1 Plan for Class

1. Stability of an equilibrium point
2. Lyapunov functions
3. Linearization about an equilibrium

2 Example from last time

$$\dot{x} = y$$
$$\dot{y} = x - x^2 + \epsilon\left[\alpha y + \beta x y\right]$$

If $\epsilon = 0$, system is Hamiltonian:

$$H(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3}$$

Note that H is constant along trajectories. In a Hamiltonian system you follow level sets.

What if $\epsilon > 0$? If $\alpha, \beta < 0$, then H always decreases.

$$\begin{aligned}\frac{d}{dt}H(x(t), y(t)) &= \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} \\ &= \epsilon y^2\left[\alpha + \beta x\right] < 0\end{aligned}$$

when $x < 0$ given $\alpha, \beta < 0$. Visually, this makes the point $(x, y) = (1, 0)$ a sink.

This example is nice because it allows us to develop and test a bunch of tools that classify if we get homoclinic orbits, limit cycles, sinks, sources, etc. In general, this is a nice system to see all the phenomenon one expects in dynamical systems.

2.1 Post class notes

Just a refresher on how to define the Hamiltonian system. For a Hamiltonian system, the equations of motion are

$$\dot{x} = \frac{\partial H}{\partial y}$$
$$\dot{y} = -\frac{\partial H}{\partial x}$$

If we use $\epsilon = 0$, then we can write

$$\dot{x} = y$$
$$\dot{y} = x - x^2$$

Given the previous 4 equations, we can set out to find $H(x, y)$. From the first equation, we have

$$\dot{x} = y = \frac{\partial H}{\partial y}$$

Integrating with respect to y, we have

$$H(x, y) = \frac{y^2}{2} + f(X)$$

where $f(x)$ is an arbitrary function of x alone. From the second equation, we can write

$$\dot{y} = x - x^2 = -\frac{\partial H}{\partial x}$$

Now differentiate our expression for H with repsect to x, we have

$$\frac{\partial H}{\partial x} = \frac{\partial}{\partial x}\left[\frac{y^2}{2} + f(x)\right] = f'(x)$$

So we need

$$x - x^2 = -f'(x)$$
$$f'(x) = -x + x^2$$

Integrating, we have

$$f(x) = -\frac{x^2}{2} + \frac{x^3}{3} + C$$

where C is a constant (which we can set to 0). Therefore

$$H(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3}$$

which matches the Hamiltonian given from class.

3 Stability of an equilibrium point

Three classes of stability:

1. Asymptotically stable (perturbations get asymptotically closer to the fixed point)
2. Lyapunov stable (orbit around the fixed point)
3. Unstable stable (perturbations get arbitrarily far away)

3.1 Equilibrium point

For $\dot{x} = f(x)$, the equilibrium point is simply

$$f(x) = 0 \iff x_0 \text{ eq. pt}$$

or also called a fixed point. In other words, x_0 is stable if "when you start close, you stay close" (also called "Lyapunov stable"). More formally, $\forall \epsilon > 0, \exists \delta > 0$ such that $||x(t) - x_0|| < \epsilon$ wherever $||x(0) - x_0|| < \delta$ for all $t > 0$. In plain english, if you start in an epsilon ball, you stay in the epsilon ball.

x_0 is asymptotically stable if it is stable and in addition $x(t) \to x_0$ as $t \to \infty$. Centers are an example of x_0 being stable but not asymptotically stable. Sinks are example of asymptotically stable.

Unstable: x_0 is unstable if it is not stable.

3.1.1 Tricky example

$$\dot{\theta} = \sin \frac{\theta}{2}$$

$\theta_0 = 0$ is a fixed point, and things will always decay to it, but it is unstable because it violates our ϵ ball definition of stability. However the following example

$$\dot{\theta} = \sin \theta$$

has a fixed point at $\theta_0 = \pi$ that is asymptotically stable.

3.1.2 Hamiltonian example

The origin is unstable because there exists at least one perturbation that blows up to infinity. The point $(x, y) = (1, 0)$ is asymptotically stable if $\epsilon = 0$.

4 Lyapunov functions

Lyapunov functions are a way to show an equilibrium point is stable in some region $U \subset \mathbb{R}^n$. Say we have a C^1 function $V(x)$ such that:

- $V(x_0) = 0$
- $V(x) > 0$ for all $x \neq x_0$
- $\dot{V}(x) \leq 0$ for $x \neq x_0$ then x_0 is stable
- Further, if $\dot{V}(x) < 0$ for $x \neq x_0$ then x_0 is asymptotically stable

where the first two conditions imply that V is "positive definite".

Here $\dot{V}(x)$ means

$$\frac{d}{dt} V(x(t)) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = f(x) \cdot \nabla V(x)$$

4.1 Proof

Say we have such as function V such that $\dot{V} < 0$. Let η be the minimum value of $V(x)$ on[1] the sphere of radius ϵ about x_0. We know that the minimum value exists from analysis because V is a continuous function on a compact set, so it must have a maximum and minimum. We want to find a δ to complete our proof.

Because V is continuous, and $V(x_0) = 0$, $\exists\delta > 0$ such that $|V(x) - V(x_0)| < \eta$ where $||x - x_0|| < \delta$.

So $V(x) < \eta$ for all $x \in B\delta$ (where this is an open ball radius δ about x_0) and $V(x)$ decreases along a trajectory $x(t)$. So $x(t)$ can never leave B_ϵ. If it did, it would need to intersect the sphere of radius ϵ, and on that sphere, $V(y) \geq \eta$, which is a contraction.

To get asymptotically stable, since $V(x)$ strictly decreases, $V(x(t))$ must go to 0 as $t \to \infty$.

4.1.1 After class notes

I prompted Claude for a more explicit proof and got the following.

Setup We want to prove that if we have a Lyapunov function V with the stated properties, then x_0 is stable (and asymptotically stable if $\dot{V} < 0$).

Step 1: Define the sphere of radius ϵ Consider a sphere S_ϵ of radius ϵ centered at x_0. We are trying to show that if we start close enough to x_0, we will stay inside the sphere.

Step 2: Find η (the minimum of V on the sphere) Let η = minimum of $V(x)$ on the sphere S_ϵ (i.e. where $\|x - x_0\| = \epsilon$). Since

- V is continuous (it's C^1)
- The sphere S_ϵ is a compact set (closed and bounded)

by the Extreme Value Theorem, V attains its minimum on S_ϵ, so η exists. The **key observation** is that since $V(x_0) = 0$ and $V(x) > 0$ for $x \neq x_0$, we have $\eta > 0$ (because all points on S_ϵ are away from x_0).

Step 3: Use continuity to find δ Since V is continuous and $V(x_0) = 0$, we can apply the definition of continuity. For our chosen $\eta > 0$, there exists $\delta > 0$ such that

$$\|x - x_0\| < \delta \Rightarrow |V(x) - V(x_0)| < \eta$$

Since $V(x_0) = 0$, this becomes

$$\|x - x_0\| < \delta \Rightarrow V(x) < \eta$$

So inside the ball B_δ of radius δ, we have $V(x) < \eta$.

[1] Where "on the sphere" means that it is on the shell of the sphere, not inside.

Step 4: The key argument (why trajectories can't escape) Now suppose we start at some point $x(0)$ with $\|x(0) - x_0\| < \delta$. Then $V(x(0)) < \eta$ from Step 3. Since $\dot{V}(x) \leq 0$ along trajectories, $V(x(t))$ is non-increasing. Therefore, we can write

$$V(x(t)) \leq V(x(0)) < \eta \qquad \text{for all } t > 0$$

Now suppose that $x(t)$ does leave B_ϵ at some time t^*. Then $x(t^*)$ must be on the sphere S_ϵ (it has to cross the boundary). But on S_ϵ, we know $V(x) \geq \eta$. This gives us

$$V(x(t^*)) \geq \eta$$

which is a contradiction given we previously showed $V(x(t)) < \eta$. Hence, $x(t)$ cannot leave B_ϵ if it starts in B_δ, completing the proof.

Asymptotic Stability If $\dot{V}(x) < 0$ (strictly negative) for $x \neq x_0$, then

- $V(x(t))$ is strictly decreasing along trajectories (except at x_0)
- $V(x(t))$ is bounded below by 0
- By the Monotone Convergence Theorem, $V(x(t)) \to$ some limit $L \geq 0$

Since V is strictly decreasing and bounded below, we must have $V(x(t)) \to 0$. Since $V(x) = 0$ only at $x = x_0$, and V is continuous, this implies

$$x(t) \to x_0 \qquad \text{as} t \to \infty$$

This proves asymptotic stability.

4.2 Simple pendulum example

From Newton, we have

$$ml^2\ddot{\theta} = -mgl\sin\theta - \mu\dot{\theta}$$

where we can rewrite it as

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= \frac{1}{ml^2}\left(-mgl\sin\theta - \mu\omega\right)\end{aligned}$$

We can imagine the phase space as a cylinder.

Let's prove that $(0,0)$ is stable for $\mu \geq 0$. We will give a Lyapunov from the energy

$$V(\theta, \omega) = \frac{1}{2}ml^2\omega^2 + mgl(1 - \cos\theta)$$

Let's make sure that $V(0,0) = 0$, and it does. Let's make sure that it is positive definite. Indeed it is since $\omega^2 > 0$ and $1 - \cos\theta$ when $\theta, \omega \in [-\pi.\pi]$ excluding 0.

Is it decreasing?

$$\begin{aligned}\dot{V}(\theta,\omega) &= \frac{\partial V}{\partial \theta}\dot{\theta} + \frac{\partial V}{\partial \omega}\dot{\omega} \\ &= (mgl\sin\theta)\omega + \omega\left[-mgl\sin\theta - \mu\omega\right] \\ &= -\mu\omega^2 \leq 0\end{aligned}$$

for all $\omega, \theta \neq 0$. This proves stability, and to prove asymptotic stability, we can use La Salle's invariance principle to conclude that starting at $\omega = 0$ for any $\theta \neq 0$ is not invariant, so is not stable.

5 Linearization about an equilibrium

Say that for $\dot{x} = f(x)$, we have an equilibrium point x_0 such that $f(x_0) = 0$. Via the taylor series, we can write

$$\dot{x} = f(x) = f(x_0) + Df(x_0) \cdot (x - x_0) + \text{Higher order terms}$$

For Linearization, we ignore the higher order terms which may or may not be justified. Now we get the linearized system $\xi = x - x_0$ that is

$$\dot{\xi} = Df(x_0) \cdot \xi$$

where $Df(x_0)$ is an $n \times n$ martix.

Let $A = Df(x_0)$. Many properties of this linear system are determined by eigenvalues of A.

$$Av = \lambda v$$

where λ is an eigenvalue and v is an eigenvector. To verify, we can write

$$\xi(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n$$
$$A\xi(t) = c_1 e^{\lambda_1 t} \lambda_1 v_1 + \cdots + c_n e^{\lambda_n t} \lambda_n v_n = \dot{\xi}(t)$$

If $\lambda_1, \cdots, \lambda_n$ all have real part strictly negative, $\xi(t) \to 0$ as $t \to \infty$, which is asymptotically stable. If any of $\lambda_1, \cdots, \lambda_n$ have real part > 0, then we get someting unstable.

5.1 Linearization theorem

(Special case of Hartman-Grobman theorem) Let x_0 be an equilibrium point for $\dot{x} = f(x)$, and let $A = Df(x_0)$.

- If all eigenvalues of A have strictly negative real parts, then x_0 is asymptotically stable.
- If any eigenvalues of A has strictly positive real parts, then x_0 is unstable.
- Otherwise, inconclusive.

We could get cases where the real part is zero, but the imaginary parts are nonzero.

5.1.1 Proof sketch

Suppose A "has only eigenvalues with negative real part" (also called Horwitz). Construct a Lyapunov function.

Result from linear systems: If A is Horwitz, then $A^\top P + PA = Q$ for any positive definite symmetric Q, solution P is positive definite symmetric.

Let $V(\xi) = \xi^\top P\dot{\xi} + \xi^\top P\xi =$