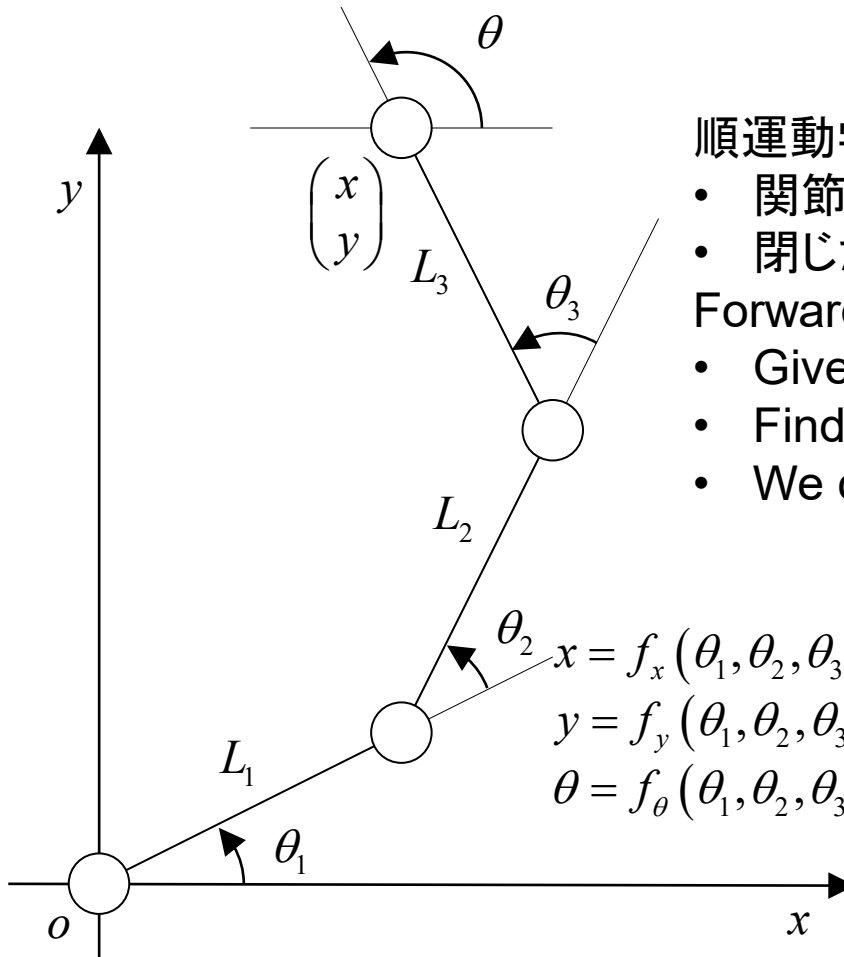


# ロボットアームの運動学: 理論 Robot Arm Kinematics: Theory

成瀬継太郎  
Keitaro Naruse

# ロボットアームの順運動学

## Forward Kinematics of Robot Arm



### 順運動学

- 関節角度から手先の姿勢(位置と向き)を求める
- 閉じた形式の解が求められる

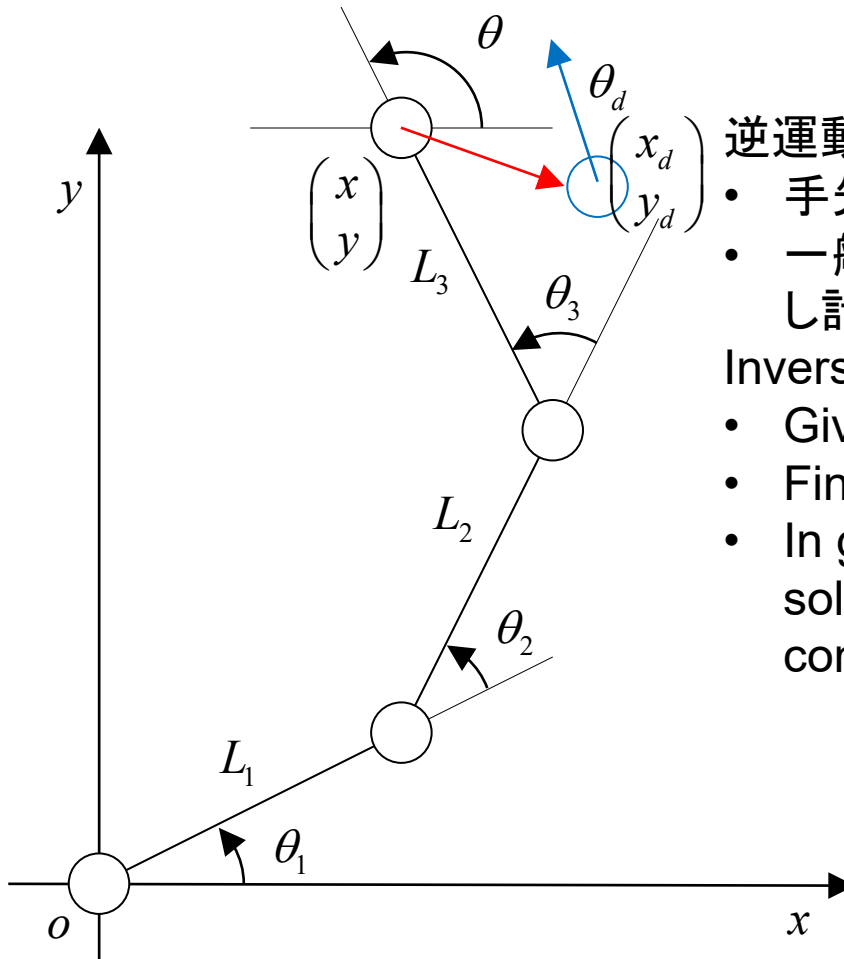
### Forward kinematics

- Given: Joint angles
- Find: A hand pose (position and orientation)
- We can find a closed form solution

$$\begin{aligned}
 x &= f_x(\theta_1, \theta_2, \theta_3) = L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\
 y &= f_y(\theta_1, \theta_2, \theta_3) = L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\
 \theta &= f_\theta(\theta_1, \theta_2, \theta_3) = \theta_1 + \theta_2 + \theta_3
 \end{aligned}$$

# ロボットアームの逆運動学

## Inverse Kinematics of Robot Arm



### 逆運動学

- 手先の位置と向きから関節角度を求める
- 一般に閉じた形式の解は存在しない。繰り返し計算で求める

### Inverse kinematics

- Given: A hand position and orientation
- Find: Joint angles
- In general, we cannot find a closed form solution. We solve it by iterative computations

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} f_1(x, y, \theta) \\ f_2(x, y, \theta) \\ f_3(x, y, \theta) \end{pmatrix} ?$$

# 手先速度と関節速度とヤコビアン Velocity and Jacobian

逆運動学解を求めるために、手先速度と関節速度の関係を考察する  
Let us consider a relation between hand velocity and joint angle velocity

全微分の原理より次が成り立つ

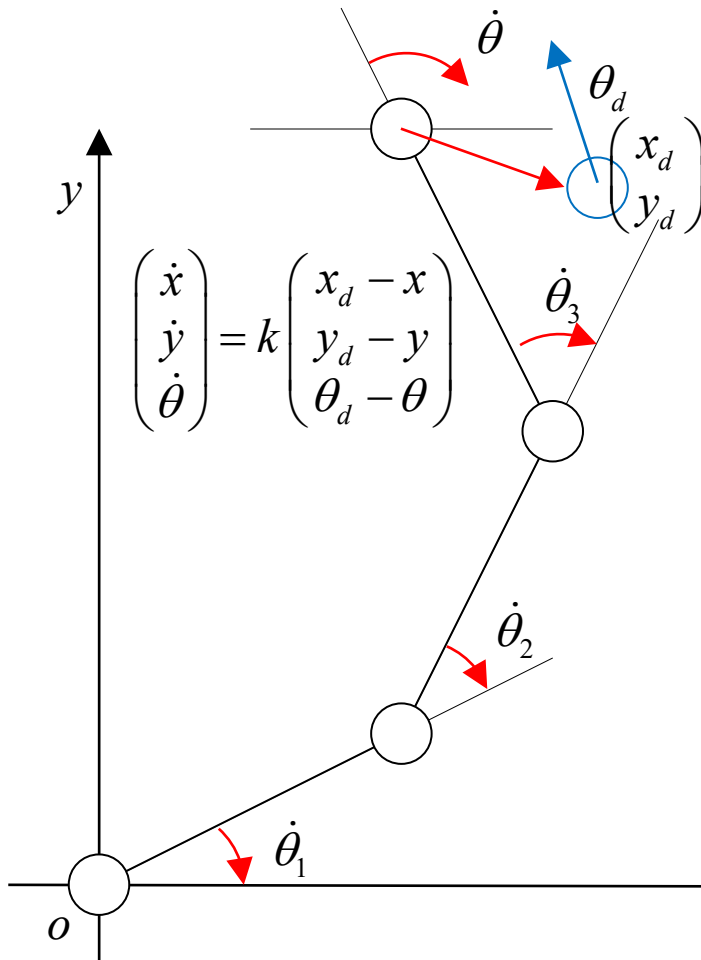
From the principle of total derivative, the following holds

$$\begin{aligned}\frac{dx}{dt} &= \frac{\partial x}{\partial \theta_1} \frac{d\theta_1}{dt} + \frac{\partial x}{\partial \theta_2} \frac{d\theta_2}{dt} + \frac{\partial x}{\partial \theta_3} \frac{d\theta_3}{dt} \\ \frac{dy}{dt} &= \frac{\partial y}{\partial \theta_1} \frac{d\theta_1}{dt} + \frac{\partial y}{\partial \theta_2} \frac{d\theta_2}{dt} + \frac{\partial y}{\partial \theta_3} \frac{d\theta_3}{dt} \\ \frac{d\theta}{dt} &= \frac{\partial \theta}{\partial \theta_1} \frac{d\theta_1}{dt} + \frac{\partial \theta}{\partial \theta_2} \frac{d\theta_2}{dt} + \frac{\partial \theta}{\partial \theta_3} \frac{d\theta_3}{dt}\end{aligned}$$
$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial \theta_3} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial \theta_3} \\ \frac{\partial \theta}{\partial \theta_1} & \frac{\partial \theta}{\partial \theta_2} & \frac{\partial \theta}{\partial \theta_3} \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = J \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix}$$

手先速度と関節速度が行列Jで関係付けられた. このJはヤコビアンと呼ばれる  
Now, hand and joint velocity is associated with J, which is called Jacobian

# 繰り返し計算による逆運動学解法

## Inverse Kinematics Solution by Iterative Computation



### 繰り返し計算による逆運動学解法

- 手先を動かしたい方向, すなわち手先速度が分かれば, それを実現する関節速度が求められる
- この繰り返しにより, 手先の目標位置に到達させることができる

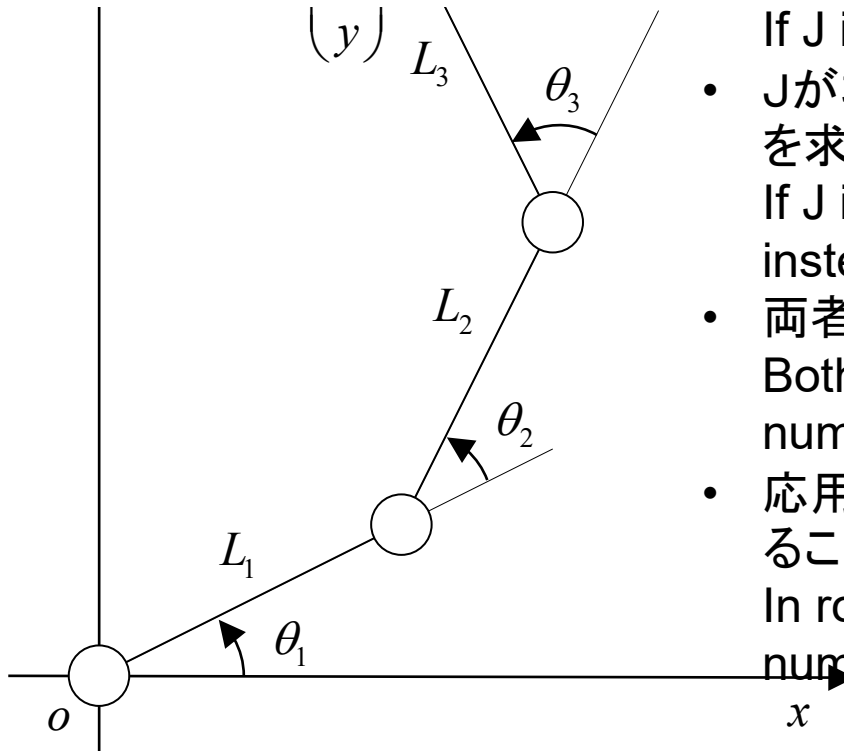
### Inverse kinematics solution by iterative computation

- If we know a hand direction to move, we can interpret it as a hand velocity, and can find joint angle velocity
- Iterating it, a hand can be reached a target eventually

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = J \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} \longrightarrow \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = J^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}$$

# 記号的ヤコビアン Instance of Symbolic Jacobian

$$J = \begin{pmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial \theta_3} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial \theta_3} \end{pmatrix} = \begin{pmatrix} -L_1 \sin \theta_1 - L_2 \sin(\theta_1 + \theta_2) - L_3 \sin(\theta_1 + \theta_2 + \theta_3) & -L_2 \sin(\theta_1 + \theta_2) - L_3 \sin(\theta_1 + \theta_2 + \theta_3) & -L_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) & L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) & L_3 \cos(\theta_1 + \theta_2 + \theta_3) \end{pmatrix}$$



- Jが正方行列なら逆行列を求めることができる  
If J is square, we can find an inverse of J
- Jが非正方行列なら、その代わりに疑似逆行列を求めることができる  
If J is not, we can find a pseudo inverse of J, instead.
- 両者は記号的にも数值的にも求めることができる  
Both can be found symbolically and numerically
- 応用事例では順運動学を用いて数值的に求めることが多い  
In robotics applications, we frequently find J numerically with forward kinematics function.

# 数値的ヤコビアンの例

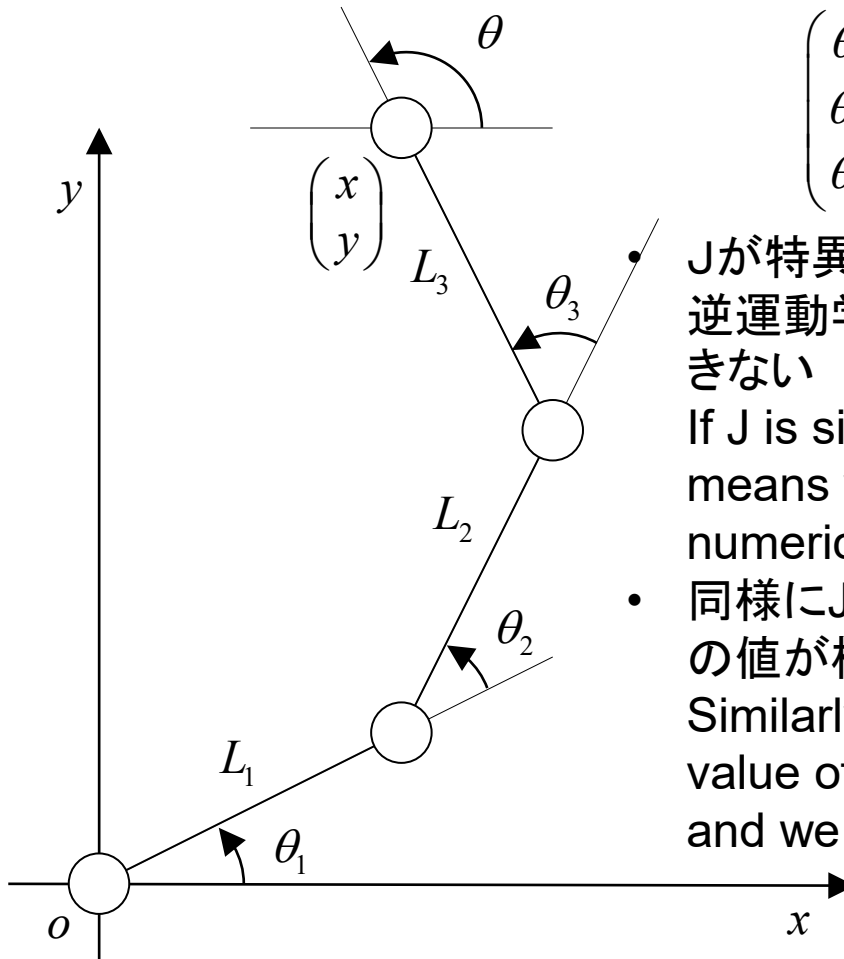
## Instance of Numerical Jacobian

$$J = \begin{pmatrix} \frac{f_x(\theta_1 + \Delta\theta_1, \theta_2, \theta_3) - f_x(\theta_1, \theta_2, \theta_3)}{\Delta\theta_1} & \frac{f_x(\theta_1, \theta_2 + \Delta\theta_2, \theta_3) - f_x(\theta_1, \theta_2, \theta_3)}{\Delta\theta_2} & \frac{f_x(\theta_1, \theta_2, \theta_3 + \Delta\theta_3) - f_x(\theta_1, \theta_2, \theta_3)}{\Delta\theta_3} \\ \frac{f_y(\theta_1 + \Delta\theta_1, \theta_2, \theta_3) - f_y(\theta_1, \theta_2, \theta_3)}{\Delta\theta_1} & \frac{f_y(\theta_1, \theta_2 + \Delta\theta_2, \theta_3) - f_y(\theta_1, \theta_2, \theta_3)}{\Delta\theta_2} & \frac{f_y(\theta_1, \theta_2, \theta_3 + \Delta\theta_3) - f_y(\theta_1, \theta_2, \theta_3)}{\Delta\theta_3} \\ \frac{f_\theta(\theta_1 + \Delta\theta_1, \theta_2, \theta_3) - f_\theta(\theta_1, \theta_2, \theta_3)}{\Delta\theta_1} & \frac{f_\theta(\theta_1, \theta_2 + \Delta\theta_2, \theta_3) - f_\theta(\theta_1, \theta_2, \theta_3)}{\Delta\theta_2} & \frac{f_\theta(\theta_1, \theta_2, \theta_3 + \Delta\theta_3) - f_\theta(\theta_1, \theta_2, \theta_3)}{\Delta\theta_3} \end{pmatrix}$$

$$\frac{\partial x}{\partial \theta_1} = \lim_{\Delta\theta \rightarrow 0} \frac{f_x(\theta_1 + \Delta\theta, \theta_2, \theta_3) - f_x(\theta_1, \theta_2, \theta_3)}{\Delta\theta_1}$$

# 数値計算の不安定性

## Unstable Numerical Solution



$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = J^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}$$

Jが特異行列のときは逆行列は求められないため、逆運動学解は存在するにも関わらず求めることができない

If J is singular, an inverse does not exist, which means we cannot solve inverse kinematics numerically even if a solution exists

- 同様にJの固有値が非常に小さいときはJの逆行列の値が極端に大きくなり、数値計算が不安定になる  
Similarly, if an eigenvalue of J is very small, a value of J inverse becomes significantly large, and we find unstable numerical solution