Linear Algebra Done Right
Solutions to Exercises 5.B

9 Zeros of p are eigenvalues of T

Problem statement

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$ with $v \neq 0$. Let p be a nonzero polynomial of smallest degree such that $p(T)v = 0$. Prove that every zero of p is an eigenvalue of T.

Solution

Suppose λ is a zero of p. We can use theorem 4.11 to write p as

$$p(z) = (z - \lambda)q(z).$$

It follows that $p(T) = (T - \lambda I)q(T)$. Given that p is the smallest degree polynomial such that $p(T)v = 0$ and $\deg p > \deg q$, it follows that $q(T)v \neq 0$.

Hence, we have

$$p(T)v = (T - \lambda I)q(T)v = 0$$

which implies $T(q(T)v) = \lambda q(T)v$. Therefore λ is an eigenvalue of T with $q(T)v$ as the corresponding eigenvector.[1]

[1]]Answer came from math.stackexchang.com.