Linear Algebra Done Right
Solutions to Exercises 5.B

1 If $T^n = 0$, then $I - T$ invertible

Problem statement

Suppose $T \in \mathcal{L}(V)$ and there exists a positive integer n such that $T^n = 0$.

(a) Prove that $I - T$ is invertible and that

$$(I - T)^{-1} = I + T + \cdots + T^{n-1}.$$

(b) Explain how you would guess the formula above.

Solution

a

Following Definition 3.5 ('invertible, inverse'), $I - T$ is invertible if there exists $S \in \mathcal{L}(V)$ such that $S(I - T) = I$ and $(I - T)S = I$. We are given a possible inverse of $I - T$, so we need only show that it satisfies the $S(I - T) = I$ and $(I - T)S = I$. For $S(I - T) = I$, we can write

$$\begin{aligned}
(I + T + \cdots + T^{n-1})(I - T) &= I(I - T) + T(I - T) + \cdots + T^{n-1}(I - T) \\
&= I^2 - T + T - T^2 + T^2 + \cdots + T^{n-1} - T^n \\
&= I^2 - T^n \\
&= I
\end{aligned}$$

where the third equality follows because T^j can be matched with $-T^j$ and the last equality follows because $T^n = 0$. For $(I - T)S = I$, the prove is similar to $S(I - T) = I$.

b

To guess the formula for $(I - T)^{-1}$, you could have guessed that $T^n = 0$ needed to be leveraged in some way. By making a creative leap, you could infer that terms in between T^0 and T^n could be 'telescoped' such that they disappear. Thus, only $T^0 = I$ and $T^n = 0$ are left.

2 Eigenvalues of T for $(T-2I)(T-3I)(T-4I)=0$

Problem statement

Suppose $T \in \mathcal{L}(V)$ and $(T-2I)(T-3I)(T-4I)=0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Solution

Suppose v is an eigenvector[1] associated with λ and $Tv = \lambda v$. Hence, it follows that

$$(T-\alpha I)v = Tv - \alpha I v = (\lambda - \alpha)v.$$

Since $(T-2I)(T-3I)(T-4I)=0$, we have

$$(T-2I)(T-3I)(T-4I)v = 0$$

and can replace each factor with $(\lambda - \alpha)$ to write

$$(\lambda-2)(\lambda-3)(\lambda-4)v = 0.$$

This implies that λ must equal $2, 3$, or 4.

[1] This implies that $v \neq 0$.

3 If $T^2 = I$ and -1 is not eigenvalue, then $T = I$

Problem statement

Suppose $T \in \mathcal{L}(V)$ and $T^2 = I$ and -1 is not an eigenvalue of T. Prove that $T = I$.

Solution

For $v, w \in V$, suppose $w = Tv - v$ where we can interpret w as the residual of $Tv - v$. We can also write the residual as $Tv = w + v$. Applying T to both sides of $w = Tv - v$, we have

$$Tw = T^2v - Tv.$$

Substituting $T^v = v$ and $Tv = w + v$, it follows that

$$Tw = v - (w + v)$$

and $Tw = -w$. However, -1 is not an eigenvalue of T. Therefore, $w = 0$ and $Tv - v = 0$, which implies $Tv = v$ and $T = I$.

4 If $P^2 = P$, then $V = \operatorname{null} P \oplus \operatorname{range} P$

Problem statement

Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.

Solution

To prove the direct sum, we need to show $V = \operatorname{null} P + \operatorname{range} P$ and $\operatorname{null} P \cap \operatorname{range} P = \{0\}$.

Suppose $v \in V$ and $v - Pv = w$ for some $w \in V$. Applying P to both sides, we have $Pv - P^2v = Pw$. Since $P^2v = Pv$, it follows that $Pw = 0$ and $w \in \operatorname{null} P$. Hence, we can write

$$v = Pv + w$$

where $Pv \in \operatorname{range} P$ and $w \in \operatorname{null} P$. Thus, we have $V = \operatorname{null} P + \operatorname{range} P$.

Suppose $v \in \operatorname{null} P \cap \operatorname{range} P$. This implies $Pv = 0$ and there exists $w \in V$ such that $Pw = v$. Applying P to both sides, we have $P^2w = Pv = 0$. However, we also have $P^2w = Pw$. Hence, we can write

$$v = Pw = P^2w = Pv = 0$$

showing that $\operatorname{null} P \cap \operatorname{range} P = \{0\}$.

5 If S is invertible, then $p(STS^{-1}) = Sp(T)S^{-1}$

Problem statement

Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Suppose $p \in \mathcal{P}(\mathbf{F})$ is a polynomial. Prove that

$$p(STS^{-1}) = Sp(T)S^{-1}.$$

Solution

Let's first examine $(STS^{-1})^n$. We can expand this expression to

$$(STS^{-1})^n = ST\mathbf{S^{-1}S}TS^{-1}\cdots STS^{-1}$$

and noticing that $S^{-1}S = I$, it follows that

$$(STS^{-1})^n = ST\mathbf{S^{-1}S}TS^{-1}\cdots STS^{-1} = ST^nS^{-1}.$$

Now we can write $p(STS^{-1})$ as

$$\begin{aligned}
p(STS^{-1}) &= a_0I + a_1STS^{-1} + a_2(STS^{-1})^2 + \cdots + a_m(STS^{-1})^m \\
&= a_0SIS^{-1} + a_1STS^{-1} + a_2ST^2S^{-1} + \cdots + a_mST^mS^{-1} \\
&= S(a_0I + a_1T + a_2T^2 + \cdots + a_mT^m)S^{-1} \\
&= Sp(T)S^{-1}
\end{aligned}$$

where the second equality is justified via our reasoning that $(STS^{-1})^n = ST^nS^{-1}$.

6 Prove U invariant under $p(T)$

Problem statement

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T. Prove that U is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbf{F})$.

Solution

First, let's show that U is invariant under T^n. For $T^0 = I$, clearly every subspace of V is invariant under I. For T^1, we already know U is invariant under T. Thus, we can use induction and assume that U is invariant under T^{n-1}. It follows that for $u \in U$, we can write

$$T^n u = T^{n-1}(Tu)$$

and given $Tu \in U$ and our induction hypothesis, it follows that $T^n u \in U$.

Suppose $p \in \mathcal{P}(\mathbf{F})$ and $u \in U$. We can write

$$p(T)u = a_0 u + a_1 Tu + a_2 T^2 u + \cdots + a_m T^m u.$$

Given $u, Tu, T^2 u, \ldots, T^m u \in U$ and U is closed under scalar multiplication and addition, it follows that $p(T)u \in U$. Hence, U is invariant under $p(T)$.

7 9 eigenvalue of T^2 iff ± 3 eigenvalue of T

Problem statement

Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigenvalue of T^2 if and only if 3 or -3 is an eigenvalue of T.

Solution

First direction

Suppose 9 is an eigenvalue of T^2. This implies there exists $v \in V$ such that $(T^2 - 9I)v = 0$. We can expand $(T^2 - 9I)$ to $(T - 3I)(T + 3I)$ to write

$$(T - 3I)(T + 3I)v = 0.$$

This implies that either $T - 3I$ or $T + 3I$ are not injective and either 3 or -3 is an eigenvalue of T.

Second direction

Suppose 3 is an eigenvalue of T. This implies there exists $v \in V$ such that $Tv = 3v$. Applying T to both sides, we have

$$T^2 v = T(3v) = 3Tv = 3(3v) = 9v$$

and 9 is an eigenvalue of T^2. The same logic works if -3 is an eigenvalue of T.

8 Example of $T \in \mathcal{L}(\mathbf{R}^2)$ s.t. $T^4 = -1$

Problem statement

Give an example of $T \in \mathcal{L}(\mathbf{R}^2)$ such that $T^4 = -1$.

Solution

Suppose T is a counterclockwise rotation by 45 deg. The operator T^4 corresponds to a counterclockwise rotation by 180 deg, which is effectively a scalar multiplication of the vector by -1.

9 Zeros of p are eigenvalues of T

Problem statement

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$ with $v \neq 0$. Let p be a nonzero polynomial of smallest degree such that $p(T)v = 0$. Prove that every zero of p is an eigenvalue of T.

Solution

Suppose λ is a zero of p. We can use theorem 4.11 to write p as

$$p(z) = (z - \lambda)q(z).$$

It follows that $p(T) = (T - \lambda I)q(T)$. Given that p is the smallest degree polynomial such that $p(T)v = 0$ and $\deg p > \deg q$, it follows that $q(T)v \neq 0$.

Hence, we have

$$p(T)v = (T - \lambda I)q(T)v = 0$$

which implies $T(q(T)v) = \lambda q(T)v$. Therefore λ is an eigenvalue of T with $q(T)v$ as the corresponding eigenvector.[2]

[2]]Answer came from math.stackexchang.com.

10 For eigenvector v, $p(T)v = p(\lambda)v$

Problem statement

Suppose $T \in \mathcal{L}(V)$ and v is an eigenvector of T with eigenvalue λ. Suppose $p \in \mathcal{P}(\mathbf{F})$. Prove that $p(T)v = p(\lambda)v$.

Solution

Notice that for $T^n v$, we have

$$T^n v = T^{n-1}Tv = T^{n-1}\lambda v = \lambda T^{n-1}v = \lambda^2 T^{n-2}v = \cdots = \lambda^n v.$$

Thus, for $p(T)v$, we can write

$$\begin{aligned} p(T)v &= a_0 v + a_1 Tv + a_2 T^2 v + \cdots + a_m T^m v \\ &= a_0 v + a_1 \lambda v + a_2 \lambda^2 v + \cdots + a_m \lambda^m v \\ &= (a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_m \lambda^m)v \\ &= p(\lambda)v. \end{aligned}$$

11 α is eigenvalue of $p(T)$ iff $\alpha = p(\lambda)$

Problem statement

Suppose $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbf{C})$ is a polynomial, and $\alpha \in \mathbf{C}$. Prove that α is an eigenvalue of $p(T)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue λ of T.

Solution

First direction

Suppose α is an eigenvalue of $p(T)$. Then there exists $v \in V$ such that

$$(p(T) - \alpha I)v = 0.$$

Given $p \in \mathcal{P}(\mathbf{C})$, there exists a factorization of $p(z) - \alpha$ such that

$$p(z) - \alpha = c(z - \lambda_1) \cdots (z - \lambda_n).$$

For any $z = \lambda_j$, we have $p(\lambda_j) - \alpha = 0$ and $p(\lambda_j) = \alpha$. If we can show that some λ_j is an eigenvalue of of T, then the desired result follows.

By substituting T into our factorization $p(z) - \alpha$, we have

$$(p(T) - \alpha I)v = c(T - \lambda_1 I) \cdots (T - \lambda_n I)v = 0$$

which implies that $T - \lambda_j I$ is not injective for at least one j. Thus, there exists an eigenvalue λ_j of T and $\alpha = p(\lambda_j)$.

Second direction

Suppose λ is an eigenvalue of T and $\alpha = p(\lambda)$. Then there exists $v \neq 0$ such that $Tv = \lambda v$. Following from Exercise 5.B(10), we know that $p(T)v = p(\lambda)v$. Hence, we have $p(T)v = \alpha v$ and α is an eigenvalue of $p(T)$.

12 Exercise 5.B(11) fails if $\mathbf{F} = \mathbf{R}$

Problem statement

Show that the result in the previous exercise does not hold if $\mathbf{C}$ is replaced with $\mathbf{R}$.

Solution

Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by a 90 deg counterclockwise rotation. From Example 5.8 we know that T has no (real) eigenvalues. Suppose $p \in \mathcal{P}(\mathbf{R})$ is defined by

$$p(z) = z^3 - z.$$

It follows that T^3 is a 270 deg rotation and $p(T)$ is a 270 deg rotated vector subtracted by a 90 deg rotated vector. Thus, $p(T) = 0$ and 0 is an eigenvalue of $p(T)$. However, there is no associated eigenvalue λ of T for the expression $0 = p(\lambda)$ to hold.