

1 Plan for Class

1. Linear systems in the plane
2. Invariant sets, attracting sets
3. Stable and unstable eigenspaces
4. Hartman-Grobman theorem 1.3.1
5. Stable manifold theorem 1.3.2

2 Linear systems

First let's defind the flow map: ϕ_t maps $x(0)$ to $x(t)$.

For linear systems, we are interested in

$$\begin{aligned}\dot{x} &= Ax \\ x(t) &= e^{At}\end{aligned}$$

where the matrix exponential is defined as

$$e^{At} = I + At + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3$$

For the flow map, we can write

$$\begin{aligned}\psi_t &= e^{At} \\ \psi_t(x_0) &= e^{At}x_0\end{aligned}$$

2.1 Special cases

Let's say we have a matrix A that is

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

which is

$$\begin{aligned}\dot{x} &= \lambda x \\ \dot{y} &= \mu y\end{aligned}$$

Our trajectories (or phase portrait) of the system will look like

$$\begin{aligned}\frac{dy}{dx} &= \frac{\mu y}{\lambda x} \\ \implies \frac{dy}{y} &= \frac{\mu}{\lambda} \frac{dx}{x} \\ \implies \log y &= \frac{\mu}{\lambda} \log x + C \\ \implies y &= x^{\mu/\lambda} \cdot c'\end{aligned}$$

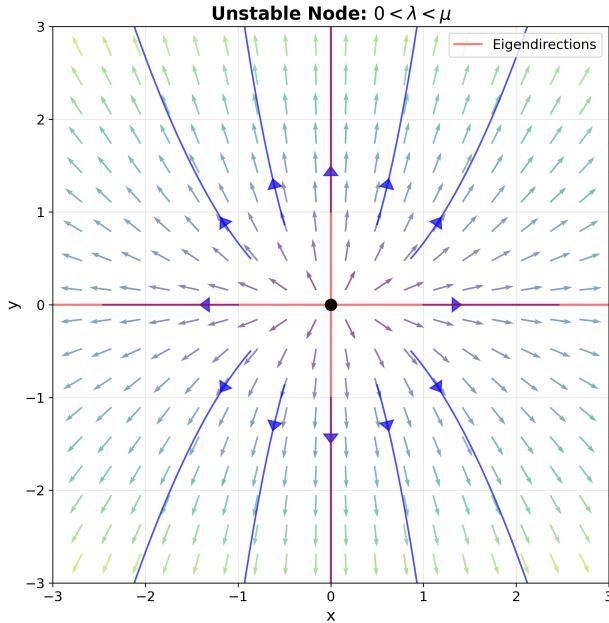
and our solution to the dynamics look like

$$\begin{aligned}x(t) &= e^{\lambda t} x_0 \\ y(t) &= e^{\mu t} y_0\end{aligned}$$

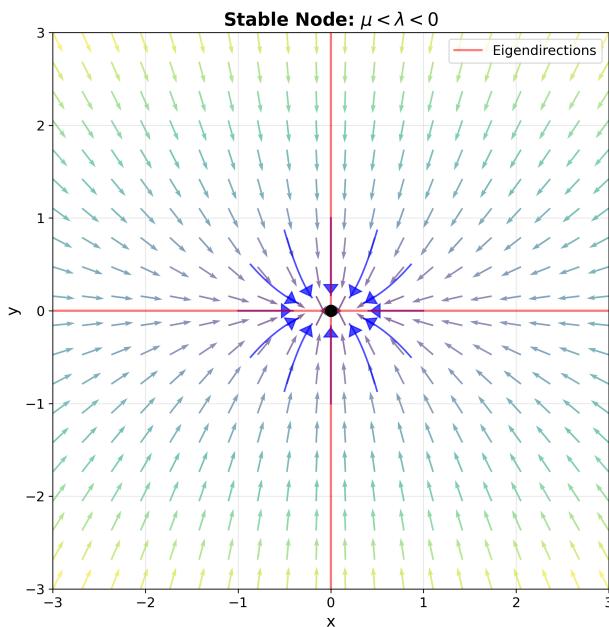
Now let's draw out the flow field for various conditions (to do after class)

- $0 < \lambda < \mu$ - unstable node
- $\mu < \lambda < 0$ - stable node
- $\lambda < 0 < \mu$ - saddle
- $\lambda = \mu < 0$ - sink (star node)
- $0 < \lambda = \mu$ - source (star node)

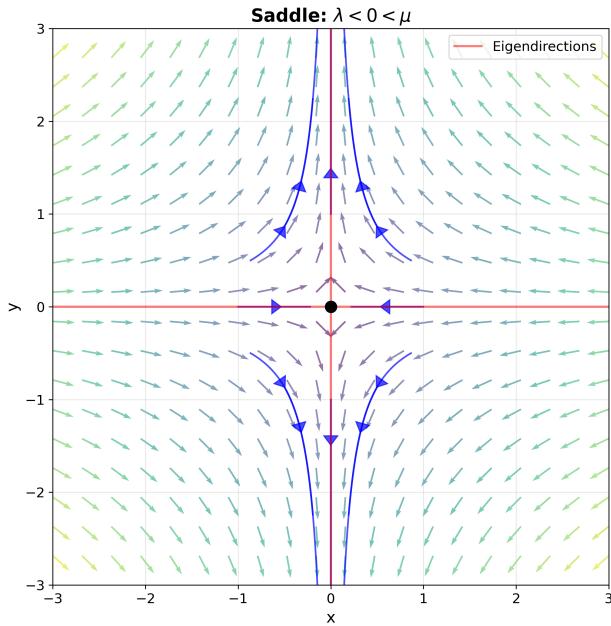
2.1.1 Unstable Node: $0 < \lambda < \mu$



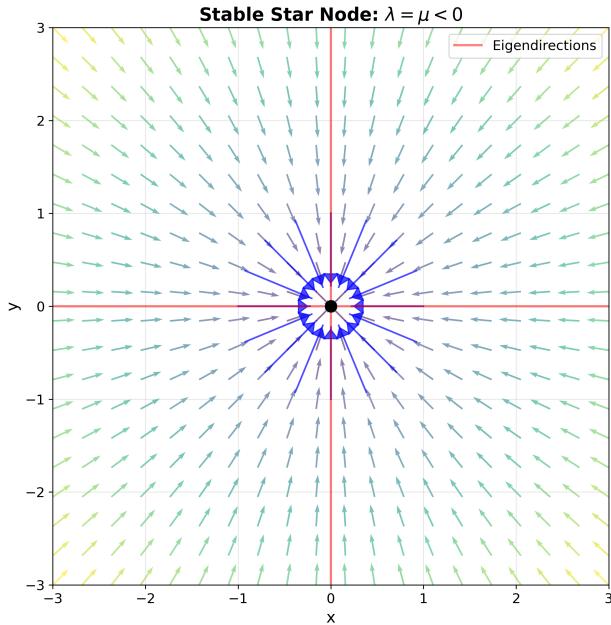
2.1.2 Stable Node: $\mu < \lambda < 0$



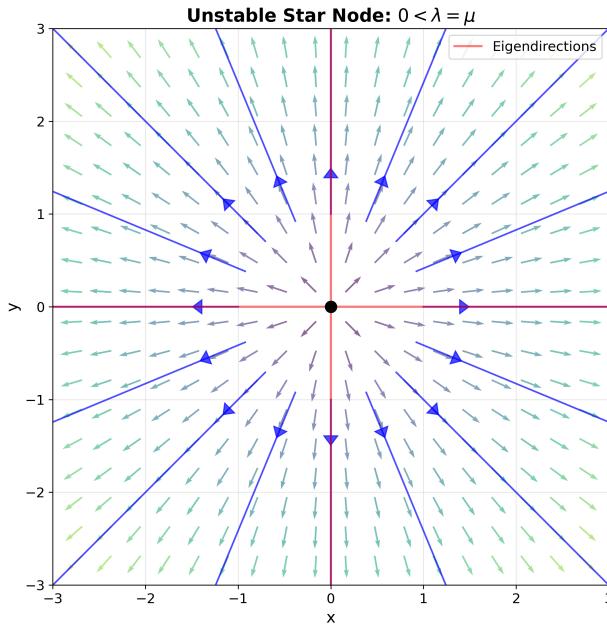
2.1.3 Saddle: $\lambda < 0 < \mu$



2.1.4 Stable Star Node: $\lambda = \mu < 0$



2.1.5 Unstable Star Node: $0 < \lambda = \mu$



2.2 Complex eigenvalues?

Let's say our A matrix looks like

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

where the eigenvalues are $\sigma \pm i\omega$.

$$\begin{aligned} \dot{x} &= \sigma x + \omega y & x &= r \cos \theta & \dot{x} &= \dot{r} \cos \theta - r \sin \theta \cdot \dot{\theta} \\ \dot{y} &= -\omega x + \sigma y & y &= r \sin \theta & \dot{y} &= \dot{r} \sin \theta + r \cos \theta \cdot \dot{\theta} \end{aligned}$$

Via some calculations that I should do later, we have

$$\begin{aligned} \dot{r} &= \sigma r \\ \dot{\theta} &= -\omega \end{aligned}$$

Now we can make some more plots!

- $\sigma < 0$ - stable spiral
- $\sigma > 0$ - unstable spiral
- $\sigma = 0$ - centers!

2.2.1 The calculations

To get \dot{r} let's first multiply \dot{x} by $\cos \theta$ and \dot{y} by $\sin \theta$, and add them to write

$$\begin{aligned} [\dot{r} \cos \theta - r \sin \theta \cdot \dot{\theta}] (\cos \theta) + [\dot{r} \sin \theta + r \cos \theta \cdot \dot{\theta}] (\sin \theta) &= \dot{x}(\cos \theta) + \dot{y}(\sin \theta) \\ \dot{r} \cos^2 \theta + \dot{r} \sin^2 \theta &= (\sigma x + \omega y) (\cos \theta) + (-\omega x + \sigma y) (\sin \theta) \\ \dot{r}(\cos^2 \theta + \sin^2 \theta) &= (\sigma r \cos \theta + \omega r \sin \theta) (\cos \theta) + (-\omega r \cos \theta + \sigma r \sin \theta) (\sin \theta) \\ \dot{r} &= \sigma r \cos^2 \theta + \sigma r \sin^2 \theta \\ \dot{r} &= \sigma r (\cos^2 \theta + \sin^2 \theta) \\ \dot{r} &= \sigma r \end{aligned}$$

To get $\dot{\theta}$, we'll multiply \dot{x} by $-\sin \theta$ and \dot{y} by $\cos \theta$, then add them:

$$\begin{aligned} (\dot{r} \cos \theta - r \sin \theta \cdot \dot{\theta}) (-\sin \theta) + (\dot{r} \sin \theta + r \cos \theta \cdot \dot{\theta}) (\cos \theta) &= \dot{x}(-\sin \theta) + \dot{y}(\cos \theta) \\ -\dot{r} \cos \theta \sin \theta + r \sin^2 \theta \cdot \dot{\theta} + \dot{r} \sin \theta \cos \theta + r \cos^2 \theta \cdot \dot{\theta} &= \dot{x}(-\sin \theta) + \dot{y}(\cos \theta) \\ r \sin^2 \theta \cdot \dot{\theta} + r \cos^2 \theta \cdot \dot{\theta} &= \dot{x}(-\sin \theta) + \dot{y}(\cos \theta) \\ r(\sin^2 \theta + \cos^2 \theta) \dot{\theta} &= \dot{x}(-\sin \theta) + \dot{y}(\cos \theta) \\ r \dot{\theta} &= \dot{x}(-\sin \theta) + \dot{y}(\cos \theta) \end{aligned}$$

Now substitute \dot{x} and \dot{y} :

$$\begin{aligned} r \dot{\theta} &= (\sigma x + \omega y)(-\sin \theta) + (-\omega x + \sigma y)(\cos \theta) \\ r \dot{\theta} &= -\sigma x \sin \theta - \omega y \sin \theta - \omega x \cos \theta + \sigma y \cos \theta \end{aligned}$$

Substitute:

$$\begin{aligned} r \dot{\theta} &= -\sigma r \cos \theta \sin \theta - \omega r \sin^2 \theta - \omega r \cos^2 \theta + \sigma r \sin \theta \cos \theta \\ r \dot{\theta} &= -\sigma r \cos \theta \sin \theta + \sigma r \sin \theta \cos \theta - \omega r \sin^2 \theta - \omega r \cos^2 \theta \\ r \dot{\theta} &= 0 - \omega r (\sin^2 \theta + \cos^2 \theta) \\ r \dot{\theta} &= -\omega r \\ \dot{\theta} &= -\omega \end{aligned}$$

2.3 What if A is non-diagonalizable?

Let's say that A looks like

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Our solution would look like

$$\begin{aligned} y(t) &= e^{\lambda t} \cdot y_0 \\ x(t) &= e^{\lambda t} \cdot x_0 + y_0 t e^{\lambda t} \end{aligned}$$

Now draw the flow field for $\lambda < 0$.

2.3.1 General cases

If A is not diagonal, find eigenvalues v_1, v_2 such that

$$Av_1 = \lambda v_1 \quad Av_2 = \lambda v_2$$

Now draw the resulting flow field.

2.4 The “big map” for linear systems

Let's say that A looks like

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

where we can find the eigenvalues via

$$\det(sI - A) = s^2 - (\lambda + \mu)s + \lambda\mu = 0$$

where the trace is $\lambda + \mu$ and the determinant of A is $\lambda\mu$.

From the trace and determinant, we can draw the “big map” for linear systems in the plane that we should supply after class.

3 Invariant sets, attracting sets

Positively invariant set: Say that we have a set of initial conditions $x \in A$. We can think of a flow map that directs the intial conditions to some new point $\phi_t(x)$. We can think of the new set of A under the flow map as $\phi_t(A)$.

If $\phi_t(A) \subset A$ for all $t > 0$, then A is positively invariant. (ϕ_t must exist for all $t > 0$)

If $\phi_t(A) \subset A$ for all $t < 0$, then A is negatively invariant.

If $\phi_t(A) \subset A$ for all t , then A is invariant.

3.1 Trapping region

Definition: A trapping region is a closed, connected set D such that $\phi_t(D) \subset \text{interior}(D)$ for all $t > 0$.

3.1.1 Example

$$\dot{x} = 1 - x^2$$

Now consider the three sets

- $(-\infty, -1]$ is negatively invariant but not positively invariant due to finite-time blow-up
- $[1, \infty)$ is positively invariant
- $[-1, 1]$ is invariant

3.1.2 Post class clarification

The definition of a positively invariant set and a trapping region look similar, but they are indeed different. Take the following region

$$D = \{1 \leq r \leq 2\}$$

for the following system

$$\dot{r} = r(2 - r)(r - 1), \quad \dot{\theta} = 1$$

The region D is positively invariant since trajectories stay in D , but it is not a trapping region because trajectories appraoch the boundary circles $r = 1$ and $r = 2$ (the equilibria). However, the slightly larger region

$$D' = \{0.5 \leq r \leq 2.5\}$$

is a trapping region because trajectories starting in D' flow to into the interior and toward D .

3.2 Attracting set

A closed, invariant set A is called an attracting set if \exists a neigborhood U of A such that $\phi_t(x) \in U \quad \forall t > 0$ and $\phi_t(x) \rightarrow A \quad x \in U$.

Definition: The domain/basin of attraction of A is $\bigcup \phi_t(U)$ for all $t \leq 0$.

3.2.1 Example 1

Let's consider the system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x - y\end{aligned}$$

where our matrix A is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

and the trace is -1 and the determinant is 1 ; this is a sink. Our attracting set A is $A = (0, 0)$ and the basin of attraction is \mathbb{R}^2 .

3.2.2 Example 2

Let's consider the system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x + \epsilon(1 - x^2)y\end{aligned}$$

where the limit cycle Γ is the attracting set and the basin of Γ is $\mathbb{R}^2 \setminus (0, 0)$

3.2.3 Example 3

Let's consider the system

$$\begin{aligned}\dot{x} &= x - x^3 \\ \dot{y} &= -y\end{aligned}$$

Is the set

$$A = \{(x, 0) : x \in [-1, 1]\}$$

an attracting set? Yes, despite the origin being an unstable fixed point. Note: we will have a different definition for an attractor, where A is not an attractor.

3.3 Stable and unstable eigenspaces

Let's say that we have a nonlinear system $\dot{x} = f(x)$ and there is an equilibrium point at x_0 (i.e. $f(x_0) = 0$). Linearization about x_0 is

$$\dot{\xi} = Df(x_0) \cdot \xi$$

Then, there are 3 invariant subspaces for the linear system

- stable subspace, E^s and $\dim E^s = n_s$ and $E^s = \text{span } \{v_1, \dots, v_{n_s}\}$ where v_1, \dots, v_{n_s} are eigenvectors
 - More explicitly, stable subspace E^s : spanned by (generalized) eigenvectors corresponding to eigenvalues with negative real part ($\text{Re}(\lambda) < 0$)
- center subspace, E^c and $\dim E^c = n_c$ and $E^c = \text{span } \{ \quad \}$
 - More explicitly, center subspace E^c : spanned by (generalized) eigenvectors corresponding to eigenvalues with zero real part ($\text{Re}(\lambda) = 0$)
- unstable subspace, E^u and $\dim E^u = n_u$ and $E^u = \text{span } \{ \quad \}$
 - More explicitly, unstable subspace E^u : spanned by (generalized) eigenvectors corresponding to eigenvalues with positive real part ($\text{Re}(\lambda) > 0$)

3.3.1 Can we have growth in E^c ?

Suppose we have

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= 0\end{aligned}$$

where

$$\begin{aligned}y &= C \\ x &= x_0 + Ct\end{aligned}$$

where $E^c = \mathbb{R}^2$ but linear growth!

3.4 Hyperbolic fixed point

Let p be a fixed point of $\dot{x} = f(x)$ such that $f(p) = 0$.

Definition: p is a hyperbolic fixed point if $Df(x_0)$ has no eigenvalues on imaginary axis (i.e., $n_c = 0$).

4 Hartman-Grobman theorem

If p is a hyperbolic fixed point, then there exists a homeomorphism h defined on a neighbourhood U of p , taking orbits of the flow ϕ_t of $\dot{x} = f(x)$ to those of the flow $e^{t \cdot Df(x_0)}$ of $\dot{\xi} = Df(p)\xi$.

h preserves the sense of orbits and can be chosen to preserve the time parameterization

Note that a homeomorphism is a continuous map with continuous inverse, but is maybe not differentiable!

1. Why is hyperbolic necessary? Suppose $\dot{x} = -x$ and $\dot{y} = y^2$. This becomes $\dot{\xi} = -\xi$ and $\dot{\eta} = 0$, and there are no mappings h between the two.
2. Why not a diffeomorphism? I forgot the argument, but I think we will come back to it later in the class.

5 Stable manifold theorem

Let p be a fixed point, and U be a neighbourhood of p .

We can define a local stable manifold of p as

$$W_{\text{loc}}^s(p) = \{x \in U | \phi_t(x) \rightarrow p \text{ as } t \rightarrow \infty \text{ and } \phi_t(x) \in U \text{ for all } t \geq 0\}$$

We can define the unstable manifold as

$$W_{\text{loc}}^u(p) = \{x \in U | \phi_t(x) \rightarrow p \text{ as } t \rightarrow -\infty \text{ and } \phi_t(x) \in U \text{ for all } t \leq 0\}$$

5.1 Now the theorem

Stable manifold theorem for a fixed point: For a p hyperbolic fixed point of $\dot{x} = f(x)$, \exists local stable and unstable manifold $W_{\text{loc}}^s(p), W_{\text{loc}}^u(p)$ such that

- same dimensions as E^s, E^u
- tangent to E^s, E^u

and $W_{\text{loc}}^s(p)$ and $W_{\text{loc}}^u(p)$ as smooth as f is.