Linear Algebra Done Right
Solutions to Exercises 6.A

1 Inhomogeneous inner products

Problem statement

Show that the function that takes $\big((x_1, x_2), (y_1, y_2)\big) \in \mathbf{R}^2 \times \mathbf{R}^2$ to $|x_1 y_1| + |x_2 y_2|$ is not an inner product on $\mathbf{R}^2$.

Solution

The inner product is not homogeneous (Definition 6.3). Consider $\lambda = -2$.

$$\begin{aligned}
\langle(-2x_1, -2x_2), (y_1, y_2)\rangle &= |-2x_1 y_1| + |-2x_2 y_2| \\
&= 2|x_1 y_1| + 2|x_2 y_2| \\
&= 2\langle(x_1, x_2), (y_1, y_2)\rangle \\
&\neq -2\langle(x_1, x_2), (y_1, y_2)\rangle
\end{aligned}$$

2 Indefinite inner products

Problem statement

Show that the function that takes $\big((x_1, x_2, x_3), (y_1, y_2, y_3)\big) \in \mathbf{R}^3 \times \mathbf{R}^3$ to $x_1y_1 + x_3y_3$ is not an inner product on $\mathbf{R}^3$.

Solution

The inner product does not obey definiteness (Definition 6.3). Consider $(x_1, x_2, x_3) = (y_1, y_2, y_3) = (0, 1, 0)$.

$$\langle(x_1, x_2, x_3), (y_1, y_2, y_3)\rangle = \langle(0, 1, 0), (0, 1, 0)\rangle = 0 \cdot 0 + 0 \cdot 0 = 0$$

3 Replacing the positivity condition

Problem statement

Suppose $\mathbf{F} = \mathbf{R}$ and $V \neq \{0\}$. Replace the positivity condition (which states that $\langle v, v\rangle \geq 0$ and for all $v \in V$) in the definition of an inner product (6.3) with the condition that $\langle v, v\rangle > 0$ for some $v \in V$. Show that this change in the definition does not change the set of functions from $V \times V$ to $\mathbf{R}$ that are inner products on V.

Solution

The set of functions from $V \times V$ to $\mathbf{R}$ that are inner products on V clearly does not grow smaller. Thus we must show that the set does not grow larger.

Suppose $v \in V$ is such that it satisfies the "new" condition that $\langle v, v\rangle > 0$. There are two cases to consider: $V = \operatorname{span}(v)$ and $V \neq \operatorname{span}(v)$.

$\mathbf{V = span(v)}$: For any vector $u \in V$ there exists $\lambda \in \mathbf{R}$ such that $u = \lambda v$. Hence

$$\langle u, u\rangle = \langle \lambda v, \lambda v\rangle = |\lambda|^2 \langle v, v\rangle$$

Given $\langle v, v\rangle > 0$ and $|\lambda|^2 \geq 0$ for all $\lambda \in \mathbf{R}$, it follows that

$$\langle u, u\rangle = |\lambda|^2 \langle v, v\rangle \geq 0$$

Thus we've shown that the positivity condition is fulfilled and the set of function that are inner products on V clearly does not grow larger.

$\mathbf{V \neq span(v)}$: Since $V \neq \{0\}$ and $V \neq \operatorname{span}(v)$, there exists $u \in V$ that is not a scalar multiple of v. Theorem 6.14 ('An orthogonal decomposition') allows us to decompose u into $u = cv + w$ where v and w are orthogonal. From the Pythagorean theorem, it follows

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

Let's assume that $||w||^2 < 0$. Now consider the vector λw where $\lambda = \frac{\|v\|}{\|w\|}$[1]

$$\|v + \lambda w\|^2 = \|v\|^2 + \|\lambda w\|^2 = \|v\|^2 + |\frac{\|v\|}{\|w\|}|^2 \|w\|^2 = \|v\|^2 - \|v\|^2 = 0$$

Thus we have a contradiction since $v + \lambda w \neq 0$ yet $||v + \lambda w||^2 = 0$, defying the property of definiteness (Definition 6.3). Therefore $||w||^2 \geq 0$.

It then follows that

$$||u||^2 = ||v + w||^2 = ||v||^2 + ||w||^2 \geq 0$$

showing that the positivity condition is fulfilled and the set of function that are inner products on V does not grow larger.

[1] The $|\frac{\|v\|}{\|w\|}|^2 \|w\|^2$ term is quite harry, but note that $\|w\|^2 < 0$ and $|\|w\||^2 > 0$, therefore $\frac{\|w\|^2}{|\|w\||^2} = -1$. Also note $|\|v\||^2 = \|v\|^2$ since $\langle v, v\rangle > 0$.

4 Diagonals of a rhombus are perpendicular

Problem statement

Suppose V is a real inner product space.

(a) Show that $\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2$ for every $u, v \in V$.

(b) Show that if $u, v \in V$ have the same norm, then $u+v$ is orthogonal to $u-v$.

(c) Use part (b) to show that the diagonals of a rhombus are perpendicular to each other.

Solution

a

Via the definition of the inner product (Definition 6.3) and Theorem 6.7 ('Basic properties of an inner product'), we can expand $\langle u+v, u-v\rangle$ to

$$\langle u+v, u-v\rangle = \langle u, u\rangle + \langle v, u\rangle + \langle u, -v\rangle + \langle v, -v\rangle.$$

Since V is a real inner product space, we have

$$\langle u, -v\rangle = -1\langle u, v\rangle \quad \text{and} \quad \langle v, -v\rangle = -\|v\|^2$$

and since $\langle u, v\rangle = \overline{\langle v, u\rangle} = \langle v, u\rangle$ for reals, we can write

$$\langle u+v, u-v\rangle = \|u\|^2 + \langle v, u\rangle - \langle v, u\rangle - \|v\|^2 = \|u\|^2 - \|v\|^2.$$

b

If $\|u\| = \|v\|$, then $\|u\|^2 = \|v\|^2$. Hence, via part (a), we can write

$$\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2 = \|u\|^2 - \|u\|^2 = 0$$

and $u+v$ is orthogonal to $u-v$.

c

In a rhombus, all sides are the same length, and hence, have the same norm. Suppose $u, v \in V$ are the sides of the rhombus and $\|u\| = \|v\|$. We can directly apply our result from part (b) to state $\langle u+v, u-v\rangle = 0$, implying that the diagonals of the rhombus are orthogonal, and thus, perpendicular to each other.

5 No eigenvalues greater than 1

Problem statement

Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

Solution

Theorem 5.6 ('Equivalent conditions to be an eigenvalue') implies that "$T - \sqrt{2}I$ is invertible" means that $\sqrt{2}$ is not an eigenvalue of T. Let's first note that $1 < \sqrt{2}$.

Suppose v is an eigenvector of T. The condition that $\|Tv\| \leq \|v\|$ implies that the eigenvalue corresponding to v, let's say λ, we have

$$|\lambda|\|v\| \leq \|v\|,$$

and thus

$$\lambda \leq 1.$$

Hence, $\sqrt{2}$ cannot be an eigenvalue and $T - \sqrt{2}I$ is invertible.

6 Condition equivalent to orthogonality

Problem statement

Suppose $u, v \in V$. Prove that $\langle u, v\rangle = 0$ if and only if

$$\|u\| \leq \|u + av\|$$

for all $a \in \mathbf{F}$.

Solution

First Direction

Suppose $\langle u, v\rangle = 0$. It follows that for all $a \in \mathbf{F}$, we have

$$\langle u, av\rangle = 0.$$

Via the Pythagorean Theorem (Theorem 6.13), we can write

$$\|u\|^2 + \|av\|^2 = \|u + av\|^2.$$

Given the positivity condition of inner products (Definition 6.3), it follows that

$$\|u\|^2 \leq \|u + av\|^2.$$

Second Direction

Suppose $\|u\| \leq \|u + av\|$ for all $a \in \mathbf{F}$. Via Theorem 6.14 ('An orthogonal decomposition'), we can decompose u into cv and w where $c = \frac{\langle u,v\rangle}{\|v\|^2}$, $w = u - cv$, and $\langle cv, w\rangle = 0$. Invoking the Pythagorean Theorem (Theorem 6.13), we can write

$$\|u\|^2 = \|cv + w\|^2 = \|cv\|^2 + \|w\|^2 = \|cv\|^2 + \|u - cv\|^2,$$

and via the positivity condition of inner products (Definition 6.3), it follows that

$$\|u\|^2 \geq \|u - cv\|^2$$

and by taking the square root of both sides

$$\|u\| \geq \|u - cv\|.$$

In order to avoid a contradiction with our hypothesis, the expression above must be an equality. The expression above can only be an equality if $c = 0$ or $v = 0$. Both cases imply that $\langle u, v\rangle = 0$.

7 $\|au + bv\| = \|bu + av\|$ iff $\|u\| = \|v\|$

Problem statement

Suppose $u, v \in V$. Prove that $\|au + bv\| = \|bu + av\|$ for all $a, b \in \mathbf{R}$ if and only if $\|u\| = \|v\|$.

Solution

As an organizational note, it's easier to work with squared norms, and the results directly translate to norms.

First Direction

Suppose $\|au + bv\|^2 = \|bu + av\|^2$ for all $a, b \in \mathbf{R}$. Expanding out $\|au + bv\|^2 = \|bu + av\|^2$, we have

$$\langle au, au\rangle + \langle bv, au\rangle + \langle au, bv\rangle + \langle bv, bv\rangle = \langle bu, bu\rangle + \langle av, bu\rangle + \langle bu, av\rangle + \langle av, av\rangle.$$

Since $a, b \in \mathbf{R}$, it follows that

$$\langle bv, au\rangle = b\bar{a}\langle v, u\rangle = ab\langle v, u\rangle,$$

and a similar result follows for $\langle au, bv\rangle, \langle av, bu\rangle, \langle bu, av\rangle$. Hence, we can write

$$a^2\|v\|^2 + ab\langle v, u\rangle + ab\langle u, v\rangle + b^2\|u\|^2 = b^2\|v\|^2 + ab\langle v, u\rangle + ab\langle u, v\rangle + a^2\|u\|^2,$$

and by subtracting out the similar terms, we have

$$a^2\|v\|^2 + b^2\|u\|^2 = b^2\|v\|^2 + a^2\|u\|^2,$$

By setting $b = 0$, it follows that

$$a^2\|v\|^2 = a^2\|u\|^2,$$

which implies $\|u\|^2 = \|v\|^2$ and $\|u\| = \|v\|$.

First Direction

Suppose $\|u\|^2 = \|v\|^2$. It follows that for all $a, b \in \mathbf{R}$,

$$a^2\|v\|^2 = a^2\|u\|^2 \quad \text{and} \quad b^2\|v\|^2 = b^2\|u\|^2.$$

By adding the two expressions together, we have

$$a^2\|v\|^2 + b^2\|u\|^2 = b^2\|v\|^2 + a^2\|u\|^2.$$

To construct $\|au + bv\|^2 = \|bu + av\|^2$, one only needs to reverse our work in the **First Direction**.

8 Properties of $u = v$

Problem statement

Suppose $u, v \in V$ and $\|u\| = \|v\| = 1$ and $\langle u, v\rangle = 1$. Prove that $u = v$.

Solution

Via the Cauchy-Schwarz Inequality (Theorem 6.15), we know that

$$|\langle u, v\rangle| \leq \|u\|\|v\|,$$

where the inequality is an equality if and only if one of u, v is a scalar multiple of the other. In our case, we have

$$1 = \langle u, v\rangle = \|u\|\|v\| = 1 \cdot 1 = 1,$$

hence, we have an equality and $u = cv$ for some $c \in \mathbf{F}$. To compute the exact c, Theorem 6.14 ('An orthogonal decomposition') tells us that

$$c = \frac{\langle u, v\rangle}{\|v\|^2} = \frac{1}{1^2} = 1.$$

Therefore, $u = v$.

9 When $\|u\| \leq 1$ and $\|v\| \leq 1$

Problem statement

Suppose $u, v \in V$ and $\|u\| \leq 1$ and $\|v\| \leq 1$. Prove that

$$\sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2} \leq 1 - |\langle u, v\rangle|.$$

Solution

Via the Cauchy-Schwarz Inequality (Theorem 6.15) and the problem statement, we have

$$|\langle u, v\rangle| \leq \|u\|\|v\| \leq 1.$$

By subtracting by 1 and multiplying by -1, we have

$$0 \leq 1 - \|u\|\|v\| \leq 1 - |\langle u, v\rangle|.$$

Thus, if we can show that $\sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2} \leq 1-\|u\|\|v\|$, then our desired result immediately follows.

By squaring the expression $\sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2} \leq 1-\|u\|\|v\|$ on both sides (which is allowed since both terms are real and greater than or equal to 0), we have

$$(1-\|u\|^2)(1-\|v\|^2) \leq (1-\|u\|\|v\|)^2$$

which can be expanded to

$$1 - \|u\|^2 - \|v\|^2 + \|u\|^2\|v\|^2 \leq 1 - 2\|u\|\|v\| + \|u\|^2\|v\|^2$$

and then reduced to

$$-\|u\|^2 - \|v\|^2 \leq -2\|u\|\|v\|.$$

By rearranging terms, we can write

$$0 \leq \|u\|^2 - 2\|u\|\|v\| + \|v\|^2,$$

and it follows that

$$0 \leq (\|u\| - \|v\|)^2$$

which, given the positivity property of inner products (Definition 6.3), is clearly true. Therefore, we can state that

$$\sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2} \leq 1 - \|u\|\|v\|$$

and our desired result immediately follows.

10 Searching for vectors in $\mathbf{R}^2$

Problem statement

Find vectors $u, v \in \mathbf{R}^2$ such that u is a scalar multiple of $(1,3)$, v is orthogonal to $(1,3)$, and $(1,2) = u + v$.

Solution

First, let's find a vector orthogonal to $(1,3)$. By inspection, the vector $(-3,1)$ will work. Now we need to find $a, b \in \mathbf{R}$ such that $u = a(1,3)$, $v = b(-3,1)$, and $(1,2) = u + v$. Using the constraint $(1,2) = u + v$, we get the following equations

$$\begin{aligned} 1 &= a - 3b \Rightarrow 3a = 9b + 3, \\ 2 &= 3a + b \Rightarrow 3a = 2 - b, \end{aligned}$$

and through combining them, we have

$$9b + 3 = 2 - b$$

and $b = -\frac{1}{10}$. Plugging in b, we have $a = \frac{7}{10}$. Hence, our vectors are

$$u = (\frac{7}{10}, \frac{21}{10}) \quad \text{and} \quad v = (\frac{3}{10}, -\frac{1}{10}).$$

11 Using the Cauchy-Schwarz Inequality: Act 1

Problem statement

Prove that

$$16 \leq (a+b+c+d)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d})$$

for all positive numbers a, b, c, d.

Solution

First notice that for *positive* numbers a, b, c, d there exist positive numbers x, y, z, w such that

$$x^2 = a, \quad y^2 = b, \quad z^2 = c, \quad w^2 = d.$$

Now we can use Example 6.17(a) of Cauchy-Schwarz Inequalities that states if $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbf{R}$, then

$$|x_1y_1 + \cdots + x_ny_n|^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$

Given the expression above, we can write

$$|x(\frac{1}{x}) + y(\frac{1}{y}) + z(\frac{1}{z}) + w(\frac{1}{w})|^2 \leq (x^2+y^2+z^2+w^2)(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{w^2})$$

where $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{w}$ are well defined since a, b, c, d are *positive* numbers[2]. Applying our substitution and reducing the left-hand side of the equation, we have

$$|1+1+1+1|^2 = 16 \leq (a+b+c+d)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}),$$

which was to be shown.

[2] 0 is not a positive number.

12 Using the Cauchy-Schwarz Inequality: Act 2

Problem statement

Prove that
$$(x_1 + \cdots + x_n)^2 \leq n({x_1}^2 + \cdots + {x_n}^2)$$
for all positive integers n and all real numbers $x_1, \ldots, x_n$.

Solution

Let's prove this by using the Cauchy-Schwarz inequality on the Euclidean inner product as specified by Example 6.4(a). Therefore, if $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbf{R}$, then
$$|x_1 y_1 + \cdots + x_n y_n|^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$
Now, suppose $(y_1, \ldots, y_n) = (1, \ldots, 1)$. Then we have
$$|x_1(1) + \cdots + x_n(1)|^2 \leq (x_1^2 + \cdots + x_n^2)(1^2 + \cdots + 1^2),$$
which is equivalently expressed as
$$(x_1 + \cdots + x_n)^2 \leq n(x_1^2 + \cdots + x_n^2),$$
which was to be shown.

13 Law of cosines computes inner products

Problem statement

Suppose u, v are nonzero vectors in $\mathbf{R}^2$. Prove that

$$\langle u, v \rangle = \|u\| \|v\| \cos\theta,$$

where θ is the angle between u and v (thinking of u and v as arrows with initial point at the origin).

Solution

Via the law of cosines, we can write

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos\theta. \tag{1}$$

By expanding the left-hand side of the above expression, we have

$$\|u - v\|^2 = \langle u - v, u - v \rangle = \|u\|^2 + \langle -v, u \rangle + \langle u, -v \rangle + \|v\|^2,$$

and since $u, v \in \mathbf{R}^2$, it follows that

$$\|u - v\|^2 = \|u\|^2 - 2\langle u, v \rangle + \|v\|^2. \tag{2}$$

Combining (1) and (2), we have

$$\|u\|^2 - 2\langle u, v \rangle + \|v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos\theta,$$

and through rearranging terms, we can write

$$\langle u, v \rangle = \|u\| \|v\| \cos\theta,$$

which was to be shown.

14 Angles and the Cauchy-Schwarz Inequality

Problem statement

The angle between two vectors (thought of as arrows with initial point at the origin) in $\mathbf{R}^2$ or $\mathbf{R}^3$ can be defined geometrically. However, geometry is not as clear in $\mathbf{R}^n$ for $n > 3$. Thus the angle between two nonzero vectors $x, y \in \mathbf{R}^n$ is defined to be

$$\arccos \frac{\langle x, y \rangle}{\|x\| \|y\|},$$

where the motivation for this definition comes from the previous exercise. Explain why the Cauchy-Schwarz Inequality is needed to show that this definition makes sense.

Solution

The arccos function is only defined for inputs in the range $[-1, 1]$. With the Cauchy-Schwarz Inequality, the fraction $\frac{\langle x,y \rangle}{\|x\| \|y\|}$ is guaranteed to be within $[-1, 1]$ since

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

implies

$$\frac{|\langle x, y \rangle|}{\|x\| \|y\|} \leq 1,$$

and thus,

$$-1 \leq \frac{\langle x, y \rangle}{\|x\| \|y\|} \leq 1.$$

15 Using the Cauchy-Schwarz Inequality: Act 3

Problem statement

Prove that

$$(\sum_{j=1}^{n} a_j b_j)^2 \leq (\sum_{j=1}^{n} j a_j^2)(\sum_{j=1}^{n} \frac{b_j^2}{j})$$

for all real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$.

Solution

Following from Example 6.17(a), we know that the expression

$$(\sum_{j=1}^{n} x_j y_j)^2 \leq (\sum_{j=1}^{n} x_j^2)(\sum_{j=1}^{n} y_j^2)$$

is a valid instance of the Cauchy-Schwarz Inequality (Theorem 6.15) for real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$. By defining $x_j = \sqrt{j} a_j$ and $y_j = \frac{b_j}{\sqrt{j}}$, we get

$$x_j y_j = \sqrt{j} a_j \frac{b_j}{\sqrt{j}} = a_j b_j, \quad x_j^2 = j a_j^2, \quad y_j^2 = \frac{b_j}{j}.$$

Substituting our expressions above into Example 6.17(a), we can write

$$(\sum_{j=1}^{n} a_j b_j)^2 \leq (\sum_{j=1}^{n} j a_j^2)(\sum_{j=1}^{n} \frac{b_j^2}{j})$$

which was to be shown.

16 Employing the parallelogram equality

Problem statement

Suppose $u, v \in V$ are such that

$$\|u\| = 3, \quad \|u+v\| = 4, \quad \|u-v\| = 6.$$

What number does $\|v\|$ equal?

Solution

Via the parallelogram equality (Theorem 6.22), we have

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2),$$

and by plugging in numbers, it follows that

$$\begin{aligned} 4^2 + 6^2 &= 2(3^2 + \|v\|^2) \\ 16 + 36 &= 18 + 2\|v\|^2 \\ 17 &= \|v\|^2 \end{aligned}$$

and $\|v\| = \sqrt{17}$.

17 The $\max$ function isn't a norm

Problem statement

Prove or disprove: there is an inner product on $\mathbf{R}^2$ such that the associated norm is given by

$$\|(x, y)\| = \max\{x, y\}$$

for all $(x, y) \in \mathbf{R}^2$.

Solution

Let's disprove that such an inner product exists. Suppose $(x, y) = (-1, 0)$. The associated norm is

$$\|(-1, 0)\| = \max\{-1, 0\} = 0$$

which violates Theorem 6.10 ('Basic properties of the norm') since only the zero vector can have a norm of 0. Thus, $\|(x, y)\| = \max\{x, y\}$ is not an inner product.

18 L^p-norm is tricky in $\mathbf{R}^2$

Problem statement

Suppose $p > 0$. Prove that there is an inner product on $\mathbf{R}^2$ such that the associated norm is given by

$$\|(x, y)\| = (x^p + y^p)^{1/p}$$

for all $(x, y) \in \mathbf{R}^2$ if and only if $p = 2$.

Solution

I'll provide two proofs. One explicit and correct, and one slick and presumptuous.

First Direction (Explicit)

There are two cases to consider: p **is odd** and p **is even**.

p **is odd**: If p is odd, then there does not exist a norm given by

$$\|(x, y)\| = (x^p + y^p)^{1/p}$$

for all $(x, y) \in \mathbf{R}^2$ because it would violate Theorem 6.10(b) ('Basic properties of the norm') which states

$$\|\lambda v\| = |\lambda|\|v\|$$

for all $\lambda \in \mathbf{F}$. More explicitly, if $\lambda < 0$ and p is odd, then it follows that

$$\begin{aligned}
\|\lambda(x, y)\| &= \|(\lambda x, \lambda y)\| \\
&= ((\lambda x)^p + (\lambda y)^p)^{1/p} \\
&= ((\lambda)^p (x^p + y^p))^{1/p} \\
&= \lambda (x^p + y^p)^{1/p} \\
&= \lambda \|(x, y)\| \neq |\lambda| \|(x, y)\|.
\end{aligned}$$

p **is even**: Suppose p is even. The squared norm is given by

$$\|(x, y)\|^2 = (x^p + y^p)^{2/p}.$$

Now suppose $u = (1, 0)$ and $v = (0, 1)$. The associated squared norms of $u, v, u + v, u - v$ are

$$\begin{aligned}
\|u\|^2 &= (1^p + 0^p)^{2/p} = 1^{2/p} = 1, \\
\|v\|^2 &= (0^p + 1^p)^{2/p} = 1^{2/p} = 1, \\
\|u + v\|^2 &= (1^p + 1^p)^{2/p} = 2^{2/p}, \\
\|u - v\|^2 &= (1^p + (-1)^p)^{2/p} = (1^p + 1^p)^{2/p} = 2^{2/p},
\end{aligned}$$

where $(-1)^p = 1^p$ because p is even. Here comes the fun part.

The squared norms must obey the Parallelogram Equality (Theorem 6.22) which states

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Plugging in the associated squared norms, we have

$$\begin{aligned} 2^{2/p} + 2^{2/p} &= 2(1+1), \\ 2(2^{2/p}) &= 2 \cdot 2, \\ 2^1 \cdot 2^{2/p} &= 4, \\ 2^{1+2/p} &= 2^2. \end{aligned}$$

Since the bases are equal and positive, the exponents must also be equal. Hence, to solve for p, we can write

$$\begin{aligned} 1 + \frac{2}{p} &= 2, \\ \frac{2}{p} &= 1, \\ 2 &= p. \end{aligned}$$

Therefore, it must be the case that $p = 2$.

Second Direction (Explicit)

If $p = 2$, then the associated norm

$$\|(x,y)\| = (x^2 + y^2)^{1/2}$$

is simply an instance of the norm associated with the Euclidean inner product.

Slick City

The squared norm is given by

$$\|(x,y)\|^2 = (x^p + y^p)^{2/p}.$$

Suppose $u = (1,0)$, $v = (0,1)$, and $u+v = (1,1)$. The associated squared norms are

$$\begin{aligned} \|u\|^2 &= (1^p + 0^p)^{2/p} = 1^{2/p} = 1, \\ \|v\|^2 &= (0^p + 1^p)^{2/p} = 1^{2/p} = 1, \\ \|u+v\|^2 &= (1^p + 1^p)^{2/p} = 2^{2/p}. \end{aligned}$$

These squared norms must obey the Pythagorean Theorem (Theorem 6.13) which states that for orthogonal vectors, we can write

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

Plugging in our values for the associated squared norms, we have

$$2^{2/p} = 1 + 1,$$
$$2^{2/p} = 2,$$
$$2^{2/p} = 2^1.$$

Since the bases are equal and positive, the exponents must also be equal. Therefore, it must be the case that $p = 2$.

Note: While this is a "slick proof", it assumes that u and v are orthogonal, which we would have to justify. We could use Exercise 6.A(19), but then we'd have to calculate $\|u - v\|^2$, which requires that we specify whether p is positive or negative. In that case, we're better off sticking with our explicit proof and using the Parallelogram Equality. Interestingly, the Parallelogram Equality is quite powerful because it's a necessary and sufficient condition for norms to have associated inner products (Jordan–von Neumann theorem).

24 Injective operators in inner products

Problem statement

Suppose $S \in \mathcal{L}(V)$ is an injective operator on V. Define $\langle \cdot, \cdot \rangle_1$ by

$$\langle u, v \rangle_1 = \langle Su, Sv \rangle$$

for $u, v \in V$. Show that $\langle \cdot, \cdot \rangle_1$ is an inner product on V.

Solution

Following the definition of an inner product (Definition 6.3), we need to show that $\langle \cdot, \cdot \rangle_1$ satisfies **positivity**, **definiteness**, **additivity in first slot**, **homogeneity in first slot**, and **conjugate symmetry**.

Positivity

Positivity of $\langle \cdot, \cdot \rangle_1$ follows from positivity of $\langle Su, Sv \rangle$.

Definiteness

Given the definiteness of $\langle \cdot, \cdot \rangle$,

$$\langle v, v \rangle_1 = \langle Sv, Sv \rangle = 0,$$

if and only if $Sv = 0$. Given S is an injective operator, $Sv = 0$ if and only if $v = 0$. Hence, $\langle \cdot, \cdot \rangle_1$ obeys definiteness.

Additivity in first slot

For $u, v, w \in V$, we have

$$\begin{aligned} \langle u + v, w \rangle_1 &= \langle S(u + v), Sw \rangle \\ &= \langle Su + Sv, Sw \rangle \\ &= \langle Su, Sw \rangle + \langle Sv, Sw \rangle \\ &= \langle u, w \rangle_1 + \langle v, w \rangle_1. \end{aligned}$$

Homogeneity in first slot

For $\lambda \in \mathbf{F}$ and $u, v \in V$, we have

$$\begin{aligned} \langle \lambda v, w \rangle_1 &= \langle S(\lambda u), Sw \rangle \\ &= \langle \lambda Sv, Sw \rangle \\ &= \lambda \langle Sv, Sw \rangle \\ &= \lambda \langle u, w \rangle_1. \end{aligned}$$

Conjugate symmetry

For $u, v \in V$, we have

$$\begin{aligned}\langle v, w\rangle_1 &= \langle Su, Sw\rangle \\ &= \overline{\langle Sw, Su\rangle} \\ &= \overline{\langle w, u\rangle}_1.\end{aligned}$$

25 Non-injective operators in inner products

Problem statement

Suppose $S \in \mathcal{L}(V)$ is not injective. Define $\langle \cdot, \cdot \rangle_1$ as in the exercise above. Explain why $\langle \cdot, \cdot \rangle_1$ is not an inner product on V.

Solution

Since S is not injective, there exists some vector $v \in V$ such that $v \neq 0$ and $Sv = 0$. Hence, it follows that

$$\langle v, v \rangle_1 = \langle Sv, Sv \rangle = 0,$$

meaning that $\langle \cdot, \cdot \rangle_1$ does not obey the property of definiteness (Definition 6.3) and $\langle \cdot, \cdot \rangle_1$ is not an inner product.

31 Apollonius's Identity

Problem statement

Use inner products to prove Apollonius's Identity: In a triangle with sides of length a, b, and c, let d be the length of the line segment from the midpoint of the side of length c to the opposite vertex. Then

$$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2.$$

Solution

First, we can write the squared norms of a and b as

$$\|a\|^2 = \|\frac{1}{2}c + d\|^2 \quad \text{and} \quad \|b\|^2 = \|-\frac{1}{2}c + d\|^2.$$

Summing our expressions together, we have

$$\begin{aligned}
\|a\|^2 + \|b\|^2 &= \|\frac{1}{2}c + d\|^2 + \|-\frac{1}{2}c + d\|^2 \\
&= \|\frac{1}{2}c\|^2 + \langle \frac{1}{2}c, d\rangle + \langle d, \frac{1}{2}c\rangle + \|d\|^2 \\
&+ \|-\frac{1}{2}c\|^2 + \langle -\frac{1}{2}c, d\rangle + \langle d, -\frac{1}{2}c\rangle + \|d\|^2.
\end{aligned}$$

Via Theorem 6.7 ('Basic properties of an inner product'), it follows that

$$\begin{aligned}
\langle \frac{1}{2}c, d\rangle + \langle -\frac{1}{2}c, d\rangle &= \langle \frac{1}{2}c - \frac{1}{2}c, d\rangle = \langle 0, d\rangle = 0, \\
\langle d, \frac{1}{2}c\rangle + \langle d, -\frac{1}{2}c\rangle &= \langle d, \frac{1}{2}c - \frac{1}{2}c\rangle = \langle d, 0\rangle = 0.
\end{aligned}$$

Via Theorem 6.10(b) ('Basic properties of the norm'), we can write

$$\begin{aligned}
\|\frac{1}{2}c\|^2 &= |\frac{1}{2}|^2\|c\|^2 = \frac{1}{4}\|c\|^2, \\
\|-\frac{1}{2}c\|^2 &= |-\frac{1}{2}|^2\|c\|^2 = \frac{1}{4}\|c\|^2.
\end{aligned}$$

Hence, putting everything all together, we have

$$\begin{aligned}
\|a\|^2 + \|b\|^2 &= \|\frac{1}{2}c\|^2 + \langle \frac{1}{2}c, d\rangle + \langle d, \frac{1}{2}c\rangle + \|d\|^2 \\
&\quad + \|-\frac{1}{2}c\|^2 + \langle -\frac{1}{2}c, d\rangle + \langle d, -\frac{1}{2}c\rangle + \|d\|^2 \\
&= \frac{1}{4}\|c\|^2 + 0 + 0 + \|d\|^2 \\
&\quad + \frac{1}{4}\|c\|^2 + 0 + 0 + \|d\|^2 \\
&= \frac{1}{2}\|c\|^2 + 2\|d\|^2.
\end{aligned}$$