A Characterization of the Esteban-Ray Polarization Measures: Supplementary Appendix

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Appendix A: Proof of Proposition 1. (Supplementary Appendix, Not for Publication)

Condition H. Take any $(\boldsymbol{\pi}, \boldsymbol{y}), (\boldsymbol{\pi}', \boldsymbol{y}') \in \mathscr{D}$ and any $\lambda > 0$. Suppose that

$$\hat{P}(\boldsymbol{\pi}, \boldsymbol{y}) \geq \hat{P}(\boldsymbol{\pi}', \boldsymbol{y}').$$

Then, by the definition of $\hat{P}$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j \hat{f}(|y_i - y_j|) \geq \sum_{i=1}^{n'} \sum_{j=1}^{n'} \pi_i'^{1+\alpha} \pi_j' \hat{f}(|y_i' - y_j'|).$$

Since $\lambda > 0$, this equation implies that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda\pi_i)^{1+\alpha} (\lambda\pi_j) \hat{f}(|y_i - y_j|) \geq \sum_{i=1}^{n'} \sum_{j=1}^{n'} (\lambda\pi_i')^{1+\alpha} (\lambda\pi_j') \hat{f}(|y_i' - y_j'|).$$

Therefore,

$$\hat{P}(\lambda\boldsymbol{\pi}, \boldsymbol{y}) \geq \hat{P}(\lambda\boldsymbol{\pi}', \boldsymbol{y}').$$

Axiom 1. Take any $p > 0$ and $x > 0$. Let us show that there exist $\varepsilon > 0$ and $\mu > 0$ such that for any $y > x$ with $y - x < \varepsilon$ and any $q > 0$ with $q < \mu p$,

$$\hat{P}\left((p, q, q), (0, x, y)\right) < \hat{P}\left((p, 2q), \left(0, \frac{x+y}{2}\right)\right).$$

Case 1 ($x < c$). Let $\varepsilon > 0$ and $\mu > 0$ be such that

$$\begin{aligned}\varepsilon &= \min\left\{\frac{1}{2}(c-x), (2^\alpha - 1)x\right\}, \\ \mu &= \frac{1}{2}.\end{aligned}$$

Take any $y > x$ with $y - x < \varepsilon$ and any $q > 0$ with $q < \mu p$. Note that $x, y, \frac{x+y}{2}, |y-x| < c$. Therefore,

$$\hat{P}\left((p,q,q),(0,x,y)\right) = K\left(\left(p^{1+\alpha}q + q^{1+\alpha}p\right)(x+y) + 2q^{2+\alpha}|y-x|\right),$$

and

$$\hat{P}\left((p,2q), \left(0, \frac{x+y}{2}\right)\right) = K\left(2p^{1+\alpha}q + 2^{1+\alpha}q^{1+\alpha}p\right)\left(\frac{x+y}{2}\right).$$

Moreover, since $|y-x| < \varepsilon$, $2q < p$ and $\varepsilon \le (2^\alpha - 1)x$, it follows that

$$2q|y-x| < \varepsilon p \le (2^\alpha - 1)xp < (2^\alpha - 1)(x+y)p.$$

Then, by the same argument as ER (line 14 on page 837),

$$\hat{P}\left((p,q,q),(0,x,y)\right) < \hat{P}\left((p,2q), \left(0, \frac{x+y}{2}\right)\right).$$

Case 2 ($x \ge c$). Let $\varepsilon > 0$ and $\mu > 0$ be such that

$$\begin{aligned}\varepsilon &= \min\left\{\frac{1}{2}c, (2^\alpha - 1)(K'x - (K'-K)c)\right\}, \\ \mu &= \frac{1}{2K}.\end{aligned}$$

Take any $y > x$ with $y - x < \varepsilon$ and any $q > 0$ with $q < \mu p$. Note that $x, y, \frac{x+y}{2} \ge c$ and $|y-x| < c$. Therefore,

$$\hat{P}\left((p,q,q),(0,x,y)\right) = \left(p^{1+\alpha}q + q^{1+\alpha}p\right)\left(K'x + K'y - 2(K'-K)c\right) + 2q^{2+\alpha}K|y-x|,$$

and

$$\begin{aligned}&\hat{P}\left((p,2q), \left(0, \frac{x+y}{2}\right)\right) = \left(2p^{1+\alpha}q + 2^{1+\alpha}q^{1+\alpha}p\right)\left(\frac{x+y}{2}K' - (K'-K)c\right) \\ &= \left(p^{1+\alpha}q + q^{1+\alpha}p\right)\left(K'x + K'y - 2(K'-K)c\right) + (2^\alpha - 1)q^{1+\alpha}p(K'x + K'y - 2(K'-K)c).\end{aligned}$$

Then, $\hat{P}\left((p,q,q),(0,x,y)\right) < \hat{P}\left((p,2q), \left(0, \frac{x+y}{2}\right)\right)$ if and only if

$$2qK|y-x| < (2^\alpha - 1)p\left(K'x + K'y - 2(K'-K)c\right).$$

Since $|y - x| < \varepsilon$, $2Kq < p$ and $\varepsilon \leq (2^\alpha - 1)(K'x - (K' - K)c)$, we have

$$2qK|y - x| < \varepsilon p \leq (2^\alpha - 1)p\big(K'x - (K' - K)c\big) < (2^\alpha - 1)p\big(K'x + K'y - 2(K' - K)c\big).$$

Therefore,

$$\hat{P}\left((p, q, q), (0, x, y)\right) < \hat{P}\left((p, 2q), \left(0, \frac{x+y}{2}\right)\right).$$

Hence $\hat{P}$ satisfies Axiom 1.

Axiom 2. Fix any $p, q, r > 0$ with $p > r$, and any $x < y$ with $x > y - x$. Let us show that for any $\Delta \in (0, y - x)$,

$$\hat{P}\left((p, q, r), (0, x, y)\right) < \hat{P}\left((p, q, r), (0, x + \Delta, y)\right).$$

Take any $\Delta \in (0, y - x)$. For simplicity, we write

$$\begin{aligned} \hat{P} &\equiv \hat{P}\left((p, q, r), (0, x, y)\right), \\ \hat{P}_\Delta &\equiv \hat{P}\left((p, q, r), (0, x + \Delta, y)\right). \end{aligned}$$

Then,

$$\begin{aligned} \hat{P} &= \hat{f}(x)\big(p^{1+\alpha}q + q^{1+\alpha}p\big) + \hat{f}(y - x)\big(q^{1+\alpha}r + r^{1+\alpha}q\big) + \hat{f}(y)\big(p^{1+\alpha}r + r^{1+\alpha}p\big), \\ \hat{P}_\Delta &= \hat{f}(x + \Delta)\big(p^{1+\alpha}q + q^{1+\alpha}p\big) + \hat{f}(y - x - \Delta)\big(q^{1+\alpha}r + r^{1+\alpha}q\big) + \hat{f}(y)\big(p^{1+\alpha}r + r^{1+\alpha}p\big). \end{aligned}$$

We shall show $\hat{P}_\Delta - \hat{P} > 0$.

Since $\hat{f}$ is convex and $x > y - x$, the slope of $\hat{f}$ at x is larger than that at $y - x$; that is,

$$\hat{f}(x + \Delta) - \hat{f}(x) \geq \hat{f}(y - x) - \hat{f}(y - x - \Delta).$$

Therefore,

$$\begin{aligned} &\hat{P}_\Delta - \hat{P} \\ &\geq \left(\hat{f}(x + \Delta) - \hat{f}(x)\right)\left(p^{1+\alpha}q + q^{1+\alpha}p\right) - \left(\hat{f}(y - x) - \hat{f}(y - x - \Delta)\right)\left(q^{1+\alpha}r + r^{1+\alpha}q\right) \\ &\geq \left(\hat{f}(y - x) - \hat{f}(y - x - \Delta)\right)\left(\left(p^{1+\alpha}q + q^{1+\alpha}p\right) - \left(q^{1+\alpha}r + r^{1+\alpha}q\right)\right). \end{aligned}$$

Hence, $\hat{P}_\Delta - \hat{P}$ is positive whenever $p > r$ since $\hat{f}(y - x) - \hat{f}(y - x - \Delta) > 0$.

Axiom 3. Fix any $p, q > 0$, and any $x, y > 0$ with $x = y - x$. Let us show that for any $\Delta \in (0, q/2)$,

$$\hat{P}\left((p, q, p), (0, x, y)\right) < \hat{P}\left((p + \Delta, q - 2\Delta, p + \Delta), (0, x, y)\right).$$

Take any $\Delta \in (0, q/2)$. For simplicity, we write

$$\hat{P} \equiv \hat{P}\left((p, q, p), (0, x, y)\right),$$
$$\hat{P}_\Delta \equiv \hat{P}\left((p+\Delta, q-2\Delta, p+\Delta), (0, x, y)\right).$$

Then,

$$\hat{P} = 2\hat{f}(d)\big(p^{1+\alpha}q + q^{1+\alpha}p\big) + 2\hat{f}(2d)\big(p^{2+\alpha}\big),$$
$$\hat{P}_\Delta = 2\hat{f}(d)\big((p+\Delta)^{1+\alpha}(q-2\Delta) + (q-2\Delta)^{1+\alpha}(p+\Delta)\big) + 2\hat{f}(2d)\big((p+\Delta)^{2+\alpha}\big).$$

We shall show $\hat{P}_\Delta - \hat{P} > 0$.

Case 1 ($2d < c$). In this case,

$$\hat{P} = K\Big(2d\big(p^{1+\alpha}q + q^{1+\alpha}p\big) + 4d\big(p^{2+\alpha}\big)\Big),$$
$$\hat{P}_\Delta = K\Big(2d\big((p+\Delta)^{1+\alpha}(q-2\Delta) + (q-2\Delta)^{1+\alpha}(p+\Delta)\big) + 4d\big((p+\Delta)^{2+\alpha}\big)\Big).$$

Then, by the same argument as ER (paragraph of verifying axiom 3 on page 837), it follows that $\hat{P}_\Delta - \hat{P} > 0$.

Case 2 ($d < c$ and $c < 2d$). By definition of $\hat{f}$, it follows that $\hat{f}(2d) \geq 2Kd$. Then, since $(p+\Delta)^{2+\alpha} \geq p^{2+\alpha}$,

$$2\hat{f}(2d)(p+\Delta)^{2+\alpha} - 2\hat{f}(2d)p^{2+\alpha} \geq 4Kd(p+\Delta)^{2+\alpha} - 4Kdp^{2+\alpha}.$$

Therefore,

$$P_\Delta - P \geq K\Big(2d\big((p+\Delta)^{1+\alpha}(q-2\Delta) + (q-2\Delta)^{1+\alpha}(p+\Delta)\big) + 4d\big((p+\Delta)^{2+\alpha}\big)\Big) - K\Big(2d\big(p^{1+\alpha}q + q^{1+\alpha}p\big) + 4d\big(p^{2+\alpha}\big)\Big). \quad \text{(A.1)}$$

Then, by the same argument as Case 1, the right hand side of (A.1) is positive. Hence $\hat{P}_\Delta - \hat{P} > 0$.

Case 3 ($c < d$). Let

$$A \equiv 2\big(p^{1+\alpha}q + q^{1+\alpha}p\big) + 4\big(p^{2+\alpha}\big),$$
$$A_\Delta \equiv 2\big((p+\Delta)^{1+\alpha}(q-2\Delta) + (q-2\Delta)^{1+\alpha}(p+\Delta)\big) + 4\big((p+\Delta)^{2+\alpha}\big),$$
$$B \equiv 2\big(p^{1+\alpha}q + q^{1+\alpha}p\big) + 2\big(p^{2+\alpha}\big),$$
$$B_\Delta \equiv 2\big((p+\Delta)^{1+\alpha}(q-2\Delta) + (q-2\Delta)^{1+\alpha}(p+\Delta)\big) + 2\big((p+\Delta)^{2+\alpha}\big).$$

Then, we can compute that

$$\hat{P} = K'dA - (K'-K)cB,$$
$$\hat{P}_\Delta = K'dA_\Delta - (K'-K)cB_\Delta.$$

Moreover, by the same argument as Case 1, it follows that $A_\Delta \geq A$. Therefore,

$$K'd\,(A_\Delta - A) \geq K'c\,(A_\Delta - A)\,,$$

and hence

$$\hat{P}_\Delta - \hat{P} \geq \Big(K'cA_\Delta - (K'-K)cB_\Delta\Big) - \Big(K'cA - (K'-K)cB\Big).$$

Therefore, it suffices to show that the derivative of the function

$$\hat{P}(\Delta) \equiv K'cA_\Delta - (K'-K)cB_\Delta,$$

evaluated at $\Delta = 0$, is non-negative and positive for all but at most one ratio $z \equiv p/q$. By a simple computation, this derivative is given by

$$\hat{P}'(\Delta) = q^{1+\alpha}\Big(2c(K'-K)(2+\alpha)z^{1+\alpha} - 4cK\varphi(z,\alpha)\Big), \tag{A.2}$$

where the function φ is defined by

$$\varphi(z,\alpha) = (1+\alpha)\left(z - \frac{z^\alpha}{2} - z^{1+\alpha}\right) - \frac{1}{2}.$$

Then, since $\alpha \in (1, \alpha^*]$, (A.2) is non-negative and is positive for all but at most one ratio z (see ER; equation (2) and subsequent arguments on page 833). Therefore, $\hat{P}$ satisfies Axiom 3. $\square$

Appendix B: Discussions on Proposition 1. (Supplementary Appendix, Not for Publication)

(i) Ordinal difference between a counterexample and ER's measure

Our counterexample and an Esteban-Ray measure generate different orderings. For example, let $(\boldsymbol{\pi}, \boldsymbol{y}) = ((1,1,1),(0,4,8))$ and $(\boldsymbol{\pi}', \boldsymbol{y}') = ((1,1,1),(0,1,7))$. Specify each parameter of $\hat{P}$ as $K = 1$, $K' = 10$ and $c = 4$; that is, let

$$P^*(\boldsymbol{\pi}, \boldsymbol{y}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i^{1+\alpha}\pi_j|y_i - y_j|,$$
$$\hat{P}(\boldsymbol{\pi}, \boldsymbol{y}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i^{1+\alpha}\pi_j \hat{f}(|y_i - y_j|),$$

where

$$\hat{f}\left(|y_i - y_j|\right) = \begin{cases} |y_i - y_j| & \text{if } |y_i - y_j| < 4, \\ 10|y_i - y_j| - 36 & \text{if } |y_i - y_j| \geq 4. \end{cases}$$

Then, for any $\alpha \in (1, \alpha^*]$,

$$P^*(\boldsymbol{\pi}, \boldsymbol{y}) = 32 > 28 = P^*(\boldsymbol{\pi}', \boldsymbol{y}'),$$

but

$$\hat{P}(\boldsymbol{\pi}, \boldsymbol{y}) = 104 < 118 = \hat{P}(\boldsymbol{\pi}', \boldsymbol{y}').$$

Hence, P^* and $\hat{P}$ yield different orderings.

(ii) Denseness of counterexamples

Many functions of the form of (1) satisfy Axioms 1–3 and Condition H, but do not take the form of (2). Indeed, let

$$F = \{f : \mathbb{R}_+ \to \mathbb{R}_+ \mid f \text{ is convex, strictly increasing, non-linear,}$$
$$\text{piecewise linear with discrete kink points, and } f(0) = 0.\}.$$

Then, for any $f \in F$, a function of the form (1)

$$P(\boldsymbol{\pi}, \boldsymbol{y}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i^{1+\alpha}\pi_j f(|y_i - y_j|)$$

satisfies Axioms 1–3 and Condition H but does not take the form of (2).

Moreover, let

$$G = \{g : \mathbb{R}_+ \to \mathbb{R}_+ \mid g \text{ is convex, strictly increasing, and } g(0) = 0\},$$

then F is dense in G with a standard metric ρ, defined as $\rho(f, g) = \sup_{x\in\mathbb{R}_+} |f(x) - g(x)|$. That is, for all $g \in G$ and any $\varepsilon > 0$, there exists $f \in F$, such that $\rho(f, g) < \varepsilon$. Denseness of F suggests that Axiom 1 does not work well to characterize the Esteban-Ray measures.

Proposition. *For any $f \in F$, a function of the form* (1)

$$P(\boldsymbol{\pi}, \boldsymbol{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j f(|y_i - y_j|)$$

where $\alpha \in (0, \alpha^*]$ satisfies Axioms 1, 2 and 3, and Condition H but does not take the form of (2)*. Moreover, F is dense in G.*

The first statement can be proven by the same way as Proposition 1. Here we show that F is dense in G. Consider any $g \in G$. Take any $\varepsilon > 0$.

Case 1 ($\boldsymbol{g}$ is non-linear). We will construct a piecewise linear function that uniformly approximates g. Consider the partition of $[0, \infty)$, $[0, \varepsilon), [\varepsilon, 2\varepsilon), [2\varepsilon, 3\varepsilon), \ldots$. Since g is strictly increasing and continuous[1], for each $k \in \{1, 2, \ldots\}$, there exists a unique real number $d_k \in \mathbb{R}_+$ with $d_k = g^{-1}(k\varepsilon)$. For example, $g(d_1) = \varepsilon$ and $g(d_2) = 2\varepsilon$. Define $d_0 = 0$. Note that $\mathbb{R}_+$ is partitioned by a family $[d_0, d_1), [d_1, d_2), [d_2, d_3), \ldots$. Let f be a piecewise linear function such that

$$f(x) = \begin{cases} \dfrac{\varepsilon}{d_1} x \ \text{ if } \ x \in [0, d_1), \\ \dfrac{\varepsilon}{d_2 - d_1}(x - d_1) + \varepsilon \ \text{ if } \ x \in [d_1, d_2), \\ \dfrac{\varepsilon}{d_3 - d_2}(x - d_2) + 2\varepsilon \ \text{ if } \ x \in [d_2, d_3), \\ \vdots \\ \dfrac{\varepsilon}{d_k - d_{k-1}}(x - d_{k-1}) + (k-1)\varepsilon \ \text{ if } \ x \in [d_{k-1}, d_k), \\ \vdots \end{cases}$$

That is, f is the piecewise linear function such that $f(d_k) = k\varepsilon = g(d_k)$ for each $k \in \{0, 1, 2, \ldots\}$. Obviously, $f \in F$.[2]

We show that g can be uniformly approximated by f. Consider any $x \in \mathbb{R}_+$. Since the family $[d_0, d_1), [d_1, d_2), [d_2, d_3), \ldots$ is a partition of $\mathbb{R}_+$, there exists $k \in \{1, 2, \ldots\}$ such that $x \in [d_{k-1}, d_k)$, and so

$$f(x) \in [(k-1)\varepsilon, k\varepsilon) \ \text{ and } \ g(x) \in [(k-1)\varepsilon, k\varepsilon).$$

Therefore, $|f(x) - g(x)| < \varepsilon$.

[1]Since g convex, strict increasing, and $g(0) = 0$, it is continuous on $[0, \infty)$.

[2]Since g is convex, $d_k - d_{k-1} > d_{k+1} - d_k$ for each $k \in \{1, 2, \ldots\}$. Therefore, $\frac{\varepsilon}{d_k - d_{k-1}}$ is increasing in k, that is, f is a convex piecewise linear function.

Case 2 (g is linear). Since g is linear, there exists $k > 0$ such that $g(x) = kx$ for any $x \in \mathbb{R}_+$. Fix any $c \in \mathbb{R}_{++}$. Let $\tilde{f}$ be a convex piecewise linear function such that

$$\tilde{f}(x) = \begin{cases} \frac{1}{c}\left(kc - \frac{1}{2}\varepsilon\right)x & \text{if } \ x < c, \\ kx - \frac{1}{2}\varepsilon & \text{if } \ x \geq c. \end{cases}$$

Obviously, $\tilde{f} \in F$. Then, $\tilde{f}(x) \in (kx - \varepsilon, kx)$ for any $x \in \mathbb{R}_+$. Therefore, $|\tilde{f}(x) - g(x)| < \varepsilon$. Hence, F is dense in G.

Appendix C: Proof of Proposition 2. (Supplementary Appendix, Not for Publication)

(Sufficiency.) We can show that P^* satisfies Axiom $1'$ as the same way in ER, so we omit the proof of this part.

(Necessity.) We show that Axiom $1'$ implies concavity of θ. since the proof of Proposition 2 is the same as that of Theorem 1 except for this point. Consider the distribution depicted in Axiom 1'. Initially, polarization is given by

$$P^1 \equiv pq[\theta(p,a) + \theta(p,b)] + pq[\theta(q,a) + \theta(q,b)] + 2q^2\theta(q, |b-a|),$$

and polarization after the distribution shifting is

$$P^2 \equiv 2pq\left[\theta\left(p, \frac{a+b}{2}\right)\right] + 2pq\left[\theta\left(2q, \frac{a+b}{2}\right)\right].$$

Axiom 1' implies that

$$\begin{aligned} 2p\left[\theta\left(p, \frac{a+b}{2}\right) + \theta\left(2q, \frac{a+b}{2}\right)\right] &> p[\theta(p,a) + \theta(p,b)] \\ &\quad + p[\theta(q,a) + \theta(q,b)] \\ &\quad + 2q\theta(q, |b-a|). \end{aligned}$$

Take the limit as $q \to 0$. Then, for any $x > 0$, there exists $\varepsilon > 0$ such that for any $a, b \in B(x, \varepsilon)$,

$$\theta\left(p, \frac{a+b}{2}\right) \geq \frac{\theta(p,a) + \theta(p,b)}{2}.$$

This means local mid-point concavity, but we show that it is sufficient for global and any convex-combination concavity; for any $a, b > 0$ and any $t \in [0,1]$,

$$\theta\left(p, ta + (1-t)b\right) \geq t\theta(p,a) + (1-t)\theta(p,b).$$

Suppose, by contradiction, that there exist $a, b > 0$ and $t \in [0, 1]$ such that

$$\theta\left(p, ta + (1-t)b\right) < t\theta(p, a) + (1-t)\theta(p, b). \tag{C.1}$$

For each $s \in [0, 1]$, define

$$h(s) = \theta\left(p, sa + (1-s)b\right) - s\theta(p, a) - (1-s)\theta(p, b).$$

Then h is locally mid-point concave. Indeed, since $\theta(p, \cdot)$ is locally mid-point concave at any point, $\theta(p, \cdot)$ is mid-point concave on $B(sa + (1-s)b, \varepsilon)$ for any $s \in [0, 1]$ and for some $\varepsilon > 0$. Moreover, $-s\theta(p, a) - (1-s)\theta(p, b) = (\theta(p, b) - \theta(p, a))s - \theta(p, b)$ is a straight line, which is globally concave. Since the sum of two concave functions is also concave, h is locally mid-point concave. Note that $h(1) = h(0) = 0$ and $h(t) < 0$ by (C.1).

Since h is continuous, by the extreme value theorem, h has the minimum value on $[0, 1]$. Let

$$m = \min\{h(s) : s \in [0, 1]\},$$
$$u = \max\{s \in [0, 1] : h(s) = m\}.$$

Then $h(s) > h(u)$ for every $s \in (u, 1]$ and $h(s') \geq h(u)$ for every $s' \in [0, u]$. Thus h is not locally concave on any epsilon ball with center u. Indeed, take any $\varepsilon' > 0$. Let $a' \in B(u, \varepsilon')$ with $a' > u$. Let $b' = u - |a' - u|$. Note that $a', b' \in B(u, \varepsilon')$, $\frac{1}{2}(a' + b') = u$, $h(a') > h(u)$, and $h(b') \geq h(u)$. This implies that

$$h\left(\frac{1}{2}(a' + b')\right) = h(u) < \frac{h(a') + h(b')}{2}.$$

This contradicts local mid-point concavity of h, as desired. Therefore, we have restored a claim: "$\theta(p, \cdot)$ must be concave" (line 11, page 835 of ER).