Linear Algebra Done Right
Solutions to Exercises 2.B

1 Vector spaces with one basis

Problem statement

Find all vector spaces that have exactly one basis.

Solution

The vector space $\{0\}$ has exactly one basis, namely $()$, the empty list, because we defined $()$ to be linearly independent (Definition 2.17) and we defined the span of the empty list to be $\{0\}$ (Definition 2.5). Hence, $()$ is linear independent and spans $\{0\}$; thus, since the empty list is unique, it is the only basis of $\{0\}$. Note that the list 0 cannot be a basis of $\{0\}$ since 0 is not linearly independent. In other words, any scalar $\lambda \in \mathbf{F}$ satisfies the equation

$$\lambda \cdot 0 = 0.$$

Also note that the 0 in $\{0\}$ need not be a scalar. The 0 could be a scalar, list of zeros, or the function that maps all inputs to zero. Also note that the 0 in $\{0\}$ need only be the **additive identity** of any vector space.

Suppose that a vector space V has at least one element $v \in V$ that is not the **additive identity** $v \neq 0$. Via Theorem 2.33 ('Linearly independent list extends to a basis'), we can extend v to be a basis of V. However, for any $\lambda \in \mathbf{F}$ such that $\lambda \neq 0$, we could also extend λv to be a basis of V, creating a different basis than before. Hence, vector spaces that have at least one element that is not the **additive identity** can have more than one basis.

2 Explorations on the subspace U of $\mathbf{R}^5$

Problem statement

(a) Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathbf{R}^5$.

(c) Find a subspace W of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$.

Solution

a

For some vector $(x_1, x_2, x_3, x_4, x_5) \in U$, via the construction of U, we can write $(x_1, x_2, x_3, x_4, x_5)$ as

$$(x_1, x_2, x_3, x_4, x_5) = (3x_2, x_2, 7x_4, x_4, x_5).$$

The obvious basis of U is the list $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$. The list is clearly linearly independent and we can show that it spans U by writing

$$x_2(3, 1, 0, 0, 0) + x_4(0, 0, 7, 1, 0) + x_5(0, 0, 0, 0, 1) = (3x_2, x_2, 7x_4, x_4, x_5)$$

for $x_2, x_4, x_5 \in \mathbf{R}$.

b

To extend our basis of U to a basis of $\mathbf{R}^5$, we can follow a similar procedure as outlined in Theorem 2.31 ('Spanning list contains a basis') and Theorem 2.33 ('Linearly independent list extends to a basis'). Let the list e_1, e_2, e_3, e_4, e_5 denote the standard basis of $\mathbf{R}^5$. Appending the standard basis to our basis of U, we have

$$(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1), e_1, e_2, e_3, e_4, e_5.$$

Now let's see which of the standard basis vectors we can remove to produce a basis of $\mathbf{R}^5$.

e_1: The basis vector e_1, also represented as $(1, 0, 0, 0, 0)$, is not in $\operatorname{span}((3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1))$, so we can leave it.

e_2: The basis vector e_2, also represented as $(0, 1, 0, 0, 0)$, is indeed in $\operatorname{span}((3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1), e_1)$, since

$$(3, 1, 0, 0, 0) - 3e_1 = (0, 1, 0, 0, 0) = e_2.$$

Thus, we can delete e_2.

e_3: The basis vector e_3, also represented as $(0,0,1,0,0)$, is not in $\operatorname{span}((3,1,0,0,0),(0,0,7,1,0),(0,0,0,0,1),e_1)$, so we can leave it.

e_4: The basis vector e_4, also represented as $(0,0,0,1,0)$, is indeed in $\operatorname{span}((3,1,0,0,0),(0,0,7,1,0),(0,0,0,0,1),e_1,e_3)$, since

$$(0,0,7,1,0) - 7e_3 = (0,0,0,1,0) = e_4.$$

Thus, we can delete e_4.

e_5: The basis vector e_4, also represented as $(0,0,0,0,1)$, is obviously already in $\operatorname{span}((3,1,0,0,0),(0,0,7,1,0),(0,0,0,0,1),e_1,e_3)$, so we can delete it.

Now that we've finished our procedure, we can confidently claim that the list

$$(3,1,0,0,0),(0,0,7,1,0),(0,0,0,0,1),e_1,e_3$$

is a basis of $\mathbf{R}^5$.

c

The obvious subspace W of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$ is the subspace defined by

$$W = \operatorname{span}(e_1, e_3)$$

where e_1 and e_3 are the standard basis vectors we identified in part (b). To prove that $\mathbf{R}^5 = U \oplus W$, we must show $\mathbf{R}^5 = U + W$ and, via Theorem 1.45 ('Direct sum of two subspaces'), $U \cap W = \{0\}$.

The proof that $\mathbf{R}^5 = U + W$ follows from the work done in part (b). Explicitly, every vector $v \in U + W$ can be written as

$$v = a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,1) + a_4 e_1 + a_5 e_3$$

where $a_1, a_2, a_3, a_4, a_5 \in \mathbf{R}$. Since the list $(3,1,0,0,0),(0,0,7,1,0),(0,0,0,0,1),e_1,e_3$ also spans $\mathbf{R}^5$, every vector $v \in \mathbf{R}^5$ can be written in a similar manner. Thus, it follows that $\mathbf{R}^5 = U + W$.

To show $U \cap W = \{0\}$, for $v \in U \cap W$, we can write

$$v = a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,1) = b_4 e_1 + b_5 e_3.$$

Rearranging terms, we have

$$0 = b_4 e_1 + b_5 e_3 - a_1(3,1,0,0,0) - a_2(0,0,7,1,0) - a_3(0,0,0,0,1),$$

but via our proof of linear independence from part (b), it necessarily follows that $a_1 = a_2 = a_3 = b_4 = b_5 = 0$ and $v = 0$.

5 Another basis of $\mathcal{P}_3(\mathbf{F})$

Problem statement

Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Solution

Consider the polynomials

$$\begin{aligned} p_0(z) &= 1, \\ p_1(z) &= z, \\ p_2(z) &= z^2 + z^3, \\ p_3(z) &= z^3. \end{aligned}$$

None of the polynomials are of degree 2, yet $z^2 \in \operatorname{span}(p_0, p_1, p_2, p_3)$ since

$$p_2 - p_3 = z^2 + z^3 - z^3 = z^2.$$

Hence, $\operatorname{span}(p_0, p_1, p_2, p_3) = \mathcal{P}_3(\mathbf{F})$. Clearly, p_0, p_1, p_2, p_3 is linearly independent, thus p_0, p_1, p_2, p_3 is a basis of $\mathcal{P}_3(\mathbf{F})$.

6 $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is also a basis of v

Problem statement

Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

Solution

We must show that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is linearly independent and spans V.

To show the list is linearly independent, we can write

$$\begin{aligned} 0 &= a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_3 + v_4) + a_4 v_4 \\ &= a_1 v_1 + (a_2 + a_1)v_2 + (a_3 + a_2)v_3 + (a_4 + a_3)v_4. \end{aligned}$$

The last equality shows that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is linearly independent if and only if v_1, v_2, v_3, v_4 is linearly independent. Since v_1, v_2, v_3, v_4 is a basis of V, then $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is linearly independent.

To show that the list spans V, we must first note that any vector $v \in V$ can be written as

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3,$$

where $a_1, a_2, a_3, a_4 \in \mathbf{F}$, and any vector $u \in \operatorname{span}(v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4)$ can be written as

$$\begin{aligned} u &= b_1(v_1 + v_2) + b_2(v_2 + v_3) + b_3(v_3 + v_4) + b_4 v_4 \\ &= b_1 v_1 + (b_2 + b_1)v_2 + (b_3 + b_2)v_3 + (b_4 + b_3)v_4, \end{aligned}$$

where $a_1, a_2, a_3, a_4 \in \mathbf{F}$. It clearly follows that $u \in V$. To show that $v \in \operatorname{span}(v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4)$, we can set $b_1 = a_1, b_2 = a_2 - a_1, b_3 = a_3 - a_2 + a_1$, and $b_4 = a_4 - a_3 + a_2 - a_1$ and write

$$\begin{aligned} u &= b_1 v_1 + (b_2 + b_1)v_2 + (b_3 + b_2)v_3 + (b_4 + b_3)v_4 \\ &= a_1 v_1 + (a_2 - a_1 + a_1)v_2 + (a_3 - a_2 + a_1 + a_2 - a_1)v_3 \\ &\quad + (a_4 - a_3 + a_2 - a_1 + a_3 - a_2 + a_1)v_4 \\ &= a_1 v_1 + a_2 v_2 + a_3 v_3 = v. \end{aligned}$$

Hence, the list $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ spans V.

7 $v_3, v_4 \notin U$ but v_1, v_2 is not a basis of U

Problem statement

Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

Solution

Let's give a counterexample. Suppose $V = \mathbf{R}^4$ and $U = \{(x, y, z, z) \in \mathbf{R}^4 : x, y, z \in \mathbf{R}\}$. Let v_1, v_2, v_3, v_4 be the standard basis vectors of $\mathbf{R}^4$. It follows that $v_1, v2 \in U$ and $v_3, v_4 \notin U$, but v_1, v_2 is not a basis of U. However, $v_1, v_2, v_3 + v_4$ is a basis of U.