Linear Algebra Done Right
Solutions to Exercises 6.B

1 Unit vectors in $\mathbf{R}^2$

Problem statement

(a) Suppose $\theta \in \mathbf{R}$. Show that $(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)$ and $(\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)$ are orthonormal bases of $\mathbf{R}^2$.

(b) Show that each orthonormal basis of $\mathbf{R}^2$ is of the form given by one of the two possibilities of part (a).

Solution

Part (a)

Suppose the inner product is the Euclidean dot product (Example 6.4(a)). We can write

$$\langle(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\rangle = -(\cos\theta)(\sin\theta) + (\sin\theta)(\cos\theta) = 0$$

to show that $(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)$ are orthogonal. To show that $\|(\cos\theta, \sin\theta)\| = 1$, we can write

$$\langle(\cos\theta, \sin\theta), (\cos\theta, \sin\theta)\rangle = \cos^2\theta + \sin^2\theta = 1$$

and to show that $\|(-\sin\theta, \cos\theta)\| = 1$, we can write

$$\langle(-\sin\theta, \cos\theta), (-\sin\theta, \cos\theta)\rangle = \sin^2\theta + \cos^2\theta = 1.$$

The same reasoning can be applied for $(\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)$.

Part (b)

If we think of vectors in $\mathbf{R}^2$ as arrows, then for $v \in \mathbf{R}^2$, the vector $v/\|v\|$ is a vector on the unit circle and can be expressed as $(\cos\theta, \sin\theta)$ for some $\theta \in \mathbf{R}$. The space of vectors orthogonal to $(\cos\theta, \sin\theta)$ is the line expressed by $\operatorname{span}((-\sin\theta, \cos\theta))$, where the vector $(-\sin\theta, \cos\theta)$ was shown to be orthogonal to $(\cos\theta, \sin\theta)$ in part (a). This line intersects with the unit circle at two and only two points, namely $(\cos\theta, \sin\theta)$ and $(\sin\theta, -\cos\theta)$.

2 Properties of vectors $v \in \text{span}(e_1, \ldots, e_m)$

Problem statement

Suppose $e_1, \ldots, e_m$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$$

if and only if $v \in \text{span}(e_1, \ldots, e_m)$.

Solution

First Direction

Proving the contrapositive is much easier, so let's do that.

Suppose $v \notin \text{span}(e_1, \ldots, e_m)$. For the list $e_1, \ldots, e_m, v$, we can apply the Gram-Schmidt Procedure (Theorem 6.31) to get $e_1, \ldots, e_m, f$. Obviously $v \in \text{span}(e_1, \ldots, e_m, f)$ since

$$\text{span}(e_1, \ldots, e_m, v) = \text{span}(e_1, \ldots, e_m, f).$$

Following from Theorem 6.30 ('Writing a vector as linear combination of orthonormal basis'), we can write

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 + |\langle v, f \rangle|^2.$$

If follows that $|\langle v, f \rangle|^2 \neq 0$ since $|\langle v, f \rangle|^2 = 0$ implies that $v \in \text{span}(e_1, \ldots, e_m)$, violating our hypothesis. Hence, we can write

$$\|v\|^2 \neq |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2.$$

Second Direction

Suppose $v \in \text{span}(e_1, \ldots, e_m)$. Thus, given $e_1, \ldots, e_m$ is an orthonormal basis of $\text{span}(e_1, \ldots, e_m)$, Theorem 6.30 implies

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2.$$

3 Applying Gram-Schmidt part 1

Problem statement

Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ has an upper-triangular matrix with respect to the basis $(1,0,0),(1,1,1),(1,1,2)$. Find an orthonormal basis of $\mathbf{R}^3$ (use the usual inner product on $\mathbf{R}^3$) with respect to which T has an upper-triangular matrix.

Solution

Theorem 6.37 ('Upper-triangular matrix with respect to orthonormal basis') implies that we can simply apply the Gram-Schmidt Procedure (Theorem 6.31) to find our desired orthonormal basis. This result takes advantage of

$$\operatorname{span}(v_1,\dots,v_j) = \operatorname{span}(e_1,\dots,e_j)$$

to show that $\operatorname{span}(e_1,\dots,e_j)$ is invariant under T. Thus, by Theorem 5.26 ('Conditions for upper-triangular matrix'), T has an upper-triangular matrix with respect to $e_1,\dots,e_n$.

e_1 With $v_1 = (1,0,0)$, then $\|v_1\| = 1$ implies $e_1 = (1,0,0)$.

e_2 With $v_2 = (1,1,1)$, let's first compute $v_2 - \langle v_2, e_1\rangle e_1$:

$$(1,1,1) - \langle (1,1,1),(1,0,0)\rangle(1,0,0) = (1,1,1) - (1,0,0) = (0,1,1).$$

Hence, we have

$$e_2 = \frac{(0,1,1)}{\|(0,1,1)\|} = \frac{(0,1,1)}{\sqrt{2}} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).$$

e_3 With $v_3 = (1,1,2)$, let's first compute $v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2$:

$$\begin{aligned}(1,1,2) - \langle (1,1,2),(1,0,0)\rangle(1,0,0) - \langle (1,1,2),(0,\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}})\rangle(0,\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}}) \\ = (1,1,2) - (1,0,0) - \frac{3}{\sqrt{2}}(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}) \\ = (0,1,2) - (0,\frac{3}{2},\frac{3}{2}) = (0,-\frac{1}{2},\frac{1}{2}).\end{aligned}$$

Hence, we have

$$e_3 = \frac{(0,-\frac{1}{2},\frac{1}{2})}{\|(0,-\frac{1}{2},\frac{1}{2})\|} = \sqrt{2}(0,-\frac{1}{2},\frac{1}{2}) = (0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}).$$

Therefore, our orthonormal basis is $((1,0,0),(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}))$.

4 An orthonormal Fourier series

Problem statement

Suppose n is a positive integer. Prove that

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\sin nx}{\sqrt{\pi}}$$

is an orthonormal list of vectors in $C[-\pi, \pi]$, the vector space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\ dx.$$

Preliminary

At the heart of this exercise is solving the integral

$$\int_{-\pi}^{\pi} \frac{\cos nx}{\sqrt{\pi}} \frac{\sin mx}{\sqrt{\pi}}\ dx$$

and other similar looking integrals. To do so, we need the so-called product-to-sum trigonometric identities:

$$\begin{aligned}
\cos\theta\cos\phi &= \frac{1}{2}\big(\cos(\theta-\phi)+\cos(\theta+\phi)\big),\\
\sin\theta\sin\phi &= \frac{1}{2}\big(\cos(\theta-\phi)+\cos(\theta+\phi)\big),\\
\sin\theta\cos\phi &= \frac{1}{2}\big(\sin(\theta+\phi)+\sin(\theta-\phi)\big),\\
\cos\theta\sin\phi &= \frac{1}{2}\big(\sin(\theta+\phi)+\sin(\theta-\phi)\big).
\end{aligned}$$

Solution

Let's prove that our list of vectors is orthonormal by computing the inner products between pairs of functions in the list and showing that

- each function has norm 1,
- any pair of distinct functions is orthogonal.

Norms of the functions

Let's start by computing the norm of each function in the list.

Constant function:

$$\left\langle \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1\, dx = \frac{1}{2\pi} \cdot 2\pi = 1.$$

Cosine functions:

$$\left\langle \frac{\cos(kx)}{\sqrt{\pi}}, \frac{\cos(kx)}{\sqrt{\pi}} \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(kx)\, dx = \frac{1}{\pi} \cdot \pi = 1,$$

since $\int_{-\pi}^{\pi} \cos^2(kx)\, dx = \pi$ for $k \in \mathbb{N}$.

Sine functions:

$$\left\langle \frac{\sin(kx)}{\sqrt{\pi}}, \frac{\sin(kx)}{\sqrt{\pi}} \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(kx)\, dx = \frac{1}{\pi} \cdot \pi = 1.$$

Orthogonality of distinct functions

Let's now check that each function in the list is orthogonal to all other functions.

Constant with sine or cosine:

$$\left\langle \frac{1}{\sqrt{2\pi}}, \frac{\cos(kx)}{\sqrt{\pi}} \right\rangle = \frac{1}{\sqrt{2\pi\pi}} \int_{-\pi}^{\pi} \cos(kx)\, dx = 0,$$

since $\cos(kx)$ integrates to zero over $[-\pi, \pi]$ for $k \neq 0$. Similarly,

$$\left\langle \frac{1}{\sqrt{2\pi}}, \frac{\sin(kx)}{\sqrt{\pi}} \right\rangle = \frac{1}{\sqrt{2\pi\pi}} \int_{-\pi}^{\pi} \sin(kx)\, dx = 0,$$

since $\sin(kx)$ is odd and integrates to zero over symmetric interval.

Cosines with different frequencies: Using the identity

$$\cos(mx)\cos(nx) = \frac{1}{2}[\cos((m-n)x) + \cos((m+n)x)],$$

we get

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\, dx = 0 \quad \text{for } m \neq n,$$

and so,

$$\left\langle \frac{\cos(mx)}{\sqrt{\pi}}, \frac{\cos(nx)}{\sqrt{\pi}} \right\rangle = 0 \quad \text{for } m \neq n.$$

Sines with different frequencies: Using the identity

$$\sin(mx)\sin(nx) = \frac{1}{2}[\cos((m-n)x) - \cos((m+n)x)],$$

we get

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = 0 \quad \text{for } m \neq n,$$

and so

$$\left\langle \frac{\sin(mx)}{\sqrt{\pi}}, \frac{\sin(nx)}{\sqrt{\pi}} \right\rangle = 0 \quad \text{for } m \neq n.$$

Sine and cosine: Using the identity:

$$\sin(mx)\cos(nx) = \frac{1}{2}[\sin((m+n)x) + \sin((m-n)x)],$$

and noting that all sine integrals over $[-\pi, \pi]$ vanish,

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0,$$

so,

$$\left\langle \frac{\sin(mx)}{\sqrt{\pi}}, \frac{\cos(nx)}{\sqrt{\pi}} \right\rangle = 0.$$

Conclusion

We have shown that all functions in the list have unit norm and are orthogonal to each other. Therefore, the list

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \ldots, \frac{\sin nx}{\sqrt{\pi}}$$

is an orthonormal list in $C[-\pi, \pi]$ with respect to the inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$.

5 Applying Gram-Schmidt part 2

Problem statement

On $\mathcal{P}_2(\mathbf{R})$, consider the inner product given by

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx.$$

Apply the Gram-Schmidt Procedure to the basis $1, x, x^2$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$.

Solution

e_1 Clearly $e_1 = 1$ since

$$\langle 1, 1 \rangle = \int_0^1 1 \cdot 1 \; dx = 1.$$

e_2 With $v_2 = x$, let's first compute $v_2 - \langle v_2, e_1 \rangle e_1$:

$$\begin{aligned} x - \langle x, 1 \rangle 1 &= x - \int_0^1 x \; dx \\ &= x - \left[\frac{1}{2}x^2\right]_0^1 \\ &= x - \frac{1}{2}. \end{aligned}$$

To normalize $x - \frac{1}{2}$, we need to evaluate $\|x - \frac{1}{2}\|$. We can write

$$\begin{aligned} \langle x - \frac{1}{2}, x - \frac{1}{2} \rangle 1 &= \int_0^1 (x - \frac{1}{2})(x - \frac{1}{2}) \; dx \\ &= \int_0^1 (x^2 - x + \frac{1}{4}) \; dx \\ &= \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x \Big|_0^1 \\ &= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}. \end{aligned}$$

Hence, it follows that

$$e_2 = \sqrt{12}(x - \frac{1}{2}).$$

e_3 With $v_3 = x^2$, let's first compute $v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2$:

$$\begin{aligned}
x^2 - &\langle x^2, 1\rangle 1 - \langle x^2, \sqrt{12}(x-\frac{1}{2})\rangle\big(\sqrt{12}(x-\frac{1}{2})\big) \\
&= x^2 - \int_0^1 x^2\ dx - 12\int_0^1 (x^3 - \frac{1}{2}x^2)\ dx\ (x - \frac{1}{2}) \\
&= x^2 - \Big[\frac{1}{3}x^3\Big]_0^1 - 12\Big[\frac{1}{4}x^4 - \frac{1}{6}x^3\Big]_0^1 (x - \frac{1}{2}) \\
&= x^2 - \frac{1}{3} - 12(\frac{1}{4} - \frac{1}{6})(x - \frac{1}{2}) \\
&= x^2 - \frac{1}{3} - x + \frac{1}{2} = x^2 - x + \frac{1}{6}.
\end{aligned}$$

To normalize $x^2 - x + \frac{1}{6}$, we need to evaluate $\|x^2 - x + \frac{1}{6}\|$. We can write[1]

$$\langle x^2 - x + \frac{1}{6}, x^2 - x + \frac{1}{6}\rangle = \int_0^1 (x^2 - x + \frac{1}{6})^2\ dx = \frac{1}{180}.$$

Hence, if follows that

$$e_3 = \sqrt{180}(x^2 - x + \frac{1}{6}).$$

Therefore, our orthonormal basis is $(1, \sqrt{12}(x - \frac{1}{2}), \sqrt{180}(x^2 - x + \frac{1}{6}))$.

[1] I got a bit lazy here.

6 Upper-triangular matrix for differentiation

Problem statement

Find an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$ (with inner product as in Exercise 5) such that the differentiation operator (the operator that takes p to p') on $\mathcal{P}_2(\mathbf{R})$ has an upper-triangular matrix with respect to this basis.

Solution

Following Theorem 5.26 ('Conditions for upper-triangular matrix'), all we need to show is that an orthonormal list e_1, e_2, e_3 is such that span(e_1), span(e_1, e_2), and span(e_1, e_2, e_3) are invariant under the differentiation operator. Conveniently, the orthonormal basis from Exercise 6.B(5) works. This basis is

$$(e_1, e_2, e_3) = (1, \sqrt{12}(x - \frac{1}{2}), \sqrt{180}(x^2 - x + \frac{1}{6})).$$

Let's call the differentiation operator $D \in \mathcal{L}(\mathcal{P}_2(\mathbf{R}))$.

D maps $\mathcal{P}_0(\mathbf{R})$ to 0, thus span(e_1) is invariant under D given $0 \subset$ span(e_1). D maps $\mathcal{P}_1(\mathbf{R})$ to $\mathcal{P}_0(\mathbf{R})$, thus span$(e_1, e_2)$ is invariant under D given $\mathcal{P}_0(\mathbf{R}) \subset$ span(e_1, e_2). Finally, D maps $\mathcal{P}_2(\mathbf{R})$ to $\mathcal{P}_1(\mathbf{R})$, thus span$(e_1, e_2, e_3)$ is invariant under D $\mathcal{P}_1(\mathbf{R}) \subset$ span(e_1, e_2, e_3). Thus, D has an upper-triangular matrix with respect to our orthonormal basis e_1, e_2, e_3.

7 Applying the Riesz Representation Theorem

Problem statement

Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$p(\frac{1}{2}) = \int_0^1 p(x)q(x)\ dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

Preliminary

This exercise is a simple application of the Riesz Representation Theorem (Theorem 6.42), but is tricky since one needs to understand how $p(\frac{1}{2})$ is a linear functional. Let's first note that $p(\frac{1}{2})$ is a map from $\mathcal{P}_2(\mathbf{R})$ to $\mathbf{R}$. To see why $p(\frac{1}{2})$ is linear, we can test of additivity and homogeneity. For some $p \in \mathcal{P}_2(\mathbf{R})$, we have $p(x) = ax^2 + bx + c$ for $a, b, c \in \mathbf{R}$ and can write

$$(\lambda p)(\frac{1}{2}) = \lambda a(\frac{1}{2})^2 + \lambda b(\frac{1}{2}) + \lambda c = \lambda(a(\frac{1}{2})^2 + b(\frac{1}{2}) + c) = \lambda(p(\frac{1}{2})).$$

For some $p, q \in \mathcal{P}_2(\mathbf{R})$, we have $p(x) = a_1x^2 + b_1x + c_1$ for $a_1, b_1, c_1 \in \mathbf{R}$, $q(x) = a_2x^2 + b_2x + c_2$ for $a_2, b_2, c_2 \in \mathbf{R}$, and can write

$$(p + q)(x) = (a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2)$$

and it follows that

$$\begin{aligned}
(p + q)(\frac{1}{2}) &= (a_1 + a_2)(\frac{1}{2})^2 + (b_1 + b_2)(\frac{1}{2}) + (c_1 + c_2) \\
&= a_1(\frac{1}{2})^2 + b_1(\frac{1}{2}) + c_1 + a_2(\frac{1}{2})^2 + b_2(\frac{1}{2}) + c_2 \\
&= p(\frac{1}{2}) + q(\frac{1}{2}).
\end{aligned}$$

But let's note that while $p(\frac{1}{2})$ is a linear functional from $\mathcal{P}_2(\mathbf{R})$ to $\mathbf{R}$, the polynomial $p(x)$ is not a linear functional from $\mathbf{R}$ to $\mathbf{R}$. Polynomials can be manipulated as mathematical objects in a vector space, but polynomials themselves are not linear functions.

Solution

To find our desired $q \in \mathcal{P}_2(\mathbf{R})$, we can use the formula

$$u = \overline{\phi(e_1)}e_1 + \cdots + \overline{\phi(e_n)}e_n$$

and the orthonormal basis from Exercise 6.B(5)

$$(e_1, e_2, e_3) = (1, \sqrt{12}(x - \frac{1}{2}), \sqrt{180}(x^2 - x + \frac{1}{6})).$$

First, let's evaluate $\overline{\phi(e_1)}, \overline{\phi(e_2)}, \overline{\phi(e_3)}$:

$$\overline{\phi(e_1)} = \overline{1} = 1$$
$$\overline{\phi(e_2)} = \overline{\sqrt{12}(1/2 - 1/2)} = 0$$
$$\overline{\phi(e_3)} = \overline{\sqrt{180}((\frac{1}{2})^2 - \frac{1}{2} + \frac{1}{6}))} = -\frac{\sqrt{180}}{12}$$

Plugging the everything together, we can write

$$\begin{aligned}
u &= 1 \cdot 1 + 0 \cdot \left(\sqrt{12}(x - \frac{1}{2})\right) - \frac{\sqrt{180}}{12}\left(\sqrt{180}(x^2 - x + \frac{1}{6})\right) \\
&= 1 - \frac{180}{12}(x^2 - x + \frac{1}{6}) \\
&= 1 - 15(x^2 - x + \frac{1}{6}) \\
&= -15x^2 + 15x - \frac{3}{2}.
\end{aligned}$$

9 The numerator in Gram-Schmidt Procedure

Problem statement

What happens if the Gram-Schmidt Procedure is applied to a list of vectors that is not linearly independent?

Solution

Suppose we have a list of linearly dependent vectors $v_1, \ldots, v_n$. Via the Linear Dependence Lemma (Theorem 2.21), there exist

$$v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$$

for the smallest j. Hence, if we apply the Gram-Schmidt Procedure on the list $v_1, \ldots, v_{j-1}$, it follows that

$$v_j \in \operatorname{span}(e_1, \ldots, e_{j-1})$$

since $\operatorname{span}(v_1, \ldots, v_{j-1}) = \operatorname{span}(e_1, \ldots, e_{j-1})$. Via Theorem 6.30 ('Writing a vectors as linear combination of orthonormal basis'), we have

$$v_j = \langle v_j, e_1 \rangle e_1 + \cdots + \langle v_j, e_{j-1} \rangle e_{j-1},$$

which implies

$$\|v_j - \langle v_j, e_1 \rangle e_1 - \cdots - \langle v_j, e_{j-1} \rangle e_{j-1}\| = \|0\| = 0.$$

Hence, the numerator of e_j is 0 and e_j can't be constructed.

17 Proving Riesz Representation Theorem

Problem statement

For $u \in V$, let Φu denote the linear functional on V defined by

$$(\Phi u)(v) = \langle v, u \rangle$$

for $v \in V$.

(a) Show that if $\mathbf{F} = \mathbf{R}$, then Φ is a linear map from V to V'.

(b) Show that if $\mathbf{F} = \mathbf{C}$ and $V \neq \{0\}$, then Φ is not a linear map.

(c) Show that Φ is injective.

(d) Suppose $\mathbf{F} = \mathbf{R}$ and V is finite-dimensional. Use parts (a) and (c) and a dimension-counting argument to show that Φ is an isomorphism from V onto V'.

Solution

a

We first need to show that Φ obeys additivity and homogeneity.

Additivity: For $u, v, w \in V$, we can write

$$(\Phi(u + w))(v) = \langle v, u + w \rangle = \langle v, u \rangle + \langle v, w \rangle = (\Phi u)(v) + (\Phi w)(v)$$

where the second equality is valid given $\mathbf{F} = \mathbf{R}$.

Homogeneity: For $u, v \in V$ and $\lambda \in \mathbf{R}$, we can write

$$(\Phi(\lambda u))(v) = \langle v, \lambda u \rangle = \lambda \langle v, u \rangle = \lambda (\Phi u)(v)$$

where the second equality is valid given $\mathbf{F} = \mathbf{R}$.

b

For homogeneity in part (a), $\mathbf{F} = \mathbf{R}$ implies $\bar{\lambda} = \lambda$. For $\mathbf{F} = \mathbf{C}$, if $\lambda = 1 + i$, then

$$(\Phi((1 + i)u))(v) = \langle v, (1 + i)u \rangle = (1 - i)\langle v, u \rangle \neq \lambda (\Phi u)(v).$$

Hence, Φ is not a linear map if $\mathbf{F} = \mathbf{C}$.

c

Note that $0 \in V$ is the zero vector while $0 \in V'$ is the zero linear functional.

Suppose $u \in \operatorname{null} \Phi$. Then $\langle v, u \rangle = 0$ for all $v \in V$. If $v = u$, then $\langle u, u \rangle = 0$. Therefore, via the property of definiteness for inner products (Definition 6.3), it follows $u = 0$ and $\operatorname{null} \Phi = \{0\}$. Hence, Φ is injective.

d

Via the Fundamental Theorem of Linear Maps

$$\dim V = \dim \operatorname{null} \Phi + \dim \operatorname{range} \Phi.$$

Via part (c), $\dim \operatorname{null} \Phi = 0$ and thus $\dim V = \dim \operatorname{range} \Phi$. From Theorem 3.95, it follows that $\dim V' = \dim V$. Hence, we can write

$$\dim \operatorname{range} \Phi = \dim V = \dim V',$$

showing Φ is surjective. Therefore, Φ is invertible and an isomorphism from V onto V'.