SUPPLEMENTARY MATERIAL FOR "ON LARGE MARKET ASYMPTOTICS FOR SPATIAL PRICE COMPETITION MODELS"

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Abstract. In this supplement, we provide proofs for Propositions 1 and 2 in the paper.

1. Proof of Proposition 1

We first consider the matrix A_n. Without loss of generality, we consider the $(1,1)$-element of A_n, say $A_n^{(1,1)}$. Also we assume $E[z_{1i}] = 0$ to simplify the presentation. By inserting the markup formula in (4) of the main paper, we can decompose

$$\begin{aligned} A_n^{(1,1)} &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} z_{1i} \sum_{j \neq i} e_1(d_{ij}) p_j \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j \neq i} z_{1i} e_1(d_{ij}) \frac{q_j}{b_{jj}^1} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j \neq i} z_{1i} e_1(d_{ij}) MC_j \gamma + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j \neq i} z_{1i} e_1(d_{ij}) u_j \\ &=: T_1 + T_2 + T_3. \end{aligned}$$

For T_1, Assumptions Q (ii) and S (i) and the law of large numbers imply

$$|T_1| \leq \left\{ \sqrt{n} \max_{1 \leq j \leq n} \frac{q_j}{b_{jj}^1} \right\} \left(\max_{1 \leq i \leq n} \sum_{j \neq i} |e_1(d_{ij})| \right) \frac{1}{n} \sum_{i=1}^{n} |z_{1i}| = o_p(1).$$

For T_2, observe that

$$\begin{aligned} E[T_2^2] &= \frac{\gamma^2}{n} \sum_{i=1}^{n} \sum_{i_1=1}^{n} \sum_{j \neq i} \sum_{j_1 \neq i_1} E[z_{1i} z_{1i_1} MC_j MC_{j_1}] e_1(d_{ij}) e_1(d_{i_1 j_1}) \\ &= \frac{\gamma^2}{n} \sum_{i=1}^{n} \sum_{j \neq i} \{E[z_{1i} MC_i] E[z_{1j} MC_j] + E[z_{1i}^2] E[MC_j^2]\} e_1(d_{ij})^2 \\ &\leq \frac{C_1}{n} \sum_{i=1}^{n} \sum_{j \neq i} e_1(d_{ij})^2 \leq C_1 \max_{1 \leq i \leq n} \sum_{j \neq i} e_1(d_{ij})^2 = O(1), \end{aligned}$$

for some $C_1 > 0$, where the inequality follows the assumption that z_{1i} and MC_i have the finite fourth moments, and the last equality follows from Assumption S (ii). Thus, Chebyshev's inequality implies $T_2 = O_p(1)$. For T_3, let $R_i = z_{1i} \sum_{j \neq i} e_1(d_{ij}) u_j$ so that $T_3 = n^{-1/2} \sum_{i=1}^{n} R_i$. Note that $E[R_i] = 0$,

$$E[R_i^2] = E[z_{1i}^2] \sum_{j \neq i} E[u_j^2] e_1(d_{1j})^2 = E[z_{1i}^2] E[u_i^2] \sum_{j \neq i} e_1(d_{1j})^2,$$

and

$$
\begin{aligned}
\mathrm{Cov}(R_1, R_2) &= \sum_{j_1 \neq 1} \sum_{j_2 \neq 2} E[z_{11} z_{12} u_{j_1} u_{j_2}] e_1(d_{1j_1}) e_1(d_{2j_2}) \\
&= \sum_{j_2 \neq 2} E[z_{11} z_{12} u_2 u_{j_2}] e_1(d_{12}) e_1(d_{2j_2}) + \sum_{j_1 \neq 1,2} \sum_{j_2 \neq 1,2} E[z_{11} z_{12} u_{j_1} u_{j_2}] e_1(d_{1j_1}) e_1(d_{2j_2}) \\
&= \sum_{j_2 \neq 2} E[z_{11} z_{12} u_2 u_{j_2}] e_1(d_{12}) e_1(d_{2j_2}) + \sum_{j_1 \neq 1,2} \sum_{j_2 \neq 1,2,j_1} E[z_{11} z_{12} u_{j_1} u_{j_2}] e_1(d_{1j_1}) e_1(d_{2j_2}) \\
&\quad + \sum_{j_1 \neq 1,2} E[z_{11} z_{12} u_{j_1}^2] e_1(d_{1j_1}) e_1(d_{1j_2}) \\
&= 0.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
Var(T_3) &= \frac{1}{n} \sum_{i=1}^n Var(R_i) = \frac{1}{n} \sum_{i=1}^n E[R_i^2] = \frac{1}{n} \sum_{i=1}^n E[z_{1i}^2] E[u_i^2] \sum_{j \neq i} e_1(d_{1j})^2 \\
&\leq C_2 \max_{1 \leq i \leq n} \sum_{j \neq i} e_1(d_{1j})^2 = O(1),
\end{aligned}
$$

for some $C_2 > 0$, where the last equality follows from Assumption S (ii). Now Chebyshev's inequality implies $T_3 = O_p(1)$. Combining these results, we obtain $A_n^{(1,1)} = O_p(1)$.

We next consider the vector b_n. Without loss of generality, we consider the first element of b_n, say

$$
b_n^{(1)} = \frac{1}{\sqrt{n}} \sum_{i=1}^n z_{1i} u_i + \frac{1}{\sqrt{n}} \sum_{i=1}^n z_{1i} r_i =: T_4 + T_5.
$$

For T_4, the i.i.d. and finite fourth moments assumptions guarantees

$$
E[T_4^2] = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n E[z_{1i} z_{1j} u_i u_j] = \frac{1}{n} \sum_{i=1}^n E[z_{1i}^2 u_i^2] = O(1).
$$

Thus, T_4 is $O_p(1)$. For T_5, note that $E[T_5^2] = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n E[z_{1i} z_{1j} r_i r_j]$ by the i.i.d. assumption, and thus we have

$$
\begin{aligned}
E[T_5^2] &= \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \sum_{\ell = L_n+1}^{\infty} \sum_{\ell' = L_n+1}^{\infty} \sum_{k \neq i} \sum_{k' \neq j} E[z_{1i} z_{1j} p_k p_{k'}] \tilde{\alpha}_\ell \tilde{\alpha}_{\ell'} e_\ell(d_{ik}) e_{\ell'}(d_{jk'}) \\
&\leq \frac{C_3}{n} \sum_{i=1}^n \sum_{j=1}^n \sum_{\ell = L_n+1}^{\infty} \sum_{\ell' = L_n+1}^{\infty} \sum_{k \neq i} \sum_{k' \neq j} |\tilde{\alpha}_\ell| |\tilde{\alpha}_{\ell'}| |e_\ell(d_{ik})| |e_{\ell'}(d_{jk'})| \\
&\leq C_3 n \left(\sum_{\ell = L_n+1}^{\infty} |\tilde{\alpha}_\ell| \right)^2 \left(\sup_{1 \leq i \leq n, \ell \in \mathbb{N}} \sum_{k \neq i} |e_\ell(d_{ik})| \right)^2 \\
&\leq C_4 n L_n^{2-2\lambda} \left(\sup_{1 \leq i \leq n, \ell \in \mathbb{N}} \sum_{k \neq i} |e_\ell(d_{ik})| \right)^2 = O(n L_n^{2-2\lambda}), \qquad (1)
\end{aligned}
$$

for some $C_3, C_4 > 0$, where the first inequality follows from the Cauchy-Schwarz inequality and finite fourth moments assumption, the third inequality follows from $\sum_{\ell = L_n+1}^{\infty} |\tilde{\alpha}_\ell| \leq$

$\sum_{\ell=L_n+1}^{\infty} C_5 \ell^{-\lambda} \leq C_5 L_n^{2-2\lambda}$ for some $C_5 > 0$ by using Assumption S (iii), and the last equality follows from Assumption S (i). Thus, Chebyshev's inequality implies $T_5 = O_p(\sqrt{n} L_n^{1-\lambda})$. Combining these results, we obtain $b_n^{(1)} = O_p(\max\{1, \sqrt{n} L_n^{1-\lambda}\})$.

2. Proof of Proposition 2

Let $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ be the maximum and minimum eigenvalues of a matrix A, respectively. It is sufficient to show that $\Pr\{(\hat{\theta}-\theta)'(\hat{\theta}-\theta) \leq M\} \to 0$ for each $M > 0$. Take any $M > 0$. Note that

$$(\hat{\theta}-\theta)'(\hat{\theta}-\theta) = b_n'(A_n^{-1})' A_n^{-1} b_n \geq \lambda_{\min}((A_n^{-1})' A_n^{-1}) b_n' b_n = \frac{b_n' b_n}{\lambda_{\max}(A_n A_n')},$$

where the last equality follows from

$$\lambda_{\min}((A^{-1})' A^{-1}) = \lambda_{\min}((AA')^{-1}) = \frac{1}{\lambda_{\max}(AA')},$$

for any invertible matrix A. Thus, we have

$$\begin{aligned}
&\Pr\{(\hat{\theta}-\theta)'(\hat{\theta}-\theta) \leq M\} \leq \Pr\left\{\frac{b_n' b_n}{\lambda_{\max}(A_n A_n')} \leq M\right\} \\
\leq\ & \Pr\left\{\frac{b_n' b_n}{\lambda_{\max}(A_n A_n')} \leq M,\ \lambda_{\max}(A_n A_n') \leq C_n\right\} + \Pr\{\lambda_{\max}(A_n A_n') > C_n\} \\
\leq\ & \Pr\{b_n' b_n \leq C_n M\} + o(1) \leq \frac{E[b_n' b_n]}{C_n M} + o(1),
\end{aligned}$$

where the third inequality follows from the assumption $\lambda_{\max}(A_n A_n') \leq C_n$ w.p.a.1, and the last inequality follows from Markov's inequality. By using the definition $b_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n z_i v_i = \frac{1}{\sqrt{n}} \sum_{i=1}^n z_i(r_i + u_i)$, we can decompose

$$\begin{aligned}
E[b_n' b_n] &= \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n E[z_i' z_j r_i r_j] + \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n E[z_i' z_j u_i u_j] + \frac{2}{n}\sum_{i=1}^n \sum_{j=1}^n E[z_i' z_j r_i u_j] \\
&=: T_1 + T_2 + 2T_3.
\end{aligned}$$

For T_1, similar arguments to (1) in the proof of Proposition 1 yield

$$\begin{aligned}
T_1 &= \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n E\left[z_i' z_j \left(\sum_{\ell=L_n+1}^{\infty} \tilde{\alpha}_\ell \sum_{h \neq i} e_\ell(d_{ih}) p_h\right)\left(\sum_{\ell=L_n+1}^{\infty} \tilde{\alpha}_\ell \sum_{k \neq j} e_\ell(d_{jk}) p_k\right)\right] \\
&\leq O(n)\left(\sum_{\ell=L_n+1}^{\infty} |\tilde{\alpha}_\ell|\right)^2 \left(\sup_{1 \leq i \leq n, \ell \in \mathbb{N}} \sum_{j \neq i} |e_\ell(d_{ij})|\right)^2 = O(nL_n^{2-2\lambda}),
\end{aligned}$$

where the inequality follows from the Cauchy-Schwarz inequality and finite fourth moments assumption, and the second equality follows from Assumptions S (i) and (iii).

For T_2, the i.i.d. assumption and Cauchy-Schwarz inequality imply

$$T_2 = \frac{1}{n}\sum_{i=1}^n E[z_i' z_i u_i^2] \leq \sqrt{E[||z_i||^4]}\sqrt{E[u_i^4]} = O(1).$$

For T_3, observe that

$$
\begin{aligned}
T_3 &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{h\neq i} E[z_i' z_j u_j p_h] \sum_{\ell=L_n+1}^{\infty} \tilde{\alpha}_\ell e_\ell(d_{ih}) \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i} E[z_i]' E[z_j u_j p_j] \sum_{\ell=L_n+1}^{\infty} \tilde{\alpha}_\ell e_\ell(d_{ij}) \\
&\leq \frac{C_1}{n}\sum_{i=1}^{n}\sum_{j\neq i} \sum_{\ell=L_n+1}^{\infty} |\tilde{\alpha}_\ell||e_\ell(d_{ij})| \\
&\leq C_1 \left(\sup_{1\leq i\leq n, \ell\in\mathbb{N}} \sum_{j\neq i} |e_\ell(d_{ij})| \right) \left(\sum_{\ell=L_n+1}^{\infty} |\tilde{\alpha}_\ell| \right) = O(L_n^{1-\lambda}),
\end{aligned}
$$

for some $C_1 > 0$, where the second equality follows from $E[z_j' z_j u_j] = E[z_j u_j] = 0$, the first inequality follows from the Cauchy-Schwarz inequality and finite fourth moments assumption, and the last equality follows from Assumptions S (i) and (iii).

Combining these results, $E[b_n' b_n] = O(nL_n^{2-2\lambda})$, and thus

$$
\Pr\{(\hat{\theta}-\theta)'(\hat{\theta}-\theta) \leq M\} \leq O(nL_n^{2-2\lambda}/C_n).
$$

Therefore, the conclusion follows by the assumption $nL_n^{2-2\lambda}/C_n \to 0$.

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