Linear Algebra Done Right
Solutions to Exercises 2.C

1 $\dim U = \dim V$ implies $U = V$

Problem statement

Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

Solution

By virtue of U being a subspace of V, we have $U \subset V$. We need to show $V \subset U$. Suppose $u_1, \ldots, u_m$ is a basis of U. It follows that $u_1, \ldots, u_m \in V$ and $u_1, \ldots, u_m$ is linearly independent in V. Via Theorem 2.39 ('Linearly independent list of the right length is a basis'), $u_1, \ldots, u_m$ is a basis of V given that the list is linearly independent and the right length ($m = \dim U = \dim V$. Hence, any vector $v \in V$ can be written as

$$v = a_1 u_1 + \cdots + a_m u_m$$

for some $a_1, \ldots, a_m \in \mathbf{F}$. Thus, $v \in U$ and $V \subset U$. Therefore, we have $U = V$.

2 Possible subspaces of $\mathbf{R}^2$

Problem statement

Show that the subspaces of $\mathbf{R}^2$ are precisely $\{0\}, \mathbf{R}^2$, and all the lines in $\mathbf{R}^2$ through the origin.

Solution

Since $(1,0),(0,1)$ is a basis of $\mathbf{R}^2$, via Definition 2.36 ('dimension'), $\dim \mathbf{R}^2 = 2$. Via Theorem 2.38 ('Dimension of a subspace'), the possible dimensions of subspaces of $\mathbf{R}^2$ are $0, 1, 2$. Let's handle each of these cases separately.

For subspaces of $\mathbf{R}^2$ with a dimension of 0, the only possible subspace is $\{0\}$. Remembering Exercise 2.B(1), the empty list () is a basis of $\{0\}$. Thus, $\dim\{0\} = 0$.

For subspaces of $\mathbf{R}^2$ with a dimension of 1, we know that the subspace, let's call it U, must contain a vector $v \in \mathbf{R}^2$ such that $v \neq 0$. Given U is closed under scalar multiplication, all vectors λv where $\lambda \in \mathbf{R}$ are in the subspace, $\lambda v \in U$. The collection of these vectors $\{\lambda v : \lambda \in \mathbf{R}\}$ can be geometrically represented as a line through the origin. The list v is linearly independent and of length 1. Therefore, v is a basis of U and

$$\{\lambda v : \lambda \in \mathbf{R}\} = \operatorname{span}(v) = U.$$

Hence, all subspaces of $\mathbf{R}^2$ with a dimension of 1 are lines through the origin. Furthermore, all lines in $\mathbf{R}^2$ through the origin can be represented as

$$\{\lambda v : \lambda \in \mathbf{R}\}$$

for some $v \in V$. Therefore, all the lines in $\mathbf{R}^2$ through the origin are subspaces of $\mathbf{R}^2$.

Suppose U is a subspace of $\mathbf{R}^2$ with a dimension of 2. Via Exercise 2.C(1), given $\dim \mathbf{R}^2 = \dim U$, it follows that $U = \mathbf{R}^2$.

4 Explorations on $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(6) = 0\}$

Problem statement

(a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(6) = 0\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Solution

a

Via Theorem 2.38 ('Dimension of a subspace'), it follows that $\dim U \leq \dim \mathcal{P}_4(\mathbf{F})$. Given U does not contain all the vectors in $\mathcal{P}_4(\mathbf{F})$, for example 1 since $1(6) \neq 0$, then a basis of U would not span $\mathcal{P}_4(\mathbf{F})$ and it necessarily follows that $\dim U < \dim \mathcal{P}_4(\mathbf{F})$. Hence the dimension of U must be $0, 1, 2, 3, 4$ but not 5.

Consider the list of polynomials $(x-6), (x-6)^2, (x-6)^3, (x-6)^4$. The list is clearly linearly independent and each polynomial satisfies the condition that $p(6) = 0$. Thus, we can state $(x-6), (x-6)^2, (x-6)^3, (x-6)^4 \in U$. Since the list is length 4 and linearly independent, Theorem 2.39 ('Linearly independent list of the right length is a basis') tells us the list is a basis of U.

b

We have already noted that $1 \notin U$. Therefore, via Exercise 2.A(11), the list $(x-6), (x-6)^2, (x-6)^3, (x-6)^4, 1$ is linearly independent. Via Theorem 2.39, it follows that this list is a basis of $\mathcal{P}_4(\mathbf{F})$.

c

The obvious choice of W is

$$W = \operatorname{span}(1).$$

Given our reasoning in part (b) concerning our basis of $\mathcal{P}_4(\mathbf{F})$, it follows that $\mathcal{P}_4(\mathbf{F}) = U + W$. Given our reasoning in part (a), namely that $1 \notin U$, it follows that $U \cap W = \{0\}$. Thus, via Theorem 1.45 ('Direct sum of two subspaces'), it follows that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

9 Dimension of $v_1 + w, \ldots, v_m + w$

Problem statement

Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

Solution

We can show $v_j - v_1 \in \operatorname{span}(v_1 + w, \ldots, v_m + w)$ for $j \in \{2, \ldots, m\}$ since $v_j + w - (v_1 + w) = v_j - v_1$. Thus, we have

$$v_2 - v_1, \ldots, v_m - v_1 \in \operatorname{span}(v_1 + w, \ldots, v_m + w)$$

and it follows that

$$\operatorname{span}(v_2 - v_1, \ldots, v_m - v_1) \subset \operatorname{span}(v_1 + w, \ldots, v_m + w)$$

which implies

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq \dim \operatorname{span}(v_2 - v_1, \ldots, v_m - v_1).$$

If we can show that $\dim \operatorname{span}(v_2 - v_1, \ldots, v_m - v_1) = m - 1$, then the desired result follows.

Clearly,

$$\operatorname{span}(v_1, \ldots, v_m) = \operatorname{span}(v_1, v_2 - v_1, \ldots, v_m - v_1)$$

since all v_j can be written as $v_j = v_j - v_1 + v_1$. Thus, we have

$$m = \dim \operatorname{span}(v_1, \ldots, v_m) = \dim \operatorname{span}(v_1, v_2 - v_1, \ldots, v_m - v_1)$$

and it follows that the list the list $v_1, v_2 - v_1, \ldots, v_m - v_1$ is linearly independent via Theorem 2.42 ('Spanning list of the right length is a basis'). Therefore, the list $v_2 - v_1, \ldots, v_m - v_1$ is also linearly independent and

$$\dim \operatorname{span}(v_2 - v_1, \ldots, v_m - v_1) = m - 1,$$

implying

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1,$$

which was to be shown.

11 Proving $\mathbf{R}^8 = U \oplus W$ with Theorem 2.43

Problem statement

Suppose that U and W are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $\mathbf{R}^8 = U \oplus W$.

Solution

Given $U + W = \mathbf{R}^8$, it follows that

$$\dim(U + W) = \dim \mathbf{R}^8 = 8.$$

Via Theorem 2.43 ('Dimension of a sum'), we can write

$$\dim(U \cap W) = \dim U + \dim W - \dim(U + W) = 3 + 5 - 8 = 0,$$

which implies $U \cap W = \{0\}$. Therefore, via Theorem 1.45 ('Direct sum of two subspaces'), we can state $\mathbf{R}^8 = U \oplus W$.

12 Proving $U \cap W \neq \{0\}$ with Theorem 2.43

Problem statement

Suppose U and W are both five-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

Solution

Given U and W are subspaces of $\mathbf{R}^9$, then it necessarily follows that $U+W \subset \mathbf{R}^9$ and

$$9 = \dim \mathbf{R}^9 \geq \dim(U + W).$$

Via Theorem 2.43 ('Dimension of a sum'), we can write

$$9 \geq \dim U + \dim W - \dim(U \cap W) = 5 + 5 - \dim(U \cap W)$$

and by rearranging terms, we get

$$\dim(U \cap W) \geq 1.$$

This implies that $U \cap W \neq \{0\}$.

16 $\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m$

Problem statement

Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V such that $U_1 + \cdots + U_m$ is a direct sum. Prove that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional and

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

Solution

Let's prove this via an algorithm.

Step 1

Given $U_1 + \cdots + U_m$ is a direct sum, it necessarily follows that $U_1 + U_2$ is a direct sum and $U_1 \cap U_2 = \{0\}$. Hence, via Theorem 2.43 ('Dimension of a sum'), we have

$$\dim(U_1 \oplus U_2) = \dim U_1 + \dim U_2.$$

Step j

Given $U_1 + \cdots + U_m$ is a direct sum, it necessarily follows that $(U_1 + \cdots + U_j) + U_{j+1}$ is a direct sum and $(U_1 + \cdots + U_j) \cap U_{j+1} = \{0\}$. Hence, via Theorem 2.43, we have

$$\begin{aligned} \dim((U_1 \oplus \cdots \oplus U_j) \oplus U_{j+1}) &= \dim(U_1 \oplus \cdots \oplus U_j) + \dim U_{j+1} \\ &= (\dim U_1 + \cdots + \dim U_j) + \dim U_{j+1} \\ &= \dim U_1 + \cdots + \dim U_{j+1}. \end{aligned}$$

Step m-1

The algorithm terminates at this step with the desired result

$$\dim(U_1 \oplus U_2) = \dim U_1 + \dim U_2,$$

which further implies that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional.

17 Theorem 2.43 fails for three subspaces

Problem statement

You might guess, by analogy with the formula for the number of elements in the union of three subset of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space then

$$\begin{aligned}\dim(U_1{+}U_2 + U_3)&\\ &= \dim U_1 + \dim U_2 + \dim U_3\\ &\quad - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3)\\ &\quad + \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

Prove this or give a counterexample.

Solution

Let's give a counterexample.

For subspaces U_1, U_2, U_3 of $\mathbf{R}^2$, let's define them as

$$\begin{aligned}U_1 &= \{(x,0) \in \mathbf{R}^2 : x \in \mathbf{R}\},\\ U_2 &= \{(y,y) \in \mathbf{R}^2 : y \in \mathbf{R}\},\\ U_3 &= \{(0,z) \in \mathbf{R}^2 : z \in \mathbf{R}\}.\end{aligned}$$

Clearly, their individual dimensions are

$$\dim U_1 = \dim U_2 = \dim U_3 = 1,$$

the dimensions of their sum is

$$\dim(U_1 + U_2 + U_3) = \dim \mathbf{R}^2 = 2,$$

and the dimensions of their intersections are

$$\dim(U_1 \cap U_2) = \dim(U_1 \cap U_3) = \dim(U_2 \cap U_3) = \dim(U_1 \cap U_2 \cap U_3) = 0.$$

Hence, substituting all our values into the formula from the **problem statement**, we have

$$2 = 1 + 1 + 1 - 0 - 0 - 0 + 0 = 3,$$

which is a contradiction.