Problem

(38 points) For the following dynamical systems, find the fixed point(s) and compute the associated stable and unstable manifolds, up to fourth order (that is, if you consider an approximation $y = h(x)$, let h be a quartic polynomial). Use these results to sketch the global phase portraits.

(a) For the following ODE, assume $\lambda > 0$.

$$\begin{aligned} \dot{x} &= x(1 - x^2) \\ \dot{y} &= -\lambda(y - x^2). \end{aligned}$$

(b) For the following map, assume $|\lambda| < 1$ and $|\mu| > 1$.

$$\begin{aligned} x &\mapsto \lambda x, \\ y &\mapsto \mu y - x^3. \end{aligned}$$

Notes

Part (A) To begin, let's find the fixed points by writing

$$\begin{aligned} 0 = \dot{x} = x(1 - x^2) &\implies x = 0, x = \pm 1 \\ 0 = \dot{y} = -\lambda(y - x^2) &\implies y = x^2 \end{aligned}$$

Hence our fixed points are $(0, 0), (-1, 1), (1, 1)$. To linearize, we can compute the Jacobian

$$J_f(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

with the quatities being

$$\begin{aligned} \frac{\partial f_1}{\partial x} &= 1 - 3x^2 \\ \frac{\partial f_1}{\partial y} &= 0 \\ \frac{\partial f_2}{\partial x} &= 2\lambda x \\ \frac{\partial f_2}{\partial y} &= -\lambda \end{aligned}$$

Now we can compute our linearized matrices

$$Df(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -\lambda \end{bmatrix}$$
$$Df(-1,1) = \begin{bmatrix} -2 & 0 \\ -2\lambda & -\lambda \end{bmatrix}$$
$$Df(1,1) = \begin{bmatrix} -2 & 0 \\ 2\lambda & -\lambda \end{bmatrix}$$

Let's first approximate the stable and unstable manifolds associated with the $(0,0)$ fixed point. Our general workflow will be to calculate the eigenvalues and vectors, construct $h(x)$ or $g(y)$, then differentiate, group coefficients together, and solve for the coefficients.

[Nudge] Hint *2026-02-11 08:41*

You're off to a great start — the fixed points, Jacobian, and linearizations all look correct. Keep going with your plan for the origin: identify the stable and unstable eigenspaces and then look for the invariant manifolds as graphs over those subspaces.

Solution
