1 Plan for Class

1. Definitions in §1.6
2. Planar systems

2 Definitions in §1.6

2.1 Non-wandering point

Definition: p is a non-wondering point for a flow φ_t / map F if $\forall$ neighborhoods $p \in U$, all $T > 0$

- $\varphi_t(U) \bigcap U$ is non-empty for some $t > T$.
- $F^t(U) \bigcap U$ is non-empty for some $t > T$.

This is just a formalization of recurrence. For example, fixed points and periodic orbits satisfy this definition. But then, why have this definition if we already have 'fixed points' and 'periodic orbits'. There are more exotic examples! Poincare recurrence[1] is an example of a non-wondering point.

2.1.1 Example

Let's say we have the following system on a torus

$$\dot{\theta} = 1$$
$$\dot{\varphi} = \sqrt{2}$$

In this system, since $\dot{\varphi}$ is irrational, then every point is non-wondering. Well actually $\dot{\varphi}$ could be rational and every point is still non-wondering, but just not densely.

2.2 Limit point

Definition: p is an $\Big|_{\alpha}^{\omega}$ limit point of x if $\exists$ points

- ω limit point is $\varphi_{t_j} \to p$ as $t_j \to \infty$
- α limit point is $\varphi_{t_j} \to p$ as $t_j \to -\infty$

where t_j is a subsequence of times.

Set of all $\Big|_{\alpha}^{\omega}$ limit points of x is the $\Big|_{\alpha}^{\omega}$ limit set of x.

[1] Did we define this?

2.3 Dense orbit

Definition: An orbit $\varphi_t(q)$ is dense in a set $S \subset \mathbb{R}^n$ if $\forall p \in S$ and $\forall \epsilon > 0$ there exists t such that $\big|\big|\varphi_t(q) - p\big|\big| < \epsilon$.

In other words $\varphi_t(q)$ comes arbitrarily close to every point in S. Look at the $\dot{\varphi} = \sqrt{2}$ for an example of dense orbits and $\dot{\varphi} = 2$ for non dense orbits.

2.4 Attractor

Provisional Definition: An attractor is an attracting set with a dense orbit.

In a previous lecture, we had an attracting set but no an attractor becuase there was no dense orbit.

3 Back to ODEs/flows (not maps)

If we are in 1 dimensions, we can't have oscillations or periodic orbits. Fixed points are the only non-wandering points. BORING! Of course, the exception is the circle where we can have periodic orbits.

In 2 dimensions, more things are possible. We can have

- periodic orbits
- homoclinic connections
- heteroclinic connections

Although, heteroclinic connections are possible in 1 dimensions, but those are stable under perturbations. Heteroclinic connections are "fragile" in 2 dimensions.

3.1 Poincare-Bendixion theorem

Theorem: A nonempty compact ω or α limit set of a planar flow, which contains no fixed points, is a closed orbit.

In other words, suppose f is C^1 and has a trajectory Γ with $\varphi_t(\Gamma)$ contained in a compact set. Then if $\omega(\Gamma)$ contains no fixed point, $\omega(\Gamma)$ is a periodic orbit.

3.1.1 Examples

Suppose there are a finite number of fixed points in a compact set S. Then $\omega(\Gamma)$ is one of the following:

(a) an equilibrium point

(b) periodic orbit

(c) collection of fixed points and orbits connecting them

3.1.2 Another example

Prove that

$$\begin{aligned}\dot{x} &= x - y - x^3 \\ \dot{y} &= x + y - y^3\end{aligned}$$

has a periodic orbit in the region $1 < r < \sqrt{2}$ where $r^2 = x^2 + y^2$.

There is only one fixed point at $(x, y) = (0, 0)$. If we show that $\dot{r} > 0$ at $r = 1$ and that $\dot{r} < 0$ at $r = \sqrt{2}$, AND that there are no fixed points in $1 < r < \sqrt{2}$, then Poincare-Bendixion theorem implies that there is a periodic orbit.

Nice.

3.2 Bendixion's criterion

This criterion will let us rule out periodic orbits. Let's say

$$\begin{aligned}\dot{x} &= f(x, y) \\ \dot{y} &= g(x, y)\end{aligned}$$

Theorem: Suppose $D \subset \mathbb{R}^2$ is simply connected (no holes). Then the system above can have periodic solution only if $\nabla \cdot (f, g) = \partial_x f + \partial_y g$ changes sign or is identically zero in D.

3.2.1 Proof

Suppose Γ is a periodic orbit in D. Let S denote the interior of Γ. Consider Green's theorem

$$\begin{aligned}\iint_S \nabla \cdot (f, g) d\sigma &= \int_\Gamma (f\, dy - g\, dx) \\ &= \int_\Gamma \left(f\, \frac{dy}{dt} - g\, \frac{dx}{dt} \right) dt \\ &= \int_\Gamma (fg - gf)\ dt = 0\end{aligned}$$

Hence the integrand $\iint_S \nabla \cdot (f, g) d\sigma$ must change sign in S or be equivalent to 0 in S.

Note Simply connected is important! For example

$$\begin{aligned}f &= x - y - x^3 \\ g &= x + y - y^3\end{aligned}$$

We have

$$f_x = 1 - 3x^2$$
$$g_y = 1 - 3y^2$$

then

$$f_x + g_y = 2 - 3(x^2 + y^2) = 2 - 3r^2 < 0$$

in our annulus!

3.3 Index Theory

If you are following along, this is page 51-53.

Suppose

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^2$$

Index of a close curve C equals the number of turns vector field f makes as C is traversed conunter-clockwise.

This is made clear by the expression

$$k(C) = \frac{1}{2\pi}\int_C d\,\arctan\left(\frac{dy}{dx}\right) = \frac{1}{2\pi}\int_C \frac{f\,dy - g\,dx}{f^2 + g^2}$$

3.3.1 Examples

- A sink has an index of $+1$
- A source has an index of $+1$
- An outward spiral has an index of $+1$
- An inward spiral has an index of $+1$

In general, a periodic orbit has an index of $+1$. Well, what doesn't have an index of $+1$?

A saddle has an index of -1. Maybe I should provide the drawing.

A region containing no fixed points has an index of 0.

The idea is that the index of C is the sum of the indices all the fixed points inside C.

3.3.2 Corollary

Inside any closed orbit (periodic orbit as(is?) limit cycle), there exists at least one fixed point (in fact $\sum$ind(fd)=1), so if there's only one, it must be a source or sink.[2]

[2]This is terribly worded.

Note: Degenerate fixed points may have indices not equal to ± 1. For example, we have the system

$$\begin{aligned}
\dot{x} &= x^2 - y^2 \\
\dot{y} &= 2xy
\end{aligned}$$

we can define $z = x + iy$ where

$$\begin{aligned}
\dot{z} = z^2 &= (x+iy)(x+iy) \\
&= (x^2 + y^2) + 2ixy
\end{aligned}$$

huh that's weird, but works! The index of the system is $+2$. In general, if $\dot{z} = z^k$, then the system has index k.

3.4 Hamiltonian systems

Let's look at Hamiltonian systems not restricted to $\mathbb{R}^2$. In general, a Hamiltonian is $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$, which is typically "energy", and satisfies

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial y} \\
\dot{y} &= -\frac{\partial H}{\partial x}
\end{aligned}$$

We can extend this to multiple dimensions as

$$\begin{aligned}
\dot{x}_i &= \frac{\partial H}{\partial y_i} \\
\dot{y}_i &= -\frac{\partial H}{\partial x_i}
\end{aligned}$$

where $i = 1, \dots, n$ and x is interpreted as position and y is interpreted as momentum.

We've already shown in class how H is conserved along trajectories because $\frac{dH}{dt} = 0$. Hence solution curves lie on level sets of H.

3.4.1 Example: simple pendulum

Note that $p =$momentum and can be written as $p = ml^2\dot{(\theta)}$. Hence we have

$$\begin{aligned}
H(\theta, p) &= \text{ total energy} \\
&= \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - \cos\theta) \\
&= \frac{p^2}{2ml^2} + mgl(1 - \cos\theta)
\end{aligned}$$

Hence we have

$$\begin{aligned}
\dot{\theta} &= \frac{\partial H}{\partial p} = \frac{p^2}{ml^2} \\
\dot{p} &= -\frac{\partial H}{\partial \theta} = -mgl\sin\theta
\end{aligned}$$