問題 2.1 (Lv.1)

次の不定積分を求めよ.

(1) $\displaystyle\int \frac{1}{x+1}\,dx$　(2) $\displaystyle\int \frac{1}{(x+1)^2}\,dx$　(3) $\displaystyle\int \frac{1}{x^2-9}\,dx$　(4) $\displaystyle\int \frac{1}{x^2+9}\,dx$

(5) $\displaystyle\int \frac{1}{3x-1}\,dx$　(6) $\displaystyle\int \frac{1}{(3x-1)^2}\,dx$　(7) $\displaystyle\int \frac{1}{x^2-2}\,dx$　(8) $\displaystyle\int \frac{1}{x^2+2}\,dx$

問題 2.2 (Lv.2)

次の不定積分を求めよ.

(1) $\displaystyle\int \frac{x}{x-1}\,dx$　(2) $\displaystyle\int \frac{2x+1}{x-1}\,dx$　(3) $\displaystyle\int \frac{x}{x^2+3}\,dx$

(4) $\displaystyle\int \frac{x^2}{(x-1)^2}\,dx$　(5) $\displaystyle\int \frac{5x-1}{x^2-1}\,dx$　(6) $\displaystyle\int \frac{3x^2+x}{x^2+3}\,dx$

問題 2.3 (Lv.2)

次の不定積分を求めよ.

(1) $\displaystyle\int \frac{1}{x^2-4x+6}\,dx$　(2) $\displaystyle\int \frac{2x-4}{x^2-4x+6}\,dx$　(3) $\displaystyle\int \frac{4x-5}{x^2-4x+6}\,dx$

(4) $\displaystyle\int \frac{1}{x^2-5x+6}\,dx$　(5) $\displaystyle\int \frac{x-5}{x^2-5x+6}\,dx$　(6) $\displaystyle\int \frac{x^2-5x}{x^2-5x+6}\,dx$

問題 2.4 (Lv.3)

次の不定積分を求めよ.

(1) $\displaystyle\int \frac{1}{(x^2+2)^2}\,dx$　(2) $\displaystyle\int \frac{2x}{(x^2+2)^2}\,dx$　(3) $\displaystyle\int \frac{3x+4}{(x^2+2)^2}\,dx$

問題 2.5 (Lv.3)

$f(x)=2x^3-3x^2-7x+9,\ g(x)=x^4-2x^3-x^2+8x+6$ とする.

このとき, 不定積分 $\displaystyle\int \frac{f(x)}{g(x)}\,dx \left(=\int \frac{2x^3-3x^2-7x+9}{x^4-2x^3-x^2+8x+6}\,dx\right)$ を求めよ.

問題 2.6 (Lv.3)

$f(x)=x^5-3x^4+3x^3+6x^2-9x+3,\ g(x)=x^4-2x^3-x^2+8x+6$ とする.

このとき, 不定積分 $\displaystyle\int \frac{f(x)}{g(x)}\,dx \left(=\int \frac{x^5-3x^4+3x^3+6x^2-9x+3}{x^4-2x^3-x^2+8x+6}\,dx\right)$ を求めよ.

問題 2.1 (解答)

(1) $\displaystyle\int \frac{1}{x+1}\,dx = \log|x+1| + C$ $\left(t = x+1 \text{ で } dt = dx \text{ から } \displaystyle\int \frac{1}{t}\,dt \text{ の形} \right)$

(2) $\displaystyle\int \frac{1}{(x+1)^2}\,dx = -\frac{1}{x+1} + C$ $\left(t = x+1 \text{ で } dt = dx \text{ から } \displaystyle\int \frac{1}{t^2}\,dt \text{ の形} \right)$

(3) $\displaystyle \frac{1}{x^2-9} = \frac{1}{6}\cdot\frac{(x+3)-(x-3)}{(x-3)(x+3)} = \frac{1}{6}\left(\frac{1}{x-3} - \frac{1}{x+3}\right)$ と分解して,

$$\int \frac{1}{x^2-9}\,dx = \frac{1}{6}\big(\log|x-3| - \log|x+3|\big) + C \ \left(= \frac{1}{6}\log\left|\frac{x-3}{x+3}\right| + C\right)$$

(4) $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\mathrm{Tan}^{-1}\frac{x}{a} + C$ (Th.1.3) より, $\displaystyle\int \frac{1}{x^2+9}\,dx = \frac{1}{3}\mathrm{Tan}^{-1}\frac{x}{3} + C$

(5) $\displaystyle\int \frac{1}{3x-1}\,dx = \frac{1}{3}\log|3x-1| + C$ $\left(t = 3x-1 \text{ で } dt = 3\,dx \text{ から } \displaystyle\frac{1}{3}\int \frac{1}{t}\,dt \text{ の形} \right)$

(6) $\displaystyle\int \frac{1}{(3x-1)^2}\,dx = -\frac{1}{3(3x-1)} + C$ $\left(t = 3x-1 \text{ で } dt = 3\,dx \text{ から } \displaystyle\frac{1}{3}\int \frac{1}{t^2}\,dt \text{ の形} \right)$

(7) $\displaystyle \frac{1}{x^2-2} = \frac{1}{2\sqrt{2}}\cdot\frac{(x+\sqrt{2}\,)-(x-\sqrt{2}\,)}{(x-\sqrt{2}\,)(x+\sqrt{2}\,)} = \frac{1}{2\sqrt{2}}\left(\frac{1}{x-\sqrt{2}} - \frac{1}{x+\sqrt{2}}\right)$ と分解して,

$$\int \frac{1}{x^2-2}\,dx = \frac{1}{2\sqrt{2}}\big(\log|x-\sqrt{2}\,| - \log|x+\sqrt{2}\,|\big) + C \ \left(= \frac{\sqrt{2}}{4}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| + C\right)$$

(8) $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\mathrm{Tan}^{-1}\frac{x}{a} + C$ (Th.1.3) より, $\displaystyle\int \frac{1}{x^2+2}\,dx = \frac{1}{\sqrt{2}}\mathrm{Tan}^{-1}\frac{x}{\sqrt{2}} + C$

問題 2.2 (解答)

(1) $\displaystyle\int \frac{x}{x-1}\,dx = \int \frac{(x-1)+1}{x-1}\,dx = \int\left(1 + \frac{1}{x-1}\right)dx = x + \log|x-1| + C$

(2) $\displaystyle\int \frac{2x+1}{x-1}\,dx = \int \frac{2(x-1)+3}{x-1}\,dx = \int\left(2 + 3\cdot\frac{1}{x-1}\right)dx = 2x + 3\log|x-1| + C$

(3) $\displaystyle\int \frac{x}{x^2+3}\,dx = \int \frac{1}{2}\cdot\frac{2x}{x^2+3}\,dx = \frac{1}{2}\int \frac{(x^2+3)'}{x^2+3}\,dx = \frac{1}{2}\log|x^2+3| + C$

(4) $\displaystyle\int \frac{x^2}{(x-1)^2}\,dx = \int \frac{\{(x-1)+1\}^2}{(x-1)^2}\,dx = \int \frac{(x-1)^2+2(x-1)+1}{(x-1)^2}\,dx$

$$= \int\left(1 + 2\cdot\frac{1}{x-1} + \frac{1}{(x-1)^2}\right)dx = x + 2\log|x-1| - \frac{1}{x-1} + C$$

(5) $5x-1 = a(x+1) + b(x-1)$ となる係数 a, b を $x = 1, -1$ を代入して決定して,

$\displaystyle \frac{5x-1}{x^2-1} = \frac{2(x+1)+3(x-1)}{(x-1)(x+1)} = 2\cdot\frac{1}{x-1} + 3\cdot\frac{1}{x+1}$ の形で分解できるので,

$$\int \frac{5x-1}{x^2-1}\,dx = 2\log|x-1| + 3\log|x+1| + C \ \left(= \log|(x-1)^2(x+1)^3| + C\,\right)$$

(6) $\displaystyle\int \frac{3x^2+x}{x^2+3}\,dx = \int \frac{3(x^2+3)+x-9}{x^2+3}\,dx = \int\left(3 + \frac{x-9}{x^2+3}\right)dx$

$$= \int\left(3 + \frac{1}{2}\cdot\frac{2x}{x^2+3} - 9\cdot\frac{1}{x^2+3}\right)dx = 3x + \frac{1}{2}\log|x^2+3| - \frac{9}{\sqrt{3}}\mathrm{Tan}^{-1}\frac{x}{\sqrt{3}} + C$$

問題 2.3 (解答)

(1) $\displaystyle\int \frac{1}{x^2-4x+6}\,dx = \int \frac{1}{(x-2)^2+2}\,dx = \frac{1}{\sqrt{2}}\mathrm{Tan}^{-1}\frac{x-2}{\sqrt{2}} + C$

$\left(t = x-2 \text{ で } dt = dx \text{ から } \displaystyle\int \frac{1}{t^2+2}\,dt = \frac{1}{\sqrt{2}}\mathrm{Tan}^{-1}\frac{t}{\sqrt{2}} + C \text{ の形}\right)$

(2) $\displaystyle\int \frac{2x-4}{x^2-4x+6}\,dx = \int \frac{(x^2-4x+6)'}{x^2-4x+6}\,dx = \log|x^2-4x+6| + C$

(3) $\displaystyle\frac{4x-5}{x^2-4x+6} = \frac{2(2x-4)+3}{x^2-4x+6} = 2\cdot\frac{2x-4}{x^2-4x+6} + 3\cdot\frac{1}{x^2-4x+6}$ と分解して,

$\displaystyle\int \frac{4x-5}{x^2-4x+6}\,dx = 2\log|x^2-4x+6| + \frac{3}{\sqrt{2}}\mathrm{Tan}^{-1}\frac{x-2}{\sqrt{2}} + C$ $\left(\because \text{上記 (1),(2)}\right)$

(4) $\displaystyle\frac{1}{x^2-5x+6} = \frac{-(x-3)+(x-2)}{(x-2)(x-3)} = -\frac{1}{x-2} + \frac{1}{x-3}$ と分解して,

$\displaystyle\int \frac{1}{x^2-5x+6}\,dx = -\log|x-2| + \log|x-3| + C \left(= \log\left|\frac{x-3}{x-2}\right| + C\right)$

(5) $x-5 = a(x-3)+b(x-2)$ となる係数 a, b を $x = 2, 3$ を代入して決定して,

$\displaystyle\frac{x-5}{x^2-5x+6} = \frac{3(x-3)-2(x-2)}{(x-2)(x-3)} = 3\cdot\frac{1}{x-2} - 2\cdot\frac{1}{x-3}$ の形で分解できるので,

$\displaystyle\int \frac{5x-1}{x^2-5x+6}\,dx = 3\log|x-2| - 2\log|x-3| + C \left(= \log\left|\frac{(x-2)^3}{(x-3)^2}\right| + C\right)$

(6) $\displaystyle\frac{x^2-5x}{x^2-5x+6} = \frac{(x^2-5x+6)-6}{x^2-5x+6} = 1 - 6\cdot\frac{1}{x^2-5x+6}$ と分解して,

$\displaystyle\int \frac{x^2-5x}{x^2-5x+6}\,dx = x + 6\log|x-2| - 6\log|x-3| + C$ $\left(\because \text{上記 (4)}\right)$

問題 2.4 (解答)

(1) $\displaystyle\int \frac{1}{(x^2+2)^2}\,dx = \int \frac{1}{2}\cdot\frac{(x^2+2)-x^2}{(x^2+2)^2}\,dx = \int \frac{1}{2}\left(\frac{1}{x^2+2} - \frac{x^2}{(x^2+2)^2}\right)dx$

$\displaystyle= \frac{1}{2}\int \frac{1}{x^2+2}\,dx - \frac{1}{2}\int \frac{x^2}{(x^2+2)^2}\,dx = \frac{1}{2}\int \frac{1}{x^2+2}\,dx - \frac{1}{4}\int \frac{2x}{(x^2+2)^2}\,x\,dx$

$\displaystyle= \frac{1}{2}\int \frac{1}{x^2+2}\,dx - \frac{1}{4}\left\{\left(-\frac{1}{x^2+2}\right)x - \int\left(-\frac{1}{x^2+2}\right)\cdot 1\,dx\right\}$ (部分積分)

$\displaystyle= \frac{1}{4}\left\{\frac{x}{x^2+2} + \int \frac{1}{x^2+2}\,dx\right\} = \frac{1}{4}\left\{\frac{x}{x^2+2} + \frac{1}{\sqrt{2}}\mathrm{Tan}^{-1}\frac{x}{\sqrt{2}}\right\} + C$

(2) $\displaystyle\int \frac{2x}{(x^2+2)^2}\,dx = \int \frac{(x^2+2)'}{(x^2+2)^2}\,dx = -\frac{1}{x^2+2} + C$ [上記の部分積分で使用]

$\left(t = x^2+2 \text{ で } dt = 2x\,dx \text{ から } \displaystyle\int \frac{1}{t^2}\,dt = -\frac{1}{t} + C \text{ の形}\right)$

(3) $\displaystyle\frac{3x+4}{(x^2+2)^2} = \frac{\frac{3}{2}(2x)+4}{(x^2+2)^2} = \frac{3}{2}\cdot\frac{2x}{(x^2+2)^2} + 4\cdot\frac{1}{(x^2+2)^2}$ と分解して,

$\displaystyle\int \frac{3x+4}{(x^2+2)^2}\,dx = -\frac{3}{2}\cdot\frac{1}{x^2+2} + \frac{x}{x^2+2} + \frac{1}{\sqrt{2}}\mathrm{Tan}^{-1}\frac{x}{\sqrt{2}} + C$ $\left(\because \text{上記 (1),(2)}\right)$

問題 2.5 (解答)

分子 $f(x)$ の次数 3 が分母 $g(x)$ の次数 4 より低いので, $g(x)$ の因数分解から行う.

$$
\begin{array}{r}
x^3-3x^2+2x+6 \\
x+1\ \overline{\big)\ x^4-2x^3-\ x^2+8x+6} \\
\underline{x^4+\ x^3} \\
-3x^3-\ x^2 \\
\underline{-3x^3-3x^2} \\
2x^2+8x \\
\underline{2x^2+2x} \\
6x+6 \\
\underline{6x+6} \\
0
\end{array}
\qquad
\begin{array}{r}
x^2-4x+6 \\
x+1\ \overline{\big)\ x^3-3x^2+2x+6} \\
\underline{x^3+\ x^2} \\
-4x^2+2x \\
\underline{-4x^2-4x} \\
6x+6 \\
\underline{6x+6} \\
0
\end{array}
\qquad
\left[\begin{array}{l}
x=-1 \text{ を代入して,} \\
\text{値が } 0 \text{ になるので,} \\
x+1 \text{ で割り切れる.}
\end{array}\right]
$$

$$\left[\ x^2-4x+6=\left(x-2-\sqrt{2}\,i\right)\left(x-2+\sqrt{2}\,i\right)\ \right]$$

実数の範囲で $g(x)=(x+1)(x^3-3x^2+2x+6)=(x+1)^2(x^2-4x+6)$ と分解できる.

$\dfrac{f(x)}{g(x)}=\dfrac{a}{x+1}+\dfrac{b}{(x+1)^2}+\dfrac{cx+d}{x^2-4x+6}$ となる実数 $a\sim d$ を求め, 部分分数に分ける.

分母を払うと, $f(x)=a(x+1)(x^2-4x+6)+b(x^2-4x+6)+(cx+d)(x+1)^2$ となり,

$f'(x)=a\{(x^2-4x+6)+(x+1)(2x-4)\}+b(2x-4)+c(x+1)^2+2(cx+d)(x+1)$

$f(x)=2x^3-3x^2-7x+9$ と $f'(x)=6x^2-6x-7$ の $x=-1$ および $x=0$ での値から,

$11b=11$ と $11a-6b=5$ および $6a+6b+d=9$ と $2a-4b+c+2d=-7$ を得る.

連立方程式を解いて, $a=1$, $b=1$, $c=1$, $d=-3$ が従い, 以下の分解が得られる.

$$\frac{f(x)}{g(x)}=\frac{1}{x+1}+\frac{1}{(x+1)^2}+\frac{x-3}{x^2-4x+6}=\frac{1}{x+1}+\frac{1}{(x+1)^2}+\frac{\frac{1}{2}(2x-4)-1}{x^2-4x+6}$$

$$=\frac{1}{x+1}+\frac{1}{(x+1)^2}+\frac{1}{2}\cdot\frac{2x-4}{x^2-4x+6}-\frac{1}{x^2-4x+6}$$ とし, 問題 2.1 (1),(2) 問題 2.3 (1),(2) から,

$$\int\frac{f(x)}{g(x)}\,dx=\log|x+1|-\frac{1}{x+1}+\frac{1}{2}\log|x^2-4x+6|-\frac{1}{\sqrt{2}}\mathrm{Tan}^{-1}\frac{x-2}{\sqrt{2}}+C$$

[線形代数によれば, 3 次以下の多項式の空間 $\mathbb{R}[x]_3$ は 4 次元のベクトル空間になる.
1 次独立な $(x+1)(x^2-4x+6)$, (x^2-4x+6), $x(x+1)^2$, $(x+1)^2$ は基底になり,
$f(x)=a(x+1)(x^2-4x+6)+b(x^2-4x+6)+cx(x+1)^2+d(x+1)^2$ と書ける.]

問題 2.6 (解答)

分子 $f(x)$ を分母 $g(x)$ で割り,

商 $h(x)$, 余り $r(x)$ を求めると,

$h(x)=x-1,\ r(x)=2x^3-3x^2-7x+9$

$f(x)=g(x)\,h(x)+r(x)$ となるので,

$$
\begin{array}{r}
x-1 \\
x^4-2x^3-x^2+8x+6\ \overline{\big)\ x^5-3x^4+3x^3+6x^2-\ 9x+3} \\
\underline{x^5-2x^4-\ x^3+8x^2+\ 6x} \\
-\ x^4+4x^3-2x^2-15x+3 \\
\underline{-\ x^4+2x^3+\ x^2-\ 8x-6} \\
2x^3-3x^2-\ 7x+9
\end{array}
$$

$\dfrac{f(x)}{g(x)}=h(x)+\dfrac{r(x)}{g(x)}=x-1+\dfrac{2x^3-3x^2-7x+9}{x^4-2x^3-x^2+8x+6}$ となり, 問題 2.5 から,

$$\int\frac{f(x)}{g(x)}\,dx=\frac{1}{2}x^2-x+\log|x+1|-\frac{1}{x+1}+\frac{1}{2}\log|x^2-4x+6|-\frac{1}{\sqrt{2}}\mathrm{Tan}^{-1}\frac{x-2}{\sqrt{2}}+C$$