1 Lecture 10: Plan for Class

Last time:

1. Normal forms §3.3

Today:

1. More normal forms §3.3

2. Back to bifurcations §3.4

2 Last time: Normal forms

Let's say that we have the system

$$\dot{x} = Ax + f_2(x) + f_3(x) + \cdots$$

and we want to find a nonlinear change of coordinates

$$x = y + h_2(y)$$

such that we can push quadratic terms to higher order to look like

$$\dot{y} = Ay - [Dh_2(y) \cdot Ay - Ah_2(y) - f_2(y)] + O(y^3)$$

where $L_A \, h_2(y) = Dh_2(y) \cdot Ay - Ah_2(y)$. Note that f_2 is a homogeneous polynomial of degree 2 and L_A is a linear operator over homogeneous polynomials of degree 2.

Now, which terms in f_2 can we eliminate? Answer, those in the range of L_A where leftover terms determine the normal form. The procedure is to find

$$R_2 = \text{Range}\,(L_A)$$

where, again, $L_A : H_2 \to H_2$. Hence, we chose a complement G_2 such that

$$H_2 = R_2 \oplus G_2$$

where the basis for G_2 determine the leftover terms.

2.1 Looking at cubic terms

We could extend this approach to even push cubic terms to higher order, after pushing quadratic terms to higher order. Again, we are looking at

$$\dot{x} = Ax + f_2(x) + f_3(x) + \cdots$$

where $f_2(x)$ contains whatever was leftover from before, and our change of coordinates

$$x = y + h_3(y)$$

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This gives us a new system

$$\dot{y} = Ay + f_2(y) - [Dh_3(y) \cdot Ay - Ah_3(y) - f_3(y)] + O(y^4)$$

where $L_A(h_3) = Dh_3(y) \cdot Ay - Ah_3(y)$. We can simply use the same procedure as before and find the range of L_A, and then find the complement G_3 such that

$$H_3 = R_3 \oplus G_3$$

3   Notation from page 142

There is some new notation in the book that can be hard to parse. On page 142, it is written: we denote the components Y^1, Y^2 of the vector field Y by first order differential operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, so that

$$\begin{pmatrix} 2xy \\ x^2 \end{pmatrix} \qquad \text{denoted} \qquad 2xy\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}$$

This is just the directional derivative in the direction of Y.

4   Normal Form Example

Let's say that we have the matrix[1]

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

where our eigenvalues are $\pm$. Looking at H_2, this is the vector space

$$H_2 = \text{span} \left\{ \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix}, \right\}$$

Now looking at L_A, we have

$$(L_A\,h)\,(z) = Dh(z) \cdot Az - Ah(z)$$

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[1]This is just the rotation matrix.

What we want to do now is to write out L_A for each basis vector in H_2 to compute L_A. Note that $Az = \begin{bmatrix} -y \\ x \end{bmatrix}$, and we can write

$$\begin{pmatrix} x^2 \\ 0 \end{pmatrix}: \quad \begin{pmatrix} 2x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2xy \\ -x^2 \end{pmatrix}$$

$$\begin{pmatrix} xy \\ 0 \end{pmatrix}: \quad \begin{pmatrix} y & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} xy \\ 0 \end{pmatrix} = \begin{pmatrix} -y^2 + x^2 \\ -xy \end{pmatrix}$$

$$\begin{pmatrix} y^2 \\ 0 \end{pmatrix}: \quad \begin{pmatrix} 0 & 2y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2xy \\ -y^2 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ x^2 \end{pmatrix}: \quad \begin{pmatrix} 0 & 0 \\ 2x & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ x^2 \end{pmatrix} = \begin{pmatrix} x^2 \\ -2xy \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ xy \end{pmatrix}: \quad \begin{pmatrix} 0 & 0 \\ y & x \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ xy \end{pmatrix} = \begin{pmatrix} xy \\ x^2 - y^2 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ y^2 \end{pmatrix}: \quad \begin{pmatrix} 0 & 0 \\ 0 & 2y \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ y^2 \end{pmatrix} = \begin{pmatrix} y^2 \\ 2xy \end{pmatrix}$$

This gives us a matrix representation of L_A that is

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -2 & 0 & 2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 2 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}$$

which has a determinant of $\det = 9 \neq 0$. Hence, L_A is invertible and is rank 6. Hence! We can eliminate all degree 2 terms. But! This means that we have to look at degree 3 terms ...

4.1 Looking at cubic terms

Looking at H_3, we can write the space as

$$H_3 = \mathrm{span}\left\{ \begin{pmatrix} x^3 \\ 0 \end{pmatrix}, \begin{pmatrix} x^2 y \\ 0 \end{pmatrix}, \begin{pmatrix} x^1 y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^3 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 y \end{pmatrix}, \begin{pmatrix} 0 \\ x^1 y^2 \end{pmatrix}, \begin{pmatrix} 0 \\ y^3 \end{pmatrix} \right\}$$

where our dimension[2] of the space is 8. To compute the new L_A, we can write we can write

$$\begin{pmatrix} x^3 \\ 0 \end{pmatrix}: \quad \begin{pmatrix} 3x^2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -3x^2 y \\ -x^3 \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} 0 \\ y^3 \end{pmatrix}$$

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[2] Aside: What is the dimension of the linear space of scalar polynomials n in higher dimension d? Answer: we can use the stars and bars trick. Look it up yourself.

yeah we are not going to write this all out. Just use a computer.

The rank of the resulting L_A is 6. Hence, it is not invertible and we will get resonance terms. Then we can find these resonance terms by finding the null space of L_A transpose to find what terms are not in our range. Rowley has a Jupyter Notebook going through this; it's quite clean!

We get that the normal form of the Hopf bifurcation as

$$\dot{x} = -y + (ax + by)\left(x^2 + y^2\right) + O\left(x^4, y^4\right)$$
$$\dot{y} = x + (ax - by)\left(x^2 + y^2\right) + O\left(x^4, y^4\right)$$

In polar coordinates, this looks really nice. If we let

$$x = r\cos\theta$$
$$y = r\sin\theta$$

then we have

$$\dot{r} = ar^3$$
$$\dot{\theta} = 1 + br^2$$

4.2 Quick note

This system becomes more interesting when we allow A to be a parameter

$$A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

This could be on the homework?

4.3 Changing coordinates

We've been looking at analytic changes of coordinates like

$$x = y + h_2(y) + h_3(y) + \cdots$$

but some good terms we couldn't eliminate, like:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{bmatrix} x^2 - 2xy + y^2 \\ x^2 - 3xy + 4y^2 \end{bmatrix}$$

kinda don't remember what that was from.

What the Hartman-Grobman theorem says is that there's a homeomorphism h such that $x = h(y)$ in which f is linear! But! A homeomorphism need not be differentiable. We could have a C^0 function h.

As an example, consider

$$\dot{x} = x^2$$

Is there a coordinate change in which this is linear? Let

$$y = e^{-\frac{1}{x}}$$

where $x \neq 0$. We have

$$\dot{y} = \left(\frac{1}{x^2}e^{-\frac{1}{x}}\right)\dot{x} = \frac{1}{x^2}e^{-\frac{1}{x}} \cdot x^2 = e^{-\frac{1}{x}} = y$$

hence we have

$$\dot{y} = y$$

WHAT!?! We did a change of coordinates and got a linear function? This is because our function y is not analytic. If we look at analytic changes of coordinates, like we have been doing with normal forms, then our solutions are more reasonable.

This is all to say, the Hartman-Grobman theorem allows us to do some weird stuff with allowing h to just be a homeomorphism. In our cases in bifurcations, we are going to stick with diffeomorphisms.

5 Back to Bifurcations §3.4

Let's return our attention to co-dimension 1 bifurcations. Recall that

$$\dot{x} = \mu - x^2$$

gives us a saddle-node bifurcation. Recall that

$$\dot{x} = \mu x - x^2$$

gives us a transcritical bifurcation. Again, recall that

$$\dot{x} = \mu x - x^3$$

gives us a supercritical pitchfork bifurcation.

What if we want to classify bifurcations for more general systems? Well, we can use some bifurcation theorems!

Remember that local bifurcations occur when an eigenvalue crosses the imaginary axis as a parameter varies. In the case of the saddle-node, transcritical, and pitchfork bifurcations, we have an eigenvalue go from negative to positive on the real axis, crossing the imaginary axis in the process. In the case of a Hopf bifurcation, we have a complex conjugate pair of eigenvalues that both cross the imaginary axis.

5.1 Bifurcation theorems

Our procedure for classifying bifurcations for more general systems will be to

1. Find the equilibrium points

2. find a parameter value where eigenvalues cross the imaginary axis

3. Find the center manifold

4. Reduce dynamics to the center manifold

5. Find normal form to simplify nonlinear terms on the center manifold

This seems pretty laborious. Is there a shortcut?

Yes! We have bifurcation theorems. To start, we have three types of hypotheses.

Let's say that we are dealing with systems

$$\dot{x} = f(x, \mu)$$

and that $\bar{x}(\mu)$ is a branch of fixed points where

$$f'(\bar{x}(\mu), \mu) = 0$$

The three types of hypotheses are

1. "$D_x f(\bar{x}) = 0$" $\implies$ μ is a bifurcation point that is non-hyperbolic

2. "$D_x^2 f(\bar{x})$, $D_x^3 f(x) \neq 0$" implies non-degeneracy of higher-order terms

3. "$D_\mu f(\bar{x}) \neq 0$" implies that eigenvalues cross with non-zero "speed"

5.2   1 dimension theorems

Again, we have

$$\dot{x} = f(x, \mu)$$

where $x, \mu \in \mathbb{R}$. Suppose $x = p$ is a fixed point for some μ.

Theorem: If

1. $\frac{\partial f}{\partial x}(p) = 0$ (eigenvalue at 0)

2. $\frac{\partial^2 f}{\partial x^2}(p) \neq 0$ (non-degeneracy of good terms)

3. $\frac{\partial f}{\partial \mu}(p) \neq 0$ (crosses with non-zero speed)

Then $x = p$ is a saddle-node bifurcation at this value of μ.

Theorem: If

1. $\frac{\partial f}{\partial x}(p) = 0$

2. $\frac{\partial^2 f}{\partial x^2}(p) \neq 0$

3. $\frac{\partial f}{\partial x \, \partial \mu}(p) \neq 0$

Then $x = p$ is a transcritical bifurcation.

Theorem: If

1. $\frac{\partial f}{\partial x}(p) = 0$

2. $\frac{\partial^3 f}{\partial x^3}(p) \neq 0$

3. $\frac{\partial f}{\partial x \, \partial \mu}(p) \neq 0$

Then we have a pitchfork bifurcation. Note that if $\frac{\partial^3 f}{\partial x^3}(p) > 0$, we have a subcritical pitchfork bifurcation. And if $\frac{\partial^3 f}{\partial x^3}(p) < 0$, we have a supercritical pitchfork bifurcation.