Numerically stable arbitrary boost matrix

Keith Pedersen

October 8, 2017

Defining an arbitrary reference frame via $\vec{\beta} = \vec{p}/E$ (using $\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$), the Lorentz boost matrix which takes $[E, \vec{p}]$ to $[m, \vec{0}]$ is the symmetric matrix

$$\Lambda(\vec{\beta}) = \mathbb{1} + \begin{bmatrix} \gamma - 1 & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ \iddots & (\gamma-1)\frac{\beta_1^2}{\beta^2} & (\gamma-1)\frac{\beta_1\beta_2}{\beta^2} & (\gamma-1)\frac{\beta_1\beta_3}{\beta^2} \\ \iddots & \iddots & (\gamma-1)\frac{\beta_2^2}{\beta^2} & (\gamma-1)\frac{\beta_2\beta_3}{\beta^2} \\ \iddots & \iddots & \iddots & (\gamma-1)\frac{\beta_3^2}{\beta^2} \end{bmatrix} . \tag{1}$$

This matrix has three general classes of term. Below, we work out the best way to calculate each, assuming we are in a momentum/mass basis (we are given $\vec{p}$ and m).

0.1 $\gamma\beta_i$

We know that $\gamma = E/m$ and $\vec{\beta} = \vec{p}/E$, so that

$$\gamma\beta_i = \frac{E}{m}\frac{p_i}{E} = \frac{p_i}{m} \tag{2}$$

0.2 $\gamma - 1$

We have

$$\gamma = \frac{\sqrt{p^2 + m^2}}{m} = \sqrt{1 + \frac{p^2}{m^2}} . \tag{3}$$

We can then use the standard trick for a more accurate subtraction from a root term

$$\gamma - 1 = \left(\sqrt{1 + \frac{p^2}{m^2}} - 1\right) \frac{1 + \sqrt{1 + \frac{p^2}{m^2}}}{1 + \sqrt{1 + \frac{p^2}{m^2}}} = \frac{p^2}{m(m + \sqrt{m^2 + p^2})} = \frac{p^2}{m^2 + \sqrt{m^2\,E^2}} . \tag{4}$$

This also gives an expression for γ which removes the systematic, $\mathcal{O}(\epsilon)$ downward bias of Eq. 3

$$\gamma = 1 + \frac{p^2}{m(m + \sqrt{m^2 + p^2})} . \tag{5}$$

0.3 $(\gamma - 1)\frac{\beta_i\beta_j}{\beta^2}$

We have already solved for $\gamma - 1$, so we can write

$$(\gamma-1)\frac{\beta_i\beta_j}{\beta^2} = (\gamma-1)\left(\frac{p_i}{E}\right)\left(\frac{p_j}{E}\right)\left(\frac{E^2}{p^2}\right) = (\gamma-1)\frac{p_i p_j}{p^2} = \frac{p_i p_j}{m(m + \sqrt{m^2 + p^2})} . \tag{6}$$