Linear Algebra Done Right
Solutions to Exercises 5.A

1 Subspaces invariant under T

Problem statement

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.

(a) Prove that if $U \subset \operatorname{null} T$, then U is invariant under T.

(b) Prove that if $\operatorname{range} T \subset U$, then U is invariant under T.

Solution

a

If $u \in U$, then $u \in \operatorname{null} T$ and $Tu = 0$. Since U is a subspace, it follows that

$$Tu = 0 \in U$$

and $Tu \in U$. Hence, via Definition 5.2 ('invariant subspaces'), U is invariant under T.

b

If $u \in U$, then clearly $Tu \in \operatorname{range} T$. Since $\operatorname{range} T \subset U$, it follows that $Tu \in U$. Hence, via Definition 5.2 ('invariant subspaces'), U is invariant under T.

2 $ST = TS$ implies $\operatorname{null} S$ is invariant under T

Problem statement

Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{null} S$ is invariant under T.

Solution

Suppose $v \in \operatorname{null} S$. This implies

$$TSv = T(0) = 0$$

and thus,

$$S(Tv) = TSv = 0.$$

Hence, we have $Tv \in \operatorname{null} S$ and $\operatorname{null} S$ is invariant under T.

3 $ST = TS$ implies $\operatorname{range} S$ is invariant under T

Problem statement

Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{range} S$ is invariant under T.

Solution

Suppose $v \in \operatorname{range} S$. This implies there exists $u \in V$ such that $Su = v$. Hence, given $ST = TS$, we can write

$$STu = TSu = Tv$$

and $Tv \in \operatorname{range} S$. Therefore, $\operatorname{range} S$ is invariant under T.

4 Prove $U_1 + \cdots + U_m$ is invariant under T

Problem statement

Suppose that $T \in \mathcal{L}(V)$ and $U_1, \ldots, U_m$ are subspaces of V invariant under T. Prove that $U_1 + \cdots + U_m$ is invariant under T.

Solution

Suppose $u \in U_1 + \cdots + U_m$. Thus there exists $u_1 \in U_1, \ldots, u_m \in U_m$ such that

$$u = u_1 + \cdots + u_m.$$

Given $U_1, \ldots, U_m$ are subspaces of V invariant under T, it follows that $Tu_1 \in U_1, \ldots, Tu_m \in U_m$ and we can write

$$Tu = T(u_1 + \cdots + u_m) = Tu_1 + \cdots + Tu_m$$

implying that $Tu \in U_1 + \cdots + U_m$. Therefore, $U_1 + \cdots + U_m$ is invariant under T.

5 When intersection of subspaces is invariant

Problem statement

Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T.

Solution

Suppose $U_1, \ldots, U_m$ is a collection of subspaces of V invariant under T. Suppose

$$u \in U_1 \cap \cdots \cap U_m.$$

It follows that $Tu \in U_j$ for $j = 1, \ldots, m$ since U_j is invariant under T. Therefore, we have

$$Tu \in U_1 \cap \cdots \cap U_m$$

and $U_1 \cap \cdots \cap U_m$ is invariant under T.

6 $\{0\}$ and V are invariant under all operators

Problem statement

Prove or give a counterexample: if V is finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Solution

Let's prove it.

Obviously $\{0\}$ and V are invariant under every operator on V. Thus, suppose U contains at least one vector $v \in V$ such that $v \neq 0$ and $U \neq V$. Suppose $u_1, \dots, u_m$ is a basis of U. Let's extend this basis to $u_1, \dots, u_m, v_1, \dots, v_n$ to be a basis of V. Note that we must append at least one vector $v_1 \in V$ to our basis of U since $U \neq V$.

Consider the list

$$v_1, \dots (n + p \text{ times}) \dots, v_1.$$

Via Theorem 3.5 ('Linear maps and basis of domain'), we can construct a unique operator $T \in \mathcal{L}(V)$ such that

$$Tu_j = v_1 \quad \text{and} \quad Tv_k = v_1$$

for all $j = 1, \dots, m$ and $k = 1, \dots, n$. It clearly follows that U is not invariant under T since $u_1 \in U$ but $Tu_1 \notin U$.

Therefore, the only two subspaces of V invariant under every operator on V are $U = \{0\}$ and $U = V$.

7 $T(x,y) = (-3y, x)$ has no eigenvalues

Problem statement

Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x,y) = (-3y, x)$. Find the eigenvalues of T.

Solution

To find the eigenvalues of T, we write $-3y = \lambda x$ and $x = \lambda y$. Substituting one equation into the other, we have $-3y = \lambda^2 y$ and $\lambda = \pm\sqrt{-3}$, which is not a 'real number'. Therefore, T has no eigenvalues.

8 Eigenvalues of $T(w, z) = (z, w)$

Problem statement

Define $T \in \mathcal{L}(\mathbf{F}^2)$ by

$$T(w, z) = (z, w).$$

Find all eigenvalues and eigenvectors of T.

Solution

To find the eigenvalues of T, we write $z = \lambda w$ and $w = \lambda z$. Substituting one equation into the other, we have $z = \lambda^2 z$ and $\lambda = \pm 1$. Via Theorem 5.13 ('Number of eigenvalues'), 1 and -1 are all the eigenvalues of T.

By observation and a quick check, the vector $(1, 1)$ is an eigenvector corresponding to $\lambda = 1$ and the vector $(-1, 1)$ is an eigenvector corresponding to $\lambda = -1$.

9 Eigenvalues of $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$

Problem statement

Define $T \in \mathcal{L}(\mathbf{F}^3)$ by

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3).$$

Find all eigenvalues and eigenvectors of T.

Solution

To find the eigenvalues of T, we write $2z_2 = \lambda z_1$, $0 = \lambda z_2$, and $5z_3 = \lambda z_3$. From $0 = \lambda z_2$ it follows that $\lambda = 0$ is an eigenvalue. From $5z_3 = \lambda z_3$, it follows that $\lambda = 5$ is an eigenvalue.

By observation and a quick check, the vector $(1, 0, 0)$ is an eigenvector corresponding to $\lambda = 0$ and the vector $(0, 0, 1)$ is an eigenvector corresponding to $\lambda = 5$.

10 $T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n)$

Problem statement

Define $T \in \mathcal{L}(\mathbf{F}^n)$ by

$$T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n).$$

(a) Find all eigenvalues and eigenvectors of T.

(b) Find all invariant subspaces of T.

Solution

a

For a vector $(x_1, \ldots, x_n) \in \mathbf{F}^n$, we can think of T as stretching each coordinate x_j by a scalar multiplication of j. Therefore, it clearly follows that the integers $1, 2, \ldots, n$ are eigenvalues with the standard basis vectors as the corresponding eigenvectors.

b

Via Exercise 5.A(6), we know $\{0\}$ and $\mathbf{F}^n$ are invariant subspaces of T. Suppose $e_1, \ldots, e_k$ is some list of standard basis vectors. Since all standard basis vectors are eigenvectors, it follows that $\text{span}(e_1, \ldots, e_k)$ is an invariant subspace of T.

11 Eigenvalues of differentiation

Problem statement

Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

Solution

Differentiation takes any constant polynomial to zero. Thus, 0 is an eigenvalue of T with $p(z) = 1$ as the corresponding eigenvector.

No other eigenvalues or eigenvectors exist for T since for $q \in \mathcal{P}(\mathbf{R})$ such that $\deg q \geq 1$, we have

$$\deg q \neq \deg q' = \deg Tq$$

and no constant λ exists such that $(Tq)(x) = \lambda q(x)$.

12 Eigenvalues of $(Tp)(x) = xp'(x)$

Problem statement

Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{R}))$ by

$$(Tp)(x) = xp'(x)$$

for all $x \in \mathbf{R}$. Find all eigenvalues and eigenvectors of T.

Solution

To understand the behavior of T, let's apply T to the polynomial $p(x) = 1 + x + x^2 + x^3 + x^4$. We can write

$$(Tp)(x) = x(1 + 2x + 3x^2 + 4x^3) = x + 2x^2 + 3x^3 + 4x^4$$

and it follows that T resembles the operator in Exercise 5.A(10). Thus, the eigenvalues of T are $0, 1, 2, 3, 4$ and the corresponding eigenvectors are the standard basis vectors of $\mathcal{P}_4(\mathbf{R})$, which are $1, x, x^2, x^3, x^4$.

13 Eigenvalues are fragile

Problem statement

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$. Prove that there exists $\alpha \in \mathbf{F}$ such that $|\alpha - \lambda| < \frac{1}{1000}$ and $T - \alpha I$ is invertible.

Solution

Via Theorem 5.6 ('Equivalent conditions to be an eigenvalue'), the phrase "$T - \alpha I$ is invertible" is equivalent to α not being an eigenvalue of T. The obvious choice of α is $\alpha = \lambda$ since $|\lambda - \lambda| = 0$, but λ could itself be an eigenvalue. Thus, suppose $\dim V = n$ and consider the following list of scalars:

$$\lambda, \lambda + (\frac{1}{10000}), \lambda + (\frac{1}{10000})^2, \ldots, \lambda + (\frac{1}{10000})^n.$$

Note that all of these scalars satisfy $|\alpha - \lambda| < \frac{1}{1000}$ and via Theorem 5.13 ('Number of eigenvalues'), at least one of the scalars is not an eigenvalue since T can have a maximum of n distinct eigenvalues.

14 Eigenvalues of a projection operator

Problem statement

Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V. Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for $u \in U$ and $w \in W$. Find all eigenvalues and eigenvectors of P.

Solution

For $u \in U$ and $0 \in W$, we have

$$P(u) = P(u + 0) = u,$$

showing that 1 is an eigenvalue of P. For $0 \in U$ and $w \in W$, we have

$$P(w) = P(0 + w) = 0,$$

showing that 0 is an eigenvalue of P. Hence, 0 and 1 are eigenvalues of P.

Let $u_1, \ldots, u_n$ be a basis of U and $w_1, \ldots, w_m$ be a basis of W. Via our reasoning in the first paragraph, it follows that $u_1, \ldots, u_n$ are eigenvectors corresponding to the eigenvalue 1 and $w_1, \ldots, w_m$ are eigenvectors corresponding to the eigenvalue 0. Given $V = U \oplus W$ and Theorem 5.10 ('Linearly independent eigenvectors'), there can be no other eigenvectors or eigenvalues of P since the list $u_1, \ldots, u_n, w_1, \ldots, w_m$ is a list of eigenvectors of P that span V.

15 Eigenvalues of T and $S^{-1}TS$

Problem statement

Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

Solution

a

Suppose λ is an eigenvalue of T. Thus, there exists a corresponding eigenvector $v \in V$ such that $Tv = \lambda v$. Given S is invertible, $v \in \operatorname{range} S$ and there exists $u \in V$ such that $Su = v$ and $S^{-1}v = u$. Hence, we can write

$$S^{-1}TSu = S^{-1}Tv = S^{-1}(\lambda v) = \lambda S^{-1}v = \lambda u$$

which shows λ is an eigenvalue of $S^{-1}TS$.

Now let's show the other direction. Suppose λ is an eigenvalue of $S^{-1}TS$. Thus, there exists a corresponding eigenvector $v \in V$ such that

$$S^{-1}TSv = \lambda v.$$

Applying S to both sides of the equation above, we have

$$TSv = \lambda(Sv)$$

which shows λ is an eigenvalue of T.

b

Following our answer in part (a), we can state that v is an eigenvector of $S^{-1}TS$ if and only if Sv is an eigenvector of T.

16 Eigenvalues for real matrices come in pairs

Problem statement

Suppose V is a complex vector space, $T \in \mathcal{L}(V)$, and the matrix of T with respect to some basis of V contains only real entries. Show that if λ is an eigenvalue of T, then so is $\bar{\lambda}$.

Solution

Suppose $\lambda \in \mathbf{C}$ is an eigenvalue of T with a corresponding eigenvector $v \in V$. Thus, we can write

$$Tv = \lambda v$$

and by taking the **matrix of** both sides, we have

$$\mathcal{M}(Tv) = \mathcal{M}(\lambda v)$$

where the basis of V used is $v_1, \ldots, v_n$ such that $\mathcal{M}(T, (v_1, \ldots, v_n), (v_1, \ldots, v_n))$ only contains real entries. We can write v as $v = c_1 v_1 + \cdots + c_n v_n$, where $c_1, \ldots, c_n \in \mathbf{C}$, and it follows from Definition 3.62 ('matrix of a vector') that

$$\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \quad \text{and} \quad \mathcal{M}(\lambda v) = \begin{pmatrix} \lambda c_1 \\ \vdots \\ \lambda c_n \end{pmatrix}.$$

Via Theorem 3.65 ('Linear maps act like matrix multiplication'), we can write

$$\mathcal{M}(T)\mathcal{M}(v) = \mathcal{M}(\lambda v).$$

Consider the entry $\mathcal{M}(\lambda v)_{k,1}$, which we can express as

$$\sum_{j=1}^{n} c_j \mathcal{M}(T)_{k,j} = \sum_{j=1}^{n} \mathcal{M}(T)_{k,j} \mathcal{M}(v)_{j,1} = \mathcal{M}(\lambda v)_{k,1} = \lambda c_k.$$

Let's focus on $\sum_{j=1}^{n} c_j \mathcal{M}(T)_{k,j} = \lambda c_k$. Taking the complex conjugate of both sides and using the **additivity and multiplicativity** properties of complex conjugates (Theorem 4.5), it follows that

$$\sum_{j=1}^{n} \overline{c_j} \mathcal{M}(T)_{k,j} = \bar{\lambda} \overline{c_k}$$

where $\overline{\mathcal{M}(T)_{k,j}} = \mathcal{M}(T)_{k,j}$ given $\mathcal{M}(T)$ has real entries. Therefore, $\bar{\lambda}$ is an eigenvalue of T with the corresponding eigenvector of

$$w = \overline{c_1} v_1 + \cdots + \overline{c_n} v_n$$

so that $Tw = \bar{\lambda} w$.

17 Example of $T \in \mathcal{L}(\mathbf{R}^4)$ with no eigenvalues

Problem statement

Give an example of an operator $T \in \mathcal{L}(\mathbf{R}^4)$ such that T has no (real) eigenvalues.

Solution

Define $T \in \mathcal{L}(\mathbf{R}^4)$ by

$$T(w, x, y, z) = (z, -w, x, y).$$

To find the eigenvalues of T, we can write

$$z = \lambda w, \quad -w = \lambda x, \quad x = \lambda y, \quad y = \lambda z.$$

Combining all our equations together, we get

$$-w = \lambda^4 w$$

and thus,

$$\lambda^4 = -1$$

which has no real solutions. Therefore, T has no (real) eigenvalues.

18 Forward shift operator has no eigenvalues

Problem statement

Show that the operator $T \in \mathcal{L}(\mathbf{C}^\infty)$ defined by

$$T(z_1, z_2, \ldots) = (0, z_1, z_2, \ldots)$$

has no eigenvalues.

Solution

To find the eigenvalues of T, we can write

$$0 = \lambda z_1, \ \ z_1 = \lambda z_2, \ \ \ldots$$

The $0 = \lambda z_1$ equation would cause any combination of equations above to have $\lambda = 0$. Thus 0 is our only candidate eigenvalue with all the possible corresponding eigenvectors being members of $\operatorname{null} T$. Yet, it clearly follows that $\operatorname{null} T = \{0\}$, and via Definition 5.5 ('eigenvalue'), 0 cannot be an eigenvector, implying that 0 is not an eigenvalue of T. Therefore, T has no eigenvalues.