A Characterization of the Esteban-Ray Polarization Measures

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Abstract

Esteban & Ray (1994) formalized the idea of polarization and developed a theory for its measurement. In their main theorem, they claimed that a class of polarization measures, called the *Esteban-Ray measures*, is characterized by a set of axioms that capture the idea of polarization. However, in this study, we show that the claim does not hold by presenting a counterexample. We amend their main theorem by strengthening the first axiom.

Keywords: Polarization, Esteban-Ray polarization measures

JEL: D31, D63

1. Introduction

A seminal work by Esteban & Ray (1994) (henceforth "ER") formalized the idea of *polarization* and developed a theory for its measurement. In their main theorem (ER, Theorem 1), they claimed that a class of allowable polarization measures, called the *Esteban-Ray measures*, is characterized by a set of axioms that capture the idea of polarization. However, by presenting a counterexample, we show that this claim does not hold. Further, we strengthen their "Axiom 1" so that the resulting characterization is reestablished.

The rest of this study is organized as follows. Section 2 introduces definitions and axioms. Section 3 presents our results. Proofs are relegated to Supplementary Appendix.

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2. Model and Axioms

Our model follows that of ER. Let $\mathbb{R}$ be the set of *attributes* (a basic perceptual variable is the natural logarithm of income). We consider population distributions on $\mathbb{R}$ with finite supports. That is, a distribution is denoted by a pair of n-dimensional vectors $(\boldsymbol{\pi}, \boldsymbol{y}) = ((\pi_1, \ldots, \pi_n), (y_1, \ldots, y_n)) \in \mathbb{R}^n_{++} \times \mathbb{R}^n$ for some $n \in \mathbb{N}$, where π_i is the population of individuals with attribute y_i and $y_i \neq y_j$ for distinct $i, j \in \{1, \ldots, n\}$. Let

$$\mathscr{D} \equiv \bigcup_{n=1}^{\infty} \mathbb{R}^n_{++} \times \left\{ \boldsymbol{y} \in \mathbb{R}^n : \ y_i \neq y_j \text{ for all distinct } i, j \in \{1, \ldots, n\} \right\}$$

be the set of distributions. A *polarization measure* is a function $P : \mathscr{D} \to \mathbb{R}_+$ that maps each distribution $(\boldsymbol{\pi}, \boldsymbol{y}) \in \mathscr{D}$ to a non-negative real number $P(\boldsymbol{\pi}, \boldsymbol{y}) \in \mathbb{R}_+$. ER's analysis focused on polarization measures that take the following functional form:

$$P\left((\boldsymbol{\pi}, \boldsymbol{y})\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i \pi_j \theta\left(\pi_i, |y_i - y_j|\right), \tag{1}$$

where θ is a function $\mathbb{R}^2_+ \to \mathbb{R}$ such that $\theta(\cdot, \cdot)$ is strictly increasing in the second argument (distance), continuous in each argument, $\theta(0, \cdot) = 0$, $\theta(\cdot, 0) = 0$, and $\theta(\pi_i, \delta) > 0$ for all $\pi_i > 0$ and $\delta > 0$.[1] An interpretation and background of this form is discussed in ER.

ER proposed three axioms that capture the idea of polarization. Figs. 1, 2 and 3 show an illustration of each axiom. Axiom 1 says that when a large mass exists at attribute 0, unifying two close masses increases polarization.

Axiom 1. *For any $p > 0$ and any $x > 0$, there exist $\varepsilon > 0$ and $\mu > 0$ such that, for any $y > x$ and any $q < p$ with $y - x < \varepsilon$ and $0 < q < \mu p$,*

$$P\left(((p, q, q), (0, x, y))\right) < P\left(\left((p, 2q), \left(0, \frac{x+y}{2}\right)\right)\right).$$

[1]The assumption $\theta(0, \cdot) = 0$ is not imposed by ER. However, this assumption is necessary to deduce Equation (6) in their proof of Theorem 1.

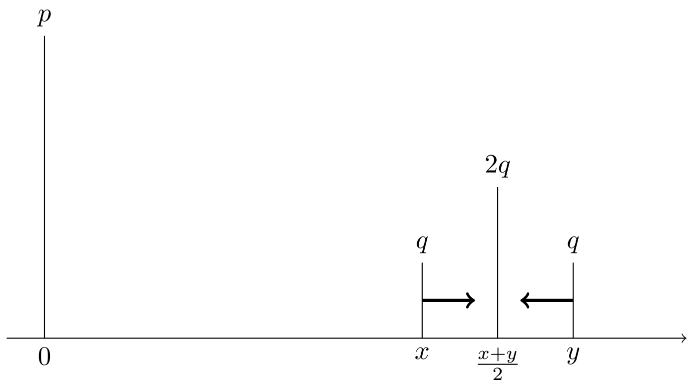

Figure 1: An Illustration of Axiom 1

Axiom 2 requires that when an intermediate mass gets closer to the right extreme mass, polarization increases.

Axiom 2. *For any* $p, q, r > 0$ *with* $p > r$*, any* $x, y > 0$ *with* $|y - x| < x < y$*, and any* $\Delta \in (0, y - x)$*,*

$$P\left(\left(p,q,r\right),\left(0,x,y\right)\right) < P\left(\left(p,q,r\right),\left(0,x+\Delta,y\right)\right).$$

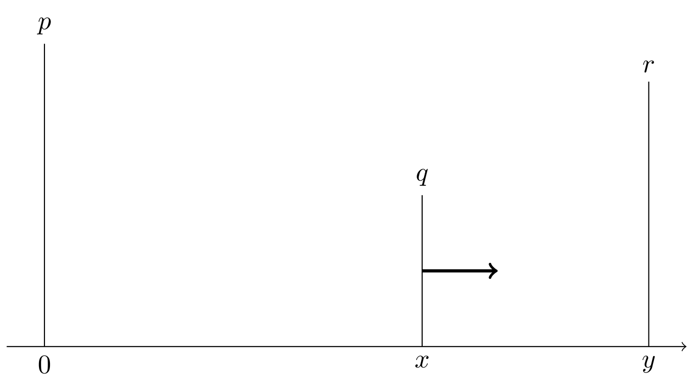

Figure 2: An Illustration of Axiom 2

Axiom 3 requires that if the population of an intermediate mass decreases, and if the population of left and right extreme masses increase equally, then polarization increases.

Axiom 3. *For any $p, q > 0$, any $x, y > 0$ with $x = y - x$, and any $\Delta \in (0, q/2)$,*

$$P\left((p, q, p), (0, x, y)\right) < P\left((p + \Delta, q - 2\Delta, p + \Delta), (0, x, y)\right).$$

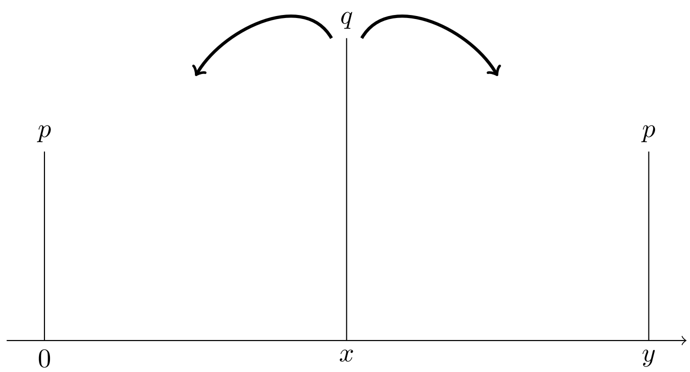

Figure 3: An Illustration of Axiom 3

Condition H is a homotheticity property requiring any bilateral comparison to be invariant to the scale of population.

Condition H. *For any $(\boldsymbol{\pi}, \boldsymbol{y}), (\boldsymbol{\pi}', \boldsymbol{y}') \in \mathscr{D}$ and $\lambda > 0$, if $P(\boldsymbol{\pi}, \boldsymbol{y}) \geq P(\boldsymbol{\pi}', \boldsymbol{y}')$, then $P(\lambda\boldsymbol{\pi}, \boldsymbol{y}) \geq P(\lambda\boldsymbol{\pi}', \boldsymbol{y}')$.*

3. Main Results

3.1. Counterexamples

ER claimed that the class of the Esteban-Ray measures is characterized by Axioms 1–3 and Condition H.

Claim 1 (ER, Theorem 1)**.** *A polarization measure P^* of the family defined in* (1) *satisfies Axioms* 1, 2, *and* 3, *and Condition H if and only if it is of the form*

$$P^*(\boldsymbol{\pi}, \boldsymbol{y}) = K \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j |y_i - y_j| \tag{2}$$

for some constants $K > 0$ and $\alpha \in (0, \alpha^*]$ where $\alpha^* \simeq 1.6$.[2]

[2] For the definition of α^*, see ER's equation (2) and subsequent arguments on page 833.

We show that Claim 1 does not hold because Axiom 1 is too weak to characterize the class of the Esteban-Ray measures. In their proof of Claim 1, ER showed that Axiom 1 and the continuity of $\theta(\cdot,\cdot)$ in Equation (1) imply that $\theta(\pi_i,\cdot)$ must be *locally concave* with respect to the distance; that is, for each $x > 0$, there exists $\varepsilon > 0$ such that $\theta(\pi_i,\cdot)$ is concave on a half-open interval $[x, x+\varepsilon)$. Then, they claim that this *local concavity* of θ implies that $\theta(\pi_i,\cdot)$ must be *concave* on $\mathbb{R}_+$. However, this claim is not correct. To see this, fix any $c \in \mathbb{R}_{++}$ and let $\hat{f} : \mathbb{R}_+ \to \mathbb{R}_+$ be such that for each $\delta \in \mathbb{R}_+$,

$$\hat{f}(\delta) = \begin{cases} K\delta & \text{if } \delta < c, \\ K'\delta - (K' - K)c & \text{if } \delta \geq c, \end{cases}$$

where $0 < K < K'$. Then, a convex piecewise linear function $\theta(\pi_i, \delta) = \pi_i^\alpha \hat{f}(\delta)$ is not *concave* on $\mathbb{R}_+$, but simply satisfies the *local concavity*.[3] Therefore, Axiom 1 cannot exclude this convex piecewise linear function. In fact, a polarization measure with convex piecewise linear function $\theta(\pi_i, \delta) = \pi_i^\alpha \hat{f}(\delta)$ satisfies Axioms 1–3 and Condition H, and hence "Claim 1" does not hold.

Proposition 1 (Counterexample to Claim 1)**.** *Let $\hat{P} : \mathscr{D} \to \mathbb{R}_+$ be such that*

$$\hat{P}(\boldsymbol{\pi}, \boldsymbol{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j \hat{f}(|y_i - y_j|), \tag{3}$$

where $\alpha \in (0, \alpha^*]$. Then, $\hat{P}$ satisfies Axioms 1, 2, *and* 3, *and Condition H, but does not take the form of* (2)*.*

Proof. See Supplementary Appendix A. □

We have two remarks on this proposition; (i) our counterexample function $\hat{P}$ and an Esteban-Ray measure generate different orderings; (ii) a set of counterexamples is *dense* in a set of polarization measures with standard properties. These discussions are relegated to Supplementary Appendix B.

[3] To confirm this, consider any $x > 0$. If $x \geq c$, then for any $\varepsilon > 0$, $\theta(\pi_i,\cdot)$ is concave on the half-open interval $[x, x+\varepsilon)$. Conversely, if $x < c$, then by letting $\varepsilon = \frac{1}{2}(c - x) > 0$, $\theta(\pi_i,\cdot)$ becomes concave on the half-open interval $[x, x+\varepsilon)$.

3.2. Modification

We provide a modified axiom that excludes the convex piecewise linear functions, and amend a characterization of the Esteban-Ray measures. For any $x > 0$ and $\varepsilon > 0$, let $B(x, \varepsilon) \equiv \{z \in \mathbb{R}_+ : |x - z| < \varepsilon\}$.

Axiom 1′. *For any $p > 0$ and any $x > 0$, there exist $\varepsilon > 0$ and $\mu > 0$ such that for any $a, b \in B(x, \varepsilon)$ and $q < p$ with $0 < q < \mu p$,*

$$P\left(\left((p, q, q), (0, a, b)\right)\right) < P\left(\left((p, 2q), \left(0, \frac{a+b}{2}\right)\right)\right).$$

In contrast to Axiom 1, Axiom 1′ implies that $\theta(\pi_i, \cdot)$ must satisfy the following stronger version of *local concavity* with respect to the second argument: for each $x > 0$, there exists $\varepsilon > 0$ such that $\theta(p, \cdot)$ is concave on an open interval $(x - \varepsilon, x + \varepsilon)$. Since convex piecewise linear function $\theta(\pi_i, \delta) = \pi_i^\alpha \hat{f}(\delta)$ with kink point c is not concave on any open interval $(c - \varepsilon, c + \varepsilon)$ with $\varepsilon > 0$, $\hat{\theta}$ does not satisfy this *strong local concavity.* This is why Axiom 1′ can exclude the convex piecewise linear functions.

Though Axiom 1′ and Axiom 1 are mathematically quite different, Axiom 1’ has almost the same interpretation as the original compelling axiom. In this sense, this modification does not change the spirit of the original axiom.[4] Now we can restore an axiomatic foundation of the Esteban-Ray polarization measures.

Proposition 2. *A polarization measure P^* of the family defined in (1) satisfies Axioms 1′, 2, and 3, and Condition H if and only if it is of the form*

$$P^*(\boldsymbol{\pi}, \boldsymbol{y}) = K \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j |y_i - y_j| \tag{1}$$

for some constants $K > 0$ and $\alpha \in (0, \alpha^*]$ where $\alpha^* \simeq 1.6$.

Proof. See Supplementary Appendix C. □

[4] Assuming differentiability of measures is one way to exclude convex piecewise linear functions. However, differentiability is irrelevant to the original Axiom 1. Moreover, we cannot find a normative reason for adopting differentiability as an axiom in this context. For these reasons, we do not assume differentiability to characterize the Esteban-Ray measures.

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