

Problem

(38 points) For the following dynamical systems, find the fixed point(s) and compute the associated stable and unstable manifolds, up to fourth order (that is, if you consider an approximation $y = h(x)$, let h be a quartic polynomial). Use these results to sketch the global phase portraits.

(a) For the following ODE, assume $\lambda > 0$.

$$\begin{aligned}\dot{x} &= x(1 - x^2) \\ \dot{y} &= -\lambda(y - x^2).\end{aligned}$$

(b) For the following map, assume $|\lambda| < 1$ and $|\mu| > 1$.

$$\begin{aligned}x &\mapsto \lambda x, \\ y &\mapsto \mu y - x^3.\end{aligned}$$

Notes

Part (A) To begin, let's find the fixed points by writing

$$\begin{aligned}0 = \dot{x} = x(1 - x^2) &\implies x = 0, x = \pm 1 \\ 0 = \dot{y} = -\lambda(y - x^2) &\implies y = x^2\end{aligned}$$

Hence our fixed points are $(0, 0)$, $(-1, 1)$, $(1, 1)$. To linearize, we can compute the Jacobian

$$J_f(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

with the quantities being

$$\begin{aligned}\frac{\partial f_1}{\partial x} &= 1 - 3x^2 \\ \frac{\partial f_1}{\partial y} &= 0 \\ \frac{\partial f_2}{\partial x} &= 2\lambda x \\ \frac{\partial f_2}{\partial y} &= -\lambda\end{aligned}$$

Now we can compute our linearized matrices

$$\begin{aligned} Df(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & -\lambda \end{bmatrix} \\ Df(-1,1) &= \begin{bmatrix} -2 & 0 \\ -2\lambda & -\lambda \end{bmatrix} \\ Df(1,1) &= \begin{bmatrix} -2 & 0 \\ 2\lambda & -\lambda \end{bmatrix} \end{aligned}$$

Let's first approximate the stable and unstable manifolds associated with the $(0,0)$ fixed point. Our general workflow will be to calculate the eigenvalues and vectors, construct $h(x)$ or $g(y)$, then differentiate, group coefficients together, and solve for the coefficients.

[Nudge] Hint

2026-02-11 08:41

You're off to a great start — the fixed points, Jacobian, and linearizations all look correct. Keep going with your plan for the origin: identify the stable and unstable eigenspaces and then look for the invariant manifolds as graphs over those subspaces.

For our fixed point $(0,0)$, we have

$$\left[\left(\text{eigenvalue } 1, \text{eigenvector } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), \left(\text{eigenvalue } -\lambda, \text{eigenvector } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right]$$

Starting with our unstable manifold with the eigenvalue of 1 and eigenvector of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we can write

$$y = h(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

From our eigenvector we can deduce that $a_0 = a_1 = 0$. Taking the time derivative, we have

$$\begin{aligned} \dot{y} &= h'(x) = (2a_2x + 3a_3x^2 + 4a_4x^3) \dot{x} \\ -\lambda(y - x^2) &= (2a_2x + 3a_3x^2 + 4a_4x^3) x(1 - x^2) \\ -\lambda(a_2x^2 + a_3x^3 + a_4x^4 - x^2) &= (2a_2x + 3a_3x^2 + 4a_4x^3) x(1 - x^2) \end{aligned}$$

Is this right? This feels like a heinous expression.

[Nudge] Hint

2026-02-11 09:07

You're on exactly the right track — that expression is correct and not as bad as it looks. Just expand both sides, collect terms by powers of x , and match coefficients order by order; you'll find each a_k is determined cleanly one at a time.

Fair enough. I'm gonna save some time though and just parse with SymPy. Hence, we get

$$(-a_2\lambda - 2a_2 + \lambda)x^2 + (-a_3\lambda - 3a_3)x^3 + (2a_2 - a_4\lambda - 4a_4)x^4 + 3a_3x^5 + 4a_4x^6 = 0$$

where we can ignore terms $3a_3x^5 + 4a_4x^6$ since we are interested in 4th order approximations. Is this right?

[Nudge] Hint

2026-02-11 11:29

You're almost there — just set the coefficient of each power of x to zero and solve sequentially for a_2, a_3, a_4 . It should come out very cleanly!

Do we get

$$\begin{aligned} a_2 &= \frac{\lambda}{\lambda + 2} \\ a_3 &= 0 \quad \text{since } \lambda > 0 \\ a_4 &= \frac{2a_2}{\lambda + 4} \end{aligned}$$

[Nudge] Hint

2026-02-11 11:38

You're on the right track — those coefficients are correct! Now simplify a_4 by substituting your expression for a_2 , then move on to the stable manifold at the origin and the other fixed points.

Simplifying a_4 , we get

$$\begin{aligned} a_4 &= \frac{\lambda}{\lambda + 2} \cdot \frac{2}{\lambda + 4} \\ a_4 &= \frac{2\lambda}{(\lambda + 2)(\lambda + 4)} \end{aligned}$$

and hence our approximated unstable manifold is

$$h(x) = \frac{\lambda}{\lambda + 2}x^2 + \frac{2\lambda}{(\lambda + 2)(\lambda + 4)}x^4$$

Moving on to the stable manifold with the eigenvalue of $-\lambda$ and eigenvector of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, note that for any point where $x = 0$, we have the following system

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= -\lambda y \end{aligned}$$

implying that trajectories that start with $x = 0$ have $\lim_{t \rightarrow \infty} y(t) \rightarrow 0$, inline with the interpretation of a stable manifold. Hence, the stable manifold is simply an extension of the eigenvector and can be described by the line

$$x = 0$$

This reasoning feels right, but is it in the spirit of the problem?

[Nudge] Hint

2026-02-11 13:28

Your reasoning is perfectly correct and well-justified — the y -axis is indeed the exact stable manifold, so there's no need for a polynomial approximation. That's absolutely in the spirit of the problem. Now tackle the manifolds at $(\pm 1, 1)$.

Let's move on to the manifolds for $(-1, 1)$. The eigenvalues and eigenvectors are

$$\left[\left(\text{eigenvalue} - 2, \text{eigenvector} \begin{bmatrix} \frac{2-\lambda}{2\lambda} \\ 1 \end{bmatrix} \right), \left(\text{eigenvalue} - \lambda, \text{eigenvector} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right]$$

For the manifold corresponding to the eigenvalue -2 , we can write

$$y = h(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

where I think $a_1 = \frac{2\lambda}{2-\lambda}$, but that doesn't feel right.

[Nudge] Hint

2026-02-11 15:24

You're on the right track thinking about the eigenvector slope, but remember you need to work in shifted coordinates centered at $(-1, 1)$ — let $u = x + 1$ and $v = y - 1$, then express the system in (u, v) and apply the same manifold computation technique there.

I guess it's a Taylor series approximation, so really it is

$$y = h(x) = 1 + a_1(x + 1) + a_2(x + 1)^2 + a_3(x + 1)^3 + a_4(x + 1)^4$$

Now let's take the derivative to get

$$\dot{y} = h'(x) = (a_1 + 2a_2(x + 1) + 3a_3(x + 1)^2 + 4a_4(x + 1)^3) \dot{x}$$

and substituting in for \dot{x}, \dot{y} , we get

$$-\lambda(y - x^2) = (a_1 + 2a_2(x + 1) + 3a_3(x + 1)^2 + 4a_4(x + 1)^3) x(1 - x^2)$$

Substituting in $h(x)$ for y , we have

$$\begin{aligned} -\lambda(1 + a_1(x + 1) + a_2(x + 1)^2 + a_3(x + 1)^3 + a_4(x + 1)^4 - x^2) \\ = (a_1 + 2a_2(x + 1) + 3a_3(x + 1)^2 + 4a_4(x + 1)^3) x(1 - x^2) \end{aligned}$$

[Nudge] Hint

2026-02-11 18:55

You're heading in the right direction, but working in the original x variable will get messy fast. It'll be much cleaner if you actually substitute $u = x + 1$ (so $x = u - 1$) and rewrite everything in terms of u before expanding — that way you're matching powers of u near the origin, just like you did for the first fixed point.

To make matters easier, we will substitute in $u = x + 1$ (also $x = u - 1$) to get

$$\begin{aligned} & -\lambda(1 + a_1(u) + a_2(u)^2 + a_3(u)^3 + a_4(u)^4 - (u - 1)^2) \\ & = (a_1 + 2a_2(u) + 3a_3(u)^2 + 4a_4(u)^3)(u - 1)(1 - (u - 1)^2) \end{aligned}$$

Welp, now let's turn to SymPy.

$$\begin{aligned} & 4a_4u^6 + u^5(3a_3 - 12a_4) + u^4(2a_2 - 9a_3 - a_4\lambda + 8a_4) \\ & + u^3(a_1 - 6a_2 - a_3\lambda + 6a_3) + u^2(-3a_1 - a_2\lambda + 4a_2 + \lambda) \\ & + u(-a_1\lambda + 2a_1 - 2\lambda) = 0 \end{aligned}$$

Gross. Now it's time to drop the higher order terms, and solve for the coefficients

$$\begin{aligned} a_1 &= \frac{2\lambda}{2 - \lambda} \\ a_2 &= \frac{3a_1 - \lambda}{4 - \lambda} \\ a_3 &= \frac{6a_2 - a_1}{6 - \lambda} \\ a_4 &= \frac{9a_3 - 2a_2}{8 - \lambda} \end{aligned}$$

I will push these through later, but this looks right?

[Nudge] Hint

2026-02-11 19:15

You're doing great — those recurrence relations look correct. Now just substitute forward to get explicit expressions in terms of λ , and then move on to the other eigenvalue's manifold at $(-1, 1)$ and the fixed point $(1, 1)$.

For the manifold corresponding to the eigenvalue $-\lambda$, we can use a similar argument as before to reason that the stable manifold is $x = -1$. Namely, if $x = -1$, then our system becomes

$$\begin{aligned} \dot{x} &= -1 \cdot (1 - (-1)^2) = 0 \\ \dot{y} &= -\lambda(y - (-1)^2) = -\lambda y - \lambda \end{aligned}$$

which will just flow to $(-1, 1)$.

[Nudge] Hint

2026-02-11 19:40

You're on the right track with that argument, but double-check your algebra — substituting $x = -1$ gives $\dot{y} = -\lambda(y - 1)$, not $-\lambda y - \lambda$.

You are right! Namely, if $x = -1$, then our system becomes

$$\begin{aligned} \dot{x} &= -1 \cdot (1 - (-1)^2) = 0 \\ \dot{y} &= -\lambda(y - (-1)^2) = -\lambda y + \lambda \end{aligned}$$

which will just flow to $(-1, 1)$ since $\lambda > 0$.

[Nudge] Hint

2026-02-11 19:41

You've corrected the algebra issue — $\dot{y} = -\lambda(y - 1)$ indeed flows to $y = 1$, confirming $x = -1$ is the stable manifold at $(-1, 1)$. Now use symmetry to handle $(1, 1)$ efficiently, and then move on to part (b).

Ah yes, intuitively we would just mirror our previous manifolds at $(-1, 1)$ to get the manifolds at $(1, 1)$. To use symmetry, don't we have to argue that our system is symmetric? Wait, no. I guess I just have to argue that the particular manifold would be symmetric.

Let's first compute our eigenvalues and eigenvectors at $(1, 1)$

$$\left[\left(\text{eigenvalue} - 2, \text{eigenvector} \begin{bmatrix} \frac{\lambda-2}{2\lambda} \\ 1 \end{bmatrix} \right), \left(\text{eigenvalue} - \lambda, \text{eigenvector} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right]$$

Notice that the eigenvector $\begin{bmatrix} \frac{\lambda-2}{2\lambda} \\ 1 \end{bmatrix}$ corresponding to the eigenvalue -2 is symmetric about the y-axis to the eigenvector corresponding to the eigenvalue -2 for the fixed point $(-1, 1)$. Hence, given that our fixed point is also symmetric about the y-axis, it follows that the manifolds are also symmetric about the y-axis. Therefore, our manifold corresponding to eigenvalue -2 is

$$y = h(-x) = 1 + a_1(-x + 1) + a_2(-x + 1)^2 + a_3(-x + 1)^3 + a_4(-x + 1)^4$$

with coefficients

$$\begin{aligned} a_1 &= \frac{2\lambda}{2 - \lambda} \\ a_2 &= \frac{3a_1 - \lambda}{4 - \lambda} \\ a_3 &= \frac{6a_2 - a_1}{6 - \lambda} \\ a_4 &= \frac{9a_3 - 2a_2}{8 - \lambda} \end{aligned}$$

and our manifold corresponding to eigenvalue $-\lambda$ is $x = +1$.

[Nudge] Hint

2026-02-11 20:04

You're making great progress — now use the symmetry of the system (note how \dot{x} is an odd function of x and \dot{y} depends on x^2) to quickly write down the manifolds at $(1, 1)$ without redoing the full computation, and then move on to part (b).

Ah yes, in general, a system is y-axis symmetric if for $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$, then $f(-x, y) = -f(x, y)$ and $g(-x, y) = g(x, y)$. For our system, we can write

$$f(-x, y) = -x(1 - (-x)^2) = -x(1 - x^2) = -f(x, y)$$

and

$$g(-x, y) = -\lambda(y - (-x)^2) = -\lambda(y - x^2) = g(x, y)$$

Hence our system is symmetric about the y -axis, therefore our manifold approximations must be as well. Therefore, our manifold corresponding to eigenvalue -2 is

$$y = h(-x) = 1 + a_1(-x + 1) + a_2(-x + 1)^2 + a_3(-x + 1)^3 + a_4(-x + 1)^4$$

with coefficients

$$\begin{aligned} a_1 &= \frac{2\lambda}{2 - \lambda} \\ a_2 &= \frac{3a_1 - \lambda}{4 - \lambda} \\ a_3 &= \frac{6a_2 - a_1}{6 - \lambda} \\ a_4 &= \frac{9a_3 - 2a_2}{8 - \lambda} \end{aligned}$$

and our manifold corresponding to eigenvalue $-\lambda$ is $x = +1$.

[Nudge] Hint

2026-02-11 20:32

You've completed part (a) nicely — now it's time to move on to part (b) with the map, where the invariant manifold condition will look a bit different since you're working with a discrete map rather than a flow.

Part (B) To begin, let's find the fixed points by writing

$$x = \lambda x$$

where since $|\lambda| < 1$, it follows that the only fixed x coord is $x = 0$. For y , we can write

$$y = \mu y$$

where since $|\mu| > 1$, it follows that the only fixed y coord is $y = 0$. Hence the only fixed point is $(0, 0)$.

[Nudge] Hint

2026-02-11 20:54

You're on the right track — now identify the stable and unstable directions at the origin and set up the invariant manifold condition using the map (i.e., $h(\lambda x) = \mu h(x) - x^3$ or similar) instead of the flow condition you used in part (a).

To linearize, we can compute the Jacobian

$$J_f(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

with the quantities being

$$\begin{aligned}\frac{\partial f_1}{\partial x} &= \lambda \\ \frac{\partial f_1}{\partial y} &= 0 \\ \frac{\partial f_2}{\partial x} &= -3x^2 \\ \frac{\partial f_2}{\partial y} &= \mu\end{aligned}$$

Now we can compute our linearized matrix

$$Df(0,0) = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

where the eigenvalues and eigenvectors are

$$\left[\left(\text{eigenvalue } \lambda, \text{eigenvector } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), \left(\text{eigenvalue } \mu, \text{eigenvector } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right]$$

[Nudge] Hint

2026-02-11 21:14

You're on the right track with the eigenvalues and eigenvectors. Now think about which direction is stable and which is unstable (given the assumptions on $|\lambda|$ and $|\mu|$), and set up the invariant manifold as a graph $y = h(x)$ over the appropriate eigenspace.

Given $|\lambda| < 1$, we can construct our stable manifold by writing

$$y = h(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

where $a_0 = 0$ since we are at the origin and $a_1 = 0$ since the corresponding eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Hence we have

$$y = h(x) = a_2x^2 + a_3x^3 + a_4x^4$$

My guess is that we have something like

$$y = h(x) = a_2x^2 + a_3x^3 + a_4x^4$$

$$y - x^3 = a_2x^2 + a_3x^3 + a_4x^4$$

$$h(x) - x^3 = a_2x^2 + a_3x^3 + a_4x^4$$

but this feels circular.

[Nudge] Hint

2026-02-11 21:22

You're close — think about what invariance means for a `*map*`: if a point $(x, h(x))$ is on the manifold, then its `*image*` under the map must also be on the manifold, giving you $h(\lambda x) = \mu h(x) - x^3$.

Oh this makes sense! It's kind of like a commute diagram. More specifically

$$\begin{array}{ccc} x_t & \xrightarrow{h} & y_t \\ \downarrow f & & \downarrow g \\ x_{t+1} & \xrightarrow{h} & y_{t+1} \end{array}$$

where $x \mapsto f(x, y)$ and $y \mapsto g(x, y)$. Hence,

$$h(f(x, y)) = g(h(x), x)$$

and plugging in our expressions, we have

$$\begin{aligned} h(\lambda x) &= \mu h(x) - x^3 \\ a_2 \lambda^2 x^2 + a_3 \lambda^3 x^3 + a_4 \lambda^4 &= \mu (a_2 x^2 + a_3 x^3 + a_4 x^4) - x^3 \end{aligned}$$

where grouping terms together gives us

$$\begin{aligned} a_2 &= 0 \\ a_3 &= \frac{1}{\mu - \lambda^3} \\ a_4 &= 0 \end{aligned}$$

because $a\lambda^n = a\mu$ implies $a = 0$.

[Nudge] Hint

2026-02-11 23:18

You've got the stable manifold for the map — nice work! Now set up the unstable manifold as a graph $x = g(y)$ over the unstable eigenspace and apply the same invariance logic.

Solution