MAE 541 / APC 571: Applied Dynamical Systems

Homework 1

Assigned: 29 Jan 2026
Due: 5 Feb 2026

Instructions

- Collaboration is permitted, and encouraged, but everything you turn in should be your own work, and reflect your understanding.
- Your work should be submitted to Gradescope (link available on Canvas). Please make sure you match your work to the correct question/subquestion, as indicated on Gradescope.

1. (10 points) Each of the following differential equations has a single equilibrium point, at $x = 0$. For each equation, find the linearization about $x = 0$, and explain what you can conclude about the stability of the equilibrium for the nonlinear system. Also determine the stability type for the nonlinear system (e.g., unstable, Lyapunov stable, asymptotically stable).
   (a) $\dot{x} = -x$
   (b) $\dot{x} = -x^2$
   (c) $\dot{x} = -x^3$

2. (10 points) Consider the differential equation
$$\dot{x} = x^{1/3}, \qquad x(0) = 0.$$
   The origin $x = 0$ is an equilibrium point, so one solution is $x(t) = 0$.
   (a) Find another solution that satisfies $x(0) = 0$, but $x(t) \neq 0$ for $t > 0$, and therefore conclude that the solution is not unique. (Why does the uniqueness theorem not apply?) [*Hint:* The ODE is separable.]
   (b) Find an infinite family of solutions that satisfy the equation with $x(0) = 0$.

3. (20 points) Consider the system of ODEs given by
$$\begin{aligned} \dot{x} &= -y + ax(x^2 + y^2) \\ \dot{y} &= x + ay(x^2 + y^2) \end{aligned} \tag{1}$$
   (a) Find all the equilibrium points for this system (for any value of the parameter a).

(b) Find the equations linearized about $(x, y) = (0, 0)$. What does the linearization let you conclude about the stability of the equilibrium point (for the nonlinear system)?

(c) Write the system (1) in polar coordinates (r, θ), with

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and determine the stability type of the equilibrium at the origin. Consider separately the cases $a < 0$, $a = 0$, and $a > 0$.

4. (30 points) The following equations describe the motion of a ball in a spinning circular hoop, where α is the (nondimensional) speed of the spinning hoop, β is the (nondimensional) damping, x is the angle of the ball in the hoop, and y is the ball's angular velocity:

$$\dot{x} = y$$
$$\dot{y} = (\alpha\cos x - 1)\sin x - \beta y,$$

(a) Find the equilibrium points, as a function of the parameters α, β (consider $x \in [-\pi, \pi]$). Make a sketch of these equilibrium points as a function of α. (Such a plot is the beginning of a bifurcation diagram).

(b) Show that, for $\beta = 0$, the system is Hamiltonian: that is, there is a function $H(x, y)$ such that

$$\dot{x} = \frac{\partial H}{\partial y}$$
$$\dot{y} = -\frac{\partial H}{\partial x}.$$

(c) Fix $\alpha = 2$, and consider the stability of each of the equilibrium points you found in part (a). Be sure to consider the cases $\beta < 0$, $\beta = 0$, and $\beta > 0$. [*Hint:* How does the Hamiltonian H vary along trajectories $(x(t), y(t))$ that satisfy the dynamics?]

5. (30 points) Use the candidate Liapunov function

$$V(x, y) = \frac{1}{2}y^2 + (1 - \cos x)$$

to show that the origin $(x, y) = (0, 0)$ is a locally asymptotically stable fixed point for the system

$$\dot{x} = y$$
$$\dot{y} = -\varepsilon y^3 - \sin x$$

whenever $\varepsilon > 0$. [*Hint:* If you find a set of points S in the neighborhood of the fixed point with $\dot{V}(x, y) = 0$ for all $(x, y) \in S$, then please explain why the trajectories always leave this set.]