Linear Algebra Done Right
Solutions to Exercises 8.D

1 Polynomials of Example 8.53

Problem statement

Find the characteristic polynomial and the minimal polynomial of the operator N in Example 8.53.

Solution

For the sake of completeness, the operator $N \in \mathcal{L}(\mathbf{F}^4)$ is defined by

$$N(z_1, z_2, z_3, z_4) = (0, z_1, z_2, z_3).$$

Via Theorem 5.32 ('Determination of eigenvalues from upper-triangular matrix'), N only has an eigenvalue of 0, and by Theorem 8.26 ('Sum of the multiplicities equals $\dim V$'), it follows that the multiplicity of 0 is 4. Hence, the characteristic polynomial is z^4.

To find the minimal polynomial, let's compute N^4, and we can stop when $N^m = 0$ for $m \in \{1, 2, 3, 4\}$. For $(z_1, z_2, z_3, z_4) \in \mathbf{F}^4$ we can write

$$\begin{aligned} N^4(z_1, z_2, z_3, z_4) &= N^3(0, z_1, z_2, z_3) \\ &= N^2(0, 0, z_1, z_2) \\ &= N^1(0, 0, 0, z_1) \\ &= (0, 0, 0, 0). \end{aligned}$$

Therefore, the minimal polynomial of N is z^4, the same as the characteristic polynomial.

2 Polynomials of Example 8.54

Problem statement

Find the characteristic polynomial and the minimal polynomial of the operator N in Example 8.54.

Solution

For the sake of completeness, the operator $N \in \mathcal{L}(\mathbf{F}^6)$ is defined by

$$N(z_1, z_2, z_3, z_4, z_5, z_6) = (0, z_1, z_2, 0, z_4, 0).$$

Via Theorem 5.32 ('Determination of eigenvalues from upper-triangular matrix'), N only has an eigenvalue of 0, and by Theorem 8.26 ('Sum of the multiplicities equals $\dim V$'), it follows that the multiplicity of 0 is 6. Hence, the characteristic polynomial is z^6.

To find the minimal polynomial, let's compute N^6, and we can stop when $N^m = 0$ for $m \in \{1, 2, 3, 4, 5, 6\}$. For $(z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbf{F}^6$ we can write

$$\begin{aligned} N^6(z_1, z_2, z_3, z_4, z_5, z_6) &= N^5(0, z_1, z_2, 0, z_4, 0) \\ &= N^4(0, 0, z_1, 0, 0, 0) \\ &= N^3(0, 0, 0, 0, 0, 0) = (0, 0, 0, 0, 0, 0). \end{aligned}$$

Therefore, since it took 3 applications of N to equal 0, the minimal polynomial of N is z^3.

3 Minimal polynomial of nilpotent is z^{m+1}

Problem statement

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Prove that the minimal polynomial of N is z^{m+1}, where m is the length of the longest consecutive string of 1's that appears on the line directly above the diagonal in the matrix of N with respect to any Jordan basis for N.

Solution

As we saw in the proof of Theorem 8.60 ('Jordan Form'), a nilpotent operator $N \in \mathcal{L}(V)$ has vectors $v_1, \dots, v_n \in V$, given by Theorem 8.55 ('Basis corresponding to a nilpotent operator'), such that list $N^{m_j} v_j, \dots, Nv_j, v_j$ has a matrix of the form

$$A_j = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}.$$

These matrices A_j lie on the diagonal of the block diagonal matrix given by the Jordan basis of N.

For A_j, the length of consecutive 1's appearing on the line directly above the diagonal is m_j[1]. We can also note that $A_j^{m_j+1} = 0$ since $N^{m_j+1} v_j = 0$.

Now suppose $N^{m_1} v_1, \dots, Nv_1, v_1, \dots, N^{m_n} v_n, \dots, Nv_n, v_n$ is the basis of N given by Theorem 8.55 and Theorem 8.60. Further suppose that k is such that $m_k > m_j$ [2] where $N^{m_k+1} v_k = 0$ and $N^{m_j+1} v_j = 0$. For every $v \in V$, we can write v as a linear combination of $N^{m_1} v_1, \dots, Nv_1, v_1, \dots, N^{m_n} v_n, \dots, Nv_n, v_n$. Applying N^{m_k} to our linear combination, we have

$$N^{m_k} v = N^{m_k} v_k$$

where all the other terms are 0 since $m_k > m_j$. Applying N^{m_k+1} to our linear combination, it follows that

$$N^{m_k+1} v = N^{m_k+1} v_k = 0$$

for all $v \in V$.

Hence, we've shown that $N^{m_k} \neq 0$ and $N^{m_k+1} = 0$, implying that z^{m_k+1} is the minimal polynomial. In the second paragraph, we showed that m_k is the length of a consecutive string of 1's appearing above the diagonal in the matrix of N with respect to the Jordan basis for N. Thus, $m_k > m_j$ implies that the consecutive string of 1's corresponding to length m_k is the longest consecutive string of 1's, which was to be shown.

[1] Note that the list $N^{m_j} v_j, \dots, Nv_j, v_j$ is of length $m_j + 1$.

[2] Okay it's possible that there is another m_i such that $m_k = m_i$, but the result still holds.

4 Matrix of T with reverse Jordan basis

Problem statement

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V that is a Jordan basis for T. Describe the matrix of T with respect to the basis $v_n, \ldots, v_1$ obtained by reversing the order of the v's.

Solution

This feels more like a LaTeX exercise than a math problem.

Suppose $v_1, \ldots, v_n$ is a basis of V that is a Jordan basis for T. The matrix of T with respect to the basis $v_n, \ldots, v_1$ is a square matrix of the form

$$\begin{pmatrix} 0 & & A_1 \\ & \iddots & \\ A_m & & 0 \end{pmatrix}$$

where $A_1, \ldots, A_m$ are square matrices lying along the line from upper right corner to the bottom left corner and all the other entries of the matrix equal 0. Each A_j is a matrix of the form

$$\begin{pmatrix} 0 & & & 1 & \lambda_j \\ & & \iddots & \iddots & \\ & & & & \\ 1 & \iddots & & & \\ \lambda_j & & & & 0 \end{pmatrix}$$

where the λ_j's lie along the line from upper right corner to the bottom left corner, the 1's lie along the line above the line of λ_j's, and all the other entries of A_j equal 0.

5 Matrix of T^2 respect to Jordan basis of T

Problem statement

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V that is a Jordan basis for T. Describe the matrix of T^2 with respect to this basis.

Solution

Following from the definition of the Jordan basis (Definition 8.59), the matrix of T^2 with respect to the Jordan basis of T is the block diagonal matrix

$$\begin{pmatrix} A_1^2 & & 0 \\ & \ddots & \\ 0 & & A_m^2 \end{pmatrix}$$

where each A_j^2 is an upper-triangular matrix of the form

$$A_j^2 = \begin{pmatrix} \lambda_j^2 & 2\lambda_j & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 2\lambda_j \\ 0 & & & \lambda_j^2 \end{pmatrix}.$$

6 $N^{m_1}v_1, \dots, N^{m_n}v_n$ is a basis of null N

Problem statement

Suppose $N \in \mathcal{L}(V)$ is nilpotent and $v_1, \dots, v_n$ and $m_1, \dots, m_n$ are as in 8.55. Prove that $N^{m_1}v_1, \dots, N^{m_n}v_n$ is a basis of $\operatorname{null} N$.

Solution

By Theorem 8.55 ('Basis corresponding to a nilpotent operator'), the list of vectors $N^{m_1}v_1, \dots, N^{m_n}v_n$ are linearly independent and

$$N^{m_1}v_1, \dots, N^{m_n}v_n \in \operatorname{null} N.$$

To show that $N^{m_1}v_1, \dots, N^{m_n}v_n$ spans $\operatorname{null} N$, we can write $v \in \operatorname{null} N$ as a linear combination of $N^{m_1}v_1, \dots, Nv_1, v_1, \dots, N^{m_n}v_n, \dots, Nv_n, v_n$. Applying N to both sides, we have $Nv = 0$, $N(N^{m_1}v_1) = \cdots = N(N^{m_n}v_n) = 0$, but vectors $N(N^{m_1-1}v_1), \dots, N(Nv_1), Nv_1, \dots, N(N^{m_n-1}v_n), \dots, N(Nv_n), N(v_n)$ do not equal zero. Since vectors
$N(N^{m_1-1}v_1), \dots, N(Nv_1), Nv_1, \dots, N(N^{m_n-1}v_n), \dots, N(Nv_n), N(v_n)$ are linearly independent, their coefficients in the linear combination equal to v are 0, implying that $v \in \operatorname{span}(N^{m_1}v_1, \dots, N^{m_n}v_n)$.

Hence, $N^{m_1}v_1, \dots, N^{m_n}v_n$ is a basis of $\operatorname{null} N$.