

Problem

(20 points) *Linear vector fields and maps.*

- (a) For the following linear ODE, compute the stable, unstable, and center subspaces E^s , E^u , E^c of the origin. Sketch these subspaces and some representative trajectories of the system in the phase space.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- (b) For the following linear map on \mathbb{R}^2 , compute the stable, unstable, and center subspaces E^s , E^u , E^c of the origin. What are the qualitative differences in the dynamics between $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$? Sketch the subspaces and some representative trajectories of the system in the phase space.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad |\lambda| < 1.$$

Notes

Part (a) Let's start with finding the eigenvalues. It's a slog, but we can write

$$A - \lambda I = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} - \lambda I = \begin{bmatrix} 1 - \lambda & -3 & 3 \\ 3 & -5 - \lambda & 3 \\ 6 & -6 & 4 - \lambda \end{bmatrix}$$

where the eigenvalues are computed as $\det(A - \lambda I) = 0$. Thankfully, we can use the formula for 3×3 matrix determinants of

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Plugging everything in we get

$$\begin{aligned} & (1 - \lambda)((-5 - \lambda)(4 - \lambda) - 3(-6)) - (-3)(3(4 - \lambda) - 3 \cdot 6) + (3)(3(-6) - (-5 - \lambda) \cdot 6) \\ & ((1 - \lambda)(-5 - \lambda)(4 - \lambda) + 18(1 - \lambda)) + (9(4 - \lambda) - 54) + (-54 - 18(-5 - \lambda)) \\ & ((-5 + 4\lambda + \lambda^2)(4 - \lambda) + 18 - 18\lambda) + (36 - 9\lambda - 54) + (-54 + 90 + 18\lambda) \\ & (-20 + 21\lambda - \lambda^3) + 18 - 18\lambda + 36 - 9\lambda - 54 - 54 + 90 + 18\lambda \\ & (-20 + 18 + 36 - 54 - 54 + 90) + (21 - 18 - 9 + 18)\lambda + \lambda^3 \\ & 16 + 12\lambda + \lambda^3 \end{aligned}$$

Setting it equal to zero and doing some factorization, we get

$$\begin{aligned} & 16 + 12\lambda + \lambda^3 = 0 \\ & (4 - \lambda)(4 + 4x + x^2) = 0 \\ & (4 - \lambda)(2 + \lambda)(2 + \lambda) = 0 \end{aligned}$$

where it follows that the eigenvalues are $\lambda_1 = 4$, $\lambda_2 = -2$, and $\lambda_3 = -2$.

To find eigenvalues, we can solve for the equation

$$\begin{bmatrix} 1 - \lambda & -3 & 3 \\ 3 & -5 - \lambda & 3 \\ 6 & -6 & 4 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for each eigenvalue. Starting with $\lambda_1 = 4$, we have

$$\begin{bmatrix} 1 - 4 & -3 & 3 \\ 3 & -5 - 4 & 3 \\ 6 & -6 & 4 - 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where the last row implies that $x_1 = x_2$. Plugging that into the expression $-3x_1 - 3x_2 + 3x_3 = 0$, we have

$$\begin{aligned} -3x_1 - 3x_1 + 3x_3 &= 0 \\ -6x_1 + 3x_3 &= 0 \\ x_3 &= 2x_1 \end{aligned}$$

Choosing $x_1 = 1$, we have an eigenvector corresponding to the eigenvalue $\lambda_1 = 4$ of

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Moving on to the eigenvalue $\lambda_{2,3} = -2$, we can write

$$\begin{bmatrix} 1 + 2 & -3 & 3 \\ 3 & -5 + 2 & 3 \\ 6 & -6 & 4 + 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and since all rows are the same, we are justified in writing

$$x_3 = -x_1 + x_2$$

where the eigenvectors corresponding to $\lambda_{2,3} = -2$ span this plane. Choosing two arbitrarily, we have

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

Compiling our work into the stable, unstable, and center subspaces E^s , E^u , E^c , we can write

$$\begin{aligned} E^s &= \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} \\ E^u &= \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \right\} \\ E^c &= \text{span} \{0\} \end{aligned}$$

Solution
