Linear Algebra Done Right
Solutions to Exercises 7.A

1 Adjoint of the forward shift operator

Problem statement

Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by

$$T(z_1, \dots, z_n) = (0, z_1, \dots, z_{n-1}).$$

Find a formula for $T^*(z_1, \dots, z_n)$.

Solution

Following Examples 7.3 and 7.4, we can write

$$\begin{aligned}
\langle (x_1, \dots, x_n), T^*(z_1, \dots, z_n) \rangle &= \langle T(x_1, \dots, x_n), (z_1, \dots, z_n) \rangle \\
&= \langle (0, x_1, \dots, x_{n-1}), (z_1, \dots, z_n) \rangle \\
&= 0(z_1) + x_1(z_2) + \cdots + x_{n-1}(z_n) \\
&= x_1(z_2) + \cdots + x_{n-1}(z_n) + 0(x_n z_1) \\
&= \langle (x_1, \dots, x_n), (z_2, \dots, z_n, 0) \rangle.
\end{aligned}$$

Thus,

$$T^*(z_1, \dots, z_n) = (z_2, \dots, z_n, 0)$$

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2 Complex conjugates of eigenvalues for adjoints

Problem statement

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. Prove that λ is an eigenvalue of T if and only if $\bar{\lambda}$ is an eigenvalue of T^*.

Solution

First, let's write

$$\operatorname{null}(T^* - \bar{\lambda} I) = \operatorname{null}(T - \lambda I)^* = (\operatorname{range}(T - \lambda I))^{\perp}$$

where the last equality comes from Theorem 7.7(a) ('Null space and range of T^*'). Thus it follows that

$$\dim \operatorname{null}(T^* - \bar{\lambda} I) = \dim(\operatorname{range}(T - \lambda I))^{\perp}.$$

To understand $\dim(\operatorname{range}(T - \lambda I))^{\perp}$, we can use the Fundamental Theorem of Linear Maps to write

$$\dim \operatorname{range}(T - \lambda I) = \dim V - \dim \operatorname{null}(T - \lambda I)$$

and it follows from Theorem 6.50 ('Dimension of the orthogonal complement') that

$$\begin{aligned} \dim(\operatorname{range}(T - \lambda I))^{\perp} &= \dim V - (\dim V - \dim \operatorname{null}(T - \lambda I)) \\ &= \dim \operatorname{null}(T - \lambda I). \end{aligned}$$

Hence, we have

$$\dim \operatorname{null}(T^* - \bar{\lambda} I) = \dim \operatorname{null}(T - \lambda I). \tag{1}$$

First Direction

Suppose λ is an eigenvalue of T. Following from (1), this implies that

$$\dim \operatorname{null}(T^* - \bar{\lambda} I) = \dim \operatorname{null}(T - \lambda I) > 0$$

and $\bar{\lambda}$ is an eigenvalue of T^*.

Second Direction

Suppose $\bar{\lambda}$ is an eigenvalue of T^*. Following from (1), this implies that

$$\dim \operatorname{null}(T - \lambda I) = \dim \operatorname{null}(T^* - \bar{\lambda} I) > 0$$

and λ is an eigenvalue of T.

3 U invariant under T iff $U^\perp$ invariant under T^*

Problem statement

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under T^*.

Solution

First Direction

Suppose U is invariant under T. It follows that for all $v \in U$ and $w \in U^\perp$, we can write $\langle Tv, w\rangle = 0$ since $Tv \in U$. Given the definition of the adjoint, we can write $0 = \langle Tv, w\rangle = \langle v, T^*w\rangle$ for all $v \in U$ and $w \in U^\perp$. Hence this implies $U^\perp$ is invariant under T^*.

Second Direction

Suppose $U^\perp$ is invariant under T^*. It follows that for all $v \in U$ and $w \in U^\perp$, we can write $\langle T^*w, v\rangle = 0$ since $T^*w \in U^\perp$. Given the definition of the adjoint, we can write $0 = \langle T^*w, v\rangle = \langle w, Tv\rangle$ for all $v \in U$ and $w \in U^\perp$. Hence this implies U is invariant under T.

4 T is injective/surjective iff T^* is surjective/injective

Problem statement

Suppose $T \in \mathcal{L}(V, W)$. Prove that

(a) T is injective if and only if T^* is surjective;

(b) T is surjective if and only if T^* is injective

Note: For Chapter 7, we can assume V and W are finite-dimensional.

Solution for (a)

First Direction

Suppose T is injective. Then $\operatorname{null} T = \{0\}$. Via Theorem 7.7(c) ('Null space and range of T^*'),

$$\{0\} = \operatorname{null} T = (\operatorname{range} T^*)^\perp.$$

Following Theorem 6.51 ('The orthogonal complement of the orthogonal complement') and Theorem 6.46(b) ('Basic properties of orthogonal complement'),

$$\operatorname{range} T^* = ((\operatorname{range} T^*)^\perp)^\perp = \{0\}^\perp = V.$$

Hence, given $T^* \in \mathcal{L}(W, V)$, it follows that T^* is surjective.

Second Direction

Suppose T^* is surjective. Then $\operatorname{range} T^* = V$. Via Theorem 7.7(b) ('Null space and range of T^*'),

$$V = \operatorname{range} T^* = (\operatorname{null} T)^\perp.$$

Following Theorem 6.51 ('The orthogonal complement of the orthogonal complement') and Theorem 6.46(c) ('Basic properties of orthogonal complement'),

$$\operatorname{null} T = ((\operatorname{null} T)^\perp)^\perp = V^\perp = \{0\}.$$

Hence, it follows that T is injective.

Solution for (b)

Solution for (b) follows a similar pattern as **solution for (a)**, except for changing injective to surjective and surjective to injective, and reasoning over W instead of V.

5 $\dim \operatorname{range} T^* = \dim \operatorname{range} T$

Problem statement

Prove that

$$\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$$

and

$$\dim \operatorname{range} T^* = \dim \operatorname{range} T$$

for every $T \in \mathcal{L}(V, W)$.

Solution for (a)

Let's first prove $\dim \operatorname{range} T^* = \dim \operatorname{range} T$ and the other proof will easily follow from the Fundamental Theorem of Linear Maps.

Via Theorem 7.7(d) ('Null space and range of T^*'), we can write

$$\dim \operatorname{range} T = \dim(\operatorname{null} T^*)^\perp$$

and via Theorem 6.50 ('Dimension of the orthogonal complement'), we can expand on our previous statement to write

$$\dim \operatorname{range} T = \dim W - \dim \operatorname{null} T^*. \tag{2}$$

Via the Fundamental Theorem of Linear Maps, we have

$$\dim W = \dim \operatorname{null} T^* + \dim \operatorname{range} T^*$$

and thus we can expand on (2) to write

$$\dim \operatorname{range} T = \dim W - \dim W + \dim \operatorname{range} T^* = \dim \operatorname{range} T^*.$$

Wonderful. Now via a simple application of the Fundamental Theorem of Linear Maps, we have

$$\dim V - \dim \operatorname{null} T = \dim W - \dim \operatorname{null} T^*$$

which is only a couple of rearrangements away from

$$\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V.$$

7 ST is self-adjoint iff $ST = TS$

Problem statement

Suppose $S, T \in \mathcal{L}(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.

Solution

First Direction

Suppose ST is self-adjoint. Following Theorem 7.6(e) ('Properties of the adjoint'), we have $ST = (ST)^* = T^*S^*$. However, since S and T are self-adjoint, it follows that $ST = S^*T^*$. Now watch this drive

$$ST = (ST)^* = (S^*T^*)^* = (T^*)^*(S^*)^* = TS$$

Second Direction

Suppose $ST = TS$ (remember that S and T are self-adjoint). For all v, $w \in V$, we can write

$$\langle STv, w\rangle = \langle v, T^*S^*w\rangle = \langle v, TSw\rangle = \langle v, STw\rangle$$

where the second equality comes from $S = S^*$ and $T = T^*$, and the third equality comes from $ST = TS$.

8 Over R, self-adjoint operators make subspace

Problem statement

Suppose V is a real inner product space. Show that the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$.

Solution

Let's call $\mathcal{A}(V)$ the set of self-adjoint operators on V. To show that it's a subspace, we need to show it contains the **additive identity**, is **closed under addition**, and is **closed under scalar multiplication**.

Additive Identity

Suppose $T = 0$. It follows that for all v, $w \in V$ we have $\langle Tv, w\rangle = 0 = \langle v, T^* w\rangle$, implying that $T^* = 0$. Hence, 0 is self-adjoint and $0 \in \mathcal{A}(V)$.

Closed under addition

Suppose S, T are self-adjoint (S, $T \in \mathcal{A}(V)$). For all v, $w \in V$, we can write

$$\langle (S+T)(v), w\rangle = \langle Sv, w\rangle + \langle Tv, w\rangle = \langle v, Sw\rangle + \langle v, Tw\rangle = \langle v, (S+T)(w)\rangle$$

Hence, $S + T$ is self-adjoint and $S + T \in \mathcal{A}(V)$.

Closed under scalar multiplication

Suppose T is self-adjoint and $\lambda \in \mathbf{R}$ (since V is a real inner product space). It follows that for all v, $w \in V$, we can write

$$\langle (\lambda T)(v), w\rangle = \lambda\langle Tv, w\rangle = \lambda\langle v, Tw\rangle = \langle v, (\bar{\lambda}\, T)(w)\rangle$$

Given $\lambda = \bar{\lambda}$, we have

$$\langle (\lambda T)(v), w\rangle = \langle v, (\bar{\lambda}\, T)(w)\rangle = \langle v, (\lambda\, T)(w)\rangle$$

and λT is self-adjoint and $\lambda T \in \mathcal{A}(V)$.

9 Over C, self-adjoint operators isn't subspace

Problem statement

Suppose V is a complex inner product space with $V \neq \{0\}$. Show that the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

Solution

Refer to our answer for the previous exercise, exercise 7.A(8). The set of self-adjoint operators on V was a subspace because for $\lambda \in \mathbf{R}$ we could write $\lambda = \bar{\lambda}$. Sadly, we cannot write such things for complex innder product spaces. Therefore the set of self-adjoints operators is not closed under scalar multiplication.