Problem

(40 points) Consider the ODE

$$\frac{dx}{dt} = v_0 + v_1 + v_2 x^2 + v_3 x^3$$

where $v_3 \neq 0$. Show that this system can be transformed into

$$\frac{dy}{d\tau} = \mu_0 + \mu_1 y - y^3$$

by setting $x = \alpha y + \beta$ and $t = \gamma \tau$ (affine rescaling of state space, and time scale change). Thus the second system may be regarded as a "universal unfolding" of the degenerate system $\dot{x} = -x^3$. Find suitable α, β, γ and derive relationships between the new (μ_j) and old (v_j) parameters. Can the number of parameters be further reduced?

Analyze the second (simpler) system an sketch bifurcation diagrams in the form of a surface over the (μ_0, μ_1)-parameter space, as well as branches of equilibria in (μ_1, y) space for "well-chosen" constant μ_0 slices. Find and indicate stability types of all equilibria.

[Hint: Start by setting $\mu_0 = 0$; then move on to small $\mu_0 \neq 0$.]

Notes