ON LARGE MARKET ASYMPTOTICS FOR SPATIAL PRICE COMPETITION MODELS

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Abstract. We study the problem of weak instruments in demand estimation of spatial price competition models by Pinkse, Slade, and Brett (2002) (hereafter, PSB). Product characteristics are employed as price instruments and have correlation with prices through the markup. We investigate whether product characteristics hold their identification power as the number of product grows in analogy with Armstrong (2016) who studied random coefficient discrete choice models. The conventional weak instruments asymptotic analysis is not applicable in PSB's model because it requires series estimation in their semiparametric two-stage least square estimator so that both the numbers of endogenous regressors and instruments grow as the number of products grows. We provide two asymptotic results that indicate lack of consistency of PSB's estimator due to weak instruments.

1. Introduction

Economists utilize instruments to solve the simultaneity problem in demand estimation. Since the influential study by Berry, Levinsohn, and Pakes (1995) (hereafter, BLP) for differentiated product markets, many papers adopt product characteristics as price instruments, which correlate with prices through the markup, especially through the market share of each product. However, Armstrong (2016) showed that under the large market asymptotics, where share of each product decays fast enough as the number of products grows, these instruments for BLP's model may lose their identifying power and lead to inconsistent estimators.

This paper studies a weak instrument problem of demand estimation for spatial price competition models by Pinkse, Slade, and Brett (2002) (hereafter, PSB) where product characteristics are employed as price instruments. In PSB's model, consumer demands are in a product space, not in a product characteristic space, and they can consume more than one good. Since BLP takes a random coefficient discrete choice approach, the demand model of PSB is considerably different from that of BLP. However, by rewriting the markup formula induced by the Bertrand equilibrium play, one can see that this formula is a function of the demand function of each product instead of the market share in BLP. Since the market size is considered to be finite in PSB's setup, we expect that the demand function collapses to zero as the number of products grows. Therefore, the instruments in PSB interact with price in a similar way to BLP.

To clarify this point, we investigate PSB's semiparametric two-stage least squares estimator $\hat{\theta}$ whose estimation error $\hat{\theta} - \theta$ is characterized as

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i w_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i v_i\right) =: A_n^{-1} b_n,$$

where z_i, w_i, v_i are vectors of instruments, regressors containing series expansion terms and exogenous variables, and regression and approximation errors, respectively. Notice that we cannot apply the conventional weak instruments asymptotics in Staiger and Stock (1997) since the dimensions of A_n and b_n are growing. Our first result characterizes the stochastic orders of each element of A_n and b_n. We find that these are not degenerate, and b_n may diverge if the number of expansion terms grows at a slower rate. Our second result provides an inconsistency result of $\hat{\theta}$ given a high-level assumption on the maximum eigenvalue of $A_n'A_n$. The proofs of these results are presented in the supplementary material.

2. Model and estimator

Our model follows that of PSB. There are n sellers of a differentiated product. For simplicity, we assume that each firm sells one product. Let q_i, p_i, and y_i be the demand, price, and product characteristic for product $i \in \{1, \ldots, n\}$. The demand function for product i is given by

$$q_i(p, y) = a_i + \sum_{j=1}^{n} (b_{ij} p_j + c_{ij} y_j),$$

where $p = (p_1, \ldots, p_n)'$, $y = (y_1, \ldots, y_n)'$, and $(\{a_i\}, \{b_{ij}\}, \{c_{ij}\})$ are parameters to be estimated. Suppose firms play the Bertrand pricing game given rival prices, i.e., firm i chooses p_i to solve

$$\max_{p_i} (p_i - \gamma MC_i) q_i(p, y) - F_i, \tag{1}$$

where MC_i and F_i are firm i's marginal and fixed costs. The best response function of firm i is

$$p_i = -\frac{1}{2\beta_{ii}} \left(a_i - b_{ii}\gamma MC_i + \sum_{j \neq i} b_{ij} p_j + \sum_{j=1}^{n} c_{ij} y_j \right).$$

PSB estimated this best response function by employing a semiparametric approach. Let x_i be a d_x-vector of MC_i, finite subset of y, and other exogenous demand and cost variables. Also let $\{e_\ell(\cdot)\}$ be a sequence of basis functions, $\{d_{ij}\}$ be measures of proximity of firms i and j, and $\tilde{\psi}_{i\ell} = \sum_{j \neq i} e_\ell(d_{ij}) p_j$. Based on this notation, PSB's semiparametric model is written as

$$p_i = \sum_{\ell=1}^{\infty} \tilde{\alpha}_\ell \tilde{\psi}_{i\ell} + x_i'\beta + u_i = \psi_i'\alpha + x_i'\beta + v_i, \tag{2}$$

where $v_i = r_i + u_i$, $r_i = \sum_{\ell=L_n+1}^{\infty} \tilde{\alpha}_\ell \tilde{\psi}_{i\ell}$, $\alpha = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{L_n})'$, and

$$\psi_i = \left(\sum_{j \neq i} e_1(d_{ij}) p_j, \sum_{j \neq i} e_2(d_{ij}) p_j, \ldots, \sum_{j \neq i} e_{L_n}(d_{ij}) p_j \right)'.$$

The number of endogenous regressors L_n grows with n. Letting $w_i = (\psi_i', x_i')'$ and $\theta = (\alpha', \beta')'$, this model can be concisely written as $p_i = w_i'\theta + v_i$.

For this model, PSB proposed to estimate θ by the (semiparametric) two-stage least squares based on K_n-dimensional vector of instruments z_i. For simplicity, we focus on the case of $K_n = L_n + d_x$. Following PSB, we adopt transformed variables of x_i as instruments for p_i. Let $z_i = g(x_i)$ be a K_n-dimensional vector-valued function of x_i. Then the semiparametric instrumental variable estimator for θ is written as

$$\hat{\theta} = \left(\sum_{i=1}^{n} z_i w_i'\right)^{-1} \sum_{i=1}^{n} z_i p_i. \tag{3}$$

This paper is concerned with the limiting behaviors of the estimator $\hat{\theta}$ when the number of products n increases to infinity under suitable conditions for the price competition models. To achieve consistency of $\hat{\theta}$, it is critical to guarantee sufficiently strong correlations between p_i's contained in the regressors w_i and x_i generating the instruments z_i. To understand the nature of the problem, observe that the first-order condition of (1) can be written as

$$p_i = \gamma MC_i - \frac{q_i(p, y)}{b_{ii}} + u_i. \tag{4}$$

Here MC_i is assumed to be an exogenous regressor included in the model (2). Thus we need to guarantee sufficiently strong correlation between the instruments $z_i = g(x_i)$ and markup $q_i(p, y)/b_{ii}$. However, in the current setup, it is common to assume that the market size is finite, i.e., $\lim_{n\to\infty} \sum_{i=1}^{n} q_i(p, y) < \infty$, which implies that $q_i(p, y)$ decays to zero as the number of products n grows. Therefore, the markup $q_i(p, y)/b_{ii}$ may not have enough variations to yield enough correlations with the instruments z_i. This phenomenon is thoroughly studied in Armstrong (2016) for the BLP model on differentiated product demands. Indeed he conjectured emergence of such a weak instruments problem in the PSB model (see, p. 1964 of Armstrong, 2016), and the next section verifies his conjecture.

3. Large market asymptotics

We now study asymptotic properties of the instrumental variables estimator $\hat{\theta}$ in (3) under the large market asymptotics. Based on the existing literature, we impose the following assumptions on the demand function $q_i(p, y)$ and market size.

Assumption Q. *(i)* $\operatorname{plim}_{n\to\infty} \sum_{i=1}^{n} q_i(y, p) < \infty$. *(ii)* $\sqrt{n} \max_{1\le i\le n} q_i(y, p)/b_{ii}^1 \xrightarrow{p} 0$.

Assumption Q (i) says that the market size $\sum_{i=1}^{n} q_i(y, p)$ remains finite as the number of products n diverges to infinity. This assumption implies that the demand $q_i(y, p)$ for each product i decays to 0. Assumption Q (ii) requires that the decay rate of $q_i(y, p)$ normalized by b_{ii}^1 should be faster than $n^{-1/2}$ uniformly over i. An analogous assumption is employed by Armstrong (2016, Theorem 1) for the BLP model.

We also impose some regularity conditions on the series expansion in (2).

Assumption S. *(i)* $\sup_{1\le i\le n, \ell\in\mathbb{N}} \sum_{j\ne i} |e_\ell(d_{ij})| = O(1)$. *(ii)* $\max_{1\le i\le n} \sum_{j\ne i} e_\ell(d_{ij})^2 = O(1)$ *for each* $\ell \in \mathbb{N}$. *(iii)* $\sup_{\ell\in\mathbb{N}} |\tilde{\alpha}_\ell \ell^\lambda| < \infty$ *for some* $\lambda > 1$.

Assumptions S (i) and (iii) are also employed by PSB. Assumptions S (i) and (ii) are on the basis functions $\{e_\ell(d)\}_{\ell\in\mathbb{N}}$. When the supports of $\{e_\ell(d)\}_{\ell\in\mathbb{N}}$ are finite, these assumptions require that the number of non-zero elements of $e_\ell(d_{ij})$ for $i, j = 1, \ldots n$ should be finite. If the supports of $\{e_\ell(d)\}_{\ell\in\mathbb{N}}$ are infinite, Assumptions S (i) and (ii) require that $e_\ell(d)$ should decay fast enough as $d \to \infty$. Assumption S (iii) can be understood as a smoothness condition for the function to be approximated by the series expansion. Intuitively, larger λ is associated with a smoother function.

From (2) and (3), the estimation error of $\hat{\theta}$ can be written as

$$\hat{\theta} - \theta = \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} z_i w_i' \right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} z_i v_i \right) =: A_n^{-1} b_n. \tag{5}$$

There are two notable features in this expression. First, the matrix A_n is normalized by $n^{-1/2}$, instead of n^{-1} for the case of the conventional instrumental variable regression with strong instruments. Such normalization is employed by Staiger and Stock (1997) for the weak instruments asymptotics. As indicated in the last section, in our setup, the markup $q_i(p, y)/b_{ii}$ (and thus w_i) may not have enough correlations with the instruments z_i, and hence we adopt the analogous normalization. Second, in contrast to the conventional or weak instruments asymptotic analysis in Staiger and Stock (1997), A_n is a $K_n \times K_n$ matrix and b_n is a $K_n \times 1$ vector so that both components have growing dimensions. In other words, we need to deal with the problem of weak instruments for semiparametric or series estimators, where not only the number of instruments K_n but also the number of endogenous regressors L_n grow with n. Such an analysis is a substantial challenge in the econometrics literature (see, e.g., Freyberger, 2017, and Han, 2020).

Although full development of the asymptotic theory for (5) by extending the random matrix theory is beyond the scope of this paper, we can present two theoretical results to indicate lack of consistency of the estimator $\hat{\theta}$. The first proposition characterizes the stochastic orders of the elements of A_n and b_n.

Proposition 1. *Suppose $\{p_i, x_i, z_i\}_{i=1}^n$ is an i.i.d. triangular array, where each element has the finite fourth moments, and Assumptions Q and S hold true. Then each element of A_n is of order $O_p(1)$, and each element of b_n is of order $O_p(\max\{1, \sqrt{n} L_n^{1-\lambda}\})$.*

This proposition says that the elements in A_n and b_n do not degenerate, and b_n may even diverge when the L_n (and thus K_n) grows at a slower rate. Although this result is not enough to characterize the stochastic order of the whole vector $\hat{\theta} - \theta = A_n^{-1} b_n$, we can observe analogous behaviors of the corresponding terms of A_n and b_n for the case of the weak instruments asymptotics in Staiger and Stock (1997).

Additionally we provide a lack of consistency result in terms of the Euclidean norm $||\hat{\theta} - \theta||$ under some high level assumption on the matrix A_n. Let $\lambda_{\max}(A)$ be the maximum eigenvalue of a matrix A.

Proposition 2. *Suppose $\{p_i, x_i, z_i\}_{i=1}^n$ is an i.i.d. triangular array, where each element has the finite fourth moments, and Assumptions Q and S hold true. If $\lambda_{\max}(A_n A_n') \leq C_n$ with probability approaching one (w.p.a.1) and $n L_n^{2-2\lambda}/C_n \to 0$ for some C_n, then $||\hat{\theta} - \theta|| \xrightarrow{p} +\infty$.*

This proposition provides sufficient conditions to induce inconsistency of the estimator $\hat{\theta}$. The additional condition $n L_n^{2-2\lambda}/C_n \to 0$ is analogous to Assumption (viii) in PSB (which requires $n L_n^{2-2\lambda}/\zeta_n \to 0$ for a sequence $\{\zeta_n\}$ associated with the minimum eigenvalue of $\sum_{i=1}^n z_i w_i'$). In our setup, it is beyond the scope of this paper to characterize the upper bound C_n for the maximum eigenvalue of the product matrix $A_n A_n'$ with growing dimension, which requires further developments of the random matrix theory.

To illustrate this point, suppose that A_n is a $K_n \times K_n$ matrix of independent standard normal random variables. Then Johnstone (2001, Theorem 1.1) showed that

$$\frac{\lambda_{\max}(A_n A_n') - \mu_n}{\sigma_n} \xrightarrow{d} \text{Tracy-Widom law of order 1},$$

where $\mu_n = k_n^2$ and $\sigma_n = k_n\{(K_n - 1)^{-1/2} + K_n^{-1/2}\}^{1/3}$ for $k_n = (K_n - 1)^{1/2} + K_n^{1/2}$. Thus, in this case, the upper bound C_n can be set as K_n. By $K_n = L_n + d_x$, the additional condition in Proposition 2 will be $nL_n^{1-2\lambda} \to 0$, which is satisfied when L_n grows fast enough and/or λ is large enough.

Finally, we mention how to test the null hypothesis $H_0 : \theta = \theta_0$. In the conventional weak identification framework, several asymptotically valid test statistics are proposed. For example, the S-statistic in Stock and Wright (2000) takes the form

$$S_{K_n}(\theta) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_i(\theta)' \left[\frac{1}{n} \sum_{i=1}^{n} g_i(\theta) g_i(\theta)' \right]^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_i(\theta),$$

where $g_i(\theta) = z_i(p_i - w_i'\theta)$. If $K_n = K$ is fixed, Stock and Wright (2000, Theorem 2) implies that $S_K(\theta)$ converges in distribution to χ_K^2. When $K_n \to \infty$, a central limit theorem yields

$$\frac{S_{K_n}(\theta) - K_n}{\sqrt{2K_n}} \xrightarrow{d} N(0, 1),$$

which may be used for constructing a valid test for the whole parameter θ. However, we cannot use this statistic if we are interested in a subset of parameters like finite dimensional parameter β in (2). Developing a general framework for inference in this situation is beyond the scope and left for future research.

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