Problem

(30 points) The following equations describe the motion of a ball in a spinning circular hoop, where α is the (nondimensional) speed of the spinning hoop, β is the (nondimensional) damping, x is the angle of the ball in the hoop, and y is the ball's angular velocity:

$$\dot{x} = y$$
$$\dot{y} = (\alpha \cos x - 1) \sin x - \beta y,$$

(a) Find the equilibrium points, as a function of the parameters α, β (consider $x \in [-\pi, \pi]$). Make a sketch of these equilibrium points as a function of α. (Such a plot is the beginning of a bifurcation diagram).

(b) Show that, for $\beta = 0$, the system is Hamiltonian: that is, there is a function $H(x, y)$ such that

$$\dot{x} = \frac{\partial H}{\partial y}$$
$$\dot{y} = -\frac{\partial H}{\partial x}.$$

(c) Fix $\alpha = 2$, and consider the stability of each of the equilibrium points you found in part (a). Be sure to consider the cases $\beta < 0$, $\beta = 0$, and $\beta > 0$.

Notes

Part (a) Let's start by setting everything equal to 0

$$0 = y$$
$$0 = (\alpha \cos x - 1) \sin x - \beta y.$$

Hence,

$$0 = (\alpha \cos x - 1) \sin x$$
$$0 = \alpha(\cos x)(\sin x) - \sin x$$
$$0 = \frac{\alpha}{2} [\sin(x + x) + \sin(x - x)] - \sin x$$
$$0 = \frac{\alpha}{2} \sin(2x) - \sin x$$

More to think about here...

Part (b) Let's start by writing

$$\frac{\partial H}{\partial y} = y$$

$$\int \partial H = \int y \partial y$$

where integrating both sides with respect to y yields

$$H(x, y) = \frac{y^2}{2} + f(x).$$

To identify $f(x)$, we can write

$$\frac{\partial H}{\partial x} = (\alpha \cos x - 1) \sin x$$

$$\int \partial H = -\int (\alpha \cos x - 1) \sin x \partial x$$

$$H(x, y) = -\int \frac{\alpha}{2} \sin(2x) - \sin x \partial x$$

$$H(x, y) = \frac{\alpha}{4} \cos(2x) - \cos(x) + f(y)$$

Where our Hamiltonian is

$$H(x, y) = \frac{\alpha}{4} \cos(2x) - \cos(x) + \frac{y^2}{2}.$$

Solution