Linear Algebra Done Right
Solutions to Exercises 8.B

1 If 0 is only eigenvalue, N is nilpotent

Problem statement

Suppose V is a complex vector space, $N \in \mathcal{L}(V)$, and 0 is the only eigenvalue of N. Prove that N is nilpotent.

Solution

Following Theorem 5.27 ('Over **C**, every operator has an upper-triangular matrix') and Theorem 5.32 ('Determination of eigenvalues from upper-triangular matrix'), we can infer that N has an upper-triangular matrix $\mathcal{M}(N)$ such that the only entries on the diagonal are zeros. Now we must prove that such a matrix is nilpotent.

Let $v_1, \ldots, v_n$ be the basis associated with $\mathcal{M}(N)$. Clearly $N^1 v_1 = 0$. To prove that $N^j(v_j) = 0$ for $j \in \{2, \ldots, n\}$, let's use induction and suppose $N^k(v_k) = 0$ for $k < j$. Since $\mathcal{M}(N)$ is upper-triangular and diagonal entries are 0's, it follows that

$$N^j(v_j) = N^{j-1}(a_1 v_1 + \cdots + a_{j-1} v_{j-1}) = a_1 N^{j-2}(N^1 v_1) + \cdots + a_{j-1} N^{j-1} v_{j-1}.$$

Via our induction hypothesis, it follows that $N^j(v_j) = 0$.

To show that $N^{\dim V} = 0$, suppose $v \in V$. Given $v_1, \ldots, v_n$ is a basis of V, we can write

$$v = a_1 v_1 + \cdots + a_n v_n$$

and applying N^n to both sides (where $N^n = N^{\dim V}$), we have

$$N^n(v) = a_1 N^n(v_1) + \cdots + a_n N^n(v_n) = a_1 N^{n-1}(N^1 v_1) + \cdots + a_n N^n(v_n) = 0$$

via our reasoning above. Hence, $N^{\dim V} = 0$ and N is nilpotent.

Notes

This exercise and Exercise 8.A(7) imply that for a complex vector space V, $N \in \mathcal{L}(V)$ is nilpotent if and only if 0 is the only eigenvalue of N.

2 Eigenvalue 0 doesn't entail nilpotent over $\mathbf{R}$

Problem statement

Give an example of an operator T on a finite-dimensional real vector pace such tht 0 is the only eigenvalue of T but T is not nilpotent.

Solution

Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ is defined by

$$T(z_1, z_2, z_3) = (-z_2, z_1, 0).$$

Clearly 0 is an eigenvalue since $T(0,0,1) = (0,0,0)$; however, there are no other eigenvalues since the operator is otherwise a rotation in the first and second coordinates. More specifically, for the subspace

$$U = \{(z_1, z_2, 0) \in \mathbf{R}^3 : z_1, z_2 \in \mathbf{R}\},$$

T is a counterclockwise rotation in U.

To show that T is not nilpotent, we need only compute T^3 via Theorem 8.18 ('Nilpotent operator raised to dimension of domain is 0'). Hence, we write

$$\begin{aligned} T^3(z_1, z_2, z_3) &= T^2(-z_2, z_1, 0) \\ &= T(-z_1, -z_2, 0) \\ &= (z_2, -z_1, 0). \end{aligned}$$

Notes

We previously noted that Exercise 8.B(1) and Exercise 8.A(7) imply that for a complex vector space V, $N \in \mathcal{L}(V)$ is nilpotent if and only if 0 is the only eigenvalue of N. This exercise shows us that for a real vector space V, we can only state that if $N \in \mathcal{L}(V)$ is nilpotent, then 0 is the only eigenvalue of N. This exercise is a counterexample against the other direction.

This result is another implication of the failure of the Fundamental Theorem of Algebra for $\mathbf{R}$.

3 Multiplicity of eigenvalues for T and $S^{-1}TS$

Problem statement

Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible. Prove that T and $S^{-1}TS$ have the same eigenvalues with the same multiplicities.

Solution

This exercise bears a strong resemblance to Exercise 5.A(15) whereby we proved that T and $S^{-1}TS$ have the same eigenvalues and that $v \in V$ is an eigenvector of $S^{-1}TS$ if and only if Sv is an eigenvector of T. To show that T and $S^{-1}TS$ have the same multiplicities, we must show

$$\dim G(\lambda, T) = \dim G(\lambda, S^{-1}TS)$$

for all eigenvalues λ.

Suppose $v \in V$ is a generalized eigenvector of T corresponding to the eigenvalue λ. Hence, we have $(T - \lambda I)^j v = 0$ for some positive integer j. Given S is invertible, it follows that S^j is invertible[1]. If follows that $v \in \operatorname{range} S^j$ and there exists $u \in V$ such that $S^j u = v$ and $S^{-j} v = u$. Now we can write

$$\begin{aligned} 0 = S^{-j}(0) &= S^{-j}(T - \lambda I)^j v \\ &= S^{-j}(T - \lambda I)^j S^j u \\ &= (S^{-1}TS - \lambda S^{-1}IS)u \\ &= (S^{-1}TS - \lambda I)u. \end{aligned}$$

Hence $u = S^{-j}v$ is a generalized eigenvector of $S^{-1}TS$.

Suppose $v \in V$ is a generalized eigenvector of $S^{-1}TS$ corresponding to the eigenvalue λ. Hence, we have $(S^{-1}TS - \lambda I)^j v = 0$ for some positive integer j. Using similar logic as before, there exists $u \in V$ such that $S^{-j}u = v$. Now we can write

$$\begin{aligned} 0 = S^j(0) &= S^j(S^{-1}TS - \lambda I)^j v \\ &= S^j(S^{-1}TS - \lambda I)^j S^{-j} u \\ &= (SS^{-1}TSS^{-1} - \lambda SIS^{-j})u \\ &= (T - \lambda I)u. \end{aligned}$$

Hence $u = S^j v$ is a generalized eigenvector of T.

To complete the proof, suppose $v_1, \ldots, v_n$ are a basis of $G(\lambda, T)$. Via our work above, it follows that $S^{-j}v_1, \ldots, S^{-j}v_n$ is linearly independent and $S^{-j}v_1, \ldots, S^{-j}v_n \in G(\lambda, S^{-1}TS)$. To show $S^{-j}v_1, \ldots, S^{-j}v_n$ spans $G(\lambda, S^{-1}TS)$,

[1] If one must be convinced of this, note that $\operatorname{range} S$ and $\operatorname{range} S^j$ are equivalent. Since S is surjective, S^j must be as well and is thus invertible.

suppose $u \in G(\lambda, S^{-1}TS)$. Via our work above, $S^j u \in G(\lambda, T)$ and there exist $a_1, \ldots a_n$ such that

$$S^j u = a_1 v_1 + \cdots + a_n v_n.$$

Applying S^{-j} to both sides, we have

$$u = a_1 S^{-j} v_1 + \cdots + a_n S^{-j} v_n$$

and it follows that $u \in \text{span}(S^{-j}v_1, \ldots, S^{-j}v_n)$. Hence $S^{-j}v_1, \ldots, S^{-j}v_n$ is a basis of $G(\lambda, S^{-1}TS)$ and it follows that

$$\dim G(\lambda, T) = n = \dim G(\lambda, S^{-1}TS).$$

Therefore, T and $S^{-1}TS$ have the same eigenvalues with the same multiplicities.

4 Distinct eigenvalues when $\operatorname{null} T^{n-2} \neq \operatorname{null} T^{n-1}$

Problem statement

Suppose V is an n-dimensional complex vector space and T is an operator on V such that $\operatorname{null} T^{n-2} \neq \operatorname{null} T^{n-1}$. Prove that T has at most two distinct eigenvalues.

Solution

Let's first note that T is not injective and 0 is an eigenvalue. If T were injective, then $\{0\} = \operatorname{null} T^0 = \operatorname{null} T^1$, and via Theorem 8.3 ('Equality in the sequence of null spaces'), it would follow that $\operatorname{null} T^{n-2} = \operatorname{null} T^{n-1}$ which is a contradiction.

Let's now think about $\dim \operatorname{null} T^{n-1}$. Following from the proof of Theorem 8.4 ('Null spaces stop growing'), we have

$$\{0\} = \operatorname{null} T^0 \subsetneq \operatorname{null} T^1 \subsetneq \cdots \operatorname{null} T^{n-2} \subsetneq \operatorname{null} T^{n-1}$$

where at each of the strict inclusions in the chain above, the dimension increases by at least 1. Hence, we have

$$n - 1 \leq \dim \operatorname{null} T^{n-1} \leq \dim \operatorname{null}(T - 0I)^n = \dim G(0, T).$$

Thus, the multiplicity of 0 is either $n - 1$ or n.

If the multiplicity of 0 is $n-1$, then Theorem 8.26 ('Sum of the multiplicities equals $\dim V$') implies another distinct eigenvalue exists with a multiplicity of 1 and T has two distinct eigenvalues. If the multiplicity of 0 is n, then Theorem 8.26 implies 0 is the only eigenvalue and T has one distinct eigenvalues. Therefore, T has at most two distinct eigenvalues.

5 Basis of eigenvectors iff they are generalized

Problem statement

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if every generalized eigenvector of T is an eigenvector of T.

Solution

First Direction

Suppose V has a basis consisting of eigenvectors of T. Via Theorem 5.41 ('Conditions equivalent to diagonalizability'), it follows that

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T). \tag{1}$$

The definition of generalized eigenspace (Definition 8.10) implies $E(\lambda_j, T) \subset G(\lambda_j, T)$, hence we have $\dim E(\lambda_j, T) \leq \dim G(\lambda_j, T)$.

To show that the inequality is an equality, let's first note that from Theorem 8.21 ('Description of operators on complex vector spaces') we can write

$$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T). \tag{2}$$

Suppose $\dim E(\lambda_j, T) < \dim G(\lambda_j, T)$. It follows that there exists $v \in V$ such that $v \notin E(\lambda_j, T)$ and $v \in G(\lambda_j, T)$. Our direct sum (1) implies that

$$v = a_1 v_1 + \cdots + a_{j-1} v_{j-1} + 0(v_j) + a_{j+1} v_{j+1} + \cdots + a_m v_m$$

where $v_k \in E(\lambda_k, T)$. Given $E(\lambda_j, T) \subset G(\lambda_j, T)$, that decomposition is also a valid decomposition in (2). However, since $v \in G(\lambda_j, T)$, direct sum (2) implies that

$$v = 0(v_1) + \cdots + 0(v_{j-1}) + a_j v_j + 0(v_{j+1}) + \cdots + 0(v_m)$$

where $a_j \neq 0$ and $v_j \neq 0$[2]. Hence we have two different representations of v and (2) is not a direct sum, leading to a contradiction.

Thus, $\dim E(\lambda_j, T) = \dim G(\lambda_j, T)$, and since $E(\lambda_j, T) \subset G(\lambda_j, T)$, it follows that $E(\lambda_j, T) = G(\lambda_j, T)$. Therefore, every generalized eigenvector of T is an eigenvector of T.

Second Direction

Suppose every generalized eigenvector of T is an eigenvector of T. Unlike the complexity of the **First Direction**, it clearly follows from Theorem 8.23 ('A basis of generalized eigenvectors') that V has a basis consisting of eigenvectors of T.

[2]Note that $v \notin E(\lambda_j, T)$ implies that $v \neq 0$.

6 Find a square root of $I+N$

Problem statement

Define $N \in \mathcal{L}(\mathbf{F}^5)$ by

$$N(x_1, x_2, x_3, x_4, x_5) = (2x_2, 3x_3, -x_4, 4x_5, 0).$$

Find a square root of $I+N$.

Solution

Via Theorem 8.31 ('Identity plus nilpotent has a square root'), we know that there must exist a square root of $I+N$, but we are only provided partial guidance for how to find it. Theorem 8.31 tells us that the square root of $I+N$ will be of the form

$$I + a_1N + a_2N^2 + a_3N^3 + a_4N^4$$

via the Taylor series for the function $\sqrt{1+x}$ and the observation that $N^5 = 0$. To solve for a_1, a_2, a_3, a_4, we can write

$$\begin{aligned}(I + a_1N + a_2N^2 + a_3N^3 + a_4N^4)^2 \\ &= I + 2a_1N + (a_1^2 + 2a_2)N^2 \\ &\quad + (2a_1a_2 + 2a_3)N^3 + (2a_1a_3 + 2a_4)N^4 + \dots\end{aligned}$$

where the other terms vanish since $N^5 = 0$. We are left to solve for the coefficients such that

$$\begin{aligned} 2a_1 &= 1, \\ a_1^2 + 2a_2 &= 0, \\ 2a_1a_2 + 2a_3 &= 0, \\ 2a_1a_3 + 2a_4 &= 0. \end{aligned}$$

Via some simple calculations, it follows that $a_1 = \frac{1}{2}, a_2 = -\frac{1}{8}, a_3 = \frac{1}{16}, a_4 = -\frac{1}{32}$ and we can write the square root of $I+N$ as

$$I + \frac{1}{2}N - \frac{1}{8}N^2 + \frac{1}{16}N^3 - \frac{1}{32}N^4.$$

Via another set of simple calculations, we can write

$$\begin{aligned}\sqrt{I+N}(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2 - \frac{3}{4}x_3 - \frac{3}{8}x_4 + \frac{3}{4}x_5, x_2 + \frac{3}{2}x_3 + \frac{3}{8}x_4 - \frac{3}{4}x_5, \\ x_3 - \frac{1}{2}x_4 + \frac{1}{2}x_5, x_4 + 2x_5, x_5).\end{aligned}$$