1 What is a dynamical system?

There are two classes. ODEs where $\dot{x} = f(x)$ where $\dot{x} = \frac{dx}{dt}$, $x \in \mathbb{R}^n$ and $x(0) = x_0$. Note that dynamical systems $\dot{x} = f(x,t)$ can be converted to $f(x)$ via a trick. There are two types of ODEs, autonomous and non-autonomous systems, where non-autonomous is time-dependent.

Iterative maps are of the form $x_{t+1} = f(x_t)$ where $x_t \in \mathbb{R}^n$ and $t = 0, 1, 2, 3, \ldots$.

2 Do solutions exist

For ODEs, do solutions even exist? If so, are they unique?

Answer: Usually. For instance, if f is Lipschitz continuous (in a region) then solutions exist (at least loccally) and are unique.

2.1 Continuity

f is continuous at x if $f(y)$ is close to $f(x)$ whenever y is close to x. "is close to" is not precise. More precisely, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $||f(y) - f(x)|| < \epsilon$ whenever $||x - y|| < \delta$.

2.2 Lipschitz Continuity

f is Lipschitz continuous in a region Ω if $\exists K > 0$ such that

$$||f(x) - f(y)|| \leq K||x - y||$$

for all $x, y \in \Omega$. Lipschitz continuous imples continuous, but continuous does not imply Lipschitz. An example is $f(x) = x^{1/3}$.

If the derivative of f exists and is continuous, this implies Lipschitz via the mean value theorem.

Also note that Ω has to be compact.

2.3 Global existence

Lipschitz continuous functions may not have global existence. For example $\dot{x} = x^2$.

$$\begin{aligned}
\dot{x} &= x^2 \\
\int_{x_0}^{x(t)} \frac{dx}{x^2} &= \int_0^t dt \\
-\frac{1}{x}\Big|_{x_0}^{x(t)} &= t - 0 \\
-\frac{1}{x(t)} + \frac{1}{x_0} &= t \\
x(t) &= \frac{1}{\frac{1}{x_0} - t}
\end{aligned}$$

This exibits a finite time blow up as $t \Rightarrow \frac{1}{x_0}$, so there is no global existence of a solution, only local.

3 Gronwall's Lemma

Suppose $u(t) \geq 0$ and $C, K \geq 0$ are constants. Suppose

$$U(t) \leq C + \int_o^t Ku(s)ds \qquad \forall t \in [o, T]$$

then $u(t) \leq Ce^{KT} \qquad \forall t \in [o, T]$.

3.1 Proof

Let $U(t) \leq C + \int_o^t Ku(s)ds \qquad \forall t \in [o, T]$. So we're given $u(t) \leq U(t)$.

$$\begin{aligned}
\frac{d}{dt}u(t) \leq Ku(t) &\Rightarrow u(t) - u(0) \leq K \int U(t) \\
\frac{d}{dt}U(t) = K(t) \leq KU(t) &\Rightarrow \frac{d}{dt}\log U(t) \leq K \\
&\Rightarrow \log U(t) - \log U(0) = K(t - o) \\
&\Rightarrow U(t) = e^{Kt} \cdot C
\end{aligned}$$

Somehow we integrate to get $u(t) \leq e^{Kt} \cdot c$.

3.2 Dependence on initial conditions

Suppose $\dot{x} = f(x)$ and $x(0) = x_0$. We integrate via the fundamental theorem of calculus to get

$$
\begin{aligned}
x(t) &= x(0) + \int_0^t f(x(s))ds \\
y(t) &= y(0) + \int_0^t f(y(s))ds \\
||x(t) - y(t)|| &\leq ||x(0) - y(0)|| + ||\int_0^t \left[f(x(s)) - f(y(s))\right] ds \text{ Via triangle inequality} \\
&\leq ||x(0) - y(0)|| + \int_0^t ||\left[f(x(s)) - f(y(s))\right] ds \\
&\leq ||x(0) - y(0)|| + \int_0^t K||x(s) - y(s)||ds \text{ Via Lipschitz}
\end{aligned}
$$

Applying Gronwall Lemma, we get

$$||x(t) - y(x)|| \leq e^{Kt}||x(0) - y(0)||$$

Given $\epsilon > 0$, choose $\delta < \frac{\epsilon}{e^{Kt}}$. Hence, solutions depend continuously on $x(0)$.

So we have abound, but it is terrible given the exponential growth of the initial condition dependence. Unfortunately, we can't do better and this is the point of chaos.

3.3 Examples

3.3.1 Global existence

Suppose $\dot{x} = 1-x^2$. We have two fixed points at $x\pm 1$, where $x = -1$ is unstable and $x = 1$ is stable. For intial conditions $x(0) < -1$, then we get finite time blow up.

Notice that $x \in [-1, 1]$ is invariant and $[-1, 1]$ is compact. This implies global existence. More generally, $x(0)$ in compact, positively invariant set implies *global* existence.

3.3.2 An introductory example

$$\ddot{x} - x + x^2 - \epsilon\left[\alpha y + \beta xy\right] = 0$$

While this is second order, we can get it into a system of first order ODEs. Let $y = \dot{x}$ and we can write

$$
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= x - x^2 + \epsilon\left[\alpha y + \beta xy\right]
\end{aligned}
$$

When $\epsilon = 0$, equations are Hamiltonian.

A "Hamiltonian" equation is $H(x,y)$ such that

$$\dot{x} = \frac{\partial H}{\partial y}$$
$$\dot{y} = -\frac{\partial H}{\partial x}$$
$$H(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3}$$

Notablly, H is conserved along trajectories.

$$\frac{d}{dt}H(x(t),y(t)) = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + \frac{\partial H}{\partial x}(-\frac{\partial H}{\partial y}) = 0$$

Hence,

$$H(x(t),y(t)) = C$$

and solutions move along level set of H (H =constant). When $H = 0$, we get a "separatrix" that deterimines whether our trajectory falls into qualitatively distinct regions.

Okay, now we have two fixed points and we can linearize around them.