1 Lecture 9: Plan for Class

Last time:

1. Center Manifolds §3.2
2. invariant manifold tangent to center eigenspaces
3. unlike stable and unstable manifold:
   (a) cannot define by asymptotic behavior $t \to \pm\infty$
   (b) non-unique
   (c) not as smooth

Today:

1. Center manifold example
2. Normal forms §3.3

2 Center manifold example

This is from the Perko book at page 156.

Let's say that we have the following system

$$\dot{x} = x^2 y - x^5$$
$$\dot{y} = -y + x^2$$

We have an equilibrium point at $(0, 0)$. Our Jacobian is

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

Where are eigenvalues are $\lambda = 0, -1$ and the eigenvectors are

$$v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

On the center subspace E^c at $y = 0$, we have

$$\dot{x} = x^2(0) - x^5 = -x^5$$

which tells us that it is stable, and we could use at Lyapunov function $V(x) = \frac{1}{2}x^2$ to show that it is asymptotically stable.

However, $x = 0$ is not strictly an invariant set! We could have some weird dynamics around the origin that are not accounted for in this picture.

Let's approximate the center manifold to make arguments as to the stability. We can write

$$\begin{aligned} y &= h(x) = ax^2 + bx^3 + O(x^4) \\ \dot{y} &= h'(x)\dot{x} = -h(x) + x^2 \\ \implies & \left(2ax + 3bx^3 + O(x^3)\right)\left[x^2\left(ax^2 + bx^3 + O(x^4)\right) - x^5\right] \\ &= -\left[ax^2 + bx^3 + O(x^4)\right] + x^2 \end{aligned}$$

From inspection, we see that $a = 1$ and $b = 0$. So the center manifold is given by

$$y = x^2 + O(x^4)$$

Now look at the dynamics on the center manifold (instead of E^c). We can write

$$\begin{aligned} \dot{x} &= x^2 h(x) - x^5 \\ &= x^2(x^2) - x^5 \\ &= x^4 - x^5 \end{aligned}$$

which is actually unstable! Note that this contradicts our earlier reasoning by just looking at $\dot{x} = x^2(0) - x^5 = -x^5$.

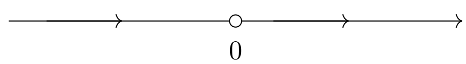

Figure 1: Phase portrait of $\dot{x} = x^4 - x^5$ showing with an unstable fixed point at $x = 0$.

3 Normal forms §3.3

If we have a linear system, say

$$\dot{x} = Ax$$

we can do a linear change of coordinates with

$$x = T\tilde{x}$$

and then our dynamics become

$$\dot{\tilde{x}} = T^{-1}AT\tilde{x}$$

where $T^{-1}AT$ can be written (if things are nice) as

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

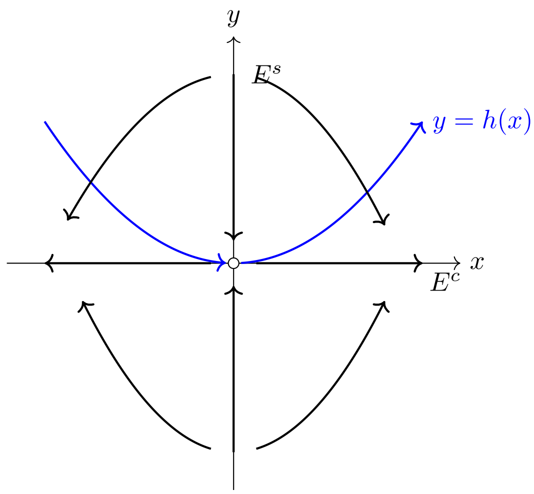

Figure 2: Phase portrait near the equilibrium at $(0,0)$ showing stable manifold E^s, unstable manifold E^c, and center manifold $y = h(x)$. (Claude generated, so there shouldn't be a crossing of the flow field and the center manifold.)

But! Things are not always this nice, and we need to use Jordan form, as in

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

3.1 Motivating example:

Let's say that we have the following systems

$$\begin{aligned} \dot{x} &= x + ax^2 + bxy + cy^2 + O(3) \\ \dot{y} &= 3y + dx^2 + exy + fy^2 + O(3) \end{aligned}$$

The linear part can be described as

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

But now our question is: How much can we simplify the quadratic terms with nonlinear change of coordinates? Ideally, we want

$$\begin{aligned} x &= \tilde{x} + h_1(\tilde{x}, \tilde{y}) \\ y &= \tilde{y} + h_2(\tilde{x}, \tilde{y}) \end{aligned}$$

where we are "near identity" with h_1, h_2 being quadratic in $\tilde{x}, \tilde{y}$.

3.2 Resonance terms

Answer: We can push all quadratic terms to higher order!

$$\begin{aligned}\dot{\tilde{x}} &= \tilde{x} + O(3)\\ \dot{\tilde{y}} &= 3\tilde{y} + O(3)\end{aligned}$$

But! Say our system is

$$\begin{aligned}\dot{x} &= x + ax^2 + \cdots\\ \dot{y} &= 2y + dx^2 + \cdot\end{aligned}$$

Now the best we can do is

$$\begin{aligned}\dot{\tilde{x}} &= \tilde{x} + O(3)\\ \dot{\tilde{y}} &= 2\tilde{y} + \tilde{d}\tilde{x}^2 + O(3)\end{aligned}$$

WHAT!!! It gets worse! Say our system is

$$\begin{aligned}\dot{x} &= x + ax^2 + \cdots\\ \dot{y} &= \frac{1}{2}y + dx^2 + \cdot\end{aligned}$$

and now the best we can do is

$$\begin{aligned}\dot{\tilde{x}} &= \tilde{x} + \tilde{c}\tilde{y}^2 + O(3)\\ \dot{\tilde{y}} &= \frac{1}{2}\tilde{y} + O(3)\end{aligned}$$

Say it isn't so!

These terms $\tilde{d}\tilde{x}^2, \tilde{c}\tilde{y}^2$ are called "resonance terms" and they will be instrumental in allowing us to study certain types of bifurcations.

3.3 Normal forms

Let's say that we have the following equation

$$\dot{x} = Ax + f_2(x) + f_3(x) + \cdots$$

where $x \in \mathbb{R}^2$, $f_2(x)$ are the quadratic terms, and $f_3(x)$ are the cubic terms.

3.3.1 Homogeneous functions

First let's talk about "homogeneous functions of degree d". These are functions

$$f\left(\lambda x_1, \lambda x_2, \ldots, \lambda x_n\right) = \lambda^d f\left(x_1, x_2, \ldots, x_n\right)$$

where λ is only a constant.

Now a homogenous polynomial is like

- degree 2: $f(x,y) = x^2 + 4xy + 2y^2$
- degree 3: $f(x,y) = 10x^3 - 2xy^2 + 3y^3$

This is to say, a homogeneous polynomial of degree d is a polynomial where all the terms are of the same degree. The following function

$$f(x,y) = x^3 + xy + y$$

is NOT homogeneous.

3.3.2 Back to $\dot{x} = Ax + f_2(x)$

Let

$$x = y + h_2(x)$$

where h_2 is a homogeneous polynomial of degree 2.

Can we push all terms in f_2 to a higher degree? Let's write

$$\begin{aligned} \dot{y} + Dh_2(y)\dot{y} &= \dot{x} \\ (I + Dh_2(y))\,\dot{y} &= A\,(y + h_2(y)) + f_2\,(y + h_2(y)) \end{aligned}$$

We can now write

$$\begin{aligned} \dot{y} &= (I + Dh_2(y))^{-1}\,[Ay + Ah_2(y) + f_2(y) + Df_2(y)\cdot h_2(y)] \\ &= (I - Dh_2(y) + O(2))\,[Ay + Ah_2(y) + f_2(y) + O(3)] \\ &= Ay + f_2(y) - [Dh_2(y)\,Ay + Ah_2(y)] + O(3) \end{aligned}$$

Cool.

Let's note that we can approximate any matrix S as

$$\begin{aligned} S &= I + X + \cdots + X^n \\ XS &= X + \cdots + X^n + X^{n+1} \\ (I - X)\,S &= I - X^{n+1} \\ \implies S &= (I - X)^{-1}\left(I - X^{n+1}\right) \\ S &= (I - X)^{-1} \end{aligned}$$

where as $n \to \infty$, it follows that $X^n \to 0$ if eigenvalue are less than 1.

Addendum: The approximation $(I+Dh_2(y))^{-1} \approx I-Dh_2(y)+O(2)$ uses the matrix geometric series $(I-X)^{-1} = I+X+X^2+\cdots$, which converges when the spectral radius $\rho(X) < 1$. This is valid here because $Dh_2(y)$ has entries that are linear in y (since h_2 is degree-2), so all entries vanish as $y \to 0$. By the Gershgorin circle theorem, the eigenvalues are trapped in discs centered at the diagonal entries with radii given by the off-diagonal row sums — as $y \to 0$, all these discs collapse to the origin, guaranteeing $\rho(Dh_2(y)) < 1$ in a sufficiently small neighborhood of the origin. Hence the approximation holds locally, consistent with normal forms being a local theory.

3.3.3 Getting rid of degree-2 terms

In order to get ride of all degree-2 terms, we can choose h_2 such that

$$Dh_2(y)\,Ay - Ah_2(y) = f_2(y)$$

where it follows (I actually don't know how tho) that

$$L_A h_2(y) = Dh_2(y)\,Ay - Ah_2(y)$$

Note that L_A is a linear operator from $H_2 \to H_2$ where H_2 is the set of all homogeneous degree-2 (vector-valued) polynomials. H_2 is essentially a vector space, where we can add and scalar multiply![1]

Hence, the operator L_A such that $L_A h_2 = f_2$ exists if and only if $f_2 \in \text{Range}(L_A)$. Moreover, we can compute L_A exactly! Cool.

In the book, the authors write

$$\begin{aligned} L_A\, h &= [A, h] \\ &= ad_A\, h \end{aligned}$$

where we are denoting a Lie bracket of vector fields.

3.4 Back to the Resonance example

We have $\vec{x} = (x, y)$ and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$$

We can write L_A as

$$\begin{aligned} L_A\, h(x) &= Dh(x) \cdot Ax - A\, h(x) \\ &= Dh(x) \begin{bmatrix} x \\ \lambda y \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} h(x) \end{aligned}$$

[1] Rowley went on a 5 minute detour catching the audience up to the properties of vector spaces. I will assume the reader knows these properties.

Let's note that H_2 can be described as

$$H_2 = \text{span}\left\{ \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix}, \right\}$$

where $\dim H_2 = 6$. What we want to do now is to write out $L_A\, h(x) = Dh(x) \cdot Ax - A\, h(x)$ for each basis vector in H_2 to compute L_A.

$$\begin{aligned}
\begin{pmatrix} x^2 \\ 0 \end{pmatrix} &: \quad \begin{pmatrix} 2x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ \lambda y \end{pmatrix} - \begin{pmatrix} 1 & \\ & \lambda \end{pmatrix}\begin{pmatrix} x^2 \\ 0 \end{pmatrix} = \begin{pmatrix} x^2 \\ 0 \end{pmatrix} \\
\begin{pmatrix} xy \\ 0 \end{pmatrix} &: \quad \begin{pmatrix} y & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ \lambda y \end{pmatrix} - \begin{pmatrix} 1 & \\ & \lambda \end{pmatrix}\begin{pmatrix} xy \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} xy \\ 0 \end{pmatrix} \\
\begin{pmatrix} y^2 \\ 0 \end{pmatrix} &: \quad \begin{pmatrix} 0 & 2y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ \lambda y \end{pmatrix} - \begin{pmatrix} 1 & \\ & \lambda \end{pmatrix}\begin{pmatrix} y^2 \\ 0 \end{pmatrix} = (2\lambda - 1) \begin{pmatrix} y^2 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ x^2 \end{pmatrix} &: \quad \begin{pmatrix} 0 & 0 \\ 2x & 0 \end{pmatrix}\begin{pmatrix} x \\ \lambda y \end{pmatrix} - \begin{pmatrix} 1 & \\ & \lambda \end{pmatrix}\begin{pmatrix} 0 \\ x^2 \end{pmatrix} = (2 - \lambda) \begin{pmatrix} 0 \\ x^2 \end{pmatrix} \\
\begin{pmatrix} 0 \\ xy \end{pmatrix} &: \quad \begin{pmatrix} 0 & 0 \\ y & x \end{pmatrix}\begin{pmatrix} x \\ \lambda y \end{pmatrix} - \begin{pmatrix} 1 & \\ & \lambda \end{pmatrix}\begin{pmatrix} 0 \\ xy \end{pmatrix} = \begin{pmatrix} 0 \\ xy \end{pmatrix} \\
\begin{pmatrix} 0 \\ y^2 \end{pmatrix} &: \quad \begin{pmatrix} 0 & 0 \\ 0 & 2y \end{pmatrix}\begin{pmatrix} x \\ \lambda y \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} 0 \\ y^2 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ y^2 \end{pmatrix}
\end{aligned}$$

This gives us a matrix representation of L_A that is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & (2\lambda - 1) & 0 & 0 & 0 \\ 0 & 0 & 0 & (2 - \lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

Now if $\lambda \neq 0, \frac{1}{2}, 2$ then $\operatorname{rank} L_A = 6$ and we can push all good terms to a higher degree.

However! What if $\lambda = 0, \frac{1}{2}, 2$?

- If $\lambda = 0$, $\operatorname{rank} L_A = 4$ and we can't eliminate $\begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix}$
- If $\lambda = 2$, $\operatorname{rank} L_A = 4$ and we can't eliminate $\begin{pmatrix} 0 \\ x^2 \end{pmatrix}$
- If $\lambda = \frac{1}{2}$, $\operatorname{rank} L_A = 4$ and we can't eliminate $\begin{pmatrix} y^2 \\ 0 \end{pmatrix}$

3.5 More general "resonance" case

Let's say we have

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where

$$Ae_i = \lambda_i e_i$$

and

$$h(x) = x_1^{m_1} \cdot x_2^{m_2} \cdots x_n^{m_n} \cdot e_i$$

Note that $m_1 + m_2 + \cdots + m_n = M$ where M is the degree.

Then $h(x)$ is an eigenvector of L_A with eigenvalue

$$\sum_j m_j \lambda_j - \lambda_j$$

and whenever this is eigenvalue is equal to 0, there are resonance terms that can't be eliminated.

3.6 General procedure

How could we find the resonance terms that can't be eliminated? First, find Range $(L_A\, h)$ for $h \in H_k$, where H_k is the set of homogeneous degree k polynomials. Let $R_k = \text{Range}\,(L_A)$ and choose a complement G_k such that

$$H_k = R_k \oplus G_k$$

where the choice of G_k is not unique. It's these terms in G_k that cannot be removed.