Problem

Consider the following second-order, periodically forced, differential equation:

$$\ddot{x} + \delta \dot{x} + f(x) = \gamma \cos(\omega t). \tag{1}$$

(a) Letting $f(x) = x$ and fixing $\omega = 1, \delta > 0, \gamma > 0$, show that there exists a 2π-periodic solution to (1).

(b) Construct a Poincaré map that takes $(x(0), \dot{x}(0))$ to $(x(T), \dot{x}(T))$, where $T = 2\pi/\omega$ is the forcing period, and use this map to investigate the stability of the periodic orbit you found in part (a).

(c) Now let $f(x) = \sin x$, and fix $\delta = \gamma = 0$. Show that (1) possess a homoclinic orbit $q^0(t)$ to a hyperbolic saddle point p_0. [*Hint:* Recognize that the system is Hamiltonian and use the conserved quantity.]

(d) Now let $\delta, \gamma > 0$, and consider how this perturbation might affect the fixed point p_0 and the homoclinic orbit. Under the periodic forcing, the fixed point p_0 is perturbed to a *periodic orbit* (of saddle type). This periodic orbit is a fixed point of the Poincaré map that takes $(x(0), \dot{x}(0))$ to $(x(T), \dot{x}(T))$. Find an approximation to this periodic orbit numerically (e.g., in Python or Matlab), by constructing a numerical approximation to the Poincaré map, and using a numerical root finder to find a fixed point close to p_0.

(e) Now investigate (numerically) what happens to the homoclinic orbit $q^0(t)$, for $\delta, \gamma > 0$. Do your best to numerically approximate the stable and unstable manifolds of the fixed point you found in the previous part, for some values of δ, γ, ω: for instance, place a bunch of points along the stable and unstable eigenspaces, and iterate them forward or backward using your Poincaré map. Can you find values for which the stable and unstable manifolds have transversal intersections? If so, can you find something resembling "chaotic" behavior?

Notes

From office hours, I learned that the Poincaré map is just the time T map.

1 Part (a)

To show that a 2π-periodic solution to (1) exists, let's use the method of undetermined coefficients where we guess that $x(t)$ is

$$x(t) = A\cos(t) + B\sin(t)$$

and then we can compute the following values

$$\dot{x} = -A\sin(t) + B\cos(t)$$
$$\ddot{x} = -A\cos(t) - B\sin(t)$$

Now we can through our values into the following equation

$$\ddot{x} + \delta\dot{x} + x = \gamma\cos(t)$$

to get

$$\begin{aligned}\gamma\cos(t) &= -A\cos(t) - B\sin(t) + \delta(-A\sin(t) + B\cos(t)) + A\cos(t) + B\sin(t)\\ &= -A\delta\sin(t) + B\delta\cos(t)\end{aligned}$$

where $A = 0$ and $B = \frac{\gamma}{\delta}$. Hence our 2π-periodic solution to (1) is

$$x(t) = \frac{\gamma}{\delta}\sin(t)$$

[Nudge] Hint *2026-02-17 17:37*

Your work on part (a) looks correct — nice job finding the particular solution. You're on the right track to continue with the remaining parts!

2 Part (b)

First, let's reparameterize our system such that it is an autonomous system of ODEs:

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\delta y - f(x) + \gamma\cos\psi\\ \dot{\psi} &= \omega\end{aligned}$$

Now let's look at the section

$$\Sigma = \{(x, y, \psi) : \psi = 0 \mod 2\pi\}$$

where $T = 2\pi/\omega$. This is convenient because our Poincaré map becomes

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\delta y - f(x) + \gamma\end{aligned}$$

where we could linearize around some closed orbit ζ to get

$$\dot{\xi} = D_x f(\zeta)\cdot\xi$$

Here, $D_x f(\zeta)$ is the Jacobian

$$J_f(x, y) = \begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{bmatrix}$$

with the quantities being

$$\frac{\partial f_1}{\partial x} = 0$$
$$\frac{\partial f_1}{\partial y} = 1$$
$$\frac{\partial f_2}{\partial x} = -f'(x)$$
$$\frac{\partial f_2}{\partial y} = -\delta$$

Putting it all together, the Jacobian is

$$J_f(x, y) = \begin{bmatrix} 0 & 1 \\ -f'(x) & -\delta \end{bmatrix}$$

and our Poincare map is

$$X(T) = e^{TJ_f}.$$

[Nudge] Hint *2026-02-17 18:05*

You're making good progress setting up the linearization, but be careful — the Jacobian J_f evaluated along your periodic orbit is *time-dependent* (since $x(t)$ varies), so e^{TJ_f} isn't quite right. Think about what the correct expression for the fundamental matrix solution looks like when the coefficient matrix depends on t.

Putting it all together, we can write the Jacobian as

$$A(t) = J_f(x(t), y) = \begin{bmatrix} 0 & 1 \\ -f'(x(t)) & -\delta \end{bmatrix}$$

and our Poincare map is

$$\dot{\xi} = A(t)\xi.$$

[Nudge] Hint *2026-02-17 18:10*

You've correctly identified that $A(t)$ is time-dependent, which is the key insight. Now, for the specific case $f(x) = x$, what does $f'(x)$ simplify to, and what does that tell you about whether $A(t)$ is actually time-dependent or constant?

Now looking at the orbit in part (a), we have $-f'(x) = -1$. Thus our Jacobian at the stable orbit is

$$J_f(x(0), y(0)) = \begin{bmatrix} 0 & 1 \\ -1 & -\delta \end{bmatrix}$$

where $x(0), y(0)$ is

$$x(0) = \frac{\gamma}{\delta} \sin 0 = 0$$

and

$$y(0) = \dot{x}(0) = \frac{\gamma}{\delta}\cos 0 = \frac{\gamma}{\delta}$$

Evaluating our eigenvalues and eigenvectors via SymPy, we have

$$\left[\left(\lambda_0 = -\frac{\delta}{2} - \frac{\sqrt{\delta^2-4}}{2},\ v_0 = \begin{bmatrix}\lambda_1\\1\end{bmatrix}\right),\ \left(\lambda_1 = -\frac{\delta}{2} + \frac{\sqrt{\delta^2-4}}{2},\ v_1 = \begin{bmatrix}\lambda_0\\1\end{bmatrix}\right)\right]$$

where our eigenvalues λ_0, λ_1 always have a negative real part since $\delta > 0$, $\delta < 2$ implies that $\sqrt{\delta^2-4}$ is imaginary, and for $\delta > 2$ we have $\delta > \sqrt{\delta^2-4}$, implying λ_1 is negative.

From this, we can conclude that the periodic orbit we found in part (a) is stable.

[Nudge] Hint *2026-02-17 19:33*

You're on the right track recognizing that the eigenvalues of A have negative real part, but remember — the stability of the periodic orbit is determined by the eigenvalues of the **Poincaré map** (i.e., $e^{\lambda_i T}$), not the eigenvalues of A directly. Make sure you connect the two explicitly to complete the argument.

Ah good point.

Note that our time-periodic linear system is

$$\dot{\xi} = A\xi$$

where A was our time-independent Jacobian evaluated at the closed orbit. To get the Poincaré map, we integrate this system to get

$$\xi(T) = \xi(0)e^{TA}$$

where our eigenvalues λ_0, λ_1 of A become eigenvalues $e^{\lambda_0}, e^{\lambda_1}$ of the Poincaré map. Hence, because λ_0, λ_1 both had negative real parts, $e^{\lambda_0}, e^{\lambda_1}$ are both less than one, implying that the map is stable around our closed orbit.

Note that even if λ is complex with a negative real part, we can write

$$e^{\lambda} = e^{\mathrm{Re}\lambda + i\,\mathrm{Im}\lambda} = e^{\mathrm{Re}\lambda}e^{i\,\mathrm{Im}\lambda} = e^{\mathrm{Re}\lambda}\left(\cos(\mathrm{Im}\lambda) + i\,\sin\left(\cos(\mathrm{Im}\lambda)\right)\right)$$

which will have a length less than 1. Hence, even if λ is complex with a negative real part, the Poincaré map is still stable.

[Nudge] Hint *2026-02-17 20:11*

You're almost there — just be careful with the exponents. The eigenvalues of the Poincaré map should be $e^{\lambda_i T}$, not e^{λ_i}. Make sure the factor of $T = 2\pi$ appears in your final argument.

Oh sure.

Note that our time-periodic linear system is

$$\dot{\xi} = A\xi$$

where A was our time-independent Jacobian evaluated at the closed orbit. To get the Poincaré map, we integrate this system to get

$$\xi(T) = \xi(0)e^{TA}$$

where our eigenvalues λ_0, λ_1 of A become eigenvalues $e^{\lambda_0 T}, e^{\lambda_1 T}$ of the Poincaré map. Hence, because λ_0, λ_1 both had negative real parts, $e^{\lambda_0 T}, e^{\lambda_1 T}$ are both less than one, implying that the map is stable around our closed orbit.

Note that even if λ is complex with a negative real part, we can write

$$e^(\lambda T) = e^{T\mathrm{Re}\lambda + i\,T\mathrm{Im}\lambda} = e^{T\mathrm{Re}\lambda}e^{i\,T\mathrm{Im}\lambda} = e^{T\mathrm{Re}\lambda}\left(\cos\left(T\mathrm{Im}\lambda\right) + i\,\sin\left(T\mathrm{Im}\lambda\right)\right)$$

which will have a length less than 1. Hence, even if λ is complex with a negative real part, the Poincaré map is still stable.

[Nudge] Hint *2026-02-17 20:17*

Your argument for part (b) now looks complete and correct — nice work! Time to move on to part (c).

3 Part (c)

Now letting $f(x) = \sin x$ and fixing $\delta = \gamma = 0$, our system is

$$\ddot{x} + \sin x = 0$$

and via reparameterizing, we have

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\sin x\end{aligned}$$

First, let's find the fixed points to show that a hyperbolic saddle point exists. We can write

$$\begin{aligned}0 = \dot{x} &= y\\ 0 = \dot{y} &= -\sin x\end{aligned}$$

where it becomes clear that our fixed points are $(\pm n\pi, 0)$ for $n \in \mathbb{Z}$. To linearize, we can compute the Jacobian

$$J_f(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

with the quatities being

$$\begin{aligned}\frac{\partial f_1}{\partial x} &= 0\\ \frac{\partial f_1}{\partial y} &= 1\\ \frac{\partial f_2}{\partial x} &= -\cos x\\ \frac{\partial f_2}{\partial y} &= 0\end{aligned}$$

Now we can compute our linearized matrices

$$Df(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$Df(+\pi,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Notice that I only calculated the Jacobian matrices for $0, +\pi$ since more multiples of π will be one of these two matrices.

Calculating the eigenvalues and eigenvectors, for equilibria $(0,0)$ and corresponding 2π multiples, we have

$$\left[\left(\lambda = -i,\ v = \begin{bmatrix} i \\ 1 \end{bmatrix}\right),\ \left(\lambda = i,\ v = \begin{bmatrix} -i \\ 1 \end{bmatrix}\right)\right]$$

implying that linearization tells us nothing since we have 0 real part to our eigenvalues.

Calculating the eigenvalues and eigenvectors, for equilibria $(+\pi,0)$ and corresponding 2π multiples, we have

$$\left[\left(\lambda = -1,\ v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right),\ \left(\lambda = 1,\ v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right]$$

where we have an unstable fixed point due to the positive eigenvalue. Specifically, it is a saddle.

Given the hint, we can recognize that the system is Hamiltonian, and go looking for such Hamiltonian. We start by solving for $\partial H/\partial y$

$$\frac{\partial H}{\partial y} = y$$
$$\partial H = y\partial y$$
$$\int \partial H = \int y\partial y$$
$$H(x,y) = \frac{y^2}{2} + f(x)$$

and by solving for $\partial H/\partial x$, we have

$$-\frac{\partial H}{\partial x} = -\sin x$$
$$\partial H = \sin x\,\partial x$$
$$\int \partial H = \int \sin x\,\partial x$$
$$H(x,y) = -\cos x + g(y)$$

Putting it all together, we have

$$H(x,y) = \frac{y^2}{2} - \cos x$$

Plugging in our saddle point $(\pi,0)$ to our Hamiltonian, we have

$$H(\pi,0) = \frac{0^2}{2} - \cos\pi = 1$$

Now to show that there is a homoclinic orbit $q^0(t)$ to the hyperbolic saddle point $p_0 = (\pi, 0)$, we can use the Hamiltonian to get an expression for y in terms of x, and then we can show that this expression goes to p_0. Hence, we can write

$$\frac{y^2}{2} - \cos x = 1$$
$$y^2 = 2(\cos x + 1)$$
$$y = \pm\sqrt{2(\cos x + 1)}$$

[Nudge] Hint *2026-02-17 20:47*

You're on the right track with the level set $H = 1$. To finish, think about what happens to $y = \pm\sqrt{2(\cos x + 1)}$ as $x \to \pm\pi$ — does this curve connect p_0 back to itself?

For $y = \pm\sqrt{2(\cos x + 1)}$, as x goes from $+\pi$ to $-\pi$, we have $(\pi, 0)$ to $(-\pi, 0)$. But aren't $(\pi, 0)$ and $(-\pi, 0)$ both different saddle points? Doesn't this mean that we have a heteroclinic orbit?

[Nudge] Hint *2026-02-17 21:40*

You're asking exactly the right question! Think about the phase portrait on the **cylinder** — since $\sin x$ is 2π-periodic, $x = -\pi$ and $x = +\pi$ represent the *same* physical state.

Ah you're right.

Given that the system is 2π-periodic in x, our saddle points at $(\pi, 0)$ and $(-\pi, 0)$ are really the same saddle point. This makes sense given that this system is a damped driven oscillator. Hence, if we start at $(\pi, 0)$, and travel along either $+\sqrt{2(\cos x + 1)}$ or $-\sqrt{2(\cos x + 1)}$, we will make it back to $(-\pi, 0)$, the same saddle point. Hence, we have a homoclinic orbit.

[Nudge] Hint *2026-02-17 21:51*

Your argument for part (c) is solid — nice work connecting the level set back to the same point on the cylinder! You're ready to tackle the numerical parts (d) and (e).

4 Part (d)

I guess I'm confused by what it means to construct a numerical approximation to the Poincaré map. Don't we already have the Poincaré map from part (b)?

[Nudge] Hint *2026-02-17 22:19*

You're right that you understand the *concept* of the Poincaré map from part (b), but now with $f(x) = \sin x$, you can't solve the system analytically — so you'll need to **numerically integrate** the ODE from $t = 0$ to $t = T$ to evaluate the map for any given initial condition.

Okay! So we can numerically approximate what trajectories will do over one T, and then use the numerical root finder to find an initial condition that is nearly (hopefully exactly) the same after T.

[Nudge] Hint *2026-02-17 22:22*

That's exactly the right idea — you've got the conceptual framework down perfectly. Go ahead and implement it!

Solution
