \\ (AB)(CD) & 20\times 2\times 30+30\times 12\times\ 8+20\times 30\times 8=\ 8,880 \\ A((BC)D) & 2\times 30\times 12+\ 2\times 12\times\ 8+20\times\ 2\times 8=\ 1,232 \\ ((AB)C)D & 20\times 2\times 30+20\times 30\times 12+20\times 12\times 8=10,320 \\ (A(BC))D & 2\times 30\times 12+20\times\ 2\times 12+20\times 12\times 8=\ 3,120 \end{array}$$

其中，以第三組是最佳的矩陣相乘順序。

An Enumeration Approach

- **Matrix Chain-Product Alg.:**
  - Try all possible ways to parenthesize $A=A_1*A_2*\ldots*A_n$
  - Calculate number of ops for each one
  - Pick the one that is best
- Running time:
  - The number of paranethesizations is equal to the number of binary trees with n nodes
  - This is **exponential**!
  - It is called the Catalan number, and it is almost 4^n.
  - This is a terrible algorithm!

Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

Recursive formula

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{i=0}^{n} C_i\, C_{n-i} \quad \text{for } n \geq 0.$$

它也滿足

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \frac{2(2n+1)}{n+2} C_n,$$

這提供了一個更快速的方法來計算卡塔蘭數。

卡塔蘭數的漸近增長為

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$$

- 六個矩陣相乘的最佳乘法順序可以分解成以下的其中一種型式：

$$A=A_1*(A_2*A_3*A_4*A_5*A_6)$$

$$A=(A_1*A_2)*(A_3*A_4*A_5*A_6)$$

$$A=(A_1*A_2*A_3)*(A_4*A_5*A_6)$$

$$A=(A_1*A_2*A_3*A_4)*(A_5*A_6)$$

$$A=A_1*(A_2*A_3*A_4*A_5*A_6)$$

- 第**k**個分解型式所需的乘法總數，為前後兩部份 **(**一為**A_1, A_2, ..., A_k** 和**A_{k+1}, ..., A_6)** 各自所需乘法數目的最小值相加，再加上相乘這前後兩部份矩陣所需的乘法數目。

$$M_{1,6}=\min_{1\le k<6}\left\{M_{i,k}+M_{k+1,6}+p_i p_{k+1} p_{j+1}\right\}$$

A "Recursive" Approach

- Define **subproblems**:
  - Find the best parenthesization of $A_i*A_{i+1}*\ldots*A_j$.
  - Let $M_{i,j}$ (or M[i][j]) denote the number of operations done by this subproblem.
  - The optimal solution for the whole problem is $\mathbf{M_{1,n}}$.
- **Subproblem optimality**: The optimal solution can be defined in terms of optimal subproblems
  - There has to be a final multiplication (root of the expression tree) for the optimal solution.
  - Say, the final multiply is at index i: $(A_1*\ldots*A_i)*(A_{i+1}*\ldots*A_n)$.
  - Then the optimal solution $M_{1,n}$ is the sum of two optimal subproblems, $M_{1,i}$ and $M_{i+1,n}$ plus the time for the last multiply.
  - If the global optimum did not have these optimal subproblems, we could define an even better "optimal" solution.

A Characterizing Equation

- The global optimal has to be defined in terms of optimal subproblems, depending on where the final multiply is at.
- Let us consider all possible places for that final multiply:
  - Recall that A_i is a $p_i \times p_{i+1}$ dimensional matrix.
  - So, a characterizing equation for $M_{i,j}$ is the following:

$$M_{i,j} = \min_{1 \le k < j} \{M_{i,k} + M_{k+1,j} + p_i p_{k+1} p_{j+1}\}$$

- Note that subproblems are not independent--the **subproblems overlap**.

A Dynamic Programming Algorithm

- Since subproblems overlap, we don't use recursion.
- Instead, we construct optimal subproblems "bottom-up."
- $M_{i,i}$'s are easy, so start with them
- Then do length 2,3,… subproblems, and so on.
- Running time: $O(n^3)$

Algorithm ***matrixChain***(***S***):

Input: sequence ***S*** of ***n*** matrices to be multiplied

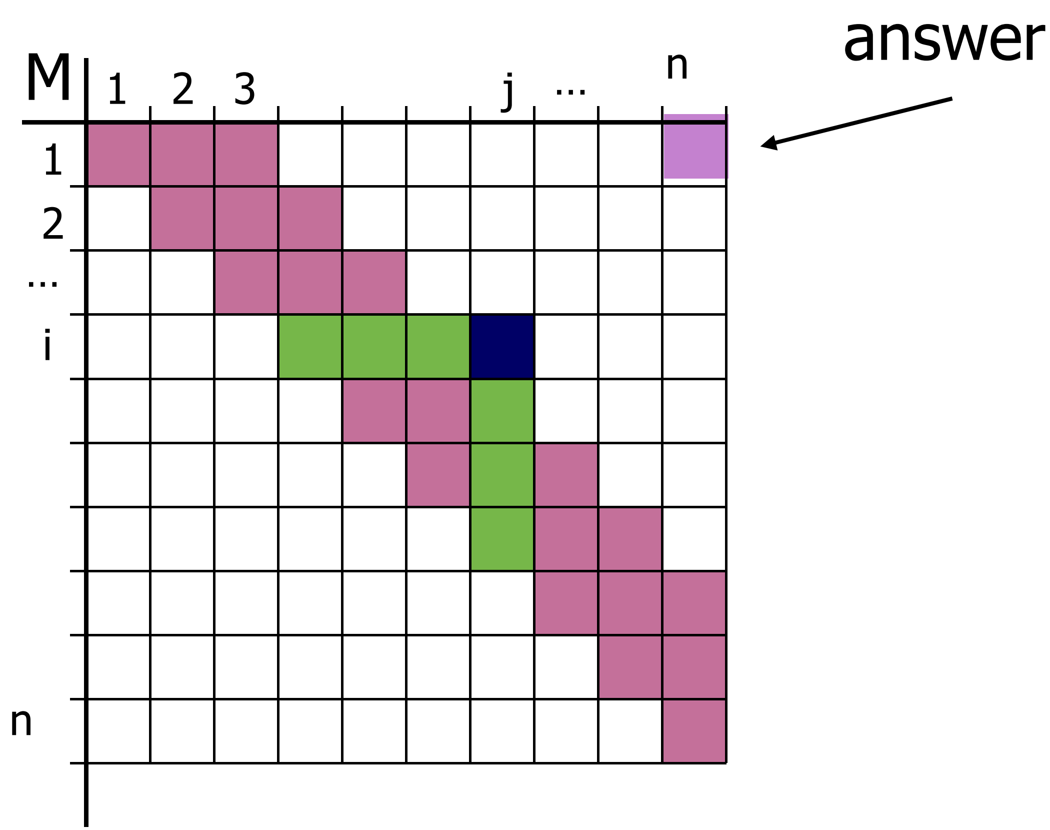
Output: number of operations in an optimal paranethization of ***S***

```
for i ← 1 to n do
    M_{i,i} ← 0
for b ← 1 to n do
    for i ← 1 to n-b do
        j ← i+b
        M_{i,j} ← +infinity
        for k ← i to j-1 do
            M_{i,j} ← min{M_{i,j} , M_{i,k} + M_{k+1,j} + p_i p_{k+1} p_{j+1}}
```

A Dynamic Programming Algorithm Visualization

$$M_{i,j} = \min_{1 \le k < j} \{M_{i,k} + M_{k+1,j} + p_i p_{k+1} p_{j+1}\}$$

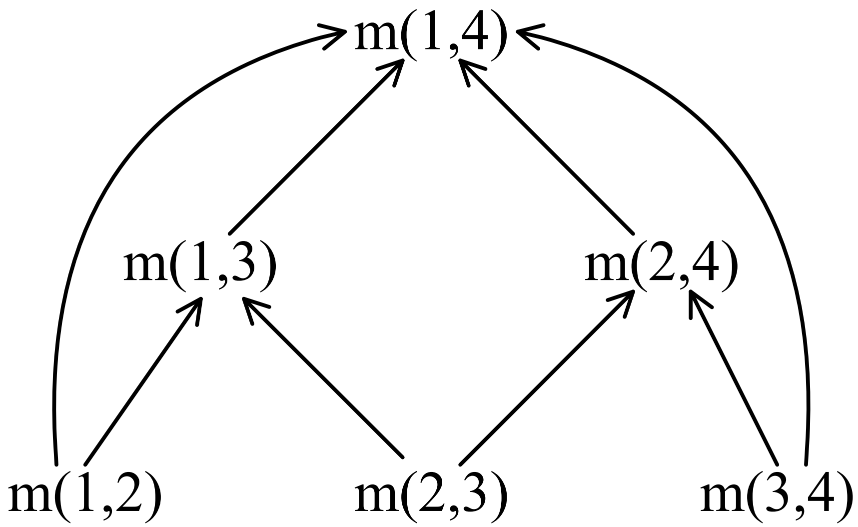
- The bottom-up construction fills in the M array by diagonals
- $M_{i,j}$ gets values from pervious entries in i-th row and j-th column
- Filling in each entry in the M table takes O(n) time.
- Total run time: $O(n^3)$
- Getting actual parenthesization can be done by remembering "k" for each M entry



- Let M(i, j) denote the minimum cost for computing $A_i \times A_{i+1} \times \ldots \times A_j$

$$m(i,j) = \begin{cases} 0 & \text{if } i = j \\ M_{i,j} = \min_{1 \le k < j} \{M_{i,k} + M_{k+1,j} + p_i p_{k+1} p_{j+1}\} & \text{if } i < j \end{cases}$$

- Computation sequence :



- Time complexity : $O(n^3)$

- **Matrix Chain**的遞迴式

$$M_{ij}=\begin{cases} 0,\ if\ i=j \\ \min\limits_{1\le k<j}\{M_{i,k}+M_{k+1,j}+p_i p_{k+1} p_{j+1}\},\ if\ i<j \end{cases}$$

- **Example:** $A^1_{3\times 3}$, $A^2_{3\times 7}$, $A^3_{7\times 2}$, $A^4_{2\times 9}$, $A^5_{9\times 4}$, 求此五矩陣的最小乘法次數。

Sol:

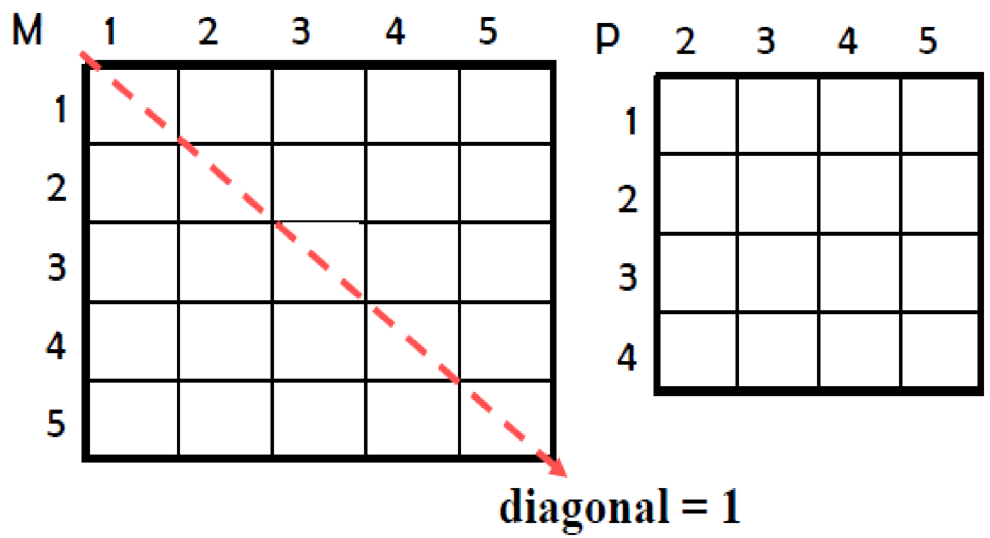
建立兩陣列 **M[1...5, 1...5]**及**P[1...4, 2...5]**

M	1	2	3	4	5
1					
2					
3					
4					
5					

P	2	3	4	5
1				
2				
3				
4				

Case ① (When diagonal = 1)

- diagonal = 1，∵只有1個矩陣，∴不會執行乘法動作
- 陣列M的中間對角線爲0，陣列P則不填任何數值



Case ② (When diagonal > 1)

- diagonal = 2，有2個矩陣相乘
- 當 i = 1及 j = 2，爲A^1及A^2矩陣相乘，此時:

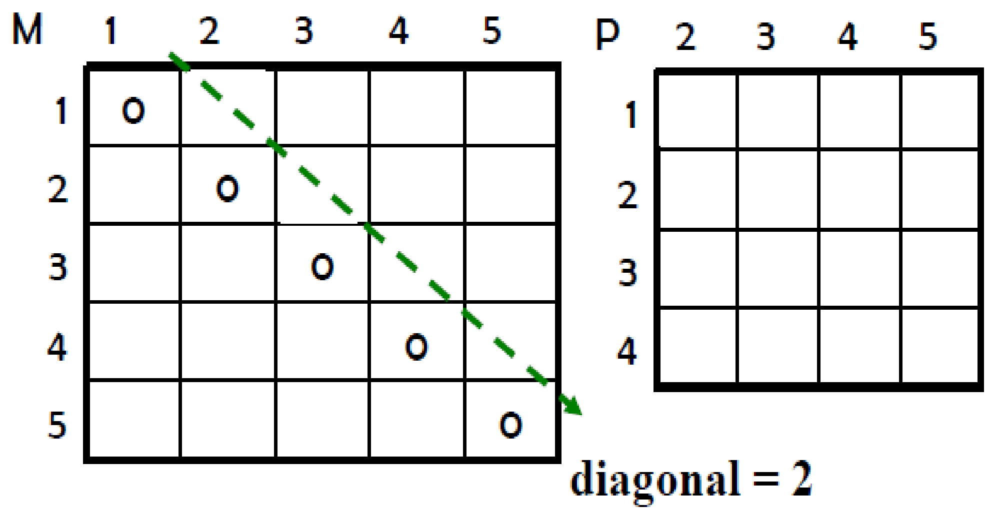
$M[1, 2] = M[1,1]+M[2,2]+3\times3\times7 = 63$，

其中 A^1 及A^2 的分割點 k 如下:

$$A^1 \times A^2$$

↑

分割點k = 1



Case ② (When diagonal > 1)

- diagonal = 3，有3個矩陣相乘
- 當 i = 2及 j = i+diagnal-1 = 2+3-1=4，爲A^2至A^4間的所有矩陣相乘，此時:

$$M[2,4] = \min\begin{cases} M[2,2]+M[3,4]+3\times 7\times 9 = 315, & \text{分割點}k=2 \\ M[2,3]+M[4,4]+3\times 2\times 9 = 96, & \text{分割點}k=3 \end{cases}$$

M	1	2	3	4	5
1	0	63	60		
2		0	42	96	
3			0	126	128
4				0	72
5					0

diagonal = 3

P	2	3	4	5
1	1	1		
2		2	3	
3			3	3
4				4

Case ② (When diagonal > 1)

- diagonal = 4，有4個矩陣
- 當 i = 1及 j = 4，爲A^1至A^4間的所有矩陣相乘，此時:

M	1	2	3	4	5
1	0	63	60	114	
2		0	42	96	138
3			0	126	128
4				0	72
5					0

diagonal = 4

P	2	3	4	5
1	1	1	3	
2		2	3	3
3			3	3
4				4

$$M[1,4] = \min\begin{cases} M[1,1] + M[2,4] + 3\times 3\times 9 = 177, & \text{分割點}k = 1 \\ M[1,2] + M[3,4] + 3\times 7\times 9 = 378, & \text{分割點}k = 2 \\ M[1,3] + M[4,4] + 3\times 2\times 9 = 114, & \text{分割點}k = 3 \end{cases}$$

Case ② (When diagonal > 1)

- diagonal = 5，有5個矩陣
- 當 i = 1及 j = 5，為A^1至A^5間所有矩陣相乘，此時:

M	1	2	3	4	5
1	0	63	60	114	156
2		0	42	96	138
3			0	126	128
4				0	72
5					0

diagonal = 5

P	2	3	4	5
1	1	1	3	3
2		2	3	3
3			3	3
4				4

$$M[1,5] = \min\begin{cases} M[1,1] + M[2,5] + 3\times 3\times 4 = 174, & \text{分割點}k = 1 \\ M[1,2] + M[3,5] + 3\times 7\times 4 = 275, & \text{分割點}k = 2 \\ M[1,3] + M[4,5] + 3\times 2\times 4 = 156, & \text{分割點}k = 3 \\ M[1,4] + M[5,5] + 3\times 9\times 4 = 222, & \text{分割點}k = 4 \end{cases}$$

◆ [Note]此演算法的概念如下:

A^1, A^2, A^3, A^4, A^5 diagonal = 1

$A^1 A^2$, A^2A^3, A^3A^4, A^4A^5 diagonal = 2

$A^1 A^2A^3$, $A^2A^3A^4$, $A^3A^4A^5$ diagonal = 3

$A^1 A^2A^3A^4$, $A^2A^3A^4A^5$ diagonal = 4

$A^1 A^2A^3A^4A^5$ diagonal = 5