Problem

(10 points) Consider the differential equation

$$\dot{x} = x^{1/3}, \qquad x(0) = 0.$$

The origin $x = 0$ is an equilibrium point, so one solution is $x(t) = 0$.

(a) Find another solution that satisfies $x(0) = 0$, but $x(t) \neq 0$ for $t > 0$, and therefore conclude that the solution is not unique. (Why does the uniqueness theorem not apply?)

(b) Find an infinite family of solutions that satisy the equation with $x(0) = 0$.

Notes

Part (a) Let's separate variables and solve the ODE:

$$\int x^{-1/3} dx = \int dt$$
$$\frac{3}{2} x^{2/3} = t + C$$
$$x(t) = \left(\frac{2}{3}(t + C)\right)^{3/2}$$

For the condition $x(0) = 0$, it follows that $C = 0$ and we have

$$x(t) = \left(\frac{2}{3} t\right)^{3/2}.$$

The uniqueness theorem doesn't apply because the derivative of $x^{1/3}$ at $x = 0$ is undefined since

$$f'(x) = \frac{1}{3x^{2/3}}.$$

(Local) uniqueness of solutions only apply for Lipschitz continuous functions, and continuously differentiable functions are synonmous with Lipschitz continuity.

Part (b) We can make an infinite family of solutions be defining the following piecewise equation

$$x(t) = \begin{cases} 0, & t < t_0 \\ \left(\frac{2}{3}(t - t_0)\right)^{3/2}, & t \geq t_0 \end{cases}$$

where we have $x(t_0) = 0$ and

$$\begin{aligned}
\frac{d}{dt}\left[\left(\frac{2}{3}(t-t_0)\right)^{3/2}\right] &= \frac{3}{2}\left(\frac{2}{3}(t-t_0)\right)^{1/2} \cdot \frac{d}{dt}\left[\frac{2}{3}(t-t_0)\right] \\
&= \frac{3}{2}\left(\frac{2}{3}(t-t_0)\right)^{1/2} \cdot \frac{2}{3} \\
&= \left(\left(\frac{2}{3}(t-t_0)\right)^{3/2}\right)^{1/3} \\
&= x^{1/3}
\end{aligned}$$

and hence, we still have $\dot{x} = x^{1/3}$.

Given that t_0 is arbitrary, our solution

$$x(t) = \begin{cases} 0, & t < t_0 \\ \left(\frac{2}{3}(t-t_0)\right)^{3/2}, & t \geq t_0 \end{cases}$$

yields and infinite number of solutions.

Solution
