Linear Algebra Done Right
Solutions to Exercises 1.A

1 Inverse of a complex number

Problem statement

Suppose a and b are real numbers, not both 0. Find real numbers c and d such that

$$1/(a+bi) = c + di$$

Solution

While we weren't given the definition of the complex conjugate[1] in Chapter 1.A, we can find c and d by multiplying $1/(a+bi)$ by its complex conjugate. Hence we have

$$\frac{1}{a+bi}\frac{a-bi}{a-bi} = \frac{a-bi}{(a^2+b^2)+(-ab+ab)i} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$$

where the first equality comes from the definition of addition and multiplication on $\mathbf{C}$ (Definition 1.1). Thus $c = a/(a^2+b^2)$ and $d = -b/(a^2+b^2)$.

[1] We'll have to wait for Chapter 4 to get Definition 4.3.

2 Cube root of 1

Problem statement

Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that is cube equals 1).

Solution

$$\begin{aligned}
\left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= \left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{-1+\sqrt{3}i}{2}\right) \\
&= \frac{1}{8}(-1+\sqrt{3}i)(-1+\sqrt{3}i)(-1+\sqrt{3}i) \\
&= \frac{1}{8}((1-3)+(-2\sqrt{3})i)(-1+\sqrt{3}i) \\
&= \frac{1}{8}(-2-2\sqrt{3}i)(-1+\sqrt{3}i) \\
&= \frac{1}{8}((2+6)+(-2\sqrt{3}+2\sqrt{3})i) \\
&= \frac{1}{8}(8+0i) \\
&= 1
\end{aligned}$$

3 Square roots of i

Problem statement

Find two distinct square roots of i.

Solution

The square roots of i concern $\frac{1}{\sqrt{2}}$.

$$
\begin{aligned}
(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)^2 &= (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) \\
&= (\frac{1}{2} - \frac{1}{2}) + (\frac{1}{2} + \frac{1}{2})i \\
&= 0 + 1i \\
&= i
\end{aligned}
$$

$$
\begin{aligned}
(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)^2 &= (-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i) \\
&= (\frac{1}{2} - \frac{1}{2}) + (\frac{1}{2} + \frac{1}{2})i \\
&= 0 + 1i \\
&= i
\end{aligned}
$$

11 System of equations to find $\lambda \in \mathbf{C}$

Problem statement

Explain why there does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i).$$

Solution

We can write $\lambda \in \mathbf{C}$ as $\lambda = a + bi$. Now we can frame our problem into a series of linear equations. We can write

$$\begin{gathered}
(a+bi)(2-3i) = 12-5i \\
(2a+3b) + (-3a+2b)i = 12-5i \\
a = -\frac{3}{2}b + 6 \quad \text{and} \quad a = \frac{2}{3}b + \frac{5}{3} \\
-\frac{3}{2}b + 6 = \frac{2}{3}b + \frac{5}{3} \\
-\frac{3}{2}b - \frac{2}{3}b = \frac{5}{3} - 6 \\
-\frac{15}{6}b = -\frac{13}{3} \\
b = \frac{26}{15}
\end{gathered}$$

and

$$\begin{gathered}
(a+bi)(5+4i) = 7+22i \\
(5a-4b) + (4a+5b)i = 7+22i \\
a = \frac{4}{5}b + \frac{7}{5} \quad \text{and} \quad a = -\frac{5}{4}b + \frac{11}{2} \\
\frac{4}{5}b + \frac{5}{4}b = -\frac{7}{5} + \frac{11}{2} \\
\frac{41}{20}b = \frac{41}{10} \\
b = 2.
\end{gathered}$$

Hence, the systems of linear equations does not have a solution and no $\lambda \in \mathbf{C}$ exists.