問題 3.1 (Lv.2)

次の定積分の値を求めよ.

(1) $\displaystyle\int_{-1}^{2}(x^4-1)\,dx$　(2) $\displaystyle\int_{0}^{1}\frac{1}{2x-3}\,dx$　(3) $\displaystyle\int_{-1}^{1}\frac{1}{x^2+1}\,dx$

(4) $\displaystyle\int_{2}^{3}\frac{1}{(x-1)^3}\,dx$　(5) $\displaystyle\int_{1}^{3}\frac{x^2}{x+1}\,dx$　(6) $\displaystyle\int_{2}^{4}\frac{1}{x^2+x}\,dx$

問題 3.2 (Lv.2)

次の定積分の値を求めよ.

(1) $\displaystyle\int_{1}^{3}\log x\,dx$　(2) $\displaystyle\int_{0}^{\log 2}e^{2x}\,dx$　(3) $\displaystyle\int_{0}^{\pi}\sin^2 x\ dx$

(4) $\displaystyle\int_{1}^{e}\frac{\log x}{x}\,dx$　(5) $\displaystyle\int_{0}^{1}\frac{x}{e^x}\,dx$　(6) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\tan x}{\cos^2 x}\,dx$

問題 3.3 (Lv.3)

次の定積分の値を求めよ.

(1) $\displaystyle\int_{0}^{3}\sqrt{x+1}\ dx$　(2) $\displaystyle\int_{0}^{\sqrt{6}}x\sqrt{x^2+2}\ dx$　(3) $\displaystyle\int_{-1}^{3}\sqrt{x^2-2x+1}\ dx$

(4) $\displaystyle\int_{0}^{3}\frac{1}{\sqrt{4-x}}\,dx$　(5) $\displaystyle\int_{1}^{\sqrt{3}}\frac{1}{\sqrt{4-x^2}}\,dx$　(6) $\displaystyle\int_{1}^{\frac{3}{2}}\frac{1}{\sqrt{2x-x^2}}\,dx$

問題 3.4 (Lv.3)

次の定積分の値を求めよ. (k, l は自然数)

(1) $\displaystyle\int_{-\pi}^{\pi}\cos kx\ \cos lx\ dx$　(2) $\displaystyle\int_{-\pi}^{\pi}\sin kx\ \sin lx\ dx$　(3) $\displaystyle\int_{-\pi}^{\pi}\cos kx\ \sin lx\ dx$

問題 3.5 (Lv.4)

c_0, a_k, b_k を実数とし, 三角多項式 $f(x)=c_0+\displaystyle\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)$ を考える.

$c_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\,dx,\ a_k=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\cos kx\ dx,\ b_k=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\sin kx\ dx$ を示せ.

問題 3.6 (Lv.5)

a, b は実数 $(a<b)$, $f_1(x)=1$ (定数関数), $f_2(x)=\begin{cases}1 & (x\text{ が有理数のとき})\\ 0 & (x\text{ が無理数のとき})\end{cases}$ とする.

(1) $\displaystyle\int_{a}^{b}f_1(x)\,dx=b-a$ となることを定積分 (リーマン積分) の定義に従って示せ.

(2) 定積分 (リーマン積分) としての $\displaystyle\int_{a}^{b}f_2(x)\,dx$ の値は定まらないことを示せ.

[ルベーグ積分と呼ばれる積分方式では, $f_2(x)$ の積分値は 0 の形で定まる.]

問題 3.1 (解答)

(1) $\displaystyle\int_{-1}^{2}(x^4-1)\,dx = \Big[\,\frac{1}{5}x^5 - x\,\Big]_{-1}^{2} = \Big(\frac{32}{5}-2\Big)-\Big(-\frac{1}{5}+1\Big) = \frac{18}{5}$

(2) $\displaystyle\int_{0}^{1}\frac{1}{2x-3}\,dx = \Big[\,\frac{1}{2}\log|2x-3|\Big]_{0}^{1} = \frac{1}{2}\log 1 - \frac{1}{2}\log 3 = -\frac{1}{2}\log 3 \quad \big(\log 1 = 0\big)$

(3) $\displaystyle\int_{-1}^{1}\frac{1}{x^2+1}\,dx = \Big[\,\mathrm{Tan}^{-1}x\,\Big]_{-1}^{1} = \mathrm{Tan}^{-1}1 - \mathrm{Tan}^{-1}(-1) = \frac{\pi}{4} - \Big(-\frac{\pi}{4}\Big) = \frac{\pi}{2}$

(4) $\displaystyle\int_{2}^{3}\frac{1}{(x-1)^3}\,dx = \int_{2}^{3}(x-1)^{-3}\,dx = \Big[-\frac{1}{2}(x-1)^{-2}\Big]_{2}^{3} = \Big(-\frac{1}{8}\Big)-\Big(-\frac{1}{2}\Big) = \frac{3}{8}$

(5) $\displaystyle\int_{1}^{3}\frac{x^2}{x+1}\,dx = \int_{1}^{3}\frac{(x+1)(x-1)+1}{x+1}\,dx = \int_{1}^{3}\Big(x-1+\frac{1}{x+1}\Big)dx$ (割り算)

$\displaystyle = \Big[\,\frac{1}{2}(x-1)^2 + \log|x+1|\Big]_{1}^{3} = (2+\log 4) - (0+\log 2) = 2+\log 2$

(6) $\displaystyle\int_{2}^{4}\frac{1}{x^2+x}\,dx = \int_{2}^{4}\frac{(x+1)-x}{x(x+1)}\,dx = \int_{2}^{4}\Big(\frac{1}{x}-\frac{1}{x+1}\Big)dx$ (部分分数分解)

$\displaystyle = \Big[\log|x| - \log|x+1|\Big]_{2}^{4} = (\log 4 - \log 5) - (\log 2 - \log 3) = \log\frac{6}{5}$

問題 3.2 (解答)

(1) 部分積分 $\displaystyle\int_{1}^{3}(x)'\log x\ dx = \Big[x\log x\Big]_{1}^{3} - \int_{1}^{3}x\,(\log x)'\,dx$ により,

$\displaystyle\int_{1}^{3}\log x\,dx = \Big[\,x\log x\Big]_{1}^{3} - \int_{1}^{3}x\,\frac{1}{x}\,dx = (3\log 3 - \log 1) - \int_{1}^{3}1\,dx$

$\displaystyle = 3\log 3 - \Big[\,x\,\Big]_{1}^{3} = 3\log 3 - (3-1) = 3\log 3 - 2$

(2) $\displaystyle\int_{0}^{\log 2}e^{2x}\,dx = \Big[\,\frac{1}{2}e^{2x}\Big]_{0}^{\log 2} = \frac{1}{2}(e^{2\log 2}-e^0) = \frac{1}{2}(e^{\log 4}-1) = \frac{1}{2}(4-1) = \frac{3}{2}$

(3) $\displaystyle\int_{0}^{\pi}\sin^2 x\ dx = \int_{0}^{\pi}\Big(\frac{1}{2}-\frac{1}{2}\cos 2x\Big)dx = \Big[\,\frac{1}{2}x - \frac{1}{4}\sin 2x\,\Big]_{0}^{\pi} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

(4) 置換 $t=\log x$ を用いると, $\dfrac{dt}{dx} = \dfrac{1}{x}$ から $\dfrac{1}{x}\,dx = dt$

x	1	$\cdots$	e
t	0	$\cdots$	1

$\displaystyle\int_{1}^{e}\frac{\log x}{x}\,dx = \int_{1}^{e}(\log x)\frac{1}{x}\,dx = \int_{0}^{1}t\,dt = \Big[\,\frac{1}{2}\,t^2\Big]_{0}^{1} = \frac{1}{2} - 0 = \frac{1}{2}$

(5) 部分積分 $\displaystyle\int_{0}^{1}x(-e^{-x})'\,dx = \Big[x(-e^{-x})\Big]_{0}^{1} - \int_{0}^{1}(x)'(-e^{-x})\,dx$ により,

$\displaystyle\int_{0}^{1}\frac{x}{e^x}\,dx = \int_{0}^{1}xe^{-x}\,dx = \Big[x(-e^{-x})\Big]_{0}^{1} - \int_{0}^{1}1\cdot(-e^{-x})\,dx = \Big[-\frac{x}{e^x}\Big]_{0}^{1} + \int_{0}^{1}e^{-x}\,dx$

$\displaystyle = \Big\{\Big(-\frac{1}{e}\Big)-0\Big\} + \Big[-e^{-x}\Big]_{0}^{1} = -\frac{1}{e} + \Big\{\Big(-\frac{1}{e}\Big)-(-1)\Big\} = 1 - \frac{2}{e}$

(6) 置換 $t=\tan x$ を用いると, $\dfrac{dt}{dx} = \dfrac{1}{\cos^2 x}$ から $\dfrac{1}{\cos^2 x}\,dx = dt$

x	$-\frac{\pi}{4}$	$\cdots$	$\frac{\pi}{3}$
t	-1	$\cdots$	$\sqrt{3}$

$\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\tan x}{\cos^2 x}\,dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{3}}(\tan x)\frac{1}{\cos^2 x}\,dx = \int_{-1}^{\sqrt{3}}t\,dt = \Big[\,\frac{1}{2}\,t^2\Big]_{-1}^{\sqrt{3}} = \frac{3}{2} - \frac{1}{2} = 1$

問題 3.3 (解答)

(1) $\displaystyle\int_0^3 \sqrt{x+1}\,dx = \int_0^3 (x+1)^{\frac{1}{2}}dx = \Big[\,\frac{2}{3}(x+1)^{\frac{3}{2}}\,\Big]_0^3 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$

(2) 置換 $t = x^2+2$ を用いると，$\dfrac{dt}{dx} = 2x$ から $2x\,dx = dt$ $\quad\begin{array}{c|ccc} x & 0 & \cdots & \sqrt{6} \\ \hline t & 2 & \cdots & 8 \end{array}$

$$\int_0^{\sqrt{6}} x\sqrt{x^2+2}\,dx = \int_0^{\sqrt{6}} \frac{1}{2}\sqrt{x^2+2}\cdot 2x\,dx = \int_2^8 \frac{1}{2}\sqrt{t}\,dt$$
$$= \int_2^8 \frac{1}{2}\,t^{\frac{1}{2}}dt = \Big[\,\frac{1}{2}\cdot\frac{2}{3}\,t^{\frac{3}{2}}\,\Big]_2^8 = \frac{16}{3}\sqrt{2} - \frac{2}{3}\sqrt{2} = \frac{14}{3}\sqrt{2}$$

(3) $\displaystyle\int_{-1}^3 \sqrt{x^2-2x+1}\,dx = \int_{-1}^3 \sqrt{(x-1)^2}\,dx = \int_{-1}^3 |x-1|\,dx$
$$= \int_{-1}^1 |x-1|\,dx + \int_1^3 |x-1|\,dx = \int_{-1}^1 (-x+1)\,dx + \int_1^3 (x-1)\,dx$$
$$= \Big[-\frac{1}{2}(x-1)^2\Big]_{-1}^1 + \Big[\,\frac{1}{2}(x-1)^2\Big]_1^3 = \big(0-(-2)\big) + \big(2-0\big) = 4$$

(4) $\displaystyle\int_0^3 \frac{1}{\sqrt{4-x}}\,dx = \int_0^3 (4-x)^{-\frac{1}{2}}dx = \Big[\,(-1)\cdot 2\,(4-x)^{\frac{1}{2}}\,\Big]_0^3 = (-2)-(-4) = 2$

(5) $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \Big[\,\mathrm{Sin}^{-1}\frac{x}{2}\,\Big]_1^{\sqrt{3}} = \mathrm{Sin}^{-1}\frac{\sqrt{3}}{2} - \mathrm{Sin}^{-1}\frac{1}{2} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

(6) 置換 $t = x-1$ を用いると，$\dfrac{dt}{dx} = 1$ から $dx = dt$ $\quad\begin{array}{c|ccc} x & 1 & \cdots & \frac{3}{2} \\ \hline t & 0 & \cdots & \frac{1}{2} \end{array}$

$$\int_1^{\frac{3}{2}} \frac{1}{\sqrt{2x-x^2}}\,dx = \int_1^{\frac{3}{2}} \frac{1}{\sqrt{1-(x-1)^2}}\,dx = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-t^2}}\,dt$$
$$= \Big[\,\mathrm{Sin}^{-1}\,t\,\Big]_0^{\frac{1}{2}} = \mathrm{Sin}^{-1}\frac{1}{2} - \mathrm{Sin}^{-1}0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

問題 3.4 (解答)

$\begin{array}{l} \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \end{array}$ から $\begin{array}{l} \cos\alpha\cos\beta = \dfrac{1}{2}\big\{\cos(\alpha-\beta)+\cos(\alpha+\beta)\big\} \\ \sin\alpha\,\sin\beta = \dfrac{1}{2}\big\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\big\} \end{array}$

(1) $\displaystyle\int_{-\pi}^{\pi} \cos kx\,\cos lx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\big\{\cos(k-l)x + \cos(k+l)x\big\}\,dx$ (積和公式)
$$= \begin{cases} \dfrac{1}{2}\Big[\dfrac{\sin(k-l)x}{k-l} + \dfrac{\sin(k+l)x}{k+l}\Big]_{-\pi}^{\pi} = \dfrac{1}{2}\big\{(0+0)-(0+0)\big\} = 0 & (k\neq l) \\ \dfrac{1}{2}\Big[\quad x \quad + \dfrac{\sin(k+l)x}{k+l}\Big]_{-\pi}^{\pi} = \dfrac{1}{2}\big\{(\pi+0)-(-\pi+0)\big\} = \pi & (k=l) \end{cases}$$

(2) $\displaystyle\int_{-\pi}^{\pi} \sin kx\,\sin lx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\big\{\cos(k-l)x - \cos(k+l)x\big\}\,dx$ (積和公式)
$$= \begin{cases} \dfrac{1}{2}\Big[\dfrac{\sin(k-l)x}{k-l} - \dfrac{\sin(k+l)x}{k+l}\Big]_{-\pi}^{\pi} = \dfrac{1}{2}\big\{(0-0)-(0-0)\big\} = 0 & (k\neq l) \\ \dfrac{1}{2}\Big[\quad x \quad - \dfrac{\sin(k+l)x}{k+l}\Big]_{-\pi}^{\pi} = \dfrac{1}{2}\big\{(\pi-0)-(-\pi-0)\big\} = \pi & (k=l) \end{cases}$$

(3) $f(x) = \cos kx\,\sin lx$ は $f(-x) = -f(x)$ となる奇関数より，$\displaystyle\int_{-\pi}^{\pi}\cos kx\,\sin lx\,dx = 0$

問題 3.5 (解答)

$f(x) = c_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$ なので, 定積分の線形性 (Th.3.3) を用いて,

$\int_{-\pi}^{\pi} f(x)\,dx = c_0 \int_{-\pi}^{\pi} 1\,dx + \sum_{k=1}^{n} \left\{ a_k \int_{-\pi}^{\pi} \cos kx\,dx + b_k \int_{-\pi}^{\pi} \sin kx\,dx \right\}$ や次の 2 式を得る.

$$\int_{-\pi}^{\pi} f(x)\cos lx\,dx = c_0 \int_{-\pi}^{\pi} \cos lx\,dx + \sum_{k=1}^{n} \left\{ a_k \int_{-\pi}^{\pi} \cos kx \cos lx\,dx + b_k \int_{-\pi}^{\pi} \sin kx \cos lx\,dx \right\}$$

$$\int_{-\pi}^{\pi} f(x)\sin lx\,dx = c_0 \int_{-\pi}^{\pi} \sin lx\,dx + \sum_{k=1}^{n} \left\{ a_k \int_{-\pi}^{\pi} \cos kx \sin lx\,dx + b_k \int_{-\pi}^{\pi} \sin kx \sin lx\,dx \right\}$$

グラフの図形的対称性などを見て, または, 原始関数 x, $\frac{\sin kx}{k}$, $-\frac{\cos kx}{k}$ で計算して,

$\int_{-\pi}^{\pi} 1\,dx = 2\pi$ (積分区間の幅), $\int_{-\pi}^{\pi} \cos kx\,dx = 0$, $\int_{-\pi}^{\pi} \sin kx\,dx = 0$ が得られて,

また, 問題 3.4 の定積分の値により, 総和のほとんどの項は 0 になって消えるので,

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi c_0,\ \int_{-\pi}^{\pi} f(x)\cos lx\,dx = \pi a_l,\ \int_{-\pi}^{\pi} f(x)\sin lx\,dx = \pi b_l \quad (l = 1, \cdots, n)$$

よって, 題意の関係式が従う. (上記の後半の 2 式は総和の $k = l$ の所の項が残った形)

$\left[\text{フーリエ解析では, 無限和の三角級数 } c_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \text{ を扱う.}\right]$

問題 3.6 (解答)

分割点 $x_0, x_1, \cdots, x_n$ $(a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b)$ による分割 Δ を考えて,

分割された各小区間から代表点 p_i $(x_{i-1} \leqq p_i \leqq x_i)$ を任意に選ぶ. $(i = 1, \cdots, n)$

分割 Δ と代表点の組 $\boldsymbol{p} = (p_1, \cdots, p_n)$ に対し, $f(x)$ のリーマン和を $R_\Delta(f, \boldsymbol{p})$ とする.

(1) $f_1(x) = 1$ (定数関数) より, 代表点 p_iでの $f_1(x)$ の値 $f_1(p_i) = 1$ なので,

$$R_\Delta(f_1, \boldsymbol{p}) = \sum_{i=1}^{n} f_1(p_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} 1 \cdot (x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1})$$
$$= (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) = x_n - x_0 = b - a$$

どの分割や代表点に対しても, リーマン和の値は $b - a$ になるので,

分割を限りなく細かくした際の極限は, 常に値 $b - a$ に収束する.

よって, リーマン和の極限の値 $b - a$ として定積分の値が定まる.

(2) 各小区間の代表点 p_iを有理数から選んだ場合は, $f_2(p_i) = 1$ だから,

$$R_\Delta(f_2, \boldsymbol{p}) = \sum_{i=1}^{n} f_2(p_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} 1 \cdot (x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1}) = b - a$$

各小区間の代表点 p_iを無理数から選んだ場合は, $f_2(p_i) = 0$ だから,

$$R_\Delta(f_2, \boldsymbol{p}) = \sum_{i=1}^{n} f_2(p_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} 0 \cdot (x_i - x_{i-1}) = \sum_{i=1}^{n} 0 = 0 + \cdots + 0 = 0$$

有理数の代表点を選んで分割を細かくした際のリーマン和の極限は $b - a$ となり,

無理数の代表点を選んで分割を細かくした際のリーマン和の極限は 0 となる.

分割や代表点の取り方に依存して極限の値が異なるので, 定積分は定まらない.