Linear Algebra Done Right
Solutions to Exercises 6.A

3 Replacing the positivity condition

Problem statement

Suppose $\mathbf{F} = \mathbf{R}$ and $V \neq \{0\}$. Replace the positivity condition (which states that $\langle v, v \rangle \geq 0$ and for all $v \in V$) in the definition of an inner product (6.3) with the condition that $\langle v, v \rangle > 0$ for some $v \in V$. Show that this change in the definition does not change the set of functions from $V \times V$ to $\mathbf{R}$ that are inner products on V.

Solution

The set of functions from $V \times V$ to $\mathbf{R}$ that are inner products on V clearly does not grow smaller. Thus we must show that the set does not grow larger.

Suppose $v \in V$ is such that it satisfies the "new" condition that $\langle v, v \rangle > 0$. There are two cases to consider: $V = \operatorname{span}(v)$ and $V \neq \operatorname{span}(v)$.

$\mathbf{V = span(v)}$: For any vector $u \in V$ there exists $\lambda \in \mathbf{R}$ such that $u = \lambda v$. Hence

$$\langle u, u \rangle = \langle \lambda v, \lambda v \rangle = |\lambda|^2 \langle v, v \rangle$$

Given $\langle v, v \rangle > 0$ and $|\lambda|^2 \geq 0$ for all $\lambda \in \mathbf{R}$, it follows that

$$\langle u, u \rangle = |\lambda|^2 \langle v, v \rangle \geq 0$$

Thus we've shown that the positivity condition is fulfilled and the set of function that are inner products on V clearly does not grow larger.

$\mathbf{V \neq span(v)}$: Since $V \neq \{0\}$ and $V \neq \operatorname{span}(v)$, there exists $u \in V$ that is not a scalar multiple of v. Theorem 6.14 ('An orthogonal decomposition') allows us to decompose u into $u = cv + w$ where v and w are orthogonal. From the Pythagorean theorem, it follows

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

Let's assume that $||w||^2 < 0$. Now consider the vector λw where $\lambda = \frac{\|v\|}{\|w\|}$[1]

$$\|v + \lambda w\|^2 = \|v\|^2 + \|\lambda w\|^2 = \|v\|^2 + |\frac{\|v\|}{\|w\|}|^2 \|w\|^2 = \|v\|^2 - \|v\|^2 = 0$$

Thus we have a contradiction since $v + \lambda w \neq 0$ yet $||v + \lambda w||^2 = 0$, defying the condition of definiteness (definition 6.3). Therefore $||w||^2 > 0$.

It then follows that

$$||u||^2 = ||v + w||^2 = ||v||^2 + ||w||^2 \geq 0$$

showing that the positivity condition is fulfilled and the set of function that are inner products on V clearly does not grow larger.

[1]The $|\frac{\|v\|}{\|w\|}|^2 \|w\|^2$ term is quite harry, but note that $\|w\|^2 < 0$ and $|\|w\||^2 > 0$, therefore $\frac{\|w\|^2}{|\|w\||^2} = -1$. Also note $|\|v\||^2 = \|v\|^2$ since $\langle v, v \rangle > 0$.