Linear Algebra Done Right
Solutions to Exercises 3.E

1 The graph of T

Problem statement

Suppose T is a function from V to W. The **graph** of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W : v \in V\}.$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

Solution

First Direction

Suppose T is a linear map. To show that the graph of T is a subspace of $V \times W$, we need to show the properties of **additive identity**, **closed under addition**, and **closed under scalar multiplication**.

additive identity: Clearly $0 \in V$. Thus, via Theorem 3.11 ('Linear maps take 0 to 0'), it follows that

$$(0, 0) = (0, T(0)) \in \text{graph of } T$$

.

closed under addition: Suppose $v, u \in V$. Thus $(v, Tv), (u, Tu) \in$ graph of T. Via the definition of addition on $V \times W$ (Definition 3.71), we have

$$(v, Tv) + (u, Tu) = (v + u, Tv + Tu) = (v + u, T(v + u)) \in \text{graph of } T$$

since $v + u \in V$. Note that the second equality follows from the **additivity** of T.

closed under scalar multiplication: Suppose $v \in V$ and $\lambda \in \mathbf{F}$. Via the definition of scalar multiplication on $V \times W$ (Definition 3.71), we have

$$\lambda(v, Tv) = (\lambda v, \lambda Tv) = (\lambda v, T(\lambda v)) \in \text{graph of } T$$

since $\lambda v \in V$. Note that the second equality follows from the **homogeneity** of T.

Second Direction

Suppose the graph of T is a subspace of $V \times W$. To show that T is a linear map, we need to show the properties of **additivity** and **homogeneity**.

additivity: Suppose $v, u \in V$. Given the graph of T is **closed under addition**, it follows that $(v, Tv) + (u, Tu) = (v + u, Tv + Tu) \in$ graph of T. Following from $v + u \in V$, we also have $(v + u, T(v + u)) \in$ graph of T. Hence, it necessarily follows that

$$(v + u, Tv + Tu) = (v + u, T(v + u))$$

and $Tv + Tu = T(v + u)$.

homogeneity: Suppose $v \in V$ and $\lambda \in \mathbf{F}$. Given the graph of T is **closed under scalar multiplication**, it follows that
$\lambda(v, Tv) = (v, \lambda Tv) \in$ graph of T. Following from $\lambda v \in V$, we also have $(\lambda v, T(\lambda v)) \in$ graph of T. Hence, it necessarily follows that

$$(v, \lambda Tv) = (\lambda v, T(\lambda v))$$

and $\lambda Tv = T(\lambda v)$.

Notes

It's kinda neat how the **graph** of T allows us to connect the properties of subspaces with the properties of linear maps.

2 The dimensionality of $V_1 \times \cdots \times V_m$ and V_j

Problem statement

Suppose $V_1, \ldots, V_m$ are vector spaces such that $V_1 \times \cdots \times V_m$ is finite-dimensional. Prove that V_j is finite-dimensional for each $j = 1, \ldots, m$.

Solution

Let $u_1, \ldots, u_n$ be a basis of $V_1 \times \cdots \times V_m$. For each vector u_k, it follows from the definition of the product of vector spaces (Definition 3.71) that we can express u_k as

$$u_k = (v_1, \ldots, v_m)$$

where each $v_j \in V_j$. Let $u_{k,j}$ be the a vector in the j-th slot of the u_k basis vector. Fix j and construct the list $(u_{1,j}, \ldots, u_{n,j})$. If this list does not span V_j, then $u_1, \ldots, u_n$ is not a basis of $V_1 \times \cdots \times V_m$. Hence, the list $(u_{1,j}, \ldots, u_{n,j})$ spans V_j and V_j is finite-dimensional.

3 Isomorphic products don't imply direct sums

Problem statement

Give an example of a vector space V and subspaces U_1, U_2 of V such that $U_1 \times U_2$ is isomorphic to $U_1 + U_2$ but $U_1 + U_2$ is not a direct sum.

Solution

Let V be the vector space $\mathbf{F}^\infty$, U_1 be the subspace defined by

$$U_1 = \{(x_1, x_2, 0, 0, \ldots) : x_1, x_2 \in \mathbf{F}\},$$

and U_2 be the subspace defined by

$$U_2 = \{(0, y_1, y_2, \ldots) : y_j \in \mathbf{F} \text{ for } j = 1, 2, \ldots\}.$$

Clearly $U_1 + U_2$ is not a direct sum, but to show that $U_1 \times U_2$ is isomorphic to $U_1 + U_2$ we need to find an isomorphism. Define $T \in \mathcal{L}(U_1 \times U_2, U_1 + U_2)$ by

$$T((x_1, x_2, 0, 0, \ldots), (0, y_1, y_2, \ldots)) = (x_1, x_2, y_1, y_2, \ldots)$$

and define $S \in \mathcal{L}(U_1 + U_2, U_1 \times U_2)$

$$S(x_1, x_2, x_3, x_4, \ldots) = ((x_1, x_2, 0, 0, \ldots), (0, x_3, x_4, \ldots)).$$

It follows that $ST = I$ on $U_1 \times U_2$ and $TS = I$ on $U_1 + U_2$. Hence, T is an isomorphism from $U_1 \times U_2$ onto $U_1 + U_2$.

4 $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$

Problem statement

Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

Solution

Via Theorem 3.61 ('$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$') and Theorem 3.76 ('Dimension of a product is the sum of dimensions'), we can write the dimension of $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ as

$$\begin{aligned} \dim \mathcal{L}(V_1 \times \cdots \times V_m, W) &= (\dim V_1 \times \cdots \times V_m)(\dim W) \\ &= (\dim V_1 + \cdots + \dim V_m)(\dim W) \\ &= (\dim V_1)(\dim W) + \cdots + (\dim V_m)(\dim W) \end{aligned}$$

and the dimension of $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ as

$$\begin{aligned} \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W) &= \dim \mathcal{L}(V_1, W) + \cdots + \dim \mathcal{L}(V_m, W) \\ &= (\dim V_1)(\dim W) + \cdots + (\dim V_m)(\dim W). \end{aligned}$$

Hence, $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ have the same dimension. Thus, via Theorem 3.59 ('Dimension shows whether vector spaces are isomorphic'), they are isomorphic vector spaces.

5 $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$

Problem statement

Suppose $W_1, \ldots, W_m$ are vector spaces. Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are isomorphic vector spaces.

Solution

Via Theorem 3.61 ('$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$') and Theorem 3.76 ('Dimension of a product is the sum of dimensions'), we can write the dimension of $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ as

$$\begin{aligned}
\mathcal{L}(V, W_1 \times \cdots \times W_m) &= (\dim V)(\dim W_1 \times \cdots \times W_m) \\
&= (\dim V)(\dim W_1 + \cdots + \dim W_m) \\
&= (\dim V)(\dim W_1) + \cdots + (\dim V)(\dim W_m)
\end{aligned}$$

and the dimension of $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ as

$$\begin{aligned}
\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m) &= \dim \mathcal{L}(V, W_1) + \cdots + \dim \mathcal{L}(V, W_m) \\
&= (\dim V)(\dim W_1) + \cdots + (\dim V)(\dim W_m).
\end{aligned}$$

Hence, $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ have the same dimension. Thus, via Theorem 3.59 ('Dimension shows whether vector spaces are isomorphic'), they are isomorphic vector spaces.

6 V^n and $\mathcal{L}(\mathbf{F}^n, V)$ are isomorphic

Problem statement

For n a positive integer, define V^n by

$$V^n = \left. V \times \cdots \times V \right\} (n \text{ times}).$$

Prove that V^n and $\mathcal{L}(\mathbf{F}^n, V)$ are isomorphic vector spaces.

Solution

Via Theorem 3.76 ('Dimension of a product is the sum of dimensions'), the dimension of V^n is

$$\begin{aligned} \dim V^n &= \left. \dim V + \cdots + \dim V \right\} (n \text{ times}) \\ &= n(\dim V). \end{aligned}$$

Via Theorem 3.61 ('$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$'), the dimension of $\mathcal{L}(\mathbf{F}^n, V)$ is

$$\dim \mathcal{L}(\mathbf{F}^n, V) = (\dim \mathbf{F}^n)(\dim V) = n(\dim V).$$

Hence, V^n and $\mathcal{L}(\mathbf{F}^n, V)$ have the same dimension. Thus, via Theorem 3.59 ('Dimension shows whether vector spaces are isomorphic'), they are isomorphic vector spaces.

7 If $v + U = x + W$, then $U + W$

Problem statement

Suppose v, x are vectors in V and U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

Solution

Given W is a subspace, it follows that $0 \in W$ and there exists some vector $u \in U$ such that

$$v + u = x + 0.$$

Rearranging terms, we have $v - x = -u$ which implies $v - x \in U$. Via theorem 3.85 ('Two affine subsets parallel to U are equal or disjoint'), it follows that $v - x \in U$ is equivalent to $v + U = x + U$. Hence, we have

$$x + W = v + U = x + U,$$

which implies that $U = W$.

8 A property of affine subsets

Problem statement

Prove that a nonempty subset A of V is an affine subset of V if and only if $\lambda v + (1-\lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbf{F}$.

Solution

First Direction

Suppose A of V is an affine subset. It follows from Definition 3.79 ('$v+U$') and Definition 3.81 ('affine subset, parallel') that we can express A as

$$A = u + U$$

for some vector $u \in V$ and some subspace U of V. Consider two vectors $v, w \in A$. We can represent these vectors as

$$v = u + u_1 \quad \text{and} \quad w = u + u_2$$

for some $u_1, u_2 \in U$. For all $\lambda \in \mathbf{F}$, we can write

$$\begin{aligned} \lambda v + (1-\lambda)w &= \lambda u + \lambda u_1 + u + u_2 - \lambda u - \lambda u_2 \\ &= u + \lambda u_1 + (1-\lambda)u_2 \in u + U = A \end{aligned}$$

where $\lambda u_1 + (1-\lambda)u_2 \in U$ since U is a subspace and is **closed under addition**.

Second Direction

Suppose A is a nonempty subset of V such that $\lambda v + (1-\lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbf{F}$. Given A is nonempty, we can choose a vector $v \in A$ and construct the set

$$U = \{u - v : u \in A\}.$$

If we can show that U is a subspace of V, then it follows that A is an affine subset since $A = v + U$.

additive identity: Clearly $0 \in U$ since $0 = v - v \in U$.

closed under addition: Consider two vectors $u_1, u_2 \in U$. Our construction of U implies there exist $w_1, w_2 \in A$ such that

$$u_1 = w_1 - v \quad \text{and} \quad u_2 = w_2 - v.$$

To show that $u_1 + u_2 \in U$, we need to find a vector $u \in A$ such that $u_1 + u_2 = u - v$. It follows that

$$u_1 + u_2 = w_1 + w_2 - v - v$$

and we need to show $w_1 + w_2 - v \in A$.

We have $\frac{1}{2}(w_1 + w_2) \in A$ since it is of the for $\lambda w_1 + (1 - \lambda)w_2$ where $\lambda = \frac{1}{2}$. Considering the vectors $\frac{1}{2}(w_1 + w_2)$ and v and the scalar $\lambda = 2$, it follows that

$$2(\frac{1}{2}(w_1 + w_2)) + (1 - 2)v = w_1 + w_2 - v \in A$$

and $u_1 + u_2 = w_1 + w_2 - v - v \in U$. Hence, U is **closed under addition**.

closed under scalar multiplication: Suppose $u \in U$ and $\lambda \in \mathbf{F}$. Via our construction of U, there exists some $w \in A$ such that $u = w - v$. To show that $\lambda u \in U$, we can write

$$\lambda u = \lambda w - \lambda v = \lambda w - (\lambda - 1)v - v = \lambda w + (1 - \lambda)v - v \in U$$

where $\lambda w + (1 - \lambda)v \in A$. Hence, U is **closed under scalar multiplication**.

9 Intersection of two affine subsets

Problem statement

Suppose A_1 and A_2 are affine subsets of V. Prove that the intersection $A_1 \cap A_2$ is either an affine subset of V or the empty set.

Solution

Let's tackle each case separately.

Intersection equals the empty set

To prove that the intersection $A_1 \cap A_2$ could be the empty set, we can find an example. Suppose $U = \{(x, 2x) \in \mathbf{R}^2 : x \in \mathbf{R}\}$, $A_1 = (1,2) + U$, and $A_2 = (-1,2) + U$. Geometrically, A_1 and A_2 are parallel lines in $\mathbf{R}^2$. Since these lines are parallel, it follows that $A_1 \cap A_2 = \emptyset$.

Intersection does not equal the empty set

Suppose $A_1 \cap A_2 \neq \emptyset$. Thus there exists a vector $v \in V$ such that $v \in A_1 \cap A_2$. Define the set U by

$$U = \{u - v : u \in A_1 \cap A_2\}$$

In a similar manner as our solution to Exercise 3.E(8), if we can show that U is a subspace of V, then it follows that $A_1 \cap A_2$ is an affine subset.

additive identity: Clearly $0 \in U$ since $0 = v - v \in U$.

closed under addition: Consider two vectors $u_1, u_2 \in U$. Our construction of U implies there exist $w_1, w_2 \in A_1 \cap A_2$ such that

$$u_1 = w_1 - v \quad \text{and} \quad u_2 = w_2 - v.$$

To show that $u_1 + u_2 \in U$, we need to find a vector $u \in A_1 \cap A_2$ such that $u_1 + u_2 = u - v$. It follows that

$$u_1 + u_2 = w_1 + w_2 - v - v$$

and we need to show $w_1 + w_2 - v \in A_1 \cap A_2$. Via our reasoning in our solution to Exercise 3.E(8), it follows that $w_1 + w_2 - v \in A_1$ and $w_1 + w_2 - v \in A_2$ since A_1 and A_2 are affine subsets. Hence, we have $w_1 + w_2 - v \in A_1 \cap A_2$ and $u_1 + u_2 \in U$. Therefore, U is **closed under addition**.

closed under scalar multiplication: Suppose $u \in U$ and $\lambda \in \mathbf{F}$. Via our construction of U, there exists some $w \in A_1 \cap A_2$ such that $u = w - v$. To show that $\lambda u \in U$, we can write

$$\lambda u = \lambda w - \lambda v = \lambda w - (\lambda - 1)v - v = \lambda w + (1 - \lambda)v - v \in U$$

where $\lambda w + (1 - \lambda)v \in A_1 \cap A_2$ via our reasoning in Exercise 3.E(8). Hence, U is **closed under scalar multiplication**.

10 Intersection of any collection of affine subsets

Problem statement

Prove that the intersection of every collection of affine subsets of V is either an affine subset of V or the empty set.

Solution

Suppose $A_1, A_2, \ldots, A_n$ is a collection of affine subsets. By arranging the intersection of the collection as

$$\big(\ldots(A_1 \cap A_2) \cap \ldots\big) \cap A_n$$

we can iteratively apply our result from Exercise 3.E(9) to show that the intersection must be an affine subset of V or the empty set.