Linear Algebra Done Right
Solutions to Exercises 6.C

1 Orthogonal complement of subset

Problem statement

Suppose $v_1, \ldots, v_m \in V$. Prove that

$$\{v_1, \ldots, v_m\}^\perp = \big(\operatorname{span}(v_1, \ldots, v_m)\big)^\perp.$$

Solution

To prove our desired result, let's first show $\{v_1, \ldots, v_m\}^\perp \subset \big(\operatorname{span}(v_1, \ldots, v_m)\big)^\perp$, and then $\big(\operatorname{span}(v_1, \ldots, v_m)\big)^\perp \subset \{v_1, \ldots, v_m\}^\perp$.

First Direction

Suppose $u \in \{v_1, \ldots, v_m\}^\perp$. This implies $\langle u, v_j \rangle = 0$ for $j = 1, \ldots, m$. We can write every $v \in \operatorname{span}(v_1, \ldots, v_m)$ as

$$v = a_1 v_1 + \cdots + a_m v_m$$

for $a_1, \ldots, a_m \in \mathbf{F}$. It follows that

$$\langle u, v \rangle = \bar{a}_1 \langle u, v_1 \rangle + \cdots + \bar{a}_m \langle u, v_m \rangle = 0.$$

Hence, $u \in \big(\operatorname{span}(v_1, \ldots, v_m)\big)^\perp$.

Second Direction

Suppose $u \in \big(\operatorname{span}(v_1, \ldots, v_m)\big)^\perp$. Since $v_1, \ldots, v_m \in \operatorname{span}(v_1, \ldots, v_m)$, it follows that $\langle u, v_j \rangle = 0$ for $j = 1, \ldots, m$. Hence, $u \in \{v_1, \ldots, v_m\}^\perp$.

2 $U^\perp = \{0\}$ iff $U = V$

Problem statement

Suppose U is a finite-dimensional subspace of V. Prove that $U^\perp = \{0\}$ if and only if $U = V$.

Solution

First Direction

Suppose $U^\perp = \{0\}$. Via Theorem 6.51 ('The orthogonal complement of the orthogonal complement') and Theorem 6.46(b) ('Basic properties of orthogonal complement'), we can write

$$U = (U^\perp)^\perp = \{0\}^\perp = V.$$

Hence, $U = V$.

Second Direction

Suppose $U = V$. Via Theorem 6.46(c) ('Basic properties of orthogonal complement'), we can write

$$U^\perp = V^\perp = \{0\}.$$

Hence, $U^\perp = \{0\}$.

3 $U^\perp = \{0\}$ iff $U = V$

Problem statement

Suppose U is a subspace of V with basis $u_1, \dots, u_m$ and

$$u_1, \dots, u_m, w_1, \dots, w_n$$

is a basis of V. Prove that if the Gram-Schmidt Procedure is applied to the basis of V above, producing a list $e_1, \dots, e_m, f_1, \dots, f_n$, then $e_1, \dots, e_m$ is an orthonormal basis of U and $f_1, \dots, f_n$ is an orthonormal basis of $U^\perp$.

Solution

Via the Gram-Schmidt Procedure (Theorem 6.31), we can state

$$\operatorname{span}(u_1, \dots, u_m) = \operatorname{span}(e_1, \dots, e_m).$$

Thus, $e_1, \dots, e_m$ is an orthonormal basis of U.

Via Theorem 6.50 (‘Dimension of the orthogonal complement’), we have

$$\dim V = \dim U + \dim U^\perp$$

which implies $\dim U^\perp = n$. Given $e_1, \dots, e_m, f_1, \dots, f_n$ is an orthonormal list, it follows that $\langle e_j, f_k \rangle = 0$ for $j = 1, \dots, m$ and $k = 1, \dots, n$. Hence, $f_j \in \{e_1, \dots, e_m\}^\perp$ and, via Exercise 6.C(1), $f_j \in \big(\operatorname{span}(e_1, \dots, e_m)\big)^\perp$. Therefore, $f_j \in U^\perp$ for $j = 1, \dots, n$.

Given $\dim U^\perp = n$ and $f_1, \dots, f_n$ is an orthonormal list of length n in $U^\perp$, it follows that $f_1, \dots, f_n$ is an orthonormal basis of $U^\perp$.

4 Finding orthonormal bases of $U, U^\perp$

Problem statement

Suppose U is the subspace of $\mathbf{R}^4$ defined by

$$U = \operatorname{span}\big((1,2,3,-4),(-5,4,3,2)\big).$$

Find an orthonormal basis of U and an orthonormal basis of $U^\perp$.

Solution

Following exercise 6.C(3), we need to expand $(1,2,3,-4),(-5,4,3,2)$ to be a basis of $\mathbf{R}^4$, and then apply the Gram-Schmidt Procedure (Theorem 6.31).

A reasonable guess at two vectors to make a basis of $\mathbf{R}^4$ is $(1,0,0,0),(0,1,0,0)$. To check if $(1,2,3,-4),(-5,4,3,2),(1,0,0,0),(0,1,0,0)$ is linearly independent, we can solve the following system of linear equations

$$\begin{aligned} x - 5y + z + 0w &= 0 \\ 2x + 4y + 0z + w &= 0 \\ 3x + 3y + 0z + 0w &= 0 \\ -4x + 2y + 0z + 0w &= 0. \end{aligned}$$

The bottom two equations can be reduced to $x = -y$ and $2x = y$, which implies $x = 0$ and $y = 0$. Substituting $x = 0$ and $y = 0$ into the top two equations, we get $z = 0$ and $w = 0$, confirming that our list is linearly independent.

By applying the Gram-Schmidt Procedure to our list[1], we obtain e_1, e_2, e_3, e_4 where e_1, e_2 is an orthonormal basis of U and e_3, e_4 is an orthonormal basis of $U^\perp$.

[1] This is lazy, but the computation is heinously messy.

5 Proving $P_{U^\perp} = I - P_U$

Problem statement

Suppose V is finite-dimensional and U is a subspace of V. Show that $P_{U^\perp} = I - P_U$, where I is the identity operator on V.

Solution

First, let's describe $P_{U^\perp}$. Following from Theorem 6.47 ('Direct sum of a subspace and its orthogonal complement'), for $v \in V$ we can write $v = u + w$ where $u \in U^\perp$ and $w \in (U^\perp)^\perp$. Given V is finite-dimensional and Theorem 6.51 ('The orthogonal complement of the orthogonal complement'), $(U^\perp)^\perp = U$ and $w \in U$. Therefore $P_{U^\perp}v = u$ and $P_U v = w$, and we can write

$$P_{U^\perp}v = Iv - P_U v.$$

Hence, it follows that $P_{U^\perp} = I - P_U$.

6 $P_U P_W = 0$ iff $\langle u, w \rangle = 0$

Problem statement

Suppose U and W are finite-dimensional subspaces of V. Prove that $P_U P_W = 0$ if and only if $\langle u, w \rangle = 0$ for all $u \in U$ and all $w \in W$.

Solution

First Direction

Suppose $P_U P_W = 0$. This implies $\operatorname{range} P_W \subset \operatorname{null} P_U$. Via Theorems 6.55(d) and 6.55(e) ('Properties of the orthogonal projection P_U'), it follows that $W \subset U^\perp$. Hence, $\langle u, w \rangle = 0$ for all $u \in U$ and all $w \in W$ since $w \in U^\perp$.

Second Direction

Suppose $\langle u, w \rangle = 0$ for all $u \in U$ and all $w \in W$. This implies $W \subset U^\perp$, and via Theorems 6.55(d) and 6.55(e), it follows that $\operatorname{range} P_W \subset \operatorname{null} P_U$. Hence, $P_U P_W = 0$.

7 Prove $\exists U$ such that $P = P_U$

Problem statement

Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in $\operatorname{null} P$ is orthogonal to every vector in $\operatorname{range} P$. Prove that there exists a subspace U of V such that $P = P_U$.

Solution

Let's first think about $\operatorname{null} P \cap \operatorname{range} P$. Suppose $v \in \operatorname{null} P \cap \operatorname{range} P$. Given every vector in $\operatorname{null} P$ is orthogonal to every vector in $\operatorname{range} P$, it follows that $\langle v, v \rangle = 0$ and $v = 0$ via the property of definiteness (Definition 6.3). Thus $\operatorname{null} P \cap \operatorname{range} P = \{0\}$. We can write

$$\dim(\operatorname{null} P + \operatorname{range} P) = \dim \operatorname{null} P + \dim \operatorname{range} P - \dim(\operatorname{null} P \cap \operatorname{range} P)$$

via Theorem 2.43 ('Dimension of a sum') and

$$\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P$$

via the Fundamental Theorem of Linear Maps. Since $\dim(\operatorname{null} P \cap \operatorname{range} P) = 0$, it follows that

$$\dim(\operatorname{null} P + \operatorname{range} P) = \dim \operatorname{null} P + \dim \operatorname{range} P = \dim V,$$

and thus

$$V = \operatorname{null} P \oplus \operatorname{range} P.$$

Via Theorem 6.47 ('Direct sum of a subspace and its orthogonal complement'), we can write

$$\operatorname{range} P \oplus \operatorname{null} P = V = \operatorname{range} P \oplus (\operatorname{range} P)^{\perp}$$

implying that $\operatorname{null} P = (\operatorname{range} P)^{\perp}$.

A reasonable guess at our desired subspace U is $\operatorname{range} P$. To show this, suppose $v \in V$. We can write v as $v = u + w$ where $u \in \operatorname{range} P$ and $w \in (\operatorname{range} P)^{\perp} = \operatorname{null} P$. It follows that

$$Pv = P(u + w) = Pu + Pw = Pu$$

and

$$P_{\operatorname{range} P} v = u$$

where u is the orthogonal projection of v onto $\operatorname{range} P$.

To complete the proof, we need to show that $Pu = u$. Since $u \in \operatorname{range} P$, there exists $x \in V$ such that $Px = u$. Applying P to both sides, we have $P^2 x = Pu$. Given $P^2 = P$, we can write

$$Pu = P^2 x = Px = u.$$

Hence, $Pv = Pu = u$ and $P_{\operatorname{range} P} v = u$ for $v \in V$.