Linear Algebra Done Right
Solutions to Exercises 9.A

1 Prove $V_{\mathbf{C}}$ is a complex vector space

Problem statement

Prove 9.3.

Solution

All we need to do is use Definition 9.3 ('Complexification of V, $V_{\mathbf{C}}$') to show that $V_{\mathbf{C}}$ satisfies the condition of a vector space in Definition 1.19 ('vector space').

commutativity

For $u_1, v_1, u_2, v_2 \in V$, we can write

$$\begin{aligned}(u_1 + iv_1) + (u_2 + iv_2) &= (u_1 + u_2) + i(v_1 + v_2) \\ &= (u_2 + u_1) + i(v_2 + v_1) = (u_2 + iv_2) + (u_1 + iv_1)\end{aligned}$$

where the first and third equalities come from the definition of addition on $V_{\mathbf{C}}$, and the second equality comes from **commutativity** on V.

associativity

For $u_1, v_1, u_2, v_2, u_3, v_3 \in V$, we can show additive **associativity** by writing

$$\begin{aligned}((u_1 + iv_1) + (u_2 + iv_2)) + (u_3 + iv_3) &= ((u_1 + u_2) + i(v_1 + v_2)) + (u_3 + iv_3) \\ &= ((u_1 + u_2) + u_3) + i((v_1 + v_2) + v_3) \\ &= (u_1 + (u_2 + u_3)) + i(v_1 + (v_2 + v_3)) \\ &= (u_1 + iv_1) + ((u_2 + u_3) + i(v_2 + v_3)) \\ &= (u_1 + iv_1) + ((u_2 + iv_2) + (u_3 + iv_3))\end{aligned}$$

where the first, second, fourth, and fifth equalities come from the definition of addition on $V_{\mathbf{C}}$, and the third equality comes from **associativity** on V.

For $a_1, b_1, a_2, b_2 \in \mathbf{R}$ and $u, v \in V$, we can show multiplicative **associativity** by writing

$$\begin{aligned}
((a_1 + b_1 i)(a_2 + b_2 i))(u + iv) &= ((a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2)i)(u + iv) \\
&= ((a_1 a_2 - b_1 b_2)u - (a_1 b_2 + b_1 a_2)v) \\
&\quad + i((a_1 a_2 - b_1 b_2)v + (a_1 b_2 + b_1 a_2)u) \\
&= (a_1 a_2 u - b_1 b_2 u - a_1 b_2 v - b_1 a_2 v) \\
&\quad + i(a_1 a_2 v - b_1 b_2 v + a_1 b_2 u + b_1 a_2 u) \\
&= (a_1(a_2 u - b_2 v) - b_1(a_2 v + b_2 u)) \\
&\quad + i(a_1(a_2 v + b_2 u) + b_1(a_2 u - b_2 v)) \\
&= (a_1 + b_1 i)((a_2 u - b_2 v) + i(a_2 v + b_2 u)) \\
&= (a_1 + b_1 i)((a_2 + b_2 i)(u + iv))
\end{aligned}$$

where the first, second, fifth, and sixth equalities come from the definition of complex scalar multiplication on $V_{\mathbf{C}}$; and the third and fourth equalities come from the **distributive properties** on V.

additive identity

The **additive identity** is $0 + i0$, which we can verify by writing

$$(u + iv) + (0 + i0) = (u + 0) + i(v + 0) = u + iv$$

for all $u, v \in V$.

additive inverse

For $u, v \in V$, the **additive inverse** of $u + iv$ is simply $-u - iv$, which we can verify by writing

$$(u + iv) + (-u - iv) = (u - u) + i(v - v) = 0 + i0.$$

multiplicative identity

The **multiplicative identity** is simply $1 + 0i$, which we can verify by writing

$$(1 + 0i)(u + iv) = (1u - 0(v)) + i(1v + 0(u)) = u + iv$$

for all $u, v \in V$.

distributive properties

For $u_1, v_1, u_2, v_2 \in V$ and $a, b \in \mathbf{R}$, we can write

$$\begin{aligned}(a+bi)((u_1+iv_1)+(u_2+iv_2)) &= (a+bi)((u_1+u_2)+i(v_1+v_2))\\ &= (a(u_1+u_2)-b(v_1+v_2))\\ &\quad\ + i(a(v_1+v_2)+b(u_1+u_2))\\ &= (au_1+au_2-bv_1-bv_2))\\ &\quad\ + i(av_1+av_2+bu_1+bu_2))\\ &= (au_1-bv_1)+i(av_1+bu_1)\\ &\quad\ + (au_2-bv_2)+i(av_2+bu_2)\\ &= (a+bi)(u_1+iv_1)+(a+bi)(u_2+iv_2).\end{aligned}$$

For $u, v \in V$ and $a_1, b_1, a_2, b_2 \in \mathbf{R}$, we can write

$$\begin{aligned}((a_1+b_1i)+(a_2+b_2i))(u+iv) &= ((a_1+a_2)+(b_1+b_2)i)(u+iv)\\ &= ((a_1+a_2)u-(b_1+b_2)v)\\ &\quad\ + i((a_1+a_2)v+(b_1+b_2)u)\\ &= (a_1u+a_2u-b_1v-b_2v)\\ &\quad\ + i(a_1v+a_2v+b_1u+b_2u)\\ &= (a_1u-b_1v)+i(a_1v+b_1u)\\ &\quad\ + (a_2u-b_2v)+i(a_2v+b_2u)\\ &= (a_1+b_1i)(u+iv)+(a_2+b_2i)(u+iv).\end{aligned}$$

2 If $T \in \mathcal{L}(V)$, then $T_{\mathbf{C}} \in \mathcal{L}(V_{\mathbf{C}})$

Problem statement

Verify that if V is a real vector space and if $T \in \mathcal{L}(V)$, then $T_{\mathbf{C}} \in \mathcal{L}(V_{\mathbf{C}})$.

Solution

To verify that $T_{\mathbf{C}} \in \mathcal{L}(V_{\mathbf{C}})$, we must demonstrate the properties of **additivity** and **homogeneity**.

additivity

For $u_1, v_1, u_2, v_2 \in V$, we can write

$$
\begin{aligned}
T_{\mathbf{C}}((u_1 + iv_1) + (u_2 + iv_2)) &= T_{\mathbf{C}}((u_1 + u_2) + i(v_1 + v_2)) \\
&= T(u_1 + u_2) + iT(v_1 + v_2) \\
&= Tu_1 + iTv_1 + Tu_2 + iTv_2 \\
&= T_{\mathbf{C}}(u_1 + iv_1) + T_{\mathbf{C}}(u_2 + iv_2)
\end{aligned}
$$

where the first and fourth equalities come from the definition of $T_{\mathbf{C}}$ (Definition 9.5).

homogeneity

For $a, b \in \mathbf{R}$ and $u, v \in V$, we can write

$$
\begin{aligned}
T_{\mathbf{C}}((a + bi)(u + iv)) &= T_{\mathbf{C}}((au - bv) + i(av + bu)) \\
&= T(au - bv) + iT(av + bu) \\
&= aTu - bTv + aiTv + biTu \\
&= (a + bi)(Tu + iTv) \\
&= (a + bi)(T_{\mathbf{C}}(u + iv))
\end{aligned}
$$

where the first and fourth equality come from the definition of complex scalar multiplication on $V_{\mathbf{C}}$ (Definition 9.2), and the second and fifth equalities come from the definition of $T_{\mathbf{C}}$ (Definition 9.5).

3 Linear independence in V and $V_{\mathbf{C}}$

Problem statement

Suppose V is a real vector space and $v_1, \ldots, v_m \in V$. Prove that $v_1, \ldots, v_m \in V$ is linearly independent in $V_{\mathbf{C}}$ if and only if $v_1, \ldots, v_m \in V$ is linearly independent in V.

Solution

First Direction

Suppose $v_1, \ldots, v_m \in V$ is linearly independent in $V_{\mathbf{C}}$. This implies that for $\lambda_1, \ldots, \lambda_m \in \mathbf{C}$ and

$$\lambda_1 v_1 + \cdots + \lambda_m v_m = 0,$$

then $\lambda_1 = \cdots = \lambda_m = 0 + 0i$. Since any scalar $\lambda \in \mathbf{C}$ can be constructed from $a, b \in \mathbf{R}$ (Definition 1.1), it follows that for $a_1, \ldots, a_m \in \mathbf{R}$ and

$$a_1 v_1 + \cdots + a_m v_m = 0,$$

then $a_1 = \cdots = a_m = 0$. Hence, $v_1, \ldots, v_m$ is linearly independent in V.

Second Direction

Suppose $v_1, \ldots, v_m \in V$ is linearly independent in V. Suppose $\lambda_1, \ldots, \lambda_m \in \mathbf{C}$ and

$$\lambda_1 v_1 + \cdots + \lambda_m v_m = 0.$$

Then the equation above implies that

$$(\operatorname{Re} \lambda_1) v_1 + \cdots + (\operatorname{Re} \lambda_m) v_m = 0 \quad \text{and} \quad (\operatorname{Im} \lambda_1) v_1 + \cdots + (\operatorname{Im} \lambda_m) v_m = 0.$$

Since $\operatorname{Re} \lambda_j, \operatorname{Im} \lambda_j \in \mathbf{R}$ (via Definition 4.2) and $v_1, \ldots, v_m$ is linearly independent in V, it follows that $\operatorname{Re} \lambda_1 = \cdots = \operatorname{Re} \lambda_m = 0$ and $\operatorname{Im} \lambda_1 = \cdots = \operatorname{Im} \lambda_m = 0$. Thus we have $\lambda_1 = \cdots = \lambda_m = 0$. Hence, $v_1, \ldots, v_m$ is linearly independent in $V_{\mathbf{C}}$.

4 Spanning list in V and $V_{\mathbf{C}}$

Problem statement

Suppose V is a real vector space and $v_1, \ldots, v_m \in V$. Prove that $v_1, \ldots, v_m \in V$ spans $V_{\mathbf{C}}$ if and only if $v_1, \ldots, v_m \in V$ spans V.

Solution

First Direction

Suppose $v_1, \ldots, v_m \in V$ spans $V_{\mathbf{C}}$. Via Theorem 2.31 ('Spanning list contains a basis'), the list $v_1, \ldots, v_m$ can be reduced to a basis of $V_{\mathbf{C}}$. Suppose that this reduced list is of length n, implying that $\dim V_{\mathbf{C}} = n$ and, following from Theorem 9.4(b) ('Basis of V is a basis of $V_{\mathbf{C}}$'), $\dim V = n$.

This basis of $V_{\mathbf{C}}$ is linearly independent and hence, following from Exercise 9.A(3), is linearly independent in V. Since it is a linearly independent list of length $n = \dim V$, Theorem 2.39 ('Linearly independent list of the right length is a basis') implies that this list is a basis of V.

Putting it all together, we have proven that $v_1, \ldots, v_m$ contains a list that is a basis of V. Therefore, $v_1, \ldots, v_m$ spans V.

Second Direction

Suppose $v_1, \ldots, v_m \in V$ spans V. Theorem 2.31 implies that $v_1, \ldots, v_m$ can be reduced to a basis of V. Following Theorem 9.4(a), this basis of V is a basis of $V_{\mathbf{C}}$. Hence, $v_1, \ldots, v_m$ contains a list that is a basis of $V_{\mathbf{C}}$. Therefore, $v_1, \ldots, v_m$ spans $V_{\mathbf{C}}$.

5 Complexification is a linear map

Problem statement

Suppose that V is a real vector space and $S, T \in \mathcal{L}(V)$. Show that $(S+T)_{\mathbf{C}} = S_{\mathbf{C}} + T_{\mathbf{C}}$ and that $(\lambda T)_{\mathbf{C}} = \lambda T_{\mathbf{C}}$ for every $\lambda \in \mathbf{R}$.

Solution

additivity

To show that $(S+T)_{\mathbf{C}} = S_{\mathbf{C}} + T_{\mathbf{C}}$, we can write

$$\begin{aligned}
(S+T)_{\mathbf{C}}(u+iv) &= (S+T)(u) + i(S+T)(v) \\
&= Su + iSv + Tu + iTv \\
&= S_{\mathbf{C}}(u+iv) + T_{\mathbf{C}}(u+iv) \\
&= (S_{\mathbf{C}} + T_{\mathbf{C}})(u+iv)
\end{aligned}$$

for $v, u \in \mathbf{R}$.

homogeneity

To show that $(\lambda T)_{\mathbf{C}} = \lambda T_{\mathbf{C}}$ for every $\lambda \in \mathbf{R}$, we can write

$$\begin{aligned}
(\lambda T)_{\mathbf{C}}(u+iv) &= (\lambda T)(u) + i(\lambda T)(v) \\
&= \lambda(Tu) + \lambda(iTv) \\
&= \lambda(Tu + iTv) \\
&= \lambda T_{\mathbf{C}}(u+iv)
\end{aligned}$$

for $v, u \in \mathbf{R}$.

Notes

This result allows us to prove Lemma 6.0.1 (‘Complexification is isomorphism from $\mathcal{L}(V)$ to $\mathcal{L}(V_{\mathbf{C}})$’).

6 Prove $T_{\mathbf{C}}$ is invertible iff T is invertible

Problem statement

Suppose V is a real vector space and $T \in \mathcal{L}(V)$. Prove that $T_{\mathbf{C}}$ is invertible if and only if T is invertible.

Solution

6.0.1 Lemma: Complexification is isomorphism from $\mathcal{L}(V)$ to $\mathcal{L}(V_{\mathbf{C}})$

Via Theorem 9.4(b) ('Basis of V is a basis of $V_{\mathbf{C}}$'), we know that $\dim V = \dim V_{\mathbf{C}}$. Hence, Theorem 3.61 ('$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$') implies

$$\dim \mathcal{L}(V) = (\dim V)(\dim V) = (\dim V_{\mathbf{C}})(\dim V_{\mathbf{C}}) = \dim \mathcal{L}(V_{\mathbf{C}}).$$

Thus, Theorem 3.59 ('Dimension shows whether vector spaces are isomorphic') tells us that $\mathcal{L}(V)$ and $\mathcal{L}(V_{\mathbf{C}})$ are isomorphic since they have the same dimension.

It is now clear to see that the **complexification** of T is an isomorphism from $\mathcal{L}(V)$ to $\mathcal{L}(V_{\mathbf{C}})$. This follows from the observation that the **complexification** of operators is injective, $T_{\mathbf{C}} = 0$ iff $T = 0$, and surjective, the **complexification** of a basis of operators in $\mathcal{L}(V)$ yields a linearly independent list of operators in $\mathcal{L}(V_{\mathbf{C}})$ of length $\dim \mathcal{L}(V_{\mathbf{C}})$[1]. Therefore, for every operator $S \in \mathcal{L}(V_{\mathbf{C}})$, there exists a unique operator $R \in \mathcal{L}(V)$ such that $R_{\mathbf{C}} = S$.

First Direction

Suppose $T_{\mathbf{C}}$ is invertible. Thus there exists a unique inverse $S \in \mathcal{L}(V_{\mathbf{C}})$ such that $ST_{\mathbf{C}} = I$ and $T_{\mathbf{C}}S = I$. From Lemma 6.0.1, there exists $R \in \mathcal{L}(V)$ such that $R_{\mathbf{C}} = S$. For $u, v \in V$, we can write

$$\begin{aligned} u + iv = I(u + iv) &= ST_{\mathbf{C}}(u + iv) \\ &= R_{\mathbf{C}}(Tu + iTv) = RTu + iRTv \end{aligned}$$

implying that $RT = I$ for all vectors in v. Since R, T are operators, Exercise 3.D(10) ('$ST = I$ iff $TS = I$') allows us to state that $TR = I$. Hence, T has an inverse and T is invertible.

Second Direction

Suppose T is invertible. Thus there exists a unique inverse $R \in V$ such that $RT = I$ and $TR = I$. We will show that $R_{\mathbf{C}}$ is the inverse of $T_{\mathbf{C}}$.

For $u, v \in V$, we can write

$$R_{\mathbf{C}}T_{\mathbf{C}}(u + iv) = R_{\mathbf{C}}(Tu + iTv) = RTu + iRTv = u + iv$$

and

$$T_{\mathbf{C}}R_{\mathbf{C}}(u + iv) = T_{\mathbf{C}}(Ru + iRv) = TRu + iTRv = u + iv.$$

Hence, $T_{\mathbf{C}}R_{\mathbf{C}} = I$ and $R_{\mathbf{C}}T_{\mathbf{C}} = I$. Therefore, $T_{\mathbf{C}}$ is invertible.

[1] This could use a more rigorous proof, but that's a labor for another time

7 Prove $N_{\mathbf{C}}$ is nilpotent iff N is nilpotent

Problem statement

Suppose V is a real vector space and $N \in \mathcal{L}(V)$. Prove that $N_{\mathbf{C}}$ is nilpotent if and only if N is nilpotent.

Solution

First Direction

Suppose $N_{\mathbf{C}}$ is nilpotent. This implies that $N_{\mathbf{C}}^{\dim V} = 0$. Via Definition 9.5 ('complexification of T'), we can express $N_{\mathbf{C}}^{\dim V}$ as

$$N_{\mathbf{C}}^{\dim V}(u + iv) = N^{\dim V}u + iN^{\dim V}v$$

for all $u, v \in V$. Since the left side of our expression above equals zero, it follows that

$$N^{\dim V}u + iN^{\dim V}v = 0 + i0$$

for all $u, v \in V$, implying that N is nilpotent.

Second Direction

Suppose N is nilpotent. This implies that $N^{\dim V} = 0$. Via Definition 9.5, we can express $N_{\mathbf{C}}^{\dim V}$ as

$$N_{\mathbf{C}}^{\dim V}(u + iv) = N^{\dim V}u + iN^{\dim V}v$$

for all $u, v \in V$. Since the right side of our expression above equals zero, it follows that

$$N_{\mathbf{C}}^{\dim V}(u + iv) = 0 + i0$$

for all $u, v \in V$, implying that $N_{\mathbf{C}}$ is nilpotent.

8 Nonreal eigenvalues of $T_{\mathbf{C}}$

Problem statement

Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and $5, 7$ are eigenvalues of T. Prove that $T_{\mathbf{C}}$ has no nonreal eigenvalues.

Solution

Theorem 9.16 states that nonreal eigenvalues of $T_{\mathbf{C}}$ come in pairs. Hence, if $T_{\mathbf{C}}$ had a nonreal eigenvalue λ, then it would also have the eigenvalue of $\bar{\lambda}$ (where $\lambda \neq \bar{\lambda}$). However, this would imply that $T_{\mathbf{C}}$ has 4 distinct eigenvalues, more than $\dim \mathbf{C}^3 = 3$, which is a contradiction of Theorem 5.13 ('Number of eigenvalues'). Therefore, if $T \in \mathcal{L}(\mathbf{R}^3)$ and $5, 7$ are eigenvalues of T, then $T_{\mathbf{C}}$ has no nonreal eigenvalues.

9 $T \in \mathcal{L}(\mathbf{R}^7)$ such that $T^2 + T + I$ is nilpotent

Problem statement

Prove there does not exist an operator $T \in \mathcal{L}(\mathbf{R}^7)$ such that $T^2 + T + I$ is nilpotent.

Solution

Suppose there exists some operator $T \in \mathcal{L}(\mathbf{R}^7)$ such that T^2+T+I is nilpotent. Via Theorem 8.18 ('Nilpotent operator raised to dimension of domain is 0'), it follows that $(T^2 + T + I)^7 = 0$ and for the polynomial $q(z) = (z^2 + z + 1)^7$ that $q(T) = 0$. Via Theorem 8.46 ('$q(T) = 0$ implies q is a multiple of the minimal polynomial'), we can infer that q is a polynomial multiple of the minimal polynomial of T. Via Theorem 9.19 ('Operator on odd-dimensional vector space has eigenvalue'), it follows that T has a real eigenvalue, and via Theorem 8.49 ('Eigenvalues are the zeros of the minimal polynomial'), that eigenvalue is a zero of the minimal polynomial of T.

However, we can see that the polynomial z^2+z+1 has no real zeros[2]. Thus, the polynomial $q(z) = (z^2+z+1)^7$ has no real zeros and cannot be a polynomial multiple of the minimal polynomial of T since the minimal polynomial of T is guaranteed to have a real zero. Therefore, we have a contradiction and it follows that no such operator $T \in \mathcal{L}(\mathbf{R}^7)$ exists such that $T^2 + T + I$ is nilpotent.

[2] This follows from Theorem 4.16 ('Factorization of a quadratic polynomial') and the simple observation that $1^2 \not\geq 4(1)$

10 $T \in \mathcal{L}(\mathbf{C}^7)$ such that $T^2 + T + I$ is nilpotent

Problem statement

Give an example of an operator $T \in \mathcal{L}(\mathbf{C}^7)$ such that $T^2 + T + I$ is nilpotent.

Solution

Let the operator $T \in \mathcal{L}(\mathbf{C}^7)$ be defined by

$$Tv = \frac{-1 + \sqrt{3}i}{2}v.$$

For every $v \in \mathbf{C}^7$, we can write

$$\begin{aligned}
(T^2 + T + I)v &= T^2v + Tv + Iv \\
&= (\frac{-1 + \sqrt{3}i}{2})^2 v + \frac{-1 + \sqrt{3}i}{2}v + v \\
&= \frac{-2 - 2\sqrt{3}i}{4}v + \frac{-2 + 2\sqrt{3}i}{4}v + v \\
&= -v + v = 0
\end{aligned}$$

and hence, $T^2 + T + I = 0$. The 0 operator is clearly nilpotent, so it follows that $T^2 + T + I$ is nilpotent.