Linear Algebra Done Right
Solutions to Exercises 1.C

1 Set or subspace?

Problem statement

For each of the following subset of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$:

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;

(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;

(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$;

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$.

Solution

a

To determine if the subset $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$ is a subspace of $\mathbf{F}^3$, we can assess if the subset satisfies the properties of a subspace: **additive identity**, **closed under addition**, and **closed under scalar multiplication**.

additive identity: Given that $0 + 2 \cdot 0 + 3 \cdot 0 = 0$, we have $(0, 0, 0) \in \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$.

closed under scalar multiplication: Suppose $(y_1, y_2, y_3) \in \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$ and $\lambda \in \mathbf{F}$. It follows that

$$\begin{aligned} y_1 + 2y_2 + 3y_3 = 0 &= \frac{1}{\lambda} \cdot 0 \\ \lambda(y_1 + 2y_2 + 3y_3) &= 0 \\ \lambda y_1 + 2(\lambda y_2) + 3(\lambda y_3) &= 0. \end{aligned}$$

Hence, we have $\lambda(y_1, y_2, y_3) \in \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$.

closed under addition: Suppose $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$. It follows that

$$x_1 + 2x_2 + 3x_3 = 0 \quad \text{and} \quad y_1 + 2y_2 + 3y_3 = 0.$$

Adding the two equations above, we have

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + y_1 + 2y_2 + 3y_3 &= 0 \\ (x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) &= 0. \end{aligned}$$

Hence, we have
$(x_1, x_2, x_3) + (y_1, y_2, y_3) \in \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$.

Since the subset $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$ satisfies all the conditions of a subspace, via Theorem 1.34 ('Conditions of a subspace'), it is a subspace of $\mathbf{F}^3$.

b

No, the subset $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$ is not a subspace of $\mathbf{F}^3$ because it does not contain the **additive identity**. This follows from the observation that

$$0 + 2 \cdot 0 + 3 \cdot 0 \neq 4.$$

c

No, the subset $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$ is not a subspace of $\mathbf{F}^3$ because it is not **closed under addition**. We can show this by noting that the vectors $(0, 1, 1)$ and $(1, 0, 1)$ are members of the subset, but $(0, 1, 1) + (1, 0, 1)$ is not a member since

$$(0, 1, 1) + (1, 0, 1) = (1, 1, 2) \quad \text{and} \quad 1 \cdot 1 \cdot 2 \neq 0.$$

d

Yes, the subset $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$ is a subspace of $\mathbf{F}^3$. This follows from the observation that the following set

$$\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 0x_2 - 5x_3 = 0\}$$

is an equivalent formulation of $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$ is a subspace of $\mathbf{F}^3$ and is of a similar form to the subspace $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$, which we showed was a subspace of $\mathbf{F}^3$. A verification that the subset $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 0x_2 - 5x_3 = 0\}$ satisfies all the conditions of a subspace is near identical to the reasoning in part **a**.

8 Set closed under multiplication not subspace

Problem statement

Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.

Solution

The subset

$$U = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 x_2 = 0\}$$

is **closed under scalar multiplication** but is not **closed under addition**; thus, it is not a subspace.

To verify that U is **closed under scalar multiplication**, for $\lambda \in \mathbf{R}$ and $(x_1, x_2) \in U$, it follows that for $\lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$ and we can write

$$(\lambda x_1)(\lambda x_2) = \lambda^2 (x_1 x_2) = \lambda^2 (0) = 0$$

since $x_1 x_2 = 0$.

To verify that U is not **closed under addition**, it follows that $(1, 0), (0, 1) \in U$ but $(1, 0) + (0, 1) = (1, 1)$ and $(1, 1) \notin U$.

10 Intersection of subspaces is a subspace

Problem statement

Suppose U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

Solution

Via Theorem 1.34 ('Conditions of a subspace'), we need to show that $U_1 \cap U_2$ satisfies the conditions of **additive identity**, **closed under scalar multiplication**, and **closed under addition**.

additive identity: Given U_1 and U_2 are subspaces, it follows that $0 \in U_1$ and $0 \in U_2$. Hence, we have $0 \in U_1 \cap U_2$.

closed under scalar multiplication: Suppose $v \in U_1 \cap U_2$ and $\lambda \in \mathbf{F}$. Since U_1 and U_2 are **closed under scalar multiplication**, it follows that $\lambda v \in U_1$ and $\lambda v \in U2$. Hence, we have $\lambda v \in U_1 \cap U_2$.

closed under addition: Suppose $v, u \in U_1 \cap U_2$. Since U_1 and U_2 are **closed under addition**, it follows that $v + u \in U_1$ and $v + u \in U2$. Hence, we have $v + u \in U_1 \cap U_2$.

Therefore, since $U_1 \cap U_2$ satisfies all the conditions of a subspace, it follows that $U_1 \cap U_2$ is a subspace V.

11 Intersection of every collection of subspaces

Problem statement

Prove that the intersection of every collection of subspaces of V is a subspace of V.

Solution

Suppose $U_1, U_2, \ldots, U_n$ is a collection of subspaces of V. By arranging the intersection of the collection as

$$\big(\ldots(U_1 \cap U_2) \cap \ldots\big) \cap U_n$$

we can iteratively apply our result from Exercise 1.C(10) to show that the intersection is a subspace of V.

21 Find W such that $\mathbf{F}^5 = U \oplus W$

Problem statement

Suppose

$$U = \{(x, y, x+y, x-y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find a subspace W of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$.

Solution

Define the subset W of $\mathbf{F}^5$ as

$$W = \{(0, 0, z, u, v) \in \mathbf{F}^5 : z, u, v \in \mathbf{F}\}.$$

To prove that $\mathbf{F}^5 = U \oplus W$, we need to show that W is a subspace of $\mathbf{F}^5$, $\mathbf{F}^5 = U + W$, and $U + W$ is a direct sum.

It obviously follows from the construction of W that W is a subspace of $\mathbf{F}^5$.

To prove $\mathbf{F}^5 = U + W$, we need to show $U + W \subset \mathbf{F}^5$ and $\mathbf{F}^5 \subset U + W$. Suppose $(x, y, x+y, x-y, 2x) \in U$ and $(0, 0, z, u, v) \in W$. It follows that

$$\begin{aligned}(x, y, x+y, x-y, 2x) &+ (0, 0, z, u, v) \\ &= (x, y, x+y+z, x-y+u, 2x+v) \in \mathbf{F}^5,\end{aligned}$$

showing $U+W \subset \mathbf{F}^5$. Suppose $(x, y, z, u, v) \in \mathbf{F}^5$. We can construct (x, y, z, u, v) from $U + W$ by choosing vectors $(x, y, x+y, x-y, 2x) \in U$ and $(0, 0, z-x-y, u-x+y, v-2x) \in W$ and writing

$$(x, y, x+y, x-y, 2x) + (0, 0, z-x-y, u-x+y, v-2x) = (x, y, z, u, v).$$

Hence, we've shown $U + W \subset \mathbf{F}^5$,[] and it follows that $\mathbf{F}^5 = U + W$.

To prove $U+W$ is a direct sum, it follows from Theorem 1.45 ('Direct sum of two subspaces') that we need only show $U \cap W = \{0\}$. This result follows from the observation that all vectors in W have a 0 in the 1^{st} and 2^{nd} coordinates. Hence, only vectors with a 0 in the 1^{st} and 2^{nd} coordinates in U can be members of $U \cap W$. Only one vector in U has a 0 in the 1^{st} and 2^{nd} coordinates, namely the 0 vector. Thus, it follows that $U \cap W = \{0\}$.