Linear Algebra Done Right
Solutions to Exercises 1.B

1 Prove that $-(-v) = v$

Problem statement

Prove that $-(-v) = v$ for every $v \in V$.

Solution

Via the properties of **additive inverse** and **commutativity** (Definition 1.19), we have

$$-v + (-(-v)) = 0 \quad \text{and} \quad v + (-v) = -v + v = 0$$

for every $v \in V$. But via Theorem 1.26 ('Unique additive inverse'), every element in a vector space has a unique additive inverse. Hence, it necessarily follows that $-(-v) = v$ for every $v \in V$.

4 Why is the empty set not a vector space?

Problem statement

The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?

Solution

The empty set fails to satisfy that property of **additive identity** (Definition 1.19). The property stipulates that an element exists, but the empty set has no elements.

5 Replacing the additive inverse condition

Problem statement

Show that in the definition of a vector space (1.19), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

Solution

We can replace the **additive inverse** condition for vector spaces (Definition 1.19) with the condition that

$$0v = 0 \text{ for all } v \in V,$$

the **additive inverse** condition for fields (Theorem 1.3), and the **distributive properties** condition for vector spaces (Definition 1.19). Via the **additive inverse** condition for fields (Theorem 1.3), we can state $1 + (-1) = 0$. Via the condition that

$$0v = 0 \text{ for all } v \in V,$$

we can substitute 0 with $1 + (-1)$ to get

$$(1 + (-1))v = 0 \text{ for all } v \in V.$$

Via the the **distributive properties** condition for vector spaces (Definition 1.19), we now have

$$(v + (-1)v = 0 \text{ for all } v \in V.$$

This result is identical to the **additive inverse** condition. Hence, we have deduced the **additive inverse** condition from our condition that

$$0v = 0 \text{ for all } v \in V.$$

6 Is $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space?

Problem statement

Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Is $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution

No, $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ is not a vector space over $\mathbf{R}$ because it does not satisfy the **distributive properties** (Definition 1.19). Consider the expression $(2-1)\infty$. We can write

$$(2-1)\infty = 2\infty + (-1)\infty = \infty + (-\infty) = 0$$

and

$$(2-1)\infty = (1)\infty = \infty.$$

Thus $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ is not a vector space over $\mathbf{R}$.