1 Plan for Class

1. Linear systems in the plane
2. Invariant sets, attracting sets
3. Stable and unstable eigenspaces
4. Hartman-Grobman theorem 1.3.1
5. Stable manifold theorem 1.3.2

2 Linear systems

First let's defind the flow map: ϕ_t maps $x(0)$ to $x(t)$.

For linear systems, we are interested in

$$\dot{x} = Ax$$
$$x(t) = e^{At}$$

where the matrix exponential is defined as

$$e^{At} = I + At + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3$$

For the flow map, we can write

$$\psi_t = e^{At}$$
$$\psi_t(x_0) = e^{At}x_0$$

2.1 Special cases

Let's say we have a matrix A that is

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

which is

$$\dot{x} = \lambda x$$
$$\dot{y} = \mu y$$

Our trajectories (or phase portrait) of the system will look like

$$\frac{dy}{dx} = \frac{\mu y}{\lambda x}$$
$$\implies \frac{dy}{y} = \frac{\mu}{\lambda}\frac{dx}{x}$$
$$\implies \log y = \frac{\mu}{\lambda}\log x + C$$
$$\implies y = x^{\mu/\lambda} \cdot c'$$

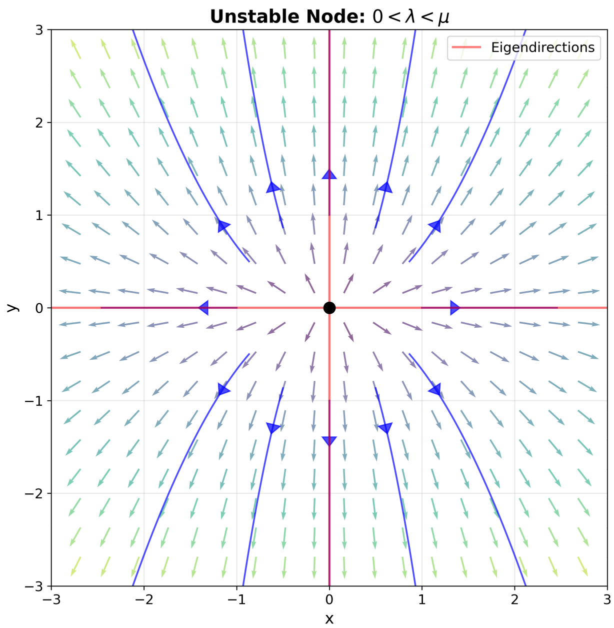
and our solution to the dynamics look like

$$x(t) = e^{\lambda t}x_0$$
$$y(t) = e^{\mu t}y_0$$

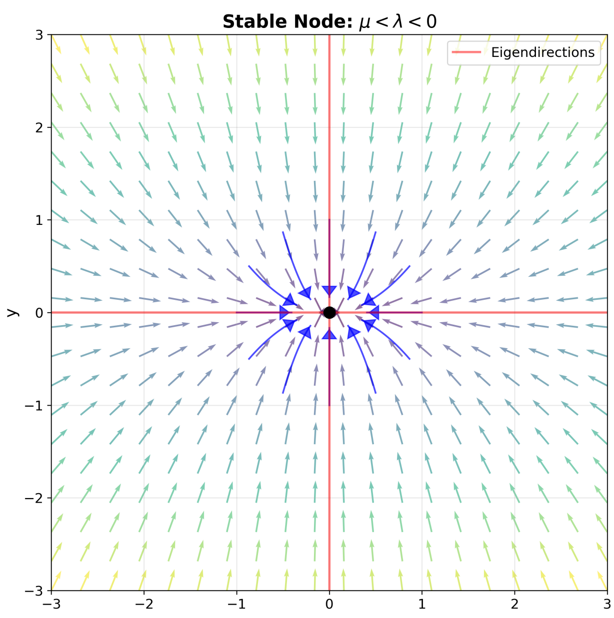
Now let's draw out the flow field for various conditions (to do after class)

- $0 < \lambda < \mu$ - unstable node
- $\mu < \lambda < 0$ - stable node
- $\lambda < 0 < \mu$ - saddle
- $\lambda = \mu < 0$ - sink (star node)
- $0 < \lambda = \mu$ - source (star node)

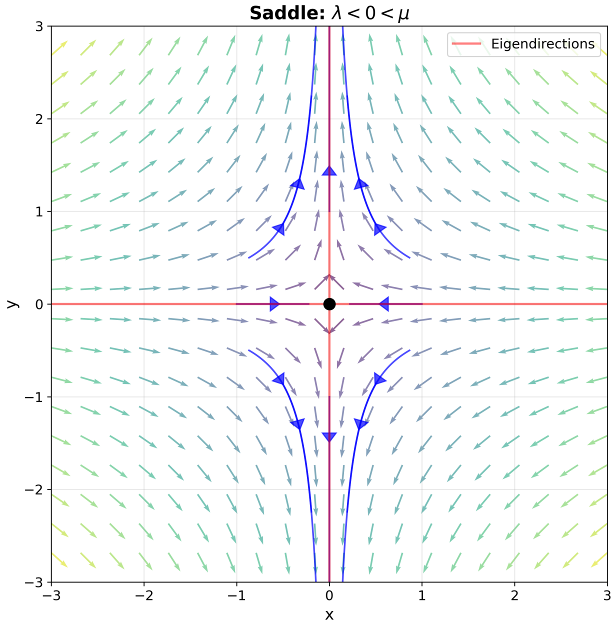
2.1.1 Unstable Node: $0 < \lambda < \mu$



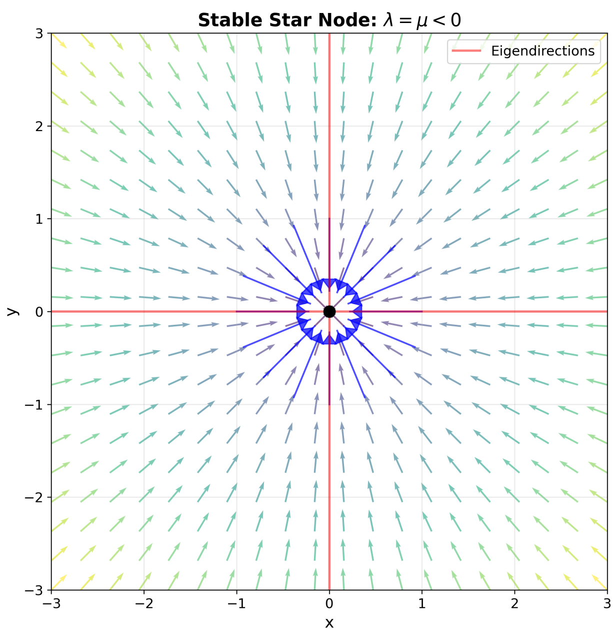
2.1.2 Stable Node: $\mu < \lambda < 0$



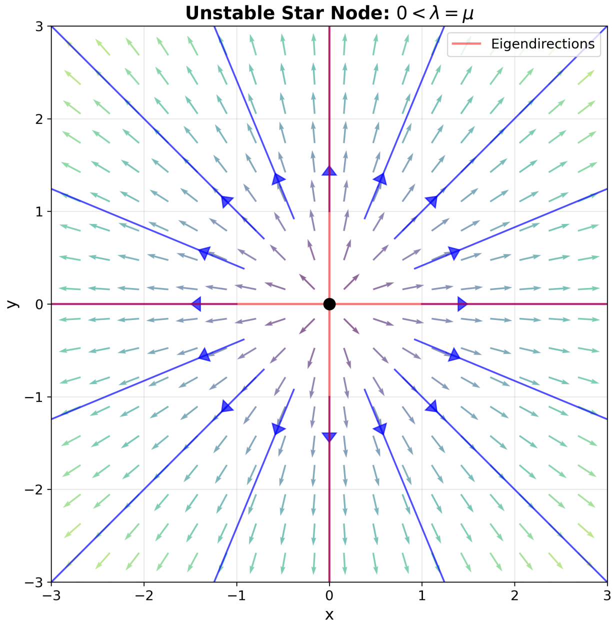
2.1.3 Saddle: $\lambda < 0 < \mu$



2.1.4 Stable Star Node: $\lambda = \mu < 0$



2.1.5 Unstable Star Node: $0 < \lambda = \mu$



2.2 Complex eigenvalues?

Let's say our A matrix looks like

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

where the eigenvalues are $\sigma \pm i\omega$.

$$\begin{aligned} \dot{x} &= \sigma x + \omega y & x &= r\cos\theta & \dot{x} &= \dot{r}\cos\theta - r\sin\theta \cdot \dot{\theta} \\ \dot{y} &= -\omega x + \sigma y & y &= r\sin\theta & \dot{y} &= \dot{r}\sin\theta + r\cos\theta \cdot \dot{\theta} \end{aligned}$$

Via some calculations that I should do later, we have

$$\begin{aligned} \dot{r} &= \sigma r \\ \dot{\theta} &= -\omega \end{aligned}$$

Now we can make some more plots!

- $\sigma < 0$ - stable spiral
- $\sigma > 0$ - unstable spiral
- $\sigma = 0$ - centers!

2.2.1 The calculations

To get $\dot{r}$ let's first multiply $\dot{x}$ by $\cos\theta$ and $\dot{y}$ by $\sin\theta$, and add them to write

$$\begin{gathered}
\left[\dot{r}\cos\theta - r\sin\theta\cdot\dot{\theta}\right](\cos\theta) + \left[\dot{r}\sin\theta + r\cos\theta\cdot\dot{\theta}\right](\sin\theta) = \dot{x}(\cos\theta) + \dot{y}(\sin\theta) \\
\dot{r}\cos^2\theta + \dot{r}\sin^2\theta = (\sigma x + \omega y)(\cos\theta) + (-\omega x + \sigma y)(\sin\theta) \\
\dot{r}(\cos^2\theta + \sin^2\theta) = (\sigma r\cos\theta + \omega r\sin\theta)(\cos\theta) + (-\omega r\cos\theta + \sigma r\sin\theta)(\sin\theta) \\
\dot{r} = \sigma r\cos^2\theta + \sigma r\sin^2\theta \\
\dot{r} = \sigma r(\cos^2\theta + \sin^2\theta) \\
\dot{r} = \sigma r
\end{gathered}$$

To get $\dot{\theta}$, we'll multiply $\dot{x}$ by $-\sin\theta$ and $\dot{y}$ by $\cos\theta$, then add them:

$$\begin{gathered}
\left(\dot{r}\cos\theta - r\sin\theta\cdot\dot{\theta}\right)(-\sin\theta) + \left(\dot{r}\sin\theta + r\cos\theta\cdot\dot{\theta}\right)(\cos\theta) = \dot{x}(-\sin\theta) + \dot{y}(\cos\theta) \\
-\dot{r}\cos\theta\sin\theta + r\sin^2\theta\cdot\dot{\theta} + \dot{r}\sin\theta\cos\theta + r\cos^2\theta\cdot\dot{\theta} = \dot{x}(-\sin\theta) + \dot{y}(\cos\theta) \\
r\sin^2\theta\cdot\dot{\theta} + r\cos^2\theta\cdot\dot{\theta} = \dot{x}(-\sin\theta) + \dot{y}(\cos\theta) \\
r(\sin^2\theta + \cos^2\theta)\dot{\theta} = \dot{x}(-\sin\theta) + \dot{y}(\cos\theta) \\
r\dot{\theta} = \dot{x}(-\sin\theta) + \dot{y}(\cos\theta)
\end{gathered}$$

Now substitute $\dot{x}$ and $\dot{y}$:

$$\begin{gathered}
r\dot{\theta} = (\sigma x + \omega y)(-\sin\theta) + (-\omega x + \sigma y)(\cos\theta) \\
r\dot{\theta} = -\sigma x\sin\theta - \omega y\sin\theta - \omega x\cos\theta + \sigma y\cos\theta
\end{gathered}$$

Substitute:

$$\begin{gathered}
r\dot{\theta} = -\sigma r\cos\theta\sin\theta - \omega r\sin^2\theta - \omega r\cos^2\theta + \sigma r\sin\theta\cos\theta \\
r\dot{\theta} = -\sigma r\cos\theta\sin\theta + \sigma r\sin\theta\cos\theta - \omega r\sin^2\theta - \omega r\cos^2\theta \\
r\dot{\theta} = 0 - \omega r(\sin^2\theta + \cos^2\theta) \\
r\dot{\theta} = -\omega r \\
\dot{\theta} = -\omega
\end{gathered}$$

2.3 What if A is non-diagonalizable?

Let's say that A looks like

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Our solution would look like

$$\begin{gathered}
y(t) = e^{\lambda t}\cdot y_0 \\
x(t) = e^{\lambda t}\cdot x_0 + y_0 t e^{\lambda t}
\end{gathered}$$

Now draw the flow field for $\lambda < 0$.

2.3.1 General cases

If A is not diagonal, find eigenvalues v_1, v_2 such that

$$Av_1 = \lambda v_1 \qquad Av_2 = \lambda v_2$$

Now draw the resulting flow field.

2.4 The "big map" for linear systems

Let's say that A looks like

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

where we can find the eigenvalues via

$$\det(sI - A) = s^2 - (\lambda + \mu)s + \lambda\mu = 0$$

where the trace is $\lambda + \mu$ and the determinant of A is $\lambda\mu$.

From the trace and determinant, we can draw the "big map" for linear systems in the plane that we should supply after class.

3 Invariant sets, attracting sets

Positively invariant set: Say that we have a set of initial conditions $x \in A$. We can think of a flow map that directs the intial conditions to some new point $\phi_t(x)$. We can think of the new set of A under the flow map as $\phi_t(A)$.

If $\phi_t(A) \subset A$ for all $t > 0$, then A is positively invariant. (ϕ_t must exist for all $t > 0$)

If $\phi_t(A) \subset A$ for all $t < 0$, then A is negatively invariant.

If $\phi_t(A) \subset A$ for all t, then A is invariant.

3.1 Trapping region

Definition: A trapping region is a closed, connected set D such that $\phi_t(D) \subset \text{interior}(D)$ for all $t > 0$.

3.1.1 Example

$$\dot{x} = 1 - x^2$$

Now consider the three sets

- $(\infty, -1]$ is negatively invariant but not positively invariant due to finite-time blow-up
- $[1, \infty)$ is positively invariant
- $[-1, 1]$ is invariant

3.1.2 Post class clarification

The definition of a positively invariant set and a trapping region look similar, but they are indeed different. Take the following region

$$D = \{1 \leq r \leq 2\}$$

for the following system

$$\dot{r} = r(2 - r)(r - 1), \quad \dot{\theta} = 1$$

The region D is positively invarient since trajectories stay in D, but it is not a trapping region because trajectories appraoch the boundary circles $r = 1$ and $r = 2$ (the equilibria). However, the slightly larger region

$$D' = \{0.5 \leq r \leq 2.5\}$$

is a trapping region because trajectories starting in D' flow to into the interior and toward D.

3.2 Attracting set

A closed, invariant set A is called an attracting set if $\exists$ a neighboorhood U of A such that $\phi_t(x) \in U \quad \forall t > 0$ and $\phi_t(x) \to A \quad x \in U$.

Definition: The domain/basin of attraction of A is $\bigcup \phi_t(U)$ for all $t \leq 0$.

3.2.1 Example 1

Let's consider the system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x - y\end{aligned}$$

where our matrix A is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

and the trace is -1 and the determinant is 1; this is a sink. Our attracting set A is $A = (0,0)$ and the basin of attraction is $\mathbb{R}^2$.

3.2.2 Example 2

Let's consider the system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x + \epsilon(1 - x^2)y\end{aligned}$$

where the limit cycle Γ is the attracting set and the basin of Γ is $\mathbb{R}^2 \backslash (0,0)$

3.2.3 Example 3

Let's consider the system

$$\begin{aligned}\dot{x} &= x - x^3 \\ \dot{y} &= -y\end{aligned}$$

Is the set

$$A = \{(x,0) : x \in [-1,1]\}$$

an attracting set? Yes, despite the origin being an unstable fixed point. Note: we will have a different definition for an attractor, where A is not an attractor.

3.3 Stable and unstable eigenspaces

Let's say that we have a nonlinear system $\dot{x} = f(x)$ and there is an equilibrium point at x_0 (i.e. $f(x_0) = 0$). Linearization about x_0 is

$$\dot{\xi} = Df(x_0) \cdot \xi$$

Then, there are 3 invariant supspaces for the linear system

- stable subspace, E^s and $\dim E^s = n_s$ and $E^s = \operatorname{span}\{v_1, \dots, v_{n_s}\}$ where $v_1, \dots, v_{n_s}$ are eigenvectors
  - More explicitly, stable subspace E^s: spanned by (generalized) eigenvectors corresponding to eigenvalues with negative real part ($\mathrm{Re}(\lambda) < 0$)
- center subspace, E^c and $\dim E^c = n_c$ and $E^c = \operatorname{span}\{\quad\}$
  - More explicitly, center subspace E^c: spanned by (generalized) eigenvectors corresponding to eigenvalues with zero real part ($\mathrm{Re}(\lambda) = 0$)
- unstable subspace, E^u and $\dim E^u = n_u$ and $E^u = \operatorname{span}\{\quad\}$
  - More explicitly, unstable subspace E^u: spanned by (generalized) eigenvectors corresponding to eigenvalues with positive real part ($\mathrm{Re}(\lambda) > 0$)

3.3.1 Can we have growth in E^c?

Suppose we have

$$\dot{x} = y$$
$$\dot{y} = 0$$

where

$$y = C$$
$$x = x_0 + Ct$$

where $E^c = \mathbb{R}^2$ but linear growth!

3.4 Hyperbolic fixed point

Let p be a fixed point of $\dot{x} = f(x)$ such that $f(p) = 0$.

Definition: p is a hyperbolic fixed point if $Df(x_0)$ has no eigenvalues on imaginary axis (i.e., $n_c = 0$).

4 Hartman-Grobman theorem

If p is a hyperbolic fixed point, then there exists a homeomorphism h defined on a neighboorhood U of p, taking orbits of the flow ϕ_t of $\dot{x} = f(x)$ to those of the flow $e^{t \cdot Df(x_0)}$ of $\dot{\xi} = Df(p)\xi$.

h perserves the sense of orbits and can be chosen to perserve the time parameterization

Note that a homeomorphism is a continuous map with continuous inverse, but is maybe not differentiable!

1. Why is hyperbolic necessary? Suppose $\dot{x} = -x$ and $\dot{y} = y^2$. This becomes $\dot{\xi} = -\xi$ and $\dot{\eta} = 0$, and there are no mappings h between the two.

2. Why not a diffeomorphism? I forget the argument, but I think we will come back to it later in the class.

5 Stable manifold theorem

Let p be a fixed point, and U be a neighboorhood of p.

We can define a local stable manifold of p as

$$W^s_{\text{loc}}(p) = \{x \in U | \phi_t(x) \to p \text{ as } t \to \infty \text{ and } \phi_t(x) \in U \text{ for all } t \geq 0\}$$

We can define the unstable manifold as

$$W^u_{\text{loc}}(p) = \{x \in U | \phi_t(x) \to p \text{ as } t \to -\infty \text{ and } \phi_t(x) \in U \text{ for all } t \leq 0\}$$

5.1 Now the theorem

Stable manifold theorem for a fixed point: For a p hyperbolic fixed point of $\dot{x} = f(x)$, $\exists$ local stable and unstable manifold $W^s_{\text{loc}}(p), W^u_{\text{loc}}(p)$ such that

- same dimensions as E^s, E^u
- tangent to E^s, E^u

and $W^s_{\text{loc}}(p)$ and $W^u_{\text{loc}}(p)$ as smooth as f is.