問題 1.1 (Lv.1)

次の不定積分を求めよ.

(1) $\displaystyle\int (x^2+1)\,dx$ (2) $\displaystyle\int \frac{x^4-1}{x}\,dx$ (3) $\displaystyle\int \frac{1}{x^2}\,dx$ (4) $\displaystyle\int x\sqrt{x}\;dx$

(5) $\displaystyle\int e^{2x}\,dx$ (6) $\displaystyle\int \frac{e^x}{e^x-1}\,dx$ (7) $\displaystyle\int \cos\frac{x}{3}\,dx$ (8) $\displaystyle\int \frac{\cos x}{\sin x}\,dx$

問題 1.2 (Lv.2)

次の不定積分を求めよ.

(1) $\displaystyle\int x^2\log x\,dx$ (2) $\displaystyle\int xe^{2x}\,dx$ (3) $\displaystyle\int x^2\sin x\;dx$

(4) $\displaystyle\int \frac{(\log x)^2}{x}\,dx$ (5) $\displaystyle\int xe^{x^2}dx$ (6) $\displaystyle\int \sin^3 x\,\cos x\;dx$

問題 1.3 (Lv.3)

次の不定積分を求めよ.

(1) $\displaystyle\int x\log(x^2+2)\,dx$ (2) $\displaystyle\int \frac{e^x}{e^{2x}+4}\,dx$ (3) $\displaystyle\int e^{2x}\cos x\;dx$

問題 1.4 (Lv.3)

次の不定積分を求めよ.

(1) $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx$ (2) $\displaystyle\int \frac{x+1}{\sqrt{x^2+2x}}\,dx$ (3) $\displaystyle\int \sqrt{2-x^2}\;dx$

問題 1.5 (Lv.3)

次の不定積分を求めよ.

(1) $\displaystyle\int \cos^2 x\;dx$ (2) $\displaystyle\int \cos^3 x\;dx$ (3) $\displaystyle\int \cos^4 x\;dx$ (4) $\displaystyle\int \cos^5 x\;dx$

問題 1.6 (Lv.4)

n は自然数とし, 不定積分 $\displaystyle\int \cos^n x\;dx$ について考える.

(1) $\cos^n x$ について, 次の関係式が成り立つことを示せ.

$$\begin{cases} n=2m \quad (\text{偶数}) \Rightarrow \cos^n x = \dfrac{1}{2^{n-1}}\left(\dfrac{1}{2}{}_n\mathrm{C}_m + \displaystyle\sum_{k=1}^{m} {}_n\mathrm{C}_{m-k}\cos 2kx\right) \\ n=2m+1\,(\text{奇数}) \Rightarrow \cos^n x = \dfrac{1}{2^{n-1}}\displaystyle\sum_{k=0}^{m} {}_n\mathrm{C}_{m-k}\cos(2k+1)x \end{cases}$$

(2) 不定積分 $\displaystyle\int \cos^n x\;dx$ を求めよ.

問題 1.1 (解答)

(1) $\displaystyle\int (x^2+1)\,dx \left(=\int x^2\,dx+\int 1\,dx\right)=\frac{1}{3}x^3+x+C$

(2) $\displaystyle\int \frac{x^4-1}{x}\,dx=\int\!\left(x^3-\frac{1}{x}\right)dx=\frac{1}{4}x^4-\log|x|+C$

(3) $\displaystyle\int \frac{1}{x^2}\,dx=\int x^{-2}\,dx=\frac{1}{-1}x^{-1}+C=-\frac{1}{x}+C$

(4) $\displaystyle\int x\sqrt{x}\;dx=\int x^{\frac{3}{2}}\,dx=\frac{1}{\frac{5}{2}}x^{\frac{5}{2}}+C=\frac{2}{5}x^2\sqrt{x}+C$

(5) $\displaystyle\left(\frac{1}{2}e^{2x}\right)'=\frac{1}{2}\cdot 2e^{2x}=e^{2x}$ となるので, $\displaystyle\int e^{2x}\,dx=\frac{1}{2}e^{2x}+C$

(6) $\displaystyle\big(\log|f(x)|\big)'=\frac{f'(x)}{f(x)}$ だから, $\displaystyle\int\frac{e^x}{e^x-1}\,dx=\int\frac{(e^x-1)'}{e^x-1}\,dx=\log|\,e^x-1|+C$

(7) $\displaystyle\left(3\sin\frac{x}{3}\right)'=3\cdot\frac{1}{3}\cos\frac{x}{3}=\cos\frac{x}{3}$ となるので, $\displaystyle\int\cos\frac{x}{3}\,dx=3\sin\frac{x}{3}+C$

(8) $\displaystyle\big(\log|f(x)|\big)'=\frac{f'(x)}{f(x)}$ だから, $\displaystyle\int\frac{\cos x}{\sin x}\,dx=\int\frac{(\sin x)'}{\sin x}\,dx=\log|\sin x\,|+C$

問題 1.2 (解答)

(1) $\displaystyle\int x^2\log x\,dx\left(=\int\!\left(\frac{1}{3}x^3\right)'\log x\;dx=\frac{1}{3}x^3\log x-\int\frac{1}{3}x^3\,(\log x)'\,dx\ (\text{部分積分})\right)$

$\displaystyle=\frac{1}{3}x^3\log x-\int\frac{1}{3}x^3\,\frac{1}{x}\,dx=\frac{1}{3}x^3\log x-\int\frac{1}{3}x^2\,dx=\frac{1}{3}x^3\log x-\frac{1}{9}x^3+C$

(2) $\displaystyle\int xe^{2x}\,dx\left(=\int x\left(\frac{1}{2}e^{2x}\right)'dx=x\,\frac{1}{2}e^{2x}-\int(x)'\,\frac{1}{2}e^{2x}\,dx\ (\text{部分積分})\right)$

$\displaystyle=x\,\frac{1}{2}e^{2x}-\int 1\cdot\frac{1}{2}e^{2x}\,dx=\frac{1}{2}xe^{2x}-\frac{1}{2}\int e^{2x}\,dx=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$

(3) $\displaystyle\int x^2\sin x\;dx=x^2(-\cos x)-\int 2x\,(-\cos x)\,dx\ (\text{部分積分})$

$\displaystyle=-x^2\cos x+2\int x\cos x\,dx=-x^2\cos x+2\left\{x\sin x-\int 1\cdot\sin x\,dx\right\}\ (\text{部分積分})$

$\displaystyle=-x^2\cos x+2x\sin x-2\int\sin x\;dx=-x^2\cos x+2x\sin x+2\cos x+C$

(4) 置換 $t=\log x$ を用いると, $\displaystyle\frac{dt}{dx}=\frac{1}{x}$ から $\displaystyle\frac{1}{x}\,dx=dt$ だから,

$\displaystyle\int\frac{(\log x)^2}{x}\,dx=\int(\log x)^2\,\frac{1}{x}\,dx=\int t^2\,dt=\frac{1}{3}\,t^3+C=\frac{1}{3}(\log x)^3+C$

(5) 置換 $t=x^2$ を用いると, $\displaystyle\frac{dt}{dx}=2x$ から $2x\,dx=dt$ だから,

$\displaystyle\int xe^{x^2}dx=\int\frac{1}{2}e^{x^2}\,2x\,dx=\int\frac{1}{2}e^t\,dt=\frac{1}{2}e^t+C=\frac{1}{2}e^{x^2}+C$

(6) 置換 $t=\sin x$ を用いると, $\displaystyle\frac{dt}{dx}=\cos x$ から $\cos x\;dx=dt$ だから,

$\displaystyle\int\sin^3x\;\cos x\;dx=\int t^3\,dt=\frac{1}{4}\,t^4+C=\frac{1}{4}\sin^4x+C$

問題 1.3 (解答)

(1) $\displaystyle\int x\log(x^2+2)\,dx = \frac{1}{2}x^2\log(x^2+2) - \int \frac{1}{2}x^2\,\frac{2x}{x^2+2}\,dx$ (部分積分)

$\displaystyle = \frac{1}{2}x^2\log(x^2+2) - \int \frac{x^3}{x^2+2}\,dx = \frac{1}{2}x^2\log(x^2+2) - \int \frac{x(x^2+2)-2x}{x^2+2}\,dx$

$\displaystyle = \frac{1}{2}x^2\log(x^2+2) - \int\left(x - \frac{2x}{x^2+2}\right)dx = \frac{1}{2}x^2\log(x^2+2) - \frac{1}{2}x^2 + \log(x^2+2) + C$

(2) 置換 $t=e^x$ を用いると, $\dfrac{dt}{dx}=e^x$ から $e^x\,dx = dt$ だから,

$\displaystyle\int \frac{e^x}{e^{2x}+4}\,dx = \int \frac{1}{(e^x)^2+4}\,e^x\,dx = \int \frac{1}{t^2+4}\,dt$ となる.

$\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\mathrm{Tan}^{-1}\frac{x}{a} + C$ (Th.1.3) より, $\displaystyle\int \frac{1}{t^2+4}\,dt = \frac{1}{2}\mathrm{Tan}^{-1}\frac{t}{2} + C$

したがって, $\displaystyle\int \frac{e^x}{e^{2x}+4}\,dx = \frac{1}{2}\mathrm{Tan}^{-1}\frac{e^x}{2} + C$ を得る.

(3) $\displaystyle\int e^{2x}\cos x\;dx = e^{2x}\sin x - \int 2e^{2x}\sin x\;dx$ (部分積分)

$\displaystyle = e^{2x}\sin x - \left\{2e^{2x}(-\cos x) - \int 4e^{2x}(-\cos x)\,dx\right\}$ (部分積分)

$\displaystyle = e^{2x}\sin x + 2e^{2x}\cos x - 4\int e^{2x}\cos x\;dx$ により, 積分の関係式を得る.

関係式を変形して, $\displaystyle\int e^{2x}\cos x\;dx = \frac{1}{5}e^{2x}\sin x + \frac{2}{5}e^{2x}\cos x + C$ が従う.

問題 1.4 (解答)

(1) 置換 $t=\sqrt{x-1}$ を用いる, $x=t^2+1$ より, $\dfrac{dx}{dt}=2t$ で $dx = 2t\,dt$ だから,

$\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{t^2+1}{t}\,2t\,dt = \int (2t^2+2)\,dt = \frac{2}{3}\,t^3 + 2t + C$

$\displaystyle = \frac{2}{3}\left(\sqrt{x-1}\right)^3 + 2\sqrt{x-1} + C \;\left(= \frac{2}{3}(x+2)\sqrt{x-1} + C\right)$

(2) 置換 $t=x^2+2x$ を用いると, $\dfrac{dt}{dx}=2x+2$ から $(2x+2)\,dx = dt$ だから,

$\displaystyle\int \frac{x+1}{\sqrt{x^2+2x}}\,dx = \int \frac{1}{2\sqrt{x^2+2x}}\,(2x+2)\,dx = \int \frac{1}{2\sqrt{t}}\,dt$

$\displaystyle = \int \frac{1}{2}\,t^{-\frac{1}{2}}\,dt = \frac{1}{2}\cdot 2\,t^{\frac{1}{2}} + C = \sqrt{t} + C = \sqrt{x^2+2x} + C$

(3) $\displaystyle\int \sqrt{2-x^2}\;dx = \int 1\cdot\sqrt{2-x^2}\;dx = x\sqrt{2-x^2} - \int x\,\frac{-2x}{2\sqrt{2-x^2}}\,dx$ (部分積分)

$\displaystyle = x\sqrt{2-x^2} + \int \frac{x^2}{\sqrt{2-x^2}}\,dx = x\sqrt{2-x^2} + \int \frac{2-(2-x^2)}{\sqrt{2-x^2}}\,dx$

$\displaystyle = x\sqrt{2-x^2} + 2\int \frac{1}{\sqrt{2-x^2}}\,dx - \int \sqrt{2-x^2}\;dx$ により, 積分の関係式を得る.

$\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \mathrm{Sin}^{-1}\frac{x}{a} + C$ (Th.1.3) より, $\displaystyle\int \frac{1}{\sqrt{2-x^2}}\,dx = \mathrm{Sin}^{-1}\frac{x}{\sqrt{2}} + C$

以上のことから, $\displaystyle\int \sqrt{2-x^2}\;dx = \frac{1}{2}x\sqrt{2-x^2} + \mathrm{Sin}^{-1}\frac{x}{\sqrt{2}} + C$ が従う.

問題 1.5 (解答)

(1) $\displaystyle\int \cos^2 x\ dx = \int\Big(\frac{1}{2}+\frac{1}{2}\cos 2x\Big)dx = \frac{1}{2}x+\frac{1}{4}\sin 2x + C$

(2) $\displaystyle\cos^3 x = \cos^2 x\ \cos x = \Big(\frac{1}{2}+\frac{1}{2}\cos 2x\Big)\cos x = \frac{1}{2}\cos x + \frac{1}{2}\cos 2x\ \cos x$

$\displaystyle\cos 2x\ \cos x = \frac{1}{2}\cos x + \frac{1}{2}\cos 3x$ (積和公式) を用いて, 式を整理して,

$$\int \cos^3 x\ dx = \int\Big(\frac{3}{4}\cos x+\frac{1}{4}\cos 3x\Big)dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C$$

(3) $\displaystyle\cos^4 x = \cos^3 x\ \cos x = \Big(\frac{3}{4}\cos x+\frac{1}{4}\cos 3x\Big)\cos x = \frac{3}{4}\cos^2 x + \frac{1}{4}\cos 3x\ \cos x$

$\displaystyle= \frac{3}{4}\Big(\frac{1}{2}+\frac{1}{2}\cos 2x\Big)+\frac{1}{4}\Big(\frac{1}{2}\cos 2x+\frac{1}{2}\cos 4x\Big)$ から式を整理して,

$$\int \cos^4 x\ dx = \int\Big(\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\Big)dx = \frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x + C$$

(4) $\displaystyle\cos^5 x = \cos^4 x\ \cos x = \Big(\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\Big)\cos x$

$\displaystyle= \frac{3}{8}\cos x+\frac{1}{2}\cos 2x\ \cos x+\frac{1}{8}\cos 4x\ \cos x$

$\displaystyle= \frac{3}{8}\cos x+\frac{1}{2}\Big(\frac{1}{2}\cos x+\frac{1}{2}\cos 3x\Big)+\frac{1}{8}\Big(\frac{1}{2}\cos 3x+\frac{1}{2}\cos 5x\Big)$ から,

$$\int \cos^5 x\ dx = \int\Big(\frac{5}{8}\cos x+\frac{5}{16}\cos 3x+\frac{1}{16}\cos 5x\Big)dx$$
$$= \frac{5}{8}\sin x+\frac{5}{48}\sin 3x+\frac{1}{80}\sin 5x + C$$

問題 1.6 (解答)

(1) オイラーの公式 $e^{i\theta} = \cos\theta + i\sin\theta$ (§14 - 4 を参照) を用いて示す.

(オイラーの公式を用いない場合は, 数学的帰納法で示すことができる.)

$\displaystyle\cos^n x = (\cos x)^n = \Big(\frac{e^{ix}+e^{-ix}}{2}\Big)^n = \frac{1}{2^n}\sum_{r=0}^{n}{}_n\mathrm{C}_r (e^{ix})^{n-r}(e^{-ix})^r$ (2 項展開)

$$= \frac{1}{2^n}\sum_{r=0}^{n}{}_n\mathrm{C}_r\, e^{i(n-2r)x} = \frac{1}{2^n}\sum_{r=0}^{n}{}_n\mathrm{C}_r\big\{\cos(n-2r)x + i\ \sin(n-2r)x\big\}$$

$\cos^n x$ は実数値だから, $\displaystyle\cos^n x = \frac{1}{2^n}\sum_{r=0}^{n}{}_n\mathrm{C}_r\cos(n-2r)x$ が従うので,

$$n\text{が偶数} \Rightarrow\ \cos^n x = \frac{1}{2^n}\begin{pmatrix} {}_n\mathrm{C}_0\cos nx + {}_n\mathrm{C}_1\cos(n-2)x+\cdots \\ \cdots + {}_n\mathrm{C}_{\frac{n-2}{2}}\cos 2x + {}_n\mathrm{C}_{\frac{n}{2}} + {}_n\mathrm{C}_{\frac{n+2}{2}}\cos(-2x)+\cdots \\ \cdots + {}_n\mathrm{C}_{n-1}\cos\big(-(n-2)x\big) + {}_n\mathrm{C}_n\cos(-nx) \end{pmatrix}$$

$$n\text{が奇数} \Rightarrow\ \cos^n x = \frac{1}{2^n}\begin{pmatrix} {}_n\mathrm{C}_0\cos nx + {}_n\mathrm{C}_1\cos(n-2)x+\cdots \\ \cdots + {}_n\mathrm{C}_{\frac{n-1}{2}}\cos x + {}_n\mathrm{C}_{\frac{n+1}{2}}\cos(-x)+\cdots \\ \cdots + {}_n\mathrm{C}_{n-1}\cos\big(-(n-2)x\big) + {}_n\mathrm{C}_n\cos(-nx) \end{pmatrix}$$

$r = 0, 1, 2, \cdots$ の項と $r = n,\ n-1,\ n-2, \cdots$ の項は等しいので, 題意の式が従う.

(2) $n = 2m$ (偶数) $\displaystyle\Rightarrow \int\cos^n x\ dx = \frac{{}_n\mathrm{C}_m}{2^n}x + \sum_{k=1}^{m}\frac{{}_n\mathrm{C}_{m-k}}{2^n k}\sin 2kx + C$

$n = 2m+1$ (奇数) $\displaystyle\Rightarrow \int\cos^n x\ dx = \sum_{k=0}^{m}\frac{{}_n\mathrm{C}_{m-k}}{2^{n-1}(2k+1)}\sin(2k+1)x + C$