1 Plan for Class

1. Linear systems in the plane

2. Invariant sets, attracting sets

3. Stable and unstable eigenspaces

4. Hartman-Grobman theorem 1.3.1

5. Stable manifold theorem 1.3.2

2 Linear systems

First let's defind the flow map: ϕ_t maps $x(0)$ to $x(t)$.

For linear systems, we are interested in

$$\dot{x} = Ax$$

$$x(t) = e^{At}$$

where the matrix exponential is defined as

$$e^{At} = I + At + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3$$

For the flow map, we can write

$$\psi_t = e^{At}$$

$$\psi_t(x_0) = e^{At}x_0$$

2.1 Special cases

Let's say we have a matrix A that is

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

which is

$$\dot{x} = \lambda x$$

$$\dot{y} = \mu y$$

Our solution to the flow field will look like

$$\frac{dy}{dx} = \frac{\mu y}{\lambda x}$$

$$\implies \frac{dy}{y} = \frac{\mu}{\lambda}\frac{dx}{x}$$

$$\implies \log y = \frac{\mu}{\lambda}\log x + C$$

$$\implies y = x^{\mu/\lambda} \cdot c'$$

and our solution to the dynamics look like

$$x(t) = e^{\lambda t} x_0$$
$$y(t) = e^{\mu t} y_0$$

Now let's draw out the flow field for various conditions (to do after class)

- $0 < \lambda < \mu$ - unstable node

- $\mu < \lambda < 0$ - stable node

- $\lambda < 0 < \mu$ - saddle

- $\lambda = \mu < 0$ - sink (straight lines)

- $0 < \lambda = \mu$ - source (straight lines)

2.2 Complex eigenvalues?

Let's say our A matrix looks like

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

where the eigenvalues are $\sigma \pm i\omega$.

$$\dot{x} = \sigma x + \omega y \qquad x = r \cos\theta \qquad \dot{x} = \dot{r}\cos\theta - r\sin\theta \cdot \dot{\theta}$$
$$\dot{y} = -\omega x + \sigma y \qquad x = r \sin\theta \qquad \dot{x} = \dot{r}\sin\theta + r\cos\theta \cdot \dot{\theta}$$

Via some calculations that I should do later, we have

$$\dot{r} = \sigma r$$
$$\dot{\theta} = -\omega$$

Now we can make some more plots!

- $\sigma < 0$ - stable spiral

- $\sigma > 0$ - unstable spiral

- $\sigma = 0$ - centers!

2.3 What if A is non-diagonalizable?

Let's say that A looks like

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Our solution would look like

$$y(t) = e^{\lambda t} \cdot y_0$$
$$x(t) = e^{\lambda t} \cdot x_0 + y_0 t e^{\lambda t}$$

Now draw the flow field for $\lambda < 0$.

2

2.3.1 General cases

If A is not diagonal, find eigenvalues v_1, v_2 such that

$$Av_1 = \lambda v_1 \qquad Av_2 = \lambda v_2$$

Now draw the resulting flow field.

2.4 The "big map" for linear systems

Let's say that A looks like

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

where we can find the eigenvalues via

$$\det(sI - A) = s^2 - (\lambda + \mu)s + \lambda\mu = 0$$

where the trace is $\lambda + \mu$ and the determinant of A is $\lambda\mu$.

From the trace and determinant, we can draw the "big map" for linear systems in the plane that we should supply after class.

3 Invariant sets, attracting sets

Positively invariant set: Say that we have a set of initial conditions $x \in A$. We can think of a flow map that directs the intial conditions to some new point $\phi_t(x)$. We can think of the new set of A under the flow map as $\phi_t(A)$.

If $\phi_t(A) \subset A$ for all $t > 0$, then A is positively invariant. (ϕ_t must exist for all $t > 0$)

If $\phi_t(A) \subset A$ for all $t < 0$, then A is negatively invariant.

If $\phi_t(A) \subset A$ for all t, then A is invariant.

3.1 Trapping region

Definition: A trapping region is a closed, connected set D such that $\phi_t(D) \subset$ interior(D) for all $t > 0$.

3.1.1 Example

$$\dot{x} = 1 - x^2$$

Now consider the three sets

- $(\infty, -1]$ is negatively invariant but not positively invariant due to finite-time blow-up

- $[1, \infty)$ is positively invariant

- $[-1, 1]$ is invariant

3.2 Attracting set

A closed, invariant set A is called an attracting set if $\exists$ a neighboorhood U of A such that $\phi_t(x) \in U \quad \forall t > 0$ and $\phi_t(x) \to A \quad x \in U$.

Definition: The domain/basin of attraction of A is $\bigcup \phi_t(U)$ for all $t \leq 0$.

3.2.1 Example 1

Let's consider the system

$$\dot{x} = y$$
$$\dot{y} = -x - y$$

where our matrix A is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

and the trace is -1 and the determinant is 1; this is a sink. Our attracting set A is $A = (0,0)$ and the basin of attraction is $\mathbb{R}^2$.

3.2.2 Example 2

Let's consider the system

$$\dot{x} = y$$
$$\dot{y} = -x + \epsilon(1 - x^2)y$$

where the limit cycle Γ is the attracting set and the basin of Γ is $\mathbb{R}^2 \backslash (0,0)$

3.2.3 Example 3

Let's consider the system

$$\dot{x} = x - x^3$$
$$\dot{y} = -y$$

Is the set

$$A = \{(x,0) : x \in [-1,1]\}$$

an attracting set? Yes, despite the origin being an unstable fixed point. Note: we will have a different definition for an attractor, where A is not an attractor.

3.3 Stable and unstable eigenspaces

Let's say that we have a nonlinear system $\dot{x} = f(x)$ and there is an equilibrium point at x_0 (i.e. $f(x_0) = 0$). Linearization about x_0 is

$$\dot{\xi} = Df(x_0) \cdot \xi$$

Then, there are 3 invariant supspaces for the linear system

4

- stable subspace, E^s and $\dim E^s = n_s$ and $E^s = \operatorname{span}\{v_1, \ldots, v_{n_s}\}$ where $v_1, \ldots, v_{n_s}$ are eigenvectors

- center subspace, E^c and $\dim E^c = n_c$ and $E^c = \operatorname{span}\{\quad\}$

- unstable subspace, E^u and $\dim E^u = n_u$ and $E^u = \operatorname{span}\{\quad\}$

3.3.1 Can have growth in E^c

Suppose we have

$$\dot{x} = y$$
$$\dot{y} = 0$$

where

$$y = C$$
$$x = x_0 + Ct$$

where $E^c = \mathbb{R}^2$ but linear growth!

3.4 Hyperbolic fixed point

Let p be a fixed point of $\dot{x} = f(x)$ such that $f(p) = 0$.

Definition: p is a hyperbolic fixed point if $Df(x_0)$ has no eigenvalues on imaginary axis (i.e., $n_c = 0$).

4 Hartman-Grobman theorem

If p is a hyperbolic fixed point, then there exists a homeomorphism h defined on a neighboorhood U of p, taking orbits of the flow ϕ_t of $\dot{x} = f(x)$ to those of the flow $e^{t \cdot Df(x_0)}$ of $\dot{\xi} = Df(p)\xi$.

h perserves the sense of orbits and can be chosen to perserve the time parameterization

Note that a homeomorphism is a continuous map with continuous inverse, but is maybe not differentiable!

1. Why is hyperbolic necessary? Suppose $\dot{x} = -x$ and $\dot{y} = y^2$. This becomes $\dot{\xi} = -\xi$ and $\dot{\eta} = 0$, and there are no mappings h between the two.

2. Why not a diffeomorphism? I forget the argument, but I think we will come back to it later in the class.

5   Stable manifold theorem

Let p be a fixed point, and U be a neighboorhood of p.

We can define a local stable manifold of p as

$$W_{\text{loc}}^s(p) = \{x \in U | \phi_t(x) \to p \text{ as } t \to \infty \text{ and } \phi_t(x) \in U \text{ for all } t \geq 0\}$$

We can define the unstable manifold as

$$W_{\text{loc}}^u(p) = \{x \in U | \phi_t(x) \to p \text{ as } t \to -\infty \text{ and } \phi_t(x) \in U \text{ for all } t \leq 0\}$$

5.1   Now the theorem

Stable manifold theorem for a fixed point: For a p hyperbolic fixed point of $\dot{x} = f(x)$, $\exists$ local stable and unstable manifold $W_{\text{loc}}^s(p), W_{\text{loc}}^u(p)$ such that

- same dimensions as E^s, E^u

- tangent to E^s, E^u

and $W_{\text{loc}}^s(p)$ and $W_{\text{loc}}^u(p)$ as smooth as f is.