Linear Algebra Done Right
Solutions to Exercises 2.A

1 $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ spans v

Problem statement

Suppose v_1, v_2, v_3, v_4 spans V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Solution

Given v_1, v_2, v_3, v_4 spans V, every vector $v \in V$ can be written as

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$$

for some $a_1, a_2, a_3, a_4 \in \mathbf{F}$. We can write a vector $u \in \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ as

$$\begin{aligned} u &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 \\ &= b_1 v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4 \end{aligned}$$

for some $b_1, b_2, b_3, b_4 \in \mathbf{F}$. To show that $v \in \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$, we can set $b_1 = a_1, b_2 = a_2 + a_1, b_3 = a_3 + a_2 + a_1, b_4 = a_4 + a_3 + a_2 + a_1$ to write

$$\begin{aligned} b_1 v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4 &= a_1 v_1 + (a_2 + a_1 - a_1)v_2 \\ &\quad + (a_3 + a_2 + a_1 - a_2 - a_1)v_3 + (a_4 + a_3 + a_2 + a_1 - a_3 - a_2 - a_1)v_4 \\ &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\ &= v. \end{aligned}$$

Hence, for any vector $v \in V$, it follows that $v \in \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$. Therefore, the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

spans V.

5 Sometimes what F is matters

Problem statement

(a) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{R}$, then the list $(1+i, 1-i)$ is linearly independent.

(b) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{C}$, then the list $(1+i, 1-i)$ is linearly dependent.

Solution

a

Via Definition 2.17 ('linearly independent'), we need to show that for

$$\lambda_1(1+i) + \lambda(1-i) = 0,$$

then $\lambda_1 = \lambda_2 = 0$. Our equation above becomes the two separate equations

$$\lambda_1 + \lambda_2 = 0 \quad \text{and} \quad \lambda_1 i + \lambda_2(-i) = 0$$

that imply $\lambda_1 = -\lambda_2$ and $\lambda_1 = \lambda_2$. Hence, it follows that $\lambda_1 = \lambda_2 = 0$ and $(1+i, 1-i)$ is linearly independent (if we restrict $\lambda_1, \lambda_2 \in \mathbf{R}$).

b

For the constant $\lambda = 0 - i \in \mathbf{C}$, we can write

$$\lambda(1+i) = -i - i^2 = -i - (-1) = 1 - i.$$

Therefore, $1-i$ is a scalar multiple of $1+i$ and the list $(1+i, 1-i)$ is linearly dependent.

6 $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is linearly independent

Problem statement

Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

Solution

Following our reasoning from Exercise 2.A(1), we can write vectors $v \in \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ as

$$\begin{aligned} v &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 \\ &= b_1 v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4. \end{aligned}$$

If we set $v = 0$, then we can write

$$0 = b_1 v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4.$$

Given v_1, v_2, v_3, v_4 is linearly independent, it follows that $b_1 = 0, b_2 - b_1 = 0, b_3 - b_2 = 0, b_4 - b_3 = 0$ and thus, $b_1 = b_2 = b_3 = b_4 = 0$. Therefore, the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

8 Scaled vectors are linearly independent

Problem statement

Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$ is linearly independent.

Solution

Let's prove it. Given $v_1, v_2, \ldots, v_m$ is linearly independent, then for the equation

$$0 = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m,$$

we have $a_1 = a_2 = \cdots = a_m = 0$ via Definition 2.17 ('linearly independent'). In a similar manner, we can show that the list $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$, where $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, is linearly independent by writing

$$0 = a_1(\lambda v_1) + a_2(\lambda v_2) + \cdots + a_m(\lambda v_m 0)) = \lambda(a_1 v_1 + a_2 v_2 + \cdots + a_m v_m)$$

and dividing either side of the equation by λ (since $\lambda \neq 0$). Hence, the proof that $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$ is linearly independent reduces to $v_1, v_2, \ldots, v_m$ being linearly independent.

9 Adding linearly independent lists

Problem statement

Prove or give a counterexample: If $v_1, \dots, v_m$ and $w_1, \dots, w_m$ are linearly independent lists of vectors in V, then $v_1 + w_1, \dots, v_m + w_m$ is linearly independent.

Solution

Let's give a counterexample. The lists $(1,0),(0,1)$ and $(0,1),(1,0)$ are both linearly independent. But the list $(1,0)+(0,1),(0,1)+(1,0)$ equals $(1,1),(1,1)$, which is clearly linearly dependent.

11 Appending vectors to linear independence

Problem statement

Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that $v_1, \ldots, v_m, w$ is linearly independent if and only if

$$w \notin \operatorname{span}(v_1, \ldots, v_m).$$

Solution

First Direction

Suppose $v_1, \ldots, v_m, w$ is linearly independent. If $w \in \operatorname{span}(v_1, \ldots, v_m)$, then there exist $a_1, \ldots, a_m \in \mathbf{F}$, that are not all zero, such that

$$w = a_1 v_1 + \cdots + a_m v_m \quad \text{and} \quad 0 = a_1 v_1 + \cdots + a_m v_m - w.$$

However, the second equation implies that $v_1, \ldots, v_m, w$ is linearly dependent, which is a contradiction. Thus, it follows that

$$w \notin \operatorname{span}(v_1, \ldots, v_m).$$

Second Direction

Suppose $w \notin \operatorname{span}(v_1, \ldots, v_m)$. If $v_1, \ldots, v_m, w$ is linearly dependent, then there exist $a_1, \ldots, a_m, a_{m+1} \in \mathbf{F}$, that are not all zero, such that[1]

$$0 = a_1 v_1 + \cdots + a_m v_m + a_{m+1} w \quad \text{and} \quad w = -\frac{1}{a_{m+1}}(a_1 v_1 + \cdots + a_m v_m).$$

However, the second equation implies that $w \in \operatorname{span}(v_1, \ldots, v_m)$, which is a contradiction. Thus it follows that $v_1, \ldots, v_m, w$ is linearly independent.

Thoughts

The result in this exercise is a restatement of the linear dependence lemma (Theorem 2.21).

[1]Note that $a_{m+1} \neq 0$ since $v_1, \ldots, v_m, w$ is linearly independent.

12 Six polynomials in $\mathcal{P}_4(\mathbf{F})$

Problem statement

Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbf{F})$.

Solution

The list of polynomials $1, x, x^2, x^3, x^5$ spans $\mathcal{P}_4(\mathbf{F})$ but is of length five. Hence, via Theorem 2.23 ('Length of linearly independent list $\leq$ length of spanning list'), there cannot exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbf{F})$.

13 Four polynomials in $\mathcal{P}_4(\mathbf{F})$

Problem statement

Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbf{F})$.

Solution

The list of polynomials $1, x, x^2, x^3, x^5$ spans $\mathcal{P}_4(\mathbf{F})$ but is of length five. Hence, via Theorem 2.23 ('Length of linearly independent list $\leq$ length of spanning list'), there cannot exist a list of four polynomials that spans in $\mathcal{P}_4(\mathbf{F})$.

17 A list of linearly dependent polynomials

Problem statement

Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_j(2) = 0$ for each j. Prove that $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

Solution

First, let's prove that $p_0, p_1, \ldots, p_m$ does not span $\mathcal{P}_m(\mathbf{F})$. Consider the polynomial 1. Clearly $1 \in \mathcal{P}_m(\mathbf{F})$, yet $1(2) = 1$. It follows that $1 \notin \operatorname{span}(p_0, p_1, \ldots, p_m)$ since $p_j(2) = 0$ for each j. Thus, $p_0, p_1, \ldots, p_m$ does not span $\mathcal{P}_m(\mathbf{F})$.

Clearly, the list $p_0, p_1, \ldots, p_m$ has a length of $m+1$. We must also note that the list $1, z, \ldots, z^m$ spans $\mathcal{P}_m(\mathbf{F})$ and has a length of $m+1$.

Let's perform a proof by contradiction and use the result from Exercise 2.A(11). Suppose $p_0, p_1, \ldots, p_m$ is linearly independent. Since $1 \notin \operatorname{span}(p_0, p_1, \ldots, p_m)$, we can append 1 to the list to get $1, p_0, p_1, \ldots, p_m$, a list of length $m+2$ that is linearly independent. However, via Theorem 2.23 ('Length of linearly independent list $\leq$ length of spanning list'), the list $1, p_0, p_1, \ldots, p_m$ cannot be linearly independent since its length is greater than the list $1, z, \ldots, z^m$, which spans $\mathcal{P}_m(\mathbf{F})$. Therefore, $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.