

**Problem**

(20 points) Consider the system of ODEs given by

$$\dot{x} = -y + ax(x^2 + y^2)$$

$$\dot{y} = x + ay(x^2 + y^2)$$

- (a) Find all the equilibrium points for this system (for any value of the parameter  $a$ ).
- (b) Find the equations linearized about  $(x, y) = (0, 0)$ . What does the linearization let you conclude about the stability of the equilibrium point (for the nonlinear system)?
- (c) Write the system (1) in polar coordinates  $(r, \theta)$ , with

$$x = r \cos \theta$$

$$y = r \sin \theta$$

and determine the stability type of the equilibrium at the origin. Consider separately the cases  $\alpha < 0$ ,  $\alpha = 0$ , and  $\alpha > 0$ .

**Notes**

**Part (a)** Let's start by setting our system equal to zero

$$0 = -y + ax(x^2 + y^2) \tag{1}$$

$$0 = x + ay(x^2 + y^2) \tag{2}$$

Now multiply equation (1) by  $x$  and equation (2) by  $y$  to get

$$0 = -xy + ax^2(x^2 + y^2)$$

$$0 = xy + ay^2(x^2 + y^2)$$

and adding them gives us

$$\begin{aligned} 0 &= -xy + ax^2(x^2 + y^2) + xy + ay^2(x^2 + y^2) \\ &= ax^2(x^2 + y^2) + ay^2(x^2 + y^2) \\ &= a(x^2 + y^2)(x^2 + y^2) \\ &= a(x^2 + y^2)^2 \end{aligned}$$

where if  $a \neq 0$ , then our only fixed point is the origin  $(0, 0)$ . If  $a = 0$ , then our earlier equations are  $\dot{x} = -y$  and  $\dot{y} = x$  and the origin is still our only fixed point.

**[Nudge] Hint**

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Your approach for part (a) is on the right track! You correctly identified that the origin  $(0, 0)$  is always an equilibrium point, and your algebraic manipulation is sound. However, you might want to double-check whether you've found \*all\* possible equilibrium points by looking more carefully at what happens when  $x^2 + y^2 = 0$  versus when  $x^2 + y^2 \neq 0$ .

**Part (b)** Our linearization becomes  $\dot{\xi} = Df(x_0)\xi$  where  $Df(x_0)$  is the jacobian

$$J_f(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

Evaluating each expression, we have

$$\begin{aligned} \frac{\partial f_1}{\partial x} &= a(x^2 + y^2) + ax(2x) = ay^2 + 3ax^3 \\ \frac{\partial f_1}{\partial y} &= -1 + 2axy \\ \frac{\partial f_2}{\partial x} &= 1 + 2axy \\ \frac{\partial f_2}{\partial y} &= a(x^2 + y^2) + ay(2y) = ax^2 + 3ay^3 \end{aligned}$$

Evaluating each of the expressions at  $(0, 0)$ , we have

$$Df(x_0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and our linearization is

$$\dot{\xi} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \xi$$

To determine stability, we can compute the eigenvalues via  $\det(A - \lambda I) = 0$ .

$$0 = \det(A - \lambda I) = \lambda^2 + 1$$

implying that  $\lambda = \pm i$  and that we cannot conclude anything about the stability of the equilibrium point.

**Part (c)** Let's start with deriving  $\dot{r}$  via the relationship  $r^2 = x^2 + y^2$ . We can write

$$\begin{aligned} \frac{d}{dt} [r^2 = x^2 + y^2] \\ 2r\dot{r} = 2x\dot{x} + 2y\dot{y} \end{aligned}$$

and substitute in the values for  $\dot{x}, \dot{y}$  to get

$$\begin{aligned} 2r\dot{r} &= 2x(-y + ax(x^2 + y^2)) + 2y(x + ay(x^2 + y^2)) \\ 2r\dot{r} &= -2xy + 2ax^2(x^2 + y^2) + 2xy + 2ay^2(x^2 + y^2) \\ 2r\dot{r} &= 2ax^2(x^2 + y^2) + 2ay^2(x^2 + y^2) \\ 2r\dot{r} &= 2a(x^2 + y^2)(x^2 + y^2) \end{aligned}$$

and substituting in  $r^2 = x^2 + y^2$ , we can write

$$\begin{aligned} 2r\dot{r} &= 2ar^4 \\ \dot{r} &= ar^3 \end{aligned}$$

To derive  $\dot{\theta}$ , let's differentiate the relationship  $\tan \theta = y/x$  to get

$$\sec^2 \theta \cdot \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{x^2}$$

and plugging in  $x^2 = r^2 \cos^2 \theta$ , we can write

$$\begin{aligned} \sec^2 \theta \cdot \dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{r^2 \cos^2 \theta} \\ \sec^2 \theta \cdot \dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{r^2} \sec^2 \theta \\ \dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{r^2} \end{aligned}$$

Substituting for  $\dot{x}, \dot{y}$ , we have

$$\begin{aligned} \dot{\theta} &= \frac{x(x + ay(x^2 + y^2)) - y(-y + ax(x^2 + y^2))}{r^2} \\ \dot{\theta} &= \frac{x^2 + axy(x^2 + y^2) + y^2 - axy(x^2 + y^2)}{r^2} \\ \dot{\theta} &= \frac{x^2 + y^2}{r^2} \\ \dot{\theta} &= \frac{r^2}{r^2} \\ \dot{\theta} &= 1 \end{aligned}$$

Hence, system (1) in polar coordinates is

$$\begin{aligned} \dot{r} &= ar^3 \\ \dot{\theta} &= 1 \end{aligned}$$

To determine the stability type of the equilibrium at the origin, let's consider separately the cases  $\alpha < 0$ ,  $\alpha = 0$ , and  $\alpha > 0$ .

For  $\alpha < 0$ , we have  $\dot{r} < 0$ , implying that

$$\|(x(t), y(t)) - (0, 0)\| = \sqrt{x(t)^2 + y(t)^2} = \sqrt{r(t)} < \sqrt{r(0)} = \sqrt{x(0)^2 + y(0)^2} = \|(x(0), y(0)) - (0, 0)\|$$

Hence, given some  $\epsilon > 0$ , we can set  $\delta = \epsilon$  and it follows that for  $\|(x(0), y(0)) - (0, 0)\| < \delta$  we have

$$\|(x(t), y(t)) - (0, 0)\| < \|(x(0), y(0)) - (0, 0)\| < \delta = \epsilon$$

and

$$\|(x(t), y(t)) - (0, 0)\| < \epsilon.$$

Furthermore, our reasoning above implies that our trajectories are strictly decreasing and bounded below by the origin. Hence, via the monotone convergence theorem, we can state that as  $t \rightarrow \infty$  we have  $(x(t), y(t)) \rightarrow (0, 0)$  and the origin is asymptotically stable.

For  $\alpha = 0$ , we have  $\dot{r} = 0$  implying that  $r(0) = r(t)$ . Hence, we can write

$$\|(x(t), y(t)) - (0, 0)\| = \sqrt{x(t)^2 + y(t)^2} = \sqrt{r(t)} = \sqrt{r(0)} = \sqrt{x(0)^2 + y(0)^2} = \|(x(0), y(0)) - (0, 0)\|$$

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and

$$\|(x(t), y(t)) - (0, 0)\| < \epsilon.$$

Thus, our system is Lyapunov stable.

For  $\alpha > 0$ , we can solve for  $r$  to show that there are initial conditions that result in finite time blow up, showing the origin is unstable. By separating variables, we can write

$$\begin{aligned} \frac{dr}{ar^3} &= dt \\ \int \frac{dr}{ar^3} &= \int dt \\ -\frac{1}{2ar^2} &= t + C \\ -\frac{1}{2ar^2} &= t + C \\ r(t) &= \frac{1}{\sqrt{-2at + D}} \end{aligned}$$

For the initial condition  $r(0) = 1$ , we have  $D = 1$  and the resulting equation

$$r(t) = \frac{1}{\sqrt{-2at + 1}}$$

experiences finite time blow up as  $t \rightarrow 1$  given that  $a > 0$ . Hence the origin is unstable.

## Solution

**Part (a)** Let's start by setting our system equal to zero

$$0 = -y + ax(x^2 + y^2) \tag{3}$$

$$0 = x + ay(x^2 + y^2) \tag{4}$$

Now multiply equation (1) by  $x$  and equation (2) by  $y$  to get

$$0 = -xy + ax^2(x^2 + y^2)$$

$$0 = xy + ay^2(x^2 + y^2)$$

and adding them gives us

$$\begin{aligned} 0 &= -xy + ax^2(x^2 + y^2) + xy + ay^2(x^2 + y^2) \\ &= ax^2(x^2 + y^2) + ay^2(x^2 + y^2) \\ &= a(x^2 + y^2)(x^2 + y^2) \\ &= a(x^2 + y^2)^2 \end{aligned}$$

where if  $a \neq 0$ , then our only fixed point is the origin  $(0, 0)$ . If  $a = 0$ , then our earlier equations are  $\dot{x} = -y$  and  $\dot{y} = x$  and the origin is still our only fixed point.

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$$\frac{\partial f_1}{\partial y} = -1 + 2axy$$

$$\frac{\partial f_2}{\partial x} = 1 + 2axy$$

$$\frac{\partial f_2}{\partial y} = a(x^2 + y^2) + ay(2y) = ax^2 + 3ay^3$$

Evaluating each of the expressions at  $(0, 0)$ , we have

$$Df(x_0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and our linearization is

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$$2r\dot{r} = -2xy + 2ax^2(x^2 + y^2) + 2xy + 2ay^2(x^2 + y^2)$$

$$2r\dot{r} = 2ax^2(x^2 + y^2) + 2ay^2(x^2 + y^2)$$

$$2r\dot{r} = 2a(x^2 + y^2)(x^2 + y^2)$$

and substituting in  $r^2 = x^2 + y^2$ , we can write

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Substituting for  $\dot{x}, \dot{y}$ , we have

$$\begin{aligned} \dot{\theta} &= \frac{x(x + ay(x^2 + y^2)) - y(-y + ax(x^2 + y^2))}{r^2} \\ \dot{\theta} &= \frac{x^2 + axy(x^2 + y^2) + y^2 - axy(x^2 + y^2)}{r^2} \\ \dot{\theta} &= \frac{x^2 + y^2}{r^2} \\ \dot{\theta} &= \frac{r^2}{r^2} \\ \dot{\theta} &= 1 \end{aligned}$$

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Hence, given some  $\epsilon > 0$ , we can set  $\delta = \epsilon$  and it follows that for  $\|(x(0), y(0)) - (0, 0)\| < \delta$  we have

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experiences finite time blow up as  $t \rightarrow 1$  given that  $a > 0$ . Hence the origin is unstable.

#### [Correct] Solution Check

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The solution is mathematically correct throughout all parts. The equilibrium analysis, linearization, polar coordinate transformation, and stability analysis for all three cases ( $a < 0$ ,  $a = 0$ ,  $a > 0$ ) are all properly executed with valid reasoning and correct conclusions.