Linear Algebra Done Right
Solutions to Exercises 4

1 Verify the properties of complex numbers

Problem statement

Verify all the assertions in 4.5 except the last one.

Solution

sum of z and $\bar{z}$

For $z \in \mathbf{C}$, we can write z and $\bar{z}$ as

$$z = \operatorname{Re} z + (\operatorname{Im} z)i \quad \text{and} \quad \bar{z} = \operatorname{Re} z - (\operatorname{Im} z)i.$$

Hence, it follows that

$$z + \bar{z} = \operatorname{Re} z + (\operatorname{Im} z)i + \operatorname{Re} z - (\operatorname{Im} z)i = 2 \operatorname{Re} z,$$

giving the desired result.

difference of z and $\bar{z}$

Following our notation for z and $\bar{z}$, we can write

$$\begin{aligned} z - \bar{z} &= \operatorname{Re} z + (\operatorname{Im} z)i - (\operatorname{Re} z - (\operatorname{Im} z)i) \\ &= \operatorname{Re} z + (\operatorname{Im} z)i - \operatorname{Re} z + (\operatorname{Im} z)i \\ &= 2(\operatorname{Im} z)i, \end{aligned}$$

giving the desired result.

product of z and $\bar{z}$

Following our notation for z and $\bar{z}$, we can write

$$\begin{aligned} z\bar{z} &= (\operatorname{Re} z + (\operatorname{Im} z)i)(\operatorname{Re} z - (\operatorname{Im} z)i) \\ &= ((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2) + ((\operatorname{Re} z)(\operatorname{Im} z) - (\operatorname{Re} z)(\operatorname{Im} z))i \\ &= (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \\ &= |z|^2, \end{aligned}$$

giving the desired result.

additivity and multiplicativity of complex conjugate

For additivity, we can write

$$\begin{aligned}\overline{w+z} &= (\operatorname{Re} w + \operatorname{Re} z) - (\operatorname{Im} w + \operatorname{Im} z)i \\ &= \operatorname{Re} w - (\operatorname{Im} w)i + \operatorname{Re} z - (\operatorname{Im} z)i \\ &= \bar{w} + \bar{z},\end{aligned}$$

giving the desired result.

For multiplicativity, we can write

$$\begin{aligned}\bar{w}\bar{z} &= (\operatorname{Re} w - (\operatorname{Im} w)i)(\operatorname{Re} z - (\operatorname{Im} z)i) \\ &= ((\operatorname{Re} w)(\operatorname{Re} z) - (\operatorname{Im} w)(\operatorname{Im} z)) + (-(\operatorname{Re} w)(\operatorname{Im} z) - (\operatorname{Im} w)(\operatorname{Re} z))i \\ &= ((\operatorname{Re} w)(\operatorname{Re} z) - (\operatorname{Im} w)(\operatorname{Im} z)) - ((\operatorname{Re} w)(\operatorname{Re} z) + (\operatorname{Im} w)(\operatorname{Im} z))i \\ &= \overline{wz},\end{aligned}$$

where the last equality comes from the observation that

$$wz = ((\operatorname{Re} w)(\operatorname{Re} z) - (\operatorname{Im} w)(\operatorname{Im} z)) + ((\operatorname{Re} w)(\operatorname{Im} z) + (\operatorname{Im} w)(\operatorname{Re} z))i.$$

conjugate of conjugate

Following our notation for z and $\bar{z}$, we can write

$$\bar{\bar{z}} = \overline{\operatorname{Re} z - (\operatorname{Im} z)i} = \operatorname{Re} z + (\operatorname{Im} z)i = z,$$

giving the desired result.

real and imaginary parts are bounded by $|z|$

Via the definition of the absolute value of a complex number (Definition 4.3), we can write

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2.$$

Given the nonnegativity of squares, it follows that

$$(\operatorname{Re} z)^2 \geq 0 \quad \text{and} \quad (\operatorname{Im} z)^2 \geq 0.$$

Hence, we can write

$$(\operatorname{Re} z)^2 \leq |z|^2 \quad \text{and} \quad (\operatorname{Im} z)^2 \leq |z|^2$$

and taking the square root of all terms gives the desired results.

absolute value of the complex conjugate

Via the definition of the complex conjugate and the definition of the absolute value of a complex number (Definition 4.3), we can write

$$|\bar{z}| = \sqrt{(\operatorname{Re} z)^2 + (-\operatorname{Im} z)^2} = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = |z|,$$

giving the desired result.

multiplicativity of absolute value

Thinking back to the **additivity and multiplicativity of complex conjugate** and our expression for wz, we can write

$$\begin{aligned}
|wz| &= \sqrt{((\operatorname{Re} w)(\operatorname{Re} z) - (\operatorname{Im} w)(\operatorname{Im} z))^2 + ((\operatorname{Re} w)(\operatorname{Im} z) + (\operatorname{Im} w)(\operatorname{Re} z))^2} \\
&= \sqrt{(\operatorname{Re} w)^2(\operatorname{Re} z)^2 + (\operatorname{Im} w)^2(\operatorname{Im} z)^2 - 2(\operatorname{Re} w)(\operatorname{Re} z)(\operatorname{Im} w)(\operatorname{Im} z)} \\
&\qquad \overline{+(\operatorname{Re} w)^2(\operatorname{Im} z)^2 + (\operatorname{Im} w)^2(\operatorname{Re} z)^2 + 2(\operatorname{Re} w)(\operatorname{Im} z)(\operatorname{Im} w)(\operatorname{Re} z)} \\
&= \sqrt{(\operatorname{Re} w)^2(\operatorname{Re} z)^2 + (\operatorname{Re} w)^2(\operatorname{Im} z)^2 + (\operatorname{Im} w)^2(\operatorname{Re} z)^2 + (\operatorname{Im} w)^2(\operatorname{Im} z)^2} \\
&= \sqrt{((\operatorname{Re} w)^2 + (\operatorname{Im} w)^2)((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2)} \\
&= \sqrt{(\operatorname{Re} w)^2 + (\operatorname{Im} w)^2}\sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} \\
&= |w||z|,
\end{aligned}$$

giving the desired result.

2 $\{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$ is not a subspace

Problem statement

Suppose m is a positive integer. Is the set

$$\{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$$

a subspace of $\mathcal{P}(\mathbf{F})$?

Solution

No and we can show it's not a subspace with a counterexample.

Suppose $m = 2$. It follows that the polynomials $p_0(z) = 1 + z + z^2$ and $p_1(z) = 2 - z^2$ are members of the set, but

$$(p_0 + p_1)(z) = 1 + z + z^2 + 2 - z^2 = 3 + z,$$

which has a degree of 1. Thus the set is not closed under addition and is not a subspace.

A similar counterexample could be constructed for a set with arbitrary m. Therefore, no sets of that form are subspaces of $\mathcal{P}(\mathbf{F})$.

3 $\{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p$ is even$\}$ is not a subspace

Problem statement

Suppose m is a positive integer. Is the set

$$\{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p \text{ is even}\}$$

a subspace of $\mathcal{P}(\mathbf{F})$?

Solution

No and we can use our counterexample from Exercise 4(2) as a counterexample for this set. A similar counterexample could be constructed for a set with arbitrary m. Therefore, no sets of that form are subspaces of $\mathcal{P}(\mathbf{F})$.

4 Existence of polynomials with specific roots

Problem statement

Suppose m and n are positive integers with $m \leq n$, and suppose $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{F})$ with $\deg p = n$ such that $0 = p(\lambda_1) = \cdots = p(\lambda_m)$ and such that p has no other zeros.

Solution

A first attempt would be to construct the polynomial $q \in \mathcal{P}(\mathbf{F})$ such that

$$q(z) = (z - \lambda_1) \cdots (z - \lambda_m).$$

However, we have $\deg q = m$ which is not necessarily equivalent to n. As a simple fix, if $m < n$, we can construct $p \in \mathcal{P}(\mathbf{F})$ such that

$$p(z) = q(z)(z - \lambda_m)^{n-m}.$$

Thus, we can compute the degree of p as

$$\deg p = \deg q + n - m = m + n - m = n.$$

To show that our polynomial p has no other zeros, we can use Theorem 4.14 ('Factorization of a polynomial over $\mathbf{C}$') to claim that the following factorization of p

$$p(z) = (z - \lambda_1) \cdots (z - \lambda_m)(z - \lambda_m)^{n-m}$$

is unique. Therefore, it follows that p has no other zeros besides $\lambda_1, \ldots, \lambda_m$.

5 Using linear maps to find unique polynomials

Problem statement

Suppose m is a nonnegative integer, $z_1, \ldots, z_{m+1}$ are distinct elements of $\mathbf{F}$, and $w_1, \ldots, w_{m+1} \in \mathbf{F}$. Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbf{F})$ such that

$$p(z_j) = w_j$$

for $j = 1, \ldots, m+1$.

Solution

To show existence and uniqueness of the polynomial $p \in \mathcal{P}_m(\mathbf{F})$ that satisfies our condition in the problem statement, we can find a linear map that is injective and surjective. Define the linear map $T \in \mathcal{L}(\mathcal{P}_m(\mathbf{F}), \mathbf{F}^{m+1})$ by

$$Tp = (p(z_1), \ldots, p(z_{m+1})).$$

To show **additivity**, suppose $p, q \in \mathcal{P}_m(\mathbf{F})$. Thus, we can write

$$\begin{aligned} T(p+q) &= ((p+q)(z_1), \ldots, (p+q)(z_{m+1})) \\ &= (p(z_1) + q(z_1), \ldots, p(z_{m+1}) + q(z_{m+1})) \\ &= (p(z_1), \ldots, p(z_{m+1})) + (q(z_1), \ldots, q(z_{m+1})) \\ &= Tp + Tq, \end{aligned}$$

which shows **additivity**. To show **homogeneity**, suppose $p \in \mathcal{P}_m(\mathbf{F})$ and $\lambda \in \mathbf{F}$. Thus, we can write

$$\begin{aligned} T(\lambda p) &= (\lambda p(z_1), \ldots, \lambda p(z_{m+1})) \\ &= \lambda(p(z_1), \ldots, p(z_{m+1})) \\ &= \lambda Tp \end{aligned}$$

which shows **homogeneity**. Now let's show that T is in injective and surjective.

For injectivity, we need to prove $\operatorname{null} T = \{0\}$. Our only worry is that the distinct scalars $z_1, \ldots, z_{m+1}$ could be the roots for some polynomial $p \in \mathcal{P}_m(\mathbf{F})$. However, following from Theorem 4.12 ('A polynomial has at most as many zeros as its degree'), polynomials $p \in \mathcal{P}_m(\mathbf{F})$ can have at most m roots. Hence, there is no polynomial $p \in \mathcal{P}_m(\mathbf{F})$ with the distinct scalars $z_1, \ldots, z_{m+1}$ as roots. Therefore, it follows that $\operatorname{null} T = \{0\}$.

For surjectivity, we can note that injectivity implies $\dim \operatorname{null} T = 0$. Hence, via the Fundamental Theorem of Linear Maps (Theorem 3.22), we can write

$$\dim \mathcal{P}_m(\mathbf{F}) = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim \operatorname{range} T.$$

Since $\dim \mathcal{P}_m(\mathbf{F}) = \dim \mathbf{F}^{m+1}$, it follows that

$$\dim \mathbf{F}^{m+1} = \dim \operatorname{range} T,$$

and T is surjective.

Putting all our results together, we've shown that there exists a surjective and injective linear map from $\mathcal{P}_m(\mathbf{F})$ to $\mathbf{F}^{m+1}$. Therefore, it follows that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbf{F})$ such that

$$p(z_j) = w_j$$

for $j = 1, \ldots, m + 1$.

Thoughts

This exercise is a good example of how powerful linear algebra can be. Simply by finding a linear map, we can prove a lot of useful properties.

7 Odd polynomials with have a real zero

Problem statement

Prove that every polynomial of odd degree with real coefficients has a real zero.

Solution

Via Theorem 4.17 ('Factorization of a polynomial over $\mathbf{R}$'), polynomials can be factored into a series of $(x - \lambda)$ and $(x^2 + bx + c)$ terms where $\lambda, b, c \in \mathbf{R}$ and $x^2 + bx + c$ has no real roots. Odd polynomials cannot solely be factored into a series of $(x^2 + bx + x)$ terms since they reduce the factored polynomial by a degree of 2. Thus, an odd polynomial must contain at least one $(x - \lambda)$ factor, implying the polynomial has a real zero.