# MAE 541 / APC 571: Applied Dynamical Systems

## Homework 3

Assigned: 12 Feb 2026
Due: 19 Feb 2026

1. Consider the equation of motion for the simple pendulum with linear damping $\delta$ and constant applied torque $\beta$:

$$\ddot{\theta} + \delta\dot{\theta} + \sin\theta = \beta,$$

where $\delta, \beta \geq 0$, and $\theta$ is on the circle $S^1$ (which can be identified with the interval $[-\pi, \pi)$).

   (a) Find conditions on $(\beta, \delta)$ for which fixed points exist, linearize at the fixed points and draw conclusions on their stability types. Allow the parameters $(\beta, \delta)$ to vary, and locate values for which the linearizations are degenerate (non-hyperbolic).

   (b) For $\delta = 0$, there is a conserved quantity (the system is Hamiltonian). Use this to deduce global phase portraits, and compare with information from (a).

   (c) For $\delta > 0$, can this system have closed orbits that **do not** encircle the 'phase cylinder'? [*Hint:* Use Bendixson's criterion.]

   (d) For $\delta > 0$ and $\beta > 0$ can the system have closed orbits that **do** encircle the phase cylinder? [*Hint:* Determine a trapping region and apply the Poincaré-Bendixson theorem.]

2. We consider the "rabbit vs. sheep" system described by the following equations:

$$\begin{aligned} \dot{x} &= x(3 - x - 2y) \\ \dot{y} &= y(2 - x - y), \end{aligned} \tag{2}$$

where $x$ represents the population of rabbits, and $y$ represents the population of sheep.

   (a) Find all the equilibria of the system, and for each determine its index.

   (b) Show that it is impossible to have a closed orbit for that system (index theory will help but you will need something else to rule out all possible closed orbit).

3. Consider the following second-order, periodically forced, differential equation:

$$\ddot{x} + \delta\dot{x} + f(x) = \gamma\cos(\omega t). \tag{3}$$

   (a) Letting $f(x) = x$ and fixing $\omega = 1, \delta > 0, \gamma > 0$, show that there exists a $2\pi$-periodic solution to (3).

(b) Construct a Poincaré map that takes $(x(0), \dot{x}(0))$ to $(x(T), \dot{x}(T))$, where $T = 2\pi/\omega$ is the forcing period, and use this map to investigate the stability of the periodic orbit you found in part (a).

(c) Now let $f(x) = \sin x$, and fix $\delta = \gamma = 0$. Show that (3) possesses a homoclinic orbit orbit $q^0(t)$ to a hyperbolic saddle point $p_0$. Find an explicit expression for this homoclinic orbit. [*Hint:* Recognize that the system is Hamiltonian and use the conserved quantity.]

(d) Now let $\delta, \gamma > 0$, and consider how this perturbation might affect the fixed point $p_0$ and the homoclinic orbit. Under the periodic forcing, the fixed point $p_0$ is perturbed to a *periodic orbit* (of saddle type). This periodic orbit is a fixed point of the Poincaré map that takes $(x(0), \dot{x}(0))$ to $(x(T), \dot{x}(T))$.

Find an approximation to this periodic orbit numerically (e.g., in Python or Matlab), by constructing a numerical approximation to the Poincaré map (e.g., with `scipy.integrate.solve_ivp`), and using a numerical root finder (e.g., `scipy.optimize.root`) to find a fixed point close to $p_0$.

(e) Now investigate (numerically) what happens to the homoclinic orbit $q^0(t)$, for $\delta, \gamma > 0$. Do your best to numerically approximate the stable and unstable manifolds of the fixed point you found in the previous part, for some values of $\delta, \gamma, \omega$: for instance, place a bunch of points along the stable and unstable eigenspaces, and iterate them forward or backward using your Poincaré map. Can you find values for which the stable and unstable manifolds have transversal intersections? If so, can you find something resembling "chaotic" behavior?

[*Hint:* In the coming weeks, we will learn a technique for predicting when such transversal intersections occur, and if $\delta, \gamma \ll 1$, this theory predicts that transversal intersections occur when

$$\frac{\gamma}{\delta} > \frac{4}{\pi} \cosh\left(\frac{\pi\omega}{2}\right). \tag{4}$$

Let this criterion guide your numerical explorations.]