1 Plan for Class

1. Approximating invariant manifolds
2. Diffeomorphisms/maps
3. Stable manifold theorem for diffeomorphisms
4. Transversal intersections
5. Hyperbolic .... automorphism

2 Points from last time

We talked about the stable manifold theorem in order to make statements about the linearization $Df(x_0) = A$. Essentially, negative real part eigenvalues tells us that those subspaces (spanned by the eigenvectors) are stable and positive real part eigenvalues tells us that those subspaces (spanned by the eigenvectors) are unstable. For the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

the eigenvectors for eigenvalues $-1, -2$ span E^s and the eigenvectors for eigenvalue 3 span E^u.

3 Approximating invariant manifolds

3.1 Example

Let's say we have the equation

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -y + x^2 \end{aligned}$$

Our only fixed point is $(0, 0)$. Linearization tells us

$$Df(x, y) = \begin{bmatrix} 1 & 0 \\ 2x & -1 \end{bmatrix}$$

Plugging in $(0, 0)$, we get

$$Df(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where we know what E^s and E^u are, but these only describe the behavior of the stable W^s_{loc} and unstable manifolds W^u_{loc} near the origin. Could we simply find these manifolds directly? Sometimes yes, but most cases no, but we can approximate them!

To approximate W^u_{loc}, we could write

$$y = h(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

where $a_0 = 0$ and $a_1 = 0$ via the stable manifold theorem (we know the eigenspace is tanget at the origin). Now we can write

$$h(x) = a_2 x^2 + a_3 x^3 + \text{ HOT}$$

We know that it's invariant via stable manifold theorem, so we can write

$$\begin{aligned} \dot{y} &= 2a_2 x\dot{x} + 3a_3 x^2 \dot{x} + \text{ HOT} \\ \implies -y + x &= 2a_2 x\dot{x} + 3a_3 x^2 \dot{x} + \text{ HOT} \\ \implies -h(x) + x &= 2a_2 x\dot{x} + 3a_3 x^2 \dot{x} + \text{ HOT} \\ \implies -\left(a_2 x^2 + a_3 x^3\right) + x &= 2a_2 x\dot{x} + 3a_3 x^2 \dot{x} + O(x^4) \end{aligned}$$

For x^2, our coefficient is

$$a_2 + 1 = 2a^2 \implies a_2 = \frac{1}{3}$$

and for x^3, our coefficient is

$$a_3 = 3a_3 \implies a_3 = 0$$

Hence, we can write

$$y = h(x) = \frac{x^2}{3}$$

and in this case, this is exact. To show this, we can write

$$\dot{y} = -y + x^2 = -\frac{x^2}{3} + x^2 = \frac{2}{3}x^2$$

and

$$\frac{d}{dt}\left(\frac{x^2}{3}\right) = \frac{2}{3}x\dot{x} = \frac{2}{3}x^2$$

and they agree.

4 Diffeomorphisms/maps

Let's remind ourselves what maps. A flow is $\dot{x} = f(x)$ and a map is $x_{t+1} = F(x_t)$. For flows, we have a version of a map

$$\phi_t : x(0) \mapsto x(t)$$

where $(\phi_t)^{-1} = \phi_{-t}$. For maps, we have something similar to ϕ_t written as $F = \phi_{\Delta t}$.

If F comes from a flow ($F = \phi_{\Delta t}$) then F is invertible, and usually differentiable (with differetiable inverse): i.e. F is a diffeomorphism.

4.1 Let's say more about maps

We can work with non-invertible maps too.

4.1.1 Example

$$x_{t+1} = \mu x_t(1 - x_t) = F(x_t)$$

where fixed points occur at $F(x_0) = x_0$. To visualize the plot, for $\mu \geq 0$ we have a parabola with roots $x = 0$ and $x > 0$.

As we can see in the plot, for $\mu > 1$, we get a fixed point along $y = x$.

There's a lot of cobweb diagrams being draw to demonstrate the stability for maps graphically. These are rather standard, so we can include these later. In general, the statements are

- $0 < f'(x) < 1$ is stable
- $f'(x) >$ is unstable
- $-1 < f'(x) < 0$ is stable
- $f'(x) < -1$ is unstable

Therefore, we are stable if $|f'(x)| < 1$ and unstable if $|f'(x)| > 1$

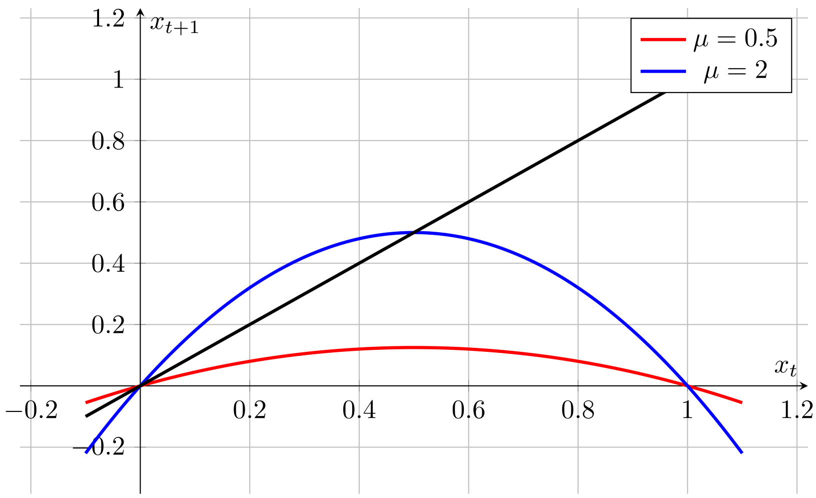

Figure 1: Logistic map for different values of μ

4.1.2 What about higher dimensions?

Let's say that $x \in \mathbb{R}^n$ and

$$x(t+1) = F(x(t))$$

Again, we have equilibrium points at x_0 for $F(x_0) = x_0$. We can do something akin to linearization by writing $x(t) = x_0 + \xi(t)$ and

$$x_0 + \xi(t+1) = x(t+1) = F(x_0 + \xi) = F(x_0) + DF(x_0)\xi + O(\xi^2)$$

where simplifing gets us

$$\xi(t+1) = DF(x_0) \cdot \xi(t) + \text{ HOT}$$

One again, we have $A = DF(x_0)$ and we can look at the eigenvalues of A. For the following eigenvalues, we can state

- $|\lambda| > 1$ is an unstable eigenvalue
- $|\lambda| < 1$ is a stable eigenvalue

Our linearization looks something like

$$\begin{aligned}
\xi(0) &= v \qquad \text{where } v \text{ is an eigenvector} \\
\xi(1) &= Av = \lambda v \\
\xi(2) &= A\xi(1) = A(\lambda v) = \lambda^2 v
\end{aligned}$$

Hence, we can write

$$\xi(t) = \lambda^t v$$

and can state

- If $|\lambda| > 1$, then $|\lambda^t| \to \infty$ as $t \to \infty$
- If $|\lambda| < 1$, then $|\lambda^t| \to 0$
- If $|\lambda| = 1$, then it's inconclusive

4.1.3 Are there things that are different with maps than flows?

As an example, let's say

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

- For $\lambda_2 > \lambda_1$, we get smooth like trajectories to the origin, just as in flows.
- For $\lambda_1 < 0 < \lambda_2$, we get zig-zag like trajectories to the origin, where z_1 is fliping across the z_2 axis at each step.
- For $\lambda_2 < \lambda_1 < 0$, we get zig-zag like trajectories to the origin, where z_1 and z_2 are constantly flipping across the axes.

Said another way, for $\det A > 0$ we have *orientation preserving* trajectories. For $\det A < 0$ we have *orientation reversing* trajectories.

4.2 Diffeomorphism

Now we'll assum F is a diffeomorphism (C^1 with C^1 inverse).

Definition: p is a fixed point of F ($F(p) = p$). If no eigenvalue of $DF(p)$ has unit modulus (i.e. on unit circle) then p is hyperbolic.

Stable manifold theorem: For diffeomorphisms, we can define W^s_{loc} and W^u_{loc} exactly as in ODE case. Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 diffeomorphism with a hyperbolic fixed point p_0. The there exists a local stable and unstable manifolds $W^{u,s}_{\text{loc}}(p_0)$ tangent at p to stable/unstable eigenspaces $E^{s,u}$ respectively. With $W^{u,s}_{\text{loc}}(p_0)$ being as smooth as G.

Also take a peak at Hartman-Grobman theorem for maps (theorem 1.4.1).

5 Transversal intersections

We can extend $W^{u,s}_{\text{loc}}(p)$ to global stable and unstable manifolds $W^{u,s}(p)$ by starting with a small piece of $W^{u,s}_{\text{loc}}(p)$ and iterating ther points forward and backward. But W^s cannot intersect itself, and W^u cannot intersect itself, but can W^s intersect W^u? Yes! This happens with homoclinic orbits.