Problem

(20 points) *Linear vector fields and maps.*

(a) For the following linear ODE, compute the stable, unstable, and center subspaces E^s, E^u, E^c of the origin. Sketch these subspaces and some representative trajectories of the system in the phase space.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) For the following linear map on $\mathbb{R}^2$, compute the stable, unstable, and center subspaces E^s, E^u, E^c of the origin. What are the qualitative differences in the dynamics between $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$? Sketch the subspaces and some representative trajectories of the system in the phase space.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad |\lambda| < 1.$$

Notes

Part (a) Let's start with finding the eigenvalues. It's a slog, but we can write

$$A - \lambda I = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} - \lambda I = \begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix}$$

where the eigenvalues are computed as $\det(A - \lambda I) = 0$. Thankfully, we can use the formula for 3×3 matrix determinants of

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Plugging everything in we get

$$\begin{gathered}
(1-\lambda)\left((-5-\lambda)(4-\lambda) - 3(-6)\right) - (-3)\left(3(4-\lambda) - 3\cdot 6\right) + (3)\left(3(-6) - (-5-\lambda)\cdot 6\right) \\
((1-\lambda)(-5-\lambda)(4-\lambda) + 18(1-\lambda)) + (9(4-\lambda) - 54) + (-54 - 18(-5-\lambda)) \\
\left((-5+4\lambda+\lambda^2)(4-\lambda) + 18 - 18\lambda\right) + (36 - 9\lambda - 54) + (-54 + 90 + 18\lambda) \\
(-20 + 21\lambda - \lambda^3) + 18 - 18\lambda + 36 - 9\lambda - 54 - 54 + 90 + 18\lambda \\
(-20 + 18 + 36 - 54 - 54 + 90) + (21 - 18 - 9 + 18)\,\lambda + \lambda^3 \\
16 + 12\lambda + \lambda^3
\end{gathered}$$

Setting it equal to zero and doing some factorization, we get

$$\begin{gathered}
16 + 12\lambda + \lambda^3 = 0 \\
(4-\lambda)(4 + 4x + x^2) = 0 \\
(4-\lambda)(2+\lambda)(2+\lambda) = 0
\end{gathered}$$

where it follows that the eigenvalues are $\lambda_1 = 4$, $\lambda_2 = -2$, and $\lambda_3 = -2$.

To find eigenvalues, we can solve for the equation

$$\begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for each eigenvalue. Starting with $\lambda_1 = 4$, we have

$$\begin{bmatrix} 1-4 & -3 & 3 \\ 3 & -5-4 & 3 \\ 6 & -6 & 4-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where the last row implies that $x_1 = x_2$. Plugging that into the expression $-3x_1 - 3x_2 + 3x_3 = 0$, we have

$$\begin{aligned} -3x_1 - 3x_1 + 3x_3 &= 0 \\ -6x_1 + 3x_3 &= 0 \\ x_3 &= 2x_1 \end{aligned}$$

Choosing $x_1 = 1$, we have an eigenvector corresponding to the eigenvalue $\lambda_1 = 4$ of

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Moving on to the eigenvalue $\lambda_{2,3} = -2$, we can write

$$\begin{bmatrix} 1+2 & -3 & 3 \\ 3 & -5+2 & 3 \\ 6 & -6 & 4+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and since all rows are the same, we are justified in writing

$$x_3 = -x_1 + x_2$$

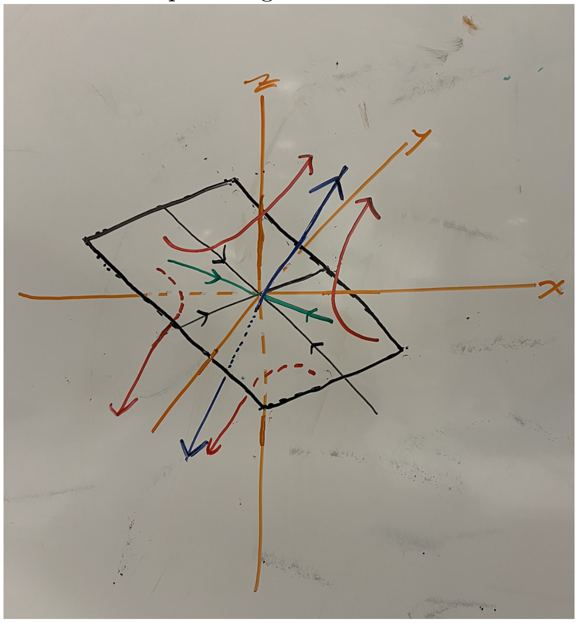
where the eigenvectors corresponding to $\lambda_{2,3} = -2$ span this plane. Choosing two arbitrarily, we have

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Compiling our work into the stable, unstable, and center subspaces E^s, E^u, E^c, we can write

$$\begin{aligned} E^u &= \operatorname{span}\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}\right\} \\ E^s &= \operatorname{span}\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\} \\ E^c &= \operatorname{span}\{0\} \end{aligned}$$

Here is the requisite figure:



Note the the black plane is E^s, the blue line is E^u, the red lines start just outside E^s and tend towards E^u, and the green lines start inside E^s and go to the origin.

Part (b) Okay so we can use SymPy to do the eigenvalues and eigenvectors. Praise be. We have

$$\left[\left(\text{eigenvalue } 1, \text{ eigenvector } \begin{bmatrix}1\\0\end{bmatrix}\right), \left(\text{eigenvalue } \lambda, \text{ eigenvector } \begin{bmatrix}-\frac{1}{\lambda-1}\\1\end{bmatrix}\right)\right]$$

Compiling our "work" into the stable, unstable, and center subspaces E^s, E^u, E^c, we can write

$$\begin{aligned} E^u &= \operatorname{span}\left\{\begin{bmatrix}-\frac{1}{\lambda-1}\\1\end{bmatrix}\right\} \\ E^s &= \operatorname{span}\{0\} \\ E^c &= \operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\} \end{aligned}$$

[Nudge] Hint *2026-02-10 22:41*

You're in great shape on part (a). For part (b), remember that the stability criteria for maps differ from flows — think carefully about what determines whether an eigenvalue corresponds to E^s, E^u, or E^c for a *discrete* map, and how the sign/magnitude of λ affects which subspace each eigenvector belongs to.

Ah yes that's right. I assigned the eigenvector to the unstable subspace instead of stable because $|\lambda| < 1$

$$\begin{aligned} E^s &= \text{span}\left\{\begin{bmatrix} -\frac{1}{\lambda-1} \\ 1 \end{bmatrix}\right\} \\ E^u &= \text{span}\left\{0\right\} \\ E^c &= \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \end{aligned}$$

Solution

Part (a) Let's start with finding the eigenvalues. It's a slog, but we can write

$$A - \lambda I = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} - \lambda I = \begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix}$$

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Plugging everything in we get

$$\begin{gathered} (1-\lambda)\left((-5-\lambda)(4-\lambda) - 3(-6)\right) - (-3)\left(3(4-\lambda) - 3\cdot 6\right) + (3)\left(3(-6) - (-5-\lambda)\cdot 6\right) \\ \left((1-\lambda)(-5-\lambda)(4-\lambda) + 18(1-\lambda)\right) + \left(9(4-\lambda) - 54\right) + \left(-54 - 18(-5-\lambda)\right) \\ \big((-5+4\lambda+\lambda^2)(4-\lambda) + 18 - 18\lambda\big) + (36 - 9\lambda - 54) + (-54 + 90 + 18\lambda) \\ (-20 + 21\lambda - \lambda^3) + 18 - 18\lambda + 36 - 9\lambda - 54 - 54 + 90 + 18\lambda \\ (-20 + 18 + 36 - 54 - 54 + 90) + (21 - 18 - 9 + 18)\,\lambda + \lambda^3 \\ 16 + 12\lambda + \lambda^3 \end{gathered}$$

Setting it equal to zero and doing some factorization, we get

$$\begin{gathered} 16 + 12\lambda + \lambda^3 = 0 \\ (4-\lambda)(4 + 4x + x^2) = 0 \\ (4-\lambda)(2+\lambda)(2+\lambda) = 0 \end{gathered}$$

where it follows that the eigenvalues are $\lambda_1 = 4$, $\lambda_2 = -2$, and $\lambda_3 = -2$.

To find eigenvalues, we can solve for the equation

$$\begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for each eigenvalue. Starting with $\lambda_1 = 4$, we have

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where the eigenvectors corresponding to $\lambda_{2,3} = -2$ span this plane. Choosing two arbitrarily, we have

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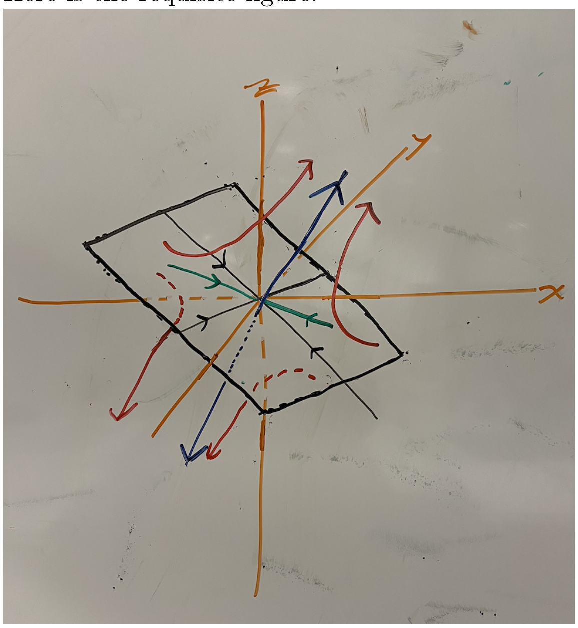
$$E^c = \operatorname{span}\{0\}$$

Here is the requisite figure:



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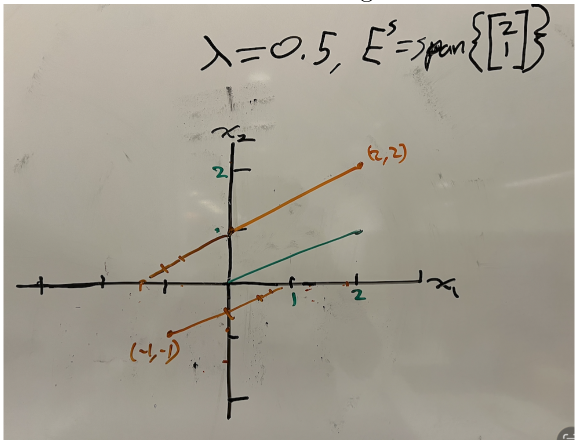
$$\left[\left(\text{eigenvalue } 1,\ \text{eigenvector } \begin{bmatrix}1\\0\end{bmatrix}\right), \left(\text{eigenvalue } \lambda,\ \text{eigenvector } \begin{bmatrix}-\frac{1}{\lambda-1}\\1\end{bmatrix}\right)\right]$$

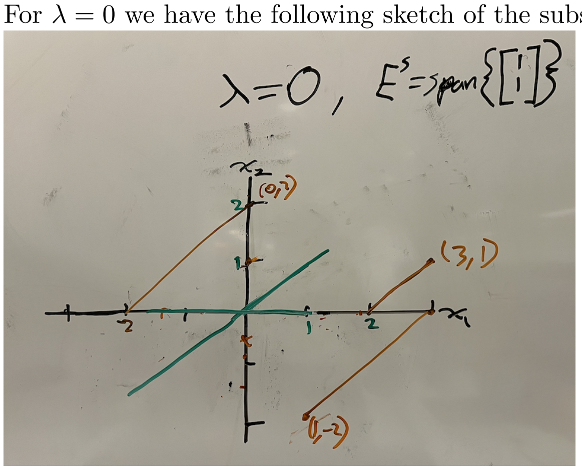
Compiling our "work" into the stable, unstable, and center subspaces E^s, E^u, E^c, we can write

$$E^s = \operatorname{span}\left\{\begin{bmatrix}-\frac{1}{\lambda-1}\\1\end{bmatrix}\right\}$$
$$E^u = \operatorname{span}\{0\}$$
$$E^c = \operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$$

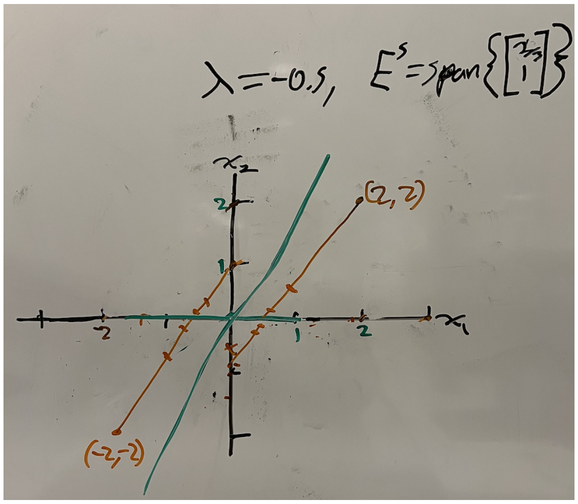
For $\lambda > 0$ we have the following sketch of the subspaces and some representative trajectories



For $\lambda = 0$ we have the following sketch of the subspaces and some representative trajectories



For $\lambda < 0$ we have the following sketch of the subspaces and some representative trajectories



The qualitative differences in the dyanmics for different values of λ are as follows:

$\lambda > 0$ — Trajectories asymptotically decay to the center subspace with a slope equal to the stable subspace.

$\lambda = 0$ — Trajectories directly fall onto the center subspace with a slop equal to the stable subspace (slope 1).

$\lambda < 0$ — Trajectories asymptotically decay to the center subspace with a slope equal to the stable subspace and oscillating across the x-axis, as we'd expect with negative λ values.