

# 1 Plan for Class

1. Linear systems in the plane
2. Invariant sets, attracting sets
3. Stable and unstable eigenspaces
4. Hartman-Grobman theorem 1.3.1
5. Stable manifold theorem 1.3.2

## 2 Linear systems

First let's define the flow map:  $\phi_t$  maps  $x(0)$  to  $x(t)$ .

For linear systems, we are interested in

$$\begin{aligned}\dot{x} &= Ax \\ x(t) &= e^{At}\end{aligned}$$

where the matrix exponential is defined as

$$e^{At} = I + At + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3$$

For the flow map, we can write

$$\begin{aligned}\psi_t &= e^{At} \\ \psi_t(x_0) &= e^{At}x_0\end{aligned}$$

### 2.1 Special cases

Let's say we have a matrix  $A$  that is

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

which is

$$\begin{aligned}\dot{x} &= \lambda x \\ \dot{y} &= \mu y\end{aligned}$$

Our trajectories (or phase portrait) of the system will look like

$$\begin{aligned}\frac{dy}{dx} &= \frac{\mu y}{\lambda x} \\ \implies \frac{dy}{y} &= \frac{\mu}{\lambda} \frac{dx}{x} \\ \implies \log y &= \frac{\mu}{\lambda} \log x + C \\ \implies y &= x^{\mu/\lambda} \cdot c'\end{aligned}$$

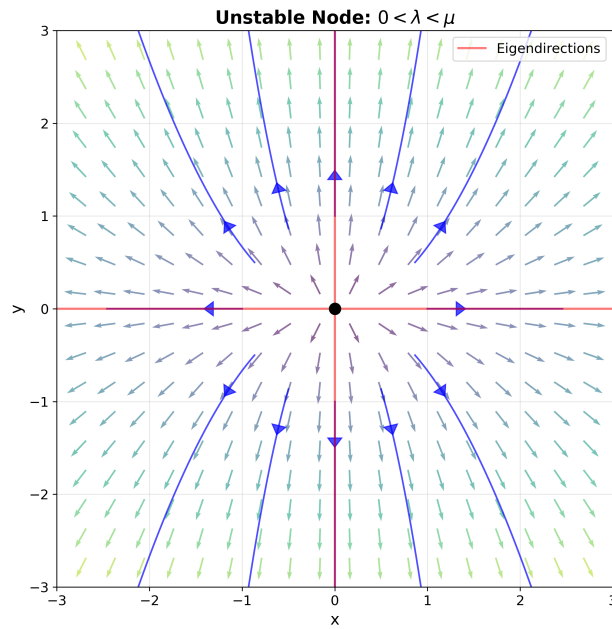
and our solution to the dynamics look like

$$\begin{aligned}x(t) &= e^{\lambda t}x_0 \\ y(t) &= e^{\mu t}y_0\end{aligned}$$

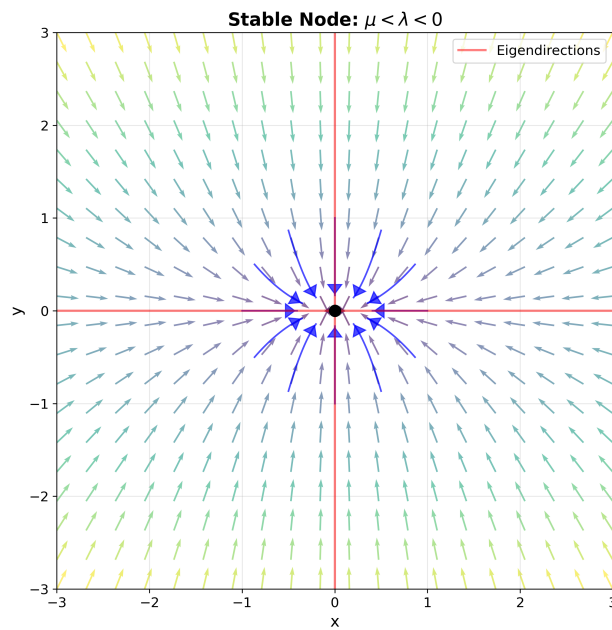
Now let's draw out the flow field for various conditions (to do after class)

- $0 < \lambda < \mu$  - unstable node
- $\mu < \lambda < 0$  - stable node
- $\lambda < 0 < \mu$  - saddle
- $\lambda = \mu < 0$  - sink (star node)
- $0 < \lambda = \mu$  - source (star node)

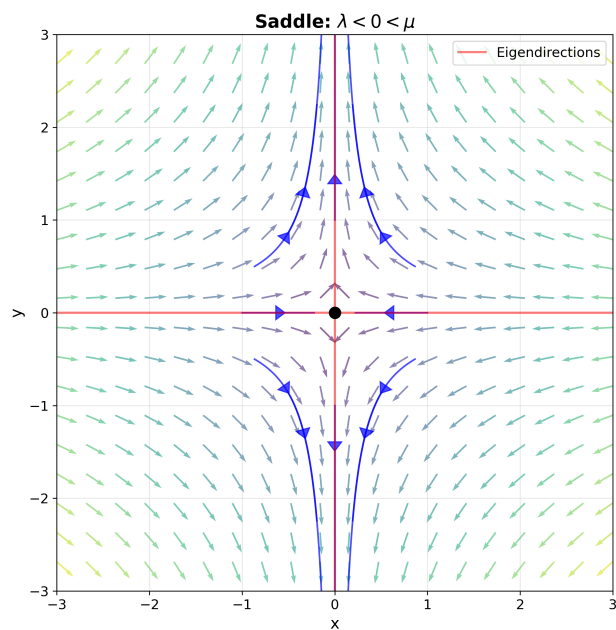
### 2.1.1 Unstable Node: $0 < \lambda < \mu$



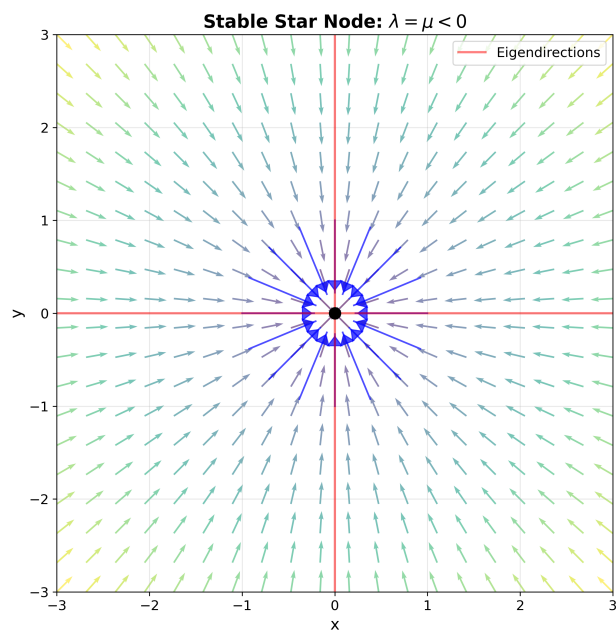
### 2.1.2 Stable Node: $\mu < \lambda < 0$



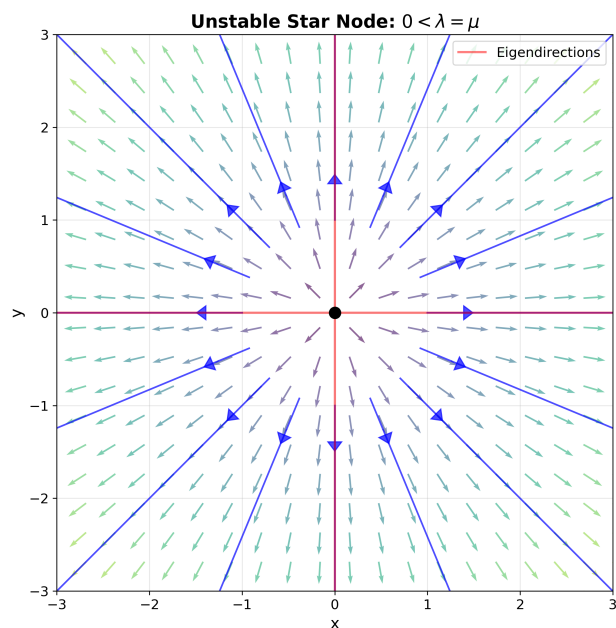
### 2.1.3 Saddle: $\lambda < 0 < \mu$



### 2.1.4 Stable Star Node: $\lambda = \mu < 0$



### 2.1.5 Unstable Star Node: $0 < \lambda = \mu$



## 2.2 Complex eigenvalues?

Let's say our  $A$  matrix looks like

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

where the eigenvalues are  $\sigma \pm i\omega$ .

$$\begin{aligned} \dot{x} &= \sigma x + \omega y & x &= r \cos \theta & \dot{x} &= \dot{r} \cos \theta - r \sin \theta \cdot \dot{\theta} \\ \dot{y} &= -\omega x + \sigma y & y &= r \sin \theta & \dot{y} &= \dot{r} \sin \theta + r \cos \theta \cdot \dot{\theta} \end{aligned}$$

Via some calculations that I should do later, we have

$$\begin{aligned} \dot{r} &= \sigma r \\ \dot{\theta} &= -\omega \end{aligned}$$

Now we can make some more plots!

- $\sigma < 0$  - stable spiral
- $\sigma > 0$  - unstable spiral
- $\sigma = 0$  - centers!

### 2.2.1 The calculations

To get  $\dot{r}$  let's first multiply  $\dot{x}$  by  $\cos \theta$  and  $\dot{y}$  by  $\sin \theta$ , and add them to write

$$\begin{aligned} \left[ \dot{r} \cos \theta - r \sin \theta \cdot \dot{\theta} \right] (\cos \theta) + \left[ \dot{r} \sin \theta + r \cos \theta \cdot \dot{\theta} \right] (\sin \theta) &= \dot{x}(\cos \theta) + \dot{y}(\sin \theta) \\ \dot{r} \cos^2 \theta + \dot{r} \sin^2 \theta &= (\sigma x + \omega y)(\cos \theta) + (-\omega x + \sigma y)(\sin \theta) \\ \dot{r}(\cos^2 \theta + \sin^2 \theta) &= (\sigma r \cos \theta + \omega r \sin \theta)(\cos \theta) + (-\omega r \cos \theta + \sigma r \sin \theta)(\sin \theta) \\ \dot{r} &= \sigma r \cos^2 \theta + \sigma r \sin^2 \theta \\ \dot{r} &= \sigma r(\cos^2 \theta + \sin^2 \theta) \\ \dot{r} &= \sigma r \end{aligned}$$

To get  $\dot{\theta}$ , we'll multiply  $\dot{x}$  by  $-\sin \theta$  and  $\dot{y}$  by  $\cos \theta$ , then add them:

$$\begin{aligned} \left( \dot{r} \cos \theta - r \sin \theta \cdot \dot{\theta} \right) (-\sin \theta) + \left( \dot{r} \sin \theta + r \cos \theta \cdot \dot{\theta} \right) (\cos \theta) &= \dot{x}(-\sin \theta) + \dot{y}(\cos \theta) \\ -\dot{r} \cos \theta \sin \theta + r \sin^2 \theta \cdot \dot{\theta} + \dot{r} \sin \theta \cos \theta + r \cos^2 \theta \cdot \dot{\theta} &= \dot{x}(-\sin \theta) + \dot{y}(\cos \theta) \\ r \sin^2 \theta \cdot \dot{\theta} + r \cos^2 \theta \cdot \dot{\theta} &= \dot{x}(-\sin \theta) + \dot{y}(\cos \theta) \\ r(\sin^2 \theta + \cos^2 \theta) \dot{\theta} &= \dot{x}(-\sin \theta) + \dot{y}(\cos \theta) \\ r \dot{\theta} &= \dot{x}(-\sin \theta) + \dot{y}(\cos \theta) \end{aligned}$$

Now substitute  $\dot{x}$  and  $\dot{y}$ :

$$\begin{aligned} r \dot{\theta} &= (\sigma x + \omega y)(-\sin \theta) + (-\omega x + \sigma y)(\cos \theta) \\ r \dot{\theta} &= -\sigma x \sin \theta - \omega y \sin \theta - \omega x \cos \theta + \sigma y \cos \theta \end{aligned}$$

Substitute:

$$\begin{aligned} r \dot{\theta} &= -\sigma r \cos \theta \sin \theta - \omega r \sin^2 \theta - \omega r \cos^2 \theta + \sigma r \sin \theta \cos \theta \\ r \dot{\theta} &= -\sigma r \cos \theta \sin \theta + \sigma r \sin \theta \cos \theta - \omega r \sin^2 \theta - \omega r \cos^2 \theta \\ r \dot{\theta} &= 0 - \omega r(\sin^2 \theta + \cos^2 \theta) \\ r \dot{\theta} &= -\omega r \\ \dot{\theta} &= -\omega \end{aligned}$$

## 2.3 What if $A$ is non-diagonalizable?

Let's say that  $A$  looks like

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Our solution would look like

$$\begin{aligned} y(t) &= e^{\lambda t} \cdot y_0 \\ x(t) &= e^{\lambda t} \cdot x_0 + y_0 t e^{\lambda t} \end{aligned}$$

Now draw the flow field for  $\lambda < 0$ .

### 2.3.1 General cases

If  $A$  is not diagonal, find eigenvalues  $v_1, v_2$  such that

$$Av_1 = \lambda v_1 \quad Av_2 = \lambda v_2$$

Now draw the resulting flow field.

## 2.4 The “big map” for linear systems

Let's say that  $A$  looks like

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

where we can find the eigenvalues via

$$\det(sI - A) = s^2 - (\lambda + \mu)s + \lambda\mu = 0$$

where the trace is  $\lambda + \mu$  and the determinant of  $A$  is  $\lambda\mu$ .

From the trace and determinant, we can draw the “big map” for linear systems in the plane that we should supply after class.

## 3 Invariant sets, attracting sets

Positively invariant set: Say that we have a set of initial conditions  $x \in A$ . We can think of a flow map that directs the initial conditions to some new point  $\phi_t(x)$ . We can think of the new set of  $A$  under the flow map as  $\phi_t(A)$ .

If  $\phi_t(A) \subset A$  for all  $t > 0$ , then  $A$  is positively invariant. ( $\phi_t$  must exist for all  $t > 0$ )

If  $\phi_t(A) \subset A$  for all  $t < 0$ , then  $A$  is negatively invariant.

If  $\phi_t(A) \subset A$  for all  $t$ , then  $A$  is invariant.

### 3.1 Trapping region

Definition: A trapping region is a closed, connected set  $D$  such that  $\phi_t(D) \subset \text{interior}(D)$  for all  $t > 0$ .

#### 3.1.1 Example

$$\dot{x} = 1 - x^2$$

Now consider the three sets

- $(\infty, -1]$  is negatively invariant but not positively invariant due to finite-time blow-up
- $[1, \infty)$  is positively invariant
- $[-1, 1]$  is invariant

#### 3.1.2 Post class clarification

The definition of a positively invariant set and a trapping region look similar, but they are indeed different. Take the following region

$$D = \{1 \leq r \leq 2\}$$

for the following system

$$\dot{r} = r(2 - r)(r - 1), \quad \dot{\theta} = 1$$

The region  $D$  is positively invariant since trajectories stay in  $D$ , but it is not a trapping region because trajectories approach the boundary circles  $r = 1$  and  $r = 2$  (the equilibria). However, the slightly larger region

$$D' = \{0.5 \leq r \leq 2.5\}$$

is a trapping region because trajectories starting in  $D'$  flow to into the interior and toward  $D$ .

### 3.2 Attracting set

A closed, invariant set  $A$  is called an attracting set if  $\exists$  a neighborhood  $U$  of  $A$  such that  $\phi_t(x) \in U \quad \forall t > 0$  and  $\phi_t(x) \rightarrow A \quad x \in U$ .

Definition: The domain/basin of attraction of  $A$  is  $\bigcup \phi_t(U)$  for all  $t \leq 0$ .

### 3.2.1 Example 1

Let's consider the system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x - y\end{aligned}$$

where our matrix  $A$  is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

and the trace is  $-1$  and the determinant is  $1$ ; this is a sink. Our attracting set  $A$  is  $A = (0, 0)$  and the basin of attraction is  $\mathbb{R}^2$ .

### 3.2.2 Example 2

Let's consider the system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x + \epsilon(1 - x^2)y\end{aligned}$$

where the limit cycle  $\Gamma$  is the attracting set and the basin of  $\Gamma$  is  $\mathbb{R}^2 \setminus (0, 0)$

### 3.2.3 Example 3

Let's consider the system

$$\begin{aligned}\dot{x} &= x - x^3 \\ \dot{y} &= -y\end{aligned}$$

Is the set

$$A = \{(x, 0) : x \in [-1, 1]\}$$

an attracting set? Yes, despite the origin being an unstable fixed point. Note: we will have a different definition for an attractor, where  $A$  is not an attractor.

## 3.3 Stable and unstable eigenspaces

Let's say that we have a nonlinear system  $\dot{x} = f(x)$  and there is an equilibrium point at  $x_0$  (i.e.  $f(x_0) = 0$ ). Linearization about  $x_0$  is

$$\dot{\xi} = Df(x_0) \cdot \xi$$

Then, there are 3 invariant subspaces for the linear system

- stable subspace,  $E^s$  and  $\dim E^s = n_s$  and  $E^s = \text{span}\{v_1, \dots, v_{n_s}\}$  where  $v_1, \dots, v_{n_s}$  are eigenvectors
  - More explicitly, stable subspace  $E^s$ : spanned by (generalized) eigenvectors corresponding to eigenvalues with negative real part ( $\text{Re}(\lambda) < 0$ )
- center subspace,  $E^c$  and  $\dim E^c = n_c$  and  $E^c = \text{span}\{ \quad \}$ 
  - More explicitly, center subspace  $E^c$ : spanned by (generalized) eigenvectors corresponding to eigenvalues with zero real part ( $\text{Re}(\lambda) = 0$ )
- unstable subspace,  $E^u$  and  $\dim E^u = n_u$  and  $E^u = \text{span}\{ \quad \}$ 
  - More explicitly, unstable subspace  $E^u$ : spanned by (generalized) eigenvectors corresponding to eigenvalues with positive real part ( $\text{Re}(\lambda) > 0$ )

### 3.3.1 Can we have growth in $E^c$ ?

Suppose we have

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= 0\end{aligned}$$

where

$$\begin{aligned}y &= C \\ x &= x_0 + Ct\end{aligned}$$

where  $E^c = \mathbb{R}^2$  but linear growth!

## 3.4 Hyperbolic fixed point

Let  $p$  be a fixed point of  $\dot{x} = f(x)$  such that  $f(p) = 0$ .

Definition:  $p$  is a hyperbolic fixed point if  $Df(x_0)$  has no eigenvalues on imaginary axis (i.e.,  $n_c = 0$ ).

## 4 Hartman-Grobman theorem

If  $p$  is a hyperbolic fixed point, then there exists a homeomorphism  $h$  defined on a neighborhood  $U$  of  $p$ , taking orbits of the flow  $\phi_t$  of  $\dot{x} = f(x)$  to those of the flow  $e^{t \cdot Df(x_0)}$  of  $\dot{\xi} = Df(p)\xi$ .

$h$  preserves the sense of orbits and can be chosen to preserve the time parameterization

Note that a homeomorphism is a continuous map with continuous inverse, but is maybe not differentiable!

1. Why is hyperbolic necessary? Suppose  $\dot{x} = -x$  and  $\dot{y} = y^2$ . This becomes  $\dot{\xi} = -\xi$  and  $\dot{\eta} = 0$ , and there are no mappings  $h$  between the two.
2. Why not a diffeomorphism? I forget the argument, but I think we will come back to it later in the class.

## 5 Stable manifold theorem

Let  $p$  be a fixed point, and  $U$  be a neighborhood of  $p$ .

We can define a local stable manifold of  $p$  as

$$W_{\text{loc}}^s(p) = \{x \in U \mid \phi_t(x) \rightarrow p \text{ as } t \rightarrow \infty \text{ and } \phi_t(x) \in U \text{ for all } t \geq 0\}$$

We can define the unstable manifold as

$$W_{\text{loc}}^u(p) = \{x \in U \mid \phi_t(x) \rightarrow p \text{ as } t \rightarrow -\infty \text{ and } \phi_t(x) \in U \text{ for all } t \leq 0\}$$

### 5.1 Now the theorem

Stable manifold theorem for a fixed point: For a  $p$  hyperbolic fixed point of  $\dot{x} = f(x)$ ,  $\exists$  local stable and unstable manifold  $W_{\text{loc}}^s(p), W_{\text{loc}}^u(p)$  such that

- same dimensions as  $E^s, E^u$
- tangent to  $E^s, E^u$

and  $W_{\text{loc}}^s(p)$  and  $W_{\text{loc}}^u(p)$  as smooth as  $f$  is.